EE 637 Midterm II

April 10, Spring 2015

Name: Key

Instructions:

- This is a 50 minute exam containing **3.1** problems with a total of 100 points.
- You may **only** use your brain and a pencil (or pen) and the included "Fact Sheet" to complete this exam.
- You **may not** use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You **may not** communicate with any person other than the official proctor during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux + vy)} dx dy \\ f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux + vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T} \right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T} \right) \end{aligned}$$

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Problem 1.(33pt)

Consider a γ-correct display with the discrete input $X(m,n)$. Assume that the output intensity at the pixel (m,n) is given by

$$I(m,n) = I_o\left(\frac{X(m,n)}{255}\right)^{\gamma},$$

where $I(m,n)$ is the output intensity in units proportional to energy and I_o is the maximum intensity output.[1]

Imagine that you view two different images on the display. The first image, $X_1(m,n)$, is given by

$$X_1(m,n) = \begin{cases} 255 & m+n \text{ is even} \\ 0 & m+n \text{ is odd} \end{cases},$$

and the second image is given by $X_2(m,n) = g$.

a) What average intensity do you see if you view the image corresponding to $X_1(m,n)$ from a large distance?

b) What average intensity do you see if you view the image corresponding to $X_2(m,n)$ from a large distance?

c) Calculate the value of g that will result in matching gray levels for $X_1(m,n)$ and $X_2(m,n)$.

d) Imagine that you adjust the value of g to achieve a match between $X_1(m,n)$ and $X_2(m,n)$. Write an equation for the value of γ given this measured value of g.

a) $I_o/2$

(b) $I_o\left(\frac{g}{255}\right)^{\gamma}$

(c) $\frac{1}{2} = \left(\frac{g}{255}\right)^{\gamma} \Rightarrow g = 255\left(\frac{1}{2}\right)^{1/\gamma}$

(d) $\log g = \log 255 + \frac{1}{\gamma}\log \frac{1}{2} \Rightarrow \gamma = \frac{\log(1/2)}{\log\left(\frac{g}{255}\right)}$

[1] Furthermore, assume that all pixels on the display are square and of uniform size.

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Problem 2.(33pt)

The approximate *Lab* color space transform is given by

$$\begin{aligned} L &= 100(Y/Y_0)^{1/3} \\ a &= 500\left[(X/X_0)^{1/3} - (Y/Y_0)^{1/3}\right] \\ b &= 200\left[(Y/Y_0)^{1/3} - (Z/Z_0)^{1/3}\right] \end{aligned}$$

where X_0, Y_0, and Z_0 corresponds to D65 white, and the chromaticity of D65 is given by $(x, y) = (0.3127, 0.3290)$, and we assume $Y_0 = 1$.

a) Determine the values of X_0 and Z_0. (It is not necessary to multiply and divide decimal numbers. You may express your result in terms of the constants.)

b) Calculate the values of Y, X, and Z for $L = 100$ and $a = b = 0$. What color is this?

c) Why are the quantities (X/X_0), (Y/Y_0), and (Z/Z_0) raised to the power $1/3$?

d) What will happen if the values of X, Y, Z are stored as 8-bit integers and then the resulting image is displayed?

e) Why are the quantities $(X/X_0)^{1/3}$ and $(Y/Y_0)^{1/3}$ subtracted for the calculation of a, and the quantities $(Y/Y_0)^{1/3}$ and $(Z/Z_0)^{1/3}$ subtracted for the calculation of b?

f) Specify an application for which the *Lab* color space is well suited. Why?

a) $z = 1 - x - y = 0.3583$

$$\begin{array}{r} 0.3127 \\ \underline{0.3290} \\ 0.6417 \end{array} \qquad \begin{array}{r} 1.0000 \\ \underline{0.6417} \\ 0.3583 \end{array}$$

$X_0 = \dfrac{0.3127}{0.3290}$

$Z_0 = \dfrac{0.3583}{0.3290}$

b) $(X, Y, Z) = (X_0, Y_0, Z_0) \Leftarrow$ D65

c) This is to account for Weber's Law/contrast in JND

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d) There will be contours in the dark regions of image.

e) This results in an approximate opponent color system.

f) Matching colors of paint. This is a good application of Lab because we are matching color over large areas.

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Problem 3.(34pt)

Let Y be a $p \times N$ matrix with $N << p$. Furthermore, let Y have a SVD with the form

$$Y = U \Sigma V^t$$

where V is an $N \times N$ orthonormal matrix, U is a $p \times N$ orthonormal matrix, and Σ is a $N \times N$ diagonal matrix, with diagonal entries that are positive and decreasing.

Furthermore, define the two matrices $A = YY^t$ and $B = Y^tY$; and let A and B have eigen decompositions with the form

$$\begin{aligned} A &= E \Lambda E^t \\ B &= T D T^t \ . \end{aligned}$$

a) Prove that both A and B are both symmetric and positive semi-definite matrices.

b) Express the eigenvalues and eigenvectors of A in terms of U, V, and Σ.

c) Express the eigenvalues and eigenvectors of B in terms of U, V, and Σ.

d) Express the eigenvalues and eigenvectors of B^{-1} in terms of U, V, and Σ.

e) Sketch a contour plot for the function

$$f(x) = \frac{1}{(2\pi)^{N/2}} B^{-1/2} \exp\left\{ \frac{1}{2} x^t B^{-1} x \right\} \ .$$

for $N = 2$ with

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \ .$$

a) $A^t = (YY^t)^t = (Y^t)^t Y^t = YY^t = A$

$\therefore$ A is symmetric.

$x^t A x = x^t (YY^t) x = x^t Y Y^t x$

$= (Y^t x)^t (Y^t x) \geq 0$

$\therefore$ A is positive semidefinite

Same for B.

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b) $A = YY^* = U\Sigma V^* (U\Sigma V^*)^*$

$= U\Sigma V^* V \Sigma^* U^*$

$= U \Sigma^2 U^*$

$\Rightarrow E = U \quad \Lambda = \Sigma^2$

c) $B = Y^* Y = (U\Sigma V^*)^* U \Sigma V^*$

$= V \Sigma U^* U \Sigma V^*$

$= V \Sigma^2 V^*$

$T = V \quad D = \Sigma^2$

d) $B^{-1} = T D^{-1} T^*$

$T = V \quad D^{-1} = (\Sigma^2)^{-1}$

e)