Lecture 3.3

Random variables and their distribution: MGFs

Moment generating functions

We already encountered $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$, i.e., the expectation of first and second powers of X. More generally, the expectations of X^k for $k \geq 1$, when they exist, are called **moments of** X and have useful theoretical properties, and some of them are used for additional summaries of a distribution.

The **moment generating function** (MGF) of a random variable X is the function, $M : I \to [0, \infty)$, given by

$$M(s) = \mathbb{E}(e^{sX}),$$

where I is any open interval containing 0 for which the above expectation a finite for all $s \in I$.

If X is bounded, i.e., there exists a constant $M > 0$ such that $X < M$, then the MGF is defined for all of $\mathbb{R}$.

Moment generating functions

Let X have pmf f given by

$$f(x) = p(1-p)^x \quad x = 0, 1, 2, \ldots$$

where $p \in (0, 1)$ is a parameter. What is the MGF of X?

Answer:

$$M(s) = \mathbb{E}(e^{sX}) = \sum_{x=0}^{\infty} e^{sx} p(1-p)^x$$

By the ratio test, the sum converges for $e^s(1-p) < 1$, i.e $s < -\ln(1-p)$. So

$$M(s) = p \sum_{x=0}^{\infty} (e^s(1-p))^x = \frac{p}{1 - e^s(1-p)},$$

where we use the geometric series.

Properties of the MGF

1. The MGFs of two random variables are the same if and only if their corresponding distribution functions are the same.
2. If $M(s)$ is finite for all s in some open interval around 0, then for each integer $n > 0$, $\mathbb{E}(X^n)$ exists and

$$\mathbb{E}(X^n) = \frac{\mathrm{d}^n M(s)}{\mathrm{d}s^n} \,|_{s=0} = M^{(n)}(0).$$

3. For any real numbers a and b,

$$M_{aX+b}(s) = e^{bs} M_X(as),$$

   for any value of s such that $M_X(as)$ is finite.

Note: For some distributions, the moments exist and yet the moment-generating function does not, e.g., log-normal distribution.

MGF example

Suppose X has MGF $M(s) = \frac{p}{1-e^s(1-p)}$ for $s < -\ln(1-p)$. What is the expected value of X?

Answer: Differentiate the MGF with respect to s,

$$M'(s) = \frac{d}{ds}\frac{p}{1-e^s(1-p)} = \frac{p(1-p)e^s}{(1-e^s(1-p))^2},$$

using the chain rule. Evaluating this derivative at $s = 0$,

$$\mathbb{E}X = M'(0) = (1-p)/p.$$

What is the MGF of $Y = X + 1$?

Answer:

$$M_Y(s) = e^s M_X(s) = \frac{pe^s}{1-e^s(1-p)}, \quad s < -\ln(1-p).$$