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Q1

(a) $Ab = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ #

(b) $b^T A^T = \begin{pmatrix} 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}$ #

(c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = B^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$B^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Such that $B^{2023} = B^3 \cdot B^{2020} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

#

Q2

(a) $\because$ U and U^{-1} is an orthogonal matrix.

$U^{-1} = U^T$, $U^T \cdot U = I$ $\therefore$ U^T is an orthogonal matrix.

for any rows and column of U

we have $(U_{ij})^2 = 1$

so $\sum_{i=1}^{n} \sum_{j=1}^{n} (U_{ij})^2 = n$ #

(b) assume $B = U \cdot U$, $B^T = U^T \cdot U^T$

we have $U \cdot U^T = I$.

we can see $B \cdot B^T = U \cdot U \cdot U^T \cdot U^T = I$

so UU is also orthogonal #

(c) we can see from (b), UU is an orthogonal matrix.

$\because$ $\det(U) = 1$ or -1

$\therefore$ $\det(U \cdot U) = 1$ #

Q3

(a) $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$A^{-1} = \left(\begin{array}{ccc|ccc} 3 & -1 & 0 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[-C_1]{R_2 \rightleftharpoons R_1} \left(\begin{array}{cccccc} 1 & -3 & 1 & 0 & -1 & 0 \\ 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-3R_1+R_2} \left(\begin{array}{cccccc} 1 & -3 & 1 & 0 & -1 & 0 \\ 0 & 8 & -3 & 1 & 3 & 0 \\ 0 & -1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \rightleftharpoons R_2]{-C_2} \left(\begin{array}{cccccc} 1 & -3 & 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 & 0 & -1 \\ 0 & 8 & -3 & 1 & 3 & 0 \end{array}\right)$$

$$\xrightarrow[-8R_2+R_3]{3R_2+R_1} \left(\begin{array}{cccccc} 1 & 0 & -8 & 0 & -1 & -3 \\ 0 & 1 & -3 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{21} & \frac{1}{7} & \frac{21}{8} \end{array}\right)$$

$$\xrightarrow[8R_3+R_1]{3R_3+R_2} \begin{pmatrix} 1 & 0 & 0 & \frac{8}{21} & \frac{1}{7} & \frac{1}{21} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{3}{7} & \frac{1}{7} \\ 0 & 0 & 1 & \frac{1}{21} & \frac{1}{7} & \frac{8}{21} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{8}{21} & \frac{1}{7} & \frac{1}{21} \\ \frac{1}{7} & \frac{3}{7} & \frac{1}{7} \\ \frac{1}{21} & \frac{1}{7} & \frac{8}{21} \end{pmatrix} \quad \#$$

(b)

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 21$$

$$C_{11} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \quad C_{12} = (-1)^{1+2}\begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix}$$

$$C_{13} = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} \quad \#$$

(c) $\det(A^{-1}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \frac{1}{21}$

$C_{11} = \begin{pmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{8}{21} \end{pmatrix}$ $C_{12} = (-1)^{1+2}\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{1}{21} & \frac{8}{21} \end{pmatrix}$

$C_{13} = \begin{pmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{1}{21} & \frac{1}{7} \end{pmatrix}$ #

Q4

(a) $||x|| = \sqrt{2}$ #

(b) $\cos\theta = \dfrac{x \cdot y}{||x|| \cdot ||y||} = \dfrac{\sqrt{2}}{3}$

$\sin\theta = \sqrt{1 - \cos^2\theta} = \dfrac{\sqrt{7}}{3}$ #

(c) assume $\varepsilon = \begin{pmatrix} a \\ b \\ d \\ e \end{pmatrix}$

$\because \varepsilon \cdot x = 0 \therefore b = e$

$y = cx + \varepsilon \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} a \\ -c+b \\ d \\ c+e \end{pmatrix}$

$a = -1$

$d = 3$

$-c + e = 1$

$c + e = 5$

$e = 3$

$c = 2$

$b = 3$ #

so $\varepsilon = \begin{pmatrix} -1 \\ 3 \\ 3 \\ 3 \end{pmatrix}$

Q5.

(a) $\sum_{i \in \mathbb{Q}n} \binom{n}{i}$

(b) $\binom{10}{2} = \frac{10!}{2!(10-2)!} = 45$ #

(c) $\exists\ n = n-k$, when $n \geq 1$, $n \in \mathbb{Z}$

$\because \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}$ $\therefore \sum_{k=0}^{n} \binom{n}{k} x^{n-k} = (1+x)^n$. #

Q6

(a) $A = 3n$, for $n \in \mathbb{Z}$. $B = 2x$, for $x \in \mathbb{Z}$.

$A \cdot B = 6n \cdot x$ $\therefore n \in \mathbb{Z}$, $x \in \mathbb{Z}$ $\therefore n \cdot x \in \mathbb{Z}$

$C = 6l$, $l \in \mathbb{Z}$ $\because l = n \cdot x \in \mathbb{Z}$.

$\because A \cap B = C$. #

(b) when $x = 2$ $B = 4$. $4 \notin C$.

so $B \not\subseteq C$ #

(c) $A = 3n = 2n + n$ $D = 2^{2n+1} + 1$

$= 2^{2n+n^0} + n^0$

$\because n \geq 1 \quad \therefore 2^{2n+n^0} \leq 2n, \quad n^0 \leq n$

$\therefore D \subseteq A$

#

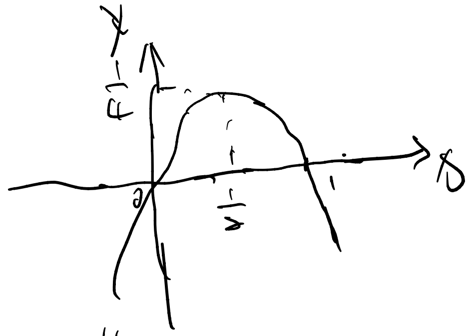
Q7

p	q	$p \Rightarrow (q \Rightarrow p)$
T	T	T
T	F	T
F	T	T
F	F	T

#

(b) $f(x) = x - x^2$

range of f: $[0, \frac{1}{4}]$



#

(c) $a < b < c$

$g(a) = a^3, \quad g(b) = b^3, \quad g(c) = c^3$

$g(a) < g(b) < g(c)$

$\therefore g(x) = x^3$ is injective.

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