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FINANCE 3

Report on A Revised Option Pricing Formula with the Underlying being Banned from Short Selling

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1 Executive Summary

1.1 Background

Short-selling is the practice of selling the financial securities that are currently not held and repurchasing the securities back to return to the lender. When it applies successful, short-selling brings the investor considerable amounts of profits in the short term as the stock prices fall faster than they grow. However, the short-sellers have always been blamed for stock market declines as they are betting on the market's falls and thereby deteriorate the downturn market. Ever since the 2008 financial crisis, many regulators around the world have imposed ban restriction on short-selling to reduce the volatility of the financial market and prevent the negative impact of the downward market from happening. Although the ban restriction on short-selling restores the well-behaved functionality of the financial market, it is still a trading constraint and certainly violates the classical Black-Scholes assumption [3] where it assumes the perfect liquidity. Under these conditions, it will result in an incomplete market and cause difficulties when pricing the derivative contract written upon the underlying asset with short-selling ban restriction. Thus, we need a new option pricing formula under this ban restriction assumption on the underlying security to resolve the problem.

Among the many proposed frameworks for option pricing under ban restrictions of the underlying asset in recent years, Guo and Zhu in 2017 [4] introduced an innovative concept of 'Equal risk pricing' framework with the following procedure by first considering the risks exposures of buyer and seller. Based on the analysis of the buyer and seller's risk exposures, they choose the optimal hedging strategy such that the two parties' risks are minimized. Consequently, the option's fair price is achieved by taking the risks faced by both the buyer and seller of the option the same so that both parties are happy if the contract is agreed upon.

1.2 Summary of the Report

The report's primary goal is to derive, develop, and examine the revised option pricing formula with the underlying being banned from short selling by He and Zhu (2020)[5]. The authors in this paper [5] adapted the equal risk framework from [4] and further defined a revised option pricing formula by assuming the existence of a financial asset that is without the trading constraints and is correlated with the underlying asset being banned from short-selling. Based on the fact that the trader will try to seek other alternatives to minimize their risks, we can use this market interaction so that un-hedgable risk by the short-selling ban can be partially hedged away with this newly introduce asset. As a result, the revised option pricing formula will only introduce one extra parameter, i.e. the correlation ρ. At the same time, it keeps the essential advantage of the Black-Scholes model[3], and the formula is easy to compute and implement according to various numerical experiments.

The financial meaning of the correlation ρ is also significant, where it builds the bridge between the Black-Scholes formula and Guo-Zhu formula[4], taking these two as two extreme cases. At the same time, the formula gives more flexibility compared with the Guo-Zhu formula and Black-Scholes formula, where the correlation ranging from 0 to 1 for the different market situations based on how much the market interaction can reduce the effect of the short-selling ban on the option of the underlying securities.

1.3 Financial Intuitions of the Findings

In the report, the newly derived revised option pricing formula under short-selling ban only requires adding one extra parameter, i.e. the correlation ρ, comparing to the classical Black-Scholes model. In this case, ρ is also called the ban-diluted factor that measures the magnitude of the effect for the short-selling ban on the option is diluted through the market interaction. In other words, the introduction of a correlated asset is just a model assumption to track the impact of the market interactions on the option price with the short-selling ban on the underlying asset. It is worthy to note that the revised formula does not depend on the volatility and initial value of the correlated asset due to the above assumption.

Moreover, we have found that when $\rho = \pm 1$, the market is complete so that the correlated asset can be used as the perfect replacement of the underlying asset with the short-selling ban. In this case, the effect of the short-selling ban on the option written on the underlying asset is totally diluted through this market interaction. When $\rho = 0$, however, the market is incomplete. The short-selling of the underlying is completely banned, and the effect for the short-selling ban on the option written on the underlying asset completely fails to dilute.

When ρ is increasing from 0 to 1, our revised price of call and put European option is increasing and decreasing accordingly. The above results correspond to the fact that as the effect of the short-selling is further diluted, the buyer of the call option (seller of a put) want to pay (offer) more (less) to purchase (sell) the derivative contract. Furthermore, the revised prices of the call option and put option (when $\rho = 0.5$) are smaller and higher than the Black-Scholes call and put prices (when $\rho = 1$) accordingly. The financial interpretation of this phenomenon is that due to the smaller dilution of the short-selling ban's effect on the option, the buyer of the call and seller of the put needs to face additional unhegable risk. In particular, the buyer of the call (the seller of a put) is willing to pay (offer) less (more) to purchase (sell) the option as the additional risk incurred. It follows that the smaller the values of the ban-dilution factor, the more compensations the buyer of a call (the seller of a put) need to pay for the additional unhegable risk. Thus, the larger the difference between our revised call (put) prices and the Black-Scholes call (put) prices.

The model's potential improvement will be extending the formula to a more complicated type of options. Since the revised option pricing formula only applies to the European option. Besides, we can consider more than one alternatives for hedging instead of assuming only one correlated asset to accommodate the real market events.

2 Abstract

The report is involved with replicating the paper "A Revised Option Pricing Formula with the Underlying being Banned from Short Selling" written by He and Zhu in 2020 [5]. Under the situation where the short-selling assumption for the underlying of a European option being banned, the authors propose a closed-form equal risk formula for a European option in this incomplete market. The main idea of the revised formula [5] is by assuming the existence of a financial asset that is without the trading constraints and is correlated with the underlying asset being ban from short-selling. Based on the fact that the trader will try to seek other alternatives to minimize their risks, we can use this market interaction to reduce (dilute) the short-selling ban's effect partially. Specifically, we can achieve the derivation of the revised option pricing formula by introducing the equal risk framework propose by Guo and Zhu (2017)[4]. This equal risk price is essentially referring to the fair price of the option written upon the short-selling ban asset by equating both option seller and buyer minimized expected risk so that both parties are happy. The resulting revised option pricing formula keeps the Black–Scholes model's essential advantage by only introducing one ban-dilution variable and involving only one one-dimensional integral that is easy to compute numerically. In this report, the general setting for the problem in Section 3 is taken from [4][5]. All theoretical results in Section 4 through 5 are taken from [5], where I have made some rearrangements of the materials based on my own understanding along with some explanations. All the numerical results in Section 6 are solely reproduced by myself using the programming software MATLAB with some financial interpretations of the numerical outputs.

3 General Setting

This section provides the general settings for a revised option pricing formula under the equal risk framework. Besides, we also include some properties of the equal risk prices for completeness and better comprehension of the framework.

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space on which the standard Brownian motion W_t^1 is defined for time $t \geq 0$, with filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ generated by the standard Brownian motion. In this case, Ω is the set of all possible realizations in the financial market within the time horizon [0, T], where T is the financial securities' maturity. $\mathbb{Q}$ is the risk-neutral measure define on the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. Let $\tilde{\mathbb{Q}}$ be an equivalent supermartingale probability measures, and assume that Y_t, the underlying asset price process with a ban on short-selling, follows the stochastic differential equation

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t^1 \tag{1}$$

where dW_t^1 is the increment of the standard BM, μ is the drift term of Y_t, $\sigma > 0$ refers to the constant volatility, and we assume no dividend-paying on the underlying asset Y_t in this case.

According to the paper [5], when the short-selling assumption of the underlying Y_t is restricted, we can assume that there exists a financial asset that is without the trading constraint and is correlated with the underlying asset Y_t so that under the equal risk framework, the effect of the short-selling ban on the option can be reduced or eliminated by this market interaction. We suppose such correlated asset price process as S_t (does not have the short-selling ban) and its dynamic under the martingale probability measure $\mathbb{Q}$ is defined as the following

$$dS_t = rS_t dt + \sigma_0 S_t dW_t^2 \tag{2}$$

$$< dW_t^1 dW_t^2 > = \rho dt, \ \rho \in [-1, 1] \tag{3}$$

where σ_0 is the volatility of the correlated asset S_t and ρ is the correlation coefficient between W_t^1 and W_t^2. In this case, r is the drift term of the S_t, i.e. the risk-free interest rate $r \geq \mu$ because the discounted price of Y_t being a supermartingale.

It is worthy to note that when $\rho = \pm 1$, the market is complete, and S_t can be used as the perfect replacement of Y_t so that $\mu = r$. In this case, the ban restriction effect has been perfectly diluted by the market interaction. On the other hand, when $\rho = 0$, Y_t and S_t are independent assets. The unhegdeable risk from Y_t can not be hedged by using the correlated asset S_t anymore. Thus, the risk can only be hedged by the underlying asset itself, and the ban restriction effect completely failed to be diluted by the correlated asset. Based on these cases, ρ can also be called the ban-dilution factor that measures the effect of the short-selling ban on the option written on the underlying with ban restriction diluted by the market interaction.

Now, we have set up the general setting of the problem. We will begin our discussions about the derivation of the revised option pricing formula. However, before that, we need to introduce the equal risk framework that lays the foundation for constructing the revised option pricing formula.

4 Equal Risk Framework

In the following section, an equal risk framework is derived and specified according to the paper [4]. In particular, we first define the self-financing portfolio with zero initial capital. We further derive the expected risk of the buyer and selling based on such a portfolio. Consequently, through analyzing the risks faced by both the buyer and seller of a European option, we can obtain an optimal trading strategy to minimize their risk.

4.1 Derivations of the Gain Process

The subsection discusses the derivations of the gain process of our trading portfolio with a self-financing strategy. This gain process, later will discuss in this subsection, consists of two separate gain processes. Each contributed solely by trading in the underlying asset Y_t with a short-selling ban and S_t without trading constraints correspondingly.

Since we want to derive a revised option pricing formula under the short-selling ban, we are interested in the gain process, i.e. the self-financing portfolio with zero initial capital. According to [5], the definition of the gain process is given as the following

Definition 4.1. *(Gain Process)*
For each $\mathbb{Q}$ admissible, self-financing strategy with zero initial capital, its discounted wealth process is defined as a gain process G. If $\mathbb{G}$ is the set of all G, it should satisfy the following conditions:

1. *For any $G \in \mathbb{G}$ is a square-integrable $\mathbb{Q}-$supermartingale with $G_0 = 0$*
2. *$0 \in \mathbb{G}$ and $\mathbb{G}$ is convex*

We consider the wealth process M_t follows the $\mathbb{Q}$ admissible, self-financing strategy with zero initial capital (i.e. $M_0 = 0$) consists of holding $\{\phi_t\}_{0\leq t\leq T} \geq 0$ units of the ban restricted underlying asset Y_t and $\{\theta_t\}_{0\leq t\leq T}$ units of the correlated asset S_t. In this case, $\{\phi_t\}_{0\leq t\leq T} \geq 0$ due the presence of the short selling ban restriction. Thus, the wealth process M_t i.e. the replicating portfolio is given by

$$M_t = \theta_t S_t + \phi_t Y_t + \eta_t B_t \tag{4}$$

where B_t is the price of the zero coupon bond at time t with its dynamic given by $dB_t = rdt$ for $0 \leq t \leq T$ and we assume holding η_t units of bonds at time t. In other words, we invest in ϕ_t amount shares of Y_t and θ_t shares of S_t, and we put the remaining part of the money $\eta_t B_t = M_t - \theta_t S_t - \phi_t Y_t$ into the money market account with risk-free interest rate r. Thus, the self-financing condition gives

$$\begin{aligned} dM_t &= (\phi_t dY_t + \theta_t dS_t) + r(M_t - \theta_t S_t - \phi_t Y_t)dt \\ &= r(M_t - \theta_t S_t - \phi_t Y_t)dt + \theta_t dS_t + \phi_t dY_t \end{aligned} \tag{5}$$

Let G_t denotes the discounted wealth process satisfying Definition 4.1., i.e. $G_t = e^{-rT}M_t$. Based on this, one can determine the gain process as

$$\begin{aligned} dG_t &= d(e^{-rT}M_t) \\ &= e^{-rT}[(\mu - r)\phi_t Y_t dt + \phi_t \sigma Y_t dW_t^1 + \theta_t \sigma_0 S_t dW_t^2] \\ &= e^{-rT}\phi_t Y_t[(\mu - r)dt + \sigma dW_t^1] + e^{-rT}\theta_t S_t[\sigma_0 dW_t^2] \end{aligned} \tag{6}$$

According to equation (6), we denote $h_t = e^{-rT}\phi_t Y_t$ and $\pi_t = e^{-rT}\theta_t S_t$ as the money invest in Y_t and S_t accordingly. So, the gain process G_t can be simplified as,

$$dG_t = \underbrace{[(\mu - r)dt + \sigma dW_t^1]h_t}_{dZ_t,\text{ with } Z_0=0} + \underbrace{\pi_t[\sigma_0 dW_t^2]}_{dX_t,\text{ with } X_0=0} \tag{7}$$

where we can observe that the dynamic of gain process dG_t consists of two separate gain dynamic dZ_t and dX_t with trading the underlying and the correlated asset accordingly. In particular, according to [5], we can write G_t as the sum of the two gain process i.e.

$$G_t = (Z_t + \alpha) + (X_t - \alpha), \text{ for some } \alpha > 0 \tag{8}$$

Now, we have obtained the gain process dynamic dZ_t by trading solely in the underlying Y_t and the gain process dynamic dX_t by trading the correlated asset S_t only. We can denote the set of all gains process obtained by trading strategy h_t as

$$\begin{aligned}\mathbb{Z} := \{Z : Z_t &= \int_0^t (\mu - r)h_s ds + \int_0^t \sigma h_s dW_s^1 \\ &= \int_0^t d(e^{-rs}Y_s)\phi_s, \phi_s \geq 0\}\end{aligned} \tag{9}$$

where we have ϕ_s[1] ≥ 0 for $0 \leq s \leq t$ due to the ban restriction imposed on Y_t and the second equality is by the fact that

$$\begin{aligned}\int_0^t d(e^{-rs}Y_s)\phi_s &= \int_0^t (-re^{-rs}Y_s\phi_s ds + e^{-rs}dY_s\phi_s) \\ &= \int_0^t (-re^{-rs}Y_s\phi_s ds + e^{-rs}(\mu Y_s ds + \sigma Y_s dW_s^1)\phi_s) \\ &= \int_0^t (e^{-rs}(\mu - r)\phi_s Y_s ds + e^{-rs}\phi_s Y_s \sigma dW_s^1), \text{ as } \mathrm{h}_s = \mathrm{e}^{-rs}\phi_s Y_s \\ &= \int_0^t h_s(\mu - r)ds + \int_0^t \sigma dW_s^1\end{aligned}$$

By similar argument, we can also denote the set of all gains process obtained by trading strategy π_t

$$\begin{aligned}\mathbb{X} := \{X : X_t &= \int_0^t \sigma_0 \pi_s dW_s^2) \\ &= \int_0^t \theta_s d(e^{-rs}S_s)\}\end{aligned} \tag{10}$$

where we can determine the second equality[2] by

$$\begin{aligned}\int_0^t \theta_s d(e^{-rs}S_s) &= \int_0^t (-re^{-rs}S_s\theta_s ds + e^{-rs}dS_s\theta_s) \\ &= \int_0^t (-re^{-rs}S_s\theta_s ds + e^{-rs}(rY_s ds + \sigma_0 S_s dW_s^2)\theta_s \\ &= \int_0^t e^{-rs}\theta_s S_s \sigma_0 dW_s^2, \text{ as } \pi_s = e^{-rs}\theta_s Y_s) \\ &= \int_0^t \pi_s \sigma_0 dW_s^2\end{aligned}$$

[1]Possible typo in paper [5] where the authors use h_s instead of ϕ_s in equation (9)
[2]Possible typo in paper [5] where the authors use π_s instead of θ_s in equation (10)

Now that we have set up everything we need for introducing the concept of the equal risk framework. Based on the model's construction from above, we will introduce and examine the expected risk of the buyer and seller of a European option in the next subsection. Later, based on both parties' expected risk analysis, we can obtain a European option's equal risk price under the equal risk framework.

4.2 The Buyer and Seller Risk

To define the expected risk of the buyer and seller of a European option, we will need to know the definition of the risk function stated as below

Definition 4.2. *(Risk Function)*
Any function R defined on $\mathbb{R} \to \mathbb{R}$ satisfying

1. *R is non-decreasing, convex and bounded below*
2. *$R(0) = 0$ and $R(x) > 0 \ \forall x > 0$*

is called a risk function

In this case, we choose the risk functions of both the buyer and seller to be in exponential form i.e. $R_a(x) = e^x - 1$ and $R_b(x) = e^x - 1$ to avoid complex solution for the revised European option formula. According to the paper [5], since for most of the time, the liabilities of individuals are random, where it is unlikely to determine the risk at maturity when the option contract is written upon it. Thus, the authors propose a natural way to measure this risk by introducing the concept of expected risk, where

Definition 4.3. *(Expected Risk)*
If R_b and R_s be two risk function satisfying Definition 4.2., Given any square integrable r.v. U represented the discounted liability, q_b:the buyer's expected risk of U and q_s:seller's expected risk of U are defined accordingly as the minimal risks when hedging X, i.e.

$$q_s(U) := \inf_{G \in \mathbb{G}} (E^{\mathbb{Q}}[R_s(U - G_T)]) \tag{11}$$

$$q_b(U) := \inf_{G \in \mathbb{G}} (E^{\mathbb{Q}}[R_b(U - G_T)]) \tag{12}$$

Now, we assume the discounted payoff of a European call option is $H(Y_T) = e^{-rT}(Y_T - k)^+$, where K is the strike price. We denote the equal price of the option as c, the risk of the buyer is as the following according to Definition 4.3.

$$q_b(c - H(Y_T)) = \inf_{G \in \mathbb{G}} E^{\mathbb{Q}}[R_b(c - H(Y_T) - G_T)|M_0 = 0, Y_0 = y] \tag{13}$$

where recalled that G_T consists of two separate gain process and thus, we can rewrite the expected risk of the buyer of a European call option as

$$\begin{aligned} q_b(c - H(Y_T)) = \inf_{Z \in \mathbb{Z}, X \in \mathbb{X}} \mathrm{E}^{\mathbb{Q}}[R_b(c - H(Y_T) - Z_T - X_T)| \\ X_0 = 0, Z_0 = 0, Y_0 = y] \end{aligned} \tag{14}$$

Similarly, the expected risk of the seller of a European call option is

$$q_s(H(Y_T) - c) = \inf_{Z\in\mathbb{Z}, X\in\mathbb{X}} \mathrm{E}^{\mathbb{Q}}[R_s(H(Y_T) - c - Z_T - X_T)| \\ X_0 = 0, Z_0 = 0, Y_0 = y] \quad (15)$$

The buyer's liability is $U = c - H(Y_T)$ because the buyer of the option is suffering from getting additional risk incurred by $c \leq H_T$, i.e. when the payoff of the derivative contract does not cover the purchase value of the derivative contract. Similarly, the seller's liability is $U = H(Y_T) - c$. At this point, the problem remains is to determine the equal risk price formula for a European option by solving the above two stochastic optimal control problems.

According to [5], we can derive the minimized expected risk of the buyer as the following theorem

Theorem 4.1. *With the risk function $R_b = e^x - 1$, the risk faced by the buyer of a European call option defined in equation (14) can be expressed as*

$$q_b = \begin{cases} e^c\{E^{\mathbb{Q}}[e^{-\frac{1}{\delta}H(Y_T)}|Y_0 = y]\}^{\delta} - 1, & \rho \neq \pm 1 \\ e^c e^{E^{\mathbb{Q}}[-H(Y_T)|Y_0=y]} - 1, & \rho = \pm 1 \end{cases} \quad (16)$$

where $\delta = \frac{1}{1-\rho^2}$, and $H(Y_T) = e^{-rT}(Y_T - K)^+$

Proof. See He and Zhu (2020) [5] □

The general scheme of the proof for Theorem 4.2. is straight forward according to He and Zhu (2020) [5]. The proof can be summarized as the following. Since the choice of π_t (money invested in S_t) is independent of h_t (money invested in Y_t) based on equation (7), then we can rewrite equation (14) as

$$q_b(c - H(Y_T)) = \inf_{h_t\geq 0} \inf_{\pi_t} \mathrm{E}^{\mathbb{Q}}[R_b(c - H(Y_T) - Z_T - X_T)|X_0 = 0, Z_0 = 0, Y_0 = y]$$

Let $f(x, y, z, t)$ denoted as the inner part of the optimal control problem from above, i.e.

$$f(x, y, z, t) = \inf_{\pi_t} \mathrm{E}^{\mathbb{Q}}[R_b(c - H(Y_T) - Z_T - X_T)|X_t = x, Z_t = y, Y_t = y]$$

Our goal is to first solve the inner part $f(x, y, z, t)$ with π_t^* and then to solve the outer part $q_b = \inf_{h_t\geq 0} f(x, y, z, t)$. For the inner part, since f is independent of S_t, we can embed the function f into a Hamilton-Jacobi-Bellman equation (HJB equation) (See Appendix A.1.) with terminal condition $f(x, y, z, T) = R_b(c - H(y) - z - x) = e^{c-H(y)-z-x} - 1$.

According to He and Zhu (2020) [5], by some manipulations, we can solve the constrained optimal control problem f(x,y,z,t) with the optimal trading strategy $\pi_t^* =$

$-\frac{\rho\sigma y \frac{\partial^2 f}{\partial x \partial y} + \rho\sigma h_t \frac{\partial^2 f}{\partial x \partial z}}{\sigma_0 \frac{\partial^2 f}{\partial x^2}}$ for difference cases (i.e. $\rho = \pm 1$ or $\rho \neq \pm 1$). For the outer part, we can easily determine the optimal trading strategy h_t^* by some designate Lemma according to [5]. Thus, the general scheme leads to Theorem 4.1.

According to [5], when $\rho \neq \pm 1$, the optimal trading strategy for the banned restricted underlying Y_t is $h_t^* = 0$ i.e. π_t^* is

$$\pi_t^*|_{\rho \neq \pm 1} = -\frac{\rho\sigma y \frac{\partial^2 f}{\partial x \partial y}}{\sigma_0 \frac{\partial^2 f}{\partial x^2}} \tag{17}$$

with

$$g(x, y, t) = f|_{h_t=0} = e^{-x}\{E^{\mathbb{Q}}[e^{\frac{1}{\delta}(c - H(Y_T))}|Y_t = y]\}^{\delta} - 1$$

Equation (17) implies that the European call option buyer can only hedge the extra risk incurred by the ban on short-selling on option written upon Y_t by using the correlated asset S_t only. Moreover, when $\rho = \pm 1$, S_t is the perfect replacement with Y_t. Thus, for any choice of trading strategy of underlying, we can perfectly hedge the risk with π_t^* as we can hedge the risk by either selling the correlated asset without trading constraints with correlation $\rho = -1$ or purchasing the correlated asset with correlation $\rho = 1$. In a special case, when $\rho = 0$, however, $\pi_t^* = 0$, which means that the unhegable risk can only be hedged by the underlying asset itself as $\pi_t^*|_{\rho=0} = 0$.

Now, we have obtained the risk faced by the buyer of the European call option. We are one step left to obtain the equal risk price of the option by determining the risk faced by the seller of the European call option, which is defined in the following Theorem 4.2

Theorem 4.2. *With the risk function $R_s = e^x - 1$, the risk faced by the seller of a European call option defined in equation (14) can be expressed as*

$$q_s = e^{-c} e^{E^{\mathbb{Q}}[H(Y_T)|Y_0 = y]} - 1 \tag{18}$$

with $H(Y_T) = e^{-rT}(Y_T - K)^+$

Proof. See He and Zhu (2020) [5] □

According to [5], through the derivation of Theorem 4.2, the optimal trading strategy for the ban restricted underlying Y_t is $h_t^* \geq 0$ and the optimal strategy for the correlated asset without trading constrain S_t is $\pi_t^* = 0$. In other words, the seller's liability, i.e. $U = H(Y_T) - c$, in this case, can be completely hedged without relying on the correlated asset S_t. The additional trading on the correlated asset will destroy the balance so that the minimized expected risk q_s is not minimized at all, which will have additional risk introduced. The fact explains the Theorem 4.2 to have only one equation along instead of several equations into cases. The above arguments complete the derivations of the expected risk framework. We can now determine the revised European option formula by equating the minimized expected risk of two parities, i.e. under the equal risk framework $q_b = q_s$ in the next section.

5 Revised Option Pricing Formula

According to Guo and Zhu (2017)[4], by equating the resultant minimized expected buyer risk and seller risk, i.e. $q_b = q_s$, we can obtain an 'Equal risk framework' to determine equal risk price formulas for a European option and a Forward contract accordingly.

5.1 Revised European Call Option Formula

Given the minimized risk of buyer and seller from Theorem 4.1 and Theorem 4.2, we can determine the equal risk price of a European call option by equating (16) & (18) and obtain the formula as the following Theorem 5.1

Theorem 5.1. *If a European call option with discounted payoff $H(Y_T) = e^{-rT}\ (Y_T - K)^+$ can be hedge through trading the underlying asset with short sell ban and a correlated asset with without trading constraints, its equal risk price c can be defined as,*

$$c = \begin{cases} \frac{1}{2}C_{bs} - \frac{\delta}{2}\ln\{E^{\mathbb{Q}}[e^{-\frac{1}{\delta}H(Y_T)}|Y_0 = y]\}, & \rho \neq \pm 1 \\ C_{bs}, & \rho = \pm 1 \end{cases} \tag{19}$$

where $E^{\mathbb{Q}}[e^{-\frac{1}{\delta}H(Y_T)}|Y_0 = y] = \int_0^{+\infty} e^{-\frac{1}{\delta}H(Y_T)}m(Y_T|Y_0)dY_T$ and $\delta = \frac{1}{1-\rho^2}$. Y_T follows a log-normal distribution, whose density function is

$$m(Y_T|Y_0) = \frac{1}{Y_T\sigma\sqrt{2\pi T}}e^{\frac{-[\ln(\frac{Y_T}{Y_0})-(\mu-\frac{1}{2}\sigma^2)T]^2}{2\sigma^2 T}},\ Y_T > 0 \tag{20}$$

Proof. See He and Zhu (2020) [5] $\square$

We need to point out that c_{bs} is the price of the same European call option under the Black-Scholes-Merton model [3]

$$c_{bs} = Y_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where $\Phi(\cdot)$is the CDF for a standard normal random variable, d_1 and d_2 are given by

$$d_{1,2} = \frac{\log(Y_0/K) + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}}$$

We can also observe that, when $\rho = \pm 1$, the revised formula degenerates to the Black-Scholes model with perfect liquidity.

5.2 Revised Forward Contract Formula

The same logic applies to the price of a forward contract under the equal risk pricing framework as equation (21) stated (in this case $H(Y_T) = e^{-rT}(Y_T - K)$ non-decreasing)

Theorem 5.2. *Under the same equal risk framework, the value of the forward contract F with strike K and discounted payoff $H(Y_T) = e^{-rT}(Y_T - K)$ can be represented as,*

$$F = \begin{cases} \frac{1}{2}(Y_0 - e^{-rT}K) - \frac{\delta}{2}\ln\{E^{\mathbb{Q}}[e^{-\frac{1}{\delta}H(Y_T)}|Y_0 = y]\}, & \rho \neq \pm 1 \\ (Y_0 - e^{-rT}K), & \rho = \pm 1 \end{cases} \tag{21}$$

where $E^{\mathbb{Q}}[e^{-\frac{1}{\delta}H(Y_T)}|Y_0 = y] = \int_0^{+\infty} e^{-\frac{1}{\delta}H(Y_T)}m(Y_T|Y_0)dY_T$ *and* $\delta = \frac{1}{1-\rho^2}$

Proof. See He and Zhu (2020) [5] □

5.3 Revised European Put Option Contract Formula

The revised European put option formula in this case can also be derived similar to the revised European call option formula. The only difference is that the liability of the buyer of the put option can be perfectly hedged without the short selling of the underlying asset, while the seller's liability can be partially hedged upon different cases for ρ. This implies that we can determine the minimized expected risk of the seller of the put option through the same rationale as in Theorem 4.1 and the minimized expected risk of the buyer of the put option through the same arguments as in Theorem 4.2. Thus, by equating the resultant minimized risk of the two parties, we can obtain the equal risk price of a European put option in the following Theorem

Theorem 5.3. *The equal risk price p for a European put option with discounted payoff $H(Y_T) = e^{-rT}(K - Y_T)^+$ can be derived as,*

$$p = \begin{cases} \frac{1}{2}p_{bs} + \frac{\delta}{2}\ln\{E^{\mathbb{Q}}[e^{\frac{1}{\delta}H(Y_T)}|Y_0 = y]\}, & \rho \neq \pm 1 \\ p_{bs}, & \rho = \pm 1 \end{cases} \tag{22}$$

where $E^{\mathbb{Q}}[e^{\frac{1}{\delta}H(Y_T)}|Y_0 = y] = \int_0^{+\infty} e^{\frac{1}{\delta}H(Y_T)}m(Y_T|Y_0)dY_T$ *and* $\delta = \frac{1}{1-\rho^2}$

Proof. See He and Zhu (2020) [5] □

Again, we should point out that p_{bs} is referring to the corresponding price of the same European put option under the Black-Scholes-Merton model[3]

$$p_{bs} = K\mathrm{e}^{-rT}\Phi(-d_2) - Y_0\Phi(-d_1) \tag{23}$$

where $\Phi(\cdot)$, d_1, and d_2 are the same as the previous section for the price of the European call option under the Black-Scholes-Merton model. We can also observe that, when $\rho = \pm 1$, the revised formula reduces to the Black-Scholes model with perfect liquidity.

In this case, it is clear that all revised pricing formulas under the equal risk framework are

1. not depending on either the volatility σ_0 or the initial values of S_t even though the dynamic of S_t is specified.

- Since this is the desired property of the equal risk pricing formula, introducing a correlated asset S_t is just a model assumption to incorporate market interactions. In practice, when the underlying of the option has a short-selling ban restriction, we can follow the optimal trading strategy of the correlated asset π_t^* and use some other derivatives for hedging this option.

2. only introducing one extra variable, i.e. the ban-dilution factor ρ to the model comparing to the classical Black-Sholes-Merton model [3]

3. involving only one one-dimensional integral so that it is easy to be implemented and evaluated. In other words, we can compute the equal risk price without any extra computational burden comparing to calculate the classical Black-Scholes price.

5.4 Greeks Related to the ban-dilution factor

According to the revised option prices formula from the previous section, the Greeks related to the ban-dilution factor (i.e. ρ) can be easily obtained as the following for $\rho \in [0, 1]$

$$\begin{aligned}\frac{\partial c}{\partial \rho} &= -\rho\delta^2\{\ln \mathrm{E}^{\mathbb{Q}}[e^{-\frac{1}{\delta}H(Y_T)}|Y_0 = y] \\ &\quad + \delta\frac{\int_0^{+\infty} \frac{1}{\delta^2}H(Y_T)e^{-\frac{1}{\delta}H(Y_T)}m(Y_T|Y_0)dY_T}{\mathrm{E}^{\mathbb{Q}}[e^{-\frac{1}{\delta}H(Y_T)}|Y_0 = y]}\} \end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\partial p}{\partial \rho} &= \rho\delta^2\{\ln \mathrm{E}^{\mathbb{Q}}[e^{\frac{1}{\delta}H(Y_T)}|Y_0 = y] \\ &\quad + \delta\frac{\int_0^{+\infty} -\frac{1}{\delta^2}H(Y_T)e^{\frac{1}{\delta}H(Y_T)}m(Y_T|Y_0)dY_T}{\mathrm{E}^{\mathbb{Q}}[e^{\frac{1}{\delta}H(Y_T)}|Y_0 = y]}\} \end{aligned} \tag{25}$$

where according to the paper [5], $\frac{\partial c}{\partial \rho} \geq 0$ and $\frac{\partial p}{\partial \rho} \leq 0$, which implies that the revised European call and put option price are both monotonic functions of ρ for $\rho \in [0, 1]$.

These monotonicities suggest that for choosing the correlated asset for $\rho \in [0, 1]$, the revised call option price will be lower than the Black-Scholes price for $\rho = 1$. Similarly, for choosing the correlated asset for $\rho \in [0, 1]$, the revised put option price will be higher than the Black-Scholes price for $\rho = 1$.

6 Numerical Experiments

In this section, we will perform some numerical experiments to show how equal risk prices are changed w.r.t different parameters. All numerical results are solely my works on replicating the results from the reference paper [5]. Firstly, before everything, to determine the terminal underlying asset price Y_T with short-selling ban, I used the

Euler discretization with N discretized time steps and increment $\Delta t = \frac{T}{N}$. The values of the parameters are: $\mu = 0.1$, r = 0.1, K = 10, $Y_0 = 10$, $\sigma = 0.2$, T = 1, N = 10^4, $\Delta t = \frac{T}{N}$, with the ban-dilution factor $\rho = 0.8$. Furthermore, in Figure 1. and 3., we are using $\rho \in [0, 1]$ instead of $\rho \in [-1, 1]$ as the effect of selling the correlated asset S_t with $\rho = -1$ is the same as purchasing the correlated asset S_t with $\rho = 1$. Thus, it is suffice to use the constraint $\rho \in [0, 1]$ for illustrative purposes.

According to Figure 1. the graph of the European call and put prices v.s the correlation. As expected, our equal risk European call and put prices are monotonic increasing and decreasing accordingly as the theoretical results suggested when rho is increasing.

In particular, when the ban-dilution factor ρ increases, it implies the short-selling ban effect on the option written on the underlying with ban restriction is further diluted. Consequently, the risk faced by the buyer of the call option written upon the short-selling ban restricted asset Y_t is reduced. Thus, the buyer of the call option is willing to pay the additional amount to purchase the derivative contract as the ban-dilution factor increases. This phenomenon explains that our equal risk price for the European call option is a monotonic increasing function of ρ. On the other hand, as

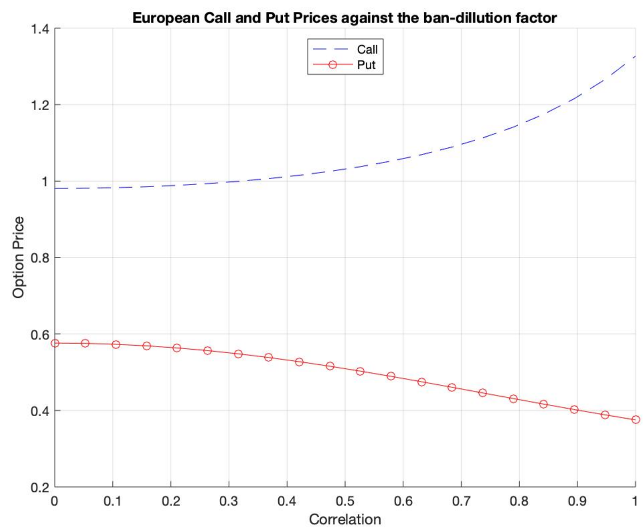

Figure 1: European Call and Put prices against the ban-dilution factor

the ban-dilution factor increases, the risk faced by the seller of the put option written upon the short-selling ban restricted asset Y_t is increased due to the reducing effect of the short-selling ban on the underlying asset. Thus, the put option seller is willing to sell the derivative contract in a lower value as the ban-dilution factor increases. This result explains that our equal risk price for the European put option is a monotonic decreasing function of ρ. Furthermore, as Figure 1. suggests, the rate of change in option price increases when the ban-dilution factor close to 1. According to [5], this is due to the choice of our risk function being of the exponential form so that people in the market are more sensitive to the risk when extra risk is introduced.

Figure 2. shows the equal risk European call and put option price w.r.t the underlying price for different values of ρ s. According to the graph, we can observe that our equal risk call prices are lower than the classical Black-Scholes call prices, whereas our equal risk put prices are higher than the classical Black-Scholes put prices. Again, this observation can be explained using the same rationales we used in interpreting the results in the previous Figure 1. In particular, as the ban-dilution factor ρ increases from 0.5 (our case) to 1 (Black-Scholes case), the buyer of the call option written upon the short-selling ban restricted asset Y_t is reduced. Therefore, the buyer of the call option is willing to pay the additional amount to purchase the derivative contract as the ban-dilution factor increases. Hence, the classical Black-Scholes call prices with higher correlations will have higher values than our equal risk call prices with lower correlations. It follows that the difference between our equal risk call prices and the Black-Scholes call prices enlarges when the ban-dilution factor decreases, as Figure 1. indicated. Moreover, as the spot price increases, Y_T increases, thereby $H(Y_T) = e^{-rT}(Y_T - K)$ monotonic increases for unit increment of the spot price Y_0 for fixed $\rho = 0.5$, i.e. fixed δ. It follows that for an non-increasing function $e^{-\frac{1}{\delta}H(Y_T)}$ of $H(Y_T)$ and $ln(\cdot)$ a monotonic increasing function, the equal risk call price c increases for unit increment of the spot price Y_0 and fixed $\rho = 0.5$ by equation (19) in Theorem 5.1.. Thus, that is why our equal risk call prices monotonic increases as the spot price Y_0 increases suggested in Figure 2. A similar argument can be applied to a European put option case using the same rationale in interpreting Figure 1.

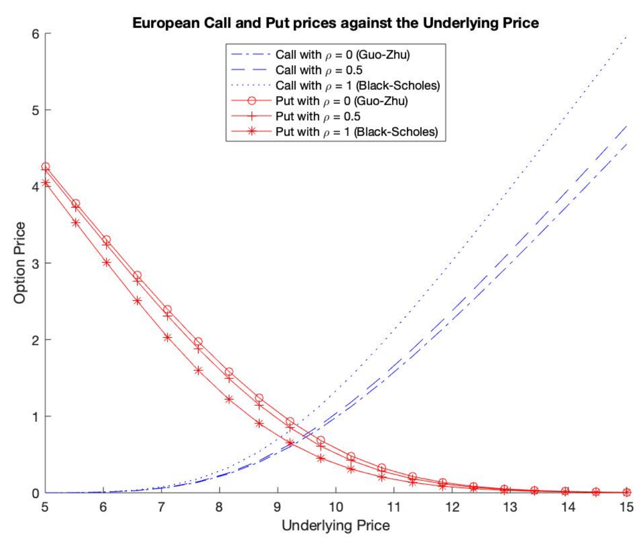

Figure 2: European Call and Put prices against the Underlying Price

Under the short-selling ban assumption on the underlying asset, the market is incomplete, so the perfect hedging is impossible in our case, i.e. the put-call-parity does not hold anymore. In this case, the authors in [5] suggest that we can calculate the deviation of the put-call parity by $c - p - F$ to see the effect of the ban-dilution factor, i.e. the market interaction, on the put-call parity. According to Figure 3., the graph of the deviation of put-call parity against the ban-dilution factor shows that the deviation of the put-call parity is a monotonic decreasing function of ρ. As we increase the diluted factor, the rate of drop increases accordingly. It is worthy to note that when the correlation goes to 1, it reaches the case of a classical Black-Scholes model.

Meanwhile, as we plot the deviation of the put-call parity against the underlying prices as in Figure 4., we can observe a fat tail on the right of the curve. When the underlying prices are much smaller than the strike K = 10, the put-call parity deviation is almost equal to 0. As the underlying price approaches the maximum deviation of the put-call parity, the option becomes at-the-money at the peak. The deviation decreases to almost 0 when the underlying price is much larger than the strike K = 10. According to [5], the rationale behind this is when deep-in or deep-out of the money, the options will have a large probability of being exercised at expiry, which makes the hedging for those options easier.

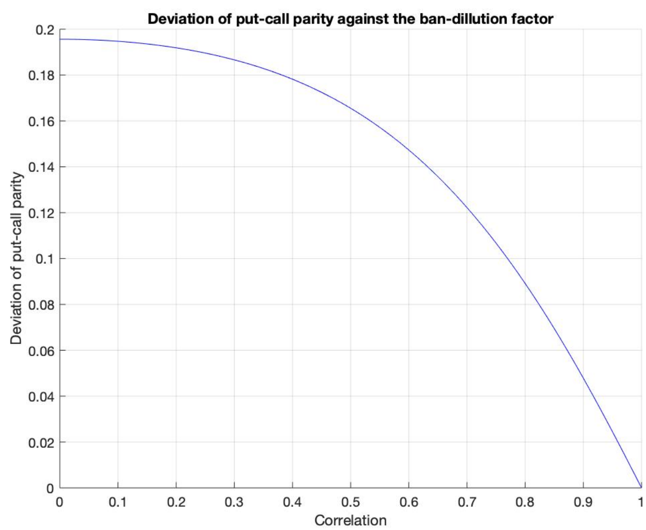

Figure 3: Deviation of Put-Call Parity against the Correlation

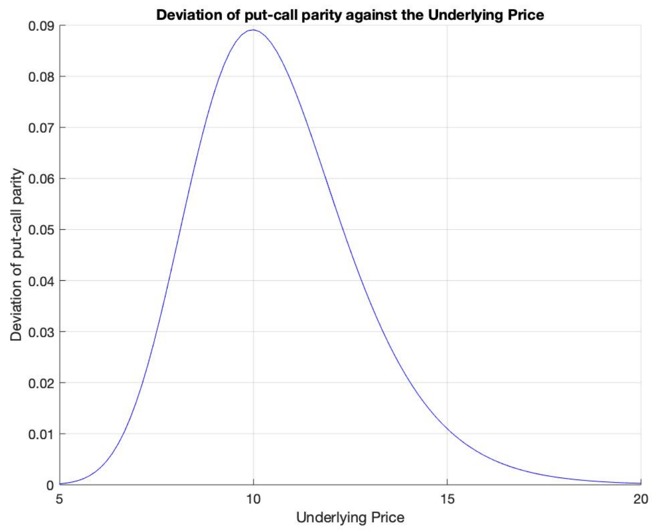

Figure 4: Deviation of Put-Call Parity against the Underlying Price

7 Conclusion and Enlightenment

In general, we have introduced a closed-form equal risk pricing formula for European options, under the assumption that the short sell of the underlying asset is abolished. The revised formulas introduce only one extra variable, i.e. the ban-dilution factor ρ, where it still keeps the essential advantage of the Black–Scholes model. Apart from that, the revised formulas are easy to be numerically evaluated as there is only one one-dimensional integral involved. We have also examined the financial intuition of the ρ. Specifically, when $\rho = \pm 1$, i.e. the Black-Sholes case, the effect of the short-selling ban on the option written on the underlying with ban restriction is perfectly diluted, i.e. the additional risk incurred by the short-selling ban has been perfectly hedged. Whereas when $\rho = 0$ Guo and Zhu case [4], the underlying asset and the correlated asset are independent, which implies the additional risk incurred by the short-selling ban has completely failed to be hedged. In the end, by various numerical experiments, we have shown that the model is applicable when unhedgable risk by the short-selling ban can be partially or perfectly hedged away by the newly introduce correlated asset.

For further works, we can apply this revised formula on real-world option data where the option's underlying has a short-selling ban to see how well the model fits the market. The potential improvement will be extending the formula to a more complicated type of options. Since the revised option pricing formula only applies to the European option. We can also change the form of the risk function for different market situations, for which traders in the market are not susceptible to the occurrences of additional risk or traders has larger sensitives than in a normal situation when the market is volatile. Besides, we can consider more than one alternatives for hedging instead of assuming only one correlated asset to accommodate the real market events.

Appendix A Prerequisite Knowledge

A.1 HJB (Hamilton-Jacobi-Bellman) Equation

HJB According to [2], for a continuous-time dynamic programming, time $t \in R_+$, state $x \in \mathbb{X}$, and control $p \in \mathbb{P}$ such that $dx = u(p(t,x))dt + \sigma(p(t,x))dW(t)$ with a cost function V(x,t), the equation

$$0 = \inf_{p \in \mathbb{P}} \{V_t(x,p) + \mu_x V_x + \frac{1}{2}(\sigma_x)^2 V_{xx}\}$$

is called the Hamilton-Jacobo-Bellman equation

A.2 Ito's Formula

Ito's Formula Assume $dX = udt + \sigma dW$, let $Z(t) = f(t, X(t))$ be a $C^{1,2}$ function, then Z has a stochastic differential given by

$$df(t, X(t)) = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma \frac{\partial f}{\partial x} dW(t)$$

Appendix B MATLAB Code

GitHub URL: Please click here to view the code

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