HodgeRank

A ranking method based on Hodge-Helmholtz decomposition

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Hodge Decomposition Described in the Language of Linear Algebra

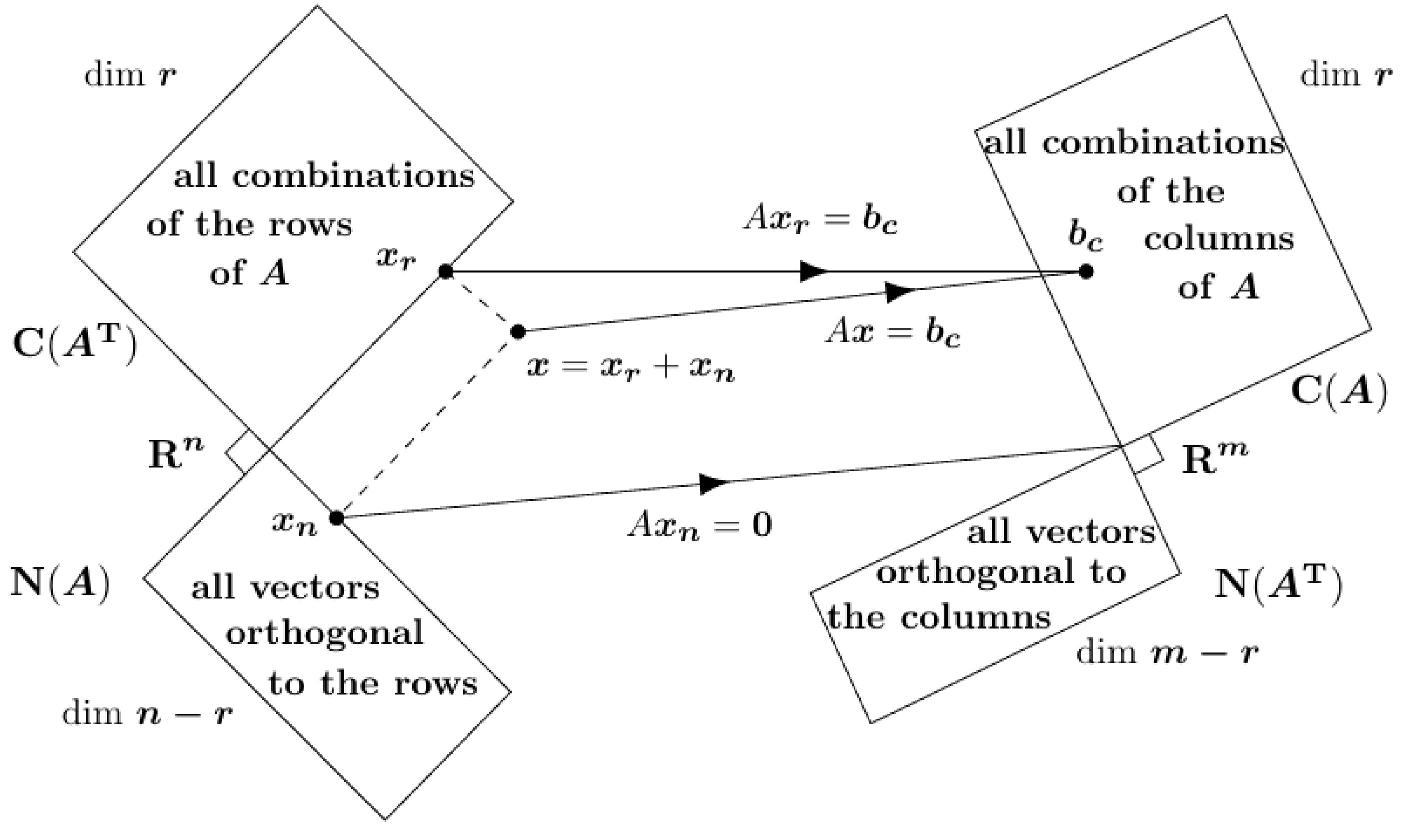


Figure 1: $Ax_r = b$ is in the column space of A and $Ax_n = \mathbf{0}$. The complete solution to $Ax = b$ is $x =$ one $x_r +$ any x_n.

$$\boldsymbol{A}_{m\times n}$$

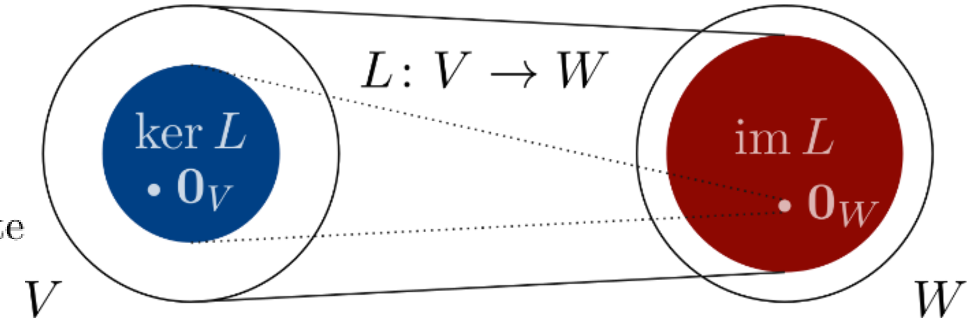
$$\operatorname{im}(\boldsymbol{A}_{m\times n}) \oplus \ker(\boldsymbol{A}_{m\times n}{}^{\boldsymbol{T}}) = \mathbb{R}^{\boldsymbol{m}}$$

$$\operatorname{im}(\boldsymbol{A}_{m\times n}{}^{\boldsymbol{T}}) \oplus \ker(\boldsymbol{A}_{m\times n}) = \mathbb{R}^{\boldsymbol{n}}$$

$$\boldsymbol{B}_{n\times p}$$

$$\operatorname{im}(\boldsymbol{B}_{n\times p}) \oplus \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) = \mathbb{R}^{\boldsymbol{n}}$$

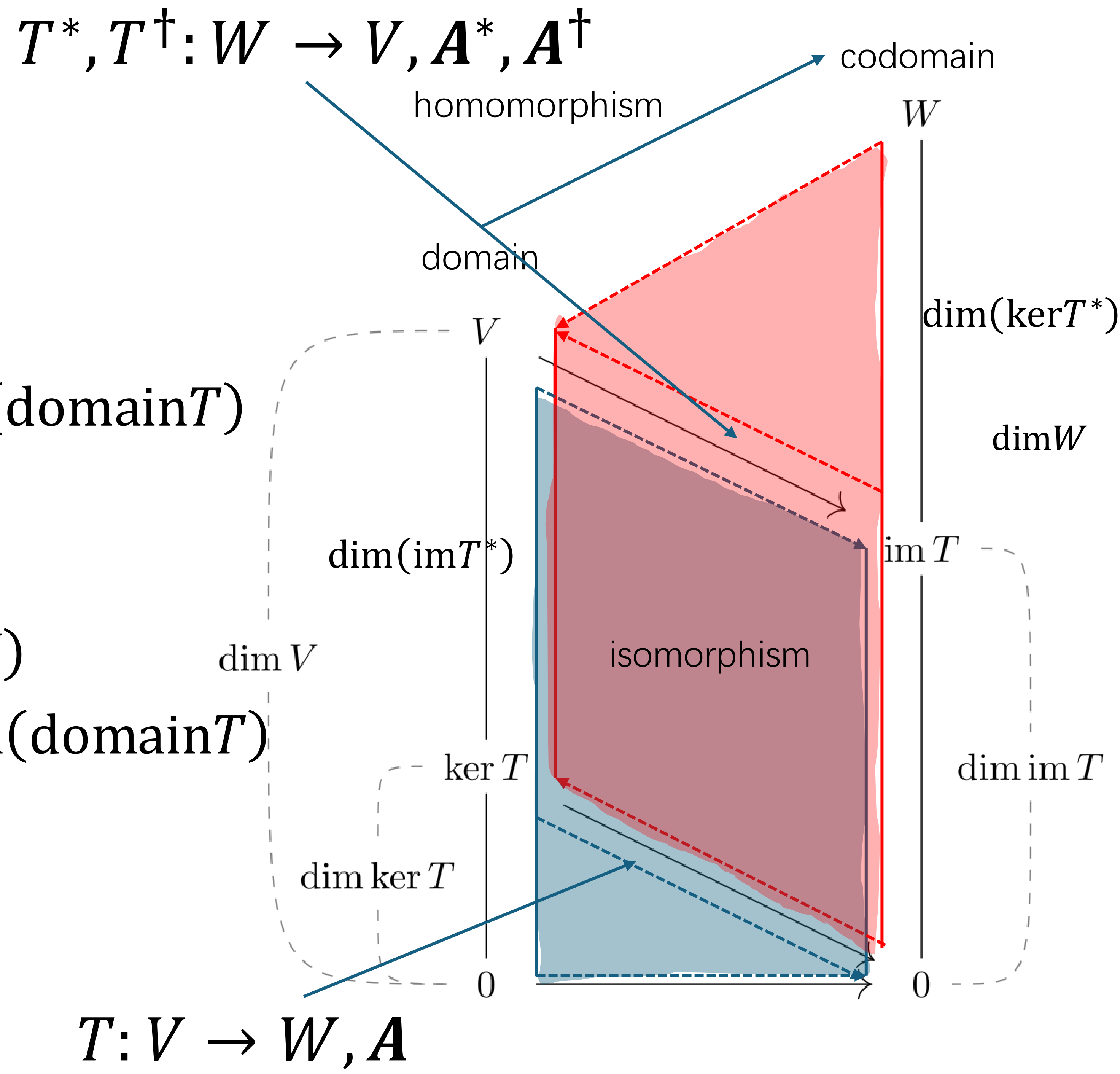
$$\operatorname{im}(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) \oplus \ker(\boldsymbol{B}_{n\times p}) = \mathbb{R}^{\boldsymbol{p}}$$

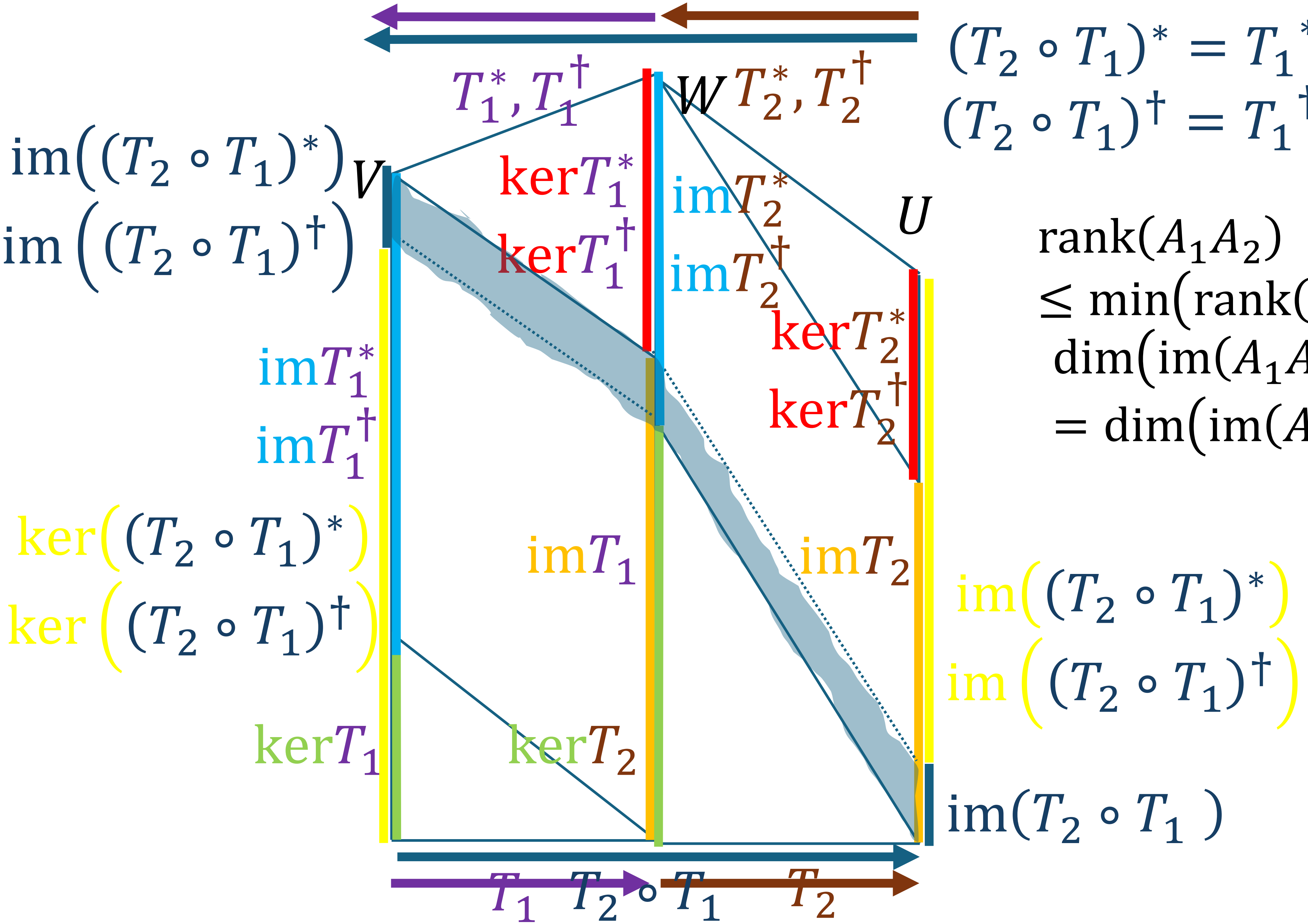


$$\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times p} = \mathbf{0}_{m\times p} \Leftrightarrow \boldsymbol{B}_{n\times p}{}^{T}\boldsymbol{A}_{m\times n}{}^{T} = \mathbf{0}_{p\times m}$$

$$\operatorname{im}(\boldsymbol{B}_{n\times p}) \subseteq \ker(\boldsymbol{A}_{m\times n}) \Leftrightarrow \operatorname{im}(\boldsymbol{A}_{m\times n}{}^{T}) \subseteq \ker(\boldsymbol{B}_{n\times p}{}^{T})$$

- Rank-nullity theorem:
- $T: V \to W$
- $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V)$
- $\dim(\mathrm{im}T) + \dim(\ker T) = \dim(\mathrm{domain}T)$
- $\mathrm{im}T \oplus \ker T \cong V$
- $T^*: W \to V$
- $\mathrm{rank}(T^*) + \mathrm{nullity}(T) = \dim(V)$
- $\dim(\mathrm{im}T^*) + \dim(\ker T) = \dim(\mathrm{domain}T)$
- $\mathrm{im}T^* \oplus \ker T \cong V$
- $\mathrm{rank}(T^*) = \mathrm{rank}(T)$
- $\dim(\mathrm{im}T^*) = \dim(\mathrm{im}T)$

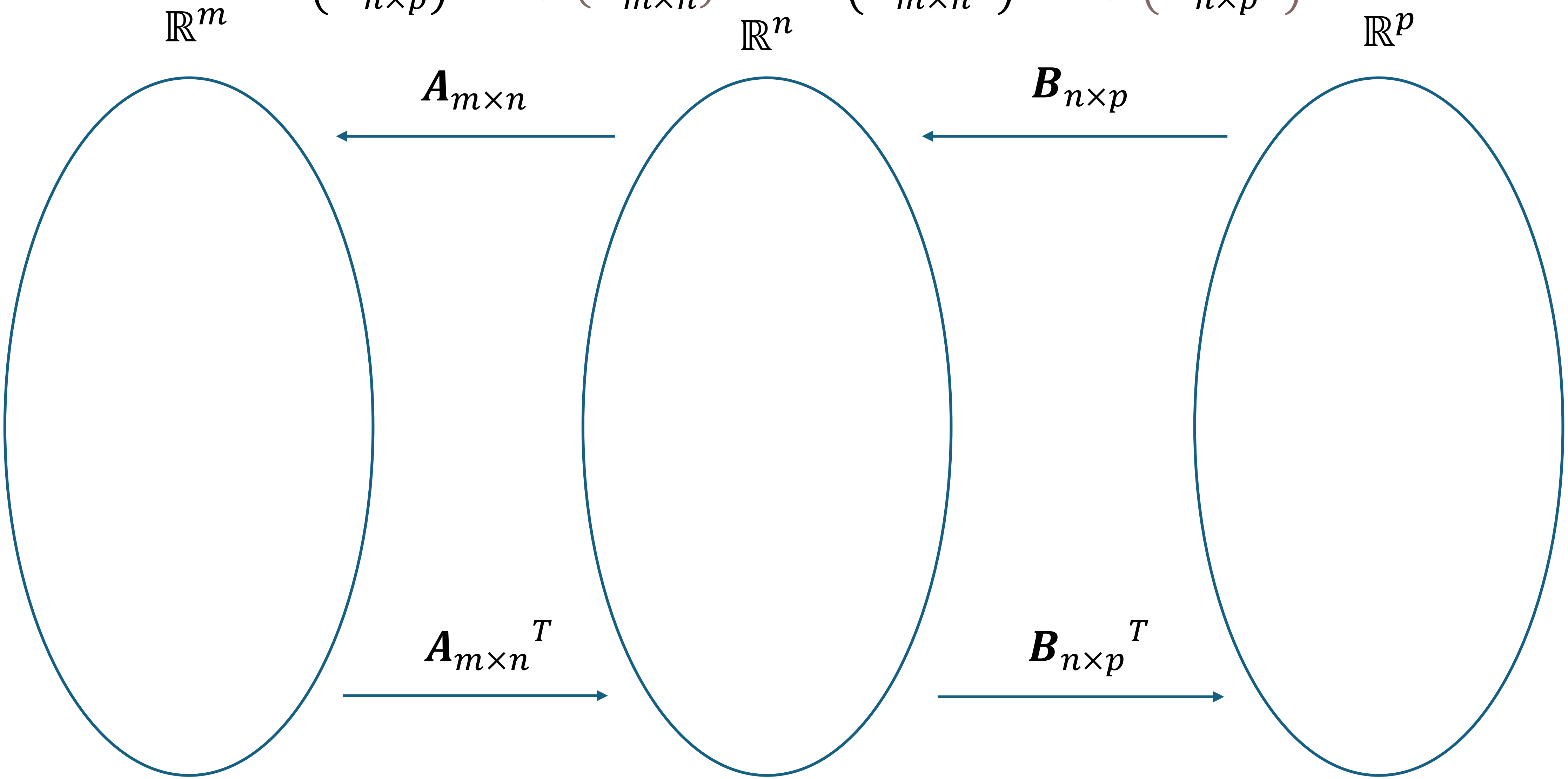




$$\mathrm{rank}(A_1 A_2)$$
$$\leq \min\left(\mathrm{rank}(A_1), \mathrm{rank}(A_2)\right)$$
$$\dim\left(\mathrm{im}(A_1 A_2)\right)$$
$$= \dim\left(\mathrm{im}(A_1) \cap \mathrm{im}(A_2^*)\right)$$

$$\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times p} = \boldsymbol{0}_{m\times p} \Leftrightarrow {\boldsymbol{B}_{n\times p}}^T {\boldsymbol{A}_{m\times n}}^T = \boldsymbol{0}_{p\times m}$$

$$\operatorname{im}\left(\boldsymbol{B}_{n\times p}\right) \subseteq \ker\left(\boldsymbol{A}_{m\times n}\right) \Leftrightarrow \operatorname{im}\left({\boldsymbol{A}_{m\times n}}^T\right) \subseteq \ker\left({\boldsymbol{B}_{n\times p}}^T\right)$$

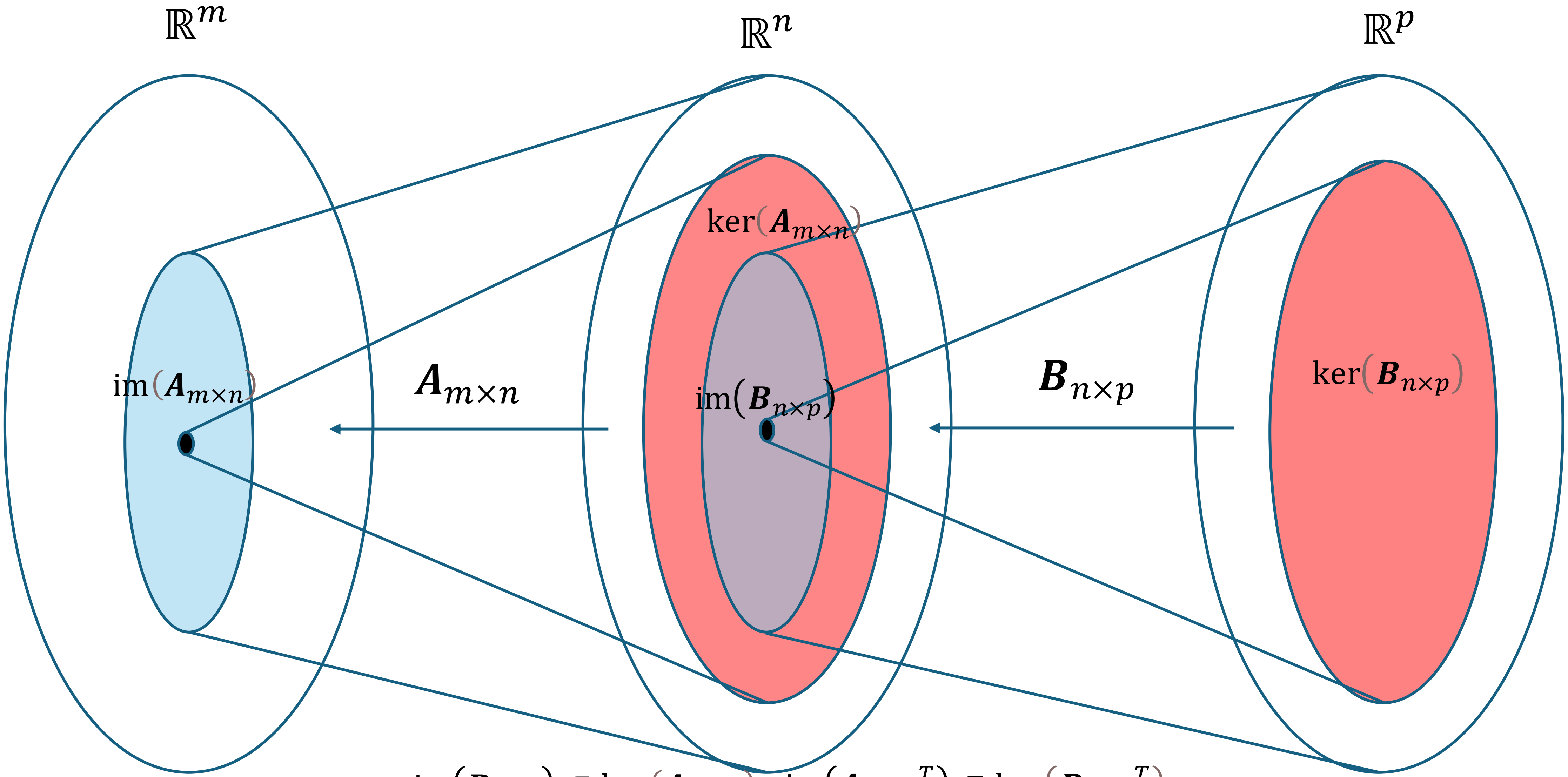


$$\operatorname{im}\left(\boldsymbol{A}_{m\times n}\right) \oplus \ker\left({\boldsymbol{A}_{m\times n}}^T\right) = \mathbb{R}^{\boldsymbol{m}}$$

$$\operatorname{im}\left({\boldsymbol{A}_{m\times n}}^{\boldsymbol{T}}\right) \oplus \ker\left(\boldsymbol{A}_{m\times n}\right) = \mathbb{R}^{\boldsymbol{n}}$$

$$\operatorname{im}\left(\boldsymbol{B}_{n\times p}\right) \oplus \ker\left({\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}}\right) = \mathbb{R}^{\boldsymbol{n}}$$

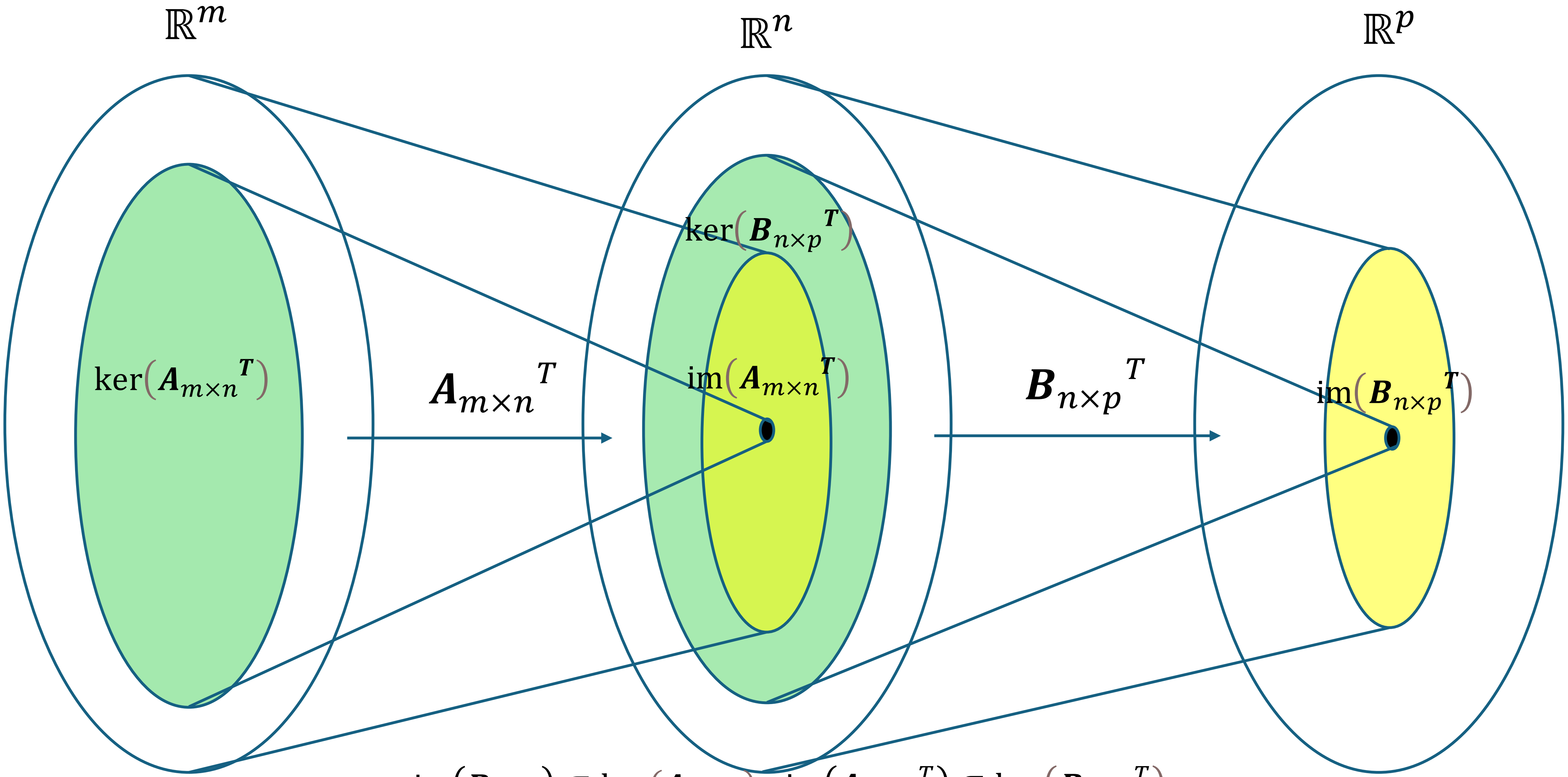
$$\operatorname{im}\left({\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}}\right) \oplus \ker\left(\boldsymbol{B}_{n\times p}\right) = \mathbb{R}^{\boldsymbol{p}}$$



$\mathrm{im}(\boldsymbol{B}_{n\times p}) \subseteq \ker(\boldsymbol{A}_{m\times n}) \quad \mathrm{im}(\boldsymbol{A}_{m\times n}{}^{T}) \subseteq \ker(\boldsymbol{B}_{n\times p}{}^{T})$

$\mathrm{im}(\boldsymbol{A}_{m\times n}{}^{\boldsymbol{T}}) \oplus \ker(\boldsymbol{A}_{m\times n}) = \mathbb{R}^{\boldsymbol{n}}$

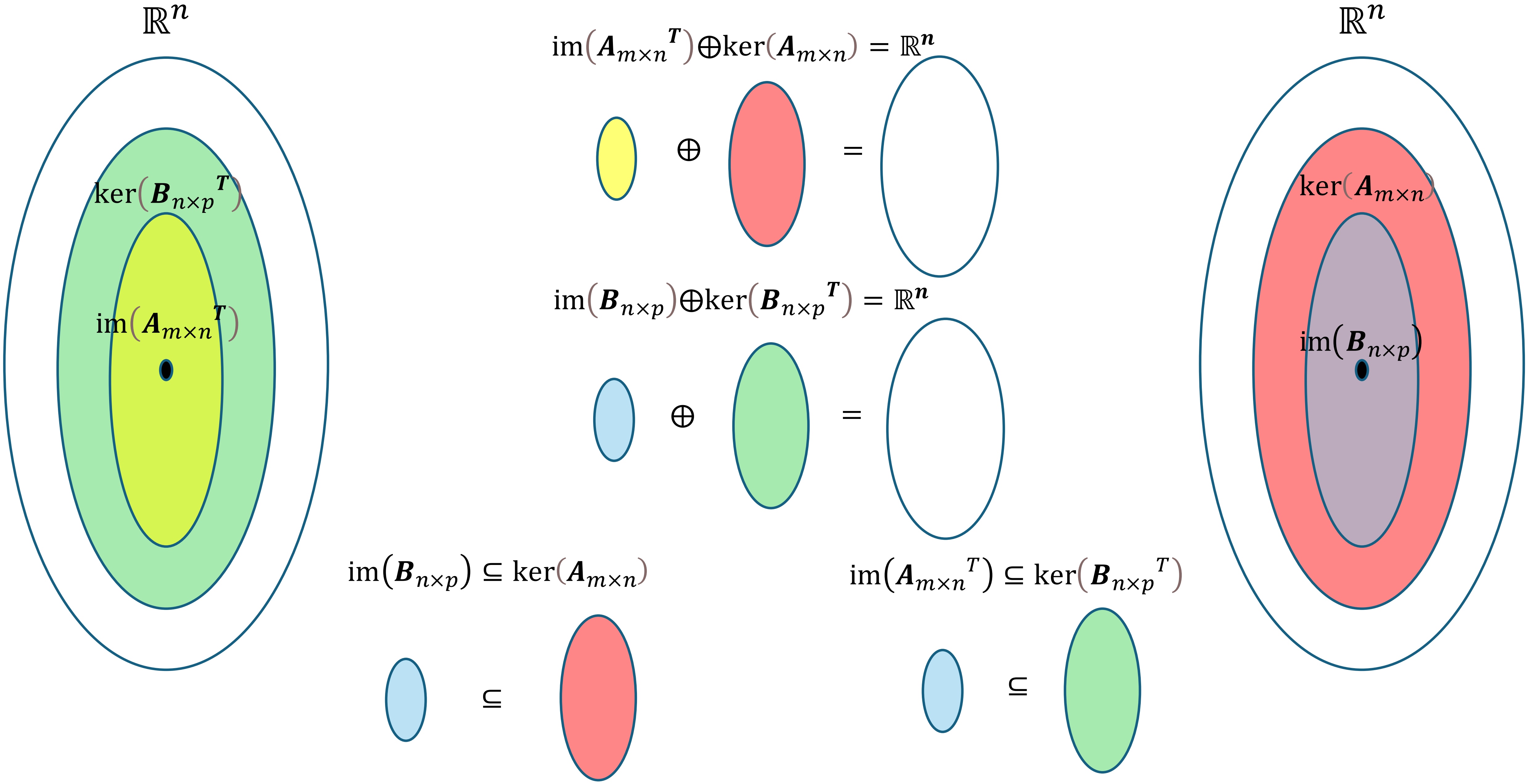
$\mathrm{im}(\boldsymbol{B}_{n\times p}) \oplus \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) = \mathbb{R}^{\boldsymbol{n}}$

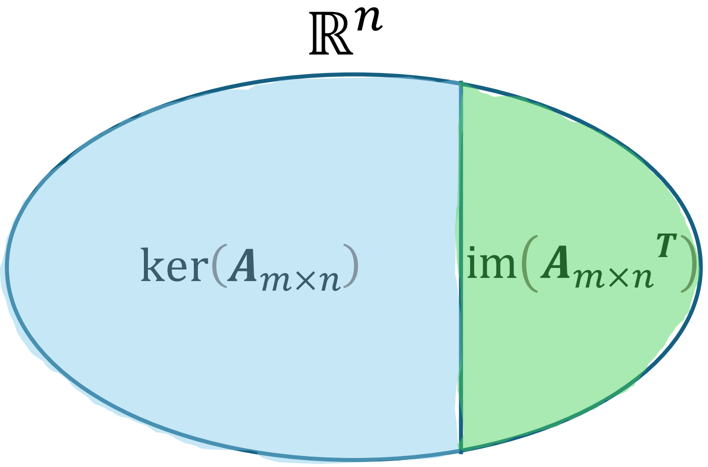


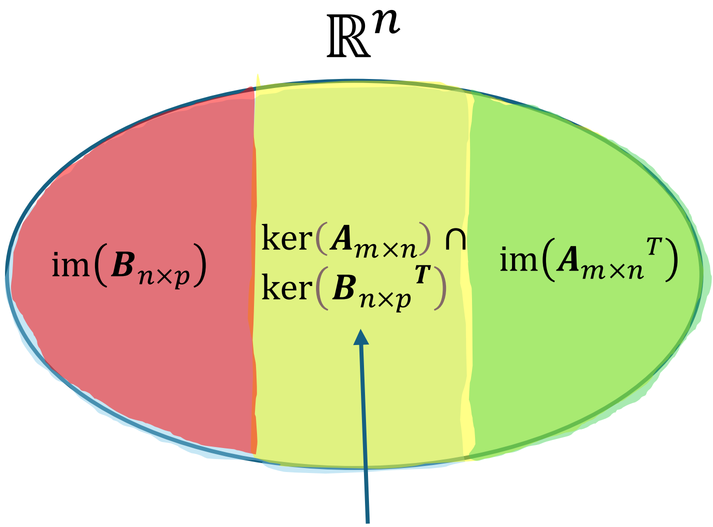
$\mathrm{im}(\boldsymbol{B}_{n\times p}) \subseteq \ker(\boldsymbol{A}_{m\times n}) \quad \mathrm{im}(\boldsymbol{A}_{m\times n}{}^{T}) \subseteq \ker(\boldsymbol{B}_{n\times p}{}^{T})$

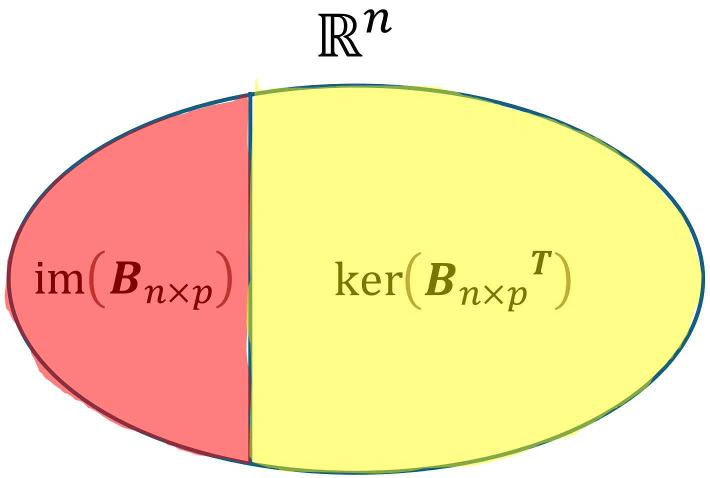
$\mathrm{im}(\boldsymbol{A}_{m\times n}{}^{\boldsymbol{T}}) \oplus \ker(\boldsymbol{A}_{m\times n}) = \mathbb{R}^{\boldsymbol{n}}$

$\mathrm{im}(\boldsymbol{B}_{n\times p}) \oplus \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) = \mathbb{R}^{\boldsymbol{n}}$


$\mathbb{R}^n$
$\ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})$
$\operatorname{im}(\boldsymbol{A}_{m\times n}{}^{\boldsymbol{T}})$
$\operatorname{im}(\boldsymbol{A}_{m\times n}{}^{\boldsymbol{T}}) \oplus \ker(\boldsymbol{A}_{m\times n}) = \mathbb{R}^{\boldsymbol{n}}$
$\oplus$ $=$
$\operatorname{im}(\boldsymbol{B}_{n\times p}) \oplus \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) = \mathbb{R}^{\boldsymbol{n}}$
$\oplus$ $=$
$\mathbb{R}^n$
$\ker(\boldsymbol{A}_{m\times n})$
$\operatorname{im}(\boldsymbol{B}_{n\times p})$
$\operatorname{im}(\boldsymbol{B}_{n\times p}) \subseteq \ker(\boldsymbol{A}_{m\times n})$
$\subseteq$
$\operatorname{im}(\boldsymbol{A}_{m\times n}{}^{T}) \subseteq \ker(\boldsymbol{B}_{n\times p}{}^{T})$
$\subseteq$



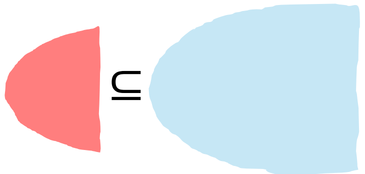




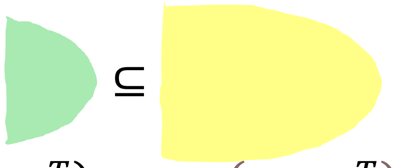
$\operatorname{im}(\boldsymbol{A}_{m\times n}{}^{\boldsymbol{T}}) \oplus \ker(\boldsymbol{A}_{m\times n}) = \mathbb{R}^{\boldsymbol{n}}$

$\ker(\boldsymbol{A}_{m\times n}) \cap \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})$
$= \ker(\boldsymbol{A}_{m\times n}) / \operatorname{im}(\boldsymbol{B}_{n\times p})$
$= \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) / \operatorname{im}(\boldsymbol{A}_{m\times n}{}^{T})$
$= \ker(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})$

$\operatorname{im}(\boldsymbol{B}_{n\times p}) \oplus \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) = \mathbb{R}^{\boldsymbol{n}}$



$\operatorname{im}(\boldsymbol{B}_{n\times p}) \subseteq \ker(\boldsymbol{A}_{m\times n})$



$\operatorname{im}(\boldsymbol{A}_{m\times n}{}^{T}) \subseteq \ker(\boldsymbol{B}_{n\times p}{}^{T})$

$$\mathbb{R}^n = \operatorname{im}(\boldsymbol{B}_{n\times p}) \oplus \left(\ker(\boldsymbol{A}_{m\times n}) \cap \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})\right) \oplus \operatorname{im}(\boldsymbol{A}_{m\times n}{}^{T})$$

$$\mathbb{R}^n = \operatorname{im}(\boldsymbol{B}_{n\times p}) \oplus \ker(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) \oplus \operatorname{im}(\boldsymbol{A}_{m\times n}{}^{T})$$

$$\boldsymbol{x} \in \mathbb{R}^n, \qquad (\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})\boldsymbol{x}_{\boldsymbol{H}} = \boldsymbol{0}, \qquad \boldsymbol{v} \in \mathbb{R}^p, \qquad \boldsymbol{w} \in \mathbb{R}^m$$

$$\boldsymbol{x} = \boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{w} + \boldsymbol{x}_{\boldsymbol{H}} + \boldsymbol{B}_{n\times p}\boldsymbol{v}, \qquad \langle \boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{w}, \boldsymbol{x}_{\boldsymbol{H}} \rangle = \langle \boldsymbol{B}_{n\times p}\boldsymbol{v}, \boldsymbol{x}_{\boldsymbol{H}} \rangle = \langle \boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{w}, \boldsymbol{B}_{n\times p}\boldsymbol{v} \rangle = \boldsymbol{0}$$

$$\boldsymbol{x}_{n\times1} \in \ker(\boldsymbol{A}_{m\times n}) \Longrightarrow \boldsymbol{A}_{m\times n}\boldsymbol{x}_{n\times1} = \boldsymbol{0}_{m\times1} \Longrightarrow \boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n}\boldsymbol{x}_{n\times1} = \boldsymbol{0}_{n\times1}$$

$$\boldsymbol{x}_{n\times1} \in \ker\left(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right) \Longrightarrow \boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{x}_{n\times1} = \boldsymbol{0}_{p\times1} \Longrightarrow \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{x}_{n\times1} = \boldsymbol{0}_{n\times1}$$

$$\Longrightarrow \left(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right)\boldsymbol{x}_{n\times1} = \boldsymbol{0}_{n\times1} \Longrightarrow \boldsymbol{x}_{n\times1} \in \ker\left(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right)$$

$$\boldsymbol{x}_{n\times1} \in \ker\left(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right) \Longrightarrow \left(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right)\boldsymbol{x}_{n\times1} = \boldsymbol{0}_{n\times1}$$

$$\Longrightarrow \boldsymbol{A}_{m\times n}{}^{T}\,\boldsymbol{A}_{m\times n}\boldsymbol{x}_{n\times1} = -\boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\,\boldsymbol{x}_{n\times1}$$

$$\boldsymbol{A}_{m\times n}\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n}\boldsymbol{x}_{n\times1} = -\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{x}_{n\times1} \Longrightarrow \boldsymbol{A}_{m\times n}\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n}\boldsymbol{x}_{n\times1} = \boldsymbol{0}_{n\times1}$$

$$\Longrightarrow \boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n}\boldsymbol{x}_{n\times1} \in \ker(\boldsymbol{A}_{m\times n})$$

$$\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n}\boldsymbol{x}_{n\times1} \in \operatorname{im}\left(\boldsymbol{A}_{m\times n}{}^{T}\right) = \ker(\boldsymbol{A}_{m\times n})^{\perp}$$

$$\Longrightarrow \boldsymbol{A}_{m\times n}{}^{T}\,\boldsymbol{A}_{m\times n}\boldsymbol{x}_{n\times1} = \boldsymbol{0} \Longrightarrow \boldsymbol{x}_{n\times1} \in \ker\left(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n}\right) = \ker(\boldsymbol{A}_{m\times n})$$

$$\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n}\boldsymbol{x}_{n\times1} = -\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{x}_{n\times1} \Longrightarrow \boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{x}_{n\times1} = \boldsymbol{0}_{n\times1}$$

$$\Longrightarrow \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{x}_{n\times1} \in \ker\left(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right)$$

$$\boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{x}_{n\times1} \in \operatorname{im}(\boldsymbol{B}_{n\times p}) = \ker\left(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right)^{\perp}$$

$$\Longrightarrow \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\boldsymbol{x}_{n\times1} = \boldsymbol{0} \Longrightarrow \boldsymbol{x}_{n\times1} \in \ker\left(\boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right) = \ker\left(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right)$$

$\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}$ is a sum of Gram matrices, symmetric, positive semi-definite

$$\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times p}=\boldsymbol{0}_{m\times p}\Longrightarrow$$

$$\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times p}=\boldsymbol{0}_{m\times p}\Leftrightarrow{\boldsymbol{B}_{n\times p}}^{T}{\boldsymbol{A}_{m\times n}}^{T}=\boldsymbol{0}_{p\times m}$$

$$\mathrm{im}(\boldsymbol{B}_{n\times p})\subseteq\ker(\boldsymbol{A}_{m\times n})\Leftrightarrow\mathrm{im}({\boldsymbol{A}_{m\times n}}^{T})\subseteq\ker({\boldsymbol{B}_{n\times p}}^{T})$$

$$\boldsymbol{A}_{m\times n}\quad\begin{array}{l}\mathrm{im}(\boldsymbol{A}_{m\times n})\oplus\ker({\boldsymbol{A}_{m\times n}}^{\boldsymbol{T}})=\mathbb{R}^{\boldsymbol{m}}\\ \mathrm{im}({\boldsymbol{A}_{m\times n}}^{\boldsymbol{T}})\oplus\ker(\boldsymbol{A}_{m\times n})=\mathbb{R}^{\boldsymbol{n}}\end{array}$$

$$\boldsymbol{B}_{n\times p}\quad\begin{array}{l}\mathrm{im}(\boldsymbol{B}_{n\times p})\oplus\ker({\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}})=\mathbb{R}^{\boldsymbol{n}}\\ \mathrm{im}({\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}})\oplus\ker(\boldsymbol{B}_{n\times p})=\mathbb{R}^{\boldsymbol{p}}\end{array}$$

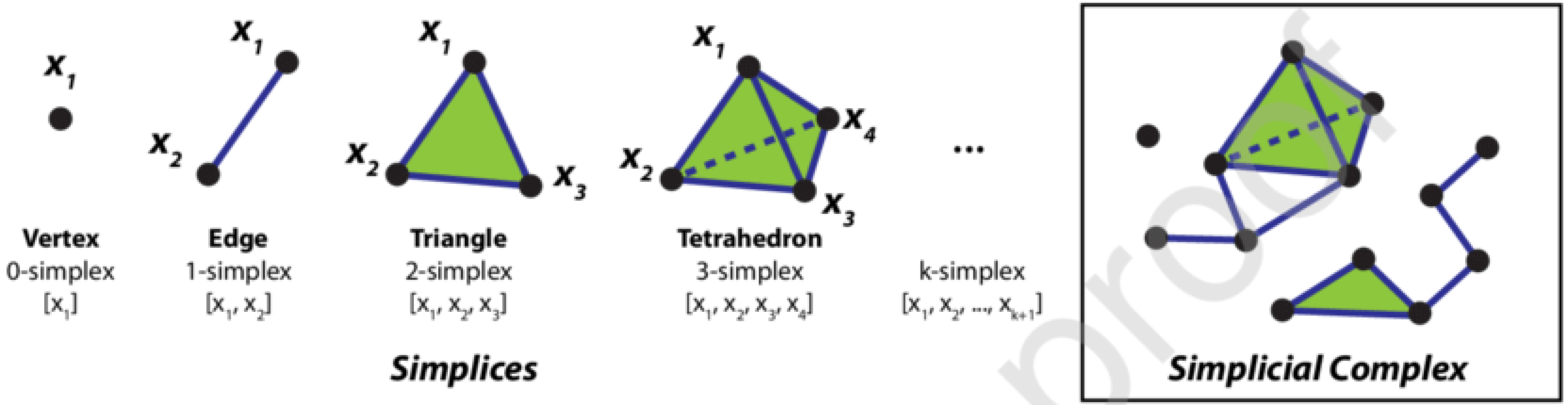
$$\Longrightarrow\mathbb{R}^{n}=\mathrm{im}(\boldsymbol{B}_{n\times p})\oplus\ker({\boldsymbol{A}_{m\times n}}^{T}\boldsymbol{A}_{m\times n}+\boldsymbol{B}_{n\times p}{\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}})\oplus\mathrm{im}({\boldsymbol{A}_{m\times n}}^{T})$$

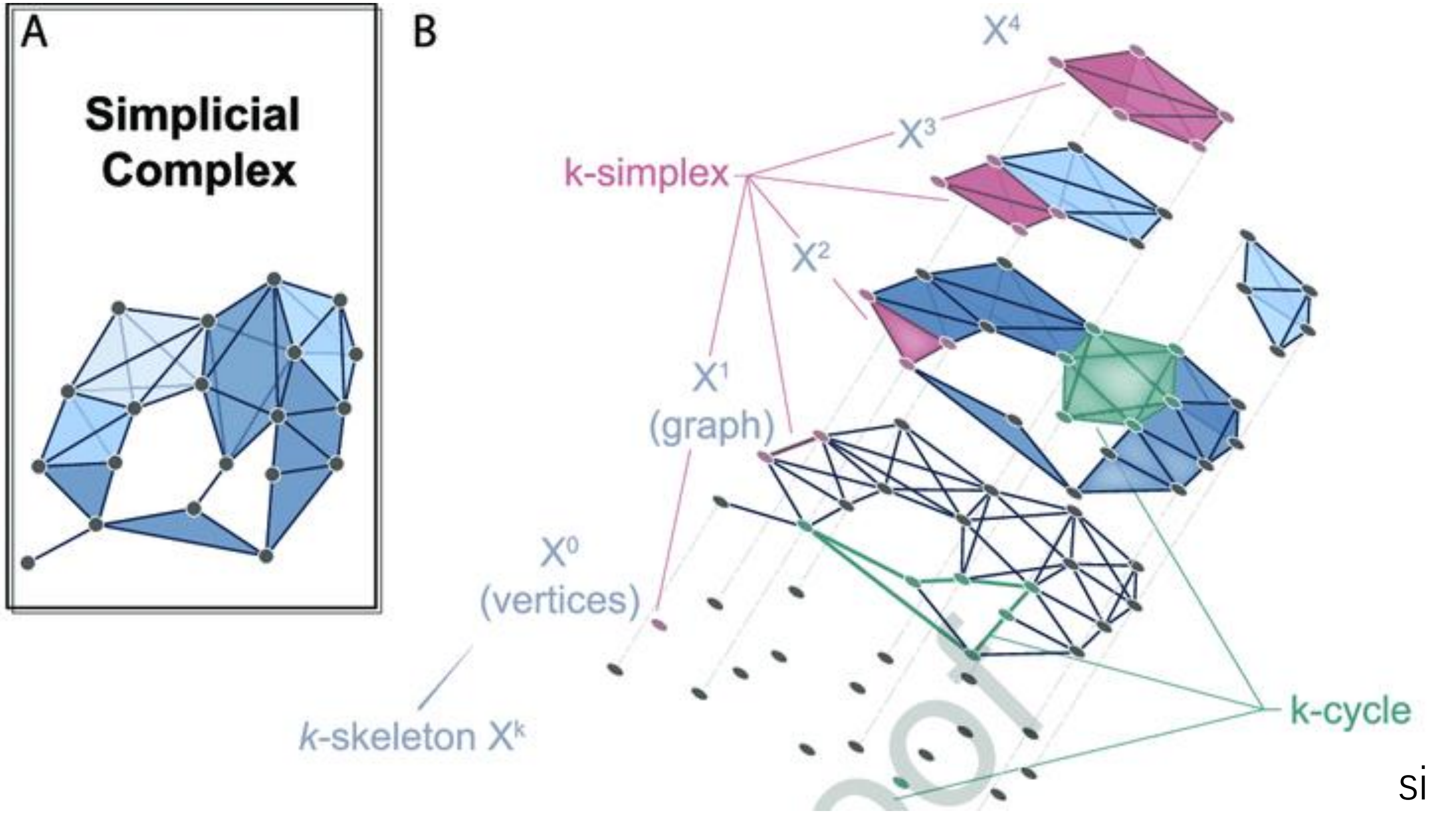
${\boldsymbol{A}_{m\times n}}^{T}\boldsymbol{A}_{m\times n}+\boldsymbol{B}_{n\times p}{\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}}$: Hodge Laplacian $\begin{cases}\text{Graph Laplacian}\\ \text{Graph Helmholtzian}\\ \text{and more}\end{cases}$

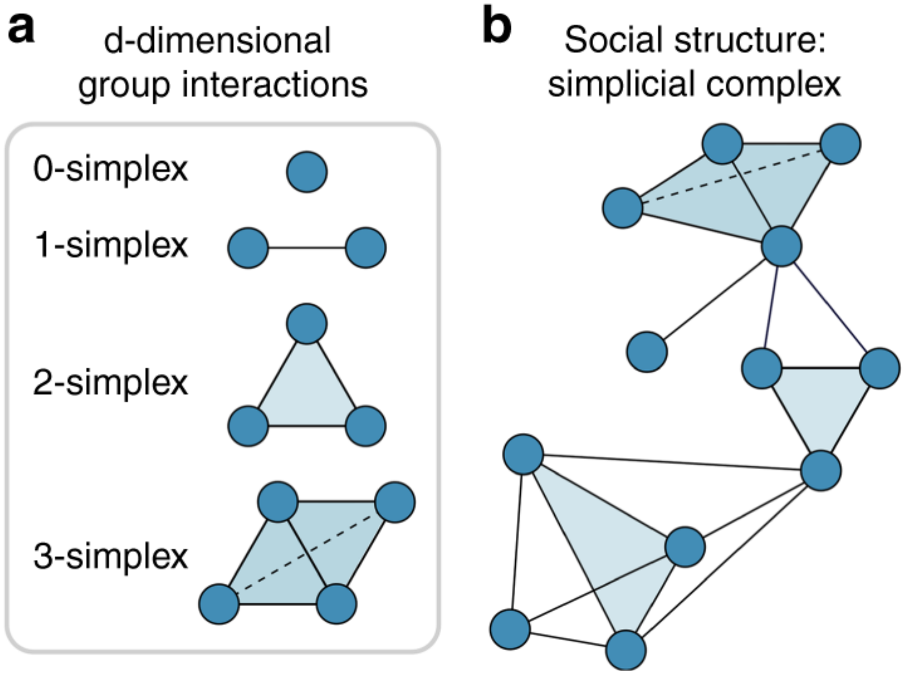
$\ker({\boldsymbol{A}_{m\times n}}^{T}\boldsymbol{A}_{m\times n}+\boldsymbol{B}_{n\times p}{\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}})$: Harmonic function

Some Concepts in Algebraic Topology

Simplex and Simplicial Complex







K-simplex: filled K+1 cliques

simplicial complex: family of subsets of vertices

- k-simplex (oriented) σ^k: $[i_0, \dots, i_k]$: $i_0 < \cdots < i_k$
- Simplicial complex X: a set of simplices that satisfies every face of a simplex from X is also in X
- Chain: linear combination of simplices (simplices as bases)

$$C_k(X) = \left\{ \sum_i a_i {\sigma^k}_i \right\}$$

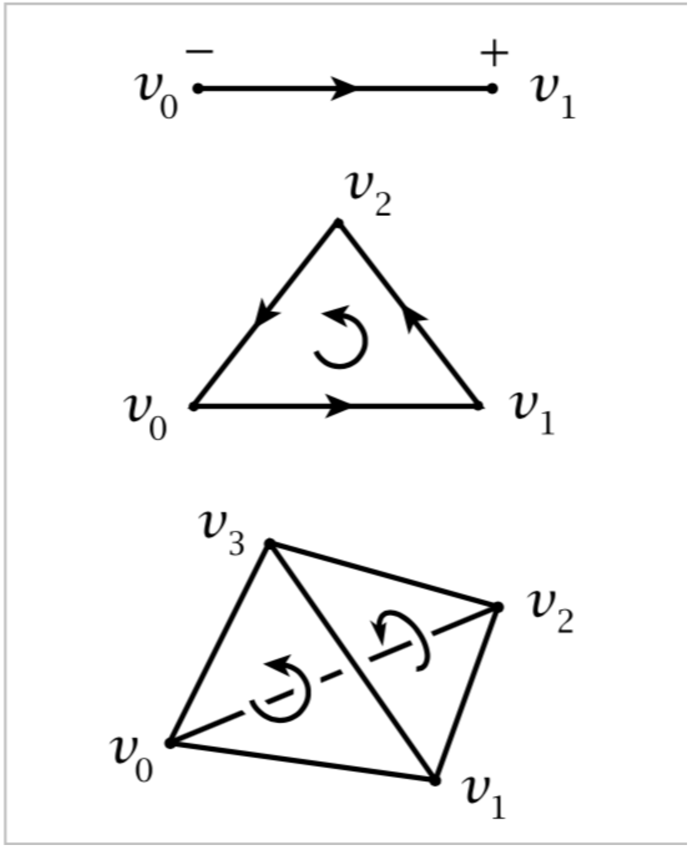
- Cochain: function on simplices $X \to \mathbb{R}$, can be represented as a vector $\mathbb{R}^{|V|}$, $\mathbb{R}^{|E|}$, $\mathbb{R}^{|T|}$, etc.

$$f\left(\left[i_{p(0)}, \dots, i_{p(k)}\right]\right) = \mathrm{sign}(p) f([i_0, \dots, i_k])$$

$i_{p(0)}, \dots, i_{p(k)}$ is a permutation of $i_0, \dots, i_k$,

Odd permutation $\mathbf{sign}(p) = -1$, $f([1,2]) = -f([2,1])$, $f([1,2,3]) = -f([2,1,3])$

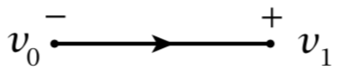
Even permutation $\mathbf{sign}(p) = 1$, $f([1,2,3]) = f([3,1,2])$



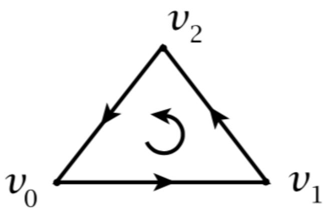
kth boundary operator $\partial_k : C_k(X) \ \rightarrow C_{k-1}(X)$ i the linear map that takes a k-chain $\in C_k$ to a $(k-1)$-chain $\in C_{k-1}$defined by

$$(\partial_k)[i_0, i_1, \ldots, i_k]:$$

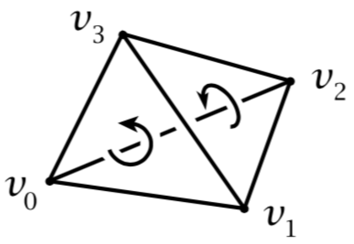
$$= \sum_{j=0}^{k} (-1)^j [i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k]$$



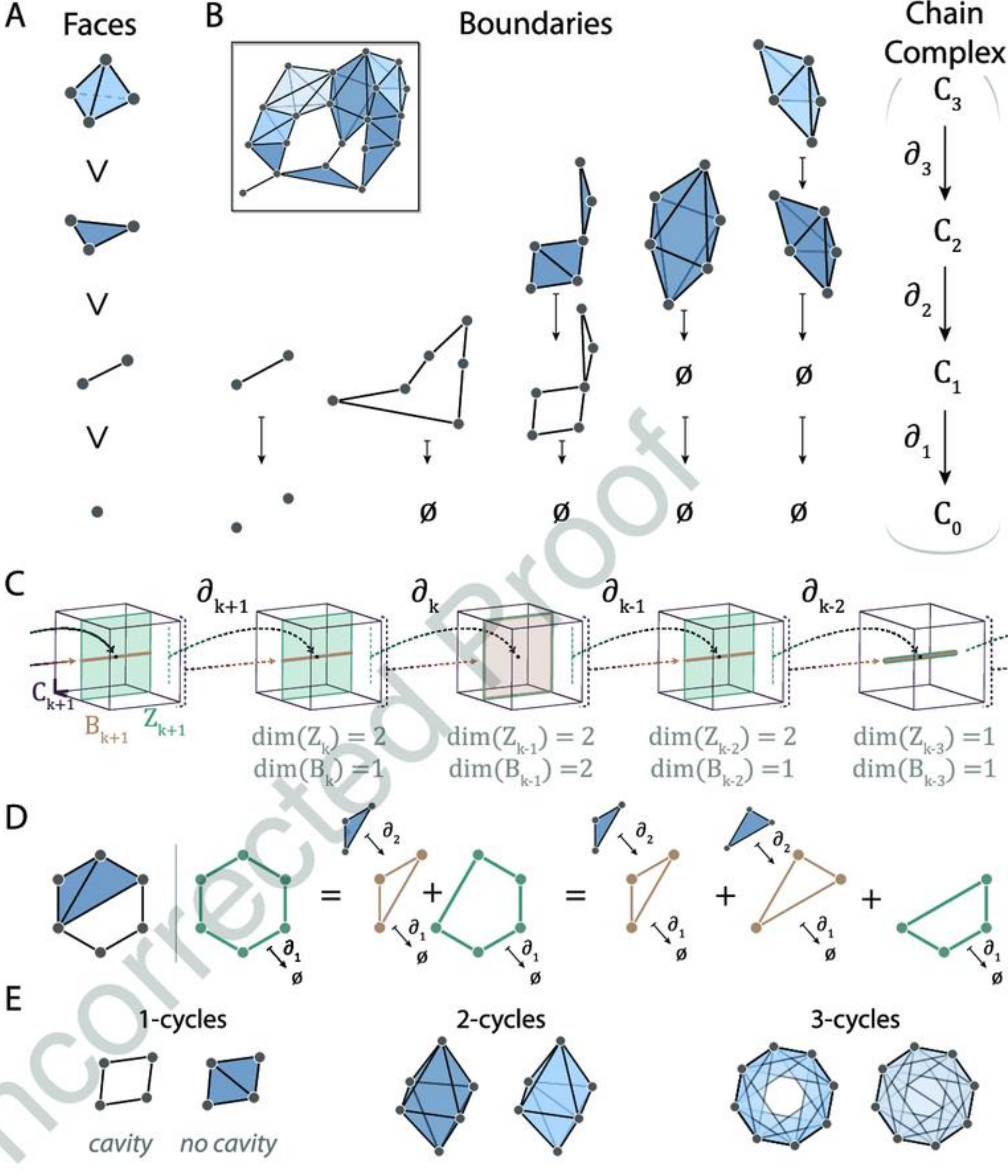
$\partial[v_0, v_1] = [v_1] - [v_0]$



$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$



$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$



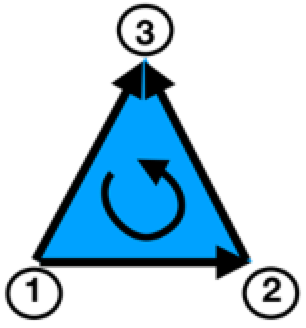
Boundary operator

Incidence matrices

(a)

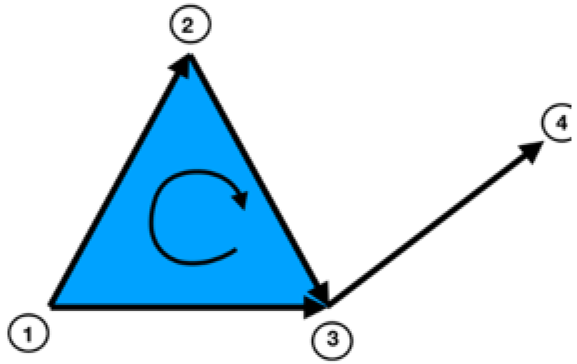
$$\partial_1[1,2] = [2] - [1]\,.$$

(b)



$$\partial_2[1,2,3] = [2,3] - [1,3] + [1,2]\,.$$

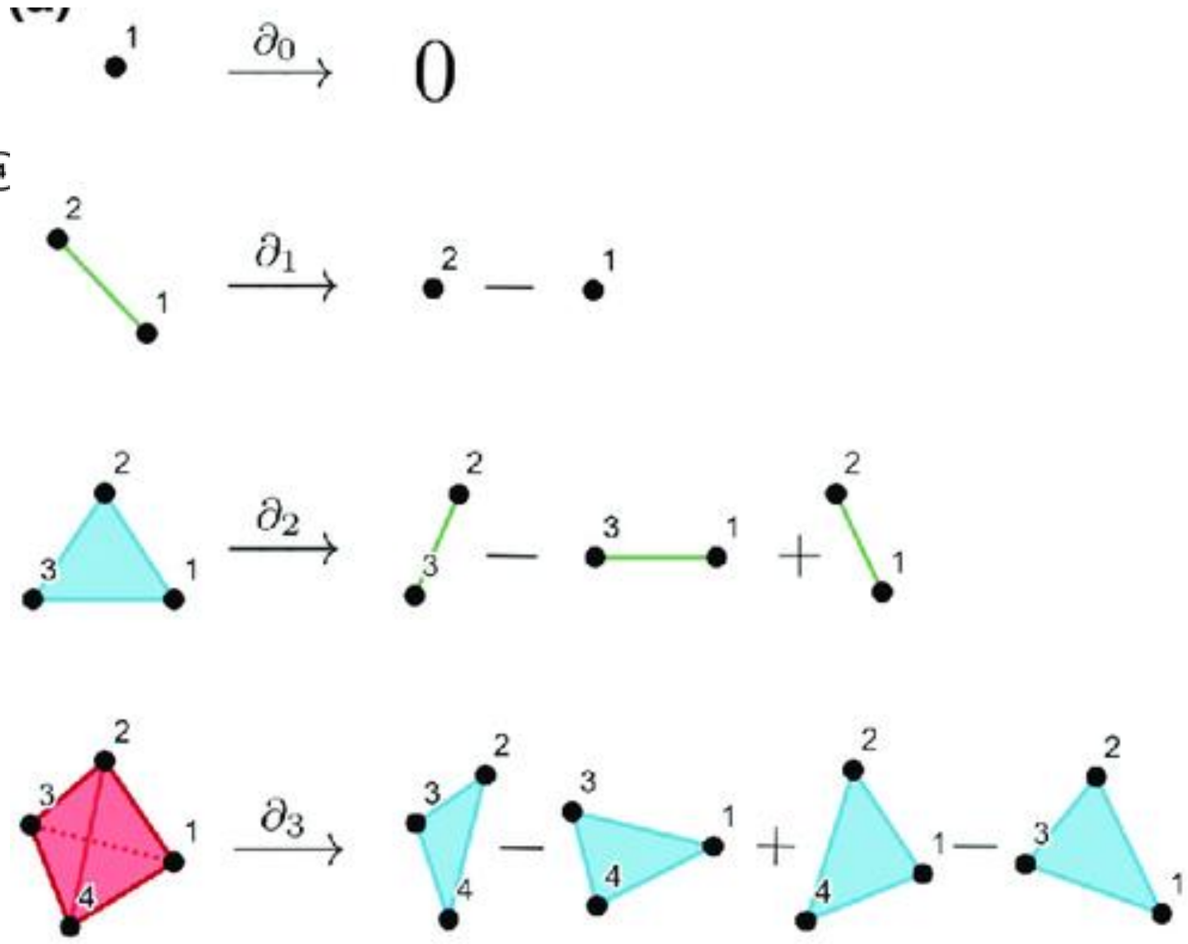
(c)



(d)

$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array},$$

$$\mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [1,3] \\ [2,3] \\ [3,4] \end{array} \begin{array}{c} [1,2,3] \\ 1 \\ -1 \\ 1 \\ 0 \end{array}.$$



- Chain complex $(A_\blacksquare, d_\blacksquare)$: a sequence of abelian groups or modules (chains) $\ldots, A_0, A_1, A_2, A_3, A_4, \ldots$ connected by homomorphism (boundary operators) $d_n: A_n \to A_{n-1}$ (decrease dimension) $(d_n \circ d_{n+1} = 0)$

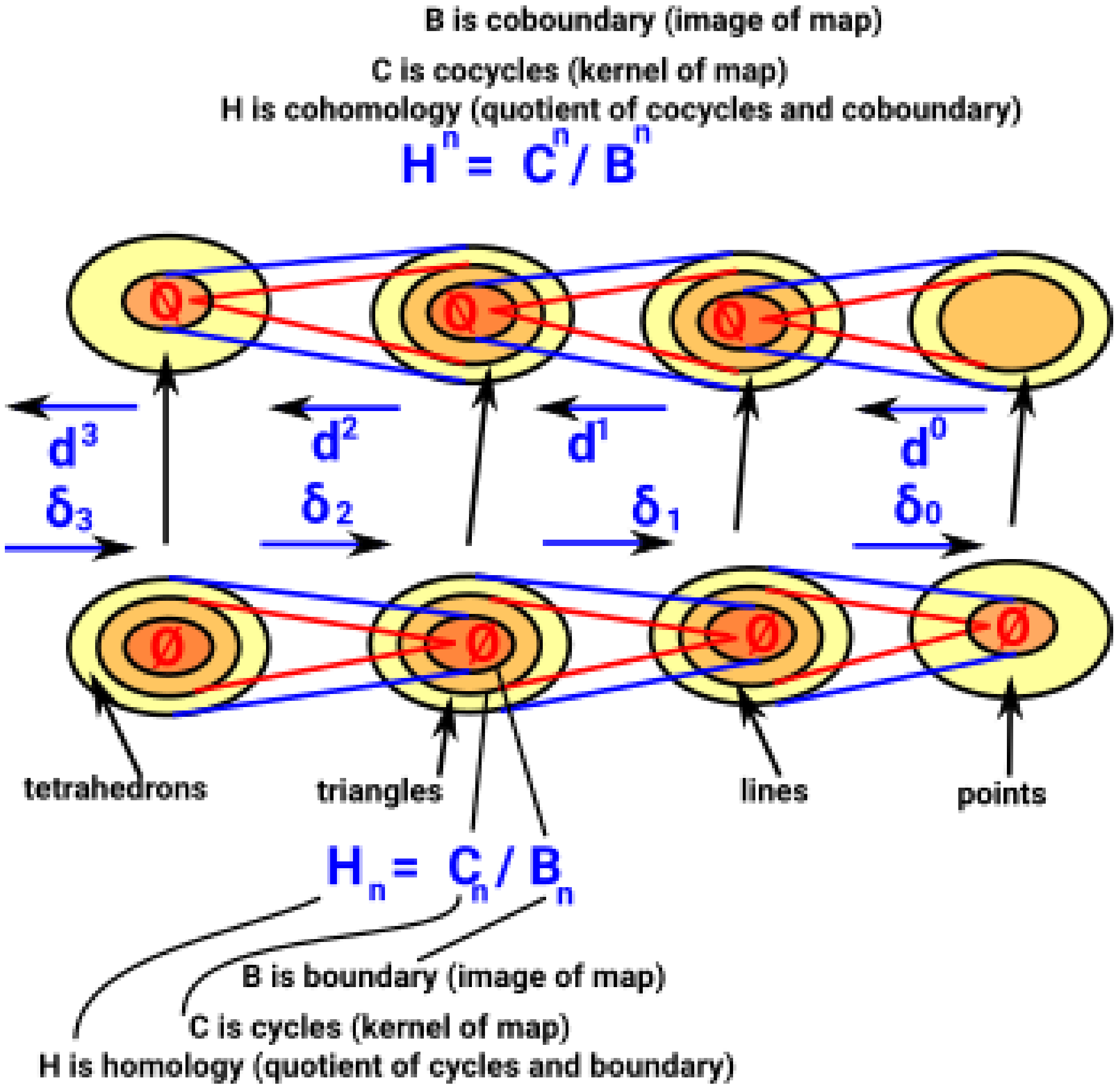
$$\ldots \overset{d_0}{\overbrace{\leftarrow}} A_0 \overset{d_1}{\overbrace{\leftarrow}} A_1 \overset{d_2}{\overbrace{\leftarrow}} A_2 \overset{d_3}{\overbrace{\leftarrow}} A_3 \overset{d_4}{\overbrace{\leftarrow}} A_4 \overset{d_5}{\overbrace{\leftarrow}} \ldots$$

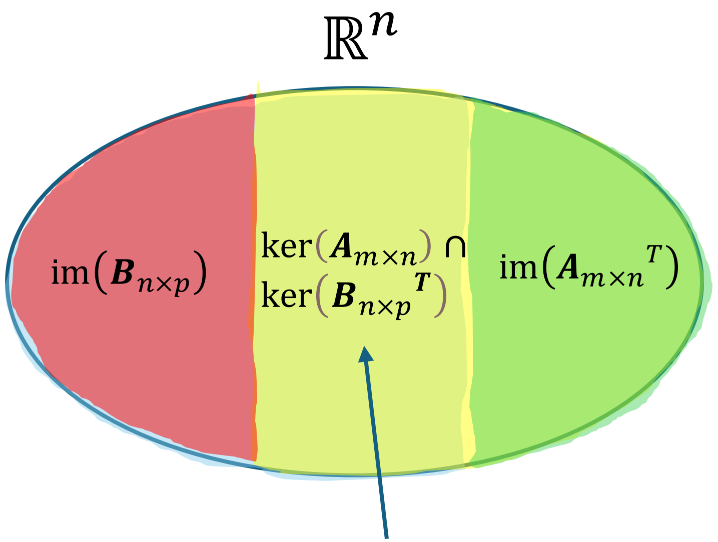
- Cochain complex $(A^\blacksquare, d^\blacksquare)$: a sequence of abelian groups or modules (cochains) $\ldots, A^0, A^1, A^2, A^3, A^4, \ldots$ connected by homomorphism $d^n : A^n \to A^{n+1}$ (increase dimension) $(d^{n+1} \circ d^n = 0)$

$$\ldots \overset{d^{-1}}{\overbrace{\rightarrow}} A^0 \overset{d^0}{\overbrace{\rightarrow}} A^1 \overset{d^1}{\overbrace{\rightarrow}} A^2 \overset{d^2}{\overbrace{\rightarrow}} A^3 \overset{d^3}{\overbrace{\rightarrow}} A^4 \overset{d^4}{\overbrace{\rightarrow}} ..$$

The boundary of a boundary is 0.

- n Boundary operator: d_n
- n Boundaries B_n : $\mathrm{im}(d_n)$
- n Cycles C_n : $\ker(d_{n-1})$
- n homology group:
  $H_n = C_n/B_n =: \ker(d_{n-1})/\mathrm{im}(d_n)$
- n Coboundary operator: d^n
- n Coboundaries B^n: $\mathrm{im}(d^{n-1})$
- n Cocycles C^n: $\ker(d^n)$
- n cohomology group:
  $H^n = C^n/B^n =: \ker(d^n)/\mathrm{im}(d^{n-1})$





$$A_{n-1} \xleftarrow{d_{n-1}} A_n \xleftarrow{d_n} A_{n+1}$$

- n Boundary operator: d_n
- n Boundaries B_n : $\mathrm{im}(d_n)$
- n Cycles C_n : $\ker(d_{n-1})$
- n homology group:
  $H_n = C_n/B_n =: \ker(d_{n-1})/\mathrm{im}(d_n)$

$$\ker(\boldsymbol{A}_{m\times n}) \cap \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})$$
$$= \ker(\boldsymbol{A}_{m\times n})/\mathrm{im}(\boldsymbol{B}_{n\times p})(homology)$$
$$= \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})/\mathrm{im}(\boldsymbol{A}_{m\times n}{}^{T})(cohomology)$$
$$= \ker(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})$$

$$A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1}$$

- n Coboundary operator: d^n
- n Coboundaries B^n: $\mathrm{im}(d^{n-1})$
- n Cocycles C^n: $\ker(d^n)$
- n cohomology group:
  $H^n = C^n/B^n =: \ker(d^n)/\mathrm{im}(d^{n-1})$

$$\mathbb{R}^n = \mathrm{im}(\boldsymbol{B}_{n\times p}) \oplus (\ker(\boldsymbol{A}_{m\times n}) \cap \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})) \oplus \mathrm{im}(\boldsymbol{A}_{m\times n}{}^{T})$$
$$\mathbb{R}^n = \mathrm{im}(\boldsymbol{B}_{n\times p}) \oplus \ker(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) \oplus \mathrm{im}(\boldsymbol{A}_{m\times n}{}^{T})$$
$$\mathbb{R}_n^k = n \text{ boundaries} \oplus (n \text{ homology/cohomology group}) \oplus n \text{ coboundaries}$$

$$C_{k-1} \overset{\overbrace{\boldsymbol{A}_{m\times n}}}{\leftarrow} C_k \overset{\overbrace{\boldsymbol{B}_{n\times p}}}{\leftarrow} C_{k+1}$$

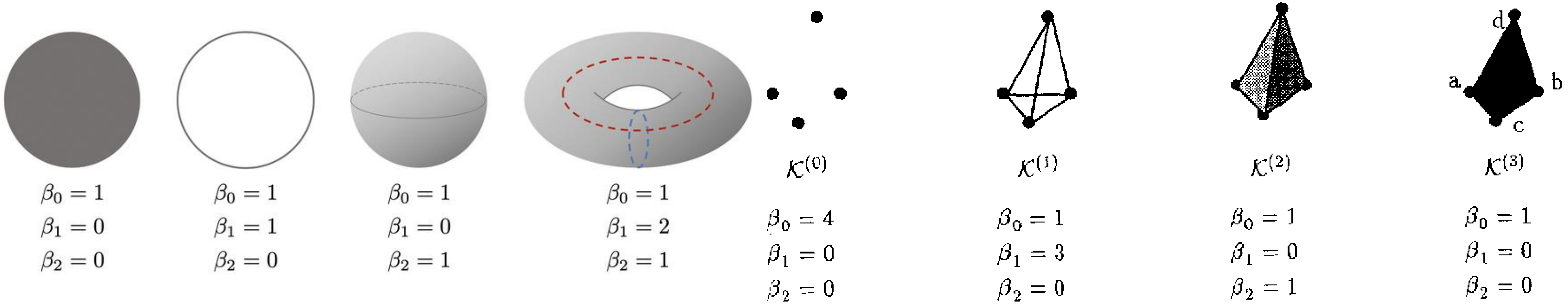
- A hole is a chain with zero boundary (a cycle) but not the boundary of anything else (this cycle is not filled) $= \ker(\boldsymbol{A}_{m\times n})/\mathrm{im}(\boldsymbol{B}_{n\times p})$
- $\text{kth homology} = H_k(X) = \ker(\boldsymbol{A}_{m\times n}) \cap \ker(\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}) = \ker(\boldsymbol{A}_{m\times n})/\mathrm{im}(\boldsymbol{B}_{n\times p}) = \ker(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})$
- $\dim\left(\ker(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}})\right) = \text{Betti number} = \text{number of k dimensional holes}$
- $H_0(X) = \ker(d_0^T d_0 + d_1 d_1^T) = \ker(d_1 d_1^T) = \ker(\boldsymbol{L}) = \textbf{connected components}$ (both continuous and discrete/in graphs)

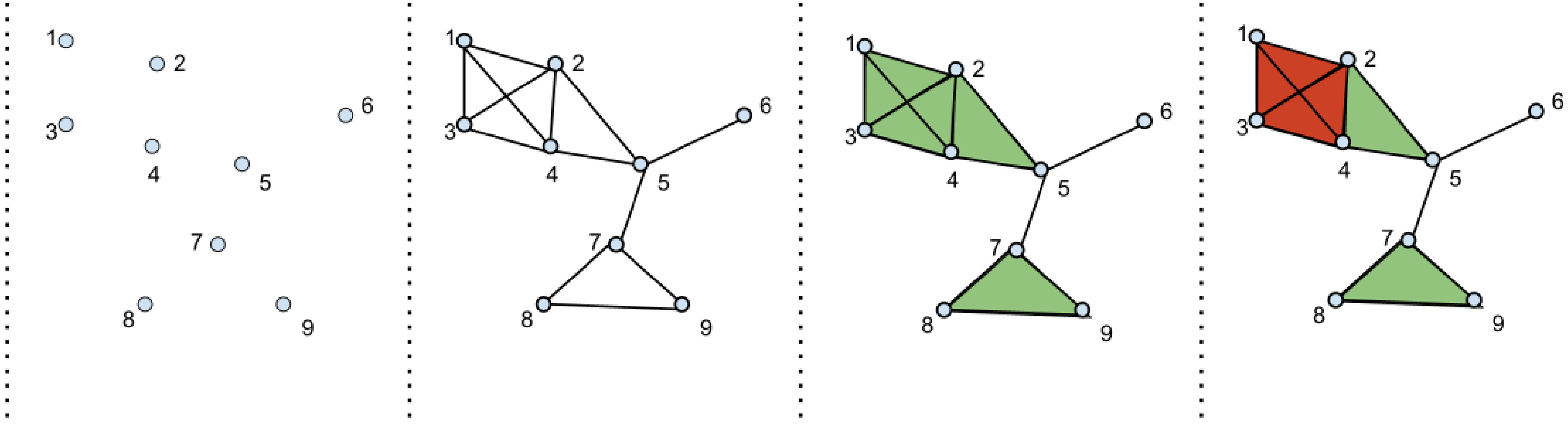
$$\boldsymbol{L}: \text{Graph Laplacian}$$

- $H_1(X) = \ker(d_1^T d_1 + d_2 d_2^T) = \ker(\boldsymbol{H}) = \textbf{non} - \textbf{contractable cycles}$ (in graphs: cycles longer than 3, cycles of length 3 are the boundaries of those triangles encircled by them, cycles longer than 3 are not boundaries of anything)

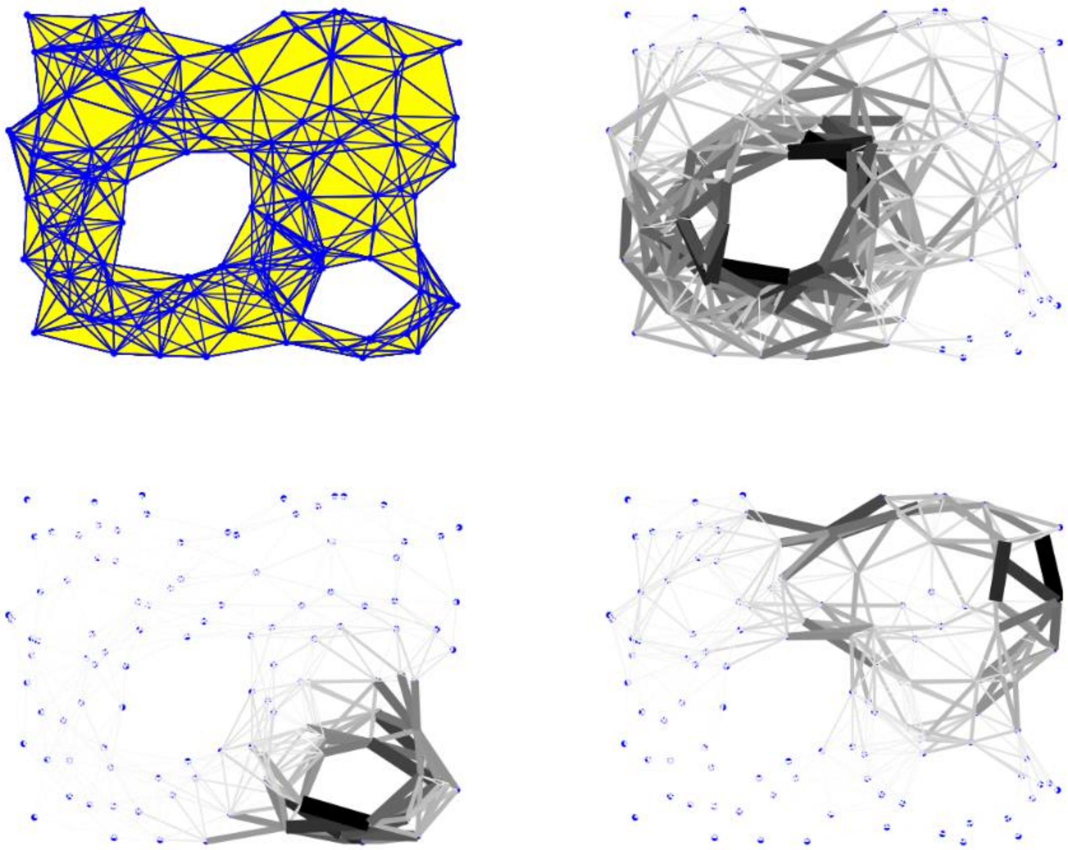
$$\boldsymbol{H}: \text{Graph Helmholtzian}$$

- $H_2(X) = \ker(d_2^T d_2 + d_3 d_3^T) = \textbf{3} - \textbf{dimensional voids}$ (in graphs: closed surface made up of more than four triangles, can not encircle a tetrahedron)





β_2	0	0	1	0
β_1	0	5	0	0
β_0	9	1	1	1



$$\dim\left(\ker\left(d_k^T d_k + d_k d_k^T\right)\right) = \text{number of k dim holes}$$

$$\ker\left(d_k^T d_k + d_k d_k^T\right) = \text{nullspace} =$$
eigenvectors corresponding to 0 eigenvalues

eigenvectors corresponding to small but non − zero eigenvalues indicate what parts of the complex are close to becoming holes. fragilities of a simplicial complex.

Spectral clustering: looking for the eigenvectors corresponding to the smallest k eigenvalues of the graph Laplacian (detect the different almost connected components)

Hodge Decomposition in Graphs

Divergence, Gradient, Curl

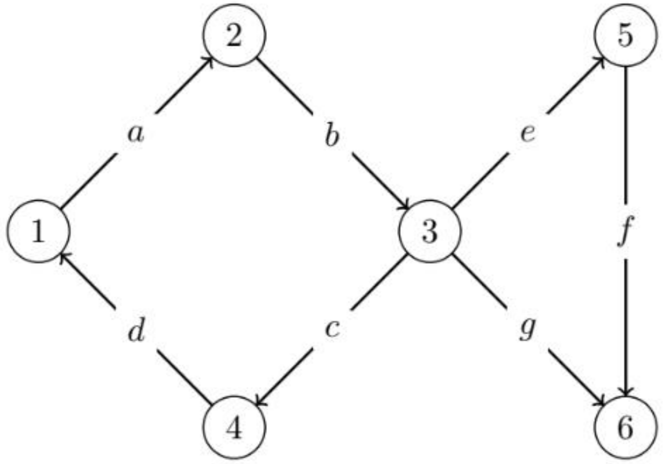
- $\text{grad}\boldsymbol{v}_{(i,j)} = v_j - v_i$, $\quad \text{grad}\boldsymbol{f}_{\mathbf{1}_{|V|\times 1}} = \boldsymbol{g}_{\mathbf{2}_{|E|\times 1}}$
- $\text{curl}\boldsymbol{e}_{(i,j,k)} = e_{(i,j)} + e_{(j,k)} + e_{(k,i)}$, $\quad \text{curl}\boldsymbol{f}_{\mathbf{2}_{|E|\times 1}} = \boldsymbol{g}_{\mathbf{3}_{|T|\times 1}}$
- $\text{div}\boldsymbol{e}_{(i)} = \sum e_{(i,j)}$, $\quad \text{div}\boldsymbol{f}_{\mathbf{2}_{|E|\times 1}} = \boldsymbol{g}_{\mathbf{1}_{|V|\times 1}}$
- $\text{curl} \circ \text{grad} = 0 \Leftrightarrow -\text{div} \circ \text{curl}^* = 0$
- $-\text{div} \circ \text{grad} = \text{graph Laplacian}$

Continuous calculus in vector field $\Longrightarrow$ Discrete calculus in a graph

$\boldsymbol{f}_{\mathbf{1}_{|V|\times 1}}$: potential function on nodes (0-simplicies)

$\boldsymbol{f}_{\mathbf{2}_{|E|\times 1}}$: flow function on edges (1-simplicies)

$\boldsymbol{f}_{\mathbf{3}_{|T|\times 1}}$: local curl function on triangles (2-simplicies)



Edge-vertex incident matrix= **grad** $A_1 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline a & -1 & 1 & 0 & 0 & 0 & 0 \\ b & 0 & -1 & 1 & 0 & 0 & 0 \\ c & 0 & 0 & -1 & 1 & 0 & 0 \\ d & 1 & 0 & 0 & -1 & 0 & 0 \\ e & 0 & 0 & -1 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 0 & -1 & 1 \\ g & 0 & 0 & -1 & 0 & 0 & 1 \end{array}$

$-\mathbf{div} = \mathbf{grad}^* = \boldsymbol{A_1^T}$

(Very sparse)

Graph Laplacian $L_1 = A_1^* A_1 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & -1 & 0 & -1 & 0 & 0 \\ 2 & -1 & 2 & -1 & 0 & 0 & 0 \\ 3 & 0 & -1 & 4 & -1 & -1 & -1 \\ 4 & -1 & 0 & -1 & 2 & 0 & 0 \\ 5 & 0 & 0 & -1 & 0 & 2 & -1 \\ 6 & 0 & 0 & -1 & 0 & -1 & 2 \end{array}$

$-\mathbf{div} \circ \mathbf{grad}$

(Very sparse)

Triangle-edge incident matrix= **curl** $B_1 = \begin{array}{c|ccccccc} & a & b & c & d & d & e & f \\ \hline T & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{array}$

$\mathbf{curl}^* = \boldsymbol{B_1^T}$

(Very sparse)

Graph Helmholtzian $H_1 = A_1 A_1^* + B_1^* B_1 = \begin{array}{c|ccccccc} & a & b & c & d & d & e & f \\ \hline a & 2 & -1 & 0 & -1 & 0 & 0 & 0 \\ b & -1 & 2 & -1 & 0 & -1 & 0 & -1 \\ c & 0 & -1 & 2 & -1 & 1 & 0 & 1 \\ d & -1 & 0 & -1 & 2 & 0 & 0 & 0 \\ e & 0 & -1 & 1 & 0 & 3 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ g & 0 & -1 & 1 & 0 & 0 & 0 & 3 \end{array}$

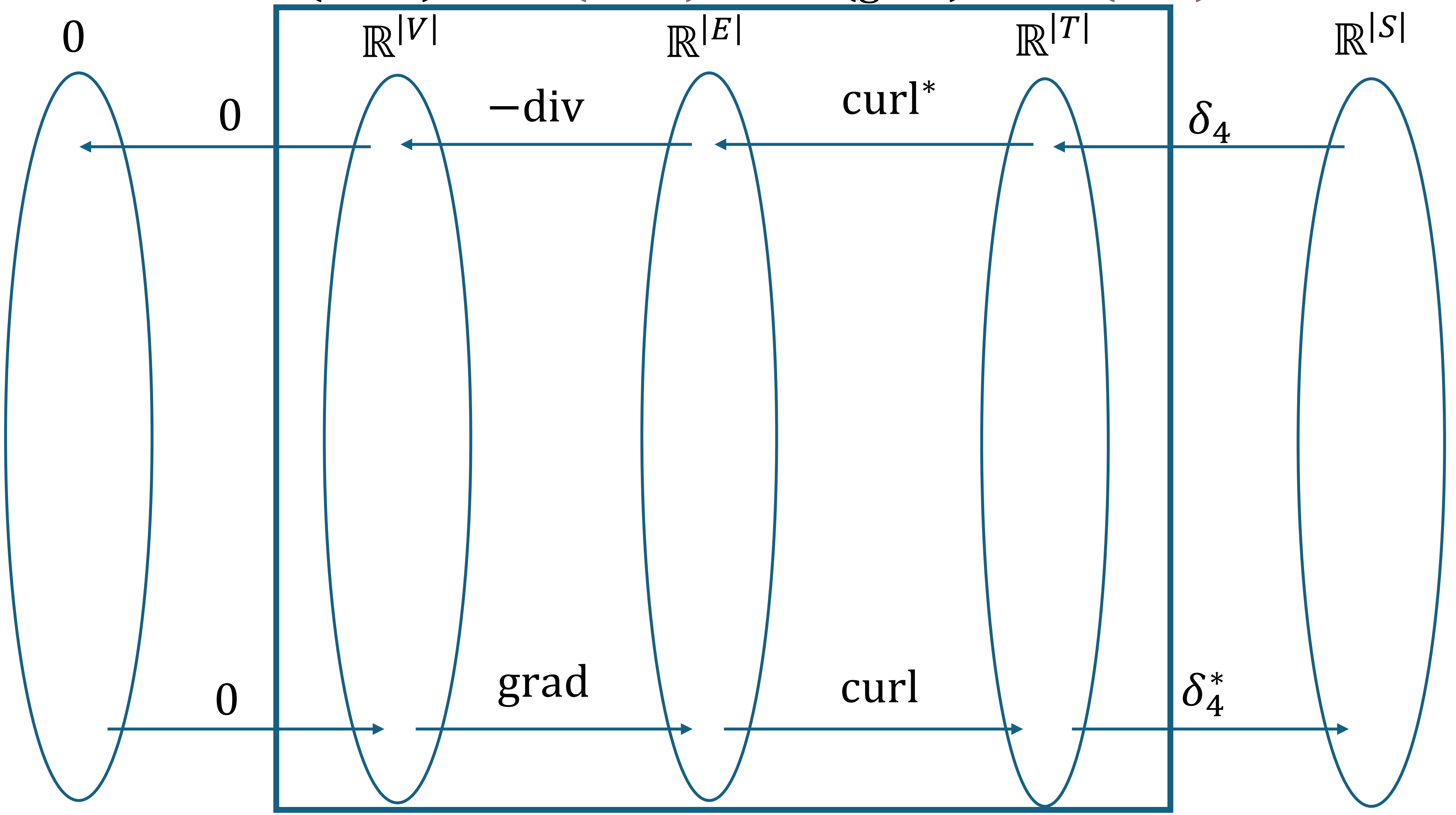
$-\mathbf{grad} \circ \mathbf{div} + \mathbf{curl}^* \circ \mathbf{curl}$

(Very sparse)

$$
T \quad \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & d & e & f \end{array} \\ \left[\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{array} \right] \end{array}
\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \left[\begin{array}{cccccc} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array} \right] \end{array}
= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]
$$

$$-\mathrm{div} \circ \mathrm{curl}^* = 0 \iff \mathrm{curl} \circ \mathrm{grad} = 0$$

$$\mathrm{im}(\mathrm{curl}^*) \subseteq \ker(-\mathrm{div}) \iff \mathrm{im}(\mathrm{grad}) \subseteq \ker(\mathrm{curl})$$



$$\mathrm{im}(-\mathrm{div}) \oplus \ker(\mathrm{grad}) = \mathbb{R}^{|V|}$$

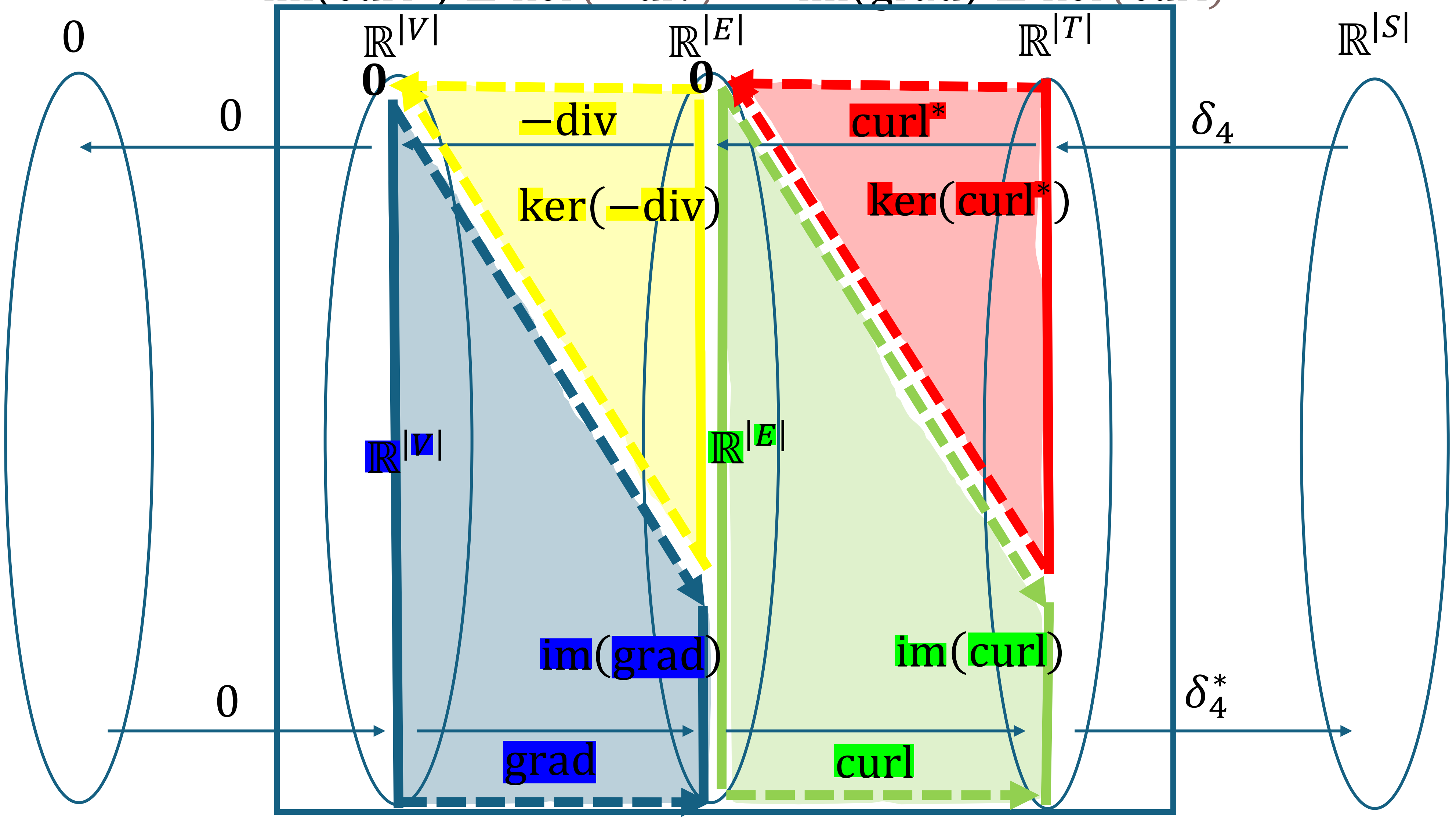
$$\mathrm{im}(\mathrm{grad}) \oplus \ker(-\mathrm{div}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl}^*) \oplus \ker(\mathrm{curl}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl}) \oplus \ker(\mathrm{curl}^*) = \mathbb{R}^{|T|}$$

$$-\mathrm{div}\circ\mathrm{curl}^* = 0 \Leftrightarrow \mathrm{curl}\circ\mathrm{grad} = 0$$

$$\mathrm{im}(\mathrm{curl}^*) \subseteq \ker(-\mathrm{div}) \Leftrightarrow \mathrm{im}(\mathrm{grad}) \subseteq \ker(\mathrm{curl})$$



$$\mathrm{im}(-\mathrm{div})\oplus\ker(\mathrm{grad}) = \mathbb{R}^{|V|}$$

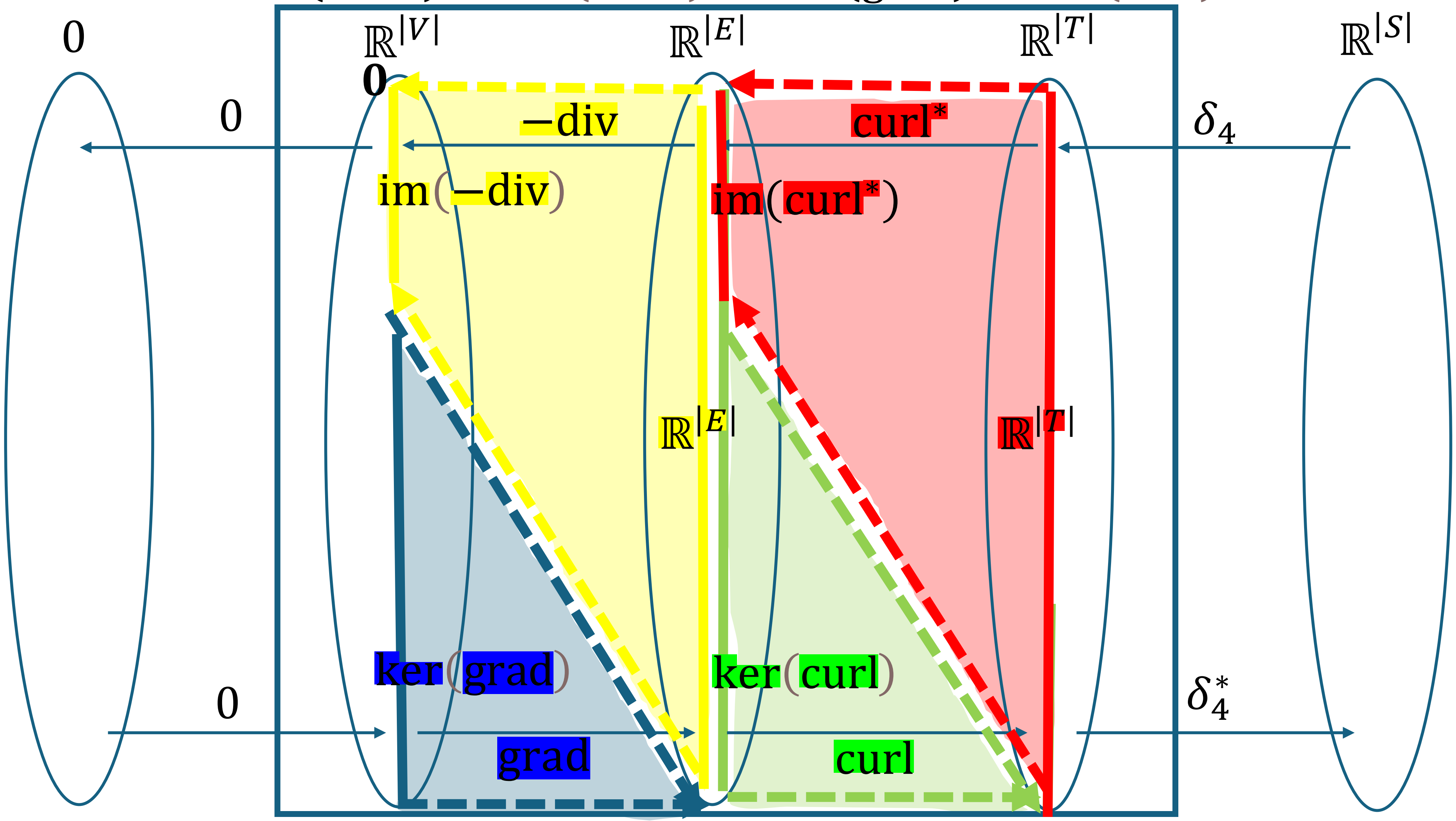
$$\mathrm{im}(\mathrm{grad})\oplus\ker(-\mathrm{div}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl}^*)\oplus\ker(\mathrm{curl}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl})\oplus\ker(\mathrm{curl}^*) = \mathbb{R}^{|T|}$$

$$-\mathrm{div} \circ \mathrm{curl}^* = 0 \Leftrightarrow \mathrm{curl} \circ \mathrm{grad} = 0$$

$$\mathrm{im}(\mathrm{curl}^*) \subseteq \ker(-\mathrm{div}) \Leftrightarrow \mathrm{im}(\mathrm{grad}) \subseteq \ker(\mathrm{curl})$$

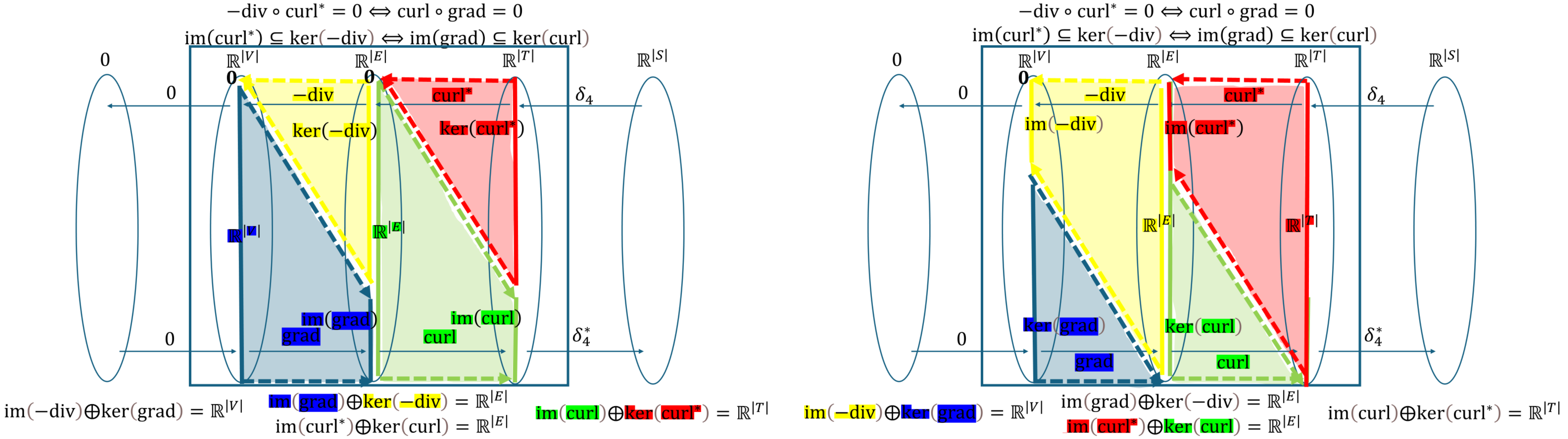


$$\mathrm{im}(-\mathrm{div}) \oplus \ker(\mathrm{grad}) = \mathbb{R}^{|V|}$$

$$\mathrm{im}(\mathrm{grad}) \oplus \ker(-\mathrm{div}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl}^*) \oplus \ker(\mathrm{curl}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl}) \oplus \ker(\mathrm{curl}^*) = \mathbb{R}^{|T|}$$

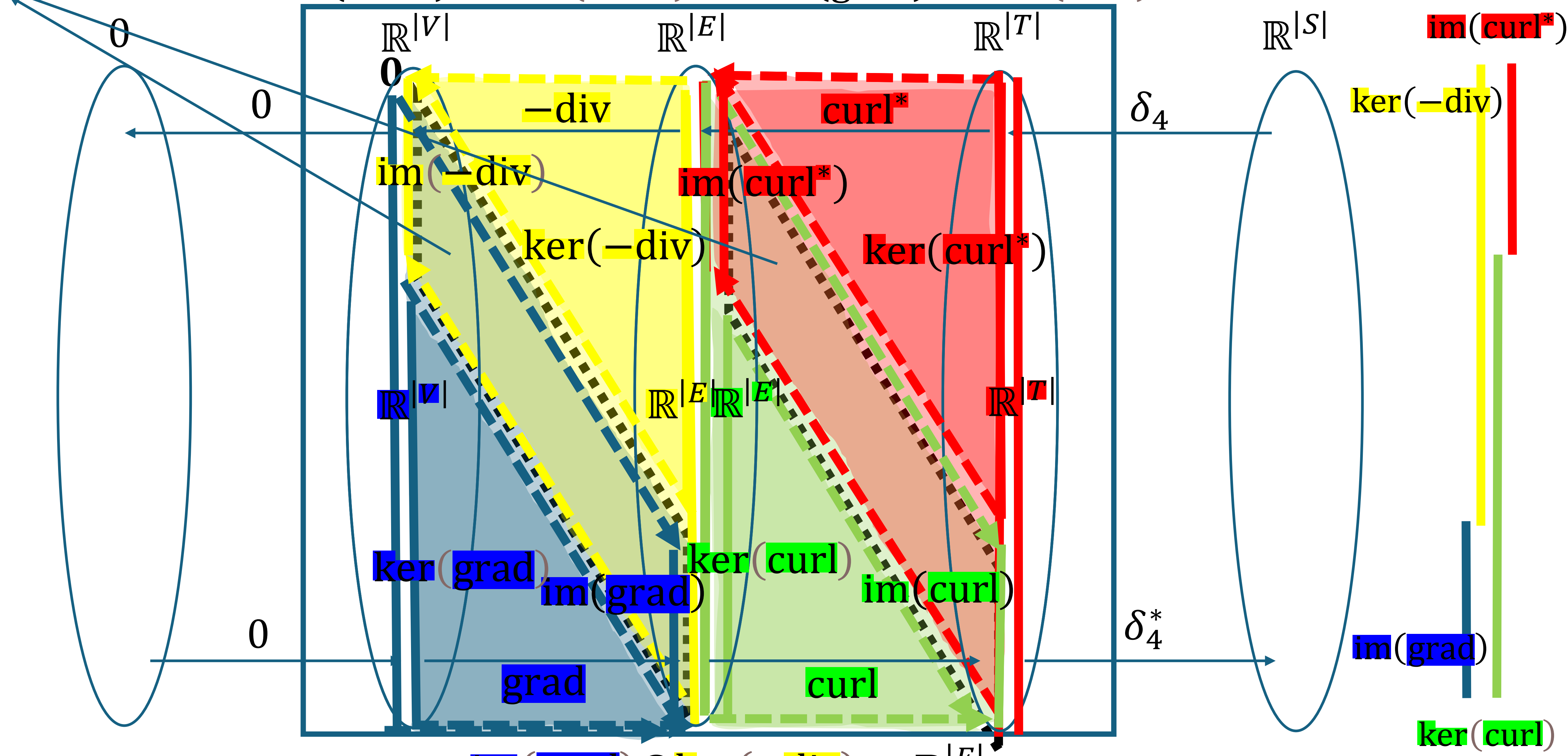


−div ∘ curl* = 0 ⟺ curl ∘ grad = 0
im(curl*) ⊆ ker(−div) ⟺ im(grad) ⊆ ker(curl)
0
ℝ^|V|
ℝ^|E|
ℝ^|T|
ℝ^|S|
−div
curl*
δ4
ker(−div)
ker(curl*)
im(grad)
grad
im(curl)
curl
δ4*
im(−div)⊕ker(grad) = ℝ^|V|
im(grad)⊕ker(−div) = ℝ^|E|
im(curl*)⊕ker(curl) = ℝ^|E|
im(curl)⊕ker(curl*) = ℝ^|T|
im(−div)
im(curl*)
ker(grad)
ker(curl)

Bijective isomorphism

$$-\mathrm{div} \circ \mathrm{curl}^* = 0 \iff \mathrm{curl} \circ \mathrm{grad} = 0$$

$$\mathrm{im}(\mathrm{curl}^*) \subseteq \ker(-\mathrm{div}) \iff \mathrm{im}(\mathrm{grad}) \subseteq \ker(\mathrm{curl})$$

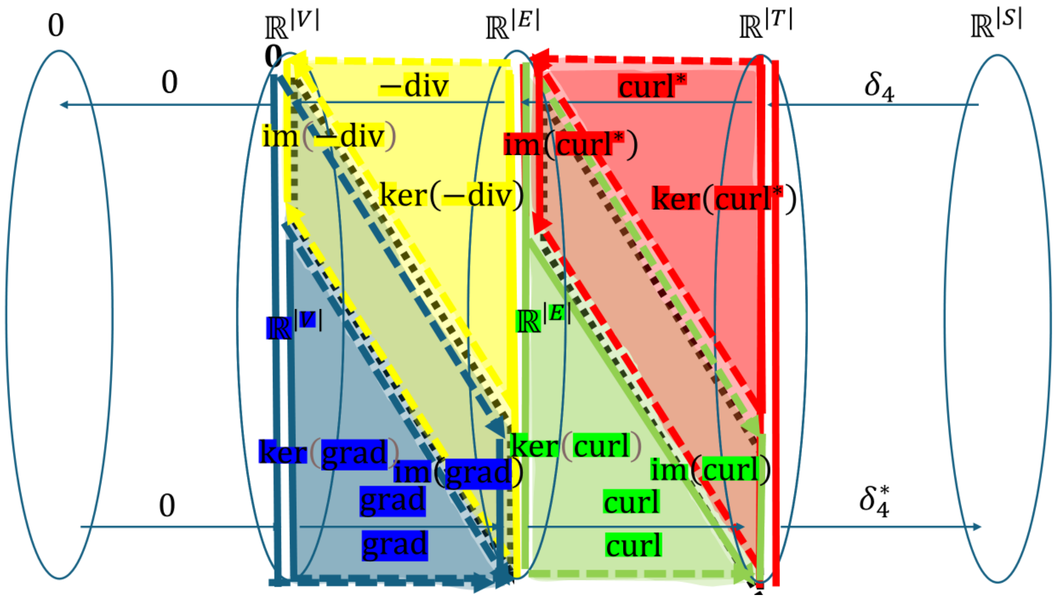


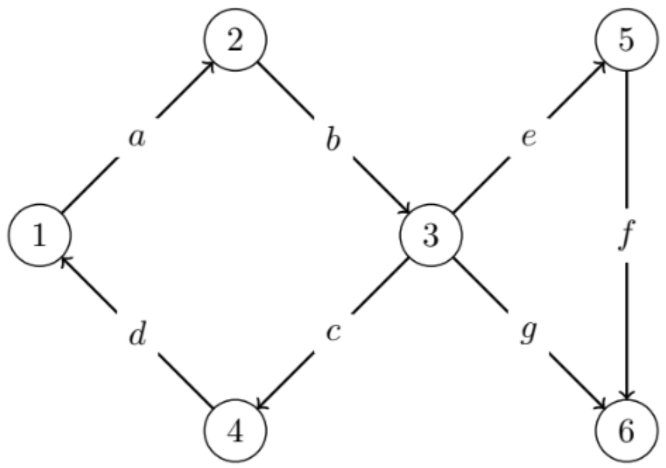
$$\mathrm{im}(-\mathrm{div}) \oplus \ker(\mathrm{grad}) = \mathbb{R}^{|V|}$$

$$\mathrm{im}(\mathrm{grad}) \oplus \ker(-\mathrm{div}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl}^*) \oplus \ker(\mathrm{curl}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl}) \oplus \ker(\mathrm{curl}^*) = \mathbb{R}^{|T|}$$





$$\ker(-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}) = \ker(-\mathrm{div}) \cap \ker(\mathrm{curl})$$

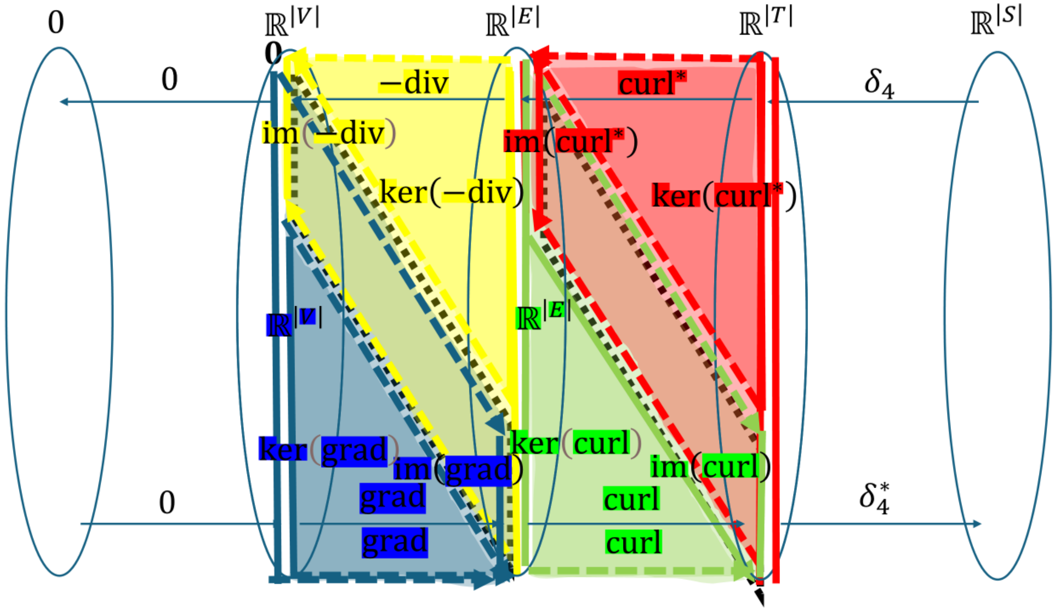
$$\mathbb{R}^{|E|} = \mathrm{im}(\mathrm{curl}^*) \oplus \ker(-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}) \oplus \mathrm{im}(\mathrm{grad})$$

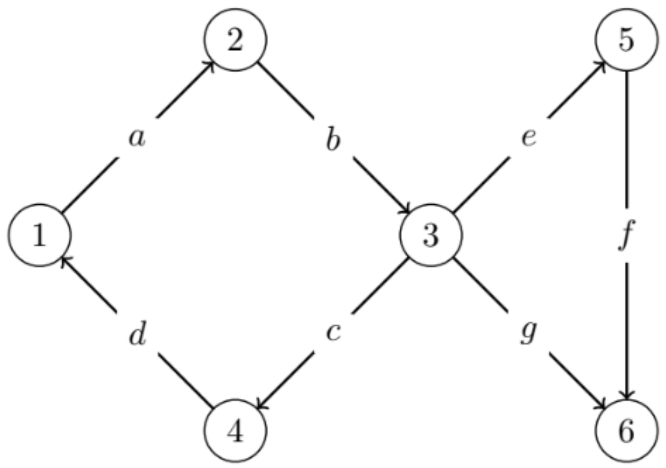
$$\mathrm{im}(\mathrm{grad}) = \text{conservative} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array} \boldsymbol{f}_{|V|\times 1}$$

(The value on every edge is the directed difference between the two end nodes)

$$\mathrm{im}(\mathrm{curl}^*) = \text{vorticity} = T \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & d & e & f \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}^T \end{array} \boldsymbol{f}_{|T|\times 1}$$

(The values on edges in a triad are the same)





$$\ker(-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}) = \ker(-\mathrm{div}) \cap \ker(\mathrm{curl})$$

$$\mathbb{R}^{|E|} = \mathrm{im}(\mathrm{curl}^*) \oplus \big(\ker(-\mathrm{div}) \cap \ker(\mathrm{curl})\big) \oplus \mathrm{im}(\mathrm{grad})$$

$\ker(-\mathrm{div}) = \text{divergence} - \text{free}$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}^T \end{array}$$

$\ker(-\mathrm{div}) = \mathbf{0}$

(The flow into a node is equal to the flow out of a node)

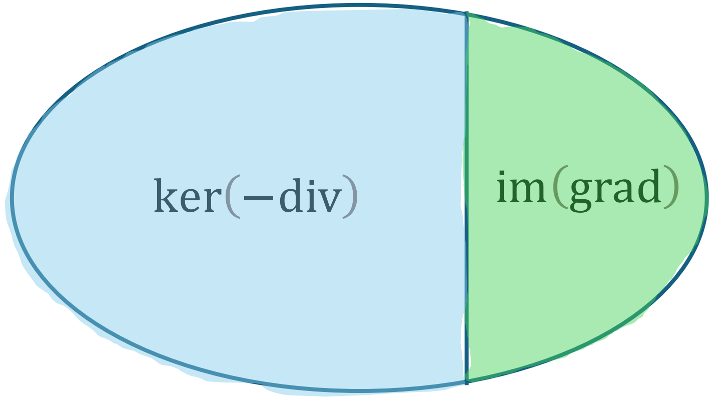
$\ker(\mathrm{curl}) = \text{curl} - \text{free}$

$$T \quad \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & d & e & f \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \end{array}$$

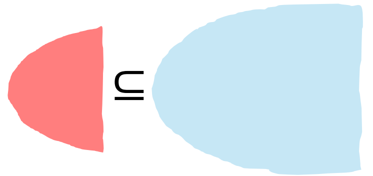
$\ker(\mathrm{curl}) = \mathbf{0}$ (The directed flows on edges in a triad must be added to 0)

$\ker(-\mathrm{div}) \cap \ker(\mathrm{curl}) = \text{divergence} - \text{free and curl} - \text{free}$

$\mathbb{R}^{|E|}$

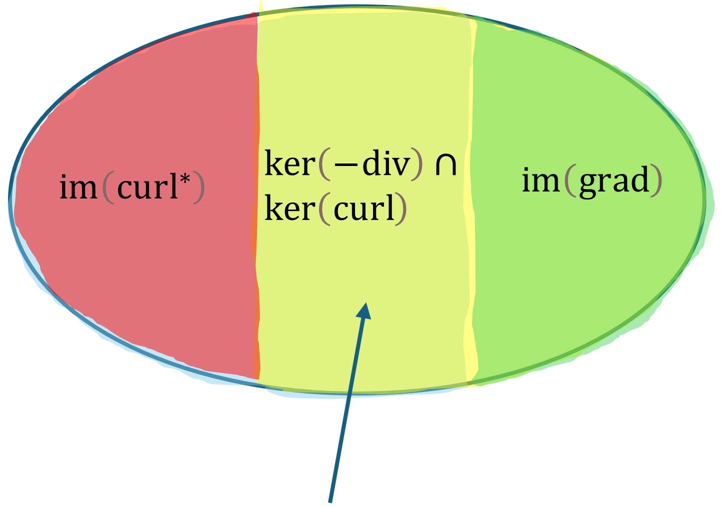


$\mathrm{im}(\mathrm{grad}) \oplus \ker(-\mathrm{div}) = \mathbb{R}^{|E|}$



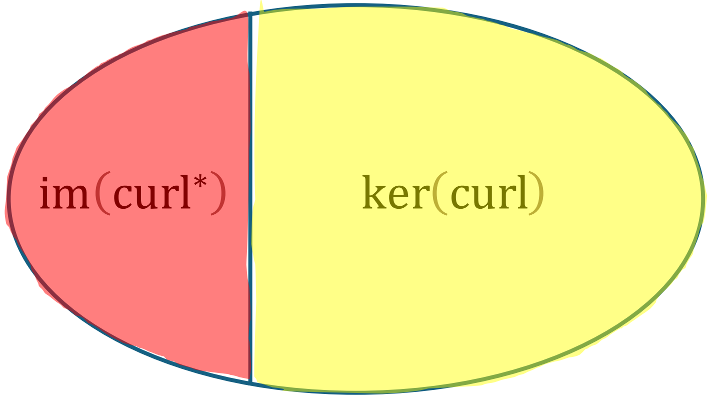
$\mathrm{im}(\mathrm{curl}^*) \subseteq \ker(-\mathrm{div})$

$\mathbb{R}^{|E|}$

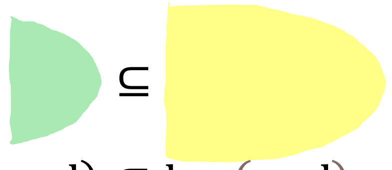


$$\begin{aligned} &\ker(-\mathrm{div}) \cap \ker(\mathrm{curl}) \\ &= \ker(-\mathrm{div}) / \mathrm{im}(\mathrm{curl}^*) \\ &= \ker(-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}) \end{aligned}$$

$\mathbb{R}^{|E|}$



$\mathrm{im}(\mathrm{curl}^*) \oplus \ker(\mathrm{curl}) = \mathbb{R}^{|E|}$



$\mathrm{im}(\mathrm{grad}) \subseteq \ker(\mathrm{curl})$

$$\mathbb{R}^{|E|} = \mathrm{im}(\mathrm{curl}^*) \oplus (\ker(-\mathrm{div}) \cap \ker(\mathrm{curl})) \oplus \mathrm{im}(\mathrm{grad})$$

$$\mathbb{R}^{|E|} = \mathrm{im}(\mathrm{curl}^*) \oplus \ker(-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}) \oplus \mathrm{im}(\mathrm{grad})$$

$$\boldsymbol{x} \in \mathbb{R}^{|E|}, \qquad (-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl})\boldsymbol{x_H} = \boldsymbol{0}, \qquad \boldsymbol{v} \in \mathbb{R}^{|T|}, \qquad \boldsymbol{w} \in \mathbb{R}^{|V|}$$

$$\boldsymbol{x} = \mathrm{grad}\boldsymbol{w} + \boldsymbol{x_H} + \mathrm{curl}^*\boldsymbol{v}, \qquad \langle \mathrm{grad}\boldsymbol{w}, \boldsymbol{x_H} \rangle = \langle \mathrm{curl}^*\boldsymbol{v}, \boldsymbol{x_H} \rangle = \langle \mathrm{grad}\boldsymbol{w}, \mathrm{curl}^*\boldsymbol{v} \rangle = \boldsymbol{0}$$

$$\mathbb{R}^{|E|} = \mathrm{im}(\mathrm{curl}^*) \oplus (\ker(-\mathrm{div}) \cap \ker(\mathrm{curl})) \oplus \mathrm{im}(\mathrm{grad})$$

$$\mathbb{R}^{|E|} = \text{vorticity} \oplus (\text{solenoidal and irrotational}) \oplus \text{conservative}$$

$$\mathbb{R}^{|E|} = \mathrm{im}(\mathrm{curl}^*) \oplus \ker\Big(\underbrace{-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}}_{\text{Graph Helmholtzian}}\Big) \oplus \mathrm{im}(\mathrm{grad})$$

$$L^2_\wedge(E) = \overbrace{\mathrm{im}(\mathrm{curl}^*) \oplus \underbrace{\ker(\Delta_1)}^{\ker(\mathrm{div})} \oplus \mathrm{im}(\mathrm{grad})}_{\ker(\mathrm{curl})}, \qquad \begin{aligned} \ker(\Delta_1) &= \ker(\mathrm{curl}) \cap \ker(\mathrm{div}), \\ \mathrm{im}(\Delta_1) &= \mathrm{im}(\mathrm{curl}^*) \oplus \mathrm{im}(\mathrm{grad}). \end{aligned}$$

$$\text{Edge Flow} = \overbrace{\text{vorticity} \oplus \underbrace{\begin{matrix}\text{solenoidal}\\ \text{irrotational}\end{matrix}}^{\text{solenoidal}} \oplus \text{conservative}}_{\text{irrotational}}$$

$$\text{Edge Flow} = \overbrace{\begin{matrix}\text{divergence} - \text{free but}\\ \text{not curl} - \text{free}\end{matrix} \oplus \underbrace{\begin{matrix}\text{divergence} - \text{free}\\ \text{and curl} - \text{free}\end{matrix}}^{\text{divergence} - \text{free}} \oplus \begin{matrix}\text{curl} - \text{free but}\\ \text{not divergence} - \text{free}\end{matrix}}_{\text{curl} - \text{free}}$$

Hodge Decomposition $\mathbb{R}^{|E|} = \boxed{\mathrm{im}(\mathrm{curl}^*) \oplus \ker\left(\underbrace{-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}}_{\text{Graph Helmholtzian}}\right)} \oplus \boxed{\mathrm{im}(\mathrm{grad})}$

solenoidal

Edge Flow $=$ vorticity $\oplus$ solenoidal irrotational $\oplus$ conservative

irrotational

divergence − free

Edge Flow $=$ divergence − free but not curl − free $\oplus$ divergence − free and curl − free $\oplus$ curl − free but not divergence − free

curl − free

Helmholtz Decomposition

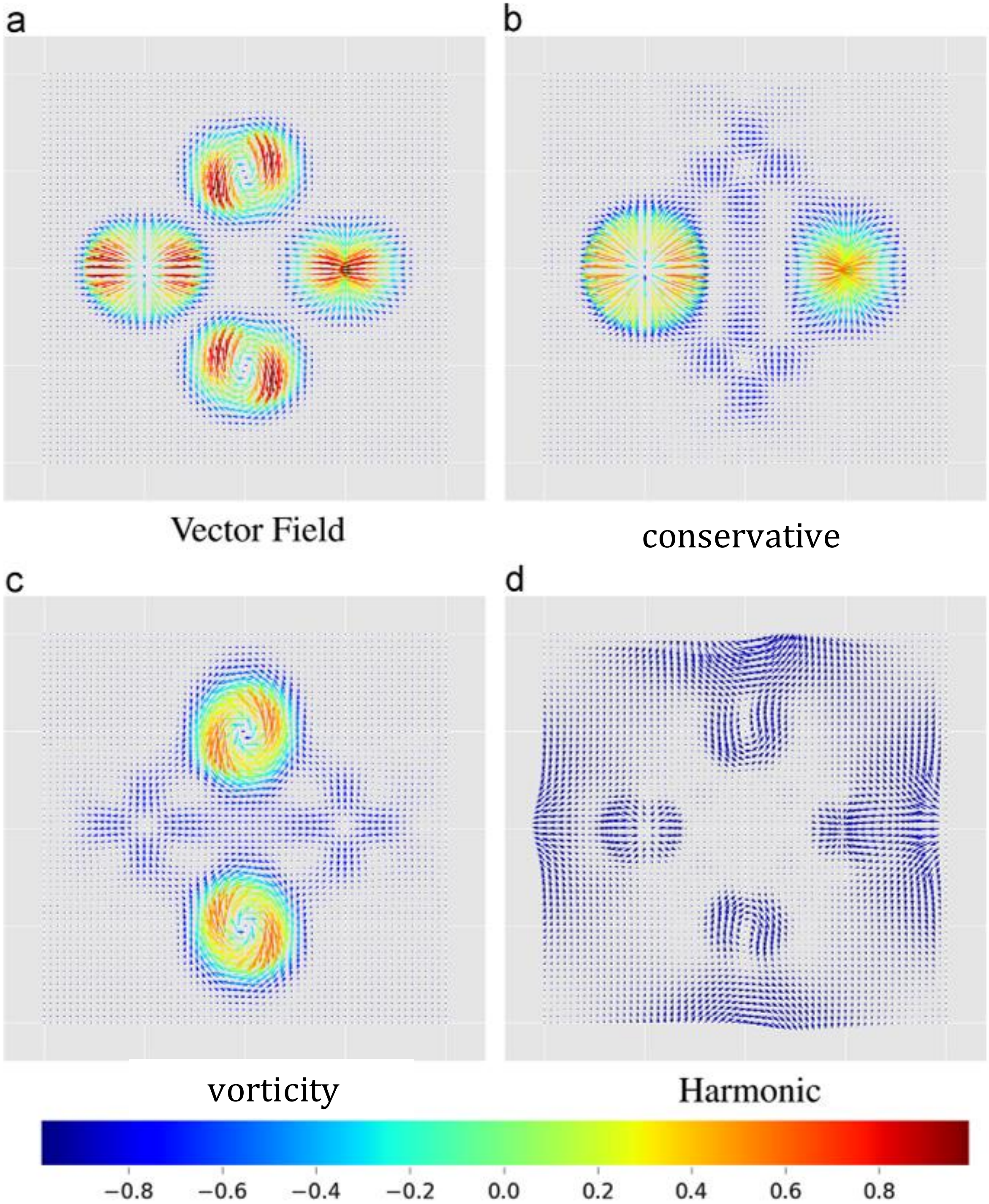
For a vector field $\boldsymbol{F} \in C^1(V, \mathbb{R}^n)$ defined on a domain $V \subseteq \mathbb{R}^n$, there exists a pair of vector field $\boldsymbol{G} \in C^1(V, \mathbb{R}^n)$ and $\boldsymbol{R} \in C^1(V, \mathbb{R}^n)$ such that

$$\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{G}(\boldsymbol{r}) + \boldsymbol{R}(\boldsymbol{r})$$

$$\boldsymbol{G}(\boldsymbol{r}) = -\nabla\Phi(\boldsymbol{r}) \quad \text{gradient field}$$

$$\nabla \cdot \boldsymbol{R}(\boldsymbol{r}) = 0 \quad \text{divergence-free}$$

Helmholtz Decomposition



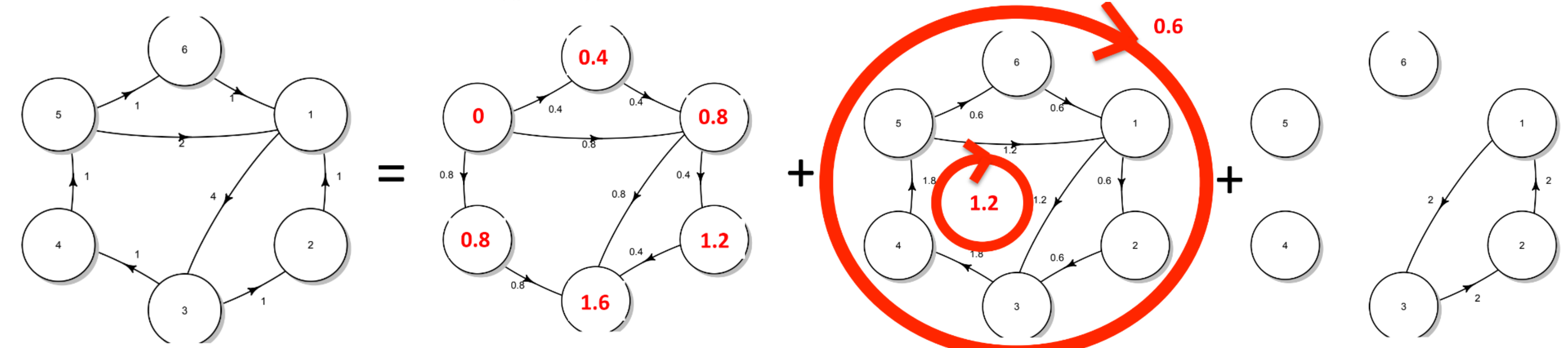
Hodge Decomposition

$$\text{Edge Flow} = \overbrace{\text{vorticity} \oplus \rlap{\underbrace{\phantom{\text{solenoidal irrotational} \oplus \text{conservative}}}_{\text{irrotational}}}\text{solenoidal irrotational}}^{\text{solenoidal}} \oplus \text{conservative}$$

$$\mathbb{R}^{|E|} = \mathrm{im}(\mathrm{curl}^*) \oplus \ker\Big(\underbrace{-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}}_{\text{Graph Helmholtzian}}\Big) \oplus \mathrm{im}(\mathrm{grad})$$

$$\text{Edge Flow} = \text{locally cyclic flow} + \text{globally cyclic flow} + \text{gradient flow}$$

$$\text{Edge Flow} = \overbrace{\text{divergence-free but not curl-free} \oplus \rlap{\underbrace{\phantom{\text{divergence-free and curl-free} \oplus \text{curl-free but not divergence-free}}}_{\text{curl-free}}}\text{divergence-free and curl-free}}^{\text{divergence-free}} \oplus \text{curl-free but not divergence-free}$$



gradient flow

$\mathrm{im}(\mathrm{grad})$

globally cyclic flow

$\ker(-\mathrm{div}) \cap \ker(\mathrm{curl})$

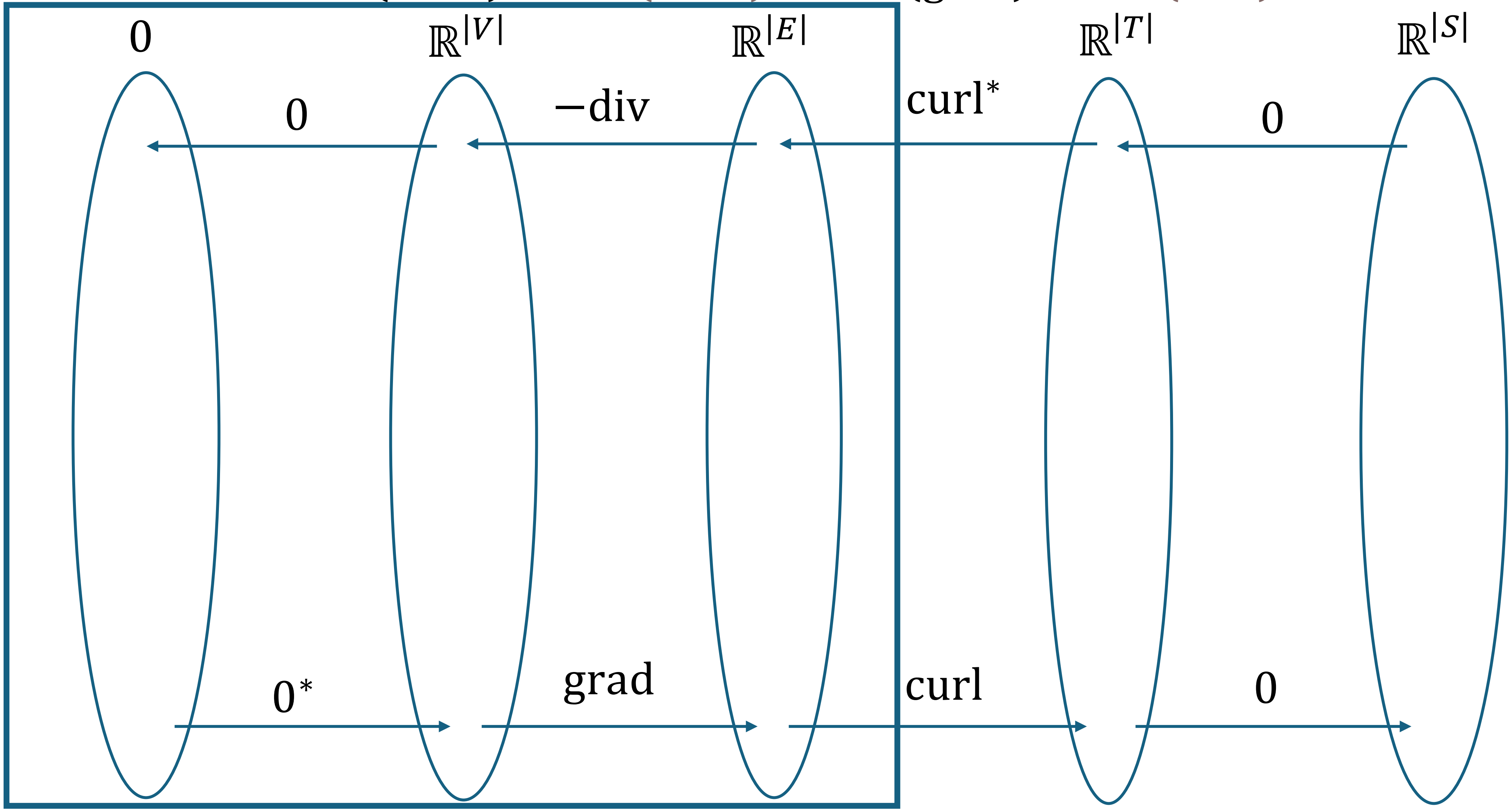
divergence − free and curl − free

locally cyclic flow

$\mathrm{im}(\mathrm{curl}^*)$

$$-\mathrm{div} \circ \mathrm{curl}^* = 0 \iff \mathrm{curl} \circ \mathrm{grad} = 0$$

$$\mathrm{im}(\mathrm{curl}^*) \subseteq \ker(-\mathrm{div}) \iff \mathrm{im}(\mathrm{grad}) \subseteq \ker(\mathrm{curl})$$

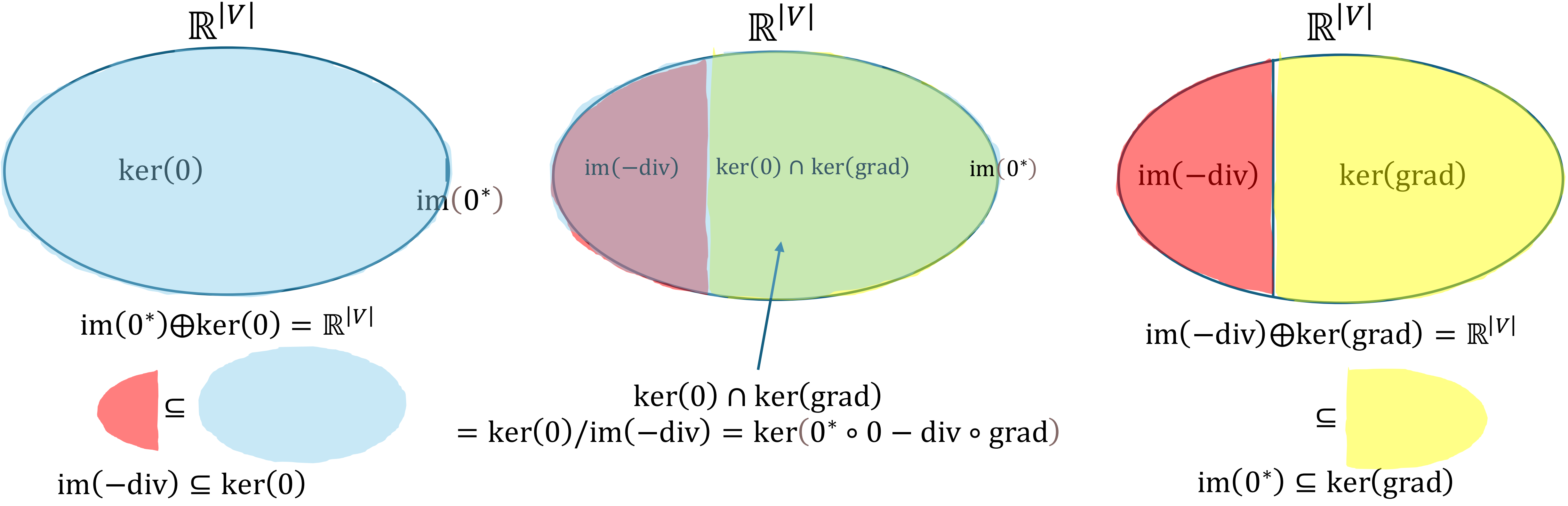


$$\mathrm{im}(-\mathrm{div}) \oplus \ker(\mathrm{grad}) = \mathbb{R}^{|V|}$$

$$\mathrm{im}(\mathrm{grad}) \oplus \ker(-\mathrm{div}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl}^*) \oplus \ker(\mathrm{curl}) = \mathbb{R}^{|E|}$$

$$\mathrm{im}(\mathrm{curl}^*) \oplus \ker(\mathrm{curl}) = \mathbb{R}^{|T|}$$



$$\mathbb{R}^{|V|} = \text{im}(-\text{div}) \oplus \big(\ker(0) \cap \ker(\text{grad})\big) \oplus \text{im}(0^*)$$

$$\mathbb{R}^{|V|} = \text{im}(\text{div}) \oplus \ker(\text{grad})$$

$$\boldsymbol{x} \in \mathbb{R}^{|V|}, \qquad \text{grad}\boldsymbol{x}_{\boldsymbol{H}} = \boldsymbol{0}, \qquad \boldsymbol{v} \in \mathbb{R}^{|E|}$$

$$\boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{H}} + \text{div}\boldsymbol{v}, \qquad \langle \text{div}\boldsymbol{v}, \boldsymbol{x}_{\boldsymbol{H}} \rangle = \boldsymbol{0}$$

$\boldsymbol{x}_{\boldsymbol{H}}$ is a multiple of all 1s vector

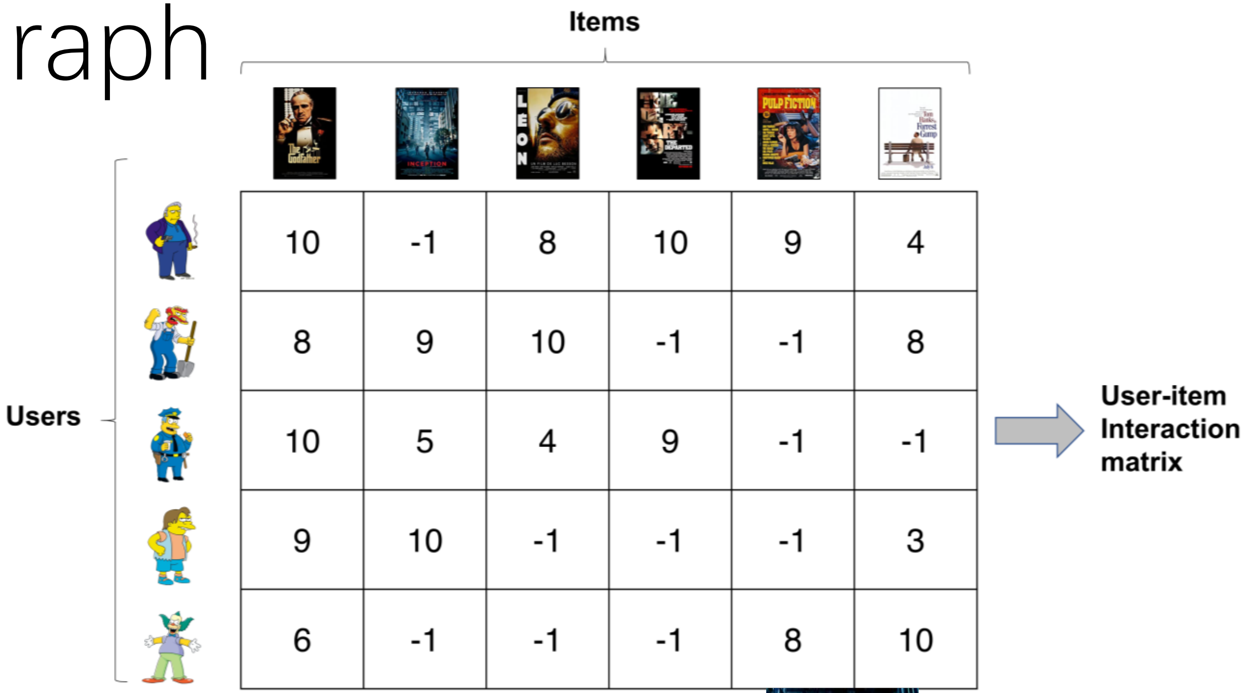
$$\mathbb{R}^{|V|} = \mathrm{im}(\mathrm{div}) \oplus \ker(\mathrm{grad}) = \mathrm{im}(\mathrm{div}) \oplus \ker(\mathrm{grad}^* \circ \mathrm{grad}) = \mathrm{im}(\mathrm{div}) \oplus \ker\Bigg(\underbrace{-\mathrm{div} \circ \mathrm{grad}}_{\text{Graph Laplacian}}\Bigg)$$

$$\mathbb{R}^{|V|} = \text{divergence} \oplus \text{gradient} - \text{free} = \text{divergence} + \text{Const}\,\mathbf{1}$$

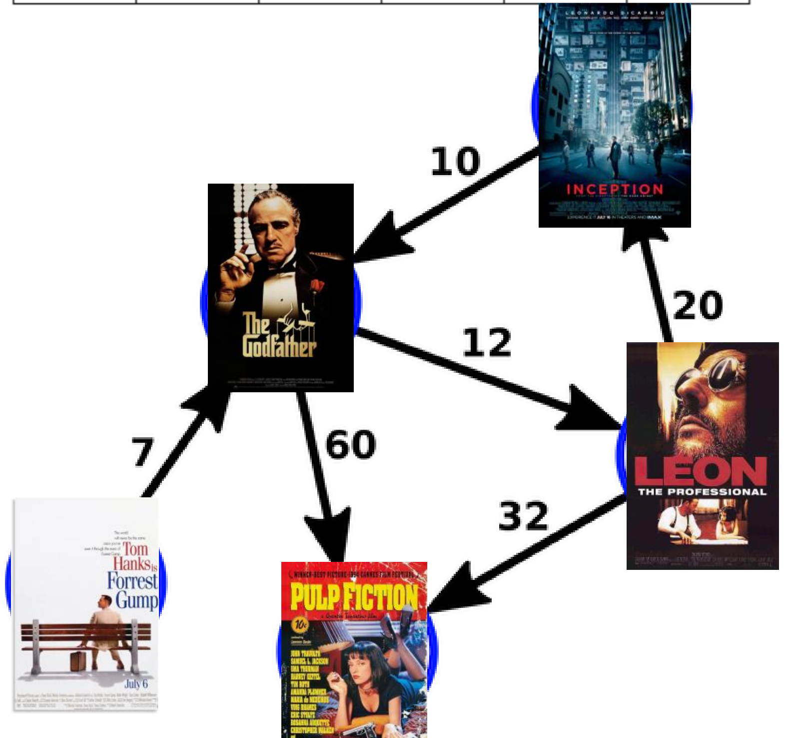
HodgeRank

Rank movies based on a graph

- Most viewers only watched a small portion of movies in the database—user-item matrix is very sparse
- However, building a weighted directed graph with movies as vertices and average score differences as edges does not involve sparsity
- Edge weights represent the number of viewers coviewing the two movies
- Average score differences between two movies are the averages of the viewers' scoring differences



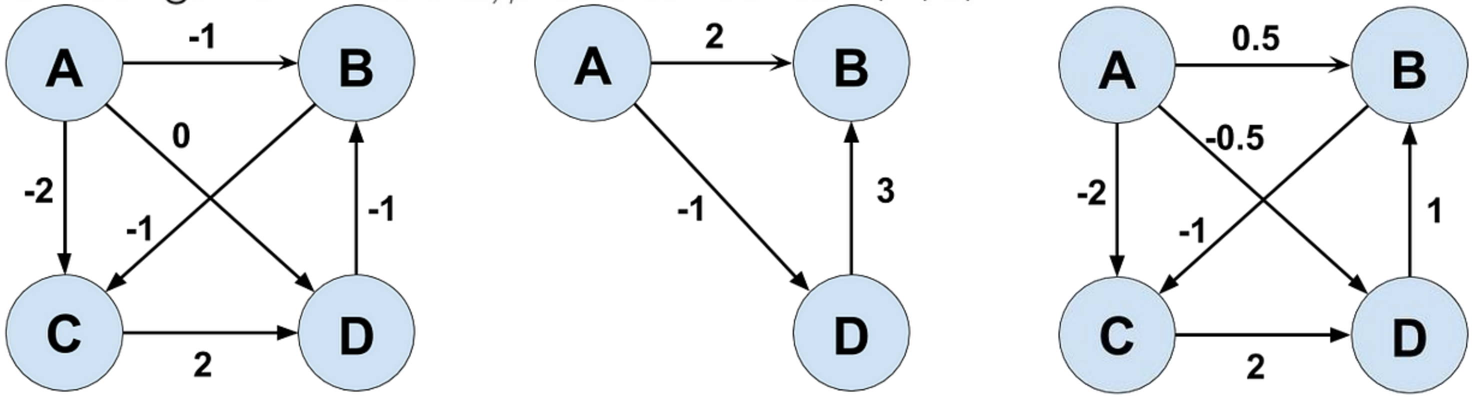
10	-1	8	10	9	4
8	9	10	-1	-1	8
10	5	4	9	-1	-1
9	10	-1	-1	-1	3
6	-1	-1	-1	8	10



	A	B	C	D
α	5	4	3	5
β	3	5	-	2

Table: Ratings from voters α, β to alternatives A,B,C,D

$$(\partial_1)_{ij} = \begin{cases} -1, & v_i \text{ is the source node in } e_j \\ +1, & v_i \text{ is the sink node in } e_j \\ 0, & \text{else} \end{cases}$$



α **β** **α+β**

	A	B	C	D
α	5	4	3	5
β	3	5	-	2
$\boldsymbol{r}$	r_A	r_B	r_C	r_D

$\boldsymbol{f} \in \mathbb{R}^{|E|}$ takes the average of all the votes between each pair of alternatives.

$$\boldsymbol{f} = \begin{pmatrix} [(5-4)+(3-5)]/2 \\ (5-3)/1 \\ (4-3)/1 \\ [(3-2)+(5-5)]/2 \\ [(4-5)+(5-2)]/2 \\ (3-5)/1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 2 \\ 1 \\ 0.5 \\ 1 \\ -2 \end{pmatrix} \in \mathbb{R}^{|E|}$$

$\boldsymbol{f} \in \mathbb{R}^{|E|}$ takes the average of all the votes between each pair of alternatives.

$$\boldsymbol{f} = \begin{pmatrix} [(5-4)+(3-5)]/2 \\ (5-3)/1 \\ (4-3)/1 \\ [(3-2)+(5-5)]/2 \\ [(4-5)+(5-2)]/2 \\ (3-5)/1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 2 \\ 1 \\ 0.5 \\ 1 \\ -2 \end{pmatrix} \in \mathbb{R}^{|E|}$$

Define $\boldsymbol{r} \in \mathbb{R}^{|V|} = \begin{pmatrix} r_A \\ r_B \\ r_C \\ r_D \end{pmatrix}$ and notice

$$\partial_1^T \boldsymbol{r} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} r_A \\ r_B \\ r_C \\ r_D \end{pmatrix} = \begin{pmatrix} r_A - r_B \\ r_A - r_C \\ r_B - r_C \\ r_A - r_D \\ r_B - r_D \\ r_C - r_D \end{pmatrix}$$

Very sparse incidence matrix

$$\partial_1^T \boldsymbol{r} = \begin{pmatrix} r_A - r_B \\ r_A - r_C \\ r_B - r_C \\ r_A - r_D \\ r_B - r_D \\ r_C - r_D \end{pmatrix} \longleftrightarrow \boldsymbol{f} = \begin{pmatrix} -0.5 \\ 2 \\ 1 \\ 0.5 \\ 1 \\ -2 \end{pmatrix}$$

$$\min_{\boldsymbol{r} \in \mathbb{R}^{|V|}} \|\boldsymbol{f} - \partial_1^T \boldsymbol{r}\|_W^2,$$

$W = \mathrm{diag}([2, 1, 1, 2, 2, 1])$

$$\partial_1 W \partial_1^T \boldsymbol{r} = \partial_1 W \boldsymbol{f}.$$

The normal equation can also be written as

$$L\boldsymbol{r} = \boldsymbol{b}$$

for weighted graph Laplaican defined as $L = \partial_1 W \partial_1^T$, $\boldsymbol{b} = \partial_1 W \boldsymbol{f}$.

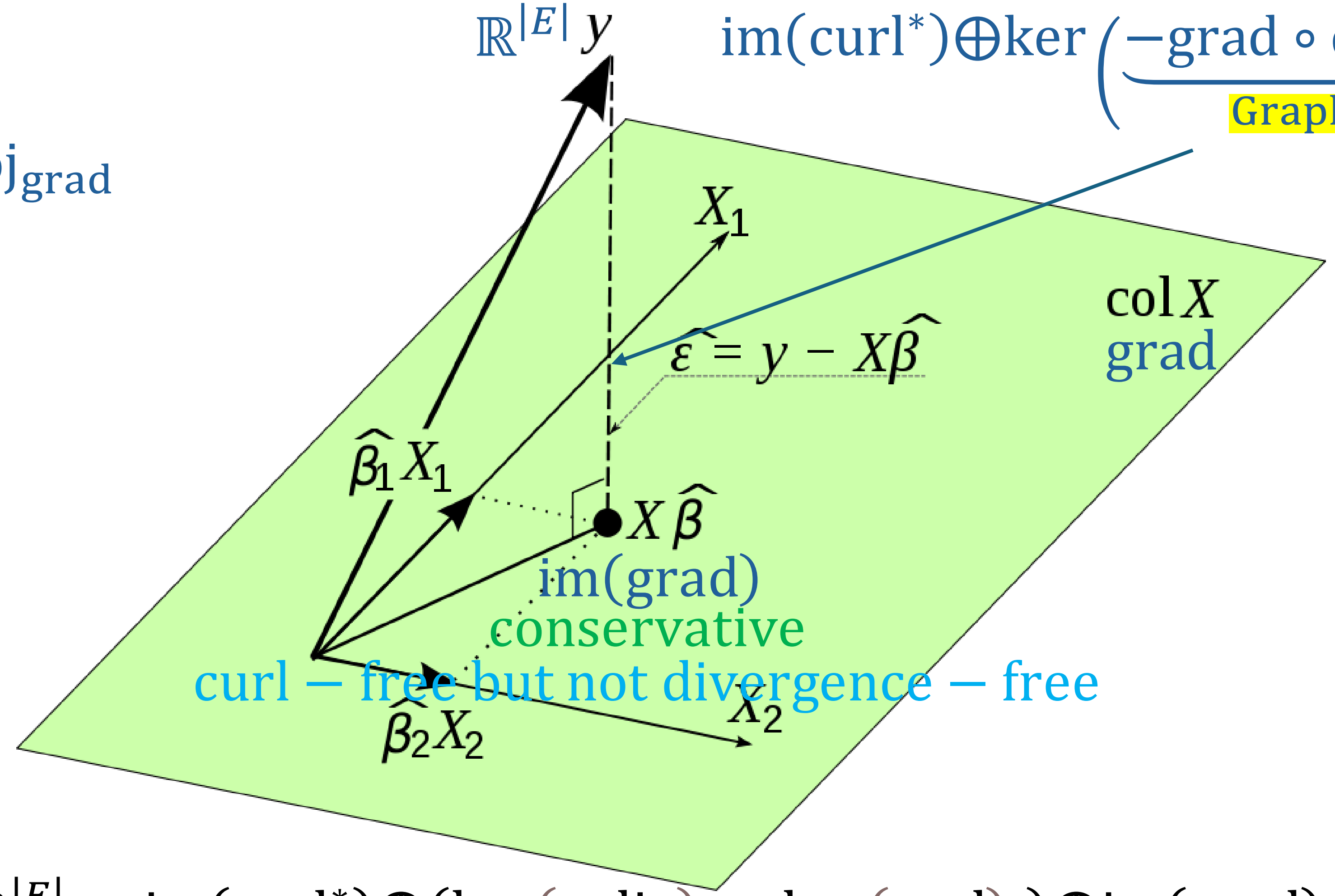
- Weighted least squares

$\text{divergence} - \text{free but not curl} - \text{free} \oplus (\text{divergence} - \text{free and curl} - \text{free})$

$\text{vorticity} \oplus (\text{solenoidal and irrotational})$

$\mathrm{im}(\mathrm{curl}^*) \oplus \ker\left(\underbrace{-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}}_{\text{Graph Helmholtzian}}\right)$

$\mathrm{proj}_{\mathrm{grad}}$



$$\mathbb{R}^{|E|} = \mathrm{im}(\mathrm{curl}^*) \oplus (\ker(-\mathrm{div}) \cap \ker(\mathrm{curl})) \oplus \mathrm{im}(\mathrm{grad})$$
$$\mathbb{R}^{|E|} = \text{vorticity} \oplus (\text{solenoidal and irrotational}) \oplus \text{conservative}$$

$$\mathbb{R}^n = \mathrm{im}\left(\boldsymbol{B}_{n\times p}\right)\oplus\ker\left(\boldsymbol{A}_{m\times n}{}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}{}^{\boldsymbol{T}}\right)\oplus\mathrm{im}\left(\boldsymbol{A}_{m\times n}{}^{T}\right)$$

$$\partial_1 = \boldsymbol{A}_{m\times n}, \qquad \boldsymbol{r} = \mathrm{proj}_{\boldsymbol{A}_{m\times n}{}^{T}}(\boldsymbol{x})$$

$$\min_{r\in\mathbb{R}^{|V|}} \left\|\boldsymbol{f} - \partial_1{}^{T}\boldsymbol{r}\right\|_{\boldsymbol{W}}^{2}$$

Normal equation

$$\partial_1\boldsymbol{W}\partial_1{}^{T}\boldsymbol{r} = \partial_1\boldsymbol{W}\boldsymbol{f}$$

$$\partial_1\boldsymbol{W}^{\frac{1}{2}}\boldsymbol{W}^{\frac{1}{2}\boldsymbol{T}}\partial_1{}^{T}\boldsymbol{r} = \partial_1\boldsymbol{W}^{\frac{1}{2}}\boldsymbol{W}^{\frac{1}{2}\boldsymbol{T}}\boldsymbol{f}$$

$$\left(\partial_1\boldsymbol{W}^{\frac{1}{2}}\right)\left(\partial_1\boldsymbol{W}^{\frac{1}{2}}\right)^{T}\boldsymbol{r} = \left(\partial_1\boldsymbol{W}^{\frac{1}{2}}\right)\left(\boldsymbol{W}^{\frac{1}{2}}\boldsymbol{f}\right)$$

∂_1 is not full rank, $\partial_1\boldsymbol{W}\partial_1{}^{T}$ is not full rank, non-invertible

Infinite solution (potential function additive invariant)

$$\boldsymbol{r} = \left(\partial_1\boldsymbol{W}^{\frac{1}{2}}\right)^{T\dagger}\boldsymbol{W}^{\frac{1}{2}}\boldsymbol{f} + \left(\boldsymbol{I} - \left(\partial_1\boldsymbol{W}^{\frac{1}{2}}\right)^{T\dagger}\left(\partial_1\boldsymbol{W}^{\frac{1}{2}}\right)^{\boldsymbol{T}}\right)\boldsymbol{w}$$

Infinite solution (potential function additive invariant)

$$\boldsymbol{r} = \left(\partial_1 \boldsymbol{W}^{\frac{1}{2}}\right)^{T\dagger} \boldsymbol{W}^{\frac{1}{2}} \boldsymbol{f} + \left(\boldsymbol{I} - \left(\partial_1 \boldsymbol{W}^{\frac{1}{2}}\right)^{T\dagger} \left(\partial_1 \boldsymbol{W}^{\frac{1}{2}}\right)^{\boldsymbol{T}}\right) \boldsymbol{w}$$

SVD of $\partial_1, {\partial_1}^T$:

$\partial_1 = \boldsymbol{U\Sigma V^T} = \boldsymbol{U}[\boldsymbol{S} \quad \boldsymbol{0}]\boldsymbol{V^T}$	$\partial_1 = \boldsymbol{U\Sigma V^T} = \boldsymbol{U}\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{V^T}$
${\partial_1}^T = \boldsymbol{V\Sigma^T U^T} = \boldsymbol{V}\begin{bmatrix}\boldsymbol{S^T}\\ \boldsymbol{0^T}\end{bmatrix}\boldsymbol{U^T}$	${\partial_1}^T = \boldsymbol{V\Sigma^T U^T} = \boldsymbol{V}[\boldsymbol{S^T} \quad \boldsymbol{0^T}]\boldsymbol{U^T}$

Moore-Penrose pseudoinverse of $\partial_1, {\partial_1}^T$:

${\partial_1}^\dagger = \left(\boldsymbol{U\Sigma V^T}\right)^\dagger = \boldsymbol{V}\begin{bmatrix}\boldsymbol{S^{-T}}\\ \boldsymbol{0^T}\end{bmatrix}\boldsymbol{U^T}$	${\partial_1}^\dagger = \left(\boldsymbol{U\Sigma V^T}\right)^\dagger = \boldsymbol{V}[\boldsymbol{S^{-T}} \quad \boldsymbol{0^T}]\boldsymbol{U^T}$
${\partial_1}^{T\dagger} = \left(\boldsymbol{V\Sigma^T U^T}\right)^\dagger = \boldsymbol{U}[\boldsymbol{S^{-1}} \quad \boldsymbol{0}]\boldsymbol{V^T}$	${\partial_1}^{T\dagger} = \left(\boldsymbol{V\Sigma^T U^T}\right)^\dagger = \boldsymbol{U}\begin{bmatrix}\boldsymbol{S^{-1}}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{V^T}$
$\partial_1{\partial_1}^\dagger = \boldsymbol{U}[\boldsymbol{S} \quad \boldsymbol{0}]\boldsymbol{V^T V}\begin{bmatrix}\boldsymbol{S^{-T}}\\ \boldsymbol{0^T}\end{bmatrix}\boldsymbol{U^T} = \boldsymbol{UU^T} = \boldsymbol{I}$	$\partial_1{\partial_1}^\dagger = \boldsymbol{U}\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{V^T V}[\boldsymbol{S^{-T}} \quad \boldsymbol{0^T}]\boldsymbol{U^T} = \boldsymbol{U}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0^T}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{U^T}$
${\partial_1}^\dagger\partial_1 = \boldsymbol{V}\begin{bmatrix}\boldsymbol{S^{-T}}\\ \boldsymbol{0^T}\end{bmatrix}\boldsymbol{U^T U}[\boldsymbol{S} \quad \boldsymbol{0}]\boldsymbol{V^T} = \boldsymbol{V}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0^T} & \boldsymbol{0}\end{bmatrix}\boldsymbol{V^T}$	${\partial_1}^\dagger\partial_1 = \boldsymbol{V}[\boldsymbol{S^{-T}} \quad \boldsymbol{0^T}]\boldsymbol{U^T U}\begin{bmatrix}\boldsymbol{S}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{V^T} = \boldsymbol{VV^T} = \boldsymbol{I}$

<table>
<tr><td></td><td></td><td>$\boldsymbol{A}$ full row rank</td><td>$\boldsymbol{A}$ not full row rank</td></tr>
<tr><td></td><td></td><td>$\boldsymbol{A}$ full row rank $\Leftrightarrow \boldsymbol{A}'$ column rank is m $\Leftrightarrow$
$\boldsymbol{A}$ can span the whole m dimensional space$\Leftrightarrow$ any $\boldsymbol{b}$ is in the column space of $\boldsymbol{A}$ $\Leftrightarrow \boldsymbol{Ax}=\boldsymbol{b}$ always has solution</td><td>$\boldsymbol{A}$ not full row rank $\Leftrightarrow \boldsymbol{A}'$ column rank is less than m $\Leftrightarrow$
$\boldsymbol{A}$ can not span the whole m dimensional space $\Leftrightarrow$ not any $\boldsymbol{b}$ is in the column space of $\boldsymbol{A}$ $\Leftrightarrow \boldsymbol{Ax}=\boldsymbol{b}$ does not always have solution</td></tr>
<tr><td>$\boldsymbol{A}$
full column rank</td><td>$\boldsymbol{A}$ full column rank$\Leftrightarrow \boldsymbol{Ax}=\boldsymbol{0}$ only one solution (only zero) $\Leftrightarrow \boldsymbol{Ax}=\boldsymbol{b}$ only one solution (when has solution)</td><td>example (A is squared matrix and full rank)
dimension of space does not change before and after the transformation<table><tr><td>$\boldsymbol{Ax}=\boldsymbol{b}$</td><td>$\boldsymbol{Ax}=\boldsymbol{0}$</td></tr><tr><td>$\boldsymbol{Ax}=\boldsymbol{b}$ has unique solutions $\boldsymbol{x}$ for any $\boldsymbol{b}$</td><td>$\boldsymbol{Ax}=\boldsymbol{0}$ has unique solutions $\boldsymbol{x}=\boldsymbol{0}$
$\mathrm{rank}(\boldsymbol{A}_{m\times n})=m=n$
$\mathrm{nullity}(\boldsymbol{A}_{m\times n})=0$</td></tr></table></td><td>example (tall and slim A full column rank)
dimension of space increases seemingly after transformation (in reality does not really increase)<table><tr><td>$\boldsymbol{Ax}=\boldsymbol{b}$</td><td>$\boldsymbol{Ax}=\boldsymbol{0}$</td></tr><tr><td>$\boldsymbol{Ax}=\boldsymbol{b}$ does not unique solution $\boldsymbol{x}$ for any $\boldsymbol{b}$
(approximation solution using projection is unique)
has unique solution $\boldsymbol{x}$ for some $\boldsymbol{b}$
does not have solution $\boldsymbol{x}$ for some $\boldsymbol{b}$</td><td>$\boldsymbol{Ax}=\boldsymbol{0}$ has unique solution $\boldsymbol{x}=\boldsymbol{0}$
$\mathrm{rank}(\boldsymbol{A}_{m\times n})=n$
$\mathrm{nullity}(\boldsymbol{A}_{m\times n})=0$</td></tr></table></td></tr>
<tr><td>$\boldsymbol{A}$
not full column rank</td><td>$\boldsymbol{A}$ not full column rank$\Leftrightarrow \boldsymbol{Ax}=\boldsymbol{0}$ infinite solutions (non-zero solutions exist) $\Leftrightarrow \boldsymbol{Ax}=\boldsymbol{b}$ infinite solution (when has solution)</td><td>example (short and fat A full row rank)
dimension of space decreases after transformation (indeed decreases)<table><tr><td>$\boldsymbol{Ax}=\boldsymbol{b}$</td><td>$\boldsymbol{Ax}=\boldsymbol{0}$</td></tr><tr><td>$\boldsymbol{Ax}=\boldsymbol{b}$ has infinite solutions for any $\boldsymbol{b}$</td><td>$\boldsymbol{Ax}=\boldsymbol{0}$ has infinite solutions
$\mathrm{rank}(\boldsymbol{A}_{m\times n})=m<n$
$\mathrm{nullity}(\boldsymbol{A}_{m\times n})=n-m>0$</td></tr></table></td><td>example (A is squared matrix and not full rank)
example (short and fat A not full row rank)
example (tall and slim A not full column rank)
dimension of space decreases after transformation (indeed decreases), but can not span the whole target space<table><tr><td>$\boldsymbol{Ax}=\boldsymbol{b}$</td><td>$\boldsymbol{Ax}=\boldsymbol{0}$</td></tr><tr><td>$\boldsymbol{Ax}=\boldsymbol{b}$ does not infinite solutions $\boldsymbol{x}$ for any $\boldsymbol{b}$
(approximation solution using projection is infinite)
has infinite solutions $\boldsymbol{x}$ for some $\boldsymbol{b}$
does not have solution $\boldsymbol{x}$ for some $\boldsymbol{b}$</td><td>$\boldsymbol{Ax}=\boldsymbol{0}$ has infinite solutions $\boldsymbol{x}$
$\mathrm{rank}(\boldsymbol{A}_{m\times n})<n$
$\mathrm{rank}(\boldsymbol{A}_{m\times n})<m$
$\mathrm{nullity}(\boldsymbol{A}_{m\times n})>0$</td></tr></table></td></tr>
</table>

$$\mathbb{R}^n = \mathrm{im}\left(\boldsymbol{B}_{n\times p}\right) \oplus \ker(\Delta) \oplus \mathrm{im}\left({\boldsymbol{A}_{m\times n}}^T\right)$$

$$\Delta = {\boldsymbol{A}_{m\times n}}^T \boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}{\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}}$$

$$\boldsymbol{x} \in \mathbb{R}^n, \qquad \boldsymbol{x} = \mathrm{proj}_{\boldsymbol{B}_{n\times p}}(\boldsymbol{x}) \oplus \mathrm{proj}_{{\Delta^T}^{\perp}}(\boldsymbol{x}) \oplus \mathrm{proj}_{{\boldsymbol{A}_{m\times n}}^T}(\boldsymbol{x})$$

$$\mathrm{proj}_{\boldsymbol{X}} = \boldsymbol{X}\boldsymbol{X}^{\dagger}, \quad \mathrm{proj}_{\boldsymbol{X}}(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{X}^{\dagger}\boldsymbol{y} + \boldsymbol{X}\left(\boldsymbol{I} - \boldsymbol{X}^{\dagger}\boldsymbol{X}\right)\boldsymbol{w}$$

$${\Delta^T}^{\perp} = \boldsymbol{I} - \Delta^T, \qquad \mathrm{proj}_{{\Delta^T}^{\perp}} = \boldsymbol{I} - \mathrm{proj}_{\Delta^T} = \boldsymbol{I} - \Delta^T {\Delta^T}^{\dagger} = \boldsymbol{I} - \Delta\Delta^{\dagger}, \quad \mathrm{proj}_{{\Delta^T}^{\perp}}(\boldsymbol{y}) = \left(\boldsymbol{I} - \Delta\Delta^{\dagger}\right)\boldsymbol{y}$$

$$\Delta^T = \Delta, \qquad \Delta = \boldsymbol{USV^T}, \qquad \Delta^{\dagger} = \left(\boldsymbol{USV^T}\right)^{\dagger} = \boldsymbol{VS^{-T}U^T},$$

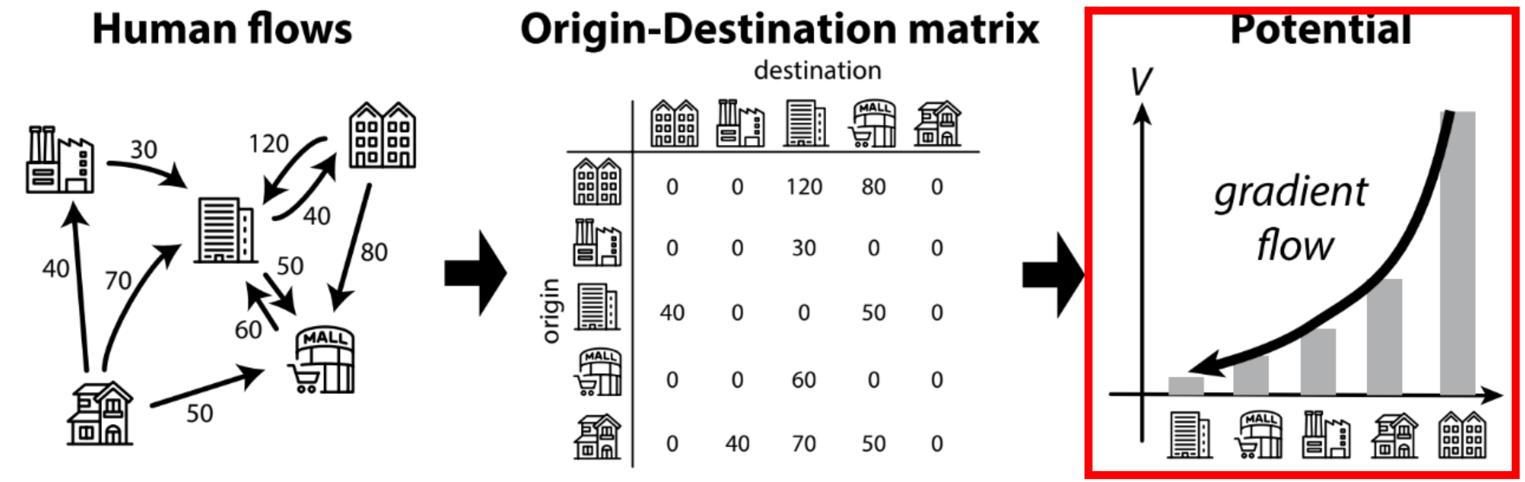
$$\Delta^{\dagger}\Delta = \boldsymbol{V}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{V^T}, \qquad \Delta\Delta^{\dagger} = \boldsymbol{U}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{U^T}$$

Δ is a symmetric square matrix, $\boldsymbol{U}, \boldsymbol{V}$ same dimension, $\boldsymbol{U} = \boldsymbol{V}$

$$\Delta^{\dagger}\Delta = \Delta\Delta^{\dagger}, \qquad \boldsymbol{U^T U} = \boldsymbol{UU^T} = \boldsymbol{V^T V} = \boldsymbol{VV^T} = \boldsymbol{I}$$

$$\left\langle \operatorname{proj}_{\boldsymbol{B}_{n\times p}}(\boldsymbol{x}), \operatorname{proj}_{{\boldsymbol{A}_{m\times n}}^{T}}(\boldsymbol{x}) \right\rangle = \left\langle \boldsymbol{B}_{n\times p}\boldsymbol{y}_{\boldsymbol{1}_{p\times 1}}, {\boldsymbol{A}_{m\times n}}^{T}\boldsymbol{y}_{\boldsymbol{2}_{m\times 1}} \right\rangle$$
$$= \left(\boldsymbol{B}_{n\times p}\boldsymbol{y}_{\boldsymbol{1}_{p\times 1}}\right)^{T}{\boldsymbol{A}_{m\times n}}^{T}\boldsymbol{y}_{\boldsymbol{2}_{m\times 1}} = {\boldsymbol{y}_{\boldsymbol{1}_{p\times 1}}}^{T}{\boldsymbol{B}_{n\times p}}^{T}{\boldsymbol{A}_{m\times n}}^{T}\boldsymbol{y}_{\boldsymbol{2}_{m\times 1}}$$
$$= {\boldsymbol{y}_{\boldsymbol{1}_{p\times 1}}}^{T}\left(\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times p}\right)^{T}\boldsymbol{y}_{\boldsymbol{2}_{m\times 1}} = {\boldsymbol{y}_{\boldsymbol{1}_{p\times 1}}}^{T}\boldsymbol{0}_{p\times m}\boldsymbol{y}_{\boldsymbol{2}_{m\times 1}} = 0$$
$$\Delta\boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} = \left({\boldsymbol{A}_{m\times n}}^{T}\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{n\times p}{\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}}\right)\boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} = \boldsymbol{0} \Longrightarrow \begin{cases} \boldsymbol{A}_{m\times n}\boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} = \boldsymbol{0} \\ {\boldsymbol{B}_{n\times p}}^{\boldsymbol{T}}\boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} = \boldsymbol{0} \end{cases}$$
$$\left\langle \operatorname{proj}_{\boldsymbol{B}_{n\times p}}(\boldsymbol{x}), \boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} \right\rangle = \left\langle \boldsymbol{B}_{n\times p}\boldsymbol{y}_{\boldsymbol{1}_{p\times 1}}, \boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} \right\rangle = {\boldsymbol{y}_{\boldsymbol{1}_{p\times 1}}}^{T}{\boldsymbol{B}_{n\times p}}^{T}\boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} = 0$$
$$\left\langle \operatorname{proj}_{{\boldsymbol{A}_{m\times n}}^{T}}(\boldsymbol{x}), \boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} \right\rangle = \left\langle {\boldsymbol{A}_{m\times n}}^{T}\boldsymbol{y}_{\boldsymbol{2}_{m\times 1}}, \boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} \right\rangle = {\boldsymbol{y}_{\boldsymbol{2}_{m\times 1}}}^{T}\boldsymbol{A}_{m\times n}\boldsymbol{x}_{\boldsymbol{3}_{n\times 1}} = 0$$

Get potentials of nodes in a flow network



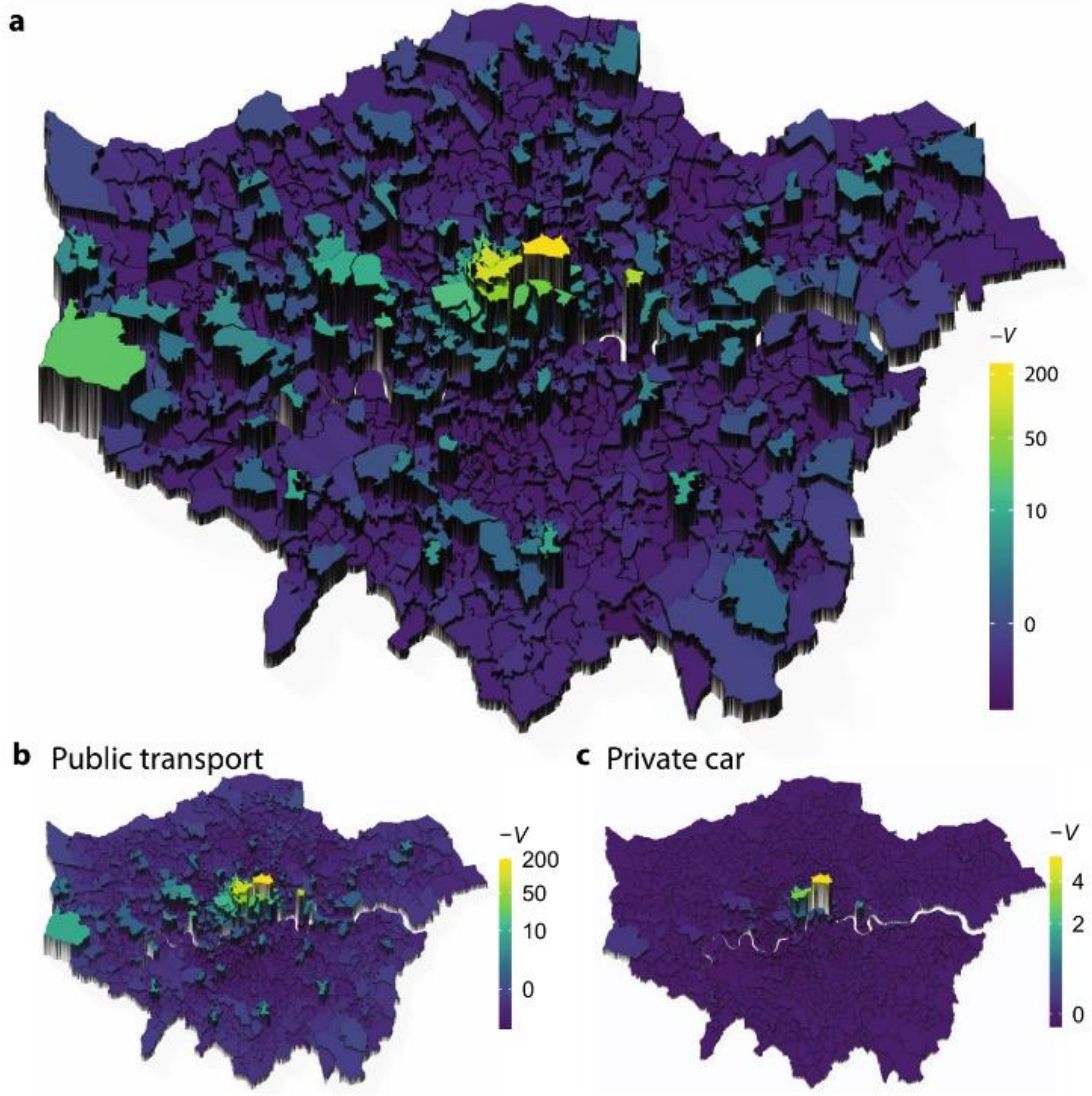
$$\mathbb{R}^{|E|} = \boxed{\mathrm{im}(\mathrm{curl}^*) \oplus \ker\left(\underbrace{-\mathrm{grad} \circ \mathrm{div} + \mathrm{curl}^* \circ \mathrm{curl}}_{\text{Graph Helmholtzian}}\right)} \oplus \boxed{\mathrm{im}(\mathrm{grad})}$$

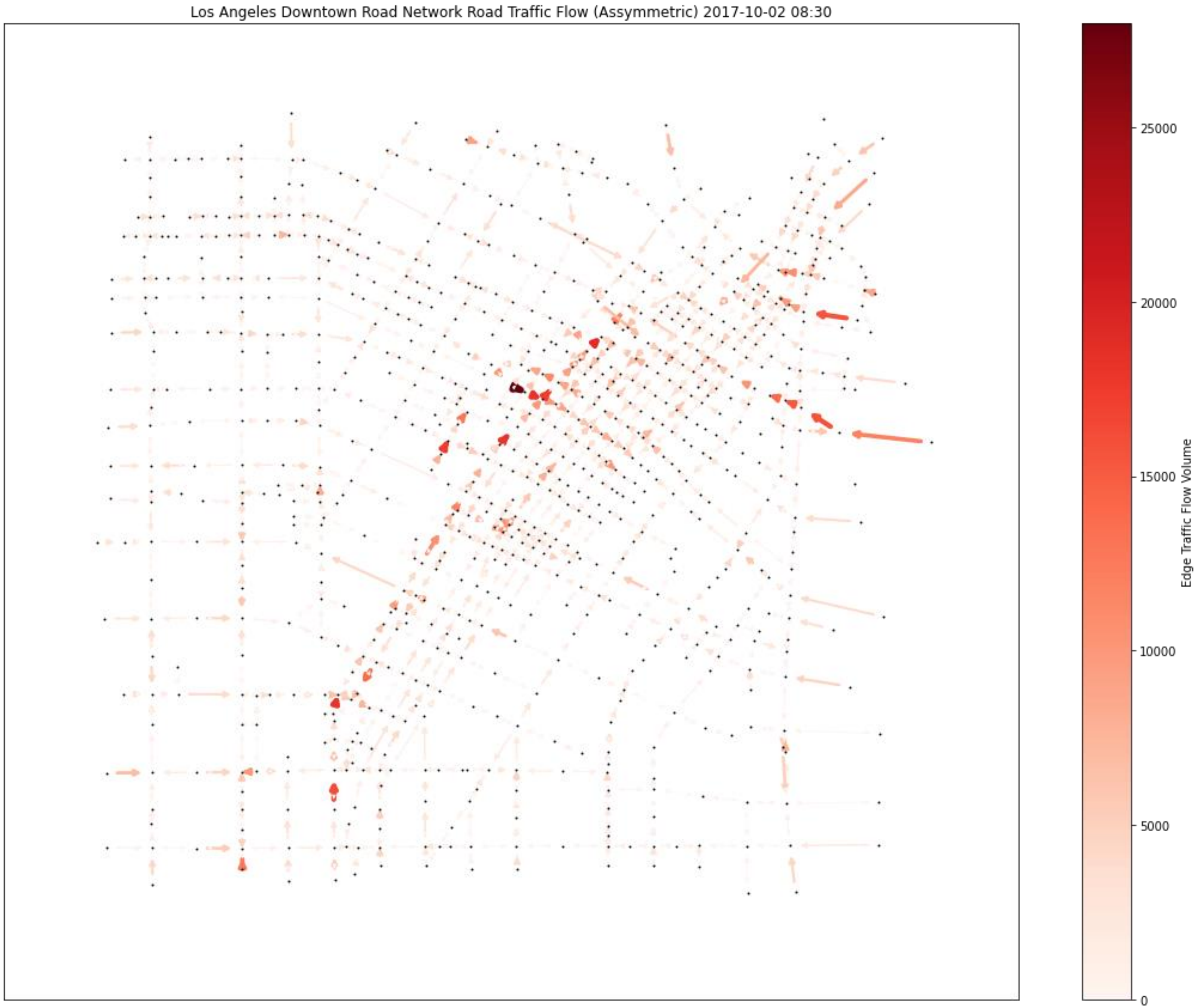
$$\text{Edge Flow} = \boxed{\overbrace{\text{vorticity} \oplus \mathrlap{\underbrace{\phantom{\text{solenoidal irrotational} \oplus \text{conservative}}}_{\text{irrotational}}}\text{solenoidal irrotational}}^{\text{solenoidal}}} \oplus \boxed{\text{conservative}}$$

$$\text{Edge Flow} = \boxed{\overbrace{\text{divergence} - \text{free but not curl} - \text{free} \oplus \mathrlap{\underbrace{\phantom{\text{divergence} - \text{free and curl} - \text{free} \oplus \text{curl} - \text{free but not divergence} - \text{free}}}_{\text{curl} - \text{free}}}\text{divergence} - \text{free and curl} - \text{free}}^{\text{divergence} - \text{free}}} \oplus \boxed{\text{curl} - \text{free but not divergence} - \text{free}}$$

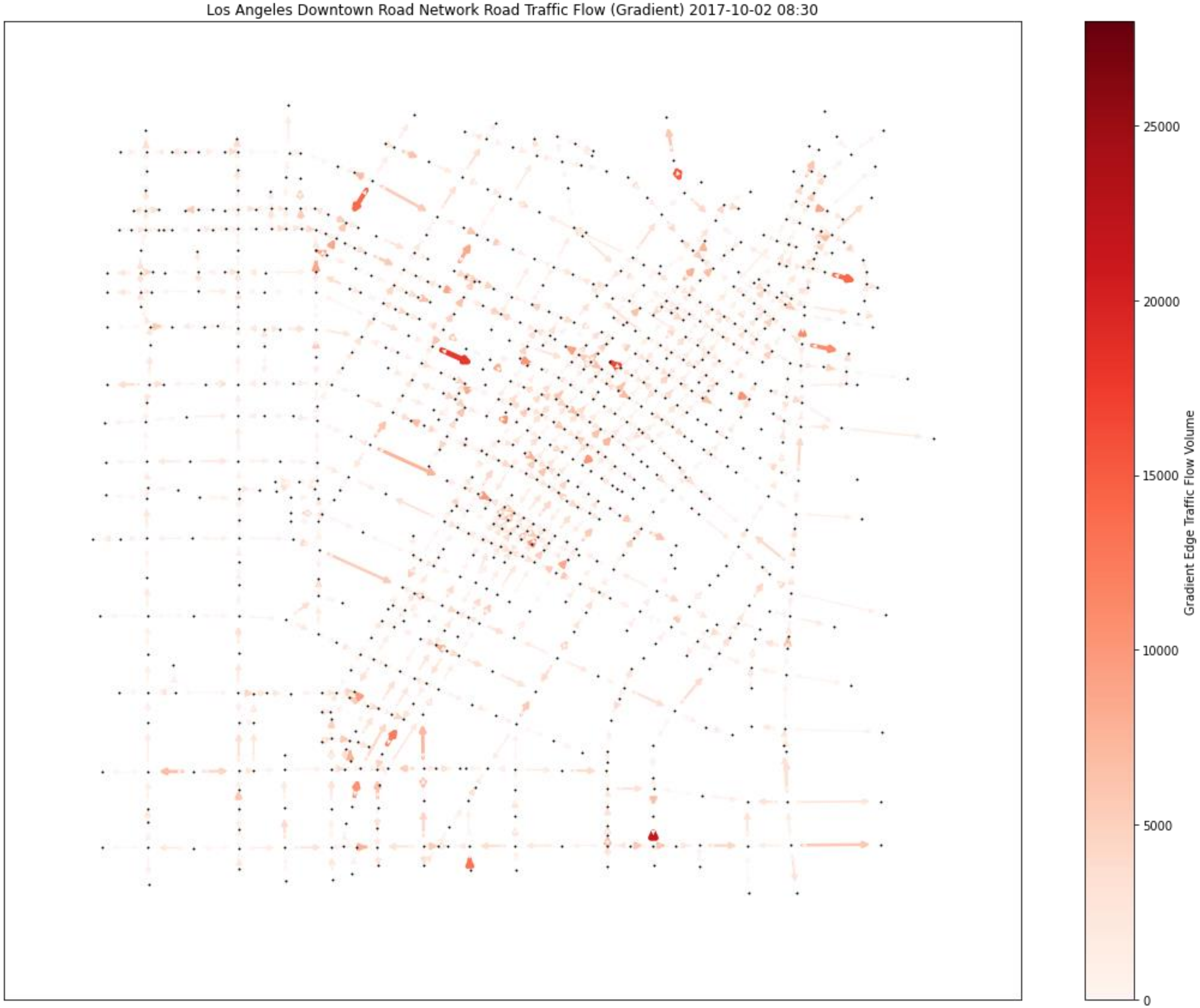
Red part denotes the gradient flow, which is used as a ranking of nodes in the flow network.

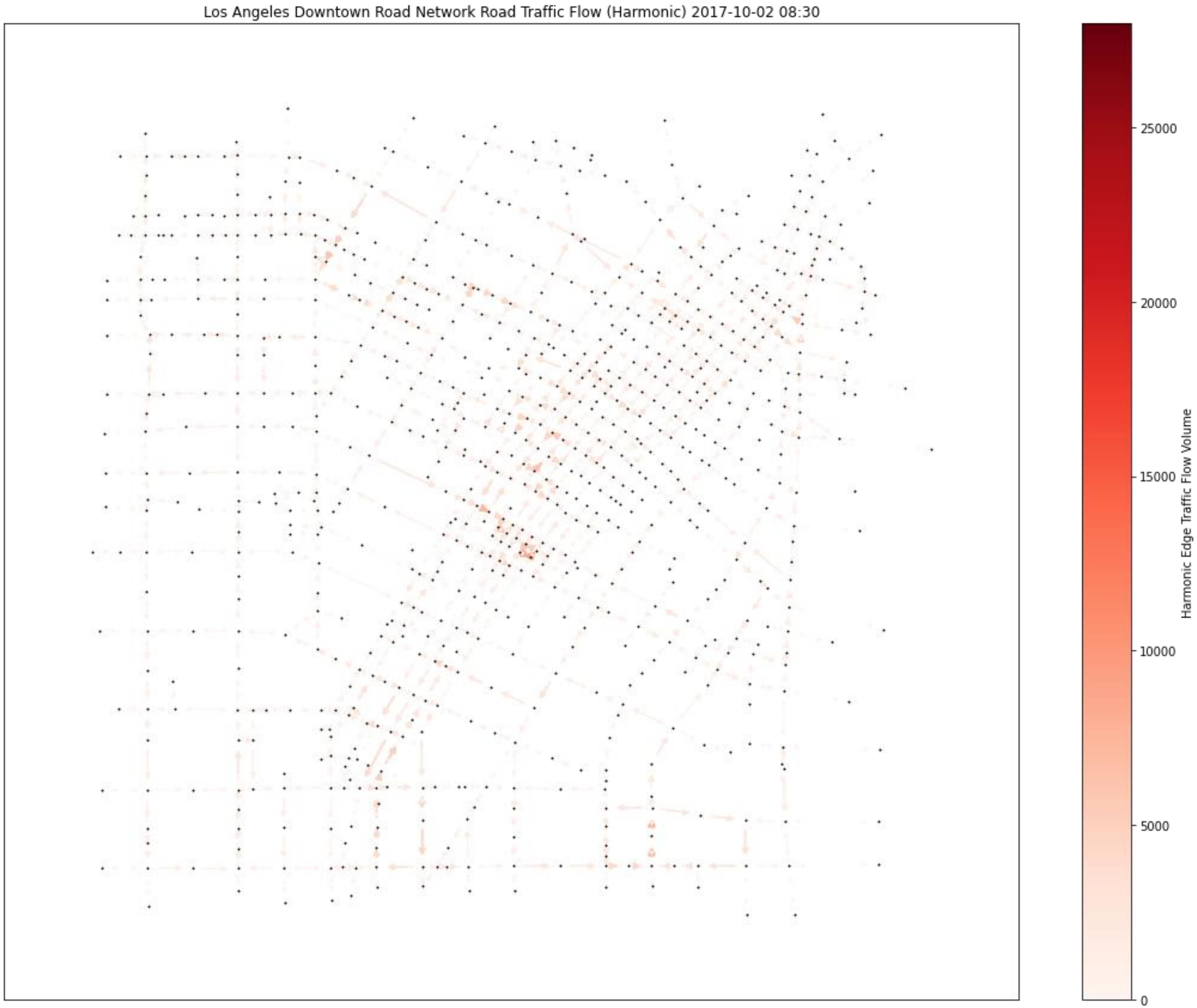
Blue part orthogonal to the gradient flow is used as inconsistency measurements. A larger blue component means this flow graph is more unrankable.


a
−V
200
50
10
0
b Public transport
−V
200
50
10
0
c Private car
−V
4
2
0

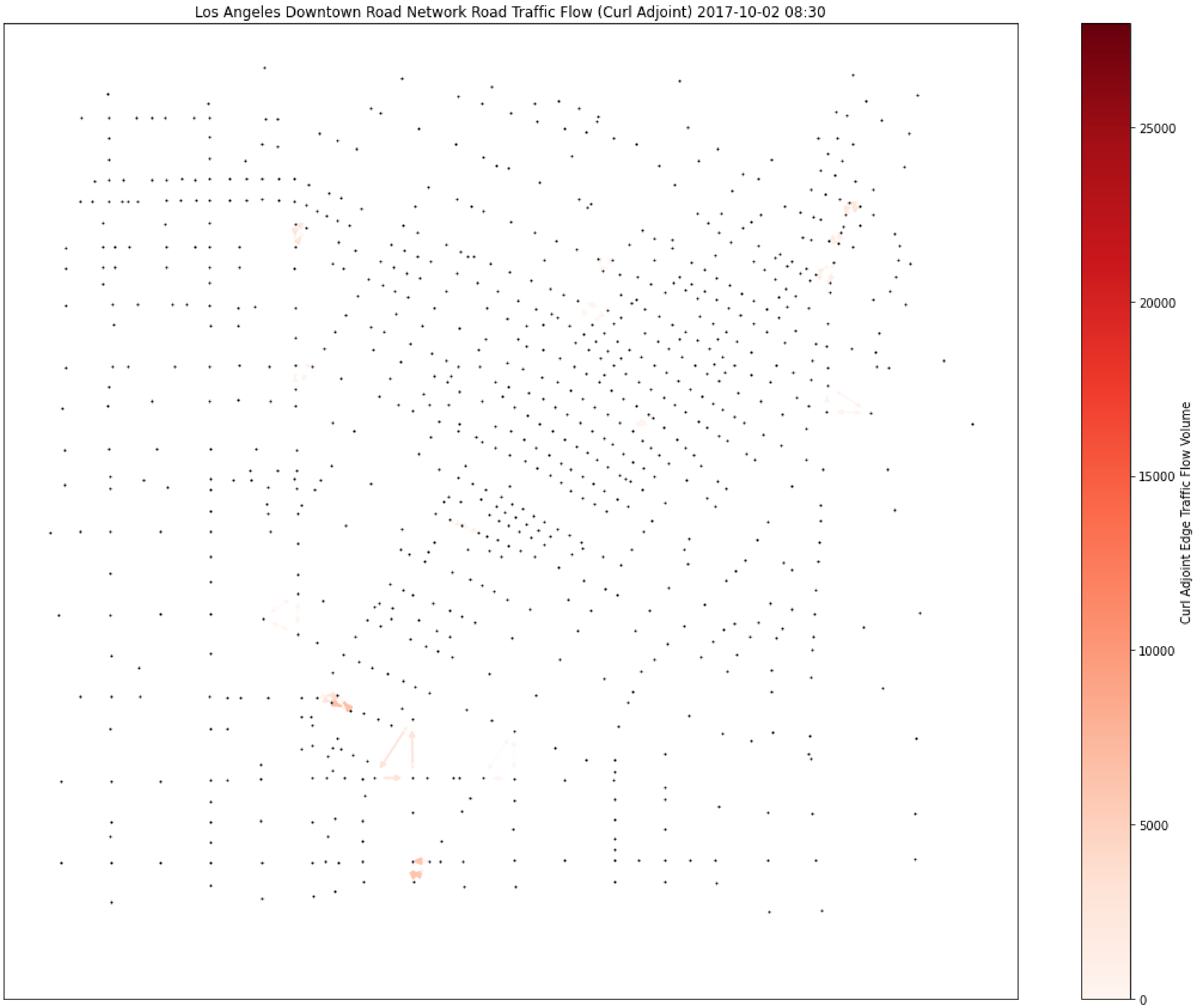


Los Angeles Downtown Road Network Road Traffic Flow (Assymmetric) 2017-10-02 08:30
Edge Traffic Flow Volume
0
5000
10000
15000
20000
25000

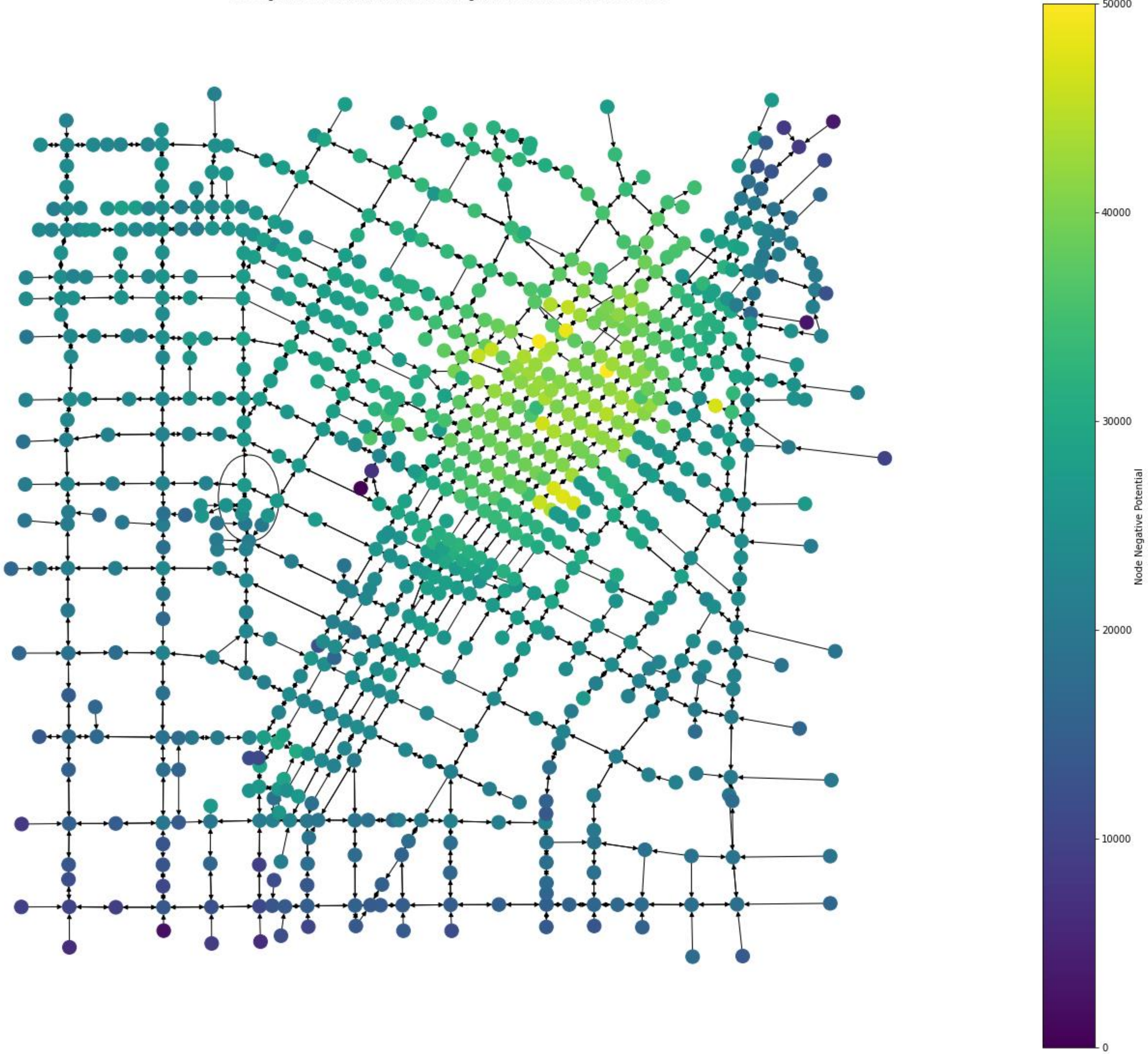

Los Angeles Downtown Road Network Road Traffic Flow (Gradient) 2017-10-02 08:30
Gradient Edge Traffic Flow Volume
25000
20000
15000
10000
5000
0



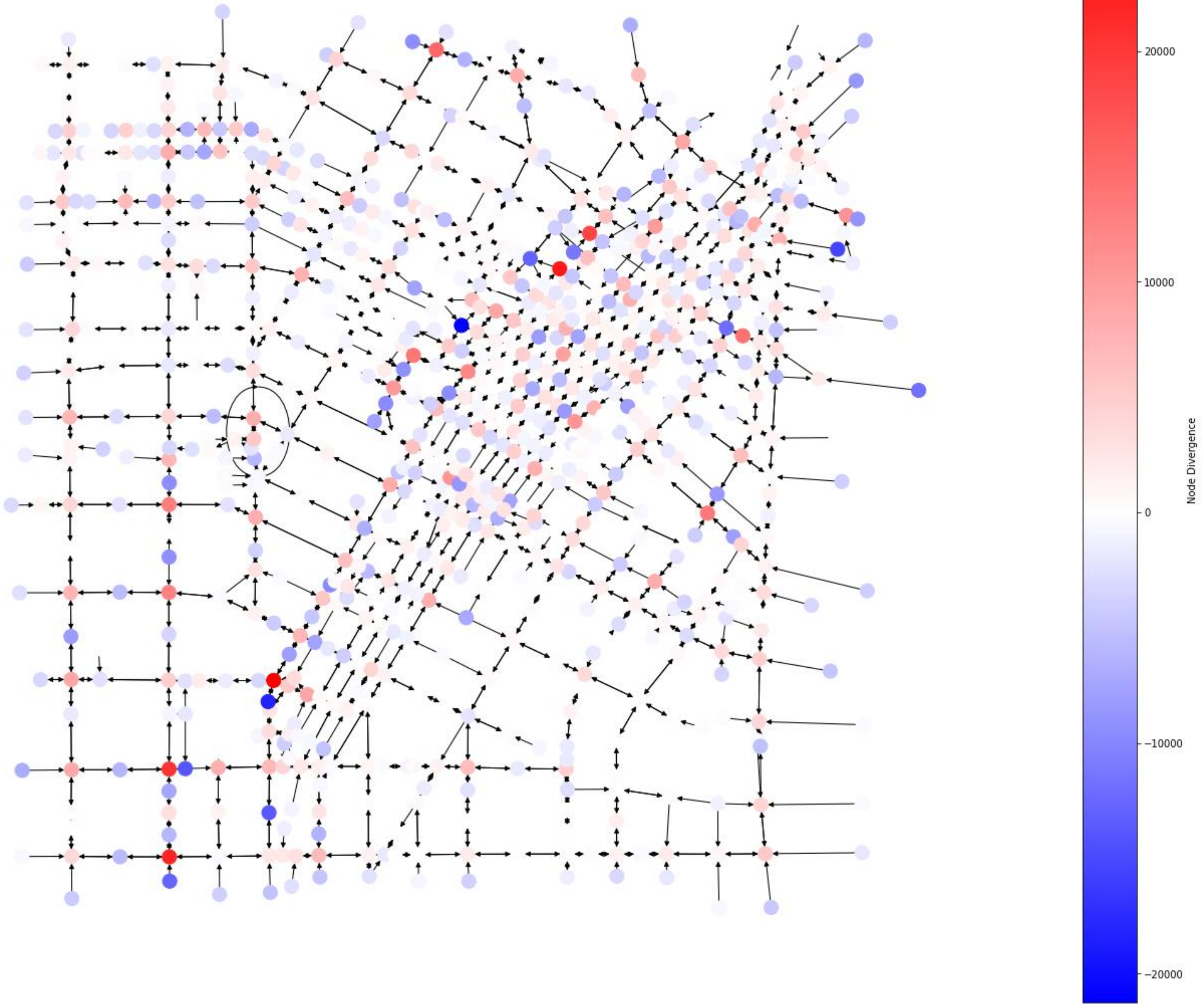
Los Angeles Downtown Road Network Road Traffic Flow (Harmonic) 2017-10-02 08:30
Harmonic Edge Traffic Flow Volume
25000
20000
15000
10000
5000
0

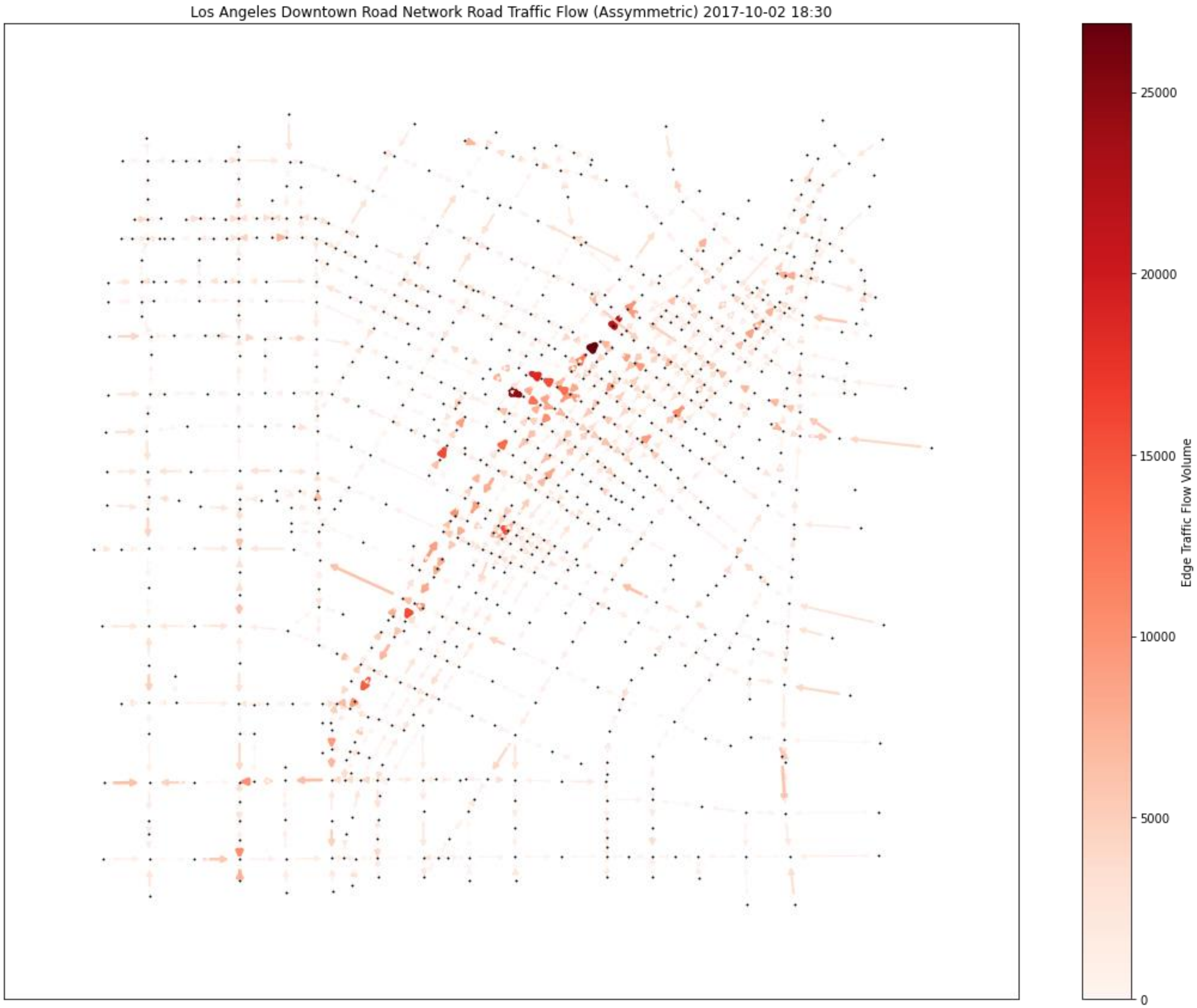

Los Angeles Downtown Road Network Road Traffic Flow (Curl Adjoint) 2017-10-02 08:30
Curl Adjoint Edge Traffic Flow Volume
25000
20000
15000
10000
5000
0

Los Angeles Downtown Road Network Negative Potential 2017-10-02 08:30

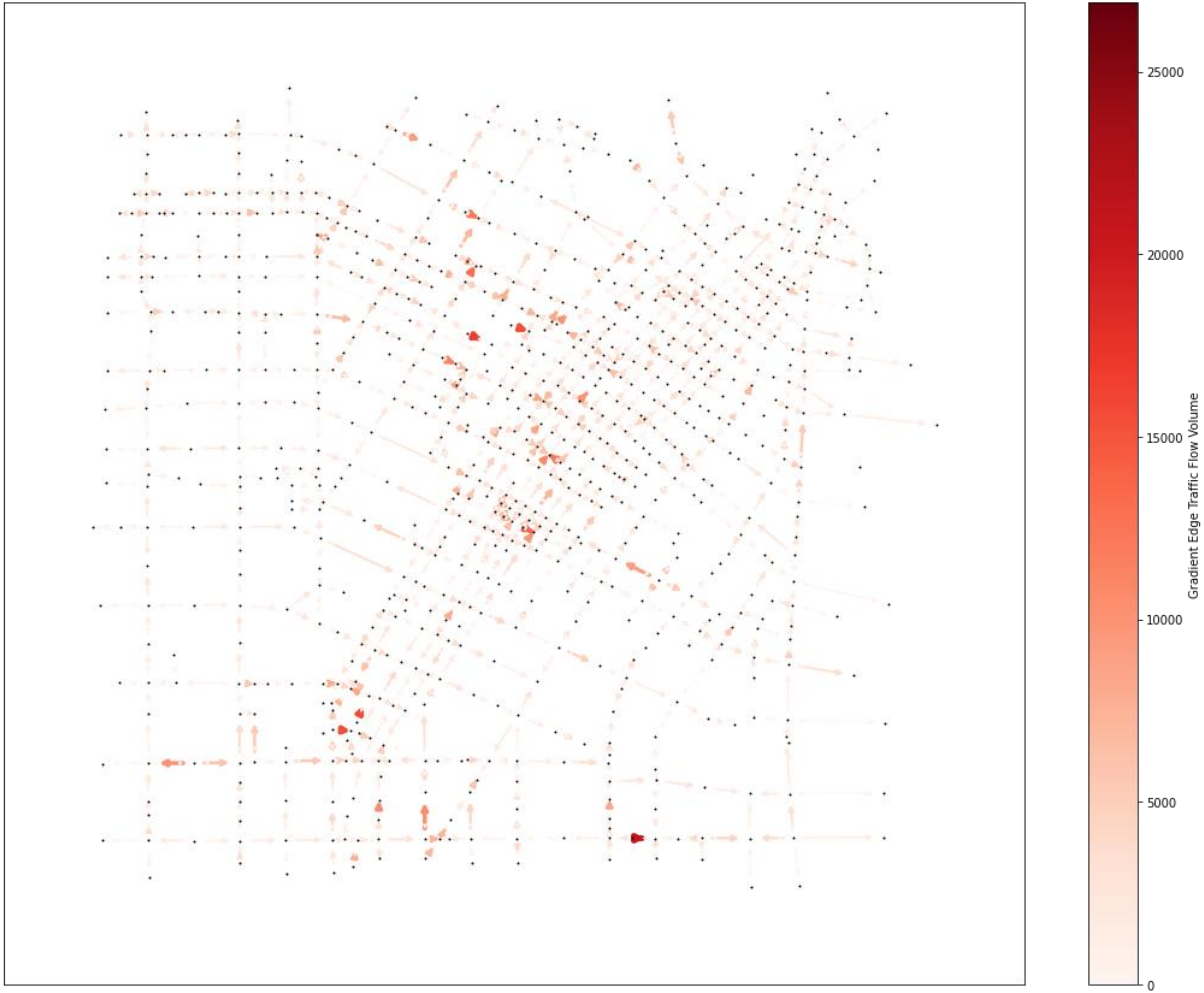


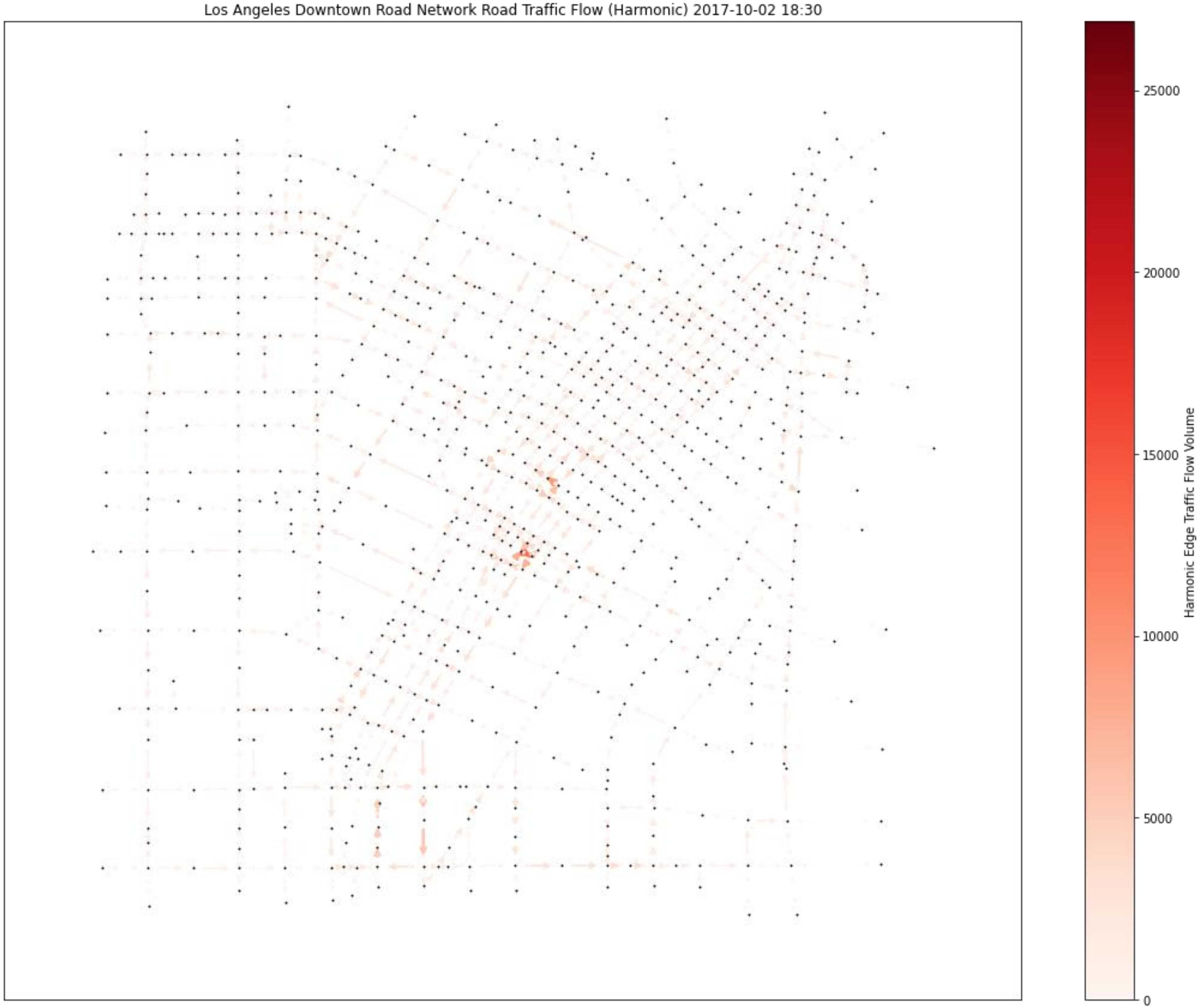
Los Angeles Downtown Road Network Divergence 2017-10-02 08:30




Los Angeles Downtown Road Network Road Traffic Flow (Assymmetric) 2017-10-02 18:30
Edge Traffic Flow Volume
0
5000
10000
15000
20000
25000

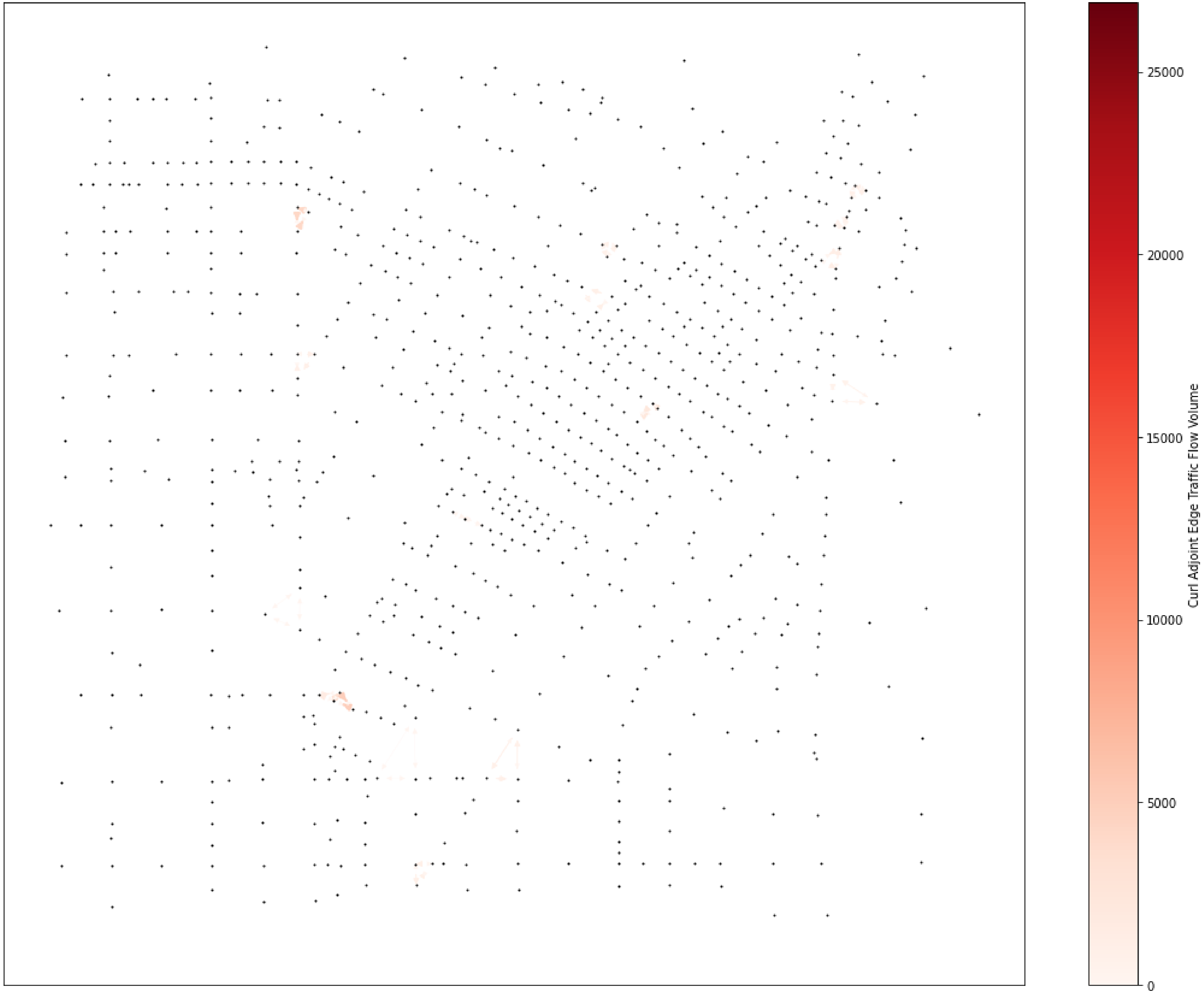
Los Angeles Downtown Road Network Road Traffic Flow (Gradient) 2017-10-02 18:30

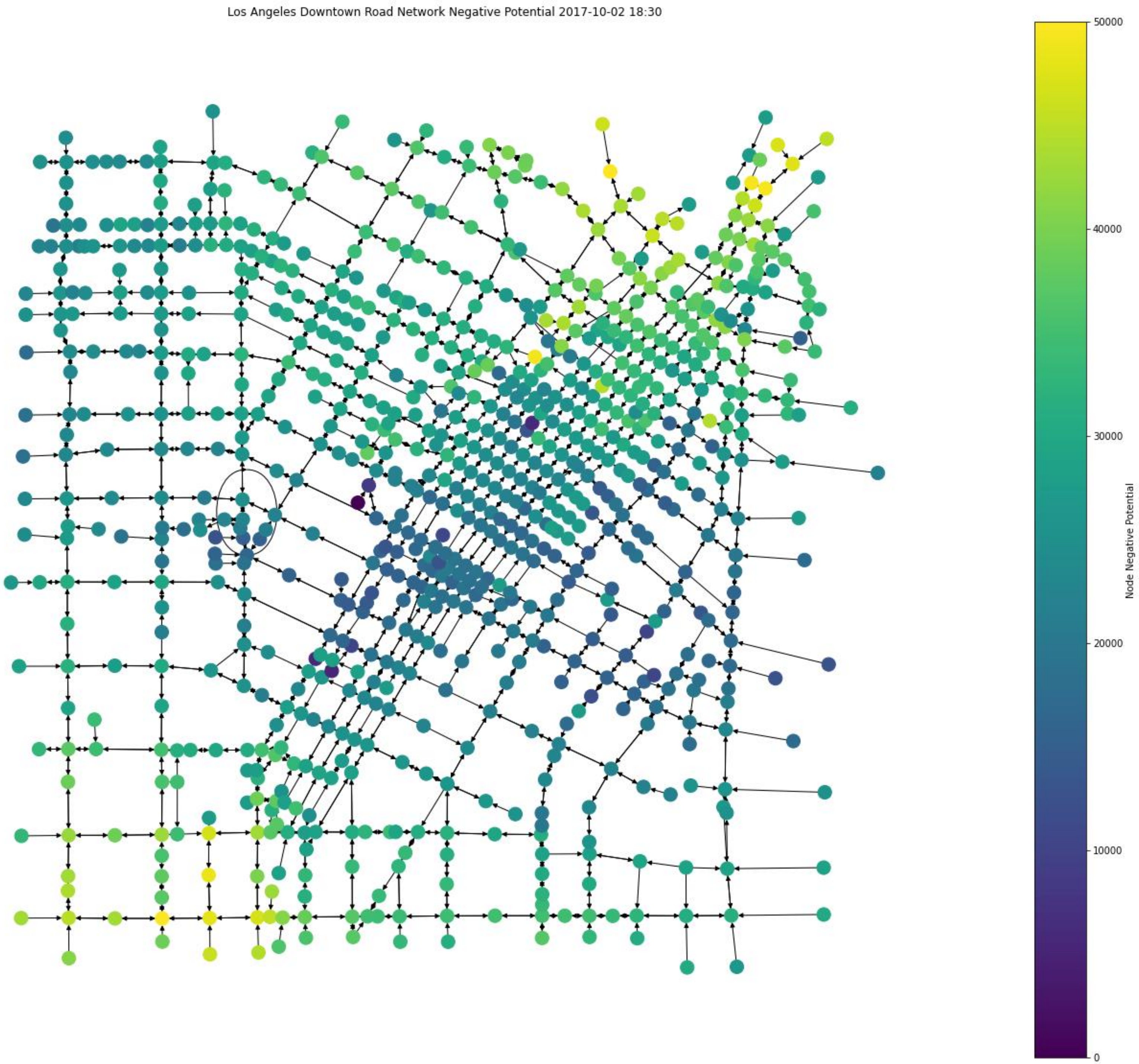


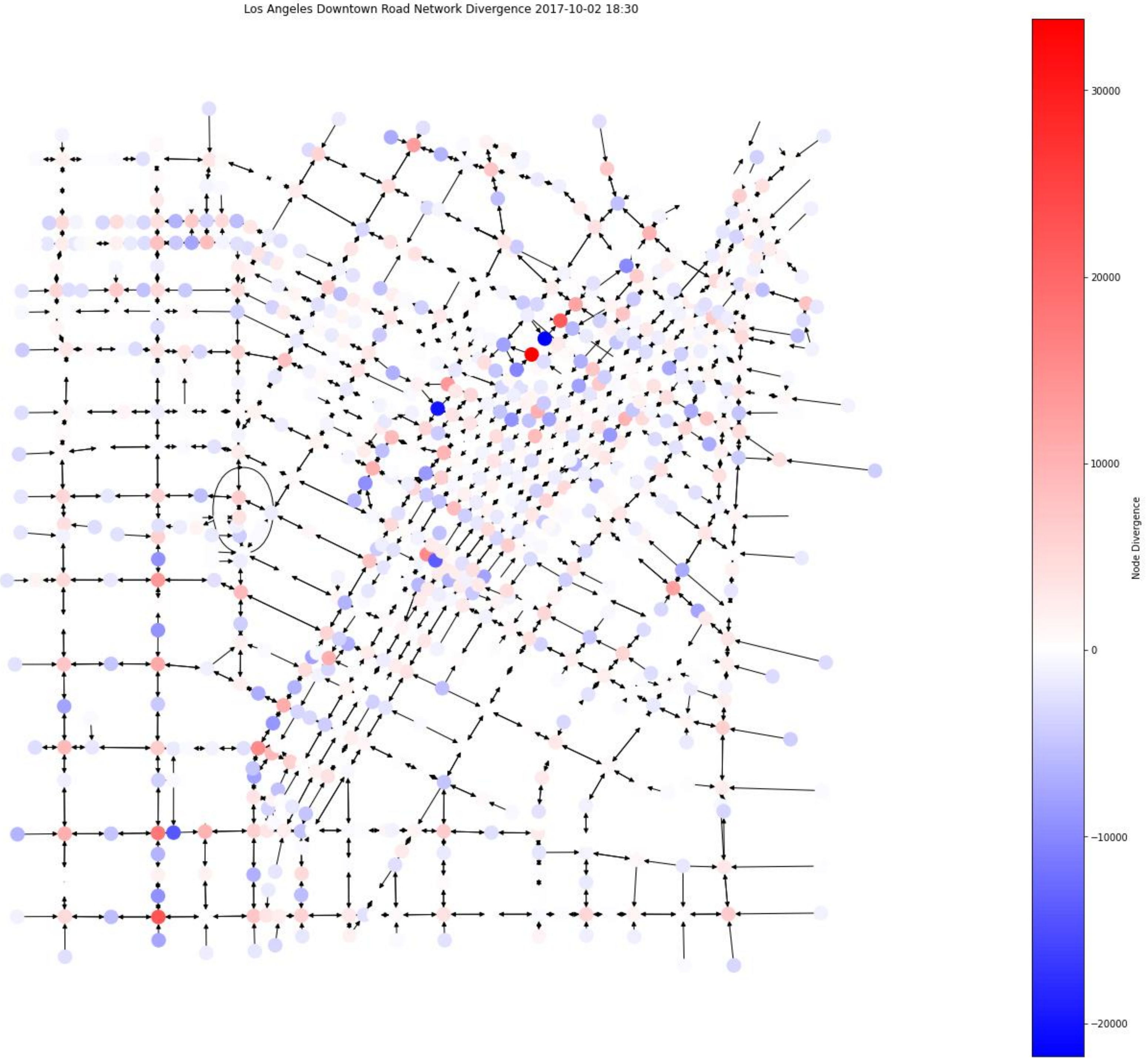


Los Angeles Downtown Road Network Road Traffic Flow (Harmonic) 2017-10-02 18:30
Harmonic Edge Traffic Flow Volume
25000
20000
15000
10000
5000
0

Los Angeles Downtown Road Network Road Traffic Flow (Curl Adjoint) 2017-10-02 18:30



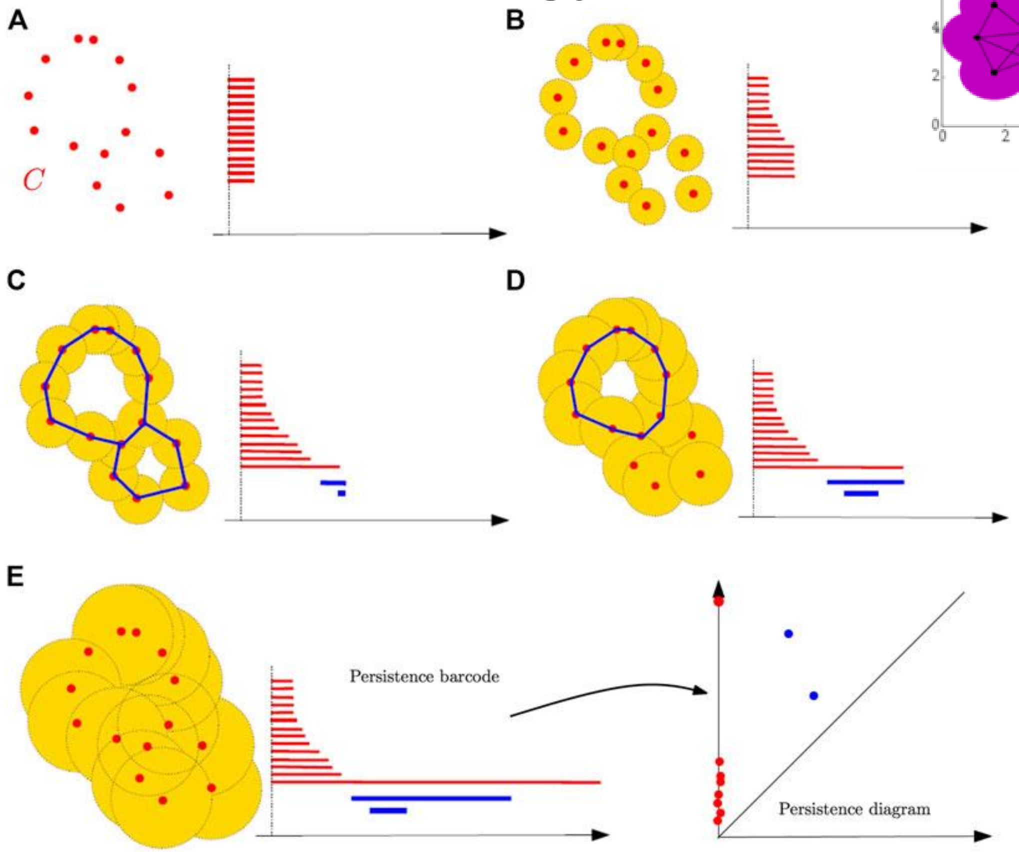

Los Angeles Downtown Road Network Negative Potential 2017-10-02 18:30
50000
40000
30000
20000
10000
0
Node Negative Potential

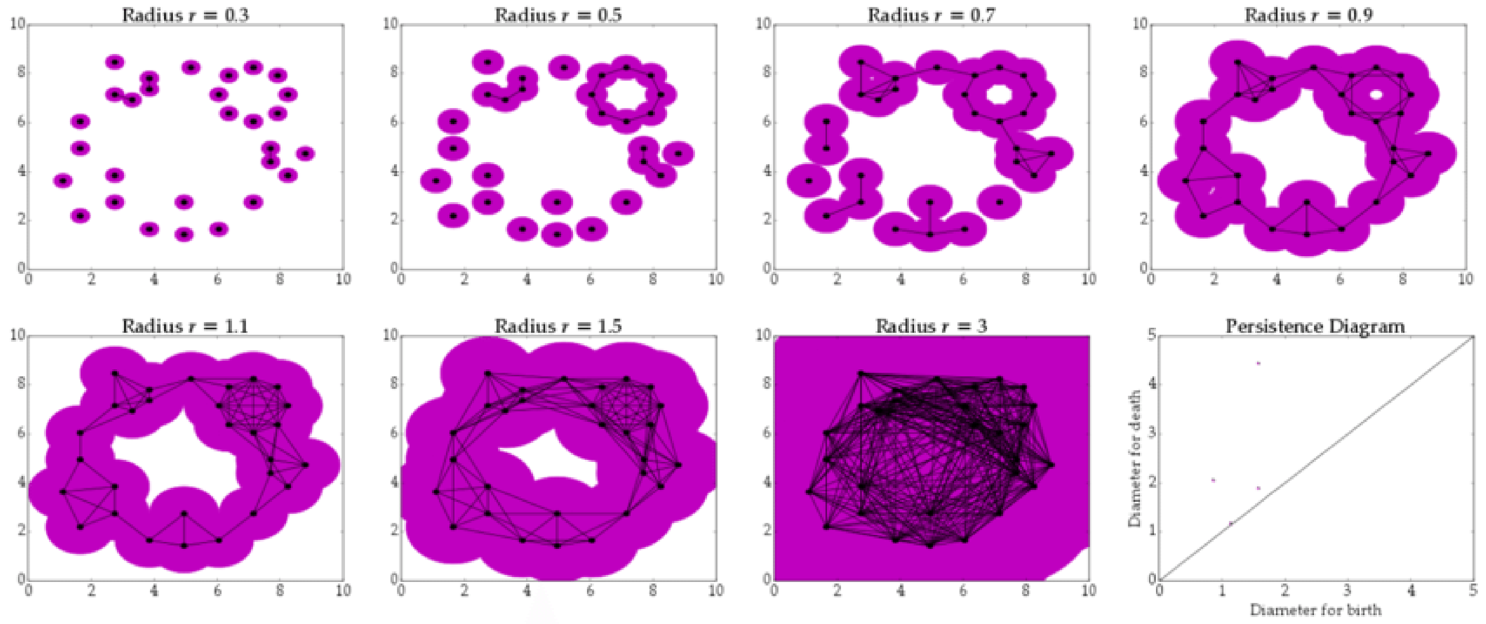

Los Angeles Downtown Road Network Divergence 2017-10-02 18:30
30000
20000
10000
0
−10000
−20000
Node Divergence

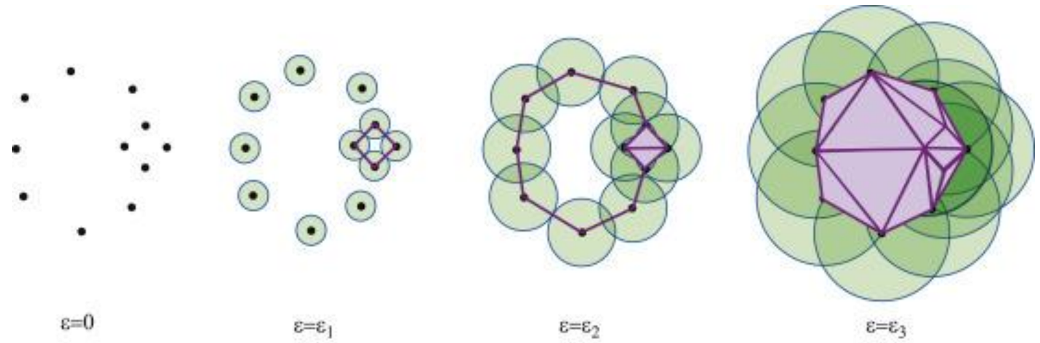
Related topics

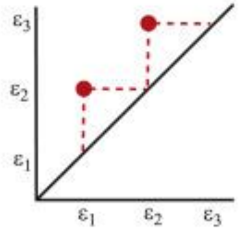
- Topological data analysis

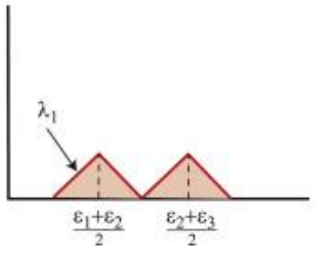
Persistent homology











Other applications

- (i) ranking
- (ii) graphics and imaging
- (iii) games and traffic flows
- (iv) brain networks
- (v) data representations
- (vi) deep learning
- (vii) denoising
- (viii) dimension reduction
- (ix) link prediction
- (x) object synchronization
- (xi) sensor network coverage
- (xii) cryo-electron microscopy
- (xiii) generalizing effective resistance to simplicial complexes
- (xiv) modeling biological interactions between a set of molecules or communication systems with group messages

Reference:

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- Lim, L. H. Hodge Laplacians on Graphs.
- Zian Jiang. HodgeRank: Generating Movie Ranking From IMDb Movie Ratings (Part 1: Problem Formulation) https://medium.com/@zj444/hodgerank-generating-movie-ranking-from-imdb-movie-ratings-part-1-2a88ec148f10
- Aoki T, Fujishima S and Fujiwara N (2022) Urban spatial structures from human flow by Hodge-Kodaira decomposition. Sci. Rep. 12(1): 11258.
- The derivative isn't what you think it is. https://www.youtube.com/watch?v=2ptFnIj71SM