CS 512, Spring 2014

Assignment 4

1 Problem 1: 2.3.11 a

$P(b) \vdash \forall x(x = b \rightarrow P(x))$

1	$P(b)$	premise
2	x_0	fresh
3	$x_0 = b$	assume
4	$P(x_0)$	$=$e $1, 3$
5	$(x_0 = b) \rightarrow P(x_0)$	$\rightarrow$i$3, 4$
6	$\forall x((x = b) \rightarrow P(x))$	$\forall x$i $2 - 5$

2 Problem 2: 2.3.12

$S \to \forall x Q(x) \vdash \forall x (S \to Q(x))$

1	$S \to \forall x Q(x)$	premise
2	x_0	fresh
3	S	assume
4	$\forall x Q(x)$	$\to$e $1, 3$
5	$Q(x_0)$	$\forall x$ e
6	$S \to Q(x_0)$	$\to$i$3, 8$
7	$\forall x (S \to Q(x))$	$\forall x$i $2 - 6$

3 Problem 3: 2.3.13(a)

$\forall x P(a,x,x), \forall x \forall y \forall z (P(x,y,z) \to P(f(x),y,f(z))) \vdash P(f(a),a,f(a))$

1	$\forall x P(a,x,x)$	premise
2	$\forall x \forall y \forall z (P(x,y,z) \to P(f(x),y,f(z)))$	premise
3	$P(a,a,a)$	$\forall x$ e1
4	$\forall y \forall z (P(a,y,z) \to P(f(a),y,f(z)))$	$\forall x$ e2
5	$\forall z (P(a,a,z) \to P(f(a),a,f(z)))$	$\forall y$ e4
6	$P(a,a,a) \to P(f(a),a,f(a))$	$\forall z$ e5
7	$P(f(a),a,f(a))$	$\to$e$3,6$

4 Problem 3: 2.3.13(b)

$\forall x P(a,x,x), \forall x \forall y \forall z (P(x,y,z) \to P(f(x),y,f(z))) \vdash \exists z P(f(a),z,f(f(a)))$

1	$\forall x P(a,x,x)$	premise
2	$\forall x \forall y \forall z (P(x,y,z) \to P(f(x),y,f(z)))$	premise
3	$P(a,f(a),f(a))$	$\forall x$ e1
4	$\forall y \forall z (P(a,y,z) \to P(f(a),y,f(z)))$	$\forall x$ e2
5	$\forall z (P(a,f(a),z) \to P(f(a),f(a),f(z)))$	$\forall y$ e4
6	$P(a,f(a),f(a)) \to P(f(a),a,f(f(a)))$	$\forall z$ e5
7	$P(f(a),f(a),f(f(a)))$	$\to$e$3,6$
8	$\exists z P(f(a),z,f(f(a)))$	$\exists z$ i7

5 Problem 4

(a) The formula $\forall x \forall y \exists z(R(x,y) \to R(y,z))$ is not true in the model M. Eg, we have $(b,a) \in R^M$, but there is no $m \in A$ with $(a,m) \in R^M$ contrary to what the formula claims. Thus, we may defeat this formula by choosing b for x and a for y to construct the contradiction to the truth of this formula over the given model.

(b) The formula $\forall x \forall y \exists x(R(x,y) \to R(y,z))$ is true in the model M. We may list all the elements of R in a cyclic way as $(a,b), (b,c), (c,b)$, the cycle being the last two pairs. Thus, for any choice of x and y we can find some z so that the implication $R(x,y) \to R(y,z)$ is true.

6 Problem 5

(i) We choose a model with A being the set of integers. We define $(n,m) \in P^M$ if and only if, n is less than or equal to m i.e $(n \leq m)$. Evidently, this interpretation of P is reflexive (every integer is less than equal to itself) and transitive: $n \leq m$ and $m \leq k$ implies $n \leq k$. However, $2 \leq 3$ and $3 \not\leq 2$ show that this interpretation cannot be symmetric.

(ii) We choose as set A the sons of a gentleman. We interpret $P(x,y)$ as "x is brother of y". Clearly, this relation is transitive and symmetric but not reflexive.

(iii) We define $A = a, b, c$ and $P^M = \{(a,a), (b,b), (c,c), (a,c), (a,b), (b,a), (c,a)\}$. Note that this interpretation is reflexive and symmetric. We also have that (b,a) and (a,c) are in P^M. Thus, we would need $(b,c) \in P^M$ to secure transitivity of P^M. Since this is not the case, we infer that this interpretation of P is not transitive.

7 Problem 6

(1) What does Lemma 'one' says? Express it in English precisely. Is it satisfiable, unsatisfiable, or valid?

Lemma 1 states:
If:
- R contains an edge between every pair of nodes.
- S contains at most one edge between every node
- R is contained in S

Then all three of these things imply: S is contained in R.

This lemma is valid.

(2) Is Lemma 'three' equivalent to Lemma 'one' and Lemma 'two' above ? Express it in English precisely. If not, is it satisfiable, unsatisfiable, or valid ?

Lemma 3 states:
If:
- R contains an edge between every pair of nodes.
- S contains at most one edge between every node
- R is contained in S.

Then all three of these things imply: S = R.

Since for S = R, S must be contained in R, Lemma 3 is equivelent to Lemma 1 and 2.
This lemma is valid.

(3) Is Lemma 'four' equivalent to Lemmas 'one', 'two' or 'three' above? Justify your answer carefully in English. If not, is it satisfiable, unsatisfiable, or valid?
Lemma 4 states:
If:
- R contains an edge between every pair of nodes.
- S has no edges.
- R is contained in S.
Then all three of these things imply: S = R.

Lemma four is not equivalent to lemmas 1,2,3 because lemma 4 states that if S(x, y) then x = y which is a more strict assertion than in lemmas 1,2,3. This lemma is valid.

(4) Is Lemma 'five' equivalent to 'one', 'two', 'three', or 'four' above? Justify your answer carefully in English. If not, is it satisfiable, unsatisfiable, or valid?

Lemma 5 specifies that there exists three distinct nodes, while the previous lemma's do not mention how many nodes there need to be. Therefore Lemma 5 is not equivelent to Lemmas 1,2,3,4. Also, the conclusion in lemma five is that S is not equal to R, which is not what lemmas one through three conclude.
The lemma is satisfiable.

(5) Is Lemma 'six' equivalent to 'one', 'two', 'three', 'four', or 'five'? Justify your answer carefully in English. If not, is it satisfiable, unsatisfiable, or valid ?
Lemma 6 is not equivelent to lemmas 1,2,3 because in lemma 1,2,3 we have that for a node x, there must be R(x,x), but in lemma 6 this is not allowed since x = x.
Lemma 6 is not equivelent to lemma 4 since in lemma 6 we could have S(x,y) where x and y are different, but according to lemma 4 S(x,y) implies that x and y are the same.
Lemma 6 is not equivelent to lemma 5 since it does not specifiy that there has to exist three distinct nodes.
Lemma is satisfiable.

8 Problem 7

(i) "there exists exactly three values of x which make $\varphi(x)$ true"
$\exists a \exists b \exists c(\varphi(a) \wedge \varphi(b) \wedge \varphi(c) \wedge \neg(a = b) \wedge \neg(b = c) \wedge \neg(a = c) \wedge \forall x(\neg(x = a) \wedge \neg(x = b) \wedge \neg(x = c)) \rightarrow \neg\varphi(x))$

(ii) "there exists at least three values of x which makes $\varphi(x)$ true"
$\exists a \exists b \exists c(\varphi(a) \wedge \varphi(b) \wedge \varphi(c) \wedge \neg(a = b) \wedge \neg(b = c) \wedge \neg(a = c))$

(iii) "there exists at most three values of x which makes $\varphi(x)$ true"
$\forall x(\exists a \exists b \exists c(\varphi(a) \wedge \varphi(b) \wedge \varphi(c)) \rightarrow ((\neg(x = a) \wedge \neg(x = b) \wedge \neg(x = c)) \rightarrow \neg\varphi(x)))$

(iv) "for all but finitely many values of x it holds that $\varphi(x)$ is true"
$\exists x \forall y(\neg\varphi(y) \rightarrow y \leq x)$

(v) "there are infinitely many values of x for which $\varphi(x)$ is true"
$\forall x(\varphi(x) \rightarrow \exists y(x \leq y \wedge \neg(x = y) \wedge \varphi(y)))$