THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

Course Reader for,

MATH7501
Mathematics
for
Data Science 1

(last revision: 15/1/2021).

Xin's office hours: Friday 3~4 pm, starting Week 4

* Technique: proof by induction.

→ To prove $P(n)$ for $n = 1, 2, \cdots$

→ prove $P(1)$

→ For any $k \geq 1$, assume $P(1), \cdots, P(k)$, and prove $P(k+1)$.

Idea: Prove $P(1)$

use $P(1)$ to prove $P(2)$

use $P(1), P(2)$, to prove $P(3)$.

use $P(1), P(2)$ & $P(3)$ to prove $P(4)$

...

They have the same format, and we may be able to process them uniformly.

Example: Prove: $2^{2n-1}+1$ is divisible by 3, $\forall\, n=1,2,\cdots$.

Proof. When $n=1$. $2^{2n-1}+1=3$, divisible by 3.

Assume $2^{2n-1}+1$ is divisible by 3, for $n=1,2,\cdots,k$.

Then, $2^{2(k+1)-1}+1=2^2\times 2^{2k-1}+4-3=4(2^{2k-1}+1)-3$.

By induction assumption, $3\mid 2^{2k-1}+1$, so $3\mid 4(2^{2k-1}+1)-3$.

The proof is completed by induction.

Example: Prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

for any $n = 1, 2, \cdots$

LHS: left-hand side

RHS: right-hand side

Proof: When $n = 1$, LHS $= 1 =$ RHS

Assume: the above equation holds

for $n = 1, 2, \cdots, k$. One has,

$$\sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k+1}{6}\left(k(2k+1) + 6(k+1)\right)$$

$$= \frac{k+1}{6}(2k+3)(k+2) = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}.$$

The proof is done by induction ※.

The statement $P(k)$ states $1+2+\cdots+k = \frac{k(k+1)}{2}$. We **assume** this is true (this is our **inductive hypothesis**).

The statement $P(k+1)$ states $1+2+\cdots+(k+1) = \frac{(k+1)(k+2)}{2}$.

We must show that IF $P(k)$ is true, then $P(k+1)$ is also true.

$$
\begin{aligned}
\text{Now L.H.S. of } P(k+1) &= 1+2+\cdots+(k+1) \\
&= 1+2+\cdots+k+(k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \text{ } (P(k) \text{ is assumed true}) \\
&= \frac{k(k+1)}{2} + \frac{(k+1)\cdot 2}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \text{R.H.S. of } P(k+1).
\end{aligned}
$$

Hence $P(k+1)$ is true.

Thus, by the principle of mathematical induction, for all integers $n \geqslant 1$, we have $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Example 71. *For all integers $n \geq 1$, use mathematical induction to prove that*

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Example 72. *For all integers $n \geq 1$, prove that*

$$\sum_{j=1}^{n}\frac{1}{j(j+1)} = \frac{n}{n+1}.$$

Example 73. *For all integers $t \geqslant 1$, use induction to prove that*

$$\sum_{j=1}^{t} 2^{j-1} = 2^t - 1.$$

3.5 Logic

In this chapter we shall consider some formal *logic*. This will enable us to determine whether the conclusion of a formal argument is true or false, given various suppositions or *premises* in the argument.

We shall see how to write truth tables for compound statements.

A **statement** or a **proposition** is a sentence that is true or false, but not both.

Which of the following are propositions?

- It is raining.
- Is it raining?
- Tom is a male and Susan is a female.
- Mary is a male.
- No smoking inside.
- The number 6 is a prime number. a false statement is also a statement.
- What comes next?
- ~~That pelican is beautiful.~~ we need to define "beautiful".
- ~~Elizabeth's favourite bird is a pelican.~~
- Hello there.

- We often use p, q, r etc. to stand for simple statements.
  If we let p denote "it is raining", then we can denote the **negation** of this statement by $\sim p$ or $\neg p$. We read this as "not p", so the negation of the statement "it is raining" is "it is not raining".

- If statement p is FALSE, then what about the statement $\sim p$?
  Can you say whether it is true or false?

- A truth table gives the truth value of a statement for all possible instances of the truth values of its component parts. A statement p has two possible truth values: true or false.

Here is a truth table to complete; it will give the truth values for $\sim p$ in terms of the truth values for the statement p.

p	$\sim p$
T	F
F	T

- If we have two statements, say p and q, we can combine them in various ways.
  Suppose p denotes "it is dark" and q denotes "it is raining".
  Then the statement "it is dark and it is raining" can be written as $p \wedge q$, read "p and q". This is known as the **conjunction** of p and q.

- We can use a truth table to determine the truth value of the conjunction $p \wedge q$ in all possible cases, whatever the truth values of p and q may be. With two statements p and q we have 2^2 or 4 possible scenarios.

Here is a truth table to determine the truth value of $p \wedge q$, according to the truth values of p and q.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- The **disjunction** of two statement forms p and q, written $p \vee q$, and read "p or q", means p or q (or possibly both). This is sometimes known as the "inclusive or".

The truth table for $p \vee q$:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- $p \wedge q$ is true when p and q are both true.
- $p \wedge q$ is false when
- $p \vee q$ is true when
- $p \vee q$ is false when

A **statement form** or **propositional form** is made up from variables such as p, q, r, and logical connectives such as $\wedge$, $\sim$, $\vee$.

Two statement forms are **logically equivalent** if and only if they have *identical* truth values for every possible combination of truth values for the variables.

Write $P \Leftrightarrow Q$, whenever P and Q are logically equivalent.

Are the statement forms $p \wedge (\sim q)$ and $(p \vee q) \wedge (\sim q)$ logically equivalent?

yes, since for all the possible values of p and q, these two statements take the same value always.

$\sim q$	$p \vee q$	$p \wedge (\sim q)$	p	q	$(p \wedge (\sim q))$	$((p \vee q) \wedge (\sim q))$
F	T	F	T	T	F	F
T	T	T	T	F	T	T
F	T	F	F	T	F	F
T	F	F	F	F	F	F

- **De Morgan's Laws** (negations of "and" and "or"):
  The statement $\sim (p \wedge q)$ is logically equivalent to the statement $(\sim p) \vee (\sim q)$.

  The statement $\sim (p \vee q)$ is logically equivalent to the statement $(\sim p) \wedge (\sim q)$.

  A **tautology** is a statement form which *always* takes the truth value TRUE, for all possible truth values of its variables.

- A **contradiction** is a statement form which *always* takes the truth value FALSE for all possible truth values of its variables.

e.g. all the possible truth values of p are: T and F.

e.g.

Construct a truth table to determine the truth values for $(p \vee q) \wedge (\sim p)$.

p	q	$(p \vee q) \wedge (\sim p)$
T	T	F
T	F	F
F	T	T
F	F	F

$p \vee (\sim p)$ is a tautology

$p \wedge (\sim p)$ is a contradiction

If a statement form P has *three* variables, such as p, q and r, how many rows will a truth table for P need? Discuss.

Is the statement form
$(p \wedge q) \vee (\sim p \vee (p \wedge (\sim q)))$ a tautology, a contradiction, or neither?

p	q	$(p \wedge q) \vee (\sim p \vee (p \wedge (\sim q)))$
T	T	T
T	F	T
F	T	T
F	F	T

The truth table for a statement form with n statement variables will have how many rows? Discuss.

When the same connective ($\wedge$ or $\vee$) is used, the commutative and associative laws hold:

Commutativity: $p \wedge q \;\Leftrightarrow\; q \wedge p$ and $p \vee q \;\Leftrightarrow\; q \vee p$.

Associativity: $p \wedge (q \wedge r) \;\Leftrightarrow\; (p \wedge q) \wedge r$, and $p \vee (q \vee r) \;\Leftrightarrow\; (p \vee q) \vee r$.

Also note the **distributive** laws:

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

Exercises:

$p \wedge p = p$

1. Construct a truth table for $(p \vee q) \wedge \sim (p \vee r)$.

2. Exclusive or: We use $\underline{\vee}$ where $p \,\underline{\vee}\, q$ means p or q but NOT both. Write out the truth table for this exclusive or.

3. Verify one of the distributive laws with a truth table.

break: 17:55 ~ 18:05.

"$\Rightarrow$"

"$\Leftrightarrow$"

Conditional statements

You have probably heard the terms "if and only if" and "necessary and sufficient". In this subsection we'll examine these carefully. We'll see truth tables for "if p then q", and for "p if and only if q"; we shall see how to replace these with the logical connectives we've met already: $\vee$, $\wedge$ and $\sim$. We'll also see what the contrapositive of a statement is.

- **if** p **then** q is denoted $p \Rightarrow q$. You can also read this as "p implies q". Here p is the **hypothesis** and q is the **conclusion**.
- "if p then q" is *false* when p is TRUE and q is FALSE. It is true in *all* other cases. We'll complete the truth table for "implies":

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If "$p \Rightarrow q$" is a tautology, then it is proved.

Translate the following statements into symbolic form. Let p denote "I will sleep", q denote "I am worried", and r denote
"I will work hard".

(a) If I am worried, I will not sleep.

(b) I will not sleep if I am worried.

(c) If I am worried, then I will both work hard and not sleep.

- "If p then q" (denoted $p \Rightarrow q$) is logically equivalent to $(\sim p) \vee q$.
  Check with a truth table.

p	q	$p \Rightarrow q$	$(\sim p) \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Example 74. *Rewrite the following sentence in "if–then" form.*
Either you do not study or else you pass the test

- The **contrapositive** of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.
  Saying "if p then q" is like saying "if not q, then not p".
  Again, a truth table shows this equivalence:

p	q	$p \Rightarrow q$	$(\sim q) \Rightarrow (\sim p)$
T	T		
T	F		
F	T		
F	F		

$$\begin{aligned}(\sim q) &\Rightarrow (\sim p)\\ &= (\sim(\sim q)) \vee (\sim p)\\ &= q \vee (\sim p)\\ &= (\sim p) \vee q\\ &= p \Rightarrow q\end{aligned}$$

Example 75. *Write the contrapositive of the following sentence:*
If you do not study, then you will fail the test.

Example 76. *Construct a truth table to determine the truth values for* $p \Rightarrow (q \wedge (\sim p))$.

p	q	$p \Rightarrow (q \wedge (\sim p))$

There is a quote from the book *Alice in Wonderland* by Lewis Carroll (who was in fact Charles Dodgson, an English author, mathematician, logician, Anglican deacon and photographer) which is part of a conversation between Alice and the March Hare and the mad Hatter:

> *"Do you mean that you think you can find out the answer to it?" said the March Hare.*
>
> *"Exactly so," said Alice.*
>
> *"Then you should say what you mean," the March Hare went on.*
>
> *"I do," Alice hastily replied; "at least—at least I mean what I say—that's the same thing, you know."*
>
> *"Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"*

Rewrite the statements "*I say what I mean*" and
"*I mean what I say*" in if–then format. Use a truth table to show that the two statements are not logically equivalent.

- Given statement variables p and q, the **biconditional** of p and q is $p \Leftrightarrow q$. Read this as "p if and only if q."

  $p \Leftrightarrow q$ is true precisely when p and q take the *same* truth values. It is false when p and q take opposite truth values.

  Complete its truth table:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- **Necessary and sufficient:**
  If p and q are statements,
  p is a **sufficient condition** for q means that if p then q. $p \Rightarrow q$
  p is a **necessary condition** for q means that if *not* p, then *not* q. $(\sim p) \Rightarrow (\sim q)$
  So if p is a **necessary condition** for q, we have: if q, then p. $q \Rightarrow p$

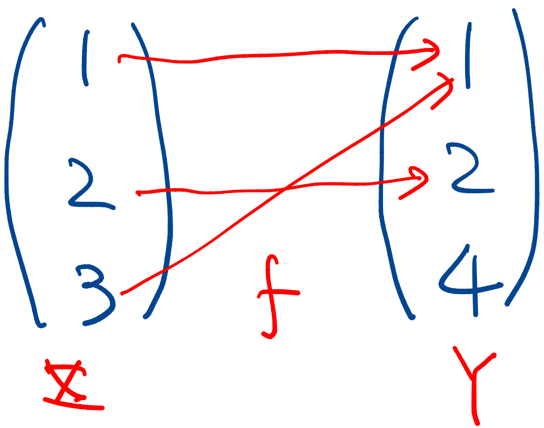
Why is necessary and sufficient the same as biconditional, according to these conditions?

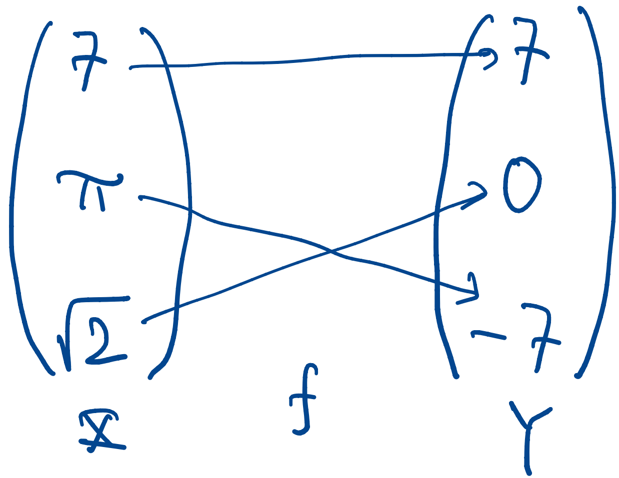
3.6 Application

§. Mappings & Functions.

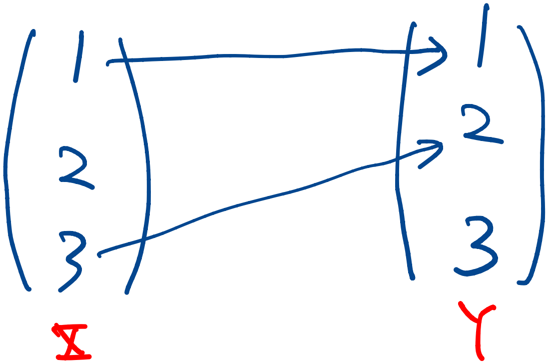
Definition [Mapping] Let X and Y be two sets. A *mapping* f from X to Y, is a rule that assigns to each element $x \in X$, exactly one element $y = f(x) \in Y$.

Examples: $f: X \to Y$

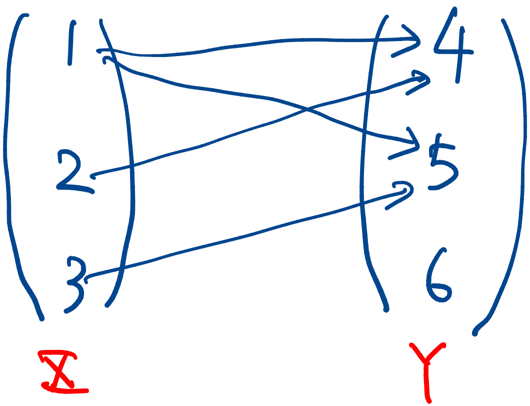




The following rules are **NOT** mappings.



Not a mapping because no rule for $2 \in X$



Not a mapping because $1 \in X$ is assigned two values 4 and 5 $\in Y$.

A mapping f from X to Y, is usually written as

$$f: X \longmapsto Y$$

$$x \longmapsto f(x)$$

Here, X: the domain of f, important

Y: the codomain of f. not important

Def: Two mappings f and g are equal, if and only if they share the same domain and the same mapping rule.

Example: If $f: \mathbb{R} \to [0, \infty)$, $x \mapsto \sqrt{x^2}$

and $g: \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$

Then $f = g$.

Example: if $f : \mathbb{R} \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$

be defined by $f(x) = x^2$ and $g(x) = x^2$,

Since f and g have different domains,

$f \neq g$.

Example: $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$,

$g(x) = \min\{x^2, 100\}$. Since $f(100) = 10^4$, and

$g(100) = 100$, $f \neq g$

"mapping", "map", "function"

☆ usually used interchangeably.

☆ Sometimes "function" is used to emphasize that the codomain is $\mathbb{R}$, or $\mathbb{C}$.

$\boxed{\text{Def}}$ [range/image] Let $f: X \to Y$ be a mapping. The range (or image) of f is defined by

$$\mathrm{Range}(f) \triangleq \{ f(x) : x \in X \}.$$

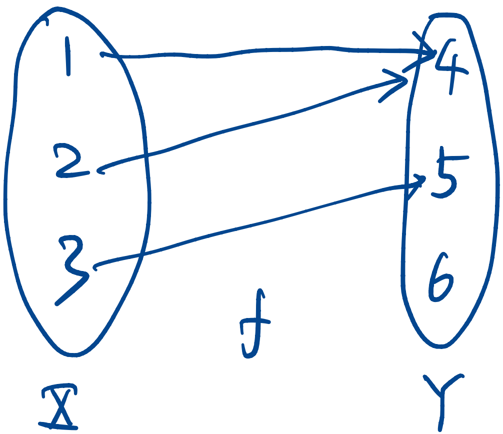
"$\triangleq$", or ":=" means: "is defined as"

Recall: Y is called the codomain.

difference between codomain & range

① $\text{Range}(f) \subseteq \text{Codomain of } f$

② Consider

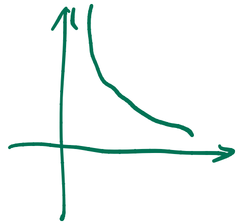


$\text{Range}(f) = \{4, 5\}$

$\text{Codomain} = Y = \{4, 5, 6\}$

$\text{Range}(f) \subsetneq \text{Codomain of } f$ in the left example.

Example: $f: (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$.



Find $\text{Range}(f)$.

<u>Sol</u>: We claim: $\text{Range}(f) = (0,\infty)$

$\forall\, x \in (0,\infty)$, $\frac{1}{x} > 0 \Rightarrow f(x) \in (0,\infty)$

$\Rightarrow \text{Range}(f) \subseteq (0,\infty)$.

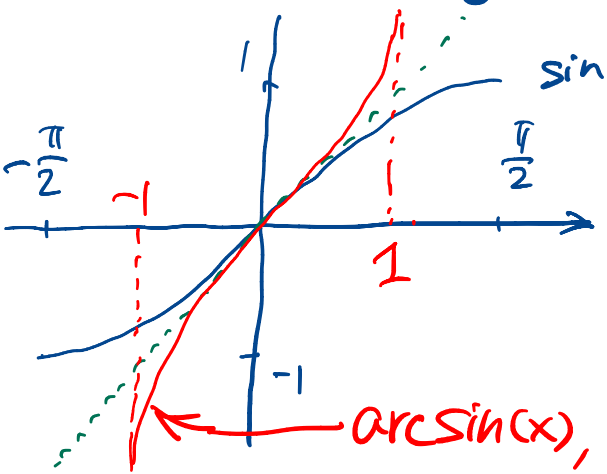
$\forall\, y \in (0,\infty)$, let $x = \frac{1}{y}$ to have $f(x) = \frac{1}{x} = y$.

So, $y \in \text{Range}(f) \Rightarrow (0,\infty) \subseteq \text{Range}(f)$

We have proved $\text{Range}(f) = (0,\infty)$.

Def [surjective/onto] A mapping $f: X \to Y$ is surjective (or, onto), if $\mathrm{Range}(f) = Y$

Example: $f: \mathbb{R} \to [-1, 1]$. $f(x) = \sin(x)$, is surjective. Indeed, $\forall\, y \in [-1, 1]$, $\sin(\arcsin(y)) = y$



$\sin(x)$, restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arcsin(x)$, $[-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

Example: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not surjective

$\boxed{\text{Def}}$ [injective/one-to-one] Let $f: X \to Y$ be a mapping. We call f one-to-one (or injective), if whenever $f(x) = f(y)$, we always have $x = y$

Example: $f, g: \mathbb{R} \to \mathbb{R}$. $f(x) = x^3$ is injective (and indeed also surjective). $g(x) = x^2$ is not injective, since $g(1) = g(-1)$ but $1 \neq -1$.

[break 1859 ~ 1909]

☆ $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, not surjective, not injective

☆ $f: \mathbb{R} \to [0, \infty)$, $x \mapsto x^2$: surjective, but not injective

☆ $f: [0, \infty) \to \mathbb{R}$, $x \mapsto x^2$: injective, but not surjective

☆ $f: (-\infty, 0] \to [0, \infty)$, $x \mapsto x^2$: both injective and surjective

$\boxed{\text{Def.}}$ [bijection / one-to-one correspondence]

$f: X \to Y$ is a bijection if f is both surjective and injective

Example: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is a bijection

Example: An injection $f: X \to Y$ with $Y = \mathrm{Range}(f)$ is a bijection.

$\boxed{\text{Thm}}$ [inverse function] Let $f: X \to Y$ be a bijection. Then $\exists$ a mapping $g: Y \to X$ such that

$$f(g(y)) = y, \quad \forall y \in Y.$$

① g is called the inverse function of f.

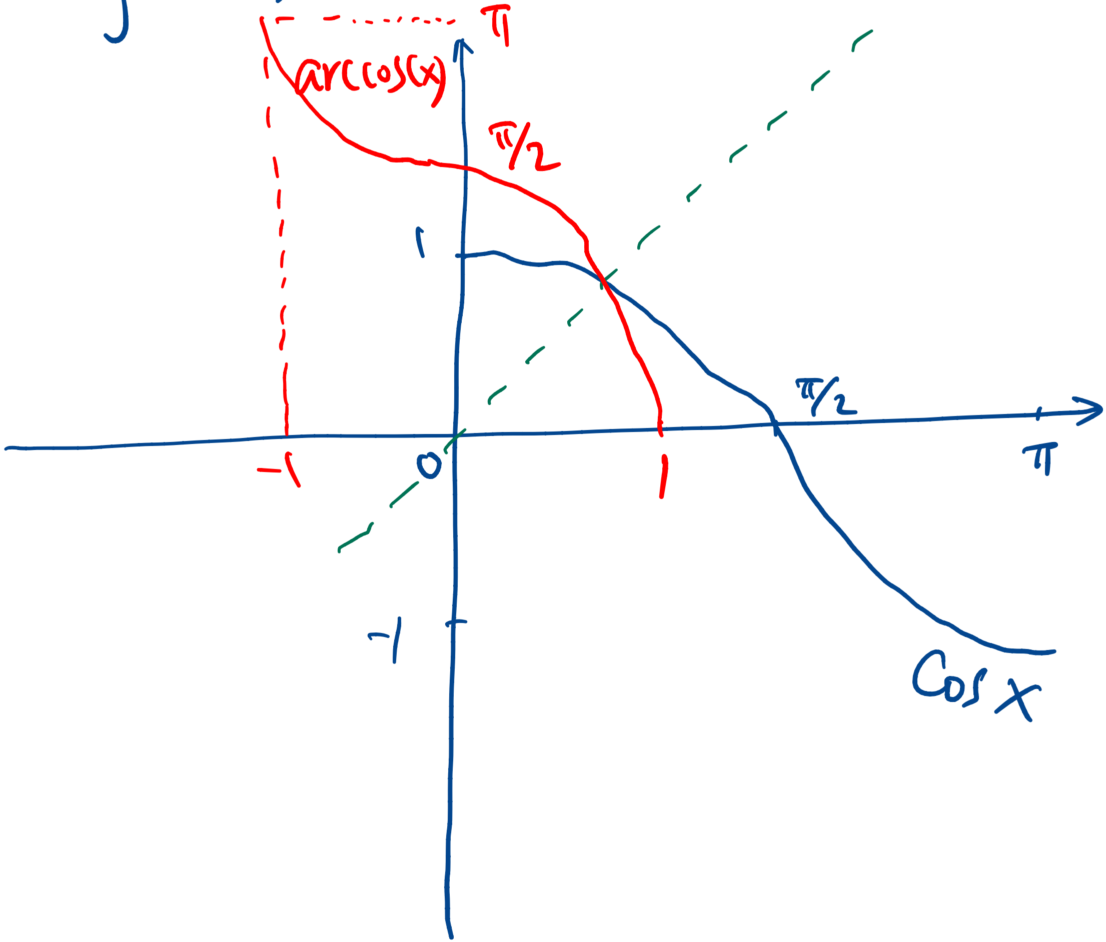
② We usually write $g = f^{-1}$

③ $f^{-1}: Y \to X$ is also a bijection, with $(f^{-1})^{-1} = f$.

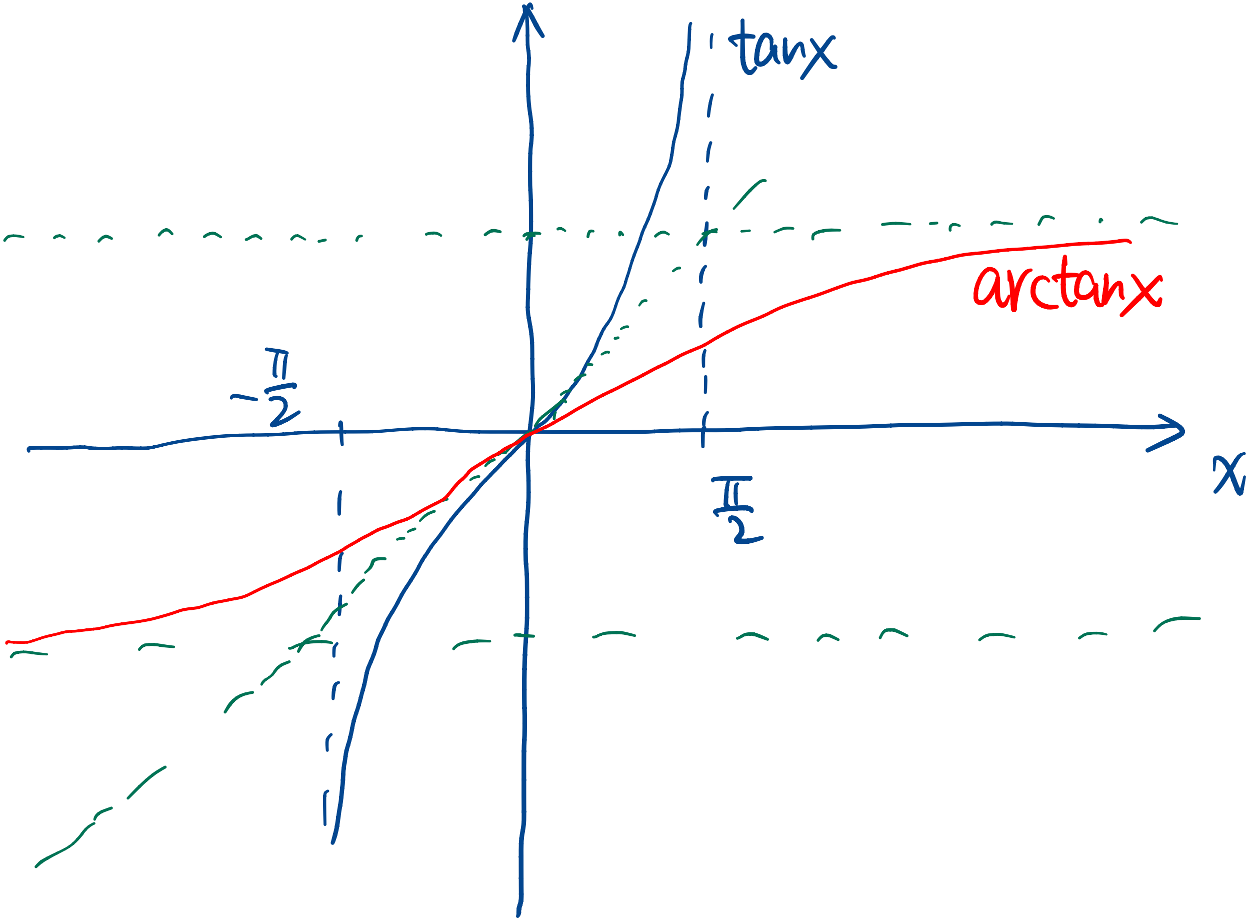
④ The notation f^{-1} is not for "reciprocal".

In general, $f^{-1}(x) \neq \dfrac{1}{f(x)}$.

Example. $f : [0, \pi] \to [-1, 1]$, $x \mapsto \cos(x)$



Example $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f(x) = \tan x$



§. Functions of Several Variables

Example. Let $c \in \mathbb{R}^n$ be a constant vector. Consider $f: \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \langle x, c \rangle$.

Write $x = (x_1, \cdots, x_n)^T$, $c = (c_1, \cdots, c_n)^T$

$$\Rightarrow f(x) = c^T x = c_1 x_1 + \cdots + c_n x_n$$

$$\frac{\partial f(x)}{\partial x_i} = c_i \,,$$

$$\nabla f(x) := \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c.$$

$$\nabla \langle x, c \rangle = c.$$

Example: $f: \mathbb{R}^n \to \mathbb{R}$ $\quad f(x) = \|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$

$$\frac{\partial f(x)}{\partial x_1} = 2x_1, \quad \frac{\partial f(x)}{\partial x_i} = 2x_i$$

$$\frac{dx^2}{dx} = 2x$$

$$\nabla f(x) = \begin{pmatrix} \partial x_1 \\ \partial x_2 \\ \vdots \\ \partial x_n \end{pmatrix} = 2x$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Example $f: \mathbb{R}^{n\times n} \to \mathbb{R}$, $f(W) = \det(W)$

$$W = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1n} \\ W_{21} & W_{22} & \cdots & W_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ W_{n1} & W_{n2} & \cdots & W_{nn} \end{pmatrix}, \quad \text{Cofactor: } C^W = \begin{pmatrix} C^W_{11} & C^W_{12} & \cdots & C^W_{1n} \\ C^W_{21} & C^W_{22} & \cdots & C^W_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ C^W_{n1} & C^W_{n2} & \cdots & C^W_{nn} \end{pmatrix}$$

Recall: ① $\det(W) = \sum_{l=1}^{n} W_{il} C^W_{il} = \sum_{l=1}^{n} W_{lj} C^W_{lj}$, $\forall\, i, j$.

② C^W_{ij} is free of W_{ij}

$$\frac{\partial \det(W)}{\partial W_{ij}} = C^W_{ij}$$