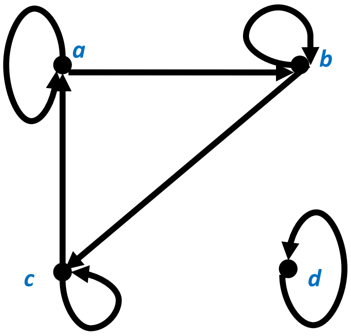
Sample Solutions on HW11 (*45 exercises in total*)

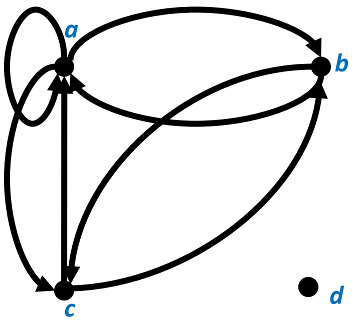
Sec. 9.4 2, 6, 9(6), 11(6), 20, 28(a), 29

2 When we add all the pairs (x,x) to the given relation we have all of $Z \times Z$; in other words, we have the relation that always holds.

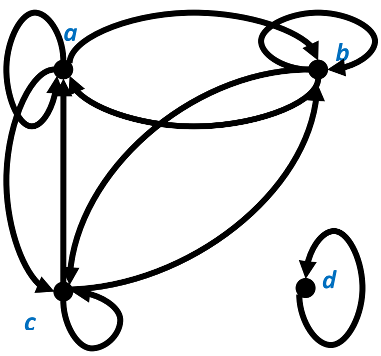
6 We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



9(6) We form the symmetric closure by taking the given directed graph and appending an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there); in other words, we append the edge (b, a) whenever we see the edge (a, b).



11(6) We are asked for the symmetric and reflexive closure of the given relation. We form it by taking the given directed graph and appending (1) a loop at each vertex at which there is not already a loop and (2) an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there).



20(a) The pair (a,b) is in R^2 precisely when there is a city c such that there is a direct flight from a to c and a direct flight from c to b – in other words, when it is possible to fly from a to b with a scheduled stop (and possibly a plane change) in some intermediate city.

20(b) The pair (a,b) is in R^3 precisely when there are cities c and d such that there is a direct flight from a to c, a direct flight from c to d, and a direct flight from d to b – in other words, when it is possible to fly from a to b with two scheduled stops (and possibly a plane change at one or both) in intermediate cities.

20(c) The pair (a,b) is in R^* precisely when it is possible to fly from a to b.

28(a) We compute the matrices Wi for i = 0, 1, 2, 3, 4, 5, and then W5 is the answer.

$$W_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = W_5$$

29(a) We need to include at least the transitive closure, which we can compute by Algorithm 1 or Algorithm 2 to be $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. All we need in addition is the pair (2,2) in order to make the relation reflexive. Note that the result is still transitive (the addition of a pair (a,a) cannot make a transitive relation no longer transitive), so our answer is $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

29(b) The symmetric closure of the original relation is represented by $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

We need at least the transitive closure of this relation, namely $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. Since it is also symmetric, we are done. Note that it would not have been correct to find first the transitive closure of the original matrix and then make it symmetric, since the pair (2,2) would be missing. What is going on here is that the transitive closure of a symmetric relation is still symmetric, but the symmetric closure of a transitive relation might not be transitive.

29(c) Since the answer to part (b) was already reflexive, it must be the answer to this part as well.

Sec. 9.5 3, 10, 16, 36(b), 39, 41

3(a) This is an equivalence relation, one of the general form that two things are considered equivalent if they have the same "something" (see Exercise 9 for a formalization of this idea). In this case the "something" is the value at 1.

3(b) This is not an equivalence relation because it is not transitive. Let $f(x) = 0$, $g(x) = x$, and $h(x) = 1$ for all $x \in Z$. Then f is related to g since $f(0) = g(0)$, and g is related to h since $g(1) = h(1)$, but f is not related to h since they have no values in common. By inspection we see that this relation is reflexive and symmetric.

3(c) This relation has none of the three properties required for an equivalence relation. It is not reflexive, since $f(x) - f(x) = 0 \neq 1$. It is not symmetric, since if $f(x) - g(x) = 1$, then $g(x) - f(x) = -1 \neq 1$. It is not transitive, since if $f(x) - g(x) = 1$ and $g(x) - h(x) = 1$, then $f(x) - h(x) = 2 \neq 1$.

3(d) This is an equivalence relation. Two functions are related if they differ by a constant. It is clearly reflexive (the constant is 0). It is symmetric, since if $f(x) - g(x) = C$, then $g(x) - f(x) = -C$. It is transitive, since if $f(x) - g(x) = C_1$ and $g(x) - h(x) = C_2$, then $f(x) - h(x) = C_3$, where $C_3 = C_1 + C_2$.

3(e) This relation is not reflexive, since there are lots of functions f (for instance, $f(x) = x$) that do not have the property that $f(0) = f(1)$. It is symmetric by inspection (the roles of f and g are the same). It is not transitive. For instance, let $f(0) = g(1) = h(0) = 7$, and let $f(1) = g(0) = h(1) = 3$; fill in the remaining values arbitrarily. Then f and g are related, as are g and h, but f is not related to h since $7 \neq 3$.

10 The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

16 This follows from Exercise 9, where f is the function from the set of pairs of positive integers to the set of positive rational numbers that takes (a,b) to a/b, since clearly $ad = bc$ if and only if $a/b = c/d$.

If we want an explicit proof, we can argue as follows. For reflexivity, $((a,b),(a,b)) \in R$ because $a \cdot b = b \cdot a$. If $((a,b),(c,d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c,d),(a,b)) \in R$; this tells us that R is symmetric. Finally, if $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$ then $ad = bc$ and $cf = de$. Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a,b),(e,f)) \in R$; this tells us that R is transitive.

36(b) The equivalence class of 4 is the set of all integers congruent to 4, modulo m.
$\{4+3n \mid n \in Z\} = \{\ldots,-2,1,4,7,\ldots\}$

39(a) We observed in the solution to Exercise 15 that (a,b) is equivalent to (c,d) if $a - b = c - d$. Thus because $1 - 2 = -1$, we have $[(1,2)]=\{(a,b)|a - b = -1\}=\{(1,2), (2,3), (3,4), (4,5), (5,6),\cdots\}$.

39(b) Since the equivalence class of (a,b) is entirely determined by the integer $a - b$, which can be negative, positive, or zero, we can interpret the equivalences classes as being the integers. This is a standard way to define the integers once we have defined the whole numbers.

41 The sets in a partition must be nonempty, pairwise disjoint, and have as there union all of the underlying set.
41(a) This is not a partition, since the sets are not pairwise disjoint (the elements 2 and 4 each appear in two of the sets).

41(b) This is a partition.

41(c) This is a partition.

41(d) This is not a partition, since none of the sets includes the element 3.

Sec. 9.6 5, 10, 23(a), (c), 32, 44, 46

5 The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

5(a) The equality relation on any set satisfies all three conditions and is therefore a partial ordering. (It is the smallest partial ordering; reflexivity insures that every partial order contains at least all the pairs (*a*,*a*).)

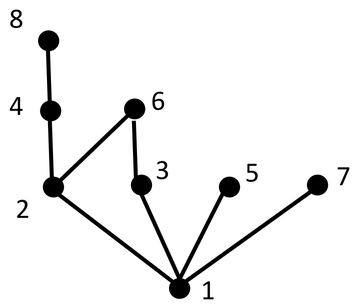
5(b) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

5(c) This is a poset, as explained in Example 1 in this section.

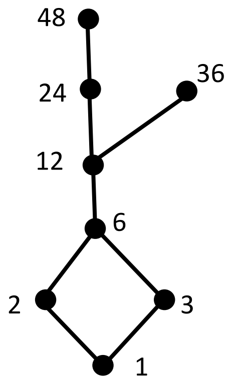
5(d) This is not a poset. The relation is not reflexive, since it is not true, for instance, that 2 does not divide 2. (It also is not antisymmetric and not transitive.)

10 This relation is not transitive (there is no arrow from *c* to *b*), so it is not a partial order.

23(a)



23(c)



32(a) The maximal elements are the ones with no other elements above them, namely *l* and *m*.

32(b) The minimal elements are the ones with no other elements below them, namely *a*, *b* and *c*.

32(c) There is no greatest element, since neither *l* nor *m* is greater than the other.
32(d) There is no least element, since none of *a*, *b*, and *c* is less than the other two.

32(e) We need to find elements from which we can find downward paths to all of *a*, *b*, and *c*. It is clear that *k*, *l*, and *m* are the elements fitting this description.

32(f) Since *k* is less than both *l* and *m*, it is the least upper bound of *a*, *b*, and *c*.

32(g) No element is less than both *f* and *h*, so there are no lower bounds.

32(h) Since there are no lower bounds, there can be no greatest lower bound.

44 In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.
44(a) This is not a lattice, since the elements 6 and 9 have no upper bound (no element in our set is a multiple of both of them).

44(b) This is a lattice; in fact it is a linear order, since each element in this list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

44(c) Again, this is a lattice because it is a linear order. The least upper bound of two numbers in this list is the smaller number (since here "greater" really means "less"), and the greatest lower bound is the larger of the two numbers.

44(d) This is similar to Example 24 of this section, with the roles of subset and superset reversed. Here the g.l.b. of two subsets A and B is $A \cup B$, and their l.u.b. is $A \cap B$.

46 By the duality in the definitions, the greatest lower bound of two elements of S under R is their least upper bound under R^{-1}, and their least upper bound under R is their greatest lower bound under R^{-1}. Therefore, if (S, R) is a lattice (i.e., all the lubs and glbs exist), then so is (S, R^{-1}).