Dijkstra's algorithm: proof

Let

- $\delta(v)$ be the weight of the shortest path from start vertex s to v,
- $\delta_{fin}(v)$ be the weight of the shortest path from start vertex s to v among paths via finished vertices only (not in PQ), and
- $p(v)$ be priority of v.

Dijkstra's algorithm maintains the loop invariants:

1. for each v in PQ, $p(v) = v.d = \delta_{fin}(v)$, i.e. considering only paths via finished vertices (vertices not in PQ),
2. for each v not in PQ, $v.d = \delta(v)$ over all paths, and $v.pred$ is the vertex before v on the shortest path.

Dijkstra's algorithm: proof

Initially (after lines 0-5):

- PQ contains all of V,
- $s.d = p(s) = 0$, and
- $v.d = p(v) = \infty$, for all $v \neq s$

so (1) and (2) are true.

Dijkstra's algorithm: proof

Suppose (1) and (2) are true on line 6.

- Line 7 adds a new finished vertex u (moves from $u \in PQ$ to $u \notin PQ$).
- Before line 7 we had $p(u) = u.d = \delta_{fin}(u)$.
- Take artibtrary vertex $v \in PQ$.
  Before line 7 we had $p(v) = v.d = \delta_{fin}(v)$.
- If v adjacent to u:
  - look at the path $p_v = p_u + (u, v)$ where p_u is shortest via finished vertices to u
  - have
    $w(p_v) = w(p_u) + w(u, v) = \delta_{fin}(u) + w(u, v) = u.d + w(u, v)$
  - if $w(p_v) < v.d$ then it is the shortest via finished vertices to v
  - then condition on line 10 is true and we set
    $p(v) = v.d = w(p_v) = \delta_{fin}(v)$
  - otherwise, no change
  - so (1) is still true after line 13

Dijkstra's algorithm: proof

(cont.)

- If v not adjacent to u:
  - Can we have a shorter path to v via finished vertices that looks like:
    $s \to ... \to x \to u \to y \to ... \to v$?
  - No, because y is finished, so path from s to y must have been shortest.
  - So no change means (1) still true after line (13)

Dijkstra's algorithm: proof

Now to show $u.d = \delta(u)$.

- consider the time just before u is dequeued on line 7
- there is some (overall) shortest path p_u from s to u
- at some point p_u crosses from $V - PQ$ (finished vertices) to PQ (not finished vertices) for the first time via some edge (x, y) with $x \notin PQ$ and $y \in PQ$

$$p_u = \underbrace{s \to \ldots \to x \to y}_{p_y} \to \ldots \to u$$

- have $w(p_y) = \delta(y) = \delta_{fin}(y) = y.d = p(y)$ from (1)
- have both $u, y \in PQ$ and u dequeued first, so $p(u) \leq p(y)$
- then $u.d \leq y.d = \delta(y) \leq \delta(u)$ (p_u has added edges)
- $\therefore u.d = \delta(u)$