Lecture 2.1

Understanding Randomness

"Education is not the learning of facts, but the training of the mind to think."

Albert Einstein

Random experiments

If we repeat the process of collecting the data, we would most likely obtain different measurements.

A **random experiment** is an experiment whose outcome cannot be determined in advance, e.g.,

- Tossing a coin three times and note the sequence of H and T
- An Airplane's both engines failing mid-flight
- Donald Trump tweeting something silly!

Note: Although the word experiment generally suggests a planned or carefully controlled laboratory testing situation, we use it here in a much wider sense.

Random experiments and probability

Even though the outcome of a random experiment is uncertain, we'd still like to analyse it.

In any situation where one of a number of possible outcomes may occur, the discipline of **probability** provides methods for quantifying the chances, or likelihoods, associated with the various outcomes.

This is achieved by analysing the underlying **probability model**.

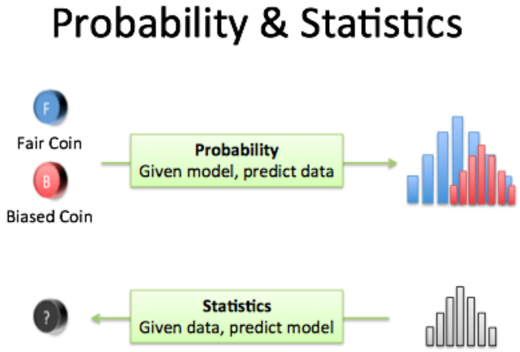
Statistics vs. Probability

What do probability and statistics have in common?

They both deal with *random experiments*.

Statistics vs. Probability

Difference between probability and statistics?



Probabilist (by making assumptions about model) would ask:

- Tossing a coin 3 times and noting the sequence of H and T
  - Q: On average, how many tosses before seeing HHT
- An Airplane's both engines failing mid-flight
  - Q: Out of 1000 sorties of an airline, what is the probability than 1 such incident happen?
- Donald Trump tweeting something silly! 
  - Q: If he types randomly, expected time of the first appearance of the word COVFEFE!!!

Statistics vs. Probability

Statistician (by observing the data) would ask

- Tossing a coin 3 times and noting the sequence of H and T
  - Q: Is the coin fair?
- An Airplane's both engines failing mid-flight
  - Q: Is it safer to fly with "AeroMaybe" or "US Scareways"?
- Donald Trump tweeting something silly! 
  - Q: Was he typing randomly or COVFEFE is the nulear lunch code? 

Probability models - Sample space

Probability model: We need three ingredients to model a random experiment:

(i) Sample Space

(ii) Collection of Events

(iii) Probability Measure

Sample space

The **sample space** Ω of a random experiment is the set of all possible outcomes of the random experiment, e.g.,

- Flip of a coin: $\Omega = \{\text{Heads}, \text{Tails}\}$
- the lifespan of a person in years: $\Omega = \{0, 1, 2, \ldots, 140\}$
- Time for App to load: $\Omega = (0, \infty)$

Events

An **event** is a <u>subset</u> of the sample space Ω.

Probabilities are assigned to events.

Events will be denoted by capital letters $A, B, C, \ldots$

An event is **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

We say that event A *occurs* if the outcome of the random experiment is one of the elements in A, e.g.,

- Heads occurs: $A = \{\text{Heads}\}$
- A person lives at least 100 years: $A = \{100, 101, \ldots, 140\}$
- App takes less than 2s to load: $A = (0, 2)$

Set notation

Since events are subsets of the sample space, we will often use set notation to describe events.

The event that both A and B occur is the intersection $A \cap B$.

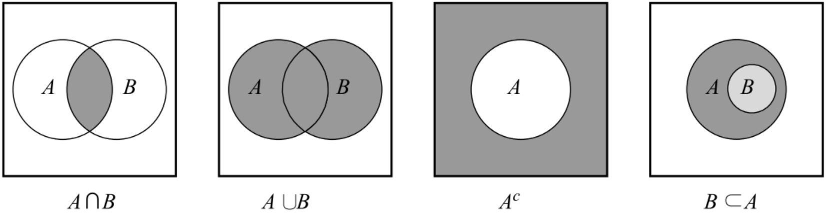
The event that either A or B (or both) occur is the union $A \cup B$.

The event that A does not occur is the complement A^c.

If B is a subset of A, i.e., $B \subseteq A$, then we say B implies A.

Two events A and B which have no outcomes in common are called *disjoint* events, that is, $A \cap B = \emptyset$.

Venn Diagrams



Question

Consider the random experiment in which a standard six-sided die is rolled and the outcome recorded. Let A be the event that the outcome is even and B be the event that the outcome is odd.

$$\Omega = \{1, 2, 3, 4, 5, 6\}, \quad A = \{2, 4, 6\}, \quad B = \{1, 3, 5\}.$$
$$A \cup B = \Omega, \quad A \cap B = \emptyset, \quad A^c = B.$$

Let $C = \{5, 6\}$ and $D = A \cap C$. What are the following events?

$$D = A \cap C = \{2, 4, 6\} \cap \{5, 6\} = \{6\}, \quad B \cap C = \{1, 3, 5\} \cap \{5, 6\} = \{5\}$$

Which of the following are true?

$$D \subseteq C \checkmark, \quad C \subseteq A \times, \quad D \subseteq A \checkmark, \quad D \subseteq B \times$$

Probablity measure

A **probability measure** tells us how likely it is that a particular event will occur.

A probability measure, $\mathbb{P}$, is a function, which satisfies the following rules:

1. $0 \leq \mathbb{P}(A) \leq 1$,
2. $\mathbb{P}(\Omega) = 1$, and
3. For any sequence $A_1, A_2, \ldots$ of disjoint events, we have

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots . \quad \text{(Sum Rule)}$$

Properties of probability measures

Property I: $\mathbb{P}(\emptyset) = 0$.

Proof: Ω and $\emptyset$ are disjoint and $\Omega \cup \emptyset = \Omega$, so

$$\mathbb{P}(\Omega) = \mathbb{P}(\Omega \cup \emptyset) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset),$$

and hence $\mathbb{P}(\emptyset) = 0$.

Properties of probability measures

Property II: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Proof: A and $A^\complement$ are disjoint and $A \cup A^\complement = \Omega$, so

$$\mathbb{P}(\Omega) = \mathbb{P}(A \cup A^\complement) = \mathbb{P}(A) + \mathbb{P}(A^\complement),$$

and hence $\mathbb{P}(A^\complement) = 1 - \mathbb{P}(A)$.

Properties of probability measures

Property III: If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Proof: We have

$$B = B \cap \Omega = B \cap \left(A^{\complement} \cup A\right) = \left(B \cap A^{\complement}\right) \cup (B \cap A) = \left(B \cap A^{\complement}\right) \cup A.$$

Since $\left(B \cap A^{\complement}\right)$ and A are disjoint, we get

$$\mathbb{P}(B) = \overbrace{\mathbb{P}\left(B \cap A^{\complement}\right)}^{\geq 0} + \mathbb{P}(A) \geq \mathbb{P}(A).$$

Properties of probability measures

Property IV: For any two events A and B, we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Proof: As before, we have

$$B = \left(B \cap A^{\complement}\right) \cup (B \cap A).$$

Since $B \cap A^{\complement}$ and $B \cap A$ are disjoint,

$$\mathbb{P}(B) = \mathbb{P}\left(B \cap A^{\complement}\right) + \mathbb{P}(B \cap A).$$

At the same time, we also have

$$A \cup B = \left(B \cap A^{\complement}\right) \cup A.$$

Since $B \cap A^{\complement}$ and A are disjoint,

$$\mathbb{P}(A \cup B) = \mathbb{P}\left(B \cap A^{\complement}\right) + \mathbb{P}(A).$$

Now, putting this all together, we get the result.