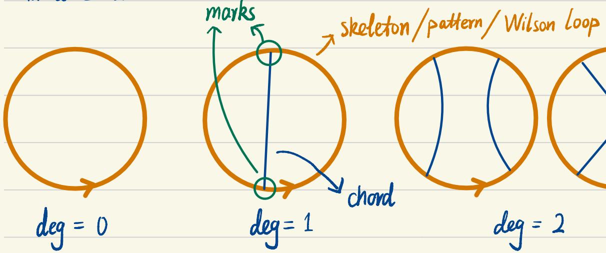
Jacobi Diagrams

Yifan Li

Recall.

- $\mathcal{C} = \bigoplus_{m\geq 0} \mathcal{C}_m$, vector space spanned by chord diagrams (of order / degree m) in a circle.



(Gauss code)

$*$: Vassiliev invariants only depend on the cyclic order of the singular points of a singular knot.

$\rightsquigarrow$ $\mathcal{C}_m$ is enough to define degree m Vassiliev invariants.

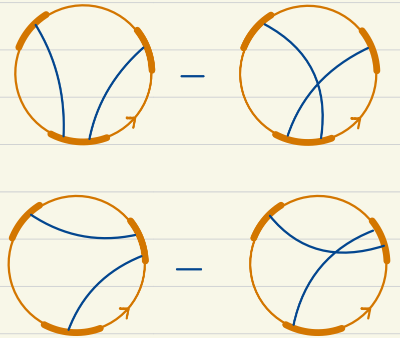
$$\begin{array}{ccccccccc} \mathcal{V}_m & & \mathcal{V}_m/\mathcal{V}_{m-1} & & & & \mathcal{V}_1/\mathcal{V}_0 & & \mathcal{V}_0 \\ \downarrow{\scriptstyle \beta_m} & = & \downarrow{\scriptstyle \alpha_m} & \oplus & \cdots & \oplus & \downarrow{\scriptstyle \alpha_1} & \oplus & \downarrow{\scriptstyle \alpha_0} \\ \mathcal{C}^*_{\leq m} & & \mathcal{C}^*_m & & & & \mathcal{C}^*_1 & & \mathcal{C}^*_0 \end{array}$$

$\star$ α_m is not surjective! $\mathcal{C}_m$ are too big, so we should find some relations in $\mathcal{C}_m$, such that

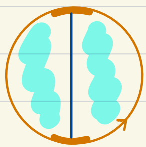
$$\exists\ \varphi_m : \mathcal{C}_m/\sim \ \xrightarrow{\cong} \left(\mathcal{V}_m/\mathcal{V}_{m-1}\right)^* \cong \mathcal{K}_m/\mathcal{K}_{m+1}$$

and $\varphi_m^* = \alpha_m$.

- (4T-relations)



- (1T-relations)

 $= 0$.

Thm. As vector spaces,

$$\varphi_m : \bar{\mathcal{A}}_m^{G} := \frac{\mathcal{C}_m}{1T, 4T} \longrightarrow \mathcal{K}_m / \mathcal{K}_{m+1},$$

$$\varphi_m^f : \mathcal{A}_m^{\mathcal{C}} := \frac{\mathcal{C}_m}{4T} \longrightarrow \mathcal{K}_m^f / \mathcal{K}_{m+1}^f.$$

However, we need to go through a long path to show φ_m and φ_m^f are isomorphisms!

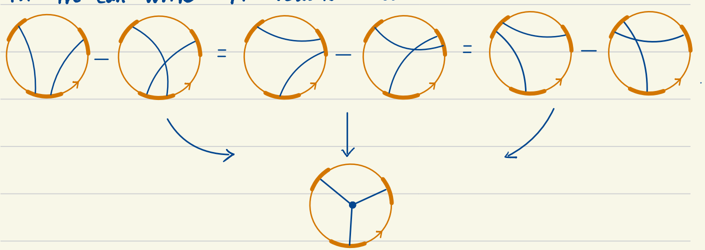
Questions

1). How to understand 4T-relations?

2). Is $\varphi^f := \oplus \varphi_m^f$ only a vector space isomorphism? Or say, Does $\mathcal{A}^{\mathcal{C}}$ carry any canonical algebraic structures?

Idea.

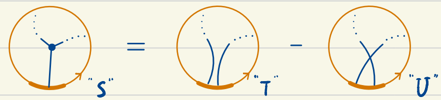
1). We can write 4T-relations as



$\boxed{\text{4T-relations are equivalences between different ways to represent a trivalent vertex.}}$

Hence, we need to consider unitrivalent graphs as the generelization of chord diagrams,

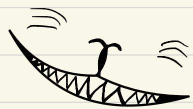
i). Trivalent vertices can be expressed by commutators of chords.



(STU-relation)

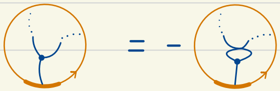
[Bar-Natan]

"S" is the smile of a Cheshire cat.



ii). Cyclic order of half edges around each trivalent vertex matters.

E.g.,

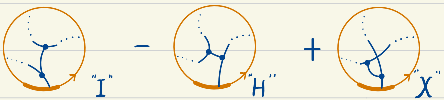


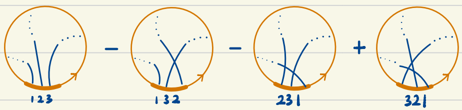
(AS - relations)

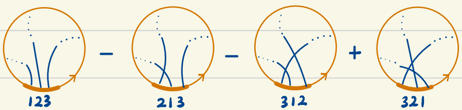
"AS" is short for "anti-symmetric".

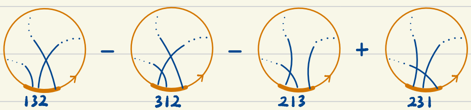
iii). New relations come in for pairs of trivalent vertices.

E.g.,

$0 =$  (IHX - relations)

$I =$ 

$H =$ 

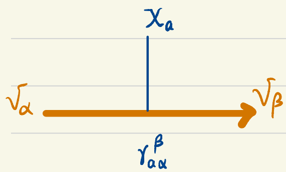
$X =$ 

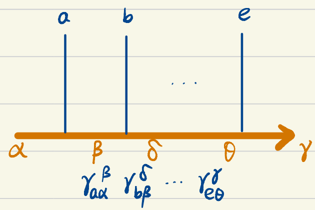
2). Via an algebraic way. Imagine we have an action

$$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V) \quad \rightsquigarrow \quad \rho : T(\mathfrak{g}) \to \mathrm{End}(V)$$

with $\mathfrak{g} = \mathrm{span}_k\{X_a, X_b, X_c \cdots\}$ and $\rho(X_a)\, v_\alpha =: \gamma^{\beta}_{a\alpha} v_\beta$

$V = \mathrm{span}_k\{v_\alpha, v_\beta, v_\gamma, \cdots\}$

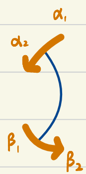




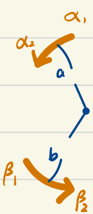
[Penrose]

n-valent vertex $\rightsquigarrow$ n-tensor $\in \mathfrak{g}^{\otimes n}$

connecting half edges $\rightsquigarrow$ contraction through all indices



a chord $\rightsquigarrow$



$\rightsquigarrow \eta = \eta^{ab} X_a \otimes X_b \in \mathfrak{g}^{\otimes 2}$

$$\eta^{ab}\, \gamma^{\alpha_2}_{a\alpha_1}\, \gamma^{\beta_2}_{b\beta_1}$$

Hence, for $\mathfrak{g}$, η and V, we define a map

$$T_{\mathfrak{g},\eta,V} : D \in \mathcal{G} \longmapsto k.$$

$\boxed{\text{Assumption I : } T_{\mathfrak{g},\eta,V} \text{ is a weight system for nice } \mathfrak{g}, \eta, V.}$

$\boxed{\text{Assumption II : } \exists!\ T_{\mathfrak{g},\eta} \text{ universal for all } V.}$

i). A chord is unoriented, so η should by symmetric, i.e.,

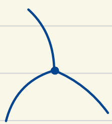
$$\eta \in \mathrm{Sym}^2(\mathfrak{g})$$

ii). η should be non-degenerate, $\exists\ t \in \mathrm{Sym}^2(\mathfrak{g}^\vee)$,

$$\eta = t^{-1}.$$

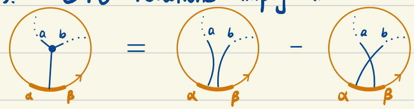
iii). $T_{\mathfrak{g},\eta,V}$ should satisfies $(4T)^*$.

$\boxed{\text{Assumption III : There is a canonical generalization of Penrose's construction for trivalent vertices.}}$



$$\rightsquigarrow \quad f = f^{abc} X_a \otimes X_b \otimes X_c \in \mathfrak{g}^{\otimes 3}$$

iv). STU relations imply that



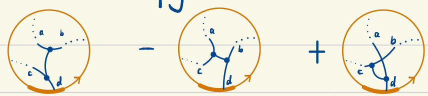
$$f_{ab}{}^{c}\, \gamma^{\beta}_{c\alpha} = \gamma^{\delta}_{a\alpha}\, \gamma^{\beta}_{b\delta} - \gamma^{\delta}_{b\alpha}\, \gamma^{\beta}_{a\delta} \iff \rho(f_{ab}{}^{c} X_c) = [\rho(X_a), \rho(X_b)].$$

v). Trivalent vertices are antisymmetric and cyclic invariant.

$$\iff \quad f \in \Lambda^3(\mathfrak{g}).$$

Denote by $[X_a, X_b] := f_{ab}{}^{c} X_c \in \mathfrak{g}$.

vi). IHX relations imply that



$$0 = f^{abe} f^{dc}{}_{e} - f^{aec} f^{bd}{}_{e} + f^{ade} f^{cb}{}_{e}$$

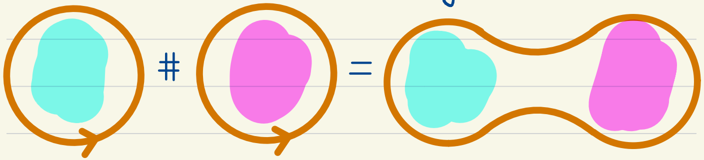
$$\iff [[X_a, X_b], X_c] + [[X_c, X_a], X_b] + [[X_b, X_c], X_a] = 0$$

vii). $f_{ab}{}^{d}\, t_{dc} = f_{abc} = f_{bca} = f_{bc}{}^{d}\, t_{da}.$

$\iff$ t is an invariant bilinear form.

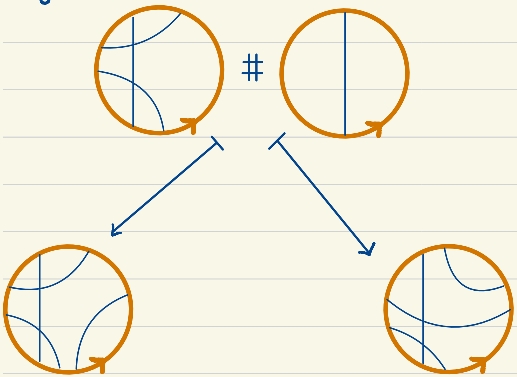
$\boxed{(\mathfrak{g}, \eta) \text{ is a Lie algebra with } \eta \in \mathrm{Sym}^2(\mathfrak{g}) \text{ a Casimir element.}}$

3). Connect sum between chord diagrams.



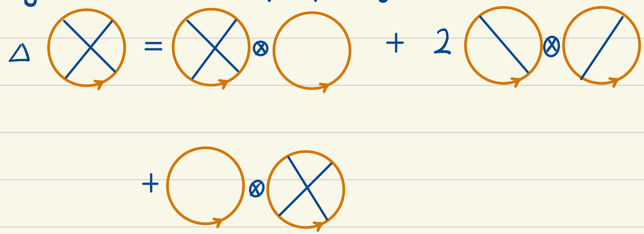
However, such an operation is not well-defined.

E.g.



But they are equivalent by 4T - relations!

Chords decomposition splits a chord diagram in all the ways then sum them up formally.



This gives a coproduct.

Plan.

i). Space of Jacobi diagrams / STU relations, $\mathcal{A}$

ii). $\mathcal{A} \cong \mathcal{A}^c$ as graded vector spaces

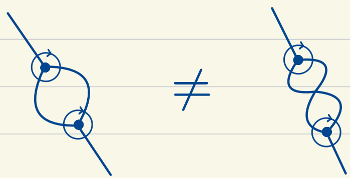
iii). $(\mathcal{A}, \#, \Delta) \cong (\mathcal{A}^c, \#, \Delta)$ as Hopf algebras.

§1. Jacobi Diagrams, STU relations.

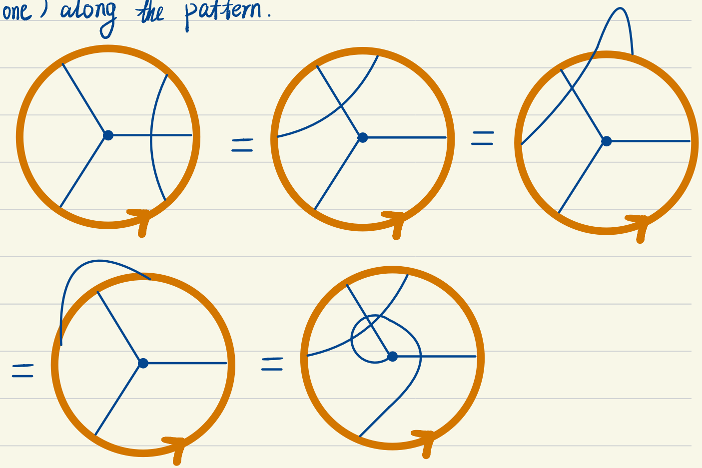
Def 1.1. (Jacobi diagrams)

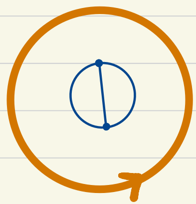
1). A unitrivalent graph $G = (V(G), E(G))$ is a graph satisfies that

- Each edge in $E(G)$ is a pair of half edges.
- $V(G) = V_{int}(G) \sqcup V_{ext}(G)$
  - i). each interior vertex in $V_{int}(G)$ is a set of three of edges with fixed cyclic order.
  - ii). each exterior vertex in $V_{ext}(G)$ is a set of a single half edge.
- Each half edge in any vertex appears only once in edges and vice verse.



2). A closed Jacobi diagram is a unitrivalent graph and an oriented circle, called the pattern/skeleton/Wilson loop, with an assignment of exterior vertices (at least one) along the pattern.





not allowed !

3). Denote by $\mathcal{D}$ the space spanned by all closed Jacobi diagrams. (CJD)

$\mathcal{D}$ has a trigrading structure.

$\forall$ closed diagram D (even with higher-valent vertices),

i). order of D,

$$\mathrm{ord}(D) := |E(D)| - |V_{int}(D)|.$$

ii). degree of D,

$$\deg(D) := 2|E(D)| - 3|V_{ext}(D)| - |V_{int}(D)|$$

iii). level of D,

$$\mathrm{level}(D) := |V_{ext}(D)|$$

If D is a CJD,

$$\mathrm{ord}(D) = \frac{1}{2}|V(D)|, \deg(D) = 0.$$

By order and level, we can write

$$\mathcal{D} = \bigoplus_{m\geq 0} \mathcal{D}_m$$

and

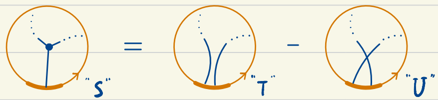
$$\mathcal{D}_m = \bigoplus_{k=1}^{2m} \mathcal{D}_m^k$$

$D \in \mathcal{D}_m^k$ iff $\mathrm{ord}(D) = m$, $\mathrm{level}(D) = k$.

Notice that $\mathcal{C}_m \cong \mathcal{D}_m^{2m}$.

Def 1.2. (Graded space $\mathcal{A}$)

1). On $\mathcal{D}_m$, $\forall m \geq 0$, the STU relations are



2). Define the space $\mathcal{A}$ as

$$\mathcal{A} := \mathcal{D}/(STU).$$

$\mathcal{A}$ has a decomposition by order,

$$\mathcal{A} = \bigoplus_{m\geq 0} \mathcal{A}_m := \bigoplus_{m\geq 0} \mathcal{D}_m/(STU).$$

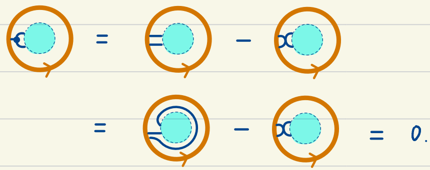
3). $\mathcal{A}_m$ has an increasing filtration

$$\mathcal{A}_m = \mathcal{A}_m^{2m} \supset \cdots \supset \mathcal{A}_m^1,$$

where $\mathcal{A}_m^k$ is spanned by $[D]$ for D in $\mathcal{D}_m^{\leq k}$.

Example.

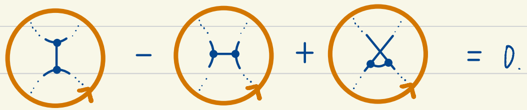
$\mathcal{A}_m^1 = 0$ in general, because



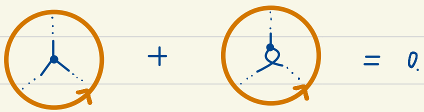
Lem 1.3. (IHX and AS relations)

STU relations imply the following two relations

1). IHX relations



2). AS relations



pf. By induction on the minimal distance of trivalent vertices from the pattern.

1°. distance = 0. Done by previous discussion.

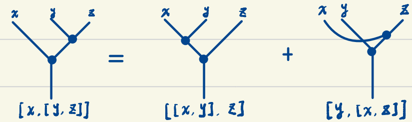
2°. Apply STU to move a trivalent vertex closer to the pattern. □

Example.

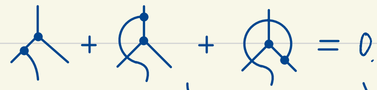
By AS relation,

Remark. IHX relations can be written as

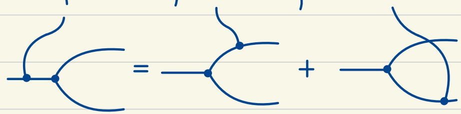
(Jacobi form)



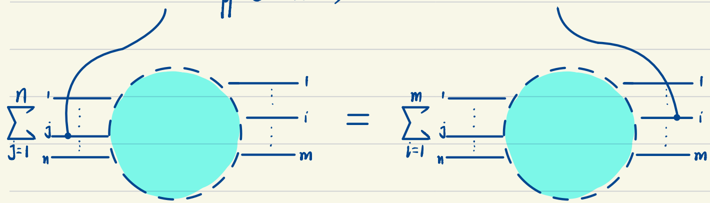
(Symmetric form)



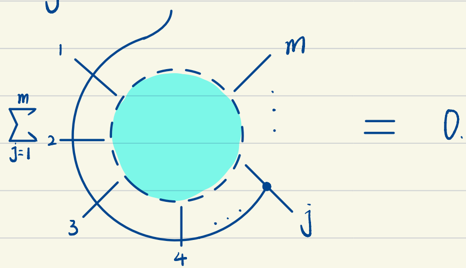
(Kirchhoff form)



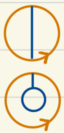
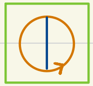
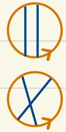
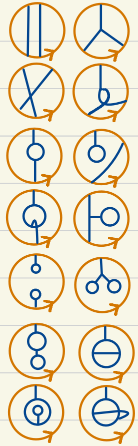
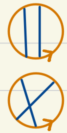
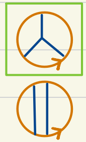
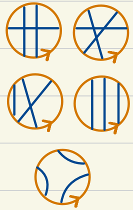
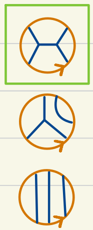
Lem 1.3′. (Kirchhoff's law)



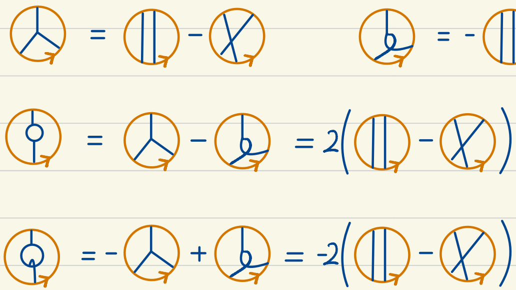
Equivalently,



Example.

m	$\mathcal{C}_m$	$\mathcal{D}_m$	$\mathcal{A}^G_m$	$\mathcal{A}_m$
0	1 	1 	1 	1 
1	1 	2 	1 	1 
2	2 	12 	2 	2 
3	5 	Too many!	3 	3 

For order 2 diagram,



Remark. The definition for $\mathcal{D}_m$ is not good!

Remark.

1). There is a better way to define diagrams. Fix dimension d.

i). Define the spaces of labelled ribbon diagraphs with V vertices and E edges, $gc_d^{V,E}$

ii). Consider relabelling action $S_V \times S_E \ltimes (S_2)^E$ on $gc_d^{V,E}$

1° If d is odd, S_V and S_2 actions change the sign

2° If d is even, S_E action changes the sign.

iii). Define the space GC_d as

$$GC_d := \prod_{V,E} \left(gc_d^{V,E}\right)_{S_V \times S_E \ltimes (S_2)^E}.$$

iv). GC_d carries a natural bigrading structure, and a differential ∂

"replacing a vertex (val ≥ 4) by $\bullet\!\!-\!\!\!-\!\!\bullet$ in all ways"

$\deg \partial = -1 \quad \Rightarrow \quad \deg(\text{edge}) + \deg(\text{vertex}) = -1.$

$\operatorname{ord} \partial = 0 \quad \Rightarrow \quad \operatorname{ord}(\text{edge}) + \operatorname{ord}(\text{vertex}) = 0.$

$$\Rightarrow \deg(\Gamma) := (d-1)\,|E(\Gamma)| - d\,|V(\Gamma)|$$

$$\operatorname{ord}(\Gamma) := |E(\Gamma)| - |V(\Gamma)|$$

Ref: T. Willwacher, 1411.2369

R. Koytcheff, B. Munson, I. Volić, 1109.0056

2). In our cases, $d = 3$,

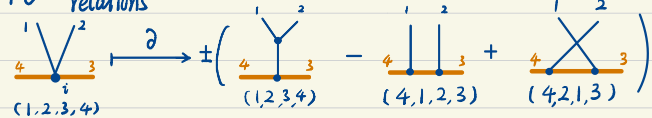
$$\deg(D) = 2(|E(D)| + |V_{ext}(D)|) - 3|V(D)|$$
$$= 2|E(D)| - 3|V_{int}(D)| - |V_{ext}(D)|.$$

and we need to some adjustments to GC_3 so that CJD and chord diagrams live in $H_0(GC_3) = \bigoplus_{k \geq 0} H_0(GC_3^k, \partial)$. order k graphs

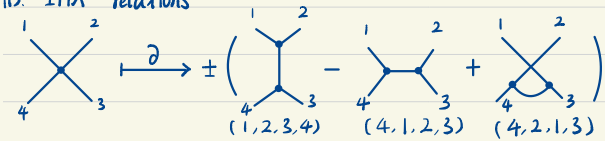
Level filtration provides a bicomplex structure.

E.g.

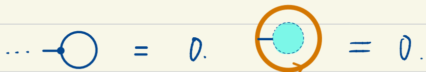
i). STU relations



ii). IHX relations



3). Since $d = 3$, vertices (and half edges) are odd and edges are even. Hence, symmetry arguments also give,



Tadpoles always vanish.

§2. $\mathcal{A} \cong \mathcal{A}^c$ as vector spaces

Since chord diagrams are all closed Jacobi diagrams, and STU relations imply 4T relations, we have an inclusion $\lambda_m^\circ : \mathcal{C}_m \longrightarrow \mathcal{D}_m$ s.t.

$\lambda_m^\circ(\mathrm{Span}_k\{4T\}) \subseteq \mathrm{Span}_k\{STU\}$,

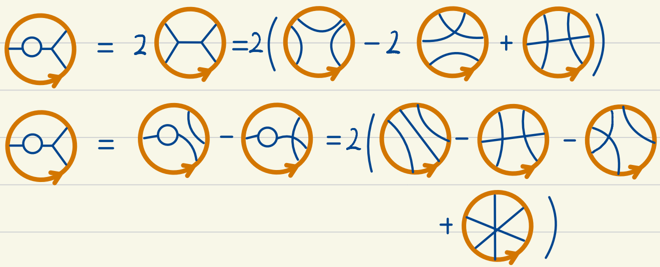
We have a well-defined map,

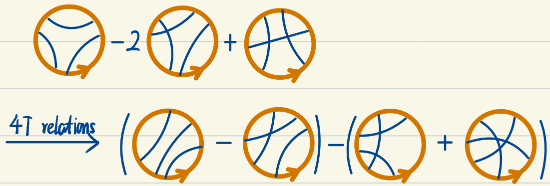
$$\lambda_m : \mathcal{A}_m^c = \mathcal{C}_m/4T \longrightarrow \mathcal{A}_m = \mathcal{D}_m/STU$$

Thm 2.1. [Bar-Natan]

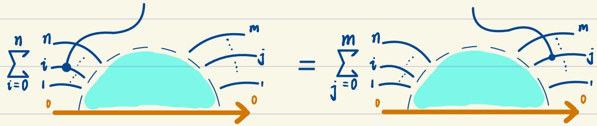
- λ_m is an isomorphism between vector spaces.
- $\lambda := \bigoplus_{m\geq 0} \lambda_m : \mathcal{A}^c \longrightarrow \mathcal{A}$ is an isomorphism between graded vector spaces.

Example

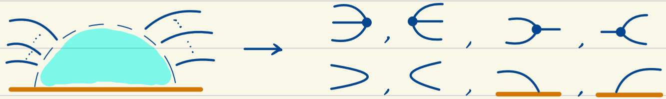




Lem 2.2. (Walking lemma)



Pf. We can split a diagram into pieces,



It is sufficient to check that walking lemma holds for each piece, and it's done by IHX, AS, STU relations. □

Remark. In the proof of Lem 1.3 and Lem 2.2, we only use STU to increase the level.

Combinatorial proof :

For each m, we construct, by induction on level, a linear map $\rho : \mathcal{A}_m \to \mathcal{A}_m^{G}$ and show that ρ is inverse to λ_m.

Define $\mathcal{D}_m^{\geq k} := \bigoplus_{j=k}^{2m} \mathcal{D}_m^{j}$ and $\mathcal{A}_m^{\geq k} := \mathcal{D}_m^{\geq k}/(STU)$.

Notice that $\mathcal{A}_m^{\geq 2m} = \mathcal{D}_m^{\geq 2m} = \mathcal{D}_m^{2m} = \mathcal{C}_m$, $\mathcal{A}_m^{\geq 1} = \mathcal{A}_m$

1° For level-$2m$ diagrams, define $\rho_0 : \mathcal{A}_m^{\geq 2m} = \mathcal{C}_m \to \mathcal{A}_m^{e}$.

For level-$(2m-1)$ element $x \in \mathcal{A}_m^{\geq 2m-1}$, choose any $D \in \mathcal{A}_m^{\geq 2m}$ such that $[D] = x$ and define $\rho_1(x) := \rho_0(D)$.

2° Assume we have defined $\rho_j : \mathcal{A}_m^{\geq 2m-j} \to \mathcal{A}_m^{e}$ for $j = 0, \cdots, k-1$ for $k \geq 2$ and $\forall x \in \mathcal{A}_m^{\geq 2m-j}$ and $D \in \mathcal{A}_m^{\geq 2m}$ if $[D] = x$, then

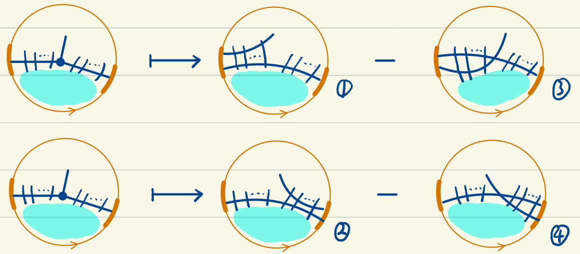
$$\rho_j(x) = \rho_0(D).$$

Then, we try to define $\rho_k : \mathcal{A}_m^{\geq 2m-k} \to \mathcal{A}_m^{e}$.

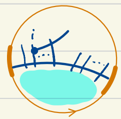
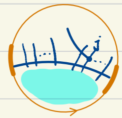
For a diagram D in $\mathcal{D}_m^{2m-k}$, use STU relation to resolve a trivalent vertex in D, we get an element $y \in \mathcal{D}_m^{2m-k+1}$.

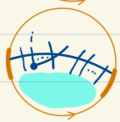
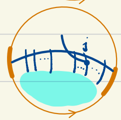
Claim: $\rho_{k-1}([[y]])$ is independent of the choice of the trivalent vertex and its leg.

case I: The same vertex, different legs.

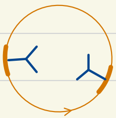


By walking lemma,

① − ② $= -\sum_i$  $+ \sum_j$ 

③ − ④ $= -\sum_i$  $+ \sum_j$ 

case II: different vertices, obviously by taking STU again in $\mathcal{A}_m^{\geq 2m-k+1}$.



By this claim, define $P_k(D) = P_{k-1}(y)$.

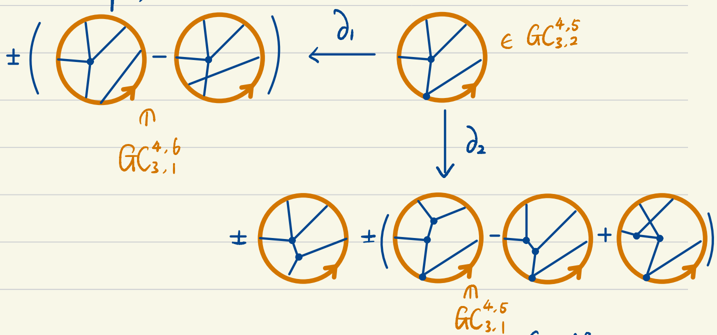
Define $P = P_{2m-1} : \mathcal{A}_m^{\geq 1} = \mathcal{A}_m \longrightarrow \mathcal{A}_m^{\ell}$.

Easy to check P is inverse to λ_m. □

"Fancy proof": Write $(\bigoplus_{*\geq 0} GC^{m}_{3,*}, \partial)$ for our "graph complex for CJD's of order m". Level defines a filtration

$$\begin{tikzcd}
GC^{m,2m}_{3,0} & GC^{m,2m-1}_{3,1} \arrow[l, "\partial_1"'] \arrow[d, "\partial_2"] & GC^{m,2m-2}_{3,2} \arrow[l, "\partial_1"'] \arrow[d, "\partial_2"] & \cdots & GC^{m,2}_{3,2m-2} \arrow[l, "\partial_1"'] \arrow[d, "\partial_2"] \\
& GC^{m,2m-1}_{3,0} & GC^{m,2m-2}_{3,1} \arrow[l, "\partial_1"'] \arrow[d, "\partial_2"] & \cdots & GC^{m,2}_{3,2m-3} \arrow[l, "\partial_1"'] \arrow[d, "\partial_2"] \\
& & GC^{m,2m-2}_{3,0} & \cdots & GC^{m,2}_{3,2m-4} \arrow[l, "\partial_1"'] \arrow[d] \\
& & & \ddots & \vdots \\
& & & & GC^{m,2}_{3,0} \\
\text{level}\geq 2m & \text{level}\geq 2m-1 & \text{level}\geq 2m-2 & & \text{level}\geq 2
\end{tikzcd}$$

For example,



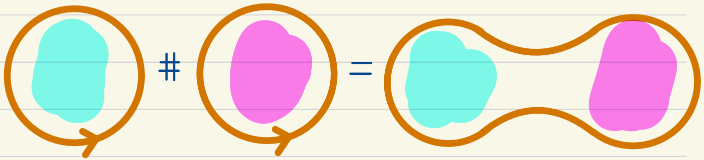
Write $E^0_{-p,q} = GC^{m,2m-p}_{3,q}$, $E^2_{-p,p} \cong E^{\infty}_{-p,p} = \begin{cases} \mathcal{A}^{\ell}_m, & \text{if } p=0, \\ 0, & \text{else.} \end{cases}$

However, if directly computation gives $H_0(GC^m_3) \cong \mathcal{A}_m$.

"□"

§3. Hopf algebra structures on $\mathcal{A}$ and $\mathcal{A}^{\ell}$

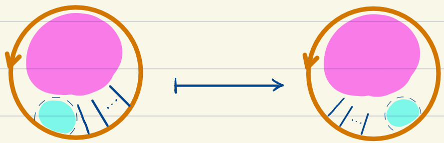
Def 3.1 (Connect sum between chord diagrams and CJDs)



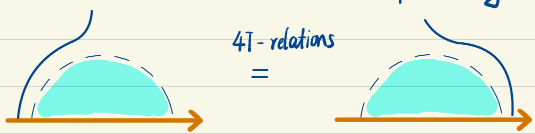
Lem 3.2 $(\bigoplus_{m\geq 0}\mathcal{A}_m, \#, \bigcirc)$ and $(\bigoplus_{m\geq 0}\mathcal{A}^{\ell}_m, \#, \bigcirc)$ are commutative graded algebra.

pf. 1). $\#$ is well-defined.

On $\mathcal{A}_m$, we have by walking lemma,



On $\mathcal{A}^{\ell}_m$, we have similar version of walking lemma,



2). $\#$ is commutative and associative.

3). $\bigcirc$ is the unit.

4). $\#$ is homogeneous w.r.t. the order. $\square$

Def 3.3

1). Coproduct on $\mathcal{A}^{\ell}$ is defined by

$$\Delta(D) := \sum_{J\subset \ell(D)} D_J \otimes D_{\ell(D)\setminus J},$$

$\ell(D) = \{\text{chords of } D\}$, $D_J = D$ forgetting chords in J.

2). Coproduct on $\mathcal{A}$ is defined by

$$\Delta(D) := \sum_{J\subset \mathrm{Comp}(D)} D_J \otimes D_{\mathrm{Comp}(D)\setminus J}$$

$\mathrm{Comp}(D) = \{$ connected components of the unitrivalent diagram of $D\}$

$D_J = D$ forgetting components in J.

Example.

Lem 3.4 $(\bigoplus_{m\geq 0}\mathcal{A}_m, \Delta, \varepsilon)$ and $(\bigoplus_{m\geq 0}\mathcal{A}^{\ell}_m, \Delta, \varepsilon)$ are cocommutative graded coalgebra, where ε maps  to 1 and other diagrams to zero.

pf. 1). $\Delta(4T) \subset (4T)$ and $\Delta(STU) \subset (STU)$.

2). Δ is coassociative and cocommutative.

3). ε is counit.

4). Δ is homogeneous w.r.t. the order. $\square$

Thm 3.5. $\lambda: \mathcal{A}^{\ell} \to \mathcal{A}$ is an isomorphism between bialgebras.

Def. A graded algebra A is connected, iff $A_0 \cong k$.
A is of finite type, if $\dim A_n < +\infty$, $\forall n$.

Thm [Milnor-Moore] Any connected, commutative and cocommutative bialgebra A of finite type is canonically isomorphic to the Hopf algebra k[primitive elements in A].

Thm 3.6. [Bar-Natan] The primitive space $\mathcal{P}$ of $\mathcal{A}$ is spanned by connected CJDs.

Example.

The primitive space is graded by the order, $\mathcal{P}_m := \mathcal{P} \cap \mathcal{A}_m$

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{P}_n$	1	1	1	2	3	5	8	12	18	27	39	55

Level introduces a filtration, $\mathcal{P}_m^k := \mathcal{P} \cap \mathcal{A}_m^k$ spanned by order m connected CJDs with at most k exterior vertices.

Connectness $\Rightarrow$ $\begin{cases} |V_{int}(D)| + |V_{ext}(D)| \leq |E(D)| + 1 \\ |V_{int}(D)| + m = |E(D)| \end{cases}$

$\mathrm{ord}(D) = m \Rightarrow$

$\Rightarrow \quad k \leq m+1$.

Hence, $\mathcal{P}_m$ is filtered as:

$$0 = \mathcal{P}_m^1 \subset \mathcal{P}_m^2 \subset \cdots \subset \mathcal{P}_m^{m+1} = \mathcal{P}_m$$

Calculate $\dim \mathcal{P}_m^k / \mathcal{P}_m^{k-1}$ gives:

$n \backslash k$	1	2	3	4	5	6	7
1		1					
2		1					
3		1					
4		1		1			
5		2		1			
6		2		2		1	

by J. Kneissler.

$\forall\, m \leq 12$.

m	$\mathcal{P}_m$		
0			
1			
2			
3			
4			
5.