Lecture 2.3

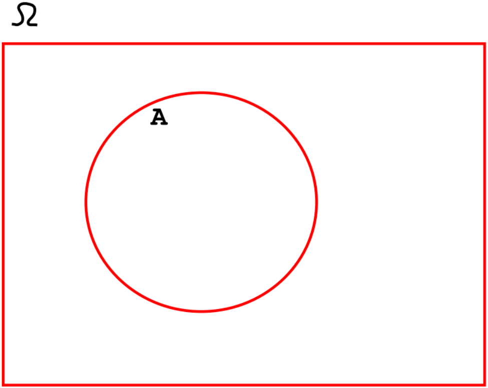
# Understanding Randomness: Conditional Probability and Law of Total Probability

# Conditional Probability

How do probabilities of future events change when we know that some event has already occurred?

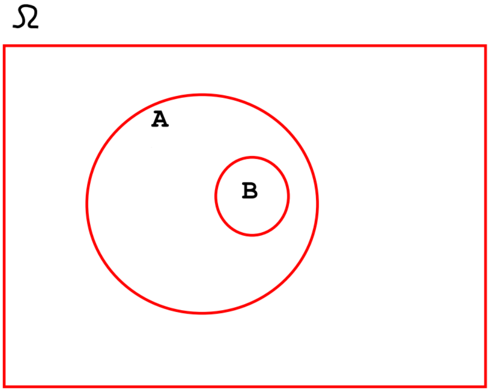
- What is the probability that Timmy has Covid?
- What is the probability that Timmy has Covid, given he is wearing jeans?
- What is the probability that Timmy has Covid, given he is coughing?

# Conditional Probability



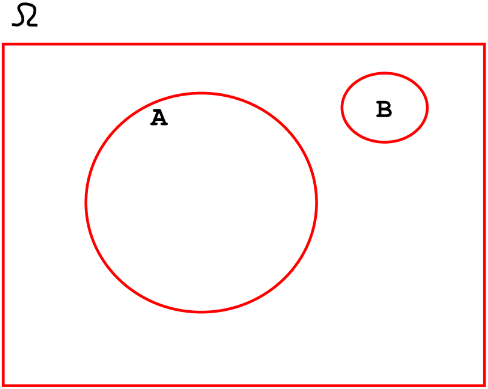

P(dart hitting A)

# Conditional Probability



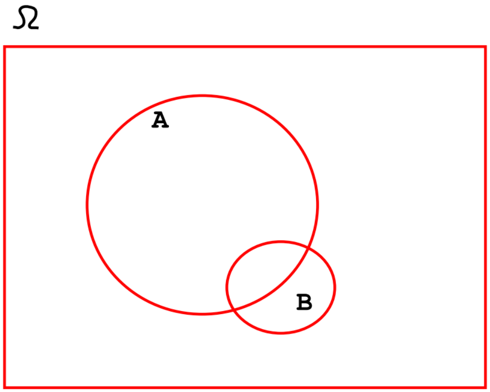

P(dart hitting A | dart has hit B)

# Conditional Probability



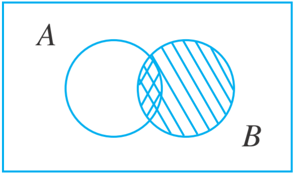

P(dart hitting A | dart has hit B)

# Conditional Probability




P(dart hitting A | dart has hit B)

# Conditional Probability




If $B$ occurs, then A will occur if and only if $A \cap B$ occurs.

**Definition:** Assuming $\mathbb{P}(B) > 0$, the conditional probability of $A$ given $B$ is

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

We can rearrange this definition to give $\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A \mid B)$.

# Independent events

When the occurrence of $B$ does not give extra information about $A$, the events $A$ and $B$ are said to be independent.

**Definition:** Events $A$ and $B$ are said to be *independent* if

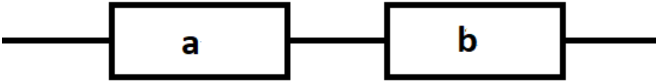
$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Multiple events $A_1, A_2, \ldots, A_n$ are independent if and only if for any subset of these events $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$,

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1})\ P(A_{i_2})\ \ldots\ P(A_{i_k}).$$

# Series systems

Consider a system comprised of two components connected in series. The systems is working iff both components are working.




Let $A$ be the event that component (a) is working and $B$ be the event that component (b) is working. Assume that the two events are independent. Suppose $\mathbb{P}(A) = 0.8$ and $\mathbb{P}(B) = 0.6$

What is the probability that the system is working?

## Series systems

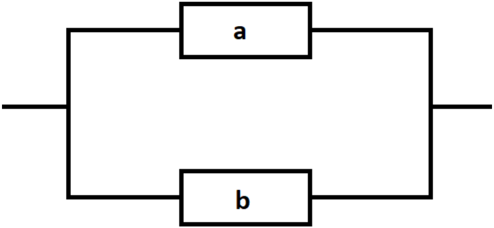
Answer: The event that the system is working is $A \cap B$. As the two events are independent,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B) = 0.8 \times 0.6 = 0.48$$

The system is working with probability 0.48

# Parallel systems

Consider a system comprised of two components connected in parallel. The systems is working iff at least one component is working.




Let $A$ be the event that component (a) is working and $B$ be the event that component (b) is working. Assume that the two events are independent. Suppose $\mathbb{P}(A) = 0.8$ and $\mathbb{P}(B) = 0.6$

What is the probability that the system is working?

# Parallel systems

Answer: The event that the system is working is $A \cup B$. Therefore,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Again, as the events $A$ and $B$ are independent

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B) = 0.8 \times 0.6 = 0.48$$

The system is working with probability

$$\mathbb{P}(A \cup B) = 0.8 + 0.6 - 0.48 = 0.92$$

# Product rule

Consider three events $A_1$, $A_2$, and $A_3$. Suppose $\mathbb{P}(A_1 \cap A_2) > 0$. By the definition of conditional probability

$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2 \cap A_3) &= \mathbb{P}(A_3 \mid A_1 \cap A_2)\, \mathbb{P}(A_1 \cap A_2) \\
&= \mathbb{P}(A_3 \mid A_1 \cap A_2)\, \mathbb{P}(A_2 \mid A_1)\, \mathbb{P}(A_1)
\end{aligned}$$

**Note:** Since $\mathbb{P}(A_1 \cap A_2) > 0$, we also have $\mathbb{P}(A_1) > 0$ (**why?**)

Let $A_1, \ldots, A_n$ be a sequence of events with $\mathbb{P}(A_1 \ldots A_{n-1}) > 0$. We can generalise the above formula to $n$ intersections as

$$\mathbb{P}(A_1 \ldots A_n) = \mathbb{P}(A_n | A_1 \cdots A_{n-1}) \mathbb{P}(A_{n-1} | A_1 \cdots A_{n-2}) \cdots \mathbb{P}(A_2 | A_1) \mathbb{P}(A_1).$$

# Example

We draw consecutively 3 balls from an urn with 5 white and 5 black balls, without putting them back. What is the probability that all drawn balls will be black?

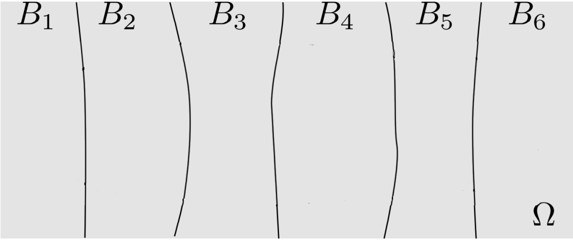
Let $A_i$ be the event that the $i$th ball is black. We wish to find the probability of $A_1A_2A_3$. So, by the product rule

$$\mathbb{P}(A_1A_2A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_1A_2) = \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} \approx 0.083.$$

# Partition

Suppose that $B_1, B_2, \ldots, B_n$ are disjoint and their union is $\Omega$, i.e.,

$$B_i \cap B_j = \emptyset, \quad \text{for all } i \neq j, \qquad \text{and} \quad \bigcup_{i=1}^{n} B_i = \Omega.$$

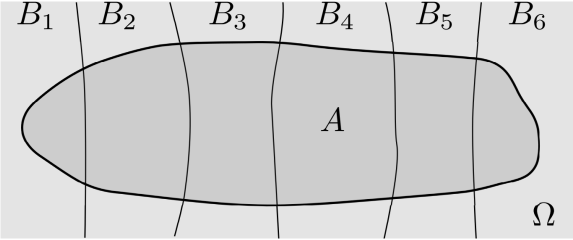



We call the collection of sets $\{B_1, B_2, \ldots, B_n\}$ a *partition* of $\Omega$.

# Law of total probability

With this partition, for any $A \subseteq \Omega$, the sets $A \cap B_i$ and $A \cap B_j$ are disjoint for $i \neq j$. Applying the sum rule we obtain the **law of total probability**

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^{n}(A \cap B_i)\right) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)\,.$$



# Bayes' Rule

From the definition of conditional probability, we have

$$\mathbb{P}(B_j \mid A) = \frac{\mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)}{\mathbb{P}(A)}$$

Combining this with the law of total probability gives **Bayes' rule**

$$\mathbb{P}(B_j \mid A) = \frac{\mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)}{\sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)}\,,$$

where $\{B_1, B_2, \ldots, B_n\}$ is a *partition* of $\Omega$

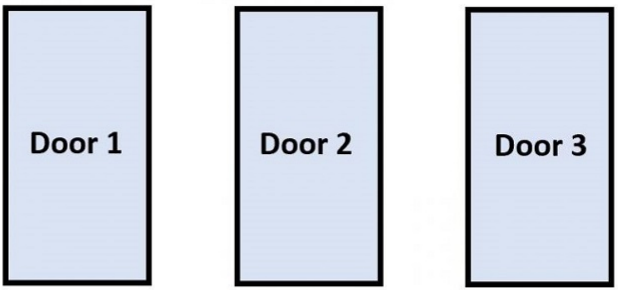
# Example: Monty Hall Problem

The television show *Let's Make a Deal*, hosted by *Monty Hall*, gave contestants the opportunity to choose one of three doors. Behind one door there is the grand prize, while behind the other two there is nothing. After the contestant chose one of the doors, Monty opened one of the other two doors that does not contain the prize. The contestant was then asked whether he or she wished to stay with the original choice or switch to the other closed door.

**Q:** What should the contestant do? Is it better to stay with the original choice or to switch to the other closed door? Or does it really matter?

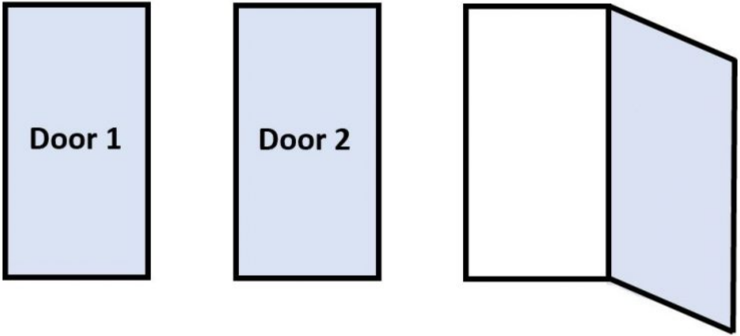
**Figure 1:** Monty Hall

# Example: Monty Hall Problem




**Let's assume the contestant initially guesses Door 1.**

# Example: Monty Hall Problem




**. . . and Monty opens Door 3. Should the contestant stick with Door 1 or switch to Door 2?**

# Example: Monty Hall Problem

$A_i$: The event that the prize is behind door $i$

$B_j$: The event that Monty opens door $j$

Initially: $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = 1/3$

After Monty reveals Door 3, we need to calculate $\mathbb{P}(A_1 \mid B_3)$ ("sticking to Door 1"), and $\mathbb{P}(A_2 \mid B_3)$ ("switching to Door 2")

# Example: Monty Hall Problem

We know that $\mathbb{P}(B_3 \mid A_3) = 0$, $\mathbb{P}(B_3 \mid A_1) = 1/2$, and $\mathbb{P}(B_3 \mid A_2) = 1$

So:

$$\begin{aligned}
\mathbb{P}(A_1 \mid B_3) &= \frac{\mathbb{P}(B_3 \mid A_1)\mathbb{P}(A_1)}{\mathbb{P}(B_3 \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B_3 \mid A_2)\mathbb{P}(A_2) + \mathbb{P}(B_3 \mid A_3)\mathbb{P}(A_3)} \\
&= \frac{1/2 \times 1/3}{1/2 \times 1/3 + 1 \times 1/3 + 0 \times 1/3} = 1/3
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(A_2 \mid B_3) &= \frac{\mathbb{P}(B_3 \mid A_2)\mathbb{P}(A_2)}{\mathbb{P}(B_3 \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B_3 \mid A_2)\mathbb{P}(A_2) + \mathbb{P}(B_3 \mid A_3)\mathbb{P}(A_3)} \\
&= \frac{1 \times 1/3}{1/2 \times 1/3 + 1 \times 1/3 + 0 \times 1/3} = 2/3
\end{aligned}$$