LECTURE NOTES ON DIFFERENCE EQUATIONS

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ABSTRACT. This is a collection of notes on a series of online lectures given by Prof. Ruan in Zhejiang University on April, 2020, based on handwriting notes of Weiqaing He and Yifan Li.

CONTENTS

1 Introduction

We shall begin with **linear differential equations**.

$$\left(a_0(z)\left(z\frac{\partial}{\partial z}\right)^n + a_1(z)\left(z\frac{\partial}{\partial z}\right)^{n-1} + \cdots + a_n(z)\right) f = 0, \tag{1.1}$$

where $a_0, \cdots, a_n$ are meromorphic functions.

Analogously, we can define ***q*-difference equations**,

$$\left(a_0(q,z){\sigma_q}^n + a_1\sigma_q^{n-1} + \cdots + a_n(q,z)\right) f = 0, \tag{1.2}$$

for $|q| < 1$ and $a_k(q,z)$ meromorphic with respect to z. The operator σ_q is defined as follows,

$$(\sigma_q f)(q,z) := f(q,qz).$$

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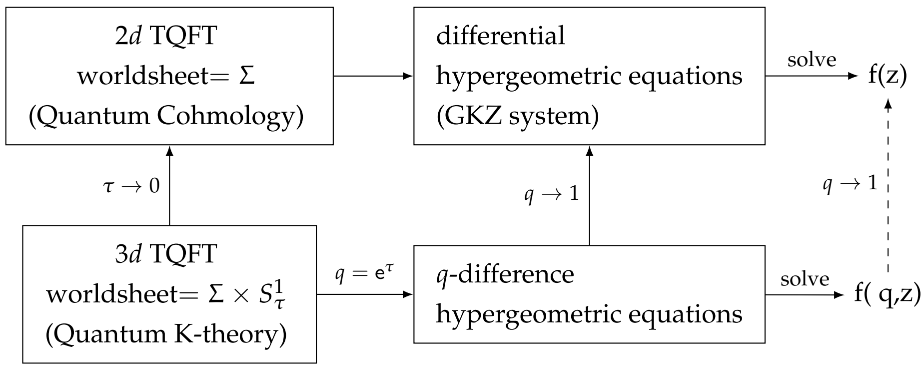
Both of them are introduced very naturally into the study of topological quantum field theory (TQFT for short) as follows,



Example 1.1. $\Delta_q f := \frac{\sigma_q - 1}{q-1} f$, then

$$\lim_{q\to 1} \Delta_q f = z\frac{\partial}{\partial z} f. \tag{1.3}$$

Here are some general remarks:

(I) Coefficients: For linear differential equations, the solution spaces are $\mathbb{C}$-linear spaces. However, for linear q-difference equations, solution spaces are K_q-linear spaces, where $K_q = \{\sigma_q f = f\}$, which is elliptic functions on elliptic curves $\mathbb{C}^\times/q\mathbb{Z}$.

(II) Analyticity: For a differential equation, we shall go through following steps to get a non-local solution,

However, for difference equations, things go really differently. Without lack of generality, assume $a_n(q,z) \neq 0$. Else, we shall set $\hat{f}(q,z) = f(q,qz)$, which will reduce the initial equation to another one with a lower order. Then,

$$f(q,z) = -\frac{1}{a_n(q,z)}\left(a_0(q,z)f(q,q^n z) + \cdots + a_{n-1}(q,z)f(q,qz)\right). \tag{1.4}$$

Hence, if $f(q,z)$ is meromorphic for $|z| < R$, then $f(q,z)$ is also meromorphic for $|qz| < R$, and so $f(q,z)$ is meromorphic on $\mathbb{C}^\times$.

In summary, there is **No issue of analytic continuation**!

(III) Change of basis: K_q linear transformations shall give purely nontrivial identities!

2 Worm-up: Review of Solutions for Differential Equations

Find local solutions at $z = 0$ for

$$\left(a(z)\frac{\partial^2}{\partial z^2} + b(z)\frac{\partial}{\partial z} + c(z)\right) f = 0. \tag{2.1}$$

Case I: $a(0) \neq 0$. Locally,

$$\frac{\partial^2}{\partial z^2} f + B(z)\frac{\partial}{\partial z} f + C(z) f = 0, \text{ with } f(0) = \alpha, \frac{\partial f}{\partial z}(0) = \beta,$$

has a unique solution along a path. The point is we can expect **multi-valued solutions**.

To be explicit, assume

$$f(z) = \sum_{n\geqslant 0} f_n z^n, \quad B(z) = \sum_{n\geqslant 0} b_n z^n, \quad C(z) = \sum_{n\geqslant 0} c_n z^n.$$

Then we have

$$\sum_{n\geqslant 2} n(n-1) f_n z^{n-2} + \sum_{n\geqslant 1}\sum_{m\geqslant 0} n b_m f_n z^{m+n-1} + \sum_{n\geqslant 0}\sum_{m\geqslant 0} c_m f_n z^{n+m} = 0$$

$$\Longrightarrow \forall n \geqslant 0, (n+2)(n+1) f_{n+2} + \sum_{k=0}^{n} (n+1-k) b_k f_{n+1-k} + \sum_{k=0}^{n} c_k f_{n-k} = 0$$

$$\Longrightarrow \forall n \geqslant 0, f_{n+2} = -\frac{1}{(n+1)(n+2)}\left(\sum_{k=0}^{n} (n+1-k) b_k f_{n+1-k} + \sum_{k=0}^{n} c_k f_{n-k}\right). \tag{2.2}$$

Given $f_0 = \alpha, f_1 = \beta$, we shall determine the whole $\{f_n\}$. Hence, the solution space is a two dimensional space.

Case II: $a(z) = z\tilde{a}(z)$ with $\tilde{a}(0) \neq 0$. Then we shall write

$$\frac{\partial^2}{\partial z^2} f + B(z)\frac{\partial}{\partial z} f + C(z) f = 0,$$

where $B(z)$ and $C(z)$ has order one pole at $z = 0$, thus,

$$B(z) = \sum_{n\geqslant -1} b_n z^n, \quad C(z) = \sum_{n\geqslant -1} c_n z^n.$$

Similarly, we have

$$\forall n \geqslant 0, (n+2)(n+1) f_{n+2} + \sum_{k=-1}^{n} (n+1-k) b_k f_{n+1-k} + \sum_{k=-1}^{n} c_k f_{n-k} = 0, \tag{2.3}$$

but while $n = -1$,

$$b_{-1} f_1 + c_{-1} f_0 = 0. \tag{2.4}$$

There exists only one (dimensional) solution in the form of power series.

Case III: $a(z) = z^2\tilde{a}(z)$ with $\tilde{a}(0) \neq 0$. We shall only work with regular singular cases,

Definition 2.1. We say the 2nd order differential equation (2.1) is **regular singular** at $z = 0$, if $a(z) = z^2\tilde{a}(z), b(z) = z\tilde{b}(z)$, such that $\tilde{a}(z), \tilde{b}(z), c(z)$ regular at $z = 0$ and that $\tilde{a}(0) \neq 0$.

In these cases, we shall rewrite (2.1) (by abuse of notations) as

$$\Big(a(z)D^2 + b(z)D + c(z)\Big) f = 0, \tag{2.5}$$

where $a(z), b(z), c(z)$ is regular at $z = 0$, $a(0) \neq 0$ and the operator D is defined as

$$D := z\frac{\partial}{\partial z}. \tag{2.6}$$

Example 2.1 (Euler's hypergeometric equations)**.** The 2nd order differential equation

$$z(1-z)\frac{\partial^2}{\partial z^2}f + (c - (a+b+1)z)\frac{\partial}{\partial z}f - abf = 0 \tag{2.7}$$

is regular singular at $z = 0, 1, \infty$. At $z = 0$, we shall rewrite it as

$$(1-z)D^2 f + \big((c-1) - (a+b)z\big)Df - abzf = 0. \tag{2.8}$$

Without lack of generality, we shall assume $a(z) \equiv 1$, and write

$$b(z) = \sum_{n \geqslant 0} b_n z^n, \quad c(z) = \sum_{n \geqslant 0} c_n z^n, \quad f(z) = z^r \sum_{n \geqslant 0} f_n z^n.$$

Then $\forall n \geqslant 1$,

$$f_n(n+r)^2 + \sum_{k=0}^{n} (n+r)b_k f_{n+r-k} + \sum_{k=0}^{n} c_k f_{n-k} = 0.$$

$$\Longrightarrow f_n = -\frac{1}{(n+r)^2 + b_0(n+r) + c_0} \sum_{k=1}^{n} \big((n+r-k)b_k + c_k\big) f_{n-k}, \tag{2.9}$$

While for $n = 0$, $f_0(r^2 + b_0 r + c_0) = 0$. We call the equation

$$X^2 + b_0 X + c_0 = 0 \tag{2.10}$$

the **characteristic polynomial**, which says that $f(z)$ defined above is a nonzero solution only if r is a root of the characteristic polynomial and $(n+r)$ is not a root for all $n \geqslant 1$. Now suppose r_1, r_2 are two roots of the characteristic polynomial.

Case III.1: $r_1 - r_2 \notin \mathbb{Z}$. There are two distinct solutions in the form as above, which are $z^{r_1}F_1$ and $z_{r_2}F_2$. F_i are power series and $z^{r_i}F_i$ is multi-valued unless $r_i \in \mathbb{Z}$. The monodromy action (along a simple loop aroud $z = 0$) is diagonal with eigenvalues $\mathrm{e}^{2\pi r_1\sqrt{-1}}, \mathrm{e}^{2\pi r_2\sqrt{-1}}$.

Case III.2: $r_1 = r_2 + n_0$ with $n_0 \in \mathbb{Z}_{\geqslant 0}$. $z^{r_1}F$ will give one solution. By Frobenius' method, another solution will be in the form of

$$z^{r_1}\log zF + z^{r_2}G = \log z \sum_{n\geqslant 0} f_n z^{n+r_1} + \sum_{n\geqslant 0} g_n z^{n+r_2},$$

which satisfies for all $n \geqslant n_0$,

$$(n+r_2)^2 g_n + \sum_{k=0}^{n} \big((n-k+r_2)b_k + c_k\big)g_{n-k} + 2(n+r_2)f_{n-n_0} + \sum_{k=0}^{n-n_0} b_k f_{n-n_0-k} = 0, \tag{2.11}$$

and for $n < n_0$,

$$(n+r_2)^2 g_n + \sum_{k=0}^{n} \big((n-k+r_2)b_k + c_k\big)g_{n-k} = 0. \tag{2.12}$$

Especially, when $n = n_0$, we have

$$\sum_{k=1}^{n_0} \big((r_1-k)b_k + c_k\big)g_{n_0-k} + 2r_1 f_0 + b_0 f_0 = 0. \tag{2.13}$$

In summary, G is determined by g_{n_0} and f_0. Especially, if $f_0 = 0$, we shall have $z^{r_1}\log zF + z^{r_2}G = z^{r_2}z^{n_0}\tilde{G}$, which is a solution in the form of $z^{r_1}F$.

Example 2.2 (Euler's hypergeometric equation)**.** For (2.7), assume $c > 1$, then the characteristic polynomial is

$$X^2 + (c-1)X = 0, \tag{2.14}$$

whose roots are $r_1 = 0, r_2 = 1-c$. With respect to r_1, the hypergeometric series

$${}_2F_1(a,b;c;z) := \sum_{n\geqslant 0} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

gives one special solution. Here, $(q)_n$ is the **Pochhammer symbol** defined as

$$(q)_n := q(q+1)\cdots(q+n-1). \tag{2.15}$$

With respect to r_2, in cases where $c \notin \mathbb{Z}$, the hypergeometric series

$$z_2^{1-c}F_1(1+a-c, 1+b-c; 2-c; z) := \sum_{n\geqslant 0} \frac{(1+a-c)_n(1+b-c)_n}{(2-c)_n}\frac{z^{1+n-c}}{n!}$$

gives another special solution.

3 q-difference Equations

A 2nd order q-difference equation is in the form of

$$\big(a(q,z)\sigma_q^2 + b(q,z)\sigma_q + c(q,z)\big)f(q,z) = 0, \tag{3.1}$$

where $c(q,z) \neq 0$. (3.1) is called **regular singular** if $a_0(q) := a(0,q) \neq 0$. In there cases, we can rewrite (3.1) as

$$\big(\sigma_q^2 + b(q,z)\sigma_q + c(q,z)\big) f(q,z) = 0, \tag{3.2}$$

with

$$b(q,z) = \sum_{n\geqslant 0} b_n(q) z^n, \quad c(q,z) = \sum_{n\geqslant 0} c_n(q) z^n.$$

For solutions in the form of

$$\sum_{n\geqslant 0} f_n(q) z^{n+r},$$

we shall have

$$\big(q^{2(n+r)} + q^{n+r} b_0 + c_0\big) f_n + \sum_{k=1}^{n} (q^{n+r-k} b_k + c_k) f_{n-k} = 0. \tag{3.3}$$

We can get a nonzero solution $f(q,z) \neq 0$ only if q^r is a root of the **characteristic polynomial** $X^2 + b_0 X + c_0$ and q^{n+r} is not a root for all $n \geqslant 1$. In summary, suppose q^{r_1}, q_2^r are roots of the characteristic polynomial, then

Case I: $r_1 - r_2 \notin \mathbb{Z}$, then there exists two linearly independent special (might be multi-valued) solutions in the form of $z^{r_1} F_1, z^{r_2} F_2$, for F_1, F_2 power series.

Case II: $r_1 - r_2 = n_0 \in \mathbb{Z}_{\geqslant 0}$. There exists at least one special (multi-valued) solution. For other solutions, we shall leave later.

Example 3.1 (q-hypergeometric equations)**.** Consider

$$\big((1-\sigma_q)(1-q^\gamma \sigma_q) - z(1-q^\alpha \sigma_q)(1-q^\beta \sigma_q)\big) f = 0, \tag{3.4}$$

for $\gamma \notin \mathbb{Z}$. The characteristic polynomial is $(1-X)(1-q^\gamma X)$, whose roots are $q^0, q^{-\gamma}$. With respect to q^0, one solution is given by the ***q*-hypergeometric series**,

$${}_2\phi_1(\alpha, \beta; 1+\gamma; q, z) := \sum_{n\geqslant 0} \frac{(q^\alpha; q)_n (q^\beta; q)_n}{(q^{1+\gamma}; q)_n} \frac{z^n}{(q;q)_n}, \tag{3.5}$$

where $(a;q)_n$ is the ***q*-Pochhammer symbol** defined as

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{3.6}$$

With respect to $q^{-\gamma}$, another solution can be given by

$$z^{-\gamma}{}_2\phi_1(\alpha-\gamma, \beta-\gamma; 1-\gamma; q; z). \tag{3.7}$$

Remark. q-difference equations of the form of

$$(\cdots) + z(\cdots) = 0$$

is essential in the study of quintics, whose q-analog of Picard-Fuchs equation is

$$\big((1-\sigma_q)^5 - z(1-\sigma_q^5)(1-q\sigma_q^5)\cdots(1-q^4\sigma_q^5)\big) f = 0. \tag{3.8}$$

Remark. The generalized q-hypergeometric series is defined as

$$ {}_r\phi_s\left(\begin{matrix}a_1, \cdots, a_r \\ b_1, \cdots, b_s\end{matrix}; q, z\right) := \sum_{n\geqslant 0} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(b_1;q)_n \cdots (b_s;q)_n} \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{1+s-r} \frac{z^n}{(q;q)_n}, \tag{3.9}$$

where the factor $\left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{1+s-r}$ cones from the contribution of the level structure.

Then, we shall consider q-difference equations

$$\sigma_q f(q,z) = q^r f(q,z), \tag{3.10}$$

$$\sigma_q f(q,z) = f(q,z) + 1. \tag{3.11}$$

They both have special solutions, which are multi-valued. To be explicit, we have

$$\sigma_q z^r = q^r z^r, \quad \sigma_q \frac{\log z}{\log q} = \frac{\log z}{\log q} + q.$$

If $r \notin \mathbb{Z}$, z^r is multi-valued. Hence, to find a single-valued solution, we shall introduced **the Jacobian Theta function**[1].

Definition 3.1. The Jacobian Theta function is defined as

$$\theta_q(z) := \sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} z^n. \tag{3.12}$$

It will satisfies the q-difference equation[1],

$$\sigma_q \theta_q(z) = \frac{1}{z}\theta_q(z). \tag{3.13}$$

With this, we can define the ***q*-character** associated to the eigenvalue $c(q) \in \mathbb{C}^\times$ and the ***q*-logarithm**[2],

Definition 3.2. Associated to $c(q) \in \mathbb{C}^\times$, the q-character is defined as

$$e_{q,c(q)}(z) := \frac{\theta_q(z)}{\theta_q(c(q)z)}. \tag{3.14}$$

The q-logarithm is defined as

$$\ell_q(z) := -\frac{z\theta_q'(z)}{\theta_q(z)}. \tag{3.15}$$

One can easily prove that both $e_{q,c(q)}(z)$ and $\ell_q(z)$ are meromorphic functions and satisfy[2],

$$\sigma_q e_{q,c(q)}(z) = c(q) e_{q,c(q)}(z), \quad \sigma_q \ell_q(z) = \ell_q(z) + 1.$$

What's more, if $c(q) = q^n$ for $n \in \mathbb{Z}$,

$$e_{q,q^n} = q^{\frac{n(n-1)}{2}} z^n.$$

For now, we will look back at the 2nd order q-difference equation (3.2). Suppose c_1, c_2 are roots of the characteristic polynomial.

Case I: $c_1/c_2 \notin q^{\mathbb{Z}}$, then we shall have two special single-valued solutions,

$$e_{q,c_1}F_1, \quad e_{q,c_2}F_2,$$

with F_1, F_2 formal power series.

Case II: $c_1 = c_2 q^{n_0}$ for some $n_0 \in \mathbb{Z}_{\geqslant 0}$. There still exists one special solution $e_{q,c_1}F$. Another solution is of the form of

$$\ell_q(z)e_{q,c_1}(z)F(q,z) + e_{q,c_2}(z)G(q,z),$$

with F, G formal power series and $e_{q,c_1}(z)F(q,z)$ a special solution as above. Then, we have

$$\begin{aligned}
&(\ell_q+2)c_1^2 e_{q,c_1}\sigma_q^2 F + c_2^2 e_{q,c_2}\sigma_q^2 G + b(\ell_q+1)c_1 e_{q,c_1}\sigma_q F \\
&+ bc_2 e_{q,c_2}\sigma_q G + c(\ell_q e_{q,c_1}F + e_{q,c_2}G) \\
=&2c_1^2 e_{q,c_1}\sigma_q^2 F + bc_1 e_{q,c_1}\sigma_q F + c_2^2 e_{q,c_2}\sigma_q^2 G + bc_2 e_{q,c_2}\sigma_q G + c e_{q,c_2} G \\
=&2q^{\frac{n_0(n_0+3)}{2}} c_2^{n_0+2} e_{q,c_2} z^{n_0}\sigma_q^2 F + bc_2^{n_0+1} q^{\frac{n_0(n_0+1)}{2}} e_{q,c_2} z^{n_0}\sigma_q F \\
&+ c_2^2 e_{q,c_2}\sigma_q^2 G + bc_2 e_{q,c_2}\sigma_q G + c e_{q,c_2} G,
\end{aligned}$$

which indicates that G is determined by f_0 and g_{n_0} (g_k for $k < n_0$ is determined by f_0). If $f_0 = 0$, this solution will be in the form of the first kind again.

There are also non-regular-singular cases, which are more problematic for analyticity.

Example 3.2 (Ramanujan Equation)**.** Consider

$$\big(qz\sigma_q^2 - \sigma_q + 1\big)f = 0. \tag{3.16}$$

The characteristic polynomial is $-x+1$, whose root is $x = 1 = q^0$. Hence, we may consider a solution in the form of

$$f = \sum_{n\geqslant 0} f_n(q)z^n.$$

By direct calculation,

$$\forall n \geqslant 1, q^{2n-1}f_{n-1} - (q^n - 1)f_n = 0,$$

and f_0 is a free variable. Assume $f_0 = 1$, we shall have

$$f_n = (-1)^n \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q)_n}\left((-1)^n q^{\frac{n(n-1)}{2}}\right)^2 (-q)^n. \tag{3.17}$$

Hence,

$$f(q,z) = {}_0\phi_1(-;0;q,-qz). \tag{3.18}$$

For another special solution, we shall consider a solution in the form of

$$\theta_q(z)\sum_{n\geqslant 0} f_n z^n.$$

By direct calculation,

$$qzq^{-1}z^{-2}\theta_q\sum_{n\geqslant 0} f_n q^{2n}z^n - z^{-1}\theta_q\sum_{n\geqslant 0} f_n q^n z^n + \theta_q\sum_{n\geqslant 0} f_n z^n = 0$$

$$\Longrightarrow \forall n\geqslant 1, (q^{2n}-q^n)f_n + f_{n-1} = 0, \tag{3.19}$$

and also f_0 is free. Assume $f_0 = 1$, we shall have

$$f_n = \frac{1}{(1;q)_n}\left(q^{\frac{n(n-1)}{2}}\right)^{-1} q^{-n}. \tag{3.20}$$

Hence

$$f(q,z) = \theta_q(z)\,{}_3\phi_1(0,0,0;0;q;-\frac{z}{q}). \tag{3.21}$$

4 General Technique

4.1 Newton polygon

For a q-difference equation in the form of

$$\mathcal{L}(z,\sigma_q)f(q,z) = 0,$$

where $\mathcal{L}$ is a polynomial of σ_q with coefficients in $\mathbb{C}[[z]]$, i.e.,

$$\mathcal{L}(z,\sigma_q) = \sum_s a_s(z)\sigma_q^s,$$

the associative **Newton polygon** in $\mathbb{R}^2$ is denoted by $N(\mathcal{L})$ and defined as

$$N(\mathcal{L}) := \text{Convex hull of } \{(s,r) \mid r\geqslant ord(a_s)\}. \tag{4.1}$$

Hence, $N(\mathcal{L})$ is bounded by two vertical lines from two sides and a series of segments of slopes $\mu_1 < \cdots \mu_l$ and lengths (of their projections onto the x-axis) $r_1,\cdots,r_l\in\mathbb{Z}_{\geqslant 1}$ from below. r_i is called the multiplicities with respect to μ_i. Up to translation, $N(\mathcal{L})$ is determined by

$$\{(\mu_1,r_1),\cdots,(\mu_l,r_l) \mid (\mu_i,r_i)\in\mathbb{Q}\times\mathbb{Z}_{\geqslant 1}, \mu_1<\cdots<\mu_l\}.$$

Since $N(z^j\sigma_q^i\mathcal{L}(z,\sigma_q)) = N(\mathcal{L}) + (i,j)$, we shall always assume that the lower boundary of $N(\mathcal{L})$ intersects with x-axis at exactly one point or along one segment $(\mu_i = 0, r_i)$. For a segment (μ_i, r_i), if $\mu_i = 0$, we shall associate it with a **characteristic polynomial**

of slope 0 by

$$\mathcal{L}(0, X). \tag{4.2}$$

In general, we shall associate it with a **characteristic polynomial of the slope μ** by

$$z^C \theta_q^{-\mu} \mathcal{L} \theta_q^{\mu}(0, X), \tag{4.3}$$

where C is some appropriate integer, and define the **exponents** of the slope μ as the **nonzero** roots of the characteristic polynomial of slope μ. Notice that

$$\sharp\{\text{exponents of the slope } \mu_i\} = r_i,$$

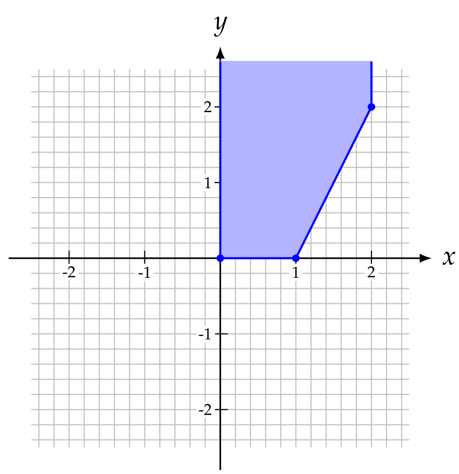
we shall denote by $c_{i,j}$ with $1 \leqslant j \leqslant r_i$, the nonzero roots of the associative characteristic polynomial. Assume that $c_{i,j}/c_{i,j'} \notin q^{\mathbb{Z}}$ unless $j = j'$. Then for each $(\mu_i, c_{i,j})$ we shall consider a solution in the form of

$$\theta_q^{\mu_i} e_{q,c_{i,j}} \sum_{n \geqslant 0} f_n z^n.$$

Example 4.1. Consider

$$(z^2 \sigma_q^2 - \sigma_q + 1) f = 0.$$

Its Newton polygon $N(\mathcal{L})$ is as follows,



Then the characteristic polynomial of the slope $\mu_1 = 0$ is

$$-X + 1,$$

and the characteristic polynomial of the slope $\mu_2 = 2$ is

$$q^{-2} X^2 - X.$$

The corresponding exponents are $1 = q^0$ and q^2 respectively. We shall consider solutions in the forms of

$$\sum_{n\geqslant 0} f_n z^n, \quad \theta_q^2 e_{q,q^2} \sum_{n\geqslant 0} g_n z^n$$

By direct calculation, we shall have the following two linear independent solutions,

$$ {}_0\phi_0(-;-;q^2,z^2) = \sum_{n\geqslant 0} (-1)^n q^{n(n-1)} \frac{z^{2n}}{(q^2;q^2)_n}, \tag{4.4}$$

$$\theta_q^2(z) e_{q,q^2}(z) {}_0\phi_1(-;0;q^{-2},-\frac{z^2}{q^6}) = \theta_q^2(z) e_{q,q^2}(z) \sum_{n\geqslant 0} \left(q^{-n(n-1)}\right)^2 \frac{\left(-\frac{z^2}{q^6}\right)^n}{(q^{-2};q^{-2})_n}. \tag{4.5}$$

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[1] David Mumford. *Tata lectures on theta. I*, volume 28 of Progress in Mathematics. Birkähuser Boston, Inc., Boston, MA, 1983. With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman.

[2] Jacques Sauloy. Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie. *Ann. Inst. Fourier (Grenoble)*, 50(4):10211071, 2000.

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