Lecture 5.3

Multiple Random Variables:

Independence of Random Variables

Independence of random variables

Recall: Events A and B are independent iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Two random variables X and Y are said to be **independent** if and only if

$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \in x)\ \mathbb{P}(Y \leq y), \quad \forall (x, y) \in \mathbb{R}^2.$$

In other words, for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, we have

$$F_{X,Y}(x, y) = F_X(x)F_Y(y).$$

More generally, random variables $X_1, X_2, \ldots, X_n$ with joint cdf F are **independent** if and only if

$$F(x_1, x_2, \ldots, x_n) = F_{X_1}(x_1)F_{X_2}(x_2)\ldots F_{X_n}(x_n), \quad \forall (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Note: The above definitions are valid for all random variables, i.e., discrete, continuous, combination, and beyond.

Independence of discrete random variables

We can also easily check the independence for discrete random variables using the their probability mass functions.

Discrete random variables $X_1, \dots, X_n$ with joint pmf f are **independent** if and only if

$$f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n), \quad \forall (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,$$

where f_{X_i} is the marginal pmf of X_i.

Example

Consider the joint probability mass function of (X, Y) below. Both X and Y only take the values $\{0, 1, 2\}$.

		y			
		0	1	2	
	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$
x	1	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$
	2	0	0	$\frac{1}{4}$	$\frac{1}{4}$
		$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

Are the event $\{X = 1\}$ and $\{Y = 1\}$ independent? Yes.

Are the random variables X and Y independent? No.

Independence of continuous random variables

Similarly, we can easily check the independence for continuous random variables using their probability density functions (if they exist).

Random variables $X_1, \ldots, X_n$ with joint pdf f are **independent** if and only if

$$f(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n), \quad \forall (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,$$

where the f_{X_i} is the marginal pdf of X_i.

Example

Consider the joint probability density function of (X, Y) below.

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{7}(1 + x^2 y), & (x,y) \in [0,1] \times [0,1] \\ 0, & \text{else.} \end{cases}$$

Are the random variables X and Y independent?

Example

We had

$$f_X(x) = \begin{cases} \frac{6}{7}(1 + x^2/2), & x \in [0,1], \\ 0, & \text{else.} \end{cases}$$

Similarly, for $x \in [0,1]$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}x = \int_0^1 \frac{6}{7}(1 + x^2 y)\mathrm{d}x = \frac{6}{7}(1 + \frac{y}{3}),$$

so,

$$f_Y(y) = \begin{cases} \frac{6}{7}(1 + y/3), & y \in [0,1], \\ 0, & \text{else.} \end{cases}$$

But for any $(x,y) \in [0,1]^2$, $f_{X,Y}(x,y) = \frac{6}{7}(1 + x^2 y) \neq f_X(x) f_Y(y)$.

Functions of Independent Random Variables

Suppose two random variable X and Y are are independent. Then, $h(X)$ and $g(Y)$ are independent no matter what the functions h and g are.

The event $\{h(X) \leq t\}$ can always be written as $\{X \in A\}$, where $A = \{x : h(x) \leq t\}$. Indeed,

$$\{\omega \in \Omega \mid h(X(\omega)) \leq t\} = \{\omega \in \Omega \mid X(\omega) \in \{x \in \mathbb{R} \mid h(x) \leq t\}\}.$$

Similarly, for any u, the event $\{g(Y) \leq u\}$ can be written as $\{Y \in B\}$, where $B = \{y : g(y) \leq u\}$. So,

$$\begin{aligned}\mathbb{P}(h(X) \leq t, h(Y) \leq u) &= \mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B) \\ &= \mathbb{P}(h(X) \leq t)\mathbb{P}(h(Y) \leq u).\end{aligned}$$

The independence of $h(X)$ and $g(Y)$ follows simply from the independence of X and Y.