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1. Sol:

(a) $2A = 2\times\begin{pmatrix} 8 & 9 & 6 \\ 5 & 7 & 4 \\ 3 & 10 & 2 \end{pmatrix} = \begin{pmatrix} 16 & 18 & 12 \\ 10 & 14 & 8 \\ 6 & 20 & 4 \end{pmatrix}$ #

(b) $A^{-1} = \left(\begin{array}{ccc|ccc} 8 & 9 & 6 & 1 & 0 & 0 \\ 5 & 7 & 4 & 0 & 1 & 0 \\ 3 & 10 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{8}R_1} \left(\begin{array}{ccc|ccc} 1 & \frac{9}{8} & \frac{3}{4} & \frac{1}{8} & 0 & 0 \\ 5 & 7 & 4 & 0 & 1 & 0 \\ 3 & 10 & 2 & 0 & 0 & 1 \end{array}\right)$

$\xrightarrow[-3R_1+R_3]{-5R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & \frac{9}{8} & \frac{3}{4} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{11}{8} & \frac{1}{4} & -\frac{5}{8} & 1 & 0 \\ 0 & \frac{53}{8} & -\frac{1}{4} & -\frac{3}{8} & 0 & 1 \end{array}\right) \xrightarrow{\frac{8}{11}R_2} \left(\begin{array}{ccc|ccc} 1 & \frac{9}{8} & \frac{3}{4} & \frac{1}{8} & 0 & 0 \\ 0 & 1 & \frac{2}{11} & -\frac{5}{11} & \frac{8}{11} & 0 \\ 0 & \frac{53}{8} & -\frac{1}{4} & -\frac{3}{8} & 0 & 1 \end{array}\right)$

$\xrightarrow[-\frac{53}{8}R_2+R_3]{-\frac{9}{8}R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{6}{11} & \frac{7}{11} & -\frac{9}{11} & 0 \\ 0 & 1 & \frac{2}{11} & -\frac{5}{11} & \frac{8}{11} & 0 \\ 0 & 0 & -\frac{16}{11} & \frac{29}{11} & -\frac{53}{11} & 1 \end{array}\right) \xrightarrow{-\frac{11}{16}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{6}{11} & \frac{7}{11} & -\frac{9}{11} & 0 \\ 0 & 1 & \frac{2}{11} & -\frac{5}{11} & \frac{8}{11} & 0 \\ 0 & 0 & 1 & -\frac{29}{16} & \frac{53}{16} & -\frac{11}{16} \end{array}\right)$

$\xrightarrow[-\frac{2}{11}R_3+R_1]{-\frac{6}{11}R_3+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{21}{8} & \frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 1 & -\frac{29}{16} & \frac{53}{16} & -\frac{11}{16} \end{array}\right)$

$B^{-1} = \left(\begin{array}{cc|cc} 5 & 6 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \xrightarrow{\frac{5}{3}R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & -\frac{2}{3} & \frac{5}{3} \end{array}\right)$

$A^{-1} = \begin{pmatrix} \frac{13}{8} & -\frac{21}{8} & \frac{3}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{29}{16} & \frac{53}{16} & -\frac{11}{16} \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{pmatrix}$ #

(c)

$A = \begin{pmatrix} 8 & 9 & 6 \\ 5 & 7 & 4 \\ 3 & 10 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$

A: $C_{11} = (-1)^{1+1} \begin{vmatrix} 7 & 4 \\ 10 & 2 \end{vmatrix} = -26$ $C_{12} = (-1)^{1+2} \begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix} = 2$ $C_{13} = (-1)^{1+3} \begin{vmatrix} 5 & 7 \\ 3 & 10 \end{vmatrix} = 29$

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{14} = -16$

B: $C_{11} = (-1)^{1+1} 3 = 3$ $C_{12} = (-1)^{1+2} \cdot 2 = -2$

$\det(B) = a_{11}C_{11} + a_{12} \cdot C_{12} = 3$ #

(d)

A^{-1}: $C_{11} = (-1)^{1+1} \begin{vmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{53}{16} & -\frac{11}{16} \end{vmatrix}$ $C_{12} = (-1)^{1+2} \begin{vmatrix} -\frac{1}{8} & \frac{1}{8} \\ -\frac{49}{16} & -\frac{11}{16} \end{vmatrix}$ $C_{13} = (-1)^{1+3} \begin{vmatrix} -\frac{1}{8} & \frac{1}{8} \\ -\frac{49}{16} & \frac{53}{16} \end{vmatrix}$

$\det(A^{-1}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = -\frac{1}{16}$

B^{-1}: $C_{11} = (-1)^{1+1} \frac{5}{3}$ $C_{12} = (-1)^{1+2} \left(-\frac{2}{3}\right)$

$\det(B^{-1}) = a_{11}C_{11} + a_{12}C_{12} = \frac{1}{3}$ #

(e) $B = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$ $\delta = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$B \cdot B^T = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 5 & 2 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 61 & 28 \\ 28 & 13 \end{pmatrix}$

$\delta^T B = (2 \quad -1) \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix} = (8 \quad 9)$ #

2.

Sol:

(a) when $n=1$, U is a 1-by-1 matrix.

$\because U^T U = 1$ $\therefore U = (1)$ or $U = (-1)$ #

(b) we have $U^TU=I$ so $(U^T \cdot U)^T = I^T$

we can get $(U^T)^T \cdot U^T = I \Rightarrow U \cdot U^T = I$.

$\because U \cdot U^T = I$. $\therefore U^T \cdot U = I$.

Such that U^T is an orthogonal matrix #

(c) For any U is an orthogonal matrix.

So rows of U are orthonormal.

$\Sigma (U_{ij})^2 = (U_{i1})^2 + (U_{i2})^2 + \cdots + (U_{in})^2$

$\because$ rows of U are orthonormal. $\therefore \Sigma (U_{ij})^2 = 1$

Such that $\sum_{j=1}^{n} (U_{ij})^2 = 1$ #

3.

Sol:

(a) $\|x\| = \sqrt{1^2+3^2} = \sqrt{10}$ #

(b) $\varepsilon \perp x$. $\varepsilon = \begin{pmatrix} a \\ b \end{pmatrix}$. $\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = a + 3b = 0 \Rightarrow a = -3b$

$y = cx \cdot \varepsilon$

$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} c - 3b \\ 3c + b \end{pmatrix} \Rightarrow c = \frac{3}{5}$. $b = -\frac{4}{5}$

$\varepsilon = \begin{pmatrix} \frac{12}{5} \\ -\frac{4}{5} \end{pmatrix}$ #

(c) $\cos\theta = \frac{x \cdot y}{\|x\| \cdot \|y\|} = \frac{6}{\sqrt{10} \cdot \sqrt{10}} = \frac{3}{5}$ #

4.

proof:

(a) $x = 5m = 5^1 \times m \quad 5^1 \times m \subseteq 5^n \times m \Rightarrow A \subseteq B.$

$y = 5^n m = 5^{(n-1)} \cdot 5m$ when $n = 1 \quad 5^{(n-1)} 5m = 5m \Rightarrow B \subseteq A$

so $A = B$ #

(b) when $m = 1 \quad A = 1 \times 5 = 5, \ 5 \notin C$

This proved $A \neq C$ #

(c) $2n = 5m, \ n = \frac{5}{2} m$, we assume $m = 2a$

$n = 5a, \ 2n = 10a$ for some integer a, $2n \subseteq 10t$.

This proved $2n \in C$ #

5.

proof:

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge (\sim p)$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

6.

sol:

(a) $C_r^n = \frac{n!}{r!(n-r)!} = \frac{5!}{3!\,2!} = 10$. so $\binom{5}{3} = 10$ #

(b)

$$(1+\delta)^{100} = \sum_{k=0}^{100} \binom{100}{k} \delta^k \qquad \binom{100}{k} = \frac{100!}{k!(100-k)!}$$

$$1+100\delta = \binom{100}{0}\delta^0 + \binom{100}{1}\delta^1 \quad \because \delta \geq 0 \quad \therefore (1+\delta)^{100} > 0$$

#

(c)

$$\lim_{n\to\infty} n\left((1+\tfrac{1}{n})^2 - 1\right) = \lim_{n\to\infty}\left(2+\tfrac{1}{n}\right)$$

$$\lim_{n\to\infty}(2) + \lim_{n\to\infty}\left(\tfrac{1}{n}\right) = 2+0 = 2$$

So $\lim_{n\to\infty} n\left((1+\tfrac{1}{n})^2-1\right) = 2$ #

(d) $$\binom{n}{k}\frac{1}{n^k} = \frac{n!}{k!(n-k)!}\cdot\frac{1}{n^k} = \frac{n(n-1)\cdot(n-2)\cdots(n-k+1)}{n^k\cdot k!}$$

$$\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) = \left(1-\frac{0}{n}\right)\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) = \frac{n(n-1)\cdots(n-k+1)}{n^k}$$

$$\frac{1}{k!}\cdot\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) = \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k\cdot k!} = \binom{n}{k}\frac{1}{n^k}$$ #

$$\binom{n+1}{k}\frac{1}{(n+1)^k} = \frac{(n+1)!}{k!(n+1-k)!}\cdot\frac{1}{(n+1)^k} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+2)}{k!(n+1)^k}$$

$$\frac{1}{k!}\prod_{i=0}^{k-1}\left(1-\frac{i}{n+1}\right) = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+2)}{(n+1)^k k!}$$

So $\binom{n+1}{k}\frac{1}{(n+1)^k} = \frac{1}{k!}\prod_{i=0}^{k-1}\left(1-\frac{i}{n+1}\right)$ #

(e)

from (d) we can get $0 \leq \binom{n}{k}\frac{1}{n^k} \leq 1 \quad 0 \leq \binom{n+1}{k}\frac{1}{(n+1)^k} \leq 1$

$$\frac{1}{k!}\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) \leq \frac{1}{k!}\prod_{i=0}^{k-1}\left(1-\frac{i}{n+1}\right) \leq \frac{1}{k!} \Rightarrow \prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) \leq \prod_{i=0}^{k-1}\left(1-\frac{i}{n+1}\right) \leq 1$$

$$\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right)=\left(1-\frac{0}{n}\right)\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)=\frac{n(n-1)\cdots(n-k+1)}{n^k}$$

$$\prod_{i=0}^{k-1}\left(1-\frac{i}{n+1}\right)=\frac{n(n+1)(n-2)\cdots(n-k+2)}{(n+1)^k}$$

$\because \frac{i}{n}>\frac{i}{n+1} \Rightarrow 1-\frac{i}{n}<1-\frac{i}{n+1}$

$\therefore \prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right)\leq\prod_{i=0}^{k-1}\left(1-\frac{i}{n+1}\right)$

$$\frac{\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right)}{\prod_{i=0}^{k-1}\left(1-\frac{i}{n+1}\right)}\leq 1$$

such that $\binom{n}{k}\frac{1}{n^k}\leq\binom{n+1}{k}\frac{1}{(n+1)^k}\leq\frac{1}{k!}$ #

(f) $\left(a_n=\left(1+\frac{1}{n}\right)^n\right)_{n=1}^{\infty}$, $\left(a_{n+1}=\left(1+\frac{1}{n+1}\right)^{n+1}\right)_{n=1}^{\infty}$

$\left(1+\frac{1}{n}\right)^n=\sum_{k=1}^{n}\binom{n}{k}\frac{1}{n^k}$, $\left(1+\frac{1}{n+1}\right)^{n+1}=\sum_{k=1}^{n}\binom{n+1}{k}\frac{1}{(n+1)^k}$

from (e) we can get $\binom{n}{k}\frac{1}{n^k}\leq\binom{n+1}{k}\frac{1}{(n+1)^k}$

So $a_n\leq a_{n+1}$ for any integer $n\geq 1$ #

(g) ① $\frac{1}{k!}=\frac{1}{k\cdot(k-1)(k-2)\cdots 1}$

② $\frac{1}{k-1}-\frac{1}{k}=\frac{1}{k^2-k}=\frac{1}{k(k-1)}$

from ① and ② we can get

$\frac{1}{k!}\leq\frac{1}{k-1}-\frac{1}{k}$ when $k\geq 2$ #

$\left(1+\frac{1}{n}\right)^n=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}\leq\sum_{k=0}^{n}\frac{1}{k!}$ $\because \frac{1}{k!}\leq\frac{1}{k-1}-\frac{1}{k}$

$\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)=1-\frac{1}{n}$

$2+1-\frac{1}{n}\geq 2+\sum_{k=2}^{n}\binom{n}{k}\frac{1}{n^k}\geq\left(1+\frac{1}{n}\right)^n$ when $n\geq 2$

So $\left(1+\frac{1}{n}\right)^n\leq 2+1-\frac{1}{n}<3$ #

(h) $\because \left(1+\frac{1}{n}\right)^n\leq 3-\frac{1}{n}$ $\lim_{n\to\infty}\left(3-\frac{1}{n}\right)=3$ $\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\leq\lim_{n\to\infty}\left(3-\frac{1}{n}\right)$

So $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists #