University Of Waterloo

STAT 906

Computer-Intensive Methods for Stochastic Models in Finance

Report on Estimating Security Price Derivatives using Simulation

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Contents

1 Abstract

The report is involved with estimating security prices derivatives using simulation. This report presents two direct methods and an indirect method, a pathwise method, a likelihood ratio method, and a resimulation method, for estimating the derivatives of security. The main advantage of the direct methods over the resimulation method is increased computational saving. Another advantage is that the direct methods yield an unbiased estimate of the derivatives, while the indirect method, the resimulation method, will always generate biased estimates for the derivatives. Besides, we have also conducted some numerical experiments for all three algorithms. Specifically, we implement the methods on an independent path type of option (a European call option) and a dependent path type of option (an Asian call option), both with dividend-paying assets. The report's primary reference is the paper 'Estimating Security Price Derivatives using Simulation' written by Broadie and Glasserman in 1996 [1]. General setting from section 3 and detailed derivations from section 4 are also referenced from the paper [1]

2 Introduction

Securities price derivatives measure security price's exposures to the movement of the market variables such as the underlying price, time and volatility. The information contains in the derivatives is important in the nowadays financial market in terms of reducing the risk of security when the closing position is not desirable. For example, the derivatives delta is the number of security one holds in the hedge portfolio, and Theta is how much value an option's price will diminish per day. Therefore, the determination of the derivative's value becomes essential in hedging the risk in the financial markets. However, there is not always a simple closed-form solution for the derivatives of security. As a method has long proved to be a valuable tool for estimating the derivatives in a fast and accurate manner, more and more people are implementing the simulation method to estimate the securities price derivatives. In particular, the author in this paper [1] has proposed three different simulation methods, one indirect method, i.e. the resimulation method that contain inherent bias and two direct methods that are unbiased, fast, and accurate.

3 General Setting

This section provides the general settings for the derivative estimates of a European option on dividend-paying assets using simulation.

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space on which the standard Brownian motion B_t is defined for time $t \geq 0$, with filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ generated by the standard Brownian motion. In this case, Ω is the set of all possible realizations in the financial market within the time horizon [0, T], where T is the financial securities' maturity. $\mathbb{Q}$ is the risk-neutral measure define on the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$.

Under the Black–Scholes–Merton model [3] and assumptions, for illustrative purpose, we consider the price p of a European Option with dividend paying asset S_t given by

$$p = \mathrm{E}^{\mathbb{Q}}[e^{-rT} \max(S_T - K, 0)] \tag{1}$$

where $\mathrm{E}^{\mathbb{Q}}[\cdot]$ is the expectation under the risk neutral measure, r is the risk free interest rate, S_T is terminal asset price, K is strike price, and the underlying asset S_t follows the stochastic differential equations

$$dS_t = (r - \delta)S_t dt + \sigma S_t dB_t$$

with dB_t defined as the increment of the standard BM, $\sigma > 0$ refers to the constant volatility, δ is the dividend paying per unit time. According to the Black-Scholes-Merton assumptions, under the risk neutral measure $\mathbb{Q}$, $ln(\frac{S_T}{S_0})$ is lognormally distributed with mean $(r - \delta - \frac{\sigma^2}{2})$ and variance $\sigma^2 T$. In particular, we assume that both the dividend δ and r is continuous compounding overtime.

Now, we have the general setting of the problem. We will begin our discussions about the first simulation algorithm for estimating the security price derivative.

4 Algorithms for Estimating Security Price Derivatives

4.1 Resimulation Method with CRN (Common Random Numbers)

Our discussion on the first algorithm is the restimulation method with common random numbers (CRN), also called the indirect method, which is one of the classical approaches for estimating security derivatives. The main idea is to use the Finite Difference Scheme (See Appendix A.1) to approximate the security derivatives using simulation.

The general scheme for constructing the resimulation method is rather simple. We first suppose the security price p depends on a parameter θ and our goal is to estimate the derivative of the security price i.e. $\frac{dp}{d\theta}$ at $\theta = \theta_0$. In this case, we denote P as the discounted payoff of the European call option on dividend paying asset i.e. P $= e^{-rT}\max(S_T - K, 0))$. We further let

$$P(\theta_0) := \text{the simulation estimate of the price at } \theta = \theta_0$$
$$P(\theta_0 \pm h) := \text{the simulation estimate of the perturbed price}$$

Then, by the finite difference method, we can obtain the resimulation estimators of the derivatives $(\frac{dp}{d\theta})$ from Table 1. One can observe that we only include part of the derivatives family that is commonly used in the financial market. In this case, we can verify that the first-order derivatives of the security are estimated through the forward difference scheme, and the second derivative is estimated through the central approximation scheme.

Greeks	Resimulation Estimators
$Vega = \frac{dp}{d\sigma}$	$\frac{(P(\sigma_0+h)-P(\sigma_0))}{h}$
$Delta = \frac{dp}{dS_0}$	$\frac{(P(S_0+h)-P(S_0))}{h}$
$Gamma = \frac{d^2p}{dS_0^2}$	$\frac{(P(S_0-h)-2P(S_0)+P(S_0+h))}{h^2}$
$Rho = \frac{dp}{dr}$	$\frac{(P(r_0+h)-P(r_0))}{h}$
$Theta = -\frac{dp}{dT}$	$-\frac{(P(T+h)-P(T))}{h}$

Table 1: The Resimulation Estimators for European Call Option

To further illustrate the method, we take the derivative Vega $(\frac{dp}{d\sigma})$ of a European call option as an example. When estimating the Vega $(\frac{dp}{d\sigma})$ of a European call option using resimulation method, we can write the resimulation estimator for Vega $(\frac{dp}{d\sigma})$ as

$$\begin{aligned}
V\hat{ega}_{re} &= \frac{1}{n}\sum_{i=1}^{n} e^{-rT}(\frac{(P(\sigma_0+h)-P(\sigma_0))}{h}) \\
&= \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\max(\frac{S_T^i(\sigma+h)-S_T^i(\sigma)}{h} - K, 0) \qquad (2)
\end{aligned}$$

where the terminal stock price S_T^i is defined as $S_T^i = S_0 e^{(r-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\Phi^{-1}(U_i)}$ with common random number generated by the random uniform distribution over the domain (0,1) as $U_i \sim i.i.d.U(0,1)$ for each simulation i = 1,...,n. In this case, we are using inversion to generate the standard normal r.v. Z_i. We can also determine the standard Error of the resimulation estimator for Vega as,

$$Std\hat{.}err_{re} = \sqrt{\frac{\sum_{i=1}^{n}(\frac{S_T^i(\sigma+h)-S_T^i(\sigma)}{h} - V\hat{ega}_{re})^2}{n(n-1)}} \tag{3}$$

In general, if we want to estimate N derivatives using the resimulation method, we will need (N+1) simulations. In this case, the estimation is computationally expensive. At the same time, Since we are implementing the Finite Difference Method, the resimulation method is inherently biased. Thus, to obtain accurate solutions, we would need to figure out the stability condition of h under some stability analysis (Von Neumann stability analysis [2], for example). In other words, accurate solutions will depend on the choice of h. The detailed discussion of finding the stability condition for h can be found in paper[1].

The advantages of the resimulation method are also prominent. Once we choose an appropriate h, the security derivatives estimates are accurate with smaller standard errors, which are indicated in the numerical experiments section. It is also worth noting that we can apply the same rationale as the European call option to a European put option and any other options to derive the resimulation estimator for different options' derivatives.

Although the resimulation method is useful in approximating the security's derivatives, the method is still biased and certainly generated a large error when the choice of h is inappropriate. Thus, we need to introduce an alternative algorithm that is unbiased and unconditionally stable in estimating the security derivatives, which leads to the second method–the pathwise method.

4.2 Pathwise Method

This subsection proposes a fast, accurate, and an unbiased method called the pathwise method to eliminate the biases inherent in the restimulation method when estimating the security's derivatives. The construction of the method is based on the dependence between the security payoff and the parameters of interest. In particular, the pathwise method will produce an unbiased and accurate estimator for security derivatives.

To illustrate the implementation of the pathwise method for the first order derivative, we consider again the Vega($\frac{dp}{d\sigma}$) of a European call option on dividend paying asset as an example. We first doing this by defining the discounted payoff function

$$P = e^{-rT}\max(S_T - K, 0) \tag{4}$$

According to the equation (4), since the only part P vary by σ is S_T, we will first examine how S_T is changing with respect to σ. Before that, we know S_T is log-normally distributed with mean $(r-\delta-\frac{\sigma^2}{2})$ and variance $\sigma^2 T$ as

$$S_T = S_0 e^{(r-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z} \tag{5}$$

where random variable Z follows a standard normal distribution and S_0 is the spot price of the underlying asset. In this case, by taking the partial derivative of S_T stated in equation(5) against σ, one can obtain

$$\frac{dS_T}{d\sigma} = S_T(-\sigma T + \sqrt{T}Z) = \frac{S_T}{\sigma}[ln(\frac{S_T}{S_0}) - (r-\delta+\frac{\sigma^2}{2})T] \tag{6}$$

where the second equality is given by the fact that $ln(\frac{S_T}{S_0}) = (r - \delta - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z$ based on equation (5). This give us information about how changes in σ affects S_T, which in terms changes the dicounted payoff P. Specifically, when $S_T \geq K$ i.e. in the money, an increases in Δ unit, a really small increment in the real space, will lead to an increase of $e^{-rT}\Delta$ in P. On the other hand, when $S_T \leq K$ i.e. out of the money, the discounted payoff P will remains 0 by definition of a European call option. Thus, based on the above observations, we achieve the following equation

$$\frac{dP}{dS_T} = e^{-rT}\mathbb{1}_{\{S_T \geq K\}} \tag{7}$$

in which $\mathbb{1}_{\{.\}}$ is referring to the indicator function. Thus, by combining the equation (6) & (7), it gives the pathwise estimate of the Vega ($\frac{dp}{d\sigma}$)

$$\begin{aligned} Vega = \frac{dP}{d\sigma} &= \frac{dP}{dS_T}\frac{dS_T}{d\sigma} \\ &= e^{-rT}\mathbb{1}_{\{S_T \geq k\}}\frac{S_T}{\sigma}(ln(\frac{S_T}{S_0}) - (r - \delta + \frac{1}{2}\sigma^2)T) \end{aligned} \tag{8}$$

According to the main reference paper's Proposition 1 in Appdendix A [1], the estimator from equation (8) is the unbiased estimate of Vega i.e.

$$\mathrm{E}^{\mathbb{Q}}[\frac{dP}{d\sigma}] = \frac{dp}{d\sigma} \tag{9}$$

for $P = f(S_T)$ satisfies some Lipschitz conditions. Therefore, we can write the unbiased pathwise estimator of Vega ($\frac{dp}{d\sigma}$) as

$$V\hat{e}ga_{path} = \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\mathbb{1}_{\{S_T^i \geq k\}}(ln(\frac{S_T^i}{S_0}) - (r - \delta + \frac{1}{2}\sigma^2)T) = \frac{1}{n}\sum_{i=1}^{n} P_i \tag{10}$$

where the terminal stock price S_T^i is defined as $S_T^i = S_0 e^{(r-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\Phi^{-1}(U_i)}$ with U_i generated by the random uniform distribution over the domain (0,1) as $U_i \sim i.i.d.U(0,1)$ for each simulation i = 1,...,n. Again, in this case, we are using inversion to generate the standard normal r.v. Z_i. Consequently, we can determine the standard error as

$$St\hat{d.e}rr_{path} = \sqrt{\frac{\sum_{i=1}^{n}(P_i - Veg\hat{a}_{path})^2}{n(n-1)}} \tag{11}$$

By similar argument, we can obtain the estimators for the other first derivatives according to Table 2. They are all proved to be unbiased for $P = f(S_T)$ satisfying some Lipschitz conditions.

Greeks	Pathwise Estimators
$Delta = \frac{dp}{dS_0}$	$e^{-rT}\mathbb{1}_{\{S_T \geq k\}}\frac{S_T}{S_0}$
$Rho = \frac{dp}{dr}$	$KTe^{-rT}\mathbb{1}_{\{S_T \geq k\}}$
$Theta = -\frac{dp}{dT}$	$-e^{-rT}\mathbb{1}_{\{S_T \geq k\}}\frac{S_T}{2T}(ln(\frac{S_T}{S_0}) + (r - \delta - \frac{1}{2}\sigma^2)T)$.

Table 2: The Pathwise Estimators for the European Call Option

However, notice that the Pathwise simulation method does not apply for Gamma ($\frac{d^2P}{dS_0^2}$) of an European Call Option. According to paper [1], by assuming considering the payoff of a European option is of exponential form and using the same rationale as deriving the first order derivative, the estimate of Gamma ($\frac{d^2P}{dS_0^2}$) is

$$Gamma = \frac{d^2P}{dS_0^2} = e^{-rT}\frac{n(d_1(K))}{S_0\sigma\sqrt{T}} \tag{12}$$

where $n(\cdot)$ is the density function of a standard normal distribution with

$$d_1(x) = \frac{[ln(\frac{S_0}{x}) + (r - \delta - \frac{1}{2}\sigma^2)T]}{\sigma\sqrt{T}}$$

The expression (12) of Gamma involved no random quantities, thereby requires no simulations and therefore, the Pathwise simulation method does not apply for Gamma($\frac{d^2P}{dS_0^2}$). A more in-depth interpretation for this is that the discounted payoff P is non-differentiable at $S_T = K$ even though P is continuous in S_T.

Overall, if we want to estimate N derivatives using the pathwise method, we will only need one simulation, which will give us an estimate of the N derivatives. So this advantage of the pathwise method certainly gives computational saving comparing to the resimulation method. In particular, it is a 1 to N+1 advantage for estimating N derivatives of security. Also, as previously discussed, the drawback of the pathwise method is evident, from which the method only applies if the payoff function is differentiable.

Now that we have introduced the pathwise method that gives unbiased estimates of the derivatives for a European call option. Apart from the pathwise method that is usually categorized as one of the direct methods, the author in record [1] has also proposed another direct method called the likelihood ratios method discussed in the next subsection.

4.3 Likelihood Ratios Method

This subsection introduces another type of direct method that estimates the derivatives based on the relationship between the parameter of interest in the underlying density, rather than in a random variable like S_T. The method is called the likelihood ratios method. Unlike the former direct method, i.e. the pathwise method, the likelihood ratios method gives an unbiased estimator of the derivatives by construction.

4.3.1 The First Order Derivative

To better illustrate the application of the likelihood ratios method on the first order derivatives, we again consider the Vega ($\frac{dp}{d\sigma}$) as an example for a European call option on dividend paying asset. Recall that under the risk neutral measure $\mathbb{Q}$, $ln(\frac{S_T}{S_0})$ follows a standard normal distribution with mean $(r - \delta - \frac{\sigma^2}{2})$ and variance $\sigma^2 T$, then we can easily obtain the density of S_T as

$$g(x) = \frac{1}{x\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d(x)^2}{2}}, \text{ x} \geq 0 \tag{13}$$

where

$$d(x) = \frac{ln(\frac{x}{S0}) - (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} \tag{14}$$

Thus, we can rewrite the price of the European call option p into an integral form using the density of S_T defined in equation (13)

$$p = E^{\mathbb{Q}}[e^{-rT}\max(x-K,0)] = \int_0^\infty e^{-rT}\max(x-K,0)g(x)\,dx \tag{15}$$

It is clear that σ, the parameter of our interest, only appears in the density of S_T and so as to the price of the European call option p. Consequently, we will first look at the derivative $\frac{dp}{d\sigma}$. Assume we can interchange the derivative and integral, then (15) becomes

$$\frac{dp}{d\sigma} = \int_0^\infty e^{-rT}\max(x-K,0)\frac{\partial g(x)}{\partial\sigma}\,dx$$

where we consider the identity $\frac{\partial ln(g(x))}{\partial\sigma} = \frac{\partial g(x)}{\partial\sigma}\frac{1}{g(x)}$ so that $\frac{\partial g(x)}{\partial\sigma} = \frac{\partial ln(g(x))}{\partial\sigma}g(x)$. Thus, using this identity, we can rewrite $\frac{dp}{d\sigma}$ as

$$\begin{aligned}\frac{dp}{d\sigma} &= \int_0^\infty e^{-rT}\max(x-K,0)\frac{\partial ln(g(x))}{\partial\sigma}g(x)\,dx\\ &= E^{\mathbb{Q}}[e^{-rT}\max(x-K,0)\frac{\partial ln(g(x))}{\partial\sigma}]\end{aligned} \tag{16}$$

Thus, we can obtain the likelihood ratios estimator of Vega ($\frac{dp}{d\sigma}$) as

$$Vega = e^{-rT}\max(x-K,0)\frac{\partial ln(g(x))}{\partial\sigma} \tag{17}$$

where we can determine $\frac{\partial ln(g(x))}{\partial\sigma}$ by identity $\frac{\partial ln(g(x))}{\partial\sigma} = \frac{\partial g(x)}{\partial\sigma}\frac{1}{g(x)}$, and $\frac{\partial d(x)}{\partial\sigma}$ from (14) as the following

$$\frac{\partial ln(g(x))}{\partial\sigma} = -d(x)\frac{\partial d(x)}{\partial\sigma} - \frac{1}{\sigma}, \text{ with } \frac{\partial d(x)}{\partial\sigma} = \frac{ln(\frac{S_0}{x}) + (r-\delta+\sigma^2)}{\sigma^2\sqrt{T}}$$

Hence, by expression (17), the unbiased likelihood ratios estimator of Vega ($\frac{dp}{d\sigma}$) is

$$\begin{aligned}\hat{Vega}_{LR} &= \frac{1}{n}\sum_{i=1}^n e^{-rT}\max(S_T^i-K,0)(-d(S_T^i)\frac{\partial d(S_T^i)}{\partial\sigma} - \frac{1}{\sigma})\\ &= \frac{1}{n}\sum_{i=1}^n L_i\end{aligned} \tag{18}$$

where $S_T^i = S_0 e^{(r-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\Phi^{-1}(U_i)}$ with U_i generated by the random uniform distribution over the domain (0,1) as $U_i \sim i.i.d.U(0,1)$ for each simulation i = 1,...,n. Again, in this case, we are using inversion to generate the standard normal r.v. Z_i. Consequently, we can calculate the standard error as

$$Std\hat{.}err_{LR} = \sqrt{\frac{\sum_{i=1}^n (L_i - \hat{Vega}_{LR})^2}{n(n-1)}} \tag{19}$$

By similar argument, the other first order derivatives of an European call option easily as Table 3. listed.

Greeks	Likelihood Ratios Estimators
$Delta = \frac{dp}{dS_0}$	$e^{-rT}\max(S_T - K, 0)\frac{1}{S_0\sigma^2 T} \times (ln(\frac{S_T}{S_0}) - (r - \delta - \frac{1}{2}\sigma^2)T)$
$Rho = \frac{dp}{dr}$	$e^{-rT}\max(S_T - K, 0)(-T + \frac{d(S_T)\sqrt{T}}{\sigma})$
$Theta = -\frac{dp}{dT}$	$e^{-rT}\max(S_T - K, 0)(r + d(S_T)\frac{\partial d(S_T)}{\partial T} + \frac{1}{2T})$

Table 3: The Likelihood Ratios Estimators of the First-Order Derivatives for European Call

Now, we have developed the first-order derivatives of a European call option using the likelihood ratios method. In the next subsection, we will discuss the derivations of the second-order derivative, i.e. Gamma ($\frac{d^2p}{dS_0^2}$).

4.3.2 The Second Order Derivative

To show the performance of the likelihood ratios method on the second order derivative of a European call option on dividend paying asset i.e. Gamma ($\frac{d^2p}{dS_0^2}$), we again use the rewrite equation (15) for price of a European call option

$$p = E^{\mathbb{Q}}[e^{-rT}\max(x - K, 0)] = \int_0^{\infty} e^{-rT}\max(x - K, 0)g(x)\,dx$$

where S_0, the parameter of our interest, only appears in the density of S_T and so as to the price of the European call option p. As a result, we will first look at the second derivative $\frac{d^2p}{dS_0^2}$. Assume we can interchange the derivative and integral, then differentiate the integral twice yields

$$\frac{d^2p}{dS_0^2} = \int_0^{\infty} e^{-rT}\max(x - K, 0)\frac{\partial^2 g(x)}{\partial S_0^2}\,dx$$

In this case, we use the identity $1 = \frac{g(x)}{g(x)}$, from which it gives

$$\begin{aligned}\frac{d^2p}{dS_0^2} &= \int_0^{\infty} e^{-rT}\max(x - K, 0)\frac{\partial^2 g(x)}{\partial S_0^2}\frac{g(x)}{g(x)}\,dx \\ &= E^{\mathbb{Q}}[e^{-rT}\max(x - K, 0)\frac{\partial^2 g(x)}{\partial S_0^2}\frac{1}{g(x)}]\end{aligned} \tag{20}$$

where we can determine $\frac{\partial^2 g(x)}{\partial S_0^2}\frac{1}{g(x)}$ base on equation (14)

$$\frac{\partial^2 g(x)}{\partial S_0^2}\frac{1}{g(x)} = \frac{d(x)^2 - d(x)\sigma\sqrt{T} - 1}{S_0^2\sigma^2 T}$$

Thus, by expression (20), the unbiased likelihood ratios estimator of Gamma ($\frac{d^2p}{dS_0^2}$) is

$$\begin{aligned}\hat{Gamma}_{LR} &= \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\max(S_T^i - K, 0)\frac{d^2(S_T^i) - d(S_T^i)\sigma\sqrt{T} - 1}{S_0^2\sigma^2 T} \\ &= \frac{1}{n}\sum_{i=1}^{n} G_i\end{aligned} \tag{21}$$

where $S_T^i = S_0 e^{(r-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\Phi^{-1}(U_i)}$ with U_i generated by the random uniform distribution over the domain (0,1) as $U_i \sim i.i.d.U(0,1)$ for each simulation i = 1,...,n. Again, in this case, we are using inversion to generate the standard normal r.v. Z_i. Consequently, we can calculate the standard error as

$$\hat{Std.err}_{LR_{Gamma}} = \sqrt{\frac{\sum_{i=1}^{n}(G_i - \hat{Gamma}_{LR})^2}{n(n-1)}} \tag{22}$$

Overall, the likelihood ratios estimators and pathwise estimators for the derivatives of parameters of interest are not the same in general. Each of them has its own merits and flaws.

We can see that by the constructions of the likelihood ratios estimators for both the first and second derivatives, the estimators are automatically unbiased estimators for parameters of our interest based on equation (16) & equation (20). This merit gives us the conveniences of no requiring extra work to determine whether the estimator is biased or not.

However, as compensation, from the derivation steps, we can find that the likelihood ratio estimator does not depend on the form of the security payoff, i.e. the parameter of our interest may not be a parameter of the density. This drawback will undoubtedly generate a large error when estimating the security derivatives using the likelihood ratio method. We can also verify this fact from the later numerical experiments section. In addition, the same conclusion can also be made as to the pathwise simulation. In particular, when we want to simulate N numbers of derivatives, we will only need one simulation. Similarly, the likelihood ratios method also yields a 1 to N+1 advantage comparing to the resimulation method.

Generally, in this section, we have introduced and developed three different methods for estimating the derivatives of a European call option. In the next section, we will apply the variance reduction technique, i.e. control variates that can be applied to the original simulation estimators of the derivative using different methods to reduce the estimations' standard error.

5 Variance Reduction Technique

In this section, we choose control variates as the variance reduction technique for the derivative estimators using different methods from the previous section. For consistency, we will use the same control variate, the terminal security price S_T, throughout the whole report.

5.1 Control Variates

Before the demonstration of implementing the control variates method on the derivative estimators using different algorithms, we first

$$D := \text{an unbiased simulation estimator of the derivatives s.th } d = E[D]$$

Let r.v. S_T represent the simulated terminal price of the underlying. We can deduce the equality $\mathrm{E}[S_T] = e^{(r-\delta)T}S_0$ from equation (5). Thus, to estimate E[D], we have found a r.v. S_T correlated with D such that $e^{(r-\delta)T}S_0$ is unbiased for $E[S_T]$. Then, we can obtained an unbiased estimator of derivative using control variate as

$$\hat{D'} = \frac{1}{n}\sum_{i=1}^{n} D_i + \beta^*(S_{T_i} - e^{(r-\delta)T}S_0) \tag{23}$$

where D_i are the unbiased estimators for the derivatives at each i simulation and the terminal security price at each simulation $S_{T_i} = S_0 e^{(r-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z_i}$ with Z_i a random standard normal distributed for i = 1,...,n. By inversion, we can approximate standard normal Z_i by $\Phi^{-1}(U_i)$ with $U_i \sim i.i.d.Unifrom(0,1)$ random uniform variables over the domain on (0,1). ($\Phi^{-1}(\cdot)$ is the CDF of standard normal distribution) In this case, we choose β^* such that it minimized the Var(D') i.e.

$$\begin{aligned}\frac{\partial}{\partial\beta}Var(D') &= \frac{\partial}{\partial\beta}[\frac{(Var(D))^2}{n} + \frac{\beta^2}{n}Var(S_T) - \frac{2\beta}{n}Cov(D,S_T)] \\ &= \frac{2\beta}{n}Var(S_T) - \frac{2}{n}Cov(D,S_T) = 0 \\ &\Rightarrow \beta^* = \frac{Cov(D,S_T)}{Var(S_T)} \end{aligned} \tag{24}$$

where $Var(S_T)$ is the sample variance of simulating the terminal security price S_T i.e. $Var(S_T) \geq 0$. One can check $Var(D')$ evaulating at β^* is a minimum: $\frac{\partial^2 Var(D')}{\partial\beta^2} = \frac{2}{n}Var(S_T) \geq 0$, Concave upward, thereby the minimum exists for β^*.

In general, we have set up everything we need for the theoretical formula of each of the three simulation methods for estimating the derivatives. We have also included the variance reduction method. We introduce and discuss implementing the control variate method on the derivative estimators generated by the three proposed algorithms. The following section will contain the numerical implementations of these three methods on two different options: the independent path type of option (a European call option on the dividend-paying asset) and the dependent path type of option (an Asian call option).

6 Numerical Experiments

6.1 European Call Option on Dividend Paying Asset

In this part, we consider the application of the resimulation method, the pathwise method, and the likelihood ratios method on estimating the Greeks of a European option on dividend paying asset. The exact value of the call option has also been included to illustrate the effectiveness of using the control variate. The simulated estimator for the price of the European Call option with dividend paying assets is given by

$$\hat{C} = \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\max(S_{t_i} - K, 0) = \frac{1}{n}\sum_{i=1}^{n} E_i$$

where, $S_{t_i} = S_0 e^{(r-\delta-\frac{\sigma^2}{2})T - \sigma\sqrt{T}\Phi^{-1}(U_i)}$. In this case, we are approximating standard normal r.v Z_i by inversion using $\Phi^{-1}(U_i)$ with $U_i \sim i.i.d.Unifrom(0,1)$ random uniform over domain on (0,1). The standard error of the estimator is Std.Err = $\sqrt{\frac{\sum_{i=1}^{n}(E_i-\hat{C})^2}{n(n-1)}}$.

6.1.1 Exact Value of Greeks for European Call Option

Before the numerical experiments of the three algorithms on estimating the derivative of a European call option, it is better to give the expression of the exact value of the Greeks for the European call option. Specifically, we can obtain the analytical solution for derivatives of European Call Option as the following:

Greeks	Exact Values
$Delta = \frac{dp}{dS_0}$	$e^{-\delta T}N(d_1(K))$
$Vega(\frac{dp}{d\sigma})$	$\sqrt{T}e^{-\delta T}S_0 n(d_1(K))$
$Gamma(\frac{d^2P}{dS_0^2})$	$e^{-\delta T}\frac{n(d_1(K))}{S_0\sigma\sqrt{T}}$
$Rho = \frac{dp}{dr}$	$KTe^{-rT}N(d_2(K))$
$Theta = -\frac{dp}{dT}$	$-\frac{\sigma e^{-\delta T}S_0 n(d_1(K))}{2\sqrt{T}} + \delta e^{-\delta T}S_0 N(d_1(K)) - rKe^{-rT}N(d_2(K))$

Table 4: The Exact Value of Greeks for European Call Option

where $d_{1,2}(x) = \frac{[ln(\frac{S_0}{x})\pm(r-\delta-\frac{1}{2}\sigma^2)T]}{\sigma\sqrt{T}}$

We can now move to the application of these algorithms on a European call option and discuss the numerical results in the next section.

6.1.2 Comments on Numerical Results

For a European call option on the dividend-paying asset, explicit equations for different methods are available in Table 1., Table 2., and Table 3. for references. Appendix B.1 contains simulation results for a European call option on dividend-paying assets. According to Table Appendix B.1, we can make the following observations:

1. The simulation estimates on Greeks of a European call option with dividend-paying assets give accurate point estimates with small standard error for all three algorithms

2. The resimulation method and pathwise method give nearly identical results in estimating all the Greeks of a European call option on dividend-paying assets.

   - This is because we choose a small and appropriate perturb parameter h in this case, so the biases incur is too small to be detected
   - In addition, for small h, by definition of the finite difference method, the resimulation method should converge to the pathwise method. Thus, there is no surprise that we obtain identical results from both methods for h = 0.0001 (h = 0.005 for Gamma)

3. The likelihood ratios method, on the other hand, gives larger standard errors comparing to the other two methods, which is typically 1.5 to 4.5 times larger than the standard error of using the resimulation method and pathwise method

   - This verifies the fact stated from Section 4.3 that larger error comparing to the other methods is because of the likelihood ratio estimators not depending on the form of the security payoff. This fact is also part of the compensations likelihood ratios method pays for being an unbiased estimator for the Greeks by construction.
   - When estimating the Gamma of the European call option, the likelihood method performs better than the resimulation method because we use a larger perturb parameter for the resimulation method for a stable solution

4. In general, the control variates method works well in reducing the variance incurred when estimating the derivatives using different simulation methods and the call price of the option. It typically reduces the standard error of all algorithms and a European call price from about 30% to 50%, which are pretty effective.

6.2 Asian Call Options on Dividend Paying Assets

In the previous section, we have seen how these simulation algorithms can be applied to a European call option with dividend-paying assets, where the analytical solution of the Greeks are known. However, in the financial market, many financial derivatives have their Greeks unknown. Therefore, in the following section, we will discuss one of those financial derivatives that do not have the analytical solution for its Greeks. In particular, we will apply the simulation methods on an Asian call option with dividend-paying assets.

Before we perform the algorithms on an Asian call option, we need to redefine some parameters and setting of the problem to accommodate an Asian call option's properties. In this case, we apply the same general setting as section 3 on the Asian call option. The Asian call option's simulated values have also been included to illustrate the effectiveness of using the control variate.

6.2.1 Applications on Asian Call Option

Consider an Asian Call option written on the average last s daily closing prices with maturity T, and the price of the Asian call option p is given by

$$p = E^{\mathbb{Q}}[e^{-rT}\max(\bar{S} - K, 0)] \tag{25}$$

where the averaged prices $\bar{S}$ is given as

$$\bar{S} = \sum_{j=1}^{s} \frac{S_j}{s} \tag{26}$$

We denote S_j as the underlying price at time $t_j = T - \frac{s-j}{365.25}$ for $T > \frac{s}{365.25}$. Let

$$\Delta t_1 = T - \frac{s-1}{365.25} \text{ and } \Delta t_j = \frac{1}{365.25} \text{ for j = 2,...,s}$$

In particular, t_1 is larger than the increment (i.e in one day) between the averaged price. The reason for the denominator of the t_j to be 365.25 is because 365.25 is the average of the calendar year where a leap year is included. In this case, we can also use 252, as the proxy for the trading day of the financial market.

Thus, with the similar argument in European Call option cases, the estimator for the vega ($\frac{dp}{d\sigma}$) of the Asian Call option using pathwise method is given by

$$\begin{aligned} V\hat{ega}^{path}_{Asian} &= \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\mathbb{1}_{\{\bar{S}\geq k\}}\frac{1}{s\sigma}\sum_{j=1}^{s} S^i_j(ln(\frac{S^i_j}{S_0}) - (r-\delta+\frac{1}{2}\sigma^2)t_j) \\ &= \frac{1}{n}\sum_{i=1}^{n} A_i \end{aligned} \tag{27}$$

where $S^i_j = S^i_{j-1}e^{(r-\delta-\frac{\sigma^2}{2})\Delta t_j + \sigma\sqrt{\Delta t_j}\Phi^{-1}(U_{ij})}$, $U_{ij} \sim i.i.dU(0,1)$ for i = 1,...,n and j = 1,...,s
The estimated standard error for the vega($\frac{dp}{d\sigma}$) is given by

$$St\hat{d.err}^{path}_{Asian} = \sqrt{\frac{\sum_{i=1}^{n}(A_i - V\hat{ega}^{path}_{Asian})^2}{n(n-1)}} \tag{28}$$

We can determine the other simulation estimators by the same scheme as above. Furthermore, we can also obtain the simulated value of an Asian call option with dividend paying assets given by

$$\hat{C}_{Asian} = \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\max(\frac{1}{s}\sum_{j=1}^{s} S^i_{t_j} - K, 0) = \frac{1}{n}\sum_{i=1}^{n} Q_i$$

where $S^i_{t_j}$ is given as this recursive form: $S^i_{t_j} = S^i_0 e^{(r-\delta-\frac{\sigma^2}{2})t_j + \sigma\sqrt{t_j}\Phi^{-1}(U_{ij})}$ with $t_j = T - \frac{(s-j)}{365.25}$, i = 1,...,n, j = 1,...s. In this case, by inversion we are approximating standard normal r.v Z by $\Phi^{-1}(U_{ij})$ with $U_{ij} \sim i.i.d.Unifrom(0,1)$ random uniform over domain on (0,1). The standard error of the estimator is $Std.Err = \sqrt{\frac{\sum_{i=1}^{n}(Q_i - \hat{C}_{Asian})^2}{n(n-1)}}$

The Pathwise Estimator of Greeks for Asian Call Option By similar arguments as the European Call option, the other pathwise estimators for Asian Call option's derivatives can be obtained as

Greeks	Pathwise Estimators
$Delta = \frac{dp}{dS_0}$	$e^{-rT}\mathbb{1}_{\{\bar{S}\geq k\}}\frac{\bar{S}}{S_0}$
$Gamma(\frac{d^2P}{dS_0^2})$	$e^{-rT}(\frac{K}{S_0})^2 sg(S_{s-1}, w_s, \Delta t_s)$
$Rho = \frac{dp}{dr}$	$e^{-rT}\mathbb{1}_{\{\bar{S}\geq k\}}(\frac{1}{s}\sum_{j=1}^{s} S_j t_j - T)$
$Theta = -\frac{dp}{dT}$	$re^{-rT}\max(\bar{S}-K,0) - \mathbb{1}_{\{\bar{S}\geq k\}}\frac{\bar{S}}{2t_1} \times (ln(\frac{S_1}{S_0}) + (r-\delta-\frac{1}{2}\sigma^2)t_1)$

Table 5: The Pathwise Estimator of Greeks for Asian Call Option

where

$$w_s = s(K - \bar{S}) + S_s, g(u,v,t) = n(d(u,v,t))/(v\sqrt{t})$$

and

$$d(u,v,t) = (ln(v/u) - (r-\delta-\frac{1}{2}\sigma^2)t)/(\sigma\sqrt{t})$$

The Likelihood Ratio Estimator of Greeks for Asian Call Option By similar arguments as the European Call option, the other Likelihood Ratio estimators for Asian Call option's derivatives can be obtained as

Greeks	Likelihood Ratios Estimators
$Delta = \frac{dp}{dS_0}$	$e^{-rT}\max(\bar{S}-K,0)\frac{1}{S_0\sigma^2\Delta t_1} \times (ln(\frac{S_1}{S_0}) - (r-\delta-\frac{1}{2}\sigma^2)\Delta t_1)$
$Vega(\frac{dp}{d\sigma})$	$e^{-rT}\max(\bar{S}-K,0)\sum_{j=1}^{s}(-d_j * \frac{\partial d_j}{\partial\sigma} - \frac{1}{\sigma})$
$Gamma(\frac{d^2P}{dS_0^2})$	$e^{-rT}\max(\bar{S}-K,0)\frac{d_1^2 - d_1\sigma\sqrt{\Delta t_1} - 1}{S_0^2\sigma^2\Delta t_1}$
$Rho = \frac{dp}{dr}$	$e^{-rT}\max(\bar{S}-K,0)(-T + \sum_{j=1}^{s}\frac{d_j\sqrt{\Delta t_j}}{\sigma})$
$Theta = -\frac{dp}{dT}$	$e^{-rT}\max(\bar{S}-K,0)(r + d_1\frac{\partial d_1}{\partial T} + \frac{1}{2\Delta t_1})$

Table 6: The Likelihood Ratio Estimator of Greeks for Asian Call Option

where, $d_j = \frac{ln(\frac{S_j}{S_{j-1}}) - (r-\delta-\frac{1}{2}\sigma^2)\Delta t_j}{\sigma\sqrt{\Delta t_j}}$ and $\frac{\partial d_1}{\partial T} = \frac{(ln\frac{S_0}{S_1} - (r-\delta-\frac{1}{2}\sigma^2)\Delta t_1)}{2\sigma\Delta t_1^{3/2}}$

6.2.2 Comments on Numerical Results

Now that we have obtained the explicit equations for different methods on estimating the Greeks for an Asian call option from Table 5. and Table 6. We can apply the simulation methods on an Asian call option. Appendix B.2 contains simulation results for an Asian call option on dividend-paying assets.

According to Appendix B.2, the simulation results on estimating the Greeks of an Asian call option are consistent with estimating the Greeks of a European call option on dividend-paying assets. For example, the resimulation method and pathwise method again give nearly identical results in estimating all the Greeks of an Asian call option on dividend-paying assets. However, this does not include the estimations of Gamma since the resimulation method uses a large perturb variable h due to the non-differentiability of the payoff function, which leads to a larger standard error than using the pathwise method. Nevertheless, there are still some additional findings that are worth to point out in the numerical results from Appendix B.2:

1. In this case, the control variate method using the terminal security price S_T does not work so well in some cases, like for Gamma. We can make a potential improvement by using the geometric mean of an Asian call option instead of the r.v S_T. Since we can obtain an analytical Black-Scholes-Merton like solution for the expected value of the geometric mean of an Asian call option, according to Kemna and Forst (1990) [4].

2. The likelihood ratio method gives a significantly larger standard error compared with the other two methods for estimating the derivative. The reason can again be attributed to the fact that the likelihood ratios does not depend on the form of the payoff function

6.3 Comments on the Computational Time

In addition to the simulation estimates of Greeks on two different types of options, we also investigate the computational saving of the direct methods comparing to the indirect method. According to the CPU time of simulating all four derivatives (Delta, Vega, Rho, Theta) on both European and Asian call option with dividend-paying assets (Appendix B.3),

1. In this case, both pathwise and Likelihood ratios method are faster than the resimulation method when estimating N = 4 derivatives. The computational saving will become more obvious for large N, i.e. increasing the numbers of derivatives for estimations. This fact verifies the 1 to N+1 simulations advantage of both the pathwise and Likelihood ratios method over the resimulation method

7 Conclusion

In general, three methods for estimating the derivatives of security price using simulation were introduced. One indirect method called resimulation method with common random numbers using the finite difference method. The first direct method is the pathwise method that uses the dependence between the parameter of interest and the random security payoff to estimate the derivatives. The second direct method is the Likelihood ratios method that puts the dependence of the underlying density on the parameter of interests to obtain the derivatives estimates.

The resimulation method gives accurate results for an appropriate choice of h satisfying the stability condition [1]. In contrast, it generates a biased estimator for the security price derivatives due to the inherent bias caused by using the finite difference method. Both direct methods have the

1 to N+1 advantage in terms of the computational speed when estimating N derivatives. In other words, for direct methods, the information from a single run of simulation can be used to estimate N derivatives, which gives a huge computational saving compare to use the resimulation method. Another advantage of direct methods is that both two direct methods give unbiased estimators for estimating the derivatives.

The numerical applications of these three methods on estimating derivatives of both a European call option and an Asian call option on dividend-paying assets give the following conclusions. The likelihood ratios method mostly gives a larger standard error than the pathwise and resimulation method since the likelihood ratios method does not depend on the payoff function. The numerical results obtained by the pathwise method and the resimulation method are nearly identical in both applications because of the appropriate choice of h. Consequently, because of the non-differentiability of the payoff function at $S_T = K$, the pathwise method does not apply for the derivative Gamma. The CPU time for simulation of N = 4 derivatives suggesting that the direct methods give significant computational saving. Therefore, in various aspects, the pathwise method is a stronger candidate among the other two methods in computational saving, unbiasedness, and accuracy.

Appendix A Prerequisite Knowledge

A.1 Finite Difference Method

This is a general review of using Finite Difference Method to approximate the first and the second derivatives of a function f(x). Consider the function of one variable f(x) shown in Figure 8., by using the forward and backward scheme of the Taylor series around point x, the point we would like to estimate for f(x)

$$\textbf{Forward scheme: } f(x+h) = f(x) + hf^{'}(x) + \frac{h^2}{2}f^{''}(x) + \frac{h^3}{3}f^{'''}(x) + \mathcal{O}(h^4) \tag{29}$$

$$\textbf{Backward scheme: } f(x-h) = f(x) - hf^{'}(x) + \frac{h^2}{2}f^{''}(x) - \frac{h^3}{3}f^{'''}(x) + \mathcal{O}(h^4) \tag{30}$$

Based on the forward and backward schemes from above, the following approximations can be derived with the order of accuracy:

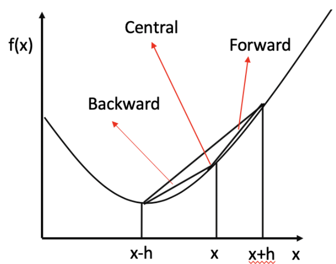

Figure 1: Illustration of PDE Discretisation

- Forward approximation of 1^{st} derivative of f(x)

$$f'(x) = \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{\mathrm{h}} + \mathcal{O}(h)$$

- Backward approximation of 1^{st} derivative of f(x)

$$f'(x) = \frac{\mathrm{f}(x) - \mathrm{f}(x-h)}{\mathrm{h}} + \mathcal{O}(h)$$

- Central approximation of 1^{st} derivative of f(x)

$$f'(x) = \frac{\mathrm{f}(x+h) - \mathrm{f}(x-h)}{2h} + \mathcal{O}(h^2)$$

- Standard approximation of 2^{nd} derivative of f(x)

$$f''(x) = \frac{\mathrm{f}(x+h) - 2f(x) + \mathrm{f}(x-h)}{\mathrm{h}^2} + \mathcal{O}(h^2)$$

Appendix B Simulation Results

B.1 European Call Option on Dividend Paying Assets

	Initial Asset Price (S_0)					
	90	(Std.Err)	100	(Std.Err)	110	(Std.Err)
Call Price						
Exact	1.220		5.126		12.327	
Simulation estimate	1.239	0.035	5.191	0.075	12.557	0.108
Estimate with control	1.210	0.245	5.150	0.033	12.325	0.0234
Delta($\frac{\mathbf{dp}}{\mathbf{dS_0}}$)						
Exact	0.222		0.568		0.844	
Resimulation Method	0.216	0.00452	0.563	0.00540	0.842	0.00402
Resimulation Method (CV)	0.220	0.00308	0.568	0.00297	0.846	0.00249
Pathwise Method	0.216	0.00452	0.563	0.00540	0.842	0.00402
Pathwise Method (CV)	0.220	0.00308	0.568	0.00297	0.846	0.00249
Likelihood Ratio Method	0.212	0.00788	0.550	0.0127	0.818	0.0172
Likelihood Ratio Method (CV)	0.217	0.00631	0.560	0.00844	0.832	0.0103
Vega($\frac{\mathbf{dp}}{\mathbf{d}\sigma}$)						
Exact	11.946		17.446		11.435	
Resimulation Method	11.546	0.267	16.974	0.294	10.933	0.388
Resimulation Method (CV)	11.760	0.176	17.240	0.155	11.273	0.220
Pathwise Method	11.543	0.267	16.974	0.294	10.928	0.388
Pathwise Method (CV)	11.757	0.176	17.239	0.155	11.268	0.220
Likelihood Ratio Method	11.480	0.683	16.469	1.101	9.748	1.561
Likelihood Ratio Method (CV)	11.805	0.611	17.020	0.972	10.532	1.377
Rho($\frac{\mathbf{dp}}{\mathbf{dr}}$)						
Exact	3.751		10.344		16.108	
Resimulation Method	3.659	0.0764	10.263	0.0979	16.091	0.0752
Resimulation Method (CV)	3.717	0.0531	10.345	0.0603	16.142	0.0573
Pathwise Method	3.658	0.0764	10.263	0.0979	16.089	0.0752
Pathwise Method (CV)	3.716	0.0531	10.345	0.0603	16.141	0.0573
Likelihood Ratio Method	3.585	0.135	9.989	0.241	15.545	0.361
Likelihood Ratio Method (CV)	3.670	0.109	10.178	0.163	15.846	0.223
Theta($-\frac{\mathbf{dp}}{\mathbf{dT}}$)						
Exact	-8.742		-14.370		-12.415	
Resimulation Method	-8.460	0.190	-14.049	0.202	-12.095	0.244
Resimulation Method (CV)	-8.614	0.124	-14.241	0.0903	-12.322	0.119
Pathwise Method	-8.576	0.194	-14.551	0.210	-13.315	0.255
Pathwise Method (CV)	-8.732	0.126	-14.751	0.0935	-13.553	0.121
Likelihood Ratio Method	-8.395	0.471	-13.639	0.763	-11.167	1.085
Likelihood Ratio Method (CV)	-8.627	0.417	-14.048	0.660	-11.759	0.931
Gamma($\frac{\mathbf{d^2p}}{\mathbf{dS_0^2}}$)						
Exact	0.029		0.035		0.019	
Resimulation Method	0.023	0.005859	0.0386	0.00715	0.0248	0.00597
Resimulation Method (CV)	0.023	0.005856	0.0385	0.00715	0.0246	0.00597
Likelihood Ratio Method	0.0283	0.00169	0.0329	0.00220	0.0161	0.00258
Likelihood Ratio Method (CV)	0.0291	0.00151	0.0340	0.00194	0.0174	0.00228

Parameters: r=0.1, K = 100, δ = 0.03, σ = 0.25,and T = 0.2.
All simulation results are based on 10,000 runs.
Resimulation method use h = 0.0001 except for gamma where we use h=0.05

B.2 Asian Call Option on Dividend Paying Assets

	Initial Asset Price (S_0)					
	90	(Std.Err)	100	(Std.Err)	110	(Std.Err)
Call Price						
Simulation estimate	0.751	0.0247	4.289	0.0618	11.570	0.0942
Estimate with control	0.786	0.0205	4.406	0.0342	11.769	0.0373
Delta($\mathbf{\frac{dp}{dS_0}}$)						
Resimulation Method	0.170	0.00411	0.561	0.00540	0.883	0.00368
Resimulation Method (CV)	0.169	0.00307	0.556	0.00338	0.871	0.00258
Pathwise Method	0.167	0.00405	0.553	0.00532	0.869	0.00362
Pathwise Method (CV)	0.169	0.00307	0.556	0.00338	0.871	0.00258
Likelihood Ratio Method	0.165	0.00699	0.540	0.0130	0.838	0.0190
Likelihood Ratio Method (CV)	0.168	0.00605	0.546	0.00996	0.847	0.0143
Vega($\mathbf{\frac{dp}{d\sigma}}$)						
Resimulation Method	8.500	0.220	14.548	0.246	7.999	0.339
Resimulation Method (CV)	8.607	0.164	14.692	0.147	8.193	0.211
Pathwise Method	8.494	0.220	14.548	0.246	7.993	0.339
Pathwise Method (CV)	8.601	0.164	14.692	0.147	8.186	0.211
Likelihood Ratio Method	8.894	0.930	17.398	2.448	12.938	4.722
Likelihood Ratio Method (CV)	9.048	0.905	17.725	2.407	13.438	4.671
Rho($\mathbf{\frac{dp}{dr}}$)						
Resimulation Method	2.274	0.0549	8.029	0.0770	13.018	0.0532
Resimulation Method (CV)	2.230	0.0422	8.070	0.0530	13.039	0.0452
Pathwise Method	2.392	0.0579	8.775	0.0845	15.165	0.0633
Pathwise Method (CV)	2.420	0.0439	8.823	0.0534	15.198	0.0447
Likelihood Ratio Method	2.258	0.0954	7.900	0.195	12.582	0.315
Likelihood Ratio Method (CV)	2.296	0.0806	8.003	0.134	12.766	0.190
Theta($\mathbf{-\frac{dp}{dT}}$)						
Resimulation Method	-8.140	0.217	-15.769	0.256	-12.316	0.329
Resimulation Method (CV)	-8.238	0.170	-15.896	0.188	-12.468	0.254
Pathwise Method	-8.138	0.217	-15.769	0.257	-12.309	0.329
Pathwise Method (CV)	-8.244	0.172	-15.896	0.188	-12.462	0.254
Likelihood Ratio Method	-8.094	0.540	-15.199	0.977	-10.945	1.473
Likelihood Ratio Method (CV)	-8.234	0.505	-15.467	0.906	-11.345	1.366
Gamma($\mathbf{\frac{d^2p}{dS_0^2}}$)						
Resimulation Method	0.0295	0.00666	0.0416	0.00740	0.0172	0.00446
Resimulation Method (CV)	0.0297	0.00665	0.0417	0.00740	0.0172	0.00445
Pathwise Method	0.0220	0.00438	0.0419	0.00650	0.0179	0.00642
Pathwise Method (CV)	0.0221	0.00419	0.0419	0.00513	0.0177	0.00313
Likelihood Ratio Method	0.0282	0.00198	0.0378	0.00288	0.0149	0.00358
Likelihood Ratio Method (CV)	0.0287	0.00186	0.0386	0.00270	0.0158	0.00337

Parameters: r=0.1, K = 100, δ = 0.03, σ = 0.25, and T = 0.2.
All simulation results are based on 10,000 runs. The average period s = 30 days.
Resimulation method use h = 0.0001 except for gamma where we use h=0.05.

B.3 CPU Time Comparison

European Call	CPU time (sec)	**Asian Call**	CPU time (sec)
Resimulation	0.225	Resimulation	1.379
Pathwise	0.037	Pathwise	0.948
LR	0.088	LR	0.967

Parameters: S_0 = 90, r= 0.1, K = 100, δ = 0.03, σ = 0.25, and T = 0.2.
All simulation results are based on 10,000 runs. The average period s = 30 days.
Resimulation method use h = 0.0001 except for gamma where we use h=0.05.

Appendix C R Code

GitHub URL: https://github.com/YifeiDeng/Estimating-Security-Price-Derivatives-Using-Simulation

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