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Q1

(a) $a_n = \frac{n^n}{n!} = \frac{n \times n \times n \times \cdots \times n}{n \times (n-1) \times (n-2) \times \cdots \times 1} = 1 \times \frac{n}{n-1} \times \frac{n}{n-2} \times \cdots \times n$

$\because n \geq 1 \quad \therefore \frac{n}{n-1}, \frac{n}{n-2} \cdots \frac{n}{n-(n-2)}$ always bigger than 1

So $a_n \geq n$ for each $n \geq 1$ #

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \left(1 \times \frac{n}{n-1} \times \frac{n}{n-2} \times \cdots \times n\right)$

$\Rightarrow \lim_{n\to\infty} \frac{n}{n-1} = \lim_{n\to\infty} \frac{n-1+1}{n-1} = 1 + \lim_{n\to\infty} \frac{1}{n-1} = 1$

Such that $\lim_{n\to\infty} \left(1 \times \frac{n}{n-1} \times \frac{n}{n-2} \times \cdots \times n\right) = \lim_{n\to\infty} n = \infty$ #

(c) $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!} \quad a_n = \frac{n^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{(n+1)^{n+1}}{(n+1)n^n} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n$$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ #

(d) $\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{1}{\frac{n^n}{n!}}$

$\because \lim_{n\to\infty} \frac{n^n}{n!} = \infty$

$\therefore \lim_{n\to\infty} \frac{1}{\frac{n^n}{n!}} = 0$

$\lim_{n\to\infty} \frac{n!}{n^n} = 0$ #

(e) $a_n = \frac{n^n}{n!}$ assume $a_k = \frac{n^k}{k!}$

$$\frac{a_n}{a_k} = n^{n-k} \cdot \frac{k!}{n!} \Rightarrow \frac{k!}{n!} = \frac{1}{\underbrace{k \cdot k+1 \cdots n}_{n-k}} \Rightarrow n^{n-k} \cdot \frac{k!}{n!} \geq 1$$

Such that $a_n \geq a_k$ $\because a_n \geq \frac{n^k}{k!}$ #

(f) to prove $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

$$\because \frac{n!}{n^n} = \frac{n \cdot (n-1) \cdots 1}{n \cdot n \cdots n} = 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{1}{n} \leq \frac{1}{n} \cdot \frac{1}{n} \cdots \frac{1}{n} = \frac{1}{n^n}$$

$\therefore \sum_{n=1}^{\infty} \frac{n!}{n^n} \leq \sum_{n=1}^{\infty} \frac{1}{n^n}$ $\because \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\therefore \sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

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2. assume $V_n = \frac{6}{7^n}$

$\lim_{n \to \infty} \frac{V_{n+1}}{V_n} = \frac{1}{7} < 1$ Such that $\sum_{n=5}^{\infty} \frac{6}{7^n}$ converges.

$$\sum_{n=5}^{\infty} \frac{6}{7^n} = \frac{6}{7^5} + \frac{6}{7^6} + \cdots + \frac{6}{7^n}$$

$\because |r| < 1$ $\therefore$ by Geometric series $\sum_{n=5}^{\infty} \frac{6}{7^n} = \frac{a}{1-r}$

$a = \frac{6}{7^5}$

Such that $\sum_{n=5}^{\infty} \frac{6}{7^n} = \frac{\frac{6}{7^5}}{(1-\frac{1}{7})} = \frac{1}{7^4} = \frac{1}{2401}$ #

3. (a) $\lim_{x \to \infty} \frac{x^2+4x-21}{x^2-5x+6} = \lim_{x \to \infty} \frac{1+\frac{4}{x}-\frac{21}{x^2}}{1-\frac{5}{x}+\frac{6}{x^2}} = 1$

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(b) $\lim_{x\to\infty} \frac{x^2-3x+2}{x^2+2x-3} = \lim_{x\to\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{2}{x}-\frac{3}{x^2}} = 1$ #

(c) $\lim_{x\to1} \frac{\sin x}{x} = \lim_{x\to1} \frac{\sin 1}{1} = \sin 1$

(d) $\lim_{x\to\infty} \ln\left(\frac{x^2+2x+1}{x^2-1}\right) = \ln \lim_{x\to\infty} \left(\frac{x^2+2x+1}{x^2-1}\right) = \ln \lim_{x\to\infty} \frac{1+\frac{2}{x}+\frac{1}{x^2}}{1-\frac{1}{x^2}} = 0$ #

(e) $\lim_{x\to\infty} \frac{x^2(\sin x+(\cos x)^3)}{(x^2+1)(x-3)} = \lim_{x\to\infty} \frac{x^2\sin x+x^2(\cos x)^3}{x^3-3x^2+x-3} = \lim_{x\to\infty} \frac{\frac{\sin x}{x}+\frac{(\cos x)^3}{x}}{1-\frac{3}{x}+\frac{1}{x^2}-\frac{3}{x^3}} = 0$ #

4. prove: $\because$ f is a continuous function, and f is one-to-one

$\therefore$ we can assume $x_1 < x_2$, $f(x_1) < f(x_2)$

must exists $c \in [x_1, x_2]$ $f(x_1) < f(c) < f(x_2)$

Or $x_1 < x_2$, $f(x_1) > f(x_2)$, muse exists $c \in [x_1, x_2]$

$f(x_1) > f(c) > f(x_2)$

and $f(x_1) = f(c)$ is not exists.

Such that f is strictly increasing or strictly decreasing #

5. $\lim_{h\to0} (f(x+h)-f(x-h)) = 0$

assume $g(h) = f(x+h)-f(x-h)$

when $h=0$ $g(0) = f(x)-f(x) = 0$.

$\Rightarrow \lim_{h\to0} (f(x+h)-f(x-h)) = \lim_{h\to0} g(h) = g(0) = 0$.

and $\lim_{h\to0} g(h)$ exsits. So we can prove f is continuous at a

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6. (a) $f(x) = x^2\cos x \quad f'(x) = 2x\cos x - x^2\sin x$ #

(b) $f(x) = \dfrac{2x+1}{3x-2} \quad f'(x) = \dfrac{(2x+1)'(3x-2)-(2x+1)(3x-2)'}{(3x-2)^2}$

$$= -\frac{7}{9x^2-12x+4}$$ #

(c) $f(x) = \cos(\sin x)$

assume $g(x) = \cos x$, $h(x) = \sin x$

$f'(x) = g'(h(x))\cdot h'(x) = -\sin(\sin x)\cdot\cos x$. #

7. $\displaystyle\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{1-(\tan x)^2}$ $\because$ when $x = \frac{\pi}{4}$, we have $\sqrt{2}\cos x - 1 = 1-(\tan x)^2 = 0$

$\therefore$ By L'Hôpital's rule

$$\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{1-(\tan x)^2} = \lim_{x\to\frac{\pi}{4}} \frac{(\sqrt{2}\cos x - 1)'}{(1-\tan(x)^2)'} = \lim_{x\to\frac{\pi}{4}} \frac{-\sqrt{2}\sin x}{-\tan x\cdot 2\sec^2 x} = \frac{1}{4}$$

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8. $f(x) = \dfrac{1}{1+|x|} + \dfrac{1}{1+|x-2|}$

$$f(x)\begin{cases} \dfrac{1}{1-x} + \dfrac{1}{3-x} & x<0 \\ \dfrac{1}{1+x} + \dfrac{1}{3-x} & 0\le x\le 2 \\ \dfrac{1}{1+x} + \dfrac{1}{x-1} & x>2 \end{cases} \qquad f'(x)\begin{cases} \dfrac{1}{(1-x)^2} + \dfrac{1}{(3-x)^2} \\ -\dfrac{1}{(1+x)^2} + \dfrac{1}{(3-x)^2} \\ -\dfrac{1}{(1+x)^2} - \dfrac{1}{(x-1)^2} \end{cases}$$

when $x<0$, $f'(x)$ is an increasing function, when $x>2$, $f'(x)$ is an descreasing function.

So when $0 \le x \le 2$ we can get $f'(x)=0$, $x=1$.

So the critical point is $x=1$.

for $x \in [0,2]$. $f(0)=\frac{4}{3}$, $f(1)=1$, $f(2)=\frac{4}{3}$.

Such that the global maximum is $\frac{4}{3}$ when $x=0$ or 2. #

9. $f(x)=x^a(1-x)^b$

$f'(x)=ax^{a-1}(1-x)^b-bx^a(1-x)^{b-1}=[a(1-x)-bx][x^{a-1}(1-x)^{b-1}]$.

when $f'(x)=0$. $x=\frac{a}{a+b}$, $x=0$, $x=1$

So we get three critical points all in the domain.

$f(0)=f(1)=0$ $f(\frac{a}{a+b})\geq 0$

So the global maximum of $f(x)$ is $f(\frac{a}{a+b})=(\frac{a}{a+b})^a(1-\frac{a}{a+b})^b$

#.