Problems for the prac of week 5

Q1 Find the range of

$$f : (-1, 1) \to \mathbb{R}, \quad x \mapsto x^2$$

$$g : [0, \infty) \to \mathbb{R} \quad x \mapsto \sqrt{x}$$

Sol. $\text{Range}(f) = [0, 1)$ (we skip the rough work)

Indeed, $\forall\, x \in (-1, 1)$, $0 \le |x| < 1$, so $0 \le x^2 < 1$. We have $\text{Range}(f) \subseteq [0, 1)$.

On the other hand, $\forall\, y \in [0, 1)$, we let $x = \sqrt{y} \in (-1, 1)$ to obtain $f(x) = y$.

So, $[0, 1) \subseteq \text{Range}(f)$. ※

$\text{Range}(g) = [0, \infty)$.

Indeed, $\forall\, x \in [0, \infty)$, $\sqrt{x} \ge 0 \Rightarrow \text{Range}(g) \subseteq [0, \infty)$

On the other hand, $\forall\, y \in [0, \infty)$, we let $x = y^2$ to obtain $x \in [0, \infty)$ and $g(x) = y$. So $[0, \infty) \subseteq \text{Range}(g)$ ※

Q2 Determine whether or not the following functions are injective

$f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin x$

$g: [0,\infty) \to \mathbb{R}, \quad x \mapsto \sqrt{x}$

$h: [-2,-1] \to \mathbb{R}, \quad x \mapsto x^2+x+1$

Sol. f is not injective. In fact, 单射

$f(0) = f(2\pi) = 0$. ※

g is injective. In fact, for any $a, b \in [0,\infty)$ with $a \neq b$, without loss of generality assume $a<b$. Then

$\sqrt{a} < \sqrt{b} \Rightarrow g(a) \neq g(b)$. ※

h is injective. In fact, $h(x) = (x+\frac{1}{2})^2 + \frac{3}{4}$. $\forall\, a,b \in [-2,-1]$, without loss of generality assume $a<b$. Then $a+\frac{1}{2} < b+\frac{1}{2} \leq 0$, so $(a+\frac{1}{2})^2 > (b+\frac{1}{2})^2$ and $h(a) \neq h(b)$. ※

Q3. Use the definition to prove

① $\lim_{n\to\infty} \frac{1}{n!} = 0$; ② $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ where $a>0$ is a constant; ③ $\lim_{n\to\infty} e^{-n^2} = 0$

Proof. ① $\forall \varepsilon>0$, let $N = \lceil 1/\varepsilon \rceil$ to have that whenever $n \geqslant N \geqslant \frac{1}{\varepsilon}$, $\left|\frac{1}{n!} - 0\right| \leqslant \frac{1}{n} \leqslant \varepsilon$

So $\lim_{n\to\infty} \frac{1}{n!} = 0$. ※

② Let $N_1 = \lceil ea \rceil \geqslant ea$. Let $C = \frac{a^{N_1}}{N_1!}$. $\forall n \geqslant N_1$,

$$0 \leqslant \frac{a^n}{n!} = C \frac{a^{n-N_1}}{n(n-1)\cdots(N_1+1)} \leqslant C \frac{a^{n-N_1}}{(ea)^{n-N_1}} = e^{N_1} C e^{-n}$$

$\forall \varepsilon>0$, let $N = \max\{N_1, \lceil \log \frac{e^{N_1} C}{\varepsilon} \rceil\}$ to have,

$\forall n \geqslant N$, $\left|\frac{a^n}{n!} - 0\right| \leqslant e^{N_1} C e^{-n} \leqslant e^{N_1} C \frac{\varepsilon}{e^{N_1} C} = \varepsilon$. ※

③ $\forall \varepsilon>0$, let $N = \lceil \log \frac{1}{\varepsilon} \rceil$ to have, $\forall n \geqslant N$,

$|e^{-n^2} - 0| \leqslant e^{-n} \leqslant e^{\log \varepsilon} = \varepsilon$ ※.

Q4. Find the following limits

① $\lim_{n\to\infty} \frac{n-500}{n+2}$ ② $\lim_{n\to\infty} \frac{n^2-700n+10^6}{2n^2+50n+500}$

③ $\lim_{n\to\infty} \frac{n+27}{n^2+9}$ ④ $\lim_{n\to\infty} \frac{n\sin(e^n)+2}{n^2+1}$

⑤ $\lim_{n\to\infty} \frac{(-1)^n}{2n+1}$

Sol. ① $\frac{n-500}{n+2} = \frac{1-\frac{500}{n}}{1+\frac{2}{n}}$, $\lim_{n\to\infty}\left(1-\frac{500}{n}\right)=1$

$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)=1$. So, $\lim_{n\to\infty} \frac{n-500}{n+2}=1$

② $\lim_{n\to\infty} \frac{n^2-700n+10^6}{2n^2+50n+500} = \lim_{n\to\infty} \frac{1-\frac{700}{n}+\frac{10^6}{n^2}}{2+\frac{50}{n}+\frac{500}{n^2}} = \frac{1}{2}$

③ $\lim_{n\to\infty} \frac{n+27}{n^2+9} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{27}{n^2}}{1+\frac{9}{n^2}} = 0$

④ From $\frac{-\frac{1}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}} \leq \frac{\frac{1}{n}\sin(e^n)+\frac{2}{n^2}}{1+\frac{1}{n^2}} \leq \frac{\frac{1}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}$,

One applies the squeeze theorem to obtain

$\lim_{n\to\infty} \frac{n\sin(e^n)+2}{n^2+1} = 0.$

⑤ From $\dfrac{-\frac{1}{n}}{2+\frac{1}{n}} \leq \dfrac{+\frac{1}{n}(-1)^n}{2+\frac{1}{n}} \leq \dfrac{\frac{1}{n}}{2+\frac{1}{n}}$,

One applies the squeeze theorem to obtain

$$\lim_{n\to\infty} \frac{(-1)^n}{2n+1} = 0.$$

Q5 Find $\lim\limits_{n\to\infty} (1+\frac{1}{n})^{n+1}$, $\lim\limits_{n\to\infty} (1+\frac{1}{n})^{-n}$

Sol. From $\lim\limits_{n\to\infty} (1+\frac{1}{n})^n = e$, $\lim\limits_{n\to\infty} (1+\frac{1}{n}) = 1$

we have $\lim\limits_{n\to\infty} (1+\frac{1}{n})^{n+1} = e \times 1 = e$

$$\lim_{n\to\infty} (1+\tfrac{1}{n})^{-n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = e^{-1}.$$

Q6 Use definition to show:

① $\lim\limits_{n\to\infty} \dfrac{n^n}{n!} = \infty$, ② $\lim\limits_{n\to\infty} \sqrt{n} = \infty$

proof. ① $\forall M>0$, let $N=\lceil M\rceil \geq M$. $\forall n \geq N \geq M$,

$$\frac{n^n}{n!} = \frac{n}{1} \times \frac{n^{n-1}}{2\times\cdots\times n} \geq n \geq M, \text{ so } \lim_{n\to\infty} \frac{n^n}{n!} = \infty.$$

②. $\forall M>0$, let $N=\lceil M^2\rceil$, $\forall n \geq N \geq M^2$, $\sqrt{n} \geq M$, so $\lim\limits_{n\to\infty} \sqrt{n} = \infty$ #