problems for week 2 pracs.

Q1 Let

$$A=\begin{bmatrix} p & 0 \\ q & r \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C=\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$$

When does $AB=BA$? When does $BC=CB$?

When does $(AB)C = A(BC)$?

Sol:

$$AB=\begin{bmatrix} p & p \\ q & q+r \end{bmatrix}, \quad BA=\begin{bmatrix} p+q & r \\ q & r \end{bmatrix}$$

So, $AB=BA$ if and only if

$q=0$ and $r=p$

$$BC=\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}, \quad CB=\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \quad \text{(by chance)}$$

By chance, $BC=CB$.

One always have $(AB)C = A(BC)$.

Q2. Use Gauss-Jordan method to compute A^{-1}

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$[A \ e_1 \ e_2 \ e_3] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \text{start}$$

$$\to \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \left(R_2 \leftarrow \frac{1}{2}R_1 + R_2\right)$$

$$\to \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 4/3 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \left(R_3 \leftarrow \frac{2}{3}R_2 + R_3\right)$$

$$\to \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{bmatrix} \left(R_2 \leftarrow \frac{3}{4}R_3 + R_2\right)$$

$$\to \begin{bmatrix} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{bmatrix} \left(R_1 \leftarrow \frac{2}{3}R_2 + R_1\right)$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} \begin{pmatrix} R_1 \leftarrow \frac{1}{2}R_1 \\ R_2 \leftarrow \frac{2}{3}R_2 \\ R_3 \leftarrow \frac{3}{4}R_3 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

Q3. Let $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$ find A^{-1}

<u>Sol</u>.

$$[A, e_1, \cdots, e_4] = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \text{Swap } R_1, R_4 \\ \text{Swap } R_2, R_3 \end{pmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} R_1 \leftarrow \frac{1}{5}R_1 \\ R_2 \leftarrow \frac{1}{4}R_2 \\ R_3 \leftarrow \frac{1}{3}R_3 \\ R_4 \leftarrow \frac{1}{2}R_4 \end{pmatrix} ※$$

~~Q4~~ (Inverse of block matrices)

Let $W = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$, where

$A \in \mathbb{R}^{n\times n}$ and $C = \mathbb{R}^{m\times m}$ are both invertible.

Find W^{-1}

<u>Sol</u>.

$$\left[\begin{array}{cccc} A & 0 & I_{n\times n} & 0 \\ B & C & 0 & I_{m\times m} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc} I & 0 & A^{-1} & 0 \\ C^{-1}B & I & 0 & C^{-1} \end{array}\right] \quad \begin{pmatrix} A^{-1}R_1 \rightarrow R_1 \\ C^{-1}R_2 \rightarrow R_2 \end{pmatrix}$$

Note: $A^{-1}R_1$, not $R_1 A^{-1}$.
row operations only!

$$\rightarrow \left[\begin{array}{cccc} I & 0 & A^{-1} & 0 \\ 0 & I & -C^{-1}BA^{-1} & C^{-1} \end{array}\right] \quad (R_2 - C^{-1}BR_1 \rightarrow R_2)$$

So, $W^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$

Q5. Find the determinants

$$a = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}, \quad b = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

<u>Sol</u>. Expand a according to the first row:

$$a = 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - (-1)^{1+2} \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 2+1 = 3$$

Expand b according to the first row

$$b = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} - (-1)^{1+2} \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= a + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = a + 2 = 5.$$

Q6. Show that

$$A_n := \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}_{n\times n}$$

$$= \begin{pmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & 0 & \\ & -\frac{2}{3} & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & -\frac{n-1}{n} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & & & 0 \\ & \frac{3}{2} & -1 & & \\ & & \frac{4}{3} & \ddots & \\ & & & \ddots & -1 \\ 0 & & & & \frac{n+1}{n} \end{pmatrix}$$

Therefore, find $\det(A_n)$

Sol. For any $i,j = 2,\cdots,n$, the (i,j) entry of RHS is

$$\begin{array}{c} (0, \cdots 0, \underset{\substack{\uparrow \\ (i-1)}}{-\frac{i-1}{i}}, \underset{\substack{\uparrow \\ (i)}}{1}, 0 \cdots 0) \end{array} \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ \frac{j+1}{j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow (j-1) \\ \leftarrow (j) \\ \\ \\ \\ \end{array}$$

$$= \begin{cases} -1 & i = j-1 \text{ or } i = j+1 \\ 2 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

The first row of RHS is $(2, -1, 0, \cdots 0)$

The first column of RHS is

$$\begin{pmatrix} 2 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This verifies the equation

$$\Rightarrow \det(A_n) = 2 \times \frac{3}{2} \times \frac{4}{3} \times \cdots \times \frac{n+1}{n} = n+1 \quad ※$$

Q7. Let $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}^n$ be column vectors.

Write $A = [a_1, \cdots, a_n] \in \mathbb{R}^{n\times n}$, $B = [b_1, \cdots, b_n] \in \mathbb{R}^{n\times n}$

Show that $AB^T = \sum_{i=1}^{n} a_i b_i^T$

proof. $(AB^T)_{rs} = \sum_{l=1}^{n} A_{rl} B_{sl}$

$$\left(\sum_{i=1}^{n} a_i b_i^T\right)_{rs} = \sum_{i=1}^{n} A_{ri} B_{si} = (AB^T)_{rs} \quad ※$$