Sample Solutions on HW8 (*41 exercises in total*)

Sec.8.1 8, 10, 26, 29, 32

8:

a) Let a_n be the number of bit strings of length n containing three consecutive 0's. In order to construct a bit string of length n containing three consecutive 0's we could start with 1 and follow with a string of length $n-1$ containing three consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n-3$ containing three consecutive 0's, or we could start with a 000 and follow with any string of length $n-3$. These four cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

b) There are no bit strings of length 0, 1, or 2 containing three consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$.

c) We will compute a_3 through a_7 using the recurrence relation:

$$\begin{aligned}
a_3 &= a_2 + a_1 + a_0 + 2^0 = 0 + 0 + +0 + 1 = 1\\
a_4 &= a_3 + a_2 + a_1 + 2^1 = 1 + 0 + 0 + 2 = 3\\
a_5 &= a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8\\
a_6 &= a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20\\
a_7 &= a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47
\end{aligned}$$

Thus there are 47 bit strings of length 7 containing three consecutive 0's.

10

First let us solve this problem without using recurrence relations at all. It is clear that the only strings that do not contain the string 01 are those that consist of a string of 1's follows by a string of 0's. The string can consist of anywhere from 0 to n 1's, so the number of such strings is $n+1$. All the rest have at least one occurrence of 01. Therefore the number of bit strings that contain 01 is $2^n - (n+1)$. However, this approach does not meet the instructions of this exercise.

a) Let a_n be the number of bit strings of length n that contain 01. If we want to construct such a string, we could start with a 1 and follow it with a bit string of length $n-1$ that contains 01, and there are a_{n-1} of these. Alternatively, for any k from 1 to $n-1$, we could start with k 0's, follow this by a 1, and then follow this by any $n-k-1$ bits. For each such k there are 2^{n-k-1} such strings, since the final bits are free. Therefore the number of such strings is $2^0 + 2^1 + 2^2 + \cdots + 2^{n-2}$, which equals $2^{n-1} - 1$. Thus our recurrence relation is $a_n = a_{n-1} + 2^{n-1} - 1$. It is valid for all $n \geq 2$.

b) The initial conditions are $a_0 = a_1 = 0$, since no string of length less than 2 can have 01 in it.

c) We will compute a_2 through a_7 using the recurrence relation:

$$\begin{aligned}
a_2 &= a_1 + 2^1 - 1 = 0 + 2 - 1 = 1\\
a_3 &= a_2 + 2^2 - 1 = 1 + 4 - 1 = 4\\
a_4 &= a_3 + 2^3 - 1 = 4 + 8 - 1 = 11\\
a_5 &= a_4 + 2^4 - 1 = 11 + 16 - 1 = 26\\
a_6 &= a_5 + 2^5 - 1 = 26 + 32 - 1 = 57\\
a_7 &= a_6 + 2^6 - 1 = 57 + 64 - 1 = 120
\end{aligned}$$

Thus there are 120 bit strings of length 7 containing 01. Note that this agrees with our nonrecursive analysis, since $2^7 - (7+1) = 120$.

26 Let a_n be the number of coverings.

(a) We follow the hint. If the right-most domino is positioned vertically, then we have a covering of the leftmost $n - 1$ columns, and this can be done in a_{n-1} ways. If the right-most domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first $n-2$ columns therefore will need to contain a covering by dominoes, and this can be done in a_{n-2} ways. Thus we obtain the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$.

(b) Clearly $a_1 = 1$ and $a_2 = 2$

(c) The sequence we obtain is just the Fibonacci sequence, shifted by one. The sequence is thus 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584. . ., so the answer to this part is 2584

29 If the codomain has only one element, then there is only one function (namely the function that takes each element of the domain to the unique element of the codomain). Therefore when ***n***=1 we have ***S(m,n)=S(m,1)=1***, the initial condition we are asked to verify. Now assume that ***m>=n>1***, and we want to count ***S***(***m,n***), the number of functions from a domain with ***m*** elements onto a codomain with ***n*** elements. The form of the recurrence relation we are supposed to verify suggests that what we want to do is to look at the non-onto functions. There are ***n^m*** functions from the ***m***-set to the ***n***-set altogether (by the product rule, since we need to choose an element from the ***n***-set, which can be done in ***n*** ways, a total of ***m*** times). Therefore we must show that there are $\sum_{k=1}^{n-1} C(n,k)S(m,k)$ functions from the domain to the codomain that are not onto. First we use the sum rule and break this count down into the disjoint cases determined by the number of elements – let us call it ***k*** – in the range of the function. Since we want the function not to be onto, ***k*** can have any value from ***1*** to ***n-1***, but ***k*** cannot equal ***n***. Once we have specified ***k***, in order to specify a function we need to first specify the actual range, and this can be done in ***C(n,k)*** ways, namely choosing the subset of ***k*** elements from the codomain that are to constitute the range; and second choose an onto function from the domain to this set of ***k*** elements. This latter task can be done in ***S(m,k)*** ways, since (and here is the key recursive point) we are defining ***S(m,k)*** to be precisely this number. Therefore by the product rule there are ***C(n,k)S(m,k)*** different functions with our original domain to our original codomain. Note that this two-dimensional recurrence relation can be used to computer ***S(m,n)*** for any desired positive integers ***m*** and ***n***. Using it is much easier than trying to list all onto functions.

We let a_n be the number of moves required for this puzzle.

a) In order to move the bottom disk off peg 1, we must have transferred the other $n-1$ disks to peg 3 (since we must move the bottom disk to peg 2); this will require a_{n-1} steps. Then we can move the bottom disk to peg 2 (one more step). Our goal, though, was to move it to peg 3, so now we must move the other $n-1$ disks from peg 3 back to peg 1, leaving the bottom disk quietly resting on peg 2. By symmetry, this again takes a_{n-1} steps. One more step lets us move the bottom disk from peg 2 to peg 3. Now it takes a_{n-1} steps to move the remaining disks from peg 1 to peg 3. So our recurrence relation is $a_n = 3a_{n-1} + 2$. The initial condition is of course that $a_0 = 0$.

b) Computing the first few values, we find that $a_1 = 2$, $a_2 = 8$, $a_3 = 26$, and $a_4 = 80$. It appears that $a_n = 3^n - 1$. This is easily verified by induction: The base case is $a_0 = 3^0 - 1 = 1 - 1 = 0$, and $3a_{n-1} + 2 = 3 \cdot (3^{n-1} - 1) + 2 = 3^n - 3 + 2 = 3^n - 1 = a_n$.

c) The only choice in distributing the disks is which peg each disk goes on, since the order of the disks on a given peg is fixed. Since there are three choices for each disk, the answer is 3^n.

d) The puzzle involves $1 + a_n = 3^n$ arrangements of disks during its solution—the initial arrangement and the arrangement after each move. None of these arrangements can repeat a previous arrangement, since if

Sec. 8.2 2, 4(g), 20, 30, 35

2(a) linear, homogeneous, with constant coefficients, degree 2

2(b) linear with constant coefficients but not homogeneous

2(c) not linear

2(d) linear, homogeneous, with constant coefficients, degree 3

2(e) linear and homogeneous, but not with constant coefficients

2(f) linear with constant coefficients, but not homogeneous

2(g) linear, homogeneous, with constant coefficients, degree 7

4(g) $r^2 + 4r - 5 = 0$ r = -5, 1

$a_n = \alpha_1(-5)^n + \alpha_2 1^n = \alpha_1(-5)^n + \alpha_2$

$2 = \alpha_1 + \alpha_2$, $8 = -5\alpha_1 + \alpha_2$

$\alpha_1 = -1$ $\alpha_2 = 3$

$a_n = -(-5)^n + 3$

20 This is a fourth degree recurrence relation. The characteristic polynomial is $r^4 - 8r^2 + 16$, which factors as $(r^2 - 4)^2$, which then further factors into $(r-2)^2 (r+2)^2$. The roots are 2 and -2, each with multiplicity 2. Thus we can write down the general solution as usual: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 (-2)^n + \alpha_4 n \cdot (-2)^n$

30(a) The associated homogeneous recurrence relation is $a_n = -5a_{n-1} - 6a_{n-2}$. To solve it we find the characteristic equation $r^2 + 5r + 6 = 0$, find that $r = -2$ and $r = -3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha(-2)^n + \beta(-3)^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = c \bullet 4^n$. We plug this into our recurrence relation and obtain $c \bullet 4^n = -5\, c \bullet 4^{n-1} - 6c \bullet 4^{n-2} + 42 \bullet 4^n$. We divide through by 4^{n-2}, obtaining $16c = -20c - 6c + 42 \bullet 16$, whence with a little simple algebra $c = 16$. Therefore the particular solution we seek is $a_n^{(p)} = 16 \cdot 4^n = 4^{n+2}$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha(-2)^n + \beta(-3)^n + 4^{n+2}$

30(b) We plug the initial conditions into our solution from part (a) to solve for α and β. So the solution is $a_n = (-2)^n + 2(-3)^n + 4^{n+2}$

35 The associated homogeneous recurrence relation is $a_n = 4a_{n-1} - 3a_{n-2}$. To solve it we find the characteristic equation $r^2 - 4r + 3 = 0$, find that $r = 1$ and $r = 3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha + \beta 3^n$. Next we need a particular solution to the given recurrence relation. By using the idea in Theorem 6 twice, we want to look for a function of the form $a_n = c \cdot 2^n + n(dn + e) = c \cdot 2^n + dn^2 + en$. We plug this into our recurrence relation an $c \cdot 2^n + dn^2 + en = 4c \cdot 2^{n-1} + 4d(n-1)^2 + 4e(n-1) - 3c \cdot 2^{n-2} - 3d(n-2)^2 - 3e \cdot (n-2) + 2^n + n + 3$ .A lot of messy algebra transforms this into the following equation, where we group by function of n: $2^{n-2}(-c-4) + n^2 \cdot 0 + n(-4d - 1) + (8d - 2e - 3) = 0$. The coefficients must therefore all be 0, whence $c = -4$, $d = -1/4$, and $e = -5/2$. Therefore the particular solution we see is $a_n^{(p)} = -4 \cdot 2^n - n^2/4 - 5n/2$. So the general solution is $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + \alpha + \beta 3^n$. We solve this system of equations to obtain $\alpha = 1/8$ and $\beta = 39/8$. So the final solution is

$a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + 1/8 + (39/8)3^n$. As a check of our work, we can compute a_2 both from the recurrence and from the solution, and we find that $a_2 = 22$ both ways.

Sec. 8.4 6(d,f), 10 (c, d, e), 16, 24(a), 30, 34, 43

6(d) The power series for the function e^x is $\sum_{n=0}^{\infty} x^n/n!$. That is almost what we have here; the difference is that the denominator is $(n+1)!$ instead of n!. So we have $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!}$ by a change of variable. This last sum is $e^x - 1$ (only the first term is missing), so our answer is $(e^x - 1)/x$.

6(f) By Table 1,

$$\sum_{n=0}^{\infty} C(10, n+1)x^n = \sum_{n=1}^{\infty} C(10, n)x^{n-1}$$
$$= \frac{1}{x}\sum_{n=1}^{\infty} C(10, n)x^n = \frac{1}{x}((1+x)^{10} - 1)$$

10(c) If we factor out as high a power of x from each factor as we can, then we can write this as $x^7 (1 + x^2 + x^3)(1 + x)(1 + x + x^2 + x^3 + \cdots)$, and so we seek the coefficient of x^2 in $(1 + x^2 + x^3)(1 + x)(1 + x + x^2 + x^3 + \cdots)$. We could do this by brute force, but let's try it more analytically. We write our expression in closed form as

$$\frac{(1+x^2+x^3)(1+x)}{1-x} = \frac{1+x+x^2+ higherorderterms}{1-x}$$

$$= \frac{1}{1-x} + x \cdot \frac{1}{1-x} + x^2 \cdot \frac{1}{1-x} + \text{irrelevant terms}$$

The coefficient of x^2 in this power series comes either from the coefficient of x^2 in the first term in the final expression displayed above, or from the coefficient of x^1 in the second factor of the second term of that expression, or from the coefficient of x^0 in the second factor of the third term. Each of these coefficients is 1, so our answer is 3.

10(d) The easiest approach here is simply to note that there are only two combinations of terms that will give us an x^9 term in the product: $x \bullet x^8$ and $x^7 \bullet x^2$. So the answer is 2.

10(e) The highest power of x appearing in this expression when multiplied out is x^6. Therefore the answer is 0.

16 The factors in the generating function for choosing the egg and plain bagels are both $x^2 + x^3 + x^4 + \ldots$ The factor for choosing the salty bagels is $x^2 + x^3$. Therefore the generating function for this problem is $(x^2 + x^3 + x^4 + \ldots)^2 (x^2 + x^3)$. We want to find the coefficient of x^{12}, since we want 12 bagels. This is equivalent to finding the coefficient of x^6 in $(1 + x + x^2 + \ldots)^2 (1 + x)$ This function is $(1+x)/(1-x)^2$, so we want the coefficient of x^6 in $1/(1-x)^2$, which is 7, plus the coefficient of x^5 in $1/(1-x)^2$, which is 6. Thus the answer is **13**.

24(a) The restriction on x_1 gives us the factor $x^3 + x^4 + x^5 + \cdots$. The restriction on x_2 gives us the factor $x + x^2 + x^3 + x^4 + x^5$. The restriction on x_3 gives us the factor $1+ x + x^2+ x^3+ x^4$. And the restriction on x_4 gives us the factor $x + x^2 + x^3 + \cdots$. Thus the answer is the product of these:

$$(x^3 + x^4 + x^5 + \cdots)(x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \cdots)$$

We can use algebra to rewrite this in closed form as $x^5(1+x+x^2+x^3+x^4)^2/ (1-x)^2$.

30(a) Multiplication distributes over addition, even when we are talking about infinite sums, so the generating function is just $2G(x)$.

30(b) What used to be the coefficient of $x0$ is now the coefficient of $x1$, and similarly for the other terms. The way that happened is that the whole series got multiplied by x. Therefore the generating function for this series is xG9x). In symbols,

$$a_0 x + a_1 x^2 + a_2 x^3 + \cdots = x(a_0 + a_1 x + a_2 x^2 + \cdots) = xG(x)$$

30(c) The terms involving a_0 and a_1 are missing, $G(x) - a_0 - a_1 x = a_2 x^2 + a_3 x^3 + \cdots$. Here, however, we want a_2 to be the coefficient of x^4, not x^2 (and similarly for the other powers), so we must throw in an extra factor. Thus the answer is $x^2(G(x)-a_0-a_1x)$.

30(d) This is just like part(c), except that we slide the powers down, Thus the answer is $(G(x)-a_0-a_1x)/x^2$.

30(e) Following the hint, we differentiate $G(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain $G'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$. By a change of variable this becomes $\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$, which is the generating function for precisely the sequence we are given. Thus G’(x) is the generating function for this sequence.

30(f) If we look at Theorem 1, it is not hard to see that the sequence shown here is precisely the coefficients of $G(x) \cdot G(x)$.

34 Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$ (by changing the name of the variable from k to k+1). Thus

$$G(x)-3xG(x)=\sum_{k=0}^{\infty}a_k x^k-\sum_{k=1}^{\infty}3a_{k-1}x^k=a_0+\sum_{k=1}^{\infty}(a_k-3a_{k-1})x^k$$

$$=1+\sum_{k=1}^{\infty}4^{k-1}x^k=1+x\sum_{k=1}^{\infty}4^{k-1}x^{k-1}=1+x\sum_{k=0}^{\infty}4^k x^k=1+x\frac{1}{1-4x}=\frac{1-3x}{1-4x}$$

Thus $G(x)(1-3x)=(1-3x)/(1-4x)$, so $G(x)=1/(1-4x)$. Therefore $a_k = 4^k$, from Table 1.

43 Following the hint, we note that $(1+x)^{m+n}=(1+x)^m(1+x)^n$. Then applying the Binomial Theorem, we have

$$\sum_{r=0}^{m+n}C(m+n,r)x^r=\sum_{r=0}^{m}C(m,r)x^r\cdot\sum_{r=0}^{n}C(n,r)x^r=\sum_{r=0}^{m+n}\left(\sum_{k=0}^{r}C(m,r-k)C(n,k)\right)x^r$$

by Theorem 1. Comparing coefficients gives us the desired identity.