$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

the number of ways to place n distinguishable objects into k indistinguishable boxes

$$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k} \left(\left(\sum_{i=0}^{j-1} (-1)^i C_j^i (j-i)^n\right) / j!\right)$$

$S(r+1,n)=S(r,n-1)+nS(r,n)$

【Theorem 1】 Let c_1, c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1, r_2. Then the Sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0,1,2,\cdots$, where α_1, α_2 are constants.

【Theorem 2】 Let c_1, c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0,1,2,\cdots$, where α_1, α_2 are constants.

【Theorem 3】 Let $c_1, c_2, \cdots, c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \cdots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$ for $n = 0,1,2,\cdots$ where $\alpha_1, \alpha_2, \cdots, \alpha_k$ are constants.

【 Theorem 4】 Let $c_1, c_2, \cdots, c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has t distinct roots $r_1, r_2, \cdots, r_t$ with multiplicities $m_1, m_2, \cdots, m_t$, respectively, so that $m_i \geq 1$ *for* $i = 1, 2, \cdots, t$ and $m_1 + m_2 + \cdots + m_t = k$
Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^{\,n} +$$
$$(\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^{\,n} + \cdots +$$
$$(\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^{\,n}$$

for $n = 0, 1, 2, \cdots$ where $\alpha_{i,j}$ are constants for

$1 \leq i \leq t, 0 \leq j \leq m_i - 1$

【 Theorem 5】 Let $\{a_n^{(p)}\}$ be a *particular solution* of the nonhomogeneous linear recurrence relation with constant coefficients $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$
Then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

【 Theorem 6】 Assume a linear nonhomogeneous recurrence equation with constant coefficients with the nonlinear part *F(n)* of the form

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$$

If *s* is not a root of the characteristic equation of the associated homogeneous recurrence equation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

If *s* is a root of multiplicity *m*, a particular solutions is of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

- **A particular solution of the form:**

$$n(p_1 n + p_0) = p_1 n^2 + p_0 n$$

- **Find p_1,p_0:**

$$p_1 n^2 + p_0 n = p_1 (n-1)^2 + p_0 (n-1) + n$$

$$n(2p_1 - 1) + (p_0 - p_1) = 0$$

$$p_0 = p_1 = 1/2,$$

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if} \quad k > 0 \\ 1 & \text{if} \quad k = 0 \end{cases}$$

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{k=0}^{\infty} (-1)^k C(n+k-1,k) x^k$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \sum_{k=0}^{\infty} (-1)^k C(n+k-1,k)(-1)^k x^k = \sum_{k=0}^{\infty} C(n+k-1,k) x^k$$

$\frac{1}{k!}$	e^x
$\frac{(-1)^{k+1}}{k}$	$\ln(1+\mathrm{x})$

【Definition】A *binary relation* **R** from a set *A* to a set *B* is a subset of $A \times B$.

A *relation on the set A* is a relation from *A* to *A*.

A relation *R* on a set *A* is *antisymmetric* if

$$\forall x \forall y((x, y) \in R \wedge (y, x) \in R \rightarrow x = y)$$

The *composite of R and S*: $S \circ R$

【 Theorem 】 The relation *R* on a set *A* is transitive if and only if $R^n \subseteq R, \ for \ n = 1,2,3,\cdots$

$$(A \times B)^{-1} = B \times A$$

$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}$$

$$(R \circ T) \circ P = R \circ (T \circ P)$$

$$(R \cup S) \circ T = R \circ T \cup S \circ T$$

The closure of a relation **R** with respect to property **P**

The smallest relation with property *P* containing *R*

$$R^* = \bigcup_{n=1}^{\infty} R^n$$ $t(R) = R^*$.

equivalence relation

- reflexive
- symmetric
- transitive

a and b are equivalent

R is an equivalence relation, and $(a, b) \in R$

Notation: $a \sim b$

【 Theorem 4】 If R_1, R_2 are equivalence relations on *A*, then $R_1 \cup R_2$ is a reflexive and symmetric relation on A. transitive X

partial ordering or partial order **if *R* is**

- **reflexive**
- **antisymmetric**
- **transitive**

【Definition】If $(S,\leqslant)$ is a poset and every two elements of *S* are comparable, *S* is called a *totally ordered* or *linearly ordered set*, $\leqslant$ is called a *total order* or *linear order*. In this case $(S,\leqslant)$ is also called a *chain*.

lexicographic ordering **on $A_1 \times A_2$ is defined by**

$(a_1, a_2) \prec (b_1, b_2)$,

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

$B \subseteq A$, **if** $\forall a,b \in B(a \neq b),(a,b) \notin R,(b,a) \notin R$
then *B* is called a *antichain* of $(A,\leq)$.

maximal element **if there does not exist an element *b* in *A* such that $a \prec b$.**

greatest element **of *A* if $b \leq a$ for every *b* in *A*,**

there exists an element *a* in *S* such that $b \leq a$ for all *b* in *A*, then *a* is called an *upper bound* of *A*.

less than every other upper bound of *A*, then *a* is the *least upper bound*, denoted by lub(*A*).

【Definition】A poset *(A, R)* is *well-ordered set* if every nonempty subset of *A* has a least element.

【Definition】A poset is called a *lattice* if every pair of elements has a lub and a glb.

We impose a total ordering $\leq$ on a poset (*A*,*R*) *compatible* with the partial order if $a \leq b$ whenever aRb .

Constructing a compatible total ordering from a partial ordering is called *topological sorting*.

Multigraph: **Graphs that may have multiple edges connecting the same vertices.**

Pseudograph: **Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices.**

If deg(v) = 1, v is called *pendant*.

Complete Graphs - K_n: **Cycles C_n (n>2)**

Wheels W_n (n>2) ***n*-Cubes Q_n (n>0)**

respectively, and *every vertex* in V_1 is connected to *every vertex* in V_2, denoted by $K_{m,n}$, where $m = | V_1 |$ and $n = | V_2 |$.

A simply graph is called *regular* if every vertex of this graph has the same degree.

A *regular graph* is called n-regular if every vertex in this graph has degree n.

H is a *spanning subgraph* of G if $W = V, F \subseteq E$

of G. Then the *incidence matrix* with respect to this ordering of V and E is $n \times m$ matrix $\mathbf{M} = [m_{ij}]_{n \times m}$ **, where**

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

A path is *simple* if it does not contain the same edge more than once.

A vertex is a *cut vertex* (or *articulation point*),

The graph is *weakly connected* if the underlying undirected graph is connected.

For directed graph, the maximal strongly connected subgraphs are called the ***strongly connected components*** or just the ***strong components***.

A directed multigraph having no isolated vertices has an Euler path but not an Euler circuit if and only if

-- the graph is weakly connected

-- the in-degree and out-degree of each vertex are equal for all but two vertices, one that has in-degree 1 larger than its out-degree and the other that has out-degree 1 larger than its in-degree.

If G is a simple graph with n vertices with n>=3 such that deg(u)+deg(v) >= n for every pair of nonadjacent vertices u and v in G , then G has a Hamilton circuit.

For any nonempty subset S of set V *in a* Hamilton graph, the number of connected components in G-S <=|S|.

Let G be a *connected planar simple* graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r=e-v+2$.

Proof:

First, we specify a planar representation of G. We will prove the theorem by constructing a sequence of subgraphs G_1, G_2, ⋯, G_e = G, successively adding an edge at each stage.

【Definition】 Suppose R is a region of a connected planar simple graph, the number of the edges on the boundary of R is called the *Degree of R* .

Notation: Deg(R)

【 Corollary 1 】 If G is a connected planar simple graph with e edges and v vertices where v≥3, then e≤3v-6

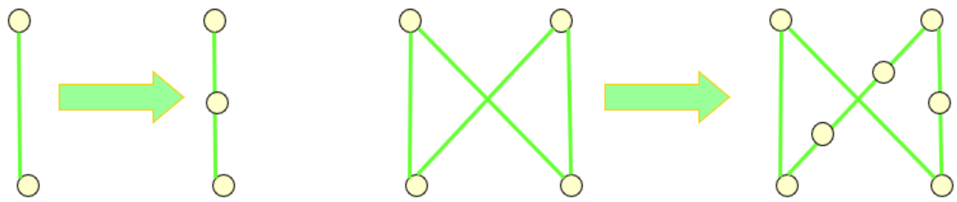
Suppose that a connected planar simple graph divides the plane into r regions, the degree of each region is at least 3.

Since $2e = \sum \deg(R_i) \geq 3r$,

Generally, if every region of a connected planar simple graph has at least *k* edges, then

$$e \le \frac{(v-2)k}{k-2}$$

Elementary subdivision



Homeomorphic

-- **The graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision.**

【Theorem 2】 A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

l leaves has $n=(ml-1)/(m-1)$ vertices and $i=(l-1)/(m-1)$ internal vertices