1. $V_1^T V_2 = [1, 0, -1]\begin{bmatrix}1\\-1\\0\end{bmatrix} = 1$

So $V_1, V_2, V_3$ can not constitute a set of orthogonal vectors. #

2. $V_2^T \cdot V_3 = [1, -1, 0]\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$

So $V_2$ and $V_3$ orthogonal #

3.

$i=1$, $\tilde{q}_1 = V_1$ $\quad q_1^T V_2 = [\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}]\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}$ $\quad q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}$

$\|\tilde{q}_1\| = \sqrt{2}$

$i=2$, $\tilde{q}_2 = V_2 - (q_1^T V_2) q_1$ $\quad \|\tilde{q}_2\| = \frac{\sqrt{3}}{\sqrt{2}}$

$$= \begin{bmatrix}1\\-1\\0\end{bmatrix} - \frac{1}{\sqrt{2}} \times \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} \qquad q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{2}{\sqrt{6}} \times \begin{bmatrix}\frac{1}{2}\\-1\\\frac{1}{2}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2}\\-1\\\frac{1}{2}\end{bmatrix} \qquad = \begin{bmatrix}\frac{\sqrt{6}}{6}\\-\frac{\sqrt{6}}{3}\\\frac{\sqrt{6}}{6}\end{bmatrix} \qquad q_2^T V_3 = \frac{\sqrt{6}}{6}$$

$i=3$.

$$\tilde{q}_3 = V_3 - (q_1^T V_3) q_1 - (q_2^T V_3) q_2 \qquad q_1^T V_3 = -\frac{1}{\sqrt{2}}$$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix} - \left(-\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}\right) - \frac{\sqrt{6}}{6}\begin{bmatrix}\frac{\sqrt{6}}{6}\\-\frac{\sqrt{6}}{3}\\\frac{\sqrt{6}}{6}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix} \qquad q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \begin{bmatrix}\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\end{bmatrix}$$

$$Q = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3}\\ 0 & -\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3}\\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3}\end{bmatrix}$$

#

4. $\because Q$ is orthogonal $\therefore Q^{-1} = Q^{T}$

$$Q^{T} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & -\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \end{bmatrix} \#$$

5.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ assume $x$ is nullspace of $A$

$Ax = 0$ $\quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} [x \ y \ z] \Rightarrow \begin{cases} x - z = 0 \\ x - y = 0 \\ z = 0 \end{cases} \Rightarrow x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ #

6. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we can get $\text{rank}(A) = 3$ #

7. $aV_1 + bV_2 + cV_3$

$\begin{cases} a + b = 0 \\ -b = 0 \\ -a + c = 0 \end{cases} \Rightarrow a = b = 0$

So we can get $V_1, V_2, V_3$ is linearly independent.
So they are basis for $R^3$ #

8.

$i = 1$, $\tilde{q}_1 = V_3$, $q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$i = 2$, $\tilde{q}_2 = V_2 - (q_1^T V_2) q_1$ $\qquad q_1^T V_2 = [0, 0, 1] \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$

$= V_2 - 0 q_1$

$= V_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\qquad q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{\sqrt{2}}{2} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$

$i = 3$, $\tilde{q}_3 = V_1 - (q_1^T V_1) q_1 - (q_2^T V_1) q_2$ $\qquad q_1^T V_1 = [0, 0, 1] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -1$

$q_2^T V_1 = [\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{\sqrt{2}}{2}$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \qquad q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix} \quad \#$$

9. $B = M \times Q$

$\because$ $M, Q$ is orthogonal matrix.

$$B \cdot B^{-1} = (MQ)(MQ)^{-1} \qquad QQ^{-1} = I$$

$$MM^{-1} = I$$

$$= MQ(Q^{-1}M^{-1})$$

$$= I$$

$\therefore$ $B$ is orthogonal matrix. #

10. $\because$ $M, Q$ is orthogonal matrix

$\therefore$ $M^{-1} = M^T$, $Q^{-1} = Q^T$

$$G^{-1} = \begin{bmatrix} M^{-1} & 0 & 0 \\ 0 & M^{-1} & 0 \\ 0 & 0 & Q^{-1} \end{bmatrix} = \begin{bmatrix} M^T & 0 & 0 \\ 0 & M^T & 0 \\ 0 & 0 & Q^T \end{bmatrix} \quad \#$$