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LOCALIZATION AND PERIODICITY IN UNSTABLE HOMOTOPY THEORY

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1. Introduction

In this paper, we develop a hierarchy of natural localizations of spaces, called the *v_n-periodizations* for $n \geq 0$, which may be used to expose and study periodic phenomena in unstable homotopy theory. These v_n-periodizations act less radically than the corresponding homological localizations [2] and respect fibrations to a very considerable extent. A major part of this paper is devoted to developing the general theory of periodizations of spaces, thereby providing a foundation for the study of the v_n-periodization and many others. Some of this general theory has been developed independently by Dror Farjoun [13], [14], and we refer the reader to his work for an alternative approach with other interesting general results. During the past decade, remarkable progress was made by Ravenel, Hopkins, Devinatz, and Smith [12], [16], [17], [33] toward a global understanding of stable periodic phenomena, and we hope that the present paper will help to prepare the way for a similar understanding of unstable periodic phenomena.

An excellent exposition of localization and periodicity in stable homotopy theory is now available in Ravenel's book [35]. Major features of stable homotopy are understood "chromatically" as manifestations of more basic periodic phenomena. These phenomena belong to a hierarchy starting with those detected rationally, followed by those detected in classical K-theory and in the successive Morava K-theories. Most fundamentally, each finite CW-spectrum has an intrinsic periodicity given by a "v_n self-map" which becomes a self-equivalence after suitable localization.

For simplicity, we describe our results in the pointed homotopy category Ho_0 of connected CW-complexes. For spaces $W, Y \in Ho_0$, we call Y *W-periodic* or *W-local* when the pointed mapping space $\mathrm{map}_*(W, Y)$ is contractible, or equivalently when $[\Sigma^t W, Y] \cong *$ for $t \geq 0$. As shown more generally in [3, Corollary 7.2], [10], or [13], there is a natural initial example $X \to P_W X$ of a map from X to a W-periodic space $P_W X$ in Ho_0, and we call this the *W-periodization* or *W-localization* of X. Our notation is derived from the classical example where $W = S^{n+1}$ and $P_W X$ is the n-th Postnikov section

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of X. The v_n-periodization $X \to P_{v_n} X$ will be given by $X \to P_{\Sigma W_n} X$ for a suitable space W_n as explained below.

Our first fibration theorem (Theorem 4.1) shows that each homotopy fiber sequence $F \to X \to B$ maps naturally to a homotopy fiber sequence $P_W F \to \overline{X} \to P_{\Sigma W} B$ where the map $X \to \overline{X}$ may be viewed as a "mixture" of the W-periodization and ΣW-periodization of X. For $n \geq 1$ and a prime p, we say that a space W satisfies the *n-supported p-torsion condition* when: $\widetilde{H}_*(W; Z)$ is p-torsion, $\widetilde{H}_i(W; Z) = 0$ for $i < n$, and $H^n(W; Z/p) \neq 0$. For such a W, we prove the key result that there is a natural homotopy fiber sequence

$$K(G, n+1) \to P_{\Sigma^2 W} Y \to P_{\Sigma W} Y$$

for some p-torsion abelian group G. This is similar to the fiber sequence relating Postnikov sections of Y, and our actual theorem (Theorem 7.2) is stated more generally so as to include such examples. When W satisfies the n-supported p-torsion condition and when $F \to X \to B$ is a homotopy fiber sequence, we deduce that the map from $P_{\Sigma W} F$ to the homotopy fiber of $P_{\Sigma W} X \to P_{\Sigma W} B$ has homotopy fiber of the form $K(G, n)$ for some p-torsion abelian group G. Thus $P_{\Sigma W} F \to P_{\Sigma W} X \to P_{\Sigma W} B$ is "almost" a homotopy fiber sequence.

We call two spaces W and W' *P-similar* when the W-periodic spaces are the same as the W'-periodic spaces, and we write $\langle W \rangle$ for the P-similarity class, or *P-class*, of W. We also write $\langle W \rangle \leq \langle W' \rangle$ when each W'-periodic space is W-periodic. Dror Farjoun has independently considered such classes. Like the stable classes in [4], the P-classes of spaces also have smash and wedge operations. For a space W satisfying the n-supported p-torsion condition, we prove that

$$\langle \Sigma W \rangle = \langle \Sigma^k W \rangle \vee \langle K(Z/p, n+1) \rangle$$

for each $k \geq 1$. This leads to a P-classification theorem (Theorem 9.15) showing that two finite p-torsion suspension spaces ΣW and $\Sigma W'$ have $\langle \Sigma W \rangle = \langle \Sigma W' \rangle$ if and only if $\mathrm{conn}(\Sigma W) = \mathrm{conn}(\Sigma W')$ and $\mathrm{type}(\Sigma W) = \mathrm{type}(\Sigma W')$, where $\mathrm{conn}(\Sigma W)$ denotes the connectivity of ΣW and $\mathrm{type}(\Sigma W)$ denotes the smallest integer m such that $\widetilde{K}(m)_*(\Sigma W) \neq 0$ for the m-th Morava K-theory $K(m)$. This result suggests that the problem of determining all possible P-classes of finite CW-complexes may not be totally out of reach.

For each $n \geq 0$, we find a canonical P-class $\langle \Sigma W_n \rangle$ whose iterated suspensions give all of the P-classes of finite p-torsion suspension spaces of type $n+1$. We then define the *v_n-periodization* $Y \to P_{v_n} Y$ to be $Y \to P_{\Sigma W_n} Y$. For each space Y, there is a natural *chromatic tower* (10.4)

$$P_{v_0} Y \leftarrow P_{v_1} Y \leftarrow P_{v_2} Y \leftarrow \cdots$$

with homotopy inverse limit equivalent to Y. This provides successive approximations to Y allowing "successively higher sorts of periodicity".

For a finite p-torsion space V_{n-1} of type n with "v_n self-map" $\omega : \Sigma^d V_{n-1} \to V_{n-1}$, the *$v_n$-periodic homotopy groups* $v_n^{-1}\pi_*(Y; V_{n-1})$ may be constructed by inverting the action of ω on $\pi_*(Y; V_{n-1})$. These groups, especially for

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$n = 1$ and $n = 2$, have figured prominently in recent work of Mahowald, Thompson, and others (see [20], [23], [24]). We show that

$$v_m^{-1}\pi_*(P_{v_n}Y; V_{m-1}) \cong \begin{cases} v_m^{-1}\pi_*(Y; V_{m-1}) & \text{for } m \le n, \\ 0 & \text{for } m > n, \end{cases}$$

$$\pi_t(P_{v_n}Y; V_{n-1}) \cong v_n^{-1}\pi_t(Y; V_{n-1}) \quad \text{for } t \ge 2.$$

Thus the v_n-periodic homotopy groups $v_n^{-1}\pi_*(Y; V_{n-1})$ are exposed as ordinary homotopy groups of $P_{v_n}Y$ with coefficients in V_{n-1}.

We establish general homotopical and homological characterizations (13.3 and 13.15) of P_{v_n}-equivalences for highly connected spaces. These lead, for instance, to a comparison of $v_1^{-1}\pi_*(-; Z/p)$-equivalences and $K_*(-; Z/p)$-equivalences in the pointed homotopy category Ho_3 of 3-connected spaces. A map $\varphi : X \to Y$ in Ho_3 is called a *durable* $K_*(-; Z/p)$-*equivalence* when $\tilde{\Omega}^k\varphi : \tilde{\Omega}^kX \to \tilde{\Omega}^kY$ is a $K_*(-; Z/p)$-equivalence for each $k \ge 0$, where $\tilde{\Omega} : Ho_3 \to Ho_3$ carries each space to the 3-connected cover of its loop space. In 14.7, we prove that the following conditions on a map $\varphi : X \to Y$ in Ho_3 are equivalent:

(i) φ is a $v_1^{-1}\pi_*(-; Z/p)$-equivalence;
(ii) φ is a durable $K_*(-; Z/p)$-equivalence;
(iii) $\tilde{\Omega}^k\varphi$ is a $K_*(-; Z/p)$-equivalence for some $k \ge 2$.

This is an unstable version of the result, proved in [5] using work of Mahowald and Miller, that a map $\theta : E \to F$ of spectra is a $v_1^{-1}\pi_*(-; Z/p)$-equivalence if and only if it is a $K_*(-; Z/p)$-equivalence. As an application, we verify the old conjecture of Miller-Snaith [26] and Mahowald-Ravenel [21] that the Snaith map

$$s : \Omega_0^{2n+1}S^{2n+1} \to Q(\mathbb{R}P^{2n})$$

is a $K_*(-; Z/2)$-equivalence for $n \ge 1$. This is now an immediate corollary of Mahowald's theorem [20] that the Snaith map is a $v_1^{-1}\pi_*(-; Z/2)$-equivalence. Since the algebra $K_*(Q(\mathbb{R}P^{2n}); Z/2)$ is known from work of Miller-Snaith [27], the hitherto inaccessible algebra $K_*(\Omega_0^{2n+1}S^{2n+1}; Z/2)$ is now also known. In very recent work, Lisa Langsetmo [19] has similarly determined all of the algebras $K_*(\Omega^jS^{2n+1}; Z/p)$ for $j < 2n$ using $v_1^{-1}\pi_*(-; Z/p)$-equivalences derived from the work of Mahowald-Thompson [24] together with K-theoretic calculations on the resulting "infinite loop space related" models for Ω^jS^{2n+1}. We believe that our results should permit other K-theoretic calculations of this sort.

This paper initially grew from our efforts to understand work of Mahowald and Thompson from a more general perspective. We also wish to thank Dror Farjoun for a useful exchange of preliminary manuscripts in 1991 after we had independently developed our main results in this area.

This paper is written simplicially so that "space" means "simplicial set". However, for convenience, many results are presented in the pointed homo-

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topy category Ho_*, whose objects may be taken as pointed CW-complexes when desired.

2. The periodization of spaces

For pointed spaces A, Y, let $\mathrm{map}_*(A, Y)$ and $\mathrm{map}(A, Y)$ respectively denote the pointed and unpointed mapping spaces, and recall that $\pi_0\mathrm{map}_*(A, Y) \cong [A, Y]$ when Y is fibrant. For a pointed map $f: A \to B$ and fibrant pointed space Y, consider the conditions:

(H1) $f^*: [B, Y] \cong [A, Y]$;
(H2) $f^*: \mathrm{map}_*(B, Y) \simeq \mathrm{map}_*(A, Y)$;
(H3) $f^*: \mathrm{map}(B, Y) \simeq \mathrm{map}(A, Y)$.

Lemma 2.1. *In general,* (H3) $\Rightarrow$ (H2) $\Rightarrow$ (H1). *When Y is connected,* (H2) $\Leftrightarrow$ (H3).

Proof. This follows using the natural fiber sequence

$$\mathrm{map}_*(A, Y) \to \mathrm{map}(A, Y) \to Y.$$

The mapping space functors may be applied in the pointed homotopy category Ho_* using fibrant versions of target spaces, and we adopt (H3) as our main "orthogonality" condition in Ho_*. For spaces $W, Y \in Ho_*$, we call Y *W-periodic* or *W-local* when $W \to *$ induces an equivalence $Y \simeq \mathrm{map}(W, Y)$. Thus, for W connected, Y is W-periodic if and only if $\mathrm{map}_*(W, Y_\alpha) \simeq *$ for each component Y_α of Y, where an arbitrary basepoint is chosen in Y_α. A map $f: A \to B$ in Ho_* will be called a *W-periodic equivalence* or *W-local equivalalence* when $f^*: \mathrm{map}(B, Y) \simeq \mathrm{map}(A, Y)$ for each W-periodic space $Y \in Ho_*$. Thus, for W connected, a map $f: A \to B$ is a W-periodic equivalence if and only if $f_*: \pi_0 A \cong \pi_0 B$ and $f_\alpha^*: \mathrm{map}_*(B_\alpha, Y) \simeq \mathrm{map}_*(A_\alpha, Y)$ for each connected W-periodic space $Y \in Ho_*$ and for each submap $f_\alpha: A_\alpha \to B_\alpha$ of components, where compatible basepoints are chosen in A_α and B_α. Note that $W \to *$ is itself a W-periodic equivalence. A map $u: X \to Y$ in Ho_* will be called a *W-periodization* or *W-localization* of X when u is a W-periodic equivalence and Y is W-periodic. The localization theorems of the author [3, Corollary 7.2], Casacuberta-Peschke-Pfenniger [10], or Dror Farjoun [13] specialize to give

Theorem 2.2. *For each $W, X \in Ho_*$, there exists a W-periodization of X.*

This periodization is clearly unique up to equivalence and will be denoted by $u: X \to P_W X$. It is the initial example of a map from X to a W-periodic space in Ho_*, and also the terminal example of a W-periodic equivalence out of X in Ho_*. General constructions of $P_W X$ will be briefly discussed in 2.8 and 2.10.

Example 2.3. If W is a point, then $P_W X \simeq X$ for each $X \in Ho_*$. If W is not connected or if $X = W$, then $P_W X \simeq *$.

Example 2.4. If $W = S^{n+1}$ for $n \geq 0$, then $u: X \to P_W X$ is the n-th Postnikov section for each $X \in Ho_*$.

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The reader should keep this Postnikov example in mind, since many of its formal properties will generalize.

For $W, X \in Ho_*$ we have

$$u \simeq P_W u : P_W X \simeq P_W P_W X,$$

and we may therefore view $P_W : Ho_* \to Ho_*$ and $u : \mathrm{Ident} \to P_W$ as an idempotent functor on Ho_*. We shall use the term *P_W-equivalence* as a synonym for *W-periodic equivalence*.

2.5. **Closure properties of P_W-equivalences.** For $W \in Ho_*$ and a map $f : A \to B$ of pointed spaces, the following conditions are equivalent:

(a) f is a P_W-equivalence;
(b) $f^* : [B, Y] \cong [A, Y]$ for each W-periodic space $Y \in Ho_*$;
(c) $f^* : \mathrm{map}_*(B, Y) \simeq \mathrm{map}_*(A, Y)$ for each W-periodic space $Y \in Ho_*$;
(d) $f^* : \mathrm{map}(B, Y) \simeq \mathrm{map}(A, Y)$ for each W-periodic space $Y \in Ho_*$.

Consequently, the P_W-equivalences are closed under homotopy colimits in both the pointed and unpointed senses. Some basic cases are:

(i) for a set $\{A_\alpha \to B_\alpha\}$ of P_W-equivalences, the wedge $\vee_\alpha A_\alpha \to \vee_\alpha B_\alpha$ is a P_W-equivalence;
(ii) for a pointed space J and P_W-equivalence $A \to B$, the maps $J \wedge B \to J \wedge B$ and $J \times A \to J \times B$ are P_W-equivalences;
(iii) for a homotopy cofiber square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

of pointed spaces, if $A \to B$ is a P_W-equivalence then so is $A' \to B'$;
(iv) for a possibly transfinite sequence of pointed spaces $A_0 \to A_1 \to \cdots$, if each $A_0 \to A_\alpha$ is a P_W-equivalence, then so is $A_0 \to \operatorname{hocolim}_\alpha A_\alpha$.

2.6. **Closure properties of W-periodic spaces.** For $W \in Ho_*$, the fibrant W-periodic pointed spaces are closed under homotopy limits. Some basic examples are:

(i) for a set $\{Y_\alpha\}$ of fibrant W-periodic pointed spaces, the product $\prod_\alpha Y_\alpha$ is W-periodic;
(ii) for a pointed space J and fibrant W-periodic pointed space Y, the mapping spaces $\mathrm{map}_*(J, Y)$ and $\mathrm{map}(J, Y)$ are W-periodic;
(iii) for a homotopy fiber square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

of pointed spaces, if X, Y, Y' are W-periodic, then so is X';

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(iv) for a possibly transfinite tower $A_0 \leftarrow A_1 \leftarrow \cdots$ of fibrant pointed spaces, if each A_α is W-periodic, then so is $\operatorname{holim}_\alpha A_\alpha$.

An immediate consequence of 2.5 (ii) and 2.6 (i) is

Proposition 2.7. *For $W \in Ho_*$ the natural map*

$$P_W(X_1 \times \cdots \times X_n) \to P_W(X_1) \times \cdots \times P_W(X_n)$$

is a homotopy equivalence for each finite set of spaces $X_1, \ldots, X_n \in Ho_*$.

We briefly describe two general constructions of $P_W X$.

2.8. **A topological construction of $P_W X$.** For connected CW-complexes $W, X \in Ho_*$, let λ be the first infinite ordinal with cardinality greater than the number of cells in W, and inductively construct an increasing sequence of CW-complexes

$$X = X(0) \subset X(1) \subset \cdots \subset X(\alpha) \subset X(\alpha+1) \subset \cdots \subset X(\lambda)$$

indexed by the ordinals $\leq \lambda$ as follows. Given $X(\alpha)$, choose a set of pointed cellular maps $\{g : \Sigma^i W \to X(\alpha)\}_{g \in G(i)}$ for each $i \geq 0$ representing all the pointed homotopy classes from $\Sigma^i W$ to $X(\alpha)$, and let $X(\alpha+1)$ be the mapping cone of the induced map

$$\bigvee_{i \geq 0} \bigvee_{g \in G(i)} \Sigma^i W \to X(\alpha).$$

Also, let $X(\beta) = \bigcup_{\alpha < \beta} X(\alpha)$ for each limit ordinal β. The map $X \to X(\lambda)$ produced by induction is a W-periodization, and thus $P_W X \simeq X(\lambda)$. An easy consequence of this construction is

Proposition 2.9. *For $W, X \in Ho_*$ with W n-connected, $u_* : \pi_i X \to \pi_i P_W X$ is bijective for $i \leq n$ and onto for $i = n+1$.*

Our second general construction of $P_W X$ involves a functor

$$P_W : \mathit{Spaces} \to \mathit{Spaces}$$

on the category of unpointed spaces (i.e., simplicial sets). We use a construction from [6] since it provides a simplicial functor in the sense of Quillen [32, II, §1], whereas other available functors on simplicial sets or topological spaces seem to lack this continuity property. For $W, Y \in \mathit{Spaces}$ with Y fibrant, we call Y *W-periodic* when $W \to *$ induces an equivalence $Y \simeq \operatorname{map}(W, Y)$.

Theorem 2.10. *For each space W, there exists a functor $P_W : \mathit{Spaces} \to \mathit{Spaces}$ together with natural maps $u : X \subset P_W X$ and $c : A \times P_W X \to P_W(A \times X)$ such that for all spaces A, B, X the following hold:*

(i) *$P_W X$ is a fibrant W-periodic space;*
(ii) *$u^* : \operatorname{map}(P_W X, Y) \simeq \operatorname{map}(X, Y)$ for each fibrant W-periodic space Y;*
(iii) *$P_W(*) = *$;*

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(iv) *the canonical map*

$$P_W X \cong * \times P_W X \xrightarrow{c} P_W(* \times X) \cong P_W X$$

is the identity;

(v) *there are equalities*

$$c = u(A \times c) : A \times B \times P_W X \to P_W(A \times B \times X),$$
$$u = c(B \times u) : B \times X \to P_W(B \times X).$$

Proof. Let $\{A_\alpha \subset B_\alpha\}$ consist of the inclusions

$$(W \times \Delta^n) \cup (CW \times \dot{\Delta}^n) \subset CW \times \Delta^n,$$
$$V_i^{n+1} \subset \Delta^{n+1}$$

for all $n \geq 0$ and $0 \leq i \leq n+1$, where $\dot{\Delta}^n$ is the boundary of the standard n-simplex Δ^n and V_i^{n+1} is the union of all faces of Δ^{n+1} except for the i-th. The construction of [6] with respect to $\{A_\alpha \subset B_\alpha\}$ now gives the desired functor and transformations.

Note. For a pointed space X, the above inclusion $u : X \subset P_W X$ represents the W-periodization of X in Ho_*, and thus P_W : *Spaces* $\to$ *Spaces* induces P_W : $Ho_* \to Ho_*$.

2.11. **A generalization.** By [3, Corollary 7.2], [10], or [13], the elementary theory of W-periodizations can be generalized to a theory of f-periodizations (equivalently called f-localizations) using an arbitrary map $f : W \to W'$ of spaces in place of $W \to *$. Thus, a space $Y \in Ho_*$ is called *f-periodic* or *f-local* when $f^* : \mathrm{map}(W', Y) \simeq \mathrm{map}(W, Y)$; a map $\varphi : A \to B$ in Ho_* is called an *f-periodic equivalence* or *f-local equivalence* when $\varphi^* : \mathrm{map}(B, Y) \simeq \mathrm{map}(A, Y)$ for each f-periodic space $Y \in Ho_*$; a map $u : X \to Y$ in Ho_* is called an *f-periodization* or *f-localization* when it is an f-periodic equivalence and Y is f-periodic. The terminology and results of 2.2, 2.5, 2.6, 2.7, and 2.10 all apply immediately with f in place of W.

3. The equivalence of $P_W(\Omega Y)$ and $\Omega(P_{\Sigma W} Y)$

For $W, Y \in Ho_*$, the space $\Omega(P_{\Sigma W} Y) \in Ho_*$ is W-periodic, and thus there is a unique map $\lambda : P_W(\Omega Y) \to \Omega(P_{\Sigma W} Y)$ in Ho_* such that

$$\begin{tikzcd} \Omega Y \arrow[d, "u"'] \arrow[dr, "\Omega u"] & \\ P_W(\Omega Y) \arrow[r, "\lambda"] & \Omega(P_{\Sigma W} Y) \end{tikzcd}$$

commutes. In this section, we shall prove that λ is an equivalence by destabilizing work of [6] on the localization of infinite loop spaces. The following theorem and corollary were obtained independently by Dror Farjoun [14].

Theorem 3.1. *For $W, Y \in Ho_*$, the natural map $\lambda : P_W(\Omega Y) \to \Omega(P_{\Sigma W} Y)$ is a homotopy equivalence.*

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Corollary 3.2. *For $W, Y \in Ho_*$ and $k \geq 1$, the natural map $\lambda^k : P_W(\Omega^k Y) \to \Omega^k(P_{\Sigma^k W} Y)$ is a homotopy equivalence.*

For the proof of 3.1 and for later use, we need

3.3. **Segal's theory of loop spaces.** In preparation for his work on infinite loop spaces, Segal [37] introduced a theory of single loop spaces. The following version will be developed more generally for iterated loop spaces in [7]. For $m \geq 0$, let $\Delta^m/\mathrm{sk}^0\Delta^m$ be the space formed by pinching the 0-skeleton of Δ^m to a point, and for a pointed space Y, let $\Omega_{\mathrm{bis}} Y$ be the pointed bisimplicial set with

$$(\Omega_{\mathrm{bis}} Y)_{m,\bullet} = \mathrm{map}_*(\Delta^m/\mathrm{sk}^0\Delta^m, Y).$$

Note that $(\Omega_{\mathrm{bis}} Y)_{0,\bullet} = *$ and $(\Omega_{\mathrm{bis}} Y)_{1,\bullet} = \Omega Y$. A bisimplicial set X is called *horizontally reduced* when $X_{0,\bullet} = *$, and is called *very special* when it is horizontally reduced and

$$\rho_1 \times \cdots \times \rho_m : X_{m,\bullet} \to X_{1,\bullet} \times \cdots \times X_{1,\bullet}$$

is a weak equivalence for each $m \geq 1$, where $\rho_i : X_{m,\bullet} \to X_{1,\bullet}$ is the horizontal simplicial operator corresponding to the monotone function $\rho_i : \{0, 1\} \to \{0, 1, \ldots, m\}$ with $\rho_i(0) = 0$ and $\rho_i(1) = i$. This is equivalent to the condition that X is special in Segal's sense with $\pi_0 X_{1,\bullet}$ a group. With our choice of operators, we do not need a separate condition on $\pi_0 X_{1,\bullet}$. When Y is fibrant, $\Omega_{\mathrm{bis}} Y$ is very special and "captures the higher multiplicative structure of ΩY". The functor Ω_{bis} from pointed spaces to horizontally reduced bisimplicial sets is right adjoint to the diagonal functor diag. Using [8, Appendix B], as explained more generally in [7], one shows:

(i) for a pointed connected fibrant space Y, the adjunction counit $\mathrm{diag}\, \Omega_{\mathrm{bis}} Y \to Y$ is a weak equivalence;
(ii) for a very special bisimplicial set X, the natural map $X_{1,\bullet} \to \Omega \mathrm{Ex}^\infty \mathrm{diag}\, X$ is a weak equivalence, where Kan's [18] functor Ex^∞ is used to "make $\mathrm{diag}\, X$ fibrant".

3.4. **The classifying space of $P_W(\Omega Y)$.** For a pointed connected fibrant space Y, we consider the pointed space $T_W Y = \mathrm{diag}\, P_W \Omega_{\mathrm{bis}} Y$ and the natural maps

$$Y \xleftarrow{\simeq} \mathrm{diag}\, \Omega_{\mathrm{bis}} Y \to \mathrm{diag}\, P_W \Omega_{\mathrm{bis}} Y = T_W Y$$

obtained from 2.10 and 3.3 (i). These determine a functor $T_W : Ho_0 \to Ho_0$ with coaugmentation $\bar{u} : \mathrm{Ident} \to T_W$, where $Ho_0 \subset Ho_*$ is the full subcategory of connected spaces. By 2.7 the bisimplicial set $P_W \Omega_{\mathrm{bis}} Y$ is very special, and thus by 3.3 (ii) there is a natural equivalence $\beta : P_W \Omega Y \simeq \Omega T_W Y$ with $\beta u = \Omega \bar{u} : \Omega Y \to \Omega T_W Y$ for $Y \in Ho_0$. Consequently, $T_W Y$ is the classifying space of $P_W \Omega Y$ and:

(i) a space $Y \in Ho_0$ has $\bar{u} : Y \simeq T_W Y$ if and only if ΩY is W-periodic in Ho_*;
(ii) a map $\varphi : X \to Y$ in Ho_0 has $T_W \varphi : T_W X \simeq T_W Y$ if and only if $\Omega \varphi : \Omega X \to \Omega Y$ is a P_W-equivalence in Ho_*.

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Thus, the maps $\overline{u}$, $T\overline{u} : T_W Y \to T_W T_W Y$ are equivalences for each $Y \in Ho_0$ and must be equal by [6, Lemma 6.6]. In other words, $T_W : Ho_0 \to Ho_0$ and $\overline{u} : \mathrm{Ident} \to T_W$ constitute an idempotent functor on Ho_0.

3.5. **Proof of Theorem 3.1.** Since $P_W \Omega Y$ and $\Omega P_{\Sigma W} Y$ depend only on the base component of Y, we may assume that Y is connected. The idempotent functors T_W and $P_{\Sigma W}$ on Ho_0 must be equivalent since they have the same local spaces, namely those with W-periodic loop spaces. Thus there is a natural equivalence $\overline{\lambda} : T_W Y \simeq P_{\Sigma W} Y$ such that the triangle

$$\begin{tikzcd} T_W Y \arrow[rr, "\overline{\lambda}"] & & P_{\Sigma W} Y \\ & Y \arrow[ul, "\overline{u}"] \arrow[ur, "u"'] & \end{tikzcd}$$

commutes. Applying Ω, we find that $\lambda : P_W \Omega Y \to \Omega P_{\Sigma W} Y$ is an equivalence since it is the composite of the equivalences $\beta : P_W \Omega Y \simeq \Omega T_W Y$ and $\Omega\overline{\lambda} : \Omega T_W Y \simeq \Omega P_{\Sigma W} Y$.

3.6. **A generalization.** We note that the terminology and results of 3.1, 3.2, and 3.4 all apply immediately when a map of spaces is used in place of W as in 2.11.

4. A GENERAL FIBRATION THEOREM

In this section, we prove the following general fibration theorem and explore some of its consequences. It will be used in Section 8 to prove our main fibration theorem.

Theorem 4.1. *For a pointed space W and a homotopy fiber sequence $F \to X \to B$ of pointed spaces with B connected, there is a natural homotopy fiber sequence $P_W F \to \overline{X} \to P_{\Sigma W} B$ such that $\overline{X}$ is ΣW-periodic. Moreover, there is a natural map*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "u"] & X \arrow[r] \arrow[d, "\overline{u}"] & B \arrow[d, "u"] \\ P_W F \arrow[r] & \overline{X} \arrow[r] & P_{\Sigma W} B \end{tikzcd}$$

in the homotopy category of homotopy fiber sequences where $\overline{u} : X \to \overline{X}$ is a P_W-equivalence.

This will be proved in 4.10. We view $\overline{u} : X \to \overline{X}$ as a "mixture" of the W-periodization and ΣW-periodization of X.

Proposition 4.2. *For a pointed space W and homotopy fiber sequence $F \to X \to B$ with B connected, the following hold*:

(i) *if F and B are W-periodic, then so is X*;
(ii) *if F is W-periodic and X is ΣW-periodic, then B is ΣW-periodic*;
(iii) *if X is W-periodic and B is ΣW-periodic, then F is W-periodic.*

This is easily proved, and Theorem 4.1 now implies

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Theorem 4.3. *For a pointed space* W *and pointed connected space* B *with* $P_{\Sigma W}B \simeq P_W B$, *the functor* P_W *carries each homotopy fiber sequence* $F \to X \to B$ *to a homotopy fiber sequence* $P_W F \to P_W X \to P_W B$.

This is due to Dror Farjoun [14] when B is W-periodic. For a pointed space X, let $X\langle n\rangle \to X$ be the n-connected cover, and let $X \to P^n X$ be the n-th Postnikov section.

Corollary 4.4. *For a pointed* n*-connected space* W *with* $n \geq 0$ *and pointed connected space* X, *there are natural equivalences*

$$P_W(X\langle n\rangle) \simeq (P_W X)\langle n\rangle,$$
$$P^n X \simeq P^n(P_W X).$$

Proof. Apply 4.3 to the homotopy fiber sequence $X\langle n\rangle \to X \to P^n X$ and note that $P_W(X\langle n\rangle)$ is n-connected by 2.9.

4.5. **Fiberwise localizations.** For a pointed space W and a homotopy fiber sequence $F \to X \to B$ of pointed spaces with B connected, there is a natural homotopy fiber sequence $P_W F \to X' \to B$ and a natural map

$$\begin{tikzcd} F \arrow[r] \arrow[d, "u"] & X \arrow[r] \arrow[d, "u'"] & B \arrow[d, "1"] \\ P_W F \arrow[r] & X' \arrow[r] & B \end{tikzcd}$$

in the homotopy category of homotopy fiber sequences obtained by pulling back the map of 4.1. The map $u' : X \to X'$ is a P_W-equivalence by 4.7 below.

The following theorem extends a result of Zabrodsky and Miller [25, Proposition 9.5] and will be proved in 4.11.

Theorem 4.6. *Let*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"] & X \arrow[r, "\varphi"] \arrow[d, "g"] & B \arrow[d, "h"] \\ F' \arrow[r] & X' \arrow[r, "\varphi'"] & B' \end{tikzcd}$$

be a map of homotopy fiber sequences with B *and* B' *connected. For a fibrant space* Y, *if* $f^* : \mathrm{map}(F', Y) \simeq \mathrm{map}(F, Y)$ *and* $(\Omega h)^* : \mathrm{map}(\Omega B', Y) \simeq \mathrm{map}(\Omega B, Y)$, *then* $g^* : \mathrm{map}(X', Y) \simeq \mathrm{map}(X, Y)$. *When* Y *is pointed and connected,* "map" *may be replaced by* "map_*". *For a spectrum* E, *if* $f : F \to F'$ *and* $\Omega h : \Omega B \to \Omega B'$ *are* E^**-equivalences or* E_**-equivalences, then so is* $g : X \to X'$.

Corollary 4.7. *For a pointed space* W *and map*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d, "g"] & B \arrow[d, "h"] \\ F' \arrow[r] & X' \arrow[r] & B' \end{tikzcd}$$

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of homotopy fiber sequences with B and B' connected, the following hold:

(i) *if f is a P_W-equivalence and h is a $P_{\Sigma W}$-equivalence, then g is a P_W-equivalence;*

(ii) *if g and h are $P_{\Sigma W}$-equivalences, then f is a P_W-equivalence;*

(iii) *if f and g are $P_{\Sigma W}$-equivalences, then h is a $P_{\Sigma W}$-equivalence.*

Proof. Part (i) follows from 4.6, and the other parts follow from (i) using 3.1.

Corollary 4.7 specializes to give

Corollary 4.8. *For a pointed space W and homotopy fiber sequence $F \to X \to B$ with B connected, the following hold:*

(i) *if $P_W F \simeq *$, then $X \to B$ is a P_W-equivalence;*

(ii) *if $X \to B$ is a $P_{\Sigma W}$-equivalence, then $P_W F \simeq *$;*

(iii) *if $P_{\Sigma W} B \simeq *$, then $F \to X$ is a P_W-equivalence;*

(iv) *if $F \to X$ is a $P_{\Sigma W}$-equivalence, then $P_{\Sigma W} B \simeq *$.*

We devote the rest of this section to proving 4.1 and 4.6. Let $\varphi : X \to B$ be a fibration of pointed spaces with B connected and fibrant, and let F be the fiber of φ. The termwise cofiber sequence of pairs

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Delta^m \cup * \arrow[r] \arrow[d, "A^m"'] & \Delta^m \cup * \arrow[d] \\
\Delta^m/\mathrm{sk}^0\Delta^m \arrow[r] & \Delta^m/\mathrm{sk}^0\Delta^m \arrow[r] & *
\end{tikzcd}$$

for $m \geq 0$ is carried by $\mathrm{map}_*(-, \varphi)$ to a fiber sequence

$$\mathrm{map}(\Delta^m, F) \to \mathrm{map}_*(A^m, \varphi) \to \mathrm{map}_*(\Delta^m/\mathrm{sk}^0\Delta^m, B)$$

of pointed spaces, and this is homotopically equivalent to a projection fiber sequence

$$F \to \Omega B \times \cdots \times \Omega B \times F \to \Omega B \times \cdots \times \Omega B.$$

Letting m vary, we obtain a termwise fiber sequence of bisimplicial sets

$$\mathrm{map}(\Delta^\bullet, F) \longrightarrow \mathrm{map}_*(A^\bullet, \varphi) \longrightarrow \Omega_{\mathrm{bis}} B.$$

A more familiar, but less natural, version of this sequence may be obtained by "turning the action of ΩB on F into a group action and applying the geometric bar construction". For a pointed space W, we apply the transformation u : Ident $\to P_W$ termwise and then take diagonals to give a diagram of pointed spaces

$$\begin{tikzcd}
\mathrm{diag}\,\mathrm{map}(\Delta^\bullet, F) \arrow[r] \arrow[d] & \mathrm{diag}\,\mathrm{map}_*(A^\bullet, \varphi) \arrow[r] \arrow[d] & \mathrm{diag}\,\Omega_{\mathrm{bis}} B \arrow[d] \\
\mathrm{diag}\,P_W\,\mathrm{map}(\Delta^\bullet, F) \arrow[r] & \mathrm{diag}\,P_W\,\mathrm{map}_*(A^\bullet, \varphi) \arrow[r] & \mathrm{diag}\,P_W\Omega_{\mathrm{bis}} B.
\end{tikzcd}$$

Lemma 4.9. *The rows in the above diagram are homotopy fiber sequences.*

Proof. This follows by Theorem B.4 of [8] whose hypotheses are straightforward to verify since P_W preserves finite products up to homotopy. In particular,

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$\Omega_{\mathrm{bis}}B$ and $P_W\Omega_{\mathrm{bis}}B$ satisfy the π_*-Kan condition by [7] since they are very special.

4.10. **Proof of Theorem 4.1.** In the natural commutative diagram

$$\begin{array}{ccccc}
\operatorname{diag}\operatorname{map}(\Delta^\bullet, F) & \longrightarrow & \operatorname{diag}\operatorname{map}_*(A^\bullet, \varphi) & \longrightarrow & \operatorname{diag}\Omega_{\mathrm{bis}}B \\
\downarrow{=} & & \downarrow{\simeq} & & \downarrow{\simeq} \\
\operatorname{diag}\operatorname{map}(\Delta^\bullet, F) & \longrightarrow & \operatorname{diag}\operatorname{map}(\Delta^\bullet, X) & \longrightarrow & \operatorname{diag}\operatorname{map}(\Delta^\bullet, B) \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\
F & \longrightarrow & X & \longrightarrow & B
\end{array}$$

the lower vertical maps are weak equivalences since their canonical right inverses are weak equivalences by [8, B.2], and thus all three rows are homotopy fiber sequences. The composite map $\operatorname{diag}\Omega_{\mathrm{bis}}B \to B$ is the adjunction counit which is a weak equivalence by 3.3 (i). Thus all vertical maps are weak equivalences. Using 3.4 and 3.5, we conclude that the diagram preceding 4.9 is equivalent to a natural map

$$\begin{array}{ccccc}
F & \longrightarrow & X & \longrightarrow & B \\
\downarrow{u} & & \downarrow{\overline{u}} & & \downarrow{u} \\
P_WF & \longrightarrow & \overline{X} & \longrightarrow & P_{\Sigma W}B
\end{array}$$

in the homotopy category of homotopy fiber sequences. Since $\overline{X}$ is ΣW-periodic by 4.2, and $\overline{u} : X \to \overline{X}$ is a P_W-equivalence by 4.7 (i), it remains to give

4.11. **Proof of Theorem 4.6.** Since the vertical maps in 4.6 induce the equivalences $\operatorname{map}(F', Y) \simeq \operatorname{map}(F, Y)$ and $\operatorname{map}(\Omega B', Y) \simeq \operatorname{map}(\Omega B, Y)$, they must also induce the equivalences

$$\operatorname{map}(\operatorname{map}_*(A^m, \varphi'), Y) \simeq \operatorname{map}(\operatorname{map}_*(A^m, \varphi), Y)$$

for $m \geq 0$. Since $\operatorname{diag}(\operatorname{map}_*(A^\bullet, \varphi)) \simeq X$ by the proof of Theorem 4.1, and since $\operatorname{diag}(\operatorname{map}_*(A^\bullet, \varphi))$ is equivalent to the homotopy colimit of $\operatorname{map}_*(A^\bullet, \varphi)$ by [9, p. 335], we conclude that $g^* : \operatorname{map}(X', Y) \simeq \operatorname{map}(X, Y)$. The statements for map_* and E^* now follow by 2.1, and imply the statement for E_* since the E_*-equivalences are the same as the $(\nabla E)^*$-equivalences, where ∇E is the "Anderson-Yosimura dual" of E with universal coefficient sequence

$$0 \to \operatorname{Ext}(E_{*-1}X, Z) \to (\nabla E)^*X \to \operatorname{Hom}(E_*X, Z) \to 0$$

as in [40]. Alternatively, the statements for E^* and E_* follow by easy spectral sequence arguments.

4.12. **A generalization.** When a map of spaces is used in place of W as in 2.11 and 3.6, we no longer have the results of 4.2 on periodicity properties of spaces in a fibration. However, 4.1 still holds without mention of a periodicity

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property for $\overline{X}$. Moreover, the results and proofs of 4.5, 4.7, 4.8, and 4.9 all apply immediately.

5. Periodizations with respect to Moore spaces

Before continuing with the theory of W-periodizations, we consider the case where W is a Moore space or a wedge of Moore spaces. Our results will involve

5.1. **The abelian group** $A//G$. For abelian groups G and X, we call X *G-reduced* when $\operatorname{Hom}(G, X) = 0$. Each abelian group A has a maximal G-reduced quotient group denoted by $A//G$, which may be constructed as the image of the homomorphism from A to the product of all its G-reduced quotient groups. The following general examples are easily verified:

(i) for a set I of primes, if G is an I-torsion abelian group with $G/pG \neq 0$ for each $p \in I$, then $A//G$ is the quotient of A by its maximal I-torsion subgroup;

(ii) for a set J of primes, if G is a uniquely J-divisible abelian group with $\operatorname{Hom}(G, Z[J^{-1}]) \neq 0$ (or equivalently with G containing a direct summand isomorphic to $Z[J^{-1}]$), then $A//G$ is the cokernel of the canonical homomorphism $\operatorname{Hom}(Z[J^{-1}], A) \to A$. Thus $A//G$ is the image of the Ext-completion map $A \to \prod_{p \in J} \operatorname{Ext}(Z_{p^\infty}, G)$ of [9].

Theorem 5.2. *For an abelian group G and $n \geq 2$, let W be the Moore space $M(G, n)$ whose only reduced homology group is G in dimension n, and let J be the set of all primes p with $p : G \cong G$. Then, for each space $Y \in Ho_*$,*

$$\pi_i P_W Y \cong \begin{cases} \pi_i Y & \text{for } i < n, \\ \pi_i Y//G & \text{for } i = n, \\ \pi_i Y \otimes Z_{(J)} & \text{for } i > n \text{ when } G \text{ is torsion.} \end{cases}$$

Moreover, there is a natural short exact sequence

$$0 \to \prod_{p \in J} \operatorname{Ext}(Z_{p^\infty}, \pi_i Y) \to \pi_i P_W Y \to \prod_{p \in J} \operatorname{Hom}(Z_{p^\infty}, \pi_{i-1} Y) \to 0$$

for $i > n$ when G is not torsion.

This theorem may be applied repeatedly to determine $\pi_* P_W Y$ when W is an arbitrary wedge of Moore spaces. For instance

Theorem 5.3. *Let $W = M(G_1, n_1) \vee M(G_2, n_2)$ with $2 \leq n_1 < n_2$. Then for $W_1 = M(G_1, n_1)$ and $W_2 = M(G_1 \oplus G_2, n_2)$, there is a natural equivalence $P_W Y \simeq P_{W_2}(P_{W_1} Y)$ for $Y \in Ho_*$.*

We devote the rest of this section to proving the above theorems. Clearly

Lemma 5.4. *For an abelian group G and $n \geq 2$, a connected space $Y \in Ho_*$ is $M(G, n)$-periodic if and only if* $\operatorname{Hom}(G, \pi_i Y) = 0 = \operatorname{Ext}(G, \pi_i Y)$ *for each $i > n$ and* $\operatorname{Hom}(G, \pi_n Y) = 0$.

For an abelian group A and set J of primes, the condition that $\operatorname{Hom}(Z[J^{-1}], A) = 0 = \operatorname{Ext}(Z[J^{-1}], A)$ is equivalent to the condition that

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$\operatorname{Hom}(Q, A) = 0 = \operatorname{Ext}(Q, A)$ with $\operatorname{Ext}(Z_{p^\infty}, A) = 0$ for each prime $p \notin J$. Such an A is called *J-cotorsion* and decomposes naturally as a product

$$A \cong \prod_{p \in J} \operatorname{Ext}(Z_{p^\infty}, A)$$

of p-cotorsion (also called Ext-p-complete) factors for $p \in J$ as in [9]. For a given abelian group G, the class of all abelian groups X with $\operatorname{Hom}(G, X) = 0 = \operatorname{Ext}(G, X)$ will be denoted by $C(G)$, and the set of all primes p such that $p : G \cong G$ will be denoted by J_G.

Lemma 5.5. *For an abelian group G, the class $C(G)$ consists of the J_G-local abelian groups when G is torsion, and consists of the J_G-cotorsion abelian groups when G is not torsion.*

Proof. Let $C(G)^\perp$ consist of all abelian groups X with $\operatorname{Hom}(X, A) = 0 = \operatorname{Ext}(X, A)$ for each $A \in C(G)$. Then $C(G)^\perp$ is one of the "special classes" determined explicitly in [1]. The result follows since $C(G)$ can be recovered from $C(G)^\perp$ as the class of all Y with $\operatorname{Hom}(B, Y) = 0 = \operatorname{Ext}(B, Y)$ for each $B \in C(G)^\perp$.

Using the E_*-localization $X \to X_E$ in the sense of [2], we have

Proposition 5.6. *If W is an E_*-acyclic space for a spectrum E, then each E_*-local space in Ho_* is W-periodic, and each P_W-equivalence in Ho_* is an E_*-equivalence. Thus there is a natural E_*-localization map $\alpha : P_W X \to X_E$ for $X \in Ho_*$.*

5.7. **Proof of Theorem 5.2.** We may assume by 4.4 that Y is $(n-1)$-connected. The isomorphism $\pi_n P_W Y \cong \pi_n Y//G$ follows since $\operatorname{Hom}(G, \pi_n P_W Y) = 0$ by 5.4 and since

$$u^* : \operatorname{Hom}(\pi_n P_W Y, B) \cong \operatorname{Hom}(\pi_n Y, B)$$

whenever $\operatorname{Hom}(G, B) = 0$, because $K(B, n)$ is then W-periodic. Let $R = Z_{(J)}$ when G is torsion, and let $R = \bigoplus_{p \in J} Z/p$ otherwise. Since W is $H_*(\ ; R)$-acyclic, there is a natural $H_*(\ ; R)$-localization map $\alpha : P_W Y \to Y_R$ by 5.6, where Y_R is the $H_*(\ ; R)$-localization of Y. By 5.4 and 5.5, $\pi_i P_W Y$ is: (a) J-local for $i > n$ when G is torsion; (b) J-cotorsion for $i > n$ when G is not torsion. Moreover, the maximal divisible subgroup D of $\pi_n P_W Y \cong \pi_n Y//G$ satisfies $\operatorname{Hom}(G, D) = 0$, and thus $D = 0$ when G is not torsion. Hence, by [2], $\alpha_* : \pi_i P_W Y \cong \pi_i Y_R$ for $i > n$, and the theorem follows from our knowledge of $\pi_i Y_R$.

5.8. **Proof of Theorem 5.3.** Since the periodization maps $Y \to P_{W_1} Y \to P_{W_2}(P_{W_1} Y)$ are clearly P_W-equivalences, it suffices to show that $P_{W_2}(P_{W_1} Y)$ is W_1-periodic. This follows by 5.2 using

Lemma 5.9. *If A is a J-cotorsion abelian group for a set J of primes, and if G is not torsion, then $A//G$ is J-cotorsion.*

Proof. Since $A//G$ is a quotient of A, $\operatorname{Ext}(Q, A//G) = 0$ and $\operatorname{Ext}(Z_{p^\infty}, A//G) = 0$ for each prime $p \notin J$. Since G is not torsion, the maximal divisible subgroup of $A//G$ is zero and $\operatorname{Hom}(Q, A//G) = 0$. Thus $A//G$ is J-cotorsion.

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6. Convergent functors, Γ-spaces, and mapping spaces

In this section, we make preparations for a comparison of $P_{\Sigma^2 W} Y$ with $P_{\Sigma W} Y$ in Section 7. We first discuss Segal's theory of Γ-spaces [37] from the viewpoint of Bousfield-Friedlander [8].

6.1. **Convergent functors and Γ-spaces.** A *convergent functor* consists of a functor $T : Sets_* \to Spaces_*$ from pointed sets to pointed spaces (i.e., pointed simplicial sets) such that $T(*) = *$ and T preserves direct limits over directed systems. A Γ-*space* consists of a functor $A : \Gamma^\circ \to Spaces_*$ with $A(0^+) = *$, where $\Gamma^\circ \subset Sets_*$ is the full subcategory of objects $n^+ = \{0, 1, \dots, n\}$ with basepoint $0 \in n^+$ for $n \geq 0$. Each Γ-space $A : \Gamma^\circ \to Spaces_*$ extends to a convergent functor $A : Sets_* \to Spaces_*$, unique up to natural equivalence, and we identify Γ-spaces with convergent functors. A Γ-space is called *special* when

$$A(p_1) \times \cdots \times A(p_n) : A(n^+) \to A(1^+) \times \cdots \times A(1^+)$$

is a weak equivalence for each $n \geq 1$, where $p_i : n^+ \to 1^+$ is defined by $p_i(i) = 1$ and $p_i(j) = 0$ for $j \neq i$. This is equivalent to the condition that

$$A(E_1 \vee E_2) \to A(E_1) \times A(E_2)$$

is a weak equivalence for each $E_1, E_2 \in Sets_*$. When A is special, $A(1^+)$ has an induced abelian monoid structure in Ho_*. A Γ-space A is called *very special* when it is special and the abelian monoid $\pi_0 A(1^+)$ is an abelian group. Finally, each Γ-space A determines a functor $A : Spaces_* \to Spaces_*$ with $AX = \operatorname{diag}(AX_\bullet)_\bullet$ and each very special Γ-space A determines a reduced homology theory $\pi_* AX$.

Example 6.2. For an abelian monoid M, there is a convergent functor

$$\widetilde{M} : Sets_* \to Sets_* \subset Spaces_*,$$
$$\widetilde{M}(E) = \left(\bigoplus_{x \in E} Mx\right) \Big/ M*,$$

where each Mx is a copy of M. Let $M \to M_{grp}$ be the group completion of M, and note that $\widetilde{M}$ is special while $\widetilde{M}_{grp}$ is very special. There is a natural isomorphism

$$\pi_*(\widetilde{M}_{grp} X) \cong \widetilde{H}_*(X; M_{grp})$$

for $X \in Spaces_*$, and Spanier [38, Corollary 5.7] showed that $\widetilde{M}X \to \widetilde{M}_{grp}X$ is a weak equivalence when X is connected.

Example 6.3. For $X, Y \in Spaces_*$, there is a Γ-space $\Phi(X, Y)$ with

$$\Phi(X, Y)(n^+) = \operatorname{map}_*(X^{n^+}, Y)$$

for $n \geq 0$ where $X^{n^+} = \operatorname{map}_*(n^+, X) = X \times \cdots \times X$ is the product of n copies of X.

For Γ-spaces A and B, let $\operatorname{map}_*(A, B)$ be the pointed space whose n-simplicies are the Γ-space maps $(\Delta^n \cup *) \wedge A \to B$ for $n \geq 0$.

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Proposition 6.4. *For a convergent functor* (*i.e.*, Γ-*space*) A *and pointed spaces* X *and* Y, *there is a natural isomorphism*

$$\mathrm{map}_*(AX, Y) \cong \mathrm{map}_*(A, \Phi(X, Y)).$$

Proof. This is an easy consequence of the natural coend isomorphism

$$\int^{n^+} X^{n^+} \wedge A(n^+) \cong A(X)$$

which follows from the elementary case where X is a pointed set.

Corollary 6.5. *Let* X *be a pointed space and* Y *be a pointed connected fibrant space. If* $\mathrm{map}_*(X, Y) \simeq *$, *then* $\mathrm{map}_*(AX, Y) \simeq *$ *for each convergent functor* A.

Proof. Using the "strict" homotopy theory of Γ-spaces [8], choose a weak equivalence $B \to A$ with B cofibrant. This induces a weak equivalence $BX \to AX$ by [8, Theorem B.2] and it suffices to show $\mathrm{map}_*(BX, Y) \simeq *$. The Γ-space $\Phi(X, Y)$ is fibrant since Y is fibrant, and $\Phi(X, Y) \to *$ is a weak equivalence since $\mathrm{map}_*(X^{n^+}, Y) \simeq *$ for each $n \geq 0$. Thus $\mathrm{map}_*(BX, Y) \simeq *$ by 6.4.

We wish to study $\mathrm{map}_*(AX, Y)$ when $\mathrm{map}_*(X, Y)$ is *homotopically discrete*, i.e., when $\mathrm{map}_*(X, Y) \to \pi_0 \mathrm{map}_*(X, Y)$ is a weak equivalence.

Lemma 6.6. *Let* J, X, *and* Y *be pointed connected spaces such that* Y *is fibrant and* $\mathrm{map}_*(X, Y)$ *is homotopically discrete. If* J *is simply connected, or if* $\pi_1 Y$ *acts trivially on* $[X, Y]$, *then the inclusion* $J \vee X \subset J \times X$ *induces an equivalence*

$$\mathrm{map}_*(J \times X, Y) \simeq \mathrm{map}_*(J, Y) \times \mathrm{map}_*(X, Y).$$

Proof. By 2.1 it suffices to show that $J \vee X \subset J \times X$ induces an equivalence

$$\mathrm{map}(J \times X, Y) \simeq \mathrm{map}(J \vee X, Y),$$

or equivalently that it induces a homotopy fiber square:

$$\begin{array}{ccc} \mathrm{map}(J \times X, Y) & \longrightarrow & \mathrm{map}(J, Y) \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\gamma} \\ \mathrm{map}(X, Y) & \xrightarrow{\beta} & Y \end{array}$$

Thus, it suffices to show, for each choice of basepoint $f \in \mathrm{map}(X, Y)$ with $f(*) = *$, that the map of fibers

$$\alpha^{-1}(f) = \mathrm{map}_*(J, \mathrm{map}(X, Y)) \xrightarrow{\beta_*} \mathrm{map}_*(J, Y) = \gamma^{-1}(*)$$

is a weak equivalence. This follows easily because $\beta : \mathrm{map}(X, Y) \to Y$ has homotopically discrete fiber $\mathrm{map}_*(X, Y)$.

A Γ-space A is called *homotopically discrete* when $A(n^+)$ is homotopically discrete for $n \geq 0$.

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Proposition 6.7. *Let X and Y be pointed connected spaces such that Y is fibrant and* $\mathrm{map}_*(X, Y)$ *is homotopically discrete. If X is simply connected or if $\pi_1 Y$ acts trivially on $[X, Y]$, then $\Phi(X, Y)$ is a homotopically discrete, very special, fibrant Γ-space, and $[X, Y]$ has a natural abelian group structure.*

Proof. By 6.6, $\Phi(X, Y)$ is homotopically discrete, special, and fibrant. Thus $\pi_0\Phi(X, Y)(1^+) \cong [X, Y]$ is an abelian monoid, and there is a weak equivalence $\Phi(X, Y) \to [X, Y]^{\sim}$. Using 6.4 as in the proof of 6.5, we obtain

$$\pi_0 \,\mathrm{map}_*(\widetilde{M}X, Y) \cong \mathrm{Hom}_{\mathscr{M}}(M, [X, Y])$$

for each M in the category $\mathscr{M}$ of abelian monoids. This combines with the weak equivalence $\widetilde{M}X \to \widetilde{M}_{grp}X$ of 6.2, to give

$$\mathrm{Hom}_{\mathscr{M}}(M_{grp}, [X, Y]) \cong \mathrm{Hom}_{\mathscr{M}}(M, [X, Y]).$$

Consequently, $[X, Y]$ is an abelian group and $\Phi(X, Y)$ is very special.

Theorem 6.8. *Under the hypotheses of* 6.7, *if A is a convergent functor, then* $\mathrm{map}_*(AX, Y)$ *is homotopically discrete, and there is a natural isomorphism*

$$[AX, Y] \cong \mathrm{Hom}(\pi_0 AS, [X, Y])$$

of abelian groups where S is the sphere spectrum.

Proof. As in the proof of 6.5, we may assume that A is cofibrant. Since $\Phi(X, Y)$ is very special and fibrant,

$$\mathrm{map}_*(AX, Y) \cong \mathrm{map}_*(A, \Phi(X, Y)) \simeq \mathrm{map}_*(AS, \Phi(X, Y)S)$$

by [8]. The weak equivalence $\Phi(X, Y) \to [X, Y]^{\sim}$ induces an equivalence of $\Phi(X, Y)S$ with the Eilenberg-MacLane spectrum $H[X, Y]$. Thus

$$\mathrm{map}_*(AX, Y) \simeq \mathrm{map}_*(AS, H[X, Y])$$

and the theorem follows.

The following corollary is a version of the "key lemma" initially used by the author to prove the main results of this paper. It involves the symmetric products

$$X = SP^1X \subset SP^2X \subset \cdots \subset SP^{\infty}X$$

of a pointed space X.

Corollary 6.9. *Under the hypotheses of* 6.7, *the natural inclusions $X \subset SP^kX$ and $X \subset \widetilde{Z}X$ induce equivalences*

$$\mathrm{map}_*(SP^kX, Y) \simeq \mathrm{map}_*(X, Y) \simeq \mathrm{map}_*(\widetilde{Z}X, Y)$$

for $1 \leq k \leq \infty$.

Proof. This follows from 6.8 since the inclusions induce isomorphisms

$$\pi_0 SP^kS \cong \pi_0 S \cong \pi_0 \widetilde{Z}S \cong Z.$$

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7. The tower of spaces $P_{\Sigma^i W} Y$

For W, $Y \in Ho_*$, there is a unique W-periodization map $\sigma : P_{\Sigma W} Y \to P_W Y$ under Y, and thus there is a natural tower

$$P_W Y \xleftarrow{\sigma} P_{\Sigma W} Y \xleftarrow{\sigma} P_{\Sigma^2 W} Y \xleftarrow{\sigma} \cdots$$

under Y. This reduces to the Postnikov tower when $W = S^0$, and we shall show much more generally that the homotopy fibers are successive Eilenberg-Mac Lane spaces. For this we need

7.1. **A homological condition.** For $n \geq 1$ and a set J of primes, we say that a space $W \in Ho_*$ satisfies the *n-supported J-torsion* (resp. *J-local*) *condition* when:

(i) $\widetilde{H}_*(W; Z)$ is J-torsion (resp. J-local);
(ii) $\widetilde{H}_i(W; Z) = 0$ for $i < n$;
(iii) $H^n(W; Z/p) \neq 0$ for each $p \in J$ (resp. $H^n(W; Z_{(J)}) \neq 0$).

We devote most of this section to proving

Theorem 7.2. *For $W \in Ho_*$ satisfying the n-supported J-torsion (resp. J-local) condition, for $Y \in Ho_*$, and for $i \geq 1$, the homotopy fiber of the map $\sigma : P_{\Sigma^{i+1} W} Y \to P_{\Sigma^i W} Y$ is an Eilenberg-Mac Lane space $K(G_i, n+i)$ for some J-torsion (resp. J-local) abelian group G_i.*

Of course, it suffices to assume $i = 1$. For later use, we give another version of this theorem with different conditions on W in 7.7 below, covering the remaining cases where $\widetilde{H}_*(W; Z)$ is p-torsion. In subsequent work, Dror Farjoun and Smith [15] have proved a generalization showing that the homotopy fiber of $\sigma : P_{\Sigma^2 W} Y \to P_{\Sigma W} Y$ is a (possibly infinite) product of Eilenberg-Mac Lane spaces when W is an arbitrary connected space. We have used some of their ideas to simplify our original proof. Theorems 7.2 and 7.7 extend easily to cover $\sigma : P_{\Sigma W} Y \to P_W Y$ when $Y \in Ho_*$ is an H-space. However, for a space W with $\widetilde{H}_*(W; Z) = 0$, the homotopy fiber of $\sigma : P_{\Sigma W} W \to P_W W$ is equivalent to W by 2.3. Finally, 7.2 and 2.9 combine to give

Corollary 7.3. *For $W \in Ho_*$ satisfying the n-supported J-torsion (resp. J-local) condition, for $Y \in Ho_*$, and for $i \geq 1$, the maps*

$$Y \xrightarrow{u} P_{\Sigma^i W} Y \xrightarrow{\sigma} P_{\Sigma W} Y$$

give the $(n+i)$-th Moore-Postnikov factorization of $u : Y \to P_{\Sigma W} Y$.

Consequently the successive periodizations $u : Y \to P_{\Sigma^i W} Y$ for $i \geq 1$ are easily obtained from $u : Y \to P_{\Sigma W} Y$.

Our proof of 7.2 will depend on two lemmas. For a set J of primes, a space $X \in Ho_*$ is called a *J-torsion* (resp. *J-local*) *GEM* when $X \simeq \prod_{m=1}^{\infty} K(G_m, m)$ for a sequence of J-torsion (resp. J-local) abelian groups G_m.

Lemma 7.4. *Let $W \in Ho_*$ satisfy the n-supported J-torsion (resp. J-local) condition, and let $Y \in Ho_*$ be connected with* $\mathrm{map}_*(W, Y) \simeq *$. *If $B \in Ho_*$*

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is an $(n-1)$*-connected* J*-torsion* (*resp.* J*-local*) *GEM, then* $\operatorname{map}_*(B, Y) \simeq *$ *and* $P_W B \simeq *$.

Proof. By 6.5, $\operatorname{map}_*(W, Y) \simeq *$ implies $\operatorname{map}_*(\tilde{Z}W, Y) \simeq *$ and hence $\operatorname{map}_*(K(H_n W, n), Y) \simeq *$. By 4.6, the class of abelian groups G with $\operatorname{map}_*(K(G, n), Y) \simeq *$ is closed under extensions, cokernels, and direct limits over directed systems. We conclude that $\operatorname{map}_*(K(G, n), Y) \simeq *$ for each J-torsion (resp. J-local) abelian group G. The lemma now follows by 4.6 and a direct limit argument.

Lemma 7.5. *Let* $W \in Ho_*$ *satisfy the n-supported* J*-torsion* (*resp.* J*-local*) *condition, and let* $X \in Ho_*$. *If* $P_W X \simeq *$, *then* $\tilde{Z}X$ *is an* $(n-1)$*-connected* J*-torsion* (*resp.* J*-local*) *GEM.*

Proof. Since $P_W X \simeq *$, the construction of $X \to P_W X$ in 2.8 shows that $\tilde{H}_*(X; Z)$ is J-torsion (resp. J-local) and $(n-1)$-connected. The lemma now follows since $\tilde{Z}X$ is a simplicial abelian group with $\pi_* \tilde{Z}X \cong \tilde{H}_*(X; Z)$.

7.6. **Proof of Theorem 7.2.** The homotopy fiber F of $P_{\Sigma^2 W} Y \to P_{\Sigma W} Y$ is n-connected by 2.9 and is $\Sigma^2 W$-periodic with $P_{\Sigma W} F \simeq *$ by 4.2 and 4.3. Thus $\Omega^2 F$ is W-periodic and $P_W \Omega F \simeq *$ by 3.1. Hence, $\operatorname{map}_*(\Omega F, \Omega^2 F) \simeq *$ and $\operatorname{map}_*(\Omega F, \Omega F)$ is homotopically discrete. The "key lemma" 6.9 now shows that ΩF is a homotopy retract of $\tilde{Z}\Omega F$, and 7.5 shows that $\tilde{Z}\Omega F$ is an $(n-1)$-connected J-torsion (resp. J-local) GEM. Since ΩF is ΣW-periodic, it is also a homotopy retract of $P_{\Sigma W}(\tilde{Z}\Omega F)$, and the theorem now follows since $P_{\Sigma W}(\tilde{Z}\Omega F)$ is of the form $K(G, n)$ for a J-torsion (resp. J-local) abelian group G by 2.7 and 7.4.

Finally, we must give a supplemental version of Theorem 7.2. For $n \geq 1$ and a prime p, we say that a space $W \in Ho_*$ satisfies the *n-supported divisible p-torsion condition* when

(i) $\tilde{H}_*(W; Z)$ is p-torsion;
(ii) $\tilde{H}_i(W; Z) = 0$ for $i < n$;
(iii) $H_n(W; Z)$ is a nontrivial divisible p-torsion abelian group.

Theorem 7.7. *For* $W \in Ho_*$ *satisfying the* n*-supported divisible* p*-torsion condition, for* $Y \in Ho_*$, *and for* $i \geq 1$, *the homotopy fiber of the map* $P_{\Sigma^{i+1} W} Y \to P_{\Sigma^i W} Y$ *is equivalent to a product* $K(G_i, n+i) \times K(G_i', n+i+1)$, *where* G_i *is a divisible* p*-torsion abelian group and* G_i' *is a* Z_{p^∞}*-reduced* p*-torsion abelian group.*

Proof. The proof is essentially the same as for 7.2. However, in place of ordinary Eilenberg-Mac Lane spaces, we must use products $K(G, m) \times K(G', m+1)$ where G is a divisible p-torsion abelian group and G' is a Z_{p^∞}-reduced p-torsion abelian group. The homotopy category of such products, for a given $m \geq 1$, is equivalent to the abelian category of Ext-p-complete abelian groups. This equivalence may be obtained by using the p-completion functor F_{p^∞} of [9] to carry each such product to an Eilenberg-Mac Lane space $K(\overline{G}, m+1)$, where $\overline{G}$ is an Ext-p-complete abelian group. Inversely, for each Ext-p-complete

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abelian group $\overline{G}$, the "p-torsion part of $K(\overline{G}, m+1)$",

$$t_pK(\overline{G}, m+1) = \mathrm{Fib}(K(\overline{G}, m+1) \to K(\overline{G}[1/p], m+1)),$$

is a product of the above sort (see 14.1).

8. The main fibration theorem

Our earlier fibration theorem in 4.1 involved a bothersome mixture of the W-periodization and the ΣW-periodization. We can now give a fibration theorem involving only the ΣW-periodization.

Theorem 8.1. *For $W \in Ho_*$ satisfying the n-supported J-torsion (resp. J-local) condition of 7.1 and for a homotopy fiber sequence $F \to X \to B$ of pointed spaces, the map from $P_{\Sigma W}F$ to the homotopy fiber of $P_{\Sigma W}X \to P_{\Sigma W}B$ has homotopy fiber of the form $K(G, n)$ for a J-torsion (resp. J-local) abelian group G. Moreover, the action of $\pi_1 P_{\Sigma W}F$ on G is trivial.*

Proof. We may assume that B is connected and apply 4.1 to give a natural map

$$\begin{tikzcd} F \arrow[r] \arrow[d, "u"] & X \arrow[r, "f"] \arrow[d, "\overline{u}"] & B \arrow[d, "u"] \\ P_{\Sigma W}F \arrow[r] & \overline{X} \arrow[r] & P_{\Sigma^2 W}B \end{tikzcd}$$

of homotopy fiber sequences, where $\overline{u}$ is a $P_{\Sigma W}$-equivalence and $\overline{X}$ is $\Sigma^2 W$-periodic. This determines a natural diagram of pointed spaces

$$\begin{tikzcd} \mathrm{Fib}\,\sigma_1 \arrow[r] \arrow[d] & \mathrm{Fib}\,\sigma_2 \arrow[r] \arrow[d] & \mathrm{Fib}\,\sigma_3 \arrow[d] \\ P_{\Sigma W}F \arrow[r] \arrow[d, "\sigma_1"] & \overline{X} \arrow[r] \arrow[d, "\sigma_2"] & P_{\Sigma^2 W}B \arrow[d, "\sigma_3"] \\ \mathrm{Fib}\,P_{\Sigma W}f \arrow[r] & P_{\Sigma W}X \arrow[r] & P_{\Sigma W}B \end{tikzcd}$$

whose rows and columns are homotopy fiber sequences. Theorem 7.2 shows that $\mathrm{Fib}\,\sigma_2 \simeq K(G_2, n+1)$ and $\mathrm{Fib}\,\sigma_3 \simeq K(G_3, n+1)$, where G_2 and G_3 are J-torsion (resp. J-local) abelian groups. Thus $\pi_*\mathrm{Fib}\,\sigma_1$ is also J-torsion (resp. J-local) abelian and $\pi_i\mathrm{Fib}\,\sigma_1 = 0$ for $i \neq n, n+1$. Since $\mathrm{Fib}\,\sigma_1$ is ΣW-periodic by 4.2, it must have $\pi_{n+1}\mathrm{Fib}\,\sigma_1 = 0$ by 7.4. Thus $\mathrm{Fib}\,\sigma_1$ has the required form and $\pi_1 P_{\Sigma W}F$ acts trivially on $\pi_n\mathrm{Fib}\,\sigma_1$ since it acts trivially on $\pi_{n+1}\mathrm{Fib}\,\sigma_3$.

We may view $K(G, n)$ as an error term measuring the failure of $P_{\Sigma W}F \to P_{\Sigma W}X \to P_{\Sigma W}B$ to be a homotopy fiber sequence. When W satisfies the n-supported divisible p-torsion condition of 7.7, this error term is of the form $K(G, n) \times K(G', n+1)$ for a divisible p-torsion abelian group G and a Z_{p^∞}-reduced p-torsion abelian group G'. In subsequent work, Dror Farjoun and

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Smith [15] have proved a generalization showing that the error term is a (possibly infinite) product of Eilenberg-Mac Lane spaces when W is an arbitrary connected space. Theorem 8.1 also generalizes to fiber squares, although the statement on trivial action must be omitted, even when $W = S^n$.

Theorem 8.2. *For $W \in Ho_*$ satisfying the n-supported J-torsion (resp. J-local) condition and a homotopy fiber square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

of pointed spaces, the map from $P_{\Sigma W}A$ to the homotopy pull-back of $P_{\Sigma W}B \to P_{\Sigma W}D \leftarrow P_{\Sigma W}C$ has homotopy fiber of the form $K(G, n)$ for a J-torsion (resp. J-local) abelian group G.

Proof. It suffices to show that the homotopy fiber of $\mathrm{Fib}(P_{\Sigma W}f) \to \mathrm{Fib}(P_{\Sigma W}g)$ has the required form $K(G, n)$. Since $\mathrm{Fib}\, f \simeq \mathrm{Fib}\, g$, this follows by 8.1 and 7.4.

Theorem 8.1 also leads to a mapping space theorem for $P_{\Sigma W}$.

Theorem 8.3. *For $W \in Ho_*$ satisfying the n-supported J-torsion (resp. J-local) condition and for connected spaces X, $Y \in Ho_*$ with X equivalent to a finite CW complex, let F_v be the homotopy fiber of the map*

$$P_{\Sigma W}\,\mathrm{map}_*(X, Y) \to \mathrm{map}_*(X, P_{\Sigma W}Y)$$

at a vertex $v \in P_{\Sigma W}\,\mathrm{map}_*(X, Y)$. Then:

(i) *$\pi_i F_v = 0$ for $i \geq n+1$;*
(ii) *$\pi_i F_v$ is a J-torsion (resp. J-local) nilpotent group for $i \geq 1$;*
(iii) *the action of $\pi_1 P_{\Sigma W}\,\mathrm{map}_*(X, Y)$ on $\pi_i F_v$ is nilpotent for each $i \geq 1$.*

Proof. The result holds for $X = S^k$ with $k \geq 1$ by 3.2 and 7.2. We may assume inductively that X is the homotopy cofiber of a map $\alpha : S^k \to X'$ where the result holds for S^k and X'. We then deduce the result for X by applying 8.1 to the homotopy fiber sequence

$$\mathrm{map}_*(X, Y) \longrightarrow \mathrm{map}_*(X', Y) \xrightarrow{\alpha^*} \mathrm{map}_*(S^k, Y)$$

and using the map

$$\begin{tikzcd} \mathrm{Fib}\, P_{\Sigma W}\alpha^* \arrow[r] \arrow[d] & P_{\Sigma W}\,\mathrm{map}_*(X', Y) \arrow[r] \arrow[d] & P_{\Sigma W}\,\mathrm{map}_*(S^k, Y) \arrow[d] \\ \mathrm{map}_*(X, P_{\Sigma W}Y) \arrow[r] & \mathrm{map}_*(X', P_{\Sigma W}Y) \arrow[r] & \mathrm{map}_*(S^k, P_{\Sigma W}Y) \end{tikzcd}$$

of homotopy fiber sequences.

9. The P-classes of spaces

In [4], we introduced equivalence classes of spectra corresponding to the natural equivalence classes of homological localization functors. In almost the

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same way, we now investigate equivalence classes of spaces corresponding to the natural equivalence classes of periodization functors. Dror Farjoun has independently considered these classes. We say that a space $W \in Ho_*$ is *P-similar* to $W' \in Ho_*$ when the W-periodic spaces in Ho_* are the same as the W'-periodic spaces. We write $\langle W \rangle$ for the P-similarity class, or *P-class*, of W. Our main results are classification theorems (9.12 and 9.15) for P-classes of p-torsion suspension spaces.

9.1. **The algebra of P-classes.** For P-classes $\langle W \rangle$ and $\langle W' \rangle$, we write $\langle W \rangle \leq \langle W' \rangle$ when each W'-periodic space is W-periodic. This happens if and only if $W \to *$ is a $P_{W'}$-equivalence, which happens if and only if $P_{W'}(W) \simeq *$. Moreover, this gives a partial ordering of P-classes. Note that $\langle * \rangle \leq \langle W \rangle \leq \langle S^0 \rangle$ for each $W \in Ho_*$. If $\langle W \rangle \leq \langle W' \rangle$, then each P_W-equivalence in Ho_* is a $P_{W'}$-equivalence, and there is a canonical W'-localization map $\sigma : P_W X \to P_{W'} X$ under X. As in [4], there are smash and wedge operations

$$\langle W \rangle \wedge \langle W' \rangle = \langle W \wedge W' \rangle,$$
$$\bigvee_\alpha \langle W_\alpha \rangle = \langle \bigvee_\alpha W_\alpha \rangle$$

with the expected properties. For instance, if $\langle V \rangle \leq \langle V' \rangle$ and $\langle W \rangle \leq \langle W' \rangle$, then $\langle V \rangle \wedge \langle V' \rangle \leq \langle W \rangle \wedge \langle W' \rangle$. Thus, if $\langle W \rangle \leq \langle W' \rangle$, then $\langle \Sigma^k W \rangle \leq \langle \Sigma^k W' \rangle$ for each $k \geq 0$. Of course, $\langle \Sigma W \rangle \leq \langle W \rangle$ for each $W \in Ho_*$. Some other elementary general properties of P-classes are:

Proposition 9.2. *If $A \to B \to C$ is a homotopy cofiber sequence, then*:

(i) $\langle B \rangle \leq \langle A \rangle \vee \langle C \rangle$;
(ii) $\langle C \rangle \leq \langle B \rangle \vee \langle \Sigma A \rangle \leq \langle B \rangle \vee \langle A \rangle$;
(iii) $\langle \Sigma A \rangle \leq \langle C \rangle \vee \langle \Sigma B \rangle \leq \langle C \rangle \vee \langle B \rangle$.

Proposition 9.3. *If $F \to X \to B$ is a homotopy fiber sequence with B connected, then*:

(i) $\langle B \rangle \leq \langle F \rangle \vee \langle X \rangle$;
(ii) $\langle X \rangle \leq \langle F \rangle \vee \langle B \rangle$;
(iii) $\langle F \rangle \leq \langle X \rangle \vee \langle \Omega B \rangle$.

Proof. Apply 4.8.

Corollary 9.4. *If $B \in Ho_*$ is connected, then $\langle B \rangle \leq \langle \Omega B \rangle$.*

Proposition 9.5. *If $F, B \in Ho_*$, then $\langle F \times B \rangle = \langle F \rangle \vee \langle B \rangle$.*

Proof. This follows by 2.7.

Proposition 9.6. *For $V, W \in Ho_*$ with V connected, the condition $\langle V \rangle \leq \langle \Sigma W \rangle$ holds if and only if $\langle \Omega V \rangle \leq \langle W \rangle$.*

Proof. This follows since the condition $P_{\Sigma W} V \simeq *$ is equivalent to $P_W(\Omega V) \simeq *$ by 3.1.

Corollary 9.7. *For $k \geq 1$ and for $V, W, W' \in Ho_*$ with W and W' $(k-1)$-connected, the following hold*:

(i) $\langle \Omega^k \Sigma^k V \rangle \leq \langle V \rangle$;

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(ii) $\langle W\rangle \leq \langle \Sigma^k\Omega^k W\rangle$;
(iii) *if* $\langle W\rangle \leq \langle W'\rangle$, *then* $\langle \Omega^k W\rangle \leq \langle \Omega^k W'\rangle$.

Proposition 9.8. *If* $W \in Ho_*$ *is connected, then* $\langle \Sigma\Omega\Sigma W\rangle = \langle \Sigma W\rangle$.
Proof. This follows from Milnor's [28] decomposition of $\Sigma\Omega\Sigma W$.

We next discuss

9.9. **The algebra of stable** P**-classes.** These classes have also been considered by Dror Farjoun. For each $W \in Ho_*$, there is a decreasing sequence of P-classes

$$\langle W\rangle \geq \langle \Sigma W\rangle \geq \langle \Sigma^2 W\rangle \geq \cdots ,$$

and we say that $W \in Ho_*$ is *stably* P*-similar* to $W' \in Ho_*$ when $\langle \Sigma^j W\rangle \leq \langle W'\rangle$ and $\langle \Sigma^k W'\rangle \leq \langle W\rangle$ for some $j, k \geq 0$. For this it is sufficient, but *not* necessary, to have $\langle \Sigma^h W\rangle = \langle \Sigma^i W'\rangle$ for some $h, i \geq 0$. For instance, the Moore space wedges $M^3(p) \vee M^4(q)$ and $M^3(p) \vee M^5(q)$ are stably P-similar for distinct primes p and q, although none of their iterated suspensions are P-similar. We write $\{W\}$ for the stable P-similarity class, or *stable* P*-class*, of W. For stable P-classes $\{W\}$ and $\{W'\}$, we write $\{W\} \leq \{W'\}$ when $\langle \Sigma^j W\rangle \leq \langle W'\rangle$ for some $j \geq 0$. This gives a partial ordering of the stable P-classes. There are smash and wedge operations

$$\{W\} \wedge \{W'\} = \{W \wedge W'\},$$
$$\{W\} \vee \{W'\} = \{W \vee W'\}$$

with the expected properties, but there is not a well-defined infinite wedge for stable P-classes. Since $\langle W\rangle \leq \langle W'\rangle$ implies $\{W\} \leq \{W'\}$, and since $\{\Sigma W\} = \{W\}$, the results 9.2, 9.3, 9.4, 9.5, 9.7 (i), 9.7 (ii), 9.8, and the "if" part of 9.6, all still hold when $\langle\ \rangle$ is replaced by $\{\ \}$. For a prime p and $m, n \geq 1$, Ravenel and Wilson [36] showed that $K(Z/p, n)$ is $K(m)_*$-acyclic (or equivalently $K(m)^*$-acyclic) if and only if $n > m$, where $K(m)$ is the m-th Morava K-theory spectrum. Thus

$$\{K(Z/p, 1)\} > \{K(Z/p, 2)\} > \{K(Z/p, 3)\} > \cdots$$

by 9.4, and consequently 9.7 (iii) and the "only if" part of 9.6 fail when $\langle\ \rangle$ is replaced by $\{\ \}$.

To go further, we need

Theorem 9.10. *For* $n \geq 1$ *and a set* J *of primes, if* $W \in Ho_*$ *satisfies the* n*-supported* J*-torsion* (*resp.* J*-local*) *condition, and if* $k \geq 1$, *then*

$$\langle \Sigma W\rangle = \langle \Sigma^k W\rangle \vee \bigvee_{p\in J} \langle K(Z/p, n+1)\rangle$$

$$(\textit{resp. } \langle \Sigma W\rangle = \langle \Sigma^k W\rangle \vee \langle K(Z_{(J)}, n+1)\rangle).$$

Proof. The inequality $\langle \Sigma W\rangle \geq \cdots$ follows by 7.4. For the inequality $\langle \Sigma W\rangle \leq \cdots$, let $Y \in Ho_*$ be a connected space which is $\Sigma^k W$-periodic and $K(Z/p, n+1)$-periodic for each $p \in J$ (resp. $K(Z_{(J)}, n+1)$-periodic). Then

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let F be the homotopy fiber of $u : Y \to P_{\Sigma W} Y$ and apply 7.2 to show that $\pi_i F$ is J-torsion (resp. J-local) for $n+1 \leq i \leq n+k-1$ and $\pi_i F = 0$ otherwise. When $i \geq n+1$, F is $K(Z/p, i)$-periodic for each $p \in J$ (resp. $K(Z_{(J)}, i)$-periodic) by 2.6 and 7.4. Consequently, $\pi_* F = 0$ and $Y \simeq P_{\Sigma W} Y$. Hence, the inequality $\langle \Sigma W \rangle \leq \cdots$ follows.

Using 7.7 in place of 7.2, we also obtain

Theorem 9.11. *For $n \geq 1$ and a prime p, if $W \in Ho_*$ satisfies the n-supported divisible p-torsion condition, and if $k \geq 1$, then*

$$\langle \Sigma W \rangle = \langle \Sigma^k W \rangle \vee \langle K(Z_{p^\infty}, n+1) \rangle.$$

We now focus on the p-torsion suspension spaces. Let $\operatorname{conn}(X)$ denote the connectivity of X, where X is a space or a graded abelian group.

Theorem 9.12. *For a prime p, let W and W' be pointed spaces such that $\widetilde{H}_*(W; Z)$ and $\widetilde{H}_*(W'; Z)$ are p-torsion. Then the following are equivalent*:

(i) $\langle \Sigma W \rangle \leq \langle \Sigma W' \rangle$;
(ii) $\{\Sigma W\} \leq \{\Sigma W'\}$, $\operatorname{conn}(\Sigma W) \geq \operatorname{conn}(\Sigma W')$, *and* $\operatorname{conn} \widetilde{H}^*(\Sigma W; Z/p) \geq \operatorname{conn} \widetilde{H}^*(\Sigma W'; Z/p)$.

Proof. Clearly (i) $\Rightarrow$ (ii). Given (ii), we have $\langle \Sigma^k W \rangle \leq \langle \Sigma W' \rangle$ for some $k \geq 1$ and $\langle K(G, n+1) \rangle \leq \langle K(G', n'+1) \rangle$, where $n = \operatorname{conn}(\Sigma W)$ with

$$G = \begin{cases} Z/p & \text{when } H^{n+1}(\Sigma W; Z/p) \neq 0, \\ Z_{p^\infty} & \text{otherwise} \end{cases}$$

and $n' = \operatorname{conn}(\Sigma W')$ with

$$G' = \begin{cases} Z/p & \text{when } H^{n+1}(\Sigma W'; Z/p) \neq 0, \\ Z_{p^\infty} & \text{otherwise.} \end{cases}$$

We deduce (i) by applying 9.10 and 9.11 to give

$$\langle \Sigma W \rangle = \langle \Sigma^k W \rangle \vee \langle K(G, n+1) \rangle \leq \langle \Sigma W' \rangle \vee \langle K(G', n'+1) \rangle = \langle \Sigma W' \rangle.$$

By Theorem 9.12, the P-classification problem for p-torsion suspension spaces is effectively reduced to a stable P-classification problem. The latter problem seems difficult in general, but has been completely solved for p-torsion finite CW-complexes using a classification theorem of Devinatz, Hopkins, and Smith, which we now recall in a form suggested by Ravenel [35]. Let $FHo_{(p)} \subset Ho_*$ be the full subcategory of all $X \in Ho_*$ such that $\bigoplus_{i=0}^{\infty} \widetilde{H}_i(X; Z)$ is a finitely generated $Z_{(p)}$-module, i.e., $\Sigma^\infty X$ is equivalent to a p-local finite spectrum. A full subcategory $C \subset F\, Ho_{(p)}$ is called *thick* when it is nonempty and satisfies the conditions:

(i) in a homotopy cofiber sequence $X \xrightarrow{f} Y \longrightarrow \operatorname{Cof} f$, if two of the spaces belong to C, then so does the third;
(ii) if a wedge of spaces $X \vee Y$ belongs to C, then so do X and Y.

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Thus, for any spectrum E, the E_*- (or E^*-) acyclic spaces in $FHo_{(p)}$ form a thick subcategory. For $m \geq 0$, let $K(m)$ denote the m-th Morava K-theory spectrum at the prime p. In particular, let $K(0)$ denote the rational spectrum HQ. As shown in [33, Theorem 2.11], for each space $X \in FHo_{(p)}$ with $\widetilde{H}_*(X; Z) \neq 0$, there exists an integer $n \geq 0$ such that $\widetilde{K(m)}_* X = 0$ for all $m < n$ and $\widetilde{K(m)}_* X \neq 0$ for all $m \geq n$. This n is called the *type* of X at the prime p, and we write $\operatorname{type}(X) = n$. A space $X \in FHo_{(p)}$ with $\widetilde{H}_*(X; Z) = 0$ is said to have type ∞. As shown by Mitchell [30] and subsequently by Hopkins and Smith [16], [17], for each prime p and $n \geq 0$, there exists a space $X \in F\,Ho_{(p)}$ with $\operatorname{type}(X) = n$.

Theorem 9.13 (Thick subcategory theorem). *If $C \subset F\,Ho_{(p)}$ is a thick subcategory, then*

$$\operatorname{obj} C = \{X \in FHo_{(p)} \mid \operatorname{type}(X) \geq n\}$$

for some n with $0 \leq n \leq \infty$.

This is proved in [16, Theorem 7], [17], [35] using the nilpotence theorem of Devinatz, Hopkins, and Smith [12]. Dror Farjoun has established the following consequence.

Theorem 9.14. *For a prime p and spaces W, $W' \in F\,Ho_{(p)}$, $\{W\} \leq \{W'\}$ if and only if* $\operatorname{type}(W) \geq \operatorname{type}(W')$.

Proof. The "only if" part follows since the conditions $\{W\} \leq \{W'\}$ and $\widetilde{K(m)}^* W' = 0$ imply $\widetilde{K(m)}^* W = 0$. For the "if" part, let $C \subset FHo_{(p)}$ be the full subcategory of all spaces $X \in FHo_{(p)}$ such that $\{X\} \leq \{W'\}$. Then C is thick by 9.2, and thus

$$\operatorname{obj} C = \{X \in FHo_{(p)} \mid \operatorname{type}(X) \geq n\}$$

for some $n \geq 0$. Using the already proved "only if" part, we find that $n = \operatorname{type}(W')$, and thus $\operatorname{type}(W) \geq \operatorname{type}(W')$ implies $\{W\} \leq \{W'\}$.

Theorems 9.12 and 9.14 imply the following P-classification theorem.

Theorem 9.15. *For a prime p and spaces W, $W' \in F\,Ho_{(p)}$ of type > 0, the following are equivalent:*

(i) $\langle \Sigma W \rangle \leq \langle \Sigma W' \rangle$;
(ii) $\operatorname{type}(\Sigma W) \geq \operatorname{type}(\Sigma W')$ *and* $\operatorname{conn}(\Sigma W) \geq \operatorname{conn}(\Sigma W')$.

This theorem shows that $\langle \Sigma W \rangle = \langle \Sigma W' \rangle$ if and only if $\operatorname{type}(\Sigma W) = \operatorname{type}(\Sigma W')$ and $\operatorname{conn}(\Sigma W) = \operatorname{conn}(\Sigma W')$. To be more explicit, we survey

9.16. **The P-classes of finite p-torsion suspension spaces.** We work over a fixed prime p, and for each $n \geq 0$ we choose a space $\overline{V}_n \in FHo_{(p)}$ of type $n+1$ with $\operatorname{conn}(\Sigma \overline{V}_n)$ minimal. *Then the possible P-classes of p-torsion suspension spaces in $FHo_{(p)}$ are precisely the $\langle \Sigma^i \overline{V}_n \rangle$ for $i \geq 1$ and $n \geq 0$.* We know that

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$\operatorname{conn}(\Sigma\overline{V}_n) \geq n+1$ for each $n \geq 0$, since the condition $\widetilde{K(n)}^*\overline{V}_n = 0$ implies $\widetilde{K(n)}^* K(Z/p, j) = 0$ for $j = \operatorname{conn}(\Sigma\overline{V}_n)$ by 7.4. Moreover,

$$\operatorname{conn}(\Sigma\overline{V}_n) \leq \operatorname{conn}(\Sigma\overline{V}_{n+1})$$

for each $n \geq 0$, since $\operatorname{type}(\overline{V}_n \vee \overline{V}_{n+1}) = n+1$. Clearly, $\operatorname{conn}(\Sigma\overline{V}_0) = 1$ since we may let $\overline{V}_0 = M^2(p) = S^1 \cup_p e^2$. Also, $\operatorname{conn}(\Sigma\overline{V}_1) = 2$ when p is odd since we may let $\overline{V}_1$ be the cofiber of the Adams map

$$A : \Sigma^{2p-2} M^3(p) \to M^3(p)$$

constructed in [11]. When $p = 2$, an Adams map

$$A : \Sigma^8 M^j(2) \to M^j(2)$$

is constructed for $j = 6$ in [29] and for $j = 5$ in [22], [31]. Thus, $2 \leq \operatorname{conn}(\Sigma\overline{V}_1) \leq 4$ when $p = 2$. We emphasize that $\overline{V}_1$ need not be constructed as the cofiber of an Adams map.

10. The v_n-periodization of spaces

We work over a fixed prime p. In view of the classification of finite p-torsion suspension spaces in 9.16, it is natural to focus on the $\Sigma\overline{V}_n$-periodization for $n \geq 0$. However, we suspect that almost all of the P-classes $\langle\Sigma\overline{V}_n\rangle$ are highly iterated suspensions of P-classes of *nonfinite* p-torsion spaces, and we incorporate that possibility into our construction of the v_n-periodization.

10.1. **The fundamental P-classes $\langle\Sigma W_n\rangle$.** For each $n \geq 0$ we choose a space $W_n \in Ho_*$ such that:

(i) the stable P-class $\{W_n\}$ equals $\{\overline{V}_n\}$;
(ii) W_n satisfies the k-supported p-torsion condition of 7.1 for some $k \geq 1$;
(iii) the above value of k is the smallest possible for choices of W_n satisfying (i) and (ii).

By 9.12, the P-class $\langle\Sigma W_n\rangle$ is uniquely determined by the above conditions, and $\langle\Sigma\overline{V}_n\rangle = \langle\Sigma^j W_n\rangle$ for some $j \geq 1$. We know that $\operatorname{conn}(\Sigma W_n) \geq n+1$ for each $n \geq 0$ by the argument of 9.16. Moreover, as we explain in 13.7 and 13.8, we conjecture for $n \geq 0$ that $\operatorname{conn}(\Sigma W_n) = n+1$ and that W_n may be chosen as $B \vee K(Z/p, n+1)$ for any space $B \in FHo_{(p)}$ of type $n+1$. This is known for $n = 0, 1$. In general,

$$\operatorname{conn}(\Sigma W_n) \leq \operatorname{conn}(\Sigma W_{n+1})$$

for each $n \geq 0$, by the argument of 9.16, and thus $\langle\Sigma W_{n+1}\rangle \leq \langle\Sigma W_n\rangle$ by 9.12. We let

$$c(n) = \operatorname{conn}(\Sigma W_n) + 1.$$

10.2. **The v_n-periodization of spaces.** For $n \geq 0$ and $Y \in Ho_*$, we let $P_{v_n} Y$ denote $P_{\Sigma W_n} Y$ and call $u : Y \to P_{v_n} Y$ the *v_n-periodization* or *v_n-localization* of Y. We also call a ΣW_n-periodic space *v_n-periodic* or *v_n-local* and call a

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$P_{\Sigma W_n}$-equivalence a P_{v_n}*-equivalence.* For any $B \in FHo_{(p)}$ of type $n+1$, we may immediately recover $u: Y \to P_{\Sigma B} Y$ from $u: Y \to P_{v_n} Y$ by using a Moore-Postnikov factorization as in 7.3. Note that the fibration theorems of Section 8 apply to v_n-periodizations for $n \geq 0$. Thus

Theorem 10.3. *For a homotopy fiber sequence $F \to X \to B$ of pointed spaces, the map from $P_{v_n} F$ to the homotopy fiber of $P_{v_n} X \to P_{v_n} B$ has homotopy fiber of the form $K(G, c(n)-1)$ for a p-torsion abelian group G.*

10.4. **The chromatic tower of spaces.** Since $\langle \Sigma W_{n+1} \rangle \leq \langle \Sigma W_n \rangle$ for $n \geq 0$, there is a natural tower

$$P_{v_0} Y \leftarrow P_{v_1} Y \leftarrow P_{v_2} Y \leftarrow \cdots$$

under each $Y \in Ho_*$, and the homotopy inverse limit is equivalent to Y by 2.9. Like the chromatic tower in stable homotopy theory, this provides successive approximations to Y allowing "successively higher sorts of periodicity" (see 11.9).

The v_n-periodization is well understood on Postnikov spaces.

Proposition 10.5. *For $n \geq 0$ and a connected space $Y \in Ho_*$ with $\pi_i Y = 0$ for sufficiently large i, there are natural isomorphisms*

$$\pi_i P_{v_n} Y \simeq \begin{cases} \pi_i Y & \text{for } i < c(n), \\ \pi_i Y//(Z/p) & \text{for } i = c(n), \\ \pi_i Y \otimes Z[1/p] & \text{for } i > c(n), \end{cases}$$

where $\pi_i Y//(Z/p)$ is the quotient of $\pi_i Y$ by its p-torsion subgroup.

Proof. For $A = M^{c(n)}(p)$, the homotopy fiber, Fib u, of $u: Y \to P_{\Sigma A} Y$ is a $(c(n)-1)$-connected p-torsion Postnikov space by 5.2. By 7.4, $P_{v_n} K(G, j) \simeq *$ for each p-torsion abelian group G and $j \geq c(n)$. Thus by 4.8, $P_{v_n}(\text{Fib } u) \simeq *$ and $u: Y \to P_{\Sigma A} Y$ is a P_{v_n}-equivalence. The result now follows from 5.2 since $P_{\Sigma A} Y$ is v_n-periodic because $\langle \Sigma W_n \rangle \leq \langle \Sigma A \rangle$ by 9.12.

11. On v_n-periodization of spaces and v_m-periodic homotopy groups

We continue to work over a fixed prime p and proceed to establish close relations between the theory of v_n-periodization of spaces and that of v_m-periodic homotopy groups. These groups, especially for $m = 1$ and $m = 2$, have figured prominently in recent work of Mahowald, Thompson, and others (see [20], [23], [24]).

We begin by recalling convenient versions of the stable nilpotence and periodicity theorems of Devinatz, Hopkins, and Smith [12], [16], [17], [35]. As in 9.13, let $FHo_{(p)} \subset Ho_*$ be the full subcategory of all $X \in Ho_*$ such that $\bigoplus_{i=0}^{\infty} \widetilde{H}_i(X; Z)$ is a finitely generated $Z_{(p)}$-module. Two maps $u, v: X \to Y$ in $FHo_{(p)}$ are called *stably homotopic* when $\Sigma^k u \simeq \Sigma^k v$ for some $k \geq 0$. For a self-map $f: \Sigma^d X \to X$ in $FHo_{(p)}$ and $i > 0$, let $f^i: \Sigma^{id} X \to X$ denote the

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composite $f \circ \Sigma^d f \circ \cdots \circ \Sigma^{(i-1)d} f$, and call f *stably nilpotent* when f^i is stably homotopic to the trivial map for some $i > 0$. Devinatz, Hopkins, and Smith [12], [16], [35] have proved

Theorem 11.1 (Nilpotence theorem). *For a self-map $f : \Sigma^d X \to X$ in $FHo_{(p)}$ with $d \geq 0$, if $\widetilde{K(m)}_* f = 0$ for all $m \geq 0$, then f is stably nilpotent.*

For a space $X \in FHo_{(p)}$ of type n with $0 < n < \infty$, a map $\omega : \Sigma^d X \to X$ with $d \geq 0$ is called a v_n *self-map* when $\widetilde{K(n)}_* \omega$ is an isomorphism and $\widetilde{K(m)}_* \omega = 0$ for all $m \neq n$. Hopkins and Smith [16], [17], [35] have proved

Theorem 11.2 (Periodicity theorem). *Let $X \in FHo_{(p)}$ be a space of type n with $0 < n < \infty$. Then:*

(i) *$\Sigma^k X$ has a v_n self-map for some $k \geq 0$;*
(ii) *if $\omega : \Sigma^d X \to X$ is a v_n self-map of X, then d is a positive multiple of $2p^n - 2$;*
(iii) *if $\omega : \Sigma^d X \to X$ and $\tau : \Sigma^e X \to X$ are v_n self-maps of X, then $\omega^i : \Sigma^{id} X \to X$ is stably homotopic to $\tau^j : \Sigma^{je} X \to X$ for some $i, j > 0$ with $id = je$.*

11.3. **The v_m-periodic homotopy groups.** For each $m \geq 1$, we may choose a pointed space V_{m-1} and a pointed map $\omega : \Sigma^d V_{m-1} \to V_{m-1}$ such that V_{m-1} is equivalent to a finite CW-complex of type m in $FHo_{(p)}$ and ω represents a v_m self-map of V_{m-1} in $FHo_{(p)}$.

For a space $Y \in Ho_*$ and integer t, we let $\pi_t(Y; V_{m-1}) = [\Sigma^t V_{m-1}, Y]$ and define the v_m*-periodic homotopy group* $v_m^{-1}\pi_t(Y; V_{m-1})$ as the colimit of the sequence

$$\pi_t(Y; V_{m-1}) \xrightarrow{\omega^*} \pi_{t+d}(Y; V_{m-1}) \xrightarrow{\omega^*} \pi_{t+2d}(Y; V_{m-1}) \longrightarrow \cdots .$$

Although $v_m^{-1}\pi_t(Y; V_{m-1})$ depends on the choice of V_{m-1}, it does not depend on the choice of ω by 11.2. Note that these groups are periodic with

$$\omega^* : v_m^{-1}\pi_t(Y; V_{m-1}) \cong v_m^{-1}\pi_{t+d}(Y; V_{m-1}),$$

and that each fiber sequence $F \to X \to B$ has a long exact sequence of v_m-periodic homotopy groups. Following Mahowald and Thompson [23], for $m \geq 1$ and a pointed fibrant space Y, we construct the homotopy colimit space $T_m Y$ of the sequence

$$\mathrm{map}_*(V_{m-1}, Y) \xrightarrow{\omega^*} \mathrm{map}_*(\Sigma^d V_{m-1}, Y) \xrightarrow{\omega^*} \mathrm{map}_*(\Sigma^{2d} V_{m-1}, Y) \longrightarrow \cdots .$$

There is a natural isomorphism

$$\pi_t T_m Y \cong v_m^{-1}\pi_t(Y; V_{m-1})$$

for $t \geq 0$ and a natural equivalence

$$\omega^* : T_m Y \simeq \Omega^d T_m Y$$

making $T_m Y$ an infinite loop space.

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Lemma 11.4. *The space $T_m Y$ is v_n-periodic for each $n \geq m$.*

Proof. Since $T_m Y$ is a periodic infinite loop space and $\langle \Sigma V_n \rangle = \langle \Sigma^j W_n \rangle$ for some $j \geq 1$, it suffices to show that $T_m Y$ is ΣV_n-periodic for $n \geq m$. Since $\Sigma V_n \in Ho_*$ is equivalent to a finite CW-complex, it now suffices by exponentiation to show that the tower

$$\Sigma V_n \wedge V_{m-1} \longleftarrow \Sigma V_n \wedge \Sigma^d V_{m-1} \longleftarrow \Sigma V_n \wedge \Sigma^{2d} V_{m-1} \longleftarrow \cdots$$

is pro-trivial in Ho_*. This follows by observing that $\widetilde{K}(j)_*(\Sigma V_n \wedge \omega) = 0$ for each $j \geq 0$, and then applying 11.1 to show that

$$\Sigma V_n \wedge \omega : \Sigma V_n \wedge \Sigma^d V_{m-1} \longrightarrow \Sigma V_n \wedge V_{m-1}$$

is stably nilpotent.

Theorem 11.5. *For each space $Y \in Ho_*$ and each $n \geq m \geq 1$, the v_n-periodization map $u : Y \longrightarrow P_{v_n} Y$ induces an isomorphism*

$$v_m^{-1}\pi_*(Y ; V_{m-1}) \cong v_m^{-1}\pi_*(P_{v_n} Y ; V_{m-1}).$$

Thus each P_{v_n}-equivalence in Ho_* is a $v_m^{-1}\pi_*(- ; V_{m-1})$-equivalence for $n \geq m \geq 1$.

Proof. We may assume that Y is connected and fibrant. The diagram

$$\begin{array}{llll}
P_{\Sigma V_n}\mathrm{map}_*(V_{m-1}, Y) & \rightarrow P_{\Sigma V_n}\mathrm{map}_*(\Sigma^d V_{m-1}, Y) & & \\
\quad\downarrow & \qquad\downarrow & & \\
P_{\Sigma V_n}\mathrm{map}_*(V_{m-1}, P_{\Sigma V_n} Y) & \rightarrow P_{\Sigma V_n}\mathrm{map}_*(\Sigma^d V_{m-1}, P_{\Sigma V_n} Y) & & \\
 & \qquad\qquad \rightarrow & P_{\Sigma V_n}\mathrm{map}_*(\Sigma^{2d} V_{m-1}, Y) & \rightarrow \cdots \\
 & & \quad\downarrow & \\
 & \qquad\qquad \rightarrow & P_{\Sigma V_n}\mathrm{map}_*(\Sigma^{2d} V_{m-1}, P_{\Sigma V_n} Y) & \rightarrow \cdots
\end{array}$$

induces a map $T_m u : T_m Y \rightarrow T_m P_{\Sigma V_n} Y$ of homotopy colimits by 11.4 and 11.6 below. Since each vertical map is a π_i-equivalence for $i \geq \mathrm{conn}(\Sigma V_n) + 2$ by 8.3, so is $T_m u : T_m Y \rightarrow T_m P_{\Sigma V_n} Y$. Thus, by periodicity, we find

$$v_m^{-1}\pi_*(Y ; V_{m-1}) \cong v_m^{-1}\pi_*(P_{\Sigma V_n} Y ; V_{m-1}) \cong v_m^{-1}\pi_*(P_{\Sigma W_n} Y ; V_{m-1}).$$

We have used

Lemma 11.6. *For a directed system $\{X_\alpha\}$ of pointed spaces and a finite CW-complex $A \in Ho_*$, there is a natural equivalence*

$$P_A(\operatorname*{hocolim}_\alpha X_\alpha) \simeq \operatorname*{hocolim}_\alpha(P_A X_\alpha).$$

Proof. This follows by 2.5 since the right side is clearly A-periodic.

Proposition 11.7. *For each $Y \in Ho_*$ and $n \geq 1$, the natural map*

$$\pi_t(P_{v_n} Y ; V_{n-1}) \longrightarrow v_n^{-1}\pi_t(P_{v_n} Y ; V_{n-1})$$

is monic when $t = 1$ and an isomorphism when $t \geq 2$.

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Proof. The map $\omega : \Sigma^d V_{n-1} \to V_{n-1}$ has homotopy cofiber, $\mathrm{Cof}\,\omega$, of type $\geq n+1$. Thus, for $t \geq 1$, $\langle \Sigma^t \mathrm{Cof}\,\omega \rangle \leq \langle \Sigma W_n \rangle$ and $[\Sigma^t \mathrm{Cof}\,\omega, P_{v_n} Y] = 0$ as required.

Proposition 11.8. *For each space $Y \in Ho_*$ and $m > n \geq 0$,*

$$\pi_t(P_{v_n} Y ; V_{m-1}) \cong v_m^{-1} \pi_t(P_{v_n} Y ; V_{m-1}) \cong 0$$

when $t \geq 1$.

Proof. This follows since $\langle \Sigma^t V_{m-1} \rangle \leq \langle \Sigma W_n \rangle$.

11.9. **Periodic homotopy groups and the chromatic tower.** For $Y \in Ho_*$, we consider the chromatic tower

$$P_{v_0} Y \longleftarrow P_{v_1} Y \longleftarrow P_{v_2} Y \longleftarrow \cdots$$

under Y as in 10.4, and let $\widetilde{P}_{v_n} Y$ denote the homotopy fiber of $P_{v_n} Y \to P_{v_{n-1}} Y$. By 11.5, 11.7, and 11.8, there are natural isomorphisms

$$v_m^{-1} \pi_*(P_{v_n} Y ; V_{m-1}) \cong \begin{cases} v_m^{-1} \pi_*(Y ; V_{m-1}) & \text{for } m \leq n, \\ 0 & \text{for } m > n, \end{cases}$$

$$v_m^{-1} \pi_*(\widetilde{P}_{v_n} Y ; V_{m-1}) \cong \begin{cases} v_n^{-1} \pi_*(Y ; V_{n-1}) & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases}$$

Moreover, for $t \geq 2$ and $n \geq 1$, there are natural isomorphisms

$$v_n^{-1} \pi_t(Y ; V_{n-1}) \cong \pi_t(P_{v_n} Y ; V_{n-1}) \cong \pi_t(\widetilde{P}_{v_n} Y ; V_{n-1}).$$

Thus, $\widetilde{P}_{v_n} Y$ captures the groups $v_n^{-1} \pi_t(Y ; V_{n-1})$ in "pure harmonic form", and exposes them as ordinary homotopy groups $\pi_t(\widetilde{P}_{v_n} Y ; V_{n-1})$. We now turn to a study of periodic homotopy equivalences.

Theorem 11.10. *For a map $\varphi : X \to Y$ in Ho_* and $n \geq 1$, the following are equivalent*:

(i) $\varphi_* : \pi_i P_{v_n} X \cong \pi_i P_{v_n} Y$ *for sufficiently large i*;

(ii) $\varphi_* : v_m^{-1} \pi_*(X ; V_{m-1}) \cong v_m^{-1} \pi_*(Y ; V_{m-1})$ *for each m with $1 \leq m \leq n$ and $\varphi_* : \pi_i X \otimes Z[1/p] \cong \pi_i Y \otimes Z[1/p]$ for sufficiently large i.*

Proof. By a homotopy fiber argument using 10.3, it suffices to show, for a connected space $F \in Ho_*$, the equivalence of:

(i)$'$ $\pi_i P_{v_n} F \cong 0$ for sufficiently large i;

(ii)$'$ $v_m^{-1} \pi_*(F ; V_{m-1}) \cong 0$ for each m with $1 \leq m \leq n$ and $\pi_i F \otimes Z[1/p] \cong 0$ for sufficiently large i.

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Given (i)$'$, we deduce that $v_m^{-1}\pi_*(P_{v_n}F; V_{m-1}) \cong 0$ for $1 \le m \le n$ and that $\pi_i P_{v_0} F \cong \pi_i P_{v_0} P_{v_n} F \cong 0$ for sufficiently large i. This implies (ii)$'$ by 11.5 and 5.2. Given (ii)$'$, we deduce that $P_{v_n}F$ is ΣV_{m-1}-periodic for $1 \le m \le n$ since

$$\pi_t(P_{v_m}F; V_{m-1}) \cong v_m^{-1}\pi_t(P_{v_m}F; V_{m-1}) \cong 0$$

for $t \ge 1$. Hence, $P_{v_m}F$ is $\Sigma^k W_{m-1}$-periodic for some $k \ge 1$, and $\pi_i P_{v_m} F \cong \pi_i P_{v_{m-1}} F$ for sufficiently large i by 7.2. Since $\pi_i P_{v_0} F \cong \pi_i F \otimes Z[1/p] \cong 0$ for sufficiently large i, condition (i)$'$ follows.

Corollary 11.11. *For a map φ in Ho_* and $n \ge 1$, the following are equivalent:*

(i) $\varphi_* : \pi_i \widetilde{P}_{v_n} X \cong \pi_i \widetilde{P}_{v_n} Y$ *for sufficiently large* i;

(ii) $\varphi_* : v_n^{-1}\pi_*(X; V_{n-1}) \cong v_n^{-1}\pi_*(Y; V_{n-1})$.

Proof. This follows by 11.9 and 11.10 since $\widetilde{P}_{v_n}X$ is v_n-periodic and simply connected with p-torsion homotopy.

This shows that *the notion of a $v_n^{-1}\pi_*(-; V_{n-1})$-equivalence does not depend on the choice of V_{n-1}*. Finally, applying the theory of v_n-periodizations, we obtain some useful general isomorphisms of v_m-periodic homotopy groups.

Theorem 11.12. *Let $A \to X \to Y$ be a homotopy cofiber sequence, and let Z be a pointed space. If $\langle A \rangle \le \langle \Sigma W_n \rangle$ for some $n \ge 1$, then the map $X \to Y$ and the inclusion $Z \subset Z \vee A$ induce isomorphisms*

$$v_m^{-1}\pi_*(X; V_{m-1}) \cong v_m^{-1}\pi_*(Y; V_{m-1}),$$
$$v_m^{-1}\pi_*(Z; V_{m-1}) \cong v_m^{-1}\pi_*(Z \vee A; V_{m-1})$$

for each m with $1 \le m \le n$.

Proof. Since $\langle A \rangle \le \langle \Sigma W_n \rangle$, the maps $* \to A$ and $A \to *$ are P_{v_n}-equivalences, and hence so are the maps $X \to Y$ and $Z \to Z \vee A$ by 2.5. The result now follows by 11.10.

This theorem may be used with

Lemma 11.13. *For $n \ge 1$, if B is a space of type $\ge n+1$ in $FHo_{(p)}$, then $\langle \Sigma B \rangle \le \langle \Sigma W_n \rangle$.*

Proof. Since $\operatorname{type}(\Sigma B) \ge n+1$, 9.14 and 10.1 imply that $\{\Sigma B\} \le \{\Sigma \overline{V}_n\} = \{\Sigma W_n\}$ and

$$\operatorname{conn} \widetilde{H}^*(\Sigma B; Z/p) = \operatorname{conn}(\Sigma B) \ge \operatorname{conn}(\Sigma W_n) = \operatorname{conn} \widetilde{H}^*(\Sigma W_n; Z/p).$$

The lemma now follows from 9.12.

Corollary 11.14. *For $n \ge 1$, if $\varphi : X \to Y$ is a $K(n)_*$-equivalence in $FHo_{(p)}$, then $\Sigma^2\varphi : \Sigma^2 X \to \Sigma^2 Y$ is a $v_m^{-1}\pi_*(-; V_{m-1})$-equivalence for each m with $1 \le m \le n$.*

Proof. Apply 11.13 with $B = \operatorname{Cof}\varphi$, and then apply 11.12 with $A = \Sigma B$.

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12. Homological properties of the v_n-periodization and a theorem of Thompson

Working over a fixed prime p, we now discuss homological properties of the v_n-periodization and give a homological criterion, based on a theorem of Thompson, for a map to be a $v_m^{-1}\pi_*(-;V_{m-1})$-equivalence. This criterion will involve a homology theory $S(m)_*$ which could be replaced by the Morava K-theories $K(i)_*$ for $0 \le i \le m$ if an unstable telescope conjecture stated in 12.3 were confirmed. The results of this section are preparatory to more definitive results in Section 13 on P_{v_n}-equivalences of highly connected spaces. We start with

Proposition 12.1. *For $n \geq 0$, the v_n-periodization $u: Y \to P_{v_n} Y$ of a space $Y \in Ho_*$ is both an E_*-equivalence and E^*-equivalence for each spectrum E with $\widetilde{E}^*(V_n) = 0$.*

Proof. Since V_n is finite, the condition $\widetilde{E}^*(V_n) = 0$ implies $\widetilde{E}_*(V_n) = 0$, and this implies $(\widetilde{\nabla E})^*(V_n) = 0$ by 4.11. Since $\{W_n\} = \{V_n\}$, we deduce that $\widetilde{E}^*(W_n) = 0$, $(\widetilde{\nabla E})^*(W_n) = 0$, and $\widetilde{E}_*(W_n) = 0$. The result now follows by the construction of $u: Y \to P_{\Sigma W_n} Y$ in 2.8.

For $m \leq n$, the BP-related spectra $v_m^{-1}BP$, $E(m)$, and $K(m)$ all satisfy the hypotheses of 12.1, and thus the associated (co)homologies are all preserved by v_n-periodizations and P_{v_n}-equivalences. We shall need the "strongest" possible example, $S(n)$, of a p-local spectrum satisfying the above hypotheses.

12.2. **The spectrum** $S(n)$. For $n \geq 0$ let $\mathscr{A}(V_n)$ be the class of all spectra E with $\widetilde{E}^*(V_n) \cong 0$. As explained more generally in [4] and [5, §3], each spectrum X has a universal (initial) map $u: X \to X//V_n$ to a spectrum $X//V_n \in \mathscr{A}(V_n)$. We let $S(n)$ denote $S_{(p)}//V_n$, where $S_{(p)}$ is the p-localized sphere spectrum, and recall that $S(n)$ is a commutative ring spectrum whose multiplication gives an equivalence $S(n) \wedge S(n) \simeq S(n)$. Moreover, there is a natural equivalence $S(n) \wedge X \simeq X_{(p)}//V_n$ for an arbitrary spectrum X. As noted by Hal Sadofsky, when X is a finite p-local spectrum of type n, the spectrum $S(n) \wedge X$ may alternatively be obtained as a Ravenel telescope $\widehat{X}$ using the construction in [33], [34], [35]. The stable class $\langle S(n) \rangle$ is the complement of $\langle \Sigma^\infty V_n \rangle$ in $\langle S_{(p)} \rangle$ by [4] and thus satisfies

$$\langle S(n) \rangle \geq \bigvee_{i=0}^{n} \langle K(i) \rangle = \langle E(n) \rangle = \langle v_n^{-1}BP \rangle$$

by [33, Theorem 2.1]. Ravenel's telescope conjecture is equivalent to the assertion that

$$\langle S(n) \rangle = \bigvee_{i=0}^{n} \langle K(i) \rangle$$

for each n. This holds for $n = 0, 1$, as shown in [5, Corollary 4.9] using work of Mahowald and Miller, but has now been refuted by Ravenel [34] for $n = 2$ when $p \geq 5$. However, the following unstable version remains plausible.

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12.3. **An unstable telescope conjecture.** *A space is $S(n)_*$-acyclic if and only if it is $K(i)_*$-acyclic for each i with $0 \le i \le n$.*

In view of the $K(i)_*$-acyclicity results of [36], a weaker and even more plausible version is

12.4. **A weak unstable telescope conjecture.** *For each $j > n \ge 0$, $K(Z/p, j)$ is $S(n)_*$-acyclic.*

We presently know 12.3 and 12.4 only for $n = 0, 1$. Note that $S(0) \simeq HQ$ and $S(1)$ is the $K_{(p)*}$-localization of the sphere spectrum (see [5], [33]). We now generalize a key result proved by Thompson [39] in the case $m = 1$. Our proof is a direct adaptation of his.

Theorem 12.5. *For $m \ge 1$ and $k \ge \dim V_{m-1}$, if $\varphi : X \to Y$ is a map in Ho_* such that $\Omega^k\varphi : \Omega^k X \to \Omega^k Y$ is an $S(m)_*$-equivalence, then*

$$\varphi_* : v_m^{-1}\pi_*(X; V_{m-1}) \cong v_m^{-1}\pi_*(Y; V_{m-1}).$$

Proof. We may assume that X and Y are k-connected and fibrant. Since φ induces an $S(m)_*$-equivalence $\Omega^k X \to \Omega^k Y$, it must induce an $S(m)_*$-equivalence $\mathrm{map}_*(F, X) \to \mathrm{map}_*(F, Y)$ for each pointed connected finite space F with $\dim F \le k$, by an inductive argument using 4.6. Using the functor T_m of 11.3, we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{map}_*(V_{m-1}, X) \arrow[r] \arrow[d, "\varphi_*"] & T_m X \arrow[d] \\ \mathrm{map}_*(V_{m-1}, Y) \arrow[r] \arrow[ur, dashed, "\lambda"] & T_m Y \end{tikzcd}$$

where φ_* is an $S(m)_*$-equivalence, and where the periodic infinite loop spaces $T_m X$ and $T_m Y$ are $S(m)_*$-local by 11.4. Hence, there is a unique map λ : $\mathrm{map}_*(V_{m-1}, Y) \to T_m X$ making both triangles commute in Ho_*. The map $\omega : \Sigma^d V_{m-1} \to V_{m-1}$ induces a map from the above solid arrow diagram to its d-fold looping, and the resulting square of diagonals

$$\begin{tikzcd} \mathrm{map}_*(V_{m-1}, Y) \arrow[r, "\lambda"] \arrow[d] & T_m X \arrow[d, "\simeq"] \\ \Omega^d \mathrm{map}_*(V_{m-1}, Y) \arrow[r, "\Omega^d\lambda"] & \Omega^d T_m X \end{tikzcd}$$

commutes since it is equalized by the $S(m)_*$-equivalence

$$\varphi_* : \mathrm{map}_*(V_{m-1}, X) \to \mathrm{map}_*(V_{m-1}, Y).$$

Thus all maps in

$$\begin{tikzcd} \pi_*\mathrm{map}_*(V_{m-1}, X) \arrow[r] \arrow[d, "\varphi_*"] & v_m^{-1}\pi_*(X; V_{m-1}) \arrow[d, "\varphi_*"] \\ \pi_*\mathrm{map}_*(V_{m-1}, Y) \arrow[r] \arrow[ur, "\lambda_*"] & v_m^{-1}\pi_*(Y; V_{m-1}) \end{tikzcd}$$

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respect the action of ω. Since the horizontal maps are algebraic localizations inverting ω, $\varphi_* : v_m^{-1}\pi_*(X; V_{m-1}) \cong v_m^{-1}\pi_*(Y; V_{m-1})$ as required.

Note. Each of the spaces V_{m-1}, chosen with $\omega : \Sigma^d V_{m-1} \to V_{m-1}$ in 11.3, is equivalent to a finite CW-complex, and $\dim V_{m-1}$ may be interpreted as the minimum possible dimension of such a complex. Thus, we may take $\dim V_0 = 3$ for p odd and $\dim V_0 = 5$ for $p = 2$ by 9.16.

The following corollary was obtained by R. Thompson [39] for $m = 1$.

Corollary 12.6. *For $m \geq 1$ and $k \geq \dim V_{m-1}$, if $\varphi : X \to Y$ is an $S(m)_*$-equivalence of pointed connected spaces, then*

$$(\Sigma^k\varphi)_* : v_m^{-1}\pi_*(\Sigma^k X; V_{m-1}) \cong v_m^{-1}\pi_*(\Sigma^k Y; V_{m-1}).$$

Proof. This follows by 12.5 since the functor $\Omega^k\Sigma^k : Ho_* \to Ho_*$ preserves generalized homology equivalences of connected spaces by the usual approximation theory (see [39]).

13. P_{v_n}-EQUIVALENCES OF HIGHLY CONNECTED SPACES

Working over a fixed prime p, we shall establish criteria in 13.3 and 13.15 for a map of $c(n)$-connected spaces to be a P_{v_n}-equivalence, where $c(n) = \operatorname{conn}(\Sigma W_n) + 1$. As explained in 13.8 below, we know that $c(n) \geq n + 2$ and conjecture that $c(n) = n+2$ for each $n \geq 0$. We first show that P_{v_n} commutes with the $c(n)$-connected cover functor.

Proposition 13.1. *For $n \geq 0$ and a space $Y \in Ho_*$, there is a natural equivalence*

$$P_{v_n}(Y\langle c(n)\rangle) \simeq (P_{v_n}Y)\langle c(n)\rangle.$$

Proof. Since $Y\langle c(n)\rangle$ is the homotopy fiber of the natural map from Y to its $c(n)$-Postnikov section, we may apply 10.3 and 10.5 to show that

$$\pi_i P_{v_n} Y\langle c(n)\rangle \to \pi_i P_{v_n} Y$$

is an isomorphism for $i \geq c(n) + 1$. The result now follows since $P_{v_n}(Y\langle c(n)\rangle)$ is $c(n)$-connected by 2.9.

13.2. **The v_n-periodization in $Ho_{c(n)}$.** For $n \geq 0$, let $Ho_{c(n)}$ be the full subcategory of Ho_* given by the $c(n)$-connected spaces, and note that P_{v_n} restricts to a functor $P_{v_n} : Ho_{c(n)} \to Ho_{c(n)}$. For a map $\varphi : X \to Y$ of pointed $c(n)$-connected spaces, let $\widetilde{\text{Fib}}\ \varphi$ denote the $c(n)$-connected cover of the homotopy fiber $\text{Fib}\ \varphi$. By 10.3 and 13.1, there is a natural equivalence

$$P_{v_n}(\widetilde{\text{Fib}}\ \varphi) \simeq \widetilde{\text{Fib}}(P_{v_n}\varphi)$$

for $n \geq 0$, and thus P_{v_n} preserves "fibrations in the sense of $c(n)$-connected homotopy theory". In particular, for $Y \in Ho_{c(n)}$, there is a natural equivalence

$$P_{v_n}(\widetilde{\Omega} Y) \simeq \widetilde{\Omega}(P_{v_n} Y),$$

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where $\widetilde{\Omega} : Ho_{c(n)} \to Ho_{c(n)}$ is the functor with $\widetilde{\Omega}X$ given by the $c(n)$-connected cover of ΩX. Thus, if $\varphi : X \to Y$ is a P_{v_n}-equivalence in $Ho_{c(n)}$, then so is $\widetilde{\Omega}^k\varphi : \widetilde{\Omega}^k X \to \widetilde{\Omega}^k Y$ for each $k \geq 0$. Our first characterization of the P_{v_n}-equivalences in $Ho_{c(n)}$ is

Theorem 13.3. *For $n \geq 0$, the following conditions on a map $\varphi : X \to Y$ in $Ho_{c(n)}$ are equivalent:*

(i) *φ is a P_{v_n}-equivalence;*

(ii) *$\varphi_* : v_m^{-1}\pi_*(X; V_{m-1}) \cong v_m^{-1}\pi_*(Y; V_{m-1})$ for each m with $1 \leq m \leq n$ and $\varphi_* : \pi_* X \otimes Z[1/p] \cong \pi_* Y \otimes Z[1/p]$.*

Proof. Since X and Y are simply connected, the condition $\varphi_* : \pi_* X \otimes Z[1/p] \cong \pi_* Y \otimes Z[1/p]$ is equivalent to $\varphi_* : H_*(X; Z[1/p]) \cong H_*(Y; Z[1/p])$. Thus (i) $\Rightarrow$ (ii) by 5.6 and 11.5. Given (ii), we deduce from 2.9 and 11.10 that the map $P_{v_n}\varphi : P_{v_n}X \to P_{v_n}Y$ has homotopy fiber $\mathrm{Fib}(P_{v_n}\varphi)$ such that $\pi_i\mathrm{Fib}(P_{v_n}\varphi)$ is: trivial for $i < c(n)$, trivial for sufficiently large i, and p-torsion for all i. Since $\mathrm{Fib}(P_{v_n}\varphi)$ is v_n-periodic, we may now apply 10.5 to deduce

$$\mathrm{Fib}(P_{v_n}\varphi) \simeq P_{v_n}\mathrm{Fib}(P_{v_n}\varphi) \simeq *,$$

and thus to deduce (i).

This theorem combines with 13.1 and 10.5 to show

Corollary 13.4. *For $n \geq 0$ and a space $Y \in Ho_*$, the condition $P_{v_n}Y \simeq *$ is equivalent to the combined conditions:*

(i) *$\pi_i Y = 0$ for $i < c(n)$;*

(ii) *$\pi_* Y$ is p-torsion;*

(iii) *$v_m^{-1}\pi_*(Y; V_{m-1}) = 0$ for each m with $1 \leq m \leq n$.*

Using Theorem 13.3, we may also reformulate Thompson's result for suspensions (12.6) as

Theorem 13.5. *For $n \geq 0$ and $k \geq \max\{2, \dim V_0, \ldots, \dim V_{n-1}\}$, if $f : X \to Y$ is an $S(n)_*$-equivalence of pointed connected spaces, then $(\Sigma^k f)_{(p)} : (\Sigma^k X)_{(p)} \to (\Sigma^k Y)_{(p)}$ is a P_{v_n}-equivalence in $Ho_{c(n)}$.*

Proof. Since f is an $S(n)_*$-equivalence, it is an $S(m)_*$-equivalence for $m \leq n$. Thus f induces isomorphisms $\pi_*(\Sigma^k X) \otimes Q \cong \pi_*(\Sigma^k Y) \otimes Q$ and

$$v_m^{-1}\pi_*(\Sigma^k X; V_{m-1}) \cong v_m^{-1}\pi_*(\Sigma^k Y; V_{m-1})$$

for $1 \leq m \leq n$ by 12.6. Using the homotopy cofiber $\mathrm{Cof}\,\omega$ of $\omega : \Sigma^d V_{n-1} \to V_{n-1}$, we find

$$\begin{aligned} k &\geq \dim V_{n-1} \geq \mathrm{conn}(\Sigma V_{n-1}) + 1 = \mathrm{conn}(\Sigma\ \mathrm{Cof}\,\omega) + 1 \\ &\geq \mathrm{conn}(\Sigma W_n) + 1 = c(n) \end{aligned}$$

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for $n \geq 1$. Thus, for $n \geq 0$, $\Sigma^k X$ and $\Sigma^k Y$ are in $Ho_{c(n)}$, and the result follows from 13.3.

This leads to a homological characterization of the stable P-class $\{V_n\}$.

Proposition 13.6. *For $n \geq 0$ and $X \in Ho_*$, the following are equivalent:*

(i) $\{X\} \leq \{V_n\}$;

(ii) *X is $S(n)_*$-acyclic and $\widetilde{H}_*(X; Z)$ is p-local.*

Proof. Since $\widetilde{S}(n)_* V_n = 0$ and $\widetilde{H}_*(V_n; Z/q) = 0$ for each prime $q \neq p$, (i) implies (ii) as in the proof of 12.1. Given (ii), 13.5 implies $P_{v_n}(\Sigma^k X) \simeq *$ for $k \geq \max\{2, \dim V_0, \dots, \dim V_{n-1}\}$. Thus $\langle \Sigma^k X \rangle \leq \langle \Sigma W_n \rangle$ and $\{X\} \leq \{W_n\} = \{V_n\}$.

We now obtain a homological characterization of the integer $c(n) = \operatorname{conn}(\Sigma W_n) + 1$.

Proposition 13.7. *For $n \geq 0$, let c_n be the integer determined by the conditions that $\widetilde{S(n)}_* K(Z/p, j) \neq 0$ for $j < c_n - 1$ and $\widetilde{S(n)}_* K(Z/p, j) = 0$ for $j \geq c_n - 1$. Then $c(n) = c_n$ and we may choose $W_n = B \vee K(Z/p, c_n - 1)$ for any space $B \in FHo_{(p)}$ of type $n+1$.*

Proof. The space W_n chosen in 10.1 must be $S(n)_*$-acyclic by the proof of 12.1, and thus $K(Z/p, c(n) - 1)$ is also $S(n)_*$-acyclic by 7.4 and 4.11. Hence, $c(n) \geq c_n$. By 13.6 and 9.14,

$$\{K(Z/p, c_n - 1)\} \leq \{V_n\} = \{B\} = \{\overline{V}_n\}$$

and thus

$$\{B \vee K(Z/p, c_n - 1)\} = \{\overline{V}_n\}.$$

This implies that $c_n \geq c(n)$ and the proposition follows easily.

13.8. **The value of $c(n)$.** The weak unstable telescope conjecture of 12.4 now implies that $c(n) = c_n = n + 2$ for all $n \geq 0$. Since this conjecture is known for $n = 0, 1$, we conclude that $c(0) = 2$ and $c(1) = 3$. We also know that $c(n) \geq n + 2$ since $\widetilde{K(n)}_* K(Z/p, n) \neq 0$ for $n \geq 0$.

13.9. **The integer $e(n)$.** Although $\widetilde{S(n)}_* K(Z/p, c(n) - 2) \neq 0$ for $n \geq 0$, it is not evident that $\widetilde{S(n)}_* K(Z_{p^\infty}, c(n) - 2) \neq 0$. We let $e(n) = 0$ when this holds and let $e(n) = 1$ when $\widetilde{S(n)}_* K(Z_{p^\infty}, c(n) - 2) = 0$. Since

$$\widetilde{K(n)}_* K(Z_{p^\infty}, n) \cong \widetilde{K(n)}_* K(Z, n+1) \neq 0,$$

we know that $\widetilde{S(n)}_* K(Z_{p^\infty}, n) \neq 0$. Thus the weak unstable telescope conjecture implies that $e(n) = 0$ for all n, and we know that $e(0) = 0$ and $e(1) = 0$. We also know that the condition $\widetilde{S(n)}_* K(Z_{p^\infty}, j) = 0$ implies $\widetilde{S(n)}_* K(Z/p, j + e(n)) = 0$.

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Theorem 13.10. *For $n \geq 0$ and $k \geq 1 + e(n)$, if $X \in Ho_*$ is an $S(n)_*$-acyclic space, then $P_{v_n}(\Sigma^k X)_{(p)} \simeq *$ or equivalently $\langle(\Sigma^k X)_{(p)}\rangle \leq \langle \Sigma W_n \rangle$.*

Proof. The result is clear for $n = 0$ since $(\Sigma X)_{(p)}$ is a 1-connected p-torsion space and $\langle \Sigma W_0 \rangle = \langle M^3(p) \rangle$. For $n \geq 1$, the condition $\widetilde{S(n)}_* X = 0$ implies that $\widetilde{H}_*(X; Q) = 0$, and thus $\widetilde{H}_*(X; Z_{(p)})$ is p-torsion. Let $\widetilde{H}_m(X; Z_{(p)})$ be the first nontrivial $Z_{(p)}$-homology group of X, let $G = Z_{p^\infty}$ when $\widetilde{H}_m(X; Z_{(p)})$ is p-divisible, and let $G = Z/p$ otherwise. Then $\widetilde{Z}_{(p)}X$ and $K(G, m)$ are $S(n)_*$-acyclic by 6.5, 4.11, and 4.6. Hence $K(Z/p, m + e(n))$ is also $S(n)_*$-acyclic and $m + e(n) \geq c(n) - 1$ by 13.7. In particular, $m \geq 2$ and $\widetilde{H}_1(X; Z_{(p)}) = 0$. Using the p-localization $X_{(p)} = (Z_{(p)})_\infty X$ of [9], we find that $X \to X_{(p)}$ is an $H_*(-; Z_{(p)})$-equivalence and $X_{(p)}$ is an $(m-1)$-connected p-torsion space with $\widetilde{H}_*(X_{(p)}; Z) \cong \widetilde{H}_*(X; Z_{(p)})$. Since $(\Sigma^i X)_{(p)} \simeq \Sigma^i(X_{(p)})$ for $i \geq 0$, we may apply 9.10 and 9.11 to show

$$\langle(\Sigma^k X)_{(p)}\rangle = \langle(\Sigma^{j+k} X)_{(p)}\rangle \vee \langle K(G, m+k)\rangle$$

for $j \geq 0$. The theorem now follows since $P_{v_n}(\Sigma^{j+k} X)_{(p)} \simeq *$ for sufficiently large j by 13.5 and $P_{v_n} K(G, m+k) \simeq *$ by 10.5 since $m + k \geq m + 1 + e(n) \geq c(n)$.

We can now prove a strengthened version of Thompson's result for suspensions (12.6, 13.5).

Corollary 13.11. *For $n \geq 0$ and $k \geq 2 + e(n)$, if $\varphi : X \to Y$ is an $S(n)_*$-equivalence of pointed spaces, then $(\Sigma^k \varphi)_{(p)} : (\Sigma^k X)_{(p)} \to (\Sigma^k Y)_{(p)}$ is a P_{v_n}-equivalence.*

Proof. Since Cof φ is $S(n)_*$-acyclic, $P_{v_n}(\Sigma^{k-1}\text{Cof } \varphi)_{(p)} \simeq *$ by 13.10, and the theorem follows by 2.5.

Corollary 13.12. *For $n \geq 0$ and $k \geq 2 + e(n)$, if $Y \in Ho_*$ is v_n-periodic with $\pi_i Y$ p-local for $i \geq k+1$, then $\Omega^k Y$ is $S(n)_*$-local.*

Proof. This follows from 13.11 by adjunction.

Theorem 13.13. *For $n \geq 0$ and $k \geq 1 + e(n)$, if $X \in Ho_*$ is a k-connected p-local space with $\widetilde{S(n)}_*(\Omega^k X) \cong 0$, then $P_{v_n} X \simeq *$.*

Proof. This follows since

$$\langle X \rangle \leq \langle \Sigma^k \Omega^k X \rangle = \langle(\Sigma^k \Omega^k X)_{(p)}\rangle \leq \langle \Sigma W_n \rangle$$

by 9.7 and 13.10.

A strengthened version of Thompson's result for loop spaces (12.5) is

Corollary 13.14. *For $n \geq 0$ and $k \geq 2 + e(n)$, if $\varphi : X \to Y$ is a map of pointed k-connected spaces such that $\Omega^k \varphi : \Omega^k X \to \Omega^k Y$ is an $S(n)_*$-equivalence and $\varphi_* : \pi_* X \otimes Z[1/p] \cong \pi_* Y \otimes Z[1/p]$, then $\varphi : X \to Y$ is a P_{v_n}-equivalence.*

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Proof. $P_{v_n}(\mathrm{Fib}\ \varphi) \simeq *$ by 4.6 and 13.13. Thus φ is a P_{v_n}-equivalence by 4.8.

Finally, we shall give a homological characterization of P_{v_n}-equivalences in $Ho_{c(n)}$ supplementing our homotopical characterization in 13.3. Using the $c(n)$-connected loop functor $\widetilde{\Omega}: Ho_{c(n)} \to Ho_{c(n)}$ of 13.2, we say that a map $\varphi : X \to Y$ in $Ho_{c(n)}$ is a *durable* E_*- (or E^*-) *equivalence* for a spectrum E when $\widetilde{\Omega}^k\varphi : \widetilde{\Omega}^k X \to \widetilde{\Omega}^k Y$ is an E_*- (or E^*-) equivalence for each $k \geq 0$.

Theorem 13.15. *For $n \geq 0$, the following conditions on a map $\varphi : X \to Y$ in $Ho_{c(n)}$ are equivalent:*

(i) *φ is a P_{v_n}-equivalence;*
(ii) *φ is a durable E_*- (and E^*-) equivalence for each spectrum E with $\widetilde{E}^*(V_n) = 0$;*
(iii) *φ is a durable $S(n)_*$-equivalence and $\varphi_* : \pi_*X \otimes Z[1/p] \cong \pi_*Y \otimes Z[1/p]$;*
(iv) *$\widetilde{\Omega}^k\varphi$ is an $S(n)_*$-equivalence for some $k \geq 2 + e(n)$ and $\varphi_* : \pi_*X \otimes Z[1/p] \cong \pi_*Y \otimes Z[1/p]$.*

Proof. If (i), then $\widetilde{\Omega}^k\varphi$ is a P_{v_n}-equivalence for each $k \geq 0$ and (ii) follows by 12.1. If (ii), then (iii) follows since $\widetilde{S}(n)^* V_n \cong 0$ and $\widetilde{H}^*(V_n; Z[1/p]) \cong 0$. If (iii), then (iv) follows immediately. If (iv), then $\varphi\langle c(n)+k\rangle$ is a P_{v_n}-equivalence by 13.14, and (i) follows by 13.3.

14. On $v_1^{-1}\pi_*(-; Z/p)$-equivalences and $K_*(-; Z/p)$-equivalences

In this final section, we use the preceding results to compare $v_1^{-1}\pi_*(-; Z/p)$-equivalences and $K_*(-; Z/p)$-equivalences. As an application, we confirm the conjecture that the Snaith map

$$s : \Omega_0^{2n+1}S^{2n+1} \to Q(\mathbb{R}P^{2n})$$

is a $K_*(-; Z/2)$-equivalence. We continue to work over a fixed prime p.

Let $M^n(p)$ denote the Z/p-Moore space $S^{n-1} \cup_p e^n$, and let $q = 2p - 2$ for p odd and $q = 8$ for $p = 2$. As in 9.16, choose Adams maps (i.e., K_*-equivalences)

$$A : M^{j+q}(p) \to M^j(p)$$

for $j \geq 3$ when p is odd and $j \geq 5$ when $p = 2$, where the maps for successive j are obtained by suspension. For a space $X \in Ho_*$, there is an induced operation

$$A : \pi_j(X; Z/p) \to \pi_{j+q}(X; Z/p)$$

on the mod-p homotopy groups

$$\pi_j(X; Z/p) = [M^j(p), X].$$

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As in 11.3, we define the *v_1-periodic homotopy groups* $v_1^{-1}\pi_*(X; Z/p)$ by inverting the action of A on $\pi_*(X; Z/p)$. The *v_1-periodization functor* $P_{v_1} = P_{\Sigma W_1} : Ho_* \to Ho_*$ may be defined using

$$W_1 = M^3(p) \underset{A}{\cup} CM^{3+q}(p)$$

for p odd and

$$W_1 = K(Z/2, 2) \vee (M^5(2) \underset{A}{\cup} CM^{5+q}(2))$$

for $p = 2$ by 13.7.

Since the $S(1)_*$-equivalences in Ho_* are the same as the $K_*(-; Z_{(p)})$-equivalences, and since $c(1) = 3$ and $e(1) = 0$, the homological results of Section 13 can immediately be reformulated in terms of $K_*(-; Z_{(p)})$. However, it is convenient to have the corresponding results for $K_*(-; Z/p)$. For these and for future reference we need

14.1. **The p-cocompletion of a nilpotent space.** For a nilpotent group G, let $\mathrm{tors}_p G \subset G$ denote the p-torsion subgroup. Let $NHo_* \subset Ho_*$ denote the full subcategory of all nilpotent spaces $X \in Ho_*$ such that $\pi_1 X/\mathrm{tors}_p \pi_1 X$ is uniquely p-divisible.

For $X \in NHo_*$, we let $\eta : t_p X \to X$ denote the homotopy fiber of the localization $X \to X[1/p]$ and call $\eta : t_p X \to X$ the *p-cocompletion* (or *p-torsion part*) of X. Note that $t_p X$ is a p-torsion nilpotent space, and $\eta_* : H_*(t_p X; Z/p) \cong H_*(X; Z/p)$. In general, if A is a pointed space with p-torsion homology $\widetilde{H}_*(A; Z)$ and if $f : Y \to Y'$ is an $H_*(-; Z/p)$-equivalence of pointed nilpotent spaces, then $\mathrm{map}_*(A, Y) \simeq \mathrm{map}_*(A, Y')$ and $[A, Y] \cong [A, Y']$ since $F_{p^\infty} f : F_{p^\infty} Y \simeq F_{p^\infty} Y'$ and $\mathrm{map}_*(A, Y) \simeq \mathrm{map}_*(A, F_{p^\infty} Y)$ by [25, Theorem 1.5]. Thus, the map $\eta : t_p X \to X$ in NHo_* is characterized, up to equivalence, by the conditions that $t_p X$ is a p-torsion nilpotent space and $\eta_* : H_*(t_p X; Z/p) \cong H_*(X; Z/p)$. We remark that the functor $t_p : NHo_* \to NHo_*$ is left adjoint to $F_{p^\infty} : NHo_* \to NHo_*$, and these functors restrict to equivalences between the full subcategories of p-torsion spaces and of p-complete spaces in NHo_*. The functor $t_p : NHo_* \to NHo_*$ has the advantage of respecting cofibrations as well as fibrations. A homotopy cofibration $A \to B \to C$ in NHo_* with $\pi_1 t_p B = 0$ is carried to a homotopy cofibration $t_p A \to t_p B \to t_p C$, and there is a natural equivalence $t_p(\Sigma A) \simeq \Sigma(t_p A)$ for all $A \in NHo_*$. A homotopy fibration $X \to Y \to Z$ in NHo_* is carried to a homotopy fibration $t_p X \to t_p Y \to t_p Z$, and there is a natural equivalence $t_p(\Omega Z) \simeq \Omega(t_p Z)$ for all simply connected $Z \in Ho_*$ with $\pi_2 Z/\mathrm{tors}_p \pi_2 Z$ uniquely p-divisible.

Lemma 14.2. *For a space $X \in NHo_*$, the p-cocompletion $\eta : t_p X \to X$ induces isomorphisms*

$$\eta_* : v_1^{-1}\pi_*(t_p X; Z/p) \cong v_1^{-1}\pi_*(X; Z/p),$$
$$\eta_* : K_*(t_p X; Z/p) \cong K_*(X; Z/p).$$

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Proof. This follows since $v_1^{-1}\pi_*(X[1/p]; Z/p) = 0$ and $\eta_* : H_*(t_p X; Z/p) \cong H_*(X; Z/p)$.

The following theorem and its three corollaries extend results of Thompson [39].

Theorem 14.3. *For $k \geq 1$, if $X \in Ho_*$ is a k-connected space with*

$$\widetilde{K}_*(\Omega^k X; Z/p) = 0,$$

then $v_1^{-1}\pi_*(X; Z/p) = 0$.

Proof. Since $\Omega^k X$ is $K_*(-; Z/p)$-acyclic and $K(Z_p^\wedge, 2)$ is $K_*(-; Z/p)$-local by [29], we have $H^i(\Omega^k X; Z_p^\wedge) = 0$ for $i = 0, 1$. Thus

$$\mathrm{Ext}(\pi_{k+1}X, Z_p^\wedge) = 0 = \mathrm{Hom}(\pi_{k+1}X, Z_p^\wedge)$$

and $\pi_{k+1}X$ is uniquely p-divisible by [1]. Thus $t_p X$ is a k-connected p-torsion space with

$$\widetilde{K}_*(\Omega^k t_p X; Z_{(p)}) \cong \widetilde{K}_*(t_p \Omega^k X; X_{(p)}) = 0;$$

we have $P_{v_1}(t_p X) \simeq *$ by 13.13 and

$$v_1^{-1}\pi_*(X; Z/p) \cong v_1^{-1}\pi_*(t_p X; Z/p) = 0$$

by 11.5.

Corollary 14.4. *For $k \geq 2$, if $\varphi : X \to Y$ is a map of pointed k-connected spaces such that $\Omega^k\varphi$ is a $K_*(-; Z/p)$-equivalence, then*

$$\varphi_* : v_1^{-1}\pi_*(X; Z/p) \cong v_1^{-1}\pi_*(Y; Z/p).$$

Proof. Since the homotopy fiber $\mathrm{Fib}\,\varphi$ is a $(k-1)$-connected space with $\widetilde{K}_*(\Omega^{k-1}\ \mathrm{Fib}\ \varphi; Z/p) = 0$, we have $v_1^{-1}\pi_*(\mathrm{Fib}\ \varphi; Z/p) = 0$ by 14.3.

Corollary 14.5. *For $k \geq 1$, if $X \in Ho_*$ is a pointed connected space with $\widetilde{K}_*(X; Z/p) = 0$, then $v_1^{-1}\pi_*(\Sigma^k X; Z/p) = 0$.*

Proof. This follows from 14.3 since $K_*(\Omega^k\Sigma^k X; Z/p) = 0$ as in 12.6.

Corollary 14.6. *For $k \geq 2$, if $\varphi : X \to Y$ is a $K_*(-; Z/p)$-equivalence of pointed connected spaces, then $\varphi_* : v_1^{-1}\pi_*(\Sigma^k X; Z/p) \cong v_1^{-1}\pi_*(\Sigma^k Y; Z/p)$.*

Proof. This follows from 14.4 since $\Omega^k\Sigma^k\varphi$ is a $K_*(-; Z/p)$-equivalence as in 12.6.

Our main result relating $v_1^{-1}\pi_*(-; Z/p)$-equivalences and $K_*(-; Z/p)$-equivalences is

Theorem 14.7. *The following conditions on a map $\varphi : X \to Y$ in Ho_3 are equivalent*:

(i) $\varphi_* : v_1^{-1}\pi_*(X; Z/p) \cong v_1^{-1}\pi_*(Y; Z/p)$;
(ii) *φ is a durable $K_*(-; Z/p)$-equivalence*;
(iii) *$\widetilde{\Omega}^k\varphi$ is a $K_*(-; Z/p)$-equivalence for some $k \geq 2$.*

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Proof. If (i), then (ii) follows since $t_p(\varphi\langle 4\rangle)$ is a durable $K_*(-;Z/p)$-equivalence in Ho_3 by 13.3 and 13.15. If (ii), then (iii) follows immediately. If (iii), then (i) follows by 14.4 since $\Omega^k(\varphi\langle 3+k\rangle)$ is a $K_*(-;Z/p)$-equivalence.

An old conjecture of Miller and Snaith [26] and Mahowald and Ravenel [21] asserts that the Snaith map

$$s:\Omega_0^{2n+1}S^{2n+1}\longrightarrow Q(\mathbb{R}P^{2n})$$

is a $K_*(-;Z/2)$-equivalence for $n\geq 1$. This conjecture was based on Mahowald's earlier insights and was strongly supported by his proof [20] that the Snaith map is a $v_1^{-1}\pi_*(-;Z/2)$-equivalence. However, Mahowald and Thompson obtained a surprising counterexample to this conjecture for $n=1$ in [23] using a theorem of Mayorquin to compute $K_*(\Omega_0^3S^3;Z/2)$. When work on the present project showed that Mahowald's $v_1^{-1}\pi_*(-;Z/p)$-equivalence result *does* in fact imply the conjecture, the author appealed to Mahowald and Thompson, who resolved the dilemma by finding a crucial error in Mayorquin's proof. This cleared the way for

Theorem 14.8. *The Snaith map*

$$s:\Omega_0^{2n+1}S^{2n+1}\longrightarrow Q(\mathbb{R}P^{2n})$$

is a $K_*(-;Z/2)$-equivalence for $n\geq 1$.

Proof. Since the map $s\langle 3\rangle$ of 3-connected covers is a $v_1^{-1}\pi_*(-;Z/2)$-equivalence by [20], it is a durable $K_*(-;Z/2)$-equivalence in Ho_3 by 14.7. Since $\pi_3\Omega_0^{2n+1}S^{2n+1}$ and $\pi_3Q(\mathbb{R}P^{2n})$ are torsion, $s\langle 2\rangle$ is also a $K_*(-;Z/2)$-equivalence. Finally, since

$$s_*:\pi_i\Omega_0^{2n+1}S^{2n+1}\longrightarrow \pi_iQ(\mathbb{R}P^{2n})$$

is an isomorphism for $i\leq 2$, we conclude that s is a $K_*(-;Z/2)$-equivalence.

This theorem implies that s is also an equivalence for $K^*(-;Z/2)$, $K_*(-;Z_{(2)})$, $K^*(-;Z_{(2)})$, $KO_*(-;Z_{(2)})$, $KO^*(-;Z_{(2)})$, etc. We may use this theorem to determine the hitherto inaccessible mod 2 K-theory of $\Omega_0^{2n+1}S^{2n+1}$.

Corollary 14.9. *There is an isomorphism*

$$K_*(\Omega_0^{2n+1}S^{2n+1};Z/2)\cong E(y_1,\dots,y_n)\otimes P(z_1,\dots,z_n)$$

of $Z/2$-algebras for $n\geq 1$, with exterior generators $y_i\in\widetilde{K}_1(\Omega_0^{2n+1}S^{2n+1};Z/2)$ and polynomial generators $z_i\in\widetilde{K}_0(\Omega_0^{2n+1}S^{2n+1};Z/2)$.

Proof. Since $s:\Omega_0^{2n+1}S^{2n+1}\to Q(\mathbb{R}P^{2n})$ is a $K_*(-;Z/2)$-equivalence and a loop map, this follows from the calculation of $K_*(Q(\mathbb{R}P^{2n});Z/2)$ by Miller and Snaith [27].

In very recent work, Lisa Langsetmo [19] has determined all of the Z/p-algebras $K_*(\Omega^jS^{2n+1};Z/p)$ for $j<2n$, using $v_1^{-1}\pi_*(-;Z/p)$-equivalences derived from the work of Mahowald and Thompson [24], and using K-theoretic calculations on the resulting "infinite loop space related" models for Ω^jS^{2n+1}. The preceding theorems should permit other calculations of this sort.

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