Problems for the prac of Week 6

Q1. Find $\sum_{n=0}^{\infty} 0.2^n \times 10^2$

<u>Sol</u>. Let $a=10^2$ and $r=0.2$ to have

$$\sum_{n=0}^{\infty} 0.2^n \times 10^2 = 10^2 + \sum_{n=1}^{\infty} ar^n = a + \frac{ar}{1-r} = \frac{a}{1-r}$$

$$= 125$$

Q2. Determine whether or not the following series

Converge

① $\sum_{n=1}^{\infty} \frac{n^3}{n^2+3}$ ② $\sum_{n=1}^{\infty} \frac{100n^5}{n^7+2}$ ③ $\sum_{n=1}^{\infty} \frac{n^{-2}}{3+9n^6}$

<u>Sol</u>. ① $\sum_{n=1}^{\infty} \frac{n^3}{n^2+3} \geq \sum_{n=1}^{\infty} \frac{n^3}{4n^2} = \frac{1}{4}\sum_{n=1}^{\infty} n$ diverges

② $\sum_{n=1}^{\infty} \frac{100n^5}{n^7+2} \leq 100 \sum_{n=1}^{\infty} \frac{1}{n^2+2} \leq 100 \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges.

③ $\sum_{n=1}^{\infty} \frac{n^{-2}}{3+9n^6} \leq \sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges.

Q3. Find the following limits

① $\lim\limits_{x\to 1} \frac{x-1}{x^2-1}$ ② $\lim\limits_{x\to\pm\infty} \frac{x-1}{x^2-1}$ ③ $\lim\limits_{x\to 0} \frac{e^x-x}{(\sin x)+2}$

④ $\lim\limits_{x\to 0} \frac{\sin(x^2)}{x^2}$ ⑤ $\lim\limits_{x\to\infty} \frac{x^2+2}{\sqrt{x^4+2}}$ ⑥ $\lim\limits_{x\to\infty} \frac{\sin(x^2)}{x^2}$

⑦ $\lim\limits_{x\to\infty} \frac{x-\cos x}{x}$ ⑧ $\lim\limits_{x\to 0} \frac{\tan x}{x}$ ⑨ $\lim\limits_{x\to\infty} \sqrt{x}\sin\frac{\pi}{x}$

Sol. ① $\lim\limits_{x\to 1} \frac{x-1}{x^2-1} = \lim\limits_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$

② $\lim\limits_{x\to\pm\infty} \frac{x-1}{x^2-1} = \lim\limits_{x\to\pm\infty} \frac{1}{x+1} = \lim\limits_{x\to\pm\infty} \frac{1}{x}\cdot\frac{1}{1+\frac{1}{x}} = 0$

③ $\lim\limits_{x\to 0} \frac{e^x-x}{(\sin x)+2} = \frac{1}{2}$

④ Let $f(x) = \begin{cases} \frac{\sin x}{x} & x\neq 0 \\ 1 & x=0 \end{cases}$

$\Rightarrow f$ is continuous at $x=0$.

$\lim\limits_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim\limits_{x\to 0} f(x^2) = f(\lim\limits_{x\to 0} x^2) = f(0) = 1.$

⑤ $$\lim_{x\to\infty} \frac{x^2+2}{\sqrt{x^4+2}} = \lim_{x\to\infty} \frac{1+\frac{2}{x^2}}{\sqrt{1+\frac{2}{x^4}}} = 1$$

⑥ Recall $-\frac{1}{x^2} \leqslant \frac{\sin(x^2)}{x^2} \leqslant \frac{1}{x^2}$ and

$\lim\limits_{x\to\infty} -\frac{1}{x^2} = 0 = \lim\limits_{x\to\infty} \frac{1}{x^2}$. So, $\lim\limits_{x\to\infty} \frac{\sin(x^2)}{x^2}$.

⑦ Recall $-1 \leqslant \cos x \leq 1$. So, from $\lim\limits_{x\to\infty} \frac{x-1}{x} = \lim\limits_{x\to\infty} \frac{x+1}{x}$

$=1$, we have $\lim\limits_{x\to\infty} \frac{x-\cos x}{x} = 1$

⑧ $$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cos x = 1$$

⑨ Define $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$

So, f is continuous at $x=0$.

$$\lim_{x\to\infty} \frac{x}{\pi} \sin\frac{\pi}{x} = \lim_{x\to\infty} f\left(\frac{\pi}{x}\right) = f\left(\lim_{x\to\infty}\frac{\pi}{x}\right) = 1$$

$$\Rightarrow \lim_{x\to\infty} \sqrt{x} \sin\frac{\pi}{x} = \lim_{x\to\infty} \frac{\pi}{\sqrt{x}} \frac{x}{\pi} \sin\frac{\pi}{x} = 0$$

Q4 Find the following limits

① $\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)$ ② $\lim_{x\to\infty}\left(\ln(x+1)-\ln x\right)$

Sol. ① $\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)$

$$= \lim_{x\to\infty}\frac{+1}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}}\left(\frac{1}{\sqrt{1+\frac{1}{x}}+1}\right) = 0$$

② $\lim_{x\to\infty}\left(\ln(x+1)-\ln x\right) = \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)$

$$= \ln\left(1+\lim_{x\to\infty}\frac{1}{x}\right) = 0$$