STAT2203/7203: Week 8 Practical Questions

1. A pair of random variables (X, Y) has a joint probability distribution in which $X \sim \mathcal{U}[0, 1]$ and the conditional probability density function of Y given $\{X = x\}$ is

$$f_{Y|X}(y|x) = \begin{cases} 1, & x \le y \le 1 + x \\ 0, & \text{else.} \end{cases}$$

   (a) Determine the marginal probability density function for Y.

   (b) Using the fact that $\mathbb{E}Y = \mathbb{E}[\mathbb{E}[Y|X]]$, compute the expectation of Y.

   (c) We can also use the conditional expectation to help computing the covariance. Note that

$$\begin{aligned} \mathbb{E}[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_{X,Y}(x, y)\, dy\, dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_{Y|X}(y|x)\, f_X(x)\, dy\, dx \\ &= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy \right) f_X(x)\, dx \\ &= \mathbb{E}[X\mathbb{E}[Y|X]]. \end{aligned}$$

   Use this fact to compute the covariance between X and Y.

2. Let $X_1, \ldots, X_{100}$ be independent random variables form a $\mathcal{U}[0, 1]$ distribution. We wish to sort $X_1, \ldots, X_{100}$ using the bucketsort algorithm:

   - The interval [0, 1] is divided into 10 sub-intervals $[(k-1)/10, k/10)$ for $k = 1, 2, \ldots, 10$.
   - For $k = 1, \ldots, 10$, the values X_i falling in $[(k-1)/10, k/10)$ are naively sorted. The naive sorting of a list of length n requires n^2 comparisons.

   Let N_k be the number of X_i's that fall in the k-th sub-interval. Then the number of comparisons required to sort the values is $\sum_{k=1}^{10} N_k^2$.

   (a) What is the marginal distribution of N_k?

   (b) What is the expected number of comparisons performed by bucketsort? (*This is the 'average-case scenario'.*)