ECE 618 Project 2

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Contents

Introduction

In this project, we use finite element method (FEM) to analyze a homogeneous waveguide. The waveguide is cylindrical and assumed to be uniform (and infinitely long) in its longitudinal direction (z-direction). So the problem can be reduced to a two-dimensional one. It is known that in such a hollow metallic waveguide, there exist two sets of modes: TE and TM modes. (Since there is only one conductor, such a waveguide does not have a TEM mode.)

For the TM modes, starting from Maxwell's equations and assuming that the fields propagate in the z-direction with a propagation constant β, we can derive the second order partial differential equation for E_z as [2]

$$\nabla_t^2 E_z + k_c^2 E_z = 0 \quad \text{on } \Omega \tag{1}$$

where Ω denotes the cross section of the waveguide. ∇_t^2 denotes the two-dimensional Laplacian, which is $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. And $k_c^2 = k^2 - \beta^2$ with $k = \omega\sqrt{\mu\epsilon}$. Obviously, once the cut-off wave number k_c is found (also the cut-off frequency $f_c = k_c/(2\pi\sqrt{\mu\epsilon})$ is found), the propagation constant β can be calculated for any frequency. The boundary condition is

$$E_z = 0 \quad \text{on } \Gamma \tag{2}$$

where Γ denotes the conduction wall of the waveguide.

The analysis for the TE modes is similar. We have

$$\nabla_t^2 H_z + k_c^2 H_z = 0 \quad \text{in } \Omega \tag{3}$$

$$\frac{\partial H_z}{\partial n} = 0 \quad \text{on } \Gamma \tag{4}$$

Note for the two modes, the only difference is the boundary condition. One is Dirichlet, another is Neumann.

TM modes, with Dirichlet BC

Let's first discuss how to apply the finite element formulation to solve the TM modes. We first discretize the solution domain Ω using a triangular mesh, and expand E_z in terms of the nodal basis functions

$$E_z = \sum E_{zi} N_i \tag{5}$$

To satisfy the essential/Dirichlet BC, here the summation for i only goes for all the nodes inside Ω or on Neumann boundary Γ_N.

And we transfer the PDE into a weak form and use Galerkin method (use the same set of bases of E_z as testing function)

$$\begin{aligned}
\int_\Omega N_j k_c^2 E_z ds &= -\int_\Omega N_j \nabla \cdot \nabla E_z ds \\
&= -(\int_\Omega \nabla \cdot (N_j \nabla E_z) ds - \int_\Omega \nabla N_j \cdot \nabla E_z ds) \\
&= -\int_\Gamma (N_j \nabla E_z) \cdot \vec{n} dl + \int_\Omega \nabla N_j \cdot \nabla E_z ds \\
&= -\int_{\Gamma_D} (N_j \nabla E_z) \cdot \vec{n} dl - \int_{\Gamma_N} (N_j \nabla E_z) \cdot \vec{n} dl + \int_\Omega \nabla N_j \cdot \nabla E_z ds \\
&= \int_\Omega \nabla N_j \cdot \nabla E_z ds
\end{aligned} \tag{6}$$

where the line integral on Γ_D vanishes because $\forall j, N_j = 0$ on Γ_D, the term on Γ_N vanishes simply because there is no Neumann boundary in this case. If there is Neumann boundary, for example H_z, since $(N_j \nabla H_z) \cdot \vec{n} = N_j \frac{\partial H_z}{\partial n}$, the integral vanishes due to the natural/Neumann boundary condition.

Plug E_z (equ. 5) into the integrals on both sides,

$$k_c^2 \sum_i [\int_\Omega N_j N_i ds] \; E_{zi} = \sum_i [\int_\Omega \nabla N_j \cdot \nabla N_i ds] \; E_{zi} \tag{7}$$

We can rewrite it into a matrix form

$$[A]\{E_z\} = k_c^2 [B]\{E_z\} \tag{8}$$

where

$$A_{ij} = \int_\Omega \nabla N_i \cdot \nabla N_j ds \tag{9}$$

$$B_{ij} = \int_\Omega N_i N_j ds \tag{10}$$

Now the problem becomes a *generalized eigenvalue problem*. We can solve it to get the eigenvalues k_c^2.

TE modes, with Neumann BC

It is very similar to solve the H_z for the TE modes. We expand

$$H_z = \sum H_{zi} N_i \tag{11}$$

Note, here the summation on i includes the nodes on Neumann boundary Γ_N. Therefore, we get different matrix A and B of larger dimensions than in TM cases, and, of course, different eigenvalues and eigenmodes. In practice, we can calculate the full matrix A and B for TE modes first, and then index the interior nodes to get the matrix A and B for TM modes.

Approach

After obtaining the mathematical framework, we are going to implement the FEM to solve the question 5 and 6 in project 2, which are an empty rectangular waveguide with $a/b = 2:1$ and an empty circular waveguide with radius r.

Discretization

We use MATLAB's pde toolbox to draw the geometry of the cross section and to generate triangular mesh on it. Due to the simple geometry, the meshes generated by this tool are good enough.

Generating Matrix A and B, Assembly Process

For a triangle element e, we learn in class how to calculate the linear nodal basis of node i on e, which is $N_i^e(x, y) = A_i + B_i x + C_i y$, also in [1]. For its contribution to matrix A, we can calculate as $\int_e \nabla N_i^e \cdot \nabla N_j^e ds = (B_i B_j + C_i C_j) S^e$, where S^e is the area of the triangle element e. For matrix B, we need to get the integral

$$\begin{aligned}
B^e_{ij} &= \int_e N^e_i N^e_j ds \\
&= \int_e (A_i + B_i x + C_i y)(A_j + B_j x + C_j y) ds \\
&= \int_e \begin{bmatrix} A_i & B_i & C_i \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} A_j \\ B_j \\ C_j \end{bmatrix} ds \\
&= \begin{bmatrix} A_i & B_i & C_i \end{bmatrix} \int_e \begin{bmatrix} 1 & x & y \\ x & x^2 & xy \\ y & xy & y^2 \end{bmatrix} ds \begin{bmatrix} A_j \\ B_j \\ C_j \end{bmatrix}
\end{aligned} \tag{12}$$

The integral in the above equation can be calculated by mapping an arbitrary triangle to the canonical element. I calculate this by hand and also check my result using Mathematica.

```
t1 = Triangle[{{x1, y1}, {x2, y2}, {x3, y3}}];
Integrate[x, {x, y} \[Element] t1] // Simplify
Integrate[x^2, {x, y} \[Element] t1] // Simplify
Integrate[x y, {x, y} \[Element] t1] // Simplify
```

In this way, we assemble the matrix A and B and store them as sparse matrix in MATLAB, since they are really sparse.

Solve the Generalized Eigenvalue Problem

To solve the generalized eigenvalue problem, I use MATLAB function eigs(), here is a link for the document of this function. The reason I choose eigs(), whoes description is *subset of eigenvalues and eigenvectors*, is because our matrices are sparse and also we are only interested in a few smallest eigenvalues, instead of all eigenvalues. When we call this function, the arguments are specified as the following: sigma='sm' (for version R2017a and before) / 'smallestabs' (for version R2017b and after) to get the eigenvalues of smallest magnitude; opts.issym = 1, opts.isreal = 1 (opts.isreal for version R2017a and before) to tell the solver both matrics are real and symmetric.

It seems like the solver eigs() uses some kind of iterative methods to get the eigenvalues and eigenvectors. [1] We use it to get the 20 smallest eigenmodes for TM and TE modes, respectively. Sometimes, the results from the solver are not sorted. To handle this, the program will always check it and sort them if needed.

Results and Discussion

In the program, we use a mesh of 2688 triangles for the rectangular waveguide, and a mesh of 4128 triangles for the circular waveguide.

We also calculate the theoretical eigenvalues of both cases. For circular waveguide, I use a Mathematica script to get the n-th zeros of Bessel function $J_m(z)$ (for TM) and the first derivative $J'_m(z)$ (for TE), with $m = 0, 1, 2, \dots$. For all $m > 0$, there are two degenerate modes for each eigenvalue of mn. For details, refer to Equ 6.2.11 and 6.2.18 in [2].

[1] I need to finish reading eigs() source code to make sure this. But just comparing its result with theory can give us the confidence in this solver.

Results

We compared the first five modes of simulation result with the theory result [1] as below:

For all TE modes, we compare the dashed line (magnetic field line) in theory with our result plot of Hz. For all TM modes, we compare the solid line (electric field line) in theory with our result plot of Ez.

Rectangular waveguide:

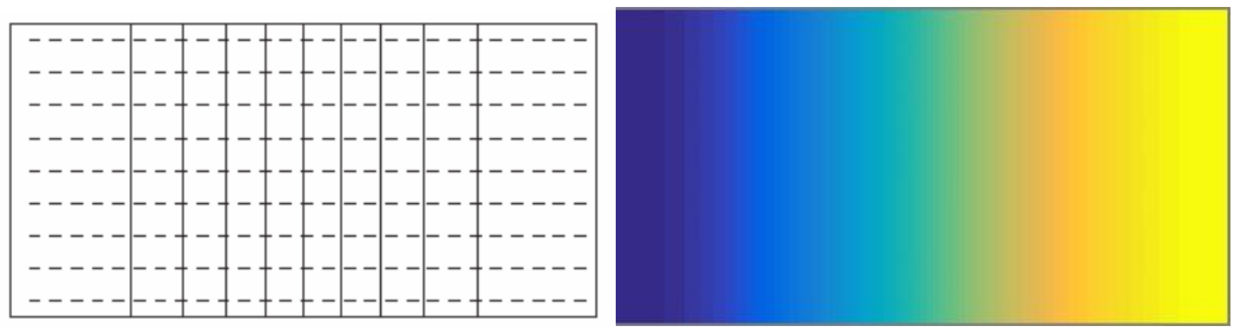

Figure 1. TE_{10}

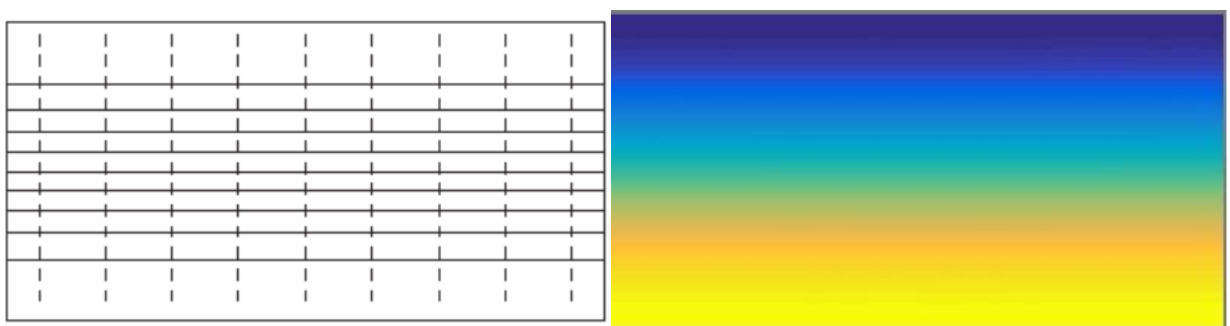

Figure 2. TE_{01}

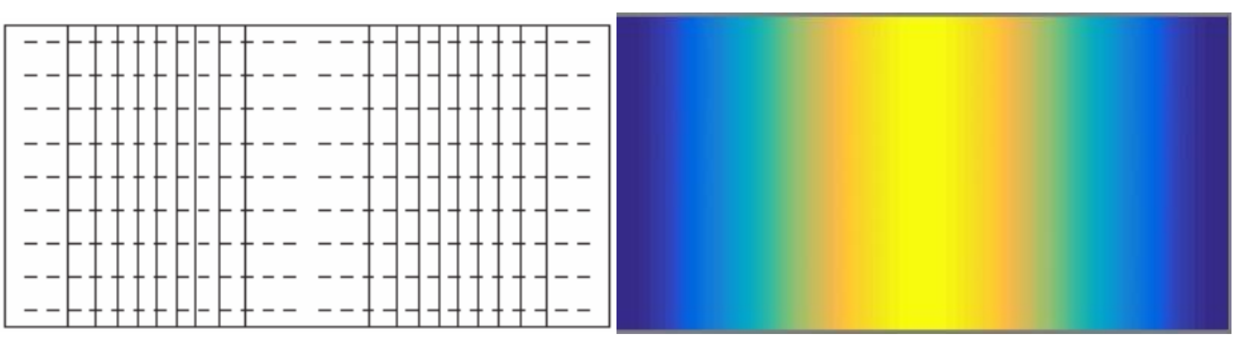

Figure 3. TE_{20}

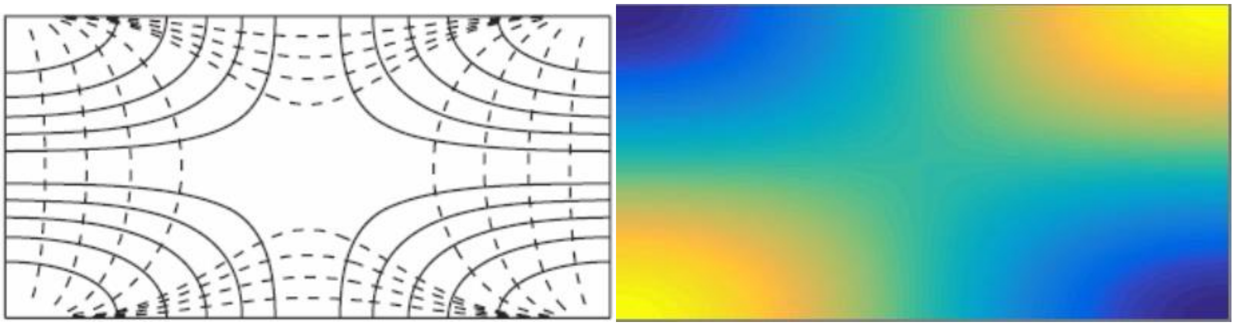

Figure 4. TE_{11}

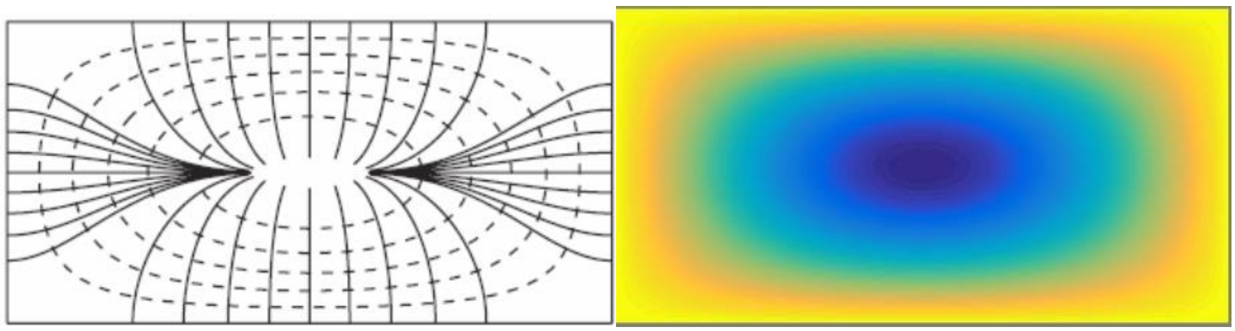

Figure 5. TM_{11}

Circular Waveguide:

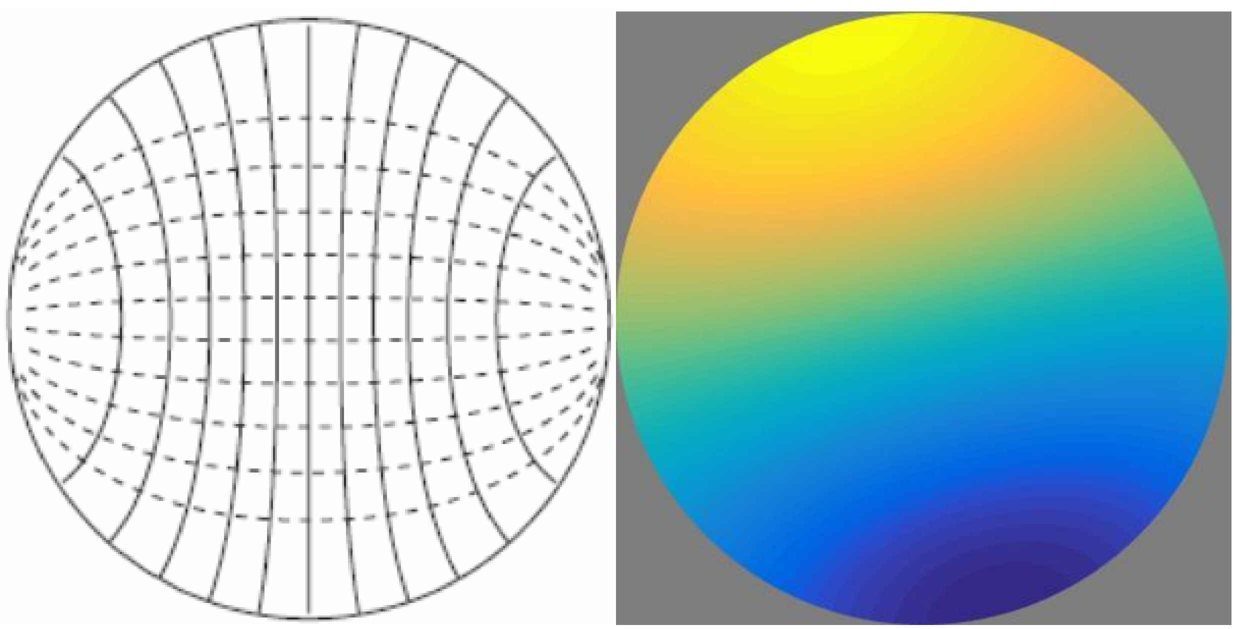

Figure 6. TE_{11}

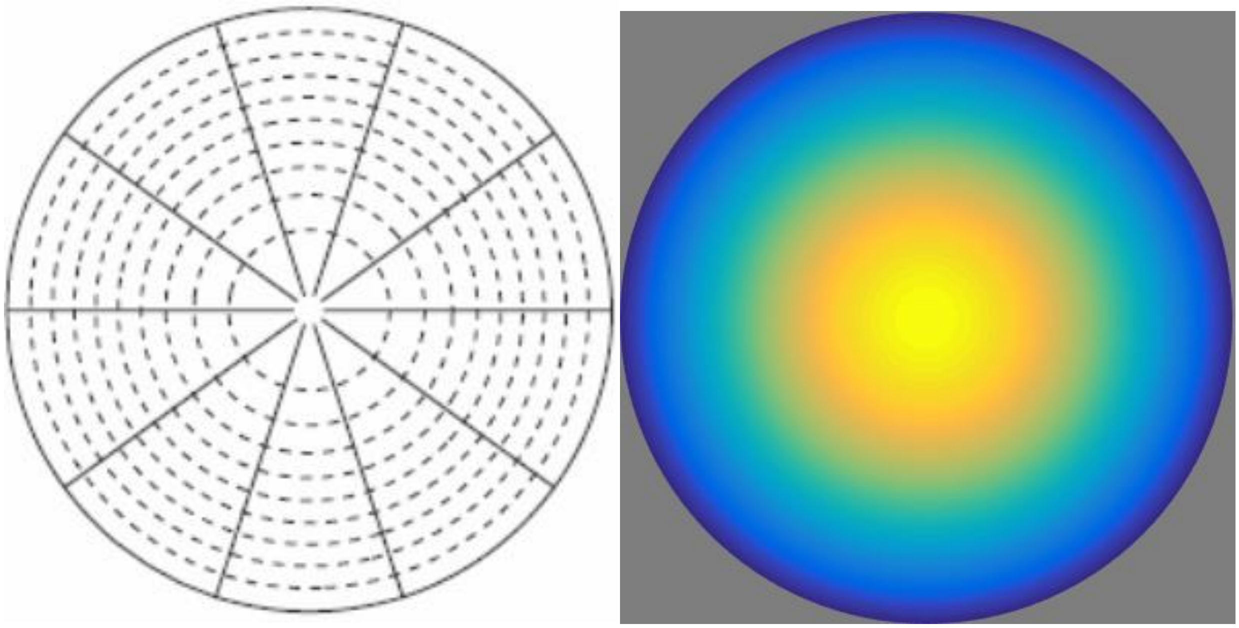

Figure 7. TM_{01}

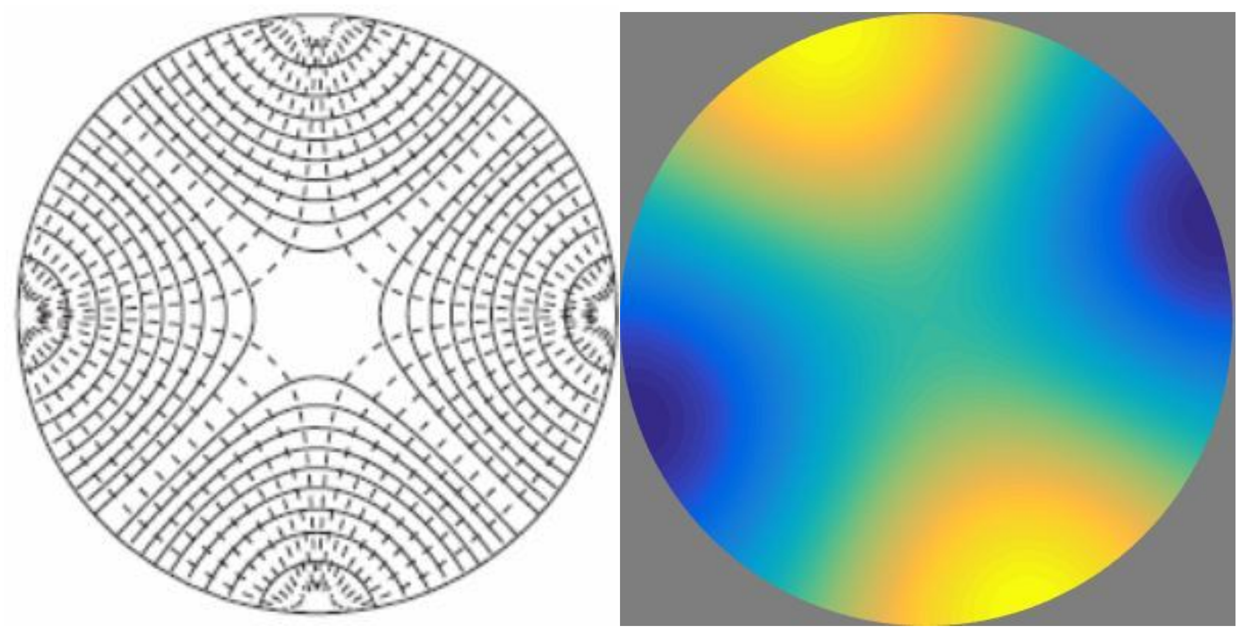

Figure 8. TE_{21}

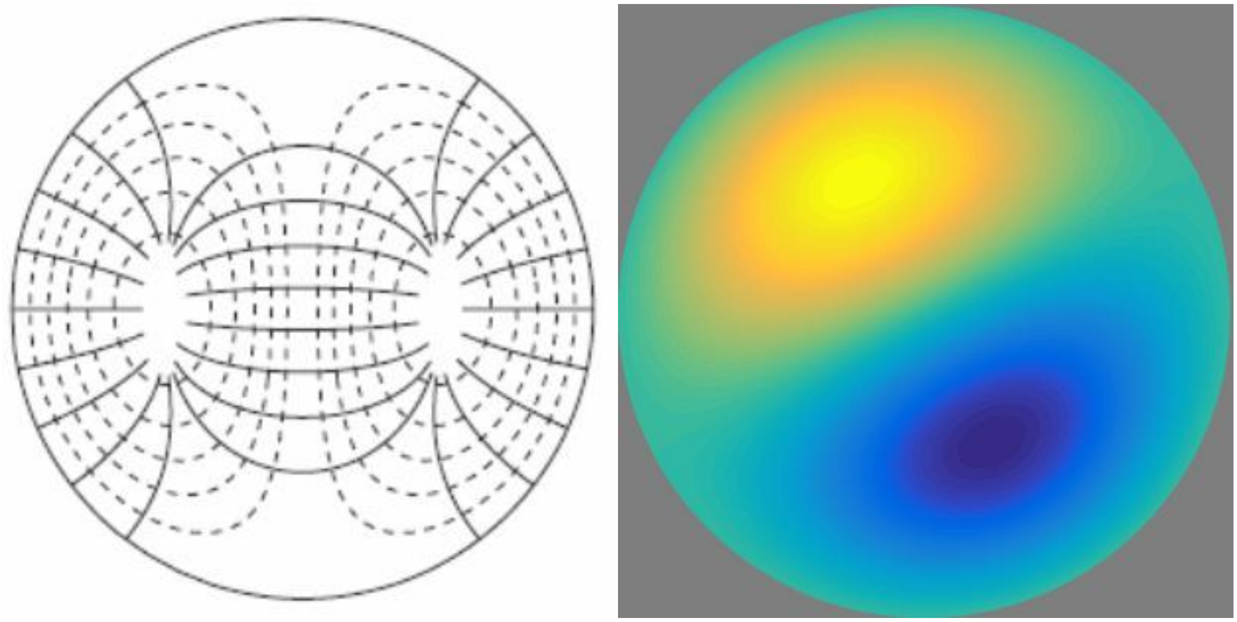

Figure 9. TM_{11}

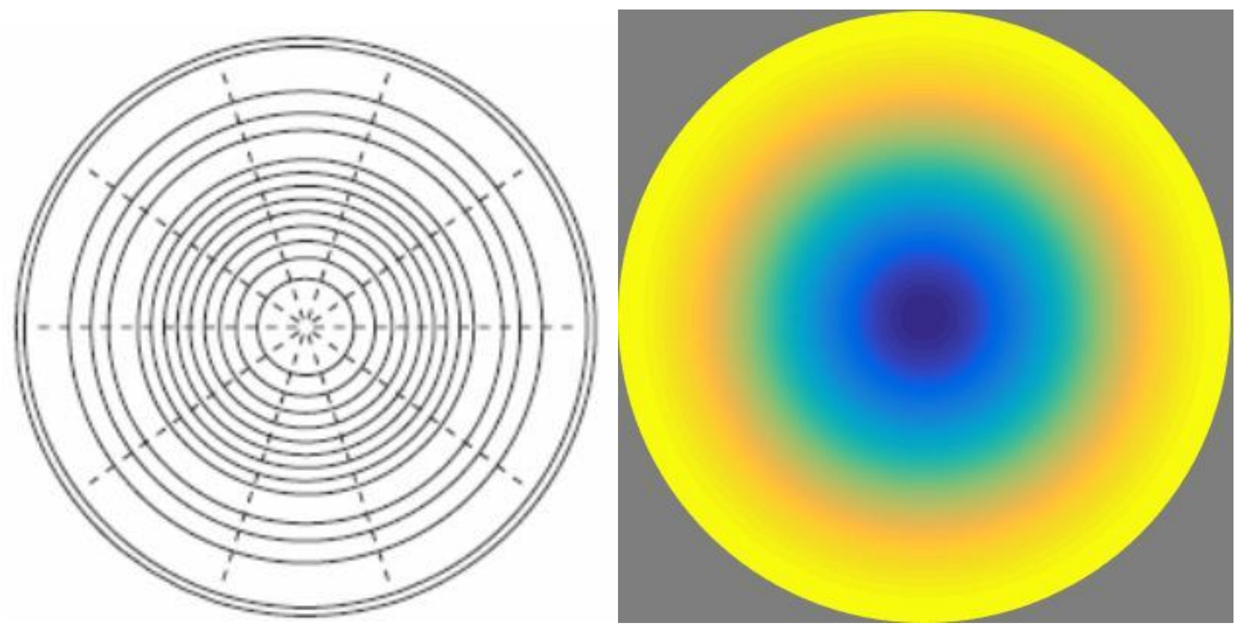

Figure 10. TE_{01}

We also compared k_c of the first 20 modes with theory result as below:

m	n	sim	theory
1	0	1.5902	1.5708
0	1	3.1059	3.1416
2	0	3.182	3.1416
1	1	3.4901	3.5124
2	1	4.4492	4.4429
3	0	4.7766	4.7124
3	1	5.7027	5.6636
0	2	6.2228	6.2832
4	0	6.3772	6.2832
1	2	6.4241	6.4766
2	2	6.995	7.0248
4	1	7.0997	7.0248
3	2	7.8577	7.854
5	0	7.9831	7.854
5	1	8.5747	8.459
4	2	8.9291	8.8858
0	3	9.3598	9.4248
6	0	9.495	9.4248
1	3	9.5977	9.5548
2	3	9.8963	9.9346

Figure 11. Rectangular TE

m	n	sim	theory
1	1	3.49014	3.5124
2	1	4.4492	4.4429
3	1	5.70237	5.6636
1	2	6.42419	6.4766
2	2	6.99542	7.0248
4	1	7.09933	7.0248
3	2	7.85779	7.854
5	1	8.57331	8.459
4	2	8.93203	8.8858
1	3	9.49724	9.5548
2	3	9.89631	9.9346
6	1	10.0951	9.9346
5	2	10.1512	10.058
3	3	10.5306	10.537
4	3	11.3641	11.327
6	2	11.4734	11.327
5	3	11.6538	12.268
1	4	12.3532	12.664
2	4	12.6382	12.953
6	3	12.8751	13.329

Figure 12. Rectangular TM

m	n	sim	theory
1	1	1.8418	1.8412
1	1	1.8418	1.8412
2	1	3.0562	3.0542
2	1	3.0563	3.0542
0	1	3.8361	3.8317
3	1	4.2057	4.2012
3	1	4.2059	4.2012
4	1	5.3266	5.3176
4	1	5.3267	5.3176
1	2	5.3419	5.3314
1	2	5.342	5.3314
5	1	6.4313	6.4156
5	1	6.4316	6.4156
2	2	6.7261	6.7061
2	2	6.7266	6.7061
0	2	7.0384	7.0156
6	1	7.5266	7.5013
6	1	7.5267	7.5013
3	2	8.0484	8.0152
3	2	8.0491	8.0152

Figure 13. Circular TE

m	n	sim	theory
0	1	2.4058	2.4048
1	1	3.8352	3.8317
1	1	3.8353	3.8317
2	1	5.1443	5.1356
2	1	5.1443	5.1356
0	2	5.5303	5.5201
3	1	6.3965	6.3802
3	1	6.3968	6.3802
1	2	7.0366	7.0156
1	2	7.0368	7.0156
4	1	7.6158	7.5883
4	1	7.6163	7.5883
2	2	8.4532	8.4172
2	2	8.4549	8.4172
0	3	8.6933	8.6537
5	1	8.8138	8.7715
5	1	8.8149	8.7715
3	2	9.8171	9.761
3	2	9.82	9.761
6	1	9.998	9.9361

Figure 14. Circular TM

From the result figures and k_c, the simulation matches the theory.

References

[1] Jian-Ming Jin. *Theory and Computation of Electromagnetic Fields*. Wiley-IEEE Press, 2015

Conclusion

In this project, we developed a Finite Element Method (FEM) solver to compute the TE/TM modes in a 2:1 rectangular waveguide or circular waveguide. We also develop a GUI that can display first 20 TE/TM modes in a listbox. By clicking a mode in the listbox, the transverse field distribution of the selected mode is shown on the right. By comparing the results of cutoff wavenumbers and each mode's transverse field distribution with theory, we are confident that our FEM solver can compute the TE/TM modes efficiently and correctly.

The GUI is developed by Bo-Wen Wei. The FEM solver is developed by Yifan Wang, and fully reviewed and discussed by both of us.

References

[1] *Finite Element Approximation*. http://www.cs.rpi.edu/~flaherje/pdf/fea4.pdf. Accessed: 2018-04.

[2] Jian-Ming Jin. *Theory and Computation of Electromagnetic Fields*. Wiley-IEEE Press, 2015.