Sample Solutions on HW6 (*22 exercises in total*)

Sec. 5.4 29

29 procedure *a* (*n*: nonnegative integer)

  if $n = 0$ **then return** 1

  else if $n = 1$ **then return** 2

  else return $a(n-1) \cdot a(n-2)$

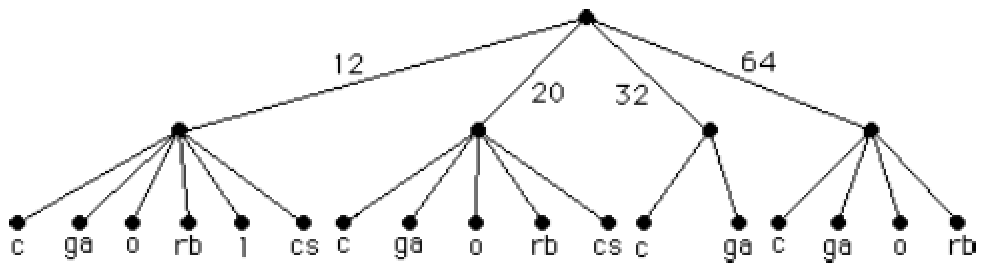
Sec. 6.1 41, 56, 68

41 If n is even, $2^{n/2}$; if n is odd, $2^{(n+1)/2}$

56 We need to compute the number of variable names of length i for i = 1, 2, . . . , 8, and add. A variable name of length i is specified by choosing a first character, which can be done in 53 ways ($2 \cdot 26$ letters and 1 underscore to choose from), and $i - 1$ other characters, each of which can be done in 53 + 10 = 63 ways. Therefore the answer is

$$\sum_{i=1}^{8} 53 \cdot 63^{i-1} = 53 \cdot \frac{63^8 - 1}{63 - 1} \approx 2.1 \times 10^{14}$$

68(a) It is more convenient to branch on bottle size first. Note that there are a different number of branches coming off each of the nodes at the second level. The number of leaves in the tree is 17, which is the answer.



68(b) We can add the number of different varieties for each of the sizes. The 12-ounce bottle has 6, the 20-ounce bottle has 5, the 32-ounce bottle has 2, and the 64-ounce bottle has 4. Therefore 6 + 5 + 2 + 4 = 17 different types of bottles need to be stocked.

Sec. 6.2 10, 38, 42

10 The midpoint of the segment whose endpoints are (a, b) and (c, d) is $((a + c) / 2, \quad (b + d) / 2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and c have the same parity (either both odd or both even) and b and d have the same parity. Thus what matters in this problem is the parities of the coordinates. There are four possible pairs of parities: (odd, odd), (odd, even), (even, odd), (even, even). Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

38 Label the computers C_1 through C_8 , and label the printers P_1 through P_4 . If we connect C_k to P_k for k = 1, 2, 3, 4 and connect each of the computers C5 through C8 to all the printers, then we have used a total of $4 + 4 \cdot 4 = 20$ cables. Clearly this is sufficient, because if computers C_1 through C_4 need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, since they are connected to all the printers. Now we must show that 19 cables are not enough. Since there are 19 cables and 4 printers, the average number of computers per printer is 19/4, which is less than 5. Therefore some printer must be connected to fewer than 5 computers (the average of a set of numbers cannot be bigger than each of the numbers in the set). That means it is connected to 4 or fewer computers, so there are at least 4 computers that are not connected to it. If those 4 computers all needed a printer simultaneously, then they would be out of luck, since they are connected to at most the 3 other printers.

42(a) Let a_j be the number of matches held during or before the j^{th} hour. Then $a_1, a_2, \ldots, a_{75}$ is an increasing sequence of distinct positive integers, since there was at least one match held every hour. Furthermore $1 \leqslant a_j \leqslant 125$, since there were only 125 matches altogether.

Moreover, a_1+2, a_2+2, …, $a_{75}+2$ is also an increasing sequence of distinct positive integers, with $3 \leqslant a_i+2 \leqslant 127$. Now the 150 positive integers a_1, a_2, …, a_{75}, a_1+2, a_2+2, …, $a_{75}+2$
are all less than or equal to 127. Hence by the pigeonhole principle two of these integers are equal. Since the integers a_1, a_2, …, a_{75} are all distinct, and the integers a_1+2, a_2+2, …, $a_{75}+2$ are all distinct, there must be distinct indices i and j such that $a_j = a_i + 2$. This means that exactly 2 matches were held from the beginning of hour i+1 to the end of hour j, precisely the occurrence we wanted to find.

42(b) The solution of part (a), with 2 replaced by 23 and 127 replaced by 148, tells us that the statement is true.

42(c) We begin in a manner similar to the solution of part (a). Look at a_1, a_2, …, a_{75}, a_1+25,

a_2+25, …, $a_{75}+25$, where a_i is the total number of matches played up through and including hour i. Then $1 \leqslant a_1 < a_2 < \ldots < a_{75} \leqslant 125$, and $26 \leqslant a_1+25 < a_2+25 < \ldots < a_{75}+25 \leqslant 150$. Now

either these 150 numbers are precisely all the number from 1 to 150, or else by the pigeonhole principle we get $a_j = a_i + 25$ for some i and j and we are done. In the former case, however, since each of the numbers a_i+25 is greater than or equal to 26, the numbers
1,2,…,25 must all appear among the a_i ‘s. But since the a_i ‘s are increasing, the only way this can happen is if $a_1 = 1$, $a_2 = 2, \ldots, a_{25} = 25$. Thus there were exactly 25 matches in the first 25 hours.

42(d) We need a different approach for this part, an approach, incidentally, that works for many numbers besides 30 in this setting. Let a_1, a_2, …, a_{75} be as before, and note that $1 \leqslant a_1$

$< a_2 < \ldots < a_{75} \leqslant 125$. By the pigeonhole principle two of the numbers among a_1, a_2, …, a_{31}
are congruent modulo 30. If they differ by 30, then we have our solution. Otherwise they differ by 60 or more, so $a_{31} \geq 61$. Similarly, among a_{31} through a_{61}, either we find a solution, or two numbers must differ by 60 or more; therefore we can assume that $a_{61} \geq 121$. But this means that $a_{66} \geq 126$, a contradiction.

Sec. 6.3 20, 42, 44

20(a) There are $C(10,3)$ ways to choose the positions for the 0’s, and that is the only choice to be made, so the answer is $C(10,3) = 120$.

20(b) There are more 0s than 1s if there are fewer than five 1’s. Using the same reasoning as in part (a), together with the sum rule, we obtain the answer $C(10,0)$ +

$C(10,1) + C(10,2) + C(10,3) + C(10,4) = 1 + 10 + 45 + 120 + 310 = 386$. Alternatively, by symmetry, half of all cases in which there are no five 0s have more 0s than 1s; therefore the answer is $(2^{10} - C(10,5)) / 2 = (1024 - 252) / 2 = 386$

20(c) We want the number of bit strings with 7, 8, 9, or 10 1s. By the same reasoning as above, there are $C(10,7) + C(10,8) + C(10,9) + C(10,10) = 120+45+10+1 = 176$ such strings.

20(d) if a string does not have at least three 1s, then it has 0, 1, or 2 1s. There are $C(10,0) + C(10,1) + C(10,2) = 1+10+45 = 56$ such strings. There are 210=1024 strings in all. Therefore there are 1024 – 56 = 968 strings with at least three 1s.

42 The only difference between this problem and the problem solved in Exercise 41 is a factor of 2. Each seating under the rules here corresponds to two seatings under the original rules, because we can change the order of people around the table from clockwise to counterclockwise. Therefore we need to divide the formula there by 2, giving us $n!/(2r(n-r)!)$. This assumes that $r\geq 3$. If $r = 1$ then the problem is trivial (there are n choices under both sets of rules). If $r = 2$, then we do not introduce the extra factor of 2, because clockwise order and counterclockwise order are the same. In this case, both answers are just $n!/(2(n-2)!)$, which is $C(n, 2)$, as one would expect.

44 We can solve this problem by breaking it down into cases depending on the number of ties. There are five cases. (1) If there are no ties, then there are clearly $P(4,4) = 24$ possible ways for the horses to finish. (2) Assume that there are two horses that tie, but the others have distinct finishes. There are $C(4,2) = 6$ ways to choose the horses to be tied; then there $P(3,3)=6$ ways to determine the order of finish for the three groups (the pair and the two single horses). Thus there are 6•6=36 ways for this to happen. (3) There might be two groups of two horses that are tied. There are $C(4,2) = 6$ ways to choose the winners (and the other two horses are the losers). (4) There might be a group of three horses all tied. There are $C(4,3)$
= 4 ways to choose which these horses will be, and then two ways for the race to end (the tied horses win or they lose), so there are 4•2 = 8 possibilities. (5) There is only one way for all the horses to tie. Putting this all together, the answer is 24+36+6+8+1=75.

Sec. 6.4 14, 22, 26, 30

14 Using the factorial formulae for computing binomial coefficients, we see that $\binom{n}{k-1} = \frac{k}{n-k+1}\binom{n}{k}$. If $k \leq n/2$, then $k/(n-k+1) < 1$, so the “less than” signs are correct. Similarly, if $k > n/2$, then $k/(n-k+1) > 1$, so the “greater than” signs are correct. The middle equality is Corollary 2 in Section 6.3, since $\lfloor n/2\rfloor + \lceil n/2\rceil = n.$ The equalities at the ends are clear.

22(a) Suppose that we have a set with n elements, and we wish to choose a subset A with k elements and another, disjoint, subset with r − k elements. The left-hand side gives us the number of ways to do this, namely the product of the number of ways to choose the r elements that are to go into one or the other of the subsets and the number of ways to choose which of these elements are to go into the first of the subsets. The right-hand side gives us the number of ways to do this as well, namely the product of the number of ways to choose the first subset and the number of ways to choose the second subset from the elements that remain.

22(b) On the other hand,

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!\,(n-r)!}\cdot\frac{r!}{k!\,(r-k)!} = \frac{n!}{k!\,(n-r)!\,(r-k)!}$$

and on the other hand

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!\,(n-k)!}\cdot\frac{(n-k)!}{(r-k)!\,(n-r)!} = \frac{n!}{k!\,(n-r)!\,(r-k)!}$$

26 First, use Exercise 25 to rewrite the right-hand side of this identity as $\binom{2n}{n+1}$. We give a combinatorial proof, showing that both sides count the number of ways to choose from collection of *n* men and *n* women, a subset that has one more man than woman. For the left-hand side, we note that this subset must have *k* men and *k*-1 women for some *k* between 1 and *n*, inclusive. For the (modified) right-hand side, choose any set of *n*+1 people from this collection of *n* men and *n* women, the desired subset is the set of men chosen and the women left behind.

30 We follow the hint. The number of ways to choose this committee is the number of ways to choose the chairman from among the n mathematicians (n ways) times the number of ways to choose the other n – 1 members of the committee from among the other 2n − 1 professors. This gives us $n\binom{2n-1}{n-1}$, the expression on the right-hand side. On the other hand, for each k from 1 to n, we can have our committee consist of k mathematicians and n − k computer scientists. There are $\binom{n}{k}$ ways to choose the mathematicians, k ways to choose the chairman from among these, and $\binom{n}{n-k}$ ways to choose the computer scientists. Since this last quantity equals $\binom{n}{k}$, we obtain the expression on the left-hand side of the identity.