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南方科技大学

SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

本科生毕业设计（论文）

题　　目：流形上的几种上同调理论，以及它们之间的等价性

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2023 年 5 月 6 日

CLC ___________ Number___________

UDC___________ Available for reference ☐Yes ☐No

Southern University of Science and Technology

Undergraduate Thesis

Thesis Title: **Cohomology Theories for Manifolds, and Their Equivalences**

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Date: May 6, 2023

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流形上的几种上同调理论，以及它们之间的等价性

王铭杰

（数学系　指导教师：朱一飞, 王博潼）

[摘要]：上同调理论是源自代数学方法的一族拓扑不变量，能测量拓扑空间的全局性质，被广泛运用于现代数学的各个分支。Čech 上同调、de Rham 上同调和奇异上同调是三种经典上同调理论，而在适当条件下，这三者可以利用导出函子的理论被统一到层上同调的框架下。同时，这种层上同调的定义方法启发了一种给上同调理论赋予更广泛研究对象的途径。本文先介绍了上述三种经典上同调理论并用经典办法证明三者在光滑流形上同构，然后介绍 Grothendieck 的基于导出函子的层上同调理论。在这个框架下，以层上同调为桥梁，上述三种上同调理论的同构变得非常自然。最后，本文给出了一个复流形的例子，展示了非平凡的 de Rham 上同调群导致层上同调非平凡，从而影响函数层的结构的现象，以此展示层上同调将空间的几何与拓扑性质联系起来的能力。

[关键词]：上同调理论, 层论, 微分流形

[**ABSTRACT**]: Cohomology is a family of topological invariants originated from the context of algebra that can detect the global structure of given topological space, which is used almost everywhere in modern mathematics. Čech cohomology, de Rham cohomology and singular cohomology are three classical cohomology theories, and under appropriate conditions, they can be unified into the framework of sheaf cohomology using the machinery of derived functor. Moreover, this way of defining a cohomology theory inspires a method to developing cohomolgy theories for more general class of objects. In this paper, these three classical cohomology theories are introduced, and are proven to coincide on a smooth manifold using the classical approach. Then we introduce Grothendieck's theory of derived-functor-based sheaf cohomolgy, which serves as a bridge to make the isomorphism between three theories become natural. An example of complex manifold is given in the end of this paper, demonstrating the phenomenon of non-trivial global topological structure making sheaf cohomology non-trivial, and therefore influencing the structural sheaf. This illustrates the ability of sheaf cohomology to relate the topological and geometrical properties of certain space.

[**Key words**]: Cohomology theoreies, Sheaf, Smooth manifold

目录

1. Introduction

Cohomology theories are a family of topological invariants originated from the context of algebra that is used to detect global structures of topological spaces (or, for a more restricted class of topological spaces), They are widely used in almost all fields of mathematics. Čech cohomology, de Rham cohomology, and singular cohomology are three classical cohomology theories. Under appropriate conditions, they can be unified under the framework of sheaf cohomology using the machinery of derived functors. Moreover, this way of defining a cohomology theory has inspired a method to develop cohomolgy theories for more general classes of objects.

This paper is divided into three sections. In section 1, we introduce singular cohomology and de Rham cohomology and prove their isomorphism on a smooth manifold, that is, the de Rham theorem. In section 2, we introduce Čech cohomology, which bears the name "sheaf cohomology" in many traditions, but we leave this terminology for derived-functor-based cohomology introduced later. In this section, we prove that Čech cohomology is also isomorphic to de Rham cohomology in the case of smooth manifolds. In section 3, we define sheaf cohomology using derived functor, and then prove that sheaf cohomolgy of constant sheaf on a topological space with appropriate assumptions is isomorphic to de Rham cohomology and singular cohomology. At the end of this section, we give the example of exponential exact sequence to demonstrate how topological and geometrical properties of a space can influence each other, using the tool of sheaf cohomology.

2. De Rham cohomology and de Rham's theorem

De Rham cohomology is a natural cohomology theory arising basically from calculus. Our starting point is to introduce de Rham cohomology and prove its equivalence to ordinary singular cohomology.

2.1 Singular and de Rham cohomology theories

The term *cohomology* is an umbrella term for a family of functors that assigns an algebraic structure to a "space", usually a topological space, but sometimes also more restricted classes of spaces like the case of de Rham cohomology upcoming later. One common ingredient of cohomology is a functor from the homotopic category of chain complexes to the category of modules, which we call it *cohomology of chain complex*:

Definition 1. Let $C^\bullet$ be a family of R-modules indexed by $\mathbb{Z}$ together with homomorphisms $d^k : C^k \to C^{k+1}$ for each k, such that $d^2 = 0$. The data of the pair $(C^\bullet, d^\bullet)$ is called a *chain complex of R-modules*, or simply, a chain complex. They form a category together with family of morphisms $f^\bullet : A^\bullet \to B^\bullet$ that makes the following diagram commute

$$\begin{tikzcd} A^k \arrow[r, "d"] \arrow[d, "f^k"'] & A^{k+1} \arrow[d, "f^{k+1}"] \\ B^k \arrow[r, "d"] & B^{k+1} \end{tikzcd}$$

We denote this category by $\mathsf{Ch}(\mathsf{Mod}_R)$.

Definition 2. We define a series of functor $H^*(-) : \mathsf{Ch}(\mathsf{Mod}_R) \to \mathsf{Mod}_R$ as follows: for each k, define $H^k(A^\bullet) := \ker(d^k)/\mathrm{im}(d^{k-1})$; while for morphism $f : A^\bullet \to B^\bullet$, it will naturally induce a map $f^* : H^*(A^\bullet) \to H^*(B^\bullet)$. We call $H^*(-)$ the *algebraic cohomology functor*.

In some since, this notion of algebraic cohomology functor can be seen as "a homotopy theory of chain complexes". With this notion in mind, we can quickly cook up all kinds of cohomology functors that works with specific classes of spaces by generating different chain complexes from the topological or geometrical data, and feed them into the algebraic cohomology functor.

The first one to introduce is *singular cohomology*:

Definition 3. Let X be a topological space. Define *standard simplex* Δ^n to be the subspace $\{\sum \lambda_i \mathbf{e}_i \in \mathbb{R}^{n+1} | \sum \lambda_i = 1\}$ of Euclidean space $\mathbb{R}^{n+1}$ where $\{\mathbf{e}_i\}$ is the standard basis. We use Δ_i^n to denote the n-th face of the standard n-simplex, which can be defined by removing $\mathbf{e}_i$ from the linear combination in the definition of Δ^n. Define $C_k(X, R)$ to be the free R-module generated by all continuous maps $\Delta^k \to X$ and define $\partial_k : C_k(X, R) \to C_{k-1}(X, R)$

by

$$\partial(\sigma) := \sum_i (-1)^i \sigma|_{\Delta_i^n}.$$

Now we can construct a chain complex $C^\bullet(X, R)$ by letting $C^k(X, R) := \mathrm{Hom}_R(C_k(X, \mathbb{Z}), R)$ and $d := \partial^*$, where $(-)^*$ means set-theoretical pullback of functions. We can check that $d^2 = 0$, so $C^\bullet(X, R)$ is indeed a chain complex, and we call it *singular chain complex of cochains on X with coefficient R.* whose cohomology we denote by $H^*(X, R)$. This is called the *singular cohomology of space X with coefficient R.*

Remark. There is actually the notion of chain complex $(C_\bullet, \partial)$ with decreasing index, and the "chain complex" we just stated should be more suitably referred to as "cochain complex". However, for the brevity and consistency we only emphasize the notion of cochain complex and just call it chain complex, since converting between them is just a matter of flipping the sign of the complex, i.e. $C_k := C^{-k}$ and $\partial_k := d^{-k}$.

The second one to introduce is de Rham cohomology, which a the cohomology theory that gives a series of $\mathbb{R}$-modules (real vector spaces) for a given smooth manifold. In this paper, when we talk about smooth manifolds, we are referring to the topological manifold equipped with a smooth atlas, with two additional requirements:

1. it should be Hausdorff;

2. it should be second countable.

It turns out that with these notions, we will be able to construct a *partition of unity* for any given open cover, which gives certain "softness" to functions (and forms) on the manifold. This is an essential ingredient for section 3 that will come later.

In some sense, the biggest tool provided by smooth manifold is the ability to consider differential, i.e. to consider tangent space, and such notion can be endured with some algebraic structure to capture some local geometric information, which leads to the notion of *differential form*.

Definition 4. Let M be a smooth manifold. We use T_p^*M to denote the cotangent space of M at point p. Consider the exterior algebra $\Lambda^k(T_p^*M)$ defined on the vector space T_p^*M. We can put them together fiber-by-fiber:

$$\Lambda^k T^*M := \coprod_{p\in M} \Lambda^k T_p^* M$$

and give it appropriate smooth structure to let it become a bundle on M. A *differential k-form on* M is defined to a section of this bundle. The set of k-forms naturally form a $\mathbb{R}$-linear space, which we denote by $\Omega^k(M)$.

Our goal is to make $\Omega^*(M)$ into a chain complex $(\Omega^\bullet, d)$, so we need a reasonably interesting d, defined as follows:

Definition 5. For $\omega \in \Omega^k(M)$, we define the *exterior differentiation* $d\omega \in \Omega^{k+1}(M)$ of form ω on local charts. First we can see that $\Omega^k(\mathbb{R}^n)$ has a natural basis $\{dx_{i_1}\wedge\cdots\wedge dx_{i_k} | i_1 < \cdots < i_k\}$. Using this, ω has a local coordinate Representation of $\sum_{i_1,\ldots,i_k} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for smooth functions $f_{i_1,\ldots,i_k}$. Define $d\omega$ locally to be

$$d\omega = \sum_{i_1,\ldots,i_k} \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$

Through a simple calculation, we can see that $d^2 = 0$, making $(\Omega^\bullet(M), d)$ a chain complex. Therefore we have the definition:

Definition 6. De Rham cohomology is the cohomology of the complex $(\Omega^\bullet(M), d^\bullet)$. We denote it by $H^*_{\text{DR}}(M)$.

With this definition, we now prove the de Rham theorem, which gives the equivalence between de Rham cohomology and singular cohomology on a smooth manifold, which in some sense serves as a bridge between the beautiful world of smooth manifold and wild landscape of general topological spaces. This proof is based on the one given in chapter 18 of[1].

2.2 Smooth approximation of singular homology

The idea of this proof is to consider the natural paring $(-,-) : H_k(M;\mathbb{R})\otimes H^k_{\text{DR}}(M;\mathbb{R}) \to \mathbb{R}$ defined by

$$(\sigma,\varphi) := \int_{\Delta^k} \sigma^*(\varphi),$$

where Δ^k is our notation for standard k-simplex. It induces a homomorphism $H^k_{\text{DR}}(M;\mathbb{R}) \to H^k(M;\mathbb{R})$ and we prove that it is an isomorphism. A technical problem is that this paring is only well defined for singular chains that happens to be smooth (because only smooth maps have the notion of pullback), but this can be fixed by approximation. Consider the following variant of singular homology: we call a simplex $\sigma : \Delta^k \to M$ to be smooth iff for any point $x \in \Delta^k$, there exists a neighborhood of x in $\mathbb{R}^{k+1}$ (here we embed Δ^n into a Euclidean space) on which σ has smooth extension. Let $C^\infty_\bullet(M;\mathbb{R})$ be the sub-complex of the original singular complex $C_\bullet(M;\mathbb{R})$ generated by smooth singular simplexes. Since boundary map only involves restricting simplexes to a closed subset, and since smoothness is defined locally, we can see that boundary map respects smoothness. This allows us to define $H^\infty_*(M;\mathbb{R})$ to be the homology of $(C^\infty_\bullet(M;\mathbb{R}),\partial)$. Let $i : C^\infty_\bullet(M;\mathbb{R}) \to C_\bullet(M;\mathbb{R})$ be the inclusion map. It is easy to see that i is a chain map, so it induces a homology-level map $i_* : H^\infty_k(M) \to H_k(M)$.

One benefit of considering "smooth stuff" is that smooth maps have enough flexibility to do approximations while having limited pathology. To make it precise, we state the following statement without proving, which is taken from[1]:

Theorem 1 (Whitney Approxibation Theorem for Functions)**.** *Suppose N is a smooth manifold with or without boundary, M is a smooth manifold (without boundary), and $F : N \to M$ is a continuous map. Then F is homotopic to a smooth map. If F is already smooth on a closed subset $A \subset N$, then the homotopy can be taken to be relative to A.*

Such theorem remind us of an approach to relate singular homology and its smooth variant, that is, we claim that any singular simplex can be approximated by a smooth simplex from the same homology class:

Theorem 2 (Smooth Approximation of Singular Homology)**.** *The homomorphism induced by inclusion* $i_* : H_k^\infty(M;\mathbb{R}) \to H_k(M;\mathbb{R})$ *is an isomorphism for any* k*.*

To do this we need to subdivide the homotopy given by Whitney's approximation theorem into boundaries?

2.3 The de Rham theorem

For homology simplex $\sigma \in C_k^\infty(M;\mathbb{R})$ and differential form $\varphi \in \Omega^k(M)$, we define $\int_\sigma \varphi := \int_{\Delta^k} \sigma^*(\varphi)$ extend it bi-linearly to chains and cochains. As mentioned before, we describe a homomorphism induced by the "integration paring": for $[\sigma] \in H_k(M;\mathbb{R})$ and $[\varphi] \in H^k_{\mathrm{DR}}(M)$, define $F : H^k_{\mathrm{DR}}(M) \to H^k(M;\mathbb{R})$ by letting

$$F([\varphi])([\sigma]) := \int_{\Delta^k} \tilde{\sigma}^*(\varphi), \tag{1}$$

where $\tilde{\sigma}$ is the smooth approximation of σ mentioned before. To check that this map is well-defined, we need the following re-packaged form of Stokes's theorem:

Theorem 3 (Stokes)**.** *For* $\sigma \in C_{k+1}(M;\mathbb{R})$ *and* $\varphi \in \Omega^k(M)$*,*

$$\int_{\partial\sigma} \varphi = \int_\sigma d\varphi.$$

Proof of this theorem is omitted for being just simply unpacking and repacking definitions. With this result, we have

$$\int_\sigma d\varphi = \int_{\partial\sigma} \varphi = 0 \tag{2}$$

for cycle σ and coboundary $d\varphi$, and

$$\int_{\partial\sigma} \varphi = \int_\sigma d\varphi = 0 \tag{3}$$

for boundary σ and cocycle φ. Moreover, by theorem 2, the choice of $\tilde{\sigma}$ is unique up to a boundary. Therefore the homomorphism F is well-defined. We call it the **de Rham homomorphism**.

For simplicity, we say that a smooth manifold M is **de Rham** if the de Rham map for it is an isomorphism on every level of cohomology groups. Then we can state our target briefly as

Theorem 4. *(de Rham) Every smooth manifold is de Rham.*

The most naïve case is that any convex open subsets of $\mathbb{R}^n$ is de Rham.

Lemma 1. *(de Rham Theorem, Baby ver.) If $U \subset \mathbb{R}^n$ is a convex set, then U is de Rham.*

Proof. By Poincaré lemma (theorem 11.49 in[1]), $H^k_{\mathrm{DR}}(U) = \mathbb{R}$ when $k = 0$ and vanishes otherwise. And we have the same result for singular cohomology since U is contractible. F is trivially isomorphic for $k \neq 0$ since both groups are trivial. For $k = 0$, F maps the generator $1 \in H^0_{\mathrm{DR}}(M)$ to $\sigma \mapsto \int_{\tilde{\sigma}} 1$, which is clearly not trivial. Since F is an endomorphism of $\mathbb{R}$ as $\mathbb{R}$-module, being nontrivial implies that F is isomorphic. $\square$

In fact, The general idea of manifolds is to record global geometric data and keep local data as trivial as possible. Our proof for de Rham theorem also follows the same "local-to-global" pattern. The tool we use to do the patching is the Mayer-Vietoris sequence. To make this work, we should first show that the de Rham map commutes with each horizontal map of the Mayer-Vietoris sequence:

Lemma 2. *Let F be the de Rham map. Then*

1. *for any smooth map $f : M \to N$, the following diagram commutes:*

$$\begin{array}{ccc} H^k_{\mathrm{DR}}(N) & \xrightarrow{F} & H^k(N;\mathbb{R}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^k_{\mathrm{DR}}(M) & \xrightarrow{F} & H^k(M;\mathbb{R}) \end{array}$$

2. *if manifold M can be decomposed into union of open subsets U and V, then*

$$\begin{array}{ccc} H^{k-1}_{\mathrm{DR}}(U \cap V) & \xrightarrow{F} & H^{k-1}(U \cap V;\mathbb{R}) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ H^k_{\mathrm{DR}}(M) & \xrightarrow{F} & H^k(M;\mathbb{R}) \end{array}$$

where the vertical maps are the differential maps in de Rham version and singular version of the Mayer-Vietoris sequence.

Proof. The proof is done by simply chasing diagrams. For the first claim, pick $\psi \in H^k_{\text{DR}}(N)$ and $\sigma \in H^k(M)$, then $f^*(F(\psi))(\sigma) = f^*(\int_\sigma f^*(\psi)) = \int_{\Delta^n} \sigma^*(f^*(\psi)) = \int_{f^*(\sigma)} \psi = F(f^*(\psi))(\sigma)$. For the second claim, again, pick $\psi \in H^{k-1}_{\text{DR}}(U \cap V)$ and $\sigma \in H^k(M)$, using theorem 3, there is $d(F(\psi))(\sigma) = F(\psi)(\partial\sigma) = \int_{\partial\sigma} \psi = \int_\sigma d\psi = F(d(\psi))$. $\square$

Now we can start proving de Rham theorem. This can be done by proving the following statements in sequence.

Lemma 3. *We have the following statement:*

1. *Disjoint union of countably many de Rham manifolds is also de Rham.*
2. *Convex open subset of $\mathbb{R}^n$ is de Rham.*
3. *If a manifold has a finite de Rham cover, then it is de Rham. Here de Rham cover is defined to be an open cover where every element and every finite intersection is de Rham.*
4. *If a manifold has a de Rham basis, then it is de Rham. Here de Rham basis is defined to be a basis that happens to be a de Rham cover.*

Proof. 1. Let $M = \coprod_i M_i$, and $\iota_i : M_i \to \coprod_i M_i$ be inclusion maps. These maps induces isomorphism between cohomology of disjoint union of spaces and product of cohomology of each space. By part 1 of lemma 2, F commutes with each ι_i, therefore the following diagram commutes:

$$\begin{tikzcd} H^k_{\text{DR}}(\coprod M_i) \arrow[r, "\cong"] \arrow[d, "F"] & \prod_i H^k_{\text{DR}}(M_i) \arrow[d, "\prod F"] \\ H^k(\coprod M_i) \arrow[r, "\cong"] & \prod_i H^k(M_i) \end{tikzcd}$$

and this implies that M is de Rham.

2. This is Lemma 1.

3. In this part we use Mayer-Vietoris sequence to do the actual patching work. Suppose M admits a finite de Rham cover consisting of k open sets. If $k = 1$, M is trivially de Rham. Now inductively suppose M is de Rham whenever it is covered by a de Rham cover of size k. Then if M is covered by the de Rham cover $M = U_1 \cup \cdots \cup M_{k+1}$, let $A := U_1 \cup \cdots \cup U_k$ and $B := U_{k+1}$, we have the following:

$$\begin{array}{ccccccccc} H^n_{\text{DR}}(A) \oplus H^n_{\text{DR}}(B) & \longrightarrow & H^n_{\text{DR}}(A \cap B) & \longrightarrow & H^n_{\text{DR}}(M) & \longrightarrow & H^{n+1}_{\text{DR}}(A) \oplus H^{n+1}_{\text{DR}}(B) & \longrightarrow & H^{n+1}_{\text{DR}}(A \cap B) \\ \downarrow^{F \oplus F} & & \downarrow^{F} & & \downarrow^{F} & & \downarrow^{F \oplus F} & & \downarrow^{F} \\ H^n(A) \oplus H^n(B) & \longrightarrow & H^n(A \cap B) & \longrightarrow & H^n(M) & \longrightarrow & H^{n+1}(A) \oplus H^{n+1}(B) & \longrightarrow & H^{n+1}(A \cap B) \end{array}$$

By assumption A is de Rham since it is the union of k de Rham sets, and B is clearly also de Rham. Notice that $A \cap B = \bigcup_i (U_i \cap U_{k+1})$ is also union of k de Rham sets, thus also de Rham. Therefore the first, second, fourth and fifth vertical maps are all isomorphisms. By five lemma, the middle vertical map is isomorphism, whish shows that M is de Rham.

4. Claim 3 only works for finite situations, which is not enough. In this part we extend the result to infinite case, which utilizes heavily the technical assumption on smooth manifolds that requires them to be paracompact. Let $\{U_\alpha\}$ be a de Rham basis of M. By proposition 2.28 of[1], we have a smooth exhaustion function $f : M \to \mathbb{R}$. Then for any integer m, define $A_m := \{q \in M | m \leq f(q) \leq m + 1\}$ and $A'_m := \{q \in M | m - \frac{1}{2} < f(q) < m + \frac{3}{2}\}$. Since A'_m is open, we can cover its subset A_m by basis elements inside A'_m. Since A_m is compact, it can be covered by a finite cover, which is still inside A'_m. We call the union of this finite cover B_m. Then B_m is de Rham since it admits a finite de Rham cover. Since each B_m is "protected" by $A'_m \supset B_m$, we can observe that if m is odd, each B_m are disjoint from each other for distinct m. This is also true if m is even. Therefore the two union

$$U := \bigcup_{m:\text{odd}} B_m \quad \text{and} \quad V := \bigcup_{m:\text{even}} B_m$$

are both disjoint union of countably many de Rham sets, therefore both de Rham by part 1. Moreover, since $U \cap V$ is a disjoint union of $B_m \cap B_{m+1}$ for m varies across all integers, it is de Rham since it is also a union of countably many de Rham sets. Now we can conclude that M is de Rham by part 3.

5. Since $\mathbb{R}^n$ admit a basis consisting of open balls, and each open ball and finite intersection of union balls are all convex, we can see that it is a de Rham basis. Any open subset of $\mathbb{R}^n$ inherits this de Rham basis, therefore becoming de Rham.

□

Using these results we can finally conclude the proof of theorem 4. Indeed, every manifold admits a basis consisting of all coordinate charts, each of them being diffeomorphic to an open set in $\mathbb{R}^n$. Such de Rham-ness can be transferred from such Euclidean open subsets to the basis elements since de Rham map commutes with smooth maps (lemma 2). Therefore every manifold is de Rham since they admit de Rham basis.

Such "local-to-global patching" is extremely cumbersome and require a lot of bookkeeping. One common practice is to develop some machinery to keep this data well-organized, and this is accomplished by the introduction of sheaf and sheaf cohomology. We will use them to re-formulate this proof in section 4.

3. Čech Cohomology

Čech cohomology is a cohomology theory that takes a presheaf on a topological space as input. So it is not surprising that it occupies the term *sheaf cohomology* until Grothendieck's notion of derived-functor-based sheaf cohomology took its place. However, Čech cohomology is still useful since it gives a practical way to compute sheaf cohomology, thanks to theorem 12 coming later.

In this section we first introduce the notion of Čech cohomology, and prove its equivalence to de Rham cohomology in the case of a smooth manifold. The approach used by this chapter is mostly based on[2].

3.1 Preliminary

Recall the Mayer-Vietoris sequence on smooth manifolds involves a cover of two open sets of a smooth manifold M, and such sequence is exact mainly thanks to the existence of partition of unity. That is, the functions on M is "soft" in some sense. In a more general setting, if we have a cover of countably many open sets, we can generalize the sequence to a double complex, namely the (p, q)-th component $K^{p,q}$ should be the group of q-th differential forms on p-fold intersections of open sets in the cover. So first we introduce some tools to handle double complexes.

3.1.1 Some homological algebra for double complexes

Definition 7. A *double complex of R-modules* is a family of R-modules $K^{p,q}$ indexed by two coordinates, together with connecting homomorphisms $d_{p,q} : K^{p,q} \to K^{p,q+1}$ and $\delta_{p,q} : K^{p,q} \to K^{p+1,q}$ such that $d^2 = 0$, $\delta^2 = 0$, and they satisfy the *anti-commutativity*

$$d\delta + \delta d = 0.$$

It can be viewed as a single complex $K^\bullet$ by letting its i-th stage to be

$$K^i := \bigoplus_{p+q=i} K^{p,q}$$

together with the connecting homomorphism $D := (-1)^p d + \delta$. The restrictions imposed on d and δ, together with the anti-commutativity, implies that $D^2 = 0$, so $K^\bullet$ is indeed a complex. We denote the cohomology of this complex by $H^*_D(E^{\bullet,\bullet})$.

We have the following result which is the proposition 3.11 on[3], whose proof is just diagram chasing:

Proposition 1. If all rows of $E^{p,q}$ are exact except the bottom row $E^{p,0}$, then $H^*_D(E^{\bullet,\bullet}) \cong H^*(E^{\bullet,0})$. Symmetrically, if all columns are exact except the left-most column $E^{0,q}$. then $H^*_D(E^{\bullet,\bullet}) \cong H^*(E^{0,\bullet})$.

For the purpose of this paper, an augmented version of this two proposition is also available, which can be obtained by simply shifting both of the indices of $K^{p,q}$.

Proposition 2. We augment $E^{p,q}$ with $r : E^{-1,q} \to E^{0,q}$ and $i : E^{p,-1} \to E^{p,0}$ to get the following augmented double complex:

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots & \\
E^{-1,2} \arrow[r] \arrow[u] & E^{0,2} \arrow[r] \arrow[u] & E^{1,2} \arrow[r] \arrow[u] & E^{2,2} \arrow[r] \arrow[u] & \cdots \\
E^{-1,1} \arrow[r] \arrow[u] & E^{0,1} \arrow[r] \arrow[u] & E^{1,1} \arrow[r] \arrow[u] & E^{2,1} \arrow[r] \arrow[u] & \cdots \\
E^{-1,0} \arrow[r] \arrow[u] & E^{0,0} \arrow[r] \arrow[u] & E^{1,0} \arrow[r] \arrow[u] & E^{2,0} \arrow[r] \arrow[u] & \cdots \\
 & E^{0,-1} \arrow[r] \arrow[u] & E^{1,-1} \arrow[r] \arrow[u] & E^{2,-1} \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Then if all rows of $E^{p,q}$ are exact (except the augmented row $E^{p,-1}$), then $H^*_D(E^{\bullet,\bullet})$ is isomorphic to the cohomology of the augmented row $H^*(E^{p,-1})$ through map r; and again symmetrically, if all columns of $E^{p,q}$ are exact (except the augmented column $E^{-1,q}$), then $H^*_D(E^{\bullet,\bullet})$ is isomorphic to the cohomology of the augmented column $H^*(E^{-1,q})$ through map i.

3.1.2 Presheaves and sheaves

A presheaf is just a contravariant functor. More precisely, in this paper we use a notion of a *presheaf of abelian groups on topological space* X to be a functor from the category of open subsets of X to the category of abelian groups, and we simply call it a presheaf. Morphisms between presheaves are just natural transformations between such type of functors. Moreover, there is a full subcategory of presheaves, defined by imposing additional axioms:

Definition 8. A presheaf $\mathcal{F}$ is a sheaf if the following is satisfied:

1. (Glueing axiom) For any open cover $\bigcup_\alpha U_\alpha$ of a randomly chosen open set U, if for each U_α we have a section $f_\alpha \in \mathcal{F}(U)$, such that $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ for any α

and β, then there exist a "glueing" of these f_α, namely a section $f \in \mathcal{F}(U)$ such that $f|_{U_\alpha} = f_\alpha$;

2. (Separation axiom) given the same setup for the open set U and its open cover $\{U_\alpha\}$, if two sections $f, g \in \mathcal{F}(U)$ agrees when restricted to each U_α, then $f = g$.

Moreover, (pre)sheaves has the notion of *stalks*, which is not used until section 4 but we put it here to avoid scattering a single topic throughout the paper.

Definition 9. For point $x \in X$ and sheaf $\mathscr{F}$ on X, define the *stalk* of $\mathscr{F}$ at x to be

$$\mathscr{F}_x := \varinjlim_{U} F(U)$$

where U varies through all open set containing x.

Although working with sheaves on the level of sections (i.e. working with $s \in \mathscr{F}(U)$) comes handy in some cases, the nature of sheaves is better described on the level of stalks. For example, it would be much more natural to define image, kernel, etc on the level of stalks, while more technical steps are required on the level of sections. Due to this reason, the early definition of sheaves are done by explicitly placing stalks in a fiber-bundle-like fashion, called *étalé space*. We will not touch this topic in this paper.

Example. Let M be a smooth manifold. Define $\underline{\mathbb{R}}$ to be the following presheaf: for each open set $U \subset M$, let $\underline{\mathbb{R}}$ to be the $\mathbb{R}$-module of all locally constant functions from U to $\mathbb{R}$. This is a sheaf since firstly the identity axiom is clearly satisfied since we are only dealing with functions and restriction of functions, and secondly if a bunch of locally constant function agrees on intersections of their domains, then the glued function will still be locally constant. Such kind of construction is called a *constant sheaf*, which is one way of bring an ordinary object (sets, groups, rings, modules, etc) into a sheaf-theoretical setting. We will see this construction again later.

Example. Differential forms on smooth manifold M also form a sheaf, in the following since: each open set $U \subset M$ is itself a smooth manifold (with obvious smooth structure), called a *open submanifold* of M. Then we have a presheaf $\Omega^p(-)$ over space M defined by $U \mapsto \Omega^p(U)$. Moreover, glueing and separation axioms are automatically satisfied since differential forms are just a bunch of smooth maps, and since smoothness is a local property.

3.2 Čech–de Rham complex

In this section we extend Mayer-Vietoris sequence to the case of a cover of countably many open sets. Given an open cover $\mathcal{U}$ of smooth manifold M of countably many sets, define a double complex

$$C^p(\mathcal{U};\Omega^q) := \prod_{\alpha_0<\cdots<\alpha_p} \Omega^p(U_{\alpha_0\ldots\alpha_p})$$

where we use the notation $U_{\alpha_0\ldots\alpha_p} := \bigcap_i U_{\alpha_i}$. We extend the indices of tuple $\omega \in C^p(\mathcal{U};\Omega^q)$ to any arrangement of $\alpha_0,\ldots,\alpha_p$ by defining $\omega_{\ldots\alpha_i\ldots\alpha_j\ldots} = -\omega_{\ldots\alpha_j\ldots\alpha_i\ldots}$. The "upward" differential $d : C^p(\mathcal{U},\Omega^q) \to C^p(\mathcal{U},\Omega^{q+1})$ is induced by the exterior differential, and the "rightward" differential $\delta : C^p(\mathcal{U},\Omega^q) \to C^{p+1}(\mathcal{U},\Omega^q)$ is defined by mapping a tuple ω to the tuple

$$(\alpha_0,\ldots,\alpha_{p+1}) \mapsto \sum_{i=0}^{p+1}(-1)^i \omega_{\alpha_0\ldots\hat{\alpha_i}\ldots\alpha_{p+1}}$$

where the restriction is understood. We can check immediately $d^2 = \delta^2 = 0$ and $d\delta+\delta d = 0$, so $C^p(\mathcal{U},\Omega^q)$ is a double complex. We can augment this complex by adding a left-most column $\Omega^q(M)$ together with restriction map $r : \Omega^q(M) \to C^0(\mathcal{U},\Omega^q)$ and by adding a bottom row consisting of kernels of maps $C^p(\mathcal{U};\Omega^0) \to C^p(\mathcal{U};\Omega^1)$ equipped with the same differential map δ. The augmented bottom row is denoted by $C^\bullet(\mathcal{U};\underline{\mathbb{R}})$ for some reason that we will explain later. We call it *Čech-de Rham complex*.

Similar to the case of Mayer-Vietoris sequence, the existence of partition of unity on a smooth manifold makes sure that the rows of the double complex are exact:

Proposition 3. For any $q \geq 0$, the sequence

$$0 \longrightarrow \Omega^q(M) \longrightarrow \prod \Omega^q(U_{\alpha_0}) \longrightarrow \prod \Omega^q(U_{\alpha_0\alpha_1}) \longrightarrow \ldots$$

is exact.

Proof. Firstly, the sequence is exact at $\Omega^q(M)$ since we noticed before $\Omega^q(-)$ is a sheaf and then it is obvious from identity axiom of sheaves. Next, suppose $\omega \in \prod \Omega^q(U_{\alpha_0\dots\alpha_p})$ is a closed q-chain, i.e. $\delta\omega = 0$. Then we have

$$0 = (\delta\omega)_{\alpha_0\dots\alpha_{p+1}} = \omega_{\alpha_1\dots\alpha_{p+1}} + \sum_{i=1}^{p+1}(-1)^i\omega_{\alpha_0\dots\hat{\alpha_i}\alpha_{p+1}}.$$

If we rename the indices, we can re-write this equation as

$$\omega_{\beta_0\dots\beta_p} = \sum_{i=0}^{p}(-1)^i\omega_{\alpha\beta_0\dots\hat{\beta}_i\dots\beta_p}$$

Inspired by this equation, we pick a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$, and define a $(q-1)$-chain τ to be

$$\tau_{\gamma_0\dots\gamma_{p-1}} := \sum_\alpha \rho_\alpha\omega_{\alpha\gamma_0\dots\gamma_{p-1}},$$

and then we have

$$(\delta\tau)_{\beta_0\dots\beta_p} = \sum_{i=0}^{p}(-1)^i\tau_{\beta_0\dots\hat{\beta}_i\dots\beta_p} = \sum_\alpha \rho_\alpha \sum_{i=0}^{p}(-1)^i\omega_{\alpha\beta_0\dots\hat{\beta}_i\dots\beta_p} = \sum_\alpha \rho_\alpha\omega_{\beta_0\dots\beta_p} = \omega_{\beta_0\dots\beta_p}$$

so we conclude that $\delta\tau = \omega$. This finishes the proof. $\square$

Given this result, using proposition 2, we know that $H^q_{\mathrm{DR}}(M) \cong H^q_D(C^\bullet(\mathcal{U};\Omega^\bullet))$.

The columns of the double complex (without the augmented column) is not necessarily exact. In fact, the failure of exactness is measured by the cohomology groups $\prod H^q(U_{\alpha_0\dots\alpha_p})$. If we restrict the cover $\mathcal{U}$ to be a good cover, i.e. all sets and all finite finite intersections of the sets in the cover are contractible, then we have the exactness of the columns. Then we get the result:

Theorem 5. *If $\mathcal{U}$ is a good cover of smooth manifold M, then $H^*_{\mathrm{DR}}(M) \cong H^*_D(C^\bullet(\mathcal{U};\Omega^\bullet)) \cong H^*(C^\bullet(\mathcal{U};\mathbb{R}))$.*

3.3 Definition of Čech cohomology

Now we define the notion of Čech cohomology. Our end product is a cohomology theory $H^*(M;\mathscr{F})$ that takes a smooth manifold and presheaf, which is called the *Čech cohomology of M with values in $\mathscr{F}$*. Fix a smooth manifold M, and let $\mathcal{U} = \{U_\alpha\}_\alpha$ be an open cover. Consider a chain $C^k(\mathcal{U};\mathscr{F}) := \prod \mathscr{F}(U_{\alpha_0\ldots\alpha_k})$ with differential map δ defined by

$$(\delta\omega)_{\alpha_0\ldots\alpha_{k+1}} = \sum_{i=0}^{k+1}(-1)^i\omega_{\alpha_0\ldots\hat{\alpha_i}\ldots\alpha_{k+1}}$$

where $\delta^2 = 0$ can be checked similarly as before, making it a chain complex, thus giving a cohomology $H^*(C^\bullet(\mathcal{U};\mathscr{F}))$. We define Čech cohomology to be the colimit

$$\check{H}^*(M;\mathscr{F}) := \varinjlim_{\mathcal{U}} H^*(C^\bullet(\mathcal{U};\mathscr{F}))$$

where the underlying directed system consists of all open covers of M, with $\mathcal{U} < \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$.

Example. We used the notation $C^\bullet(\mathcal{U};\underline{\mathbb{R}})$ to denote the augmented bottom row in the Čech-de Rham complex defined to be the kernel of the map $d : C^p(\mathcal{U};\Omega^0) \to C^p(\mathcal{U};\Omega^1)$. This notation makes sense since we can see that such kernel consists of exactly tuples of locally constant functions on $U_{\alpha_0\ldots\alpha_p}$.

We take the following proposition from[2] without proving:

Proposition 4. The good covers are cofinal in the set of all covers of a manifold M. In other words, any open cover of M has a refinement being a good cover.

The colimit of a directed system can be replaced by the colimit of a cofinal subsystem. By Theorem 5, we know that $H^*(C^\bullet(\mathcal{U};\underline{\mathbb{R}}))$ is constantly isomorphic to $H^*_{\mathrm{DR}}(M)$ whenever $\mathcal{U}$ is a good cover. Therefore we have the following, which is the main result of this section:

Theorem 6. *For a smooth manifold M,*

$$H^*_{\mathrm{DR}}(M) \cong \check{H}^*(M;\underline{\mathbb{R}}).$$

Such isomorphism should not be surprising, since in this case, Čech cohomology extracts combinatorical data from the arrangement of open covering with limited complexity locally (i.e. being a good cover) to give a cohomological description of the topological structure of given manifold. This still follows the "trivial locally and patch to global" principal mentioned before. Moreover, Čech cohomology somehow gives a solid way to compute sheaf cohomology, which will be introduced in the next section.

4. Sheaf cohomology and reformulations of classical cohomology theories

In some sense, the geometric information of certain space and the sheaf allowable on it will influence each other. Sheaf cohomology, which is the topic in this section, is a cohomology theory that extract information from both a given space and a given sheaf on top of that space. After introducing the machinery of sheaf cohomology, we first demonstrate that if we consider sheaf cohomology of the "sheaf containing least information", that is, the constant sheaf, then sheaf cohomology will degenerate into these "classical" cohomology theories introduced before. In the end of this section, we give an example of a complex manifold, which shows the phenomenon of underlying space influencing the structure of sheaves. Hence we demonstrate the modern viewpoint toward geometry: the geometrical data of certain space is determined by its structural sheaf.

4.1 Abelian categories and derived functors

In some sense, sheaves of abelian groups acts just like modules, and it is reasonable to consider extending the beautiful theory of homological algebra of modules, whose foundation is laid by Cartan and Eilenberg[4] to the category of sheaves. This idea is systematically explained in Grothendieck's famous Tohoku paper[5], which introduces the notion of abelian category to put both category of modules and sheaves into a same framework. This approach is much more general beyond its original goal since it is about extracting certain information from a functor with minimal abelian enrichment structure, not related with any particular geometric or algebraic context. This makes such approach extremely elegant and powerful, and at the same time, having a lot more technical issue. Therefore the purpose of this paper, we are going to slightly touch this topic and use it as a tool, omitting most of the proof.

The following is mainly based on[6].

Definition 10. A category C is called an *pre-abelian category* if the following are satisfied

1. C has an object that is initial and final at the same time, called a *zero object* $0 \in \mathsf{C}$;

2. The product $A \times B$ for every pair of object $A, B \in \mathsf{C}$ exists;

3. Each hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B) := \mathrm{Mor}_{\mathsf{C}}(A, B)$ has a abelian group structure.

A functor between additive categories preserving abelian group structure of hom-sets is called a *additive functor*.

In such category, we have the notion of *kernel* and *cokernel* defined using equalizers and coequalizers. For some morphism $f \in \mathrm{Hom}(A, B)$, the kernel is an object $\mathrm{ker} f$ together with a morphism $\mathrm{ker} f \to A$ that is final among all morphisms $t : T \to A$ satisfying $f \circ t = 0$. In other words, kernel is characterized by the following universal property:

$$\begin{tikzcd}
 & T \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left, "0"] & \\
 & \mathrm{ker} f \arrow[dl] \arrow[dr, "0"] & \\
A \arrow[rr, "f"] & & B
\end{tikzcd}$$

Cokernel is just a dual definition, that is, defined by the following universal property:

$$\begin{tikzcd}
 & T & \\
 & \mathrm{coker} f \arrow[u, dashed] & \\
A \arrow[rr, "f"] \arrow[ur, "0"] \arrow[uur, bend left, "0"] & & B \arrow[ul] \arrow[uul, bend right]
\end{tikzcd}$$

If we impose an additional requirement, we get the following:

Definition 11. An abelian category is an additive category satisfying the following requirements:

1. Every morphism has kernel and cokernel;
2. every categorical monomorphism is the kernel of its cokernel;
3. every categorical epimorphism is the cokernel of its kernel.

The homological algebra of modules over a ring is always the canonical model of abelian category. Therefore we have the counterpart of injective module from module theory in a general abelian category:

Definition 12. An object I in abelian category C is called *injective* if for any monomorphism $m : A \to B$, an morphism $A \to I$ can be lifted (not necessarily uniquely) to a morphism $B \to I$. That is, we have the diagram

$$\begin{tikzcd} & X \arrow[d, tail, "m"] \\ I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Moreover, C is said to have *enough injective* if for any object $A \in \mathsf{C}$, there exist an injective object I together with a monomorphism $A \rightarrowtail I$. Given this requirement, it turns out that thanks to the additional requirement of being an abelian category, for any $A \in \mathsf{C}$, there exists a long exact sequence

$$A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

which is called the *injective resolution* of A. (A proof of this can be found in lemma 2.3.6 of[6])

Using these terminology we can define the notion of derived functor to measure the failure of exactness of an additive functor: (in our case, it is actually right derived functor)

Definition 13 (Derived functor)**.** Let $F : \mathsf{A} \to \mathsf{B}$ be a covariant left exact additive functor. Define a series of new functors $R^iF : \mathsf{A} \to \mathsf{B}$ as follows: for each object $X \in \mathsf{A}$, consider a injective resolution of X:

$$X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

and then apply F it to get a chain complex with FX replaced with 0: (not necessarily exact anymore):

$$0 \longrightarrow FI^0 \longrightarrow FI^1 \longrightarrow FI^2 \longrightarrow \cdots$$

and we define $R^iF(X) := H^i(FI^\bullet)$.

The whole point of introducing the heavy machinery of derived functor is concluded in the following theorem, which we cite from[7] without proving:

Theorem 7. *Let $F : \mathsf{A} \to \mathsf{B}$ be an left exact additive functor between abelian categories. For an short exact sequence of objects in A,*

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

there is a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & FX' \arrow[r] & FX \arrow[r] & FX'' \arrow[dll] \\
& R^1F(X') \arrow[r] & R^1F(X) \arrow[r] & R^1F(X'') \arrow[dll] \\
& R^2F(X') \arrow[r] & R^2F(X) \arrow[r] & R^2F(X'') \arrow[r] & \cdots
\end{tikzcd}$$

That is, derived functor gives a way to “mend” the incomplete short exact sequence produced by a functor that is only left exact by extending it into a long exact sequence, and we can easily see that F becomes exact if and only if $R^1F(X') = 0$.

By definition we can easily see that if an object I is already injective, then for any left exact additive functor F, we have $R^kF(I) = 0$ for all $k \geq 1$. However, in general case, a injective resolution is hard to obtain, making sheaf cohomology being nearly impossible to be calculated by definition. Luckily, it turns out that there is a larger class of objects, larger than the class of injective objects, that can replace injective objects in the resolution while produce the same sheaf cohomology.

Definition 14. Given additive left exact functor $F : \mathsf{C} \to \mathsf{D}$, an object $A \in \mathsf{C}$ is said to be an *F-acyclic object* if $R^k F(A) = 0$ for all $k \geq 1$. It is clear that injective objects are acyclic with respect to all choices of F.

Theorem 8. *Derived functors can be calculated using acyclic resolution. More precisely, for some left exact additive functor F and an object B, if $B \leftarrow A^\bullet$ is an F-acyclic resolution, then $R^k F(B) \cong H^k(F(A^\bullet))$ for all k.*

Proof. See theorem 4.1 of[8]. □

4.2 Sheaf cohomology

In this section, we apply the machinery of derived functor to the category of sheaves. Before that, we need to demonstrate that this category is indeed an abelian category with enough injectives.

We want the notion of kernel, image and quotient for sheaves, in order to treat it as an abelian category. Given a presheaf $\mathscr{F}$ of R-modules, we can consider the closest sheaf approximation, which is called the sheafification $\mathscr{F}^{\mathrm{sh}}$ of $\mathscr{F}$:

Definition 15. The sheafification $\mathscr{F}^{\mathrm{sh}}$ of presheaf $\mathscr{F}$ is a sheaf (together with a morphism $\mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$) such that for any sheaf $\mathscr{G}$, we have the following diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr, dashed] & \mathscr{F}^{\mathrm{sh}} \arrow[d] \\ & \mathscr{G} \end{tikzcd}$$

We may define the kernel, cokernel and image as straightforward as how we define them in the category of abelian groups, but the resulting presheaf may not always be a sheaf. But we can fix this by using sheafification to "force" the presheaf to be a sheaf. That is to say,

Definition 16. Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Define the kernel $\ker(f)$ of f to be the sheafification of the presheaf $U \mapsto \ker(f(U))$. Similarly, the cokernel and image of f are defined to be the sheafification of $U \mapsto \mathrm{coker}(f(U))$ and $U \mapsto \mathrm{im}(f(U))$, respectively.

The essence of the notion of sheafification is the fact that sheafification preserves the most important viewpoint of sheaves: the stalks.

Proposition 5. For any presheaf $\mathscr{F} \in \mathsf{Sh}(X)$ and point $x \in X$, there is $\mathscr{F}_x = (\mathscr{F}^{\mathrm{sh}})_x$.

Theorem 9. *The category* $\mathsf{Sh}(X)$ *of sheaves on space* X *is an abelian category.*

Proof. Just check that cokernel of kernel and kernel of cokernel for sheaves coincide on the level of stalks, on which the problem reduces to checking the same statement for modules.

□

Proposition 6. The category of sheaves of abelian groups has enough injectives.

Proof. See □

Another notation for the sections $\mathscr{F}(U)$ of presheaf $\mathscr{F}$ over open set U is $\Gamma(U, \mathscr{F})$. This notation comes handy if we let U to be the whole space X, and let the sheaf $\mathscr{F}$ vary and consider the functor $\Gamma(X, -)$, which is called the *global section functor*. This functor is easily seen to be an additive functor. Moreover, we have the following property:

Proposition 7. The functor $\Gamma(X, -)$ from the category of sheaves of abelian groups to the category of abelian groups is left exact.

Proof. Consider the following exact sequence of sheaves:

$$0 \longrightarrow \mathscr{F}' \xrightarrow{f} \mathscr{F} \xrightarrow{g} \mathscr{F}'' \longrightarrow 0$$

Using the naturalality of colimit, we can form the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathscr{F}') \arrow[r, "f"] \arrow[d, hook] & \Gamma(\mathscr{F}) \arrow[r, "g"] \arrow[d, hook] & \Gamma(\mathscr{F}'') \arrow[d, hook] & \\
0 \arrow[r] & \prod_{x\in X} \mathscr{F}'_x \arrow[r, "\prod f"] & \prod_{x\in X} \mathscr{F}_x \arrow[r, "\prod g"] & \prod_{x\in X} \mathscr{F}''_x \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are injective by identity axiom of sheaves, and the lower row is exact since taking stalks and product both preserves exactness. Our goal is to prove that the upper row is exact. First it is clearly a chain complex. For $s \in \Gamma(\mathscr{F}')$ such that $f(s) = 0$, we have $(s_x)_x = 0$ by exactness of the lower row, therefore $s = 0$. For $t \in \Gamma(\mathscr{F})$ such that $g(t) = 0$, by commutativity we have $g((t_x)_x) = 0$, so there exist a family of stalks $(s_x)_x$ such that $f(s_x) = t_x$ for all x. Since $(t_x)_x$ comes from a global section t, it is compatible for gluing, that is, if we let $t_{\alpha(x)} \in \mathscr{F}(U_{\alpha(x)})$ be the representative of t_x, then for any x, y, there is

$$\mathrm{res}_{U_{\alpha(x)}, U_{\alpha(x)} \cap U_{\alpha(y)}}(t_{\alpha(x)}) = \mathrm{res}_{U_{\alpha(y)}, U_{\alpha(x)} \cap U_{\alpha(y)}}(t_{\alpha(y)})$$

since $f(s_x) = t_x$, this becomes

$$f(\mathrm{res}(s_{\alpha(x)})) = \mathrm{res}(f(s_{\alpha(x)})) = \mathrm{res}(f(s_{\alpha(y)})) = f(\mathrm{res}(s_{\alpha(y)}))$$

so we have $\mathrm{res}(s_{\alpha(x)}) = \mathrm{res}(s_{\alpha(y)})$, which is the gluing compatibility for s_x. Therefore we can glue them to $s \in \Gamma(\mathscr{F}')$, and we know that $f(s)$ and t is mapped to the same thing in $\prod \mathscr{F}_x$, so $f(s) = t$. Then the proof is done. $\square$

Now we are allowed to consider the derived functors of $\Gamma(X, -)$.

Definition 17 (sheaf cohomology)**.** For a sheaf $\mathscr{F}$ on a space X, we define the k-th sheaf cohomology of $\mathscr{F}$, denoted by $H^k(X, \mathscr{F})$, to be $R^k\Gamma(X, -)(\mathscr{F})$.

The notion of sheaf cohomology is a powerful tool: on the one hand, since algebraic geometry also cares about sheaves and structural sheaves of certain spaces (that is, the notion of *schemes*), we can use the sheaf cohomology to explain some theorems about structure of structural sheaf on certain space, such as in the famous Riemann-Roch theorem; on the other hand, the way we define sheaf cohomology using derived functor inspires a more general method of giving a cohomology theory out of a left-exact additive functor to measure its failure of exactness, whose application includes cohomology theories even in some seemingly disparate fields, for example, the notion of *group cohomology* and *algebraic K-theory*.

4.3 Equivalences between sheaf cohomology and other cohomology theories

Our first bounus of developing such a complicated machinery is its natrual equivalence to de Rham cohomology.

Theorem 10. *On a smooth manifold M, there is an isomorphism $H^*_{\mathrm{DR}}(M) \cong H^*(M, \underline{\mathbb{R}})$.*

Proof. Previously we have mentioned in example 3.1.2 that for an smooth manifold M, the differential forms on open subsets of M form a sheaf $\Omega^k(-)$. Consider the chain complex

$$0 \longrightarrow \underline{\mathbb{R}} \xrightarrow{d} \Omega^0(-) \xrightarrow{d} \Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \cdots$$

It is clear that $\underline{\mathbb{R}}$ is the kernel of the map $\Omega^0(-) \to \Omega^1(-)$, so the sequence is exact at $\underline{\mathbb{R}}$. For the rest of the sequence, we can check the exactness on stalk level. Since M is locally homeomorphic to $\mathbb{R}^n$, a contractible space, and by Poincaré's lemma, which is the theorem 11.49 in[1], we can see that locally a closed form is always exact. So the exactness of the rest of the complex is proved. Moreover, $\Omega^k(-)$ turns out to be a $\Gamma(M, -)$-cyclic for each k, whose proof can be found in Theorem 10.22 of[9]. Therefore we have $H^*(M, \underline{\mathbb{R}}) \cong H^*(\Omega^\bullet(M))$, where the latter cohomology coincide with the definition of de Rham cohomology. This finishes the proof. $\square$

We can apply the same method to Similarly, for a smooth manifold M and ring R, we can treat singular cochains as a presheaf by considering the contravariant functor $C^k(-; R)$. We take the sheafification of this presheaf and still use the same notation. According to pp. 42 of[10], we know that the complex

$$0 \longrightarrow \underline{R} \xrightarrow{d} C^0(-; R) \xrightarrow{d} C^1(-; R) \xrightarrow{d} C^2(-; R) \xrightarrow{d} \cdots$$

is acyclic with respect to the global section functor. The exactness at each step of this chain is again guaranteed by the local contractibility provided by M being a manifold. Now if we use this resolution to calculate sheaf cohomology of M, what we get is $H^*(C^\bullet(M; R))$, which is exactly singular cohomology. So we have the theorem

Theorem 11. *On a smooth manifold M, there is an isomorphism $H^*(M; R) \cong H^*(M, \underline{R})$. Moreover, this result can be generalized to any locally contractible space X.*

The two above theorems gracefully completes the long and tedious proof of theorem 4 presented in section 2.3, demonstrating the power of this derived-functor-based sheaf cohomology.

However, as we mentioned before, the difficulty of calculating sheaf cohomology itself by definition is an obstacle to putting this machinery into actual use. The follwoing theorem solves this problem by stating that Čech cohomology is equivalent to sheaf cohomology, where the former is much easier to calculate. This result is first given by Jean Leray.

Theorem 12. *Sheaf cohomology and Čech cohomology coincide for any sheaf on a paracompact and Hausdorff space.*

Proof. See section II.5.10 of[11]. □

4.4 An example of sheaf cohomology measuring local-to-global obstacles

In this section we give an elementary example of using sheaf cohomology to detect "the obstacle to local-to-global problem" mentioned as a slogan before. We fix the notation of *n-dimensional complex manifold* to be a smooth manifold M with atlas being $\mathbb{C}^n$, and with transition maps being holomorpic. Consider the presheaf of all holomorphic functions $\mathcal{O}_M$ on M, defined by $U \mapsto \{f : U \to \mathbb{C} | f : \text{holomorphic}\}$, and it is immediately seen to be a sheaf. It has a subsheaf $\mathcal{O}_M^* \subset \mathcal{O}_M$ consisting of nowhere vanishing functions. Our example start with the chain of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{j} \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}_M^* \longrightarrow 1$$

where $j(n) := 2n\pi i$ and exp is defined by post-compose $f : U \to \mathbb{C}$ with the complex exponential e^z. The exactness of the chain can be checked on the level of stalks.

Proposition 8. This is an exact sequence.

Proof. First it is clear that $j : \underline{\mathbb{Z}} \to \mathcal{O}_M$ is the kernel of exp, so the exactness follows. For the exactness at $\mathcal{O}_M$, pick a stalk s_z at $z \in M$ represented by section $s : U \to \mathbb{C}$ that is mapped to constant function 1 by exp. Since the pre-image of 1 of function e^z is $N := \{2n\pi i | n \in \mathbb{Z}\}$, we know that there exist some neighborhood U of z such that $\mathrm{im}(s|_U) \subset N$. Since s is holomorphic, N is discrete and M is locally contractible (thus locally connected), we can see that s is the constant function to $2k\pi i$ for some k when restricted in an even smaller neighborhood of z, so $s_z = (2k\pi i)_z \in \underline{\mathbb{Z}}$. The exactness at $\mathcal{O}_M^*$ lays heavily on the local nature of stalks: for any stalk s'_z represented by $s' \in \mathcal{O}_M^*(U)$, we have $\mathrm{im}(s') \subset \mathbb{C}^*$. Therefore we can find some small disk D that avoids 0 and pass to another representative $s'' := s'|_{s'^{-1}(D)}$ which still represent the same germ. Since then we can safely choose a ray l that avoids D and take the corresponding branch of complex logarithm, and we get the desired preiamge in $\mathcal{O}_M$. $\square$

After applying the global section functor, according what we have done before, there is a long exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(M, \underline{\mathbb{Z}}) \arrow[r, "j"] & \Gamma(M, \mathcal{O}_M) \arrow[r, "\exp"] & \Gamma(M, \mathcal{O}_M^*) \arrow[dll] & \\ & H^1(M, \underline{\mathbb{Z}}) \arrow[r] & H^1(M, \mathcal{O}_M) \arrow[r] & H^1(M, \mathcal{O}_M^*) \arrow[r] & \cdots \end{tikzcd}$$

As we can see, the global section of exp is surjective if and only if $H^1(M, \underline{\mathbb{Z}}) = 0$, and this is exactly the vanishing of first singular cohomology group, according to the result we have proved. Under this guidance, we can easily cook up some example for failure of surjectiveness of $\Gamma(M, \exp)$: the existence of "one-dimensional holes" on M, such as the case of $M := \mathbb{C}^*$, will make it impossible to choose a reasonable branch for complex logarithm for some sections whose image goes arounds the origin, therefore ruin the surjectiveness. This is also a strong example of the slogan of geometry: "the sheaf of functions on certain space contains the same amount of data", since here existence of certain holomorphic functions is binded with topological data on M.

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致谢

感谢朱一飞和王博潼两位老师的在学术上和论文写作上的指导，以及梁桐桐和李昀升两位学长在写作过程中给予的帮助和鼓励。