Problems for the pracs in week 3.

Problem 1. Let $x = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$. Find $\|x\|$

Sol. $\|x\| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$

Problem 2. Find the angle θ between x and y, where

$x = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $y = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Sol. We have $\cos\theta = \dfrac{\langle x, y \rangle}{\|x\| \cdot \|y\|} = \dfrac{\overset{-1\cdot 2+2\cdot -1}{-4}}{\sqrt{1+4}\sqrt{4+1}} = -\dfrac{4}{5}$

So, $\theta = \arccos(-\frac{4}{5}) \approx 143^\circ$

Problem 3. Prove that $A = B$, where

$A = \{(x,y) : x \in \mathbb{R},\ y \leq \frac{1}{2}x + 1\}$, and

$B = \{(x,y) : y \in \mathbb{R},\ x \geqslant 2y - 2\}$.

Proof. $\forall (x,y) \in A$, $x \geqslant 2y - 2 \Rightarrow (x,y) \in B. \Rightarrow A \subseteq B$

$\forall (x,y) \in B$, $y \leqslant \frac{1}{2}x + 1$, so $(x,y) \in A. \Rightarrow B \subseteq A$.

Therefore, $A = B$.

problem 4. Use the binomial expansion

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad \forall \text{ integer } n \geq 0$$

and the fact $(1+x)^m (1+x)^n = (1+x)^{m+n}$, to prove the Vandermonde's identity

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

for any integers $m \geq 0$, $n \geq 0$, and $0 \leq r$, $r \leq m$, $r \leq n$.

proof. From $(1+x)^m (1+x)^n = (1+x)^{m+n}$, we have

$$\left[\sum_{i=0}^{m} \binom{m}{i} x^i\right]\left[\sum_{j=0}^{n} \binom{n}{j} x^j\right] = \sum_{l=0}^{m+n} \binom{m+n}{l} x^l.$$

Compare the coefficient of x^r to give

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}. \quad \divideontimes$$

problem 5. Let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$.

Find $A \times B - B \times A$

Sol. $A \times B = \{(1,2), (1,1), (2,1), (2,2), (3,1), (3,2)\}$

$B \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

So, $A \times B - B \times A = \{(3,1), (3,2)\}$

problem 6. Use Vandermonde's identity

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}, \quad 0 \leq r \leq m, \quad 0 \leq r \leq n.$$

to prove $$\sum_{l=0}^{k} \binom{k}{l}^2 = \binom{2k}{k} \text{ for any } k \geq 0.$$

proof. In Vandermonde's identity, let $m = n = r = k$

to give

$$\sum_{l=0}^{k} \binom{\overset{k}{\cancel{m}}}{l} \binom{k}{k-l} = \sum_{l=0}^{k} \binom{k}{l}^2 = \binom{2k}{k} \quad \#$$