Present your answers in order, showing the working for each answer.

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined via

$$f([u \ \ v]^T) = \begin{bmatrix} u^3 v \\ uv^2 - e^{u+v} \end{bmatrix}.$$

   Further, define the function $g : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$,

$$g(x) = f(x)[2 \ \ 1].$$

   Note here that $[2 \ \ 1]$ is a row vector. Also let $z = [-1 \ \ 1]^T$ be a (column) vector in $\mathbb{R}^2$.

   (a) Evaluate $f(z)$. [1]

   (b) Evaluate $g(z)$. [1]

   (c) Evaluate $||g(z)z||$. [1]

   (d) Evaluate the inner product between the two columns of $g(z)$. [1]

   (e) Find $\det(g(x))$ for any $x \in \mathbb{R}^2$. Explain why the answer does not depend on x. [2]

   (f) Find the Jacobian matrix $Df(u, v)$ associated with the function $f(\cdot)$. [2]

   (g) Consider now the linear approximation around z at a point $x \in \mathbb{R}^2$,

$$\hat{f}(x) = f(z) + Df(z)(x - z).$$

   Find a point $x^0 \in \mathbb{R}^2$ such that $\hat{f}(x^0) = 0$. [3]

2. Let A and B be two lower triangular $n \times n$ matrices. That is for $i < j$, $A_{i,j} = 0$ and $B_{i,j} = 0$. Consider now the unit vector $e_1 \in \mathbb{R}^n$ with 0 entries everywhere except the first entry which is 1. Determine the value of $e_1^T AB e_1$. [2]

3. Let $u, v \in \mathbb{R}^n$. Use the definition of the 2-norm $|| \cdot ||$ to prove that if $u^T v = 0$ then [2]

$$\|u + v\|^2 - \|u\|^2 - \|v\|^2 = 0\,.$$

Total: [15]