THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

Student Number |__|__|__|__|__|__|__|__|

Family Name ______________________

First Name ______________________

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School of Mathematics & Physics

EXAMINATION

Semester One Final Examinations, 2020

MATH7501 Mathematics for Data Science 1

Examination Duration: 120 minutes (+ additional 30 mins encompassing reading time, and time to scan and upload solutions). You must commence your exam at the time listed in your personalised timetable within a 24 hour window.

Materials Permitted While Completing The Exam:

Any course materials including the course reader, assignments, their solution, and personal notes you have taken during the course.

You may not make use of any other material. This includes web-sites, books, or any other material or software.

Instructions To Students: *Answer all questions.*

You can print the exam and write on the exam paper, or write your answers on blank paper, or write electronically on a suitable device. Scan or photograph your work if necessary and upload your answers to Blackboard as a single pdf file.

Who to Contact

Since students may not all undertake the online exam at the same time, or in the same time zone, and that some questions may be randomised, responding to student queries and/or relaying corrections to exam content during the exam will not be feasible. Course coordinators will not be able to respond to academic queries during the exam.

If you have any concerns or queries about a particular question, or need to make any assumptions to answer the question, state these at the start of your solution to that question. You may also include queries you may have made with respect to a particular question, should you have been able to 'raise your hand' in an examination room.

If you experience any technical difficulties during the exam, contact the Library AskUs service via the Live Chat or Phone for advice (open 7:00am – 10:00pm AEST every day during the final exam period). You should ask the library staff for an email documenting the advice provided so you can provide this to the course coordinator.

Certification (must be signed before submission):

I certify that my submitted answers are entirely my own work and that I have neither given nor received any unauthorised assistance on this assessment item.

Signed:______________________________Date:_____________________

Instructions:

The number of points per question is marked next to the question.

The total number of points is 100.

Answer all questions on this exam paper.

(1) Assume you have a grayscale image represented in a 50×200 matrix A where A_{ij} represents the pixel intensity of the matrix as a number in the range $[0, 1]$, with $A_{ij} = 0$ implying dark and $A_{ij} = 1$ bright.

Let $\mathbf{1}_n$ denote a column vector of n 1's and set,

$$m = \mathbf{1}_{50}^T \, A \, \mathbf{1}_{200}.$$

(1a) Represent the minimizer of,

$$L(x) = \sum_{i=1}^{50} \sum_{j=1}^{200} (x - A_{ij})^2,$$

in terms of m.
Comment: The minimizer is the value of x that minimizes $L(x)$.

(15 points)

(1b) Say that for demonstration purposes, you wish to find the minimizer of $L(x)$ as defined above via a one dimensional gradient descent algorithm. For this you set $x_0 = 0.5$, and then for $n \geq 1$, you proceed via the recursion:

$$x_{n+1} = x_n - \eta L'(x_n),$$

where L' is derivative of L. Here, η, the learning rate, is a positive value. Establish a range for η over which the sequence $\{x_n\}_{n=1}^{\infty}$ converges to your solution of (1a).

(20 points)

(2) Let γ be a positive quantity. Consider the probability density function of a distribution defined over $[0, \infty)$:

$$f(x) = K\frac{x}{\gamma}e^{-\frac{x^2}{\gamma}},$$

where K is some positive constant (depending on γ).

(2a) Determine K such that the integral,

$$\int_0^\infty f(x)\,dx = 1.$$

(15 points)

(2b) The median is the value $\mu > 0$ such that,

$$\int_0^{\mu} f(x)\,dx = \frac{1}{2}.$$

Find the median in terms of γ.

(15 points)

(3) Let θ be a real value parameter and consider the matrix $B_\theta(x)$ represented as a function of x and parameterized by θ:

$$B_\theta(x) = \begin{bmatrix} x^2 & 0 \\ e^{\cos(x)} & e^{\theta x} \end{bmatrix}.$$

You now wish to calculate the definite improper integral

$$M_\theta = \int_{-10}^{\infty} |B_\theta(u)| \, du.$$

where $|\cdot|$ represents the determinant. Determine the set of values of θ for which the integral converges.

(15 points)

(4) Assume $\lambda > 0$ and let $f(x) = \lambda e^{-\lambda x}$. Now for $k = 0, 1, 2, \ldots$, define,

$$M_k = \int_0^\infty x^k f(x)\, dx.$$

Prove that,

$$M_k = \frac{k!}{\lambda^k}.$$

(20 points)

END OF EXAMINATION