Pattern formation through spatial segregation

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joint works with M. Conti, D. De Silva, B. Noris, N. Soave, H. Tavares, G. Verzini and A. Zilio

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SISSA webinar
April 28, 2020

Complex patterns

My researches focus on nontrivial solutions of systems of differential equations characterized by strongly nonlinear interactions.

In these cases, the configuration space is typically multi-dimensional or even infinite-dimensional, and we are interested in the effect of the nonlinearities on the emergence of non trivial self-organized structures. Such patterns correspond to selected solutions of the differential system possessing special symmetries or shadowing particular shapes.

- We want to understand, from the mathematical point of view, what are the main mechanisms involved in the aggregation process in terms of the global structure of the problem.

Therefore we will consider cases where

(a) the interaction becomes the prevailing mechanism,

(b) the equations are very far from being solved explicitly,

(c) the problems can not be seen in any extent as perturbations of simpler systems.

Following this common thread, we deal with a number of different type of strong interactions.

Attractive interactions

As in the classical N-body problem of Celestial Mechanics, where the balance between attraction and centrifugal effects produces solutions showing complex patterns. More precisely, we are interested in periodic and bounded solutions and parabolic trajectories with the final intent of proving density of periodic solutions and the occurrence of chaos. This will be achieved through the intermediate, but still fundamental, goal of detecting the presence of symbolic dynamics, through the study of symmetric and complex periodic solutions and theire Morse indices. The classification of periodic solutions will be related, through the ζ-function and the trace formula, to the spectrum of the assocuated Schrödinger operator.

Repulsive interactions

As in competition-diffusion systems, where pattern formation is driven by strongly repulsive forces. Our ultimate goal is to capture the geometry and analysis of the phase segregation, including its asymptotic aspects and the classification the solutions of the related PDE's. We deal with elliptic, parabolic and hyperbolic systems of differential equations with strongly competing interaction terms, modeling both the dynamics of competing populations (Lotka-Volterra systems) and other relevant physical phenomena, among which the phase segregation of solitary waves of Gross-Pitaevskiĭ systems arising in the study of multicomponent Bose-Einstein condensates.

We approach all these different problems with the same basic methodology which relies on the following steps

- asymptotic analysis
- analysis of special self-similar simple solutions
- interface analysis
- gluing techniques to build complex solutions.

Basic methodology
Asymptotics

Asymptotic analysis. The study of the effect of singularities (or singular limits) on the profiles of the solution shows striking similarities between classical and quantum systems and free boundary problems, and it draws, in the essential points, the most crucial elements of the classical theory of minimal surfaces. The monotonicity formulæ, adjusted for the different cases, the blow-up analysis, the classification of the limiting (conic) solutions equivariant by dialation, along with the appropriate tools of dimensional reduction, underpin the asymptotic analysis of solutions.

Basic methodology

Special solutions

Entire solutions. Equilibrium configurations, of course, play a fundamental role. Other simple, yet nontrivial, patterns also appear naturally as symmetric extremals of the associated energies. Symmetries are the key tool for this exploration. On the other hand, entire solutions also carry transitions from one configuration to another: this is the case of parabolic trajectories in Celestial Mechanics and entire solutions of competition-diffusion systems. Entire solutions also heavily enter in the blow-up analysis, as they represent the limiting profiles in some scaling process.

Basic methodology

Building complex solutions

Interface analysis. Asymptotic limiting profiles arise as blow-ups at different scales. They may show sharp transitions of the gradients, obeying different refraction of reflection rules. Here we shall take advantage of tools from free boundary theory in order to describe the geometric features of the interface.

Gluing techniques. Having gathered different types of elementary solutions, the next step consists of gluing them to build more complex patterns. Gluing can be done, once more, using global variational techniques, or other methods. This can be done, e.g., by the broken geodesics argument, in the case of trajectories of Classical and Quantum Mechanics, or by other types of reductions, e.g. by solving optimal partition problems, as in the case of competition-diffusion systems.

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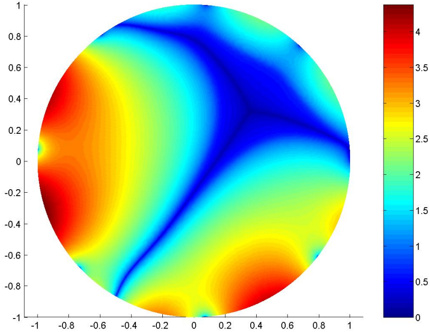
Competition diffusion systems with Lotka-Volterra interactions: symmetric competition rates

With large and symmetric interspecific competition rates $\beta_{ij} = \beta_{ji}$ and three populations:

$$\frac{\partial u_i}{\partial t} - \mathrm{div}(d_i \nabla u_i) = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j\neq i}}^{k} \beta_{ij} u_j \text{ in } \Omega,$$



Competition diffusion systems with Lotka-Volterra interactions : asymmetric competition rates

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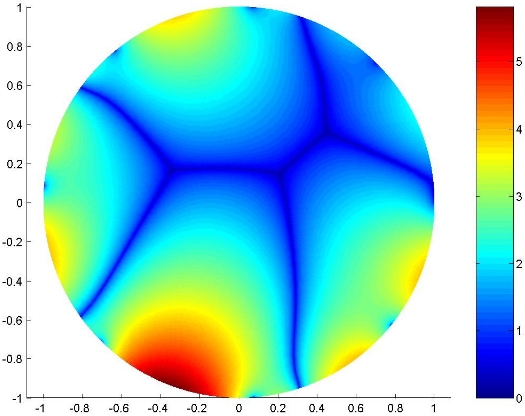
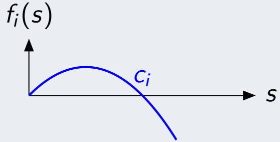


Competition diffusion systems with Lotka-Volterra interactions: symmetric competition rates

With large and symmetric interspecific competition rates $\beta_{ij} = \beta_{ji}$ and three populations:

$$\overbrace{\frac{\partial u_i}{\partial t}}^{\text{evolution}} - \overbrace{\mathrm{div}(d_i \nabla u_i)}^{\text{diffusion}} = \overbrace{f_i(u_i)}^{\text{reaction}} - \overbrace{u_i \sum_{\substack{j=1 \\ j\neq i}}^{k} \beta_{ij} u_j}^{\text{competition}} \text{ in } \Omega,$$

u_i is the density of the ith population,
$d_i > 0$ diffusion rates,
$\beta_{i,j}$ interspecific competition rates,
$f_i(s) = u(c_i - u)$ internal forces (logistic)



Diffusion vs strong competition

For the the sake of simplicity, assuming the system be already in equilibrium, we consider only stationary cases, with all equal diffusions, namely we deal with the semilinear ellipitc system:

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i}^{k} a_{ij} u_j \qquad \text{in } \Omega, \quad +\text{B.C.}, \quad i = 1, \dots, k, \ \text{(P)}$$

subject to diffusion, reaction and competitive interaction ($a_{ij}, \beta > 0$).

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Gause's law

Two species competing for the same limiting resource cannot coexist at constant population values. When one species has even the slightest advantage over another, the one with the advantage will dominate in the long term. This leads either to the extinction of the weaker competitor or to an evolutionary or behavioral shift toward a different ecological niche.

Gause, Georgii Frantsevich (1934). The Struggle For Existence (1st ed.). Baltimore: Williams & Wilkins. Archived from the original on 2016-11-28

If similar competing species cannot coexist, then how do we explain the great patterns of diversity that we observe in nature? If species living together cannot occupy the same niche indefinitely, then how do competitors coexist?

Mimura'sresult

M. Mimura, Asymptotic behaviors of a parabolic system related to a planktonic prey and predator model, SIAM J. Appl. Math. 37(3) (1979) 499-512
Mimura considered predator-prey system with no flux boundary condition in a bounded set:

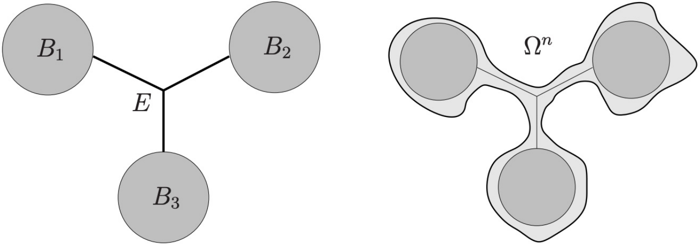
$$\begin{cases} u_t = d_1 \Delta u + f(u)u - uv \\ v_t = d_2 \Delta v + g(u)u + uv \end{cases}$$

Showing that if $f'(u) \leq 0$ and $g'(v) \geq 0$ for $u \geq 0$, $v \geq 0$, and if there is a positive, spatially constant, steady state then every uniformly bounded, nonnegative solution becomes spatially homogeneous as $t \to +\infty$.

Berestycki and A. Zilio, Predators-prey models with competition I-IV

Ecological niches

Coexistence needs complex geometries, that allow the presence of niches, or strongly inhomogenous environments (modeled by strongly varying diffusion functions d_i).



(a) the set $\Omega^0 = B_1 \cup B_2 \cup B_3$ and segments E joining the balls

(b) sets Ω obtained by small perturbation of Ω^0.

Felli, V. and Conti, M. (2008). Coexistence and segregation for strongly competing species in special domains. INTERFACES AND FREE BOUNDARIES, 10(2), 173-195.

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Segregation phenomena

We say that a family of solutions $\{\mathbf{u}_\beta\}_\beta$ segregates if

$$u_{i,\beta} \to u_i, \qquad u_i \cdot u_j \equiv 0, \qquad \text{a.e. as } \beta \to +\infty$$

for nontrivial limits (with some abuse, we talk about "disjoint supports").

Several questions to be addressed:

- what kind of convergence (in terms of function spaces);
- features properties of the limiting profiles;
- geometry of the nodal set $\Gamma = \{x \ : \ u_i(x) = 0, \ \forall i = 1, \ldots, k\}$.

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For standard diffusions, this has been studied in a number of papers by different teams:

- M. Conti, B. Noris, H. Tavares, S. Terracini, G. Verzini, N. Soave, A. Zilio
- J. Wei, T. Weth
- L.A. Caffarelli, A. Karakhanyan, F. Lin, JM. Roquejoffre, V. Quitalo, S. Patrizi
- E.N. Dancer, Y. Du, K. Wang, Z. Zhang
- A.R. Domingos, B. Noris, M. Ramos

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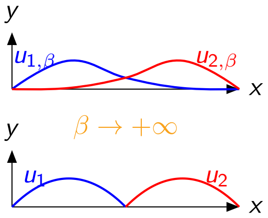
Also the one-dimensional problem is significant:

$$\begin{cases} -u_1'' = f(u_1) - \beta u_1 u_2 & \text{in } (a,b) \\ -u_2'' = f(u_2) - \beta\gamma u_1 u_2 & \text{in } (a,b) \\ u_i(a) = u_i(b) = 0 \end{cases}$$



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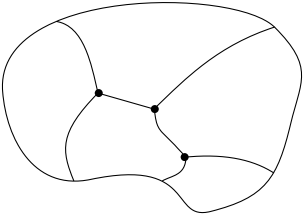
Even though, only in dimension $N \geq 2$ true free boundaries arise.



We consider different models:

Symmetric quadratic interactions (Lotka-Volterra)

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j, \qquad i = 1, \dots, k,$$

with

$$a_{ij} = a_{ji}$$

We consider different models:

Symmetric cubic interactions (Groß-Pitaevskii energies)

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j^2, \qquad i = 1, \dots, k,$$

Variational structure iff

$$a_{ij} = a_{ji}$$

We consider different models:

Asymmetric quadratic interactions (Lotka-Volterra)

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j, \qquad i = 1, \dots, k,$$

with

$$a_{ij} \neq a_{ji}$$

We consider different models:

Anomalous diffusions $s \in (0,1)$ (all range of exponents)

$$-(\Delta)^s u_i = f_i(x, u_i) - \beta u_i^p \sum_{j \neq i} a_{ij} u_j^q, \qquad i = 1, \dots, k,$$

Variational structure iff

$$a_{ij} = a_{ji} \qquad p = q - 1$$

We consider different models:

Interaction at a distance (different range of exponents)

$$-\Delta u_i = f_i(x, u_i) - \beta u_i^p \sum_{j \neq i} a_{ij}(\mathbb{1}_{B_1} \star u_j^q), \qquad i = 1, \dots, k,$$

Variational structure iff

$$a_{ij} = a_{ji} \qquad p = q - 1$$

Basic questions:

1. does the particular expression of the interaction matter? (quadratic vs cubic interactions)
2. do the diffusion rules matter? (standard $s = 1$ vs anomalous diffusion $0 < s < 1$)
3. what is the role of distance? (pointwise interaction vs interaction at a distance)

Variational Principles

One of the winning ideas of modern science, and mechanics in particular, is that nature proceeds trying to optimize certain quantities (energies).

Fundamental questions:

1. are there dissipated quantities during the evolutions?
2. is there an underlying variational principle for the equilibrium limiting profiles? (symmetric vs asymmetric interactions).

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Competition in the Lotka-Volterra model

We consider the semilinear system:

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j \qquad \text{in } \Omega, \qquad i = 1, \dots, k, \qquad \text{(LV)}$$

where $u_i \geq 0$, $\beta > 0$, $a_{ij} > 0$ (+ boundary conditions).

(LV) is the stationary version of the
competition-diffusion system with Lotka-Volterra interactions:

$$\partial_t u - \Delta u_i = f_i(u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j .$$

(LV) is never variational. It can be
either symmetric ($a_{ij} = a_{ji}$) or asymmetric ($a_{ij} \neq a_{ji}$).

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The symmetric case for $k \geq 3$ populations

We assume $a_{ij} = a_{ji} (= 1$ w.l.o.g.). The system becomes

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \textstyle\sum_{j \neq i} u_j \qquad \text{in } \Omega, \qquad i = 1, \dots, k,$$

Theorem (Conti, Terracini, Verzini '05)

Let U_β be a family of H^1–bounded solutions. For every $\alpha < 1$ there exists $L_\alpha > 0$ such that

$$\sup_{x,y \in \Omega} \frac{|u_{i,\beta}(x) - u_{i,\beta}(y)|}{|x - y|^\alpha} < L_\alpha$$

for all $i = 1, \dots, k$ and for all $\beta > 0$.

This allows to pass to the limit as $\beta \to +\infty$.

Optimal uniform Lipschitz bounds have been obtained [Soave-Zilio, *ARMA* 2015]

Structure of the nodal set

Theorem (Conti-Terracini-Verzini '05, Caffarelli-Karakanyan-Lin '08, Tavares-Terracini '12)

Let U be any of these limiting profiles, and let $\mathcal{Z} = \{x \in \Omega \ : \ U(x) = 0\}$. Then, there exists a set $\mathcal{Z}_2 \subseteq \mathcal{Z}$ = the regular part, relatively open in $\mathcal{Z}$, such that

- *$\mathcal{Z}_2$ is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0 < \alpha < 1$). Furthermore for every $x_0 \in \mathcal{Z}_2$*

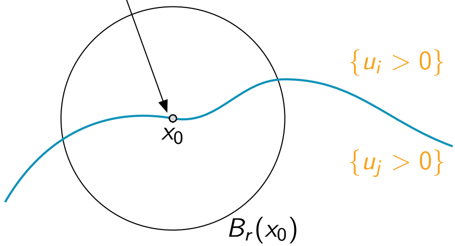
$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

  where the limits as $x \to x_0^{\pm}$ are taken from the opposite sides of the hyper-surface;
- *$\mathcal{H}_{dim}(\mathcal{Z} \setminus \mathcal{Z}_2) \leq N - 2$, and $\lim_{x \to x_0} |\nabla U(x)| = 0$.*

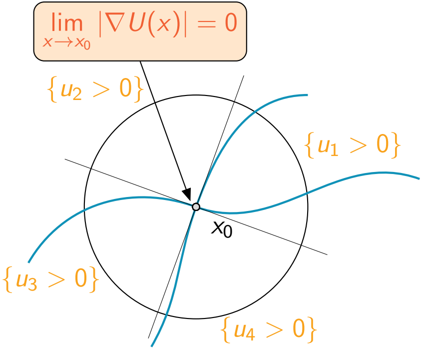
Furthermore, if $N = 2$ then $\mathcal{Z}$ consists in a locally finite collection of curves meeting with equal angles at a locally finite number of singular points.

Nodal set: regular points

$$\lim_{\substack{x \to x_0 \\ x \in \{u_i > 0\}}} \nabla u_i(x) = - \lim_{\substack{x \to x_0 \\ x \in \{u_j > 0\}}} \nabla u_j(x)$$



Nodal set: singular points ($N = 2$)



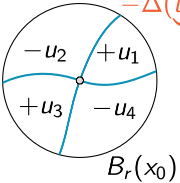
A fundamental optimization property

A non trivial consequence of this theorem is the following priciple.

For symmetric inter-specific competition rates, even though the system does not possess a variational nature, it fulfills a minimization principle in the segregation limit. The limiting partition is optimal with respect to the sum of the Lagrangian energies.

Asymptotic expansion near multiple points

An heuristic argument without reactions:



$$-\Delta(\underbrace{u_1 - u_2 + u_3 - u_4}_{w}) = f_1 - f_2 + f_3 - f_4$$

Then $w(r, \vartheta) = \sum_{k \in \mathbb{Z}} [a_k \cos(k\vartheta) + b_k \sin(k\vartheta)]\, r^k$

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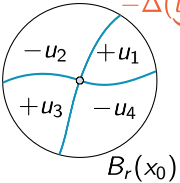

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Then $w(r,\vartheta) = \sum_{k\in\mathbb{Z}} [a_k \cos(k\vartheta) + b_k \sin(k\vartheta)]\, r^k$ and

- $a_k^2 + b_k^2 = 0$ for $k < 0$ as w is not singular in x_0,
- $a_k^2 + b_k^2 = 0$ for $k = 0, 1$ as $m(x_0) = 4$,

$$w(r,\vartheta) = r^2 \cos(2\vartheta + \vartheta_0) + o(r^2) \text{ as } r \to 0.$$

In general, $w \sim r^{m(x_0)/2}$, also in the odd case.

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The limiting profiles

Back to the original problem

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j \qquad \text{in } \Omega, \qquad i = 1, \ldots, k,$$

assume now $a_{ij} \neq a_{ji}$

- **Passing to the limit as $\beta \to \infty$ we find a new class of limiting profiles.**

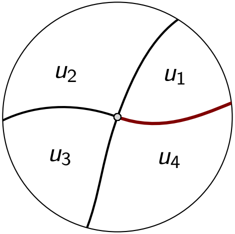
Proportionality of the gradients:

$$\lim_{\substack{x \to x_0 \\ x \in \{u_i > 0\}}} a_{ji} \nabla u_i(x) = - \lim_{\substack{x \to x_0 \\ x \in \{u_j > 0\}}} a_{ij} \nabla u_j(x)$$

Main changes

Local expansion at multiple points (in dimension $N = 2$).
Near an isolated point x_0 with (e.g.) $m(x_0) = 4$
we have that

$$w = u_1 - \frac{a_{12}}{a_{21}} u_2 + \frac{a_{12}a_{23}}{a_{21}a_{32}} u_3 - \frac{a_{12}a_{23}a_{34}}{a_{21}a_{32}a_{43}} u_4$$



satisfies

$$-\Delta w = 0 \qquad \text{in } B_{r_0}(x_0) \setminus \underbrace{\left(\overline{\{u_1 > 0\}} \cap \overline{\{u_4 > 0\}}\right)}_{\tilde{\Gamma}}.$$

A theorem

Let $(u_1, \dots, u_k)$ be a segregated limiting profile in the asymmetric case.

Theorem (S. T. , G. Verzini, A. Zilio, CPAM 2019)

Let $\mathcal{Z}$ be a compact connected component of $\{x : m(x) \geq 3\}$. Then $\mathcal{Z} = \{x_0\}$.

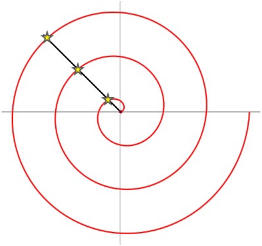
Theorem (S. T. , G. Verzini, A. Zilio, CPAM 2019)

Let $x_0 \in \Omega$ with $m(x_0) = h \geq 3$. Then there exists $\alpha \in \mathbb{R}$ and ϑ_0 such that

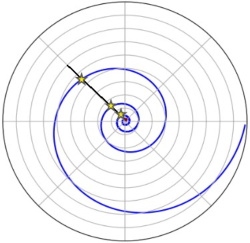
$$w(r, \vartheta) = C r^{h/2 + 2\alpha^2/h} \exp(\alpha\theta) \cos\left(\frac{h}{2}\theta - \alpha \log r + \vartheta_0\right) + o(r^{h/2+2\alpha^2/h})$$

as $r \to 0$, where (r, θ) denotes a system of polar coordinates about x_0 and $\tilde{U}$ is a suitably weighted sum of the components u_i.

Asymptotics and nodal set



Archimedean Spiral



Logarithmic Spiral

Conclusions

In the asymmetric case, the nodal lines of the limiting profiles meet by forming, asymptotically, spirals.

However, the limiting profiles still share with the symmetric case the following fundamental features:

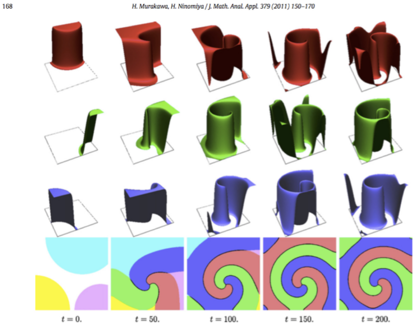
- singular points are isolated and have a finite vanishing order;
- the possible vanishing orders are quantized;
- the regular part is smooth.

In the asymmetric case, the nodal partition determined by the supports of the components can not be optimal with respect to any Lagrangian energy. Indeed, it is known that boundaries of optimal partitions share the same nodal properties of the energy minimizing configurations. Hence they can not exhibit logarithmic spirals. This fact is in striking contrast with the picture for symmetric inter-specific competition rates: indeed, in such a case, we know that solutions are unique, together with their limit profiles.

The evolution problem

With asymmetric interspecific competition rates $\beta_{i,j} \neq \beta_{j,i}$ large and three populations:

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j\neq i}}^{h} \beta_{i,j} u_j \text{ in } \Omega,$$



H. Murakawa and H. Ninomiya, *Fast reaction limit of a three-component reaction-diffusion system*. J. Math. Anal. Appl. 379 (2011), no. 1, 150-170,

The spiralling wave ansatz in two-dimension

But... do rotating spirals really exist in mathematics? Seeking them desperately!

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - \beta u_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} u_j \text{ in } \mathbb{C},$$

Ansatz:

$$u_i(t,x) = v_i\left(e^{i\omega t}x\right) \ , \qquad x \in \mathbb{C}$$

The spiralling wave ansatz in two-dimension

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Ansatz:

$$u_i(t,x) = v_i\left(e^{i\omega t}x\right) , \qquad x \in \mathbb{C}$$

Then $(v_1, \dots, v_i)$ solve

$$\omega x^{\perp} \cdot \nabla v_i - \Delta v_i = f_i(v_i) - \beta v_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} v_j \text{ in } \mathbb{C}.$$

Spiralling limiting profiles:

$$\omega x^{\perp} \cdot \nabla v_i - \Delta v_i = f_i(v_i) - \beta v_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} v_j \text{ in } \mathbb{C}, \qquad (*)$$

Next we pass to the limit as $\beta \to +\infty$.

Theorem (Salort,Terracini,Verzini, Zilio 2019)

For every ω, for a codimension two set of boundary traces, there exists a unique solution in the class $\mathcal{S}$ associated with $(*)$ in the unit disk. Furthermore, there exists $\alpha \in \mathbb{R}$ and θ_0 such that

$$\tilde{V}(r,\theta) = Cr^{h/2} \exp\left(\alpha\vartheta\right) \left| \cos\left(\frac{h}{2}\vartheta - \alpha \log r + \vartheta_0\right) \right| + o(r^{h/2})$$

as $r \to 0$, where (r,θ) denotes a system of polar coordinates about 0 and $\tilde{U}$ is a suitably weighted sum of the components v_i.

Some numerical simulations (by courtesy of Alessandro Zilio)

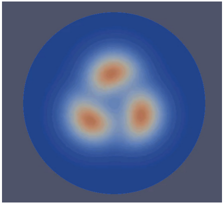


Figure 1

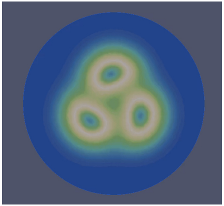


Figure 2

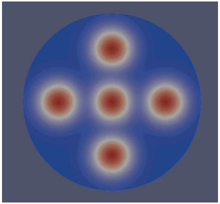
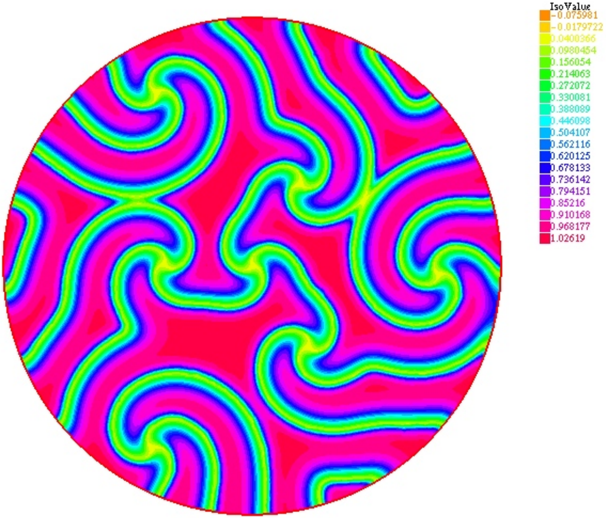


Figure 3

Final remarks

- Rotating spirals appear to exist and being stable for three populations under extreme asymmetric competitive interaction.
- This contradicts Gause's exclusion principle.
- Simulations show that multiple rotating spirals form complex patterns persisting for long times.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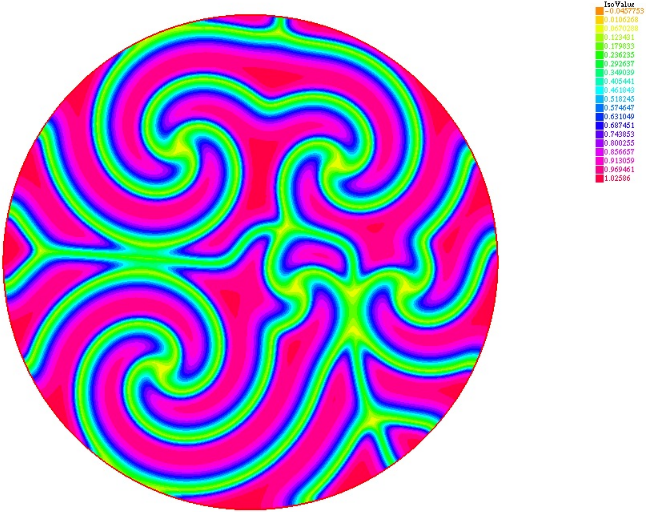
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0.574647
0.631049
0.687451
0.743853
0.800255
0.856657
0.913059
0.969461
1.02586

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