2.6 Elimination = Factorization: $A = LU$

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Introduction

Goal: Describe Gaussian elimination in the most useful way.

Observe: Many key ideas of linear algebra, when you look at them closely. are really factorizations of a matrix. The first factorization comes now from elimination.

The factors L and U are triangular matrices. The factorization that comes from elimination is $A = LU$

For a basic 3×3 matrix without row exchange, we have:

$$(E_{32}E_{31}E_{21})A = U \Rightarrow A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U$$
$$\Rightarrow A = LU$$

Explanation and Examples

First point: Every inverse matrix E^{-1} is lower triangular. Its oof-diagonal entry is L_{ij} to undo the subtraction produced by $-l_{ij}$

Second point: Lower triangular product is L

Third Point: Each multiplier l_{ij} goes directly into its i, j position-unchanged-in the product of inverses which is L.

In general:

$$A = LU$$

1. This is elimination without row exchanges.
2. The upper triangular U has the pivots on it diagonal.
3. The lower triangular L has all 1's on its diagonal.
4. The multipliers l_{ij} are below the diagonal of L.

example:

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} = LU$$

Note:

1. When a row of A starts with zeros, so does that row of L

2. When a column of A starts with zeros, so does that column of U

Better balance with from LDU

Divide U by a diagonal matrix D that contains the pivots.

$$\text{Split U into} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

Therefore, we have:

$$A = LU \textbf{ or } A = LDU$$

One Square System = Two Triangular System

Forward and backward Solve $Lc = b$ and then solve $Ux = c$

example:

$$\mathbf{Ax = b} \quad \begin{matrix} u + 2v = 5 \\ 4u + 9v = 21 \end{matrix} \quad \textbf{becomes} \quad \begin{matrix} u + 2v = 5 \\ v = 1 \end{matrix} \quad \mathbf{Ux = c}$$

$Lc = b$ The lower triangular system $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{c} \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix}$ gave $c = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$Ux = c$ The lower triangular system $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ gave $c = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The Cost of Elimination

Elimination on A requires about $\frac{1}{3}n^3$ **multiplications and** $\frac{1}{3}n^3$ **subtractions**

Solve Each right side need n^2 **multiplications and** n^2 **subtractions**