MATH7501: Week 9 Practical Questions

April 23, 2023

Problem 1. Compute the second order Taylor polynomial of $f(x) = 2(\ln x)^2 - \ln x$, at $a = 1$.

Problem 2. Use a linear approximation to estimate the quantity: $\sqrt{1.01} \times \left(\sqrt{0.97}\right)^3$.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$, and suppose that there exists a finite $B > 0$, such that $f''(x) \leq B$, for all x. Prove that for each $y \in \mathbb{R}$, the function

$$g(x) = f(y) + f'(y)(x - y) + \frac{1}{2}B(x - y)^2 \geq f(x)$$

for all $x \in \mathbb{R}$.

Problem 4. We say that the function $f : \mathbb{R} \to \mathbb{R}$ is convex if for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$, the statement

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

holds for every $\lambda \in [0, 1]$. Suppose that f is differentiable on $\mathbb{R}$. Show that if

$$f(y) \geq f(x) + f'(x)(y - x),$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, then f is convex.

Problem 5. For each $n \in \{1, 2, 3, \dots\}$, prove that

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!} > 0,$$

for all $x \in \mathbb{R}$.

Problem 6. Prove that for all $n \in \{1, 2, \dots\}$, the function $f : (0, \infty) \to \mathbb{R}$, given by $f(x) = x^n \ln x$ has nth derivative:

$$f^{(n)}(x) = n!\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \ln x\right).$$

Problem 7. Let $f : [0, 1] \to \mathbb{R}$ be twice differentiable, and suppose that $|f''(x)| \leq M$, for each $x \in [0, 1]$. Prove that if $\sum_{k=1}^{\infty} f(1/k)$ converges, then $f(0) = f'(0) = 0$.

Problem 8. Recall that a positive number c is rational if there exists $a \in \mathbb{N}$ and $b \in \mathbb{N}$, such that $c = a/b$. Prove that e is irrational (i.e., e is not rational).