Sample Solutions on HW2 (*49 exercises in total)*

Sec. 1.4 6(c,d,e,f), 9(b,d), 20(e), 24(b,d), 40(b), 44, 49(a), 60

6: ***The answers given here are not unique.***

(c) No student in your school has visited North Dakota.

(d) Some student in your school has not visited North Dakota.

(e) Not all students in your school have visited North Dakota.

(f) No student in your school has visited North Dakota. ("All students in your school have not visited North Dakota," in common English usage means the answer to part (e))

9(b) $\exists x(P(x) \wedge \neg Q(x))$

9(d) $\neg\exists x(P(x) \vee Q(x)) \equiv \forall x(\neg P(x) \wedge \neg Q(x))$

20(e)

$$(\neg P(-5) \vee \neg P(-3) \vee \neg P(-1) \vee \neg P(1) \vee \neg P(3) \vee \neg P(5)) \wedge (P(-1) \wedge P(-3) \wedge P(-5))$$

This can be simplified further:

$$(\neg P(1) \vee \neg P(3) \vee \neg P(5)) \wedge (P(-1) \wedge P(-3) \wedge P(-5))$$

24 Let C(x) be "x is in your class."

(b) Let F(x) be "x has seen a foreign movie." Then we have $\exists x F(x)$ the first way, or $\exists x(C(x) \wedge F(x))$ the second way.

(d) Let Q(x) be "x can solve quadratic equations." Then we have $\forall x Q(x)$ the first way, or $\forall x(C(x) \rightarrow Q(x))$ the second way.

40(b) There are many ways to write this, depending on what we use for predicates. Let $O(x)$ be "Directory x can be opened," let $C(x)$ be "File x can be closed," and let E be the proposition "System errors have been detected." Then we have $E \rightarrow ((\forall x \neg O(x)) \wedge (\forall x \neg C(x)))$.

44 We want propositional functions P and Q that are sometimes, but not always, true (so that the second biconditional is $F \leftrightarrow F$ and hence true), but such that there is an x making one true and the other false. For example, we can take $P(x)$ to mean that x is an even number (a multiple of 2) and $Q(x)$ to mean that x is a multiple of 3. Then an example like x = 4 or x = 9 shows that $\forall x(P(x) \leftrightarrow Q(x))$ is false.

49(a)
$$\begin{aligned} &\forall x(P(x) \rightarrow A) \\ &\equiv \forall x(\neg P(x) \vee A) \\ &\equiv \forall x \neg P(x) \vee A \\ &\equiv \neg \exists x P(x) \vee A \\ &\equiv \exists x P(x) \rightarrow A \end{aligned}$$

60

a) $\forall x\ (P(x) \rightarrow Q(x))$

b) $\exists x(R(x) \wedge \neg Q(x))$

c) $\exists x(R(x) \wedge \neg P(x))$

d) Yes. The unsatisfactory excuse guaranteed by part (b) cannot be a clear explanation by part (a).

Sec. 1.5 6(e, f), 12(d, h, k, n), 14(c, d, e, f), 24(a, d), 32(d), 34, 38(b, d), 42

6(e) There exist two distinct people, the second of whom is enrolled in every course that the first is enrolled in.

6(f) There exist two distinct people enrolled in exactly the same courses.

12 ***The answers to this exercise are not unique. There are many ways of expressing the same propositions symbolically.***

(d) $\neg\exists x C(x, Bob)$

(h) $\exists x \forall y (x = y \leftrightarrow I(y))$

(k) $\exists x (I(x) \wedge \forall y (x \neq y \rightarrow \neg C(x, y)))$

(n) $\exists x \exists y (x \neq y \wedge \forall z \neg (C(x, z) \wedge C(y, z)))$

14. The answers to this exercise are not unique. There are many ways of expressing the same propositions symbolically. Assume the domain for persons consists of all people in this class.

(c) Let V(x,y) mean that person x has visited state y. Then

$$\exists x (V(x, Alaska) \wedge \neg V(x, Hawaii))$$

(d) Let L(x,y) mean that person x has learned programming language y. Then $\forall x \exists y L(x, y)$

(e) Let T(x,y) mean that person x has taken course y, and let O(y, z) mean that course y is offered by department z. Assume the domain for courses consists of all courses offered in this school, and the domain for departments consists of all departments in this school. Then

$$\exists x \exists z \forall y (O(y, z) \rightarrow T(x, y))$$

(f) Let G(x,y) mean that persons *x* and *y* grew up in the same town. Then

$$\exists x \exists y (x \neq y \wedge G(x, y) \wedge \forall z (G(x, z) \rightarrow (x = z \vee y = z)$$

24(a) There exists some real number such that when added to every real number does not change its value.

24(d) The product of two numbers is nonzero if and only if both factors are nonzero.

32(d) $\exists y \forall x \forall z(\neg T(x, y, z) \wedge \neg Q(x, y))$

34. The logical expression is asserting that the domain consists of at most two members. (It is saying that whenever you have two unequal objects, any object has to be one of those two. Note that this is vacuously true for domains with one element.) Therefore any domain having one or two members will make it true (e.g. female and male), and any domain with more than two members will make it false.

38(b)

The negation is "Every student in this class has seen a computer."

Let S(x) be "student x has seen a computer", where the domain of x consists of all students in this class. Then $\forall x S(x)$

38(d)

Let P(z, y) be "Room z is in building y," and let Q(x,z) be "student x has been in room z." Then the original statement is $\exists x \forall y \exists z(P(z, y) \wedge Q(x, z))$. the negation is therefore $\forall x \exists y \forall z(\neg P(z, y) \vee \neg Q(x, z))$. In English, it means "For every student there is a building such that for every room in that building, the student has not been in that room."

42. The distributive law is just the statement that x(y+z)=xy+xz for all real numbers. Therefore the expression we want is $\forall x \forall y \forall z (x(y+z) = xy + xz)$, where the quantifiers are assumed to range over the real numbers.

Sec. 1.6 12, 14(d), 18, 24, 29, 34(a)

12. Exercise 11 states that $((p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q) \to r) \to ((p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to (q \to r))$

Applying exercise 11, we just need to show that the conclusion r follows

from the five premises

$(p \wedge t) \to (r \vee s)$, $q \to (u \wedge t)$, $u \to p$, $\neg s$, and q

Step	Reason
1. q	Premise
2. $q \to (u \wedge t)$	Premise
3. $u \wedge t$	Modus ponens, 1
4. u	Simplification, 3
5. t	Simplification, 3
6. $u \to p$	Premise
7. p	Modus ponens, 4,,6
8. $p \wedge t$	Conjunction, 5,7
9. $(p \wedge t) \to (r \vee s)$	Premise

10. $r \vee s$	Modus ponens, 8,9
11. $\neg s$	Premise
12. r	Disjunctive syllogism, 10,11

Q.E.D.

14(d) Let C(x) be "x is in this class," let F(x) be "x has been to France," and let L(x) be "x has visited the Louvre." We are given premises $\exists x(C(x) \wedge F(x))$, $\forall x(F(x) \to L(x))$, and we want to conclude $\exists x(C(x) \wedge L(x))$.

In the following proof, y represents an unspecified particular person.

Step	Reason
1. $\exists x(C(x) \wedge F(x))$	Premise
2. $C(y) \wedge F(y)$	EI, 1
3. F(y)	Simplification,2
4. $C(y)$	Simplification,2
5. $\forall x(F(x) \to L(x))$	Premise
6. $F(y) \to L(y)$	UI, 5

7. $L(y)$	Modus ponens, 3,6
8. $C(y) \wedge L(y)$	Conjunction, 4,7
9. $\exists x(C(x) \wedge L(x))$	EG, 8

Q.E.D.

18. We know that some *s* exists that makes *S*(*s*, Max) true, but we cannot conclude that Max is one such *s*. Therefore this first step is invalid.

24. Steps 3 and 5 are incorrect; simplification applies to conjunction, not disjunction.

29.

Step	Reason
1. $\exists x \neg P(x)$	Premise
2. $\neg P(c)$	EI, 1
3. $\forall x(P(x) \vee Q(x))$	Premise
4. $P(c) \vee Q(c)$	UI, 3
5. $Q(c)$	Disjunctive syllogism, 4,2

6. $\forall x(\neg Q(x) \vee S(x))$	Premise
7. $\neg Q(c) \vee S(c)$	UI, 6
8. $S(c)$	Disjunctive syllogism, 5,7
9. $\forall x(R(x) \rightarrow \neg S(x))$	Premise
10. $R(c) \rightarrow \neg S(c)$	UI, 9
11. $\neg R(c)$	Modus tollens, 8,10
12. $\exists x \neg R(x)$	EG, 11

Q.E.D.

34(a) Let d be "logic is difficult," s be "many students like logic," and e for "mathematics is easy.". Then the assumptions are $d \vee \neg s$ and $e \to \neg d$.

Note that the first of these is equivalent to

$s \to d$. This exercise asks whether we can conclude $s \to \neg e$

Step	Reason
1. $d \vee \neg s$	Premise
2. $s \to d$	Implication Rule, 1
3. $e \to \neg d$	Premise

4. $d \to \neg e$ Contrapositive, 3

5. $s \to \neg e$ Hypothetical syllogism, 2,4

Q.E.D.

Problems on Normal Forms

1. Give the simplest DNF and CNF of the following statements:

1) $((P \to Q) \leftrightarrow (\neg Q \to \neg P)) \wedge R$

$\because \neg Q \to \neg P$ is the contrapositive of $P \to Q$

$\therefore (P \to Q) \leftrightarrow (\neg Q \to \neg P)$ is always true

The original expression is equivalent to $T \wedge R \equiv R$, which is the simplest DNF and CNF

2) $P \vee (\neg P \vee (Q \wedge \neg Q))$

$\equiv P \vee (\neg P \vee F) \equiv P \vee \neg P \equiv T$, which is the simplest DNF and CNF.

3) $(P \wedge (Q \wedge S)) \vee (\neg P \wedge (Q \wedge S))$

$\equiv (Q \wedge S) \wedge (P \vee \neg P) \equiv Q \wedge S$, which is the simplest DNF and CNF.

2. Give the full DNF of the following statements:

You can always make use of truth tables to obtain the full DNF. Here I only give the solutions by using logical equivalences.

1) $(\neg P \vee \neg Q) \to (P \leftrightarrow \neg Q)$

$\equiv \neg(\neg P \vee \neg Q) \vee ((\neg P \vee \neg Q) \wedge (P \vee Q))$

$\equiv (P \wedge Q) \vee (\neg(P \wedge Q) \wedge (P \vee Q))$

$\equiv ((P \wedge Q) \vee (\neg(P \wedge Q))) \wedge ((P \wedge Q) \vee (P \vee Q))$

$\equiv P \vee Q$

$\equiv (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P)$

$\equiv (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P) \equiv (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge \neg P)$

2) $(\neg R \wedge (Q \to P)) \to (P \to (Q \vee R))$

$\equiv \neg(\neg R \wedge (\neg Q \vee P)) \vee (\neg P \vee (Q \vee R))$

$\equiv (R \vee (Q \wedge \neg P)) \vee (\neg P \vee Q \vee R)$

$\equiv R \vee R \vee (Q \wedge \neg P) \vee \neg P \vee Q$

$\equiv R \vee \neg P \vee Q \equiv (R \wedge (P \vee \neg P) \wedge (Q \vee \neg Q)) \vee (\neg P \wedge (R \vee \neg R) \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P) \wedge (R \vee \neg R))$

$\equiv (R \wedge P \wedge Q) \vee (R \wedge P \wedge \neg Q) \vee (R \wedge \neg P \wedge Q) \vee (R \wedge \neg P \wedge \neg Q) \vee (\neg P \wedge \neg R \wedge Q) \vee (\neg P \wedge \neg R \wedge \neg Q) \vee (Q \wedge P \wedge \neg R)$

3) $(P \to (Q \wedge R)) \wedge (\neg P \to (\neg Q \wedge \neg R))$

$\equiv (\neg P \vee (Q \wedge R)) \wedge (P \vee (\neg Q \wedge \neg R))$

$\equiv (\neg P \wedge (P \vee (\neg Q \wedge \neg R))) \vee ((Q \wedge R) \wedge (P \vee (\neg Q \wedge \neg R)))$

$\equiv (\neg P \wedge \neg Q \wedge \neg R) \vee ((Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg Q \wedge \neg R))$

$\equiv (\neg P \wedge \neg Q \wedge \neg R) \vee (Q \wedge R \wedge P)$

3. Give the prenex normal forms of the following statements:

1) $(\forall x)(P(x) \to (\exists y)Q(x,y))$

$\equiv \forall x(\neg P(x) \vee \exists y Q(x,\ y)) \equiv \forall x \exists y(\neg P(x) \vee Q(x,\ y))$

2) $(\forall x)(\forall y)(((\exists z)P(x,y,z) \wedge (\exists u)Q(x,u)) \to (\exists v)Q(y,v))$

$\equiv \forall x \forall y((\neg \exists z P(x, y, z) \vee \neg \exists u Q(x,u) \vee \exists v Q(y, v))$

$\equiv \forall x \forall y((\forall z \neg P(x, y, z) \vee \forall u \neg Q(x,u) \vee \exists v Q(y, v))$

$\equiv \forall x \forall y \forall z \forall u \exists v(\neg P(x, y, z) \vee \neg Q(x,\ u) \vee Q(y, v))$

4. Give the prenex DNF and CNF of the following statements:

1) $(\exists x P(x) \vee \exists x Q(x)) \rightarrow \exists x\,(P(x) \vee Q(x))$

$\equiv \forall x \forall y \exists z((\neg P(x) \wedge \neg Q(y)) \vee (P(z) \vee Q(z)))$

$\equiv \forall x \forall y \exists z((\neg P(x) \wedge \neg Q(y)) \vee P(z) \vee Q(z))$ (Prenex DNF)

$\equiv \forall x \forall y \exists z((\neg P(x) \vee P(z)) \wedge (\neg Q(y) \vee P(z)) \vee Q(z))$

$\equiv \forall x \forall y \exists z((\neg P(x) \vee P(z) \vee Q(z)) \wedge (\neg Q(y) \vee P(z) \vee Q(z)))$ (Prenex CNF)

2) $\forall x\, \forall y(P(x) \rightarrow Q(x,y)) \rightarrow \exists y(\mathrm{P}(y) \wedge \exists z Q(y,z))$

$\equiv \exists x \exists y \exists u \exists v((P(x) \wedge \neg Q(x, y)) \vee (P(u) \wedge Q(u,v)))$ (Prenex DNF)

$\equiv \exists x \exists y \exists u \exists v\,(((P(x) \wedge \neg Q(x, y)) \vee P(u)) \wedge ((P(x) \wedge \neg Q(x, y)) \vee Q(u, v)))$

$\equiv \exists x \exists y \exists u \exists v\,((P(x) \vee P(u)) \wedge (\neg Q(x, y) \vee P(u)) \wedge (P(x) \vee Q(u, v)) \wedge (\neg Q(x, y)) \vee Q(u, v)))$

(Prenex CNF)