Sample Solutions on HW5 (*13 exercises in total*)

Sec. 3.3 7, 10

7 Linear search is faster.

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10(a) By the way that $S - 1$ is defined, it is clear that $S \wedge (S - 1)$ is the same as S except that the rightmost 1 bit has been changed to a 0. Thus we add 1 to ***count*** for every one bit (since we stop as soon as $S = 0$, i.e., as soon as S consists of just 0 bits).

10(b) Obviously the number of bitwise AND operations is equal to the final value of ***count***, i.e., the number of one bits in S.

Sec. 5.1 46, 47, 48

46 This proof will be similar to the proof in Example 10.
Basis step: since for $n = 3$, the set has exactly one subset containing exactly three elements, and $3(3 - 1)(3-2)/6 = 1$.
Inductive step: Assume the inductive hypothesis, that a set with n elements has $n(n - 1)(n - 2)/6$ subsets with exactly three elements; we want to prove that a set S with $n + 1$ elements has $(n + 1)n(n - 1)/6$ subsets with exactly three elements. Fix an element a in S , and let T be the set of elements of S other than a. There are two varieties of subsets of S containing exactly three elements. First there are those that do not contain a. These are precisely the three-element subsets of T, and by the inductive hypothesis, there are $n(n - 1)(n - 2)/6$ of them. Second, there are those that contain a together with two elements of T. Therefore there are just as many of these subsets as there are two-element subsets of T. By Exercise 45, there are exactly $n(n-1)/2$ such subsets of T; therefore there are also $n(n-1)/2$ three-element subsets of S containing a. Thus the total number of subsets of S containing exactly three elements is $(n(n - 1)(n - 2)/6) + n(n - 1)/2$, which simplifies algebraically to $(n + 1)n(n -1)/6$, as desired.

47 Reorder the locations if necessary so that $\mathbf{x}_1 \le \mathbf{x}_2 \le \mathbf{x}_3 \le \cdots \le \mathbf{x_d}$. Place the first tower at position $\mathbf{t}_1 = \mathbf{x}_1+1$. Assume tower $\mathbf{k}$ has been placed at position $\mathbf{t_k}$. Then place tower $\mathbf{k}+1$ at position $\mathbf{t_{k+1}} = \mathbf{x}+1$, where $\boldsymbol{x}$ is the smallest $\mathbf{x_i}$ greater than $\mathbf{t_k} + 1$.

48 We will show that any minimum placement of towers can be transformed into the placement produced by the algorithm. Although it does not strictly have the form of a proof by mathematical induction, the spirit is the same. Let $\boldsymbol{s}_1 < \boldsymbol{s}_2 < \cdots < \boldsymbol{s_k}$ be an optimal locations of the towers (i.e., so as to minimize $\boldsymbol{k}$), and let $\boldsymbol{t}_1 < \boldsymbol{t}_2 < \cdots < \boldsymbol{t_l}$ be the locations produced by the algorithm from Exercise 47. In order to serve the first building, we must have $\boldsymbol{s}_1 \le \boldsymbol{x}_1 +1 = \boldsymbol{t}_1$. If $\boldsymbol{s}_1 \ne \boldsymbol{t}_1$, then we can move the first tower in the optimal solution to position $\boldsymbol{t}_1$ without losing cell service for any building. Therefore we can assume that $\boldsymbol{s}_1 = \boldsymbol{t}_1$. Let $\boldsymbol{x_j}$ be smallest location of a building out of range of the tower at $\boldsymbol{s}_1$; thus $\boldsymbol{x_j} > \boldsymbol{s}_1 + 1$. In order to serve that building there must be a tower $\boldsymbol{si}$ such that $\mathbf{s}_i \le \boldsymbol{x_j} + 1 = \boldsymbol{t}_2$. If $\boldsymbol{i} > 2$, then towers at positions $\boldsymbol{s}_2$ through $\boldsymbol{s}_{i-1}$ are not needed, a contradiction. As before, it then follows that we can move the second tower from $\boldsymbol{s}_2$ to $\boldsymbol{t}_2$. We continue in this manner for all the towers in the given minimum solution; thus $\boldsymbol{k} = \mathrm{l}$. This proves that the algorithm produces a minimum solution.

Sec. 5.2 8, 18, 39

8 Since both 25 and 40 are multiples of 5, we cannot form any amount that is not a multiple of 5. So let's determine for which values of n we can form $5n$ dollars using these gift certificates, the first of which provides 5 copies of \$5, and the second of which provides 8 copies. We can achieve the following values of n: 5 = 5, 8 = 8, 10 = 5+5, 13 = 8+5, 15 =5+5+5, 16 = 8+8, 18 = 8+5+5, 20 = 5+5+5+5+5, 21 = 8+8+5, 23 = 8 + 5 + 5 + 5, 24 = 8 + 8+ 8, 25 = 5 + 5 + 5 + 5 + 5, 26 = 8 + 8 + 5 + 5, 28 = 8 + 5 + 5 + 5 + 5, 29 = 8 + 8 + 8 + 5, 30= 5 + 5 + 5 + 5 + 5 + 5, 31 = 8 + 8 + 5 + 5 + 5, 32 = 8 + 8 + 8 + 8. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form total amounts of the form $5n$ for all $n \geq 28$ using these gift certificates. (In other words, \$135 is the largest multiple of \$5 that we cannot achieve.)

To prove this by strong induction, let $P(n)$ be the statement that we can form $5n$ dollars in gift certificates using just 25-dollar and 40-dollar certificates. We want to prove that $P(n)$ is true for all n ≥ 28. From our work above, we know that $P(n)$ is true for n = 28, 29, 30, 31,32. Assume the inductive hypothesis, that $P(j)$ is true for all j with $28 \le j \le k$, where k is a fixed integer greater than or equal to 32. We want to show that $P(k+1)$ is true. Because $k-4 \geq 28$, we know that $P(k-4)$ is true, that is, that we can form $5(k-4)$ dollars. Add one more \$25-dollar certificate, and we have formed $5(k + 1)$ dollars, as desired.

18 If a n-gon whose vertices are labeled consecutively as $v_m, v_{m+1}, \ldots, v_{m+n-1}$ is triangulated, then the triangles can be numbered from m to $m+n-3$ so that v_i is a vertex of triangle i for $i = m, m+1, \ldots, m+n-3$. (The statement we are asked to prove is the case $m = 1$.) The basis step is $n = 3$, and there is nothing to prove.

For the inductive step, assume the inductive hypothesis that the statement is true for polygons with fewer than n vertices, and consider any triangulation of a convex n-gon whose vertices are labeled consecutively as $v_m, v_{m+1}, \ldots, v_{m+n-1}$. One of the diagonals in the triangulation must have either v_{m+n-1} or v_{m+n-2} as an endpoint (otherwise, the region containing v_{m+n-1} would not be a triangle). So there are two cases. If the triangulation uses diagonal v_kv_{m+n-1}, then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-1} as v_{k+1} in the polygon that contains v_m. This gives us the desired numbering of the triangles, with numbers v_m through v_{k-1} in the first polygon and numbers v_k through v_{m+n-3} in the second polygon. If the triangulation uses diagonal v_kv_{m+n-2} , then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-2} as v_{k+1} and v_{m+n-1} as v_{k+2} in the polygon that contains v_{m+n-1} , and renumbering v_k as v_{m+n-1} in the other polygon. This gives us the desired numbering of the triangles, with numbers v_m through v_k in the first polygon and numbers v_{k+1} through v_{m+n-3} in the second polygon. Note that we did not need the convexity of our polygons.

39 This is a paradox caused by self-reference. The answer is clearly “no.” There are a finite number of English words, so only a finite number of strings of 15 words or fewer; therefore, only a finite number of positive integers can be so described, not all of them.

Sec. 5.3 6(a,d), 14, 29(a)

6(a) This is valid, since we are provided with the value at $n = 0$, and each subsequent value is determined by the previous one. Since all that changes from one value to the next is the sign, we conjecture that $f(n) = (-1)^n$. This is true for $n = 0$, since $(-1)^0 = 1$. If it is true for $n = k$, then we have $f(k + 1) = -f(k + 1 - 1) = -f(k) = -(-1)^k$ by the inductive hypothesis, when $f(k + 1) = (-1)^{k+1}$

6(d) This is invalid, because the value at $n = 1$ is defined in two conflicting ways—first as $f(1) = 1$ and then as $f(1) = 2f(1 - 1) = 2f(0) = 2 \cdot 0 = 0$.

14 The basis step ($n = 1$) is clear, since $f_2 f_0 - f_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1$. Assume the inductive hypothesis. Then we have

$$f_{n+2}f_n - f_{n+1}^2 = (f_{n+1} + f_n)fn - f_{n+1}^2$$

$$
\begin{aligned}
&= f_{n+1}f_n + f_n^2 - f_{n+1}^2 \\
&= -f_{n+1}(f_{n+1} - f_n) + f_n^2 \\
&= -f_{n+1}f_{n-1} + f_n^2 \\
&= -(f_{n+1}f_{n-1} - f_n^2) \\
&= -(-1)^n = (-1)^{n+1} .
\end{aligned}
$$

29(a) Define S by $(1, 1) \in S$, and if $(a, b) \in S$, then $(a + 2, b) \in S$, $(a, b + 2) \in S$, and $(a + 1, b + 1) \in S$. All elements put in S satisfy the condition, because $(1, 1)$ has an even sum of coordinates, and if (a, b) has an even sum of coordinates, then so do $(a+2, b)$, $(a, b+2)$, and $(a+1, b+1)$. Conversely, we show by induction on the sum of the coordinates that if $a + b$ is even, then $(a, b) \in S$. If the sum is 2, then $(a, b) = (1, 1)$, and the basis step put (a, b) into S. Otherwise the sum is at least 4, and at least one of $(a - 2, b)$, $(a, b - 2)$, and $(a - 1, b - 1)$ must have positive integer coordinates whose sum is an even number smaller than $a+b$, and therefore must be in S. Then one application of the recursive step shows that $(a, b) \in S$.