MONOIDAL STRUCTURES

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Our goal in this talk is to build the stable homotopy category as a symmetric monoidal category with the smash product. We define monoidal structures in Section 1. We then talk about the symmetric monoidal structure on various point set models and their comparison in Section 2 and give an explicit formula for orthogonal spectra in Section 3. Finally we talk about how to build Sp as a symmetric monoidal ∞-category.

1. MONOIDAL CATEGORY AND MONOIDS

We start with the 1-category land.

Definition 1.1. A symmetric monoidal category is a category $(\mathcal{C}, \otimes, I)$, with bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$ that is called the unit, three natural isomorphisms $\alpha_a : a \otimes I \cong a$, $\beta_{a,b} : a \otimes b \cong b \otimes a$, $\gamma_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$ such that the following is true / diagrams commute:

(1) Commutativity: $\beta_{a,b} \circ \beta_{b,a} = \mathrm{id}$;
(2) Unital (also known as the triangle identity):

$$\begin{tikzcd} a \otimes (I \otimes c) \arrow[r, "\gamma"] \arrow[d, "1\otimes\beta"] & (a \otimes I) \otimes c \arrow[d, "\alpha\otimes 1"] \\ a \otimes (c \otimes I) \arrow[r, "1\otimes\alpha"] & a \otimes c \end{tikzcd};$$

(3) The associativity pentagon: the vertices are the five possible orders to multiply four elements in binary ways and the edges are one-step associativities;
(4) "C_3-equivariant" (also known as the hexagon identity):

$$\begin{tikzcd} a \otimes (b \otimes c) \arrow[r, "\gamma"] \arrow[d, "1\otimes\beta"] & (a \otimes b) \otimes c \arrow[r, "\beta"] & c \otimes (a \otimes b) \arrow[d, "\gamma"] \\ a \otimes (c \otimes b) \arrow[r, "\gamma"] & (a \otimes c) \otimes b \arrow[r, "\beta\otimes 1"] & (c \otimes a) \otimes b \end{tikzcd}.$$

Remark 1.2. Alternatively, we can define a symmetric monoidal category $\mathcal{C}$ to be an algebra over the commutative operad in the category of categories. This is to say, there is one way up to canonical isomrphisms to multiply k objects in $\mathcal{C}$ for each $k \geq 0$. The above diagrams simply the infinitly many canonical isomorphisms and diagrams to finitely many.

Example 1.3. If $\mathcal{C}$ admits finite products, then $(\mathcal{C}, \times)$ is a symmetric monoidal category.

Definition 1.4. Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. A monoid in $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with unit and multiplication maps

$$\eta : I \to M \text{ and } \mu : M \otimes M \to M,$$

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such that the following diagrams are commutative:

$$\begin{tikzcd} I \otimes M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\alpha"'] & M \otimes M \arrow[d, "\mu"] & M \otimes I \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\alpha"] \\ & M & \end{tikzcd},$$

$$\begin{tikzcd} (M \otimes M) \otimes M \arrow[rr, "\gamma"] \arrow[d, "\mu\otimes\mathrm{id}"'] & & M \otimes (M \otimes M) \arrow[d, "\mathrm{id}\otimes\mu"] \\ M \otimes M \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd}.$$

A commutative monoid in $\mathcal{C}$ is a monoid M satisfying the addition diagram:

$$\begin{tikzcd} M \otimes M \arrow[rr, "\beta"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

Example 1.5. A (commutative) monoid in the category of set is a (commutative) monoid. Here, the second "monoid" means a group without inverse.

Remark 1.6. Terminology: Sometimes people use the words "monoid" and "algebra" are used interchangeably to mean the same thing in a symmetric monoidal category. The reason is probably because a commutative monoid is the same thing as an algebra over the commutative opeard. They only slightly differ in that "algebra" is used for a general symmetric monoidal category while "monoid" is used for a Cartesian monoidal category.

However, we point out that an algebra in algebra has two binary operations while an algebra in topology has only one.

In topology it is too restrictive to work with commutative monoids. For example, ΩX is an H-space rather than a group, and if we pass to set by taking π_0, we get a group. To work on the space level, a relaxed version where the associativity and commutatively are only held up to all higher coherent homotopies is more useful. Such an object is called an E_∞-monoid (or E_∞-algebra) in $\mathcal{C}$. For the precise definition using operads, see [1, Chapter 1-3].

Example 1.7. The 0-th space of an Ω-spectra is an E_∞-space.

Historically, E_∞-spaces (the E_∞-monoids in $(Top_*, \wedge, S^0)$)) have been studied by May for the purpose of recognizing when a space can be delooped into a spectra, called infinite loop space machine.

Theorem 1.8. *(*[1, Thm 13.1]*) If X is a grouplike E_∞-space, then $X \simeq \Omega^\infty E$ for some connective spectra E.*

Alternatively, Segal has build another infinite loop space machine using what he call Γ-spaces and what we call Segal spaces. For a Segal space A, he constructed a classifying space $\mathbf{B}A$ that is again a Segal space. In this way he was able to deloop a Segal space to a spectra. We give the definition of Segal space, as it motivates the theory of a symmetric monoidal category in ∞-category.

Definition 1.9. Let Fin_* be the category of finite based sets and based maps. We write its objects as $\langle n\rangle = \{0, 1, \cdots, n\}$ where 0 is the base point. We say a morphism $f : \langle m\rangle \to \langle n\rangle$ is inert if for each $1 \leq i \leq n$, the inverse $f^{-1}(i)$ has exactly one element. In particular, we specify the following special inert morphisms $\rho^i : \langle n\rangle \to \langle 1\rangle$ that sends i to 1 and everything else to 0. We also specify an active morphism $\alpha \in \mathrm{Fin}_*(\langle 2\rangle, \langle 1\rangle)$ that sends both $1, 2$ to 1.

Notice that in Segal's paper, $\Gamma \cong \mathrm{Fin}_*^{op}$.

Definition 1.10. ([2, 1.2]) A Segal space is a functor $A : \mathrm{Fin}_* \to \mathcal{S}$, $\langle n\rangle \mapsto A_n$, such that

(1) A_0 is contractible;
(2) For $n \geq 1$, the inert morphism $\rho^i : \langle n\rangle \to \langle 1\rangle$ incudes homotopy equivalence

$$\prod_i \rho^i : A_n \simeq (A_1)^n.$$

A Segal space can be understood in analogy to a commutative monoid. The space A_1 is tempting to be a monoid, where $A_0 \to A_1$ gives the unit and $(A_1)^2 \simeq A_2 \overset{A(\alpha)}{\to} A_1$ gives the $\otimes$. However, the diagrams for a commutative monoid are only satisfied up to coherent homotopy. These homotopy data are packaged in the seemingly-redundant-but-not functor A.

Example 1.11. A commutative monoid M gives rise to a Segal space A by setting $A_n = M^n$. On morphisms, the functor A insert I in the appropriate coordinates for injections and multiply in the appropriate coordinates for surjections.

We now go to the ∞-category land. By design, all notions here are automatically homotopy notions. It is worth pointing out that the literature has chosen the same words for the notions, but they lie in different settings and it is non-trivial work to compare them.

Definition 1.12. [3, 2.0.0.7] A symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p : C^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ with the following property:

(*) For each $n \geq 0$, the innert morphisms $\{\rho^i : \langle n\rangle \to \langle 1\rangle\}_{1\leq i\leq n}$ induce functors $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle}$ which determine an equivalence $\mathcal{C}^{\otimes}_{\langle n\rangle} \to (\mathcal{C}^{\otimes}_{\langle 1\rangle})^n$.

Here, $\mathcal{C}^{\otimes}_{\langle n\rangle}$ is the fiber of $\mathcal{C}^{\otimes}$ over $\langle n\rangle$.

We explain how to understand the definition. A coCartesian fibration over $\mathrm{N}(\mathrm{Fin}_*)$ is a functor $\mathrm{N}(\mathrm{Fin}_*) \to \mathbf{Cat}_\infty$ by the Gronthdieck construction (Ang Li will explain this in the evening). This functor sends the object $\langle n\rangle$ to $\mathcal{C}^{\otimes}_{\langle n\rangle}$. The word "coCartesian fibration" is to ensure a homotopically well defined $f_! : \mathcal{C}^{\otimes}_{\langle m\rangle} \to \mathcal{C}^{\otimes}_{\langle n\rangle}$ for each $f : \langle m\rangle \to \langle n\rangle$. The condition (*) exibits this funcor as in analogy to the condition 2 of a Segal space, thus also called the Segal condition.

The underlying ∞-category of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is $\mathcal{C}^{\otimes}_{\langle 1\rangle}$.

Definition 1.13. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and denote by $\mathcal{C}$ its underlying ∞-category $\mathcal{C}^{\otimes}_{\langle 1\rangle}$. Then we define the functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

by the composite:

$$(\mathcal{C}^{\otimes}_{\langle 1\rangle})^2 \simeq \mathcal{C}^{\otimes}_{\langle 2\rangle} \overset{\alpha_!}{\to} \mathcal{C}^{\otimes}_{\langle 1\rangle}.$$

Here, the first map is an inverse to the canonical equivalence and the second map is induced by $\alpha \in \mathrm{Fin}_*(\langle 2\rangle, \langle 1\rangle)$.

Example 1.14. ([3, 2.1.1.7]) Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. Define a category $\mathcal{C}^{\otimes}$ as follows:

(1) The objects of $\mathcal{C}^{\otimes}$ are finite sequences of objects $X_1, \cdots, X_n \in \mathcal{C}$;
(2) A morphism from $\{X_i\}_{1 \le i \le m}$ to $\{Y_j\}_{1 \le j \le n}$ is a $(n+1)$-tuple $(\alpha, \phi_1, \cdots, \phi_n)$ where

$$\alpha \in \mathrm{Hom}_{\mathrm{Fin}_*}(\langle m \rangle, \langle n \rangle), \;\; \phi_j \in \mathrm{Hom}_{\mathcal{C}}(\otimes_{i \in \alpha^{-1}(j)} X_i, Y_j) \text{ for } 1 \le j \le n.$$

(3) Composition of morphisms in $\mathcal{C}^{\otimes}$ is by the composition laws on Fin_* and in $\mathcal{C}$.

Notice that the category $\mathcal{C}^{\otimes}$ comes naturally with a map $\pi : \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$. Taking the nerves will give us a symmetric monoidal ∞-category $\mathrm{N}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$. Its underlying ∞-category is $\mathrm{N}(\mathcal{C}^{\otimes})_{\langle 1 \rangle} \cong \mathrm{N}(\mathcal{C})$.

Example 1.15. In Example 1.14, we may take $(\mathcal{C}, \otimes)$ to be $(*, \times)$, the category with one object. In this case $\pi : \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ is an isomorphism. This establishes $\mathrm{N}(\mathrm{Fin}_*)$ as a symmetric monoidal ∞-category with underlying ∞-category $*$.

To define the commutative monoids in a symmetric monoidal ∞-category, we would need the notion of an ∞-operad ([3, 2.1.1.10]). It is an ∞-categorical version of a multicategory, which generalizes both a symmetric monoidal category and an opeard. For the sake of time, we will skip the definition here. It is useful to bear in mind that a symmetric monoidal ∞-category is an example of an ∞-operad.

Remark 1.16. Example 1.15 gives the symmetric monoidal ∞-category $\mathrm{N}(\mathrm{Fin}_*)$, thus an ∞-operad, is called the commutative ∞-operad. As in the 1-category case, it is the terminal object among operads.

Definition 1.17. ([3, 2.1.2.7]) Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be ∞-operads. An ∞-operad map from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ is a map of simplicial sets $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ such that

(1) The map f commutes with projection to $\mathrm{N}(\mathrm{Fin}_*)$.
(2) The functor f carries inert morphisms in $\mathcal{O}^{\otimes}$ to inert morphisms in $\mathcal{O}'^{\otimes}$. (An inert morphism in $\mathcal{O}^{\otimes}$ is a coCartesian morphism that covers an inert morphism in $\mathrm{N}(\mathrm{Fin}_*)$.)

We denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ the full ∞-subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by the maps of ∞-operads. It is refered to as the ∞-category of $\mathcal{O}$-algebras in $\mathcal{O}'$.

Note that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ is an abuse of notation. It actually depends on $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$.

The conditions are to impose that after Gronthendieck construction we have a natural transformation of funcotrs, i.e. a morphism of Segal objects.

Remark 1.18. In Definition 1.17, if we take the special ase of $\mathcal{O}^{\otimes} = \mathrm{N}(\mathrm{Fin}_*)$, then we denote the category of $\mathcal{O}$-algebras in $\mathcal{C}^{\otimes}$ by $\mathrm{CAlg}(\mathcal{C})$. We refer to it as the ∞-category of commutative algebra objects of $\mathcal{C}$.

Example 1.19. ([3, 2.1.3.3]) Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. From Example 1.14, we can regard $\mathrm{N}(\mathcal{C}^{\otimes})$ as a symmetric monoidal ∞-category. Then $\mathrm{CAlg}(\mathrm{N}(\mathcal{C}^{\otimes}))$ can be identified with the nerve of the category of commutative algebra objects in $\mathcal{C}$.

Let $\mathbf{Cat}_\infty$ be the ∞-category of (small) ∞-categories. It has a Cartesian monoidal structure.

Proposition 1.20. *The ∞-category of symmetric monoidal ∞-categories can be identified with* $\mathrm{CAlg}(\mathbf{Cat}_\infty)$.

The identification here are in two steps.

Step 1: Define the commutative monoid object in $\mathbf{Cat}_\infty$ to be a functor M : $\mathrm{N}(\mathrm{Fin}_*) \to \mathbf{Cat}_\infty$ satisfying some properties. It is also know as a Segal object in $\mathbf{Cat}_\infty$.

Any functor $M : \mathrm{N}(\mathrm{Fin}_*) \to \mathbf{Cat}_\infty$ is classified by a coCartesian fibration $\mathcal{C}^\otimes \to \mathrm{N}(\mathrm{Fin}_*)$. The functor M is a commutative monoid if and only if $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category [3, 2.4.2.4].

Step 2: For the Cartesian monoidal structure $\mathbf{Cat}_\infty^\times$ on $\mathbf{Cat}_\infty$, $\mathrm{CAlg}(\mathbf{Cat}_\infty)$ is equivalent to the ∞-category of commutative monoids in $\mathbf{Cat}_\infty$ [3, 2.4.2.5].

2. MONOIDAL STRUCTURE ON Sp, INTRO

In the last talk we introduced the Adams' category of spectra and the stable ∞-category of spectra. There have been other point set models of spectra developed. It is shown that as closed symmetric monoidal model categories they are Quillen equivalent ([4, 5]), so that they have isomorphic homotopy categories. Figure 1 is a diagram summarizing the comparision, due to May and texed by Weinan Lin. Here, $\mathscr{S}$ is the Lewis-May spectra, $\mathscr{P}$ is the prespectra, $\Sigma\mathscr{S}$ is the symmetric spectra (Hovey-Shipley-Smith), $\mathscr{I}\mathscr{S}$ is the orthogonal spectra (Mandell-May), $F\mathscr{T}$ is the Segal space, $W\mathscr{T}$ is the W-spaces (Anderson), M_S is the EKMM S-modules (Elmendorf-Kriz-Mandell-May). The interested may refer to [4] for the left lower patagon and for [5] for the rest.

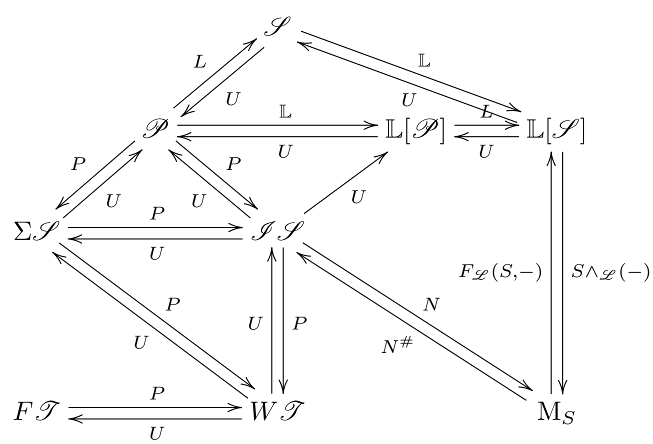

FIGURE 1. comparision of models

However, none of them has all the good properties we want, as predicted by the following theorem of Lewis:

Theorem 2.1. ([6, Thm 11.1]) *There is no category $\mathscr{S}$ that is enriched in based topological spaces $\mathscr{T}$ and sateisfies the following three properties:*

(1) *$\mathscr{S}$ is closed symmetric monoidal under continuous smash product and function spectrum functors $\wedge$ and F that satisfy the topological adjuntion*

$$\mathscr{S}(E \wedge E', E'') \cong \mathscr{S}(E, F(E', E'')).$$

(2) *There are continuous adjunctions*

$$\Sigma^\infty : \mathscr{T} \rightleftharpoons \mathscr{S} : \Omega^\infty.$$

(3) *The unit for the smash product in $\mathscr{S}$ is $\mathsf{S} = \Sigma^\infty S^0$.*

3. MONOIDAL STRUCTURES ON PRESHEAVES, ORTHOGONAL SPECTRA

We define a topological category $\mathbf{O}$([7, 3.1.1]). The objects are finite dimensional inner product spaces. For two objects V and W, the morphism is the Thom space of a vector bundle: $\mathbf{O}(V, W) = \mathrm{Th}(\xi(V, W))$. Here, $\xi(V, W)$ is a vector bundle over $\mathbf{L}(V, W)$, the space of linear isometric embeddings $V \to W$, and it is defined by:

$$\xi(V, W) = \{(w, \varphi) \in W \times \mathbf{L}(V, W) | w \perp \varphi(V)\}.$$

Definition 3.1. ([7, 3.1.3]) An orthogonal spectrum is a continuous functor $\mathbf{O}$ to $\mathscr{T}$. A morphism of orthogonal spectra is a natural transformation of functors.

In other words, the orthogonal spectra is the category of covariant presheaves on $\mathbf{O}$. This perspective offers to build the smash product formally in category theory.

Definition 3.2. ([4, 21.1]) For $X, Y : \mathcal{D} \to \mathscr{T}$, define the external smash product $X \bar{\wedge} Y$ by

$$X \bar{\wedge} Y = \wedge \circ (X \times Y) : \mathcal{D} \times \mathcal{D} \to \mathscr{T}.$$

If $(\mathcal{D}, \otimes)$ is a symmetric monoidal topological category with small skeleton, then it is formal that presheaves on $\mathcal{D}$ has a closed symmetric monoidal structure ([8]). This symmetric monoidal strcture, written as $\wedge$ in our case, is called Day convolution and given as follows:

Definition 3.3. ([4, 21.4]) For $X, Y : \mathcal{D} \to \mathscr{T}$, define the internal smash product $X \wedge Y$ to be the topological left Kan extension of $X \bar{\wedge} Y$ along $\otimes$. It is characterized by the universal property that for any $X : \mathcal{D} \to \mathscr{T}$, there is

$$\mathrm{Nat}_{\mathrm{Fun}(\mathcal{D}, \mathscr{T})}(X \wedge Y, Z) \cong \mathrm{Nat}_{\mathrm{Fun}(\mathcal{D} \times \mathcal{D}, \mathscr{T})}(X \bar{\wedge} Y, Z \circ \otimes).$$

Taking $\mathcal{D} = \mathbf{O}$, we get a symmetric monoidal smash product on the category of orthogal spectra.

4. MONOIDAL STRUCTURE ON THE ∞-CATEGORY Sp

In Secion 2 we have seen different point set models with symmetric monoidal smash product, whose homotopy categories all give the stable homotopy category. Moreover, one can endow the ∞-category Sp with the structure of a symmetric monoidal ∞-category in the sense of Section 1, such that ho(Sp) also gives the stable category with the smash product. We describe the ideas following [3, 4.8.2]. There are three ways to understand/build this smash product.

The first way is to start with a simplicial model category with a compatible symmetric monoidal structure, whose underlying ∞-category is equivalent to Sp. Then the symmetric monoidal structure will be inherited to Sp via the operadic coherent nerve.

The second way is to utilize universal property in the last talk: $\mathrm{ev}_{\mathsf{S}} : \mathrm{LFun}(\mathrm{Sp}, \mathrm{Sp}) \to \mathrm{Sp}$ is an equivalence. Recall that LFun is spanned functors preserving small colimits. There is the composition monoidal structure on $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$, which restricts to the monoidal structure on $\mathrm{LFun}(\mathrm{Sp}, \mathrm{Sp})$. Intuitively, the inverse to ev_{S} can be thought of as smashing with $E \in \mathrm{Sp}$. Via this perspective, the algebra objects of $\mathrm{LFun}(\mathrm{Sp}, \mathrm{Sp})$ are precisely the colimit preserving monads on the ∞-category Sp.

The third way is to first build a symmetric monoidal structure on Pr^{L}, the ∞-category of presentable ∞-categories and colimit preserving functors. The tensor product on Pr^{L} has the property to preserve small colimits in each variable. (Colimits in Pr^{L} are computed as limits in Pr^{R}, which are limits in $\mathbf{Cat}_{\infty}$.) Let $\mathrm{Pr}^{\mathrm{St}}$ be the full subcategory spanned by the objects that are stable. Then it inherits a symmetric monoidal structure, with Sp being the unit object. By [3, 3.2.1.9], Sp is an initial object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{St}})$. Similar to Proposition 1.20, the commutative algebra objects in $\mathrm{Pr}^{\mathrm{St}}$ can be identified with those symmetric monoidal ∞-categories $\mathcal{C}$ which are presentable, stable and such that $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits in each variable. The smash product $\otimes$ on Sp is characterized, up to contractible choice, by the following universal property (for a concrete description, see [3, 4.8.2.19]):

(1) The bifunctor $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ preserves small colimits in each variable;
(2) The unit object of Sp is the sphere spectra $\mathbb{S}$.

So the symmetric monoidal ∞-category of spectra is unique, up to contractible choice, with these properties.

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