Basic Homotopy Theory Review

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1 Introduction

We will review classical category theory and space-level homotopy theory. We will talk about motivations for ∞-categories. We avoid using simplicial language in the treatments.

2 Classical theory reviews

First we review the language of categories. Recall the following definition:

A *category* $\mathcal{C}$ consists of the following data:
(1). A collection of *objects* of $\mathcal{C}$. If X is an object of $\mathcal{C}$, we usually write $X \in \mathcal{C}$.
(2). For every pair of objects $X, Y \in \mathcal{C}$, a set of *morphisms* from X to Y, the set is usually denoted as $\mathcal{C}(X, Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.
(3). For every $X \in \mathcal{C}$, an *identity morphism* $\mathrm{id}_X \in \mathcal{C}(X, X)$.
(4). For every triple $X, Y, Z \in \mathcal{C}$, a *composition* map

$$\circ : \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z).$$

(5). The composition is associative and unital.

Example. The category of sets, denotes by Set. The category of R-modules, denoted by RMod.
A morphism $f : X \to Y$ is said to be an *isomorphism*, if there exists $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. Isomorphisms are bijections in the category Set.

A *functor* is a morphism between categories. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, a functor $F : \mathcal{C} \to \mathcal{D}$ assigns an object $FX \in \mathcal{D}$ for any $X \in \mathcal{C}$, and a morphism $Ff : FX \to FY$ for any $f : X \to Y$ in a way that respects both unitality and associativity. For example, for $f : X \to Y, g : Y \to Z$, the associativity is preserved as

$$Fg \circ Ff = F(g \circ f).$$

A functor is *full*/*faithful* if it induces surjection/injection on hom-sets.

A *natural transformation* is a morphismn between functors. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors, a natural transformation $F \Rightarrow G$ assigns for any $X \in \mathcal{C}$ a morphism $FX \to GX$, in a natural way. Given two categories $\mathcal{C}, \mathcal{D}$, there is a functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, also denoted by $\mathcal{D}^{\mathcal{C}}$. Its objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and morphisms are natural transformations between functors. Isomorphisms in functor categories are called *natural isomorphisms*. Two categories $\mathcal{C}, \mathcal{D}$ are *equivalent*, if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $GF \Rightarrow \mathrm{Id}_{\mathcal{C}}$, $FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$.

Two functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ are said to be *adjoint*, if there are natural bijections $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$. An adjoint, if it exists, is unique up to natural isomorphism.

A category is *small* if its collection of objects form a set. Let $\mathcal{B}, \mathcal{C}$ be categories, and $\mathcal{B}$ be small. A $\mathcal{B}$-shaped diagram in $\mathcal{C}$ is a functor $F : \mathcal{B} \to \mathcal{C}$. There is a generalized diagonal functor $\Delta_{\mathcal{B}} : \mathcal{C} \to \mathcal{C}^{\mathcal{B}}$. The *colimit* is the left adjoint of $\Delta_{\mathcal{B}}$.

The *duality* plays an important role in category theory. The *opposite category* reverses all arrows, and produces dual notions, including *contravariant functors*, *limits*, etc. Left adjoint preserves colimits and right adjoint preserves limits.

Example. (Co)product, pushout, pullback, sequential (co)limit.

Let Top be the category of compactly generated weak Hausdorff (CGWH) topological spaces, with continuous maps as morphisms. We usually write $*$ for the basepoint. We denote the category of based CGWH spaces by $\mathrm{Top}^{*/}$, whose morphisms are continuous maps that preserve basepoints. We usually use capital letters $X, Y, Z, ...$ for spaces. A pair of space $A \subset X$ is denoted as (X, A).

The topology of CGWH spaces is determined by compact Hausdorff subsets. The category Top is a full subcategory of the original category of topological spaces. It is bi-complete (has all small limits and colimits) and Cartesian closed (the following adjunction holds):

$$Z^{X \times Y} \cong (Z^Y)^X.$$

All CW complexes (to be revisited later) are CGWH.

The category $\mathrm{Top}^{*/}$ enjoys similar properties. The smash product is defined by $X \wedge Y := X \times Y / X \vee Y$. The function space $F(X, Y)$ is the set of based maps from X to Y. The adjunction reads as

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)).$$

There is a closed symmetric monoidal structure $(\mathrm{Top}^{*/}, \wedge, S^0)$.

A homotopy is a map $X \times I \to Y$, and a based homotopy is a map $X \wedge I_+ \to Y$. We will denote the homotopy categories respectively by $h\mathrm{Top}$ and $h\mathrm{Top}^{*/}$, the morphisms are *homotopy classes of maps*. However, these homotopy categories are not complete or cocomplete.

The set of based homotopy classes of maps between X and Y will be denoted by $[X, Y]$. The *homotopy groups*, are $\pi_n(X) = [S^n, X]$. Since

$$\pi_0 F(X, Y) = \mathsf{Top}(I, F(X, Y)) = \mathsf{Top}^{*/}(I_+, F(X, Y)) = \mathsf{Top}^{*/}(X \wedge I_+, Y) = [X, Y],$$

the adjunction passes to homotopy categories.

Some more constructions and examples: In $\mathrm{Top}^{*/}$. The *reduced cone*, is $CX = X \wedge I$; the *path space*, is $PX = F(I, X)$. The *reduced suspension*, is $\Sigma X = X \wedge S^1$; the *loop space*, is $\Omega X = F(S^1, X)$. We have natural bijection

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

A *cofibration* is a map $i : A \to X$ that satisfies homotopy extension property (HEP). A cofibration is always a closed inclusion. It could be tested by the neighborhood-deformation (NDR) criterion. In $\mathrm{Top}^{*/}$, there is a parallel definition of *based cofibration*, where all spaces and maps are based.

Let $f : X \to Y$ be a based map, the *homotopy cofiber* Cf is

$$Cf = Y \cup_f CX.$$

The inclusion $i : Y \to Cf$ is a cofibration (since it is a pushout of $X \to CX$). Let $\pi : Cf \to Cf/Y = \Sigma X$ be the quotient map, we have the following *based cofiber sequence*:

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \to ...$$

Lemma. If $i : A \to X$ is a based cofibration, then the quotient map

$$q : Ci \to Ci/CA \cong X/A$$

is a based homotopy equivalence.

Theorem. For any based space Z, the induced sequence

$$... \to [\Sigma X, Z] \to [Cf, Z] \to [Y, Z] \to [X, Z].$$

is exact. Since S^n for $n \geq 1$ is a cogroup object and is abelian for $n \geq 2$, the sequence is exact as groups to the left of $[\Sigma X, Z]$ and exact as abelian groups to the left of $[\Sigma^2 X, Z]$.

Dually, a *fibration* is a surjective map $p : X \to Y$ that satisfies covering homotopy property (CHP). This includes the notions of covering spaces, vector bundles, and fiber bundles. Given a based map $f : X \to Y$, there is a dual *fiber sequence*, but we only give a special case of it here:

Theorem. Let $p : E \to B$ be a fibration, choose a basepoint from B and let $F = p^{-1}(*) \subset E$ be a fiber. There is a long exact sequnce of homotopy groups:

$$... \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to ... \to \pi_0(E).$$

Another example you may know is the long exact sequence of homotopy groups for a pair of space.

A map $f : X \to Y$ is called a *weak homotopy equivalence* if $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n and all choices of basepoints in X.

Now we introduce an important type of spaces: CW complexes. Let X^0 be a discrete set. Inductively, X^{n+1} is the pushout obtained from X^n via attatching map $\coprod S^n \to X^n$ and cofibration $\coprod S^n \to \coprod D^{n+1}$. A *CW complex* is the colimit (union with weak topology) of any such expanding sequence $X^0 \to X^1 \to ....$ If $X = X^n$, we say X is of *dimension n*.

A *subcomplex* $A \subset X$ is a subspace and a CW complex whose cells are also cells of X. We use $\mathrm{Top}_{\mathrm{CW}}$ to denote the full subcategory of spaces homeomorphic to CW complexes.

A CW complex is Hausdorff, locally contractible and paracompact. A map $f : X \to Y$ is said to be *cellular* if $f(X^n) \subset Y^n$.

Examples. S^n, $\mathbb{R}P^n$ and their CW structures.

We record the following facts:

(a). The wedge sum of CW complexes (X_i, x_i) based at vertices is a CW complex which contains each X_i as a subcomplex.

(b). If (X, A) is a CW pair, then the quotient X/A is a CW complex.

(c). If A is a subcomplex of X, Y is a CW complex, and $f : A \to Y$ is a cellular map, then the pushout $Y \cup_f X$ is a CW complex which contains Y as a subcomplex.

(d). The colimit of inclusions $X_i \to X_{i+1}$ of subcomplexes is a CW complex that contains each X_i as a subcomplex.

(e). The product $X \times Y$ of CW complexes is a CW complex.

CW complexes are nice: The definition is in some sense combinatorial; and it turns out that any space can be replaced functorially by weakly equivalent CW complexes. As a result we would turn to study the (homotopy) category of CW complexes. In fact, most spaces are homotopy equivalent to CW complexes.

Lemma. If (X, A) is a CW pair, then the inclusion $i : A \to X$ is a cofibration.

Theorem. (Whitehead) A weak equivalence between CW complexes is a homotopy equivalence.

Theorem. (Cellular approximation) Any map $f : (X, A) \to (Y, B)$ between CW pairs is homotopic to a cellular map relative to A.

Theorem. (CW approximation) There is a functor $\Gamma : \mathrm{Top} \to h\mathrm{Top}$ and a natural transformation $\gamma : \Gamma \Rightarrow \mathrm{Id}$ that assigns a CW complex ΓX and a weak equivalence $\Gamma X \to X$ to each space X.

There is a more general notion (model category) of *homotopy categories*. If a category $\mathcal{C}$ is equipped with a collection of morphisms called *weak equivalences*, satisfying certain axioms, one may form a universal *localization* by formally inverting all weak equivalences. After inverting weak homotopy equilvances in Top, the homotopy category is equivalent to the naive homotopy category of the full subcategory $\mathrm{Top}_{\mathrm{CW}}$.

The familiar constructions: homotopy groups, homology and cohomology, are all weak homotopy equivalence invariants. If is favorable that all that we concern behave well with respect to weak equivalences. Note that the homotopy category is not bi-complete.

Given a pair of maps $f : A \to X$ and $g : A \to Y$, the *homotopy pushout*, is the pushout of maps $A \coprod A \to A \times I$ and $A \coprod A \to X \coprod Y$. The based counterpart construction is analogous. Note that the homotopy cofiber of $f : X \to Y$ is the homotopy pushout of $* \leftarrow X \to Y$.

Given a seqeunce of maps of based spaces

$$X_0 \to X_1 \to X_2 \to ...,$$

the *mapping telescope* is the quotient space of the disjoint union of reduced cylinders:

$$\coprod_{n \in \mathbb{N}} (X_n \wedge [n, n+1]_+) / \sim$$

where the equivalent relation quotiented out is $(x_n, n) \sim (f(x_n), n+1)$.

The two examples above are special *homotopy colimits*. The basic idea is to replace gluing in colimit construction by gluing up to homotopy. We won't explain the general construction for all diagrams, but we record that homotopy colimit preserves weak equivalences, and satisfies certain universal property as derived functors.

3 Higher categorical point of view

In this section we introduce the idea of infinity categories.

Consider the category of all (small) categories, its morphisms are functors. For two categories $\mathcal{C}, \mathcal{D}$, the hom-set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$. It forms a functor category, whose morphisms are natural transformations. Thus natural transformations are morphisms between morphisms (functors). This sheds light on the idea of higher categories. Roughly speaking, they are categories equipped with higher k-morphisms for each $k = 1, 2, 3, ...$

Example. Let X be a space, the *fundamental groupoid* of X, is a groupoid whose objects are the points of X, and morphisms are homotopy classes of paths of X. Clearly all morphisms are invertible.

Now let $0 < n < \infty$, we define an n-category $\Pi_n X$. The objects are the points of X. If $x, y \in X$, the morphisms from x to y are paths from x to y; the 2-morphisms are homotopies of paths; the 3-morphisms are homotopies of homotopies, etc. Two n-morphisms are identified if they are homotopic. There should of course be requirements for coherence of higher morphisms, but we won't bother ourselves with them. For example, when $n = 1$, the definition reduces to the fundamental groupoids of spaces we've defined.

We can also define $\Pi_\infty X$. It has all higher morphisms. We call it an ∞-*groupoid*, as all morphisms are invertible.

We've seen an example of a 2-category. Its hom-sets are all 1-categories. It turns out that to define ∞-categories, most of the higher morphisms should be invertible. Hence we use the term (∞, n)-*category* to refer to ∞-categories whose all k-morphisms are invertible for $k > n$. Therefore for $(\infty, 1)$-categories as an example, each hom-set should be an $(\infty, 0)$-category, i.e., an ∞-groupoid.

There are many models to strictly define ∞-categories. Despite different methods, a key principle is the *homotopy hypothesis*. The examples $\Pi_\infty X$ are all ∞-groupoids. Conversely, homotopy hypothesis asserts that every ∞-groupoid has the form $\Pi_\infty X$ for some topological space X.

In detail, it proves an equivalence between the category of ∞-groupoids and the category of topological spaces. This leads to a naive definition of $(\infty, 1)$-categories: A *topological category* is a category enriched over Top. Explicitly speaking, a topological category $\mathcal{C}$ consists of objects together with a space $\mathcal{C}(X, Y)$ for any pair of objects. The composition law is given by continuous maps.

From now on by ∞-categories we mean $(\infty, 1)$-categories. One thing to keep in mind is that spaces replace sets in the new context. An alternative definition, using simplicial methods, will be covered in today's evening session.

Let $f : X \to Y$ be a morphism in a topological category $\mathcal{C}$, f is an *equivalence* if one of the following equivalent condition holds:

(a). The morphism f has a homotopy inverse.

(b). For every object $Z \in \mathcal{C}$, the induced map $\mathcal{C}(Z, X) \to \mathcal{C}(Z, Y)$ is a homotopy equivalence.

(c). For every object $Z \in \mathcal{C}$, the induced map $\mathcal{C}(Z, X) \to \mathcal{C}(Z, Y)$ is a weak homotopy equivalence.

We omit the definition of functors between ∞-categories. We present informally some results in ∞-categoy theory. We will use $\mathcal{S}$ to denote the ∞-category of CW complexes.

One central result in category theory is the Yoneda lemma. Let $\mathcal{C}$ be a category, the functor category $\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Set})$ is also called the (set-valued) *presheaf* category of $\mathcal{C}$, denoted as $\mathsf{PSh}(\mathcal{C})$. Any object $c \in \mathcal{C}$ determines a presheaf $\mathcal{C}(-, c) : c' \mapsto \mathcal{C}(c', c)$.

Proposition. The Yoneda embedding $j : \mathcal{C} \to \mathsf{PSh}(\mathcal{C})$ is fully faithful.

Lemma. If $F : \mathcal{C} \to \mathsf{Set}$ is a functor and $c \in \mathcal{C}$. The set of natural transformations from $\mathcal{C}(c, -)$ to F is isomorphic to Fc.

Let $\mathcal{C}$ be an ∞-category, an ∞-*presheaf* on $\mathcal{C}$ is an ∞-functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, $\mathcal{P}(\mathcal{C})$ will denote the ∞-category of ∞-presheaves. The following results hold:

Proposition. The ∞-category $\mathcal{P}(\mathcal{C})$ admits all small limits and colimits.

Proposition. The Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is fully faithful.

Let $\mathcal{D}$ be an ∞-category which admits small colimits. There is an equivalence of ∞-categories

$$\mathsf{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}^L(P(S), \mathcal{C})$ is the full subcategory of $\mathrm{Fun}(P(S), \mathcal{C})$ spanned by functors preserving small colimits.

Proposition. Let $\mathcal{C}$ be a small ∞-category, the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ generates $\mathcal{P}(\mathcal{C})$ under small colimits.

Thus the ∞-Yoneda embedding determines the ∞-presheaf category to a much greater extent. As an example, if $\mathcal{C}$ is a point, $\mathcal{P}(\mathcal{C})$ is equivalent to $\mathcal{S}$. We conclude that the category of spaces is generated by one point under colimits. And the universality reads as: a functor from a point to a category which has small colimits extends uniquely up to homotopy to a functor from the category of spaces.