3+1D Dissipative hydrodynamics

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I. FIRST ORDER DERIVATIVE EXPANSION IN HYDRODYNAMICS

The conservation equation for energy momentum tensor $T^{\mu\nu}$ is:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1}$$

Where ∇ is the geometric covariant derivative and $T^{\mu\nu} = \varepsilon u^\mu u^\nu + T_\perp^{\mu\nu}$. $T_\perp^{\mu\nu}$ is the spatial part. u^μ is the boosted velocity in energy frame. ε is the energy density in the local rest frame.

$$T_\perp^{\mu\nu} = -P\Delta^{\mu\nu} - \eta_v \sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}(\nabla \cdot u) \tag{2}$$

where P is the pressure density, $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu\nu}$ is the symmetric and transverse tensor which satisfies $u_\mu \Delta^{\mu\nu} = 0$ and $\Delta^{\mu\nu}\Delta_{\mu\nu} = 3$, η_v is the shear viscous coefficient, ζ is the bulk viscous coefficient, and $\sigma^{\mu\nu}$ is the symmetric and traceless shear tensor which is defined as:

$$\sigma^{\mu\nu} = 2\nabla^{<\mu}u^{\nu>} \tag{3}$$

Where

$$A^{<\mu\nu>} = \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(A_{\alpha\beta} + A_{\beta\alpha}) - \frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta} \tag{4}$$

$$\begin{aligned}
\frac{1}{2}\sigma^{\mu\nu} &= \nabla^{<\mu}u^{\nu>} & (5)\\
&= \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right) - \frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}(\nabla_\alpha u_\beta) & (6)\\
&= \frac{1}{2}\left(\Delta^{\nu\beta}\nabla_\perp^\mu u_\beta + \Delta^{\mu\alpha}\nabla_\perp^\nu u_\alpha\right) - \frac{1}{d-1}\Delta^{\mu\nu}(\nabla_\perp^\beta u_\beta) & (7)\\
&= \frac{1}{2}\left(\nabla_\perp^\mu u^\nu + \nabla_\perp^\nu u^\mu\right) - \frac{1}{3}\Delta^{\mu\nu}\theta & (8)
\end{aligned}$$

Where $\nabla_\perp^\mu = \Delta^{\mu\nu}\nabla_\nu$ is the spatial part of the covariant derivative. Notice that we used $\Delta^{\nu\beta}\nabla_\perp^\mu u_\beta = (g^{\nu\beta} - u^\nu u^\beta)\nabla_\perp^\mu u_\beta = \nabla_\perp^\mu u^\nu - \frac{1}{2}u^\nu\nabla_\perp(u^\beta u_\beta) = \nabla_\perp^\mu u^\nu$. $d = 4$ is used in this note. $\Delta^{\alpha\beta}\nabla_\alpha u_\beta = (g^{\alpha\beta} - u^\alpha u^\beta)\nabla_\alpha u_\beta = \nabla \cdot u = \theta$.

The identity that the covariant derivative of metric is zero is used many times through our this note:

$$\nabla_k g_{ij} = \partial_k g_{ij} - g_{jl}\Gamma^l_{ki} - g_{il}\Gamma^l_{kj} = 0 \tag{9}$$

Energy density in the local rest frame can be calculated from $\varepsilon = u_\mu T^{\mu\nu} u_\nu$.

First order shear viscosity is $\pi^{\mu\nu} = -\eta_v \sigma^{\mu\nu}$.

First order bulk viscosity is $\Pi = -\zeta\nabla \cdot u$.

And $P + \Pi = -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$(Notice that $\Delta_{\mu\nu}\pi^{\mu\nu} = \pi^\mu_\mu = 0$.)

For conformal fluid we have $\varepsilon = 3P$ and $T^\mu_\mu = g_{\mu\nu}T^{\mu\nu} = \varepsilon - 3(P + \Pi) = 0$. Which means $\zeta = 0$ in this case.

First order dissipative hydrodynamics is proved to have causality problems which will introduce instabilities in numerical simulation. We will use the second order dissipative hydrodynamic equations derived by Israel and Stewart.

II. SECOND ORDER DISSIPATIVE HYDRODYNAMIC EQUATIONS.

In one arbitrary frame, we choose the time-like and normalized 4 velocity, $u \cdot u = 1$. The spatial projector is $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. The net charge and energy momentum can be decomposed as:

$$
\begin{aligned}
N_i^\mu &= n_i u^\mu + j^\mu && (10)\\
T^{\mu\nu} &= \varepsilon u^\mu u^\nu - (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu && (11)
\end{aligned}
$$

Where n_i is the net charge density for species i and h^μ is the heat flux. In particle frame $j^\mu = 0$, while in energy frame, $h^\mu = 0$.

The off-equilibrium identity can be wrote as:

$$
S^\mu = P(\alpha,\beta)\beta^\mu - \alpha N^\mu + \beta_\nu T^{\mu\nu} - Q^\mu(\delta N^\mu, \delta T^{\mu\nu}) \tag{12}
$$

where $\alpha = \mu/T$, $\beta = 1/T$ and $\beta^\mu = \beta u^\mu$. This identity is a covariant extension from $\varepsilon + P = Ts + \mu n$.

So the entropy derivative is:

$$
\nabla_\mu S^\mu = \nabla_\mu(P\beta^\mu) - N^\mu \nabla_\mu \alpha + T^{\lambda\mu}\nabla_\mu \beta_\lambda - \nabla_\mu Q^\mu \tag{13}
$$

We can also rewrite the Gibbs-Duhem equation:

$$
\begin{aligned}
d(P/T) &= dP/T - PdT/T^2\\
&= (sdT + nd\mu)/T - (-\varepsilon + Ts + \mu n)dT/T^2\\
&= nd\mu/T - \mu n dT/T^2 + \varepsilon dT/T^2\\
&= nd(\mu/T) - \varepsilon dT^{-1}
\end{aligned} \tag{14}
$$

as:

$$
\nabla_\mu(P\beta^\mu) = N_0^\mu \nabla_\mu \alpha - T_0^{\lambda\mu}\nabla_\mu \beta_\lambda \tag{15}
$$

Where N_0^μ and $T_0^{\lambda\mu}$ are the equilibrium part of charge flow and energy momentum tensor.

Then we get:

$$
\begin{aligned}
\nabla_\mu S^\mu &= -(N^\mu - N_0^\mu)\nabla_\mu \alpha + (T^{\lambda\mu} - T_0^{\lambda\mu})\nabla_\mu \beta_\lambda - \nabla_\mu Q^\mu\\
&= -j^\mu \nabla_\mu \alpha + (\pi^{\mu\nu} - \Pi\Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu)\nabla_\mu \beta_\nu - \nabla_\mu Q^\mu
\end{aligned} \tag{16}
$$

The second order dissipative part can be decomposed as:

$$
TQ^\mu = \frac{1}{2}u^\mu(\beta_0 \Pi^2 + \beta_1 q_\lambda q^\lambda + \beta_2 \pi^{\lambda\nu}\pi_{\lambda\nu}) - \alpha_0 \Pi q^\mu - \alpha_1 \pi^{\lambda\mu} q_\lambda + TR^\mu \tag{17}
$$

Where the second order spatial vector $q^\mu = h^\mu - (\varepsilon + P)j^\mu/n$ $(q \cdot u = 0)$ and

$$
R^\mu = \gamma_0 u^\mu h^\lambda h_\lambda + \gamma_1 \pi^{\mu\lambda} h_\lambda + \gamma_2 \Pi h^\mu \tag{18}
$$

At energy frame and considering baryon free case where $h^\mu = n = j^\mu = 0$. We have

$$
\begin{aligned}
TQ^\mu &= \frac{1}{2}u^\mu(\beta_0 \Pi^2 + \beta_2 \pi^{\lambda\nu}\pi_{\lambda\nu}) && (19)\\
\nabla_\mu Q^\mu &= \Pi^2 \nabla_\mu \frac{\beta_0 u^\mu}{2T} + \pi^{\lambda\nu}\pi_{\lambda\nu}\nabla_\mu \frac{\beta_2 u^\mu}{2T} + \frac{1}{T}(\beta_0 \Pi D\Pi + \beta_2 \pi^{\lambda\nu} D\pi_{\lambda\nu}) && (20)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\mu S^\mu &= \frac{1}{T}\pi^{\mu\nu} < \nabla_\mu u_\nu > -\frac{1}{T}\Pi\nabla_\mu u^\mu - \nabla_\mu Q^\mu && (21)\\
&= -\frac{1}{T}\Pi(\theta + \beta_0 D\Pi + T\Pi\nabla_\mu \frac{\beta_0 u^\mu}{2T}) + \frac{\pi^{\lambda\nu}}{T}(< \nabla_\nu u_\lambda > -\beta_2 D\pi_{\nu\lambda} - \pi_{\nu\lambda}\nabla_\mu \frac{\beta_2 u^\mu}{2T}) && (22)
\end{aligned}
$$

To make $\nabla_\mu S^\mu > 0$ we use a phenomenological method where we set:

$$\Pi = -\zeta(\theta + \beta_0 D\Pi + T\Pi\nabla_\mu \frac{\beta_0 u^\mu}{2T}) \tag{23}$$

$$\pi^{\nu\lambda} = 2\eta_v(< \nabla_\nu u_\lambda > -\beta_2 D\pi_{\nu\lambda} - \pi_{\nu\lambda}\nabla_\mu \frac{\beta_2 u^\mu}{2T}) \tag{24}$$

Where $\theta = \nabla_\mu u^\mu$ and the temporal derivative is $D = \frac{\nabla X^\mu}{\nabla \tau_f}\frac{\nabla}{\nabla X^\mu} = u^\mu \nabla_\mu$.
The upper equation can be rewritten as:

$$D\Pi \approx -\frac{1}{\zeta\beta_0}(\Pi + \zeta\theta) \tag{25}$$

$$D\pi^{\nu\lambda} \approx -\frac{1}{2\eta_v\beta_2}(\pi^{\nu\lambda} - 2\eta_v < \nabla^\nu u^\lambda >) \tag{26}$$

And the relaxation time $\tau_\Pi = \zeta\beta_0$ and $\tau_\pi = 2\eta_v\beta_2$. In the upper equations, we did not consider the vorticity $\omega^{\mu\nu} = \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_{\alpha\perp}u_\beta - \nabla_{\beta\perp}u_\alpha)$. And $\pi_{\nu\lambda}\nabla_\mu\frac{\beta_2 u^\mu}{2T}$ term are kept to keep the conformal invariance in the recent work given by D.T.Son.
The simplest form that can be used in dissipative hydrodynamic simulation should be:

$$\pi^{\nu\lambda} \approx \eta_v\sigma^{\nu\lambda} - \tau_\pi(D\pi^{<\nu\lambda>} + \frac{4}{3}\pi^{\nu\lambda}\theta) \tag{27}$$

Notice that in order to keep the symmetric, traceless and transverse properties of $\pi^{\mu\nu}$, a projection must be done on $D\pi^{\nu\lambda}$.

III. ISRAEL-STEWART EQUATIONS IN (τ, x, y, η_s) COORDINATES.

The main task of the numerical simulation is to solve the flowing equations in (τ, x, y, η_s) coordinates:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{28}$$

$$D\pi^{<\mu\nu>} = -\frac{1}{\tau_\pi}(\pi^{\mu\nu} - \eta\sigma^{\mu\nu}) - \frac{4}{3}\pi^{\mu\nu}\theta \tag{29}$$

$\pi^{\mu\nu}$ is symmetric, transverse $u_\mu\pi^{\mu\nu} = 0$ (4 constrains), traceless $g_{\mu\nu}\pi^{\mu\nu} = 0$ (1 constrain). So there are only $10 - 4 - 1 = 5$ independent variables.
We have the following definitions:

$$\tau = \sqrt{t^2 - z^2} \tag{30}$$

$$\eta_s = \frac{1}{2}\ln(\frac{t+z}{t-z}) \tag{31}$$

$$Y_L = \frac{1}{2}\ln(\frac{1+v_z}{1-v_z}) \tag{32}$$

$$\tau_f = \sqrt{t^2 - x^2 - y^2 - z^2} \tag{33}$$

$$ds^2 = d\tau^2 - dx^2 - dy^2 - \tau^2 d\eta_s^2 \tag{34}$$

$$g_{\mu\nu} = \mathbf{diag}(1, -1, -1, -\tau^2) \tag{35}$$

$$g^{\mu\nu} = \mathbf{diag}(1, -1, -1, -1/\tau^2) \tag{36}$$

$$X^\mu = (\tau, x, y, \eta_s) \tag{37}$$

$$u^\mu = \frac{dX^\mu}{d\tau_f} \tag{38}$$

$$v_x = v_1\cosh(Y_L)/\cosh(Y_L - \eta_s) \tag{39}$$

$$v_y = v_2\cosh(Y_L)/\cosh(Y_L - \eta_s) \tag{40}$$

$$v_{\eta_s} = \tanh(Y_L - \eta_s) \tag{41}$$

Christoffel symbols in this coordinates are calculated through metric:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) \tag{42}$$

And there are only three non zero terms left: $\Gamma^{\eta_s}_{\eta_s \tau} = \Gamma^{\eta_s}_{\tau \eta_s} = \frac{1}{\tau}$, $\Gamma^{\tau}_{\eta_s \eta_s} = \tau$.
The covariant derivative for vectors and tensors in Riemannian space is:

$$\nabla_k A^i = \partial_k A^i + \Gamma^i_{jk} A^j \tag{43}$$

$$\nabla_s T^{lk}_{ij} = \partial_s T^{lk}_{ij} - \Gamma^r_{is} T^{lk}_{rj} - \Gamma^r_{js} T^{lk}_{ir} + \Gamma^l_{rs} T^{rk}_{ij} + \Gamma^k_{rs} T^{lr}_{ij} \tag{44}$$

A. Calculate $\sigma^{\mu\nu}$ in the (τ, x, y, η_s) coordinates

We have $D = u^\mu \nabla_\mu$ and $\nabla^\mu_\perp = \Delta^{\mu\alpha} \nabla_\alpha = \nabla^\mu - u^\mu D$. $\nabla_\perp \cdot u = \nabla \cdot u - u^\mu D u_\mu = \nabla \cdot u - \frac{1}{2} D u^2 = \nabla \cdot u = \theta$.

$$\begin{aligned} \sigma^{\mu\nu} &= 2 \nabla^{<\mu}_\perp u^{\nu>} \\ &= \nabla^\mu_\perp u^\nu + \nabla^\nu_\perp u^\mu - \frac{2}{3} (\nabla \cdot u) \Delta^{\mu\nu} \\ &= (g^{\mu\alpha} \nabla_\alpha u^\nu + g^{\nu\alpha} \nabla_\alpha u^\mu) - (u^\mu D u^\nu + u^\nu D u^\mu) - \frac{2}{3} \Delta^{\mu\nu} \theta \\ &= (g^{\mu\alpha} \nabla_\alpha u^\nu + g^{\nu\alpha} \nabla_\alpha u^\mu) - D(u^\mu u^\nu) - \frac{2}{3} \Delta^{\mu\nu} \theta \end{aligned} \tag{45}$$

The explicit form of shear tensor is:

$$\begin{aligned} \sigma^{\mu\nu} = & \begin{pmatrix} 2\nabla_\tau u^\tau & \nabla_\tau u^x - \nabla_x u^\tau & \nabla_\tau u^y - \nabla_y u^\tau & \nabla_\tau u^{\eta_s} - \frac{1}{\tau^2} \nabla_{\eta_s} u^\tau \\ \nabla_\tau u^x - \nabla_x u^\tau & -2\nabla_x u^x & -(\nabla_x u^y + \nabla_y u^x) & -(\nabla_x u^{\eta_s} + \frac{1}{\tau^2} \nabla_{\eta_s} u^x) \\ \nabla_\tau u^y - \nabla_y u^\tau & -(\nabla_x u^y + \nabla_y u^x) & -2\nabla_y u^y & -(\nabla_y u^{\eta_s} + \frac{1}{\tau^2} \nabla_{\eta_s} u^y) \\ \nabla_\tau u^{\eta_s} - \frac{1}{\tau^2} \nabla_{\eta_s} u^\tau & -(\nabla_x u^{\eta_s} + \frac{1}{\tau^2} \nabla_{\eta_s} u^x) & -(\nabla_y u^{\eta_s} + \frac{1}{\tau^2} \nabla_{\eta_s} u^y) & -\frac{2}{\tau^2} \nabla_{\eta_s} u^{\eta_s} \end{pmatrix} \\ & - \begin{pmatrix} D(u^\tau u^\tau) & D(u^\tau u^x) & D(u^\tau u^y) & D(u^\tau u^{\eta_s}) \\ D(u^\tau u^x) & D(u^x u^x) & D(u^x u^y) & D(u^x u^{\eta_s}) \\ D(u^\tau u^y) & D(u^x u^y) & D(u^y u^y) & D(u^y u^{\eta_s}) \\ D(u^\tau u^{\eta_s}) & D(u^x u^{\eta_s}) & D(u^y u^{\eta_s}) & D(u^{\eta_s} u^{\eta_s}) \end{pmatrix} \\ & + \begin{pmatrix} u^\tau u^\tau - 1 & u^\tau u^x & u^\tau u^y & u^\tau u^{\eta_s} \\ u^x u^\tau & u^x u^x + 1 & u^x u^y & u^x u^{\eta_s} \\ u^y u^\tau & u^y u^x & u^y u^y + 1 & u^y u^{\eta_s} \\ u^{\eta_s} u^\tau & u^{\eta_s} u^x & u^{\eta_s} u^y & u^{\eta_s} u^{\eta_s} + \frac{1}{\tau^2} \end{pmatrix} \times \frac{2}{3} \theta \end{aligned}$$

There are some covariant derivatives we should take care of:

$$\theta = \nabla \cdot u = \partial_\tau u^\tau + \partial_x u^x + \partial_y u^y + \partial_{\eta_s} u^{\eta_s} + \frac{u^\tau}{\tau} \tag{46}$$

$$D = u^\tau \nabla_\tau + u^x \nabla_x + u^y \nabla_y + u^{\eta_s} \nabla_{\eta_s} \tag{47}$$

$$\nabla_{\eta_s} u^\tau = \partial_{\eta_s} u^\tau + \tau u^{\eta_s} \tag{48}$$

$$\nabla_{\eta_s} u^{\eta_s} = \partial_{\eta_s} u^{\eta_s} + \frac{1}{\tau} u^\tau \tag{49}$$

$$\nabla_\tau u^{\eta_s} = \partial_\tau u^{\eta_s} + \frac{1}{\tau} u^{\eta_s} \tag{50}$$

so

$$\nabla_\tau u^{\eta_s} - \frac{1}{\tau^2} \nabla_{\eta_s} u^\tau = \partial_\tau u^{\eta_s} - \frac{1}{\tau^2} \partial_{\eta_s} u^\tau \tag{51}$$

$$-\frac{2}{\tau^2} \nabla_{\eta_s} u^{\eta_s} = -\frac{2}{\tau^2} \partial_{\eta_s} u^{\eta_s} - \frac{2}{\tau^3} u^\tau \tag{52}$$

and

$$\begin{aligned}
Du^\tau &= (u^\tau\partial_\tau + u^x\partial_x + u^y\partial_y + u^{\eta_s}\partial_{\eta_s})u^\tau + \tau u^{\eta_s}u^{\eta_s} && (53)\\
Du^x &= (u^\tau\partial_\tau + u^x\partial_x + u^y\partial_y + u^{\eta_s}\partial_{\eta_s})u^x && (54)\\
Du^y &= (u^\tau\partial_\tau + u^x\partial_x + u^y\partial_y + u^{\eta_s}\partial_{\eta_s})u^y && (55)\\
Du^{\eta_s} &= (u^\tau\partial_\tau + u^x\partial_x + u^y\partial_y + u^{\eta_s}\partial_{\eta_s})u^{\eta_s} + \frac{2}{\tau}u^\tau u^{\eta_s} && (56)
\end{aligned}$$

While for other terms in the first matrix $\nabla_\mu u^\nu = \partial_\mu u^\nu$. We can assume $\pi_0^{\mu\nu} = -\eta_v\sigma^{\mu\nu}$ at τ_0 for the initial condition. The $\sigma^{\mu\nu}$ can be calculated from u^μ which is given by some model assumptions. For $u^\mu = (1,0,0,0)$ initial setting, we can simply get:

$$\tau_0^2\pi_0^{\eta_s\eta_s} = -2\pi^{xx} = -2\pi^{yy} = \frac{4\eta_v}{3\tau_0} \tag{57}$$

The $D\pi^{<\mu\nu>}$ term in Eq. 29 can be expanded as:

$$\begin{aligned}
D\pi^{<\mu\nu>} &= \frac{1}{2}\Delta^\mu_\alpha\Delta^\nu_\beta(D\pi^{\alpha\beta} + D\pi^{\beta\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}D\pi_{\alpha\beta}\\
&= \Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta} - \frac{1}{3}\Delta^{\mu\nu}\left(D(\Delta^{\alpha\beta}\pi_{\alpha\beta}) - \pi_{\alpha\beta}D\Delta^{\alpha\beta}\right)\\
&= \Delta^\mu_\alpha\left(D(\pi^{\alpha\nu}) - \pi^{\alpha\beta}D\Delta^\nu_\beta\right) + \frac{1}{3}\Delta^{\mu\nu}\pi_{\alpha\beta}D\Delta^{\alpha\beta}\\
&= D\pi^{\mu\nu} - \pi^{\alpha\nu}D\Delta^\mu_\alpha - \pi^{\mu\beta}D\Delta^\nu_\beta + \frac{1}{3}\Delta^{\mu\nu}\pi_{\alpha\beta}Dg^{\alpha\beta}\\
&= \nabla_\lambda(u^\lambda\pi^{\mu\nu}) - \pi^{\mu\nu}\theta - \pi^{\alpha\nu}D\Delta^\mu_\alpha - \pi^{\mu\beta}D\Delta^\nu_\beta + \frac{1}{3}\Delta^{\mu\nu}\pi_{\alpha\beta}Dg^{\alpha\beta}\\
&= \nabla_\lambda(u^\lambda\pi^{\mu\nu}) - \pi^{\mu\nu}\theta + u^\mu\pi^{\alpha\nu}Du_\alpha + u^\nu\pi^{\mu\beta}Du_\beta\\
&= \nabla_\lambda(u^\lambda\pi^{\mu\nu}) - \pi^{\mu\nu}\theta + (u^\mu\pi^{\nu\alpha} + u^\nu\pi^{\mu\alpha})Du_\alpha. && (58)
\end{aligned}$$

Where $\Delta^\beta_\alpha = \delta^\beta_\alpha - u_\alpha u^\beta$ and $Dg^{\alpha\beta} = u^\tau\nabla_\tau g^{\eta_s\eta_s} = u^\tau\left(\partial_\tau g^{\eta_s\eta_s} + \Gamma^{\eta_s}_{\eta_s\tau}g^{\eta_s\eta_s} + \Gamma^{\eta_s}_{\tau\eta_s}g^{\eta_s\eta_s}\right) = 0$.

The Eq. 29 can be expanded as follows with the $(u^\mu\pi^{\nu\alpha} + u^\nu\pi^{\mu\alpha})Du_\alpha$ term to keep the traceless and transverse properties.

$$D\pi^{\mu\nu} = -\frac{1}{\tau_\pi}(\pi^{\mu\nu} - \eta\sigma^{\mu\nu}) - \frac{4}{3}\pi^{\mu\nu}\theta - (u^\mu\pi^{\nu\alpha} + u^\nu\pi^{\mu\alpha})Du_\alpha = B^{\mu\nu} \tag{59}$$

Where the covariant derivative for $\pi^{\mu\nu}$ is:

$$\begin{aligned}
D\pi^{\mu\nu} &= \gamma(\nabla_\tau + v_x\nabla_x + v_y\nabla_y + v'_{\eta_s}\nabla_{\eta_s})\pi^{\mu\nu}\\
&= \gamma\left((\pi^{\mu\nu})_{,\tau} + \Gamma^\mu_{\lambda\tau}\pi^{\lambda\nu} + \Gamma^\nu_{\lambda\tau}\pi^{\mu\lambda} + v_x(\pi^{\mu\nu})_{,x} + v_y(\pi^{\mu\nu})_{,y} + v'_{\eta_s}\left((\pi^{\mu\nu})_{,\eta_s} + \Gamma^\mu_{\lambda\eta_s}\pi^{\lambda\nu} + \Gamma^\nu_{\lambda\eta_s}\pi^{\mu\lambda}\right)\right)\\
&= \gamma((\pi^{\mu\nu})_{,\tau} + v_x(\pi^{\mu\nu})_{,x} + v_y(\pi^{\mu\nu})_{,y} + v'_{\eta_s}(\pi^{\mu\nu})_{\eta_s}) + \gamma(\Gamma^\mu_{\lambda\tau}\pi^{\lambda\nu} + \Gamma^\nu_{\lambda\tau}\pi^{\mu\lambda}) + \gamma v'_{\eta_s}(\Gamma^\mu_{\lambda\eta_s}\pi^{\lambda\nu} + \Gamma^\nu_{\lambda\eta_s}\pi^{\mu\lambda}) && (60)
\end{aligned}$$

The upper equation can be wrote alternatively for further numerical convenient:

$$\begin{aligned}
(\pi^{\mu\nu})_{,\tau} + (v_x\pi^{\mu\nu})_{,x} + (v_y\pi^{\mu\nu})_{,y} + (v'_{\eta_s}\pi^{\mu\nu})_{\eta_s} &= \frac{1}{\gamma}B^{\mu\nu} + \pi^{\mu\nu}\left((v_x)_{,x} + (v_y)_{,y} + (v'_{\eta_s})_{,\eta_s}\right)\\
&\quad -(\Gamma^\mu_{\lambda\tau}\pi^{\lambda\nu} + \Gamma^\nu_{\lambda\tau}\pi^{\mu\lambda}) - v'_{\eta_s}(\Gamma^\mu_{\lambda\eta_s}\pi^{\lambda\nu} + \Gamma^\nu_{\lambda\eta_s}\pi^{\mu\lambda}) && (61)
\end{aligned}$$

Where $\gamma = u^\tau = \frac{1}{\sqrt{1-v_x^2-v_y^2-v_{\eta_s}^2}}$ and:

$$B^{\mu\nu} = -\frac{1}{\tau_\pi}(\pi^{\mu\nu} - \eta\sigma^{\mu\nu}) - \frac{4}{3}\pi^{\mu\nu}\theta - g_{\alpha\beta}(u^\mu\pi^{\nu\alpha} + u^\nu\pi^{\mu\alpha})Du^\beta \tag{62}$$

Notice that these kind of IS equations can be solved by using the same transport algorithm we developed for ideal hydrodynamics. The dimension of the formula must be self consistent. We have $\left[\frac{1}{\tau_\pi}\right] = fm^{-1}$ and $[\eta_v] = GeVfm^{-2}$.

B. (3+1)D Viscous hydrodynamic equations

In rapidity coordinates, the hydrodynamic equations $\nabla_\mu T^{\mu\nu} = 0$ can be expand as:

$$
\begin{aligned}
T^{\tau\tau}_{,\tau} + T^{\tau x}_{,x} + T^{\tau y}_{,y} + T^{\tau\eta_s}_{,\eta_s} + \frac{1}{\tau}T^{\tau\tau} + \tau T^{\eta_s\eta_s} &= 0 &(63)\\
T^{\tau x}_{,\tau} + T^{xx}_{,x} + T^{xy}_{,y} + T^{x\eta_s}_{,\eta_s} + \frac{1}{\tau}T^{\tau x} &= 0 &(64)\\
T^{\tau y}_{,\tau} + T^{xy}_{,x} + T^{yy}_{,y} + T^{y\eta_s}_{,\eta_s} + \frac{1}{\tau}T^{\tau y} &= 0 &(65)\\
T^{\tau\eta_s}_{,\tau} + T^{x\eta_s}_{,x} + T^{y\eta_s}_{,y} + T^{\eta_s\eta_s}_{,\eta_s} + \frac{3}{\tau}T^{\tau\eta_s} &= 0 &(66)
\end{aligned}
$$

Where $T^{\mu\nu}_{,\mu} = \partial_\mu T^{\mu\nu}$. And the energy pressure tensor is the dissipative one $T^{\mu\nu} = T^{\mu\nu}_0 - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} = \varepsilon u^\mu u^\nu - (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$. The Π and $\pi^{\mu\nu}$ are bulk and shear viscous pressure tensor respectively. These equations can be simplified as:

$$
\begin{aligned}
\tilde{T}^{\tau\tau}_{,\tau} + (v_x\tilde{T}^{\tau\tau})_{,x} + (v_y\tilde{T}^{\tau\tau})_{,y} + (v'_{\eta_s}\tilde{T}^{\tau\tau})_{,\eta_s} &= \tilde{S}^{\tau\tau} &(67)\\
\tilde{T}^{\tau x}_{,\tau} + (v_x\tilde{T}^{\tau x})_{,x} + (v_y\tilde{T}^{\tau x})_{,y} + (v'_{\eta_s}\tilde{T}^{\tau x})_{,\eta_s} &= \tilde{S}^{\tau x} &(68)\\
\tilde{T}^{\tau y}_{,\tau} + (v_x\tilde{T}^{\tau y})_{,x} + (v_y\tilde{T}^{\tau y})_{,y} + (v'_{\eta_s}\tilde{T}^{\tau y})_{,\eta_s} &= \tilde{S}^{\tau y} &(69)\\
\tilde{T}^{\tau\eta_s}_{,\tau} + (v_x\tilde{T}^{\tau\eta_s})_{,x} + (v_y\tilde{T}^{\tau\eta_s})_{,y} + (v'_{\eta_s}\tilde{T}^{\tau\eta_s})_{,\eta_s} &= \tilde{S}^{\tau\eta_s} &(70)
\end{aligned}
$$

Notice that these equations are of the same form as the ideal hydrodynamic equations except that $T^{\mu\nu}$ are dissipative energy pressure tensor and some dissipative source terms are introduced in $\tilde{S}^{\mu\nu}$. Where

$$
\begin{aligned}
\tilde{S}^{\tau\tau} &= -\tau(v_xP)_{,x} - \tau(v_yP)_{,y} - \tau(v'_{\eta_s}P)_{,\eta_s} - \tau v'^2_{\eta_s}(\tilde{T}^{\tau\tau} + \tau P) - P\\
&\quad -\tau(-v_x\pi^{\tau\tau} + \pi^{\tau x})_{,x} - \tau(-v_y\pi^{\tau\tau} + \pi^{\tau y})_{,y} - \tau(-v'_{\eta_s}\pi^{\tau\tau} + \pi^{\tau\eta_s})_{,\eta_s} - \tau^2\pi^{\eta_s\eta_s} &(71)\\
\tilde{S}^{\tau x} &= -\tau P_{,x} - \tau(\pi^{xx} - v_x\pi^{\tau x})_{,x} - \tau(\pi^{xy} - v_y\pi^{\tau x})_{,y} - \tau(\pi^{x\eta_s} - v'_{\eta_s}\pi^{\tau x})_{,\eta_s} &(72)\\
\tilde{S}^{\tau y} &= -\tau P_{,y} - \tau(\pi^{xy} - v_x\pi^{\tau y})_{,x} - \tau(\pi^{yy} - v_y\pi^{\tau y})_{,y} - \tau(\pi^{\eta_s y} - v'_{\eta_s}\pi^{\tau y})_{,\eta_s} &(73)\\
\tilde{S}^{\tau\eta_s} &= -\frac{1}{\tau}P_{,\eta_s} - \frac{2}{\tau}\tilde{T}^{\tau\eta_s} - \tau(\pi^{\eta_s x} - v_x\pi^{\tau\eta_s})_{,x} - \tau(\pi^{\eta_s y} - v_y\pi^{\tau\eta_s})_{,y} - \tau(\pi^{\eta_s\eta_s} - v'_{\eta_s}\pi^{\tau\eta_s})_{,\eta_s} &(74)
\end{aligned}
$$

Notice that the red components are introduced for dissipative hydrodynamics, we can isolate them and make it easy to switch the code from viscous to ideal. It's quite simple to add bulk viscosity to the upper equations by substituting P with $P+\Pi$. 10 components of $\pi^{\mu\nu}$ are used in the upper equations, we need to solve 10 Israel-Stewart equations. Some of the equations are abundant, however the abundantly solved $\pi^{\mu\nu}$ can be used to check the traceless and transverse properties of the shear pressure tensor. It provides a way to check the numerical accuracy in solving $\pi^{\mu\nu}$.

Please take care of $v'_{\eta_s} = v_{\eta_s}/\tau$ and associated terms. Since we have $g_{\mu\nu}u^\mu u^\nu = 1$ and $g_{\eta_s\eta_s} = \tau^2$, we can set $u^{\eta_s} = \gamma\frac{v_{\eta_s}}{\tau} = \gamma v'_{\eta_s}$, where $\gamma = \frac{1}{\sqrt{1-v_x^2-v_y^2-v_{\eta_s}^2}}$ and $v'_{\eta_s} = \frac{v_{\eta_s}}{\tau}$. The energy pressure tensor component $T^{\tau\eta_s} = (\varepsilon+P)u^\tau u^{\eta_s} - Pg^{\tau\eta_s} = \gamma^2(\varepsilon+P)v'_{\eta_s}$ and the same thing happens to $\pi^{\mu\eta_s}$.

The shear viscous deviation from equilibrium distribution function is:

$$
\delta f(p,x) = f_{eq,i}(p,x)(1 \mp f_{eq,i})\frac{p_\mu p_\nu \pi^{\mu\nu}(x)}{2T^2(x)(\varepsilon(x)+P(x))} \tag{75}
$$

Where

$$
\begin{aligned}
p_\mu p_\nu \pi^{\mu\nu} &= m_T^2\left(\cosh^2(Y_L-\eta_s)\pi^{\tau\tau} + \tau^2\sinh^2(Y_L-\eta_s)\pi^{\eta_s\eta_s}\right)\\
&\quad -2m_T\cosh(Y_L-\eta_s)\left(p_x\pi^{\tau x} + p_y\pi^{\tau y} + \tau\sinh(Y_L-\eta_s)\pi^{\tau\eta_s}\right)\\
&\quad +\left(p_x^2\pi^{xx} + 2p_xp_y\pi^{xy} + p_y^2\pi^{yy}\right)\\
&\quad +2\tau m_T\sinh(Y_L-\eta_s)\left(p_x\pi^{x\eta_s} + p_y\pi^{y\eta_s}\right) &(76)
\end{aligned}
$$

Notice that $p^\mu = (m_T\cosh(Y_L-\eta_s), p^x, p^y, \frac{1}{\tau}m_T\sinh(Y_L-\eta_s))$ and $p_\mu = (m_T\cosh(Y_L-\eta_s), -p^x, -p^y, -\tau m_T\sinh(Y_L-\eta_s))$.

C. Numerical simulation

The viscous part is wrote in one independent class CViscous. This class will give out viscous associated source terms, will evolove the Israel-Stewart equations. We do not need to change the ideal part too much.

1. Initialize $T_0^{\mu\nu}$, $\pi^{\mu\nu}$ and $T^{\mu\nu}$.
2. Half time step:
   - Calc source terms for IS equations and $S^{\mu\nu}$.
   - Update $T^{\mu\nu}$ at half time step $\frac{1}{2}dt$.
   - Update $\pi^{\mu\nu}$ at half time step $\frac{1}{2}dt$.
   - Update $U0 = U1$ here.
   - Solve e, u^μ at time step $\frac{1}{2}dt$.
   - Update $E1 = e, U1 = u$.
   - UpdateSetting(tau+1/2 dt, halfStep);
3. Full time step:
   - Calc source terms for IS equations and $S^{\mu\nu}$ at $\frac{1}{2}dt$.
   - Update $T^{\mu\nu}$ at full time step.
   - Update $\pi^{\mu\nu}$ at full time step.
   - Solve e, u^μ at full time step.
   - Update $E1 = e, U1 = u$.
   - UpdateSetting(tau+dt, fullStep);

The declearation of the viscous class is as follows:

```
/* Version viscous01
 * time 08/02/2011
 * author LongGang Pang
 */
#ifndef __CVISCOUS__
#define __CVISCOUS__

#include<iostream>
#include<cmath>
#include<cassert>
#include<cstdarg>
#include<cstdlib>
#include"Algorithm.h"
#include"CEos.h"
#include"CMatrix.h"

using namespace std;
typedef const int CI;
typedef const double CD;

class CViscous{
        /*Calc \pi^{\mu\nv} for initial condition */
        /*Solve Israel-Stewart Equations */
        /*Calc viscous associated source terms in \nabla_{\mu}T^{\mu\nu}=0 */
        /*Calc off-equilibrium deviation for the distribution function.*/

        public:
                /* etav=shear viscous coefficient;
```

```
 * etaos = \eta_v / s(entropy density);
 * taupi = 3*etaos/T; is the relaxtion time*/
double etaos,tau0, dt, dx, dy, de;
/*eta_v/s \tau0, dtau, dx, dy, d\eta_s */
int NX0, NX, NY0, NY, NE0, NE;
/* The lower and upper index of array*/
int IEOS;

CEos   eos;                        /* Equation of state */
double tau;
int bool_half;
/* If      bool_half==0    it's a full time step evolution   */
/*else if bool_half==1    it's a half time step evolution   */
/* The time derivative is affected by this parameter        */
double g[4][4];         /*g^{\mu\nu}= diag{1, -1, -1, -1/(tau*tau)}*/
double Gamma[4][4][4]; /*\Gamma^{k}_{i j} is the christoffel symbol*/


M3D U0[4];                    /*  U^mu at proper time tau */
M3D U1[4];                    /*  U^mu at proper time tau+dt */
M3D V1[4];                    /*  V_i at proper time tau+dt */
M3D E1;                       /*  Energy density at time tau+dt*/
M3D pi0[10];
/*  The upper right triangle components of pi^{mu nu} at tau */
M3D pi1[10];
/*  The upper right triangle components of pi^{mu nu} at tau+dt */
/*In array pi0[n] and pi1[n], n=0 to 9 corresponds
 * tt, tx, ty, te, xx, xy, xe, yy, ye, ee
 * which can be expressed as (j>=i)? 3*i+(j-i) : 3*j+(i-j)*/

M3D STT, STX, STY, STE;
/*Viscous associated source terms for T^{mu nu}  */


CViscous();
CViscous(double etaos_, double tau0_, \
         double dt_, double dx_, double dy_, double de_, \
         int NX0_, int NX_, int NY0_, int NY_,\
         int NE0_, int NE_, int IEOS_);
/*Initialization for the shear viscous associated terms */

void UpdateSetting(CD & tau_, CI& bool_half_);
/*Update g_{\mu\nu}, Gamma^{k}_{i j} and time_step */
double etav(CI& i, CI& j, CI& k);
/* \eta_v at grid (i,j,k) */
double taupi(CI& i,CI& j, CI& k);
/*relaxtion time at grid (i,j,k) */

/*Some derivative functions */

/* \partial_{mu}A at grid (i,j,k) */
double Partial(CI& mu, M3D& A0, M3D& A1, CI &i, CI& j, CI& k);
/* \Nabla_{mu}u^{nu} at grid (i,j,k)  */
double NablaU( CI& mu, CI& nu, CI& i, CI& j, CI& k);
/*   Du^{\mu} at grid (i,j,k) */
double DU(CI &mu, CI& i, CI& j, CI& k);
/*   D(u^{\mu}u^{\nu}) at grid (i,j,k) */
double DUU(CI &mu, CI&nu, CI& i, CI& j, CI& k);
```

```
        /*   \theta = \Nabla\cdot u   */
        double Theta(CI& i, CI& j, CI& k);
        /*   \Delta^{\mu \nu} = g^{\mu\nu}-u^{\mu}u^{\nu}   */
        double Delta(CI& mu, CI& nu, CI& i, CI& j, CI& k);
        /*   sigma^{\mu\nu} = (g[mu][mu]\Nabla_{mu}u^{nu}
         *   +g[nu][nu]\Nabla_{nu}u^{mu})
         *   -D(u^mu u^nu)
         *   - 2.0/3.0*theta*Delta^{mu nu} */
        double Sigma(CI& mu, CI& nu, CI& i, CI& j, CI& k);

        double B(CI& mu, CI& nu, CI& i, CI& j, CI& k);

        //////Calc source terms for Israel-Stewart equations ///
        double SPI(CI& mu, CI& nu, CI& i, CI& j, CI& k);

        //Calc viscous part source terms for \nabla_{\mu} T^{\mu\nu}=0
        double PPI(CI& mu, CI& nu, CI& i, CI& j, CI& k);
        void Calc_Source_Tmn( M3D& STT, M3D& STX, M3D& STY, M3D& STE );

        //////Calc pi0^{mu nu} from given U^{mu} ////////////////
        void Calc_Ini_Pimn(M3D& ed, M3D u0[4], M3D u1[4]);

};

/*Solve Israel-Stewart evolution equation in 1 dimension */
void IS_Evolve(CI & IW, M3D PIMN[10], M3D &VX, M3D &VY, M3D &VZ,\
               CD &DT, CD &DX, CD &DY, CD &DZ, CD &Diff);

/* Time split to extend IS_Evolve to 3D   */
void IS_Tsplit( M3D PIMN[10], M3D &ED, M3D &BD, M3D &VX, M3D &VY, M3D & VZ, \
               CD & DT, CD & DX, CD & DY, CD & DZ, CD &Diff);
#endif
```