EE 637 Final

May 5, Spring 2015

Name: ____

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\mathrm{comb}_T\left[x(t)\right] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\mathrm{comb}_T\left[x(t)\right] \quad \overset{CTFT}{\Leftrightarrow} \quad \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}\left[X(f)\right]$$

$$\mathrm{rep}_T\left[x(t)\right] \quad \overset{CTFT}{\Leftrightarrow} \quad \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}\left[X(f)\right]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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Problem 1.(20pt)

Consider a color imaging device that takes input values of (r, g, b) and produces output (X, Y, Z) values given by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = A \begin{bmatrix} r^\alpha \\ g^\alpha \\ b^\alpha \end{bmatrix} .$$

where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} .$$

a) Calculate the white point of the device in chromaticity coordinates.

b) Determine the chromaticity coordinates of the three primaries associated with the r, g, and b components.

c) What is the gamma of the device?

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Problem 2.(20pt)

Let X be a scalar random variable and Z be a $M \times 1$ random vector. Assume that X and Z are zero mean with

$$\begin{aligned} b &= E[XZ] \\ R &= E[ZZ^t] \ . \end{aligned}$$

Further define the estimator

$$\hat{X} = \theta Z$$

where θ is a $1 \times M$ parameter vector.

a) **Derive** an expression for the value of θ that results in the minimum mean squared error **linear** estimator of X given Z.

b) What estimator always results in the minimum mean squared error estimate of X given Z?

c) Is the minimum mean squared error estimator of X always a linear function of Z? Justify your answer.

d) What estimator, $\hat{X} = T(Z)$, minimizes the following expression?

$$E[|X - \hat{X}|]$$

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Problem 3.(20pt)

Consider the 1-D error diffusion algorithm specified by the equations

$$\begin{aligned} b_n &= Q(y_n) \\ e_n &= y_n - b_n \\ y_n &= x_n + e_{n-1} \end{aligned}$$

where x_n is the input, b_n is the output, and $Q(\cdot)$ is a binary quantizer with the form

$$Q(y) = \begin{cases} 1 & \text{if } y > 0.5 \\ 0 & \text{if } y \leq 0.5 \end{cases} .$$

where we assume that $e_0 = 0$ and the algorithm is run for $n \geq 1$.

Furthermore, define $d_n = x_n - b_n$.

a) Draw a flow diagram for this algorithm. Make sure to label all the signals in the flow diagram using the notation defined above.

b) Calculate b_n for $n = 1$ to 10 when $x_n = 0.25$ and $e_0 = 0$.

c) Calculate an expression for d_n in terms of the quantization error e_n.

d) Calculate an expression for $\sum_{n=1}^{N} d_n$ in terms of the quantization error e_n.

e) What does the result of part d) above tell you about the output of error diffusion?

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Problem 4.(20pt)

Consider a discrete-time random process X_n with i.i.d. samples that are Gaussian with mean 0 and variance $\sigma^2 > 0$.

The rate distortion relation for this source is then given by

$$\begin{aligned} R(\Delta) &= \max\left\{ \frac{1}{2}\log_2\left(\frac{\sigma^2}{\Delta^2}\right), 0 \right\} \\ D(\Delta) &= \min\left\{ \sigma^2, \Delta^2 \right\} \end{aligned}$$

a) Plot the minimum possible rate (y-axis) versus distortion (x-axis) required to code this source when $\sigma^2 = 1$.

b) Explain the meaning of the distortion-rate function.

c) How many bits per sample are required in order to achieve zero distortion?

d) How many bits per sample are required in order to achieve a distortion of $D = \sigma^2$?

e) Let $\{X_n\}_{n=1}^N$ be a set of independent Gaussian random variables each with zero mean and variance σ_n^2 where $\sigma_{n-1}^2 \geq \sigma_n^2$. Write an expression for the rate and distortion.

f) Calculate the distortion-rate function for a Gaussian vector X with mean zero and covariance $R = E\Lambda E^t$ where $EE^t = I$ and Λ is diagonal?

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Problem 5.(20pt)

Let $x(t) = \text{sinc}(at)$ for some positive constant a, and let $y(n) = x(nT)$ where $f_s = 1/T$ is the sampling frequency of the system. Further assume that a has units of sec^{-1}, T has units of sec, and f_s has units of $Hz = sec^{-1}$.

a) Calculate $X(f)$, the CTFT of $x(t)$.

b) What is the cutoff frequency of $X(f)$ in Hz? (Your answer will be in terms of a.)

c) Calculate $Y(e^{j\omega})$, the DTFT of $y(n)$.

d) Calculate f_q, the Nyquist sampling frequency for the signal. (Your answer will be in terms of a.)

e) Sketch the function $Y(e^{j\omega})$ on the interval $[-\pi, \pi]$ when $f_s = (3/2)f_q$.

f) Sketch the function $Y(e^{j\omega})$ on the interval $[-\pi, \pi]$ when $f_s = (2/3)f_q$.

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