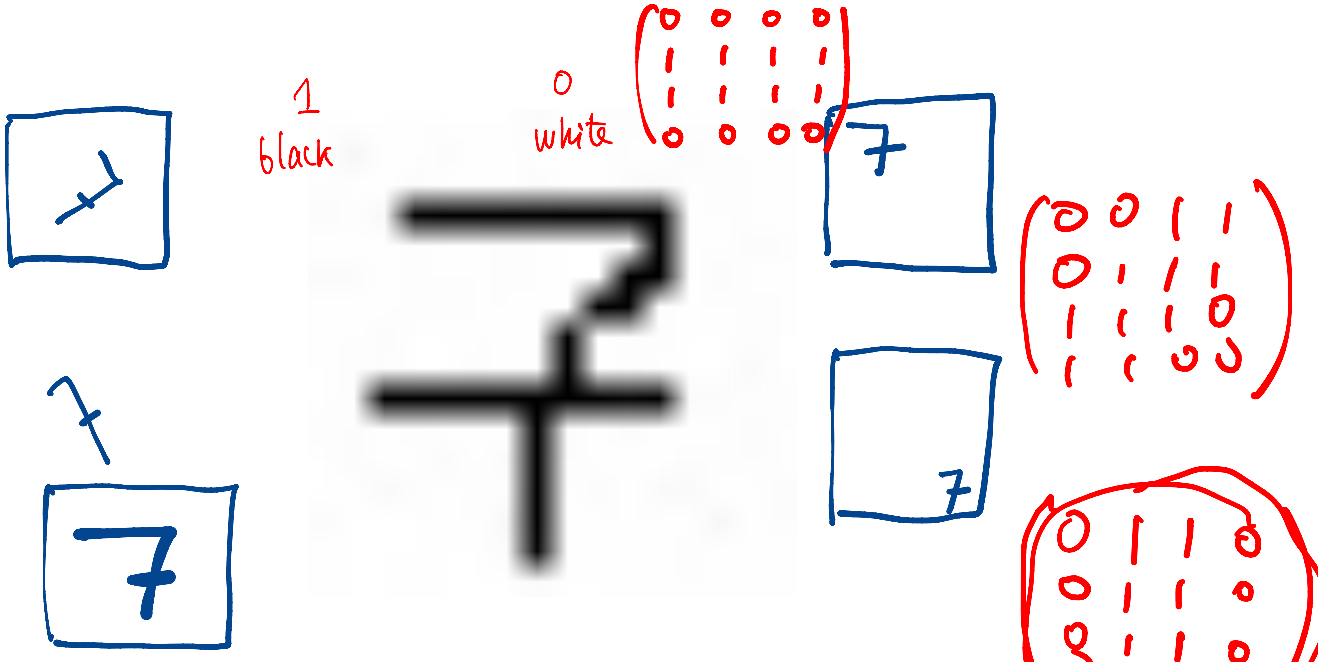


Figure 2: A 16 by 16 pixel black and white image of the number 7.

The matrix corresponding to the pixel data in Figure 2 is given in the following matrix B.

$$B = \left(\begin{array}{cccccccccccccccc}
0.99 & 1. & 1. & 0.99 & 1. & 1. & 1. & 0.97 & 1. & 0.98 & 0.99 & 1. & 1. & 0.99 & 1. & 0.98 \\
1. & 0.96 & 1. & 1. & 0.98 & 1. & 0.96 & 1. & 1. & 0.98 & 1. & 1. & 0.97 & 0.98 & 1. & 1. \\
0.99 & 1. & 1. & 0.98 & 1. & 1. & 1. & 0.98 & 1. & 1. & 0.97 & 0.98 & 1. & 0.99 & 0.98 & 0.98 \\
0.98 & 0.95 & 0.97 & 0.019 & 0. & 0. & 0. & 0.01 & 0. & 0. & 0.01 & 0.01 & 1. & 0.99 & 1. & 1. \\
1. & 1. & 1. & 0.99 & 1. & 1. & 0.97 & 1. & 1. & 1. & 1. & 0. & 0.99 & 1. & 0.97 & 0.98 \\
0.98 & 0.98 & 1. & 0.98 & 0.99 & 1. & 0.98 & 0.99 & 1. & 1. & 0. & 0. & 1. & 0.98 & 1. & 1. \\
1. & 1. & 1. & 0.99 & 0.98 & 0.99 & 1. & 1. & 1. & 0. & 0.01 & 1. & 0.96 & 1. & 0.99 & 0.96 \\
0.99 & 1. & 0.98 & 1. & 1. & 1. & 0.99 & 0.99 & 0. & 1. & 0.99 & 0.97 & 1. & 1. & 0.96 & 1. \\
1. & 0.99 & 0.99 & 1. & 1. & 0.99 & 1. & 0.99 & 0. & 1. & 0.99 & 1. & 1. & 1. & 0.98 & 1. \\
1. & 0.99 & 0.01 & 0. & 0. & 0.01 & 0. & 0.01 & 0.01 & 0. & 0.02 & 0. & 0.98 & 0.98 & 1. & 0.99 \\
0.99 & 0.98 & 1. & 0.99 & 1. & 1. & 1. & 0. & 0.98 & 1. & 0.98 & 1. & 1. & 1. & 0.98 & 0.99 \\
1. & 0.97 & 1. & 1. & 0.98 & 1. & 0.98 & 0.01 & 0.99 & 1. & 1. & 1. & 0.99 & 0.99 & 0.99 & 1. \\
1. & 1. & 0.98 & 0.99 & 1. & 0.99 & 1. & 0. & 1. & 0.98 & 1. & 0.98 & 0.98 & 1. & 1. & 0.98 \\
0.98 & 1. & 1. & 1. & 0.97 & 1. & 0.97 & 0.01 & 1. & 1. & 0.99 & 1. & 1. & 0.97 & 1. & 0.99 \\
1. & 1. & 1. & 1. & 1. & 1. & 0.98 & 0.02 & 0.99 & 0.99 & 1. & 1. & 0.96 & 1. & 1. & 1. \\
0.99 & 1. & 1. & 0.99 & 1. & 1. & 1. & 0.98 & 1. & 1. & 0.96 & 1. & 0.99 & 0.99 & 1. & 0.98
\end{array}\right)$$

Conversely, we can store data from a matrix as an image. This provides a means of storing a large data set a concise form. The matrix A is displayed as an image in Figure 3.

1.3 Matrix operations

Addition

The *sum* of two $m \times n$ matrices A and B is defined to be the $m \times n$ matrix $A + B$ with entries

$$(A + B)_{ij} = a_{ij} + b_{ij}$$

We add matrices element wise, as for vectors.

Addition is only defined between matrices of the same size.

Connection with inverses

An important property of determinants is the following.

Fact (product of determinants)

Let $A,\ B$ be $n \times n$ matrices. Then

$$|AB| = |A| \cdot |B| \,.$$

Theorem 1 (Invertible matrices)**.**

$$A \textit{ is invertible } \iff |A| \neq 0.$$

Proof $\quad \Longrightarrow$ If A is invertible, then

$$\begin{aligned} & I = AA^{-1} \\ \Rightarrow \quad & 1 = |I| = |AA^{-1}| = |A| \cdot |A^{-1}| \,, \\ & \text{So } |A| \neq 0. \end{aligned}$$

$\Longleftarrow$ Follows from below.

Remark 1. *It follows immediately from the proof that if A is invertible, then*

$$|A^{-1}| = \frac{1}{|A|},$$

i.e., the inverse of the determinant is the determinant of the inverse.

1.5 Vectors operations (in 2 and n dimensions)

- A *vector* quantity has both a magnitude and a direction. Force and velocity are examples of vector quantities.
- A *scalar* quantity has only a magnitude (it has no direction). Time, area and temperature are examples of scalar quantities.

geometric diff.

A vector is represented geometrically in the (x, y) plane (or in (x, y, z) space) by a directed line segment (arrow). The direction of the arrow is the direction of the vector, and the length of the arrow is proportional to the magnitude of the vector. Only the length and direction of the arrow are significant: it can be placed anywhere convenient in the (x, y) plane (or (x, y, z) space).

Vector: from one point, to another point

If P, Q are points, $\overrightarrow{PQ}$ denotes the vector from P to Q.

A vector $\boldsymbol{v}$ $=\overrightarrow{PQ}$ in the (x, y) plane may be represented by a pair of numbers

Coordinates

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \end{pmatrix}$$

$P = \begin{pmatrix} x_P \\ y_P \end{pmatrix}, \quad Q : \begin{pmatrix} x_Q \\ y_Q \end{pmatrix}$

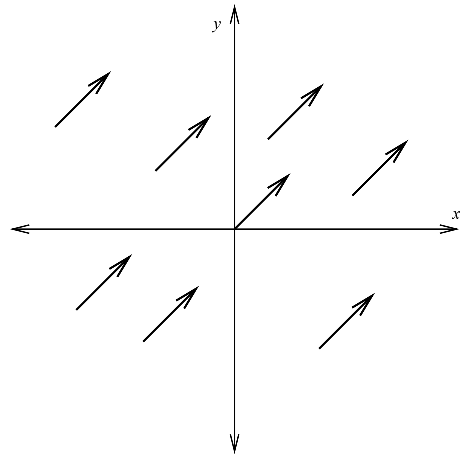


Figure 4: The vector of unit length at $45°$ to the x-axis has many representations.

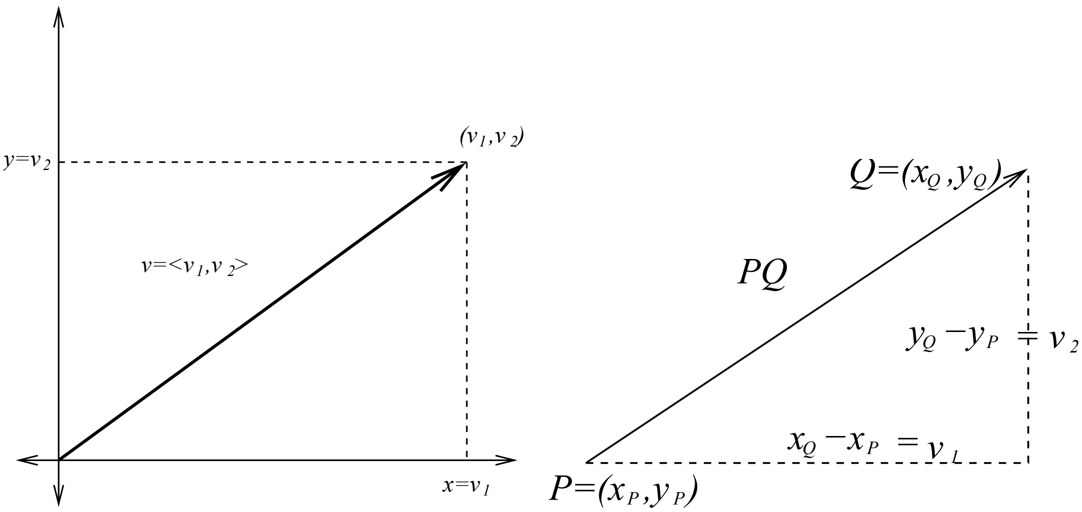


Figure 5: Geometric representation of a vector.

which is the same for all representations $\overrightarrow{PQ}$ of $\boldsymbol{v}$. We call v_1, v_2 the *components* of the vector $\boldsymbol{v}$.

We call the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the *zero vector*. It is denoted by $\mathbf{0}$.

Vectors will be indicated by bold lowercase letters $\boldsymbol{v}$, $\boldsymbol{w}$ etc. Writing by hand you may use $\underline{v}$ or $\vec{v}$ or $\widetilde{v}$.

Position vectors Special case, from the origin to a point

Let $P = (x_p, y_p)$ be a point in the (x, y) plane. The vector $\overrightarrow{OP}$, where O is the origin, is called the *position vector* of P. Obviously

$$\overrightarrow{OP}= \begin{pmatrix} x_p \\ y_p \end{pmatrix}.$$

origin: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

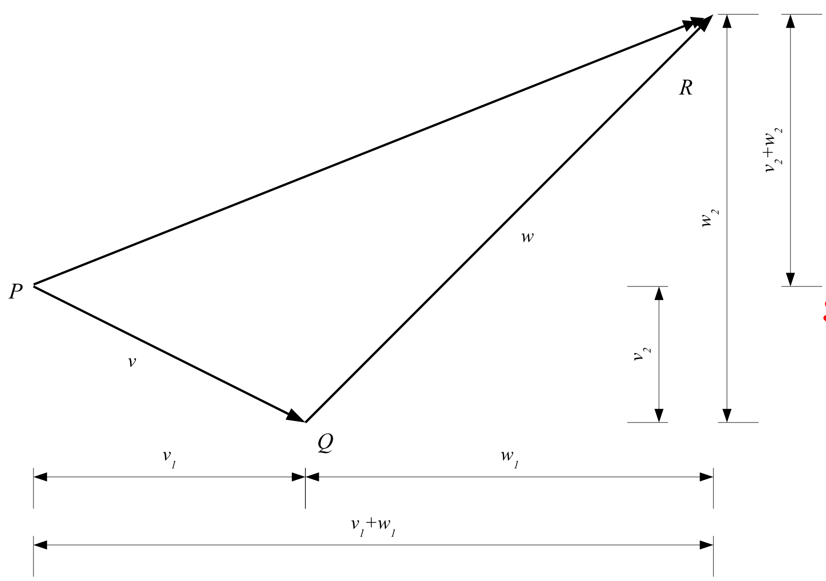
Norm (length, distance)

For vector $\boldsymbol{v} = \overrightarrow{PQ}$, the *norm*(or *length* or *magnitude*) of $\boldsymbol{v}$, written $\|\boldsymbol{v}\|$, is the distance between P and Q. Thus for $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we have

$$\boxed{\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2}}$$

Vector addition

We add vectors by the triangle rule.



$\overrightarrow{PQ} + \overrightarrow{QR}$
$= \overrightarrow{PR}$

Figure 6: $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

Consider the triangle PQR with $\boldsymbol{v} = \overrightarrow{PQ}$, $\boldsymbol{w} = \overrightarrow{QR}$. Then $\boldsymbol{v} + \boldsymbol{w} = \overrightarrow{PR}$; see Figure 6. In terms of components, if

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

then

$$\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}.$$

It follows from the component description that vector addition satisfies the following properties:

$$\begin{aligned} \boldsymbol{v} + \boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{v} && \text{(commutative law)} \\ \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) &= (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} && \text{(associative law)} \\ \boldsymbol{v} + \boldsymbol{0} &= \boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v} \end{aligned}$$

Scalar multiplication

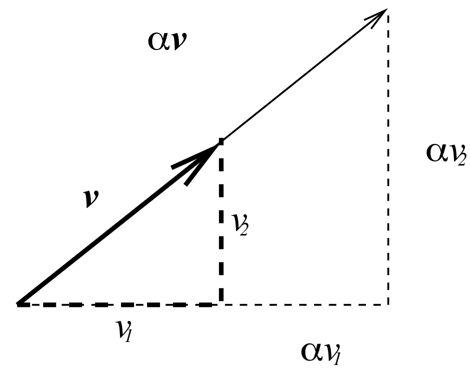


Figure 7: We can multiply the vector $\boldsymbol{v}$ by a number α (scalar).

If α is a real number (called a *scalar*), we define $\alpha\boldsymbol{v}$ to be the vector of norm

$$\|\alpha\boldsymbol{v}\| = |\alpha| \cdot \|\boldsymbol{v}\|$$

in the same direction as $\boldsymbol{v}$ if $\alpha > 0$, and opposite direction if $\alpha < 0$.

Using similar triangles it follows that if $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $\alpha\boldsymbol{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix}$.

If we multiply any vector $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ by zero we obtain the zero vector:

$$0 \cdot \boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \boldsymbol{0}$$

let $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

$\vec{w} \neq \vec{0}$ if and only if at least one of the numbers $w_1, \dots, w_n$ is not zero.

Unit vectors

A *unit vector* is a vector of norm 1. If $\boldsymbol{v} \neq \boldsymbol{0}$ is a vector, then

$$\frac{1}{\|\boldsymbol{v}\|}\,\boldsymbol{v}$$

is a unit vector in the direction of $\boldsymbol{v}$.

In particular

$j \neq 0,$

$\|i\| = 1$

$\|j\| = 1$

$$\boldsymbol{i} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{}\ i \neq 0, \quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

determine unit vectors along the x and y axes respectively.

For any vector $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we have

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j},$$

Hence we can decompose $\boldsymbol{v}$ into a vector $v_1\boldsymbol{i}$ along the x-axis and $v_2\boldsymbol{j}$ along the y-axis. The numbers v_1 and v_2 are called the *components* of $\boldsymbol{v}$ with respect to $\boldsymbol{i}$ and $\boldsymbol{j}$.

$$\begin{aligned}\|\boldsymbol{v}\| &= \sqrt{(10\sqrt{2}+5)^2+(10\sqrt{2}-5\sqrt{3})^2} \\ &= \sqrt{500+100\sqrt{2}(1-\sqrt{3})} \\ &\approx 19.91.\end{aligned}$$

To calculate the angle θ we use

$$\begin{aligned}\tan\theta &= \frac{10\sqrt{2}-5\sqrt{3}}{10\sqrt{2}+5} \\ \Rightarrow \theta &\approx 16^\circ\end{aligned}$$

So $\boldsymbol{v}$ = $19.9 kmh^{-1}$ at $E16^\circ N$.

Vectors in $\mathbb{R}^n$

We are familiar with vectors in two and three dimensional space, $\mathbb{R}^2$ and $\mathbb{R}^3$. These generalize to n dimensional space, denoted $\mathbb{R}^n$. A vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is specified by n components:

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad v_i \in \mathbb{R}.$$

We define addition of vectors and multiplication by a scalar component-wise. Thus if $\alpha \in \mathbb{R}$ is a scalar and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ is another vector then

$$\boldsymbol{v}+\boldsymbol{w} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{pmatrix} = \boldsymbol{w}+\boldsymbol{v}, \qquad \alpha\boldsymbol{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix} \quad \alpha \in \mathbb{R},$$

We define the *dot* (or *scalar*) *product* between $\boldsymbol{v}$ and $\boldsymbol{w}$ as

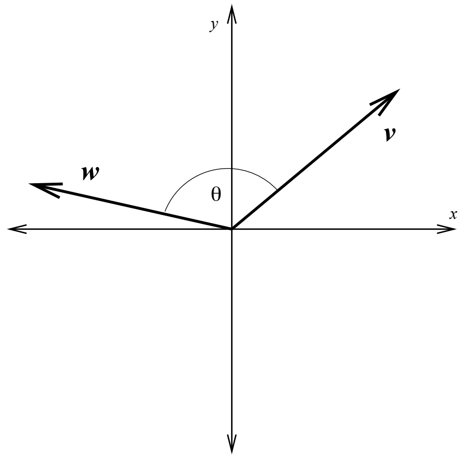
inner product

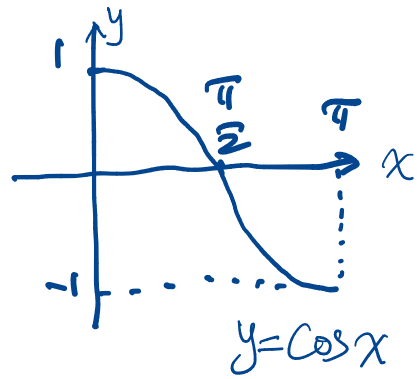
$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{w} &= v_1w_1+v_2w_2+\cdots\cdots+v_nw_n \\ &= \sum_{k=1}^{n} v_kw_k.\end{aligned}$$

The *length* or *norm* of the vector $\boldsymbol{v}$ is

$$\begin{aligned}\|\boldsymbol{v}\| &= \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} \\ &= \sqrt{v_1^2+v_2^2+\cdots+v_n^2}\end{aligned}$$

$\theta \in [0, \pi]$





The function $f(x) = \cos x$ is monotone on $x \in [0, \pi]$.

$\Rightarrow \cos^{-1} x$, the inverse function, exists.

Figure 10: θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

The *dot* (or *scalar* or *inner*) product of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, denoted by $\boldsymbol{v} \cdot \boldsymbol{w}$, is the *number* given by

must remember

$$\boxed{\boldsymbol{v} \cdot \boldsymbol{w} = \begin{cases} 0\,, & \text{if } \boldsymbol{v} \text{ or } \boldsymbol{w} = \boldsymbol{0} \\ \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \cos\theta\,, & \text{otherwise} \end{cases}}$$

$\vec{v} \neq 0$ and $\vec{w} \neq 0$.

where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

If $\boldsymbol{v}, \boldsymbol{w} \neq \boldsymbol{0}$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ then $\boldsymbol{v}$ and $\boldsymbol{w}$ are said to be *orthogonal* or *perpendicular*.

If $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ and $\boldsymbol{w} = w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k}$ are two vectors, then $\boldsymbol{v} \cdot \boldsymbol{w}$ is given by:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3,$$

In particular, for $\boldsymbol{v} \in \mathbb{R}^3$,

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = v_1^2 + v_2^2 + v_3^2.$$

Example 12. *Find the angle θ between the vectors:*

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$$\boxed{\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\|}.}$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = 1 \cdot 3 - 5 \cdot 3 + 4 \cdot 3 = 0.$$

So $\cos\theta = 0$, so the vectors are perpendicular, i.e. $\theta = \dfrac{\pi}{2}$.

Example 13. *If $P = (2, 4, -1)$, $Q = (1, 1, 1)$, $R = (-2, 2, 3)$, find the angle $\theta = PQR$.*

Find vectors joining Q to P, and Q to R:

$$\overrightarrow{QP} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix},$$

$$\overrightarrow{QR} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}.$$

$$\therefore \quad \left\|\overrightarrow{QP}\right\| = \left\|\overrightarrow{QR}\right\| = \sqrt{1+9+4} = \sqrt{14}.$$

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 + 3 - 4 = -4$$

$$\therefore \quad \cos\theta = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left\|\overrightarrow{QP}\right\| \cdot \left\|\overrightarrow{QR}\right\|} = \frac{-4}{14} \Rightarrow \theta \simeq 107^\circ.$$

The projection formula

orthogonal projection

Fix a vector $\boldsymbol{v}$. Given another vector $\boldsymbol{w}$ we can write $\boldsymbol{w}$ as

$\vec{v} \neq 0.$

$\vec{w}$ can either be zero, or not zero vector.

$$\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2$$

where:

- $\boldsymbol{w}_1$ is ~~in the direction of~~ parallel to $\boldsymbol{v}$ meaning that $\vec{w}_1 = \alpha \vec{v}$, $\alpha \in \mathbb{R}$
- $\boldsymbol{w}_2$ is perpendicular to $\boldsymbol{v}$. meaning: $W_2 \cdot v = 0$

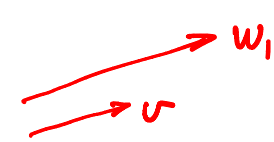


Fig 1: W_1 parallel to v

See Figure 11. We want to find $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ in terms of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Since $\boldsymbol{w}_1$ is in the direction of $\boldsymbol{v}$, let

$$\boldsymbol{w}_1 = \alpha\boldsymbol{v} \qquad \text{for some } \alpha \in \mathbb{R}.$$

Then $\boldsymbol{w}_2 = \boldsymbol{w} - \underbrace{\alpha\boldsymbol{v}}_{\vec{w}_1}$.

We need to choose α to make $\boldsymbol{v}$ and $\boldsymbol{w}_2$ orthogonal.

$$0 = \boldsymbol{w}_2 \cdot \boldsymbol{v} = (\boldsymbol{w} - \alpha\boldsymbol{v}) \cdot \boldsymbol{v} = \boldsymbol{w} \cdot \boldsymbol{v} - \alpha\boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{w} \cdot \boldsymbol{v} - \alpha\|\boldsymbol{v}\|^2.$$

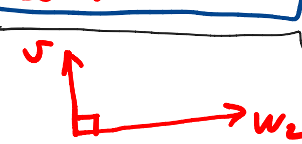


Fig 2 W_2 perpendicular to v

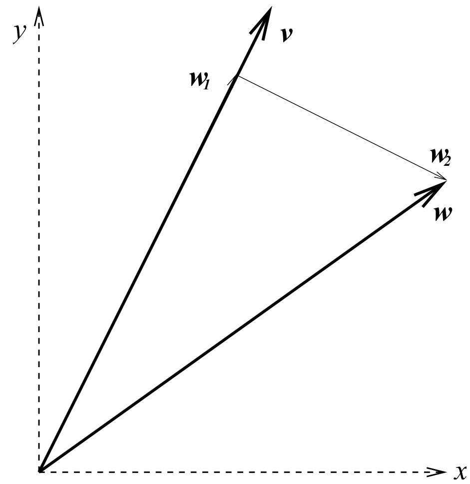


Figure 11: $\boldsymbol{w}$ can be decomposed into a component $\boldsymbol{w}_1$ in the direction of $\boldsymbol{v}$ and a component $\boldsymbol{w}_2$ perpendicular to $\boldsymbol{v}$.

$$\cos \angle w, v = \frac{w \cdot v}{\|w\| \cdot \|v\|}$$

when $w, v \neq 0$

So we need

$$\alpha = \frac{\boldsymbol{w} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|^2}.$$

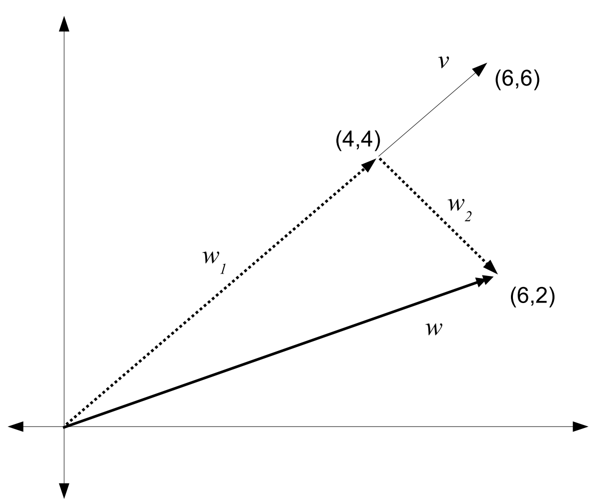
We have derived the *projection formula:*

$$\boldsymbol{w}_1 = \frac{(\boldsymbol{w} \cdot \boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v}\ , \qquad \boldsymbol{w}_2 = \boldsymbol{w} - \frac{(\boldsymbol{w} \cdot \boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v}. \tag{1}$$

Example 14. *Find the projection of* $\boldsymbol{w} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ *onto* $\boldsymbol{v} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$.

$$\boldsymbol{w}_1 = \frac{\boldsymbol{w} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \frac{36 + 12}{6^2 + 6^2}\begin{pmatrix} 6 \\ 6 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

$$\boldsymbol{w}_2 = \boldsymbol{w} - \boldsymbol{w}_1 = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2. \end{pmatrix}$$



Properties of dot product

If $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then

(1) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.

(2) $\boldsymbol{v} \cdot (\boldsymbol{w} + \boldsymbol{u}) = \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{u}$.

(3) $(\alpha \boldsymbol{v}) \cdot \boldsymbol{w} = \alpha(\boldsymbol{v} \cdot \boldsymbol{w}) = \boldsymbol{v} \cdot (\alpha \boldsymbol{w})$.

(4) $\boldsymbol{v} \cdot \boldsymbol{v} \geq 0$ and $\boldsymbol{v} \cdot \boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$.

1.7 Application

singular value decomposition

Concepts missing:

linear dependency, ranks of matrices

eigen decomp. SVD, range & null space of matrix,

trace, projection & orthogonal projection.

quadratic forms, positive (semi-) definite matrices, ...

Ref: Gilbert Strang's books

break: 17:48–17:58.

2 Sets, counting and cardinality

The terms *set* and *element* ~~won't~~ can't be formally defined here. Roughly speaking, a **set** will be a collection of objects called **elements**, and given any element and any set, we should be able to say whether the element belongs to the set or not.

2.1 Sets

- Recall that the notation $a \in S$ means that a is an element of the set S, or a belongs to S.
- We can list elements in a set using braces or curly brackets: $\{x_1, x_2, x_3\}$. The order in which we list the elements of a set is irrelevant, so $\{x_1, x_2, x_3\} = \{x_3, x_1, x_2\}$, etc.
  The number of times that each element is listed is also irrelevant; $\{a, b, a\} = \{a, b\}$ for example.
- The **empty** set is the set containing NO elements, denoted by $\emptyset$.
  (We'll see soon that "the" empty set is the right terminology, because we'll show that there is only one empty set.)
- For any set A, the **cardinality** of A is the number of elements in the set A; we shall denote this as $|A|$. $|\{0,1\}| = 2$ $\quad |(0,1]| = \infty$

Note that in *Mathematica*, we can calculate with sets provided we use the Union function.

```
Union[{a,b,a}] = = Union[{a, b}]
MemberQ[{2,{2}},2]
```

Otherwise these will behave like vectors.

Example 15. **(a)** *How many elements does the set* $\{2, 2, \{2\}\}$ *have?*

(b) *Is it true that* $\{1, 1, 2\} = \{1, 2\}$ *?*

(c) *Is it true that* $1 \in \{1\}$ *?*

(d) *Is it true that* $1 \in \{\{1\}\}$ *?*

$\{\{1\}\}$: a set, of which the only element is the set $\{1\}$.

Example: *Describe the following sets in words.*

(a) $\{1, 2, \ldots, 100\}$

(b) $\{x \in \mathbb{R} : x > 0\}$

(c) $\{y \in \mathbb{Z}^+ : -3 \leqslant y \leqslant 3\}$

- *If A and B are any sets, A is called a* **subset** *of B, written $A \subseteq B$, if and only if every element of A is also an element of B.*

$B \supseteq A$

Set

Concept, not a definition

Set: a collection of objects satisfying:

1. Rigorous membership: whether an object is in one set is well defined.
2. Uniqueness: all the objects in a set are distinct.
3. No order: there is not any order among the objects in a set.

Examples:

- $A = \{$All the students in the class MATH7501 in Fall 2023$\}$
- $B = \{\sqrt{2}, 34, \pi\}$
- $C = \{x : 2x^2 - 5x - 3 = 0\} = \{3, -\frac{1}{2}\}$
- $D = \{x : -1.5 < x \leq 3\} = (-1.5, 3]$

Counterexamples: *The following examples are NOT sets*

- $\{$All the nice people$\}$ (ambiguous membership: Am I nice? Always so?)
- $\{1, 3, 3, 4\}$ (The number "3" appears twice.)

- *For sets A and B, we say A is a* **proper subset** *of B if and only if $A \subseteq B$ and $A \neq B$. So A is a proper subset of B if and only if every element of A is also an element of B, and there is some element of B which is not in A.*

- *For sets A and B, we say sets A and B are* **equal**, *$A = B$, if and only if every element of A is in B, and also every element of B is in A.*

  So $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

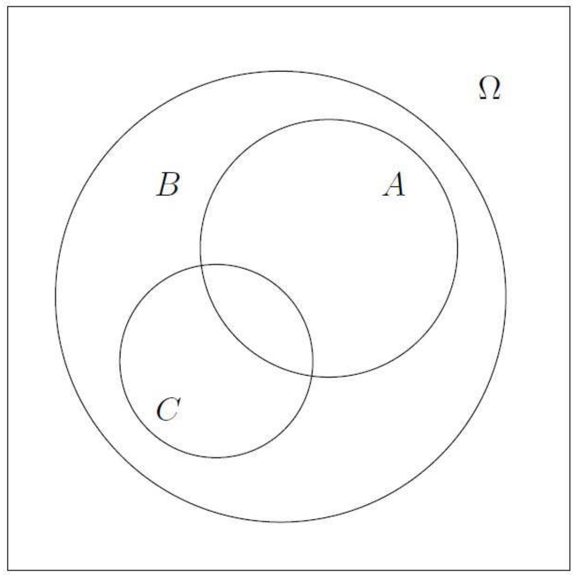


Figure 12: A Venn diagram with universe Ω.

Example 16. *Suppose $A = \{a, b, c, d\}$, $B = \{a, b, e\}$ and $C = \{a, b, c, d, e\}$. Answer the following, and give reasons.*

(a) *Is it true that $B \subseteq A$?* No, since $e \notin A$

(b) *Is it true that $A \subseteq C$?* yes

(c) *Is A a proper subset of C?* yes since $A \subseteq C$, $e \in C$, but $e \notin A$.

(d) *Is it true that $B \subseteq B$?* yes & trivial.

Example 17. *Draw a Venn diagram to represent the relationship between the following sets: $A = \{1, 2, 3\}$, $B = \{1, 4\}$, $C = \{2, 3\}$.*

Example 18. *True or false?*
(a) $\{4\} \in \{1, \{3\}, 4\}$

(b) $\{4\} \subseteq \{1, \{3\}, 4\}$

(c) $\{3\} \in \{1, \{3\}, 4\}$

(d) $1 \subseteq \{1, \{3\}, 4\}$

Exam requirement: verify or disprove A=B.

Note the set notation used here; verbally read ":" as "such that".

Example 19. *Let* $A = \{x \in \mathbb{Z} : x = 4p - 1$ *for some* $p \in \mathbb{Z}\}$,
$B = \{y \in \mathbb{Z} : y = 4q - 5$ *for some* $q \in \mathbb{Z}\}$. *Prove that* $A = B$.

We investigate some **operations** on sets now, obtaining new sets from existing sets. Let U be some **universal set**, depending on the context.
(So U could perhaps be $\mathbb{R}$ in some contexts.)

In the following, suppose A and B are some subsets of a universal set U.

- The **union** of sets A and B, denoted $A \cup B$, is the set of all elements x in U such that $X \in A$ *or* $X \in B$ *(or both)*.
- The **intersection** of sets A and B, denoted $A \cap B$, is the set of all elements x in U such that $x \in A$ *and* $x \in B$.
- The **set difference** of B minus A, denoted $B - A$, and sometimes also called the relative complement of A in B, is the set of all x in U such that $x \in B$ and $x \notin A$. Some texts write $B \setminus A$ instead of $B - A$.
- The **complement** of A, denoted A^c, is the set of all x in U such that $x \notin A$.

Summarising the above:

$$\begin{aligned} A \cup B &= \{x \in U : x \in A \text{ or } x \in B\}; \\ A \cap B &= \{x \in U : x \in A \text{ and } x \in B\}; \\ B - A &= \{x \in U : x \in B \text{ and } x \notin A\}; \\ A^c &= \{x \in U : x \notin A\} \end{aligned}$$

Example 20. *Let the universal set be* $\{1, 2, \ldots, 10\}$, *and let*
$A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8, 10\}$ *and* $C = \{1, 3, 5, 7, 9\}$.
Write down the following sets:

(a) $A \cap B =$ $\{2,4\}$

(b) $A \cup B =$ $\{1,2,3,4,6,8,10\}$

(c) $B \cup C =$

(d) $B - A =$ $\{6,8,10\}$

(e) $A - C =$ $\{2,4\}$

(f) $B^c =$ $\{1,3,5,7,9\}$

(g) $A^c =$ $\{5,6,7,8,9,10\}$

(h) $A^c \cup B =$ $\{2,4,5,6,7,8,9,10\}$.

Let $A = \{x \in \mathbb{Z} : x = 4p - 1, \text{ for some } p \in \mathbb{Z}\}$.

$B = \{y \in \mathbb{Z} : y = 4q - 5 \text{ for some } q \in \mathbb{Z}\}$

Is $A = B$?

Sol: We check if $A \subseteq B$ and $B \subseteq A$.

① $\forall x \in A$. $x = 4p - 1 = 4(p+1) - 5$ for integer $p+1$.

for any

$\Rightarrow x \in B$. $\Rightarrow A \subseteq B$.

②. $\forall x \in B$, $x = 4q - 5 = 4(q-1) - 1$ for integer $q-1$. $\Rightarrow x \in A$. $\Rightarrow B \subseteq A$

Therefore, $A = B$.

Question: Let $A = \{(x, y) : 0 \leq x \leq 1,\ x \leq y \leq 1\}$.

$B = \{(x, y) : 0 \leq y \leq 1,\ 0 \leq x \leq y\}$.

Prove that $A = B$.

Definition 7. *Let n be a positive integer (so $n \in \mathbb{Z}^+$), and let $x_1, x_2, \ldots, x_n$ be n not necessarily distinct elements. The* **ordered n-tuple***, denoted $(x_1, x_2, \ldots, x_n)$, consists of the n elements with their ordering: first x_1, then x_2, and so on up to x_n. (Note round brackets, not braces)*

e.g., $(1,1,2,4)$ is a 4-tuple.

An ordered 2-tuple is an **ordered pair**.

An ordered 3-tuple is an **ordered triple**.

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal if and only if $x_i = y_i$ for all i with $1 \leqslant i \leqslant n$.

Example 21. *Is it true that $(3, 1) = (1, 3)$?* no.

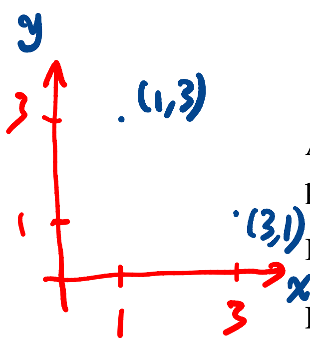
Example 22. *If $((-2)^2,\ y,\ \sqrt{9}) = (4,\ 3,\ z)$, find y and z.*

Definition 8. *Given two sets A and B, the* **Cartesian product** *of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. (Note the word "all") So*

$$A \times B \ = \ \{(a, b) : a \in A \ \text{ and } \ b \in B\}.$$

Similarly

$$A_1 \times A_2 \times \cdots \times A_n \ = \ \{(a_1, a_2, \ldots, a_n) : a_i \in A_i,\ 1 \leqslant i \leqslant n\}.$$



A Cartesian product with which you are probably already familiar is the xy-plane. The points in the real xy-plane are the elements of the set $\mathbb{R} \times \mathbb{R}$.

If you plot $(3, 1)$ and $(1, 3)$ you'll see these ordered pairs are not equal.

Example 23. *Let $A = \{-1, 0, 1\}$ and $B = \{x, y\}$. Write out the set $A \times B$. How many elements are in $A \times B$?*

$\{(-1, x), (-1, y), (0, x), (0, y), (1, x), (1, y)\}$

Recall that we denote a set with NO elements in it by $\emptyset$.

what is $\emptyset \times A = \emptyset$

- The empty set is a subset of every set. So if S is any set, we have $\emptyset \subseteq S$.
- Every set is a subset of itself. So if S is any set, we have $S \subseteq S$.

Theorem 3. *The empty set $\emptyset$ is unique.*

Proof We use a contradiction argument. So suppose that $\emptyset_1$ and $\emptyset_2$ are each sets with no elements. Since $\emptyset_1$ has no elements, it is a subset of $\emptyset_2$, that is, $\emptyset_1 \subseteq \emptyset_2$. Also since $\emptyset_2$ has no elements, we have $\emptyset_2 \subseteq \emptyset_1$. Thus $\emptyset_1 = \emptyset_2$, by definition of set equality.

Partitions of sets

[break: 1843~1853].

- Two sets are called **disjoint** if and only if they have no elements in common.

  So A and B are disjoint if and only if $A \cap B = \emptyset$.

having nothing in common.

- Sets $A_1, A_2, \ldots, A_n$ are **mutually disjoint** (or **pairwise disjoint**) if and only if for all pairs of sets A_i and A_j with $i \neq j$, their intersection is empty; that is, if and only if $A_i \cap A_j = \emptyset$ for all $i, j = 1, 2, \ldots, n$ with $i \neq j$.

- A collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set A if and only if

  (1) $A = A_1 \cup A_2 \cup \cdots \cup A_n$ and

  (2) the sets $A_1, A_2, \ldots, A_n$ are mutually disjoint.

  like cutting a pizza into 8 slices.

Example 24. *Determine whether the following statements are true or false.*

(a) $\emptyset = \{\emptyset\}$

ϕ: empty set. $\{\phi\}$: a set with one element ϕ, so $\{\phi\}$ is not empty.

(b) $A \cup \emptyset = A$

(c) $A \cap A^c = \emptyset$

(d) $A \cup A^c = \emptyset$

(e) $A \cap \emptyset = \emptyset$

(f) $(A - B) \cap B = \emptyset$

(g) $\{a, b, c\}$ *and* $\{d, e\}$ *are disjoint sets.*

(h) $\{1, 2\}$, $\{5, 7, 9\}$ *and* $\{3, 4, 5\}$ *are mutually disjoint sets.*

Exam

Example 25. *Let*

$0 \notin A_1 \cup A_2$

$$A_1 = \{n \in \mathbb{Z} : n < 0\}, \qquad A_2 = \{n \in \mathbb{Z} : n > 0\}.$$

Is $\{A_1, A_2\}$ a partition of $\mathbb{Z}$? If so, explain why; if not, see if you can turn it into a partition with a small change. → no, because $A_1 \cup A_2 \neq \mathbb{Z}$

Example 26. *Find a partition of $\mathbb{Z}$ into four parts such that none of the four parts is finite in size.* $A = \{2^n : n \geq 1\}$, $B = \{3^n : n \geq 1\}$, $C = \{5^n : n \geq 1\}$, $D = \mathbb{Z} - (A \cup B \cup C)$

Definition 9. *Given a set X, the* **power set** *of X is the set of all subsets of X. It is denoted by $\mathcal{P}(X)$.*

Example 27. *If $B = \{1, 2, 3\}$, write down the set $\mathcal{P}(B)$.*

Example 28. *Let $X = \emptyset$. Write down $\mathcal{P}(X)$, and $\mathcal{P}(\mathcal{P}(X))$.*

If $|S| = n$, how many elements does the power set $\mathcal{P}(S)$ have?

Subset relations

- For all sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

- For all sets A and B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

- For all sets A, B and C,

  if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

In the following, U denotes some universal set, and A, B and C are any subsets of U.

Set Identities

- $A \cup B = B \cup A$ and $A \cap B = B \cap A$. **(commutative)**
- $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$. **(associative)**
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. **(distributive)** try to prove this.
- $A \cup \emptyset = A$; $A \cap U = A$; $A \cup A^c = U$; $A \cap A^c = \emptyset$;
  $(A^c)^c = A$; $A \cup A = A$; $A \cap A = A$.
- $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. **(De Morgan's laws)**

To prove $X = Y$:

First show that $X \subseteq Y$, and then that $Y \subseteq X$.

So take any x in X, and show that then $x \in Y$. This shows $X \subseteq Y$.

Next take any $y \in Y$, and show that then $y \in X$. This shows $Y \subseteq X$.

From these results we conclude that $X = Y$.

Note that: $(A \cap B)^c = A^c \cup B^c$ (one of De Morgan's laws).

Use De Morgan's law to show that: For all sets A and B, $(A \cap B)^c = A^c \cup B^c$.

First, let $x \in (A \cap B)^c$. Then $x \notin A \cap B$.

So it is *false* that "x is in A and x is in B".

Thus $x \notin A$ or $x \notin B$ (since $\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$).

So $x \in A^c$ or $x \in B^c$.

Therefore $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

To complete this example, we must show that $A^c \cup B^c \subseteq A \cap B)^c$:

- Note that $(x, y) \in A \times B$ is equivalent to "$(x \in A)$ and $(y \in B)$".

Example 29. *If $A \subseteq C$ and $B \subseteq D$, prove that $A \times B \subseteq C \times D$.*

$(A \cup B)^c = A^c \cap B^c$.

proof: ① For any $x \in (A \cup B)^c \Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$. $\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \in A^c \cap B^c \Rightarrow (A \cup B)^c \subseteq A^c \cap B^c$

② For any $x \in A^c \cap B^c \Rightarrow x \in A^c$ and

$x \in B^c \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin A \cup B \Rightarrow x \in (A \cup B)^c$.

$\Rightarrow A^c \cap B^c \subseteq (A \cup B)^c$

Therefore, $(A \cup B)^c = A^c \cap B^c$ ※

2.2 Complex numbers

why? Solving equations $x^2+1=0$.

You may have heard that it is not possible to take the square root of a negative number. This is false. More precisely,

> The square root of a negative number is not real; e.g. $\sqrt{-17} \notin \mathbb{R}$.

how?

We deal with such numbers differently and refer to them as **imaginary numbers**. We use i to denote the square root of -1. For example, $\sqrt{-17} = \sqrt{17}i$, where $i^2 + 1 = 0$. A natural extension of the set of real numbers $\mathbb{R}$ is the set of **complex numbers**, $\mathbb{C}$, given by

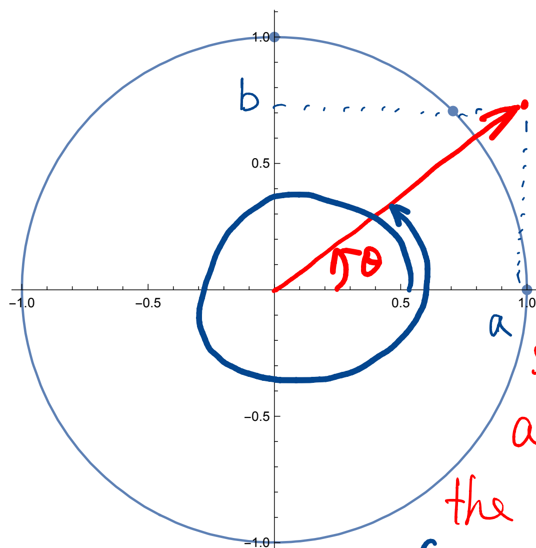
what?

$$\mathbb{C} = \{a + bi \ : \ a, b \in \mathbb{R}, i^2 + 1 = 0\}. \tag{2}$$

This is an extension of degree 2 over the real numbers, essentially meaning that this is structurally equivalent to the Cartesian product $\mathbb{R} \times \mathbb{R}$. To facilitate the description of complex numbers and their properties, we often represent these numbers geometrically as points in a 2-dimensional space with the real part of $z = a + bi$, a on the horizontal axis and the imaginary part b on the vertical axis. See Figure 13.

modulus: $|z| = \sqrt{a^2+b^2}$

Conjugate $\bar{z} = a - bi$



$(a+bi) = z$

Arguments of z: $\theta, \theta \pm 2\pi, \theta \pm 4\pi, \theta \pm 6\pi, \dots$

principal value of argument: the argument in $(-\pi, \pi]$ (Someone prefers $[0, 2\pi)$)

Figure 13: The complex plane where we plot $a + bi$, the circle $a^2 + b^2 = 1$, and the three complex numbers $1 + 0i$, $0 + 1i$, and $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, all on the unit circle.

Definition 10. *The modulus and argument of the complex number $z = a + bi$ are respectively $|z| = \sqrt{a^2 + b^2}$, the distance from 0 also the absolute value of z, and the angle θ between the position of the point (a, b) and the real axis. We denote $\Re(z) = a$ and $\Im(z) = b$, the real and imaginary parts of z respectively. The* **conjugate** *of $z = a + bi$ is $\overline{z} = a - bi$.*

$R(z)=a$, $I(z)=b \in \mathbb{R}$

Complex numbers have the following important properties among several more:

1. $z_1 = z_1$ if and only if $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$;

$$a+bi=re^{i\theta} \qquad r\geqslant 0 \qquad e=2.71828\cdots$$

$$e^{i\theta}=\cos\theta+i\sin\theta$$

$$e^{x}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

$$e^{i\theta}=1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+i\frac{\theta^5}{5!}-\frac{\theta^6}{6!}$$

2. $z + \overline{z} = 2\Re(z)$;

3. $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$;

4. Euler's formula: If $r = |z|$ and θ is the argument of z, then $z = re^{i\theta} = r\cos\theta + ir\sin\theta$. This is also known as polar form.

An important result in algebra states essentially that are all we need when solving equations

Theorem 4 (Fundamental theorem of algebra)**.** *Every non-constant polynomial $f(x)$ with complex coefficients has at least one complex root. Further, if the degree of $f(x)$ is equal to n, then there are exactly n such roots of $f(x)$, provided we count these as distinct when they are repeated.*

Example 30. *Find all of the roots in $\mathbb{C}$ of the polynomial $f(x) = x^5 - 3x^4 - 7x^3 - 13x^2 - 9x - 5$, determine the number of such roots according to Theorem 4. How many real roots of $f(x)$ are there? Can $f(x)$ be factorised over $\mathbb{R}$? Express each of the complex roots of $f(x)$ in polar form.*

$x^2 - 2x + 1 = 0$

root = 1, w/ multiplicity 2

2.3 Counting and elementary combinations

In this chapter we give an introduction to counting and probability. At first thought counting may seem a fairly simple exercise. However, when we are counting certain things it can turn out to be quite an involved process.

Counting subsets of a set: combinations

In this section we'll investigate questions of the following form.

> Given a set S with n elements, how many subsets of size r can be chosen from S?

- Let n and r be nonnegative integers with $r \leqslant n$. An r**-combination** of a set of n elements is a subset of the n elements of size r.

  The symbol $\binom{n}{r}$, which is read "n **choose** r," denotes the number of subsets of size r (so the number of r-combinations) which can be chosen from a set of n elements.

  Alternative notation for $\binom{n}{r}$ includes: $C(n,r)$, or ${}_nC_r$, or $C_{n,r}$, or ${}^nC_r$.

Theorem 5. *The number of subsets of size r (or r-combinations) which can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula*

$$\binom{n}{r} = \frac{P(n,r)}{r!}$$

$P(n,r) = \frac{n!}{(n-r)!}$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where n and r are nonnegative integers with $r \leqslant n$.

- An **ordered selection** of r elements from a set of n elements is an r-permutation of the set (use $P(n, r)$).

- An **unordered selection** of r elements from a set of n elements is the same as a subset of size r, or an r-combination of the set (use $\binom{n}{r}$).

Example 31. *Calculate the value of:*

(a) $\binom{9}{3}$; $\frac{9!}{3!(9-3)!}$

(b) $\binom{200}{198}$;

(c) $\binom{8}{4}$.

Example 32. *A student has a maths assignment with five questions on it, but only has enough time to complete three of them. How many combinations of questions could the student complete?* $\binom{5}{3} = \frac{5!}{3!\,2!} = 10$

Example 33. *Imagine a word game in which a sentence has to be made using three words drawn out of a bag containing ten words.*

(a) *How many possible ways are there to choose three words from a bag of ten words?*

(b) *Suppose that the rules of the game change so that the sentence has to use the three words in the order in which they are chosen. How many possible combinations are there now?*

(c) *What is the relationship between the answers to parts (a) and (b)?*

Example 34. *In a game of straight poker, each player is dealt five cards from an ordinary deck of 52 cards, and each player is said to have a* 5*-card hand.*

(a) *How many* 5*-card poker hands contain four cards of the one denomination? (So e.g. four aces, or four threes, etc.)*

(b) *Find the error in the following calculation of the number of* 5*-card poker hands which contain at least one jack. Then calculate the true number of* 5*-card poker hands which contain at least one jack.*

Consider this in two steps:

Choose one jack from the four jacks.

Choose the other four cards in the hand.

Thus there are $\binom{4}{1}\binom{51}{4} = 999600$ *such hands.*

$n \geq 1$: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ $\in$ exam

Leibniz rule: f, g sufficiently differentiable

$$\frac{d^n}{dx^n}\left(f(x)g(x)\right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d^k}{dx^k} f(x)\right)\left(\frac{d^{n-k}}{dx^{n-k}} g(x)\right)$$

Watch out for the common error of counting things twice.

Wi will now work with some useful relationships involving $\binom{n}{r}$.

$n-(n-r)=r$

Exam

Theorem 6. *Let n and r be positive integers with $r \leqslant n$. Then*

$$\binom{n}{r} = \binom{n}{n-r}.$$

$LHS = \dfrac{n!}{r!(n-r)!} = RHS$

~~Example 35.~~ *Given that* $\binom{n}{2} = \dfrac{n(n-1)}{2}$, *find an expression for* $\binom{x+3}{x+1}$.

Pascal's Formula Let n and r be positive integers with $r \leqslant n$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}. \tag{3}$$

Proof: $RHS = \dfrac{n!}{(r-1)!(n-r+1)!} + \dfrac{n!}{r!(n-r)!}$

$= \dfrac{n!\,r}{r!(n-r+1)!} + \dfrac{n!(n-r+1)}{r!(n-r+1)!}$

$= \dfrac{n!(n+1)}{r!(n+1-r)!} = \dfrac{(n+1)!}{r!(n+1-r)!} = \binom{n+1}{r}$

Example 36. *Use Pascal's formula* (3) *to calculate:*

(a) $\binom{7}{5} + \binom{7}{6}$

(b) $\binom{9}{6} + \binom{9}{5}$

(c) $\binom{4}{2} + \binom{4}{3}$

(d) $\binom{6}{1} + \binom{6}{2}$

2.4 Cardinality

In this section we shall investigate the concept of the *cardinality* of a set and show that there are *infinite* sets that are larger than other infinite sets. This concept has applications in determining what can and what cannot be computed on a computer.

- A **finite** set is either one which has no elements at all, or one for which there exists a one-to-one correspondence (bijection) with a set of the form $\{1, 2, 3, \ldots, n\}$ for some fixed positive integer n.

- An **infinite** set is a nonempty set for which there does *not* exist any one-to-one correspondence (bijection) with a set of the form $\{1, 2, 3, \ldots, n\}$ for any positive integer n.

- Let $\mathcal{A}$ and $\mathcal{B}$ be any sets. Sets $\mathcal{A}$ and $\mathcal{B}$ are said to have the **same cardinality** if and only if there exists a one-to-one correspondence (bijection) from $\mathcal{A}$ to $\mathcal{B}$.

  In other words, $\mathcal{A}$ has the **same cardinality** as $\mathcal{B}$ if and only if there is a function f from $\mathcal{A}$ to $\mathcal{B}$ that is one-to-one (injective) and onto (surjective).

Denote $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ the set of all the positive integers.

Example 1: there is a bijection between $\mathbb{Z}^+$ and $\mathbb{Z}$.

Proof: consider $f: \mathbb{Z}^+ \to \mathbb{Z}$.

x	1	2	3	4	5	6	7	$\cdots$
$f(x)$	0	1	-1	2	-2	3	-3	$\cdots$

Example 2: there is a bijection between $\mathbb{Z}^+$ and $\mathbb{Q}^+$ (i.e., the set of all the rational numbers > 0)

proof: (ideas only)

$$
\begin{array}{cccccc}
\overset{(1)}{\frac{1}{1}} & \overset{(2)}{\frac{2}{1}} & \overset{(5)}{\frac{3}{1}} & \overset{(6)}{\frac{4}{1}} & \frac{5}{1} & \cdots \\[2ex]
\overset{(3)}{\frac{1}{2}} & \cancel{\frac{2}{2}} & \frac{3}{2}\,(7) & \cancel{\frac{4}{2}} & \frac{5}{2} & \cdots \\[2ex]
\overset{(4)}{\frac{1}{3}} & \frac{2}{3}\,(8) & \cancel{\frac{3}{3}} & \frac{4}{3} & \frac{5}{3} & \cdots \\[2ex]
\overset{(9)}{\frac{1}{4}} & \cancel{\frac{2}{4}} & \frac{3}{4} & \cancel{\frac{4}{4}} & \frac{5}{4} & \cdots
\end{array}
$$

Example: There is no bijection between $\mathbb{Z}^+$ and $[0,1]$

proof: Cantor's diagonalization.

Suppose $\exists$ a bijection $f: \mathbb{Z}^+ \to [0,1]$.

write $x \in [0,1]$ as $0.d_1 d_2 d_3 \cdots$

(not unique, $0.\dot{9} = 1$)

$$
\begin{array}{lllllll}
1 \longmapsto & 0. & 3 & 2 & 9 & 5 & 7 \\
2 \longmapsto & 0. & 1 & 0 & 1 & 0 & 2 \\
3 \longmapsto & 0. & 6 & 4 & 3 & 0 & 2 \\
4 \longmapsto & 0. & 7 & 5 & 2 & 0 & 1 \\
\cdots & & & & & & \\
\hline
 & 0. & 3 & 0 & 3 & 0 & \cdots
\end{array}
$$

map : $k \to k+5 \bmod 10.$

$$\begin{array}{ccccccc} 0 & 1 & 2 & & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow \\ 5 & 6 & 7 & & 2 & 3 & 4 \end{array}$$

$0.3030\cdots$

$\downarrow$

$0.8585\cdots \notin f(\mathbb{Z}^+)$

a contradiction, #