BASICS OF SPECTRA

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In the previous talks, we have already studied the category Top, compactly generated weak Hausdorff spaces.

People discovered that the suspension functor seemed to interact with the homotopy theory, as seen in the Fraudenthal's suspension theorem:

Theorem 0.1. *If Y is $(n-1)$-connected, then $[X, Y] \to [\Sigma X, \Sigma Y]$ is isomorphism if* $\dim(X) < 2n-1$ *and surjection if* $\dim(X) \leqslant 2n-1$.

In particular, taking $Y = S^n$, it implies that $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism if $k < n-1$. So the limit in the definition of stable homotopy groups of the sphere stablizes at a finite stage.

Definition 0.2. The stable homotopy groups of the sphere are $\pi_k^s(\mathrm{S}) = \mathrm{colim}_n \pi_{n+k}(S^n)$.

The stable morphisms between two spaces X and Y are $[X, Y]_s = \mathrm{colim}_n[\Sigma^n X, \Sigma^n Y]$,

Notice that the stable morphisms form an abelian group. We are interested in a category where the morphisms are stable morphisms in Top. This category is called spectra, or the stable homotopy category.

1. THE STABLE CATEGORY: CLASSICAL

Adams, based on the work of Boardman, worked out the following point set model of the category of spectra.

Definition 1.1. ([1, page 131,140]) A spectrum X is a sequence of pointed spaces X_n with structure maps $\epsilon_n : \Sigma X_n \to X_{n+1}$. A function $f : X \to Y$ of degree r between two spectra is a sequence of maps $f_n : X_n \to Y_{n-r}$ that is strictly compatible with the structure maps.

Remark 1.2. By adjointness of Σ and Ω this is equivalent to giving maps $\epsilon_n' : X_n \to \Omega X_{n+1}$.

Definition 1.3. X is a Ω-spectrum if all of the ϵ_n' are homotopy equivalences.

Example 1.4. For a based space X, the suspension spectrum $\Sigma^\infty X$ is $\{\Sigma^n X\}$ with the canonical structure maps $\Sigma\Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X$.

The suspension spectrum is usually not an Ω-spectrum. The function defined above is too restrictive (when Y is not an Ω-spectrum) to be useful. For example, the Hopf map does not extend to a function. So Adams came up with the "finally defined maps":

Definition 1.5. ([1, page 142]) Consider all pairs (E', f') of a cofinal subspectrum $E' \subset E$ and a function $f' : E' \to F$. Two pairs are equivalent if they agree on a cofinal subspectrum. A map of spectra $E \to F$ is an equivalence class of the pairs.

Date: August 13, 2019.

Definition 1.6. The category of CW-spectra Sp is defined as follows: An object is a spectrum X as defined in Definition 1.1 whose spaces are all CW-complexes and ϵ_n are inclusions of subcomplexes. A morphism between spectra is a map defined in Definition 1.5.

To get the homotopy category, we need to define homotopy between maps. The cylinder spectrum $Cyc(E)_n = I_+ \wedge E_n$.

Definition 1.7. ([1, page 144]) Two maps $f_0, f_1 : X \to Y$ are homotopic if there is a map $h : Cyc(X) \to Y$ such that $hi_0 = f_0$, $hi_1 = f_1$. The homotopy classes of maps are deoted $[X, Y]$. The homotopy groups of a spectra are $\pi_k(X) = [S^k, X]$.

Remark 1.8. For a category $\mathcal{C}$ with a collection of weak equivalences W, the homotopy category of $\mathcal{C}$ is $\mathrm{ho}(\mathcal{C}) = \mathcal{C}[W^{-1}]$ obtained by inverting the weak equivalences / Bousfield localization with respect to W. For example, inverting quasi-isomorphisms in chain complexes, we get the derived category $D(\mathbb{Z})$.

The homotopy category is easier to handle when we have a model structure on $\mathcal{C}$.

Here, the weak equivalences in Sp are the maps that induce π_*-isomorphisms. The homotopy category ho(Sp) has the same objects as Sp and morphisms $[X, Y]$ between two spectrum. We have the Whithead theorem:

Theorem 1.9. *([1, Cor 3.5]) Let $f : E \to F$ be a morphism between CW-spectra such that $f_* : \pi_*(E) \to \pi_*(F)$ is an isomorphism. Then f is an equivalence in* Sp*.*

We give some properties of ho(Sp).

Theorem 1.10. *([1, Thm 3.7]) The map $[X, Y] \to [\Sigma X, \Sigma Y]$ is a bijection.*

This implies that Σ is fully-faithful, furthermore it is an equivalence of categories.

With Theorem 1.10, we can show that for a cofier sequence $X \xrightarrow{f} Y \xrightarrow{i} Cf$ and any $W \in Sp$, there is exact sequence:

$$[W, X] \xrightarrow{f_*} [W, Y] \xrightarrow{i_*} [W, Cf]$$

by examining

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "i"] & Cf \arrow[r, "-\Sigma f"] & \Sigma X \\ W \arrow[r, "\mathrm{id}"] & W \arrow[u, "h|_W"] \arrow[r, "i"] & CW \arrow[u, "h"] \arrow[r] & \Sigma W. \end{tikzcd}$$

Theorem 1.11. *([1, Prop 3.11]) In* ho(Sp)*, finite coproducts are products.*

This implies that ho(Sp) is an additive category.

Theorem 1.12. *In* ho(Sp)*, a sequence $X \to Y \to Z$ is a cofiber sequence if and only if it is a fiber sequence.*

For a commutative ring R, the derived category $D(R)$ does not have cokernals in general. Instead it has non-canonical mapping cones. Verdier introduced the notion of triangulated category to axiomize this structure.

Definition 1.13. ([3, 1.1.2.5]) A triangulated category $\mathcal{D}$ is an additive category $\mathcal{D}$ with the following data:

(1) A translation functor $\Sigma : \mathcal{D} \to \mathcal{D}$ that is an equivalence of category. It is usually written as $X \mapsto X[1]$;

(2) A collection of distinguished triangles $X \to Y \to Z \to X[1]$ satisfying some axioms (The axioms are non-trivial to give, but we omit them here).

Theorem 1.14. *The stable homotopy category* $\mathrm{ho}(\mathrm{Sp})$ *is a triangulated category.*

The translation functor is Σ. The distinguished triangles are the cofiber sequences.

We mention here another construction of the stable homotopy category. It is in two steps: first invert Σ on $\mathrm{Top}^{\mathrm{fin}}$ to get the Spanier-Whitehead category of finite spectra, then complete under colimts.

2. More Examples

The complex K-theory spectrum.

Theorem 2.1. *(Bott periodicity) There is a homotopy equivalence*

$$\beta : BU \times \mathbb{Z} \to \Omega^2 BU.$$

Definition 2.2. The complex K-theory spectrum has spaces

$$K_n = \begin{cases} BU \times \mathbb{Z} & n \text{ even;} \\ U & n \text{ odd.} \end{cases}$$

The adjoints of structure maps are given by the canonical equivalence $U \simeq \Omega(BU)$ and β.

Proposition 2.3. *If X is a finite CW complex, $[X, K] \cong K^0(X)$. Here, $K^0(X)$ is the Grothendieck group of isomorphism classes of complex vector bundles over X with direct sum.*

The complex cobordism spectrum MU. Let γ^n be the universal bundle over $BU(n)$, the classifying space of n-dimentional complex vector bundle. The inclusions $i : BU(n) \to BU(n+1)$ pulls back the universal bundle over $BU(n+1)$ to the universal bundle over $BU(n)$ plus a trivial complex line bundle

$$i^*(\gamma^{n+1}) \cong \gamma^n \oplus 1.$$

Let $T(n) = \mathrm{Th}(\gamma^n)$ be the Thom space, then the bundle map gives on Thom spaces

$$i : \Sigma^2 T(n) \to T(n+1).$$

Definition 2.4. The complex cobordism spectrum MU has spaces

$$MU_k = \begin{cases} T(n) & k = 2n; \\ \Sigma T(n) & k = 2n+1. \end{cases}$$

The structure maps are given by the above defined i.

3. The stable category: the ∞-land

The definition of the stable ∞-category is motivated by the properties of the classical stable category.

Throughout this part let $\mathcal{C}, \mathcal{D}$ be ∞-categories. A triangle in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ that sends $(0,1)$ to a zero object 0 of $\mathcal{C}$, depicted as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z. \end{tikzcd}$$

We say it is a fiber sequence if it is a pullback square; a cofiber sequence if it is a pushout square. Note that this picture is only a simplified visualization (see [3, 1.1.1.5]). In particular, it consists more data than the two 1-cells f, g and it is only homotopy commutative.

Definition 3.1. [3, 1.1.1.9] An ∞-category $\mathcal{C}$ is stable if the followings are true:

(1) It has a zero object;
(2) Every morphism admits a fiber and a cofiber;
(3) A triangle is a fiber sequence if and only if it is a cofiber sequence.

It can be shown that in a stable ∞-category general pushout squares and pullback squares are the same thing. This can be seen as the defining property by the following proposition:

Proposition 3.2. [3, 1.1.3.4] *Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is stable if and only if $\mathcal{C}$ has all finite limits, finite colimits and that a square in $\mathcal{C}$ is a pushout if and only if it's a pullback.*

Given an ∞-category $\mathcal{C}$, we now define the functor Σ (or dually Ω). Let $\mathcal{M}^{\Sigma}$ be all the pushout triangles of the form

$$\begin{array}{ccc} X & \longrightarrow & 0' \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & Y. \end{array}$$

Evalutating at the initial object gives a trivial fibration $\mathcal{M}^{\Sigma} \to \mathcal{C}$. So it has a section s. Then $\Sigma : \mathcal{C} \to \mathcal{C}$ is the composite of this section with evaluating at the final object. Conceptually Σ sends X to Y. When $\mathcal{C}$ is stable, pushouts and pullbacks are the same, so that $\mathcal{M}^{\Sigma} = \mathcal{M}^{\Omega}$. If follows that Σ and Ω are homotopy inverses.

Theorem 3.3. *([3, 1.1.2.14]) Let $\mathcal{C}$ be a stable ∞-category. Then* $\mathrm{ho}(\mathcal{C})$ *has the structure of a triangulated category.*

We remark that for an ∞-category to be stable is a *property*, whereas a triangulation of an additive 1-category is a *structure*. If it exists, a triangulated structure is (typically) not unique, but is something that one has to choose and to remember how one has chosen. This is difficult and easily leads to (sign) mistakes. By contrast, working with stable ∞-categories, there are no choices to be made, so this makes life much easier!

Now we construct the stable ∞-category of spectra Sp. There are two perspectives, one regarding a spectrum as a cohomology theory and the other as a homology theory. We will use the first one to build Sp and treat the second one as a universal property.

We first take the cohomology viewpoint. Let $\mathcal{S}_*^{\mathrm{fin}}$ be the ∞-category of finite pointed spaces and $\mathcal{S}_*$ be the ∞-category of spaces.

Definition 3.4. The ∞-category Sp is the homotopy limit of $\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$.

This relates back to Definition 1.1: An object in the the homotopy limit will consists of a sequence of spaces $X_n \in \mathcal{S}_*$ and maps $X_n \xrightarrow{\simeq} \Omega X_{n+1}$.

Let us elaborate a little bit. For any $X \in \mathrm{Sp}$, the functors $\pi_0(\mathrm{Hom}_{\mathrm{Sp}}(-, \Sigma^r X))$ form a comology theory. However, cohomology theories via Brown representability are only well defined as objects in $\mathrm{ho}(\mathrm{Sp})$, since the homotopy equivalence $X_n \xrightarrow{\simeq} \Omega X_{n+1}$ is defined in $\mathrm{ho}(\mathcal{S}_*)$.

Next we switch to the homology viewpoint.

Definition 3.5. For a functor $F : \mathcal{E} \to \mathcal{C}$ between ∞-categories which amdit the following mentioned things, we say

(1) F is excisive if it sends pushout squares in $\mathcal{E}$ to pullback squares in $\mathcal{C}$;
(2) F is reduced if it sends a final object in $\mathcal{E}$ to a final object in $\mathcal{C}$.

Let $\mathcal{E}$ admit pushouts and a final object. We denote by $\mathrm{Exc}_*(\mathcal{E}, \mathcal{C})$ the full subcategory spanned by reduced, excisive functors $F : \mathcal{E} \to \mathcal{C}$.

Definition 3.6. [3, 1.4.2.8] Let $\mathcal{C}$ be an ∞-category with finite limits. Define the ∞-category of spectrum objects in $\mathcal{C}$ to be:

$$\mathrm{Sp}(\mathcal{C}) = \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}).$$

An object of $\mathrm{Sp}(\mathcal{C})$, or a reduced and excisive functor $X : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$, is called a spectrum object in $\mathcal{C}$.

It turns out that the ∞-category of spectrum objects is stable:

Proposition 3.7. *(*[3, 1.4.2.16]*) Let $\mathcal{C}, \mathcal{D}$ be ∞-categories such that $\mathcal{C}$ is pointed and admits colimits and that $\mathcal{D}$ admits finite limits. Then* $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ *is stable.*

As an example, the functor $X \mapsto \mathrm{Z}\mathit{Sing}(X)_*$ sends homotopy pushout squares of spaces to homotopy pullback sqaures of Kan complexes. Notice that $\pi_*(\mathrm{Z}\mathit{Sing}(X)_\bullet) \cong \tilde{H}_*(X, \mathbb{Z})$. The above claim is then equivalent to the theorem/axiom of the Mayer-Vietoris sequence.

The stable ∞-category Sp as in Definition 3.4 agrees with the ∞-category of the spectrum objects in $\mathcal{S}_*$ as in Definition 3.6.

Definition 3.8. Evaluation at $S^0 \in \mathcal{S}_*^{\mathrm{fin}}$ gives a functor, denoted by $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$.

Proposition 3.9. [3, 1.4.2.24] *The map $\Omega^\infty : \mathrm{Sp}(\mathcal{S}_*) \to \mathcal{S}_*$ lifts to an equivalence*

$$\mathrm{Sp}(\mathcal{S}_*) \to \mathrm{Sp}.$$

We introduce some terminology.

Definition 3.10. Let $F : \mathcal{C} \to \mathcal{D}$ be functors between ∞-categories. We say that

(1) F is left exact if F commutes with finite limits. Denote the full subcategory spanned by left exact functors by $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$.
(2) F is right exact if F commutes with finite colimits. Denote the full subcategory spanned by right exact functors by $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})$.
(3) F is exact if it carries zero objects to zero objects and fiber sequences to fiber sequences.

Remark 3.11. If $\mathcal{C}$ is stable, then $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) = \mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$; If $\mathcal{D}$ is stable, then $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D}) = \mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$; If both $\mathcal{C}$ and $\mathcal{D}$ are stable, the three kinds of exactness for F are equivalent ([3, 1.1.4.1]), as well as being both reduced and excisive (Proposition 3.2).

Proposition 3.12. [3, 1.4.2.22] *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories such that $\mathcal{C}$ has finite colimits and $\mathcal{D}$ has finite limits. Then composing with $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces equivalence*

$$\mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}).$$

Proof. (sketch) First, identify $\mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) = \mathrm{Sp}(\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}))$ and the functor in question to $\Omega^\infty : \mathrm{Sp}(\mathcal{E}) \to \mathcal{E}$ for $\mathcal{E} = \mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$. Second, Ω^∞ is an equivalence here since $\mathcal{E}$ is stable. □

We introduce some more terminology.

Definition 3.13. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories.

(1) Denote the full ∞-subcategory spanned by functors admitting right adjoints by $\mathsf{LFun}(\mathcal{C}, \mathcal{D})$.
(2) Denote the full ∞-subcategory spanned by functors admitting left adjoints by $\mathsf{RFun}(\mathcal{C}, \mathcal{D})$.

Proposition 3.14. *([3, 1.4.4.4]) Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories such that $\mathcal{D}$ is stable. Then $\Omega^\infty : \mathsf{Sp}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\Sigma^\infty_+ : \mathcal{C} \to \mathsf{Sp}(\mathcal{C})$.*

Definition 3.15. Taking $\mathcal{C} = \mathcal{S}_*$ in Proposition 3.14, the sphere spectrum is defined to be $\mathsf{S} := \Sigma^\infty_+(*)$.

Theorem 3.16. *([3, 1.4.4.5]) Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories such that $\mathcal{D}$ is stable. Then there are equivalences:*

$$\mathsf{LFun}(\mathsf{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{-\circ\Sigma^\infty_+} \mathsf{LFun}(\mathcal{C}, \mathcal{D});$$
$$\mathsf{RFun}(\mathcal{D}, \mathsf{Sp}(\mathcal{C})) \xrightarrow{\Omega^\infty\circ-} \mathsf{RFun}(\mathcal{D}, \mathcal{C}).$$

Proof. (sketch) The two assertions are equivalent. To prove the second assertion, we first assume that $\mathcal{C}$ is pointed. By Proposition 3.14, we have the commutative diagram:

$$\begin{tikzcd} \mathsf{Exc}_*(\mathcal{D}, \mathsf{Sp}(\mathcal{C})) \arrow[r, "\Omega^\infty\circ-"] & \mathsf{Exc}_*(\mathcal{D}, \mathcal{C}) \\ \mathsf{RFun}(\mathcal{D}, \mathsf{Sp}(\mathcal{C})) \arrow[u, "\subset"] \arrow[r, "\Omega^\infty\circ-"] & \mathsf{RFun}(\mathcal{D}, \mathcal{C}) \arrow[u, "\subset"] \end{tikzcd} \tag{1}$$

If we can show that it is a pullback, the assertation will follow from Proposition 3.12. The proof breaks into the following steps:

(1) Define an auxilory ∞-category Pr^R. An object is a presentable ∞-category. A morphism is a functor that is accessible and preserves small limits. ([2, 5.5.3.1]) Between presentable ∞-categories, the adjoint functor theorem shows that a functor has a left adjoint if and only if it is accessible and preserves small limits ([2, 5.5.2.9]), so that we have $\mathsf{RFun} = \mathsf{Pr}^R$.
(2) By the general case of 3.9, $\mathsf{Sp}(\mathcal{C}) \simeq \lim(\cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C})$. Since $\mathcal{C}$ is pointed, Ω has left adjoint Σ. So we can view the diagram in the braket as a diagram in Pr^R.
    The theorem [2, 5.5.3.18] says that it has a limit in Pr^R and $\mathsf{Pr}^R \to \hat{\mathsf{Cat}}_\infty$ preserves the limit. That is, $\mathsf{Sp}(\mathcal{C})$ is presentable, and
$$\mathsf{RFun}(\mathcal{D}, \mathsf{Sp}(\mathcal{C})) \simeq \lim(\cdots \to \mathsf{RFun}(\mathcal{D}, \mathcal{C}) \xrightarrow{\Omega\circ-} \mathsf{RFun}(\mathcal{D}, \mathcal{C})).$$
(3) Since $\mathcal{D}$ is stable, an excisive functor $G : \mathcal{D} \to \mathcal{C}$ is in RFun if and only of ΩG is in RFun. That is, each square is a pullback in the following diagram:
$$\begin{tikzcd} \cdots \arrow[r] & \mathsf{Exc}_*(\mathcal{D}, \mathcal{C}) \arrow[r, "\Omega"] & \mathsf{Exc}_*(\mathcal{D}, \mathcal{C}) \\ \cdots \arrow[r] & \mathsf{RFun}(\mathcal{D}, \mathcal{C}) \arrow[u, "\subset"] \arrow[r, "\Omega"] & \mathsf{RFun}(\mathcal{D}, \mathcal{C}) \arrow[u, "\subset"] \end{tikzcd}$$

(4) By steps 2,3 and the identification:

$$\mathrm{Exc}_*(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) \simeq \mathrm{Sp}(\mathrm{Exc}_*(\mathcal{D}, \mathcal{C})) \simeq \lim(\cdots \to \mathrm{Exc}_*(\mathcal{D}, \mathcal{C}) \xrightarrow{\Omega} \mathrm{Exc}_*(\mathcal{D}, \mathcal{C})),$$

we get that (1) is a pullback square.

For a not-necessarily-pointed $\mathcal{C}$, replace it by the pointed ∞-category $\mathcal{C}_* := \mathcal{C}_{1/}$ for a terminal obejct 1. We have the following facts: $\mathcal{C}_*$ is presentable ([2, 5.5.3.11]). There is a canonical equivalence $\mathrm{Sp}(\mathcal{C}_*) \simeq \mathrm{Sp}(\mathcal{C})_*$ and $\mathrm{Sp}(\mathcal{C})_* \to \mathrm{Sp}(\mathcal{C})$ is also an equivalence since $\mathrm{Sp}(\mathcal{C})$ is pointed ([3, 1.4.2.18]). In particular, they are all presentable by step 2.

It suffices to show that the two vertical maps in the following commutative diagram are equivalences:

$$\begin{tikzcd} \mathrm{RFun}(\mathcal{D}, \mathrm{Sp}(\mathcal{C}_*)) \arrow[r, "\Omega^\infty \circ -"] \arrow[d, "\simeq"] & \mathrm{RFun}(\mathcal{D}, \mathcal{C}_*) \arrow[d, "\simeq"] \\ \mathrm{RFun}(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) \arrow[r, "\Omega^\infty \circ -"] & \mathrm{RFun}(\mathcal{D}, \mathcal{C}) \end{tikzcd}$$

Indeed, the left arrow is an equivalence by the same machanism as the right arrow, where $\mathrm{Sp}(\mathcal{C})$ plays the role of $\mathcal{C}$. So we focus on the right arrow. By the adjoint functor theorem stated in step 1, it suffices to show that a functor $\mathcal{D} \xrightarrow{G} \mathcal{C}_*$ is accessible and preserves small limits if and only if the composition $\mathcal{D} \xrightarrow{G} \mathcal{C}_* \to \mathcal{C}$ does. The "limit" part follows from the dual statement of [2, 1.2.13.8]. The "colimit" part follows from [2, 4.4.2.9] and the fact that $\mathcal{C}_{1/} \to \mathcal{C}$ is a left fibration. □

Corollary 3.17. *Let $\mathcal{D}$ be a presentable stable ∞-category. Then evaluating at the sphere spectrum* S *gives equivalence*

$$\mathrm{ev}_{\mathrm{S}} : \mathrm{LFun}(\mathrm{Sp}, \mathcal{D}) \to \mathcal{D}.$$

Proof. The evaluation map factors as:

$$\mathrm{ev}_{\mathrm{S}} : \mathrm{LFun}(\mathrm{Sp}, \mathcal{D}) \to \mathrm{LFun}(\mathcal{S}, \mathcal{D}) \xrightarrow{\mathrm{ev}_*} \mathcal{D}.$$

The first map is an equivalence by taking $\mathcal{C} = \mathcal{S}$ in Corallary 3.16. The second map is an equivalence by taking $S = *$ in [2, 5.1.5.6]. □

We would like to point out that the equivalence ev_* in the proof shows that $\mathcal{S}$ is freely generated under colimit by the object of one point $*$; and the equivalence ev_{S} shows that Sp is freely generated under colimit by the object of the sphere spectrum S.

4. Duality

In this section we deal with various notions of duality. We first look at Alexander duality concerning a subspace of a sphere. Passing to the stable category, Spanier-Whitead duality is able to eliminate the choice of the sphere in Alexander duality. Specializing Spanier-Whitehead duality to manifolds, one can recover the Atiyah duality relating a manifold and its tangent bundle. If furthur this manifold is orientable, the Thom isomorphism simplifies the Atiyah duality to give Poincaré duality of homology and cohomology of the manifold.

We start withe the Alexander duality. Let $X \subset S^n$ be a proper subspace. For a subspace $A \subset S^n$, let A^c denote the complement of A in S^n.

Theorem 4.1 (Alexander duality)**.** *There is isomorphism for all r:*

$$\tilde{\check{H}}^r(X) \cong \check{H}^r(X, \mathrm{pt}) \cong H_{n-r}(\mathrm{pt}^c, X^c) \cong \tilde{H}_{n-r-1}(X^c).$$

The theorem implies that the homology of X^c is determined only by the (Čech) cohomology of X. The Čech cohomology is for dealing with local pathologies. However, the homotopy type of X^c clearly involves the embedding. (Otherwise, knot theory would not exist.)

Assume that $X \xrightarrow{i} S^n$ is a proper compact subset. Spanier-Whitehead proved that the **stable** homotopy type of X^c is determined by the stable homotopy type of X.

Their strategy is as follows: Let S denote the unreducd suspension. Then $SX \xrightarrow{Si} S^{n+1}$ is also a proper compact embedding. This procedure gives canonical homotopy equivalence on the compliment. By manipulation on the dimenstion, they give for $f : X \to Y$ a functorial stable class of maps $f^* : Y^c \to X^c$, where X and Y are embedded in different spheres. Using this map, they show that if f is a homotopy equivalence, then f^* is a stable homotopy equivalence.

Moreover, Spanier eliminates the choice of embeddings of X and Y in a sphere. What matters in the proof is a map $\mu : X \wedge Y \to S^{n-1}$. This gives the modern treatment.

Definition 4.2. In the symmetric monoidal category $(\mathrm{Sp}, \wedge, \mathrm{S})$ (defined in the next talk), an object X is dualizable if there is an object DX and maps $\mu : X \wedge DX \to \mathrm{S}$, $\eta : \mathrm{S} \to DX \wedge X$ such that the following diagrams commute (triangle identities):

$$\begin{tikzcd} & DX \wedge X \wedge DX \arrow[dr, "\mathrm{id}\wedge\mu"] & \\ DX \arrow[ur, "\eta\wedge\mathrm{id}"] \arrow[rr, equal] & & DX \end{tikzcd}, \qquad \begin{tikzcd} X \arrow[rr, equal] \arrow[dr, "\mathrm{id}\wedge\eta"'] & & X \\ & X \wedge DX \wedge X \arrow[ur, "\mu\wedge\mathrm{id}"'] & \end{tikzcd}.$$

Theorem 4.3. *Let $X \subset S^n$ be a proper compact subspace. Then $\Sigma^{1-n}X^c$ is the (Spanier-Whitehead) dual of X.*

Proposition 4.4. *If X is a CW spectrum, then it is dualizable. If moreover X is finite (built from finitely many cells), then DX is equivalent to a finite spectrum.*

Proposition 4.5. $T : [W, Z \wedge DX] \to [W \wedge X, Z]$ *is an isomorphism if Z is finite or both W and X are finite.*

Corollary 4.6. *If X is a finite spectrum, then for any spectrum E and integer r, $E_{-r}(DX) \cong E^r(X)$.*

Theorem 4.7 (Atiyah duality)**.** *If M is a compact manifold (possibly with boundary), then $D(M/\partial M) \simeq Th(-\mathrm{T}M)$. (Note that when M is closed, $M/\partial M = M_+$.)*

Proof. We only proof the case when M is closed. The general case is similar. By Whitney embedding theorem, there exists a number N such that there is an embedding $i : M \to \mathbb{R}^N$. By tubular neighborhood theorem, i has a tubular neighborhood that is diffeomorphic to the normal bundle ν of M in $\mathbb{R}^N$. We can put a Riemannian metric on ν and denote by $\nu_{\leqslant 1}$ and $\nu_{=1}$ the unit disk bundle and unit sphere bundle.

Embedding the disjoint point at infinity, we get an embeeding $M_+ \to S^N$. By Spanier-Whitehead duality,

$$D(M_+) \simeq D(\nu_{<1,+}) \simeq \Sigma^{1-N}(\mathbb{R}^N \backslash \nu_{<1}).$$

Consider the cofiber sequence

$$\mathbb{R}^N \backslash \nu_{\leqslant 1} \to \mathbb{R}^N \to \mathbb{R}^N/(\mathbb{R}^N \backslash \nu_{<1}) \simeq \nu_{\leqslant 1}/\nu_{=1} \simeq \mathrm{Th}(\nu).$$

Since $\mathbb{R}^N$ is contractible, we have $\mathrm{Th}(\nu) \simeq \Sigma(\mathbb{R}^N \backslash \nu_{<1})$.

The embedding gives $\nu \oplus \mathsf{T}M \cong M \times \mathbb{R}^N$, so that we have $\Sigma^{-N}\mathsf{Th}(\nu) \simeq \mathsf{Th}(-\mathsf{T}M)$. This completes the proof. □

Theorem 4.8. *(Poincaré duality) Let M be a compact manifold of dimension n (possibly with boundary) and E be a spectrum. Assume that M is E-orientable. Then there is isomorphism $E^r(M, \partial M) \cong E_{n-r}(M^+)$ for all integers r.*

Proof. (sketch) The manifold M being E-orientable will imply the Thom isomorphism:

$$E_*(\mathsf{Th}(-\mathsf{T}M)) \cong E_*(\Sigma^{-n}(M^+)).$$

The claim follows from Atiyah duality (Theorem 4.7) and Corollary 4.6. □

References

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