CS 591, Fall 2014

Assignment 1

Shan Sikdar

(1) Recall from class that for a matrix $A = (a_{jk} \in C^{mxN})$, the operator norm of A from l_p into l_p is defined as: $||A||_{p\to p} := max_{||x||_p=1}||Ax||_p$

(a) $||A||_{1\to 1} = max_{k\in[N]} \sum_j^m |a_{jk}|$
proof.

If A = $\begin{bmatrix} a_{11} & ... & a_{1N} \\ . & . & . \\ . & . & . \\ a_{m1} & ... & a_{mN} \end{bmatrix}$, and x = $\begin{bmatrix} x_1 \\ . \\ . \\ x_N \end{bmatrix}$, then $Ax = \begin{bmatrix} a_{11}x_1 + ... + a_{1N}x_N \\ . \\ . \\ a_{m1}x_1 + ... + a_{mN}x_N \end{bmatrix}$

Therefore:

$$\|A\|_{1\to 1} = max_{||x||_1=1} \|Ax\| = max_{||x||_1=1} \left\| \begin{bmatrix} a_{11}x_1 + ... + a_{1N}x_N \\ . \\ . \\ a_{m1}x_1 + ... + a_{mN}x_N \end{bmatrix} \right\|_1$$

$$= max_{||x||_1=1} |a_{11}x_1 + ... + a_{1N}x_N| + ... + |a_{m1}x_1 + ... + a_{mN}x_N|$$

Since this is the max over all $||x||_1 = 1$, this would happen when $x_i = 1$ for some index i and 0 for all other indices.
The result would then be $\sum_{j=1}^m |a_{jk}|$ for an index k. We would have to go through each index k from 1 to N to see which would give the greatest result for the maximum. Then:
$max_{||x||_1=1} |a_{11}x_1 + ... + a_{1N}x_N| + ... + |a_{m1}x_1 + ... + a_{mN}x_N|$

$= max_{k\in[N]} \sum_{j=1}^m |a_{jk}|$

(b) $||A||_{\infty\to\infty} = max_{j\in[m]} \sum_{k=0}^{N} |a_{jk}|$

proof.

Using A, x and Ax as defined in part (a):

$$\|A\|_{\infty\to\infty} = max_{||x||_\infty=1} \|Ax\|_\infty = max_{||x||_\infty=1} \left\| \begin{bmatrix} a_{11}x_1 + ... + a_{1N}x_N \\ . \\ . \\ a_{m1}x_1 + ... + a_{mN}x_N \end{bmatrix} \right\|_\infty$$

$= max_{||x||_\infty=1} max(|a_{11}x_1 + ... + a_{1N}x_N|, ..., |a_{m1}x_1 + ... a_{mN}x_N|)$

Since this is the max over all $||x||_\infty = 1$, this will happen when we take x to be a vector made up of 1's and -1's.

$= max(|a_{11}| + ... + |a_{1N}|, ..., |a_{m1}| + ... + |a_{mN}|)$

$= max(\sum_{k=1}^{N} |a_{1k}|, ..., \sum_{k=1}^{N} |a_{mk}|)$

So the result is the maximum of the L1 norms of every row:
$= max_{j\in[m]} \sum_{k=1}^{N} |ajk|$

(c) $\|A\|_{2\to 2} = \sigma_{max}(A) = \sqrt{\lambda_{max}(A^*A)}$ where $\sigma_{max}(A)$ denotes the largest singular value of A and $\sqrt{\lambda_{max}(A^*A)}$ is the largest eigenvalue of A^*A.
proof.

Let $\Gamma = A^*A$, let $\lambda_1....\lambda_n$ be the eigenvalues of Γ.
Let $e_1, ...e_n$ be the orthonormal basis of the Euclidian Space.
Let $v \in C^N$ be $v = a_1e_1 + ... + a_ne_n = \sum_i^n a_ie_i$. So that:
$\|v\|^2 = |<v,v>| = |<\sum_i^n a_ie_i, \sum_i^n a_ie_i>| = \sum_i^n a_i^2$

Now look at Γv:
$\Gamma v = \Gamma(\sum_i^n a_ie_i) = \sum_i a_i\Gamma(e_i) = \sum a_i\lambda_ie_i$.
Denote the largest λ_i value to be λ_{max}.

Now want to show $\|A\| \leq \sqrt{\lambda_{max}(A^*A)}$,
$\|Av\|^2 =< Av, Av >$
$= (Av)^T(Av) = v^TA^TAv =< v, A^TAv >$
$=< v, \Gamma v >=< \sum a_ie_i, \sum \lambda_ia_ie_i >$
$= \sum(a_ie_i)(\lambda_ia_ie_i)^T = \sum a_ie_ie_i^Ta_i^T\lambda_i^T$
$= \sum a_ia_i^T\lambda_i^T \leq \sum a_ia_i^T\lambda_{max} = \lambda_{max}\|v\|^2$
So $\|A\|^2 \leq \lambda_{max}$.
So $\|A\| \leq \sqrt{\lambda_{max}(A^*A)}$

Now want to show $\|A\| \geq \sqrt{\lambda_{max}(A^*A)}$,
Consider $x_0 = e_{max}$. $\|x\| = 1$. Where e_{max} is the eigenvector corresponding to λ_{max} .
$< x, \Gamma x >=< e_{max}, \Gamma(e_{max}) >=< e_{max}, \lambda_{j_0}e_{max} >= |\lambda_{max}|$.
But $< x_0, \Gamma x_0 >\leq \|A\|^2$

So $\|A\| \geq \sqrt{\lambda_{max}(A^*A)}$.

If $\|A\| \geq \sqrt{\lambda_{max}(A^*A)}$ and $\|A\| \leq \sqrt{\lambda_{max}(A^*A)}$,

Then $\|A\| = \sqrt{\lambda_{max}(A^*A)}$

(2) Given a

(3) Let V be a normed vector space with an inner product. Prove Cauchy Shqartz Inequality.
Proof.

Let $u, v \in V$. If v = 0 then the equality holds trivially. So assume $v \neq 0$.
Look at its orthogonal decomposition.
$u = \frac{<u,v>}{\|v\|^2} v + (u - \frac{<u,v>}{\|v\|^2} v)$
$= \frac{<u,v>}{\|v\|^2} v + w$ here w is orthognal to v.
Using Pythgorean Theorem:
$\|u\|^2 = \left\| \frac{<u,v>}{\|v\|^2} v \right\|^2 + \|w\|^2$
$\geq \frac{|<u,v>|^2}{\|v\|^2}$.

So: $\|u\|^2 \geq \frac{|<u,v>|^2}{\|v\|^2}$

Then: $\|v\|^2 \|u\|^2 \geq | < u, v > |^2$

Then: $\|v\| \|u\| \geq | < u, v > |$