Lecture 5.2

Multiple Random Variables: Joint Distributions

Multiple random variables

Most random processes of interest involve multiple random variables. Examples include:

- Rolling two dice and recording the outcome of each as X and Y.
- The number of people a person follows on twitter X and the number of people who follow them Y.
- Consider a portfolio of three stocks where each stock's daily return is represented by a random variable, say X, Y, and Z.
- We randomly select 20 people and ask their age and let $X_1, \ldots, X_{20}$ represent the answers.

We need to extend the notions of cdf, pdf and pmf to deal with multiple random variables and to capture the possible interplay, e.g., correlation or dependence, among them.

Joint cumulative distribution function

Let $X_1, \ldots, X_n$ be random variables describing some random experiment. The **joint cdf** of $X_1, \ldots, X_n$ is the function $F : \mathbb{R}^n \to [0, 1]$ defined by

$$F(x_1, \ldots, x_n) = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) \, .$$

We have used the abbreviation

$$\mathbb{P}(\{X_1 \leq x_1\} \cap \cdots \cap \{X_n \leq x_n\}) = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

to denote the probability of the intersection of events.

The distribution of a single variable X_i from a collection of random variables $(X_1, \ldots, X_n)$ is the **marginal distribution** X_i.

Note: The above definitions are valid for all random variables, i.e., discrete, continuous, and beyond.

Joint pmf (discrete random variables)

The **joint pmf** of discrete random variables $X_1, \ldots, X_n$ is a function $f : \mathbb{R}^n \to [0, 1]$ given by

$$f(x_1, \ldots, x_n) = \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) .$$

To save on notation, we can refer to the sequence $X_1, \ldots, X_n$ simply as a random "vector" $\boldsymbol{X} = (X_1, \ldots, X_n)$.

If the joint pmf f is known, we can calculate the probability of any event B via summation as

$$\mathbb{P}(\boldsymbol{X} \in B) = \sum_{\boldsymbol{x} \in B} f(\boldsymbol{x}) .$$

For example, if $\boldsymbol{X} = (X, Y)$ and $B = \{(x, y) \mid x + y \leq 1\}$, then $\mathbb{P}(\boldsymbol{X} \in B) = \mathbb{P}(X + Y \leq 1) = \sum_{(x,y) \in B} f(x, y) = \sum_{x+y \leq 1} f(x, y)$.

Joint pmf (discrete random variables)

To evaluate the joint cdf for a given $(t_1, \ldots, t_n) \in \mathbb{R}^n$, we do

$$F(t_1, \ldots, t_n) = \mathbb{P}(X_1 \leq t_1, \ldots, X_n \leq t_n) = \sum_{\substack{x_1 \leq t_1 \\ \cdots \\ x_n \leq t_n}} f(x_1, \ldots, x_n).$$

In other words,

$$F(t_1, \ldots, t_n) = \sum_{(x_1, \ldots, x_2) \in B} f(x_1, \ldots, x_n),$$

where $B = \{(x_1, \ldots, x_n) : x_1 \leq t_1, \ldots, x_n \leq t_n\}$.

More compactly, we can write

$$F(\boldsymbol{t}) = \sum_{\boldsymbol{x} \in B} f(\boldsymbol{x}),$$

where $B = \{\boldsymbol{x} : \boldsymbol{x} \leq \boldsymbol{t}\}$, where the inequality is element-wise.

Example

Consider the joint probability mass function of (X, Y) below. Both X and Y only take the values $\{0, 1, 2\}$.

		y		
		0	1	2
x	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$
	1	0	$\frac{1}{4}$	$\frac{1}{8}$
	2	0	0	$\frac{1}{16}$

What is $\mathbb{P}(X = 1, Y = 2)$? $1/8$

What is $\mathbb{P}(Y = 1)$? $1/2$

What is $\mathbb{P}(X + Y \leq 2)$? $13/16$

Marginal pmf (discrete random variables)

As demonstrated in the previous example, given the join pmf of (X, Y) we can find the marginal pmf of X and Y. For example:

$$f_X(x) = \mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y) = \sum_y f_{X,Y}(x, y)\,.$$

In general, given the joint pmf of $(X_1, \ldots, X_n)$, we find the **marginal pmf** of X_i by summing the joint pmf over all possible values of the other variables:

$$f_{X_1}(x) = \sum_{\substack{-\infty \le x_2 \le \infty \\ \cdots \\ -\infty \le x_n \le \infty}} f(x, x_2, \ldots, x_n)\,.$$

The converse is not true: it is not possible to construct a joint pmf from the marginal distributions without additional information.

Example

Consider the joint probability mass function of (X, Y) below. Both X and Y only take the values $\{0, 1, 2\}$.

		y		
		0	1	2
	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$
x	1	0	$\frac{1}{4}$	$\frac{1}{8}$
	2	0	0	$\frac{1}{16}$

What is marginal pmf of X?

x	0	1	2
$\mathbb{P}(X = x)$	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

Example

Consider a pair of random variables (X, Y), where the marginal distribution of X is $\text{Ber}(1 - e^{-\lambda})$ and the marginal distribution of Y is $\text{Poisson}(\lambda)$. Suppose that

$$\mathbb{P}(X = 0, Y = 0) = e^{-\lambda}.$$

What is the joint pmf of (X, Y)?

		y					
		0	1	2	3	...	
x	0	$e^{-\lambda}$	0	0	0	...	$e^{-\lambda}$
	1	0	$\lambda e^{-\lambda}$	$\frac{\lambda^2 e^{-\lambda}}{2!}$	$\frac{\lambda^3 e^{-\lambda}}{3!}$	...	$1 - e^{-\lambda}$
		$e^{-\lambda}$	$\lambda e^{-\lambda}$	$\frac{\lambda^2 e^{-\lambda}}{2!}$	$\frac{\lambda^3 e^{-\lambda}}{3!}$	...	1

Joint pdf (continuous random variables)

The **joint pdf** of continuous random variables $\mathbf{X} = (X_1, \ldots, X_n)$ is the non-negative function $f : \mathbb{R}^n \to [0, \infty)$ with total integral 1 such that

$$\mathbb{P}(\mathbf{X} \in B) = \int_{\mathbf{x} \in B} f(\mathbf{x})\, d\mathbf{x} \quad \text{for all sets } B \subset \mathbb{R}^n .$$

In particular,

$$F(t_1, \ldots, t_n) = \int_{x_1 \le t_1} \ldots \int_{x_n \le t_n} f(x_1, \ldots, x_n)\, \mathrm{d}x_n \ldots \mathrm{d}x_1$$

Iterated integrals (continuous random variables)

For example, consider just two random variables (X, Y) with joint pdf $f_{X,Y}$. Suppose $B = [a, b] \times [c, d] \subset \mathbb{R}^2$. Then

$$\mathbb{P}((X, Y) \in B) = \int_a^b \int_c^d f_{X,Y}(x, y)\, dy\, dx.$$

The order in which we perform the integration is not important since it can be shown that

$$\int_a^b \int_c^d f_{X,Y}(x, y)\, dy\, dx = \int_c^d \int_a^b f_{X,Y}(x, y)\, dx\, dy.$$

Example

Consider the joint probability density function of (X, Y) below.

$$f_{X,Y}(x,y) = \begin{cases} c(1+x^2y), & (x,y) \in [0,1] \times [0,1] \\ 0, & \text{else.} \end{cases}$$

What is the constant $c > 0$ that makes this a valid pdf?

Example

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}y\mathrm{d}x = \int_0^1\int_0^1 c(1+x^2y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left[\int_0^1 c(1+x^2y)\mathrm{d}y\right]\mathrm{d}x = c\int_0^1 \left[y+\frac{1}{2}x^2y^2\right]_0^1 \mathrm{d}x \\
&= c\int_0^1 \left(1+\frac{1}{2}x^2\right)\mathrm{d}x = c\left[x+\frac{1}{6}x^3\right]_0^1 = c\frac{7}{6}.
\end{aligned}
$$

So, $c = 6/7$.

Example

Consider the joint probability density function of (X, Y) below.

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{7}(1 + x^2 y), & (x,y) \in [0,1] \times [0,1] \\ 0, & \text{else.} \end{cases}$$

What is the probability that $X + Y \geq 1$?

Example

We have

$$
\begin{aligned}
B &= \left\{(x, y) \in [0,1]^2 : x + y \geq 1\right\} \\
&= \left\{(x, y) \in [0,1]^2 : 0 \leq x \leq 1, \quad 1 - x \leq y \leq 1\right\} \\
&= \left\{(x, y) \in [0,1]^2 : 0 \leq y \leq 1, \quad 1 - y \leq x \leq 1\right\}
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathbb{P}(X + Y \geq 1) &= \int_0^1 \int_{1-x}^1 \frac{6}{7}(1 + x^2 y)\mathrm{d}y\mathrm{d}x \\
&= \frac{6}{7} \int_0^1 \left[y + \frac{1}{2}x^2 y^2\right]_{1-x}^1 \mathrm{d}x \\
&= \frac{6}{7} \int_0^1 \left(x + x^3 - \frac{1}{2}x^4\right) \mathrm{d}x = \ldots = 39/70.
\end{aligned}
$$

Marginal pdf (continuous random variables)

Given the joint pdf of (X, Y), we can find the marginal pdf of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx,$$

and the marginal pdf of X:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy.$$

This extends naturally to more than two variables. Given the joint pdf of $(X_1, \ldots, X_n)$, we find the **marginal pdf** of X_i by integrating the joint pdf with respect to the other variables over $\mathbb{R}^{n-1}$.

It is not possible to construct a joint pdf from the marginal pdfs without additional information.

Example

Consider the joint probability density function of (X, Y) below.

$$f_{X,Y}(x, y) = \begin{cases} \frac{6}{7}(1 + x^2 y), & (x, y) \in [0, 1] \times [0, 1] \\ 0, & \text{else.} \end{cases}$$

What is marginal pdf of X?

Example

For $x \in [0,1]$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = \int_0^1 \frac{6}{7}(1+x^2y)\,\mathrm{d}y = \frac{6}{7}(1+x^2/2).$$

For $x \notin [0,1]$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = \int_0^1 0\;\mathrm{d}y = 0.$$