AVL delete

1. find the node to delete; call it v
   - complexity: $\Theta(\log n)$
2. if v has no children, delete v, update height of v's parent
   - complexity: $\Theta(1)$
3. if v has one child, v's parent adopts v's child, delete v, update height of v's parent
   - complexity: $\Theta(1)$
4. if v has two children
   - 4.1 find the successor s of v (complexity: $\Theta(\log n)$)
   - 4.2 move the key/value pair of s into v (complexity: $\Theta(1)$)
   - 4.3 delete s, s's parent adopts s's (right) child if it exists, update height of s's parent (complexity: $\Theta(1)$)
5. starting from the parent of deleted node, go up to root, updating heights and rebalancing as necessary
   - complexity: $\Theta(\log n)$

AVL tree height

These two questions are equivalent:

- in a tree with n nodes, what is the maximum possible height h?
- if the tree height is h, what is the minimum possible number of nodes n?

Let $minsize(h)$ denote the minimum size (number of nodes) of a tree of height h. Then:

$$
\begin{aligned}
minsize(0) &= 0 \\
minsize(1) &= 1 \\
minsize(h+2) &= minsize(h) + minsize(h+1) + 1
\end{aligned}
$$

Does this look familiar?

AVL tree height

Exercise: prove by induction that

$$minsize(h) = fib(h + 2) - 1$$

Now recall the "golden ratio" and how it relates to Fibonacci numbers:

$$\begin{aligned}
\phi &= (1 + \sqrt{5})/2 \\
\psi &= (1 - \sqrt{5})/2 \\
fib(n) &= (\phi^n - \psi^n)/\sqrt{5}
\end{aligned}$$

We therefore have

$$minsize(h) = \frac{\phi^{h+2} - \psi^{h+2}}{\sqrt{5}} - 1$$

AVL tree height

$$
\begin{aligned}
n = minsize(h) =& \frac{\phi^{h+2} - \psi^{h+2}}{\sqrt{5}} - 1 = \frac{\phi^{h+2}}{\sqrt{5}} - \frac{\psi^{h+2}}{\sqrt{5}} - 1 \\
>& \frac{\phi^{h+2}}{\sqrt{5}} - 1 - 1 = \frac{\phi^{h+2}}{\sqrt{5}} - 2
\end{aligned}
$$

$$
\begin{aligned}
&\phi^{h+2} < \sqrt{5}(n+2) \\
&h + 2 < \log_\phi\left(\sqrt{5}(n+2)\right) \\
&h < \frac{\log_2 \sqrt{5}}{\log_2 \phi} + \frac{\log_2 (n+2)}{\log_2 \phi} - 2 \in \mathcal{O}(\log n)
\end{aligned}
$$

Thus we have height of an AVL tree with n nodes is $\in \mathcal{O}(\log n)$.