# CSCB63 – Design and Analysis of Data Structures

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[1]with huge thanks to Anna Bretscher and Albert Lai

# introduction

Today we begin studying a different way to look at complexity of algorithms.

So far we saw

- worst case analysis of individual operations
- amortised analysis over a sequence of operations

Today we add

- expected running time analysis
- need to review our probabilities!

# quicksort: idea

A divide-and-conquer algorithm.

1. if $|A| < 2$: stop
2. pick an element $x$, called "pivot"
3. partition $A$ into:
   - $L$: array of elements $\leq x$
   - $G$: array of elements $> x$

   ($\Theta(|A|)$ time; $|A| - 1$ comparisons against pivot)
4. recurse: sort $L$
5. recurse: sort $G$
6. concatenate sorted $L$, sorted $G$

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We write $L$, $G$ as new arrays to show the idea.
In reality, "partition" is done in-place in $A$, so $L$ occupies the left side, $G$ occupies the right side, and there is no "concatenate" step.

# quicksort: algorithm

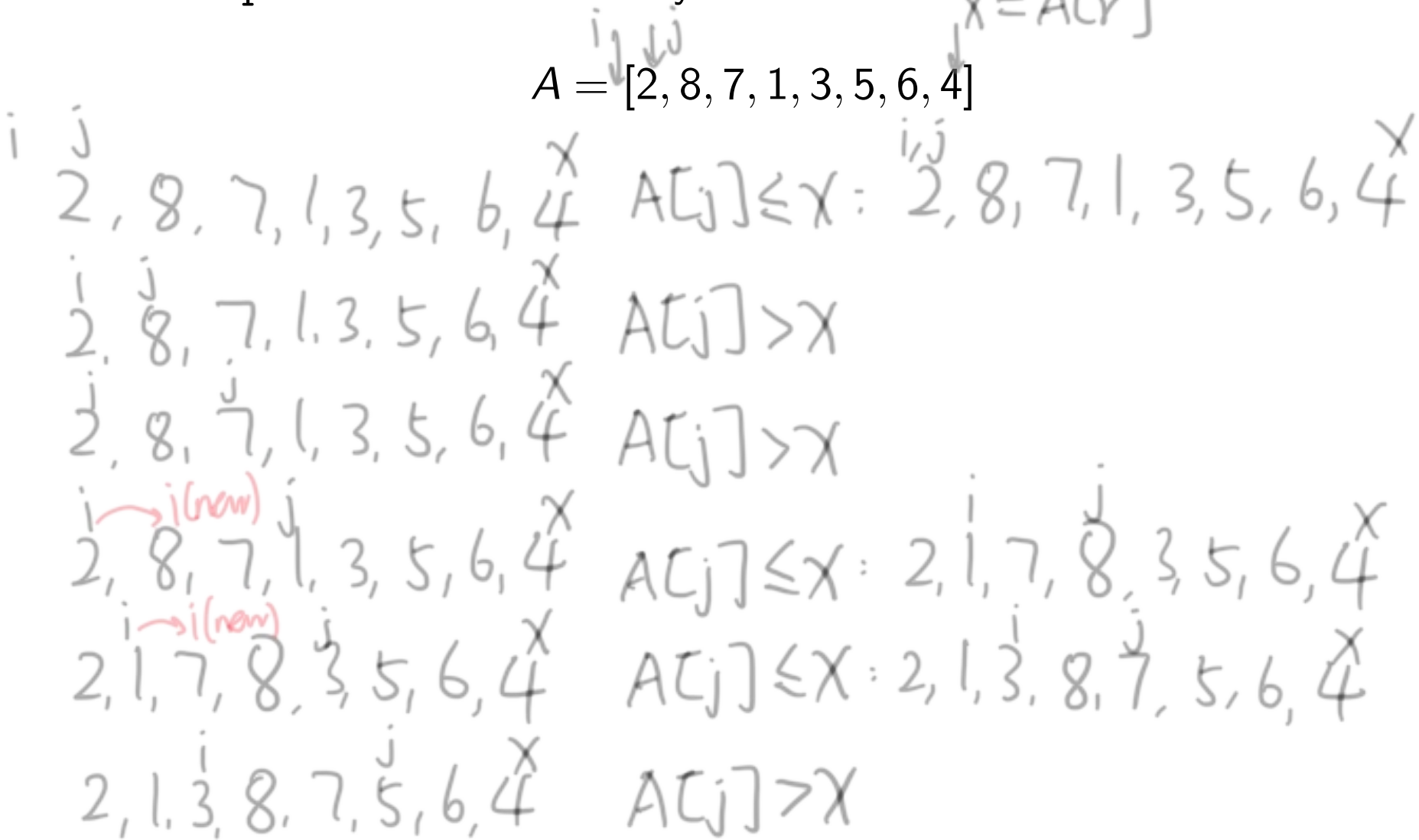
```
quicksort(A, p, r):
0. if p < r:
1.    q = partition(A, p, r)
2.    quicksort(A, p, q-1)
3.    quicksort(A, q+1, r)

partition(A, p, r):
0. x := A[r]    // pivot
1. i := p-1
2. for j in p, ..., r-1:
3.    if A[j] <= x:
4.       i := i + 1
5.       exchange A[i] with A[j]
6. exchange A[i+1] with A[r]
7. return i + 1
```

# quicksort: example

Trace `quicksort` on the array

$$A = [2, 8, 7, 1, 3, 5, 6, 4]$$



2, 1, 3, 8, 7, 5, 6, 4 (i, j, x)  A[j] > x

□ : done

⇒ 2, 1, 3, [4], 7, 5, 6, 8 (i, x, j)

recursive left: i, j, x: 2, 1, 3

A[j] ≤ x: 2, 1, 3 (i, j, x)

recursive left: i → i(new), j, x: 2, 1, 3

A[j] ≤ x: 2, 1, [3] (i, j, x)

recursive left: i, j, x: 2, 1

A[j] > x

⇒ [1], [2] (i, j, x)

A[i+1] ↔ A[r]

recursive right:

i j x
7, 5, 6, 8

A[j] ≤ x: i → i, j
7, 5, 6, 8

i → i(new) j x
7, 5, 6, 8

A[j] ≤ x: i → i, j
7, 5, 6, 8

i → i(new) j x
7, 5, 6, 8

A[j] ≤ x: i → i, j
7, 5, 6, 8

recursive right

i j x
7, 5, 6

A[j] > x

i j x
7, 5, 6

A[j] ≤ x: i → i(new) j
5, 7, 6

⇒ i j, x
5, 6, 7

A[i+1] ↔ A[r]

# quicksort: correctness

The loop invariants of `partition`:

- $A[k] \leq x$ for $p \leq k \leq i$
- $A[k] > x$ for $i + 1 \leq k \leq j - 1$
- $A[k] = x$ for $k = r$

Proof: exercise.

# quicksort: complexity

- running time of `partition` is $O(n)$
- running time depends on partitioning
  -
- worst case when:
  - $O(n^2)$
- best case when:
  - $O(n \log n)$
- average case:
  - $O(n \log n)$

# randomised quicksort: idea

- remove "bias" in the input that may cause too many unbalanced partitions
- by selecting pivots at random
- we expect the split to be reasonably well balanced on average
- technique called random sampling

# randomised quicksort: algorithm

```
r-quicksort(A, p, r):
0. if p < r:
1.    q = r-partition(A, p, r)
2.    r-quicksort(A, p, q-1)
3.    r-quicksort(A, q+1, r)

r-partition(A, p, r):
0. exchange A[r] with A[random(p, r)]
1. x := A[r]       // pivot is now random from A[p..r]
2. i := p-1
3. for j in p, ..., r-1:
4.    if A[j] <= x:
5.       i := i + 1
6.       exchange A[i] with A[j]
7. exchange A[i+1] with A[j]
8. return i + 1
```

# randomised quicksort: complexity

Idea:

- `partition` selects a pivot
- pivot never included in any future recursive calls
- at most _____ calls to `partition` over entire execution of `quicksort`
- `partition` is:
- focus on line 4 in for-loop
- can count number of times line 4 runs
- will give us bound on time spent in the for-loop over entire execution of `quicksort`

# randomised quicksort: complexity

Idea:

- `partition` selects a pivot
- pivot never included in any future recursive calls
- at most $n$ calls to `partition` over entire execution of `quicksort`
- `partition` is:
- focus on line 4 in for-loop
- can count number of times line 4 runs
- will give us bound on time spent in the for-loop over entire execution of `quicksort`

Formally:

-
-
-

# randomised quicksort: complexity

Notice:

- each pair of elements is compared at most once:
- elements are compared only to the pivot
- pivot is never used after call to `partition`

# randomised quicksort: complexity

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Let:

- elements of $A$: $\{z_1, z_2, ..., z_n\}$ where $z_i$ is the $i^{th}$ smallest element of $A$
- $Z_{ij} = \{z_i, z_{i+1}, ..., z_j\}$ be set of elements between $z_i$ and $z_j$
- $X_{ij} = I\{z_i \text{ is compared to } z_j\}$ be indicator random variable

# stats flashback: indicator variable

An indicator random variable $X$ is

$$X_i = \begin{cases} 1 & \text{if an event } i \text{ occurs} \\ 0 & \text{if it does not} \end{cases}$$

# randomised quicksort: expected time

Total number of comparisons:

$$E(X) = E\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}\right)$$

$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E(X_{ij})$$

$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P(z_i \text{ compared to } z_j)$$

$z_1 n \leftarrow X_{12} + X_{13} + X_{14} + \cdots + X_{1n}$

$z_2 n \leftarrow + \quad X_{23} + X_{24} + \cdots + X_{2n}$

$+ \cdots +$

$z_{(n-1)n} \leftarrow X_{(n-1)n}$

# randomised quicksort: expected time

When is $z_i$ compared to $z_j$?

In $z_i < \ldots < x < \ldots < z_j$,

- if $x$ becomes pivot first, $z_i$ is not compared to $z_j$
  ($z_i$ moved to left partition, $z_j$ moved to right partition)
- if $z_i$ becomes pivot first, $z_i$ is compared to $z_j$
- if $z_j$ becomes pivot first, $z_i$ is compared to $z_j$

$$P(z_i \text{ is pivot}) = P(z_j \text{ is pivot}) = \frac{1}{j-i+1} \text{ (random)}$$

total # of elements in $z_{ij}$

# randomised quicksort: expected time

$\Pr(z_i \text{ is compared to } z_j) =$

$$P(z_i \text{ is pivot}) + P(z_j \text{ is pivot}) = \frac{2}{j-i+1}$$

disjoint events

# randomised quicksort: expected time

Total number of comparisons:

$$E(X) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= 2 \cdot \sum_{i=1}^{n-1} \sum_{j-i=1}^{n-i} \frac{1}{j-i+1}$$

$$< 2 \cdot \sum_{i=1}^{n-1} \boxed{\sum_{j-i=1}^{n} \frac{1}{j-i}}$$

$< \ln n$

$$< 2 \sum_{i=1}^{n-1} \ln n$$

$$= 2(n-1) \ln n \in O(n \log n)$$