EE 637 Midterm I

February 20, Spring 2015

Name: (4 pt) Key

Instructions:

- This is a 50 minute exam containing **three** problems.
- You may **only** use your brain and a pencil (or pen) and the included "Fact Sheet" to complete this exam.
- You **may not** use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You **may not** communicate with any person other than the official proctor during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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Problem 1.(32pt)

Consider the following 1D system with input $x(n)$ and output $y(n)$.

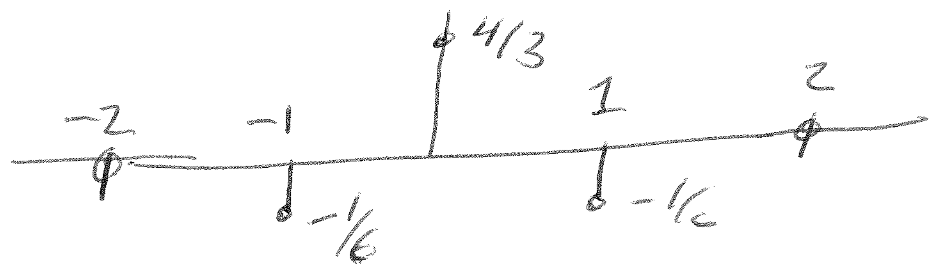
$$y(n) = x(n) + \lambda \left(x(n) - \frac{1}{3} \sum_{k=-1}^{1} x(n-k) \right) .$$

a) Is this a linear system? Is this a space invariant system?
b) Calculate and sketch the impulse response, $h(n)$ for $\lambda = 0.5$.
c) Calculate and sketch the frequency response, $H(e^{j\omega})$ for $\lambda = 0.5$.
d) Describe how the filter behaves when λ is positive and large.

a) linear ⇒ yes
Space invariant ⇒ yes

b) $\lambda = 0.5 = 1/2$

$$h(n) = \frac{4}{3}\delta(n) - \frac{1}{6}\left(\delta(n-1) + \delta(n+1)\right)$$



c) $$H(e^{j\omega}) = \frac{4}{3} - \frac{1}{6}\left(e^{j\omega} + e^{-j\omega}\right)$$

$$= \frac{4}{3} - \frac{1}{3}\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)$$

$$= \frac{4}{3} - \frac{1}{3}\cos(\omega)$$

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$$H(e^{j\omega}) = 1 + \frac{1}{3}\left(1 - \cos(\omega)\right)$$

d) It sharpens the image by adding a high pass component to the image

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Problem 2.(32pt)

Consider the 2D discrete space signal $x(m,n)$ with the DSFT of $X(e^{j\mu}, e^{j\nu})$ given by

$$X(e^{j\mu}, e^{j\nu}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(m,n) e^{-j(m\mu+n\nu)} .$$

Then define

$$\begin{aligned} p_0(n) &= \sum_{m=-\infty}^{\infty} x(m,n) \\ p_1(m) &= \sum_{n=-\infty}^{\infty} x(m,n) \end{aligned}$$

with corresponding DTFT given by

$$\begin{aligned} P_0(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} p_0(n) e^{-jn\omega} \\ P_1(e^{j\omega}) &= \sum_{m=-\infty}^{\infty} p_1(m) e^{-jm\omega} \end{aligned}$$

a) Derive an expression for $P_0(e^{j\omega})$ in terms of $X(e^{j\mu}, e^{j\nu})$.

b) Derive an expression for $P_1(e^{j\omega})$ in terms of $X(e^{j\mu}, e^{j\nu})$.

c) Find a function $x(m,n)$ that is **not zero everywhere** such that $p_0(n) = p_1(m) = 0$ for all m and n.

d) Do the functions $p_0(n)$ and $p_1(m)$ together contain sufficient information to uniquely reconstruct the function $x(m,n)$? Justify your answer.

a) $P_0(e^{j\omega}) = \sum_{m} p_0(n) e^{j\omega n} = \sum_{n,m} x(m,n) e^{j\omega n}$

$= X(e^{j0}, e^{j\omega})$

b) $P_1(e^{j\omega}) = X(e^{j\omega}, e^{j0})$

c)

$$\tilde{X}(m,n) = \begin{cases} 1 & m=1,\ n=1 \\ 1 & m=-1,\ n=-1 \\ -1 & m=-1,\ n=+1 \\ -1 & m=1,\ n=-1 \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

d) No, because $\alpha \tilde{X}(m,n)$ has the property that $p_0(m)\, p_1(m) = 0$ for all $\alpha \in \mathbb{R}$. So the inverse can not be unique.

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Problem 3.(32pt)

Let $x(t) = \text{sinc}(t/a)$ for some positive constant a, and let $y(n) = x(nT)$ where $f_s = 1/T$ is the sampling frequency of the system. Further assume that a has units of sec, T has units of sec, and f_s has units of $Hz = sec^{-1}$.

a) Calculate and sketch $X(f)$, the CTFT of $x(t)$.

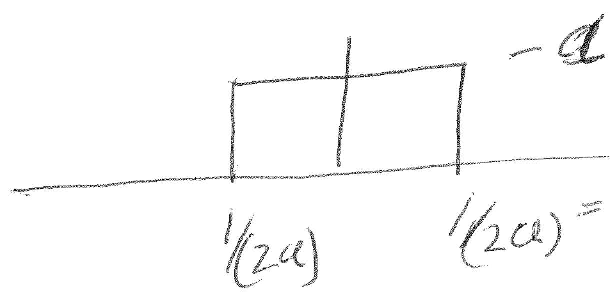
b) Calculate $Y(e^{j\omega})$, the DTFT of $y(n)$.

c) What is the minimum sampling frequency, f_s, that ensures perfect reconstruction of the signal?

d) Sketch the function $Y(e^{j\omega})$ on the interval $[-\pi, \pi]$ when $T = a$.

e) Sketch the function $Y(e^{j\omega})$ on the interval $[-\pi, \pi]$ when $T = \frac{5a}{4}$.

a) $X(f) = a \; \text{rect}(af)$



b) $$Y(e^{j\omega}) = \frac{1}{T} \sum_{K=-\infty}^{\infty} X\left(\frac{\omega - 2\pi K}{2\pi T}\right)$$

$$= \frac{1}{T} \sum_{K=-\infty}^{\infty} a \, \text{rect}\left(a \frac{\omega - 2\pi K}{2\pi T}\right)$$

$$= \frac{a}{T} \sum_{K=-\infty}^{\infty} \text{rect}\left(\frac{a}{T} \frac{\omega - 2\pi K}{2\pi}\right)$$

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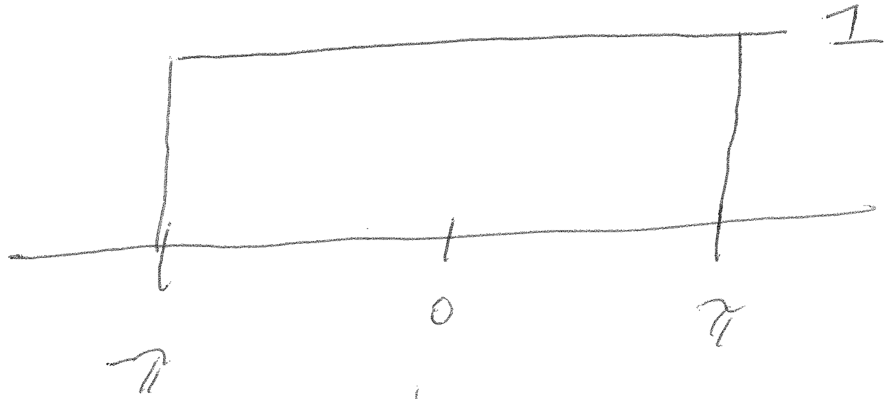
c) $f_s \geq 2\left(\frac{1}{2a}\right) = \frac{1}{a}$

$T \leq a$

d)



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