MATH7501: Week 7 Practical Questions

1. By definition. for every $\varepsilon > 0$, $\exists$ a $\delta(\varepsilon) > 0$ s.t. if $0 < |x| < \delta(\varepsilon)$, then $|f(x) - \ell| < \varepsilon$.

Problem 1. Suppose that $a > 0$ is a fixed constant. Using the definition of a limit, prove that if

$$\lim_{x\to 0} f(x) = \ell \text{ then } \lim_{x\to 0} f(ax) = \ell.$$

Problem 2. For some constant a, compute

$$\lim_{x\to a} \frac{x^2 - a^2}{x - a} \text{ and } \lim_{x\to a} \frac{x^3 - a^3}{x - a}.$$

Problem 3. Using the squeeze principle, compute

$$\lim_{x\to 0} \left(x^4 + x^2\right) \cos\left(\frac{1}{x^2}\right) \sin\left(x^3 + x\right).$$

Problem 4. For some constant a, suppose that $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = 2$, for functions $f(x)$ and $g(x)$. Evaluate:

(a) $\lim_{x\to a} \left[3f(x) + (g(x))^2\right]$.

(b) $\lim_{x\to a} (g(x))^{-1}$.

(c) $\lim_{x\to a} \sqrt{3f(x) + 8g(x)}$.

Problem 5. Let $f(x) = e^{1/x}$. Argue that $\lim_{x\to 0^+} f(x) = \infty$ and evaluate $\lim_{x\to 0^-} f(x)$.

Problem 6. Let $f(x) = e^x$. Prove that $\lim_{x\to -\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$.

Problem 7. Compute $\lim_{x\to\infty} f(x)$ for

$$f(x) = \frac{4x^{2023} + 4}{3x^{2023} + x^{1011}}.$$

Problem 8. Consider the function

$$f(x) = \begin{cases} x^2 - x + 1 & \text{if } x \le 1, \\ ax^2 + 1 & \text{if } x > 1. \end{cases}$$

Determine the value of a that makes $f(x)$ continuous for all $x \in \mathbb{R}$.

Problem 9. Let $f(x)$ and $g(x)$ be continuous functions from domain $[a, b]$ to co-domain $\mathbb{R}$. Suppose that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there exists a $c \in (a, b)$ such that $f(c) = g(c)$.

Problem 10. Use the intermediate value theorem to prove that there exists a $c \in (0, 1)$ such that $f(x) = xe^x - 2 = 0$ and approximate the value c using three loops of the bisection method algorithm with initial values $a = 0$ and $b = 1$.