THE UNIVERSITY OF QUEENSLAND
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Course Reader for,

MATH7501 Mathematics for Data Science 1

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1 Basic operations with matrices vectors

1.1 Matrices as a way of organizing data

There are many ways to obtain data. It may be collected from an image, a video, a music file, the Australian Bureau of Statistics website (ABS), come from an MRI image, a topographical survey, among countless many other sources. Collected data may have various degrees of structure. Data downloaded from the ABS website is called structured data since it is presented in the form of a spreadsheet with labeled columns and rows. Text documents having no such structure are known as unstructured data. Consider the following example.

1		Earnings; Males; Full Time; Adult; Ordinary time earnings ;	Earnings; Males; Full Time; Adult; Total earnings ;	Earnings; Males; Total earnings ;	Earnings; Females; Full Time; Adult; Ordinary time earnings ;	Earnings; Females; Full Time; Adult; Total earnings ;	Earnings; Females; Total earnings ;	Earnings; Persons; Full Time; Adult; Ordinary time earnings ;	Earnings; Persons; Full Time; Adult; Total earnings ;	Earnings; Persons; Total earnings ;
2	**Unit**	$	$	$	$	$	$	$	$	$
3	**Series Type**	Trend	Trend	Trend	Trend	Trend	Trend	Trend	Trend	Trend
4	**Data Type**	RATIO	RATIO	RATIO	RATIO	RATIO	RATIO	RATIO	RATIO	RATIO
5	**Frequency**	Biannual	Biannual	Biannual	Biannual	Biannual	Biannual	Biannual	Biannual	Biannual
6	**Collection Month**	2	2	2	2	2	2	2	2	2
7	**Series Start**	May-2012	May-2012	May-2012	May-2012	May-2012	May-2012	May-2012	May-2012	May-2012
8	**Series End**	May-2016	May-2016	May-2016	May-2016	May-2016	May-2016	May-2016	May-2016	May-2016
9	**No. Obs**	9	9	9	9	9	9	9	9	9
10	**Series ID**	A84990042R	A84990045W	A84990048C	A84990043T	A84990046X	A84990049F	A84990044V	A84990047A	A84990050R
11	May-2012	1447.10	1538.10	1286.00	1194.00	1211.30	822.80	1353.30	1417.40	1054.30
12	Nov-2012	1488.50	1576.60	1324.50	1226.40	1242.80	838.40	1392.80	1454.40	1081.00
13	May-2013	1515.70	1603.70	1351.30	1252.10	1268.90	852.20	1420.50	1483.10	1103.40
14	Nov-2013	1533.90	1621.80	1352.40	1267.60	1284.50	870.60	1437.20	1499.00	1114.60
15	May-2014	1560.70	1650.60	1362.90	1277.30	1295.00	881.20	1455.30	1518.70	1122.50
16	Nov-2014	1584.60	1675.10	1370.60	1291.20	1309.30	889.90	1474.50	1537.40	1129.00
17	May-2015	1593.20	1679.30	1370.20	1308.50	1326.80	905.70	1484.90	1545.60	1136.70
18	Nov-2015	1603.20	1686.30	1376.90	1327.60	1345.60	915.60	1500.00	1558.30	1146.70
19	May-2016	1613.60	1696.60	1393.50	1352.50	1370.10	925.10	1516.00	1575.40	[illegible]
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Figure 1: An example of structured data from the ABS website on weekly earnings.

We are able to download such files as shown in Fig. 1 and save as a CSV file. It is generally simple to import a CSV file in other software and then make use of that. Doing just that to the data in Fig. 1 gives the matrix

$$A = \begin{pmatrix} 1447.1 & 1538.1 & 1286. & 1194. & 1211.3 & 822.8 & 1353.3 & 1417.4 & 1054.3 \\ 1488.5 & 1576.6 & 1324.5 & 1226.4 & 1242.8 & 838.4 & 1392.8 & 1454.4 & 1081. \\ 1515.7 & 1603.7 & 1351.3 & 1252.1 & 1268.9 & 852.2 & 1420.5 & 1483.1 & 1103.4 \\ 1533.9 & 1621.8 & 1352.4 & 1267.6 & 1284.5 & 870.6 & 1437.2 & 1499. & 1114.6 \\ 1560.7 & 1650.6 & 1362.9 & 1277.3 & 1295. & 881.2 & 1455.3 & 1518.7 & 1122.5 \\ 1584.6 & 1675.1 & 1370.6 & 1291.2 & 1309.3 & 889.9 & 1474.5 & 1537.4 & 1129. \\ 1593.2 & 1679.3 & 1370.2 & 1308.5 & 1326.8 & 905.7 & 1484.9 & 1545.6 & 1136.7 \\ 1603.2 & 1686.3 & 1376.9 & 1327.6 & 1345.6 & 915.6 & 1500. & 1558.3 & 1146.7 \\ 1613.6 & 1696.6 & 1393.5 & 1352.5 & 1370.1 & 925.1 & 1516. & 1575.4 & 1160.2 \end{pmatrix}$$

The rows and columns of the matrix A have various meanings. In this case, as we see in Fig. 1, the rows correspond to a point in time, while the columns correspond to weekly earnings in various categories. The data contains columns representing the same quantities for males and females. Anyone who wishes to examine the data further may want extract rows or columns, perform operations on specific rows and columns such as taking a difference of two columns, or even model the data in a specific column through

examine data $\longleftrightarrow$ matrix operations.

some form of interpolation, fitting a function to it. These tasks become easier when we understand the meaning of indexing in a matrix.

Definition 1. *A rectangular array of numbers*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix}$$

is called an $m \times n$ matrix. It is made up of m rows and n columns.

Entries of the jth column may be assembled into a vector

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

called the jth *column vector* of A. Similarly the ith *row vector* of A is

$$(a_{i1}\, a_{i2} \cdots\cdots a_{in}).$$

Note that an $m \times 1$ matrix is a column vector and a $1 \times n$ matrix is a row vector. The number a_{ij} in the ith row and j column of A is called the $(i,\, j)$th *entry* of A. For brevity we write $A = (a_{ij})$.

Example 1. *The* 2×3 *matrix* A *with entries* $a_{ij} = i - j$ *is*

$$A = \begin{array}{cc} \begin{array}{ccc} j=1 & j=2 & j=3 \end{array} & \\ \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix} & \begin{array}{l} \rightarrow i=1 \\ \rightarrow i=2 \end{array} \end{array}.$$

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Example 2. *The* $m \times n$ *matrix* A *whose entries are all zero is called the* zero matrix, *denoted* 0*; e.g. the zero* 3×2 *matrix is*

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Definition 2. *Matrices* A, B *are said to be* **equal**, *written* $A = B$, *if they have the same number of rows and columns and* $a_{ij} = b_{ij}$, *for all* $i,\, j$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \end{pmatrix}$$

1.2 Matrices as images

An image file stores information on the quantity of colour or hue in each pixel. This is illustrated below in Figure 2.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}$$

black 0 — 1 white

Figure 2: A 16 by 16 pixel black and white image of the number 7.

The matrix corresponding to the pixel data in Figure 2 is given in the following matrix B.

$$B = \begin{pmatrix}
0.99 & 1. & 1. & 0.99 & 1. & 1. & 1. & 0.97 & 1. & 0.98 & 0.99 & 1. & 1. & 0.99 & 1. & 0.98 \\
1. & 0.96 & 1. & 1. & 0.98 & 1. & 0.96 & 1. & 1. & 0.98 & 1. & 1. & 0.97 & 0.98 & 1. & 1. \\
0.99 & 1. & 1. & 0.98 & 1. & 1. & 1. & 0.98 & 1. & 1. & 0.97 & 0.98 & 1. & 0.99 & 0.98 & 0.98 \\
0.98 & 0.95 & 0.97 & 0.019 & 0. & 0. & 0. & 0.01 & 0. & 0. & 0.01 & 0.01 & 1. & 0.99 & 1. & 1. \\
1. & 1. & 1. & 0.99 & 1. & 1. & 0.97 & 1. & 1. & 1. & 1. & 0. & 0.99 & 1. & 0.97 & 0.98 \\
0.98 & 0.98 & 1. & 0.98 & 0.99 & 1. & 0.98 & 0.99 & 1. & 1. & 0. & 0. & 1. & 0.98 & 1. & 1. \\
1. & 1. & 1. & 0.99 & 0.98 & 0.99 & 1. & 1. & 1. & 0. & 0.01 & 1. & 0.96 & 1. & 0.99 & 0.96 \\
0.99 & 1. & 0.98 & 1. & 1. & 1. & 0.99 & 0.99 & 0. & 1. & 0.99 & 0.97 & 1. & 1. & 0.96 & 1. \\
1. & 0.99 & 0.99 & 1. & 1. & 0.99 & 1. & 0.99 & 0. & 1. & 0.99 & 1. & 1. & 1. & 0.98 & 1. \\
1. & 0.99 & 0.01 & 0. & 0. & 0.01 & 0. & 0.01 & 0.01 & 0. & 0.02 & 0. & 0.98 & 0.98 & 1. & 0.99 \\
0.99 & 0.98 & 1. & 0.99 & 1. & 1. & 1. & 0. & 0.98 & 1. & 0.98 & 1. & 1. & 1. & 0.98 & 0.99 \\
1. & 0.97 & 1. & 1. & 0.98 & 1. & 0.98 & 0.01 & 0.99 & 1. & 1. & 1. & 0.99 & 0.99 & 0.99 & 1. \\
1. & 1. & 0.98 & 0.99 & 1. & 0.99 & 1. & 0. & 1. & 0.98 & 1. & 0.98 & 0.98 & 1. & 1. & 0.98 \\
0.98 & 1. & 1. & 1. & 0.97 & 1. & 0.97 & 0.01 & 1. & 1. & 0.99 & 1. & 1. & 0.97 & 1. & 0.99 \\
1. & 1. & 1. & 1. & 1. & 1. & 0.98 & 0.02 & 0.99 & 0.99 & 1. & 1. & 0.96 & 1. & 1. & 1. \\
0.99 & 1. & 1. & 0.99 & 1. & 1. & 1. & 0.98 & 1. & 1. & 0.96 & 1. & 0.99 & 0.99 & 1. & 0.98
\end{pmatrix}$$

Conversely, we can store data from a matrix as an image. This provides a means of storing a large data set a concise form. The matrix A is displayed as an image in Figure 3.

1.3 Matrix operations

Addition

The *sum* of two $m \times n$ matrices A and B is defined to be the $m \times n$ matrix $A + B$ with entries

$$(A + B)_{ij} = a_{ij} + b_{ij}$$

We add matrices element wise, as for vectors.

Addition is only defined between matrices of the same size.

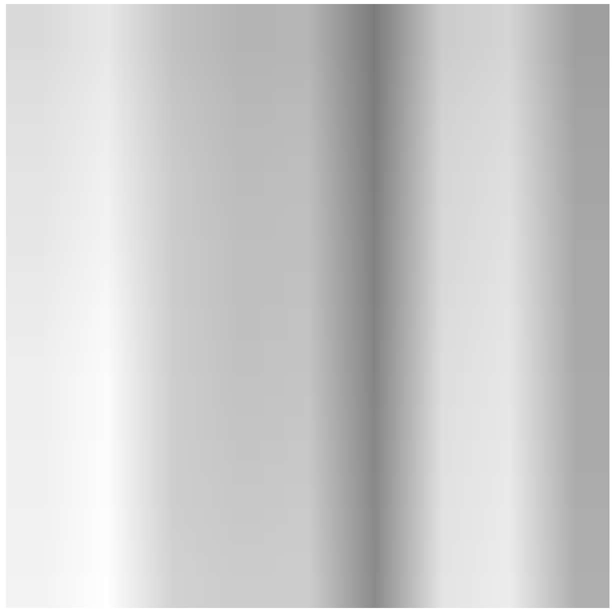

Figure 3: The data from Fig. 1 displayed as an image, after dividing throughout A by 1696.6, the maximum entry so that $0 \le a_{ij} \le 1$.

Properties

For matrices A, B, C of the same size we have

$$\begin{aligned} A + B &= B + A, \qquad \text{Commutative} \\ (A+B)+C &= A+(B+C). \qquad \text{associative} \end{aligned}$$

Scalar multiplication

Let A be an $m \times n$ matrix and $\alpha \in \mathbb{R}$. We define αA to be the $m \times n$ matrix with entries

$$(\alpha A)_{ij} = \alpha \cdot a_{ij} \quad , \quad \text{for all } i,\ j.$$

Write $-A$ for $(-1) \cdot A$. Define subtraction between matrices of the same size by

$$A - B = A + (-B).$$

Example 3. *Find $A+B$ and $A-B$ if*

$$A = \begin{pmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{pmatrix} \quad , \quad A - B = \begin{pmatrix} -9 & 7 & 3 \\ -3 & 0 & 2 \end{pmatrix}.$$

Example 4. *If $A = \begin{pmatrix} 6 & 0 \\ 2 & -1 \end{pmatrix}$ then $A + A = \begin{pmatrix} 12 & 0 \\ 4 & -2 \end{pmatrix} = 2A$.*

Matrix multiplication

We generalize the multiplication of 2×2 matrices. Let $A = (a_{ij})$ be an $m \times n$ and $B = (b_{jk})$ an $n \times r$ matrix. Then AB is the $m \times r$ matrix with ik entry

$$(ab)_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots\cdots + a_{in}b_{nk} \ :$$

$$(AB)_{ik} = \sum_{l=1}^{n} A_{il} B_{lk}$$

$$\underset{(m \times n)}{\underset{A}{\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots\cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix}}} \underset{(n \times r)}{\underset{B}{\begin{pmatrix} b_{11} & \cdots & b_{1k} & \cdots & b_{1r} \\ b_{21} & \cdots & b_{2k} & \cdots & b_{2r} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ b_{n1} & \cdots & b_{nk} & \cdots & b_{nr} \end{pmatrix}}} = \underset{(m \times r)}{AB}$$

where $(ab)_{ik} =$ dot product between ith row of A and kth column of B.

AB is only defined if the number of <u>columns</u> of $A =$ the number of <u>rows</u> of B.

Example 5.

- *Let* $A = \underset{1 \times 3}{(-1\ 0\ 1)}$ *and* $B = \underset{3 \times 2}{\begin{pmatrix} 0 & -1 \\ 1 & 2 \\ -2 & 5 \end{pmatrix}}$
- *Then AB is defined, but $\underset{3\times 2}{B}\ \underset{1\times 3}{A}$ is not defined.*

Calculate AB.

$$\begin{aligned} AB &= (-1\ 0\ 1)\begin{pmatrix} 0 & -1 \\ 1 & 2 \\ -2 & 5 \end{pmatrix} \\ &= (-1 \cdot 0 + 0 \cdot 1 + 1 \cdot -2 \qquad -1 \cdot \overset{(-1)}{\cancel{1}} + 0 \cdot 2 + 1 \cdot 5) \\ &= (-2\ \ \underset{6}{\cancel{4}}) \end{aligned}$$

Example 6.

$$A = (1\ 3\ 9)\,, \ \ B = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}.$$

Calculate AB and BA.

$$AB = \underset{1\times 3}{(1\ \ 3\ \ 9)} \underset{3\times 1}{\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}} = 2 + 3 + 63 = 68$$

$$BA = \underset{3\times 1}{\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}} \underset{1\times 3}{\left(\ 1\ \ \ 3\ \ \ 9\ \right)} = \begin{pmatrix} 2 & 6 & 18 \\ 1 & 3 & 9 \\ 7 & 21 & 63 \end{pmatrix}.$$

Properties

For matrices of appropriate size

No Commutativity

(1) $(AB)C = A(BC)$, Associativity

(2) $(A+B)C = AC + BC$, $A(B+C) = AB + AC$ Distributive Laws

Another unusual property of matrices is that $AB = 0$ does *not* imply $A = 0$ or $B = 0$. It is possible for the product of two non-zero matrices to be zero:

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Also, if $AC = BC$, or $CA = CB$ then it is not true in general that $A = B$. (If $AC = BC$, then $AC - BC = 0$ and $(A-B)C = 0$, but this does **not** imply $A - B = 0$ or $C = 0$.)

Transposition

Definition 3. *The* transpose *of an $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix A^T with entries*

$$a_{ji} = a^T_{ij} \quad , \quad \textit{for all } i,\, j$$

ie.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots\cdots & a_{m1} \\ a_{12} & a_{22} & \cdots\cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots\cdots & a_{mn} \end{pmatrix}$$

So the row vectors of A become column vectors of A^T and vice versa.

Example 7 (transposition)**.** *If* $A = \begin{pmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$, $C = (7 \;\; 5 \;\; -2)$ *then find* A^T, B^T, C^T.

$$A^T = \begin{pmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$C^T = (7 \;\; 5 \;\; -2)^T = \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix}.$$

Properties

For matrices of appropriate size

(1) $(\alpha A)^T = \alpha \cdot A^T \quad , \quad \alpha \in \mathbb{R}$

(2) $(A+B)^T = A^T + B^T$

(3) $(A^T)^T = A$

(4) $(AB)^T = B^T A^T$ (*not* $A^T B^T$!).

$(ABCDE)^T = E^T D^T C^T B^T A^T$

Dot product expressed as matrix multiplication

A column vector

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \qquad w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{R}^n$$

may be interpreted as an $n \times 1$ matrix. Then the dot product (for two column vectors $\boldsymbol{v}$ and $\boldsymbol{w}$) may be expressed using matrix multiplication: or "inner product".

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{w} &= \underbrace{v_1 w_1 + v_2 w_2 + \cdots + v_n w_n}_{\in \mathbb{R}} = \underset{1 \times n}{(v_1\ v_2\ \cdots\ v_n)} \underset{n \times 1}{\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}} \\ &= \boldsymbol{v}^T \boldsymbol{w}. = \langle v, w \rangle \end{aligned}$$

$$\parallel \sum_{i=1}^{n} v_i w_i$$

Also

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{v} &= v_1^2 + v_2^2 + \cdots + v_n^2 = \|\boldsymbol{v}\|^2 \\ &= \boldsymbol{v}^T \boldsymbol{v}. \end{aligned}$$

$$\|v\| = \sqrt{v \cdot v}$$

1.4 Identity matrix, inverses and determinants

The identity matrix

An $n \times n$ matrix $(m = n)$ is called a *square matrix* of *order* n. The diagonal containing the entries

$$a_{11},\ a_{22},\ \cdots\ ,\ a_{nn}$$

$\mathbb{R}$ = the set of real numbers

$= (-\infty, \infty)$

$\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$

$\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$

" $\in$ " $\Rightarrow$ "belongs to"

is called the *main diagonal* (or *principal diagonal*) of A. If the entries above this diagonal are all zero then A is called *lower* triangular. If all the entries below the diagonal are zero, A is called *upper triangular*

all zeros

$$\underset{\text{lower triangular}}{\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ \vdots & a_{22} & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{pmatrix}} \qquad \underset{\text{upper triangular}}{\begin{pmatrix} a_{11} & \cdots & & \cdots & a_{1n} \\ 0 & a_{22} & & & \vdots \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix}}$$

all zeros

If elements above *and* below the principal diagonal are zero, so

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ both lower triangular and upper triangular

$$a_{ij} = 0\,, \quad i \neq j$$

then A is called a diagonal matrix. Note that if A is diagonal then $A = A^T$.

$$\begin{pmatrix} 0 & 0 & 5 \\ 0 & 6 & 0 \\ 7 & 0 & 0 \end{pmatrix}$$

This is NOT a diagonal matrix!

Not upper triangular

Not lower triangular

Example 8. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *is a diagonal* 3×3 *matrix.*

Definition 4 (Identity matrix)**.** *The* $n \times n$ identity matrix $I = I_n$, *is the diagonal matrix whose entries are all* 1*:*

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}.$$

Remarks on the identity matrix

Recall from linear transformations of $\mathbb{R}^2$: $A\boldsymbol{i}$ is the first column of A and $A\boldsymbol{j}$ is the second column.

For example, for the 2×2 case we have

$$AI = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A = IA.$$

In general the jth column vector of I is the coordinate vector

$$\boldsymbol{e}_j = \begin{array}{c@{\quad}l} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ j. \\ \\ \\ \\ \end{array} \end{array}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

Thus

$$Ae_j = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_j} \begin{matrix} \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{matrix}$$

$$= \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} = \text{jth column vector of } A.$$

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So $AI = A$ for all A.

Replacing A by A^T, $A^T I = A^T$ also. Transpose both sides: $A = (A^T)^T = (A^T I)^T = I^T(A^T)^T = IA$, so $IA = A$ for all A.

We have proved that for all square matrices A

$$IA = AI = A.$$

Definition 5 (Inverse)**.** *Let I denote the identity matrix.*

A square matrix A is **invertible** *(or* **non-singular***) if there exists a matrix B such that*

$$AB = BA = I.$$

Then B is called the inverse *of A and is denoted A^{-1}. A matrix that is not invertible is also said to be* singular.

$II = I, \quad I^{-1} = I$

Inverse for the 2×2 case

this trick doesn't work for $n \times n$ matrices with $n \geq 3$

Let A be a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Set $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \underbrace{\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}}_{= \Delta I} = BA.$$

Let

$$\Delta = ad - bc.$$

If $\Delta \neq 0$, then

$$A^{-1} = \frac{1}{\Delta} B = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$AB = \Delta I$

We assume $\Delta \neq 0$

$A(\frac{1}{\Delta}B) = I$

We call Δ the *determinant* of A.

$$F_k(\alpha) = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & \text{\huge O} & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & \text{\huge O} & & & \alpha & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \qquad \alpha \neq 0.$$

k-th row

k-th column

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & & O \\ & \alpha & \\ O & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2\alpha & 3 \\ 4 & 5\alpha & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & O \\ & \alpha & \\ O & & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3\alpha & 4\alpha \\ 5 & 6 \end{pmatrix}$$

$$F_k(\alpha) \cdot F_k\left(\frac{1}{\alpha}\right) = I \qquad F_k(\alpha)^{-1} = F_k\left(\frac{1}{\alpha}\right)$$

$\alpha \in \mathbb{R}$

$$G_{ij}(\alpha) = \begin{array}{cc} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & \alpha & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow i\text{-th row} \\ \\ \\ \end{array} \\ \begin{array}{l} \quad\ \ \uparrow \\ \quad j\text{-th column} \end{array} & \end{array}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha+2 \\ 3 & 3\alpha+4 \\ 5 & 5\alpha+6 \end{pmatrix}$$

$$G_{ij}(\alpha)\, G_{ij}(-\alpha) = I$$

$$G_{ij}(\alpha)^{-1} = G_{ij}(-\alpha)$$

$$H_{ij} = \begin{array}{c} \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & 1 & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & 1 & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \begin{array}{l} \\ \\ i\text{-th row} \\ \\ \\ j\text{-th row} \\ \\ \\ \end{array} \\ \begin{array}{cc} i\text{-th col} & j\text{-th col} \end{array} \end{array} \qquad i \neq j$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{H_{23}} = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \end{pmatrix}$$

$$H_{23} H_{23} = I \qquad H_{23}^{-1} = H_{23}$$

Example 9. *Find the inverse matrix of* $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$. *Check your answer.*

$\Delta = 1 \times 5 - 2 \times 3 = -1.$

$$\Rightarrow A^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

Check whether $AA^{-1} = A^{-1}A = I$:
Multiplying the two matrices A and A^{-1} gives

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So A is invertible with inverse

$$\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

Example 10 (Inverse of a 3×3 matrix)**.** *Let*

$$A = \begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix}, \qquad B = \begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix}.$$

Show that $B = A^{-1}$.

An easy way to do this is to multiply the two matrices together.

$$AB = \begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix}\begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and, similarly

$$BA = \begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

not necessary

$(A^{-1})^{-1} = A$

Properties of inverses

(1) A has at most **one** inverse

$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

(2) If A, B are invertible so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1} \quad (\textit{not } A^{-1}B^{-1}!)$$

(3) A is invertible if and only if A^T is invertible

(4) If A is invertible then $(A^T)^{-1} = (A^{-1})^T$.

$(ABC)^{-1}(ABC) = C^{-1}B^{-1}(A^{-1}A)BC = C^{-1}(B^{-1}B)C = I$

Determinants

Definition 6. *Associated to each square matrix A is a number called the* determinant *of A and denoted $|A|$ or $\det(A)$. It is defined as follows.*

$\det((a)) = a$

- *If A is a 1×1 matrix, say $A = (a)$ then $|A|$ is defined to be a.*
- *If $A = (a_{ij})$ is 2×2 then we define* $\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

- *In general if $A = (a_{ij})$ is $n \times n$, first set* $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$$C_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\ j-1} & a_{1\ j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\ j-1} & a_{2\ j+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1\ 1} & a_{i-1\ 2} & \cdots & a_{i-1\ j-1} & a_{i-1\ j+1} & \cdots & a_{i-1\ n} \\ a_{i+1\ 1} & a_{i+1\ 2} & \cdots & a_{i+1\ j-1} & a_{i+1\ j+1} & \cdots & a_{i+1\ n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\ j-1} & a_{n\ j+1} & \cdots & a_{nn} \end{vmatrix}$$

called the cofactor *of a_{ij}. The $(n-1) \times (n-1)$ determinant is obtained by* omitting *the ith row and jth column from A (indicated by the horizontal and vertical lines in the matrix).*

$= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

$= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

We then define

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots\cdots + a_{1n}C_{1n}$$

which gives a *recursive* definition of the determinant.

Observe that $(-1)^{i+j}$ gives the pattern $\begin{pmatrix} + & - & + & - & \cdot & \cdot & \cdot & \cdot \\ - & + & - & + & \cdot & \cdot & \cdot & \cdot \\ + & - & + & - & \cdot & \cdot & \cdot & \cdot \\ \vdots & & & & & & & \end{pmatrix}$

Thus for a 3×3 matrix A we have

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

where

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Properties of determinants

Property (1): $|A| = |A^T|$

Consider for example the 2×2 case. Then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \Rightarrow \quad A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

Thus $|A| = a_{11}a_{22} - a_{12}a_{21} = |A^T|$. It can be shown that this holds for any square matrix, not just in the 2×2 case.

This means that any results about the *rows* in a general determinant is also true about the *columns* (since the rows of A^T are the columns of A). In particular, any statement about the effect of row operations on determinants is also true for column operations.

Warning: We used row operations to simplify systems of equations, because they do not change the solution. Column operations may change the solution of linear systems, so we should not use column operations on such systems.

Property (2): The determinant may by taking cofactors along any row (not just the first) or down any column. Eg. for a 3×3 matrix A

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{(definition, expansion along 1st row)} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{(expansion along 2nd row)} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} && \text{(expansion down 3rd column) etc.} \end{aligned}$$

This is useful if one row or column contains a larger number of zeros.

Determinants of triangular matrices

$$\text{Suppose } A = \begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & 0 & & \vdots \\ \vdots & & \ddots & & \vdots \\ & & & & 0 \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{pmatrix}$$

$(-1)^{1+1}$

is lower triangular. Then by repeated expansion along $\boldsymbol{r}_1$ we obtain

$$|A| = a_{11} \underbrace{\begin{vmatrix} a_{22} & & \\ \vdots & \ddots & \\ \vdots & & \vdots \\ a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}}_{\text{Another triangular matrix}} = \ldots = a_{11}\, a_{22} \cdot \cdots \cdot a_{nn},$$

which is the product of the diagonal entries. The same result holds for upper triangular and diagonal matrices.

In particular

$$|I| = 1.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$i \neq j$

$\tilde{A}_{ij}$ is obtained by swapping the i-th row and the j-th row of A $\Rightarrow$ $\tilde{A}_{ij} = H_{ij} A$

$$\det(\tilde{A}_{ij}) = -\det(A)$$

$$\det(H_{ij}) = -1$$

$\Rightarrow$ if A has two identical rows, then $\det(A) = 0$

if A has two identical columns, then $\det(A) = 0$

$\alpha \in \mathbb{R}$

$A^*_{ij}(\alpha)$: obtained by $A^*_{ij}(\alpha) = G_{ij}(\alpha)\, A$

$$\det(A^*_{ij}(\alpha)) = \det(A)$$

$$\det(F(\alpha)\, A) = \alpha \det(A)$$

$\alpha \neq 0$

Connection with inverses

An important property of determinants is the following.

Fact (product of determinants)

Let A, B be $n \times n$ matrices. Then

$$|AB| = |A| \cdot |B| \, .$$

Theorem 1 (Invertible matrices)**.**

$$A \text{ is invertible} \iff |A| \neq 0.$$

Proof $\implies$ If A is invertible, then

$$\begin{aligned} & I = AA^{-1} \\ \Rightarrow \quad & 1 = |I| = |AA^{-1}| = |A| \cdot |A^{-1}| \, , \\ & \text{So } |A| \neq 0. \end{aligned}$$

$\Longleftarrow$ ~~Follows from below~~.

Remark 1. *It follows immediately from the proof that if A is invertible, then*

$$|A^{-1}| = \frac{1}{|A|},$$

i.e., the inverse of the determinant is the determinant of the inverse.

1.5 Vectors operations (in 2 and n dimensions)

- A *vector* quantity has both a magnitude and a direction. Force and velocity are examples of vector quantities.
- A *scalar* quantity has only a magnitude (it has no direction). Time, area and temperature are examples of scalar quantities.

A vector is represented geometrically in the (x, y) plane (or in (x, y, z) space) by a directed line segment (arrow). The direction of the arrow is the direction of the vector, and the length of the arrow is proportional to the magnitude of the vector. Only the length and direction of the arrow are significant: it can be placed anywhere convenient in the (x, y) plane (or (x, y, z) space).

If P, Q are points, $\overrightarrow{PQ}$ denotes the vector from P to Q.

A vector $\boldsymbol{v} = \overrightarrow{PQ}$ in the (x, y) plane may be represented by a pair of numbers

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \end{pmatrix}$$

Assume $\det(A) \neq 0$.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \text{Cofactor}: \tilde{C} = \begin{pmatrix} \tilde{C}_{11} & C_{12} \cdots & C_{1n} \\ \vdots & \vdots & \cdots \vdots \\ C_{n1} & C_{n2} \cdots & C_{nn} \end{pmatrix}$$

Recall: $\det(A) = \sum_{k=1}^{n} a_{sk} \underbrace{\tilde{C}_{sk}}_{(\tilde{C}^T)_{ks}} \neq 0.$

$\det(A) = (A\tilde{C}^T)_{ss}$

what about $\underbrace{(A\tilde{C}^T)_{st} \text{ when } s \neq t}_{\parallel \atop 0}$

$$A\,C^T = \begin{pmatrix} \det(A) & & & 0 \\ & \det(A) & & \\ & & \ddots & \\ 0 & & & \det(A) \end{pmatrix}$$

$$= \det(A)\, I$$

$$A\left(\frac{1}{\det(A)}\,C^T\right) = I$$

$\Rightarrow A$ is invertible