Cohomology Theories and Naive Spectra

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1 Introduction

We will review Eilenberg-Steenrod axioms for cohomology. We construct Eilenberg-MacLanes spaces, which represent cohomology theories. This leads to naive Ω-spectra and Brown representability.

2 Axiomatic cohomology

In this section we review Eilenberg-Steenrod axioms for cohomology theories. Note that CW complexes are precisely constructed via colimits starting from a point, by universality, the dimension axiom determines the cohomology theory.

A *cohomology theory* E consists of $\mathbb{Z}$-graded contravariant functors E^n, from the category of pairs of CW complexes to the category of abelian groups, and natural transformations $\delta : E^n(A) := E^n(A, \varnothing) \Rightarrow E^{n+1}(X, A)$, such that:

- **Exactness.** The following sequence is exact:

$$... \to E^n(X, A) \to E^n(X) \to E^n(A) \to E^{n+1}(X, A) \to ...$$

- **Homotopy.** If $f : (X, A) \to (Y, B)$ is a homotopy equivalence, then

$$f^* : E^n(Y, B) \xrightarrow{\cong} E^n(X, A).$$

- **Excision.** If X is the union of subcomplexes A and B, then the inclusion $(A, A \cap B) \to (X, B)$ induces an isomorphism

$$E^n(X, B) \xrightarrow{\cong} E^n(A, A \cap B).$$

- **Additivity.** If $(X, A) = \coprod_i (X_i, A_i)$, then

$$E^n(X, A) \xrightarrow{\cong} \prod_i E^n(X_i, A_i).$$

Conventionally, if $E^n(*) = 0$ for $n \neq 0$, we call E an *ordinary* cohomology theory.

There is a based variant: a *reduced cohomology theory* consists of $\mathbb{Z}$-graded functors $\tilde{E}^n$, from the category of based CW complexes to the category of abelian groups, and natural isomorphisms

$$\delta : \tilde{E}^n(X) \xrightarrow{\cong} \tilde{E}^{n+1}(\Sigma X),$$

such that:

- **Exactness.** If A is a subcomplex of X then the following sequence is exact:

$$\tilde{E}^n(X/A) \to \tilde{E}^n(X) \to \tilde{E}^n(A).$$

- **Homotopy.** If $f \simeq g : X \to Y$ are based homotopic, then $f^* = g^*$.
- **Wedge.** If $X = \bigvee_i X_i$, then

$$\tilde{E}^n(X) \xrightarrow{\cong} \prod_i \tilde{E}^n(X_i).$$

In this context the dimension axiom reads as $\tilde{E}^n(S^0) = 0$ for $n \neq 0$.

The relation between reduced and unreduced cohomology is the following:

$$\tilde{E}^*(X) = E^*(X, *), \quad E^*(X) = E^*(X_+), \quad E^*(X, A) = \tilde{E}^*(X/A).$$

Example. Cellular/singular cohomology theory HG.
Example. Ordinary cohomology of S^n.

Cup product and homology.

3 Eilenberg-MacLane spaces

Given $n > 0$ and a discrete group G, the *Eilenberg-MacLane space* $K(G, n)$, is characterized by the following property: $\pi_n K(G, n) = G$, while $\pi_k K(G, n) = 0$ for $k \neq n$. Of course if $n > 1$ we require that G is abelian.

One way to construct Eilenberg-MacLanes spaces is by attaching cells. Say $n \geq 1$. Present G with generators and relations:

$$G = \langle g_1, ..., g_\alpha / r_1, ..., r_\beta \rangle.$$

The homotopy group $\pi_n(\vee_i S_i^n)$ is free abelian with α generators. Each relation r_i is represented by a based map $S^n \to \vee_i S_i^n$. One could attach a $(n+1)$-cell via this attaching map to realize the relation r_i. The result is a space X with trivial homotopy groups $\pi_i(X)$ for $i < n$ and $\pi_n(X) = G$.

The same method could be used to kill all higher homotopy groups. Starting with π_{n+1}, we attach $(n+2)$-cells via attaching maps $S^{n+1} \to X$ that generate π_{n+1}, and this won't affect lower homotopy groups. This finishes the construction.

Eilenberg-MacLane spaces are unique up to weak homotopy equivalence, some examples are

$$K(\mathbb{Z}, 1) \simeq S^1, \quad K(\mathbb{Z}/2, 1) \simeq \mathbb{R}P^\infty, \quad K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty.$$

Eilenberg-MacLane spaces represent cohomology theories. Recall that $[X, Y]$ denotes the set of based homotopy classes of maps between X and Y, and $\pi_0 F(X, Y) = [X, Y]$. The construction above reveals that Eilenberg-MacLane spaces are naturally based.

Theorem. For CW complexes X, abelian groups G, and integers $n \geq 0$, there are natural isomorphisms

$$\widetilde{H}^n(X; G) \cong [X, K(G, n)].$$

It is not hard to prove that for any based space Z, the functor $[-, Z]$ from based CW complexes to pointed sets satisfies **Homotopy**, **Exactness** and **Wedge** conditions given in the Eilenberg-Steenrod axioms for reduced cohomology theory. For the functor to take value in Abelian groups, we have to impose more structures on Z, for example if Z is a double loop space. Milnor proved that the loop space of a CW complex has the homotopy type of a CW complex. Hence we have a homotopy equivalence

$$\widetilde{\sigma}_n : K(G,n) \to \Omega K(G,n+1).$$

By iterating, Eilenberg-MacLane spaces are infinity loop spaces.

An Ω-*spectrum* is a sequence of based spaces E_n, $n \geq 0$, and based weak homotopy equivalences $\widetilde{\sigma} : E_n \to \Omega E_{n+1}$. For an abelian group G, the Eilenberg-MacLane spectrum is $\{K(G,n), \widetilde{\sigma_n}\}$.

Proposition. Let $E = \{E_n\}$ be an Ω-spectrum. Define

$$\widetilde{E}^n(X) = \begin{cases} [X, E_n] & \text{if } n \geq 0 \\ [X, \Omega^{-n} E_0] & \text{if } n < 0. \end{cases} \tag{3.1}$$

Then the functors $\widetilde{E}^n$ define a reduced cohomology theory on based CW complexes. We only need to verify the suspension isomorphism, which is induced my $\tilde{\sigma}$:

$$\widetilde{E}^n(X) = [X, E_n] \to [X, \Omega E_{n+1}] \cong [\Sigma X, E_{n+1}] = \widetilde{E}^{n+1}(\Sigma X).$$

Now we have proved the theorem.

Cohomology could as well be generated to the ∞-categorical setting. The idea is that: given an ∞-category $\mathcal{C}$. For two objects X, A of $\mathcal{C}$, the *degree 0 cohomology of X with coefficients in A*, is the set of connected components of the hom space $\mathcal{C}(X, A)$.

We shall see Eilenberg-Maclane spaces also produce ordinary homology theories. By adjunction $[\Sigma X, Y] \cong [X, \Omega Y]$, $\tilde{\sigma}_n : K(G,n) \to \Omega K(G,n+1)$ corresponds to map

$$\sigma_n : \Sigma K(G,n) \to K(G,n+1).$$

We may smash with a based CW complex X to obtain

$$\pi_{n+k}(X \wedge K(G,n))) \xrightarrow{\Sigma} \pi_{n+k+1}(X \wedge \Sigma K(G,n)) \xrightarrow{(\mathrm{Id} \wedge \sigma_n)_*} \pi_{n+k+1}(X \wedge K(G,n+1)).$$

Theorem. For based CW complexex X, abelian groups G, and integers $n \geq 0$, there are natural isomorphisms

$$\tilde{H}_k(X,G) \cong \mathrm{colim}_n \pi_{n+k}(X \wedge K(G,n)).$$

A *spectrum* is a sequence of based spaces E_n, $n \geq 0$, and based maps $\sigma_n : \Sigma E_n \to E_{n+1}$. Given nice conditions, one expect similar results. But we won't go into details here. You will see an example at the beginning of next talk.

Now we build the Eilenberg-MacLane spaces into the construction of Postnikov towers which can be expressed as tower of fibrations with Eilenberg-MacLane spaces as fibers. We say a topological space is *n-truncated* if the homotopy groups of X vanish in dimensions larger than n. Recall that the Postnikov tower of path-connected X, is a sequence of spaces

$$X \to \ldots \to X_n \xrightarrow{p_n} X_{n-1} \ldots \to X_1 \xrightarrow{p_1} X_0$$

such that

(1). $\pi_i(X_n) \cong \pi_i(X)$ for $i \leq n$,
(2). X_n is n-truncated, i.e., $\pi_i(X_n) = 0$ for $i > n$.

We could construct a Postnikov tower by attaching cells when X is a CW complex. The Postnikov tower, if it exists, is unique up to homotopy.

Furthermore, one could successively replace each map p_n by a fibration: given a map $f : X \to Y$, define the *path space* $Nf = X \times_f Y^I$. Nf consists of pairs (x, γ) such that $f(x) = \gamma(0)$. Now f could be decomposed as

$$X \xrightarrow{\nu} Nf \xrightarrow{p} Y,$$

where $\nu(x) = (x, \gamma_{f(x)})$ and $\rho(x, \gamma) = \gamma(1)$. It is not hard to check that Nf deformation retracts to X and p is a fibration.

By examining the homotopy long exact sequence, the new map p'_n is a fibration with fiber $K(\pi_n(X), n)$. One recovers the space X by taking the homotopy limit of the tower. This kind of tower resolution construction is both theoretically and computationally important.

4 Brown Representability

On the other hand, the representability of ordinary cohomology is a consequence of a general result called the Brown representability theorem.

Recall that if $\mathcal{C}$ is a category and $F : C^{\mathrm{op}} \to \mathrm{Set}$ is said to be *representable* if there exists $X \in \mathcal{C}$ and an isomorphism $F \to \mathrm{Hom}_{\mathcal{C}}(-, X)$.

There is a notion of *presentable* categories, as well as a notion of *presentable* ∞*-categories*.

Proposition. Let $\mathcal{C}$ be a presentable category, and $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. F is representable if and only if F preserves limits.

Proposition. Let $\mathcal{C}$ be a presentable ∞-category, and $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be a functor. F is representable if and only if F preserves small limits.

There are also nice criteria (Adjoint Functor Theorem (**elaborate?**)) to determine whether a functor between presentable (∞-)categories has left/right adjoints.

A contravariant functor from the homotopy category of based connected CW complexes to the category of pointed sets is called a *Brown functor* if it satisfies the following conditions:
(1). it takes coproducts to products,
(2). it takes weak pushouts to weak pullbacks.

Theorem. (Brown representability) Brown functors are representable. Every reduced cohomology theory on the category of based CW complexes is represented by an Ω-spectrum.