LOCALIZATION WITH RESPECT TO CERTAIN PERIODIC HOMOLOGY THEORIES

By Douglas C. Ravenel*

This paper represents an attempt, only partially successful, to get at what appear to be some deep and hitherto unexamined properties of the stable homotopy category. This work was motivated by the discovery of the pervasive manifestation of various types of periodicity in the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres. In section 3 of [34] and section 8 of [41], we introduced the chromatic spectral sequence, which converges to the above E_2-term. Unlike most spectral sequences, its input is in some sense more interesting than its output, as the former displays many appealing patterns which are somewhat hidden in the latter (see section 8 of [41] for a more detailed discussion). It is not so much a computational aid (although it has been used [34] for computing the Novikov 2-line) as a conceptual tool for understanding certain qualitative aspects of the Novikov E_2-term.

Since the Novikov E_2-term is a reasonably good approximation to stable homotopy itself, one is led to hope that the periodicity in it displayed by the chromatic spectral sequence is more than just an artifice of the algebra of complex cobordism. Hopefully, there is some sort of geometric periodicity behind the algebraic periodicity of the chromatic E_1-term. More specifically, we conjecture (5.8) that certain short exact sequences of BP_*BP-comodules (5.6) used to construct the chromatic spectral sequence can be realized by cofibrations (5.7) and that the spectra involved enjoy a similar sort of periodicity (5.9).

In attempting to prove this conjecture, we soon became aware of Bousfield's work on localization with respect to generalized homology, the relevant portions of which are described in section 1. For each generalized homology theory E_*, Bousfield [10] defines an idempotent functor L_E on the stable homotopy category whose image is equivalent to the category of fractions defined by Adams in section III 14 of [4]. If both X and E (the

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spectrum representing the homology theory E_*) are connective spectra, then $L_E X$ is simply the appropriate arithmetic localization or completion of X (1.12), but if either X or E fail to be connective, then $L_E X$ is much harder to predict. For example if $E = K$, the spectrum representing complex K-theory, then $L_K S^0$, which is described in section 8, is not connective and $\pi_{-2} L_K S^0 \simeq Q/Z$.

In [11] Bousfield defines an equivalence relation on spectra by $E \sim F$ if $L_E = L_F$ (1.19). We call the resulting equivalence class $\langle E \rangle$ the Bousfield class of E. The set **A** of all such classes is partially ordered (1.20) and has wedge and smash product operations. It has a subset **BA** which is a Boolean algebra (1.21), whose structure we conjecture in 10.8.

In section 2 we study the Bousfield classes of various spectra associated with BP including $K(n)$, the spectra for the nth Morava K-theory, and certain spectra $E(n)$ with $\langle E(n) \rangle = \langle v_n^{-1} BP \rangle$. In particular we show $\langle E(n) \rangle = \vee_{i=0}^{n} \langle K(i) \rangle$. Localization with respect to $E(n)$ is a natural tool for getting at the periodicity referred to above. $E(0)$ is the rational Eilenberg-MacLane spectrum and $E(1)$ is one of $p - 1$ isomorphic summands of complex K-theory localized at the prime p. For $n \geq 2$, the spectra $E(n)$ represent periodic homology theories which at present have no known geometric interpretation comparable to the description of K-theory in terms of vector bundles.

In section 3 we construct some Thom spectra X_n for $n \geq 0$ with $X_0 = S^0_{(p)}$ and $\langle X_n \rangle > \langle X_{n+1} \rangle$ and $\langle X_n \rangle > \langle BP \rangle$ for all n.

In section 4 we define a spectrum to be harmonic if it is local with respect to $E = \vee_{n \geq 0} K(n)$. Harmonic spectra include all finite spectra (4.5) and all connective spectra with torsion free homology (4.6). If, on the other hand, X is E_*-acyclic we say that X is dissonant. An example of such a spectrum is $H/(p)$, the mod (p) Eilenberg-MacLane spectrum (4.7). It follows from the definitions that there are no nontrivial maps from a dissonant spectrum to a harmonic spectrum (4.9) and this fact leads to easy proofs that $BP{*}H/(p) = 0$ (4.10), that there are no nontrivial maps from $H/(p)$ to a finite spectrum (4.11) and that each finite spectrum has infinitely many nontrivial homotopy groups (4.12). The last of these was proved earlier by Joel Cohen using entirely different methods. These three results were entirely unexpected side effects of our investigation.

In section 5 we derive the elementary properties of the localization functors with respect to $E(n)$, which we denote by L_n. In particular we find natural transformations $L_n \to L_{n-1}$. We conjecture (5.9) that the fibres of these maps are in a certain sense periodic.

In section 6, we describe $L_n BP$ and related spectra for all n.

In sections 7–9, we discuss the functor L_1, which is the same thing as localization with respect to K-theory at a prime p. In section 7, which is purely algebraic, we show that for p odd the category of torsion $E(1)_*E(1)$-comodules (or K_*K-comodules) is equivalent to a certain category of modules over the ring $\Lambda = Z_p[[t]]$. These modules have been studied in a different context by Iwasawa. In section 8, we show that $L_1 X = X \wedge L_1 S^0$ (8.4), $BP \wedge L_1 X = X \wedge L_1 BP$ (8.6) and we compute $\pi_* L_1 S^0$ (8.13 and 8.18). In section 9, we describe $L_1 RP^\infty$ (9.1) and $L_1 BP^\infty$ (9.2).

In section 10 we give some conjectures concerning these topics. These include various nilpotence statements (10.1) inspired by Nishida's theorem [39], the existence of finite spectra realizing certain cyclic BP_*-modules (10.2), a description of the Bousfield class of any finite spectrum (10.4 and 10.5) and the structure of Bousfield's Boolean algebra of spectra **BA** (10.8). Known special cases of these conjectures (10.9 and 10.10) are also given.

Throughout, we will be working in Boardman's stable homotopy category [48] and the reader should be warned that nearly all of the spectra we shall consider are nonconnective.

This paper supersedes a preprint of the same title which I had planned to publish along with [41] in the proceedings of the 1977 Evanston conference. I withdrew the manuscript when some serious errors were found in it by Zen-ichi Yosimura and others. In particular I claimed to prove that $E(n)$ is a retract of $v_n^{-1}BP$, which I now believe to be false. It is likely that such a splitting of $v_n^{-1}BP$ exists only after a suitable completion. An analogous splitting of a completion of $B(n)$ into a wedge of suspensions of $K(n)$ has been established by Würgler [50], along with the result that $B(n)$ itself does not so split.

Bousfield's work [11] did not exist then and has since provided a convenient language for expressing many of the ideas here, e.g. the results of section 2. The results of section 3 are new as is most of section 10. I am grateful to Z. Yosimura, D. C. Johnson, H. R. Miller, P. S. Landweber, and J. F. Adams for many helpful conversations. In particular I am indebted to Yosimura for the present definition of harmonic spectra and to Landweber for the proof of 6.1. I apologize to all interested parties for my delay in publishing this paper.

The ten sections of the paper are as follows.

1. Some results of Bousfield on localization in the stable homotopy category.

1. Some results of Bousfield on localization in the stable homotopy category. Let E_* be a generalized homology theory.

1.1. *Definition.* A spectrum X is **E**$_*$*-acyclic* if $E_*X = 0$.

1.2. *Definition.* A map $f: X \to Y$ is an **E**$_*$*-equivalence* if it induces an isomorphism in E_*-homology.

1.3. *Definition.* A spectrum Y is E_*-local if for each E_*-acyclic spectrum X, $[X, Y] = 0$.

1.4. *Definition.* An E_*-localization functor L_E is a covariant functor from S, the stable homotopy category, to itself along with a natural transformation η from the identify functor to L_E such that $\eta_X : X \to L_EX$ is the terminal E_*-equivalence (i.e., map inducing an isomorphism in $E_*(\cdot))$ from X, i.e.

(i) $\eta_X : X \to L_EX$ is an E-equivalence, and
(ii) for any E_*-equivalence $f: X \to Y$ there is a unique $r: Y \to L_EX$ such that $rf = \eta_X$.

The following elementary results are left to the reader.

1.5. Proposition. *If the functor L_E exists,*

(i) *it is unique,*
(ii) *it is idempotent, i.e. $L_EL_E = L_E$, and*
(iii) *for any map $g: X \to Y$ where Y is E_*-local, there is a unique map $\tilde{g}: L_EX \to Y$ such that $\tilde{g}\eta_x = f$.* □

1.6. Proposition. *If L_E exists and $W \to X \to Y$ is a cofibre sequence, so is $L_EW \to L_EX \to L_EY$.*

1.7. Proposition. *The homotopy inverse limit (see* [12] *Chapter* XI *or* [4] *p.* 325) *of E_*-local spectra is E_*-local.* □

1.8. PROPOSITION. *Let E^1_* and E^2_* be generalized homology theories such that $E^1_*X = 0$ implies $E^2_*X = 0$. Then if a spectrum Y is E^2_*-local it is E^1_*-local. In particular, if $E^1_*X = 0$ iff $E^2_*X = 0$, then the functors L_{E^1} and L_{E^2} are the same.* □

1.9. *Example.* The direct limit of E_*-local spectra need not be local. Let $M(p)$ denote the mod p Moore spectrum, where p is a prime number. In [1] Adams constructs a map $\alpha : M(p) \to \Sigma^{-q}M(p)$ (where $q = 8$ for $p = 2$ and $q = 2p - 2$ for $p > 2$) which is a K_*-equivalence, K_* being the homology theory associated with complex K-theory. Let $X = \varinjlim \Sigma^{-qi}M(p)$ and let H_* denote ordinary homology with integer coefficients. Then clearly $H_*X = 0$ since H_* commutes with direct limits. We will see below (1.12), that every connective spectrum is H_*-local. On the other hand, X is H_*-local only if it is contractible, but $K_*X \neq 0$, so it is not.

1.10. *Example.* Despite 1.7, L_E need not commute with inverse limits. Let $\{X_i\}$ be the Postnikov tower for $M(p)$, so $M(p) = \varprojlim X_i$ and each X_i has only finitely many nontrivial homotopy groups, each of which are finite. Let $H/(p)$ denote the mod (p) Eilenberg-MacLane spectrum. From [6] we know $K_*H/(p) = 0$, so $K_*X_i = 0$ for each i. Hence L_KX_i (whose existence is given by 1.11) and $\varprojlim L_KX_i$ are contractible, but $L_KM(p)$ is not since $K_*M(p) \neq 0$.

Now we come to Bousfield's main result. An unstable form of this theorem appeared in [9].

1.11. LOCALIZATION THEOREM (Bousfield [10]). *For every generalized homology theory E_*, there is a localization functor $L_E : \mathbf{S} \to \mathbf{S}$ (1.2), where $\mathbf{S}$ is the stable homotopy category.* □

Examples 1.9 and 1.10 indicate that L_EX is somewhat unpredictable if either X or E fail to be connective. However, if both E and X are connective, then L_EX is easily described.

Let J be a set of primes, finite or infinite. Let $Z_{(J)}$ denote the subring of Q in which a prime p is invertible iff $p \notin J$, and let $Z_J = \Pi_{p \in J} Z_p$, where Z_p denotes the p-adic integers. Let $X_{(J)} = X \wedge M(Z_{(J)})$, where $M(Z_{(J)})$ is the Moore spectrum for the group $Z_{(J)}$, and $X^\wedge_J = \Pi_{p \in J} X^\wedge_p$ where $X^\wedge_p$ denotes the p-adic completion of X, i.e. $X^\wedge_p = \lim_{\overleftarrow{n}} X \wedge M(Z/p^n)$. Then Bousfield has shown

1.12. THEOREM [10]. *Let E_* be a connective homology theory and X a connective spectrum. Let J be complementary to the set of primes p such that E_i is uniquely p-divisible for each i. Then $L_EX = X^\wedge_J$ if each element of E_* has finite order, and $L_EX = X_{(J)}$* otherwise. □

Next we need to consider the E_*-Adams spectral sequence. Let E be a ring spectrum, not necessarily connective. Consider the Adams tower $X = D_0X \leftarrow D_1X \leftarrow D_2S \leftarrow \cdots$ where $D_{n+1}X$ is the fibre of $D_nX \to E \wedge D_nX$. Let K_nX be the cofibre of $D_nX \to X$ and consider the associated tower $pt = K_0X \leftarrow K_1X \leftarrow K_2X \leftarrow \cdots$.

1.13. *Definition.* The *E-nilpotent completion* of X, $E^\wedge X$, is $\varprojlim K_nX$.

1.14. *Definition.* The *E_*-Adams spectral sequence for X $\{E_r^{s,t}X\}$* is the spectral sequence associated with the homotopy exact couple $D_{*+1}X \to D_*X \to E \wedge D_*X$.

1.15. THEOREM [10]. *If for each s, t there is a finite r such that $E_r^{s,t}X = E_\infty^{s,t}X$, then the Adams spectral sequence* 1.14 *converges to $\pi_*E^\wedge X$, i.e. the terms $\{E_\infty^{s,s+i}X\}_{s\geq 0}$ are the quotients of a complete Hausdorff filtration of $\pi_iE^\wedge X$.* □

1.16. *Example.* If X is connective and $E = H\mathbf{F}_p$, the mod p Eilenberg-MacLane spectrum, $E^\wedge X = X_p^\wedge$ and 1.12 is the classical mod p Adams spectral sequence. If $E = MU$ or $E = BP$, then $E^\wedge X = X$ or $E^\wedge X = X_{(p)}$ respectively, and 1.14 gives the Adams-Novikov spectral sequence [41].

If X or E fail to be connective, then 1.15 converts what is usually called a convergence question to the problem of describing $E^\wedge X$. The following result gives some information about $E^\wedge X$.

1.17. PROPOSITION.

(a) *If E is a ring spectrum, then any E-module spectrum M (e.g. $E \wedge X$ for any X) is E_*-local.*

(b) *If E is a ring spectrum, $E^\wedge X$ is E_*-local.*

Proof. (a) Let W be a spectrum which is E_*-acyclic, i.e. $E \wedge W = $ pt. By definition 1.3, M is E_*-local if $[W, M] = 0$. For any map $W \to M$ we have a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & M \\ pt = E \wedge W \arrow[r] & E \wedge M \arrow[u] \end{tikzcd}$$

so the map is trivial.

(b) By 1.7, it suffices to show K_nX is E_*-local, which we do by induction on n. Since $K_0X = pt$, we can start the induction. The fibre of $K_{n+1}X \to K_nX$ is $E \wedge D_nX$ which is E_*-local by (a), so $K_{n+1}X$ is E_*-local by 1.6. □

Unfortunately, 1.7(b) does not imply that $E^\wedge X = L_EX$ even though $E^\wedge X$ is E_*-local. We do not know that the map $X \to E^\wedge X$ is an E_*-equivalence because in general, we cannot compute $E_*E^\wedge X$ since smash products (and hence generalized homology) need not commute with homotopy inverse limits. Nor does one know that $E^\wedge E^\wedge X = E^\wedge X$ in general.

However Bousfield has shown that under certain conditions, the above equalities hold.

1.18. THEOREM [10]. *Let E be a ring spectrum such that π_*E is countable and such that for some r_0 and $s_0 \leq \infty$, E_r^{s*}, X vanishes for all $r \geq r_0$, $s \geq s_0$ and all finite complexes X. Then for any spectrum X, $E^\wedge X = X \wedge E^\wedge S = L_EX$, $E^\wedge X = E^\wedge E^\wedge X$, and the Adams spectral sequence converges to π_*L_EX.* □

The following definitions are due to Bousfield [11].

1.19. *Definition.* For a spectrum E, $\langle E\rangle$ denotes the equivalence class of E under the following equivalence relation. $E \sim G$ if for any spectrum X, $E_*X = 0 \Leftrightarrow G_*X = 0$. Equivalently, $E \sim G$ if a map is an E_*-equivalence (1.2) iff it is a G_*-equivalence. We will refer to $\langle E\rangle$ as the Bousfield class of E.

1.20. *Definition.* $\langle E\rangle \leq \langle G\rangle$ if each G_*-acyclic (1.1) spectrum is E_*-acyclic. $\langle E\rangle < \langle G\rangle$ if $\langle E\rangle \leq \langle G\rangle$ and $\langle E\rangle \neq \langle G\rangle$. $\langle E\rangle \vee \langle G\rangle = \langle E \vee G\rangle$ and $\langle E\rangle \wedge \langle G\rangle = \langle E \wedge G\rangle$. (We leave it to the reader to verify that these classes are well defined.) A class $\langle E\rangle$ has a complement $\langle E\rangle^c$ if $\langle E\rangle \wedge \langle E\rangle^c = \langle 0\rangle$ and $\langle E\rangle \vee \langle E\rangle^c = \langle S\rangle$, where S is the sphere spectrum and $\langle 0\rangle$ is the class of a point.

1.21. *Definition.* **A** is the class of all classes $\langle E\rangle$, **DL** $\subseteq$ **A** the subclass consisting of classes satisfying $\langle E\rangle \wedge \langle E\rangle = \langle E\rangle$, and **BA** $\subseteq$ **DL** is the subclass of classes having complements.

DL is a distributive lattice and **BA** is a Boolean algebra. Bousfield [11] shows that both of the above inclusions are proper. If E is any wedge of ring spectra and finite spectra then $\langle E\rangle \in$ **DL**. In [11] Bousfield defines a subalgebra **MBA** $\subset$ **BA** consisting of classes represented by Moore spectra and shows that any wedge of finite spectra represents a class in **BA**. In [10] he shows that $\langle K \rangle \in \mathbf{BA}$, where K is the spectrum representing complex K-theory. In section 10 we will discuss some possible generalizations of this fact.

1.22. PROPOSITION. *If $\langle E \rangle = \langle G \rangle$ then $L_E = L_G$, and conversely. If $\langle E \rangle \leq \langle G \rangle$ then $L_E L_G = L_E$ and there is a natural transformation $L_G \to L_E$.* □

1.23. PROPOSITION. *If $W \to X \to Y \to \Sigma W$ is a cofibre sequence, then each of the three Bousfield classes $\langle W \rangle$, $\langle X \rangle$, and $\langle Y \rangle$ is $\leq$ the wedge of the other two.* □

1.24. PROPOSITION. *If E is a ring spectrum and M a module spectrum over E, then $\langle E \rangle \geq \langle M \rangle$.*

Proof. By definition the composite $M \to E \wedge M \to M$ is the identity, where the first map is induced by the unit in E. Hence M is a retract of $E \wedge M$, so $\langle M \rangle \leq \langle E \wedge M \rangle \leq \langle E \rangle$. □

1.25. PROPOSITION. *Let E be any spectrum and $T = L_E S$. Then T is a commutative ring spectrum.*

Proof. $T \wedge T$ is E_*-equivalent to S so localization gives a multiplication $T \wedge T \to L_E(T \wedge T) = L_E S = T$. The unit $S \to T$ is given by localization on S. The commutativity of T follows from that of S. □

1.26. *Example.* Let $E = S/(p)$, the mod p Moore spectrum for a prime p. Then $T = SZ_p$, the p-adic completion of the sphere (see section 2 of [10]) and $T \wedge T \neq T$ since $Z_p \otimes Z_p \neq Z_p$. Also we have $\langle T \rangle = \langle SZ_{(p)} \rangle$ so $L_T S \neq T$. On the other hand if $E = SZ_{(p)}$ then $T \wedge T = T$ and $L_T S = T$.

1.27. PROPOSITION. *For E and T as in* 1.25, *the following are equivalent.*

(a) $\langle E \rangle = \langle T \rangle$,

(b) $X \xrightarrow{1 \wedge \eta} X \wedge T$ *is an E-localization* (*in particular* $T \xrightarrow{1 \wedge \eta} T \wedge T$ *is an equivalence*),

(c) *every direct limit of E_*-local spectra is E_*-local, and*

(d) *L_E commutes with direct limits.*

Proof. First we show (a) $\Leftrightarrow$ (b). Since T is a ring spectrum (1.25), $X \wedge T$ is T_*-local (1.17). If $\langle E \rangle = \langle T \rangle$, $X \wedge T$ is therefore E_*-local and in any event is E_*-equivalent to X, so $L_E X = X \wedge T$. Conversely if $X \wedge T =$

$L_E X$ and $T = T \wedge T$, then the T_*-local spectrum $X \wedge T$ is T_*-equivalent to X, so $X \wedge T = L_T X$ and $\langle T \rangle = \langle E \rangle$ (1.22).

We complete the proof by showing (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (b). For (b) $\Rightarrow$ (c), let $\{X_i\}$ be a directed system of E_*-local spectra. Then $L_E \varinjlim X_i = (\varinjlim X_i) \wedge T = \varinjlim (X_i \wedge T) = \varinjlim X_i$ so $\varinjlim X_i$ is E_*-local. For (c) $\Rightarrow$ (d), let $\{X_i\}$ be any directed system. Then $L_E \varinjlim X_i = (\varinjlim X_i) \wedge T = \varinjlim (X_i \wedge T) = \varinjlim L_E X_i$. For (d) $\Rightarrow$ (b) any spectrum X is the direct limit of its finite subspectra X_i. For X_i finite $X_i \wedge T$ is E_*-local and E_*-equivalent to X_i so $L_E X_i = X_i \wedge T$. Then we have $L_E X = L_E \varinjlim X_i = \varinjlim L_E X_i = \varinjlim (X_i \wedge T) = (\varinjlim X_i) \wedge T = X \wedge T$. □

1.28. *Definition.* A spectrum E is *smashing* if it satisfies the conditions of 1.27.

1.29. PROPOSITION. *If E is a ring spectrum for which the multiplication $E \wedge E \to E$ is an equivalence then E is smashing.*

Proof. Since E is a ring spectrum $X \wedge E$ is E_*-local (1.17) and since $E \wedge E = E$, $X \wedge E$ is E_*-equivalent to X, so $X \wedge E = L_E X$. □

1.30. PROPOSITION. *If spectra F and G are smashing, so is $F \wedge G$.*

Proof. Let $U = L_F S$ and $V = L_G S$. Then $\langle F \wedge G \rangle = \langle U \wedge V \rangle$. $U \wedge V$ is a ring spectrum and hence is $U \wedge V_*$-local (1.17). $F \wedge G \wedge U \wedge V = F \wedge G$, so $U \wedge V$ is $F \wedge G_*$-equivalent to S. Hence $U \wedge V = L_{F \wedge G} S$ and $F \wedge G$ satisfies 1.27(a). □

1.31. PROPOSITION. *If E is smashing then $\langle E \rangle \in$* **BA**.

Proof. Let $C_E S$ be the fibre in $C_E S \to S \to L_E S = T$. By 1.23 we have $\langle S \rangle = \langle C_E S \rangle \vee \langle T \rangle$. $C_E S$ is E_*-acyclic and therefore T_*-acyclic so $\langle C_E S \rangle \wedge \langle T \rangle = \langle 0 \rangle$ and $\langle C_E S \rangle = \langle E \rangle^c$. □

From 1.26 we see that $S/(p)$ is a counterexample to the converse of 1.31.

Now let B be a wedge (possibly infinite) of finite spectra. Combining 2.9 of [11] and 3.5 of [10] we get

1.32. PROPOSITION. *Let B be as above. Then $\langle B \rangle \in$* **BA** (1.21). *Let $\langle E \rangle = \langle B \rangle^c$. Then E is smashing* (1.28). □

The following conjecture is equivalent to one due to Bousfield.

1.33. *Conjecture* ([10], 3.4). A spectrum E is smashing iff $\langle E \rangle = \langle B \rangle^c$ with B as in 1.32.

The following result about telescopes will be useful.

1.34. LEMMA. *Let X be a spectrum, $g: \Sigma^d X \to X$ a map with cofibre Y, and $\hat{X} = \varinjlim_g \Sigma^{-id} X$. Then $\langle X \rangle = \langle \hat{X} \rangle \vee \langle Y \rangle$.*

Proof. Let C be the cofibre of $X \to \hat{X}$. We will show below that $\langle C \rangle = \langle Y \rangle$. 1.23 implies $\langle Y \rangle \leq \langle X \rangle$ and $\langle X \rangle \leq \langle \hat{X} \rangle \vee \langle C \rangle$. The construction of $\hat{X}$ guarantees $\langle \hat{X} \rangle \leq \langle X \rangle$. Combining these facts gives

$$\langle \hat{X} \rangle \vee \langle Y \rangle \leq \langle X \rangle \leq \langle \hat{X} \rangle \vee \langle Y \rangle.$$

To show $\langle C \rangle = \langle Y \rangle$ let C_i be the cofibre of $g^i : X \to \Sigma^{-id} X$. Then $C_1 = \Sigma^{-d} Y$ and there is a cofibre sequence $C_1 \to C_i \to \Sigma^{-d} C_{i-1}$. Using 1.23 and induction on i we get $\langle C_i \rangle \leq \langle Y \rangle$ for all i. Since $C = \varinjlim C_i$ this gives $\langle C \rangle \leq \langle Y \rangle$. Letting i go to ∞ in the above cofibre sequence gives $C_1 \to C \to \Sigma^{-d} C$, so $\langle Y \rangle \leq \langle C \rangle$ and the result follows. $\square$

2. The structure of $\langle BP \rangle$. In this section we will discuss the Bousfield class of the Brown-Peterson spectrum BP and various related spectra. The basic properties of BP are given in Part II of [4] and in section 3 of [41]. The related spectra we will discuss are described by Johnson-Wilson in [21] and [22]. Recall that for each prime number p there is a spectrum BP with $\pi_* BP = Z_{(p)}[v_1, v_2 \cdots]$ with dim $v_n = 2(p^n - 1)$. We denote this ring by BP_*. For each $n \geq 0$ there are BP-module spectra $BP\langle n \rangle$, $P(n)$ and $k(n)$ with $\pi_* BP\langle n \rangle = BP_*/(v_{n+1}, v_{n+2}, \ldots)$, $\pi_* P(n) = BP_*/(p, v_1, \ldots v_{n-1})$ and $\pi_* k(n) = BP_*/(p, v_1, \ldots v_{n-1}, v_{n+1}, v_{n+2} \cdots)$. In particular, $P(0) = BP$, and $k(0) = BP\langle 0 \rangle = H_{(p)}$, the Eilenberg-MacLane spectrum for $Z_{(p)}$, the integers localized at p. $H/(p)$ will denote the mod (p) Eilenberg-MacLane spectrum. If E is $BP\langle n \rangle$, $P(n)$ or $k(n)$, there is a map $\Sigma^{2(p^n-1)} E \to E$ which induces multiplication by v_n in π_* (where v_0 is understood to be p) and we can form $v_n^{-1} E = \varinjlim \Sigma^{-2i(p^n-1)} E$, denoted respectively by $E(n)$, $B(n)$ and $K(n)$. $E(0) = K(0) = H\mathbf{Q}$ the rational Eilenberg-MacLane spectrum. $BP\langle 1 \rangle$, and $E(1)$, are summands of the connective and periodic complex K-theory localized at p, while $k(1)$ and $K(1)$ are the analogous mod (p) spectra.

We also denote $\varinjlim_{v_n} \Sigma^{-2i(p^n-1)} BP$ by $v_n^{-1} BP$. If $J = (q_0, q_1, \ldots q_{n-1}) \in BP_*$ is an invariant regular ideal, BPJ will denote the BP-module spectrum with $\pi_* BPJ = BP_*/J$ (see Johnson-Yosimura [23]).

Most of these spectra have multiplications; we will discuss this point at the end of the section.

The main results of this section are the following two theorems.

2.1. THEOREM. *With notation as above,*

(a) (*Johnson-Wilson* [22]) $\langle B(n)\rangle = \langle K(n)\rangle$,
(b) (*Johnson-Yosimura* [23]) $\langle v_n^{-1}BP\rangle = \langle E(n)\rangle$,
(c) $\langle P(n)\rangle = \langle K(n)\rangle \vee \langle P(n+1)\rangle$,
(d) $\langle E(n)\rangle = \vee_{i=0}^{n} \langle K(i)\rangle$,
(e) $\langle k(n)\rangle = \langle K(n)\rangle \vee \langle H/(p)\rangle$,
(f) $\langle BP\langle n\rangle\rangle = \langle E(n)\rangle \vee \langle H/(p)\rangle$,
(g) *if* J *has* n *generators then* $\langle BPJ\rangle = \langle P(n)\rangle$,
(h) *for* $\langle E\rangle = \langle H/(p)\rangle$ *or* $\langle K(n)\rangle$ *and any* $\langle X\rangle$, $\langle E\rangle \wedge \langle X\rangle = \langle E\rangle$ *or* $\langle 0\rangle$, *and*
(i) $\langle K(m)\rangle \wedge \langle K(n)\rangle = \langle 0\rangle$ *for* $m \neq n$ *and* $\langle K(n)\rangle \wedge \langle H/(p)\rangle = \langle 0\rangle$. □

2.2. THEOREM. *Let* $E = \vee_{i\geq 0} K(i)$ *or* $H/(p)$. *Then* $\langle E\rangle$ *has no complement in* $\langle BP\rangle$, *i.e. there is no Bousfield class* $\langle G\rangle$ *such that* $\langle E\rangle \wedge \langle G\rangle = \langle 0\rangle$ *and* $\langle E\rangle \vee \langle G\rangle = \langle BP\rangle$. *Consequently*, $\langle E\rangle \notin \mathbf{BA}$. □

After proving these results we will discuss some classes $\langle G\rangle \leq \langle BP\rangle$ with $\langle G\rangle \wedge \vee_{i\geq 0}\langle K(i)\rangle = \langle 0\rangle$.

Proof of 2.1. For (a) we use Lemma 3.5 of [22] which gives a natural isomorphism

$$B(n)_*X \otimes_{\pi_*B(n)} \pi_*K(n) \xrightarrow{\approx} K(n)_*X$$

for all finite X, and the fact that a $\pi_*K(n)$-basis of $K(n)_*X$ pulls back to a $\pi_*B(n)$-basis of $B(n)_*X$, which is a free module.

Any spectrum X is a direct limit of finite spectra X_α. Since homology and tensor products commute with direct limits we can deduce that $K(n)_*X = 0$ if $B(n)_*X = 0$, so $\langle B(n)\rangle \geq \langle K(n)\rangle$.

For the reverse inequality suppose $K(n)_*X = 0$. Then for each α there is a β with $X_\alpha \subset X_\beta$ such that $K(n)_*X_\alpha$ has trivial image in $K(n)_*X_\beta$. If $F_{\alpha\beta}$ is the fibre of the inclusion map then the map $K(n)_*F_{\alpha\beta} \to K(n)_*X_\alpha$ is surjective. The above lemma implies the same for the map in $B(n)$-homology, so $B(n)_*X_\alpha$ maps trivially to $B(n)_*X_\beta$ and $B(n)_*X = \varinjlim B(n)_*X_\alpha = 0$.

(a) implies that (c) is equivalent to $\langle P(n)\rangle = \langle B(n)\rangle \vee \langle P(n+1)\rangle$, which follows from 1.34.

For (d) first we have $\langle E(n)\rangle \in \mathbf{DL}$ by 2.16. Then we iterate (c) to get $\langle BP\rangle = \langle P(0)\rangle = \vee_{i=0}^{n} \langle K(i)\rangle \vee \langle P(n+1)\rangle$. Since $\langle E(n)\rangle = \langle v_n^{-1}BP\rangle$ and $v_n^{-1}BP = \lim_{\vec{v_n}} \Sigma^{-2i(p^n-1)}BP$, we have $\langle E(n)\rangle \le \vee_{i=0}^{n} \langle K(i)\rangle \vee \langle P(n+1)\rangle$. We will show below (2.3) that $E(n) \wedge P(n+1) = \text{pt.}$, which implies $\langle E(n)\rangle \le \vee_{i=0}^{n} \langle K(i)\rangle$. For the opposite inequality, Theorem 0.1 of [23] implies that $\langle E(n)\rangle \ge \langle E(i)\rangle$ for $0 \le i \le n$. Since $B(n)$ can be obtained from $v_n^{-1}BP$ (or $K(n)$ from $E(n)$) by a finite sequence of cofibrations, $\langle E(i)\rangle \ge \langle K(i)\rangle$, so $\langle E(n)\rangle \ge \vee_{i=0}^{n} \langle K(i)\rangle$ and (d) follows.

For (e) we have a cofibre sequence $\Sigma^{2(p^n-1)}k(n) \to k(n) \to H/(p)$ with $K(n) = \lim_{\vec{v_n}} \Sigma^{-2i(p^n-1)}k(n)$ so 1.34 gives the result.

For (f) we have a cofibre sequence $\Sigma^{2(p^n-1)} BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \to BP\langle n-1\rangle$ with $E(n) = \lim_{\vec{v_n}} \Sigma^{-2i(p^n-1)} BP\langle n\rangle$, so 1.34 gives $\langle BP\langle n\rangle\rangle = \langle E(n)\rangle \vee \langle BP\langle n-1\rangle\rangle$. Iterating this and using (d) gives $\langle BP\langle n\rangle\rangle = \langle E(n)\rangle \vee \langle BP\langle -1\rangle\rangle$ where $BP\langle -1\rangle = H/(p)$.

To prove (g), let J_k $(0 \le k \le n)$ be the ideal obtained from J by replacing q_i by v_i for $i < k$ (where v_0 is understood to be p). We will show below (2.4) that each J_k is invariant regular and that in it q_k can be replaced by v_k^m for some $m > 0$. Then we have $J_0 = J$ and $BPJ_n = P(n)$ and we will show $\langle BPJ_k\rangle = \langle BPJ_{k+1}\rangle$ for each k. We have a cofiber sequence

$$\Sigma^{2(p^k-1)} BPJ_k' \xrightarrow{v_k} BPJ_k \xrightarrow{j_k} BPJ_{k+1}$$

where J_k' is obtained from J_k by replacing v_k^m by v_k^{m-1} (we ignore the case $m = 1$ because then $J_k = J_{k+1}$); J_k' is invariant by 2.4 below. We can assume by induction on m that $\langle BPJ_k'\rangle = \langle BPJ_{k+1}\rangle$, so 1.23 gives $\langle BPJ_k\rangle \le \langle BPJ_{k+1}\rangle$. For the opposite inequality, Proposition 5.5 of [23] give multiplications with unit on BPJ_k and BPJ_{k+1}. Hence we get maps

$$S^0 \wedge BPJ_{k+1} \to BPJ_k \wedge BPJ_{k+1} \to BPJ_{k+1} \wedge BPJ_{k+1} \to BPJ_{k+1}$$

whose composite is the identity, showing that BPJ_{k+1} is a retract of $BPJ_k \wedge BPJ_{k+1}$, so $\langle BPJ_{k+1}\rangle \le \langle BPJ_k\rangle$.

To prove (h) we show that for any X, $E \wedge X$ is a wedge of suspensions of E, where $E = K(n)$ or $H/(p)$. E is known to be a ring spectrum [49] (albeit a noncommutative one for $p = 2$) and $\pi_*(E)$ is a graded field in the sense that every graded module over it is free. $E \wedge X$ is an E-module spectrum so $\pi_*E \wedge X = E_*X$ is a free π_*E-module. Choose a π_*E-basis for $\pi_*E \wedge X$ and let $W \to E \wedge X$ be the corresponding map from a wedge of spheres. Then the composite

$$E \wedge W \to E \wedge E \wedge X \to E \wedge X$$

is an equivalence.

For the first part of (i) we can assume $m > n$. Using (c) we have $\langle K(m) \rangle \leq \langle P(m) \rangle \leq \langle P(n+1) \rangle$, and by (d) $\langle K(n) \rangle \leq \langle v_n^{-1}BP \rangle$, so the result follows from 2.3 below. For the second assertion we have $H/(p) = \varinjlim P(m)$ so $P(n+1)_*X = 0 \Rightarrow H_*(X; X/(p) = 0$ so $\langle P(n+1) \rangle \geq \langle H/(p) \rangle$ and the above argument applies. $\square$

2.3. LEMMA. $v_n^{-1}BP \wedge P(n+1) = pt$.

Proof. In $\pi_*P(n) \wedge BP = BP_*BP/I_n$, the maps induced by $v_n \wedge 1$ and $1 \wedge v_n$ are the same since $\eta_L(v_n) \equiv \eta_R(v_n) \bmod (v_0, \ldots, v_{n-1})$. Consequently the map $v_n \wedge 1$ on $P(n) \wedge v_n^{-1}BP$ is an equivalence, so its cofibre $P(n+1) \wedge v_n^{-1}BP$ is contractible. $\square$

2.4. LEMMA. *Let J_k be as in the proof of 2.1(g). Then J_k is an invariant regular ideal in which q_k can be replaced by some power of v_k.*

Proof. According to Landweber [27] Proposition 2.5, an invariant regular ideal with k generators is primary with radical $I_k = (v_0, \ldots, v_{k-1})$. Since $I_k \supset (q_0, \ldots, q_{k-1})$, J_k is invariant since J is. By Proposition 2.11 of Landweber [25] the only elements which are invariant modulo I_k are polynomials in v_k, so we can replace q_k by some power of v_k. For the regularity of J_k we use Proposition 2.7 of [27], which says that any invariant ideal with n generators having radical I_n is regular; J_k clearly satisfies these conditions.

Our main tool for proving 2.2 (and for constructing most counterexamples in this theory) is Brown-Comenetz duality [14]. Their main result is the following.

2.5. THEOREM [14]. *Let Y be a spectrum with finite homotopy groups. Then there is a spectrum cY (the Brown-Comenetz dual of Y) such that for any spectrum X, $[X, cY]_{-i} = \mathrm{Hom}(\pi_i X \wedge Y, \mathbf{R}/Z)$. In particular $\pi_{-i}cY = \mathrm{Hom}(\pi_i Y, \mathbf{R}/Z)$ and $cH/(p) = H/(p)$. Moreover c is a contravariant functor on spectra with finite homotopy groups which preserves cofibre sequences and satisfies $ccY = Y$.* $\square$

2.6. LEMMA. *Let Y be a spectrum with finite homotopy groups.*

(a) *If Y is connective and p-local, then $\langle cY \rangle \leq \langle H/(p) \rangle$.*
(b) *If $[X, Y] = 0$ then $\pi_*X \wedge cY = 0$.*
(c) *If Y is a ring spectrum then $\langle cY \rangle \leq \langle Y \rangle$.*

(d) *If Y is a noncontractible ring-spectrum and X is a Y-module spectrum with* $[X, Y] = 0$ *then* $\langle X \rangle < \langle Y \rangle$.

Proof. (a) $\pi_* cY$ is bounded above so cY is the direct limit of its connective covers. Each connective cover cY_i has a finite Postnikov decomposition so $\langle cY_i \rangle \leq \langle H/(p) \rangle$ and the result follows.

(b) Since $Y = ccY$ we have $0 = [X, ccY]_{-i} = \mathrm{Hom}(\pi_i X \wedge cY, \mathbf{R}/Z)$, so $\pi_* X \wedge cY = 0$.

(c) By 1.24 it suffices to show that cY is a Y-module spectrum. The multiplication on Y induces a monomorphism $\mathrm{Hom}(\pi_* Y \wedge cY, \mathbf{R}/Z) \to \mathrm{Hom}(\pi_* Y \wedge Y \wedge cY, \mathbf{R}/Z)$, which by definition corresponds to a monomorphism

$$[cY, cY] \to [Y \wedge cY, cY].$$

The image of the identity of cY is the desired module structure map; in particular it is a retraction of $Y \wedge cY$ onto cY.

(d) $\langle X \rangle \leq \langle Y \rangle$ by 1.24. Since $[X, Y] = 0$, $\langle X \rangle \wedge \langle cY \rangle = \langle 0 \rangle$ by (b), so cY is X_*-acyclic. However $\langle 0 \rangle < \langle cY \rangle \leq \langle Y \rangle$ by (c) so cY is not Y_*-acyclic. It follows that $\langle X \rangle \neq \langle Y \rangle$. □

Proof of 2.2. For $E = H/(p)$, we claim $[E, P(1)] = 0$. This can be shown either by an Adams spectral sequence argument using the methods of section 3, or, more easily by the results of section 4 (4.2, 4.7, and 4.9). Hence $\langle H/(p) \rangle \wedge \langle cP(1) \rangle = \langle 0 \rangle$ by 2.6(b) and $\langle cP(1) \rangle \leq \langle H/(p) \rangle$ by 2.6(a). Also $\langle cP(1) \rangle \leq \langle P(1) \rangle \leq \langle BP \rangle$ by 2.6(c) since $P(1)$ is a ring spectrum. (For p odd this is Theorem 5.1 of Würgler [49]. For $p = 2$ a unitary map $P(1) \wedge P(1) \to P(1)$ is given in Proposition 5.5 of Johnson-Yosimura [23] and such a map is sufficient for the argument in 2.6(c) and 1.24.)

If $\langle G \rangle$ exists we have $\langle BP \rangle = \langle E \rangle \vee \langle G \rangle$. Smashing with $cP(1)$ gives

$$(BP \wedge cP(1)\rangle = \langle E \wedge cP(1) \rangle \wedge \langle G \wedge cP(1) \rangle$$

i.e. $\langle cP(1) \rangle = \langle G \wedge cP(1) \rangle$. On the other hand, $\langle cP(1) \rangle \leq \langle E \rangle$ so $\langle G \wedge cP(1) \rangle \leq \langle E \wedge G \rangle = \langle 0 \rangle$. Hence we get $\langle cP(1) \rangle = \langle 0 \rangle$ which is a contradiction.

For $E = \vee_{0 \leq i} K(i)$ we have $\langle E \rangle \wedge \langle H/(p) \rangle = \langle 0 \rangle$ by 2.1(i). Consequently $\langle E \rangle \wedge \langle cP(1) \rangle = \langle 0 \rangle$ so $\langle cP(1) \rangle \leq \langle G \rangle$ if $\langle G \rangle$ exists. We will see in section 4 (4.2 and 4.9) that any spectrum G which is E_*-acyclic satisfies $BP^*G = 0$. Hence $[G, P(1)] = 0$ so by 2.6(b) $\langle G \rangle \wedge \langle cP(1) \rangle = \langle 0 \rangle$. Again this implies $\langle cP(1) \rangle = \langle 0 \rangle$ which is a contradiction. □

Now we will describe some classes $\langle E \rangle < \langle BP \rangle$ satisfying $\langle E \rangle \wedge \langle K(n) \rangle = \langle 0 \rangle$ for all $n \geq 0$ and $\langle E \rangle > \langle H/(p) \rangle$. We will consider spectra of the form BPJ where J is an invariant regular sequence of infinite length. If $J = \{p, v_1, v_2 \cdots\}$, we have $BPJ = H/(p)$. Let $I(J)$ denote the corresponding infinitely generated invariant regular ideal.

2.7. *Definition.* Let $J = (p^{j_0}, v_1^{j_1}, \ldots)$ and $K = (p^{k_0}, v_1^{k_1}, \ldots)$ be invariant regular sequences (IRS's). Let $J \wedge K = (p^{\min(j_0,k_0)}, v_1^{\min(j_1,k_1)}, \ldots)$ and $J \vee K = (p^{\max(j_0,k_0)}, v_1^{\max(j_1,k_1)}, \ldots)$. $J \sim K$ if $BP_*/I(J \wedge K)$ is finitely presented as a module over $BP_*/I(J \vee K)$. $J \geq K$ if $K \sim J \wedge K$.

2.8. *Conjecture.* Let J and K be IRS's of infinite length.

(a) $\langle BPJ \rangle = \langle BPK \rangle \Leftrightarrow J \sim K$.
(b) $\langle BPJ \rangle \geq \langle BPK \rangle \Leftrightarrow J \geq K$.
(c) $\langle BPJ \rangle \vee \langle BPK \rangle = \langle BP(J \vee K) \rangle$.
(d) $\langle BPJ \rangle \wedge \langle BPK \rangle = \langle BP(J \wedge K) \rangle$. □

This conjecture implies that there are uncountably many distinct Bousfield classes $\langle BPJ \rangle$. Similar statements may hold for noninvariant sequences J. Now we will prove part of 2.8.

2.9. THEOREM. *Let J and K be as above.*

(a) *If $J \sim K$ then $\langle BPJ \rangle = \langle BPK \rangle$.*
(b) *If $J \geq K$ then $\langle BPJ \rangle \geq \langle BPK \rangle$.*
(c) $\langle BPJ \rangle \leq \langle BP(J \vee K) \rangle$.
(d) $\langle BPJ \rangle \geq \langle BP(J \wedge K) \rangle$.

Proof. For (c), BPJ is a module spectrum over $BP(J \vee K)$ so the result follows from 1.24. The argument for (d) is similar. For (a) it suffices to show that if $J \sim K$ then $\langle BPJ \rangle = \langle BP(J \wedge K) \rangle$, which can be shown by an argument similar to that used for 2.1(g). (b) follows from (a) and (d). □

Additional evidence for 2.8 is contained in the following.

2.10. THEOREM. *Let $K = \{p, v_1, v_2, \ldots\}$ so $BPK = H/(p)$ and let J be as above with $J \not\sim K$. In fact $J \not\sim K$ whenever $J \neq K$.) Then $\langle BPJ \rangle > \langle BPK \rangle$.*

Proof. By 2.6(d) it suffices to show $[H/(p), BPJ] = 0$. Using 3.10 one can show that the Adams E_2-term for this group vanishes. The Adams spectral sequence converges by 3.4. □

Conceivably this argument can be generalized to other K with $J > K$ by using an Adams spectral sequence based on BPK_*-homology.

For future reference (section 10) we record the following property of Morava K-theories.

2.11. THEOREM. *Let X be a finite spectrum. Then* $\dim K(n)_*X \leq \dim K(n+1)_*X$ *for every n. In particular* $K(n)_*X = 0$ *if* $K(n+1)_*X = 0$, *and if X is $K(n)_*$-acyclic then it is $E(n)_*$-acyclic.*

Proof. Consider the functor $E(n+1)_* \otimes_{BP_*} P(n)_*X$. By Johnson-Yosimura [23] Lemma 3.5, it is a homology theory which we denote by E_*. We will show that $K(n)_*X$ and $K(n+1)_*X$ can both be computed in terms of E_*X, namely that there is a short exact sequence

$$0 \to K(n+1)_* \otimes_{E_*} E_*X \to K(n+1)_*X \to \mathrm{Tor}_1^{E_*}(K(n+1)_*, E_*X) \to 0$$

and $v_n^{-1}E_*X$ is a free $v_n^{-1}E_*$-module having the same rank as $K(n)_*X$. We will also show that E_* is a graded principal ideal domain (*PID*) in the sense that every finitely generated graded module over it is a direct sum of cyclic modules of the form $E_*/(v_n^k)$ for $0 < k \leq \infty$. Then the rank of $K(n)_*X$ is the number of free summands in this decomposition, while the rank of $K(n+1)_*X$ is that number plus twice the number of torsion summands (each torsion summand gives a summand of Tor_0 and Tor_1), thereby proving the theorem. The statement about $E(n)_*$-acyclicity follows from the earlier statements along with 2.1(d).

Note that if X were not finite, E_*X could have summands of the form $v_n^{-1}E_*$. These would contribute to the rank of $K(n)_*X$ but not of $K(n+1)_*X$, so the theorem would be false.

Now we need to verify the facts used above. For the assertion about $K(n+1)_*X$, in [2] it is shown that given a suitable pairing $E_*X \otimes K(n+1)_* \to K(n+1)_*X$, there is a spectral sequence converging to $K(n+1)_*X$ with

$$E_2 = \mathrm{Tor}^{E_*}(E_*X, K(n+1)_*),$$

so we need to show that this Tor_i vanishes for $i > 1$. To get the pairing note that the standard pairing

$$P(n)_*X \otimes_{BP_*} K(n+1)_* \to K(n+1)_*X$$

factors through $E_*X \otimes K(n+1)_*$. For the vanishing of the higher Tor groups it suffices to prove a similar vanishing for $\mathrm{Tor}^{BP_*}(P(n)_*X, K(n+1)_*)$. It follows from the Landweber Filtration Theorem ([23] 1.16) that it suffices to consider

$$\mathrm{Tor}_i^{BP_*}(BP_*/I_m, K(n+1)_*)$$

for $m \geq n$. (Here I_m is the invariant prime ideal $(p, v_1, \ldots v_{m-1})$). A routine calculation shows that this group vanishes unless $i = 0$ and $m = n$ or $n+1$ or $i = 1$ and $m = n+1$.

For the assertion about $K(n)_*X$, note that $v_n^{-1}E_*X = E(n+1)_* \otimes_{BP_*} B(n)_*X$. It is known (Johnson-Wilson [22] 3.1) that $B(n)_*X$ is a free $B(n)_*$-module having the same rank as $K(n)_*X$, so our assertion follows.

Finally we need to show that E_* is a graded PID. Note that for $n > 0$ $E_* = \mathbf{F}_p[v_n, v_{n+1}, v^{-1}_{n+1}]$, so multiplication by v_{n+1} induces an isomorphism in any graded E_*-module which raises degree by $2(p^{n+1} - 1)$. It follows that the category of Z-graded E_*-modules is equivalent to the category of $Z/(2p^{n+1} - 2)$-graded modules over $\mathbf{F}_p[v_n]$. This latter ring is a PID, so our assertion follows. $\square$

Now we will discuss the existence of multiplications on the various spectra above. A multiplication on a spectrum E is a map $\mu : E \wedge E \to E$ which is associative, commutative and unitary (with respect to a given map $S^0 \to E$) up to homotopy. Hence it corresponds to a class $m \in E^0(E \wedge E)$ with appropriate properties. If E is countable (as in the case in all examples of this section) then $E \wedge E$ is a countable direct limit of finite spectra X_α and we compute $E^0(E \wedge E)$ by means of the Milnor short exact sequence [36] $0 \to \varprojlim^1 E^{-1}(X_\alpha) \to E^0(E \wedge E) \to \varprojlim E^0(X_\alpha) \to 0$. Generally one can compute the righthand term and show that it contains an appropriate m. The real problem is to show that the $\varprojlim^1$ group on the left vanishes. Equivalently, we need to show that $E^0(E \wedge E)$ is Hausdorff in the topology induced by the maps to the $E^0(X_\alpha)$.

For $E = k(n)$, $BP\langle n \rangle$, BPJ or $P(n)$, the results of Shimada-Yagita [53] are relevant. They show that if E is a spectrum obtained from MU by the Sullivan-Baas construction (e.g. any of those listed above) then there is an external multiplication $E_*X \otimes E_*Y \to E_*(X \wedge Y)$. This product may not be commutative when $p = 2$. Contrary to their claim, this multiplication does not imply E is a ring spectrum, but merely that there is an appropriate element in $\varprojlim E^0(X_\alpha)$ above. The following argument for the vanishing of the $\varprojlim^1$ term and the extension of the multiplication to $v_n^{-1}BP$, $E(n)$, $B(n)$, and $K(n)$ is due to Yosimura.

2.12. LEMMA (Yosimura [51]). *Given an associative BP-module spectrum E of finite type there exists a similar spectrum* ∇E with $\nabla\nabla E = E$ satisfying $0 \to \mathrm{Ext}(\nabla E_{k-1}X, Z_{(p)}) \to E^kX \to \mathrm{Hom}(\nabla E_kX, Z_{(p)}) \to 0$. □

(The notation of [51] for ∇E is $\hat{E}(Z_{(p)})$.)

2.13. LEMMA. *If E is as in* 2.12 *with* $\pi_kE \otimes \mathbf{Q} = 0$ if $(2p-2) \nmid k$ *and X is a countable CW-spectrum with* $\pi_kX \otimes \mathbf{Q} = 0$ if $(2p-2) \nmid k$, *then* E^kX *in Hausdorff unless* $k \equiv 1 \bmod (2p-2)$.

Proof. Suppose we know that $E^k(X\mathbf{Q}) = 0$ for $(2p-2) \nmid (k-1)$. Then $E^k(X)$ is a subgroup of $E^{k+1}(X\mathbf{Q}/Z)$. X is a countable direct limit of finite spectra X_α so $X\mathbf{Q}/Z = \varinjlim(X_\alpha\mathbf{Q}/Z)$. Consider the commutative diagram

$$\begin{array}{ccccc}
E^kX & \longrightarrow & E^{k+1}(X\mathbf{Q}/Z) & \longleftarrow & \mathrm{Ext}(\nabla E_k(X\mathbf{Q}/Z); Z_{(p)}) \\
\downarrow & & \downarrow & & \downarrow \\
\varprojlim E^kX_\alpha & \to & \varprojlim E^{k+1}(X_\alpha\mathbf{Q}/Z) & \leftarrow & \varprojlim \mathrm{Ext}(\nabla E_k(X_\alpha\mathbf{Q}/Z), Z_{(p)}).
\end{array}$$

Since $\pi_*(X\mathbf{Q}/Z)$ and $\pi_*(X\mathbf{Q}/Z)$ are all torsion, so are $\nabla E_*(X\mathbf{Q}/Z)$ and $\nabla E_*(X_\alpha\mathbf{Q}/Z)$, so the right hand horizontal maps are isomorphisms. The right hand vertical map is an isomorphism since $\nabla E_*(X\mathbf{Q}/Z) = \varinjlim \nabla E_*(X_\alpha\mathbf{Q}/Z)$ and Ext converts $\varinjlim$ to $\varprojlim$. It follows that the left hand vertical map is monic and $E^k(X)$ is Hausdorff for $(2p-2) \nmid (k-1)$.

We still need to show $E^k(X\mathbf{Q}) = 0$ for $(2p-2) \nmid (k-1)$. Using 2.12 and the fact that $\mathrm{Hom}(\nabla E_k(X\mathbf{Q}), Z_{(p)}) = 0$ since $\nabla E_*(X\mathbf{Q})$ is a rational vector space, it suffices to show $\nabla E_k(X\mathbf{Q}) = 0$ for $(2p-2) \nmid k$. Now $\nabla E_*(X\mathbf{Q}) = \pi_*(\nabla E) \otimes \pi_*X \otimes \mathbf{Q}$, so we need to show $\pi_k(\nabla E) \otimes \mathbf{Q} = 0$ for $(2p-2) \nmid k$. Reversing the roles of E and ∇E in 2.12, setting $X = S^0$ and tensoring with $\mathbf{Q}$ we get a short exact sequence

$$0 \to \mathrm{Ext}(\pi_{k-1}E, Z_{(p)}) \otimes \mathbf{Q} \to \pi_k(\nabla E) \otimes \mathbf{Q} \to \mathrm{Hom}(\pi_kE, Z_{(p)}) \otimes \mathbf{Q} \to 0.$$

Now π_kE is torsion by assumption so the Hom vanishes, and the Ext is torsion since $\pi_{k-1}E$ is finitely generated. □

If $E = BP\langle n\rangle$, $P(n)$, BPJ or $k(n)$ and $X = E \wedge E$, then 2.13 shows $E^0(E \wedge E)$ is Hausdorff so we get

2.14. COROLLARY. *$BP\langle n\rangle$, $P(n)$, BPJ and $K(n)$ are associative, commutative (if $p > 2$) ring spectra.*

2.15. LEMMA. *Let E and X be as in 2.13. Then $(v_n^{-1}E)^0X$ is Hausdorff.*

Proof. See Lemma 1.1 of Yosimura [52]. □

In [52] an argument is given for a multiplication on $v_n^{-1}BP$ and $E(n)$ which depends on the fact that their coefficient rings are flat BP_*-modules in the sense of Landweber [26]. The argument applies to $B(n)$ and $K(n)$ since they are flat as modules over $P(n)$, although the later may not be commutative at the prime 2. Hence we have

2.16. COROLLARY (Yosimura [52]). *$E(n)$, $v_n^{-1}BP$, $K(n)$ and $B(n)$ are all ring spectra.* □

3. Some Bousfield classes larger than $\langle BP\rangle$. In this section we will construct for each prime p Thom spectra X_n for $n \geq 0$ satisfying

$$\langle S_{(p)}\rangle = \langle X_0\rangle > \langle X_1\rangle > \langle X_2\rangle > \cdots > \langle BP\rangle.$$

In proving this result we will develop some Adams spectral sequence techniques which were needed for some proofs in the previous section.

To construct the X_n recall that $BU = \Omega SU$ by Bott periodicity, where SU is the stable special unitary group. The map $\Omega SU(p^n) \to BU$ defines a stable complex vector bundle over $\Omega SU(p^n)$, and we define X_n to be the corresponding Thom spectrum localized at p. Since $SU(1)$ is the trivial group, X_0 is by definition the sphere localized at p. It can be shown that X_n splits into a wedge of suspensions of T_n where $BP_*T_n = BP_*[t_1, \ldots t_n]$ as BP_*BP-comodules, but we will not pursue this matter here. The main result of this section is the following:

3.1. THEOREM. *Let X_n be as above.*

(a) *$\langle X_n\rangle > \langle BP\rangle$ for each $n \geq 0$.*
(b) *$\langle X_n\rangle > \langle X_{n+1}\rangle$ for each $n \geq 0$.*
(c) *Neither $\langle X_n\rangle$ nor $\langle BP\rangle$ are in* **BA**, *although both are in* **DL** (1.21).

Proof. Since $\Omega SU(p^n) \to BU$ is a double loop map, X_n is an associative homotopy commutative ring spectrum. The inclusion $SU(p^n) \to SU(p^{n+1})$ leads to an X_n-module structure on X_{n+1}, while the inclusion $SU(p^n) \to SU$ leads to an X_n-module structure on $MU_{(p)}$ and hence on BP. We will show below (3.2) that $[X_{n+1} \wedge M, X_n] = 0$ and $[P(1), X_n] = 0$, where M is the mod (p) Moore spectrum (note that $P(1) = M \wedge BP$). For $p > 2$, M and hence $X_n \wedge M$ are ring spectra so 2.6(d) implies $\langle X_n \wedge M\rangle > \langle X_{n+1} \wedge M\rangle$ and $\langle X_n \wedge M\rangle > \langle P(0)\rangle$; for $p = 2$ we can use the mod (4) Moore spectrum instead. (a) and (b) follow since $\langle S_{(p)}\rangle = \langle M\rangle \vee \langle S\mathbf{Q}\rangle$.

For (c) we have $\langle cX_n\rangle \leq \langle H/(p)\rangle \leq \langle X_{n+1} \wedge M\rangle$ by 2.6(a). By 2.6(b) and 3.2 $X_{n+1} \wedge M \wedge cX_n = pt$. Hence if $\langle X_{n+1}\rangle \in \mathbf{BA}$, $\langle cX_n\rangle \leq \langle X_{n+1} \wedge M\rangle^c$. These two inequalities lead to the contradiction $cX_n = pt$, so $\langle X_n\rangle$ cannot be in $\mathbf{BA}$. The argument for $\langle BP\rangle$ is similar. The classes are in $\mathbf{DL}$ since they are represented by ring spectra. $\square$

We still need to prove

3.2. LEMMA. *With notation as above,*

(a) $[X_{n+1} \wedge M, X_n] = 0$ *and*
(b) $[P(1), X_n] = 0$. $\square$

Multiplication by p in either X_n on X_{n+1} induces the same endomorphism in $[X_{n+1}, X_n]$, from which it follows that $[M \wedge X_{n+1}, X_n] = [X_{n+1}, \Sigma^{-1} M \wedge X_n]$. Similarly, $[P(1), X_n] = [BP, \Sigma^{-1} M \wedge X_n]$. We will show that $[BP, M \wedge X_n]$ and $[X_{n+1}, M \wedge X_n]$ vanish by means of the classical Adams spectra sequence (ASS) based on ordinary mod (p) cohomology (see [4]). Standard convergence results (3.3) on the ASS for $[X, Y]$ require X to be a finite spectrum. We will prove convergence for X connective and of finite type provided Y is connective and each $\pi_i(Y)$ is a finite p-group (3.4). Then we will show that the relevant E_2-terms vanish (3.9).

We refer the reader to [4] for the construction of the ASS and the proof of the following result.

3.3. THEOREM. *Let X be a finite spectrum and Y a connective spectrum with each $\pi_i(Y)$ a finite p-group. Then there is a spectral sequence which is natural in X and Y with $E_2^{s,t} = \operatorname{Ext}_A^{s,t}(H^*(Y), H^*(X))$ (where all cohomology groups have* mod (p) *coefficients), $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$, and $[X, Y]_*$ has a decreasing filtration $[X, Y]_* = F_*^0 \supset F_*^1 \supset \cdots$ such that for each n $F_n^s = 0$ for large s, and $F_{t-s}^s/F_{t-s}^{s+1} = E_\infty^{s,t} \equiv \bigcap_{r>s} E_r^{s,t}$. (The last defining equality makes sense because $E_{r+1}^{s,t} = \ker d_r \subset E_r^{s,t}$ for $s > r$. The grading of $[X, Y]_*$ is defined by $[X, Y]_n = [\Sigma^n X, Y]$.)* $\square$

The filtration above on $[X, Y]$ is defined by saying that $f : X \to Y$ is in F^s if it can be written as the composite of s maps each of which is trivial in mod (p) cohomology.

3.4. THEOREM. *Let Y be as in* 3.3 *and let X be a connective spectrum with H^*X of finite type. Then the ASS for $[X, Y]$ converges, i.e.*

(a) *every element of E_∞^{**} is represented by a map $f : X \to Y$ and*

(b) *there are no nontrivial maps of infinite filtration. Moreover*

(c) $E_2^{s,t} = \mathrm{Ext}_A(H^*Y, H^*X) = \varprojlim \mathrm{Ext}_A(H^*Y, H^*X^n)$ *where X^n denotes the n-skeleton of X.*

Proof. We will prove (a) after proving (b) and (c). There is a short exact sequence due to Milnor [36], $0 \to \varprojlim{}^1[X^n, Y]_{*+1} \to [X, Y]_* \to \varprojlim[X^m, Y]_* \to 0$. Our hypothesis on Y insure that $[X^n, Y]_*$ is finite, so the $\varprojlim{}^1$ above vanishes and we have

$$[X, Y]_* = \varprojlim[X^n, Y]_* \tag{3.5}$$

and similarly

$$H^*X = \varprojlim H^*X^n. \tag{3.6}$$

(b) follows from 3.5 because a map $X \to Y$ of infinite Adams filtration must restrict trivially to each X^n and therefore be trivial.

For (c) it suffices by 3.6 to show that

$$\mathrm{Ext}_A(H^*Y, \varprojlim H^*X^n) = \varprojlim \mathrm{Ext}_A(H^*Y, H^*X^n). \tag{3.7}$$

In general Ext does not commute with $\varprojlim$ in this way so some special argument is required. We will use the finiteness of H^*X^n and H^*X to convert the Ext to a Tor and the inverse limit to a direct limit.

First we claim that given left and right A-modules L and M,

$$\mathrm{Ext}_A(L, M^*) = \mathrm{Tor}^A(M, L)^* \tag{3.8}$$

where $(\ )^*$ denotes the linear dual. To see this let

$$0 \leftarrow L \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots$$

be a projective resolution of L. Then we have

$$\operatorname{Hom}(M \otimes_A C_s, Z/(p)) \simeq \operatorname{Hom}_A(C_s, M^*),$$

so the corresponding complexes have isomorphic cohomologies. These cohomologies are respectively $\operatorname{Tor}^A(M, L)^*$ and $\operatorname{Ext}^A(L, M^*)$.

Since $\varprojlim H^*X^n$ has finite type, we have

$$\begin{aligned}
\operatorname{Ext}_A(H^*Y, \varprojlim H^*X^n) &= \operatorname{Ext}_A(H^*Y, (\varprojlim H^*X^n)^{**}) \\
&= \operatorname{Tor}^A((\varprojlim H^*X^n)^*, H^*Y)^* \quad \text{by} \quad 3.8 \\
&= \operatorname{Tor}^A(\varprojlim (H^*X^n)^*, H^*Y)^* \\
&= (\varinjlim \operatorname{Tor}^A(H^*X^n, H^*Y))^* \\
&= \varprojlim \operatorname{Tor}^A(H^*X^n, H^*Y)^* \\
&= \varprojlim \operatorname{Ext}_A(H^*Y, H^*X^n) \quad \text{by} \quad 3.8.
\end{aligned}$$

This proves 3.7 and hence (c).

To prove (a) let $\{E_*^{*,*}(n)\}$ and $\{E_*^{*,*}(\infty)\}$ denote the ASS for $[X^n, Y]$ and $[X, Y]$ respectively. We will need to know that $E_r^{*,*}(\infty) = \varprojlim E_r^{*,*}(n)$ for each r. We have just shown this for $r = 2$, so we can start an induction on r. E_{r+1} is related to E_r by a long exact sequence, and inverse limits preserve exactness when all of the terms are finite dimensional vector spaces, so we have the inductive step.

Now suppose $x \in E^{s,t}(\infty)$. It follows from the above that $E_\infty^{s,t}(\infty) = \varprojlim E_\infty^{s,t}(n)$. Let x_n be the projection of x in $E_\infty^{s,t}(n)$. By 3.3 x_n is represented by each element of a suitable coset of the finite group $[X^n, Y]_{t-s}$. These cosets form an inverse system of nonempty finite sets and a standard argument shows that such a system has a nonempty inverse limit. Hence we can choose a compatible set of maps $f_n : X^n \to Y$ representing x_n, so 3.5 gives a map $f : X \to Y$ representing x. $\square$

The proof of 3.2 and 3.1 will be complete once we have proved

3.9. THEOREM. *With notation as above,*

(a) $\operatorname{Ext}_A(H^*M \wedge X_n, H^*X_{n+1}) = 0$
(b) $\operatorname{Ext}_A(H^*M \wedge X_n, H^*BP) = 0.$

Proof. Since H^*X_n is concentrated in even dimensions, it is related to $H^*M \wedge X_n$ by an obvious short exact sequence, from which it follows that $M \wedge X_n$ can be replaced by X_n in the statement of the theorem. We will simplify these Ext groups with a change of rings isomorphism and then use a result of Moore-Peterson [37] to show that they vanish. We will give the details for (a) only, as the proof of (b) is quite similar.

It is convenient to dualize and work in homology rather than in cohomology. We are trying to show

$$\mathrm{Ext}_{A_*}(H_*X_{n+1}, H_*X_n) = 0.$$

Obviously we will need to know how the dual Steenrod algebra A_* coacts on H_*X_n. To simplify notation we assume p is odd; the same argument works for $p = 2$ with the obvious changes in notation. Recall that

$$H_*SU(p^n) = E(x_3, x_5, \ldots x_{2p^n-1})$$

with $\dim x_{2i+1} = 2i + 1$. It follows by easy calculation that

$$H_*\Omega SU(p^n) = P(b_1, \ldots b_{p^n-1})$$

with $\dim b_i = 2i$ and b_i maps to that standard generator of $H_*\Omega SU = H_*BU$ (see [4] page 47). Recalling that X_n is the Thom spectrum of the induced bundle over $\Omega SU(p^n)$, it follows that H_*X_n is the corresponding subring of H_*MU. Recall also that $A_* = E(\tau_0, \tau_1, \ldots) \otimes P(\xi_1, \xi_2, \ldots)$ with $\dim \xi_n = \dim \tau_n - 1 = 2(p^n - 1)$. H_*X_n is actually a comodule over the sub-Hopf algebra $P_n = P(\xi_1, \ldots \xi_n)$ since it is concentrated in even dimensions and has no generators in dimensions greater than that of ξ_n.

The vanishing of the Ext group in question will follow from that of

$$\mathrm{Ext}_{P_{n+1}}(H_*X_{n+1}, H_*X_n)$$

by an appropriate Cartan-Eilenberg ([15] page 349) spectral sequence.

To calculate this group we claim H_*X_n is free over P_n. It is known that the composite $X_n \to MU \to H/(p)$ sends b_{p^i-1} to ξ_i for $i \leq n$ ([4] page 76). It follows that in each even dimension $2j \neq 2(p^i - 1)$ there is a primitive generator equal to b_j modulo decomposables. Let R_n be the ring generated by these primitives, so $H_*X_n = R_n \otimes P_n$ as a comodule over P_n or over P_{n+1}. It follows that

$$\mathrm{Ext}_{P_{n+1}}(H_*X_{n+1}, H_*X_n) = \mathrm{Ext}_{P_{n+1}}(P_{n+1}, P_n) \otimes \mathrm{Hom}(R_{n+1}, R_n).$$

Using the Cartan-Eilenberg spectral sequence one can show

$$\mathrm{Ext}_{P_{n+1}}(P_{n+1}, P_n) = \mathrm{Ext}_{P(\xi_{n+1})}(P(\xi_{n+1}), Z/(p)) \otimes \mathrm{Hom}(P_n, Z/(p)).$$

(Both of the above Hom's are over $Z/(p)$.)

At this point it is convenient to convert back to cohomology. Let K_n be the algebra dual to $P(\xi_n)$, so we want to show

$$\mathrm{Ext}_{K'_{n+1}}(Z/(p), K_{n+1}) = 0.$$

If we were proving (b) instead of (a) we would have $P(\xi_{n+1}, \xi_{n+2}, \ldots)$ instead of $P(\xi_{n+1})$. In either case the dual algebra is injective over itself by Theorem 2.7 of Moore-Peterson [37]. Hence we have

$$\mathrm{Ext}_{k_{n+1}}(Z/(p), K_{n+1}) = \mathrm{Hom}_{K_{n+1}}(Z/(p), K_{n+1}).$$

This group is trivial because K_{n+1} (being a divided power algebra) has infinitely many generators almost all of which act nontrivially on any given basis element. This completes the proof of 3.9, 3.2 and 3.1. □

3.10. *Remark.* In the proofs of 2.2 and 2.10 we need to know $[H/(p), P(1)] = 0$ and $[H/(p), BPJ] = 0$ for $J \neq (p, v_1, v_2, \ldots)$. Similar arguments to those above can be used; one ends up with $\mathrm{Ext}(E, Z/(p))$ where $E = E(\tau_n, \tau_{n+1}, \ldots)$ where n the largest integer with $I_n \subset J$. The fact that E is dual to an infinitely generated Hopf algebra insures the triviality of this group.

4. Harmonic spectra.

4.1. *Definition.* For each prime p let $E_p = \vee_{n\geq 0} K(n)$ and let $E = \vee_p E_p$. A spectrum is *harmonic* if it is E_*-local (1.3) and *dissonant* if it is E_*-acyclic (1.1).

This terminology will be discussed in the next section. In the first edition of this paper we gave a different definition which proved later to be unworkable. The present definition was suggested by Z. Yosimura.

In this section we will show that many spectra are harmonic, including finite spectra (4.5) and connective spectra with torsion free homology (4.6). On the other hand $H/(p)$, the mod (p) Eilenberg-MacLane spectrum, is dissonant. First we show that BP is harmonic. We are indebted to Dave Johnson for the proof; see Theorem 1.3 of [20].

4.2. THEOREM. *BP is harmonic.*

Proof. It follows from the definitions that it suffices to show $BP^*X = [X, BP] = 0$ whenever $K(n)_*X = 0$ for all $n \geq 0$. By 2.1(d) and (b), the latter condition is equivalent to $v_n^{-1}BP_*X = 0$ for all $n \geq 0$.

According to Theorem 13.6 (page 285) of [4], BP^*X may be computed from BP_*X by means of a universal coefficient spectral sequence with

$$E_2 = \mathrm{Ext}_{BP_*}(BP_*X, BP_*),$$

so it suffices to show that this Ext group vanishes. In [34] we considered the chromatic resolution, a certain long exact sequence of modules over BP_*,

$$0 \to BP_* \to M^0 \to M^1 \to M^2 \cdots$$

These modules are defined by $M^n = v_n^{-1}N^n$ where the N^n are defined inductively by the short exact sequences

$$0 \to N^n \to M^n \to N^{n+1} \to 0,$$

where $N^0 = BP_*$ and $v_0 = p$. The long exact sequence above is obtained by splicing together all of these short exact sequences.

There is an easily constructed spectral sequence, analogous to the chromatic spectral sequence of [34], converging to our Ext group with

$$E_1^{n,s} = \mathrm{Ext}^s_{BP_*}(BP_*X, M^n),$$

so it suffices to show that these Ext groups vanish.

Each M^n is uniquely v_n-divisible, i.e. multiplication by v_n is an isomorphism, so we can write $M^n = \varprojlim_{v_n^{-1}} M^n$. Hence we have

$$\begin{aligned}\mathrm{Ext}_{BP_*}(BP_*X, M^n) &= \mathrm{Ext}_{BP_*}(BP_*X, \varprojlim_{v_n^{-1}} M^n)\\ &= \mathrm{Ext}_{BP_*}(\varinjlim_{v_n} BP_*X, M^n)\\ &= \mathrm{Ext}_{BP_*}(v_n^{-1}BP_*X, M^n).\end{aligned}$$

This group vanishes since we are assuming $v_n^{-1}BP_*X = 0$. □

4.3. *Definition.* For a graded BP_*-module M, hom $\dim_{BP_*} M$ is the minimal length of a resolution of M by projective graded BP_*-modules. For M an MU_*-module, hom $\dim_{MU_*} M$ is similarly defined.

It is known ([18] 3.2) that every projective graded bounded below BP_*-module is free. The usefulness of this invariant for $M = BP_*X$ (known sometimes as the ugliness number of X) has been amply demonstrated by Johnson-Wilson [21].

4.4. THEOREM. (a) *If X is a connective spectrum of finite type with* hom $\dim_{MU_*} MU_*X$ *finite then X is harmonic.*

(b) *If X is a p-local connective spectrum with $\pi_i X$ finitely generated over $Z_{(p)}$ for each i and* hom $\dim_{BP_*} BP_*X$ *is finite, then X is harmonic.*

(c) *If $\pi_i X$ is a vector space over $\mathbf{Q}$ for each i, then X is harmonic.*

Proof. We will first prove (b), the local form, as the proof of (a) is analogous. Suppose we know $BP \wedge X$ is harmonic. Consider a BP_*-Adams resolution for X (1.14). It displays $X = BP^{\wedge}X$ (1.16) as a homotopy inverse limit of a tower of spectra in which the successive fibres are $X \wedge BP \wedge I^s(BP)$ where $I(BP)$ is the fibre of the unit $S^0 \to BP$ and $I^s(BP)$ is its s-fold smash product. $BP \wedge I^s(BP)$ is a wedge of suspensions of BP (see Lemma 11.1, page 88 of [4] of increasing connectivity so each $X \wedge BP \wedge I^s(X)$ is harmonic by assumption. Hence 1.7 implies that X is harmonic.

We still have to show that $BP \wedge X$ is harmonic. Let $d =$ hom $\dim_{BP_*} BP_*X$. If $d = 0$, BP_*X is a free BP_*-module so $BP \wedge X$ is a wedge of suspensions of BP and is harmonic by 4.2. Now we argue by induction on d. Assume hom $\dim_{BP_*} BP_*X > 0$ and pick a BP_*-basis of BP_*X and consider the corresponding map $W \to BP \wedge X$ where W is an appropriate wedge of spheres. Smashing with BP and using its ring structure we get a map $BP \wedge W \to BP \wedge X$ which is surjective in homotopy and in BP-homology. $BP \wedge W$ is harmonic by 4.2, so it suffices to show that the fibre F of this map is harmonic. The short exact sequence

$$0 \to BP_*F \to BP_*BP \wedge W \to BP_*BP \wedge X \to 0$$

is the start of a projective BP_*-resolution of $BP_*BP \wedge X$, so hom $\dim_{BP_*} BP_*F = d - 1$. Hence $BP \wedge F$ and therefore F are harmonic by the inductive hypothesis and the previous paragraph.

For (c) we know that such an X is a wedge of rational spheres $S\mathbf{Q}$. To show $S\mathbf{Q} = E(0)$ is harmonic, suppose X is dissonant. Then $X \wedge S\mathbf{Q}$ is contractible by assumption so $[X, S\mathbf{Q}] = 0$. $\square$

4.5. Corollary. *If X is finite then it is harmonic.*

Proof. Conner-Smith ([18] Theorem 1.6) have shown that hom $\dim_{MU_*} MU_*X$ is finite. □

4.6. Corollary. *If X is connective and $H_*(X, Z)$ is free abelian and of finite type then X is harmonic. The same holds if X is connective, p-local and $H_*(X; Z)$ is free and of finite type over $Z_{(p)}$.*

Proof. In Corollary 3.10 of [18] it is shown that MU_*X is a free MU_*-module. For the local analogue see [21]. □

Now we turn our attention to some dissonant spectra.

4.7. Theorem. *The* mod (p) *Eilenberg-MacLane spectrum $H/(p)$ is dissonant.*

Proof. From the definition 4.1 and 2.1 it suffices to show $v_n^{-1}BP_*H/(p) = 0$ for all $n \geq 0$. Since $H/(p)$ is a BP-module spectrum we have

$$\begin{aligned} BP_*H/(p) &= BP_*BP \otimes_{BP_*} Z/(p) \\ &= BP_*BP/(\eta_R(v_0), \eta_R(v_1), \ldots), \end{aligned}$$

where as usual $v_0 = p$. One knows $\eta_R(v_n) \equiv v_n \bmod (v_0, v_1, \ldots v_{n-1})$ (see [25] or 3.14 of [41]), so multiplication by v_n is trivial in $BP_*H/(p)$ and $v_n^{-1}BP_*H/(p) = 0$ as desired. □

4.8. Theorem. *Let X be a spectrum (not necessarily connective) such that π_*X is all torsion and is bounded above, i.e. $\pi_*X = 0$ for $i > k$ for some finite k. Then X is dissonant.*

Proof. We will show that X is a direct limit of a system in which successive cofibres are wedges of suspensions of $H/(p)$ for various p. Since generalized homology commutes with direct limits, X is E_*-acyclic and therefore dissonant. To construct the directed system, choose a prime p and a basis for the subgroup of π_kX of exponent p. The corresponding map to X from a wedge of spheres extends to one from a wedge K_0 of $H/(p)$'s. Let X_1 be the cofibre of this map. It satisfies the same hypothesis as X so the procedure can be repeated, giving a sequence

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

with $\pi_k(\varinjlim X_i)_{(p)} = 0$. This construction can be repeated for all p and for all descending values of k. The resulting direct limit is contractible. Taking the fibres of the map from X to each spectrum in the system gives a system of dissonant spectra whose limit is X. □

The next result follows immediately from the definitions.

4.9. PROPOSITION. *There are no essential maps from a dissonant spectrum to a harmonic spectrum.*

Combining this with 4.2 and 4.7 we get

4.10. COROLLARY. $BP^*H/(p) = 0$. □

Combining 4.9, 4.7 and 4.5 we get

4.11. COROLLARY (MARGOLIS [31], LIN [29]). *There are no essential maps from $H/(p)$ to a finite spectrum.* □

In [16] Joel Cohen proved that a nontrivial suspension spectrum of finite type has infinitely many nontrivial homotopy groups. The following result overlaps with his.

4.12. THEOREM. *If X is a nontrivial harmonic spectrum of finite type then X has infinitely many nontrivial homotopy groups. The same is true if X is p-local and π_*X is of finite type over $Z_{(p)}$.*

Proof. If $X\mathbf{Q}$ denotes the $E(0)$ localization of X and TX is the fibre of $X \to X\mathbf{Q}$ then π_*TX is the direct sum of the torsion subgroup of π_*X and the desuspension of $\pi_*X \otimes \mathbf{Q}/Z$ (note that tensoring with $\mathbf{Q}/Z$ kills torsion). Hence if X has only finitely many nontrivial homotopy groups then TX satisfies the hypothesis of 4.8 and is therefore dissonant. Since X and $X\mathbf{Q}$ are harmonic, TX is also harmonic and therefore contractible, so $X = X\mathbf{Q}$, i.e. π_*X is a vector space over $\mathbf{Q}$. Since π_*X has finite type, the assumption that X has only finitely many nontrivial homotopy groups leads to the conclusion that X is contractible. □

This result would imply Cohen's if we could answer the following affirmatively.

4.13. *Question.* Are all suspension spectra harmonic?

5. The chromatic filtration. In this section all spectra are assumed to be p-local.

5.1. *Definition.* L_nX is the $E(n)_*$-localization (1.4 and 2.1) of X and $L_\infty X$ is the E_*-localization where E is the spectrum of 4.1. C_nX is the fibre of $X \to L_nX$.

1.22 and 2.1 give compatible natural transformations $L_\infty \to L_n$ and $L_{n+1} \to L_n$. We do not know whether $L_\infty X = \varprojlim L_n X \equiv \hat{L}_\infty X$ in general.

5.2. *Definition.* The *chromatic filtration* of the stable homotopy category $\mathbf{S}$ (localized at p) is the tower of categories and functors $L_0\mathbf{S} \leftarrow L_1\mathbf{S} \leftarrow L_2\mathbf{S} \leftarrow \cdots L_\infty\mathbf{S} \leftarrow \mathbf{S}$.

The categories $L_n\mathbf{S}$ for $0 \le n \le \infty$ are closed under cofibre sequences (1.6) and homotopy inverse limits (1.7). We conjecture (10.6) that for $n < \infty L_n\mathbf{S}$ is also closed under direct limits. $L_0\mathbf{S}$ is the rational stable homotopy category, which is well known to be equivalent to the category of graded vector spaces over $\mathbf{Q}$. In section 8 we will analyze the category $L_1\mathbf{S}$.

5.3. THEOREM. (a) *C_n is an idempotent exact functor and $C_n\mathbf{S}$ is the category of $E(n)_*$-acyclic spectra for $n < \infty$, and $C_\infty\mathbf{S}$ is the category of dissonant spectra. Both are closed under direct limits.*

(b) *For $m \le n$, $C_mC_n = C_nC_m = C_n$, $L_mC_n = C_nL_m = 0$ (i.e. the trivial functor) and $L_nC_m = C_mL_n$.*

Proof. (a) For the functoriality of C_n let $f : X \to Y$ be any map. Then we have

$$\begin{tikzcd} C_nX \arrow[r] \arrow[d, dashed, "C_nf"] & X \arrow[r] \arrow[d, "f"] & L_nX \arrow[d, "L_nf"] \\ C_nY \arrow[r] & Y \arrow[r] & L_nY \end{tikzcd}$$

The possible choices of C_nf are in one-to-one correspondence with the group $[C_nX, L_nY]_1$. Since C_nX is $E(n)_*$-acyclic and L_nY is $E(n)_*$-local, this group is trivial, so C_nf is well defined and C_n is a functor.

Since L_n is exact and idempotent, C_m is also

(b) If we apply L_m to $C_nX \to X \to L_nX$ we get the diagram

$$\begin{tikzcd} C_nX = C_mC_nX \arrow[r] \arrow[d] & C_mX \arrow[r] \arrow[d] & C_mL_nX \arrow[d] \\ C_nX \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & L_nX \arrow[d] \\ * = L_mC_nX \arrow[r] & L_mX \arrow[r] & L_mL_nX = L_mX \end{tikzcd} \tag{5.4}$$

where $L_mL_nX = L_mX$ by 1.22 and 2.1.

If we apply L_n to $C_mX \to X \to L_mX$ (written vertically) we get the diagram

$$
\begin{tikzcd}
C_nX = C_nC_mX \arrow[r] \arrow[d] & C_mX \arrow[r] \arrow[d] & L_nC_mX \arrow[d] \\
C_nX \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & L_nX \arrow[d] \\
* = C_nL_mX \arrow[r] & L_mX \arrow[r] & L_nL_mX = L_mX
\end{tikzcd}
\tag{5.5}
$$

Equating 5.4 and 5.5 gives the desired result. □

Note that C_n is not a localization functor because $C_n\mathbf{S}$ is not closed under inverse limits, although it is closed under direct limits.

We will now explain our use of the words 'chromatic' and 'harmonic' in this context. Recall ([34] section 3 or [41] section 8) that the chromatic spectral sequence for the sphere is based on the short exact sequences of BP_*BP-comodules

$$0 \to N^n \to M^n \to N^{n+1} \to 0 \tag{5.6}$$

defined inductively by $N^0 = BP_*$ and $M^n = v_n^{-1}N^n$. In a similar spirit, we define cofibrations

$$N_nX \to M_nX \to N_{n+1}X \tag{5.7}$$

inductively by setting $N_0X = X$ and $M_nX = L_nN_nX$. Then we have

5.8. *Localization Conjecture for S^0.* For $X = S^0$, the cofibrations 5.7 realize (in BP_*-homology) the exact sequences 5.6. □

In Proposition 8.28 of [41], we showed that for $n > 0$ M^n is a direct limit of comodules with periodic Novikov Ext groups, with the maps in the directed system being themselves periodic. This leads us to

5.9. *Periodicity Conjecture.* For each finite spectrum X and positive integer n there is a directed system $\{X_\alpha\}$ of finite $K(n-1)_*$-acyclic spectra such that

(i) $M_nX = \varinjlim M_nX_\alpha$,

(ii) for each X_α there is a homotopy equivalence $M_nX_\alpha \to \Sigma^{2p^i(p^n-1)}M_nX_\alpha$ for some i depending on α, and

(iii) suitable iterates of these equivalences commute with the maps in the directed system. □

The status of 5.8 will be discussed in section 10. In section 8, we will prove 5.9 for $n = 1$. If 5.8 is true, then the chromatic filtration of 5.2 is analogous to the chromatic filtration of the Adams-Novikov E_2-term discussed in section 8 of [41]. If 5.9 is true, 5.10 below implies that each fibre in the harmonic tower 5.2 has a weak form of periodicity. Hence the harmonic tower separates X into its weakly periodic components and a harmonic spectrum is one which can be described completely in terms of its weakly periodic pieces. The word dissonant is intended to be the opposite of harmonic.

5.10. THEOREM. *$N_nX = \Sigma^n C_{n-1}X$ and the fibre of $L_nX \to L_{n-1}X$ is $\Sigma^{-n}M_nX$.*

Proof. We argue by induction on n, both statements being obvious for $n = 0$ ($C_{-1}X = X$ and $L_{-1}X = pt.$) Diagram 5.4 or 5.5 with $m = n - 1$ yields a cofibration (the top row) $C_nX \to C_{n-1}X \to L_nC_{n-1}X$, and $C_{n-1}X = \Sigma^{-n}N_nX$ by assumption, so $L_nC_{n-1}X = \Sigma^{-n}M_nX$ and $C_nX = \Sigma^{-1-n}N_{n+1}X$ by definition. The right hand column is $\Sigma^{-n}M_nX = L_nC_{n-1}X \to L_nX \to L_{n-1}X$. □

5.11. *Definition.* M_nX is the *nth monochromatic component of* $\mathbf{X}$ and $M_n\mathbf{S}$ is the *nth monochromatic subcategory of* $\mathbf{S}$.

This explains our choice of the letter M.

Using standard arguments, we can get an Adams-type spectral sequence out of the harmonic tower, i.e.

5.12. PROPOSITION. *There is a spectral sequence converging to $\pi_*\hat{L}_\infty X$ with $E_1^{s,t} = \pi_t M_s X$.* □

If 5.8 is true, then for $X = S^0$, $E_1^{n,*}$ is closely related to the nth column of the chromatic spectral sequence described in section 3 of [34] and section 8 of [41].

6. The $E(n)_*$-localization of BP. In this section, we describe L_nBP and compute $\pi_*(X \wedge L_nBP)$ in terms of BP_*X. We will state our main results first and then give the proofs.

6.1. THEOREM. *The spectra N_nBP and M_nBP* (5.7) *are BP-module spectra* (*and the canonical maps between them are BP-module maps*) *with $\pi_*N_nBP = N^n$ and $\pi_*M_nBP = M^n$* (5.6). □

From 5.7, we get maps $\Sigma^{-1}N_{n+1}BP \to N_nBP$ which can be composed to give a map $i : \Sigma^{-1-n}N_{1+n}BP \to N_0BP = BP$.

6.2. THEOREM. *L_nBP is the cofibre of $\Sigma^{-1-n}N_{1+n}BP \xrightarrow{i} BP \to L_nBP$, so $\pi_*L_0BP = p^{-1}BP_*$ and $\pi_*L_nBP = BP_* \oplus \Sigma^{-n}N^{1+n}$ for $n \geq 1$.* □

The short exact sequences 5.6 give connecting homomorphisms $\mathrm{Tor}_{n+1}^{BP_*}(BP_*X, N^{n+1}) \to \mathrm{Tor}_n^{BP_*}(BP_*X, N^n)$ which can be composed to give a homomorphism $i_* : \mathrm{Tor}_{n+1}^{BP_*}(BP_*X, N^{n+1}) \to \mathrm{Tor}_0^{BP_*}(BP_*X, N^0) = BP_*X$.

6.3. THEOREM. (a) *$\pi_*(X \wedge N_nBP)$ can be computed by means of a spectral sequence with $E_2^{*,*} = \mathrm{Tor}_{*,*}^{BP_*}(BP_*X, N^n)$ and $\pi_*(X \wedge M_nBP)$ can be similarly computed,*

(b) *$\mathrm{Tor}_{s,*}(BP_*X, N^n) = 0$ for $s > n$ so there is an upper edge homomorphism $\pi_*(X \wedge N_nBP) \to \Sigma^n \mathrm{Tor}_{n,*}^{BP_*}(BP_*X, N^n)$ and $\pi_*(X \wedge M_nBP)$ admits a similar edge homomorphism.*

6.4. THEOREM. *In the cofibration $X \wedge \Sigma^{-1-n}N_{1+n}BP \xrightarrow{i} X \wedge BP \to X \wedge L_nBP$, the map π_*i is the composite*

$$\pi_*(X \wedge \Sigma^{-1-n}N_{1+n}BP) \to \mathrm{Tor}_{1+n}^{BP_*}(BP_*X, N^{1+n}) \xrightarrow{i_*} BP_*X$$

where i_* is the map described above and the first map is the edge homomorphism given by 6.3(b). □

To prove 6.1, we will inductively construct module spectra $N_n'BP$ and $M_n'BP$ having the desired homotopy groups and then show that they coincide with N_nBP and M_nBP respectively. The spectrum $M_n'BP$ will be the representing spectrum for the functor $(N_nBP)_*(\cdot) \otimes_{BP_*} v_n^{-1}BP_*$.

6.5. LEMMA. (a) *Let $E_*(\cdot)$ be a covariant functor from finite spectra to graded abelian groups which converts cofibrations into long exact sequences in the usual fashion. Then $E_*(\cdot)$ is a generalized homology theory represented by a spectrum E.*

(b) *Let $F_*(\cdot)$ be another such functor and let $\theta_* : E_*(\cdot) \to F_*(\cdot)$ be a natural transformation. Then θ_* is represented by a map $\theta : E \to F$ whose composition with any map from a finite complex to E is unique up to homotopy.*

Proof. (a) is proved by Adams in [5] as the cohomological analogue of (b). Since we are working with finite complexes, we can freely interchange homology and cohomology via Spanier-Whitehead duality. □

6.6. LEMMA. *Let E be a BP-module spectrum and M a flat BP_*-module. Then the functor $F_*(\cdot) = M \otimes_{BP_*} E_*(\cdot)$ is a homology theory represented by a spectrum F which is a retract of $BP \wedge F$.*

Proof. The flatness of M insures that $F_*(\cdot)$ is a homology theory and that F exists. The BP-module structure of E gives a natural transformation $\alpha : (BP \wedge E)_*(\cdot) \to E_*(\cdot)$ having appropriate properties. By 6.5(b), a similar natural transformation $\beta : (BP \wedge F)_*(\cdot) \to F_*(\cdot)$ will give a retraction of $BP \wedge F$ to F. $\square$

Proof of 6.1. Assume inductively that $N_n BP$ has the desired properties. $v_n^{-1}BP_*$ is a flat BP_*-module since it is a direct limit of desuspensions of BP_*, so $(N_n BP)_* \otimes_{BP_*} v_n^{-1}BP$ is represented by a spectrum $M_n' BP$ by 6.5(a). By 6.5(b) the inclusion $BP_* \to v_n^{-1}BP_*$ induces a map (not necessarily unique) $\lambda : N_n BP \to M_n' BP$ which is a $v_n^{-1}BP_*$-equivalence.

We will prove the theorem by showing

(i) $M_n' BP$ is $E(n)_*$-local, so λ is an $E(n)_*$-localization and therefore unique, and $M_n' BP = M_n BP$;

(ii) $M_n BP$ is a BP-module spectrum;

(iii) $N_{n+1} BP$ is a BP-module spectrum.

For (i), 6.6 shows $M_n' BP$ is a retract of $BP \wedge M_n' BP$, so it suffices to show that the latter is $E(n)_*$-local, i.e. that $[X, BP \wedge M_n' BP] = 0$ for any $E(n)_*$-acyclic spectrum X. The triviality of this group follows by an argument similar to that of 4.2.

Now we will prove (ii). Let $\mu_N : BP \wedge N_n BP \to N_n BP$ be the module structure map and consider

$$\begin{array}{ccc} BP \wedge N_n BP & \xrightarrow{\mu_N} & N_n BP \\ {\scriptstyle BP\wedge\lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ BP \wedge M_n BP & \xrightarrow{\mu_M} & M_n BP. \end{array}$$

$\lambda \cdot \mu_N$ is a map to an $E(n)_*$-local spectrum and $BP \wedge \lambda$ is an $E(n)_*$-equivalence, so by 1.4 μ_M can be chosen to make the diagram commute. The unit and associativity conditions on μ_M follow by similar arguments.

To prove (iii) consider the diagram

$$\begin{tikzcd}
BP \wedge M_nBP \arrow[r] \arrow[d, "\mu_M"'] & BP \wedge N_{n+1}BP \arrow[r] \arrow[d, dashed, "m"] & BP \wedge \Sigma N_nBP \arrow[d, "BP\wedge\Sigma\mu_N"] \\
M_nBP \arrow[r] & N_{n+1}BP \arrow[r] & \Sigma N_nBP
\end{tikzcd}$$

where the rows are cofibre sequences. The map m exists by standard arguments and we claim it is unique: two choices of m differ by an element in $[BP \wedge N_{n+1}BP, M_nBP]$, but $BP \wedge N_{n+1}BP$ is $E(n)_*$-acyclic and M_nBP is $E(n)_*$-local so this group is trivial. The unit and associativity conditions on m follow by similar arguments. □

Proof of 6.2. We argue by induction on n. By 5.10 we have the diagram

$$\begin{tikzcd}
L_nBP \arrow[r] & \Sigma^{-n}N_{1+n}BP \arrow[r, "i"] & \Sigma BP \\
L_{n+1}BP \arrow[r] \arrow[u] & \Sigma^{-1-n}N_{2+n}BP \arrow[r, "i"] \arrow[u] & \Sigma BP \arrow[u, equal] \\
\Sigma^{-1-n}M_{1+n}BP \arrow[r, equal] \arrow[u] & \Sigma^{-1-n}M_{1+n}BP \arrow[r] \arrow[u] & pt. \arrow[u]
\end{tikzcd}$$

in which rows and columns are cofibrations. Since $N_{1+n}BP$ is a BP-module spectrum the map $N^{1+n} = \pi_*N_{1+n}BP \to \Sigma^{1+n}BP_*$ is a BP_*-module homomorphism and therefore is trivial. Hence the fibration in the top row above gives a short exact sequence

$$0 \to BP_* \to \pi_*L_nBP \to \Sigma^{-n}N^{1+n} \to 0$$

which we need to show is split for $n \geq 1$. Since $\pi_*L_{n+1}BP$ maps to π_*L_nBP, it suffices to find a splitting for $n = 1$, but in that case $\Sigma^{-n}N^{1+n}$ is concentrated in odd dimensions while BP_* is even dimensional. □

Proof of 6.3(a). This has been shown by Adams [2], Lecture 1, provided one knows in advance that $BP_*X = 0$ implies $\pi_*(X \wedge N_nBP) = 0$. This implication follows easily from the construction of N_nBP. □

Proof of 6.3(b). We will show that for any BP_*-module M, $\mathrm{Tor}_{s,*}^{BP_*}(M, N^n) = 0$ for $s > n$ by induction on n. The statement is true for

$n = 0$ since $N^0 = BP_*$. Since $M^n = N^n \otimes_{BP_*} v_n^{-1}BP_*$ and $v_n^{-1}BP_*$ is flat, we have

$$\mathrm{Tor}^{BP_*}(M, M^n) = \mathrm{Tor}^{BP_*}(M, N^n) \otimes_{BP_*} v_n^{-1}BP_*.$$

Examining the long exact sequence in $\mathrm{Tor}(M, -)$ induced by the short exact sequence 5.6 completes the induction. □

Proof of 6.4. It follows from 5.3 and 5.10 that the map $\Sigma^{-1-n}N_{1+n}BP \to BP$ can be factored

$$\begin{tikzcd}
\Sigma^{-1-n}N_{1+n}BP \arrow[r] & \Sigma^{-n}N_nBP \arrow[r] \arrow[d] & \Sigma^{1-n}N_{n-1}BP \arrow[r] \arrow[d] & \cdots \arrow[r] & N_0BP = BP \arrow[d] \\
& \Sigma^{-n}M_nBP & \Sigma^{1-n}M_{n-1}BP & & M_0BP
\end{tikzcd}$$

where each horizontal map followed by a vertical one is a cofibre sequence. Smashing each of these cofibrations with X, taking homotopy and applying the edge homomorphism of 6.3(b) we get

$$\begin{tikzcd}
\pi_*(X \wedge \Sigma^{-1-n}N_{1+n}BP) \arrow[r] \arrow[d] & \pi_*(X \wedge \Sigma^{-n}N_nBP) \arrow[r] \arrow[d] & \pi_*(X \wedge \Sigma^{-n}M_nBP) \arrow[d] \\
\mathrm{Tor}^{BP_*}_{1+n,*}(BP_*X, N^{1+n}) \arrow[r] & \mathrm{Tor}^{BP_*}_{n,*}(BP_*X, N^n) \arrow[r] & \mathrm{Tor}^{BP_*}_{n,*}(BP_*X, M^n)
\end{tikzcd}$$

where the lower left-hand horizontal map is the connecting homomorphism. □

7. Torsion $E(1)_*E(1)$-comodules. This section consists of some algebraic preliminaries to our study in section 8 of $E(1)_*$-localization. Let $E(1)_*E(1) = E(1)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(1)$.

7.1. PROPOSITION. *For any spectrum X, $E(1)_*X$ is a comodule over $E(1)_*E(1)$.* □

We wish to study the category **TE** of torsion $E(1)_*E(1)$ comodules, i.e. comodules in which each element is annihilated by some power of p. If X is a spectrum in $C_0\mathbf{S}$, i.e. a spectrum with trivial rational homotopy type, then $E(1)_*X$ is an object in **TE**. Our main result 7.6 is that for $p > 2$, **TE** is equivalent to a certain category of modules. This result is used by Bousfield [54] to give an algebraic description of the category $L_1\mathbf{S}$ for an odd prime.

A heuristic argument, which we will not try to make precise, can be given for 7.6. We know (section 2) that $E(1)_*$ is intimately related to complex K-theory and that the latter has Adams operations ψ^k. If $p \nmid k$, these can be made into stable operations since we have localized at p. Since the spectra we are dealing with have p-torsion homotopy groups, they are p-adically complete and in the spirit of Sullivan [46], [47] we can define Adams operations ψ^k for k any unit in the p-adic integers Z_p. Our restriction from K_* to its summand $E(1)_*$ (section 2) corresponds to requiring that $k \equiv 1 \bmod p$. Hence for $X \in C_0\mathbf{S}$, $E(1)_*X$ admits a continuous action of Γ, the group of units in Z_p congruent to 1 mod p, via Adams operations, and this action is functorial. The fact that $\Gamma \cong Z_p$ for $p > 2$ but $\Gamma \cong Z/2 \oplus Z_2$ for $p = 2$ accounts for our restriction to $p > 2$.

To be more precise, let Γ_n denote the multiplicative group of units congruent to 1 mod p in $Z/(p^{n+1})$, let $Z_p[\Gamma_n]$ denote the group algebra of Γ_n over the p-adic integers Z_p, and let $\Lambda = \lim_{\overleftarrow{n}} Z_p[\Gamma_n]$.

7.2. LEMMA. *For $p > 2$, $\Lambda = Z_p[[t]]$ and the isomorphism can be chosen so that each of the maps $\Lambda \to Z_p[\Gamma_n]$ sends $t + 1$ to a generator of Γ_n. (Note that the generator and therefore the isomorphism are not canonical.)*

Proof. First notice that $\Gamma_n \cong Z/(p^n)$, so the statement makes sense. Let $\Gamma = \varprojlim \Gamma_n \cong Z_p$ and let $\gamma \in \Gamma$ be a generator. Then we have $\Lambda = \lim_{\overleftarrow{n}} Z_p[\gamma]/(\gamma^{p^n} - 1)$. The statement is now that of Lemma 1.6 of [30], to which we refer the reader for the rest of the argument. $\square$

7.3. *Remark.* Note that for $p = 2$, Γ_n is not cyclic, so 7.2 is false in that case.

7.4. *Definition.* A *torsion Λ-module* is a discrete p-torsion group on which Λ acts continuously (with respect to the inverse limit topology on Λ).

7.5. LEMMA. *Let M be a torsion Λ-module. Then for each element $m \in M$ there is an integer n such that $t^n m = 0$.*

Proof. The sequence $\{t, t^2, \ldots\}$ converges to 0 in Λ, so by continuity the sequence $\{tm, t^2m, \ldots\}$ must converge to 0 in M. Since M is discrete $t^n m = 0$ for some n. $\square$

Our main goal in this section is to prove

7.6. THEOREM. *For $p > 2$ the category* **TE** *of p-torsion $E(1)_*E(1)$ comodules is equivalent to that of $Z/(q)$-graded torsion Λ-modules where $q = 2p - 2$.* $\square$

Amusingly enough, these same modules have been studied extensively by Iwasawa [19] in connection with the class number of cyclotomic fields. He has classified such modules (with a certain finiteness condition) up to isogeny, a certain equivalence relation weaker than isomorphism. A concise, readable account of his work has been given by Manin [30] section 1. We will not make use of this classification here, but is gratifying to know that it exists.

To prove 7.6, note that if M is an $E(1)_*E(1)$-comodule concentrated in dimensions divisible by q, then M_0 is a comodule over $E(1)_0E(1)$ with $M = M_0 \otimes E(1)_*$. Let T_0E denote the category of nongraded torsion $E(1)_0E(1)$-comodules. Then 7.6 is clearly equivalent to

7.7. THEOREM. *The category T_0E is equivalent to the category* $\mathbf{T}\Lambda$ *of Torsion Λ-modules.*

To prove 7.7, it is convenient to replace $E(1)_*$ by $E(1)_*^\wedge = E(1)_* \otimes Z_p$ and $E(1)_*E(1)$ by $E(1)_*^\wedge E(1) = \lim_{\overleftarrow{i}} E(1)_*/p^i \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(1)_*$. An important difference between $E(1)_*^\wedge E(1)$ and $E(1)_*E(1)$ is that $v_1^{-1} - \eta_R v_1^{-1}$ is divisible by p in the former but not in the latter. The convenience of having $v_1^{-1} \equiv \eta_R v_1^{-1} \bmod p$ will become apparent below.

7.8. PROPOSITION. *If M is a torsion $E(1)_*E(1)$-comodule then it is an $E(1)_*^\wedge$-module and $M \otimes_{E(1)_*} E(1)_*E(1) = M \otimes_{E(1)_*^\wedge} E(1)_*^\wedge E(1)$.* □

Now let $K(1)_* = E(1)_*/(p)$ and $K(1)_*K(1) = K(1)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(1)_* = K(1)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(1)_*$. This object was studied extensively in [42], [40)] and [35]. In section 2 of [42] we showed (in slightly different language).

7.9. THEOREM. *As Hopf algebras over $\mathbf{F}_p$, $K(1)_0K(1) \simeq \mathrm{Hom}_c(\Lambda, \mathbf{F}_p)$ where $\mathrm{Hom}_c(\ ,\ )$ denotes the group of continuous homomorphisms (the topology on Λ is that induced by the ideal (p, t)) and coproduct on Λ is given by $\Delta(t) = 1 \otimes t + t \otimes 1 + t \otimes t$, i.e. the coproduct it has as a completed group ring.* □

Corollaries of this result are mod (p) analogues of 7.7 and 7.6. 7.7 and 7.6 themselves are corollaries of

7.10. THEOREM. *Let $A = E(1)_0^\wedge E(1)$. As coalgebras over Z_p, $A \simeq \mathrm{Hom}_c(\Lambda, Z_p)$.*

Before we can prove 7.10, we need

7.11. LEMMA.

(a) *A is a Hopf algebra over Z_p which is free as a Z_p-module.*

(b) *There is a short exact sequence of A-comodules*

(7.12)
$$0 \to Z_p \xrightarrow{\eta} A \xrightarrow{\rho} A \to 0$$

(c) $A = \lim_{\overrightarrow{i}} \ker \rho^i$.

Proof. (a) Since $E(1)^\wedge_* E(1)$ is a Hopf algebroid over $E(1)^\wedge_*$, A is a Hopf algebroid over $E(1)^\wedge_0 = Z_p$ and hence a Hopf algebra. To see that it is free over Z_p, observe that $BP_*BP \otimes_{BP_*} E(1)_* = BP_*E(1)$ which is a summand of the flat BP_* module $BP_*v_n^{-1}BP = \lim_{\overrightarrow{\eta_R v_n}} BP_*BP$ and is therefore torsion free. Hence $E(1)_*/(p^i) \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(1)_*$ is a free $Z/(p^i)$-module for each i and A is free over Z_p.

(b) From 7.9, we get a sequence

(7.13)
$$0 \to \mathbf{F}_p \xrightarrow{\eta} K(1)_0K(1) \xrightarrow{\rho} K(1)_0K(1) \to 0$$

where ρ is dual to multiplication by t.

Let $C = \operatorname{coker} \rho$ in 7.12. 7.13 is the mod (p) form of 7.12 so we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "p"] & C \arrow[r] \arrow[d, dashed, "\tilde{i}"] & C/p \arrow[r] \arrow[d, "i"', "\simeq"] & 0 \\
0 \arrow[r] & A \arrow[r, "p"] & A \arrow[r] & K(1)_0K(1) \arrow[r] & 0
\end{tikzcd}$$

If we can lift i to an A-comodule map $\tilde{i}$, a form of Nakayama's Lemma will imply that $\tilde{i}$ is a comodule isomorphism. The obstruction to constructing $\tilde{i}$ lies in the group $\operatorname{Ext}^1_A(C, A) = \operatorname{Ext}^1_{Z_p}(C, Z_p)$ which is trivial since C is a free Z_p-module.

(c) The mod (p) analogue of this follows from 7.9, so the statement itself follows from Nakayama's Lemma. □

Proof of 7.10. Let $A^* = \operatorname{Hom}_{Z_p}(A, Z_p)$. Let $x \in A^*$ be the composite $A \xrightarrow{\rho} A \xrightarrow{\epsilon} Z_p$ where ϵ is the augmentation. Then $\operatorname{Hom}_{Z_p}(\ker \rho^i, Z_p) = Z_p[x]/(x^i)$ as an algebra, so by 6.11(c)

$$\begin{aligned}
A^* &= \operatorname{Hom}(\varinjlim \ker \rho^i, Z_p) \\
&= \varprojlim \operatorname{Hom}(\ker \rho^i, Z_p) \\
&= Z_p[[x]] \simeq \Lambda. \qquad \square
\end{aligned}$$

7.14. COROLLARY. *Every comodule in* **TE** *can be constructed as follows. Let* N_* *be a graded torsion* Λ*-module with* $N_i \neq 0$ *only if* $0 \leq i < 2p - 2$. *Make* $N \otimes_{Z_p} E(1)^{\wedge}_{*}E(1)$ *into a comodule by setting* $\psi(n \otimes e) = n \otimes \Delta(e)$ *for* $n \in N$ *and* $e \in E(1)^{\wedge}_{*}E(1)$. *Then let* M *be the kernel in*

$$0 \to M \to N \otimes E(1)_* E(1) \xrightarrow{\mu} N \otimes E(1)^{\wedge}_{*}E(1) \to 0 \tag{7.15}$$

where $\mu(n \otimes e) = n \otimes \rho(e) - tn \otimes e$ *and the tensor products are over* Z_p.

Moreover, $M \simeq N \otimes E(1)_*$ as $E(1)_*$-modules.

Proof. Let $0 \leq i < 2p - 2$.

In dimension i 7.15 becomes

$$0 \to M_i \to N_i \otimes A \xrightarrow{\mu} N_i \otimes A \to 0 \tag{7.16}$$

It follows from 7.10 that A is a Λ-module isomorphic to Φ, the cokernel in $0 \to \Lambda \to t^{-1}\Lambda \to \Phi \to 0$ so 7.16 can be rewritten as

$$0 \to M_i \to N_i \otimes_{Z_p} \Phi \xrightarrow{\mu} N_i \otimes_{Z_p} \Phi \to 0, \tag{7.17}$$

a short exact sequence of torsion Λ-modules with $\mu(n \otimes t^{-j}) = n \otimes t^{1-j} - tn \otimes t^j$. It follows from 7.5 that any torsion Λ module M_i can arise in this manner, so 7.6 implies that M can be any torsion $E(1)_*E(1)$-comodule.

It is evident that $M_i \cong N_i$ as Z_p-modules for $0 \leq i < 2p - 2$, so $M \cong N \otimes_{Z_p} E(1)^{\wedge}_{*}$ as $E(1)_*$-modules. □

8. Localization with respect to K-theory. The main object of this section is to prove the Smash Product Theorem (8.1), identifying the functor L_1, localization with respect to p-local complex K-theory, with smashing with L_1S^0, which is described in 8.10 and 8.15.

8.1. SMASH PRODUCT THEOREM. *For* $p > 2$, $L_1X = X \wedge L_1S^0$.

We will describe L_1S^0 below (8.10). We will need several lemmas in order to prove 8.1. First we derive some corollaries.

From now on in this section assume $p > 2$ *unless otherwise indicated.*

8.2. COROLLARY. *The functor* L_1 *commutes with direct limits and the category* $L_1\mathbf{S}$ *is closed under direct limits.* □

8.3. COROLLARY. $BP \wedge L_1X = X \wedge L_1BP$. □

8.4. Corollary. *If* $BP_*X \otimes \mathbf{Q} = 0$ (*i.e. if* $E(0)_*X = 0$) *then* $BP_*L_1X = v_1^{-1}BP_*X$.

Proof. We have a cofibration $L_1BP \to L_0BP \to M_1BP$ and by assumption $X \wedge L_0BP = pt$, so $BP \wedge L_1X = X \wedge L_1BP = X \wedge \Sigma^{-1}M_1BP$ and $\pi_*(\Sigma^{-1}M_1BP \wedge X) = v_1^{-1}BP_*X$. □

Now we proceed to prove 8.1.

8.5. *Definition.* Let $M(p^i)$ denote the Moore spectrum for $Z/(p^i)$, i.e. the cofibre in

$$S^0 \xrightarrow{p^i} S^0 \to M(p^i),$$

and let $M(p^\infty) = \varinjlim M(p^i)$.

8.6. Lemma. $M_1E(1) = E(1) \wedge M(p^\infty)$.

Proof. From 5.7, we have a cofibration $E(1) \to L_0E(1) \to N_1E(1)$. Since $L_0E(1) = E(1) \wedge L_0S$, $N_1E(1) = E(1) \wedge M(p^\infty)$. By 1.17(a) then $M_1E(1) = L_1N_1E(1) = N_1E(1)$. □

8.7. Lemma. *For* $p > 2$ *there is a cofibration*

$$M_1S \to M_1E(1) \xrightarrow{r} M_1E(1) \tag{8.8}$$

Proof. The map r corresponds to the map ρ of 7.11(b). Its fibre is $E(1)_*$-local by 1.6 and has the same $E(1)$-homology as $M(p^\infty)$. Moreover, the composite $M(p^\infty) \to M(p^\infty) \wedge E(1) = M_1E(1) \xrightarrow{r} M_1E(1)$ is trivial so there is an $E(1)_*$-equivalence from $M(p^\infty)$ to the fibre of r. □

8.9. Lemma. *For any spectra* X, Y, $L_1X \wedge C_1Y = pt.$ *for* $p > 2$.

Proof. $C_1Y \wedge E(0) = pt.$, and $L_0X = E(0) \wedge X$, so the cofibration $L_1X \to L_0X \to M_1X$ gives $L_1X \wedge C_1Y = \Sigma M_1X \wedge C_1Y$. If we smash M_1X with 8.8, we get

$$M_1X \to M_1X \wedge E(1) \to M_1X \wedge E(1)$$

so $M_1X \wedge C_1Y = pt.$ since $C_1Y \wedge E(1) = pt.$ □

Proof of 8.1. By 8.9 the maps $L_1X = L_1X \wedge S^0 \to L_1X \wedge L_1S^0$ and $X \wedge L_1S^0 \to L_1X \wedge L_1S^0$ are both equivalences. □

Next we investigate L_1S^0, the K-theoretic localization of the sphere. I understand that some results on this topic were obtained earlier by Frank Adams and David Baird. (see [3] I).

8.10. THEOREM. *For* $p > 2$

(a) L_1S^0 *is a ring spectrum with* $L_1S^0 \wedge L_1S^0 = L_1S^0$

(b)

$$\pi_i L_1 S^0 = \begin{cases} Z_{(p)} & \text{for} \quad i = 0 \\ Q/Z_{(p)} & \text{for} \quad i = -2 \\ Z/(p^j) & \text{for} \quad i = sp^k q - 1 \quad \text{with} \quad p \nmid s \\ & \qquad q = 2p - 2, \quad \text{and} \quad i \neq -1. \\ 0 & \text{otherwise} \end{cases}$$

(c) *The positive dimensional summand of* $\pi_* L_1 S^0$ *is the isomorphic image of the subgroup of* $\pi_* S^0$ *detected by* $\mathrm{Ext}^1_{BP_*BP}(BP_*, BP_*)$ *in the Adams-Novikov spectral sequence* (*see section* 5 *of* [41] *or section* 4 *of* [34]).

(d) *The multiplication map* $\pi_i L_1 S^0 \otimes \pi_{-2-i} L_1 S^0 \to \pi_{-2} L_1 S^0$ *is injective.*

Proof. (a) By 8.1, the map $S^0 \wedge L_1S^0 \to L_1S^0 \wedge L_1S^0$ is an equivalence.

(b) We have the fibration $L_1S^0 \to L_0S^0 \xrightarrow{j} M_1S^0$ and we know $L_0S^0 = E(0)$, so we need to compute $\pi_* M_1 S^0$. From 8.6, we see that this amounts to computing $\mathrm{Ext}_{E(1)_*E(1)}(E(1)_*, E(1)_*/(p^\infty))$. By [35] this is isomorphic to $\mathrm{Ext}_{BP_*BP}(BP_*, M^1)$ which was computed in section 4 of [34]. The result is that $\pi_0 M_1 S^0 = \pi_{-1} M_1 S^0 = \mathbf{Q}/Z_{(p)}$ and $\pi_{jq} M_1 S^0$ is the stated value of $\pi_{jq-1} L_1 S^0$ for $j \neq 0$. It is easy to see that $\pi_0(j)$ is the standard map $\mathbf{Q} \to \mathbf{Q}/Z_{(p)}$, which gives the indicated value of $\pi_* L_1 S^0$.

(c) and (d). It is evident from the above discussion that $\pi_* L_1 S^0$ is in effect computed by the first two columns of the chromatic spectral sequence (see section 8 of [41] or section 3 of [33]) for the Novikov E_2-term $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$. Hence (c) is obvious and (d) is an easy computation for anyone familiar with [34]. □

8.11. *Remark.* The methods used in the proof of 8.10 could be used to compute $\mathrm{Ext}_{E(1)_*E(1)}(E(1)_*, E(1)_*)$. One finds that $\mathrm{Ext}^0 = Z_{(p)}$ and $\mathrm{Ext}^2 = Q/Z_{(p)}$, both concentrated in dimension zero. Ext^1 gives the remaining homotopy groups of 8.10(b) and $\mathrm{Ext}^i = 0$ for $i \geq 3$. This accounts for the theorem stated in [3] I.

8.12. *Remark.* With similar techniques, one can compute $\pi_* L_{K(1)} S^0$, where $K(1)_* = E(1)_*/(p)$ and $L_{K(1)}$ is localization with respect to mod (p) K-theory. In this case, one finds that $\pi_0 = \pi_{-1} = Z_p$, $\pi_{-2} = 0$ and the remaining groups are as in 8.10(b). The behavior of π_{-1} and π_{-2} can best be understood by taking the local arithmetic square

$$\begin{array}{ccc} Z_{(p)} & \longrightarrow & Z_p \\ \downarrow & & \downarrow \\ \mathbf{Q} & \longrightarrow & \mathbf{Q}_p \end{array} \tag{8.13}$$

where $\mathbf{Q}_p$ is the field of p-adic numbers, and considering the fibre square of localizations of S^0 with respect to K-theory with coefficients in the four rings in 8.13. For $\mathbf{Q}_p$, we get $\pi_0 = \pi_{-1} = \mathbf{Q}_p$ and $\pi_i = 0$ for $i \neq 0, -1$.

We now sketch the situation for $p = 2$ without giving detailed proofs. The Smash Product Theorem (8.1) is true for $p = 2$ and we will state an analogue of 8.10. The main ingredient of the proof of 8.1 is 8.7, which is false as stated for $p = 2$. However, we get a true statement if we replace $E(1)$ by $KO_{(2)}$, where KO is the spectrum representing real K-theory. Then to proceed further, one needs

8.14. THEOREM. *Let K and KO be the spectra representing complex and real K-theory respectively. Then $K \wedge X = pt.$ iff $KO \wedge X = pt$ so (by 1.8) the functors L_K and L_{KO} are the same and $\langle K \rangle = \langle KO \rangle$.*

Proof. We use the fact ([4] p. 206) that $K = KO \wedge CP^2$, i.e. there is a cofibration

$$\Sigma KO \xrightarrow{KO \wedge \eta} KO \longrightarrow K$$

where η is the generator of $\pi_1 S^0$. Hence $KO \wedge X = pt.$ implies $K \wedge X = pt.$ Conversely, if $K \wedge X = pt.$, then η induces an automorphism of $KO_* X$. But η is nilpotent (in fact $\eta^4 = 0$), so this implies $KO_* X = 0$. □

To compute $\pi_* M_1 S^0$ and thereby $\pi_* L_1 S^0$, one must compute $\mathrm{Ext}_{BP_*BP}(BP_*, M^1)$ for $p = 2$, which was done in section 4 of [33]. The

Novikov spectral sequence for $\pi_* L_1 S^0$ at $p = 2$ does converge but it has a pattern of differentials and nontrivial group extensions closely related to that described in section 5 of [41]. The result of these calculations is

8.15. THEOREM. *For $p = 2$*

(a) *L_1S^0 is a ring spectrum with $L_1S^0 \wedge L_1S^0 = L_1S^0$.*
(b)

$$\pi_i L_1 S^0 = \begin{cases} Q/Z_{(2)} & \text{for } i = -2 \\ Z_{(2)} \oplus Z/(2) & \text{for } i = 0 \\ Z_{(2)}/(2s) & \text{for } i = 8s-1, \quad s \neq 0 \\ Z/(2) & \text{for } i = 8s, \quad s \neq 0 \\ Z/(2) \oplus Z/(2) & \text{for } i = 8s+1 \\ Z/(2) & \text{for } i = 8s+2 \\ Z/(8) & \text{for } i = 8s+3 \\ 0 & \text{otherwise} \end{cases}$$

(c) *The 'first order part' of $\pi_* S^0$ (i.e.* Im *J and related elements; see section* 5 *of* [41]*) maps injectively into $\pi_* L_1 S^0$. The map $\pi_* S^0 \to \pi_* L_1 S^0$ has a cokernel of $Z/(2)$ in dimensions* 1 *and* 2.

(d) *Let ρ_s, α_{4s+1} and $\alpha_{4s+2/2}$ denote the generators of π_{8s-1} $(s \neq 0)$, π_{8s+1} and π_{8s+3} respectively. Then $\alpha_1^2 \rho_s \neq 0$ and $\alpha_1^2 \alpha_{4s+1} = 4\alpha_{4s+2/2} \neq 0$. There are relations $\alpha_{4s+1}\rho_t = \alpha_1 \rho_{s+t}$, $\alpha_{4s+1}\alpha_{4t+1} = \alpha_1 \alpha_{4s+4t+1}$, $\alpha_{4s+1}\alpha_{4t+2/2} = 0$, $\rho_s \alpha_{4t+2/2} = 0$. The map $\pi_{8s-1} \otimes \pi_{-8s-1} \to \pi_{-2}$ is injective and $\pi_{8s+3} \otimes \pi_{-8s-5} \to \pi_{-2}$ has kernel $Z/2$. The element of order* 2 *in π_0 is $\rho_t \alpha_{1-4t}$ $(t \neq 0)$ and π_1 is generated by α_1 and $\alpha_1 \alpha_{-3} \rho_1$.* □

We conclude this section with an amusing example, namely p-adic suspensions of $L_1 M(p^\infty)$. We have $E(1)_* M(p^\infty) = E(1)_*/(p^\infty)$. Under the equivalence of 7.6, this comodule corresponds to the Λ-module $\mathbf{Q}_p/Z_p$ with $t(x) = 0$. It is not hard to show that $E(1)_* \Sigma^{kq} M(p^\infty)$ corresponds to $\mathbf{Q}_p/Z_p$ with $t(x) = (\gamma^k - 1)x$ where $\gamma \in Z_p^\times$ is the chosen generator. It is also easy to construct an $E(1)_*$-local spectrum X_k with $E(1)_* X_k = E(1)_*/(p^\infty)$ and such that the corresponding Λ-module is as before, but now k need not be an integer in Z. We can think of X_k as the qk-th suspension of $L_1M(p^\infty)$ for $k \in Z_p$. However, if $k \notin Z$, $[L_0S^0, X_k] = 0$ so there is no corresponding qk-th suspension L_1S^0.

Section 9. $\boldsymbol{L_1RP^\infty}$ **and** $\boldsymbol{L_1CP^\infty}$**.** In this section, we will describe the K-theoretic localizations of RP^∞ and CP^∞ (regarded as suspension spectra, not as spaces). We include this material primarily in hopes of stimulating other such calculations. The proofs are very computational in nature and will probably be of interest only to those who want to apply those techniques to other spectra.

9.1. THEOREM. $L_1RP^\infty = \Sigma^{-1}M_1S^0$.

Proof. We use the map $f: RP^\infty \to S^0$ considered by Kahn-Priddy [24]. Since $E(0)_*RP^\infty = E(0)^*RP^\infty = 0$, f lifts uniquely to a map $\tilde{f}: RP^\infty \to C_0S^0 = \Sigma^{-1}M(2^\infty)$. It suffices to show that $\tilde{f}$ is $K(1)_*$ equivalence (where $K(1)_*(\cdot)$ is $(E(1)/2)_*(\cdot))$ and hence an $E(1)_*$-equivalence. Since $C_0S^0 \to S^0$ is a $K(1)_*$-equivalence, it suffices to show f is a $K(1)_*$-equivalence.

$K(1)_*RP^\infty$ was computed (certainly not for the first time) in [44], where we showed that it was a one-dimensional $K(1)_*$-module concentrated in even dimensions such that the skeletal inclusion $\Sigma M(2) \to RP^\infty$ induces a surjection in $K(1)_*$-homology.

We also know that the composite $S^1 \to RP \xrightarrow{f} S^0$ is η, the generator of π_1S^0. From this fact, it is elementary to show that the composite

$$\Sigma M(2) \to RP^\infty \xrightarrow{f} S^0$$

induces an isomorphism in the even dimensional part of $K(1)_*(\cdot)$. □

We now describe L_1CP^∞. Our main result is

9.2. THEOREM. *Let bu be the spectrum representing connective complex K-theory, let $f: CP^\infty \to \Sigma^2 bu$ be the map induced by the standard map of spaces $CP^\infty \to BU$, and let F denote the fiber of f. Then $L_0(f)$ is an equivalence and L_1F is a wedge of odd dimensional suspensions of $M_1E(1)$* (8.9). *L_1bu will be described below* (9.21). □

As a step toward proving 9.2 we will describe M_1CP^∞.

9.3. THEOREM. (a) *M_1CP^∞ is a wedge of even dimensional suspensions of $M_1E(1)$'s.*

(b) $\mathrm{Ext}^0_{K(1)_*K(1)}(K(1)_*, K(1)_*CP^\infty)$ *has basis*

$$\left\{b^j_{p^n} \prod_{0 \le i < n} (b^{p-1}_{p^i} - v_1^{p^i}) : 0 < j < p, n \ge 0\right\}.$$

(*See* 9.6(a) *for the definition of* b_{p^i}.)

9.4. LEMMA. *If* $K(1)_*X$ *is concentrated in even dimensions and* $\mathrm{Ext}^i_{K(1)_*K(1)}(K(1)_*, K(1)_*X) = 0$ *for* $i > 0$, *then* M_1X *is equivalent to a wedge of odd dimensional suspensions of* $M_1E(1)$.

Proof. The map $C_0X \to X$ is as $K(1)_*$-equivalence and $E(1)_*C_0X$ can be computed from $K(1)_*C_0X$ via a Bockstein spectral sequence. This spectral sequence collapses because $K(1)_*C_0X$ is concentrated in even dimensions. Since $E(1)_*C_0X$ is all torsion, it must be divisible.

Hence there is a short exact sequence

$$0 \to K(1)_*C_0X \to E(1)_*C_0X \xrightarrow{p} E(1)_*C_0X \to 0$$

and there is a Bockstein spectral sequence going from $\mathrm{Ext}_{E(1)_*E(1)}(E(1)_*, K(1)_*X) = \mathrm{Ext}_{K(1)_*K(1)}(K(1)_*, K(1)_*X)$ to $\mathrm{Ext}_{E(1)_*E(1)}(E(1)_*, E(1)_*C_0X)$. Since the input is concentrated in degree zero, the spectral sequence collapses and $\mathrm{Ext}^i_{E(1)_*E(1)}(E(1)_*, E(1)_*C_0X) = 0$ for $i > 0$.

We can compute $[C_0X, M_1E(1)] = [\Sigma^{-1}M_1X, M_1E(1)]$ with the Adams Universal Coefficient Theorem ([4] III 13.6) which says it can be computed with a spectral sequence whose E_2-term is $\mathrm{Ext}^{i,*}_{E(1)_*}(E(1)_*C_0X, \pi_*M_1E(1))$. Since both variables are direct sums of $E(1)_* \otimes \mathbf{Q}/Z$, this Ext vanishes for $i > 0$ and we have

$$[\Sigma^{-1}M_1X, M_1E(1)] = \mathrm{Hom}_{E(1)_*}(E(1)_*C_0X, E(1)_*/(p^\infty))$$

and there is no obstruction to constructing the desired equivalence. $\square$

We shall need the following facts about $K(1)_*K(1) = K(1)_* \otimes_{BP_*} BP_*BP \otimes K(1)_*$, which are proved in [40] and [42].

9.5. THEOREM. (a) *As an algebra* $K(1)_*K(1) = K(1)_*[t_i : i > 0] /(t_i^p - v_1^{(p^i-1)}t_i)$.

(b) *For* $p > 2$, $\mathrm{Ext}_{K(1)_*K(1)}(K(1), K(1)_*) = E(h_{1,0})$ *where* $E(\cdot)$ *denotes an exterior algebra over* $K(1)_*$ *and* $h_{1,0} \in \mathrm{Ext}^1$ *is represented by* t_1.

(c) *For* $p = 2$, $\mathrm{Ext}_{K(1)_*K(1)}(K(1)_*, K(1)_*) = E(h_{2,0}) \otimes P(h_{1,0})$ *where* $P(\cdot)$ *denotes a polynomial algebra over* $K(1)_*$ *and* $h_{2,0} \in \mathrm{Ext}^1$ *is represented by* t_2. □

The following facts about $K(1)_* CP^\infty$ can be found in [43] and [44].

9.6. THEOREM. (a) $K(1)_* CP^\infty$ *has basis* $\{b_i : i \geq 0\}$ *with* $b_i \in K(1)_{2i} CP^\infty$.

(b) *The coaction* $\psi(b) = b(c(t^F))$ *where* $b = \Sigma_{i\geq 0} b_i$, $b(x) = \Sigma_{i \geq 0} x^i \otimes b_i$, $t^F = \Sigma^F_{i\geq 0} t_i \in K(1)_* K(1)$ (*where* Σ^F *denotes the sum in the sense of the formal group law, and* $t_0 = 0$), *and* c *is the canonical anti-automorphism of* $K(1)_*K(1)$. (*This formula comes from an identical formula for the coaction of* BP_*BP *on* BP_*CP^∞. *In that case, the formula gives a finite sum in each dimension, and the formula for the* $K(1)_*K(1)$ *coaction is to be interpreted in the same way.*) *In particular* $\psi b_{p^n} = 1 \otimes b_{p^n} + v_1^{p^{n-1}} t_1 \otimes b_{p^{n-1}}$ *modulo the ideal generated by all* b_i *with* $p^{n-1} \nmid i$.

(c) *In the multiplication on* $K(1)_* CP^\infty$ *induced by the H-space structure on* CP^∞, $b_i^p = v_1^i b_i$ *and* $b_i b_j \equiv \binom{i+j}{i} b_{i+j}$ *modulo lower b's. Hence if we write* $n = \Sigma\, a_i p^i$ *with* $0 \leq a_k < p$ *we have* (*up to multiplication by a nonzero scalar*) $\Pi_i\, b_{p^i}^{a_i} \equiv b_n$ *modulo lower b's.* □

Proof of 9.3. Certainly $K(1)_* CP^\infty = BP_* CP^\infty \otimes_{BP_*} K(1)_*$ is concentrated in even dimensions, so by 9.4, it suffices to show that $\mathrm{Ext}^i_{K(1)_*K(1)}(K(1)_*, K(1)_*CP^\infty) = 0$ for $i > 0$. This Ext group is the target of a first quadrant homology spectral sequence, obtained by filtering CP^∞ by skeleta, with

$$E^2_{s,t,*} = H_s CP^\infty \otimes \mathrm{Ext}^{t,*}_{K(1)_*K(1)}(K(1)_*, K(1)_*). \tag{9.7}$$

$\mathrm{Ext}_{K(1)_*K(1)}(K(1)_*, K(1)_*)$ is described in 9.5 and the differentials in the spectral sequence are determined by the comodule structure of $K(1)_* CP^\infty$, which is described in 9.6.

We now analyze the spectral sequence for $p > 2$. In order to avoid the nuisance of having to keep track of powers of v_1, we pass to the corresponding $Z/(q)$-graded object by setting $v_1 = 1$. Then we claim

$$E^2 = E^q \quad \text{and} \quad d^q b_{i+p-1} = \pm i h_{1,0} b_i. \tag{9.8}$$

$$E^{2p^n-1} = E^{2p^{n+1}-2} \tag{9.9$_n$}$$

with basis

$$\{b_{p^k j-1} : 0 < j < p-1,\ 0 \le k < n\} \cup \{b_{ip^n} h_{1,0},\ b_{ip^n-1} : i > 0\}$$

$$(9.10_n) \qquad d^{2p^{n+1}-2} b_{ip^n+p^{n+1}-1} = \pm i h_{1,0} b_{ip^n} \quad \text{for} \quad n > 0.$$

Together these imply that E^∞ has basis $\{b_{jp^n-1} : 0 \le n,\ 2 \le j \le p\}$. Hence $\mathrm{Ext}^1 = 0$ and (a) follows. An easy calculation based on 9.6 shows that the elements listed in (b) are in Ext^0, and the calculation of E^∞ shows that they span Ext^0.

Now 9.8 follows easily from 9.6(b), and 9.9_1 follows by dimensional reasons, i.e. the structure of E^{2p-1} is such that no nontrivial differentials d^r can occur for $r < 2p^2 - 2$. Similarly, 9.9_{n+1} follows from 9.9_n and 9.10_n, so it suffices to prove the latter.

Let $x_n = \Pi_{0 \le i < n} (b_{p^i}^{p-1} - 1)$, so $b_i x_n = 0$ if $p^n \nmid i$ by 9.6(c). The spectral sequence element $b_{ip^n+p^{n+1}-1}$ is represented in $K(1)_* CP^\infty$ by $b_{(i+p-1)p^n} x_n$ by 9.6(c). To compute the desired differentials it suffices to compute the coaction in $K(1)_*(CP^\infty / CP^{ip^n-1}) = K(1)_* CP^\infty_{ip^n}$. We let $\bar\psi(u) = \psi(u) - 1 \otimes u$. Then we have

$$(9.11) \qquad \bar\psi b_{(i+p-1)p^n} x_n = \pm i t_1 \otimes b_{ip^n} x_n \in K(1)_* K(1) \otimes K(1)_* CP^\infty_{ip^n}.$$

We can write

$$(9.12) \qquad x_n = \sum_{0 < t \le n} (-1)^{1+t} b_{p^{n-t}}^{p-1} x_{n-t} + (-1)^n \quad \text{where} \quad x_0 = 1.$$

From 9.6, we can also deduce that

$$(9.13) \qquad \bar\psi b_{ip^n+p^{n+1-t}} b_{p^{n-t}}^{p-2} x_{n-t} = t_1 \otimes b_{ip^n} b_{p^{n-t}}^{p-1} x_{n-t} \in K(1)_* K(1) \otimes K(1)_* CP^\infty_{ip^n}.$$

for $0 < t \le n$.

Combining 9.11, 9.12 and 9.13, we get

$$(9.14) \qquad \bar\psi\Big(b_{(i+p-1)p^n} x_n \pm i \sum_{0 < t \le n} (-1)^t b_{ip^n+p^{n-t+1}} b_{p^{n-t}}^{p-2} x_n\Big) = \pm i t_1 \otimes b_{ip^n} \in K(1)_* K(1) \otimes K(1)_* CP^\infty_{ip^n}.$$

Then 9.9 and 9.10 follow from 9.14.

We now treat the case $p = 2$. We claim

(9.15) $$d^2 b_{2i+2} = h_{1,0} b_{2i+1} \quad \text{for} \quad i \geq 0,$$

and

(9.16) $$E^3 = E^8 \quad \text{with basis} \quad \{b_{2i+1}, b_{2i+1} h_{2,0} : i \geq 0\},$$

(9.17$_n$) $$\text{for} \quad n \geq 0,\, E^{1+2^{n+2}} = E^{2^{n+3}}$$

with basis

$$\{b_{2^k-1} : 0 < k \leq n\} \cup \{b_{2^{n+1}i+2^{n+1}-1}, b_{2^{n+1}i+2^{n+1}-1} h_{2,0} : i \geq 0\},$$

(9.18$_n$) $$\text{for } i, n \geq 0,\, d^{2^{n+3}} b_{2^{n+2}i+3\cdot 2^{n+1}-1} = h_{2,0} b_{2^{n+2}i+2^{n+1}-1}$$

and

$$d^{2^{n+3}} b_{2^{n+2}i+2^{n+2}-1} = 0.$$

As in the odd primary case, the result follows from these four statements. 9.15 and the structure of E^3 follow from 9.6(b) and 9.5(c). In this case $h_{1,0}$ is a polynomial rather than an exterior generator, so $h_{1,0} b_{2i}$ is not a cycle. The spectral sequence is concentrated in even dimensions so $E^{2i-1} = E^{-2i}$.

In $K(1)_* K(1) \otimes K(1)_* CP_{4i}^\infty$ we have $\bar{\psi} b_{4i+4} = t_2 \otimes b_{4i+2} x_1 + i t_1 \otimes b_{4i}$ by direct calculation. It follows that

(9.19) $$\bar{\psi}(b_1 b_{4i+4} + i b_{4i+2}) = t_2 \otimes b_{4i+1}$$

in $K(1)_* K(1) \otimes K(1)_* CP_{4i+1}^\infty$, so $d^8 b_{4i+5} = h_{2,0} b_{4i+1}$. It also follows that $\bar{\psi} b_{4i} x_2 = 0$ in $K(1)_* K(1) \otimes K(1)_* CP_{4i+1}^\infty$, so b_{4i+3} is a cycle in E^8 and we have proved 9.18_0.

To prove 9.18_n for $n > 0$ we take 9.19 and multiply the subscripts on the left by 2^n. The right hand side will then have some terms involving some b_j's with $2^n \nmid j$. These can be eliminated by multiplying by the primitive element x_n, so we get

(9.20) $$\bar{\psi} x_n (b_{2^n} b_{2^n(4i+4)} + i b_{2^n(4i+2)}) = t_2 \otimes b_{2^n(4i+1)}$$

in $K(1)_*K(1) \otimes K(1)_* CP^{\infty}_{2^n(4i+1)}$. Then modifying the argument for 9.18_0 we get 9.18_n. $\square$

Now as a prelude to the proof of 9.2, we describe $L_1 bu$.

9.21. THEOREM. $L_0 bu = \vee_{i\geq 0} \Sigma^{2i} E(0)$,

$$M_1 bu = \vee_{0\leq i<p-1} \Sigma^{2i} M_1 E(1)$$

and the fibration $L_1 bu \to L_0 bu \to M_1 bu$ *is such that*

$$\pi_i L_1 bu = \begin{cases} Z_{(p)} & \text{if } i \geq 0 \quad \text{and} \quad i \text{ is even} \\ Q/Z_{(p)} & \text{if } i \leq -3 \quad \text{and} \quad i \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The homotopy type of $L_0 bu$ is obvious. There is a map $bu \to \vee_{0\leq i\leq p-1} \Sigma^{2i} E(1)$ and we claim that $f: N_1 bu \to \vee_{0\leq i\leq p-1} N_1 \Sigma^{2i} E(1)$ is an $E(1)_*$-equivalence. The p-component of the cofibre of f has torsion homotopy groups which are trivial in positive dimensions, so it is $E(1)_*$-acyclic by 4.8. $N_1 E(1)$ is $E(1)_*$-local so $M_1 bu$ is as described. The behavior of the map $L_0 bu \to M_1 bu$ in homotopy can be read off from the diagram

$$\begin{array}{ccccc} bu & \longrightarrow & L_0 bu & \longrightarrow & N_1 bu \\ \downarrow & & \downarrow & & \downarrow \\ \bigvee\limits_{0\leq i<p-1} \Sigma^{2i} E(1) & \to & L_0 \bigvee\limits_{0\leq i<p-1} \Sigma^{2i} E(1) & \to & N_1 \bigvee\limits_{0\leq i<p-1} \Sigma^{2i} E(1). \end{array} \quad \square$$

Proof of 9.2. First we show that f is a rational equivalence. Since $BU = \Omega SU$ by Bott periodicity, we have a map $g: \Sigma CP^\infty \to SU$, giving a diagram

$$\begin{array}{ccccc} \Sigma CP^{n-1} & \longrightarrow & \Sigma CP^n & \longrightarrow & S^{2n+1} \\ \downarrow & & \downarrow & & \| \\ SU(n) & \longrightarrow & SU(n+1) & \to & S^{2n+1} \end{array}$$

where the top row is a cofibre sequence and the bottom row is a fibre sequence. ΣCP^∞ and SU are rationally equivalent to a wedge and product of odd dimensional spheres respectively, and g is the obvious map between them. Hence stabilizing (i.e. suspending the source and delooping the target) gives a stable rational equivalence.

It follows that $L_1F = \Sigma^{-1}M_1F$. Both M_1CP^∞ and $M_1\Sigma^2bu$ are wedges of even suspensions of $M_1E(1)$, and we will show that M_1f is a retraction. It suffices to show that the map

$$M_1f_*: \operatorname{Ext}^0_{K(1)_*K(1)}(K(1)_*, K(1)_*CP^\infty) \to \operatorname{Ext}^0_{K(1)_*K(1)}(K(1)_*, K(1)_*\Sigma^2bu)$$

is onto. The latter group has a $K(1)_*$-basis with $(p-1)$ elements which can be chosen to span (over $\mathbf{F}_p$) the image of the Hurewicz map to $K(1)_*\Sigma^2bu$, which is isomorphic under suspension to the Hurewicz image in $K(1)_*BU$, which is easily seen to be spanned by the image from $K(1)_*CP^\infty$ of b_i^i for $0 < i \leq p-1$.

10. Some conjectures. In this section we get to the heart of the matter. In hopes of stimulating further work in this area we list some conjectures with possible lines of proof suggested by the results of this paper. We list them in what appears to be descending order of difficulty. The first concerns nilpotence and consists of three statements, one of which has Nishida's Theorem as a special case, which we believe are related or perhaps special cases of some more general statement. 10.2 enables one to construct generalizations of Toda's $V(n)$-spectra, while 10.4 and 10.5 concerns their localization properties. 10.6 asserts that $E(n)$ is smashing (1.28) and 10.7 determines the BP_*-homology of an $E(n)_*$-localization. We describe how the localization (5.8) and periodicity (5.9) conjectures of section 5 may follow from the above. Finally, 10.8 describes the Boolean algebra **BA** (1.21); although this is our last conjecture it is probably not the easiest. At the end of the section we summarize the relations between and partial proofs of these conjectures.

10.1. *Nilpotence conjecture.* Let W, X and Y be finite spectra.

(a) Any map $f: X \to \Sigma^kX$ with $MU_*(f) = 0$ is nilpotent, i.e. some iterate of f is inessential.

(b) Let $W \to X \to Y \xrightarrow{f} \Sigma W$ be a cofibre sequence with $MU_*(f) = 0$. Then $\langle X\rangle = \langle W\rangle \vee \langle Y\rangle$.

(c) (H. R. Miller) Let X be p-adically complete such that

$K(n)_*X \neq 0$ and $K(n-1)_*X = 0$ for some $n > 0$. Then there is a $K(n)_*$-equivalence $g : X \to \Sigma^k X$ with $H_*(f) = 0$, where H_* is ordinary mod(p) homology. Moreover if X is a ring spectrum, g can be chosen so that the cofibre of any of its iterates is also a ring spectrum. □

Note that Nishida's Theorem [39] which states that any positive dimensional element of $\pi_* S$ is nilpotent, is a special case of 10.1(a). 10.1(b) says the map f behaves (in terms of the Bousfield classes of the spectra involved) as if it were inessential. Bousfield ([11], 2.11) has shown that the same is true if f is smash nilpotent, i.e. if some smash power of it is inessential. However the f in 10.1(b) need not be smash nilpotent, e.g. it is not in the sequence $S^0 \xrightarrow{p} S^0 \to S^0/(p) \xrightarrow{f} S^1$.

10.1(a) and (b) are false for infinite spectra. Consider the cofibre sequence

$$S^0 \to MU \to \overline{MU} \xrightarrow{f} S^1$$

where the first map is the inclusion of the bottom cell. Then $MU_*(f) = 0$ but $\langle MU \rangle \neq \langle S^0 \rangle$ by the results of section 3 so the analogue of (b) fails here. Next we claim $\langle \overline{MU} \rangle = \langle S^0 \rangle$. Since $MU \wedge \overline{MU}$ is a wedge of suspensions of MU, $\langle \overline{MU} \rangle \geq \langle MU \rangle$. The above cofibration and 1.23 imply $\langle S^0 \rangle \leq \langle MU \rangle \vee \langle \overline{MU} \rangle$, so $\langle \overline{MU} \rangle = \langle S^0 \rangle$ as claimed. In particular $\langle \overline{MU} \rangle \in \mathbf{DL}$, so the above result of Bousfield (which applies to maps between spectra representing classes in **DL**) implies that f is not smash nilpotent. This fact enables us to construct a counterexample to the infinite analogue of 10.1(a) as follows. Let $X = \bigvee_{s \geq 0} \Sigma^{-s} \overline{MU}^{(s)}$ and let $g : X \to X$ be defined by $g = \bigvee_{s \geq 0} \Sigma^{-s} \overline{MU}^{(s)} \wedge f$. Then $MU_*(g) = 0$ but g is not nilpotent.

In 10.1(c) the assumption of p-adic completeness is included merely for simplicity. A finite spectrum is p-adically complete if its homotopy is all p-torsion. Any spectrum whose homotopy is all torsion (i.e. whose rational homotopy type is trivial) is equivalent to the wedge of its p-adic completions for various primes p. If X has nontrivial rational homotopy type then the obvious analogue of 10.1(c) is true; multiplication by p gives a rational equivalence which vanishes in mod(p) homology.

Also note that if X is not contractible, then for some n $K(n)_*X \neq 0$ by 4.5. Also by 2.11, $K(n-1)_*X = 0$ implies $K(i)_*X = 0$ for all $i < n$ and $K(n)_*X \neq 0$ implies $K(i)_*X \neq 0$ for all $i > n$.

One may ask why 10.1(c) is included here at all since it concerns the existence of maps which are *not* nilpotent. In a moment we will discuss a possible derivation of (c) from (a), but first consider the following. Let $\overline{BP}$ be the cofibre of the inclusion of the bottom cell in BP. Consider the composite

$$S^0 \longrightarrow BP \xrightarrow{v_n^k} \Sigma^{-d}BP \longrightarrow \Sigma^{-d}\overline{BP}$$

where $d = 2k(p^n - 1)$. It can be shown that if X is as in 10.1(c) and $k = p^i$ for i sufficiently large, then smashing this map with X gives a map which is trivial in BP_*-homology. Now suppose we know that for $k = p^{i+j}$ for some j, the resulting map is null homotopic. (This is a sort of nilpotence condition). Then we can lift the map $X \to \Sigma^{-d}BP \wedge X$ to $\Sigma^{-d}X$ and this lifting is the desired g.

Now we give a possible derivation of 10.1(c) from 10.1(a). Consider the Adams spectral sequence (based on ordinary mod(p) homology) for $[X, X]$. It will have a pairing based on composition and there will be a class $x \in E_2^{k,d+k}$ with $x^i \neq 0$ for all $i > 0$ and such that if it were a permanent cycle it would be represented by the map g. No power of x can be the target of a differential because the corresponding class in the BP_*-Adams spectral sequence for $[X, X]$ would have filtration zero.

The difficulty is that x and conceivably all of its powers could support nontrivial differentials. However, there are only finitely many multiplicative generators in $E_2^{s,t}$ with $t < s(2p^n - 1)$ (we assume the bottom cell or cells of X are in dimension 0), i.e. lying above the line where x and its powers lie. Any products of these generators which are permanent cycles will correspond to self-maps of X which are nilpotent in BP_* homology and therefore nilpotent by 10.1(a). From this fact it should be possible to deduce that for some finite r and j, $E_r^{s,t} = 0$ for $t < s(2p^n - 1) - j$, i.e. E_r has a vanishing line of slope $1/2(p^n - 1)$. Since $x \in E_2$, x^{p^i} survives at least to E_{2+i} and for i large enough any nontrivial differential supported by x^{p^i} would have its target above our vanishing line. Therefore this x^{p^i} is a permanent cycle corresponding to the desired $K(n)_*$-equivalence of X.

10.1(c) is useful for constructing complexes realizing cyclic BP_*-modules, specifically quotients of BP_* by invariant regular ideals. Such ideals are studied by Landweber [27] and Johnson-Yosimura [23]. Such complexes are known to be useful for constructing elements in π_*S; see [41] pp. 445–447, 451–458 for a discussion of this topic. At present such complexes are known to exist only for certain ideals with few (≤ 4) generators.

There are no known examples of nontrivial finite $K(n)_*$-acyclic spectra for large n. (If the ideal has n generators then the corresponding spectrum is $K(n-1)_*$ acyclic but not $K(n)_*$-acyclic.) Such spectra can be obtained by induction on n starting with S using 10.1(c); the cofibre of the $K(n)_*$-equivalence on the $K(n-1)_*$-acyclic complex is $K(n)_*$ acyclic. In this way we get

10.2. *Realizability conjecture.* Given an invariant regular ideal $I \subset BP_*$ with n generators, I contains a similar ideal J such that there is a finite ring spectrum X with $BP_*X = BP_*/J$ and this X supports a $K(n)_*$-equivalence as in 10.1(c). □

10.3. PROPOSITION. *If* 10.1(a) *and* 10.2 *are true, so is* 10.1(c).

Proof. Our strategy is to show that a finite $K(n-1)_*$-acyclic spectrum Y is a module over some X as in 10.2. Then the composite

$$Y \longrightarrow X \wedge Y \xrightarrow{g \wedge Y} X \wedge Y \longrightarrow Y$$

is a $K(n)_*$-equivalence f on Y if g is a $K(n)_*$-equivalence on X. Suppose inductively that Y is a module over a $K(n-2)_*$-acyclic X' given by 10.2 with $K(n-1)_*$-equivalence g'. Using g' as above we get a $K(n-1)_*$-equivalence f' on Y. Y is $K(n-1)_*$-acyclic and therefore $E(n-1)_*$-acyclic by 2.11 and 2.1. From [27] we know that some iterate of g' induces multiplication by a power of v_{n-1} in BP_*-homology, so $BP_*(g' \wedge Y)$ is nilpotent. Then by 10.1(a) $(g')^k \wedge Y$ is null homotopic for some k. This k can be chosen so that the cofibre X of $(g')^k$ is a ring spectrum. Then $X \wedge Y$ has $X' \wedge Y$ as a retract, so the X'-module structure on Y gives the desired X-module structure. □

Now we consider the Bousfield class (1.19) of a finite complex. For this we will need 10.1(b) as well as (a) and (c). We have

10.4. *Class invariance conjecture.* Let X be a finite spectrum. If π_*X is not all torsion then $\langle X\rangle = \langle S\rangle$. If π_*X is all p-torsion, then $\langle X\rangle$ depends only on p and the smallest n such that $K(n)_*X \neq 0$. □

The case when $\langle X\rangle = \langle S\rangle$ can be derived from 10.1(b) as follows. Using standard methods one constructs an $E(0)_*$- (i.e. rational) equivalence $W \to X$ from a wedge of spheres W. The fibre F of this map has torsion homotopy and MU_*-homology, so the map $F \to W$ is trivial in MU_*-homology, so 10.1(b) gives $\langle X\rangle = \langle W\rangle \vee \langle F\rangle = \langle S\rangle$.

One might prove 10.4 for p-adically complete X as follows. First it will be necessary to prove it for the ring spectra provided by 10.2. For these an argument similar to that of 2.1(g) may be possible after proving certain lemmas about invariant regular ideals. If I and J are realizable in the sense of 10.2, show that their intersection contains an invariant regular ideal K which is also realizable. Then show it is possible to interpolate between I and K as we interpolated in the proof of 2.1(g) between I_n and J.

Once 10.4 has been proved for spectra X given by 10.2, one shows as in the proof of 10.3 that any finite p-adically complete Y is a module over some X, so $\langle X\rangle \geq \langle Y\rangle$ by 1.24. Then one constructs a $K(n)_*$-equivalence $W \to Y$ where W is a wedge of suspensions of X. As above, the inclusion of the fibre F in W is trivial in MU_*-homology, so $\langle Y\rangle = \langle W\rangle \vee \langle F\rangle$ by 10.1(b), so $\langle Y\rangle = \langle X\rangle$.

Now we turn to the problem of identifying the Bousfield classes given by 10.4. Let X be a p-adically complete finite $K(n-1)_*$-acyclic spectrum with $K(n)_*X \neq 0$, let $g : X \to \Sigma^{-k}X$ be a $K(n)_*$-equivalence given by 10.1(c) let Y be its cofibre and $\hat{X} = \lim_{\overrightarrow{g}} \Sigma^{-ki}X$.

10.5. *Telescope conjecture.* Let $\hat{X}$ be as above. Then $\langle\hat{X}\rangle$ depends only on n and $\langle\hat{X}\rangle = \langle K(n)\rangle$. □

Before discussing a possible proof we give some consequences. Even without 10.5 we have $\langle X\rangle = \langle\hat{X}\rangle \vee \langle Y\rangle$ by 1.34. Hence 10.5 and 2.1(d) give $\langle S_{(p)}\rangle = \langle E(n)\rangle \vee \langle Y\rangle$. Y is $K(n)_*$-acyclic by construction and therefore $E(n)_*$-acyclic by 2.11, so 1.32 gives

10.6. *Smashing conjecture.* For each n and p, $E(n)$ is smashing (1.28), i.e. $L_nX = X \wedge L_nS$ and the category of $E(n)_*$-local spectra is closed under direct limits.

This conjecture gives $BP \wedge L_nX = BP \wedge X \wedge L_nS = X \wedge L_nBP$, i.e.

10.7. *Localization conjecture.* For any spectrum X, $BP \wedge L_nX = X \wedge L_nBP$. In particular if X is $E(n-1)_*$-acyclic, then $BP_*L_nX = v_n^{-1}BP_*X$. □

The two statements in 10.7 are actually equivalent. If $E(n-1)_*X = 0$ then $L_{n-1}X = pt.$, $C_{n-1}X = X$ and $L_nX = L_n\Sigma^{-n}N_nX = \Sigma^{-n}M_nX$ (5.10). We also can show $X \wedge L_{n-1}BP = pt.$ so $X \wedge BP = X \wedge C_{n-1}BP = X \wedge \Sigma^{-n}N_nBP$ and $v_n^{-1}BP_* = \pi_*X \wedge \Sigma^{-n}M_nBP$. Hence the second part of 10.7 is equivalent to

$$BP \wedge M_nX = X \wedge M_nBP$$

To show this equivalent to the first part of 10.7 we compare the cofibre sequences

$$BP \wedge N_n X \to BP \wedge M_n X \to BP \wedge N_{n+1} X$$

and

$$X \wedge N_n BP \to X \wedge M_n BP \to X \wedge N_{n+1} BP$$

If the first part of 10.7 is true, then $X \wedge C_n BP = BP \wedge C_n X$ and by 5.10 $X \wedge N_n BP = BP \wedge N_n X$, so $BP \wedge M_n X = X \wedge M_n BP$. If the second part is true, then $BP \wedge N_{n+1} X = X \wedge N_{n+1} BP$ by induction on n so $BP \wedge C_n X = X \wedge C_n BP$ by 5.10 and the first part follows.

The localization conjecture 5.8 is a special case of 10.7, and we have a program for deriving the periodicity conjecture 5.9 from the above statements. First note that if a finite p-adically complete spectrum X is $K(n-1)_*$-acyclic then $M_n X$ is the telescope $\hat{X}$ by 10.5, which is a suspension of $L_n X$, so the $K(n)_*$-equivalence on X given by 10.1(c) induces an equivalence on $M_n X$, so it is periodic. Now we wish to derive 5.9 for $K(n-i-1)_*$-acyclic spectra by induction on i. Consider the $K(n-i)_*$-equivalence $g: X \to \Sigma^k X$ given by 10.1(c). Let Y_j be the cofibre of g^j. Each Y_j is $K(n-i)_*$-acyclic so $M_n Y_j$ satisfies 5.9 by the inductive hypothesis. We have a cofibration $X \to \hat{X} \to \varinjlim Y_j$. Since $\langle \hat{X} \rangle = \langle K(n-1) \rangle$ (10.5) and $K(n) \wedge K(n-1) = pt.$, we have $M_n X = \Sigma^{-1} M_n \varinjlim Y_j$, M_n commutes with direct limits by 10.6, so we have shown that $M_n X$ is a direct limit of spectra which are periodic.

We still need to show the periodicity equivalences commute with the maps in the directed system. Suppose $X_\alpha \to X_\beta$ is such a map. Extending the argument of 10.3, show this is a map of E-module spectra, where E is a $K(n-1)_*$-acyclic finite p-adically complete ring spectrum given by 10.2. As in the proof of 10.3, the $K(n)_*$-equivalences of X_α and X_β can be derived from that on E. Applying M_n gives the periodicity equivalences and this construction commutes with the map.

Next we consider the structure of the Boolean algebra of spectra **BA** (1.21). 10.4 and 10.5 determine the Bousfield class of any finite spectrum and therefore give a subalgebra **FBA** of **BA** which we will describe below. We are tempted to conjecture that **FBA** is all of **BA**, but have not the slightest idea of a proof. Note that 1.32 asserts that an infinite wedge of finite spectra also gives a class in **BA**, but in general a Boolean algebra need not have infinite $\wedge$ or $\vee$ products. For example it follows from 10.4 and 10.5 that $\langle K(n) \rangle \in \mathbf{BA}$ for each n and p, but $\langle \vee_{i \geq 0} K(i) \rangle \notin \mathbf{BA}$ by 2.2.

10.8. *Boolean algebra conjecture.* (a) Let **FBA** $\subset$ **BA** be the Boolean subalgebra generated by finite spectra and their complements. Let $\mathbf{FBA}_{(p)} \subset$ **FBA** denote the subalgebra of p-local spectra and their complements in $\langle S_{(p)} \rangle$. $\mathbf{FBA}_{(p)}$ is the free (under finite union and intersection) Boolean algebra generated by the $\langle K(n) \rangle$ for $n \geq 0$ (2.1) and $\langle E(n) \rangle^c = \wedge_{i+0}^{n} \langle K(i) \rangle^c$ is represented by a finite spectrum. In other words $\mathbf{FBA}_{(p)}$ is isomorphic to the Boolean algebra of finite and cofinite sets of natural numbers, with $\langle K(n) \rangle$ corresponding to the set $\{n\}$.

(b) A class $\langle X \rangle \in$ **FBA** is determined by the value of $\langle X \rangle \wedge \langle S_{(p)} \rangle \in \mathbf{FBA}_{(p)}$ for all primes p. Any sequence of values may occur provided that if $\langle K(0) \rangle \leq \langle X \wedge S_{(p)} \rangle$ for some prime then the same holds for all primes. $\square$

We will show that (b) follows from (a), which is equivalent to showing that $\langle X \rangle \in$ **FBA** if $\langle X \wedge S_{(p)} \rangle \in \mathbf{FBA}_{(p)}$ for all p. We have $\langle X \rangle = \vee_p \langle X \wedge S_{(p)} \rangle$, so $\langle X \rangle^c = \wedge_p \langle X \wedge S_{(p)} \rangle^c$ and we need to show that this infinite smash product is defined. Let $\langle Y_p \rangle$ be the complement of $\langle X \wedge S_{(p)} \rangle$ in $\langle S_{(p)} \rangle$. Then we have $\langle X \wedge S_{(p)} \rangle^c = \langle Y_p \rangle \vee \vee_{q \neq p} \langle S/(q) \rangle$. We can use the distributivity law for $\vee$ and $\wedge$ to smash these together for all p and get $\vee_p \langle Y_p \rangle$.

The smashing and localization conjectures (10.6 and 10.7) both follow from the telescope conjecture 10.5. Preparatory to discussing a possible proof of 10.5 we have

10.9. THEOREM. 10.6 *is true for* $n < p - 1$.

Proof. We will show

$$\mathrm{Ext}^{s,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*X) = 0 \tag{10.10}$$

for $s > n^2 + n$ and X any finite spectrum. Then the result will follow from Bousfield's Convergence Theorem 1.18. One can show that for finite X, $E(n)_*X$ has a Landweber filtration (see [28]) i.e. a finite filtration in which each subquotient has the form $E(n)_*/I_k$ for $0 \leq k \leq n$, where $I_k = (p, v_1, \ldots, v_{k-1}) \subset E(n)_*$. Then a routine exact sequence argument shows that 10.10 follows from

$$\mathrm{Ext}^{s,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_k) = 0 \tag{10.11}$$

for $s > n^2 + N$ and $0 \leq k \leq n$. 10.11 for general k follows from the case $k = 0$. For $\mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, E(n)_*)$ one can set up a chromatic spectral sequence (see [41] section 8 or [34] section 3) in which the 0th through nth columns will be isomorphic to those in the chromatic spectral sequence for $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, and all columns to right of the nth will be trivial. The triviality of Ext^s for $s > n^2 + n$ will then follow from the Morava Vanishing Theorem (3.16 of [34] or 8.26 of [41]). □

Now consider the simplest case in which the above argument fails, i.e. $n = 1$, $p = 2$. Here it is known that the Ext group 10.10 for $X = S^0$ contains an element $\alpha_1 \in \mathrm{Ext}^{1,2}$ all of whose powers are nontrivial, so the $E(1)_*$-Adams E_2-term does not have the vanishing line required by 1.18. However it is known (Theorem 5.8 of [41]) that α_1^4 is killed by a d_3 in the spectral sequence. From this fact we can conclude that the E_4-term for any finite X has the requisite vanishing property, so 10.6 follows in this case.

More generally, Morava has shown that the ith column of the chromatic spectral sequence (used in the proof of 10.9) has finite cohomological dimension whenever i is not divisible by $p - 1$. When $(p - 1) \mid i$, he has shown that $\mathrm{Ext}_{BP_*BP}(BP_*, v_i^{-1}BP_*/I_i)$ is periodic in the cohomological sense, i.e. there is an element x (such as α_1 above) such that multiplication by it gives an isomorphism above some cohomological dimension. Conceivably this x or some power of it comes from an element in π_*S^0 and is therefore nilpotent by Nishida's theorem, i.e. some higher power of x (after it is fed into the chromatic spectral sequence) is killed by a differential. Then it would be possible to show that some E_r-term for $X = S^0$ has finite cohomological dimension and we can use 1.18 as in 10.9.

Now consider the telescope conjecture 10.5. We have a finite $K(n-1)_*$-acyclic complex X with a $K(n)_*$-equivalence $g: X \to \Sigma^{-d}X$ and $K(n)_*X \neq 0$. We have $\hat{X} = \lim_{\overrightarrow{g}} \Sigma^{-id}X$ and we wish to show $\langle \hat{X} \rangle = \langle K(n) \rangle$. Suppose we know that the BP_*-Adams spectral sequence for $\pi_*\hat{X}$ converges; we will come back to this point below. We have $BP_*\hat{X} = v_n^{-1}BP_*X$, and it has a finite Landweber filtration (see [28] 1.16) with all subquotients isomorphic to a suspension of $v_n^{-1}BP_*/I_n$. Hence the E_2-term $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*\hat{X})$ is related to $\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$ by certain long exact sequences. It follows that it is finitely generated as a module over $\mathrm{Ext}^0_{BP_*BP}(BP, v_n^{-1}BP/I)$ for a suitable invariant regular ideal I with n generators. Moreover using 10.2 this I can be chosen so there is a finite ring spectrum Y with $BP_*Y = BP_*/I$ and such that X and $\hat{X}$ are Y-module spectra. Using arguments similar to those sketched above, one might show that some E_r-term in the spectral sequence has finite cohomological dimension. This would mean that X can be constructed from $K(n)$ by a finite number of cofibrations, i.e. X has a finite Postnikov tower if we use $K(n)$ instead of $H/(p)$ as our basic building block. It follows that $\langle X\rangle \leq \langle K(n)\rangle$. We have $\langle K(n)\rangle \leq \langle X\rangle$ by 2.1(h) since $K(n)_*X \neq 0$.

To show $\langle K(n)\rangle \leq \langle\hat{X}\rangle$, first show $\langle\hat{X}\rangle = \langle\hat{Y}\rangle$ (where $\hat{Y}$ is the telescope associated with Y, the finite ring spectrum over which X is a module with $BP_*Y = BP_*/I$) using an argument similar to that given above for the class invariance conjecture 10.4. Then we have $\langle\hat{X}\rangle \geq \langle\hat{Y}\wedge BP\rangle = \langle v_n^{-1}BPI\rangle$ and $\langle v_n^{-1}BPI\rangle = \langle K(n)\rangle$ by 2.1. Hence $\langle\hat{X}\rangle = \langle K(n)\rangle$ as asserted in 10.5 if all goes well.

The above argument depends on the convergence of the BP_*-Adams spectral sequence for $\pi_*\hat{X}$. We know the spectral sequence for π_*X converges since X is finite, and we know $\pi_*\hat{X} = \lim_{\substack{\to \\ g_*}} \pi_*\Sigma^{-id}X = g_*^{-1}\pi_*X$. If the spectral sequence for $\pi_*\hat{X}$ fails to converge it is because there are some g_*-torsion free elements in π_*X not detected by the spectral sequence, i.e. elements u such that the elements $\pi_*(g^i)u$ are all nontrivial and have unbounded (with respect to i) BP_*-Adams filtration. We will refer to such a sequence of elements as an exotic family. In order to prove such exotic families cannot exist one might use the classical Adams spectral sequence for π_*X as follows.

At this point we must ask the reader to recall the discussion of a possible derivation of 10.1(c) from 10.1(a). We had an element x in the classical Adams spectral sequence for $[X, X]$ and we argued that some E_r-term of this spectral sequence must have a vanishing line parallel to the line through the origin on which x lies. The same must be true of the classical Adams spectral sequence for π_*X. In [32] H. R. Miller developed machinery helpful for computing the localization of the classical Adams E_2-term obtained by inverting x. In general this localization will be larger than the corresponding localization of the BP_*-Adams E_2-term. In [33] Miller discusses the problem of computing certain differentials in the classical spectral sequence which will reduce the localization to something comparable to that of the BP_*E_2-term. In particular he proves the convergence for the case $X = S^0 \cup_p e^1$, the mod(p) Moore spectrum for $p > 2$.

If an exotic family exists then the corresponding sequence of elements in the classical spectral sequence must contain an infinite subset of elements annihilated by composition with x (since the corresponding composition vanishes infinitely often in the BP_*-Adams spectral sequence), i.e. the family contains infinitely many elements whose composition with g has higher Adams filtration than one would expect. However, the parallel vanishing line in E_r precludes this possibility. Thus we conclude, modulo many gaps in the argument, that the BP_*-Adams spectral sequence for

$\pi_* \hat{X}$ converges and that the telescope conjecture is true. In any case we have

10.12. THEOREM. *The nilpotence conjecture* 10.1(c) *and the telescope, smashing and localization conjectures* (10.5, 10.6 *and* 10.7) *are true for* $n = 1$.

Proof. The proof of 10.1(c) in this case is due to J. F. Adams. The statement is that any finite spectrum X whose homotopy is all p-torsion has a self-map $g : X \to \Sigma^{-d} X$ $(d > 0)$ which is a $K(1)_*$-equivalence. Any such X is a module spectrum over the mod (p^m) Moore spectrum $Y_m = S^0 u_{p^m} e^1$ for some m. It suffices to construct a $K(1)_*$-equivalence $h : Y_m \to \Sigma^{-d} Y_m$, because we can define g to be the composite

$$X \longrightarrow Y_m \wedge X \xrightarrow{h \wedge X} \Sigma^{-d} Y_m \wedge X \longrightarrow \Sigma^{-d} X.$$

The map h was constructed for $p > 2$ by Adams in [1], Lemmas 12.4 and 12.5. For $p = 2$ one needs the methods of [1] along with the knowledge that the e-invariant detects a direct summand of π_{8k-1}; this follows either from the Adams Conjecture or the 'more delicate arguments' referred to on page 21 of [1]. These give elements x in $\pi_* S$ with order 2^i and e-invariant 2^{-i}. Adams' Lemma 12.5 requires in addition the triviality of the Toda bracket $\{2^i, x, 2^i\}$. For $i > 1$ this is automatic given that $2^i x = 0$; one can use the multiplication on the mod(2^i) Moore spectrum to construct the appropriate map. Then 12.5 gives the map h.

For the telescope conjecture, let X and Y_m be as above, let $\hat{X}$ and $\hat{Y}_m$ be the corresponding telescopes, and assume $K(1)_* X \neq 0$. Bousfield ([10], 4.2) has shown that $\langle \hat{Y}_1 \rangle = \langle K(1) \rangle$. For $\langle \hat{Y}_m \rangle$ we have cofibre sequences $\hat{Y}_1 \to \hat{Y}_m \to \hat{Y}_{m-1}$ so $\langle \hat{Y}_m \rangle \leq \langle \hat{Y}_1 \rangle \vee \langle \hat{Y}_{m-1} \rangle = \langle K(1) \rangle$ by induction on m. If we can show $\hat{Y}_m$ is a ring spectrum over which $\hat{Y}_{m-1}$ is a module, we will have $\langle \hat{Y}_m \rangle \geq \langle K(1) \rangle$ by 1.24. Since $\hat{Y}_m$ is $K(1)_*$-equivalent to Y_m and $\langle \hat{Y}_m \rangle \leq \langle K(1) \rangle$, we have $Y_m \wedge \hat{Y}_m = \hat{Y}_m \wedge \hat{Y}_m$ so the Y_m-module structure on $\hat{Y}_m$ makes $\hat{Y}_m$ a ring spectrum. The $\hat{Y}_m$-module structure on Y_{m-1} follows by a similar argument, so $\langle \hat{Y}_m \rangle = \langle K(1) \rangle$ for all m.

Similarly $\hat{X}$ is a module over some $\hat{Y}_m$ so $\langle \hat{X} \rangle \leq \langle K(1) \rangle$ by 1.24. Since X and $\hat{X}$ are not $K(1)_*$-acyclic, $\langle X \wedge K(1) \rangle = \langle K(1) \rangle$ by 2.1(h), so $\langle K(1) \rangle \leq \langle X \rangle$ and 10.5 follows for $n = 1$. We have already seen that 10.5 implies 10.6 and 10.7. $\square$

Next we have

10.13. THEOREM. *If the realizability conjecture* (10.2) *is true for* $n + 1$ *then so is the localization conjecture for* n (10.7).

Proof. Let $J \subset BP_*$ be an invariant regular ideal with $n + 1$ generators such that there is a finite ring spectrum X with $BP_*X = BP_*/J$. Then we claim $BP \wedge X = BPJ$ (see 2.1(g)), and will verify this at the end of the proof. By smashing X with finite skeleta of BP, we see that BPJ is a direct limit of finite $E(n)_*$-acyclic spectra.

Using the methods of section 5 and section 6, one can show that $L_nBP \wedge C_nX = pt.$, which implies that $L_nBP \wedge X = L_nBP \wedge L_nX$. If we can show that $C_nBP \wedge L_nX = pt.$, then we will have $BP \wedge L_nX = L_nBP \wedge L_nX$ and 10.7 will follow.

By 6.3 and 5.10, $\pi_*C_nBP = \Sigma^{-n-1}N^{n+1}$. N^{n+1} can be written as a direct limit in which each subquotient is a suspension of BP_*/J. It can be shown analogously that C_nBP is a direct limit in which every cofibre is a suspension of BPJ. Hence it suffices to show $BPJ \wedge L_nX = pt.$

We have seen above that $BPJ = \varinjlim Y_i$ with Y_i finite and $E(n)_*Y_i = 0$. By Proposition 2.10 of [11], this implies $E(n)_*DY_i = 0$, where DY_i is the Spanier-Whitehead dual of Y_i. If we regard L_nX as the representing spectrum of a generalized homology theory, we have $\pi_*Y_i \wedge L_nX = (L_nX)_*Y_i = (L_nX)^*DY_i = [DY_i, L_nX]$. This last group is trivial by Definition 1.3 since DY_i is $E(n)_*$-acyclic.

Finally, we will verify our claim that $BP \wedge X = BPJ$ by induction on n. Let J' be the invariant regular ideal generated by the first n generators of J, and assume there is a finite ring spectrum X' with $BP \wedge X' = BPJ'$. By definition BPJ is the cofibre of composite h

$$S^d \wedge BPJ \xrightarrow{q_n \wedge BPJ} BP \wedge BPJ \longrightarrow BPJ$$

where q_n is the last generator of J. Similarly X is the cofibre of the composite g

$$S^d \wedge X' \xrightarrow{q_n \wedge X'} X' \wedge X' \longrightarrow X'$$

where $q \in \pi_dX'$ maps to $q_n \in BP_*X' = BP_*/J'$ under the Hurewicz homomorphism. We want to show that the equivalence $X' \wedge BP \xrightarrow{\simeq} BPJ'$ sends $g \wedge BP$ to h. We get this from the commutativity of the following diagram

$$
\begin{tikzcd}
 & BP \wedge BPJ \arrow[d] \arrow[dr] & \\
S^d \wedge BPJ \arrow[ur] \arrow[r] \arrow[dr] & BPJ \wedge BPJ \arrow[r] & BPJ \\
 & X \wedge BPJ \arrow[u] \arrow[ur] & \\
S^d \wedge X \wedge BP \arrow[uu, "\simeq"] \arrow[r] & X \wedge X \wedge BP \arrow[u, "\simeq"'] \arrow[r] & X \wedge BP. \arrow[uu, "\simeq"']
\end{tikzcd}
\qquad \square
$$

For the reader's convenience we summarize our conjectures, the possible relation between them and the special cases known to be true. The conjectures are

10.1. (a), (b) and (c) Nilpotence,
10.2. Realizability,
10.4. Class Invariance,
10.5. Telescope,
10.6. Smashing,
10.7. Localization and
10.8. Boolean Algebra.

Possible arguments involving the Adams spectral sequence are given to show that 10.1(a) implies 10.1(c) and 10.5. Nishida's Theorem [39] is a special case of 10.1(a). In 10.3 it is shown that 10.1(a) and 10.2 together imply 10.1(c). The periodicity conjecture 5.9 is derived from 10.1(c) and 10.5. A possible derivation of 10.4 from 10.1(b) is described. In 10.13 it is shown that 10.2 implies 10.7. It is easy to see that 10.5 implies 10.6 which in turn implies 10.7. 10.9 shows that 10.6 is true for $n < p - 1$, and 10.10 shows that 10.1(c), 10.5, 10.6 and 10.7 are all true for $n = 1$.

UNIVERSITY OF WASHINGTON

REFERENCES

[1] J. F. Adams, On the groups $J(X)$, IV, *Topology*, **5** (1966), 21-71.

[2] ______, *Lectures on Generalized Cohomology*, Lecture Notes in Math., **99**, Springer-Verlag (1969).

[3] ______, Operations of nth kind in K-theory and what we don't know about RP^∞, New Developments in Topology, London Math. Soc. Lecture Note Series, **11** (1974).

[4] ______, *Stable homotopy and Generalized Homology*, University of Chicago, 1974.

[5] ______, A variant of E. H. Brown's representability theorem, *Topology*, **10** (1971), 185-198.

[6] D. W. Anderson and L. Hodgkin, The K-theory of Eilenberg-MacLane spaces, *Topology*, **7** (1968), 317-330.

[7] S. Araki, *Typical Formal Groups in Complex Cobordism and K-theory*, Kinokuniya Book-Store, Kyoto, 1974.

[8] N. A. Baas, On bordism theory of manifolds with singularities, *Math. Scand.*, **33** (1973), 279-302.

[9] A. K. Bousfield, The localization of spaces with respect to homology, *Topology*, **14** (1975), 133-150.

[10] ______, The localization of spectra with respect to homology, *Topology*, **18** (1979), 257-281.

[11] ______, The Boolean algebra of spectra, *Comment. Math. Helv.*, **54** (1979), 368-377.

[12] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math., **304**, Springer-Verlag (1972).

[13] E. H. Brown, Cohomology theories, *Ann. of Math.*, **75** (1962), 467-484.

[14] E. H. Brown and M. Comenetz, Pontrjagin duality for generalized homology and cohomology theories, *Amer. J. Math.*, **98** (1976), 1-27.

[15] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.

[16] J. M. Cohen, Coherent graded rings and the non-existence of spaces of finite stable homotopy type, *Comm. Math. Helv.*, **44** (1969), 217-228.

[17] P. E. Conner and E. E. Floyd, Differentiable Periodic Maps, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., Band 33, Springer-Verlag (1964).

[18] P. E. Conner and L. Smith, On the complex bordism of finite complexes, *Publ. Math. Inst. Hautes Etudes Sci.*, **37** (1969), 117-221.

[19] K. Iwasawa, On some properties of Γ-finite modules, *Ann. of Math.*, **70** (1959), 291-312.

[20] D. C. Johnson, P. S. Landweber and Z. Yosimura, Injective BP_*BP-comodules and localizations of Brown-Peterson homology, *Illinois J. of Math.*, **25** (1981), 599-610.

[21] D. C. Johnson and W. S. Wilson, Projective dimension and Brown-Peterson homology, *Topology*, **12** (1973), 327-353.

[22] ______, BP operations and Morava's extraordinary K-theories, *Math. Zeit.*, **144** (1975), 55-75.

[23] D. C. Johnson and Z. Yosimura, Torsion in Brown-Peterson homology and Hurewicz homomorphisms, *Osaka J. Math.*, **17** (1980), 117-136.

[24] D. S. Kahn and S. B. Priddy, Applications of the transfer to stable homotopy theory, *Bull. Amer. Math. Soc.*, **78** (1972), 981-987.

[25] P. S. Landweber, Annihilator ideals and primitive elements in complex bordism, *Ill. J. Math.*, **17** (1973), 273-284.

[26] ______, Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$, *Amer. J. Math.*, **98** (1976), 591-610.

[27] ______, Invariant regular ideals in Brown-Peterson homology, *Duke Math. J.*, **42** (1975), 499-505.

[28] ______, Associated prime ideals and Hopf algebras, *J. Pure and Applied Algebra*, **3** (1973), 43-58.

[29] T. Y. Lin, Duality and Eilenberg-MacLane spectra, *Proc. Amer. Math. Soc.*, **56** (1976), 291-299.

[30] J. I. Manin, Cyclotomic fields and modular curves, *Russian Math. Surveys* 26, no. 6 (1971), 7-78.

[31] H. R. Margolis, Eilenberg-MacLane spectra, *Proc. Amer. Math. Soc.*, **43** (1974), 409-415.

[32] H. R. Miller, A localization theorem in homological algebra, *Math. Proc. Camb. Phil. Soc.*, **84** (1978), 73-84.

[33] ______, On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space, *J. of Pure and Applied Alg.*, **20** (1981), 287-312.

[34] H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, *Ann. of Math.*, **106** (1977), 469-516.

[35] H. R. Miller and D. C. Ravenel, Morava stabilizer algebras and the localization of Novilov's E_2-term, *Duke Math. J.*, **44** (1977), 443-447.

[36] J. W. Milnor, On axiomatic homology theory, *Pacific J. Math.*, **12** (1962), 337-341.

[37] J. C. Moore and F. P. Peterson, Nearly Frobenius algebras, Poincâre algebras and their modules, *J. Pure and Applied algebra*, **3** (1973), 83-93.

[38] J. Morava, Structure theorems for cobordism comodules, to appear in *Amer. J. Math.*

[39] G. Nishida, The nilpotency of elements of the stable homotopy groups of spheres, *J. Math. Soc. Japan*, **25** (1973), 707-732.

[40] D. C. Ravenel, The cohomology of the Morava stabilizer algebras, *Math. Z.*, **152** (1977), 287-297.

[41] ______, *A Novice's Guide to the Adams-Novikov Spectral Sequence*, Lecture Notes in Math., **658** (1978), 404-475, Springer-Verlag.

[42] ______, The structure of Morava stabilizer algebras, *Inv. Math.*, **37** (1976), 109-120.

[43] D. C. Ravenel and W. S. Wilson, The Hopf ring for complex cobordism, *J. Pure and Applied Algebra*, **9** (1977), 241-280.

[44] ______, The Morava K-theories of Eilenberg-MacLane spaces and the Conner-Floyd conjecture, *Amer. J. Math.*, **102** (1980), 691-748.

[45] G. Segal, The stable homotopy of complex projective space, *Quart. J. Math.*, Oxford (2), **24** (1973), 1-5.

[46] D. P. Sullivan, Genetics of homotopy theory and the Adams conjecture, *Ann. of Math.*, **100** (1974), 1-79.

[47] ______, *Geometric Topology*, M.I.T. (1970).

[48] R. Vogt, *Boardman's Stable Homotopy Category*, Lecture Notes Series No. 21, Matematisk Institut, Aarhus Universitet (1970).

[49] U. Würgler, On products in a family of cohomology theories associated to the invariant prime ideals of $\pi_*(BP)$, *Comment. Math. Helv.*, **52** (1977), 457-481.

[50] ______, A splitting theorem for certain cohomology theories associated to $BP^*(-)$, *Manuscripta Math.*, **29** (1979), 93-111.

[51] Z. Yosimura, Universal coefficient sequences for cohomology theories of CW-spectra, *Osaka J. Math.*, I **12** (1975), 305-323 and II **16** (1979), 201-217.

[52]______, Localization of BP-module spectra with respect to BP-related homologies, to appear in *Osaka J. Math.*

[53] N. Shimada and N. Yagita, Multiplication in the complex cobordism theory with singularities, *Publ. Res. Inst. Math. Sci.*, **12** (1976), 259-293.

[54] A. K. Bousfield, On the homotopy theory of K-local spectra at an odd prime, to appear.

N.B. We have proved 10.7 and filled a gap in the proof of 10.9. Details will appear in the proceedings of the J. C. Moore Conference, Princeton, 1983.