Monte Carlo Sampler to get hadrons from freeze out hypersurface

LongGang Pang

December 11, 2015

1 Overview

The number of particle species i emmited from freeze out hypersurface element Σ^μ is:

$$dN_i = \frac{p \cdot \Sigma}{(2\pi)^3} \frac{d^3 p}{p^0} f(\mathrm{p}) \Theta(p \cdot \Sigma)$$

Where $\Theta(p \cdot \Sigma)$ is a heavyside step function that will throw particles when $p \cdot \Sigma < 0$. This function should work for both (t, x, y, z) coordinates and (τ, x, y, η_s) coordinates, except that the definitions of p^μ and Σ^μ are different,

$$\begin{aligned} p^\mu &= (E, p^x, p^y, p^z) \; in \; (t, z) \\ p^\mu &= (m_T \cosh(Y - \eta_s), p^x, p^y, m_T \sinh(Y - \eta_s)) \; in \; (\tau, \eta_s) \end{aligned}$$

The $f(\mathrm{p}) = \frac{1}{\exp((p \cdot u - \mu)/T) + \lambda}$ is the distribution function which can be Bose-Einstein distribution for mesons with $\lambda = -1$, Fermi-Dirac distribution for baryons with $\lambda = 1$ or Juttner distribution with $\lambda = 0$ when $\exp((p \cdot u - \mu)/T) \gg 1$ for massive hadrons. There is big difference between Bose-Einstein distribution and Juttner distribution for pion, while for other hadrons the difference is quite negligible and we can use Juttner distribution as an approximation of Bose-Einstein or Fermi-Dirac distribution for all the hadrons except pion.

In Local Rest Frame of (t, x, y, z) coordinates, $u^\mu = (u^t, u^x, u^y, u^z) = (1, 0, 0, 0)$ and $f(\mathrm{p}) = \exp(-(E - \mu)/T)$.

In Local Rest Frame of (τ, x, y, η_s) coordinates,

$$u^\mu = (u^\tau, u^x, u^y, u^\eta) = \left(u^t \cosh \eta_s - u^z \sinh \eta_s, u^x, u^y, -u^t \sinh \eta_s + u^z \cosh \eta_s\right) = (1, 0, 0, 0)$$

which means $v_z = \tanh \eta_s$ and the rapidity of fluid velocity becomes $Y_v = \eta_s$ and the 4 momentum

$$p^\mu = (m_T \cosh(Y' + Y_v - \eta_s), p^x, p^y, m_T \sinh(Y' + Y_v - \eta_s))$$

becomes $\tilde{p}^{\mu} = (m_T \cosh Y', p^x, p^y, m_T \sinh Y') = (E, p^x, p^y, p^z)$ at local rest frame. The distribution function again becomes $f(\mathrm{p}) = \exp\left(-(p^0 - \mu)/T\right) = \exp\left(-(m_T \cosh(Y) - \mu)/T\right) = \exp\left(-(E - \mu)/T\right)$. It is not straight forward, but finally we see even in (τ, x, y, η_s) coordinates, the distribution function in local rest frame equals to thermal distribution in (t, x, y, z) coordinates.

dN_i is Lorentz invariant quantity, so it is straight foward to do the 3D momentum phase space integration in local rest frame where the integration is anlytically equals to modified Bessel function for isotime freeze out [H. Petersen] or simplified to 1D numerical integration for arbitrary freeze out hyper surface [Hirano]. Once we know the total number of hadrons dN where $dN = \sum_i dN_i$, we may do poisson sampling with dN as the probability to determine how many hadrons will actually be created. One may wonder whether the results are different if we do poisson sampling for each species with probability dN_i (obviousely this is much slower). There is a proof tells us that the results are equal to each other,

Theorem: Let x_1, x_2 are independent Poisson random variable where x_i has Poisson probability λ_i, then $x_1 + x_2$ has a Poisson distribution with $\lambda_1 + \lambda_2$.

$$
\begin{aligned}
P(x_1 + x_2 = z) &= f_Z(z) = \sum_{x=0}^{z} f_{x_1}(x) f_{x_2}(z - x) \\
&= \sum_{x=0}^{z} \frac{\lambda_1^x}{x!} e^{-\lambda_1} \frac{\lambda_2^{z-x}}{(z-x)!} e^{-\lambda_2} \\
&= e^{-(\lambda_1 + \lambda_2)} \sum_{x=0}^{z} \frac{\lambda_1^x}{x!} \frac{\lambda_2^{z-x}}{(z-x)!} \\
&= \frac{(\lambda_1 + \lambda_2)^z}{z!} e^{-(\lambda_1 + \lambda_2)}
\end{aligned}
$$

Where binormial formule has been used here $(a+b)^z = \sum_{x=0}^{z} C_z^x a^x b^{z-x} = \sum_{x=0}^{z} \frac{z!}{x!(z-x)!} a^x b^{z-x}$.

After the real number of hadrons is determined, the next step is to determine the particle type, which can be done by using the discrete distribution with probabilities given by dN_i. And there is sampling functions in c++11 for poisson distribution and discrete distribution which can be used directly.

2 Mometum sampling

It is easy to sample the 4 momentum from distribution function $f(p) = p^2 \exp(-\sqrt{p^2 + m^2}/T)$ by rejection method.This is a good idea. Scott Pratt introduces a smart way to sample 4 momentum from thermal distributions, the math trick is used here: for probability distribution $x^{n-1} e^{-x}$, one can sample the distribution by taking the natural log of n random numbers $x = -\ln(r_1 r_2 ... r_n)$ where r_i are random numbers uniformly distributed between zero and one. For three dimentional thermal distribution for massless particle whose distribution function reads $f(p) = p^2 e^{-p/T}$,

$$
\begin{aligned}
p &= -T \ln(r_1 r_2 r_3) \\
\cos\theta &= \frac{\ln(r_1) - \ln(r_2)}{\ln(r_1) + \ln(r_2)} \\
\phi &= \frac{2\pi \left[\ln(r_1 r_2)\right]^2}{\left[\ln(r_1 r_2 r_3)\right]^2}
\end{aligned}
$$

By checking the Jacobian, indeed

$$
\begin{aligned}
dp d\cos\theta d\phi &= |J| \; dr_1 dr_2 dr_3 \\
&= \frac{8\pi T}{r_1 r_2 r_3 \left[\ln(r_1 r_2 r_3)\right]^2} dr_1 dr_2 dr_3 \\
&= \frac{8\pi T}{e^{-p/T} p^2 / T^2} dr_1 dr_2 dr_3
\end{aligned}
$$

And $dr_1 dr_2 dr_3 = \frac{1}{8\pi T^3} p^2 e^{-p/T} dp d\cos\theta d\phi$.

For massive hadrons with Juttner distribution,

$$
p^2 e^{-(E-\mu)/T} = p^2 e^{-p/T} e^{(p-E+\mu)/T}
$$

Use acceptance-rejection method, draw p from $p^2 e^{-p/T}$ distribution, accept or reject with weight function $\omega(p) = e^{(p-E)/T} = e^{(p-\sqrt{p^2+m^2})/T}$. Notice that $\omega(p) < 1$ is always satisfied, while for heavy hadrons $\omega(p) \ll 1$ and many samplings are rejected which makes this method inefficient.

For π

$$
p^2/(e^{(E-\mu)/T} - 1) = p^2 e^{-p/T} * e^{p/T}/(e^{(E-\mu)/T} - 1)
$$

Use acceptance-rejection method, draw p from $p^2 e^{-p/T}$ distribution, accept or rejection with weight function $\omega(p) = e^{p/T}/(e^{(E-\mu)/T} - 1)$. However in this case we don't know when $\omega(p) < 1$ will break down. Do a simple test with $mass = 0.139$, $T = 0.150$ shows that $\omega(p) < 1$ will break down soon at large p for $\mu = 0$ and every where for non-zero μ. This is frustrating because most of the hadrons produced are pions and we already know that there are big difference for Juttner distribution and Bose-Einstein distribution for pion. However if we know one simple relationship betwen the maximum value of $\omega(p)$ and p, μ, we can scale the weight function by $\omega(p)/\omega_{max}(\mu)$ to fix it. For big μ the probability for rejection is around 50% which is still acceptable as shown in Fig 2. One remaining question is to find the maximum value of $\omega(p)$, as you can see there are 2 points with $\frac{d\omega(p)}{dp} = 0$ for $\mu \neq 0$ and one can not determine if it is maximum. For $\mu = 0$, non of these 2 points around $p = 0.1$ produce maximum since $\omega(p)$ may exceed unity at large p. So further check should be done if $\omega_{max}(p \sim 0.1) > \omega(p_{max})$ where p_{max} is the biggest momentum cut that will be used. Another way to keep $\omega(p) < 1$ is to rescale it by $\exp(a * \mu/T)$ however the optimised value of a is unknow.

For heavy hadrons there is no difference between BoseEinstein, FermiDirac and Juttner distribution, so Juttner distribution can be used to sample the 4 momentum of all hadrons except π. The weight function $\omega(p) = e^{(p-E)/T}$ is much smaller for heavier hadrons, so there are many rejections which will make the program slow. However, The philosophy is for $T/m < 0.6$ do variable transformation:

$$
\begin{aligned}
p &= \sqrt{E^2 - m^2},\ dp = E/pdE && (1)\\
dpp^2e^{-E/T} &= dE\frac{E}{p}p^2e^{-E/T} && (2)\\
&= dEpEe^{-E/T} && (3)\\
&= dk\frac{p}{E}(k+m)^2e^{-k/T}e^{-m/T} && (4)\\
&= dk(k+m)^2e^{-k/T}\omega(p) && (5)\\
&= dk(k^2+2mk+m^2)e^{-k/T}\omega(p) && (6)
\end{aligned}
$$

where $k = E - m$ and $\omega(p) = \frac{p}{E}e^{-m/T}$ is weight function. The excelent part of this algorithm is that $E - m > 0$ and $p/E < 1$. The $e^{-m/T}$ and $e^{-\mu/T}$ terms are not important and can be dropped. By split the upper distributions into 3 parts and determine which part is domainant by their integrated weight.

$$
\begin{aligned}
\int dkk^2e^{-k/T} &= 2T^3 && (7)\\
\int dk2mke^{-k/T} &= 2mT^2 && (8)\\
\int dkm^2e^{-k/T} &= m^2T && (9)
\end{aligned}
$$

Once picked k one can do rejection and repeat with weight p/E.

3 Adaptive Rjection Sampling

Since the Bose-Einstein distribution for pion is quite special, and the 4-momentum is not easy to sample with finite chemical potential μ where weight function $\omega(p) < 1$ is not satisfied, it is possible to further rescale $\omega(p)$ so that it is smaller than unity. But the best scale factor is arbitrary with unkown parameter a. It is possible to use Adaptive Rejection Sampling (ARS) which will construct one upper bound and refine this bound with rejected points. ARS asks for the probability distribution function $f(x)$ to be log concave, which means if we set $h(x) = \log f(x)$, $h''(x) < 0$ should be true for any x, fortunatelly all the Juttner, Bose-Einstein and Fermi-Dirac distributions obey this rule. The ARS method is developed further to isolate the distribution function to concave and convex part which will produce upper bounds separately.

The philosophy of ARS is to generate a piecewise exponential distribution upper bound for $f(x)$. Where the distribution function $q(x)$ is piecewise exponential distribution q(x)∝ exp(g(x))if $g(x)$ is piecewise linear. The ordered change points are $z_0 < z_1 < z_2 ... < z_n$ and $g(x)$ has slope m_i in (z_{i-1}, z_i). The area under each piece of exponential segment $\exp(g(x_i))$ is,

$$A_i = \int_{z_i - 1}^{z_i} e^{g(x)} dx = \frac{1}{m_i}\left(e^{g(z_i)} - e^{g(z_{i-1})}\right)$$

First sample j from discrete_distribution($\{A_i\}$), then sampling $x \in (z_{j-1,} z_j)$ from distribution function $q(x) = \exp(a + m_i x)$. By inversely sampling uniform distribution $r \in [0, 1]$ from the cumulative propability

$$Q(x) = \int_{z_{i-1}}^{x} q(y)dy = \frac{q(x) - q(z_{i-1})}{q(z_i) - q(z_{i-1})} = r$$

we get x from the exponential distribution,

$$x = \frac{1}{m_i} \ln\left(r e^{m_i z_i} + (1-r) e^{m_i z_{i-1}}\right)$$

With this x we can do rejection test: $ran() < \frac{f(x)}{q(x)} = \exp(h(x) - g(x))$.

If a point is rejected, it will be used to refine the upper bound which will make the upper bound close to $f(x)$ as soon as possible. In squeezing test step, lower bound is also needed which we call $l(x)$. Squeezing test is true if $ran() < \frac{l(x)}{q(x)}$.

There are 3 kinds of different upper bounds, one is from tangent lines, the other two are from scants, the upper bounds given by scants is looser but there will less derivtive calculations.

4 (3+1)D viscous hydrodynamics hydrodynamics

For ideal hydrodynamics, sample $nd\Sigma_{max}$ number of hadrons, determine the particle type using discrete distribution with the probabilities given by their equilibrium density n_i, where $n = \sum_i n_i$.

Once the particle type is determine, we sample the four-momenta of the particle in the local rest frame of the fluid cell, and keep the particle if $r < \frac{p^* \cdot d\Sigma^*}{p^{0*} d\Sigma_{max}}$.

For viscous hydrodynamics, one needs to consider the non-equilibrium contributions. However, the $\pi^{\mu\nu}$ terms does not contribute to the total number since the particle flow on freeze out hypersurface $n^\mu = nu^\mu$ is perpendicular to $\pi^{\mu\nu}$. This property is really helpful since we don't need to calculate equilibrium density for each freeze out hyper-surface.

And we can sample N hadrons, where N equals to,

$$N = nd\Sigma_{max} K_{max}$$

where the non-equilibrium scale factor K_{max} is used to get more hadrons for the additional rejection procedure,

$$K_{max} = \left[1 + (1 \mp f_0) \frac{p_\mu^* p_\nu^* \pi^{\mu\nu *}}{2T^2(\varepsilon + P)}\right]_{max}$$

After N is determine, sample how many particles will be produced with Poisson distribution whose mean probability is N. Then determine the particle type as before, sample four-momenta in LRF, do keep or rejection according to,

$$r < \frac{p^* \cdot d\Sigma}{p^{0*} d\Sigma_{max}} w_{visc}$$

where the rejection weight w_{visc} from non-equilibrium corrections is,

$$w_{visc} = \frac{A + (1 \mp f_0) p_\mu^* p_\nu^* \pi^{\mu\nu *}}{\left[A + (1 \mp f_0) p_\mu^* p_\nu^* \pi^{\mu\nu *}\right]_{max}}$$

where $A = 2T^2(\epsilon + P)$ is constant on the freeze out hyper-surface.

Then for Fermions,

$$K_{max} = 1 + \left[p_\mu^* p_\nu^* \pi^{\mu\nu *}\right]_{max} / A \tag{10}$$

$$w_{visc} = \frac{A + (1 - f_0) p_\mu^* p_\nu^* \pi^{\mu\nu *}}{A + \left[p_\mu^* p_\nu^* \pi^{\mu\nu *}\right]_{max}} \tag{11}$$

and for Bosons,

$$K_{max} = 1 + 2\left[p_\mu^* p_\nu^* \pi^{\mu\nu *}\right]_{max} / A \tag{12}$$

$$w_{visc} = \frac{A + (1 + f_0) p_\mu^* p_\nu^* \pi^{\mu\nu *}}{A + 2\left[p_\mu^* p_\nu^* \pi^{\mu\nu *}\right]_{max}} \tag{13}$$

The easiest way to get $\left[p_\mu^* p_\nu^* \pi^{\mu\nu *}\right]_{max}$ is as follows,

$$p_\mu^* p_\nu^* \pi^{\mu\nu *} \leq \sum_{\mu\nu} |p^{\mu *} p^{\nu *} \pi^{\mu\nu *}| \tag{14}$$

$$\leq E^{*2} \sum_{\mu\nu} |\pi^{\mu\nu *}| \tag{15}$$

thus we have $\left[p^*_\mu p^*_\nu \pi^{\mu\nu *}\right]_{max} = E^{*2}_{max} \sum_{\mu\nu} |\pi^{\mu\nu *}|$.

Now the only difficulty is how to boost $\pi^{\mu\nu}$ tensor to the comoving frame of the fluid.