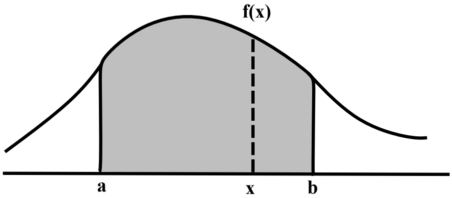
Lecture 4.3

Common Probability Distributions: Uniform, Exponential

Continuous Distributions

The distribution of a continuous random variable is specified by its *probability density function* (pdf), if it exists.



$$\mathbb{P}(a \le X \le b) = \mathbb{P}(a \le X < b) = \mathbb{P}(a < X \le b)$$
$$= \mathbb{P}(a < X < b) = F(b) - F(a) = \int_a^b f(x)\,dx$$

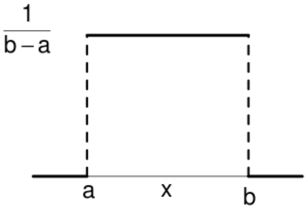
We will examine three common continuous distributions: *Uniform*, *Exponential* and *Normal*.

Uniform Distribution

A random variable X is said to have a **uniform** distribution on the interval $[a, b]$ if its pdf is given by

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b \qquad (\text{and } f(x) = 0 \text{ otherwise}).$$

We write $X \sim \mathcal{U}[a, b]$.



The uniform distribution is rarely used to model data, but it has a number of important theoretical properties.

Uniform Distribution

Suppose $X \sim \mathcal{U}[a, b]$. Then for $a \leq c \leq d \leq b$,

$$\mathbb{P}(c \leq X \leq d) = \int_c^d \frac{1}{b-a} \mathrm{d}x = \frac{d-c}{b-a}.$$

In particular,

$$F_X(x) = \mathbb{P}(X \leq x) = \int_a^x \frac{1}{b-a} \mathrm{d}x = \frac{x-a}{b-a}.$$

So, if $a = 0$ and $b = 1$, then $F_X(x) = x$.

Question

Suppose $X \sim \mathcal{U}[0, 10]$, that is, it has pdf

$$f(x) = \frac{1}{10}, \quad 0 \leq x \leq 10 \qquad \text{(and } f(x) = 0 \text{ otherwise).}$$

What is $\mathbb{P}(1 < X < 4)$?

(a) 0.1

(b) 0.25

(c) 0.3 ✓

(d) 0.4

Uniform distribution and p-values

Informally, a p-value of a hypothesis test is the probability of observing data as extreme or more exterme than the data we observed, assuming the null hypothesis is true.

As a p-value is just a function of our data (which is random), the p-value is in fact itself a random variable, i.e., if we repeat the same experiment and collect a new set of data, our p-value might change.

So, let's use P to denote the underlying random variable and p for the actual value we calculate from our data.

When the null hypothesis is true (and all other assumptions are met), P has a $\mathcal{U}[0, 1]$ distribution.

Uniform distribution and p-values

When we run an experiment, we make an observation of certain random variable, T, which we can denote by $T(\omega) = t$.

Under null hypothesis H_0, we assume that T has a certain distribution. Suppose the CDF of this assumed distribution is F_{H_0} and it is strictly increasing.

With this observation, we compute the p-value as $p = \mathbb{P}_{H_0}(T \leq t) = F_{H_0}(t) = F_{H_0}(T(\omega))$, so $P = F_{H_0}(T)$.Then

$$\mathbb{P}(P \leq p) = \mathbb{P}(F_{H_0}(T) \leq p) = \mathbb{P}(T \leq F_{H_0}^{-1}(p)) = F_T(F_{H_0}^{-1}(p)).$$

If the null hypothesis is in fact true, i.e., $F_T = F_{H_0}$, then

$$\mathbb{P}(P \leq p) = F_T(F_T^{-1}(p)) = p.$$

Exercise: Try to show this for when $p = \mathbb{P}(T \geq t)$.

Simulating from different distributions

Suppose we want to draw from a distribution with strictly increasing cdf F. Recall that, in this case, the quantile function is just $Q(p) = F^{-1}(p)$ for $p \in (0, 1)$.

Let $U \sim \mathcal{U}[0, 1]$ and define $X = Q(U)$. Then

$$\mathbb{P}(X \leq x) = \mathbb{P}(Q(U) \leq x) = \mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x).$$

This way of simulating random variables from a given distribution is called the **inverse-transform method**.

Note: Using the general definition of quantile function, one can show that the inverse-transform methods also works for any random variable with any cdf, and not just those with strictly increasing cdf.

Properties of the Uniform distribution

Suppose $X \sim \mathcal{U}[0,1]$.

$$\mathbb{E}X = \tfrac{1}{2}, \quad \text{and} \quad \text{Var}(X) = \tfrac{1}{12}.$$

For $a, b \in \mathbb{R}$, define $Y = a + (b-a)X$. Then

- $Y \sim \mathcal{U}[a,b]$.
- $\mathbb{E}Y = \mathbb{E}(a + (b-a)X) = a + (b-a) \times \dfrac{1}{2} = \dfrac{b+a}{2}$
- $\text{Var}(Y) = \text{Var}(a + (b-a)X) = (b-a)^2 \text{Var}(X) = \dfrac{(b-a)^2}{12}$

Exponential Distribution

A random variable X is said to have an **exponential** distribution with parameter $\lambda > 0$ if its pdf is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \quad \text{(and } f(x) = 0 \text{ otherwise)}.$$

We write $X \sim \mathrm{Exp}(\lambda)$.

The exponential distribution is often used to model the time until a specific event occurs, i.e., waiting time, e.g.,

- in reliability theory as a model for the life time of equipment
- the time for the next customer to arrive at a service station
- the time until the next investment firm failure
- the time until the next outbreak
- the time until the next earthquake

Question

Suppose $X \sim \text{Exp}(\lambda)$. How does the probability of $\{X > 2\}$ change as λ increases?

(a) increases as λ increases

(b) decreases as λ increases ✓

(c) coud increase or decrease

Question

Suppose $X \sim \text{Exp}(1)$. What is $\mathbb{E}(X)$?

Recall: Integration by parts

$$\int_a^b f(x)\, g'(x) \mathrm{d}x = [f(x)\, g(x)]_a^b - \int_a^b f'(x)\, g(x) \mathrm{d}x.$$

$$\begin{aligned} EX &= \int_0^\infty x\, e^{-x} \mathrm{d}x = [-xe^{-x}]_0^\infty + \int_0^\infty e^{-x} \mathrm{d}x \\ &= \int_0^\infty e^{-x} \mathrm{d}x = 1. \end{aligned}$$

Properties of the Exponential Distribution

Suppose $X \sim \text{Exp}(1)$.

$$\mathbb{E}X = 1, \quad \text{Var}(X) = 1, \quad \text{and} \quad M_X(s) = \frac{1}{1-s}, \; s < 1.$$

For $\lambda > 0$, define $Y = X/\lambda$. Then

- $Y \sim \text{Exp}(\lambda)$

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) = \mathbb{P}(X \le \lambda y) = F_X(\lambda y) \\ \implies f_Y(y) &= F_Y'(y) = \lambda f_X(\lambda y) = \lambda e^{-\lambda y} \end{aligned}$$

- $\mathbb{E}Y = \mathbb{E}(X/\lambda) = \lambda^{-1}\mathbb{E}X = \lambda^{-1}$
- $\text{Var}(Y) = \text{Var}(X/\lambda) = \lambda^{-2}\text{Var}(X) = \lambda^{-2}$
- $M_Y(s) = \mathbb{E}e^{sY} = \mathbb{E}e^{sX/\lambda} = \frac{1}{1-s/\lambda} = \frac{\lambda}{\lambda-s}, \; s < \lambda$

Memoryless property

Let X have an exponential distribution with parameter λ. Then

$$\mathbb{P}(X > x + y \mid X > x) = \mathbb{P}(X > y), \quad \text{for all } x, y \geq 0.$$

Example: Suppose that the lifetime of your laptop has an $\text{Exp}(\frac{1}{3})$ distribution. Given that it is still working after 2 years, the remaining lifetime of your laptop has an $\text{Exp}(\frac{1}{3})$ distribution.

Note: One can prove that the exponential distributions are the only continuous distributions with the memoryless property.