CMU 21-241 Linear Algebra (For Review)

[Yifan Su]

30 april 2022

1 Basic

2 Systems of Linear Equations

What is linear equation

A collection of m linear equations with n unknowns $x_1, x_2 ... x_n$

Different forms of linear equation system

1. Linear system can be written in the matrix form (as a matrix-vector equation)

$$Ax = b$$

   To solve the above equation is to find all vectors $x \in \mathbb{F}^n$ satidfying $Ax = b$

2. as a vector equation

$$x_1 a_1 + x_2 a_2 + ... + x_n a_n = b$$

   where a_k is the kth column of matrix A.

2.1 Augmented Matrix

$$\left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{array}\right)$$

2.2 Solution of a linear system

Gauss-Jordan elimination (row reduction) $\Rightarrow$ *echelon form*

1. Row operation
   a) Row exchange : intercahnge two rows of the matrix;
   b) Scaling : multiply a row by a non-zero scalar a;
   c) Row replacement : replace a row $\#k$ by its sum with a constant multiple of row $\#j$; all other rows remain same.

   Performing a row operation on the augmented matrix of the system $Ax = b$ is equivalent to the multiplication of the system (from the left) by a special invertible matrix E.

   $$Ax = b \Rightarrow EAx = Eb \Rightarrow E^{-1}EAx = E^{-1}Eb$$

   A row operation does not change the solution set of a system

2. Row reduction
   a) Find the leftmost non-zero column of the matrix;
   b) if upperleft element is zero, need do row exchange first;
   c) Using type (c) row operation to kill all non-zero entries below the pivot by adding (subtracting) an appropriate multiple of the first row from the rows number 2,3,...,m.

   Leave it alone after doing substraction, don't interchange rows
   At most m subtraction, it will be the echolon form

2.3 Echelon form

A matrix is in echelon form if it satisfies the following two conditions:

1. All zero rows (i.e. the rows with all entries equal 0), if any, are below all non-zero entries.

2. For any non-zero row its leading entry (pivot entry, or pivot) is strictly to the right of the leading entry in the previous row.

2.4 Reduced echelon form

Reduced echelon form is **coeffcient matrix equal I**, and has two properties:

1. All pivot entries are equal 1.

2. All entries above and below the pivots are 0.

$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \end{array}\right)$$

boxed entries: pivots
without boxed : free variables
Move the *free variables* to the right side

$$X = \begin{bmatrix} 1-2x_2 \\ x_2 \\ 2-5x_4 \\ x_4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}$$

Analyzing the pivots

(1) Consistency
• A system is **inconsistent** (does not have a solution) iff there is a pivot in the last column of an echelon formof the augmented matrix, i.e. iff an echelon form of the augemented matrix has a zero row with a $b \neq 0$ at the last element.
• A system is **consistent** for all right sides b iff the echelon form of the *coeffcient matrix* has a pivot in every row.

(2) Uniqueness
Equation $Ax = b$ has a *unique solution* for any right side b idd echelon form of the coefficient matrix A has a pivot in every column and every row.

2.5 Corollaries about linear independence and bases. Dimension.

(3) Let us have a system of vectors $v_1, v_2, ..., v_m \in \mathbb{F}^n$, and let $A = [v_1, v_2, ..., v_m]$ be an $n \times m$ matrix with columns $v_1, v_2, ..., v_m$:
• The system $v_1, v_2, ..., v_m$ is linearly independent iff echelon form of A has a pivot in every column;
• The system $v_1, v_2, ..., v_m$ is complete in $\mathbb{F}^n$ (spanning, generating) iff echelon form of A has a pivot in every row.
• The system $v_1, v_2, ..., v_m$ is a basis of $\mathbb{F}^n$ (spanning, generating) iff echelon form of A has a pivot in every row and every column.

(4) Any linear independent system of vectors in $\mathbb{F}^n$ cannot have more than n vectors in it.

(5) Any two bases in a vector space V have the same number of vectors in them.

(6) Any basis in $\mathbb{F}^n$ must have exactly n vectors in it.

(7) Any spanning (generating) set in $\mathbb{F}^n$ must have at least n vectors.

(8) Any spanning (generating) set in $\mathbb{F}^n$ must have at least n vectors.

2.6 Corollaries about invertible matrices

(9) A matrix A is invertible iff its echelon form has pivot in every column and every row.

(10) An invertible matrix must be square $(n \times n)$

(11) If a **square matrix** is left invertible, or if it is right inveritble, then it is inveritble.

Finding A^{-1} by row reduction

Any invertible matrix is row equivalent (i.e. can be reduced by row operations) to the identify matrix.
Simple algorithm of finding the inverse of an $n \times n$ matrix :

1. From an augmented $n \times 2n$ matrix $(A|I)$ by writing the $n \times n$ identity matrix right of A.

2. Performing row operations on the augmented matrix transform A to the identity matrix I;

3. The matrix I that we added will be automatically transformed to A^{-1};

4. If it is impossible to transform A to the identity by row operation, then A is not invertible.

Any *invertible matrix* can be represented as a product of elementary matrices.

Dimention. Finite-dimensional spaces

Definition: The dimension of a vector space V ($\dim V$) is the number of vectors in a basis.
• For a vector space consisting only of zero vector 0 we put $\dim V = 0$. If V does not have a (finite) basis, we put $\dim V = \infty$

(12) A *vector space* V is finite-dimensional iff it has finite spanning system.

(13) Any linearly independent system in a finite-dimensional vector space V cannot have more than $\dim V$ vectors in it.

(14) Any generating system in a finite-dimensional vector space V must have as least $\dim V$ vectors in it.

2.7 Completing a linearly independent system to a basis

(15) **A *linearly independent* system of vectors in a finite-dimensional space can be completed to a basis,** i.e. if $v_1, v_2, ..., v_r$ are linearly independent vectors in a finite-dimensional vector space V then one can find vectors $v_{r+1}, v_{r+2}, ..., v_n$ such that the system of vectors $v_1, v_2, ..., v_n$ is a basis in V.

2.8 Subspace of finite dimensional spaces

Let V be a subspace of a vector space W, $\dim W \leq \infty$. Then V is finite dimensional and $\dim V \leq \dim W$.
Moreover, if $\dim V = \dim W$, then $V = W$ (we are still assuming that V is a subspace of W here)

2.9 General solution of a linear system

We call a system $Ax = b$ homogeneous if the right side b = 0,i.e. a homogeneous system is a system of form $Ax = 0$.
With each system

$$Ax = b$$

, we can associate a homogeneous system just by putting b = 0.

(16) (General solution of a linear equation). Let a vector x_i satisfy the equation $Ax = b$, and let H be the set of all solutions of the associate homogeneous system

$$Ax = 0$$

. Then the set

$$\{x = x_1 + x_h : x_h \in H\}$$

is the set of all solutions of the equation $Ax = b$.

Example:

$$x = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + S \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

The simplist way to check that gives us solutions is to check that the first vector $(3, 1, 0, 2, 0)^T$ satisfies the equation $Ax = b$, and that the other two should satisfy the associated homogeneous equation $Ax = 0$

Fundamental subspaces of a matrix. Rank.

The four Fundamental subspaces : $C(A), N(A), C(A^T), N(A^T)$, they are column space, null space, row space, left null space of A, respectively.

(17) Given a linear transformation (matrix) A its rank, rank A, is the dimension of the range of A

$$rankA := dimC(A)$$

2.10 Computing fundamental subspaces and rank

After row reduction of matrix A, we get the echelon form A_e

1. The pivot columns of the **original** matrix A give us a basis in C(A)
2. The pivot rows of the echelon form A_e give us a basis in the $C(A^T)$
3. To find a basis in the N(A), we need to solve the homogeneous equation $Ax = 0$

2.11 The Rank Theorem. Dimensions of Fundamental subspaces

(18) (The Rank Theorem). For a matrix A

$$rankA = rankA^T$$

Let A be an $m \times n$ matrix, i.e. a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$. Then,

(19) $\dim N(A) + \dim C(A) = \dim N(A) + rankA = n$ (dimensino of the domain of A)

(20) $\dim N(A^T) + \dim C(A^T) = \dim N(A^T) + rankA^T = \dim N(A^T) + rankA = m$

Representation of a linear transformation in arbitrary bases. Change of coordinates formula

2.12 Coordinate vector

Let V be a vector space with a basis $B := \{b_1, b_2, ..., b_n\}$. Any vector $v \in V$ admits a unique representation as a linear combination .

$$v = x_1b_1 + x_2b_2 + \cdots + x_nb_n = \sum_{k=1}^{n} x_kb_k$$

The number $x_1, x_2, ..., x_n$ are called the coordinates of the vector v in the basis B.

2.13 Matrix of a linear transformation

Let $T : V- > W$ be a linear transformation, and let A = $\{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_m\}$ be bases in V and W respectively
A matrix of the transformation T in the bases A and B is an $m \times n$ matrix, denoted by $[T]_{BA}$, which related the coordinate vetors $[Tv]_B$ and $[v]_A$,

$$[Tv]_B = [T]_{BA}[v]_A$$

Finding the matrix $[T]_{BA}$: its kth column is just the coordinate vector $[Ta_k]_B$

2.14 Change of coordinate matrix

Let A = $\{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_m\}$ be bases in V. Consider the identity transformation $I = I_v$ and its matrix $[I]_{BA}$ in these bases. By the definition

$$[v]_B = [I]_{BA}[v]_A, \forall v \in V$$

$[I]_{BA}$, the change of coordinates matrix, transforms its coordinates in the basis A into coordinates in the basis of B.
Finding the matrix: its kth column is the coordinate representation $[a_k]_B$ of kth element of the basis A.
NOTE:

$$[I]_{AB} = ([I]_{AB})^{-1}$$

2.15 Matrix of a transformation and change of coordinates

Let $T : V- > W$ be a linear transformation, and let A, $\widetilde{A}$ be two bases in V and let B and $\widetilde{B}$ be two bases in W. Suppose we know the matrix $[T]_{BA}$, and we would like to find the matrix representation with respect to new bases $\widetilde{A}, \widetilde{B}$, i.e. the matrix $[T]_{\widetilde{B}\widetilde{A}}$.

$$[T]_{\widetilde{B}\widetilde{A}} = [I]_{\widetilde{B}B}[T]_{BA}[I]_{A\widetilde{A}}$$

2.16 Case of one basis: similar matrices

We say that a matrix A is similar to a matrix B if there exists an invertible matrix Q such that $A = Q^{-1}BQ$
Similar matrices as different matrix representation of the same linear operator (trnasformation)

3 Determinants

For 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The determinant is $ad - bc$. (Remember!)
We can imagine the determinant of a matrix A is the volume of the column vectors of

A form.

The volume of a parallelepiped is always a determinant

Properties of determinant

(21) Linearity in each argument:

Scalar:

$$D(\alpha v_1, v_2, ..., v_n) = \alpha D(v_1, v_2, ..., v_n)$$

Addition:

$$D(v_1, ..., v_k + u_k, ..., v_n) = D(v_1, ..., u_k, ..., v_n) + D(v_1, ..., v_k, ..., v_n)$$

(22) Preservation under "column replacement":

$$D(v_1, ..., v_j + \alpha v_k, ...v_k, ..., v_n) = D(v_1, ..., v_j, ..., v_k, ..., v_n)$$

(23) Antisymmetry.

$$D(v_1, ..., v_k, ..., v_j, ..., v_n) = -D(v_1, ..., v_j, ..., v_k, ..., v_n)$$

(24)

$$\det(I) = 1$$

(Volume of a parallelepiped formed by unit vector is 1) – Normalization

(25) If A has a zero column, then $\det A = 0$

(26) If A has two qual columns, then $\det A = 0$

(27) If one column of A is a multiple of another, then $\det A = 0$

(28) If columns of A are linearly dependent, then $\det A = 0$

(29) Determinant of a diagonal matrix equal the product of the diagonal entries,

$$\det(diag a_1, a_2, ..., a_n) = a_1 a_2 ... a_n$$

(30) Determinant of a triangular matrix equals to the product of the diagonal entries,

$$\det A = a_{1,1} a_{2,2} ... a_{n,n}$$

(31) $\det A = 0$ iff A is not invertible.

(32) (determinant of a transpose) For a square matrix A,

$$\det A = \det(A^T)$$

(33) (determinant of a product) For $n \times n$ matrices A and B

$$\det(AB) = (\det A)(\det B)$$

(34) If A is an $n \times n$ matrix, then $\det(aA) = a^n \det A$

3.1 Cofactor expansion

For an $n \times n$ matrix $A = \{a_{j,k}\}_{j,k=1}^n$ denotes the $(n-1) \times (n-1)$ matrix obtained from A by crossing out row number j and column number k.

(35) (Cofactor expansino of determinant). Let A be an $n \times n$ matrix. For each j, $1 \leq j \leq n$, determinant of A can be expanded in the row number j as

$$\det A = \\ a_{j,1}(-1)^{j+1} \det A_{j,1} + a_{j,2}(-1)^{j+2} \det A_{j,2} + \cdots + a_{j,n}(-1)^{j+n} \det A_{j,n} \\ = \sum_{k=1}^n a_{j,k}(-1)^{j+k} \det A_{jk}$$

Similarly, for each k, $1 \leq k \leq n$, the determinant can be expanded in the column number k,

$$\det A = \sum_{j=1}^n a_{j,k}(-1)^{j+k} \det A_{j,k}$$

Definition: The number

$$C_{j,k} = (-1)^{j+k} \det A_{j,k}$$

are called cofactors.

Hence, the formula for expansion of the determinant can be written as: –row expansion

$$\det A = a_{j,1}C_{j,1} + a_{j,2}C_{j,2} + \ldots + a_{j,n}C_{j,n} = \sum_{k=1}^n a_{j,k}C_{j,k}$$

–column expansion

$$\det A = a_{1,k}C_{1,k} + a_{2,k}C_{2,k} + \ldots + a_{n,k}C_{n,k} = \sum_{j=1}^n a_{j,k}C_{j,k}$$

3.2 Cofactor formula for the inverse matrix

Definition: The matrix $C = \{C_{j,k}\}_{j,k=1}^{n}$ whose entries are cofactors of a given matrix A is called the cofactor matrix of A.

(36) Let A be an invertible matrix and let C be its cofactor matrix, then:

$$A^{-1} = \frac{1}{\det A} C^T$$

(37) (Cramer's rule). For an invertible matrix A the entry number k of the solution of the equation $Ax = b$ is given by the formula

$$x_k = \frac{\det B_k}{\det A}$$

where the matrix B_k is obtained from A by replacing column number k of A by the vector b.

4 Eigenvalues and Eigenvector

Main definitions

4.1 Eigenvalues, Eigenvectors, Spectrum

A scalar λ is called an **eigenvalue** of an operator $A : V- > V$ if there exists a non-zero vector $v \in V$ such that

$$Av = \lambda v$$

The vector v is called the **eigenvector** of A (corresponding to the eigenvalue λ)
The set of all eigenvalues of an operator A is called *spectrum* of A, and is usually denoted $\sigma(A)$
eigenvectors x that don't change direction when youmultiply by A

4.2 Finding eigenvalues

A scalar σ is an eigenvalue iff the $N(A - \sigma I)$ is non-trivial

(38)

$$\lambda \in \sigma(A), \text{ i.e. } \lambda \text{ is an eigenvalue of A} \Leftrightarrow \det(A - \lambda I) = 0$$

Elimination does not preserve the λ

4.3 Multiplicity

Definition: Let λ be eigenvalue of A :
• The geometric multiplicity of λ is dimension of eigenspace E_λ
• The algebric multiplicity of λ is the multiplicity of roots in $\det(A - \lambda I)$

(39) geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity $(AM \geq GM)$

(40) when GM $\leq$ AM means that A is not diagonalizable.

4.4 Trace and determinant

(41) Let A be $n \times n$ matrix, and let $\lambda_1, \lambda_2, ..., \lambda_n$ be its(complex) eigenvalues (counting multiplicities). Then

1. $\text{tra}(A) = \lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n = a_{1,1} + a_{2,2} + ... + a_{n,n}$
2. $\det A = \lambda_1 \lambda_2 ... \lambda_n$

4.5 Eigenvalues of a triangular matrix

(42) Eigenvalues of a triangular matrix (counting multiplicities) are exactly the diagonal entries $a_{1,1}, a_{2,2}, ..., a_{n,n}$

Diagonalization

Suppose an operator A in a vector space V is such that V has a basis $B = b_1, b_2, ..., b_n$ of eigenvectors of A, with $\lambda_1, \lambda_2, ..., \lambda_n$ being the corresponding eigenvalues. Then the matrix of A in this basis is the diagonal matrix with $\lambda_1, \lambda_2, ..., \lambda_n$ on the diagonoal.

(43) A matrix A (with values in $\mathbb{F}$) admits a representaiton $A = SDS^{-1}$, where D is a diagonoal matrix and S is an invertible one (both with entires in $\mathbb{F}$) iff there exists a basis in $\mathbb{F}$ of eigenvectors of A
•A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.
•Let $\lambda_1, \lambda_2, ...\lambda_r$ be distinct eigenvalues of A, and let $v_1, v_2, ..., v_r$ be the corresponding eigenvectors. Then those vectors are linearly independent.
•Any matrix that has no repeated eigenvalues can be diagonalized.

Moreover, in this case diagonal entries of D are the eigenvalues and the columns of S are the corresponding eigenvectors (column number k corresponds to kth diagonal entry of D)

(44) $A^n v = \lambda^n v$

(45) $A^N = SD^N S^{-1}$

4.6 Other properties of eigenvalues and eigenvectors

(46) 0 is an eigenvalue of A $\Leftrightarrow$ A is singular / not invertible

(47) Eigenvalues (A) = Eigenvalues (A^T)

(48) $\frac{1}{\lambda}$ is eigenvalue of A^{-1}

(49) Projection matrices satisfy $P^2 = P$(property of projection matrix). All possible eigenvalues are 0,1

(50) Permutation and Reflection matrices, $Q^2 = I$, $\lambda = 1, -1$ (Orthogonal matrix)

4.7 Symmetric Matrices

All symmetric matrices are diagonalizable

(51) A symmetric matrix has only real eigenvalues.

(52) The eigenvectors can be chosen orthonormal.

(53) (Spectral Theorem) Every symmetric matrix has the factorization $S = Q\Lambda Q^T$ with real eigenvalues in Λ and orthonormal eigenvectors in the columns of Q:

$$S = Q\Lambda Q^{-1} = Q\Lambda Q^T$$

(54) Eigenvectors of a real symmetric matrix (when they correspond to different Λ's) are always perpendicular.

(55) Every symmetric matrix: $S = Q\Lambda Q^T = \lambda_1 q_1 q_1^T + \cdots + \lambda_n q_n q_n^T$

(56) let $\lambda = a + ib$ and $\bar{\lambda} = a - ib$.

$$Ax = \lambda x \Rightarrow A\bar{x} = \bar{\lambda}\bar{x}$$

4.8 Positive Definite Matrices

properties of PD :

(57) All n pivots of S are Positive

(58) All n upper left determinants are positive

(59) All n eigenvalues of S are positive

(60) $x^T Sx$ is positive except at $x = 0$.

(61) S equals $A^T A$ for a matrix A with independent columns.

(62) diagonal entries of PD are positive

5 Inner product spaces

Inner product spaces

5.1 Inner product and norm

The word *norm* is just a fancy replacement for the word length.
Hence, the norm of the vector $x \in \mathbb{R}^n$ as : (Pythangorean rule)

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Similarly, the dot product in $\mathbb{R}^n$ was defined as ::

$$\begin{aligned} x \cdot y = x^\top y &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \\ &= y^\top x \end{aligned}$$

For a complex number $z = a + bi$, we have $|z|^2 = a^2 + b^2 = z\bar{z}$ $(\bar{z} = a - bi)$
If $z \in \mathbb{C}^n$ is given by :

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \\ a_n + b_n i \end{bmatrix}$$

It is natural to define its norm $||z||$ by :

$$|z|^2 = \sum_{k=1}^{n} \left(x_k^2 + y_k^2 \right) = \sum_{k=1}^{n} |z_k|^2$$

inner product of (z, w):

$$\begin{aligned}(z, w) &= z_1\bar{w}_1 + z_2\overline{w_2} + \cdots + z_n\bar{w}_n \\ &= \sum_{k=1}^{n} z_k\overline{w_k}\end{aligned}$$

5.2 Hermitian adjoint

$A^* = \bar{A}^T$
meaning that we take the transpose of the matrix A, $A^* = A^T$.
Using the notion of A^*, one can write the standard inner product in $\mathbb{C}^n$:

$$(z, w) = w^* z$$

5.3 Properties of Inner Product

(63) Symmetry: $(x, y) = \overline{(y, x)}$

(64) Linearity: $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$, for all vector x, y, z and all scalar α, β

(65) Non-negativity: $(x, x) \geq 0, \forall x$

(66) Non-degeneracy: $(x, x) = 0$ iff $x = 0$

5.4 Gram-Schmist

• Let $x_1, ..., x_k$ be linearly independent vectors in some vector space V.
• Find: othonormal vectors $q_1, ..., q_k$ s.t. $span\{x_1, ..., x_k\} = span\{q_1, ..., q_k\}$
• Method:

1. Let $q_1 = \frac{x_1}{||x_1||}$
2. For i = 2, ...,k:

   a) Subtract the projection of x_i onto $span\{q_1, ..., q_{i-1}\}$ from x_i, **and then normalize**.
   b) Since $\{q_2, ..., q_{i-1}\}$ is orthonormal, the projection of x_i onto $span\{q_1, ..., q_{i-1}\}$ decouple: it is the sum of the projection od x_i onto each of $q_1, ..., q_{i-1}$

$$\hat{q}_1 = x_i - proj_{q_1,...,q_{i-1}}(x_1) = x_i - proj_{q_1}(x_i) - \cdots - proj_{q_{i-1}}$$

5.5 Orthogonal

(67) The product of two orthogonal matrices is also an orthogonal matrix.

(68) The inverse of a orthogonal matrix is its transpose

(69) The determinant of the orthogonal matrix has a value of ± 1.

(70) It is symmetric in nature