Problems for the prac of Week 4

Q1 Let

$$A_n = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & \\ 0 & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{n\times n}$$

① Show that for any $n \geq 3$,

$$\det(A_n) = 2\det(A_{n-1}) - \det(A_{n-2})$$

② Find $\det(A_1)$ and $\det(A_2)$

③ Based on the results above, use mathematical induction to prove $\det(A_n) = n+1$.

① proof. We expand the determinant according to the first column to give

$$\det(A_n) = 2\det(A_{n-1}) + (-1)^{2+1} \begin{vmatrix} -1 & 0 & & 0 \\ -1 & 2 & -1 & \\ & -1 & 2 & \\ 0 & & & \ddots \end{vmatrix} \times (-1)$$

$$= 2\det(A_{n-1}) - \det(A_{n-2})$$

② Sol. $A_1 = (2)$, $A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$,

So $\det(A_1) = 2$ and $\det(A_2) = 4 - 1 = 3$

③ proof. (Note: usually for mathematical induction, we only verify the claim with $n=1$ in the first step. However, in this problem, the inductive relation $\det(A_n) = 2\det(A_{n-1}) - \det(A_{n-2})$ holds only for $n \geq 3$. So, we need to verify the claim for both $n=1$ and $n=2$ in the first step of the mathematical induction)

Step 1: we have shown $\det(A_n) = n+1$ in "②" above

Step 2: For any fixed $n \geq 3$, assume $\det(A_k) = k+1$ for any $k = 1, 2, \cdots, n-1$. We use the inductive relation proved above to give

$k = n-1$

$2\det(A_{n-1}) - \det(A_{n-2})$

$2\det(A_k) - \det(A_{k-1})$

$\det(A_n) = 2n - (n-1) = n+1$

$2n - (n-1) = n+1$

This completes the proof by induction.

Q2 Prove or disprove the following statements. Assume that d and n are positive integers.

① if $d \mid n$, then $d \mid n^2$

② if $d \nmid n$, then $d \nmid n^2$

Sol. ① $d \mid n \Rightarrow \frac{n}{d}$ is an integer, so

$\frac{n^2}{d} = n \times \frac{n}{d}$ is an integer, so $d \mid n^2$

② The statement doesn't hold true. For example, $9 \nmid 6$ but $9 \mid 36$.

Q3 Translate the following statement into English and avoid the notations "$\forall, \exists, \mathbb{Q}$".

"$\forall a, b \in \mathbb{Q}$ with $a < b$, $\exists\, c \in \mathbb{Q}$ s.t. $a < c < b$."

Sol. For any rational numbers a and b with $a<b$, there exists a rational number c such that $a<c<b$.

Q4. Prove by Contradiction:

① $\forall n \in \mathbb{Z}$, if n^3+5 is odd, then n is even

$n=2k+1$ not even

$(2k+1)^3+5 = \cdots$

② There exist no integers a and b for which $6a-21b=7$

③ Let $a \in \mathbb{Q}$ and $b \in \mathbb{R}-\mathbb{Q}$. Then $a+b \in \mathbb{R}-\mathbb{Q}$.

proofs ①. Suppose n is not even, then $n=2k+1$ for some $k \in \mathbb{Z}$. We have

$$n^3+5 = 8k^3+12k^2+6k+6$$
$$=2(4k^3+6k^2+3k+3)$$

which is even. The Contradiction Completes the proof.

② Assume $\exists\, a, b \in \mathbb{Z}$ s.t $6a-21b=7$.

Then $7 = 3(2a-7b)$, so $3(2a-7b)=7$

$$\frac{7}{3} = 2a-7b \in \mathbb{Z}$$ $2a-7b=\frac{7}{3}\notin\mathbb{Z}$

The contradiction completes the proof

③ Assume $a+b \in \mathbb{Q}$. Then since $a\in\mathbb{Q} \Rightarrow -a\in\mathbb{Q}$,

One has $b=(a+b)-a \in \mathbb{Q}$.

The contradiction completes the proof.

Q5. Construct a truth table for $p \Rightarrow (q \wedge (\sim p))$

Sol. Recall that $p \Rightarrow (q \wedge (\sim p)) = (\sim p) \vee (q \wedge (\sim p))$

p	q	$\sim p$	$q \wedge (\sim p)$	$(\sim p) \vee (q \wedge (\sim p))$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	T
F	F	T	F	T