Knots and q-Series

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Abstract

Generalizations are given for the q-series identities arising from the 3_1, 4_1, and 6_3 knots of Garoufalidis, Le and Zagier. Indeed a trio of new parameters can be added in each case while preserving the infinite product side of the identities.

1 Introduction

In [3, p. 7], Garoufalidis and Le prove two surprising q-series/infinite product identities related to the 3_1 and 4_1 respectively:

$$(q)_\infty^{-2} = \sum_{a,b,c\geq 0} (-1)^a \frac{q^{\frac{3a^2}{2}+ab+ac+bc+\frac{a}{2}+b+c}}{(q)_a(q)_b(q)_c(q)_{a+b}(q)_{a+c}}. \tag{1.1}$$

$$(q)_\infty^{-3} = \sum_{\substack{a,b,c,d,e\geq 0\\ a+b=d+e}} (-1)^{b+d} \frac{q^{\frac{b^2}{2}+\frac{d^2}{2}+bc+ac+ad+be+\frac{a}{2}+c+\frac{e}{2}}}{(q)_{b+c}(q)_a(q)_b(q)_c(q)_d(q)_e(q)_{c+d}}. \tag{1.2}$$

They then state: "... for the amphicheiral knot 6_3, we conjecture that

$$(q)_\infty^{-4} = \sum_{\substack{a,b,c,d,e,f\geq 0\\ a+e\geq b, b+f\geq a}} \frac{(-1)^{a-b+e} q^{\frac{a}{2}+\frac{3a^2}{2}+\frac{b}{2}+\frac{b^2}{2}+c+ac+d+ad+cd+\frac{e}{2}+2ae-2be+de+\frac{3e^2}{2}-af+bf+f^2}}{(q)_a(q)_b(q)_c(q)_{a+c}(q)_d(q)_{a+d}(q)_e(q)_{a-b+e}(q)_{a-b+d+e}(q)_f(q)_{-a+b+f}}. \tag{1.3}$$

It is indicated in [2, p. 13] that a knot-theoretic proof of (1.1), (1.2), and (1.3) may be deduced from results in [1].

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In these identities,

$$(A)_n = (A\colon q)_n = \prod_{m=0}^{\infty} \frac{(1 - Aq^m)}{(1 - Aq^{m+n})}. \tag{1.4}$$

Note that now $(A)_n$ is defined for all integers n and $1/(q)_n = 0$ if $n < 0$.

Our object in this paper is to prove the following identities which reduce to (1.1), (1.2), and (1.3) respectively when $x = y = z = 1$.

Theorem 1.

$$\frac{1}{(yq)_\infty (zq)_\infty} = \sum_{a,b,c\geq 0} \frac{(-1)^a q^{\frac{3a^2}{2}+\frac{a}{2}+ab+ac+bc+b+c} x^a y^{a+b} z^c}{(q)_a (q)_b (q)_c (xq)_{a+b} (yq)_{a+c}} \tag{1.5}$$

$$\frac{1}{(xq)_\infty (yq)_\infty (zq)_\infty} = \sum_{\substack{a,b,c,d,e\geq 0\\ a+b=d+e}} \frac{(-1)^{b+d} q^{\frac{b^2}{2}+\frac{d^2}{2}+bc+ac+ad+be+\frac{a}{2}+c+\frac{e}{2}} x^q y^b z^c}{(yq)_{b+c} (q)_a (q)_b (q)_c (q)_d (q)_e (xq)_{c+d}} \tag{1.6}$$

and

$$\frac{1}{(wq)_\infty (yq)_\infty (zq)_\infty (z^{-1}q)_\infty}$$

$$= \sum_{a,b,c,d,e,f\geq 0} \frac{q^{f^2+(b-a)f+\frac{a}{2}+\frac{3a^2}{2}+\frac{b}{2}+\frac{b^2}{2}+c+ac+d+ad+cd+\frac{e}{2}+2ae-2be+\frac{3e^2}{2}} y^e z^{c-d} w^f}{(q)_f (wq)_{b+f-a} (q)_a (q)_b (q)_c (xq)_{a+c} (q)_d (zq)_{a+d} (q)_e (q)_{a-b+e} (yq)_{a-b+d+e}} \tag{1.7}$$

In Section 2, we provide the necessary results from the literature. Sections 3, 4, and 5 are devoted to the proofs of (1.5), (1.6), and (1.7) respectively. Finally in Section 6 we examine the q-series related to the 8_5 knot.

2 Background

We begin with some classical identities. The first two are due to Euler [`?Gasper1990`, (II.1) and (II.2), p. 236].

$$\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty}, \tag{2.1}$$

and

$$\sum_{n=0}^{\infty}\frac{(-1)^n t^n q^{n(n-1)/2}}{(q)_n} = (t)_\infty. \tag{2.2}$$

Next the q-binomial theorem [?Gasper1990, (II.4), p. 236]

$$\sum_{j=0}^{N}\frac{(-x)^j q^{\binom{j}{2}}}{(q)_j(q)_{N-j}} = \frac{(x)_N}{(q)_N}. \tag{2.3}$$

The following lemma is well-known although perhaps not in the following generality [?Gasper1990, eq. (1.6.3), p. 12].

Lemma 2. *For any integer A, $-\infty < A < \infty$,*

$$\sum_{n=0}^{\infty}\frac{q^{n^2+An}z^n}{(q)_n(zq)_{n+A}} = \frac{1}{(zq)_\infty}. \tag{2.4}$$

Proof. If $A \geq 0$, this follows immediately from the corrected [?Gasper1990, eq. (1.6.3), p. 12, $z \to zq^A$]. If $A < 0$, set $A = -B$, and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2-Bn}z^n}{(q)_n(zq)_{n-B}} &= (zq^{1-B})_B\sum_{n=0}^{\infty}\frac{q^{n^2}(zq^{-B})^n}{(q)_n(zq^{-B}q)_n}\\ &= (zq^{1-B})_B\frac{1}{(zq^{-B})_\infty} \qquad \text{(by the } A=0 \text{ case)}\\ &= \frac{1}{(zq)_\infty}.\end{aligned}$$

□

Lemma 3. *For integers M and N with $M > 0$,*

$$\sum_{j=0}^{\infty}\frac{q^{Mj}\lambda^j}{(q)_j(\mu q)_{j+N}} = \frac{1}{(\lambda q^M)_\infty}\sum_{j=0}^{\infty}\frac{q^{j^2+(M+N)j}\lambda^j\mu^j}{(q)_j(\mu q)_{j+N}}. \tag{2.5}$$

Proof.

$$\sum_{j=0}^{\infty}\frac{q^{Mj}\lambda^i}{(q)_j(\mu q)_{j+N}} = \frac{1}{(\mu q)_N}\lim_{\tau\to 0}\sum_{j=0}^{\infty}\frac{(\tau)_j(\tau)_j q^{Mj}\lambda^j}{(q)_j(\mu q^{N+1})_j}$$

$$= \frac{1}{(\mu q)_N} \lim_{\tau\to 0} \frac{(\tau^2 q^{M-N-1}\lambda\mu^{-1})_\infty}{(\lambda q^M)_\infty} \sum_{j=0}^{\infty} \frac{\left(\frac{\mu q^{N+1}}{\tau}\right)_j^2 \left(\tau^2 q^{M-N-1}\lambda\mu^{-1}\right)^j}{(q)_j(\mu q^{N+1})_j}$$

(by [?Gasper1990, eq. (1.4.6), p. 107])

$$= \frac{1}{(\lambda q^M)_\infty} \sum_{j=0}^{\infty} \frac{q^{j^2+(M+N)j}\lambda^j\mu^j}{(q)_j(\mu q)_{j+N}}.$$

□

Lemma 4. *For integers B and C,*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+1)/2+(B+C)n}\lambda^n\mu^n}{(q)_n(\lambda q)_{n+B}(\mu q)_{n+C}} = \frac{1}{(\lambda q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2+Bn}\lambda^n}{(q)_n(\mu q)_{n+C}}. \quad (2.6)$$

Proof.

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+1)/2+(B+C)n}\lambda^n\mu^n}{(q)_n(\lambda q)_{n+B}(\mu q)_{n+C}}$$

$$= \frac{1}{(\lambda q)_B(\mu q)_C} \lim_{\tau\to 0} \sum_{n=0}^{\infty} \frac{\left(\left(\frac{1}{\tau}\right)_n^3 \tau^{3n}\lambda^n\mu^n q^{n(2+B+C)}\right)}{(q)_n(\lambda q^{B+1})(\mu q^{C+1})_n}$$

$$= \frac{1}{(\lambda q)_B(\mu q)_C} \lim_{\tau\to 0} \frac{\left(\tau\lambda q^{B+1}\right)_\infty (\lambda\mu q^{B+C+2}\tau^2)_\infty}{(\lambda q^{B+1})_\infty(\tau^3\lambda\mu q^{B+C+2})_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{\tau}\right)_n (\tau\mu q^{C+1})_n^2 (\tau\lambda q^{B+1})^n}{(q)_n(\mu q^{C+1})_n(\lambda\mu q^{B+C+2}\tau^2)_n}$$

(by [?Gasper1990, p. 241, eq. (III.9), $d = \mu q^{C+1}, e = \lambda q^{B+1}$])

$$= \frac{1}{(\lambda q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2+Bn}\lambda^n}{(q)_n(\mu q)_{n+C}}$$

□

3 Proof of (1.5)

Proof.

$$\sum_{a,b,c\geq 0} \frac{(-1)^a q^{\frac{3a^2}{2}+\frac{a}{2}+ab+ac+bc+b+c} x^a y^{a+b} z^c}{(q)_a(q)_b(q)_c(xq)_{a+b}(yq)_{a+c}}$$

$$
\begin{aligned}
&= \frac{1}{(xq)_\infty} \sum_{b,c\geq 0} \frac{q^{bc+b+c} y^b z^c}{(q)_b (q)_c} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2+bn} x^n}{(q)_n (yq)_{n+c}} \\
&\hspace{6cm} \text{(by Lemma 4, } B=b, C=c, \lambda=x, \mu=y) \\
&= \frac{1}{(xq)_\infty} \sum_{c,n\geq 0} \frac{(-1)^n q^{c+n(n+1)/2} z^c x^n}{(q)_c (q)_n (yq)_{n+c}} \cdot \frac{1}{(yq^{1+c+n})_\infty} \\
&\hspace{10cm} \text{(by (2.1))} \\
&= \frac{1}{(xq)_\infty (yq)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2} x^n}{(q)_n} \sum_{c\geq 0} \frac{q^c z^c}{(q)_c} \\
&= \frac{1}{(xq)_\infty (yq)_\infty} \cdot (xq)_\infty \frac{1}{(zq)_\infty} \\
&= \frac{1}{(yq)_\infty (zq)_\infty}.
\end{aligned}
$$

□

4 Proof of (1.6)

Proof.

$$
\sum_{\substack{a,b,c,d.e\geq 0\\ a+b=d+e}} \frac{(-1)^{b+d} q^{\frac{b^2}{2}+\frac{d^2}{2}+bc+ac+ad+be+\frac{a}{2}+c+\frac{e}{2}} x^a y^b z^c}{(yq)_{b+c}(q)_a(q)_b(q)_c(q)_d(q)_e(xq)_{c+d}}
$$

$$
\begin{aligned}
&= \sum_{a,b,c,d,e\geq 0} c \frac{(-1)^{a+e} q^{\frac{3a^2}{2}+b^2+\frac{e^2}{2}+2ab+bc+ac-2ae+\frac{a}{2}+c+\frac{e}{2}} x^a y^b z^c}{(yq)_{b+c}(q)_a(q)_b(q)_c(q)_{a+b-e}(q)_e(xq)_{a+b+c-e}} \\
&\hspace{8cm} \text{(replacing } d \text{ by } a+b-e) \\
&= \sum_{b,c,e\geq 0} \frac{(-1)^e q^{b^2+\frac{e^2}{2}+bc+c+\frac{e}{2}} y^b z^c}{(yq)_{b+c}(q)_b(q)_c(q)_e} \times \frac{1}{(xq)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j x^j q^{j(j+1)/2+(b+c-e)j}}{(q)_j (q)_{j+b-e}} \\
&\hspace{4cm} \text{(by (2.6) with } B=b+c-e, C=b-e, \lambda=x, \mu=1) \\
&= \frac{1}{(xq)_\infty} \sum_{b,c,j\geq 0} \frac{q^{b^2+bc+c} y^b z^c (-1)^j x^j q^{j(j+1)/2+(b+c)j}}{(yq)_{b+c}(q)_b(q)_c(q)_j} \times \sum_{e\geq 0} \frac{(-1)^e q^{e(e+1)/2-ej}}{(q)_e (q)_{j+b-e}}
\end{aligned}
$$

$$= \frac{1}{(xq)_\infty} \sum_{c\geq 0} \frac{z^c q^c}{(q)_c} \sum_{b\geq 0} \frac{q^{b^2+bc} y^b}{(q)_b (yq)_{b+c}}$$

(by (2.3) applied to e-sum, annihilating all j terms except $j=0$)

$$= \frac{1}{(xq)_\infty} \frac{1}{(zq)_\infty} \frac{1}{(yq)_\infty},$$

by (2.4) $\square$

5 Proof of (1.7)

Proof. We start with four additional parameters, x, y, z, and w. This allows us to obtain a more general result than (1.7). Subsequently we obtain (1.7) by setting $x = 1/z$.

$$\sum_{a,b,c,d,e,f\geq 0} \frac{(-1)^{a-b+e} x^d y^e z^c w^f q^{f^2+(b-a)c+\frac{a}{2}+\frac{3a^2}{2}+\frac{b}{2}+\frac{b^2}{2}+c+ac+d+ad+cd+\frac{e}{2}+2ae-2be+\frac{3e^2}{2}}}{(q)_f (wq)_{b+f-a} (q)_a (q)_b (q)_c (xq)_{a+c} (q)_d (zq)_{a+d} (q)_e (q)_{a-b+e} (yq)_{a-b+d+e}}$$

$$= \frac{1}{(wq)_\infty} \sum_{a,b,d,e\geq 0} \frac{(-1)^{a-b+e} q^{\frac{a}{2}+\frac{3a^2}{2}+\frac{b}{2}+\frac{b^2}{2}+d+ad+\frac{e}{2}+2ae-2be+de+\frac{3e^2}{2}} x^d y^e}{(q)_a (q)_b (q)_d (zq)_{a+d} (q)_e (q)_{a-b+e} (yq)_{a-b+d+e}} \sum_{c\geq 0} \frac{q^{c(1+a+d)} z^c}{(q)_c (xq)_{a+c}}$$

(by (2.4) applied to the w sum)

$$= \frac{1}{(wq)_\infty} \sum_{a,b,c,d,e\geq 0} \frac{(-1)^{a-b+e} q^{\frac{a}{2}+\frac{3a^2}{2}+\frac{b}{2}+\frac{b^2}{2}+d+ad+\frac{e}{2}+2ae-2be+de+\frac{3e^2}{2}} x^d y^e}{(q)_a (q)_b (q)_d (zq)_{a+d} (q)_e (q)_{a-b+e} (yq)_{a-b+d+e}}$$

$$\times \frac{1}{(zq^{1+a+d})_\infty} \sum_{j\geq 0} \frac{(-1)^j q^{j^2+j(1+2a+d)} z^j x^j}{(q)_j (xq)_{j+a}}$$

(by (2.4) applied to c-sum)

$$= \frac{1}{(wq)_\infty (zq)_\infty} \sum_{a,b,d,j\geq 0} \frac{(-1)^{a-b} q^{\frac{a}{2}+\frac{3a^2}{2}+\frac{b}{2}+\frac{b^2}{2}+d+ad+j^2+j+2aj+dj} x^d z^j x^j}{(q)_a (q)_b (q)_d (q)_j (xq)_{j+a}}$$

$$\times \sum_{e\geq 0} \frac{(-1)^e q^{\frac{e^2}{2}+\frac{e}{2}+(2a-2b+d)e} y^e}{(q)_e (q)_{a-b+e} (yq)_{a-b+d+e}}$$

$$= \frac{1}{(wq)_\infty(zq)_\infty} \sum_{a,b,d,j\geq 0} \frac{(-1)^{a-b} q^{\frac{a}{2}+\frac{3a^2}{2}+\frac{b}{2}+\frac{b^2}{2}+d+ad+j^2+j+2aj+dj} x^d z^j x^j}{(q)_a(q)_b(q)_d(q)_j(xq)_{j+a}}$$

$$\times \frac{1}{(yq)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n y^n q^{\binom{n+1}{2}+(a-b+d)n}}{(q)_n(q)_{n+a-b}}$$

(by (2.6), $\lambda = y, \mu = 1, B = a-b+d, c = e-b$)

$$= \frac{1}{(wq)_\infty(zq)_\infty(yq)_\infty} \sum_{a,d,j,n\geq 0} \frac{(-1)^{a+n} y^n x^{d+j} z^j q^{\frac{3a^2}{2}+\frac{a}{2}+d+ad+j^2+j+2aj+dj+an+dn+\frac{n^2}{2}+\frac{n}{2}}}{(q)_n(q)_d(q)_j(xq)_{j+a}(q)_n}$$

$$\times \sum_{b\geq 0} \frac{(-1)^b q^{\binom{b+1}{2}-bn}}{(q)_b(q)_{a+n-b}}$$

$$= \frac{1}{(wq)_\infty(zq)_\infty(yq)_\infty} \sum_{a,d,j\geq 0} \frac{(-1)^a q^{\frac{a}{2}+\frac{3a^2}{2}+d+ad+j^2+j+2aj+dj} x^{d+j} z^j}{(q)_a(q)_d(q)_j(q)_{j+a}}$$

(because by (2.3)) the b-sum was 0 unless $n = 0$ when it was 1)

$$= \frac{(wq)_\infty(zq)_\infty(yq)_\infty(xq)_\infty}{\sum} {}_{a,j\geq 0} \frac{(-1)^a q^{\frac{a}{2}+\frac{3a^2}{2}+j^2+j+2aj} z^j x^j}{(q)_a(q)_j}$$

(by (2.1)) applied to the d-sum

$$= \frac{1}{(wq)_\infty(zq)_\infty(yq)_\infty(xq)_\infty} \sum_{a,j\geq 0} \frac{(-1)^a q^{\frac{a^2}{2}-\frac{a}{2}+j^2+j}(xz)^{j-a}}{(q)_a(q)j-a}$$

(shifting j to $j-a$ and noting the comment following (1.4))

$$= \frac{1}{(wq)_\infty(zq)_\infty(yq)_\infty(xq)_\infty \sum_{j\geq 0}} \sum_{j\geq 0} q^{j^2+j}(xz)^j \sum_{a\geq 0} \frac{(-1)^a q^{\frac{a^2}{2}-\frac{a}{2}}(xz)^{-a}}{(q)_a(q)_{j-a}}$$

$$= \frac{1}{(wq)_\infty(zq)_\infty(yq)_\infty(xq)_\infty} \sum_{j=0}^{\infty} \frac{q^{j^2+j}(xz)^j \left((xz)^{-1}\right)_j}{(q)_j}, \qquad \text{(by (2.3))}$$

To obtain (1.7) we need only set $x = z^{-1}$ in the above identity and note that the sum on j collapses to 1. $\square$

6 The q-series for the 8_5 knot.

The q-series for the 8_5 knot [2, p. 13] is given by

$$\sum_{a,b,c,d,e,f,g,h\geq 0} \mathcal{S}(a,b,c,d,e,f,g,h)$$

where

$$\begin{aligned}
\mathcal{S} &= \mathcal{S}(a,b,c,d,e,f,g,h) \\
&:= (-1)^{b+f} \frac{q^{2a+3a^2-\frac{b}{2}-2ab+\frac{3b^2}{2}+c+ac+d+ad}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h} \\
&\quad \times \frac{q^{cd+e+ae+de+\frac{3f}{2}+4af-4bf+ef+\frac{5f^2}{2}+g+ag-bg+eg+fg+h+ah-bh+fh+gh}}{(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a-b+f}(q)_{a-b+e+f}(q)_{a-b+f+g}(q)_{a-b+f+h}}.
\end{aligned}$$

Next we note that the sum on f over nonnegative integers may be replaced by $-\infty < f < \infty$ because of $5f^2/2$ in the numerator of q and the fact that (as noted before) $1/(q)_f = 0$ if $f < 0$. Hence it is valid to replace f by $f - a + b$ in the above 8-fold sum, and consequently

$$\begin{aligned}
&\sum_{a,b,c,d,e,f,g,h\geq 0} \mathcal{S}(a,b,c,d,e,f,g,h) \\
&= \sum_{\substack{a,b,c,d,e,g,h \\ -\infty<f<\infty}} \mathcal{S}(a,b,c,d,e,f-a+b,g,h) \\
&= \sum_{a,b,c,d,e,g,f,h\geq 0} (-1)^{f+a} q^{\frac{3a^2}{2}+\frac{a}{2}+b+ab+c+ac+d+ad} \\
&\quad \times \frac{q^{cd+e+be+de+\frac{5f^2}{2}+\frac{3f}{2}-af+bf+ef+g+eg+fg+h+fh+gh}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+c}(q)_{a+e}(q)_{f+e}(q)_{f+g}(q)_{f+h}(q)_{f-a+b}}
\end{aligned}$$

We now prove a final lemma that is applicable to the g and h sums and also applicable to the a and d sums.

Lemma 5. *For nonnegative integers N, M, T,*

$$\sum_{r,s\geq 0} \frac{q^{r(1+N)+n(1+T+M)+rs}\lambda^r\mu^s}{(q)_r(q)_s(\mu q)_{r+M}\,(\lambda q)_{s+N}} = \frac{1}{(\lambda q)_\infty(\mu q)_\infty}\sum_{j=0}^{\infty}\frac{q^{j^2+(1+N+M)j}\lambda^j\mu^j\left(\mu q^{j+M+1}\right)_T}{(q)_j}.$$

Proof.

$$\sum_{r,s\geq 0} \frac{q^{r(1+N)+n(1+T+M)+rs}\lambda^r\mu^s}{(q)_r(q)_s(\mu q)_{r+M}(\lambda q)_{s+N}}$$

$$= \sum_{s\geq 0}\frac{q^{s(1+T+M)}\mu^s}{(q)_s(\lambda q)_{s+N}}\sum_{r\geq 0}\frac{q^{r(1+N+s)}\lambda^r}{(q)_r(\mu q)_{r+M}}$$

$$= \sum_{s>0}\frac{q^{s(1+T+M)}\mu^s}{(q)_s(\lambda q)_{s+N}}\frac{1}{(\lambda q^{1+N+s})_\infty}\sum_{j=0}^{\infty}\frac{q^{j^2+(1+N+M+s)j}\lambda^j\mu^j}{(q)_j(\mu q)_{j+M}}$$

(by (2.5))

$$= \frac{1}{(\lambda q)_\infty}\sum_{j=0}^{\infty}\frac{q^{j^2+(1+N+M)j}\lambda^j\mu^j}{(q)_j(\mu q)_{j+M}}\sum_{s\geq 0}\frac{q^{s(1+T+M+j)}\mu^s}{(q)_s}$$

$$= \frac{1}{(\lambda q)_\infty}\sum_{j=0}^{\infty}\frac{q^{j^2+(1+N+M)j}\lambda^j\mu^j}{(q)_j(\mu q)_{j+M}(\mu q^{1+M+j+T})_\infty}$$

(by (2.1))

$$= \frac{1}{(\lambda q)_\infty(\mu q)_\infty}\sum_{j=0}^{\infty}\frac{q^{j^2+(1+M+N)j}\lambda^j\mu^j\left(\mu q_T^{j+M+1}\right)}{(q)_j}$$

$\square$

We now note that Lemma 5 reduces a double sum to a single sum. So if we apply Lemma 5 to the c and d sums and also to the g and h sums we can reduce the original 8-fold sum to a 6-fold sum.

7 Conclusion

It is natural to ask whether there are knot-theoretic implications arising from the extra parameters that generalize (1.1), (1.2), (1.3) to (1.5), (1.6), and (1.7) respectively. If a knot-theoretic interpretation of Theorem 1 is possible, it might lead to the insertion of parameters into the series for the 8_5 knot. That in turn might lead to possible further simplifications of the series for the 8_5 knot.

References

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