1.

$$A = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 15 \\ 1 & 5 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & 15 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 15 \\ 1 & 5-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)(5-\lambda) - 15 \Rightarrow \lambda = 0 \text{ or } \lambda = 8$$

So eigenvalues of A are 0 and 8 #

2.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 10 & 10^2 & 10^3 & 10^4 \end{bmatrix} = \begin{bmatrix} 1 & 10 & 10^2 & 10^3 & 10^4 \\ 2\times1 & 2\times10 & 2\times10^2 & 2\times10^3 & 2\times10^4 \\ 3\times1 & \cdots & & & \vdots \\ 5\times1 & \cdots & & & \vdots \\ 7\times1 & \cdots & & & 7\times10^4 \end{bmatrix}$$

we can find rank of A is 1. So the eigenvalues are zero expect one.

eigenvalues = sum of diagonal $= u^T v = 75321$ and four 0 #

3.

$B(B^TB)^{-1}B^T$ is a Projection matrix $\because$ rank$(B) = 2$, is full rank

$\therefore B^+ = (B^TB)^{-1}B^T$

$\min \|Bx - C\| = x^* = (B^TB)^{-1}B^T \cdot C$

$$Bx^* = B \cdot (B^TB)^{-1}B^T \cdot C$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/4 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = C$$

$\because \|Bx^* - C\| = 0$, $Bx^* = C = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ #

4.

$x^T B^T B x = (Bx)^T \cdot Bx = \|Bx\|^2$, when $x \neq 0$

This is always a positive matrix #

5.

$\because$ P is a Projection martix $\therefore P^2 = P$, eigenvalue is 0 or 1

$Px = \lambda x \Rightarrow P^2 x = \lambda^2 x \Rightarrow Px = \lambda^2 x \Rightarrow \lambda x = \lambda^2 x$

$\Rightarrow \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0$ or 1

$\because$ rank (B) $= 2$ $\therefore$ P has tow eigenvalues 1 and one eigenvalue 0

#.

6.

$P = B(B^T B)^{-1} B^T$

$x(k+1) = Px(k) - \frac{1}{2}x(k)$

$= (P - \frac{1}{2}I)\, x(k)$

$P - \frac{1}{2}I = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & -\frac{1}{4} \end{pmatrix} \Rightarrow$ the sum of diagonal is $\frac{1}{2}$

$\det((P - \frac{1}{2}I) - \lambda I) = \dfrac{(1-2\lambda)(4\lambda^2 - 1)}{8}$

$\Rightarrow \lambda = \frac{1}{2}$ or $-\frac{1}{2} \Rightarrow$ So eigenvalues of $(P - \frac{1}{2}I)$ are $\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$

we can get $(P - \frac{1}{2}I)$ is a stable matrix.

$\Rightarrow$ when k to infinite the $\|x(k)\|$ is to 0

$\Rightarrow \lim\limits_{x \to \infty} \|x(k)\| = 0 \Rightarrow$ So it doest depend on $x(0)$

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