2.3

By Bayes' rule,

$$P(c \mid x, \theta, \pi) = \frac{P(c, x \mid \theta, \pi)}{P(x \mid \theta, \pi)}$$

Now take log to $P(c \mid x, \theta, \pi)$, we have

$$\log P(c \mid x, \theta, \pi) = \log \frac{P(c, x \mid \theta, \pi)}{P(x \mid \theta, \pi)}$$

$$= \log P(c, x \mid \theta, \pi) - \log P(x \mid \theta, \pi)$$

$$= \log P(c \mid \pi) P(x \mid c, \theta) - \log \left[\sum_{c=0}^{9} P(c \mid \pi) P(x \mid c, \theta) \right]$$

$$= \log P(c \mid \pi) + \log P(x \mid c, \theta) - \log \left[\sum_{c=0}^{9} P(c \mid \pi) P(x \mid c, \theta) \right]$$

$$= \log \left[P(c \mid \pi) \prod_{j=1}^{784} P(x_j \mid c, \theta) \right] - \log \left[\sum_{c=0}^{9} P(c \mid \pi) P(x \mid c, \theta) \right]$$

$$= \log P(c \mid \pi) + \sum_{j=1}^{784} \log P(x_j \mid c, \theta) - \log \left[\sum_{c=0}^{9} P(c \mid \pi) \prod_{j=1}^{784} P(x_j \mid c, \theta, \pi) \right]$$

$$= \log \pi_c + \sum_{j=1}^{784} \left[x_j \log \theta_{jc} + (1-x_j) \log (1-\theta_{jc}) \right] - \log \left[\sum_{c=0}^{9} P(c \mid \pi) \prod_{j=1}^{784} P(x_j \mid c, \theta, \pi) \right]$$

For the following function log-likelihood :

$$\text{log-likelihood} = \log \pi_c + \sum_{j=1}^{784} \left[x_j \log \theta_{jc} + (1-x_j) \log (1-\theta_{jc}) \right] - \log \left[\sum_{c=0}^{9} P(c \mid \pi) \prod_{j=1}^{784} P(x_j \mid c, \theta, \pi) \right]$$

π_c is given by MLE $\hat{\pi}_c$,

θ_{jc} is given by MAP estimator $\hat{\theta}_{MAP_{jc}}$

x_j is given by images