Lecture 4.1

Random variables and their distribution: Transformations

Transformations of random variables

We often need to construct new random variables by transforming old ones, e.g.,

- Changing the units of measurement
- Output of deterministic algorithms with random inputs
- Estimators of parameters

We will only consider transformations of a single random variable.

Transformations of discrete random variables

Recall that X is a discrete random variable if it only takes on discrete values.

Let $Y = g(X)$ where g is some arbitrary function. Obviously, Y is also a discrete rv that takes on values $\{y_1, y_2, \ldots\}$ such that $\{x_1, x_2, \ldots\} \overset{g}{\mapsto} \{y_1, y_2, \ldots\}$.

So, we can express the pmf of Y, i.e., f_Y, in terms of the pmf of X, i.e., f_X, as

$$
\begin{aligned}
f_Y(y_j) &= \mathbb{P}(Y = y_j) = \mathbb{P}(g(X) = y_j) \\
&= \sum_{x:g(x)=y_j} \mathbb{P}(X = x) = \sum_{x:g(x)=y_j} f_X(x) \,.
\end{aligned}
$$

Transformations of continuous random variables

Suppose X is a continuous random variable and let $Y = g(X)$ where g is some arbitrary function.

Similar to the case of the discrete rv, the cdf of Y can be easily calculated as follows

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \int_{\{x: g(x) \leq y\}} f_X(x) \, \mathrm{d}x.$$

Transformations of continuous random variables

Even if g is continuous, $Y = g(X)$ need not be a continuous random variable.

- **Necessary Condition:** If $g(X)$ is a continuous random variable for any continuous random variable X, then necessarily g is not constant on any interval.
- **Sufficient Condition:** If g is differentiable and its derivative is zero at only finitely many points (e.g., g is linear or strictly monotone), then $Y = g(X)$ will be a continuous rv.

If $Y = g(X)$ is also a continuous random variable, then we can also calculate the pdf of Y as $f_Y(y) = F'_Y(y)$, at every point y at which F_Y is differentiable.

Transformations of continuous random variables

Suppose the continuous random variable X has pdf

$$f_X(x) = \tfrac{1}{2}\exp(-|x|), \quad x \in \mathbb{R}.$$

Define $Y = X^2$. What is the pdf of Y?

Answer: Note that Y can only take values in $[0, \infty)$.Also, $g(x) = x^2$ satisfy the sufficient condition on the previous slide. So, for any $y \geq 0$

$$\begin{aligned}
F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \mathbb{P}(X \leq \sqrt{y}) - \mathbb{P}(X \leq -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})
\end{aligned}$$

$$\begin{aligned}
f_Y(y) = F_Y'(y) &= \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right) \\
&= \begin{cases} \dfrac{1}{2\sqrt{y}}\exp(-\sqrt{y}), & y \in [0, \infty) \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

Linear transformations of continuous random variables

Suppose X is a continuous rv with cdf F_X and pdf f_X. Define $Y = aX + b$, where $a > 0$ and $b \in \mathbb{R}$. Then, for any $y \in \mathbb{R}$,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(aX + b \leq y) \\ &= \mathbb{P}\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right). \end{aligned}$$

We can now find f_Y, the pdf of Y, by differentiating the cdf F_Y:

$$f_Y(y) = F_Y'(y) = \frac{1}{a}\, f_X\left(\frac{y-b}{a}\right).$$

Linear transformations of continuous random variables

Suppose X has pdf

$$f_X(x) = \tfrac{1}{2}\exp(-|x|), \quad x \in \mathbb{R}.$$

Define $Y = 3X + 2$. What is the pdf of Y?

Answer: For any $y \in \mathbb{R}$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(3X + 2 \leq y) = F_X\left(\frac{y-2}{3}\right)$$

Differentiating the cdf of Y, for all $y \in \mathbb{R}$,

$$f_Y(y) = F_Y'(y) = \frac{1}{3} f_X\left(\frac{y-2}{3}\right) = \frac{1}{6}\exp\left(-\left|\frac{y-2}{3}\right|\right).$$

Linear transformations of continuous random variables

In general, for a continuous random variable X with pdf f_X, the pdf of $Y = aX + b$, where $a \neq 0$ and $b \in \mathbb{R}$, is given by

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

Family of distributions can be constructed by a linear transformations. Such a family is sometimes called location–scale family.

Monotone transformations of continuous random variables

Suppose $Y = g(X)$, where g is strictly increasing on the range of values that X takes. Then g^{-1}, the inverse of g, is also strictly increasing.

We can express the cdf of Y in terms of F_X using similar reasoning to what we used for the linear transformation.

If the pdf of X is non-zero on (a, b), then the pdf of Y is non-zero on $(g(a), g(b))$. For all $y \in (g(a), g(b))$,

$$F_Y(y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq g^{-1}(y)) = F_X(g^{-1}(y)).$$

We obtain the pdf of Y by differentiating its cdf (using chain rule).

Monotone transformations of continuous random variables

Suppose X has pdf

$$f_X(x) = \tfrac{1}{2}\exp(-|x|), \quad x \in \mathbb{R}.$$

Define $Y = e^X$. What is the pdf of Y?

Answer: Note that Y only takes values in $(0, \infty)$. For any $y > 0$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(e^X \leq y) = \mathbb{P}(X \leq \ln y) = F_X(\ln y).$$

Differentiating the cdf of Y, for $y > 0$,

$$f_Y(y) = F_Y'(y) = \frac{1}{y} f_X(\ln y) = \frac{1}{2y}\exp(-|\ln y|)$$

and $f_Y(y) = 0$ for $y \leq 0$.

Monotone transformations of continuous random variables

If g is strictly decreasing, then its inverse is also strictly decreasing.

If the pdf of X is non-zero on (a, b), then the pdf of Y is non-zero on $(g(b), g(a))$. Using the same reasoning as in the case of g being strictly increasing then shows that for all $y \in (g(b), g(a))$,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) = \mathbb{P}(X \ge g^{-1}(y)) = 1 - \mathbb{P}(X < g^{-1}(y)) \\ &= 1 - \mathbb{P}(X \le g^{-1}(y)) = 1 - F_X(g^{-1}(y)). \end{aligned}$$

Again, we obtain the pdf of Y by differentiating its cdf.