Q1 Taylor series $f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$

a)

$f(x) = 1 + x^2 - x^3, \quad f(1) = 1$

$f'(x) = 2x - 3x^2, \quad f'(1) = -1$

$f''(x) = 2 - 6x, \quad f''(1) = -4$

$f^{(3)}(x) = -6, \quad f^{(3)}(1) = -6$

$f^{(n)}(x) = 0, \quad f^{(n)}(1) = 0$

$n \geqslant 4$

$$f(x) = 1 - (x-1) - \frac{4(x-1)^2}{2!} - \frac{6(x-1)^3}{3!}$$

$$= 1 - (x-1) - 2(x-1)^2 - (x-1)^3$$ converges for all $x \in \mathbb{R}$

b) $f(x) = \ln(e+x), \quad f(0) = \ln(e) = 1$

$f'(x) = \frac{1}{e+x}, \quad f'(0) = 1/e$

$n \geqslant 1 \quad f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(e+x)^{-n}, \quad f^{(n)}(0) = (-1)^{n-1} e^{-n} (n-1)!$

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!\, x^n}{e^n\, n!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n\, e^n}$$ converges for $|x| < e$ by ratio test. - not covered.

c) $f(x) = \frac{1}{5+x} \quad f(1) = 1/6$

$f^{(n)}(x) = (-1)^n n!\,(5+x)^{-(n+1)}, \quad f^{(n)}(1) = (-1)^n n!\, 6^{-(n+1)}$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!\, 6^{-(n+1)}}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{6^{n+1}}$$

$$= \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{1-x}{6}\right)^n$$ converges for $|x-1| < 6$ as it is a geometric series.

Q2 $f(x,y) = \dfrac{1}{1+x^2+y^2}$

(i) $f(x,y) = c \implies 1+x^2+y^2 = \dfrac{1}{c}$

$$x^2+y^2 = \frac{1}{c} - 1$$

As $\frac{1}{c} - 1 < 0$ for $c > 1$, the contour for $c > 1$ is the emptyset $\emptyset$.

For $c = 1$, the contour is $\{(0,0)\}$.

For $c \in (0,1)$, the contour is $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = \frac{1}{c} - 1\}$.

(ii) $$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

$$= \left(\frac{-2x}{(1+x^2+y^2)^2}, \frac{-2y}{(1+x^2+y^2)^2} \right)$$

(iii) $$\|\nabla f\| = \sqrt{\frac{4x^2}{(1+x^2+y^2)^4} + \frac{4y^2}{(1+x^2+y^2)^4}} = \frac{2\sqrt{x^2+y^2}}{(1+x^2+y^2)^2}$$

let $R = x^2+y^2$. Then $\|\nabla f\|^2 = \dfrac{4R}{(1+R)^4}$

we find the maximum in terms of R

$$\frac{d}{dR} \frac{4R}{(1+R)^4} = \frac{4-12R}{(1+R)^5}$$

The only critical point is $R = 1/3$. As the derivative is decreasing, this occurs at a maximum.

$\|\nabla f\|$ is maximised for (x,y) such that $x^2+y^2 = 1/3$.

Q3 $\quad f(x) = 2e^{-2x}, \quad x \geq 0$

(i) The antiderivative of $2e^{-2x}$ is $-e^{-2x}$. So

$$\int_0^\infty 2e^{-2x}\,dx = \lim_{t\to\infty} \int_0^t 2e^{-2x}\,dx$$

$$= \lim_{t\to\infty} \left[-e^{-2t} - (-e^{-2\times 0})\right]$$

$$= 1 - \lim_{t\to\infty} e^{-2t} = 1$$

(ii)

$$P_n = \int_n^{n+1} 2e^{-2x}\,dx = \left[-e^{-2(n+1)} - e^{-2n}\right]$$

$$= e^{-2n}(1-e^{-2}) \qquad \text{for } n = 0, 1, 2, \ldots$$

(iii)

$$\sum_{n=0}^\infty P_n = \sum_{n=0}^\infty (1-e^{-2})e^{-2n}$$

$$= (1-e^{-2}) \sum_{n=0}^\infty (e^{-2})^n$$

This is a geometric series and it converges as $|e^{-2}| < 1$. As

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x}, \quad \Rightarrow \sum_{n=0}^\infty (e^{-2})^n = \frac{1}{1-e^{-2}}$$

and we see

$$\sum_{n=0}^\infty P_n = (1-e^{-2}) \sum_{n=0}^\infty (e^{-2})^n = \frac{1-e^{-2}}{1-e^{-2}} = 1,$$

which we would expect from part (i).

Q4 $\quad x = r\cos(\varphi)\sin(\theta) \qquad y = r\sin(\varphi)\sin(\theta)$

$z = r\cos(\theta)$

(i) $\dfrac{\partial x}{\partial r} = \cos(\varphi)\sin(\theta)$, $\dfrac{\partial x}{\partial \theta} = r\cos(\varphi)\cos(\theta)$

$\dfrac{\partial x}{\partial \varphi} = -r\sin(\varphi)\sin(\theta)$

$\dfrac{\partial y}{\partial r} = \sin(\varphi)\sin(\theta)$, $\dfrac{\partial y}{\partial \theta} = r\sin(\varphi)\cos(\theta)$

$\dfrac{\partial y}{\partial \varphi} = r\cos(\varphi)\sin(\theta)$.

$\dfrac{\partial z}{\partial r} = \cos(\theta)$, $\dfrac{\partial z}{\partial \theta} = -r\sin(\theta)$, $\dfrac{\partial z}{\partial \varphi} = 0$

So

$$A = \begin{bmatrix} \cos(\varphi)\sin(\theta) & r\cos(\varphi)\cos(\theta) & -r\sin(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) & r\sin(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{bmatrix}$$

(ii)

$$\det(A) = \cos(\varphi)\sin(\theta)\Big(r\sin(\varphi)\cos(\theta)\times 0 - r\cos(\varphi)\sin(\theta)\times(-r\sin(\theta))\Big)$$
$$-r\cos(\varphi)\cos(\theta)\Big(\sin(\varphi)\sin(\theta)\times 0 - r\cos(\varphi)\sin(\theta)\times\cos(\theta)\Big)$$
$$+(-r\sin(\varphi)\sin(\theta))\Big(\sin(\varphi)\sin(\theta)\times(-r\sin(\theta)) - r\sin(\varphi)\cos(\theta)\times\cos(\theta)\Big)$$

$$= r^2\cos^2(\varphi)\sin^3(\theta) + r^2\cos^2(\varphi)\cos^2(\theta)\sin(\theta)$$
$$+ r^2\sin^2(\varphi)\sin^3(\theta) + r^2\sin^2(\varphi)\cos^2(\theta)\sin(\theta)$$

$$= r^2\sin^3(\theta) + r^2\cos^2(\theta)\sin(\theta) = r^2\sin(\theta)$$

where we have used $\cos^2(\omega) + \sin^2(\omega) = 1$ for all ω.

Q4 (iii)

A^{-1} does not exist if and only if $\det(A) = 0$.

From part (ii) $\det(A) = r^2 \sin(\theta)$. So $\det(A) = 0$ if $r = 0$ or if $\theta = m\pi$ for some $m \in \mathbb{Z}$.