Lecture 7.2

Confidence Intervals

"*It's easy to lie with statistics, but it's hard to tell the truth without them.*"

Charles Wheelan

Estimators and Estimates

Suppose $X_1, \ldots, X_n$ are **iid** random variables. This is sometimes called a **simple random sample** of size n. A realisation of this simple random sample forms our **sample data**, denoted $x_1, \ldots, x_n$.

We would like to estimate the parameter μ, that is the expectation of the X_i. A natural **estimate** of μ is the sample mean $\bar{x}$,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

The random variable leading to our estimate is called the **estimator**. In this instance our estimator is

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

So, the estimate is a realisation of the estimator.

Uncertainty in point estimates

Assume the mean and standard deviation of exercise time among UQ students is μ and σ. Suppose we take a sample of $n = 4$ students from UQ and record their exercise times. We can use the sample mean of these 4 students to estimate the population mean for exercise time.

How accurate is this estimate?

Obviously, different samples will give different estimates . . . How can we describe the variability we see in the sample means?

What happens when we take larger samples ($n = 16$, $n = 64$)?

Uncertainty in point estimates

Although point estimates such as the sample mean, sample variance, sample proportion, etc, approximate corresponding population parameters but, in general due to sampling variability, they are different from true underlying values.

Generally speaking, reporting only the value of a point estimate is not as informative as it can be because

1. it does not quantify how accurate the estimate is, and
2. it does not relate the level of accuracy to the sample size.

Can we make these “uncertainties” more precise?

Confidence intervals

A **confidence interval** is a tool to address the inherent uncertainty in reporting only a point estimate.

A confidence interval is an interval for which we can assert, with a given degree of confidence/certainty, that it contains the true underlying value of the parameter being estimated.

The length of such an interval gives us an idea of how closely we can estimate such true parameter.

The construction of confidence intervals relies on the exact distribution of the point estimator, or an approximation to it provided by the CLT.

Confidence intervals: Example

Suppose the exercise time among UQ students is normally distributed as $\mathcal{N}(\mu, 1)$ with unknown μ.

Recall that if $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, 1)$, then

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\mu, \frac{1}{n}\right) \implies (\bar{X} - \mu)/(1/\sqrt{n}) \sim \mathcal{N}(0, 1).$$

Since $\mathbb{P}\left(-1.96 \leq (\bar{X} - \mu)/(1/\sqrt{n}) \leq 1.96\right) = 0.95$, then

$$\mathbb{P}\Big(\bar{X} - \frac{1.96}{\sqrt{n}} \leq \mu \leq \bar{X} + \frac{1.96}{\sqrt{n}}\Big) = 0.95.$$

So, the *random* interval $\left[\bar{X} - 1.96/\sqrt{n}\,,\, \bar{X} + 1.96/\sqrt{n}\right]$ is an 95% confidence interval for the mean in this case.

Confidence intervals: Example

When we sample n UQ students and calculate the corresponding sample mean, $\bar{x}$, we are in fact observing just **one** outcome, or realization, of the random variable, $\bar{X}$.

So, what can we say about **numerical confidence interval**, $\left[\bar{x} - 1.96/\sqrt{n}, \bar{x} + 1.96/\sqrt{n}\right]$, which is actually calculated.

A correct interpretation of "95% confidence" relies on the long-run relative frequency interpretation of probability.

Interpretation: In 95% of samples, μ will be within $1.96/\sqrt{n}$ of $\bar{x}$. That is, if we repeat this experiment multiple times, and calculate the numerical confidence intervals $\left[\bar{x} - 1.96/\sqrt{n}, \bar{x} + 1.96/\sqrt{n}\right]$, then μ will be contained in 95% of these intervals.

https://wise.cgu.edu/portfolio2/demo-confidence-interval-creation/

Poll Question

Increasing the sample size would give a confidence interval for the population mean that is

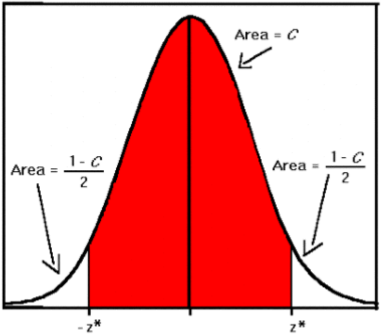
1. narrower
2. wider
3. not sure

Poll Question

Compared to a 95% confidence interval, a 99% confidence interval for the population mean would be

1. narrower
2. wider
3. not sure

Question



How much wider is a 99% confidence interval than a 95% confidence interval?

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> (qnorm(0.995)-qnorm(0.975))/qnorm(0.975)
[1] 0.3142228
```

Poll Question

How much wider is a 100% confidence interval than a 95% confidence interval?

1. About 5% wider
2. About 33% wider
3. About twice as wide
4. Infinitely wider

Say, we collected a sample of 4 UQ students and the sample mean of exercise times was $\bar{x} = 2$. It is *completely incorrect* to say "$\mathbb{P}(2 - 1.96/2 \leq \mu \leq 2 + 1.96/2) = 0.95$"!

This is somewhat unsatisfactory to many. ☹

There is another approach to quantify the uncertainties in the point estimates, known as *credible intervals*, which admits the above interpretation. These constructions use the notion of subjective probability and Bayes' theorem, but the technical details are beyond the scope of this course.