Equivariant Spaces

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1 G-spaces and G-CW complexes

The main objects in equivariant homology theory and homotopy theory are G-spaces which are spaces equipped with an action by a topological group G. In details, here is the definition.

Definition 1.1. A G-space is a topological space X with continuous actions $G \times X \to X$ such that $ex = x$ and $g(g'x) = (gg')x$.

Definition 1.2. A G-map $f : X \to Y$ is an continuous map f such that $f(gx) = gf(x)$ (We call maps satisfying this by equivariant maps).

They togeother form the category of G-spaces, Top_G. If we take all spaces to be conmpactly generated and weak Hausdorff, we have a G-homeomorphism

$$\mathrm{Map}(X \times Y, Z) \equiv \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

Definition 1.3. A G-CW complex X is the union of sub G-spaces X^n such that X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching G-cells $G/H \times D^{n+1}$ along the attaching G-maps $G/H \times S^n \to X^n$.

Remark: Compare the pushout diagrams of non-equivariant CW complex and of equivariant CW complex. To give a sense that G/H plays the role of points in nonequivariant case.

$$\begin{tikzcd}
\coprod_H G/H \times S^n \arrow[r] \arrow[d, hook] & X^n \arrow[d] \\
\coprod_H G/H \times D^{n+1} \arrow[r] & X^{n+1}
\end{tikzcd}$$

The attaching map $G/h \times S^n \to X^n$ is determined by its restriction $S^n \to (X^n)^H$.

In equivariant theory orbits, G/H play the role of points.

Example 1.4. Use the examples of S^1 with reflection action of $\mathbb{Z}/2$ and S^1 with antipodal action of $\mathcal{Z}/2$. Maybe $\mathbb{C}P^n$ by conjugation. Insert pictures.

2 Geometric fixed points and orbits

For $H \subset G$, we write the set of fixed points of H by $X^H = \{x|hx = x \text{ for } h \in H\} = \mathrm{Hom}_G(G/H, X)$ and Weyl group $W_G H = N_G H/H = \mathsf{Hom}_G(G/H, G/H)$. We denote the orbit space by $X_G = X/G$.

Consider the functor $F : \mathrm{Top} \to \mathrm{Top}_G$, which sends a space Y to its underlying space with trivial G-action. F has both left and right adjoints, which are geometric fixed points and orbits. In particular,

$$\mathsf{Hom}_G(Y, X) \cong Hom(Y, X^G)$$

$$\mathsf{Hom}_G(X, Y) \cong Hom(X_G, Y)$$

In general, if we have a map of groups $G \to K$, it induces a functor $f^* : \mathrm{Top}_K \to \mathrm{Top}_G$. f^* has both right adjoint $f_!(X) = K \times_G X$ and left adjoint $f_*(X) = \mathrm{Hom}_G(K, X)$. If we take $K = *$, it gives the above adjuntions.

Geometric fixed points and orbits are also limits and colimits. Let BG be the category with one object and G for morphisms. We can naively regard Top_G as the functor category $F(BG, \mathrm{Top})$. A map $f : G \to K$ induces a functor $F : BG \to BK$ and a functor $f^* : \mathrm{Top}_K \to \mathrm{Top}_G$, $f_!$ is the left Kan extension along F and f_* is the right Kan extension along F.

$$\begin{tikzcd} BG \arrow[r] \arrow[d, "F"'] & \mathrm{Top} \\ BK \arrow[ur, dotted, "f_!"'] \arrow[ur, phantom, bend left, "\Downarrow"] & \end{tikzcd} \qquad \begin{tikzcd} BG \arrow[r] \arrow[d, "F"'] & \mathrm{Top} \\ BK \arrow[ur, dotted, "f_*"'] \arrow[ur, phantom, bend left, "\Uparrow"] & \end{tikzcd}$$

The diagrams do not commute. There are natural transformation $\mathrm{id} \Rightarrow f^* f_!$ and $f^* f_* \Rightarrow \mathrm{id}$. In particular, if we take $K = *$, left and right Kan extensions of a functor X along F to the trivial category give the colimits and limits of X, we have

$$X^G = \lim_{BG} X.$$

$$X_G = \mathrm{colim}_{BG} X.$$

3 Homotopy homotopy fixed points and orbits

Now we define homotopy fixed points to be $\mathsf{Map}_G(EG, X) := X^{hG}$ and homotopy orbits to be $EG \times_G X := X_{hG}$.

Recall from last section, a map $f : G \to K$ induces a map $f^* : \mathrm{Top}_K \to \mathrm{Top}_G$. Qe consider homootpy right Kan extension functor, the right derived functor Rf_* of the right Kan extension f_*. The idea of derived functor here is choosing a fibrant replacement functor $Q : \mathrm{Top}_G = \mathrm{Fun}(BG, \mathrm{Top}) \to \mathrm{Top}_G$ and setting $Rf_* = f_* \circ Q$. Taking $K = *$, it defines the homotopy limits:

$$\mathsf{holim}_{BG} X = X^{hG}.$$

By the properties of right derived functors, there is a natural map from a functor to its right derived functor. We get a natural map from limits to homotopy limits and in particular,

$$X^G \to X^{hG}.$$

If we do the entire process for left homotopy derived functors $Lf_!$, the left derived functor of left Kan extension $f_!$, we have

$$\mathrm{hocolim} X = X_{hG}.$$

and we have a map

$$X_{hG} \to X/G.$$

As one expects,

$$\pi_0 \mathrm{Hom}_G(Y, X) \cong \pi_0 \mathrm{Hom}(Y, X^{hG})$$

$$\pi_0 \mathrm{Hom}_G(X, Y) \cong \pi_0 \mathrm{Hom}(X_{hG}, Y)$$

4 Homotopies

Definition 4.1. For a topological G-space X, $H \subset G$ a closed subgroup of G, its nth H-equivariant homotopy groups are

$$\pi_n^H(X) = \mathrm{Hom}_G(G/H_+ \wedge S^n, X) = \pi_n(X^H).$$

Definition 4.2. A G-map $f : X \to Y$ is weak homotopy equivalence if $f^H : X^h \to Y^H$ is a weak equivalence for all $H \subset G$.

Recall that in non-equivariant case, a map $f : Y \to Z$ is an n-equivalence if $\pi_p(f)$ is a bijection for $q < n$ and a surjection for $q = n$ (for any choice of basepoint). Now we give a analogous definition for equivariant case.

Definition 4.3. Let ν be a function from conjugacy classes of subgroups of G to the integers$\geqslant 1$. We say that a map $e : Y \to Z$ is a ν-equivalence if $e^H : Y^H \to Z^H$ is a $\nu(H)$-equivalence for all H.

Theorem 4.4 (Homotopy extension and lifting property)**.** *Let A be a subcomplex of a G-CW complex X of dimension ν and let $e; Y \to Z$ be a ν-equivalence. Suppose given maps $g : A \to Y$, $h : A \times I \to Z$, and $f : X \to Z$ such that $eg = hi_i$ and $fi = hi_0$ in the following diagram: then there exists maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commutes.*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll] & \\
X \arrow[rr, "i_0"] \arrow[ur, "f"] & & X \times I \arrow[ul, dotted, "\tilde{h}"] & & X \arrow[ll, "i_1"'] \arrow[ul, dotted, "\tilde{g}"']
\end{tikzcd}$$

Proof. We construct $\tilde{g}$ and $\tilde{h}$ on $A \cup X^n$ by induction on n. Passing from the n-skeleton to the $(n+1)$-skeleton, we may work one cell of X not in A at a time. By considering attaching mas, we quickly reduce the proof to the case when $(X, A) = (G/H \times D^{n+1}, G/H \times S^n)$ and this reduces to the nonequivariant case of (D^{n+1}, S^n) □

Theorem 4.5 (Whitehead Theorem)**.** *Let $e : Y \to Z$ be a ν-equivalence and X be a G-CW complex. Then $e_* : \mathrm{Hom}_G(X, Y) \to \mathrm{Hom}_G(X, Z)$ is a bijection if X has dimension less than ν and a surjection if X has dimension ν.*

Proof. Apply Theorem 4.4 to $(X, \varnothing)$ for the surjectivity and to $(Z \times I, X \times \partial I)$ for the injectivity. □

Corollary 4.6. *If $e : Y \to Z$ is a ν-equivalence between G-CW complexes of dimension less than ν, then e is a G-homotopy equivalence.*

References

[EHCT] J. P. May. Equivariant Homotopy and Cohomology Theory.