Lecture 3.x

Random variables and their distribution: Week 3 Review

Question - Cumulative Distribution Function

A random variable X has cdf F given by

$$F(x) = \begin{cases} 0, & x < 0 \\ (x/10)^3, & x \in [0, 10] \\ 1, & x > 10 \end{cases}$$

The random variable X is

(a) Discrete

(b) Continuous ✓ (as F is continuous)

(c) Neither

(d) Cannot be determined from the cdf

Question - Cumulative Distribution Function

A random variable X has cdf F given by

$$F(x) = \begin{cases} 0, & x < 0 \\ (x/10)^3, & x \in [0, 10] \\ 1, & x > 10 \end{cases}$$

The probability that X is in the interval $(1, 3]$ is

(a) 0.02

(b) 0.026 ✓ Calculate $F(3) - F(1)$

(c) 0.027

(d) 0.2

Question - Probability Mass Function

Let X be the lifetime of a certain component in years with pmf

x	0	1	2	3	4
$f(x)$	0.1	0.25	0.30	0.20	0.15

The probability that the component has a lifetime of at least 2 years is

(a) 0.30

(b) 0.35

(c) 0.65 ✓ Calculate $f(2) + f(3) + f(4)$

(d) 0.85

Question - Probability Mass Function

Let X be the lifetime of a certain component in years with pmf

x	0	1	2	3	4
$f(x)$	0.1	0.25	0.30	0.20	0.15

What is the probability that the component has a lifetime of 4 years, given it has a lifetime of at least 2 years?

(a) 0.15

(b) 0.231 ✓ Calculate $\mathbb{P}(X = 4|X \geq 2) = \mathbb{P}(X = 4)/\mathbb{P}(X \geq 2)$

(c) 0.429

(d) 0.5

Question - Probability Mass Function

Let X be the lifetime of a certain component in years with pmf

x	0	1	2	3	4
$f(x)$	0.1	0.25	0.30	0.20	0.15

The expected lifetime of the component (in years) is

(a) 2

(b) 2.05 ✓ Use $\mathbb{E}X = \sum_{x=0}^{4} x\, f(x)$

(c) 2.8

(d) 3

Question - Probability Mass Function

Let X be the lifetime of a certain component in years with pmf

x	0	1	2	3	4
$f(x)$	0.1	0.25	0.30	0.20	0.15

The standard deviation of the lifetime of the component (in years) is

(a) 1.203 ✓ $\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$ and $sd(X) = \sqrt{\text{Var}(X)}$.

(b) 1.448

(c) 2.05

(d) 5.65

Question - Probability Density Function

A continuous random variable X has cdf F given by,

$$F(x) = \begin{cases} 0, & x < 0 \\ \sqrt{x}, & x \in [0,1] \\ 1, & x > 1 \end{cases}$$

The pdf of X is

(a) Cannot be determine from the cdf

(b) $f(x) = \frac{1}{2}\sqrt{x},\ x \in [0,1]$ and 0 elsewhere.

(c) $f(x) = \frac{1}{2}x^{-1/2},\ x \in [0,1]$ and 0 elsewhere. ✓ $f(x) = F'(x)$.

(d) $f(x) = \frac{2}{3}x^{3/2},\ x \in [0,1]$ and 0 elsewhere.

Question - Quantile Function

A continuous random variable X has cdf F given by,

$$F(x) = \begin{cases} 0, & x < 0 \\ \sqrt{x}, & x \in [0,1] \\ 1, & x > 1 \end{cases}$$

The quantile function of X is

(a) Cannot be determine from the cdf

(b) $Q(p) = p^{1/2}$, for $p \in (0,1)$

(c) $Q(p) = p^{-1/2}$, for $p \in (0,1)$

(d) $Q(p) = p^2$, for $p \in (0,1)$ ✓ Find $F(Q(p)) = p$.

Question - Expectations

A continuous random variable X has pdf f given by

$$f(x) = 6x^5, \quad x \in (0,1)$$

and $f(x) = 0$ elsewhere. The expected value of X is

(a) $\frac{5}{6}$

(b) $\frac{6}{7}$ ✓ Find $\mathbb{E}X = \int x\, f(x)\, dx$.

(c) $\frac{6}{7}X^7$

(d) 1

Question - Variance

A continuous random variable X has pdf f given by

$$f(x) = 6x^5, \quad x \in (0,1)$$

and $f(x) = 0$ elsewhere. The variance of X is

(a) $\frac{6}{8}$

(b) $\sqrt{\frac{6}{8}}$

(c) $(\frac{6}{7})^2$

(d) $\frac{3}{196}$ ✓ $\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$

Question - Linear Transformations

Suppose X is a random variable with expectation 3 and variance 2. Let $Y = 4X + 2$. The expected value and variance of Y is:

(a) $\mathbb{E}Y = 12$ and $\text{Var}(Y) = 8$.

(b) $\mathbb{E}Y = 12$ and $\text{Var}(Y) = 10$.

(c) $\mathbb{E}Y = 14$ and $\text{Var}(Y) = 32$. ✓

(d) $\mathbb{E}Y = 14$ and $\text{Var}(Y) = 34$.

Question - Moment Generating Function

The random variable X has MGF

$$M(s) = \frac{e^s}{1 - 9s^2}, \quad s \in (-1/3, 1/3).$$

The expected value of X is

(a) 0
(b) 1 ✓ $\mathbb{E}X = M'(0)$
(c) 10
(d) 19

Question - Moment Generating Function

The random variable X has MGF

$$M(s) = \frac{e^s}{1 - 9s^2}, \quad s \in (-1/3, 1/3).$$

Define $Y = \frac{1}{3}(X - 1)$. The MGF of Y is

(a) $M_Y(s) = \frac{e^{2s}}{1-9s^2}, \quad s \in (-1/3, 1/3).$

(b) $M_Y(s) = \frac{e^{2s}}{1-s^2}, \quad s \in (-1/3, 1/3).$

(c) $M_Y(s) = \frac{1}{1-s^2}, \quad s \in (-1, 1).$ ✓

(d) $M_Y(s) = \frac{1}{1-s^2}, \quad s \in (-1/3, 1/3).$

$$
\begin{aligned}
\mathbb{E}(e^{sY}) &= \mathbb{E}(e^{\frac{s}{3}(X-1)}) = e^{-\frac{1}{3}s}\mathbb{E}(e^{(\frac{1}{3}s)X}) \\
&= e^{-\frac{1}{3}s}M_X(\tfrac{1}{3}s) = e^{-\frac{1}{3}s} \times \frac{e^{\frac{1}{3}s}}{1-9(s/3)^2} \\
&= \frac{1}{1-s^2}
\end{aligned}
$$

The transformation had the MGF of X evaluated at $s/3$. Since $s/3$ is in $(-1/3, 1/3)$ we can take s in $(-1, 1)$.