54686474
Yifan zhang

1.

(a) $\begin{pmatrix} 5 & 3 & -2 \\ 4 & -2 & 3 \\ 1 & 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 24 \end{pmatrix}$ #

(b)

```
>> A = [5, 3, -2;
        4, -2, 3;
        1, 1, -5;]

A =

     5     3    -2
     4    -2     3
     1     1    -5

>> b = [4; 12; 24];
>> w = A \ b

w =

    3.3043
   -8.0000
   -5.7391
```

#

(c) $\begin{pmatrix} 5 & 3 & -2 & 4 \\ 4 & -2 & 3 & 12 \\ 1 & 1 & -5 & 24 \end{pmatrix} \xrightarrow[R_1 \text{ and } R_3]{\text{switch}} \begin{pmatrix} 1 & 1 & -5 & 24 \\ 4 & -2 & 3 & 12 \\ 5 & 3 & -2 & 4 \end{pmatrix}$

$\xrightarrow[-5R_1+R_3]{-4R_1+R_2} \begin{pmatrix} 1 & 1 & -5 & 24 \\ 0 & -6 & 23 & -84 \\ 0 & -2 & 23 & -116 \end{pmatrix} \xrightarrow{-\frac{1}{6}R_2} \begin{pmatrix} 1 & 1 & -5 & 24 \\ 0 & 1 & -\frac{23}{6} & 14 \\ 0 & -2 & 23 & -116 \end{pmatrix}$

$\xrightarrow{2R_2+R_3} \begin{pmatrix} 1 & 1 & -5 & 24 \\ 0 & 1 & -\frac{23}{6} & 14 \\ 0 & 0 & \frac{46}{3} & -88 \end{pmatrix} \xrightarrow{\frac{3}{46}R_3} \begin{pmatrix} 1 & 1 & -5 & 24 \\ 0 & 1 & -\frac{23}{6} & 14 \\ 0 & 0 & 1 & -\frac{132}{23} \end{pmatrix}$

$\Longrightarrow z = -\frac{132}{23}$ $\quad y - \frac{23}{6}z = 14 \Rightarrow y = -8$ $\quad x + y - 5z = 24$

$x = \frac{76}{23}$

$$W = \left[\frac{76}{23} \quad -8 \quad -\frac{132}{23} \right]^T \#$$

(d)

$$\left(\begin{array}{ccc|ccc} 5 & 3 & -2 & 1 & 0 & 0 \\ 4 & -2 & 3 & 0 & 1 & 0 \\ 1 & 1 & -5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_2 \text{ and } R_1]{\text{switch}} \left(\begin{array}{ccc|ccc} 1 & 1 & 5 & 0 & 0 & 1 \\ 4 & -2 & 3 & 0 & 1 & 0 \\ 5 & 3 & -2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-5R_1+R_3]{-4R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & -5 & 0 & 0 & 1 \\ 0 & -6 & 23 & 0 & 1 & -4 \\ 0 & -2 & 23 & 1 & 0 & -5 \end{array}\right) \xrightarrow{-\frac{1}{6}R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & -5 & 0 & 0 & 1 \\ 0 & 1 & -\frac{23}{6} & 0 & -\frac{1}{6} & \frac{2}{3} \\ 0 & -2 & 23 & 1 & 0 & -5 \end{array}\right)$$

$$\xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{7}{6} & 0 & \frac{1}{6} & \frac{1}{3} \\ 0 & 1 & -\frac{23}{6} & 0 & -\frac{1}{6} & \frac{2}{3} \\ 0 & -2 & 23 & 1 & 0 & -5 \end{array}\right) \xrightarrow{-2R_2-R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{7}{6} & 0 & \frac{1}{6} & \frac{1}{3} \\ 0 & 1 & -\frac{23}{6} & 0 & -\frac{1}{6} & \frac{2}{3} \\ 0 & 0 & \frac{46}{3} & 1 & -\frac{1}{3} & -\frac{11}{3} \end{array}\right)$$

$$\xrightarrow{\frac{3}{46}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{7}{6} & 0 & \frac{1}{6} & \frac{1}{3} \\ 0 & 1 & -\frac{23}{6} & 0 & -\frac{1}{6} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{3}{46} & -\frac{1}{46} & -\frac{11}{46} \end{array}\right) \xrightarrow[-\frac{23}{6}R_3-R_2]{-\frac{6}{7}R_3-R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{92} & \frac{13}{92} & \frac{5}{92} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{46} & -\frac{1}{46} & -\frac{11}{46} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{92} & \frac{13}{92} & \frac{5}{92} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{46} & -\frac{1}{46} & -\frac{11}{46} \end{pmatrix}$$

$$W = A^{-1} \cdot b = \begin{pmatrix} \frac{7}{92} & \frac{13}{92} & \frac{5}{92} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{46} & -\frac{1}{46} & -\frac{11}{46} \end{pmatrix} \begin{pmatrix} 4 \\ 12 \\ 24 \end{pmatrix} = \begin{pmatrix} \frac{76}{23} \\ -8 \\ -\frac{132}{23} \end{pmatrix}$$

\#

2. Proof: $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$A = u \cdot v^T = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}$$

$$\det(A) = u_1v_1(u_2v_2 \cdot u_3v_3 - u_2v_3 \cdot u_3v_2) - u_1v_2(u_2v_1 \cdot u_3v_3 - u_2v_3 \cdot u_3v_1$$
$$+ u_1v_3(u_2v_1 \cdot u_3v_2 - u_2v_2 \cdot u_3v_1)$$

$\because$ u and v is linear relation $\therefore$ $u = \lambda v$ or $v = \lambda u$.

So we can get $\det(A) = 0$ #

3.

(a)

$\|u+v\|^2 = (u+v)^T \cdot (u+v) = (u^T + v^T)(u+v) = \|u\|^2 + u^T v + u v^T + \|v\|^2$

By vectors dot Product, we can get $u^T v = u v^T$

So $\|u+v\|^2 = \|u\|^2 + 2u^T v + \|v\|^2$ #

(b)

$(u+v)^T (u-v) = (u^T + v^T)(u-v) = \|u\|^2 - u^T v + v^T u - \|v\|^2$

By (a), we can get $u^T v = u v^T$

So $(u+v)^T (u-v) = \|u\|^2 - \|v\|^2$. #

(c) $u^T v = \|u\| \cdot \|v\| \cdot \cos\theta \Rightarrow |u^T v| = \left| \|u\| \cdot \|v\| \cdot \cos\theta \right|$

$\Rightarrow \left| \|u\| \cdot \|v\| \right| \geq \left| \|u\| \cdot \|v\| \cdot \cos\theta \right|$

So we can get $|u^T v| \le \|u\| \cdot \|v\|$ #

(d)
$① \|u+v\|^2 = \|u\|^2 + 2u^T v + \|v\|^2$

by (c) we can get $① \le \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$

So that $\|u+v\| \le \|u\| + \|v\|$ #

(e) $① \|u+v\|^2 = \|u\|^2 + 2u^T v + \|v\|^2$

$② \|u-v\|^2 = \|u\|^2 - 2u^T v + \|v\|^2$

$① + ② = 2(\|u\|^2 + \|v\|^2)$ #

(f) $\because u^T v = 0$ $\therefore$ u and v is vertical

$\|u+v\|^2 = \|u\|^2 + 2u^T v + \|v\|^2$ $\because u^T v = 0$ $\therefore \|u+v\|^2 = \|u\|^2 + \|v\|^2$ #

4.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

$$AB = BA = \begin{pmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{pmatrix}$$ #

5.

$$\mathrm{std}(x)^2 = \frac{\|x - (\mathbf{1}^T \frac{x}{n})\mathbf{1}\|^2}{n}$$

$$= \frac{x^T x - 2x^T(\mathbf{1}^T \frac{x}{n})\mathbf{1} + \overbrace{((\mathbf{1}^T \frac{x}{n})\mathbf{1})^T((\mathbf{1}^T \frac{x}{n})\mathbf{1})}^{n}}{n}$$

$$= \frac{x^T x}{n} - \frac{\frac{2}{n}(\mathbf{1}^T x)^2}{n} + \frac{n \cdot (\mathbf{1}^T \frac{x}{n})^2}{n}$$

$$\underset{\mathrm{rms}(x)^2}{\downarrow} \quad - \quad \underset{\mathrm{avg}(x)^2}{(\mathbf{1}^T \tfrac{x}{n})^2}$$

when we expand $\mathrm{std}(x)^2$ and break it down into three terms with n as the denominator, we find that the first term is $\mathrm{rms}(x)^2$, the second term can be written as $\frac{\frac{2}{n}(\mathbf{1}^T x)^2}{n}$ through matrix multiplication transformations and after expanding the product of the third term, we end up with n terms.
Therefore, when we combine the second and third terms we obtain exactly $\mathrm{avg}(x)^2$

$$\mathrm{std}(x)^2 = \mathrm{rms}(x)^2 - \mathrm{avg}(x)^2$$ #.

6.

(a)

$$\tilde{a} = \begin{pmatrix} x_1-\mu \\ x_2-\mu \\ \vdots \\ x_T-\mu \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \tilde{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_1-\mu \\ \vdots \\ x_T-\mu \end{pmatrix} \quad R(\tau) = \frac{\tilde{a}^T \tilde{b}}{\|\tilde{a}\| \cdot \|\tilde{b}\|}$$

when $\tau = 0$, $\tilde{a} = \begin{pmatrix} x_1-\mu \\ \vdots \\ x_T-\mu \end{pmatrix}$ $\tilde{b} = \begin{pmatrix} x_1-\mu \\ \vdots \\ x_T-\mu \end{pmatrix}$

$$R(0) = \frac{\tilde{a}^T \tilde{b}}{\|\tilde{a}\| \cdot \|\tilde{b}\|} = 1$$

when $\tau \geq T$, $\tilde{a} = \begin{pmatrix} x_1-\mu \\ \vdots \\ x_T-\mu \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\tilde{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_1-\mu \\ \vdots \\ x_T-\mu \end{pmatrix}$

$$R(\tau) = \frac{0}{\|\tilde{a}\| \cdot \|\tilde{b}\|} = 0$$

#

(b) $\mu = (1^T x)/T \Rightarrow avg(x)$

$$z = \frac{1}{std(x)} (x - avg(x)\mathbb{1})$$

$$R(\tau) = \frac{\tilde{a}}{\|\tilde{a}\|} \cdot \frac{\tilde{b}}{\|\tilde{b}\|}$$

$$= \frac{(x-\mu, 0_\tau)}{\sqrt{T}\|\tilde{a}\|} \cdot \frac{(0_\tau, x-\mu)}{\sqrt{T}\|\tilde{b}\|}$$

$$= \frac{1}{T} (z, 0_\tau)^T \cdot (0_\tau, z)$$

So we can get $R(\tau) = \frac{1}{T} \sum_{t=1}^{T-\tau} z_t z_{t+\tau}$.

#

(c) $\because$ mean is zero, norm is $\sqrt{T}$ $\therefore$ $\frac{(x-\mu\ 0_\tau)}{\sqrt{T}\|\tilde{a}\|} = \frac{(x, 0_\tau)}{T}$

$$R(\tau) = \frac{1}{T} \sum_{t=1}^{T-\tau} x_t x_{t+\tau} = \frac{T-\tau}{T} (-1)^\tau \Rightarrow \begin{cases} \frac{T-\tau}{T} & \tau \text{ is even} \\ \frac{\tau-T}{T} & \tau \text{ is odd} \end{cases}$$

#

(d) We can assume R(7) is the most busy day all the week.
So R(7) is likely R(7) but can not exceed it. #

7.

```
function result = matrix_multiply(A, B, flag)
    [rowsA, colsA] = size(A);
    [rowsB, colsB] = size(B);

    if colsA ~= rowsB
        error('Matrix dimensions are not same.');
    end

    result = zeros(rowsA, colsB);

    if flag == 1
        disp('Dot product method')
        for i = 1:rowsA
            for j = 1:colsB
                result(i, j) = dot(A(i, :), B(:, j));
            end
        end
    elseif flag == 2
        disp('Linear combination of columns method')
        for i = 1:colsB
            result(:, i) = A * B(:, i);
        end
    elseif flag == 3
        disp('Linear combination of rows method')
        for i = 1:rowsA
            result(i, :) = A(i, :) * B;
        end
    elseif flag == 4
        disp('Sum of outer products method')
        for i = 1:colsB
            result = result + A(:, i) * B(i, :);
        end
    end
end
```

#

8.

(a)

```
>> num_matrices = 10^6;
matrix_size = 3;
singular_count = 0;
>> for i = 1:num_matrices
    random_matrix = rand(matrix_size);
    if rank(random_matrix) < matrix_size
        singular_count = singular_count + 1;
    end
end
>> disp(['Out of ', num2str(num_matrices), ' random matrices with uniform entries, ', num2str(singular_count), ' were singular.']);
Out of 1000000 random matrices with uniform entries, 0 were singular.
```

```
>> while true
    random_matrix = randi([1, 3], matrix_size);
    if rank(random_matrix) < matrix_size
        break;
    end
end

disp('A singular matrix with entries from {1, 2, 3}:');
disp(random_matrix);
A singular matrix with entries from {1, 2, 3}:
     3     3     1
     2     2     1
     3     3     3
```

#

(b)

```
>> num_matrices = 100000;
matrix_size = 4;
l_values = [1, 2, 3];

rank_counts = zeros(1, matrix_size + 1);

for i = 1:num_matrices
    random_matrix = zeros(matrix_size);
    for row = 1:matrix_size
        for col = 1:matrix_size
            random_index = randi(length(l_values));
            random_matrix(row, col) = l_values(random_index);
        end
    end

    rank_value = rank(random_matrix);
    rank_counts(rank_value + 1) = rank_counts(rank_value + 1) + 1;
end

rank_distribution = rank_counts / num_matrices;

disp('Rank Distribution: ');
disp(rank_distribution);
Rank Distribution:
         0         0    0.0024    0.1696    0.8280
```

#