using limits to prove Big-O

Assume. $\exists n_0 \in \mathbb{N}\colon \forall n \geq n_0\colon f(n) \geq 0$ and $g(n) > 0$.

Theorem. If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists and is finite, then $f \in O(g)$.

Example. Prove $n(n+1)/2 \in O(n^2)$

$$\lim_{n\to\infty} \frac{n(n+1)/2}{n^2} = \frac{1}{2}$$

Example. Prove $\ln(n) \in O(n)$

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$$

using limits to disprove Big-O

Assume. $\exists n_0 \in \mathbb{N}\colon \forall n \geq n_0\colon f(n) \geq 0$ and $g(n) > 0$.

Theorem. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $f \notin O(g)$.

Example. Disprove $n^2 \in O(n)$

$$\lim_{n\to\infty} \frac{n^2}{n} = \lim_{n\to\infty} n = \infty$$

Example. Disprove $n \in O(\ln(n))$

$$\lim_{n\to\infty} \frac{n}{\ln(n)} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$$

when limits don't help

Theorem. If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists and is finite, then ...

Theorem. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then ...

Q. Which case is not covered?

A. If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist and is not ∞, then no conclusion. (Hopefully this happens rarely.)

Q. Can you think of a function *crazy* where limits do not help to show $crazy \notin O(1)$?

when limits don't help

Q. Can you think of a function *crazy* where limits do not help to show $crazy \notin O(1)$?

A. Define

$$crazy(n) = \begin{cases} 1 & \text{if n is even} \\ n & \text{if n is odd} \end{cases}$$

Then $crazy \in O(n)$ and $crazy \notin O(1)$, but

$$\lim_{n\to\infty} \frac{crazy(n)}{n} \quad \text{does not exist and is not } \infty$$
$$\lim_{n\to\infty} \frac{crazy(n)}{1} \quad \text{does not exist and is not } \infty$$

using limits for Θ

Theorem. $f \in \Theta(g)$ iff $f \in O(g)$ and $g \in O(f)$.

(Handy when you want to use limits!)

Example. $n^2 + n^{3/2} \in \Theta(n^2)$

- prove $n^2 + n^{3/2} \in O(n^2)$ by using a limit
- prove $n^2 \in O(n^2 + n^{3/2})$ by using a limit

Example. $\ln(n) \notin \Theta(n)$

- prove $n \notin O(\ln(n))$ by using a limit

Big-O, Big-Θ may miss something

Q. Can the Big-O definition be not at all useful?

A.

$$n + 10^{100} \in \Theta(n)$$

$$10^{100} n \in \Theta(n)$$

Can't say these are practical algorithm times, but O, Θ can't tell.

This is a price for ignoring constants (which we want to account for machine differences!)

Such pathological cases are rare. O and Θ are usually informative.

Myth Buster

Myth: O means worst-case time, Ω means best-case.

Truth: O, Ω, Θ classify functions, do not say what the functions stand for.

$9n^2 + 4n + 13$ may be best-case time, or worst-case time, or best-case space, or worst-case space, or just a polynomial from nowhere.

"Best case time is in $O(n^2)$" means:
Best case time is some function, that function is in $O(n^2)$.
Clearly a sensible statement and possible scenario.
O, Ω, Θ are good for any function from natural to non-negative real.