Lecture 2.2

# Understanding Randomness: Counting

# Equally likely outcomes

An important case where $\mathbb{P}$ is easily specified is where the sample space has a *finite* number of outcomes that are all *equally likely*.

The probability of an event $A \subseteq \Omega$ is in this case simply

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\text{Number of elements in } A}{\text{Number of elements in } \Omega} \, .$$

The calculation of such probabilities thus reduces to *counting*.

**Question:** What is the probability of rolling a die and getting a 6?

**Question:** What is the probability of rolling two dice and getting 'doubles'?

**Fundamental Principle of Counting:** Suppose a task can be completed in two stages such that

1. Stage 1 has $n_1$ outcomes, and
2. Stage 2 has $n_2$ outcomes (regardless of the actual outcome in Stage 1).

Then, the task has $n_1 n_2$ outcomes. This can be generalizes to $k$ stages in a straightforward manner.

# Example (counting with replacement)

A multiple choice exam has 20 questions; each question has 4 choices. In how many possible ways can the exam be completed?

- There are 20 questions, i.e., 20 stages to go over all questions.
- For each question, we have 4 choices $(a, b, c, d)$
- The answer to one question will not influence the answer to the next one.
- The number of ways to complete the exam is $\underbrace{4 \times \ldots \times 4}_{\text{20 times}} = 4^{20}$.

# Example (counting without replacement)

The lineup or batting order is a list of the 9 baseball players for a team in the order they will bat during the game. How many lineups are possible?

- There are 9 players to select, i.e., 9 stages.
- First stage, we have 9 choices.
- Regardless of who we pick first, the second stage involves 8 choices.
- So on ...
- The number of ways to form a lineup is $9 \times 8 \times \ldots \times 1 = 9!$
- This is the number of permutations or ordered arrangements of 9 players.

# Example (counting without replacement)

Now, suppose we can pick our 9 lineup players from a pool of 30. Now, how many lineups are possible?

- Again, there are 9 players to select, i.e., 9 stages.
- First stage, we have 30 choices.
- Regardless of who we pick first, the second stage involves 29 choices.
- So on ...
- The number of ways to form a lineup is $30 \times 29 \times \ldots \times 22 = 30!/(30-9)!$
- This is the number of permutations or ordered arrangements of 9 players selected from a groups of 30 players.

# Example (counting without replacement)

Now, suppose we are trying a new (crazy!) tactic where players can freely play in any position they like throughout the game, i.e., payers are not assigned to any particular position. In how many ways can we select 9 players from a pool of 30?

- When the order in the lineup was important, i.e., every payer was assigned to a particular position, we had $\frac{30!}{(30-9)!}$ choices.
- But for every group of 9 players, we are over-counting by including all of their 9! ordered arrangements.
- So overall, the number pf possibilities are $\frac{30!}{(30-9)!9!}$
- This is the number of combinations or unordered arrangements of 9 players selected from a groups of 30 players.

## Counting without replacement

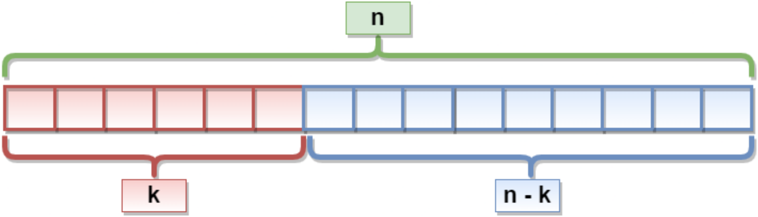
If the $k$ stages of a task involve sampling one unit each, **without replacement**, from the same group of $n$ units, then:

- If a distinction is not made between the outcomes of the stages, we say the outcomes are **unordered**. Otherwise we say the outcomes are **ordered**.
- The **unordered** outcomes are called **combinations** of $k$ units. The number of combinations of $k$ units selected from a group of $n$ units is denoted by ${}^{n}C_{k} = \binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$
- The **ordered** outcomes are called **permutations** of $k$ units. The number of permutations of $k$ units selected from a group of $n$ units is denoted by ${}^{n}P_{k} = {}^{n}C_{k} \cdot k! = \dfrac{n!}{(n-k)!}$.

## Counting without replacement

**More generally:** To order/arrange $n$ items, we can

1. select $k$ unordered items,
2. then order/arrange the $k$ elements,
3. and finally order/arrange the $n - k$ elements.



### Universal Formula for Counting Without Replacement:

$$\underbrace{n!}_{\substack{\text{Number of ways to}\\ \text{arrange } n \text{ items}}} = \overbrace{\underbrace{{}^nC_k}_{\substack{\text{Number of ways to}\\ \text{select unordered}\\ k \text{ items}}} \times \underbrace{k!}_{\substack{\text{Number of ways to}\\ \text{arrange } k \text{ items}}}}^{{}^nP_k} \times \underbrace{(n-k)!}_{\substack{\text{Number of ways to}\\ \text{arrange the remaining}\\ \text{items}}}$$