Effective resistance/Resistance distance

Imagine a potential function on nodes $\boldsymbol{p}_{|V|\times 1}$

This potential function causes the voltage $\boldsymbol{v}_{|E|\times 1} = \boldsymbol{M}_{|E|\times|V|}\boldsymbol{p}_{|V|\times 1}$

This voltage function causes current $\boldsymbol{i}_{|E|\times 1} = \operatorname{diag}(\boldsymbol{r}_{|E|\times 1})^{-1}_{|E|\times|E|}\boldsymbol{v}_{|E|\times 1}$

If we specify the sink and source nodes in the graph $\boldsymbol{d}_{|V|\times 1} = \boldsymbol{M}^T_{|E|\times|V|}\boldsymbol{i}_{|E|\times 1}$

Assume there is only one sink node and one source node in the network, we can get the effective resistance between those two nodes

$$\boldsymbol{d}_{|V|\times 1} = \boldsymbol{M}^T_{|E|\times|V|}\operatorname{diag}(\boldsymbol{r}_{|E|\times 1})^{-1}_{|E|\times|E|}\boldsymbol{M}_{|E|\times|V|}\boldsymbol{p}_{|V|\times 1}$$

For source node i, $(\boldsymbol{d}_{|V|\times 1})_i = d$, for sink node j, $(\boldsymbol{d}_{|V|\times 1})_j = -d$

$$\boldsymbol{p}_{|V|\times 1} = \left(\boldsymbol{M}^T_{|E|\times|V|}\operatorname{diag}(\boldsymbol{r}_{|E|\times 1})^{-1}_{|E|\times|E|}\boldsymbol{M}_{|E|\times|V|}\right)^{\dagger}\boldsymbol{d}_{|V|\times 1}$$

Let $\left(\boldsymbol{M}^T_{|E|\times|V|}\operatorname{diag}(\boldsymbol{r}_{|E|\times 1})^{-1}_{|E|\times|E|}\boldsymbol{M}_{|E|\times|V|}\right)^{\dagger} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$ be a generalized graph Laplacian, which does not only take the connection and adjacency into account, but also the resistance of every connection. Every 1 in graph Laplacian is replaced by $1/r_i$. This generalized graph Laplacian also satisfies the property of graph Laplacian, $\tilde{\boldsymbol{L}}_{|V|\times|V|}\boldsymbol{1}_{|V|\times 1} = \boldsymbol{0}_{|V|\times 1}$

Effective resistance between i and j is

$$\tilde{r}_{ij} = \boldsymbol{m_k}_{|V|\times 1}{}^T\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m_k}_{|V|\times 1} = \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{ii}} + \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{jj}} - \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{ij}} - \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{ji}}$$

$\boldsymbol{m_k}_{|V|\times 1}$ is a $|V|\times 1$ vector with entries i and j being 1 and -1

Defined on nodes	Defined on edges
Potential $\boldsymbol{p}_{|V|\times 1}$	Voltage $\boldsymbol{v}_{|E|\times 1}$
Divergence $\boldsymbol{d}_{|V|\times 1}$	Current $\boldsymbol{i}_{|E|\times 1}$

$$\begin{tikzcd}
\text{Potential } \boldsymbol{p}_{|V|\times 1} \arrow[r, leftrightarrow, "\boldsymbol{v}_{|E|\times 1} = \boldsymbol{M}_{|E|\times|V|}\boldsymbol{p}_{|V|\times 1}"] \arrow[d, leftrightarrow] & \text{Voltage } \boldsymbol{v}_{|E|\times 1} \arrow[d, leftrightarrow] \\
\text{Divergence } \boldsymbol{d}_{|V|\times 1} \arrow[r, leftrightarrow, "\boldsymbol{d}_{|V|\times 1} = \boldsymbol{M}^T_{|E|\times|V|}\boldsymbol{i}_{|E|\times 1}"] & \text{Current } \boldsymbol{i}_{|E|\times 1}
\end{tikzcd}$$

Resistance Laplacian matrix/Resistance Graph Laplacian

$$\tilde{\boldsymbol{L}}_{|V|\times|V|} = \boldsymbol{M}^T_{|E|\times|V|}\operatorname{diag}(\boldsymbol{r}_{|E|\times 1})^{-1}_{|E|\times|E|}\boldsymbol{M}_{|E|\times|V|} = \begin{bmatrix}\boldsymbol{m_1}_{|V|\times 1} & \boldsymbol{m_2}_{|V|\times 1} & \cdots & \boldsymbol{m}_{|E|}{}_{|V|\times 1}\end{bmatrix}\begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{|E|}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{m_1}^{\boldsymbol{T}}_{|V|\times 1} \\ \boldsymbol{m_2}^{\boldsymbol{T}}_{|V|\times 1} \\ \vdots \\ \boldsymbol{m}_{|E|}{}^{\boldsymbol{T}}_{|V|\times 1}\end{bmatrix}$$

$$= \begin{bmatrix}\boldsymbol{m_1}_{|V|\times 1} & \boldsymbol{m_2}_{|V|\times 1} & \cdots & \boldsymbol{m}_{|E|}{}_{|V|\times 1}\end{bmatrix}\begin{bmatrix} \frac{1}{r_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{r_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{r_{|E|}}\end{bmatrix}\begin{bmatrix}\boldsymbol{m_1}^{\boldsymbol{T}}_{|V|\times 1} \\ \boldsymbol{m_2}^{\boldsymbol{T}}_{|V|\times 1} \\ \vdots \\ \boldsymbol{m}_{|E|}{}^{\boldsymbol{T}}_{|V|\times 1}\end{bmatrix} = \sum_{i=1}^{|E|}\frac{1}{r_i}\boldsymbol{m}_{i}{}_{|V|\times 1}\boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}$$

$$\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$$

$$\boldsymbol{m}_{i|V|\times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}^{\boldsymbol{T}}_{i|V|\times 1} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & 0 & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & 0 & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\boldsymbol{L}_{|V|\times|V|} = \boldsymbol{M}^{\boldsymbol{T}}_{|E|\times|V|}\boldsymbol{M}_{|E|\times|V|}$$

$\tilde{\boldsymbol{L}}_{|V|\times|V|}$ is semi-positive definite (same as $\boldsymbol{L}_{|V|\times|V|}$): $\boldsymbol{x}^{T}_{|V|\times 1}\tilde{\boldsymbol{L}}_{|V|\times|V|}\boldsymbol{x}_{|V|\times 1} \geq 0$

$$\tilde{\boldsymbol{L}}_{|V|\times|V|} = \boldsymbol{Q}\begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{Q}^{\boldsymbol{T}} = \sum_{i=1}^{k} \lambda_i \boldsymbol{q}_i \boldsymbol{q}_i^T, \qquad \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = \boldsymbol{Q}\begin{bmatrix} \boldsymbol{\Lambda}^{-\boldsymbol{1}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{Q}^{\boldsymbol{T}} = \sum_{i=1}^{k} \frac{1}{\lambda_i} \boldsymbol{q}_i \boldsymbol{q}_i^T$$

$$\tilde{\boldsymbol{L}}_{|V|\times|V|}\boldsymbol{1}_{|V|\times 1} = \boldsymbol{0}_{|V|\times 1} \ \text{(same as } \boldsymbol{L}_{|V|\times|V|}\text{)}$$

Kirchhoff index

Kirchhoff index is the sum of effective resistances of all pairs of i and j in the network

$\tilde{r}_{ij}$ effective resistance between node i and j

$$\tilde{r}_{ij} = \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{ii}} + \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{jj}} - \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{ij}} - \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{ji}}$$

$$\tilde{r}_{ii} = 0, \qquad \forall i$$

Kirchhoff index i_K

$$i_K = \sum_{i\neq j} \tilde{r}_{ij} = 2(|V|-1)\mathrm{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) - 2\sum_{i\neq j}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{ij}} = 2|V|\mathrm{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) - 2\sum_{i,j}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{ij}} = \sum_{i,j} \tilde{r}_{ij}$$

$$\sum_{i,j}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}\right)_{\boldsymbol{ij}} = \boldsymbol{1}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}_{|V|\times|V|}\boldsymbol{1}_{|V|\times 1} = \boldsymbol{1}^{\boldsymbol{T}}_{|V|\times 1}\boldsymbol{M}^{T}_{|E|\times|V|}\mathrm{diag}\left(\boldsymbol{r}_{|E|\times 1}\right)^{-1}_{|E|\times|E|}\boldsymbol{M}_{|E|\times|V|}\boldsymbol{1}_{|V|\times 1} = \boldsymbol{0}^{T}_{|E|\times 1}\mathrm{diag}\left(\boldsymbol{r}_{|E|\times 1}\right)^{-1}_{|E|\times|E|}\boldsymbol{0}_{|E|\times 1} = 0$$

$$\tilde{\boldsymbol{L}}_{|V|\times|V|}\boldsymbol{1}_{|V|\times 1} = \boldsymbol{0}_{|V|\times 1}$$

$\boldsymbol{1}_{|V|\times 1}$ is in the kernel of $\tilde{\boldsymbol{L}}_{|V|\times|V|}$, $\boldsymbol{1}_{|V|\times 1}$ is also in the kernel of $\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$ ($\tilde{\boldsymbol{L}}_{|V|\times|V|}$ and $\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$ have the same bases, eigenvalues are inversed)

$$\tilde{\boldsymbol{L}}_{|V|\times|V|} = \boldsymbol{Q}\begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{Q}^{\boldsymbol{T}} = \sum_{i=1}^{k} \lambda_i \boldsymbol{q}_i \boldsymbol{q}_i^T, \qquad \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = \boldsymbol{Q}\begin{bmatrix} \boldsymbol{\Lambda}^{-\boldsymbol{1}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{Q}^{\boldsymbol{T}} = \sum_{i=1}^{k} \frac{1}{\lambda_i} \boldsymbol{q}_i \boldsymbol{q}_i^T$$

$$\sum_{i,j}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{\boldsymbol{ij}} = \boldsymbol{1}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{1}_{|V|\times 1} = 0$$

$$i_K = 2|V|\operatorname{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) = 2|V|\sum_{i=1,\lambda_i\neq 0}^{|V|}\frac{1}{\lambda_i}$$

This is the sum of inverses of all non-zero eigenvalues of the graph Laplacian $\boldsymbol{L}_{|V|\times|V|}$.

The inverses of eigenvalues of zero are infinities. One infinity indicates a pair of components that are not connected.

The number of zero eigenvalues indicates the number of connected components in the graph.

Arrange all $\tilde{r}_{ij}$ into a matrix $\widetilde{\boldsymbol{R}}$,

$$\widetilde{\boldsymbol{R}} = \operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}}_{|V|\times 1} + \boldsymbol{1}_{|V|\times 1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}} - 2\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$$

$\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)$ is the vertical vector gotten from the diagonal values of the matrix $\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$

$$\boldsymbol{1}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{1}_{|V|\times 1} = 0$$

$$i_K = \sum_{i\neq j}\tilde{r}_{ij} = \boldsymbol{1}^{\boldsymbol{T}}\widetilde{\boldsymbol{R}}\boldsymbol{1} = \boldsymbol{1}^{\boldsymbol{T}}\left(\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}} + \boldsymbol{1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}} - 2\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}$$

$$= \boldsymbol{1}^{\boldsymbol{T}}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}}\boldsymbol{1} + \boldsymbol{1}^{\boldsymbol{T}}\boldsymbol{1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{1} - 2\boldsymbol{1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{1} = \boldsymbol{1}^{\boldsymbol{T}}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)|V| + |V|\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{1}$$

$$= |V|\left(\boldsymbol{1}^{\boldsymbol{T}}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) + \operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{1}\right) = 2|V|\boldsymbol{1}^{\boldsymbol{T}}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) = 2|V|\operatorname{tr}\left(\boldsymbol{L}^{\dagger}_{|V|\times|V|}\right)$$

Average Effective Resistance of All Pairs of Two Nodes in a Graph

$$\bar{\tilde{r}} = \frac{i_K}{|V|^2} = \frac{\sum_{i\neq j}\tilde{r}_{ij}}{|V|^2} = \frac{2}{|V|}\sum_{i=1,\lambda_i\neq 0}^{|V|}\frac{1}{\lambda_i} = \frac{2}{|V|}\operatorname{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)$$

$$\bar{\tilde{r}} = \boldsymbol{w}^{\boldsymbol{T}}\widetilde{\boldsymbol{R}}\boldsymbol{w} = \frac{1}{|V|}\boldsymbol{1}^{\boldsymbol{T}}\widetilde{\boldsymbol{R}}\frac{1}{|V|}\boldsymbol{1} = \frac{2}{|V|}\operatorname{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)$$

$$\frac{\operatorname{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)}{|V|} = \frac{1}{|V|}\sum_{i=1,\lambda_i\neq 0}^{|V|}\frac{1}{\lambda_i}$$

can be used as a measurement for the value of the average effective resistance of all pairs of two nodes in a graph

Technically, graphs with different numbers of components can not be compared in terms of Kirchhoff index or overall resistance. Number of components indicates the number of infinities in resistances. Number of connected components is like the degree or dimension of connectedness of the graph.

Weighted Average Effective Resistance of All Pairs of Two Nodes in a Graph 1

If the sum is weighted with respect to the product of weights of the two end nodes of an edge, assume $\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{1} = 1$
the weight is exerted on nodes, the weight of pair of two nodes is the product of weights if two nodes, i.e. $w_{ij} = w_i w_j$
the weight on a node could be interpreted as the importance of that node, i.e. population of that node in the Thiessen polygon
$w_{ij} = w_i w_j$ means that the probability of a random start point being i is w_i and the probability of a random end point being j is w_j, the

probability of trip $i \to j$ is $w_{ij} = w_i w_j$, $\boldsymbol{W} = \boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}$

$$\bar{\tilde{r}}_{\text{weighted 1}} = \sum_{i \neq j} w_i w_j \tilde{r}_{ij} = \mathbf{1}^{\boldsymbol{T}}(\boldsymbol{W} \odot \widetilde{\boldsymbol{R}})\mathbf{1} = \mathbf{1}^{\boldsymbol{T}}\left((\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}) \odot \widetilde{\boldsymbol{R}}\right)\mathbf{1} = \boldsymbol{w}^{\boldsymbol{T}}\widetilde{\boldsymbol{R}}\boldsymbol{w} = \boldsymbol{w}^{\boldsymbol{T}}\left(\operatorname{diag}(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|})\mathbf{1}^{\boldsymbol{T}} + \mathbf{1}\operatorname{diag}(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|})^{\boldsymbol{T}} - 2\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{w}$$

$$= \boldsymbol{w}^{\boldsymbol{T}}\operatorname{diag}(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|})\mathbf{1}^{\boldsymbol{T}}\boldsymbol{w} + \boldsymbol{w}^{\boldsymbol{T}}\mathbf{1}\operatorname{diag}(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|})^{\boldsymbol{T}}\boldsymbol{w} - 2\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w} = 2\boldsymbol{w}^{\boldsymbol{T}}\mathbf{1}\operatorname{diag}(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|})^{\boldsymbol{T}}\boldsymbol{w} - 2\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}$$

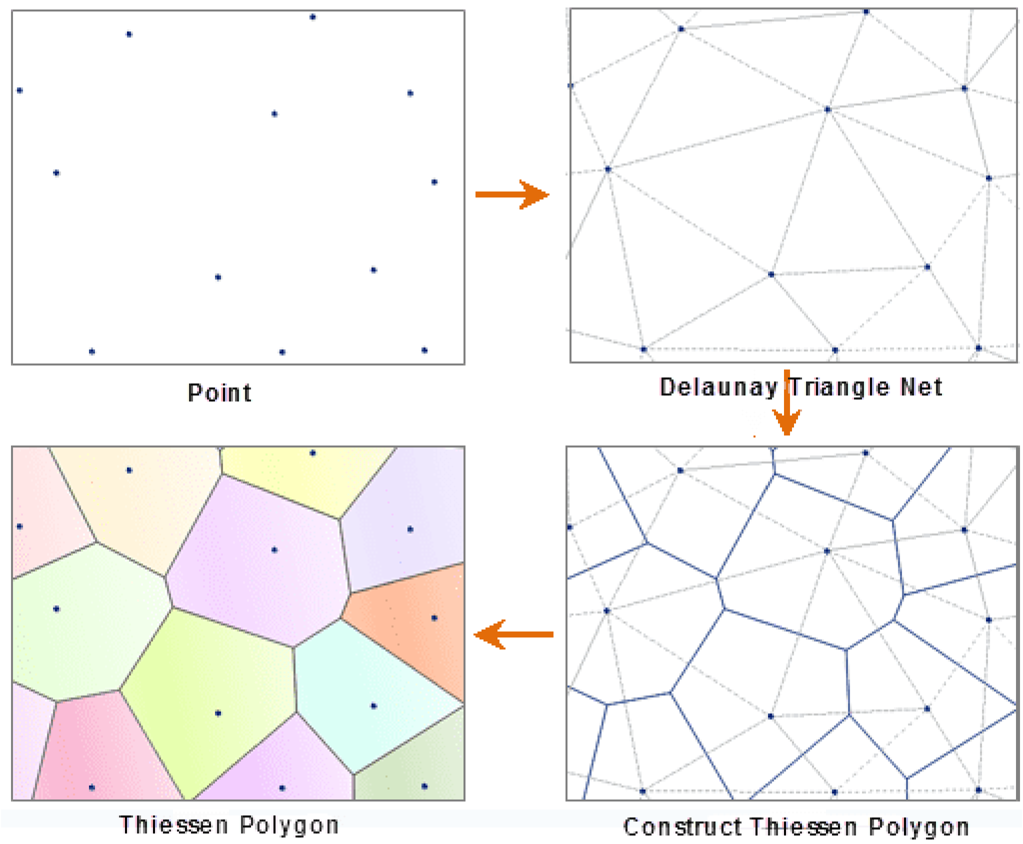
$$= 2\operatorname{diag}(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|})^{\boldsymbol{T}}\boldsymbol{w} - 2\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}$$

$\odot$ is Hadamard product

$\operatorname{diag}(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|})^{\boldsymbol{T}}\boldsymbol{w} - \boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}$ can be used as a measurement for the value of the weighted average effective resistance of all pairs of two nodes in a graph

We consider this weighted average because the unweighted sum tends to be affected by peripheral edges in the graph, which are areas with low population (in later part of rank-one update of resistance graph Laplacian/remove one edge, edges creating large increases in unweighted average effective resistance tend to be peripheral edges). We intend to use this weighted average to rectify this and identify edges not on the periphery.

Thiessen polygon



Weighted Average Effective Resistance of All Pairs of Two Nodes in a Graph 2

However the real scenario is that people do not travel uniformly with respect to the population distribution. The assumption of the weighted matrix $\boldsymbol{W}$ being a rank-one matrix $\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}$ is not well guaranteed. We could use the origin-destination matrix $\boldsymbol{A}$ to get the weight of every entry of two nodes (one origin node and one destination node).

$$\boldsymbol{W} = \frac{\boldsymbol{A}}{\boldsymbol{1}^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{1}}$$

$$\boldsymbol{1}^{\boldsymbol{T}}\boldsymbol{W}\boldsymbol{1} = 1$$

$$\sum_{i\neq j} w_{ij}\tilde{r}_{ij} = \boldsymbol{1}^{\boldsymbol{T}}(\boldsymbol{W}\odot\widetilde{\boldsymbol{R}})\boldsymbol{1} = \boldsymbol{1}^{\boldsymbol{T}}\left(\boldsymbol{W}\odot\left(\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}} + \boldsymbol{1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}} - 2\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\right)\boldsymbol{1}$$

$$= \boldsymbol{1}^{\boldsymbol{T}}\left(\boldsymbol{W}\odot\left(\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}}\right)\right)\boldsymbol{1} + \boldsymbol{1}^{\boldsymbol{T}}\left(\boldsymbol{W}\odot\left(\boldsymbol{1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\right)\right)\boldsymbol{1} - 2\boldsymbol{1}^{\boldsymbol{T}}\left(\boldsymbol{W}\odot\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}$$

$$\boldsymbol{1}^{\boldsymbol{T}}\left(\boldsymbol{W}\odot\left(\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}}\right)\right)\boldsymbol{1} = \operatorname{tr}\left(\boldsymbol{I}\boldsymbol{W}\boldsymbol{I}\left(\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}}\right)^{\boldsymbol{T}}\right) = \operatorname{tr}\left(\boldsymbol{W}\boldsymbol{1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\right)$$

$$\boldsymbol{1}^{\boldsymbol{T}}\left(\boldsymbol{W}\odot\left(\boldsymbol{1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\right)\right)\boldsymbol{1} = \operatorname{tr}\left(\boldsymbol{I}\boldsymbol{W}\boldsymbol{I}\left(\boldsymbol{1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}}\right) = \operatorname{tr}\left(\boldsymbol{W}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}}\right)$$

If assume $\boldsymbol{W}$ is symmetric (this is reasonable in a commute network in a city because going to work and going home, etc. are symmetric)

$$\operatorname{tr}\left(\boldsymbol{W}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}}\right) = \operatorname{tr}\left(\boldsymbol{W}^{\boldsymbol{T}}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1}^{\boldsymbol{T}}\right) = \operatorname{tr}\left(\boldsymbol{1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W}\right) = \operatorname{tr}\left(\boldsymbol{W}\boldsymbol{1}\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\right) = \operatorname{tr}\left(\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W}\boldsymbol{1}\right)$$

$$= \operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W}\boldsymbol{1}$$

$$\boldsymbol{1}^{\boldsymbol{T}}\left(\boldsymbol{W}\odot\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{1} = \operatorname{tr}\left(\boldsymbol{I}\boldsymbol{W}\boldsymbol{I}\tilde{\boldsymbol{L}}^{\dagger\,\boldsymbol{T}}_{|V|\times|V|}\right) = \operatorname{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)$$

$$\sum_{i\neq j} w_{ij}\tilde{r}_{ij} = 2\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W}\boldsymbol{1} - 2\operatorname{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)$$

the weighted matrix $\boldsymbol{W}$ can be interpreted as the needs of every trip in the graph

$\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W}\boldsymbol{1} - \operatorname{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)$ can be used as a measurement for the value of the weighted average effective resistance of all pairs of two nodes in a graph

average effective resistance

$$\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W}\boldsymbol{1} - \operatorname{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) \overset{\boldsymbol{W}=\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}}{\Rightarrow} \operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{w} - \boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w} \overset{\boldsymbol{w}=\frac{1}{|V|}\boldsymbol{1}}{\Rightarrow} \frac{\operatorname{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)}{|V|} = \frac{1}{|V|}\sum_{i=1,\lambda_i\neq 0}^{|V|}\frac{1}{\lambda_i}$$

$$\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W}\boldsymbol{1} - \operatorname{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) = \operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{1} - \operatorname{tr}\left(\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) = \operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{w} - \operatorname{tr}\left(\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}\right)$$

$$= \operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{w} - \boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}$$

$$\operatorname{diag}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{w}-\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{w}=\operatorname{diag}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)^{\boldsymbol{T}}\frac{1}{|V|}\mathbf{1}-\frac{1}{|V|}\mathbf{1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\frac{1}{|V|}\mathbf{1}=\frac{1}{|V|}\operatorname{tr}\left(\boldsymbol{L}_{|V|\times|V|}^{\dagger}\right)$$

Properties of Hadamard product

$$\boldsymbol{x}^{\boldsymbol{H}}(\boldsymbol{A}\odot\boldsymbol{B})\boldsymbol{y}=\operatorname{tr}\left(\boldsymbol{D}_{\boldsymbol{x}}^{\boldsymbol{H}}\boldsymbol{A}\boldsymbol{D}_{\boldsymbol{y}}\boldsymbol{B}^{\boldsymbol{T}}\right)$$

$\boldsymbol{D}_{\boldsymbol{x}}$ is the diagonal matrix created from vector $\boldsymbol{x}$

Explanation of eigenvalues in topology

$$\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times p}=\mathbf{0}_{m\times p}\ \rightarrow \mathbb{R}^{n}=\operatorname{im}(\boldsymbol{B}_{n\times p})\oplus\ker\left(\boldsymbol{A}_{m\times n}^{T}\boldsymbol{A}_{m\times n}+\boldsymbol{B}_{n\times p}\boldsymbol{B}_{n\times p}^{T}\right)\oplus\operatorname{im}(\boldsymbol{A}_{m\times n}^{T})$$

$$\text{curl grad}=\mathbf{0}_{m\times p}\ \rightarrow$$

$$\mathbb{R}^{|\mathrm{E}|}=\operatorname{im}(\text{grad})\oplus\ker(\text{curl}^{*}\circ\text{curl}+\text{grad}\circ(-\text{div}))\oplus\operatorname{im}(\text{curl}^{*})=\operatorname{im}(\text{grad})\oplus\ker(-\text{grad}\circ\text{div}+\text{curl}^{*}\circ\text{curl})\oplus\operatorname{im}(\text{curl}\,*)$$

$$\text{graph Laplacian}:\ -\text{div}\circ\text{grad}$$

$$\text{graph Helmholtzian}:\ -\text{grad}\circ\text{div}\ +\text{curl}^{*}\circ\text{curl}$$

The dimension of the kernel of the graph Laplacian, which is the multiplicity of eigenvalue 0, is the number of 0-holes in the graph (the number of connected components). The number of zero eigenvalues indicates the number of components of the graph. For a well-connected graph, its non-zero eigenvalues are far from 0, while graphs with several obvious clusters will have some eigenvalues very close to 0. Eigen vectors associated with eigenvalues close to 0 indicate the almost components in the graph, and are a good way of identifying clusters in a graph. A graph with several eigenvalues close to zero will result in extremely large Kirchhoff index, meaning the graph has several clusters and is not well connected (has a very large overall resistance).

Rank-1 update of Moore-Penrose Inverse

The following is a rank-1 update for the Moore-Penrose pseudo-inverse of real valued matrices

$$(\boldsymbol{A}+\boldsymbol{c}\boldsymbol{d}^{\boldsymbol{T}})^{\dagger}=\boldsymbol{A}^{\dagger}+\boldsymbol{G}$$

Using the notation

$$\beta=1+\boldsymbol{d}^{\boldsymbol{T}}\boldsymbol{A}^{\dagger}\boldsymbol{c}$$
$$\boldsymbol{v}=\boldsymbol{A}^{\dagger}\boldsymbol{c}$$
$$\boldsymbol{n}=\left(\boldsymbol{A}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{d}$$
$$\boldsymbol{w}=\left(\boldsymbol{I}-\boldsymbol{A}\boldsymbol{A}^{\dagger}\right)\boldsymbol{c}$$
$$\boldsymbol{m}=\left(\boldsymbol{I}-\boldsymbol{A}^{\dagger}\boldsymbol{A}\right)^{\boldsymbol{T}}\boldsymbol{d}$$

the solution is given as six different cases, depending on the entities $\|\boldsymbol{w}\|,\|\boldsymbol{m}\|$, and β. Please note, that for any (column) vector $\boldsymbol{v}$ it holds that $\boldsymbol{v}^{\dagger}=\boldsymbol{v}^{\boldsymbol{T}}(\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{v})^{-1}=\frac{\boldsymbol{v}^{\boldsymbol{T}}}{\|\boldsymbol{v}\|^{2}}$. The solution is:

$$\boldsymbol{v}^{\dagger}=\left(\begin{bmatrix}\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|} & \left(\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right)_{\text{unitary}}^{\perp}\end{bmatrix}\begin{bmatrix}\|\boldsymbol{v}\|\\ 0\\ \vdots\\ 0\end{bmatrix}[1]\right)^{\dagger}=1\begin{bmatrix}\dfrac{1}{\|\boldsymbol{v}\|} & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}\left(\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right)^{\boldsymbol{T}}\\ \left(\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right)^{\perp\boldsymbol{T}}\end{bmatrix}=\frac{\boldsymbol{v}^{\boldsymbol{T}}}{\|\boldsymbol{v}\|^{2}}$$

Case 1 of 6: If $\|\boldsymbol{w}\|\neq 0$ and $\|\boldsymbol{m}\|\neq 0$. Then

$$\boldsymbol{G}=-\boldsymbol{v}\boldsymbol{w}^{\dagger}-\left(\boldsymbol{m}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{n}^{\boldsymbol{T}}+\beta\left(\boldsymbol{m}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{w}^{\dagger}=-\frac{\boldsymbol{v}\boldsymbol{w}^{\boldsymbol{T}}}{\|\boldsymbol{w}\|^{2}}-\frac{\boldsymbol{m}\boldsymbol{n}^{\boldsymbol{T}}}{\|\boldsymbol{m}\|^{2}}+\frac{\beta\boldsymbol{m}\boldsymbol{w}^{\boldsymbol{T}}}{\|\boldsymbol{m}\|^{2}\|\boldsymbol{w}\|^{2}}$$

Case 2 of 6: If $\|\boldsymbol{w}\|=0$ and $\|\boldsymbol{m}\|\neq 0$ and $\beta\neq 0$. Then

$$\boldsymbol{G}=-\boldsymbol{v}\boldsymbol{v}^{\dagger}\boldsymbol{A}^{\dagger}-\left(\boldsymbol{m}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{n}^{\boldsymbol{T}}=-\frac{1}{\|\boldsymbol{v}\|^{2}}\boldsymbol{v}\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{A}^{\dagger}-\frac{1}{\|\boldsymbol{m}\|^{2}}\boldsymbol{m}\boldsymbol{n}^{\boldsymbol{T}}$$

Case 3 of 6: If $\|\boldsymbol{w}\|=0$ and $\beta\neq 0$. Then

$$\boldsymbol{G}=\frac{1}{\beta}\boldsymbol{m}\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{A}^{\dagger}-\frac{\beta}{\|\boldsymbol{v}\|^2\|\boldsymbol{m}\|^2+|\beta|^2}\left(\frac{\|\boldsymbol{v}\|^2}{\beta}\boldsymbol{m}+\boldsymbol{v}\right)\left(\frac{\|\boldsymbol{m}\|^2}{\beta}\left(\boldsymbol{A}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{v}+\boldsymbol{n}\right)^{\boldsymbol{T}}$$

Case 4 of 6: If $\|\boldsymbol{w}\|\neq 0$ and $\|\boldsymbol{m}\|=0$ and $\beta=0$. Then

$$\boldsymbol{G}=-\boldsymbol{A}^{\dagger}\boldsymbol{n}\boldsymbol{n}^{\dagger}-\boldsymbol{v}\boldsymbol{w}^{\dagger}=-\frac{1}{\|\boldsymbol{n}\|^2}\boldsymbol{A}^{\dagger}\boldsymbol{n}\boldsymbol{n}^{\boldsymbol{T}}-\frac{1}{\|\boldsymbol{w}\|^2}\boldsymbol{v}\boldsymbol{w}^{\boldsymbol{T}}$$

Case 5 of 6: If $\|\boldsymbol{m}\|=0$ and $\beta\neq 0$. Then

$$\boldsymbol{G}=\frac{1}{\beta}\boldsymbol{A}^{\dagger}\boldsymbol{n}\boldsymbol{w}^{\boldsymbol{T}}-\frac{\beta}{\|\boldsymbol{n}\|^2\|\boldsymbol{w}\|^2+|\beta|^2}\left(\frac{\|\boldsymbol{w}\|^2}{\beta}\boldsymbol{A}^{\dagger}\boldsymbol{n}+\boldsymbol{v}\right)\left(\frac{\|\boldsymbol{n}\|^2}{\beta}\boldsymbol{w}+\boldsymbol{n}\right)^{\boldsymbol{T}}$$

Case 6 of 6: If $\|\boldsymbol{w}\|=0$ and $\|\boldsymbol{m}\|=0$ and $\beta=0$. Then

$$\boldsymbol{G}=-\boldsymbol{v}\boldsymbol{v}^{\dagger}\boldsymbol{A}^{\dagger}-\boldsymbol{A}^{\dagger}\boldsymbol{n}\boldsymbol{n}^{\dagger}+\boldsymbol{v}^{\dagger}\boldsymbol{A}^{\dagger}\boldsymbol{n}\boldsymbol{v}\boldsymbol{n}^{\dagger}=-\frac{1}{\|\boldsymbol{v}\|^2}\boldsymbol{v}\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{A}^{\dagger}-\frac{1}{\|\boldsymbol{n}\|^2}\boldsymbol{A}^{\dagger}\boldsymbol{n}\boldsymbol{n}^{\boldsymbol{T}}+\frac{\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{A}^{\dagger}\boldsymbol{n}}{\|\boldsymbol{v}\|^2\|\boldsymbol{n}\|^2}\boldsymbol{v}\boldsymbol{n}^{\boldsymbol{T}}$$

Delete one edge: rank one update (unweighted)

Removing one edge $\boldsymbol{m}_{i|V|\times 1}$ equals change the resistance r_i to infinity, and $\frac{1}{r_i}$ to 0, and equals to subtracting a rank one matrix from the original graph Laplacian

$$\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}-\frac{1}{r_i}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\right)^{\dagger}=\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}+\boldsymbol{G}$$

Average effective resistance

$$\bar{\tilde{r}}=\frac{\operatorname{tr}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)}{|V|}=\frac{1}{|V|}\sum_{i=1,\lambda_i\neq 0}^{|V|}\frac{1}{\lambda_i}$$

Average effective resistance update after deleting an edge

$$(\bar{\tilde{r}})_{\text{remove an edge}}=\frac{1}{|V|}\operatorname{tr}\left(\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}-\frac{1}{r_i}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\right)^{\dagger}\right)=\frac{1}{|V|}\operatorname{tr}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}+\boldsymbol{G}\right)=\frac{1}{|V|}\operatorname{tr}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)+\frac{1}{|V|}\operatorname{tr}(\boldsymbol{G})$$

$$\Delta\bar{\tilde{r}}=(\bar{\tilde{r}})_{\text{remove an edge}}-\bar{\tilde{r}}=\frac{1}{|V|}\operatorname{tr}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)+\frac{1}{|V|}\operatorname{tr}(\boldsymbol{G})-\frac{1}{|V|}\operatorname{tr}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)=\frac{1}{|V|}\operatorname{tr}(\boldsymbol{G})$$

$$\frac{\Delta\bar{\tilde{r}}}{\bar{\tilde{r}}}=\frac{\operatorname{tr}(\boldsymbol{G})}{\operatorname{tr}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)}$$

Using rank one update to calculate the average effective resistance after removing edges is significantly cost efficient

$$\tilde{\boldsymbol{L}}_{|V|\times|V|}^{2}=\left(\sum_{i=1}^{k}\lambda_i\boldsymbol{q}_i\boldsymbol{q}_i^T\right)^{\mathbf{2}}=\sum_{i=1}^{k}\lambda_i^2\boldsymbol{q}_i\boldsymbol{q}_i^T$$

$$\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)^{\mathbf{2}}=\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{2}\right)^{\dagger}=\left(\sum_{i=1}^{k}\frac{1}{\lambda_i}\boldsymbol{q}_i\boldsymbol{q}_i^T\right)^{\mathbf{2}}=\sum_{i=1}^{k}\frac{1}{\lambda_i^2}\boldsymbol{q}_i\boldsymbol{q}_i^T$$

$$\tilde{\boldsymbol{L}}_{|V|\times|V|}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}=\left(\sum_{i=1}^{k}\lambda_i\boldsymbol{q}_i\boldsymbol{q}_i^T\right)\left(\sum_{i=1}^{k}\frac{1}{\lambda_i}\boldsymbol{q}_i\boldsymbol{q}_i^T\right)=\sum_{i=1}^{k}\lambda_i\boldsymbol{q}_i\boldsymbol{q}_i^T\frac{1}{\lambda_i}\boldsymbol{q}_i\boldsymbol{q}_i^T=\sum_{i=1}^{k}\boldsymbol{q}_i\boldsymbol{q}_i^T\boldsymbol{q}_i\boldsymbol{q}_i^T=\sum_{i=1}^{k}\boldsymbol{q}_i\boldsymbol{q}_i^T$$

$$\boldsymbol{I}-\tilde{\boldsymbol{L}}_{|V|\times|V|}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}=\sum_{i=k+1}^{|V|}\boldsymbol{q}_i\boldsymbol{q}_i^T$$

$$\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\tilde{\boldsymbol{L}}_{|V|\times|V|}=\left(\sum_{i=1}^{k}\frac{1}{\lambda_i}\boldsymbol{q}_i\boldsymbol{q}_i^T\right)\left(\sum_{i=1}^{k}\lambda_i\boldsymbol{q}_i\boldsymbol{q}_i^T\right)=\sum_{i=1}^{k}\frac{1}{\lambda_i}\boldsymbol{q}_i\boldsymbol{q}_i^T\lambda_i\boldsymbol{q}_i\boldsymbol{q}_i^T=\sum_{i=1}^{k}\boldsymbol{q}_i\boldsymbol{q}_i^T\boldsymbol{q}_i\boldsymbol{q}_i^T=\sum_{i=1}^{k}\boldsymbol{q}_i\boldsymbol{q}_i^T$$

$$I - \tilde{L}^{\dagger}_{|V|\times|V|}\tilde{L}_{|V|\times|V|} = \sum_{i=k+1}^{|V|} q_i q_i^T$$

$$\beta = 1 + d^T A^{\dagger} c = 1 + m_{i|V|\times 1}^T \tilde{L}^{\dagger}_{|V|\times|V|}\left(-\frac{1}{r_i} m_{i|V|\times 1}\right) = 1 - \frac{1}{r_i} \overbrace{m_{i|V|\times 1}^T \tilde{L}^{\dagger}_{|V|\times|V|} m_{i|V|\times 1}}^{\text{effective resistance of edge } i\leq r_i} > 0$$

$$v = A^{\dagger} c = -\frac{1}{r_i}\tilde{L}^{\dagger}_{|V|\times|V|} m_{i|V|\times 1}$$

$$n = \left(A^{\dagger}\right)^T d = \left(\tilde{L}^{\dagger}_{|V|\times|V|}\right)^T m_{i|V|\times 1}$$

$$w = \left(I - AA^{\dagger}\right)c = -\frac{1}{r_i}\left(I - \tilde{L}_{|V|\times|V|}\tilde{L}^{\dagger}_{|V|\times|V|}\right) m_{i|V|\times 1} = -\frac{1}{r_i}\left(I - \tilde{L}_{|V|\times|V|}\tilde{L}^{\dagger}_{|V|\times|V|}\right) m_{i|V|\times 1}$$

$$m = \left(I - A^{\dagger}A\right)^T d = \left(I - \tilde{L}^{\dagger}_{|V|\times|V|}\tilde{L}_{|V|\times|V|}\right)^T m_{i|V|\times 1} = \left(I - \tilde{L}^{\dagger}_{|V|\times|V|}\tilde{L}_{|V|\times|V|}\right)^T m_{i|V|\times 1}$$

If there are multiple components in a graph, q_i for $\lambda_i = 0$ should be similar to an one-hot encoding,

$$q_i = \frac{e_i}{\|e_i\|}$$

e_i is one hot encoding, entries are either 0 or 1, $\|e_i\|$ is the square root of the number of 1 entries in e_i

$$q_i^T q_i = \frac{e_i^T}{\|e_i\|}\frac{e_i}{\|e_i\|} = \frac{e_i^T e_i}{\|e_i\|^2}$$

$$q_i q_i^T = \frac{e_i}{\|e_i\|}\frac{e_i^T}{\|e_i\|} = \frac{e_i e_i^T}{\|e_i\|^2}$$

$\sum_{i=k+1}^{|V|} q_i q_i^T$ would be a diagonal block matrix, columns of nodes of the same components are the same, if the 1 entry and -1 entry of $m_{i|V|\times 1}$ are in the same component, then $\left(\sum_{i=k+1}^{|V|} q_i q_i^T\right) m_{i|V|\times 1} = \mathbf{0}$

For an edge in the graph, the two end nodes must be in one component, therefore $\left(\sum_{i=k+1}^{|V|} q_i q_i^T\right) m_{i|V|\times 1} = \mathbf{0}$ is true for any $m_{i|V|\times 1}$ in matrix $M_{|E|\times|V|}$. Therefore we have

$$\left(I - \tilde{L}_{|V|\times|V|}\tilde{L}^{\dagger}_{|V|\times|V|}\right) m_{i|V|\times 1} = \mathbf{0}$$

$$\left(I - \tilde{L}^{\dagger}_{|V|\times|V|}\tilde{L}_{|V|\times|V|}\right)^T m_{i|V|\times 1} = \mathbf{0}$$

We have $\|m\| = 0$ and $\|w\| = 0$ and $\beta \neq 0$

Case 3 of 6: If $\|w\| = 0$ and $\beta \neq 0$. Then

$$G = \frac{1}{\beta} m v^T A^{\dagger} - \frac{\beta}{\|v\|^2\|m\|^2 + |\beta|^2}\left(\frac{\|v\|^2}{\beta} m + v\right)\left(\frac{\|m\|^2}{\beta}\left(A^{\dagger}\right)^T v + n\right)^T$$

$$G = -\frac{\beta}{|\beta|^2} v n^T$$

Case 5 of 6: If $\|m\| = 0$ and $\beta \neq 0$. Then

$$G = \frac{1}{\beta} A^{\dagger} n w^T - \frac{\beta}{\|n\|^2\|w\|^2 + |\beta|^2}\left(\frac{\|w\|^2}{\beta} A^{\dagger} n + v\right)\left(\frac{\|n\|^2}{\beta} w + n\right)^T$$

$$G = -\frac{\beta}{|\beta|^2} v n^T$$

$$\beta > 0$$

$$\boldsymbol{G} = -\frac{\beta}{|\beta|^2}\boldsymbol{v}\boldsymbol{n}^{\boldsymbol{T}} = -\frac{1}{\beta}\boldsymbol{v}\boldsymbol{n}^{\boldsymbol{T}}$$

$$\boldsymbol{G} = -\frac{1}{\beta}\boldsymbol{v}\boldsymbol{n}^{\boldsymbol{T}} = \frac{1}{r_i}\frac{1}{1-\frac{1}{r_i}\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}$$

$$= \frac{1}{r_i - \underbrace{\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}_{\text{effective resistance of edge } i \leq r_i}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger} = \frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i - \underbrace{\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}_{\text{effective resistance of edge } i \leq r_i}}$$

$$\text{given } \tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}, \quad \tilde{\boldsymbol{L}}_{|V|\times|V|\,\text{remove edge i}}{}^{\dagger} = \left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger} - \frac{1}{r_i}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\right)^{\dagger} = \tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger} + \frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i - \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}$$

$$\Delta\tilde{\bar{r}} = \frac{1}{|V|}\mathrm{tr}(\boldsymbol{G}) = \frac{1}{|V|}\mathrm{tr}\left(\frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i - \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\right) = \frac{1}{|V|}\frac{1}{r_i - \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\mathrm{tr}\left(\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}\right)$$

$$= \frac{1}{|V|}\frac{1}{r_i - \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\mathrm{tr}\left(\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)$$

$$= \frac{1}{|V|}\frac{1}{r_i - \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\mathrm{tr}\left(\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)$$

$$= \frac{1}{|V|}\frac{1}{r_i - \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\mathrm{tr}\left(\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\boldsymbol{m}_{i|V|\times 1}\right)$$

$$= \frac{1}{|V|}\frac{1}{r_i - \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\mathrm{tr}\left(\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\left(\sum_{i=1}^{k}\frac{1}{\lambda_i^2}\boldsymbol{q}_i\boldsymbol{q}_i^{T}\right)\boldsymbol{m}_{i|V|\times 1}\right)$$

$$\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\boldsymbol{m}_{i|V|\times 1} = \left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\right)_{jj} + \left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\right)_{jk}$$

$$\Delta\tilde{\bar{r}} = \frac{1}{|V|}\frac{\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\boldsymbol{m}_{i|V|\times 1}}{r_i - \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}} = \frac{1}{|V|}\frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\right)_{jj} + \left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\right)_{jk}}{r_i - \left(\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{jj} + \left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{jk}\right)}$$

The increase of the average effective resistance $\Delta\tilde{\bar{r}}$ is related to the resistance distance of the about to be removed edge r_i (haven't been removed yet) and the effective resistance between the two end nodes of the about to be removed edge $\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\boldsymbol{L}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}$ in the current network (edge has not been removed yet). The effective resistance is always smaller than or equal to the edge resistance, because it is like circuits in parallel. The closer the effective resistance of the two end nodes of the about to be removed edge is in the current network $\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}$ to the edge resistance r_i is, the bigger the increase in average effective resistance will be. If removing the edge results in disconnect the network into different components, the increase would be infinite.

$$\frac{\Delta\tilde{\bar{r}}}{\tilde{\bar{r}}} = \frac{\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\boldsymbol{m}_{i|V|\times 1}}{\mathrm{tr}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)\left(r_i - \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)} = \frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\right)_{jj} + \left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}{}^{\boldsymbol{2}}\right)_{jk}}{\mathrm{tr}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)\left(r_i - \left(\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{jj} + \left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{jk}\right)\right)}$$

When $\tilde{\boldsymbol{L}}_{|V|\times|V|}$ is known for the current graph, $\Delta\tilde{\bar{r}}$ is only dependent on $\boldsymbol{m}_{i|V|\times 1}$, namely the two end nodes, namely choosing elements from the pseudo inverse of graph Laplacian $\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}$, so does $\frac{\Delta\tilde{\bar{r}}}{\tilde{\bar{r}}}$

$\boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}$ is the effective resistance between the two end nodes of the edge, when it is very close to the edge resistance r_i, it means that that edge is close to the sole connecting path between the two end nodes. Deleting this edge would greatly increase the Kirchhoff index. $\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{jj} + \left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{jk}$ is the effective resistance between the two end nodes in the squared graph Laplacian graph.

When removing the edge causes the number of components to increase (i.e. the edge connecting a single node to the rest of nodes in the component, the edge acting like a bridge connecting two components), the effective resistance of that edge $\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{jj} + \left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)_{kk} -$

$2\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{jk}$ is the same as r_i, the denominator $r_i-\left(\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{jj}+\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{kk}-2\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{jk}\right)$ is 0, average effective resistance increase $\Delta\bar{\tilde{r}}$ becomes infinity. We should exclude such cases.

Delete one edge: rank one update (weighted 1)

$$\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{remove edge i}}}{}^{\dagger}=\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}-\frac{1}{r_i}\boldsymbol{m}_{i_{|V|\times 1}}\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\right)^{\dagger}=\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}+\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}$$

$$\begin{aligned}
\Delta\bar{\tilde{r}}_{\text{weighted}}&=\left(2\operatorname{diag}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{remove edge i}}}{}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{w}-2\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{remove edge i}}}{}^{\dagger}\boldsymbol{w}\right)-\left(2\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{w}-2\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}\right)\\
&=\left(2\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}+\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\right)^{\boldsymbol{T}}\boldsymbol{w}\right.\\
&\quad\left.-2\boldsymbol{w}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}+\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\right)\boldsymbol{w}\right)-\left(2\operatorname{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{w}-2\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}\right)\\
&=2\left[\operatorname{diag}\left(\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\right)^{\boldsymbol{T}}\boldsymbol{w}-\boldsymbol{w}^{\boldsymbol{T}}\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\boldsymbol{w}\right]\\
&=\frac{2}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\left[\operatorname{diag}\left(\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}}\boldsymbol{w}-\left(\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\mathbf{2}}\right]\\
&=\frac{2}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\left[\operatorname{tr}\left(\operatorname{diag}(\boldsymbol{w})\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}\right)-\left(\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\mathbf{2}}\right]\\
&=\frac{2}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\left[\operatorname{tr}\left(\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}\operatorname{diag}(\boldsymbol{w})\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)\right)-\left(\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\mathbf{2}}\right]\\
&=\frac{2}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\left[\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}\operatorname{diag}(\boldsymbol{w})\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)-\left(\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\mathbf{2}}\right]\\
&=\frac{2}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\left[\boldsymbol{m}_{i_{|V|\times 1}}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\operatorname{diag}(\boldsymbol{w})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i_{|V|\times 1}}-\boldsymbol{m}_{i_{|V|\times 1}}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i_{|V|\times 1}}\right]\\
&=\frac{2}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\boldsymbol{m}_{i_{|V|\times 1}}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\operatorname{diag}(\boldsymbol{w})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}-\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i_{|V|\times 1}}\\
&=\frac{2}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\boldsymbol{m}_{i_{|V|\times 1}}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\left(\operatorname{diag}(\boldsymbol{w})-\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\right)\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i_{|V|\times 1}}\\
&=\frac{2}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}\left[\sum_{k=1}^{|V|}w_k\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)_k^{\mathbf{2}}-\left(\sum_{k=1}^{|V|}w_k\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)_{\boldsymbol{k}}\right)^{\mathbf{2}}\right]
\end{aligned}$$

Let $\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\left(\operatorname{diag}(\boldsymbol{w})-\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\right)\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}=\boldsymbol{A}_{|V|\times|V|}$

$$\Delta\bar{\tilde{r}}_{\text{weighted}}=\frac{\boldsymbol{m}_{i_{|V|\times 1}}{}^{\boldsymbol{T}}\boldsymbol{A}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}=\frac{\left(\boldsymbol{A}_{|V|\times|V|}\right)_{jj}+\left(\boldsymbol{A}_{|V|\times|V|}\right)_{mm}-2\left(\boldsymbol{A}_{|V|\times|V|}\right)_{mj}}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}$$

When $\boldsymbol{w}=\frac{1}{|V|}\mathbf{1}$

$$\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\operatorname{diag}(\boldsymbol{w})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}-\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}=\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\frac{1}{|V|}\boldsymbol{I}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}-\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\frac{1}{|V|^2}\mathbf{1}\mathbf{1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}=\frac{1}{|V|}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}-\mathbf{0}\mathbf{0}^{\boldsymbol{T}}=\frac{1}{|V|}\boldsymbol{L}^{\dagger}_{|V|\times|V|}{}^{\mathbf{2}}$$

Delete one edge: rank one update (weighted 2)

$$\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{remove edge i}}}{}^{\dagger}=\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}-\frac{1}{r_i}\boldsymbol{m}_{i_{|V|\times 1}}\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\right)^{\dagger}=\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}+\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}\right)^{\boldsymbol{T}}}{r_i-\boldsymbol{m}_{i_{|V|\times 1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times 1}}}$$

$$\Delta\bar{\tilde{r}}_{\text{weighted}} = \left(2\text{diag}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{remove edge i}}}{}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{W1} - 2\text{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{remove edge i}}}{}^{\dagger}\right)\right) - \left(2\text{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W1} - 2\text{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\right)$$

$$= \left(2\,\text{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} + \frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)^{\boldsymbol{T}}}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}\right)^{\boldsymbol{T}}\boldsymbol{W1}\right.$$

$$\left. - 2\text{tr}\left(\boldsymbol{W}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} + \frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)^{\boldsymbol{T}}}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}\right)\right)\right) - \left(2\text{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W1} - 2\text{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\right)$$

$$= 2\left[\text{diag}\left(\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)^{\boldsymbol{T}}}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}\right)^{\boldsymbol{T}}\boldsymbol{W1} - \text{tr}\left(\boldsymbol{W}\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)^{\boldsymbol{T}}}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}\right)\right]$$

$$= \frac{2}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}\left[\text{diag}\left(\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}}\boldsymbol{W1}\right.$$

$$\left. - \boldsymbol{m}_{i_{|V|\times1}}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i_{|V|\times1}}\right]$$

$$= \frac{2}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}\left[\text{tr}\left(\boldsymbol{m}_{i_{|V|\times1}}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\text{diag}(\boldsymbol{W1})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}\right) - \boldsymbol{m}_{i_{|V|\times1}}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i_{|V|\times1}}\right]$$

$$= \frac{2}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}\left[\boldsymbol{m}_{i_{|V|\times1}}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\text{diag}(\boldsymbol{W1})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}} - \boldsymbol{m}_{i_{|V|\times1}}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i_{|V|\times1}}\right]$$

$$= \frac{2}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}\boldsymbol{m}_{i_{|V|\times1}}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\text{diag}(\boldsymbol{W1}) - \boldsymbol{W})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i_{|V|\times1}}$$

Let $\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\text{diag}(\boldsymbol{W1}) - \boldsymbol{W})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = \boldsymbol{A}_{|V|\times|V|}$

$$\Delta\bar{\tilde{r}}_{\text{weighted}} = \frac{\boldsymbol{m}_{i_{|V|\times1}}{}^{\boldsymbol{T}}\boldsymbol{A}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}} = \frac{\left(\boldsymbol{A}_{|V|\times|V|}\right)_{jj} + \left(\boldsymbol{A}_{|V|\times|V|}\right)_{mm} - 2\left(\boldsymbol{A}_{|V|\times|V|}\right)_{mj}}{r_i - \boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i_{|V|\times1}}}$$

When $\boldsymbol{W} = \boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}$

$$\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\text{diag}(\boldsymbol{W1}) - \boldsymbol{W})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\text{diag}(\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{1})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} - \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\text{diag}(\boldsymbol{w}) - \boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$$

Add one edge: rank one update

Adding one edge $\boldsymbol{m}_{i_{|V|\times1}}$ equals change the resistance r_i from infinity to r_i, and $\frac{1}{r_i}$ from 0 to $\frac{1}{r_i}$, and equals to adding a rank one matrix to the original graph Laplacian

$$\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} + \frac{1}{r_i}\boldsymbol{m}_{i_{|V|\times1}}\boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\right)^{\dagger} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} + \boldsymbol{G}$$

Average effective resistance

$$\bar{\tilde{r}} = \frac{1}{|V|}\sum_{i=1}^{k}\frac{1}{\lambda_i} = \frac{1}{|V|}\text{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)$$

Average effective resistance update after adding an edge

$$(\bar{\tilde{r}})_{\text{add an edge}} = \frac{1}{|V|}\text{tr}\left(\left(\tilde{\boldsymbol{L}}_{|V|\times|V|} + \frac{1}{r_i}\boldsymbol{m}_{i_{|V|\times1}}\boldsymbol{m}_{i_{|V|\times1}}^{\boldsymbol{T}}\right)^{\dagger}\right) = \frac{1}{|V|}\text{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} + \boldsymbol{G}\right) = \frac{1}{|V|}\text{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) + \frac{1}{|V|}\text{tr}(\boldsymbol{G})$$

$$\Delta\bar{\tilde{r}} = (\bar{\tilde{r}})_{\text{remove an edge}} - \bar{\tilde{r}} = \frac{1}{|V|}\mathrm{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) + \frac{1}{|V|}\mathrm{tr}(\boldsymbol{G}) - \frac{1}{|V|}\mathrm{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right) = \frac{1}{|V|}\mathrm{tr}(\boldsymbol{G})$$

$$\frac{\Delta\bar{\tilde{r}}}{\bar{\tilde{r}}} = \frac{\mathrm{tr}(\boldsymbol{G})}{\mathrm{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)}$$

$$\beta = 1 + \boldsymbol{d}^{\boldsymbol{T}}\boldsymbol{A}^{\dagger}\boldsymbol{c} = 1 + \boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\frac{1}{r_i}\boldsymbol{m}_{i|V|\times 1} = 1 + \frac{1}{r_i}\overbrace{\boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}^{\text{effective resistance of edge } i \leq r_i} > 0$$

$$\boldsymbol{v} = \boldsymbol{A}^{\dagger}\boldsymbol{c} = \frac{1}{r_i}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}$$

$$\boldsymbol{n} = \left(\boldsymbol{A}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{d} = \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{m}_{i|V|\times 1}$$

$$\boldsymbol{w} = \left(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\dagger}\right)\boldsymbol{c} = \frac{1}{r_i}\left(\boldsymbol{I} - \tilde{\boldsymbol{L}}_{|V|\times|V|}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i|V|\times 1} = \frac{1}{r_i}\left(\boldsymbol{I} - \tilde{\boldsymbol{L}}_{|V|\times|V|}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i|V|\times 1}$$

$$\boldsymbol{m} = \left(\boldsymbol{I} - \boldsymbol{A}^{\dagger}\boldsymbol{A}\right)^{\boldsymbol{T}}\boldsymbol{d} = \left(\boldsymbol{I} - \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\tilde{\boldsymbol{L}}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{m}_{i|V|\times 1} = \left(\boldsymbol{I} - \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\tilde{\boldsymbol{L}}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{m}_{i|V|\times 1}$$

Same as the aforementioned deleting an edge discussion, if the two end nodes of edge $\boldsymbol{m}_{i|V|\times 1}$ are already in the same component, then we have

$$\left(\boldsymbol{I} - \tilde{\boldsymbol{L}}_{|V|\times|V|}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i|V|\times 1} = \boldsymbol{0}$$

$$\left(\boldsymbol{I} - \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\tilde{\boldsymbol{L}}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{m}_{i|V|\times 1} = \boldsymbol{0}$$

We have $\|\boldsymbol{m}\| = 0$ and $\|\boldsymbol{w}\| = 0$ and $\beta \neq 0$

$$\boldsymbol{G} = -\frac{1}{\beta}\boldsymbol{v}\boldsymbol{n}^{\boldsymbol{T}} = -\frac{1}{r_i}\frac{1}{1 + \frac{1}{r_i}\boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$$

$$= -\frac{1}{r_i + \underbrace{\boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}_{\substack{\text{effective resistance of}\\ \text{the two end nodes}}}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = -\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i + \underbrace{\boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}_{\substack{\text{effective resistance of}\\ \text{the two end nodes}}}}$$

$$\Delta\bar{\tilde{r}} = \mathrm{tr}(\boldsymbol{G}) = -\mathrm{tr}\left(\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i + \boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\right) = -\frac{1}{r_i + \boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\mathrm{tr}\left(\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\boldsymbol{m}_{i|V|\times 1}\right)$$

$$\text{given } \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}, \quad \tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{add edge i}}}{}^{\dagger} = \left(\tilde{\boldsymbol{L}}_{|V|\times|V|} + \frac{1}{r_i}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\right)^{\dagger} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} - \frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i + \boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}$$

$$\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\boldsymbol{m}_{i|V|\times 1} = \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\right)_{jj} + \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\right)_{jk}$$

$$\Delta\bar{\tilde{r}} = -\frac{1}{|V|}\frac{\boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\boldsymbol{m}_{i|V|\times 1}}{r_i + \boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}} = -\frac{1}{|V|}\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\right)_{jj} + \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\right)_{jk}}{r_i + \left(\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{jj} + \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{jk}\right)}$$

The decrease of the average effective resistance $-\Delta\bar{\tilde{r}}$ is related to the resistance distance of the newly added edge r_i (haven't been added yet) and the effective resistance between the two end nodes of the newly added edge $\boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}$ in the current network (edge has not been added yet). The bigger of both the resistance distance of the newly added edge and the current effective resistance of the two end nodes of the edge, the smaller the decrease in average effective resistance will be.

$$\frac{\Delta\bar{\tilde{r}}}{\bar{\tilde{r}}} = -\frac{\boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\boldsymbol{m}_{i|V|\times 1}}{\mathrm{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\left(r_i + \boldsymbol{m}_{i}{}^{\boldsymbol{T}}_{|V|\times 1}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)} = -\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\right)_{jj} + \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{}^{2}\right)_{jk}}{\mathrm{tr}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\left(r_i + \left(\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{jj} + \left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{kk} - 2\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)_{jk}\right)\right)}$$

If the two end nodes are not in the same component originally, then this decrease is also meaningless, as the average effective resistance decreases from infinity.

Add one edge: rank one update (weighted 1)

$$\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{add edge i}}}{}^{\dagger}=\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}+\frac{1}{r_i}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i|V|\times 1}^{T}\right)^{\dagger}=\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}-\frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}$$

$$\begin{aligned}
\Delta\bar{\tilde{r}}_{\text{weighted}}&=\left(2\operatorname{diag}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{add edge i}}}{}^{\dagger}\right)^{T}\boldsymbol{w}-2\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{add edge i}}}{}^{\dagger}\boldsymbol{w}\right)-\left(2\operatorname{diag}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)^{T}\boldsymbol{w}-2\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{w}\right)\\
&=\left(2\operatorname{diag}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}-\frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\right)^{T}\boldsymbol{w}\right.\\
&\quad\left.-2\boldsymbol{w}^{T}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}-\frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\right)\boldsymbol{w}\right)-\left(2\operatorname{diag}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)^{T}\boldsymbol{w}-2\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{w}\right)\\
&=-2\left[\operatorname{diag}\left(\frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\right)^{T}\boldsymbol{w}-\boldsymbol{w}^{T}\frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\boldsymbol{w}\right]\\
&=-\frac{2}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\left[\operatorname{diag}\left(\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}\right)^{T}\boldsymbol{w}-\left(\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{2}\right]\\
&=-\frac{2}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\left[\operatorname{tr}\left(\operatorname{diag}(\boldsymbol{w})\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}\right)-\left(\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{2}\right]\\
&=-\frac{2}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\left[\operatorname{tr}\left(\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}\operatorname{diag}(\boldsymbol{w})\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\right)-\left(\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{2}\right]\\
&=-\frac{2}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\left[\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}\operatorname{diag}(\boldsymbol{w})\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)-\left(\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{2}\right]\\
&=-\frac{2}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\left[\boldsymbol{m}_{i|V|\times 1}{}^{T}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\operatorname{diag}(\boldsymbol{w})\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)\boldsymbol{m}_{i|V|\times 1}-\boldsymbol{m}_{i|V|\times 1}{}^{T}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{w}\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)\boldsymbol{m}_{i|V|\times 1}\right]\\
&=-\frac{2}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\boldsymbol{m}_{i|V|\times 1}{}^{T}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\operatorname{diag}(\boldsymbol{w})\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}-\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{w}\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)\boldsymbol{m}_{i|V|\times 1}\\
&=-\frac{2}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\boldsymbol{m}_{i|V|\times 1}{}^{T}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\left(\operatorname{diag}(\boldsymbol{w})-\boldsymbol{w}\boldsymbol{w}^{T}\right)\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\right)\boldsymbol{m}_{i|V|\times 1}\\
&=-\frac{2}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}\left[\sum_{k=1}^{|V|}w_k\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)_k^{2}-\left(\sum_{k=1}^{|V|}w_k\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)_{\boldsymbol{k}}\right)^{2}\right]
\end{aligned}$$

Let $\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\left(\operatorname{diag}(\boldsymbol{w})-\boldsymbol{w}\boldsymbol{w}^{T}\right)\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}=\boldsymbol{A}_{|V|\times|V|}$

$$\Delta\bar{\tilde{r}}_{\text{weighted}}=\frac{\boldsymbol{m}_{i|V|\times 1}{}^{T}\boldsymbol{A}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}{r_i-\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}=\frac{\left(\boldsymbol{A}_{|V|\times|V|}\right)_{jj}+\left(\boldsymbol{A}_{|V|\times|V|}\right)_{mm}-2\left(\boldsymbol{A}_{|V|\times|V|}\right)_{mj}}{r_i-\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}$$

When $\boldsymbol{w}=\frac{1}{|V|}\mathbf{1}$

$$\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\operatorname{diag}(\boldsymbol{w})\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}-\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{w}\boldsymbol{w}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}=\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\frac{1}{|V|}\boldsymbol{I}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}-\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\frac{1}{|V|^{2}}\mathbf{1}\mathbf{1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}=\frac{1}{|V|}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}-\mathbf{0}\mathbf{0}^{T}=\frac{1}{|V|}\boldsymbol{L}_{|V|\times|V|}^{\dagger}{}^{2}$$

Add one edge: rank one update (weighted 2)

$$\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{add edge i}}}{}^{\dagger}=\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}+\frac{1}{r_i}\boldsymbol{m}_{i|V|\times 1}\boldsymbol{m}_{i|V|\times 1}^{T}\right)^{\dagger}=\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}-\frac{\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}\right)^{T}}{r_i+\boldsymbol{m}_{i|V|\times 1}^{T}\tilde{\boldsymbol{L}}_{|V|\times|V|}^{\dagger}\boldsymbol{m}_{i|V|\times 1}}$$

$$\Delta\bar{\tilde{r}}_{\text{weighted}} = \left(2\text{diag}\left(\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{add edge i}}}^{\dagger}\right)^{\boldsymbol{T}}\boldsymbol{W1} - 2\text{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}_{|V|\times|V|_{\text{add edge i}}}^{\dagger}\right)\right) - \left(2\text{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W1} - 2\text{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\right)$$

$$= \left(2\text{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} - \frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\right)^{\boldsymbol{T}}\boldsymbol{W1}\right.$$

$$\left. - 2\text{tr}\left(\boldsymbol{W}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} - \frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\right)\right)\right) - \left(2\text{diag}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)^{\boldsymbol{T}}\boldsymbol{W1} - 2\text{tr}\left(\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\right)$$

$$= -2\left[\text{diag}\left(\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\right)^{\boldsymbol{T}}\boldsymbol{W1} - \text{tr}\left(\boldsymbol{W}\frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\right)\right]$$

$$= -\frac{2}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\left[\text{diag}\left(\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right)^{\boldsymbol{T}}\right)^{\boldsymbol{T}}\boldsymbol{W1}\right.$$

$$\left. - \boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i|V|\times 1}\right]$$

$$= -\frac{2}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\left[\text{tr}\left(\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\text{diag}(\boldsymbol{W1})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}\right) - \boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i|V|\times 1}\right]$$

$$= -\frac{2}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\left[\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\text{diag}(\boldsymbol{W1})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1} - \boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{W}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i|V|\times 1}\right]$$

$$= -\frac{2}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\text{diag}(\boldsymbol{W1}) - \boldsymbol{W})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\right)\boldsymbol{m}_{i|V|\times 1}$$

Let $\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\text{diag}(\boldsymbol{W1}) - \boldsymbol{W})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = \boldsymbol{A}_{|V|\times|V|}$

$$\Delta\bar{\tilde{r}}_{\text{weighted}} = -\frac{\boldsymbol{m}_{i|V|\times 1}{}^{\boldsymbol{T}}\boldsymbol{A}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}} = -\frac{\left(\boldsymbol{A}_{|V|\times|V|}\right)_{jj} + \left(\boldsymbol{A}_{|V|\times|V|}\right)_{mm} - 2\left(\boldsymbol{A}_{|V|\times|V|}\right)_{mj}}{r_i + \boldsymbol{m}_{i|V|\times 1}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{m}_{i|V|\times 1}}$$

When $\boldsymbol{W} = \boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}$

$$\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\text{diag}(\boldsymbol{W1}) - \boldsymbol{W})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\text{diag}(\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{1})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} - \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\text{diag}(\boldsymbol{w}) - \boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$$

A quick method of getting the average effective resistance increase

Old method:

after deleting/adding some edges, do eigen-decomposition on both the graph Laplacian before the deleting/adding and after the deleting/adding. Sum up the inverses of the non-zero eigenvalues before and after the deleting/adding. Get the difference. This requires two eigen-decompositions.

$$\begin{array}{ccc}
\text{old graph} & \overset{\text{delete/add edges}}{\rightrightarrows} & \text{new graph} \\
\Downarrow & & \Downarrow \\
\text{eigen decomposition} & & \text{eigen decomposition} \\
\Downarrow & & \Downarrow \\
\begin{array}{c}\text{sum of inverses}\\ \text{of eigenvalues}\end{array} & & \begin{array}{c}\text{sum of inverses}\\ \text{of eigenvalues}\end{array}
\end{array}$$

$$\text{Increase=new}\begin{array}{c}\text{sum of inverses}\\ \text{of eigenvalues}\end{array} - \text{old}\begin{array}{c}\text{sum of inverses}\\ \text{of eigenvalues}\end{array}$$

New method:

increase = 0

Get $\boldsymbol{L}^{\dagger}_{|V|\times|V|}$ of the current graph

For edge i in deleted_edges:

$$\boldsymbol{A}_{|V|\times|V|} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\mathrm{diag}(\boldsymbol{W}\mathbf{1}) - \boldsymbol{W})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$$

$$(\Delta\bar{\tilde{r}})_{\text{remove an edge } i} = \frac{{\boldsymbol{m}_{i}}_{|V|\times 1}^{T}\boldsymbol{A}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}}{r_i - {\boldsymbol{m}_{i}}_{|V|\times 1}^{T}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}}$$

increase = increase+$(\Delta\bar{\tilde{r}})_{\text{remove an edge } i}$

$$\boldsymbol{L}^{\dagger}_{|V|\times|V|} = \boldsymbol{L}^{\dagger}_{|V|\times|V|} + \frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}\right)^{T}}{r_i - \underbrace{{\boldsymbol{m}_{i}}_{|V|\times 1}^{T}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}}_{\text{effective resistance of edge } i \leq r_i}}$$

increase = 0

Get $\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$ of the current graph

For edge i in added_edges:

$$\boldsymbol{A}_{|V|\times|V|} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}(\mathrm{diag}(\boldsymbol{W}\mathbf{1}) - \boldsymbol{W})\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}$$

$$(\Delta\bar{\tilde{r}})_{\text{add an edge } i} = -\frac{{\boldsymbol{m}_{i}}_{|V|\times 1}^{T}\boldsymbol{A}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}}{r_i + {\boldsymbol{m}_{i}}_{|V|\times 1}^{T}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}}$$

increase = increase+$(\Delta\bar{\tilde{r}})_{\text{add an edge } i}$

$$\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} = \tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|} - \frac{\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}\right)\left(\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}\right)^{T}}{r_i + \underbrace{{\boldsymbol{m}_{i}}_{|V|\times 1}^{T}\tilde{\boldsymbol{L}}^{\dagger}_{|V|\times|V|}{\boldsymbol{m}_{i}}_{|V|\times 1}}_{\text{effective resistance of edge } i \leq r_i}}$$

rank one update of graph Laplacian

change of average effective resistance (unweighted)

pseudo invert the graph Laplacian (Moore Penrose pseudo inverse) before removing edge once and then retrieve entries' values and do addition and division

Moore Penrose pseudo inverse is done by using singular value decomposition in NumPy

	Number of components stays the same	Number of components changes
Add an edge	Two end nodes in the same component before adding the edge	Two end nodes not in the same component before adding the edge After adding the edge, the two end nodes must be in the same component $\Delta i_K = -\inf$

	$$\Delta i_K = -\frac{1}{\vert V\vert}\frac{{\boldsymbol{m}_i}_{\vert V\vert\times 1}^{\boldsymbol{T}}{\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}}^{2}{\boldsymbol{m}_i}_{\vert V\vert\times 1}}{r_i + {\boldsymbol{m}_i}_{\vert V\vert\times 1}^{\boldsymbol{T}}\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}{\boldsymbol{m}_i}_{\vert V\vert\times 1}}$$ $$= -\frac{1}{\vert V\vert}\frac{\left({\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}}^{2}\right)_{jj} + \left({\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}}^{2}\right)_{kk} - 2\left({\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}}^{2}\right)_{jk}}{r_i + \left(\left(\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}\right)_{jj} + \left(\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}\right)_{kk} - 2\left(\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}\right)_{jk}\right)}$$	$\boldsymbol{L}_{\vert V\vert\times\vert V\vert}$ kernel dimension decreases
Remove an edge	Two end nodes in the same component after removing the edge $$\Delta i_K = \frac{1}{\vert V\vert}\frac{{\boldsymbol{m}_i}_{\vert V\vert\times 1}^{\boldsymbol{T}}{\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}}^{2}{\boldsymbol{m}_i}_{\vert V\vert\times 1}}{r_i - {\boldsymbol{m}_i}_{\vert V\vert\times 1}^{\boldsymbol{T}}\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}{\boldsymbol{m}_i}_{\vert V\vert\times 1}}$$ $$= \frac{1}{\vert V\vert}\frac{\left({\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}}^{2}\right)_{jj} + \left({\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}}^{2}\right)_{kk} - 2\left({\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}}^{2}\right)_{jk}}{r_i - \left(\left(\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}\right)_{jj} + \left(\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}\right)_{kk} - 2\left(\boldsymbol{L}_{\vert V\vert\times\vert V\vert}^{\dagger}\right)_{jk}\right)}$$	Two end nodes not in the same component after removing the edge Before removing the edge, they must be in the same component $\Delta i_K = \inf$ $\boldsymbol{L}_{\vert V\vert\times\vert V\vert}$ kernel dimension increases

r_i and ${\boldsymbol{m}_i}_{|V|\times 1}$ are the properties of the about to be added/removed edge and its two end nodes

$\boldsymbol{L}_{|V|\times|V|}$ is the description of the current whole graph

r_i is the resistance of the edge, ${\boldsymbol{m}_i}_{|V|\times 1}$ represents it two end nodes as 1 and -1, the rest are 0s, ${\boldsymbol{m}_i}_{|V|\times 1}^{\boldsymbol{T}}\boldsymbol{L}_{|V|\times|V|}^{\dagger}{\boldsymbol{m}_i}_{|V|\times 1}$ is the effective resistance between the two end nodes.

Normally what we need to do is to do eigen decomposition twice on graph Laplacians before removing edge and after removing edge, get the difference of inverse sum of eigenvalues of two decompositions.

Here adding and removing one edge is a rank one update, which enables us to get the pseudo inverse after the rank one update relatively easily without having to pseudo invert the graph Laplacian after removing/adding one edge again

$$\text{given } \boldsymbol{L}_{|V|\times|V|}^{\dagger}, \quad {\boldsymbol{L}_{|V|\times|V|}}_{\text{remove edge i}}^{\dagger} = \left(\boldsymbol{L}_{|V|\times|V|} - \frac{1}{r_i}{\boldsymbol{m}_i}_{|V|\times 1}{\boldsymbol{m}_i}_{|V|\times 1}^{\boldsymbol{T}}\right)^{\dagger} = \boldsymbol{L}_{|V|\times|V|}^{\dagger} + \frac{\left(\boldsymbol{L}_{|V|\times|V|}^{\dagger}{\boldsymbol{m}_i}_{|V|\times 1}\right)\left(\boldsymbol{L}_{|V|\times|V|}^{\dagger}{\boldsymbol{m}_i}_{|V|\times 1}\right)^{\boldsymbol{T}}}{r_i - {\boldsymbol{m}_i}_{|V|\times 1}^{\boldsymbol{T}}\boldsymbol{L}_{|V|\times|V|}^{\dagger}{\boldsymbol{m}_i}_{|V|\times 1}}$$

$$\text{given } \boldsymbol{L}_{|V|\times|V|}^{\dagger}, \quad {\boldsymbol{L}_{|V|\times|V|}}_{\text{add edge i}}^{\dagger} = \left(\boldsymbol{L}_{|V|\times|V|} + \frac{1}{r_i}{\boldsymbol{m}_i}_{|V|\times 1}{\boldsymbol{m}_i}_{|V|\times 1}^{\boldsymbol{T}}\right)^{\dagger} = \boldsymbol{L}_{|V|\times|V|}^{\dagger} - \frac{\left(\boldsymbol{L}_{|V|\times|V|}^{\dagger}{\boldsymbol{m}_i}_{|V|\times 1}\right)\left(\boldsymbol{L}_{|V|\times|V|}^{\dagger}{\boldsymbol{m}_i}_{|V|\times 1}\right)^{\boldsymbol{T}}}{r_i + {\boldsymbol{m}_i}_{|V|\times 1}^{\boldsymbol{T}}\boldsymbol{L}_{|V|\times|V|}^{\dagger}{\boldsymbol{m}_i}_{|V|\times 1}}$$

$$\text{given } {\boldsymbol{L}_{|V|\times|V|}}_{\text{remove edge i}}^{\dagger},$$

$$\begin{aligned}{\boldsymbol{L}_{|V|\times|V|}}_{\text{remove edge i,j}}^{\dagger} &= \left({\boldsymbol{L}_{|V|\times|V|}}_{\text{remove edge i}} - \frac{1}{r_j}{\boldsymbol{m}_j}_{|V|\times 1}{\boldsymbol{m}_j}_{|V|\times 1}^{\boldsymbol{T}}\right)^{\dagger} \\ &= {\boldsymbol{L}_{|V|\times|V|}}_{\text{remove edge i}}^{\dagger} + \frac{\left({\boldsymbol{L}_{|V|\times|V|}}_{\text{remove edge i}}^{\dagger}{\boldsymbol{m}_j}_{|V|\times 1}\right)\left({\boldsymbol{L}_{|V|\times|V|}}_{\text{remove edge i}}^{\dagger}{\boldsymbol{m}_j}_{|V|\times 1}\right)^{\boldsymbol{T}}}{r_j - {\boldsymbol{m}_j}_{|V|\times 1}^{\boldsymbol{T}}{\boldsymbol{L}_{|V|\times|V|}}_{\text{remove edge i}}^{\dagger}{\boldsymbol{m}_j}_{|V|\times 1}}\end{aligned}$$

$$\text{given } {\boldsymbol{L}_{|V|\times|V|}}_{\text{add edge i}}^{\dagger},$$

$$\begin{aligned}{\boldsymbol{L}_{|V|\times|V|}}_{\text{add edge i,j}}^{\dagger} &= \left({\boldsymbol{L}_{|V|\times|V|}}_{\text{add edge i}} + \frac{1}{r_j}{\boldsymbol{m}_j}_{|V|\times 1}{\boldsymbol{m}_j}_{|V|\times 1}^{\boldsymbol{T}}\right)^{\dagger} \\ &= {\boldsymbol{L}_{|V|\times|V|}}_{\text{add edge i}}^{\dagger} - \frac{\left({\boldsymbol{L}_{|V|\times|V|}}_{\text{add edge i}}^{\dagger}{\boldsymbol{m}_j}_{|V|\times 1}\right)\left({\boldsymbol{L}_{|V|\times|V|}}_{\text{add edge i}}^{\dagger}{\boldsymbol{m}_j}_{|V|\times 1}\right)^{\boldsymbol{T}}}{r_j + {\boldsymbol{m}_j}_{|V|\times 1}^{\boldsymbol{T}}{\boldsymbol{L}_{|V|\times|V|}}_{\text{add edge i}}^{\dagger}{\boldsymbol{m}_j}_{|V|\times 1}}\end{aligned}$$

change of average effective resistance (weighted 1)

pseudo invert the graph Laplacian (Moore Penrose pseudo inverse) before removing edge once and then retrieve entries' values and do addition and division

the weight is exerted on nodes, the weight of pair of two nodes is the product of weights if two nodes, i.e. $w_{ij} = w_i w_j$

the weight on a node could be interpreted as the importance of that node, i.e. population of that node in the Thyssen polygon

$w_{ij} = w_i w_j$ means that the probability of a random start point being i is w_i and the probability of a random end point being j is w_j, the probability of trip $i \to j$ is $w_{ij} = w_i w_j$

Moore Penrose pseudo inverse is done by using singular value decomposition in NumPy

	Number of components stays the same	Number of components changes
Add an edge	Two end nodes in the same component before adding the edge $$\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert} = \boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}(\mathrm{diag}(\boldsymbol{w}) - \boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}})\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}$$ $$\Delta i_K = -\frac{\boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}}{r_i + \boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}} = -\frac{(\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{jj} + (\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{mm} - 2(\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{mj}}{r_i + \boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}}$$	Two end nodes not in the same component before adding the edge After adding the edge, the two end nodes must be in the same component $\Delta i_K = -\mathrm{inf}$ $\boldsymbol{L}_{\lvert V\rvert\times\lvert V\rvert}$ kernel dimension decreases
Remove an edge	Two end nodes in the same component after removing the edge $$\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert} = \boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}(\mathrm{diag}(\boldsymbol{w}) - \boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}})\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}$$ $$\Delta i_K = \frac{\boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}}{r_i - \boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}} = \frac{(\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{jj} + (\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{mm} - 2(\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{mj}}{r_i - \boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}}$$	Two end nodes not in the same component after removing the edge Before removing the edge, they must be in the same component $\Delta i_K = \mathrm{inf}$ $\boldsymbol{L}_{\lvert V\rvert\times\lvert V\rvert}$ kernel dimension increases

change of average effective resistance (weighted 2)

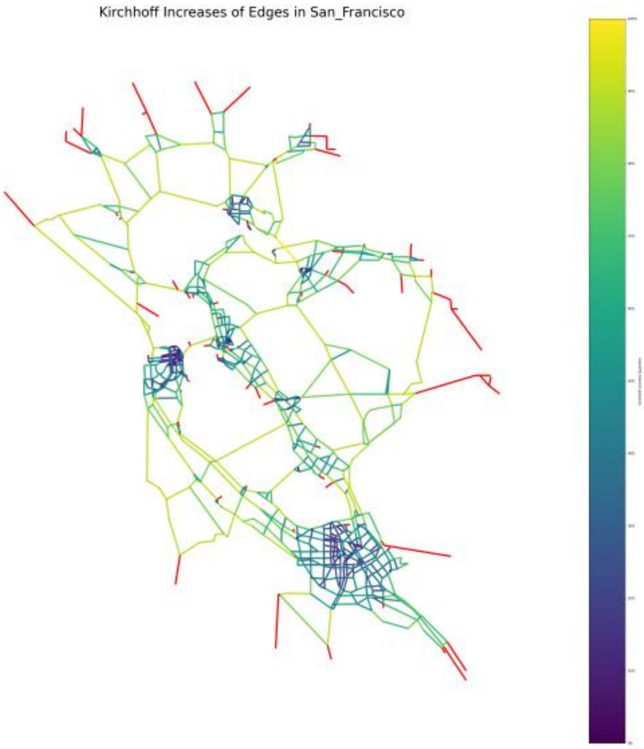
pseudo invert the graph Laplacian (Moore Penrose pseudo inverse) before removing edge once and then retrieve entries' values and do addition and division

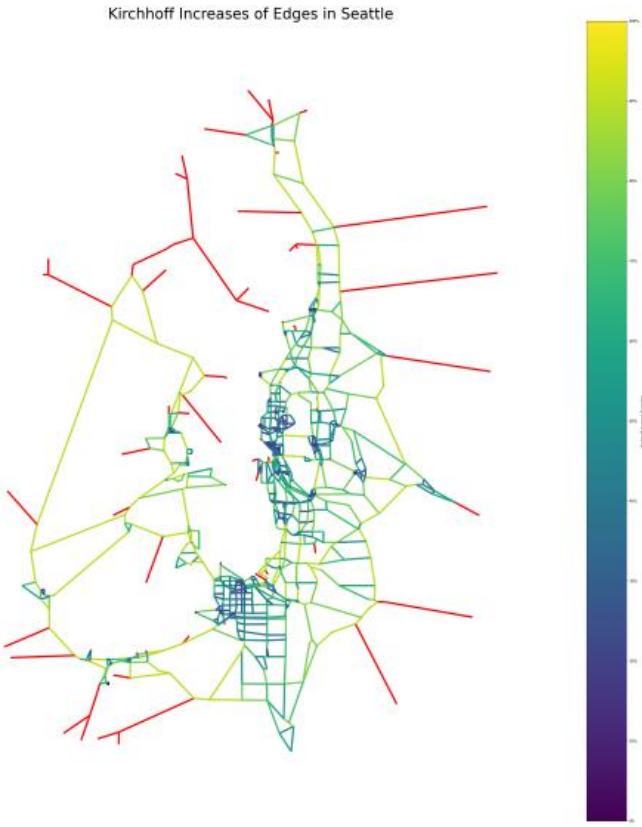
the weight is exerted on pairs of two nodes, the weight is determined by origin-destination matrix

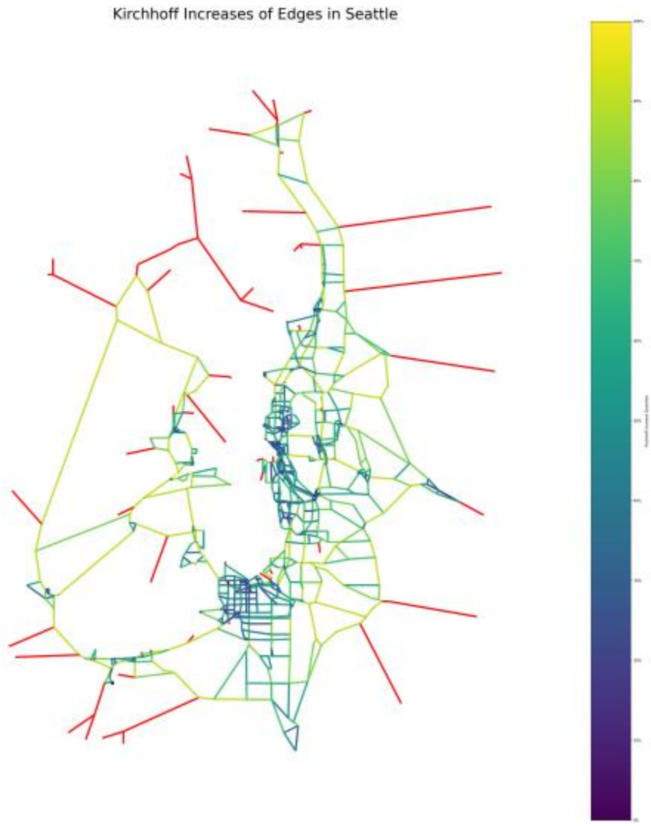
Moore Penrose pseudo inverse is done by using singular value decomposition in NumPy

	Number of components stays the same	Number of components changes
Add an edge	Two end nodes in the same component before adding the edge $$\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert} = \boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}(\mathrm{diag}(\boldsymbol{W}\boldsymbol{1}) - \boldsymbol{W})\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}$$ $$\Delta i_K = -\frac{\boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}}{r_i + \boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}} = -\frac{(\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{jj} + (\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{mm} - 2(\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{mj}}{r_i + \boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}}$$	Two end nodes not in the same component before adding the edge After adding the edge, the two end nodes must be in the same component $\Delta i_K = -\mathrm{inf}$ $\boldsymbol{L}_{\lvert V\rvert\times\lvert V\rvert}$ kernel dimension decreases
Remove an edge	Two end nodes in the same component after removing the edge $$\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert} = \boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}(\mathrm{diag}(\boldsymbol{W}\boldsymbol{1}) - \boldsymbol{W})\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}$$ $$\Delta i_K = \frac{\boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}}{r_i - \boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}} = \frac{(\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{jj} + (\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{mm} - 2(\boldsymbol{A}_{\lvert V\rvert\times\lvert V\rvert})_{mj}}{r_i - \boldsymbol{m}_i{}^{\boldsymbol{T}}_{\lvert V\rvert\times 1}\boldsymbol{L}^{\dagger}_{\lvert V\rvert\times\lvert V\rvert}\boldsymbol{m}_{i\lvert V\rvert\times 1}}$$	Two end nodes not in the same component after removing the edge Before removing the edge, they must be in the same component $\Delta i_K = \mathrm{inf}$ $\boldsymbol{L}_{\lvert V\rvert\times\lvert V\rvert}$ kernel dimension increases

Unweighted Weighted (OD matrix)


Kirchhoff Increases of Edges in San_Francisco


Kirchhoff Increases of Edges in Seattle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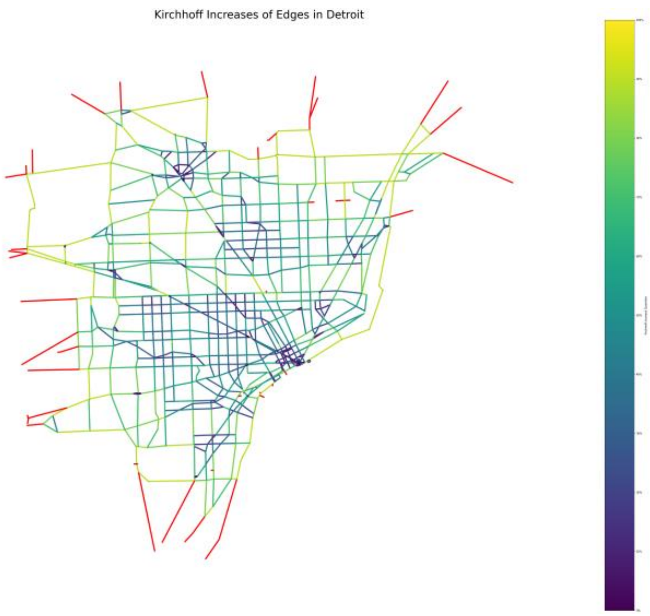
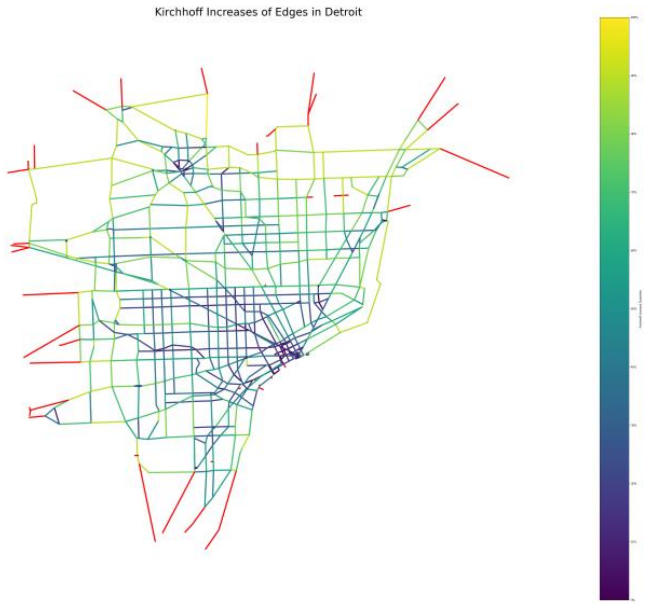

Kirchhoff Increases of Edges in Detroit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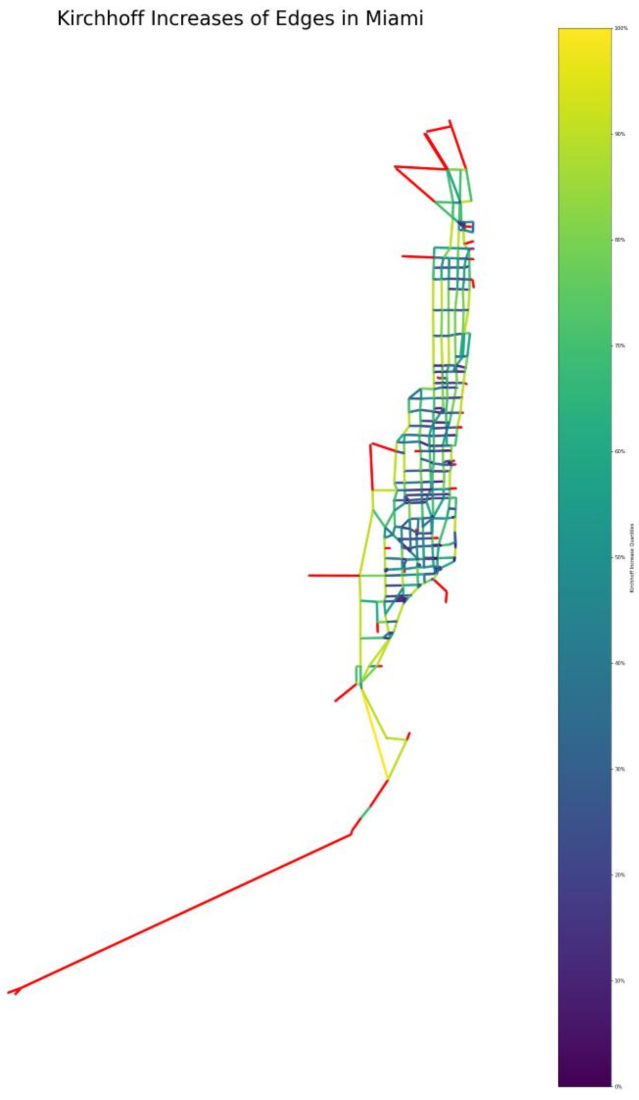
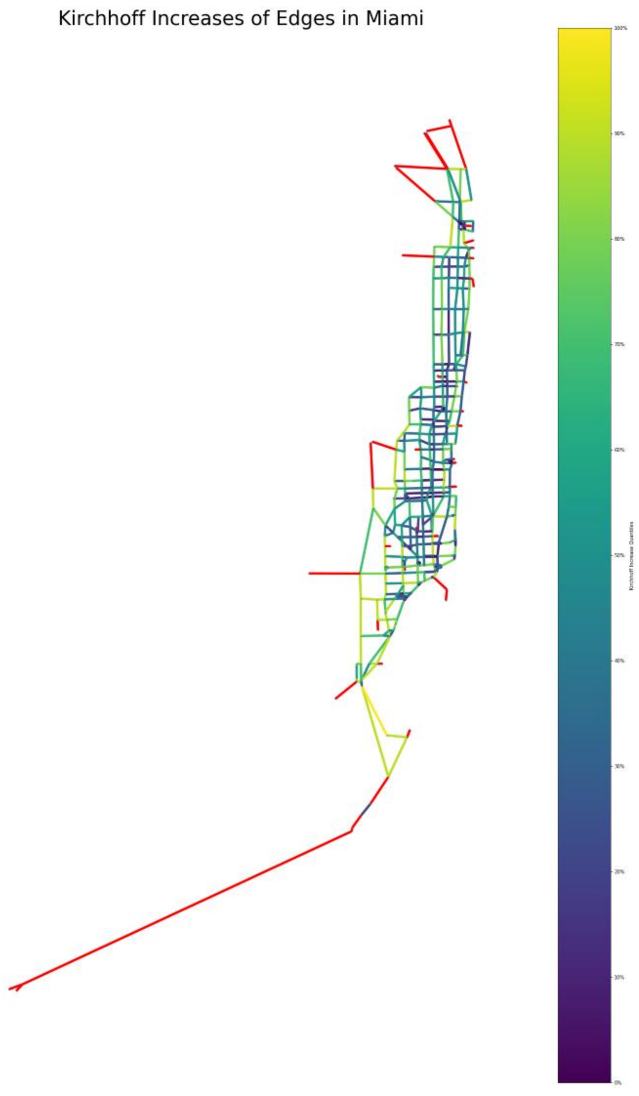

Kirchhoff Increases of Edges in Miami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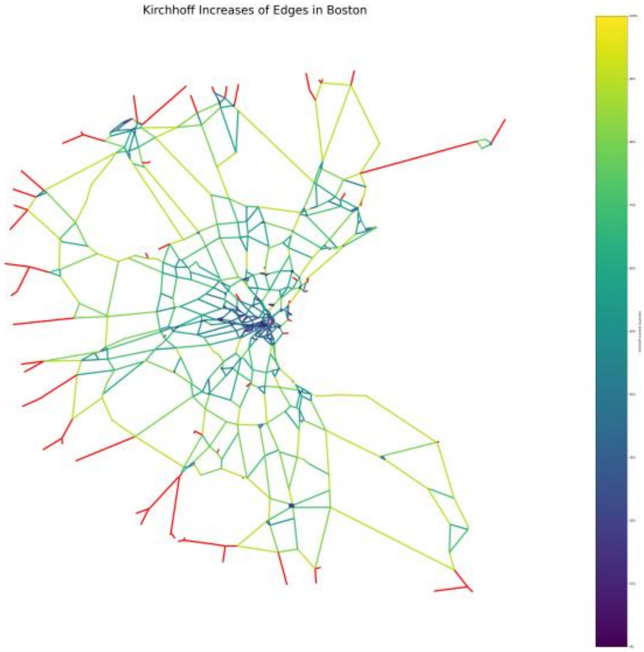
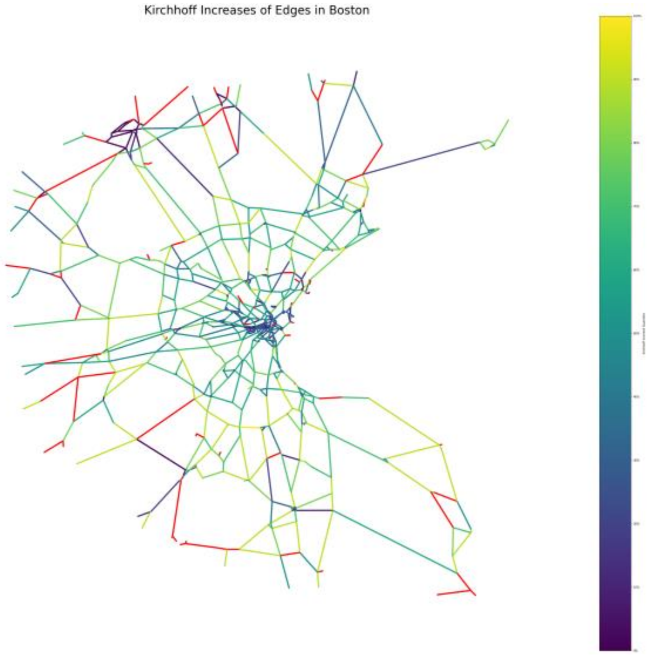
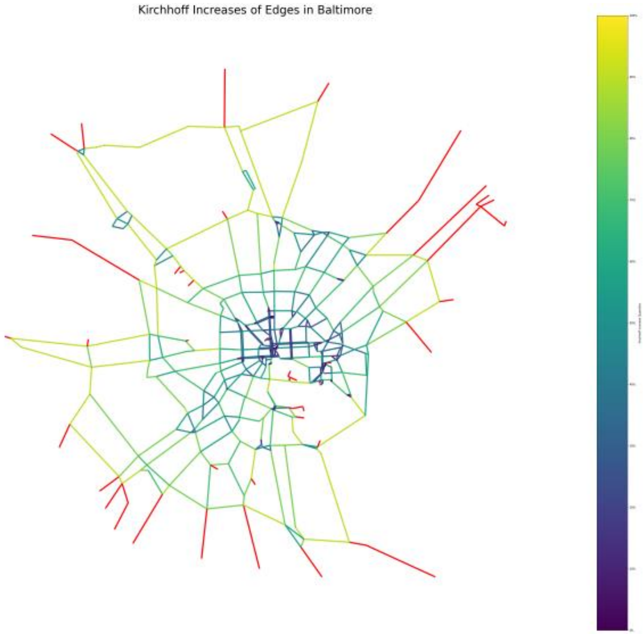
Kirchhoff Increases of Edges in Boston



Kirchhoff Increases of Edges in Boston



Kirchhoff Increases of Edges in Baltimore



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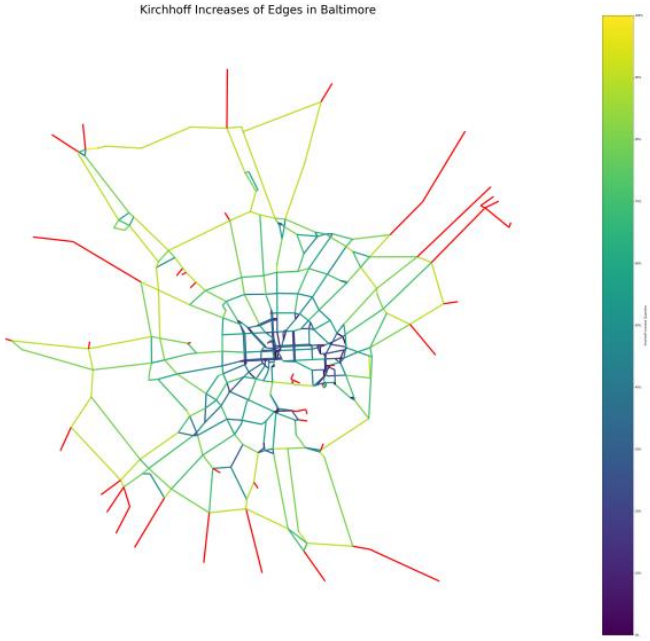
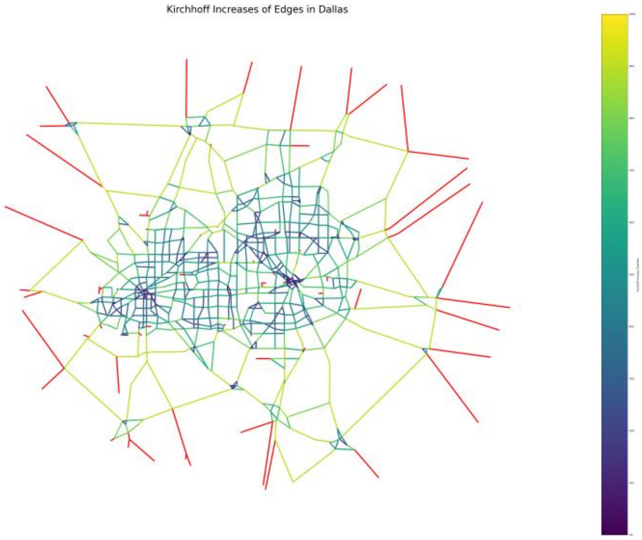
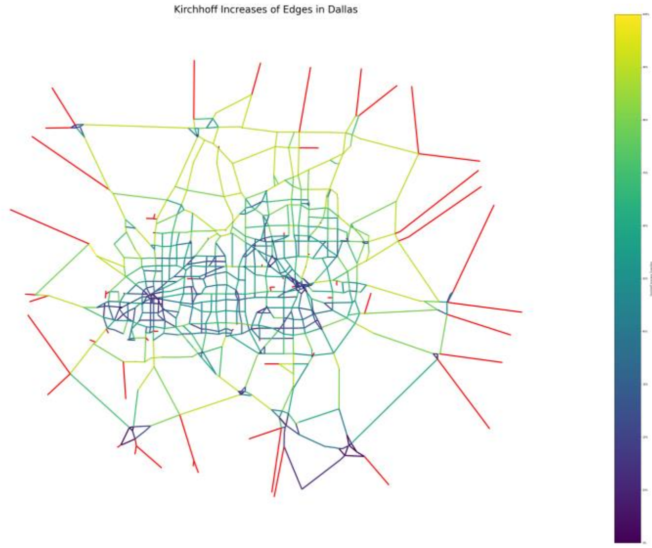
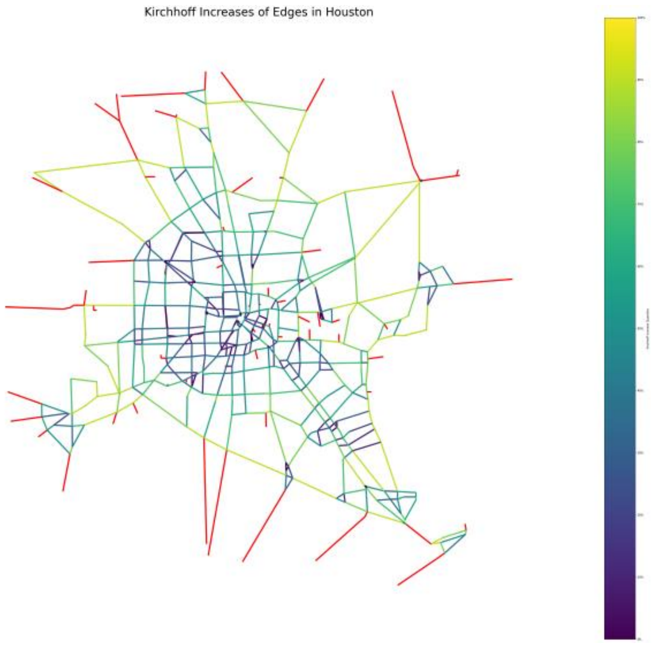


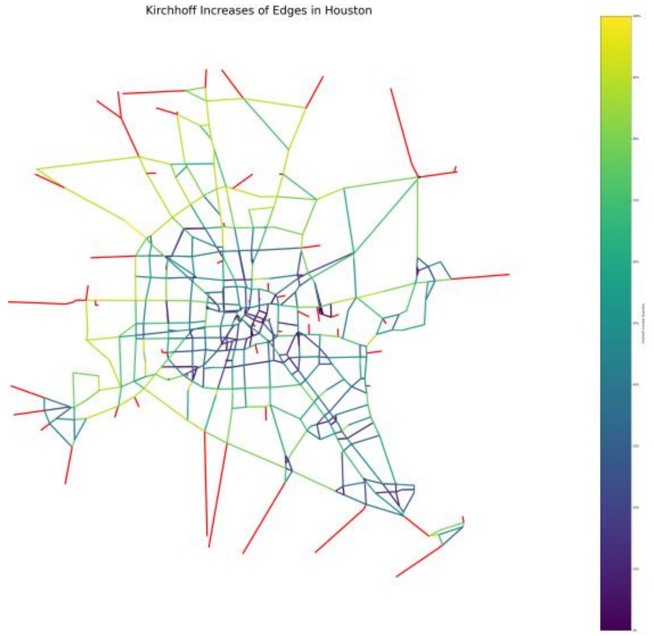
Kirchhoff Increases of Edges in Dallas



Kirchhoff Increases of Edges in Dallas




Kirchhoff Increases of Edges in Houston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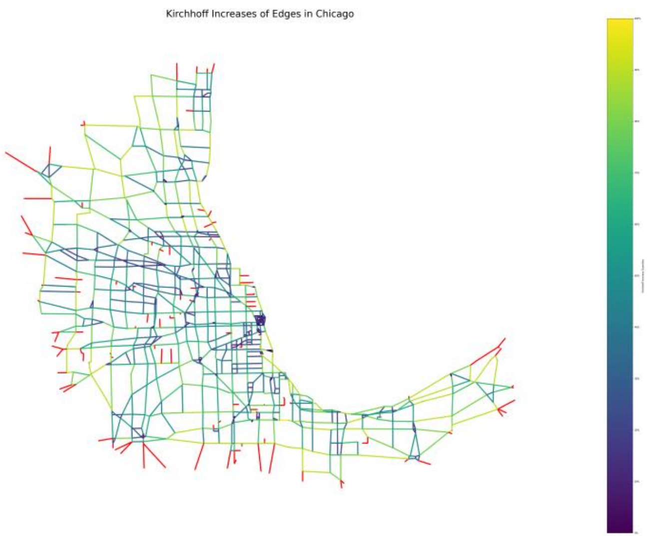
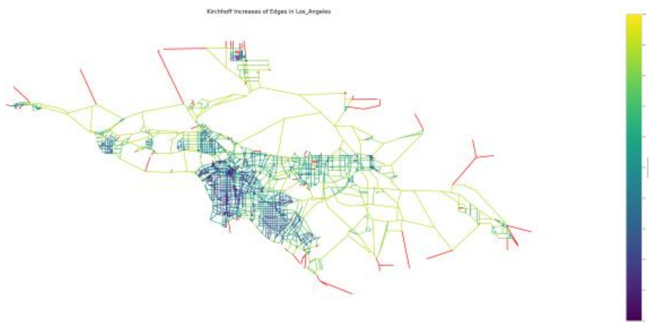
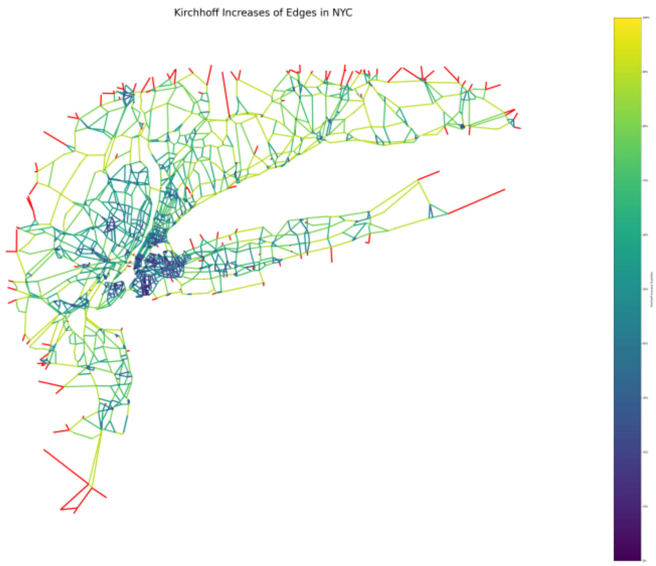

Kirchhoff Increases of Edges in Chicago


Kirchhoff Increases of Edges in Los_Angeles


Kirchhoff Increases of Edges in NYC