21241 CHEATSHEET

Determinant

1. 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$: det = ad - bc

2. Linearity : - Scalar : $D(\alpha v_1, v_2, \cdots, v_n) = \alpha D(v_1, v_2, \cdots, v_n)$

   - addition : $D(v_1, \cdots, v_k + u_k, \cdots, v_n) = D(v_1, \cdots, v_k, \cdots v_n) + D(v_1, \cdots, u_k, \cdots, v_n)$

3. preservation under "column replacement"

$$D(v_1, \cdots, v_j + \alpha v_k, \cdots, v_k, \cdots v_n) = D(v_1, \cdots, v_j, \cdots, v_k, \cdots v_n)$$

4. Antisymmetric. (change of ~~col~~ row).

$$D(v_1, \cdots, v_k, \cdots, v_j, \cdots v_n) = -D(v_1, \cdots, v_j, \cdots v_k, \cdots v_n).$$

5. $\det(I) = 1$

6. $\det(A) = 0$: i) not invertible ii) have a zero column iii) two equal columns iv) linearly dependent

7. det of diagonal matrix : ~~prod.~~ product of diagonal entries

   " triangular matrix : same above

8. $\det(A) = \det(A^T)$ 9. $\det(AB) = \det(A)\det(B)$ 10. for $n \times n$ matrix $\det(\alpha A) = \alpha^n \det(A)$

9. cofactor : $C_{j,k} = (-1)^{j+k} \det A_{j,k}$

   - $\det A = \cancel{a_{j,k}}\ a_{j,1} C_{j,1} + a_{j,2} C_{j,2} + \cdots + a_{j,n} C_{j,n}$

   - $\det A = a_{1,k} C_{1,k} + a_{2,k} C_{2,k} + \cdots + a_{n,k} C_{n,k}$.

10. cofactor matrix for inverse matrix $A^{-1} = \frac{1}{\det A} C^T$

11. Cramer's Rule $x_k = \frac{\det B_k}{\det A}$

Gram-Schmist

- Find ~~q~~ $w_1 = v_1$, $q_1 = \frac{v_1}{\|v_1\|}$
- $w_2 = v_2 - \frac{v_2^T w_1}{\|w_1\|^2} w_1$ $q_2 = \frac{w_2}{\|w_2\|}$ normalize!
- $w_3 = v_3 - \frac{v_3^T w_1}{\|w_1\|^2} w_1 - \frac{v_3^T w_2}{\|w_2\|^2} w_2$ $q_3 = \frac{w_3}{\|w_3\|}$

Orthogonality (Q).

1. the product of 2 orthogonal matrices is also an orthogonal matrix.
2. $Q^{-1} = Q^T$;
3. $N(A) \perp C(A^T)$. $C(A) \perp N(A^T)$.

→ Projection.

4. to-line : $P = \frac{a^T b}{a^T a} a$ → $a \to a$: $Pa = a$
   (b → a) if $b \perp a$, $b \to a$: $Pa = 0$

⇔ projection matrix : $P = \frac{aa^T}{a^T a}$; $P^2 = P$.

5. to-subspace : projection matrix : $P = A(A^T A)^{-1} A^T$
   if A has L.I. columns

Linear Transformation

1. ~~Linear~~ Linear : $T(v+w) = T(v) + T(w)$
2. change of basis matrix : $B = W^{-1} V = B_{out}^{-1} B_{in}$
3. $T: V \to W$ is surjective if $\forall w \in W$, there exists a $v \in V$ s.t. $T(v) = w$ or $\mathrm{Im}\, T = W$
   " is injective if $T(v_1) = T(v_2) \Rightarrow v_1 = v_2$
4. Injective ⇔ $\ker(T) = \{0\}$

Eigenvectors & Eigenvalues

$* \boxed{Av = \lambda v} *$

- finding → calculate $\det(A - \lambda I) = 0$

1. trace : $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$; det : $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$ } for $n \times n$ matrix

   $= a_{11} + a_{22} + \cdots + a_{nn}$

2. eigenvalues of triangular matrix : diagonal entries

3. if A diagonalizable ⟺ A has n linearly independent eigenvectors

4. let $\lambda_1 \cdots \lambda_r$ are distinct eigenvalues, ⇒ $V_1 \cdots V_r$ are linearly independent

5. Any matrix that has NO REPEATED eigenvalues can be diagonalized.

6. SDS^{-1} 7. $A^n v = \lambda^n v$ 8. $A^n = SD^nS^{-1}$

eigenvectors ↗ eigenvalues ↑

9. 0 is one of eigenvalues of A ⟺ A is singular / not invertible

10. Eigenvalues (A) = Eigenvalues (A^T)

11. $\frac{1}{\lambda}$ is eigenvalue of A^{-1}

12. · projection matrices ($P^2 = P$) ⇒ all eigenvalues <u>0 or 1</u>
    · permutation matrices & orthogonal matrices <u>1 or -1</u>
    ($Q^* = I$)

→ Symmetric Matrices

12. has only real eigenvalues 13. eigenvectors can be chosen orthonormal

14. Spectral Theorem : $S = Q\Lambda Q^{-1} = Q\Lambda Q^T$

15. eigenvectors are always perpendicular

16. $S = Q\Lambda Q^T = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$

17. let $\lambda = a + ib$ and $\bar{\lambda} = a - ib$: $Ax = \lambda x \Rightarrow A\bar{x} = \bar{\lambda}\bar{x}$

→ **Positive Definite Matrices**

18. all n pivots are positive *19. all n upper-left determinants are positive

Ref'n 20. all n eigenvalues are positive 21. * $x^TAx \geq 0$ except $x = 0$.

22. diagonal entries of PD are positive

Other:

→ fundamental subspaces

→ pivot columns of original matrix A gives us a basis in $C(A)$

→ pivot rows of echelon form gives us a basis in $C(A^T)$.

→ $N(A) \Rightarrow$ find $Ax = 0$.