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Q_1

(a) $\left(\begin{array}{ccc|ccc} 1 & x & 0 & 1 & 0 & 0 \\ 0 & 1 & x & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-xr_3+r_2} \left(\begin{array}{ccc ccc} 1 & x & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -x \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$\xrightarrow{-xr_2+r_1} \left(\begin{array}{ccc ccc} 1 & 0 & 0 & 1 & -x & x^2 \\ 0 & 1 & 0 & 0 & 1 & -x \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$A^{-1} = \begin{pmatrix} 1 & -x & x^2 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}$ #

(b) $\det(A^{-1}) = 1$ $\lim_{x\to\infty} \det(A^{-1}) = 1$

#

(c) $A = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$ $A \times A = \begin{pmatrix} 1 & 2x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 1 \end{pmatrix} \times A = \begin{pmatrix} 1 & 3x & 6x^2 \\ 0 & 1 & 3x \\ 0 & 0 & 1 \end{pmatrix} \ldots$

we can see $A^n = \begin{pmatrix} 1 & nx & \frac{n(n-1)x^2}{2} \\ 0 & 1 & nx \\ 0 & 0 & 1 \end{pmatrix}$ #

Q_2

(a) $\binom{100}{2} = \frac{100!}{2!(100-2)!} = 4950$

#

(b) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-n+k)!}$ when $0 \leq k \leq n$

So $\binom{n}{k} = \binom{n}{n-k}$

#

(c) $(1+\frac{1}{n})^{100} = \sum_{k=0}^{100} \binom{100}{k}(\frac{1}{n})^k = \binom{100}{0} + \binom{100}{1}\frac{1}{n} + \cdots + \binom{100}{k}(\frac{1}{n})^k$

$\lim_{n\to\infty} n((1+\frac{1}{n})^{100} - 1)$

$= \lim_{n\to\infty} (\binom{100}{1} + \binom{100}{2}\frac{1}{n} + \cdots + \binom{100}{k}\frac{1}{n^{k-1}}) = 100$ #

Q3

(a) $\lim_{x\to 0} \frac{\sin(\tan x)}{\sin x}$ $\because$ $\lim_{x\to 0} \sin(\tan x) = 0$ $\lim_{x\to 0} \sin x = 0$

So use L' Hopital's rules. $\lim_{x\to 0} \frac{(\sin(\tan x))'}{(\sin x)'} = \frac{\cos(\tan x)\sec^2(x)}{\cos x} = 1$

(b) $\lim_{x\to\infty} \frac{x^2+4x+4}{x^2+x-2} = \lim_{x\to\infty} \frac{1+\frac{4}{x}+\frac{4}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}} = 1$ #

(c) $\lim_{x\to 1} (x-1)^2 \sin(x^2)\cos(1+e^x)$

$\Rightarrow (1-1)^2 \sin(1)\cdot\cos(1+e) = 0$

So $\lim_{x\to 1} (x-1)^2 \sin(x^2)\cos(1+e^x) = 0$ #

Q4

(a) $f'(x) = \frac{2x}{1+x^2} + 8x$ $f'(0) = 0$ So $f(x)$ has a critical point #

(b) $f''(x) = \frac{2-2x^2}{(1+x^2)^2} + 8$ $f''(0) = 10$ $\because f''(0) > 0$ So $x=0$ is a local minimun #

(c) $\because f''(x) > 0$ for all x

$\therefore x=0$ is a global minimum #

Q5

(a) $\frac{\partial f}{\partial x} = e^{x} + e^{x} \cdot \left(\frac{1}{1+x^2} \cdot 2x \right)$

$\frac{\partial f}{\partial y} = e^{y} \ln(1+x^2)$ #

(b) $f(1,1) = e + e\ln 2$

$f(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= e + e\ln 2 + 2e(x-1) + e\ln 2(y-1)$

$= 2ex + e\ln 2\, y - e$

(c) $f(1.05, 1.05) = 1.1e + 1.05e\ln 2$

$\Delta f = f(1.05, 1.05) - f(1,1)$

$= 0.1e + 0.05e\ln 2$ #

Q6

(a) $\int (e^{3x} + \cos(2x))\,dx$

$= \frac{1}{3}e^{3x} + \frac{\sin(2x)}{2} + C$

#

(b) assume $u = x^2$ $v' = (1+x)^{100}$ $\Rightarrow u' = 2x$

$v = \frac{1}{101}(1+x)^{101} + C$

(c) $\int (\sin x)^2 \, dx = \int \frac{(1-\cos(2x))}{2} dx = \int (\frac{1}{2} - \frac{\cos(2x)}{2}) dx$

$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$ #

Q7

(a) $\frac{df}{dx} = -\frac{2x}{1+x^2+y^2}$ $\frac{df}{dy} = -\frac{2y}{1+x^2+y^2}$

$\nabla f = -\frac{2x}{1+x^2+y^2} i - \frac{2y}{1+x^2+y^2} j$ #

(b) $f_u = \nabla f \cdot \frac{u}{||u||}$ $\frac{u}{||u||} = \frac{(-1,-1)}{\sqrt{2}}$

$= (-\frac{2}{3}, -\frac{2}{3}) \cdot \frac{(-1,-1)}{\sqrt{2}}$

$= \frac{2\sqrt{2}}{3}$ #

(c) $\nabla f(-1,-1) = (\frac{2}{3}, \frac{2}{3})$ $||\nabla f(-1,-1)|| = \frac{2\sqrt{2}}{3}$ #