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THE KERVAIRE INVARIANT OF EXTENDED POWER MANIFOLDS

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INTRODUCTION

SUPPOSE M is a compact smooth manifold whose stable normal bundle νM is trivial. Let F be a specific isomorphism of νM with the trivial bundle. The pair (M, F) will be referred to as a framed manifold. Suppose the dimension of M is $2n$ where n is odd, then, in [9] (see also [10]), Kervaire has defined a $Z/2$ valued invariant $K(M, F)$. Browder has shown[2] that $K(M^{2n}, F) = 0$ unless $n = 2^k - 1$, and has given a necessary and sufficient homotopy theoretic condition for the existence of $2^{k+1} - 2$ dimensional framed manifolds with Kervaire invariant one. It is of much interest, both in differential topology and, via Browder's result, in homotopy theory to decide in which dimensions of the form 2^{k+1}-2 such manifolds can exist.

In dimensions 2, 6 and 14, $S^1 \times S^1$, $S^3 \times S^3$ and $S^7 \times S^7$ can be framed to have Kervaire invariant one. In dimensions 30 and 62 the necessary homotopy theory has been done: the 30-dimension case is published in [13], the 62-dimension case is due to Barratt and Mahowald but is not yet published. The problem is unsolved in dimensions greater than 62.

Extended power manifolds are constructed in the following manner: Let Y be a manifold on which the group $G \subset \Sigma_t$ acts freely (Σ_t is the group of permutations of a set with t elements). Let N be another manifold, then G acts on the cartesian product $(N)^t$ by permuting factors. We may form the "extended power" $Y \times_G (N)^t$ which will always be denoted $Y_G(N)$. This is obviously based on the extended power construction in homotopy theory, a construction which has been extensively studied. Some of this work, and references to the origins of the ideas involved, may be found in Milgram's papers[16–18] and Nishida's papers[19–21].

The purpose of this paper is to examine the Kervaire invariant of extended powers of S^7, a project suggested some time ago by Barratt.

First it is necessary to decide when $Y_G(N)$ can be framed. Define $\bar{Y} = Y/G$ and let ξ be the t-dimensional bundle $Y_G(R) \to \bar{Y}$.

THEOREM A. *Let N^n be a framed manifold. Then $Y_G(N)$ can be framed if and only if $\tau\bar{Y} + n\xi$ is stably trivial. (Here τX stands for the tangent bundle of the manifold X.) Given a framing F of N and a stable trivialisation α of $\tau\bar{Y} + n\xi$ then there is an associated framing of $Y_G(N)$, denoted $\alpha_G(F)$.*

This theorem, in statement and proof, is a straightforward generalisation of results due to Milgram[15]. Milgram considers the case $Y = S^m$ and $G = \Sigma_2$.

From now on assume Y, G and N have been chosen so that $Y_G(N)$ can be framed. We go on to study framings of $Y_G(N)$ induced by framings of N. Let $g: N \to O$ be a map (g will be identified with an element of $KO^{-1}N$) and let F be a framing of N. Then, as described in [23, §2] we may twist F by g and obtain a new framing gF.

THEOREM B. *Suppose G is a 2-group. Then there is a homomorphism $h: KO^{-1}N \to KO^{-1}Y_G(N)$ such that*

$$\alpha_G(gF) = h(g)\alpha_G(F).$$

The homomorphism h will be described in detail in §3.

We come now to the main results of the paper, concerning the Kervaire invariant.

We will use the wreath product $\Sigma_2 \int \Sigma_2 \subset \Sigma_4$. This is a Sylow-2-subgroup of Σ_4. It is conjugate in Σ_4 to the dihedral group D_4, that is the full symmetry group of the square.

THEOREM C. *Let X be an orientable surface of genus 5, and $G = \Sigma_2 \int \Sigma_2$. Then G can act freely on X so that*

(i) *$X_G(S^7)$ can be framed.*

Let H be the Cayley number framing of S^7 and let α be any choice of stable trivialisation of $\tau\bar{X} + 7\xi$ (the notation is as in Theorem A). Then

(ii) *$K(X_G(S^7), \alpha_G(H) = 1$.*

Notes. (i) The Cayley number framing of S^7 may be described as follows. Let F be a framing of S^7 which extends over D^8, and let $g: S^7 \to SO$ be the map obtained from multiplication (on the left) by unit Cayley numbers. Then $H = gF$.

(ii) Using the description of G as the dihedral group D_4 one may give an explicit construction of the surface X.

(iii) The theorem remains true if S^7 is replaced by S^3 and the Cayley number framing replaced by the Quaternionic framing. If S^7 is replaced by S^1 then $X_G(S^1)$ cannot be framed.

In view of Theorem C we go on to consider the group $G = \Sigma_2 \int \Sigma_2 \int \Sigma_2 \subset \Sigma_8$ and ask whether there is a 6 manifold Y on which G acts freely so that $Y_G(S^7)$ with framing induced by the Cayley number framing of S^7 has Kervaire invariant one. The next theorem shows this cannot happen. In the statement of the theorem $G_k \subset \Sigma_{2^k}$ is the iterated wreath product $G_k = \Sigma_2 \int \ldots \int \Sigma_2$ (k copies of Σ_2).

THEOREM D. *Let Y be a d-dimensional manifold where $d = 2^{l+1} - 2 - 7 \cdot 2^k$. Suppose G_k acts freely on Y and that $Y_{G_k}(S^7)$ can be framed. Let F be any framing of S^7 and α a stable trivialisation of $\tau\bar{Y} + 7\xi$. Then if $d \neq 2$ the Kervaire invariant of the $2^{l+1} - 2$ dimensional manifold $Y_{G_k}(S^7)$ equipped with the framing $\alpha_{G_k}(F)$ is zero.*

Note that Theorem D does not assert that the Kervaire invariant of $Y_{G_k}(S^7)$ is zero for all framings of this manifold, only for the "natural" framings, that is those induced by framings of S^7. Another way of expressing this is to say that we are considering the *framed* manifold $Y_{G_k}(S^7)$ as a function of the *framed* manifold S^7.

The line of proof of Theorem D is to use Theorem B and the change of framing formula for the Kervaire invariant to show that if $d \neq 2$ then the Kervaire invariant of $Y_{G_k}(S^7)$ with framing $\alpha_{G_k}(F)$ is independent of the choice of framing F of S^7. It is easy to see from the proof of Theorem A that we may choose F so that $K(Y_{G_k}(S^7), \alpha_{G_k}(F)) = 0$ and so Theorem D will follow.

This paper is set out as follows: §1 contains the necessary generalities on the Kervaire invariant, §2 contains a discussion of the extended power construction, including the proof of Theorem A, §3 contains the proof of Theorem B. In §4 there is a plan of the proof of Theorems C and D, and a good deal of preparatory calculation is done, §5 contains the proof of Theorem C and §6 that of Theorem D.

A large proportion of this work is contained in my Oxford D.Phil. thesis, written under the supervision of Elmer Rees. It is a great pleasure to thank Elmer Rees for introducing me to this subject and for his constant help and encouragement.

§1. GENERALITIES ON THE KERVAIRE INVARIANT

Suppose (M^{2n}, F) is a closed framed manifold. Following Pontryagin[19], Kervaire[9], Kervaire and Milnor[10], Browder[2] and Brown[3] we may use the framing to construct a quadratic function

$$q_F: H^n M \to Z/2$$

(all homology groups in this paper will have $Z/2$ coefficients), that is

$$q_F(x + y) = q_F(x) + q_F(y) + x \cdot y$$

where $x \cdot y$ stands for the mod 2 intersection number of x and y. There is a mod 2 invariant, the Arf invariant $A(q_F)$, associated to q_F [1]. A thorough account of this invariant is given in [20, Appendix pp. 411–413]; $A(q_F)$ is zero if and only if q_F sends the majority of elements of H^nM to zero.

The Kervaire invariant is now defined by

$$K(M, F) = A(q_F).$$

It is a framed bordism invariant.

The first result we require is the change of framing formula due essentially to Brown [3, Thm 1.18, 4, p. 299, Thm 3.3]. A proof of this key result is also given in [6]. To state the result requires some notation. Let $v_{n+1} \in H^{n+1}BO$ be the universal $(n+1)$-th Wu-class (see [2, p. 164]). Let $y_n = \Omega v_{n+1}$ in H^nO.

1.1 THEOREM. *Let (M^{2n}, F) be a framed manifold and $g: M \to O$. Let gF be the framing obtained by twisting F by g. Then*

(i) $q_{gF}(x) = q_F(x) + x \cdot g^*y_n$

(ii) $K(M, gF) = K(M, F) + q_F(g^*y_n)$.

The second result required concerns the evaluation of the quadratic form q_F on a class of the form Sq^ky.

1.2 THEOREM. *Let (M^{2n}, F) be a framed manifold where $n = 2^l - 1$ with $l \geq 4$. Suppose $k = 1, 2$ or 4. Then if $y \in H^{n-k}M$*

$$q_F(Sq^ky) = \sum_{i=0}^{k-1} (Sq^iy) \cdot (Sq^{2k-i}y).$$

Some comments on this theorem are necessary. The hypotheses on n and k are sufficient for the use of this theorem in this paper. However, the theorem, as stated, is valid for many more values of n and k. A proof of a more general version of this theorem will be given in a forthcoming paper [8].

§2. THE EXTENDED POWER CONSTRUCTION

We will take for granted throughout this section the notation and assumptions of Theorem A. We begin with a proof of Theorem A following Milgram's proof in [15] for the special case $Y = S^m$, $G = \Sigma_2$. Let $\pi: Y_G(N) \to \bar{Y}$ be the projection.

2.1 LEMMA. *$\tau Y_G(N)$ and $\pi^*(\tau\bar{Y} + n\xi)$ are stably isomorphic.*

Proof. Since N^n can be framed there is an embedding $N \times R^L \subset R^{n+L}$. This gives an embedding j

$$Y_G(N) \to Y_G(N \times R^L) \to Y_G(R^{n+L}).$$

The normal bundle of this embedding is $\nu = Y_G(N \times R^L) = \pi^*L\xi$. Since $Y_G(R^{L+n})$ is the total space of the bundle $(L+n)\xi$ it follows that $\tau(Y_G(R^{L+n})) = p^*(\tau\bar{Y} + (L+n)\xi)$ where $p: Y_G(R^{L+n}) \to \bar{Y}$ is the projection. From the definition of the normal bundle of an embedding

$$\tau Y_G(N) + \pi^*L\xi = j^*p^*(\tau\bar{Y} + (L+n)\xi).$$

However $pj = \pi$, so solving this equation stably gives

$$\tau Y_G(N) = \pi^*(\tau\bar{Y} + n\xi).$$

The projection π has a section, namely $\bar{Y} = Y_G(x) \to Y_G(N)$ where x is any point of N. Thus $\tau Y_G(N)$ is stably trivial if and only if $\tau\bar{Y} + n\xi$ is stably trivial. Therefore $Y_G(N)$ can be framed if and only if $\tau\bar{Y} + n\xi$ is stably trivial. This proves the first assertion of Theorem A.

From now on assume that Y, G and the framed manifold N are chosen so that $Y_G(N)$ can be framed. We now study framings of $Y_G(N)$ induced by framings of N. The framing F of N gives, as in the proof of 2.1, a stable isomorphism of $\tau Y_G(N)$ with $\pi^*(\tau\bar{Y} + n\xi)$. Combining this with a stable trivialisation of $\tau\bar{Y} + n\xi$ gives a stable trivialisation of $\tau Y_G(N)$, that is a framing of $Y_G(N)$. We require rather detailed information about this framing and so we give a more explicit description of it.

Since ξ is a bundle with finite structural group over a finite complex, it follows that we may choose a large integer L and an isomorphism $\beta: L\xi \to \epsilon^{tL}$ (ϵ^q will always denote the trivial q-dimensional bundle). The stable trivialisation α of $\tau\bar{Y} + n\xi$ gives an isomorphism, also denoted α

$$\alpha: \nu^k\bar{Y} \to \epsilon^{k-nt} + n\xi$$

where $\nu^k\bar{Y}$ is the k-dimensional normal bundle of $\bar{Y}$ with k large. We now get an embedding $i: R^{k-nt} \times Y_G(R^{L+n}) \to R^{d+k+tL}$ ($d = \dim Y$) as follows:

$$R^{k-nt} \times Y_G(R^{L+n}) = (\epsilon^{k-nt} + n\xi) + L\xi \xrightarrow[\alpha^{-1}+\beta]{} \nu^k\bar{Y} + \epsilon^{tL} \subset R^{d+k+tL}. \tag{2.2}$$

Let F be a framing of N, then F gives an embedding $j_F: N \times R^L \to R^{L+n}$, and so an embedding

$$R^{k-nt} \times Y_G(N \times R^L) \to R^{k-nt} \times Y_G(R^{L+n}). \tag{2.3}$$

Note that $Y_G(N \times R^L) = \pi^* L\xi$ so there is an isomorphism h

$$R^{k-nt} \times R^{tL} \times Y_G(N) \xrightarrow[1\times(\pi^*\beta)^{-1}]{} R^{k-nt} \times Y_G(N \times R^L). \tag{2.4}$$

The composite of these three embeddings gives a framed embedding of $Y_G(N)$ in R^{d+k+tL}. Denote the associated framing by $\alpha_G(F)$.

2.5 LEMMA. *The framing $\alpha_G(F)$ does not depend on the choice of trivialisation $\beta: L\xi \to \epsilon^{tL}$.*

Proof. Suppose we alter β by an automorphism $g: \epsilon^{tL} \to \epsilon^{tL}$. Stably this alters the embedding i of (2.2) by adding ϵ^{tL} and replacing i by $i + g$. The embedding of (2.3) remains unaltered. The isomorphism h of (2.4) is altered by adding ϵ^{tL} and replacing h by $h + g^{-1}$. The composite remains unaltered and so, stably, the embedding is independent of β. However k and L can be chosen large enough so that the embedding is already stable. This completes the proof.

Next we examine how the framed bordism class of $Y_G(N)$ with framing $\alpha_G(F)$ depends on the framed bordism class of (N, F).

2.6 LEMMA. *Suppose (N, F) is framed bordant to (P, H). Then $(Y_G(N), \alpha_G(F))$ is framed bordant to $(Y_G(P), \alpha_G(H))$.*

Proof. Let Y^+ denote Y with a disjoint base point. For any pointed space X let $X^{(t)}$ denote the t-fold smash product of X with itself. Given (N, F), then when the framed embedding corresponding to the framing $\alpha_G(F)$ is compactified the resulting map between spheres factorises as follows:

$$S^{d+l+tL} \xrightarrow[A]{} S^{k-nt} \wedge Y^+ \wedge_G (S^{L+n})^{(t)} \xrightarrow[1\wedge 1\wedge_G \Phi^{(t)}]{} S^{k-nt} \wedge Y^+ \wedge (S^L)^{(t)} \xrightarrow[B]{} S^{k-nt+tL}$$

where A is the compactification of (2.2), Φ is obtained from compactifying the embedding $j_F: N \times R^L \to R^{n+L}$ and B is obtained from compactifying (2.4). Once we have fixed α the homotopy class of the above map depends only on the homotopy class of Φ. By transversality the lemma follows.

We next show that the framed bordism class of $(Y_G(N), \alpha_G(F))$ depends only on a suitable bordism class of (Y, α). Let $r: G \to O(t)$ be the permutation representation, that is the representation of G obtained by allowing G to permute the basis of R^t. Let ρ be the bundle over BG classified by Br. Giving an isomorphism $\alpha: \nu^k\bar{Y} \to \epsilon^{k-nt} + n\xi$ is equivalent to giving a bundle map $\nu^k\bar{Y} \to \epsilon^{k-nt} + n\rho$, which on base spaces classifies the principal G covering $Y \to Y/G = \bar{Y}$. We may classify d-dimensional manifolds whose stable normal bundle admits such a structure up to the evident bordism relation. The resulting group is denoted by $\Omega_d(BG; n\rho)$. Transversality shows that $\Omega_d(BG; n\rho) \cong \pi^s_{d+nt}(T(n\rho))$ where $T(\eta)$ stands for the Thom complex of the bundle η. The proof of the next lemma follows directly from the definitions.

2.7 LEMMA. *Suppose (Y, α) and (Z, β) define the same element of $\Omega_d(BG; n\rho)$. Then $(Y_G(N), \alpha_G(F))$ is framed bordant to $(Z_G(N), \beta_G(F))$ for all framed manifolds (N, F).*

§3. THE PROOF OF THEOREM B

We now examine how the framing $\alpha_G(F)$ depends on the framing F, and hence give a proof of Theorem B. The first task is to describe the homomorphism $h: KO^{-1}N \to KO^{-1}Y_G(N)$. Let $\gamma: Y \to EG$ be the equivariant map classifying the free G-action on Y. Given $g \in KO^{-1}N$ form the map

$$Y \times_G (N)^t \xrightarrow[\gamma \times_G (1)^t]{} EG \times_G (N)^t \xrightarrow[1 \times_G (g)^t]{} EG \times_G (O)^t \xrightarrow[D]{} O \tag{3.1}$$

where D is the Dyer–Lashof map for the infinite loop space O, see [5, pp. 36–41]. It follows from the properties of the Dyer–Lashof map (see the diagram on page 39 of [5]) that h is a homomorphism.

To prove Theorem B we require a concrete description of the Dyer–Lashof map D when G is a 2-group. This is provided by a straightforward generalisation of an observation in [12]. Some notation is required. Let $G_k \subset \Sigma_{2^k}$ be the wreath product $G_k = \Sigma_2 \int \Sigma_2 \ldots \int \Sigma_2$ (k copies of Σ_2), and let $r_{k,L}: G_k \to O(2^{k+L})$ be the representation defined by allowing G_k to permute the factors of $(R^{2^L})^{2^k}$.

3.2 LEMMA. *Let E^lG_k be the l-skeleton of EG_k. Then for each $L \geq l + 2$ there exists a map $f_{k,L}: E^lG_k \to O(2^{k+L})$ such that*

(i) *$f_{k,L}(gx) = f_{k,L}(x) r_{k,L}(g)^{-1}$ for $g \in G_k$, $x \in E^lG_k$*

(ii) *The following diagram commutes, up to homotopy.*

$$\begin{array}{ccc}
E^lG_k \times_{G_k} (O(2^L))^{2^k} & \xrightarrow{\hat{f}_{k,L}} & O(2^{L+k}) \\
\cap & & \cap \\
EG_k \times_{G_k} (O)^{2^k} & \xrightarrow[D]{} & O
\end{array}$$

Here $\hat{f}_{k,L}$ is defined by

$$\hat{f}_{k,L}(x; A_1, \ldots, A_{2^k}) = f_{k,L}(x)(A_1 \oplus \cdots \oplus A_{2^k}) f_{k,L}(x)^{-1}$$

for $x \in E^lG_k$ and $A_1, \ldots, A_{2^k} \in O(2^L)$.

Proof. The proof is by induction on k. When $k = 1$, $G = \Sigma_2$ and Madsen's description of the Dyer–Lashof map, [12, pp. 237–241], shows the result is true. We summarize Madsen's work. As usual S^l with the antipodal action of Σ_2 will be identified with $E^l\Sigma_2$. For each $L \geq l + 1$, we know that S^l is contained in the units of the real Clifford algebra C_L. As a vector space C_L has dimension 2^L. The map $f_{1,L}: S^l \to O(2^{L+1})$ is defined as follows: Regard $R^{2^{L+1}}$ as $C_L \oplus C_L$ and for $(u, v) \in C_L \oplus C_L$, and $x \in S^l \subset C_L$ define $f_{1,L}(x)(u, v) = (1/\sqrt{2})(x(u+v), x(u-v))$. Here the product on the right hand side is the product in the Clifford algebra. It is clear that $f_{1,L}$ satisfies property (i) of the lemma, Madsen points out, in [12], that it also satisfies property (ii).

Now assume as inductive hypothesis, that for each $s < k$ the map $f_{s,L}: E^lG_s \to O(2^{L+s})$ has been defined for each $L \geq l + 2$, with the required properties. Since $G_k = \Sigma_2 \int G_{k-1}$ it follows that $E^lG_k \subset S^l \times E^lG_{k-1} \times E^lG_{k-1}$. For each $L \geq l + 2$, $L + k - 1 \geq l + 2$ since $k \geq 1$. Therefore by the inductive hypothesis we have defined $f_{1,L+k-1}: S^l \to O(2^{L+k})$. Also by the inductive hypothesis we have defined

$f_{k-1,L}\colon E^lG_{k-1}\to O(2^{L+k-1})$. Now define $\bar{f}_{k,L}\colon S^l\times E^lG_{k-1}\times E^lG_{k-1}\to O(2^{L+k})$ by $\bar{f}_{k,L}(x,y,z)=f_{1,L+k-1}(x)(f_{k-1,L}(y)\oplus f_{k-1,L}(z))$. Finally define $f_{k,L}$ to be the restriction of $\bar{f}_{k,L}$ to the subspace E^lG_k. Property (i) is verified directly from the definition of $f_{k,L}$ and property (ii) is verified by using the inductive hypothesis and the commutative diagram in Dyer and Lashof's paper [5, p. 39].

3.3 Corollary. *Let $G\subset\Sigma_t$ be any 2-group, and let $r_L\colon G\to O(2^L\cdot t)$ be the representation obtained by allowing G to permute the factors of $(R^{2^L})^t$. Let E^lG be the l-skeleton of EG. Then for each $L\geq l+2$ there exists a map $f_L\colon E^lG\to O(2^L\cdot t)$ such that*

(i) *$f_L(gx)=f_L(x)r_L(g)^{-1}$ for $g\in G$, $x\in E^lG$.*

(ii) *The following diagram commutes up to homotopy.*

$$\begin{array}{ccc} E^lG\times_G(O(2^L))^t & \xrightarrow{\hat{f}_L} & O(2^L\cdot t)\\ \cap & & \cap\\ EG\times_G(O)^t & \xrightarrow[D]{} & O \end{array}$$

Here $\hat{f}_L(x;A_1,\ldots,A_t)=f_L(x)(A_1\oplus\cdots\oplus A_t)f_L(x)^{-1}$.

Proof. Let $t=2^{k_1}+\cdots+2^{k_n}$ be the 2-adic expansion of t. Then $G_{k_1}\times\cdots\times G_{k_n}$ is a Sylow-2-subgroup of Σ_t. Since G is a 2-group G is contained in some Sylow-2-subgroup and since all such subgroups are conjugate we may assume $G\subset G_{k_1}\times\cdots\times G_{k_n}$. For $L\geq l+2$ define $\bar{f}_L\colon E^lG_{k_1}\times\cdots\times E^lG_{k_n}\to O(2^L\cdot t)$ by

$$\bar{f}_L(x_1,\ldots,x_n)=f_{k_1,L}(x_1)\oplus f_{k_2,L}(x_2)\oplus\cdots\oplus f_{k_n,L}(x_n)$$

for $x_i\in E^lG_{k_i}$ where the f_{k_i}'s are given by 3.2. Now define the map f_L by restricting $\bar{f}_L$ to the subspace E^lG. Property (i) is verified directly from the definition and property (ii) is verified using the commutative diagram in Dyer and Lashof's paper [5, p. 39].

Proof of Theorem B. Y is a d-dimensional manifold with a free G action and so the equivariant map $\gamma\colon Y\to EG$ factorises as

$$Y\xrightarrow[\gamma']{} E^{d+1}G\subset EG.$$

Choose $L\geq d+3$ such that $2^L\geq n$ where $n=\dim N$. The map $f_L\cdot\gamma'\colon Y\to O(2^L\cdot t)$ provides a trivialisation β of the bundle $2^L\xi$ over $\bar{Y}$ as follows:

$$\beta\colon 2^L\xi=Y_G(R^{2^L})\to\bar{Y}\times R^{2^L\cdot t}$$

is defined by $\beta(y;v_1,\ldots,v_t)=f_L\cdot\gamma'(v_1\oplus\cdots\oplus v_t)$ where $y\in Y$ and $v_1,v_2,\ldots,v_t\in R^{2^L}$.

We will use this choice of β in forming the embedding (2.2) and isomorphism (2.4). Let $j_1\colon N\times R^{2^L}\to R^{2^L+n}$ be the embedding corresponding to the framing F and let $j_2\colon N\times R^{2^L}\to R^{2^L+n}$ be that corresponding to the framing gF. Regard g as a map of N into $O(2^L)$, then $j_2(x,v)=j_1(x,g(x)^{-1}v)$. Choose a stable trivialisation α of $\tau\bar{Y}+n\xi$ and let $i_1,i_2\colon R^{k-nt}\times R^{t\cdot 2^L}\times Y_G(N)\to R^{d+k+t\cdot 2^L}$ be the embeddings formed using α, j_1 and β, and α, j_2 and β respectively. Then

$$i_2(v_1,v_2,z)=i_1(v_1,\bar{h}(g)^{-1}(z)(v_2),z)$$

where $v_1\in R^{k-nt}$, $v_2\in R^{2^L\cdot t}$, $z\in Y_G(N)$ and $\bar{h}(g)\colon Y_G(N)\to O(2^L\cdot t)$ is given by

$$\bar{h}(g)(y;x_1,\ldots,x_t)=f_L\cdot\gamma'(y)(g(x_1)\oplus\cdots\oplus g(x_t))f_L\cdot\gamma'(y)^{-1}.$$

Corollary 3.2 shows that $\bar{h}(g)$ thought of as an element of $KO^{-1}Y_G(N)$ is equal to the element described in (3.1). This completes the proof of Theorem B.

§4. PLAN OF THE PROOF OF THEOREMS C AND D AND SOME PRELIMINARY CALCULATIONS

We begin by fixing some notation which will be used for the rest of this paper. Let Y be a d-dimensional closed manifold, where $d = 2^{l+1} - 2 - 7 \cdot 2^k$, with a free G_k action. In order that d be strictly positive we will assume $l \geqslant 4$. Assume $\tau\bar{Y} + 7\xi$ is stably trivial (the notation is as in Theorem A) and fix once and for all a stable trivialisation α of this bundle. Let F be a framing of S^7 which extends over the disc D^8 and let $g: S^7 \to SO$ be the map obtained by multiplication, on the left, by unit Cayley numbers. We deduce from Theorem A that the $2^{l+1}-2$ dimensional manifold $Y_{G_k}(S^7)$ can be framed. Let B be the framing of this manifold induced by F, that is $B = \alpha_{G_k}(F)$. Then from (2.6) we see that $(Y_{G_k}(S^7), B)$ is a framed boundary and therefore

$$K(Y_{G_k}(S^7), B) = 0. \tag{4.1}$$

Note that $KO^{-1}S^7 \cong Z$ and the map g represents a generator of this group. Recall the homomorphism $h: KO^{-1}S^7 \to KO^{-1}Y_{G_k}(S^7)$ described in §3 and the definition of y_{2^l-1} in $H^{2^l-1}O$ given before (1.1).

4.2 LEMMA. *The following are equivalent*:

(i) *There is a framing Φ of S^7 such that $K(Y_{G_k}(S^7), \alpha_{G_k}(\Phi)) = 1$.*

(ii) $K(Y_{G_k}(S^7), \alpha_{G_k}(gF)) = 1$.

(iii) $q_B(h(g)^* y_{2^l-1}) = 1$.

Proof. Every framing Φ of S^7 is of the form $(mg)F$ for some integer m. Using Theorem B, 1.1, and 4.1 we deduce

$$K(Y_{G_k}(S^7), \alpha_{G_k}(mgF)) = q_B(h(mg)^* y_{2^l-1}).$$

since by definition $B = \alpha_{G_k}(F)$. Now $y_n = \Omega v_{n+1} \in H^nO$ where v_{n+1} is the Wu-class. Therefore if $f: X \to O$ is a map it follows that $(mf)^* y_n = m(f^* y_n)$. Therefore if m is even $h(mg)^* y_{2^l-1} = 0$ and if m is odd $h(mg)^* y_{2^l-1} = h(g)^* y_{2^l-1}$. The lemma now follows.

The plan of the proof of Theorems C and D is as follows: Lemma 4.2 shows we only need compute $q_B(h(g)^* y_{2^l-1})$. A calculation due to Kochman[11] enables us to compute $h(g)^* y_{2^l-1}$. We next show that there are classes $u \in H^{2^l-2}Y_{G_k}(S^7)$ and $v \in H^{2^l-5}Y_{G_k}(S^7)$ such that $h(g)^* y_{2^l-1} = Sq^1u + Sq^4v$. We now use (1.2) to compute $q_B(Sq^1u + Sq^4v)$. Lemma 4.2 shows that we will have dealt with all induced framings of $Y_{G_k}(S^7)$.

The main purpose of this section is to prove the following theorem.

4.3 THEOREM. *Assume $l \geqslant 4$, then there exist classes $u \in H^{2^l-1}Y_{G_k}(S^7)$ and $v \in H^{2^l-5}Y_{G_k}(S^7)$ such that*

$$h(g)^*(y_{2^l-1}) = Sq^1u + Sq^4v.$$

It is convenient to factor $h(g)$. For spaces X define E_2X to be the space $S^\infty \times_{\Sigma_2} (X \times X)$ where Σ_2 acts antipodally on S^∞ and by switching factors on $X \times X$. The space E_2X is a functor of X in an obvious way. We make the following identification

$$\begin{aligned} EG_k \times_{G_k} (X)^{2^k} &= E_2 \ldots E_2X \qquad (k \quad E_2\text{'s}) \\ &= E_2{}^kX. \end{aligned}$$

Write $E_2{}^0X = X$. Now define maps

$$g_k: E_2{}^kS^7 \to SO \tag{4.4}$$

inductively by setting $g_0 = g: S^7 \to SO$ and defining g_k to be the composite

$$E_2{}^kS^7 = E_2(E_2{}^{k-1}S^7) \xrightarrow[E_2(g_{k-1})]{} E_2SO \xrightarrow[D]{} SO.$$

where D is the Dyer–Lashof map for SO. Then using the diagram on p. 39 of Dyer and Lashof's paper[5] it follows that $h(g)$ factors as

$$Y \times_{G_k} (S^7)^{2^k} \xrightarrow[\gamma \times_{G_k}(1)^{2^k}]{} EG_k \times_{G_k} (S^7)^{2^k} = E_2{}^k S^7 \xrightarrow[g_k]{} SO \tag{4.5}$$

where $\gamma: Y \to EG_k$ is the equivariant map classifying the free G_k action on Y.

Most of our subsequent calculations are carried out in the ring $H^*E_2{}^kS^7$. First we record the structure of H^*E_2X. This consists simply of well known results see [5, 14, 15, 19–21].

Let W be the usual resolution over $Z/2$ of Σ_2 by free Σ_2 modules, that is $W_n = Z/2[\Sigma_2]$ with generator e_n, and the differential is determined by $\partial e_n = e_{n-1} + Te_{n-1}$ where $T \in \Sigma_2$ is the non-trivial element. Let X be a c.w. complex then form the cochain complex $C = W^* \otimes_{\Sigma_2} H^*X \otimes H^*X$, where Σ_2 acts on $H^*X \otimes H^*X$ by $T(a \otimes b) = b \otimes a$. The co-boundary homomorphism is determined by $\delta(e^n \otimes x \otimes y) = e^{n+1} \otimes (x \otimes y + y \otimes x)$, where $e^n \in \mathrm{Hom}(W_n, Z/2)$ is dual to e_n. Then $H^*C \cong H^*E_2X$, see [5, Lemma 2.4]. We abuse notation and write $e^n \otimes x \otimes x$ for the cohomology class of this cocycle. We write $[x, y]$ for the cohomology class of the cocycle $1 \otimes (x \otimes y + y \otimes x)$. Note that $[x, y] = [y, x]$ and $[x, x] = 0$.

Let $\{x_1, x_2, \ldots\}$ be a basis, of homogeneous elements, for H^*X, then a basis for H^*E_2X is provided by the following elements;

$$e^n \otimes x_i \otimes x_i, \qquad n \geq 0; \qquad [x_i, x_j], \qquad i \neq j.$$

Products in H^*E_2X are determined by the following formulae; (see for example [15, p. 39],

$$\begin{aligned} (e^n \otimes x \otimes x)(e^m \otimes y \otimes y) &= e^{n+m} \otimes xy \otimes xy \\ (e^n \otimes x \otimes x)[y, z] &= 0, \qquad \text{if } n \geq 1 \\ (1 \otimes x \otimes x)[y, z] &= [xy, xz] \\ [x, y][u, v] &= [xu, yv] + [xv, yu]. \end{aligned} \tag{4.6}$$

Let $f: X \to Y$ be a map, then $(E_2f)^*: H^*E_2Y \to H^*E_2X$ is determined by the formulae;

$$\begin{aligned} (E_2f)^*(e^n \otimes x \otimes x) &= e^n \otimes f^*x \otimes f^*x \\ (E_2f)^*[y, z] &= [f^*y, f^*z]. \end{aligned} \tag{4.7}$$

There are various natural maps associated to E_2X;

$$\begin{array}{lll} j\colon X \times X \to E_2X & \text{is the natural inclusion} & \\ \pi\colon E_2X \to B\Sigma_2 & \text{is the natural projection} & (4.8) \\ t\colon H^*(S^\infty \times X \times X) \to H^*E_2X & \text{is the transfer map associated to} & \\ & \text{the covering } S^\infty \times X \times Y \to E_2X. & \end{array}$$

In cohomology these maps are determined by the following formulae;

$$\begin{aligned} j^*(e^n \otimes x \otimes x) &= 0, \qquad \text{if } n > 0 \\ j^*(1 \otimes x \otimes x) &= x \otimes x \\ j^*[x, y] &= x \otimes y + y \otimes x \\ \pi^*e^n &= e^n \otimes 1 \otimes 1 \\ t(1 \otimes x \otimes y) &= [x, y]. \end{aligned} \tag{4.9}$$

Finally the action of the Steenrod algebra is determined by the following formulae; (see [15, p. 40] and also [19, 21, p. 716])

for $x \in H^nX$ and $k \geq 1$

$$Sq^t(e^k \otimes x \otimes x) = \sum_{i \geq 0} \binom{k+n-i}{t-2i} e^{t+k-2i} \otimes Sq^ix \otimes Sq^ix,$$

for $x \in H^n X$

$$Sq^t(1 \otimes x \otimes x) = \sum_{i \geq 0} \binom{n-i}{t-2i} e^{t-2i} \otimes Sq^i x \otimes Sq^i x + \sum_{0 \leq i < t/2} [Sq^{t-i}x, Sq^i x], \quad (4.10)$$

for $x, y \in H^*X$

$$Sq^t[x, y] = \sum_{0 \leq i \leq t} [Sq^{t-i}x, Sq^i y].$$

In (4.10) where the limits of the summation are not explicitly given then the sum is to be taken over all values of i for which the summand makes sense.

Now suppose X is an infinite loop space, then there exists a map D: $E_2X \to X$ such that $Dj: X \times X \to X$ is the multiplication in the H-space X. Therefore $(Dj)^*: H^*X \to H^*X \otimes H^*X$ is the comultiplication Ψ in the Hopf algebra H^*X. Since X is homotopy commutative

$$\Psi(x) = \bar{x} \otimes \bar{x} + \sum_i (x_i' \otimes x_i'' + x_i'' \otimes x_i'), \qquad \text{for } x \in H^*X.$$

Dual to the Dyer–Lashof homology operations in H_*X there are the Dyer–Lashof cohomology operations Q_*^k: $H^nX \to H^{n-k}X$. They satisfy the following formula. Let $x \in H^nX$, then

$$D^*x = \sum_{n \leq 2k \leq 2n} e^{2k-n} \otimes Q_*^k x \otimes Q_*^k x + \sum_i [x_i', x_i'']. \quad (4.11)$$

Kochman has computed these operations in H^*SO. To state his results let $w_{n+1} \in H^{n+1}BSO$ be the universal Stiefel–Whitney class and let $a_n = \Omega w_{n+1}$ in H^nSO.

4.12 THEOREM. (See Kochman's paper [11, p. 107, Thm 5.2].

$$Q_*^r a_k = \binom{r-1}{k-r} a_{k-r}.$$

Our first task is to compute $h(g)^* y_{2^l-1}$ where $y_{2^l-1} = \Omega v_{2^l}$ (v_n is the universal Wu-class). From (4.5) this involves computing $g_k^* y_{2^l-1}$ and so, from the definition of g_k (see 4.4), computing $D^* y_{2^l-1}$ where D is the Dyer–Lashof map for SO. We therefore need to compute $Q_*^r y_{2^l-1}$. Now it is known that

(i) $v_{2^l} = w_{2^l}$ + decomposables
(ii) v_n is decomposable if $n \neq 2^l$
(iii) If $a \in H^*BSO$ is decomposable then $\Omega a = 0$.

Therefore $y_{2^l-1} = a_{2^l-1}$. The computation of $Q_*^r y_{2^l-1}$ is completed using the following lemma whose proof is left to the reader.

4.13 LEMMA. *Let* $k = 2^p - 1$, *then* $\binom{r-1}{k-r} \equiv 1 \bmod 2$ *if and only if* $k - r = 2^q - 1$ *with* $q < p$.

4.14 COROLLARY. *Let* D: $E_2SO \to SO$ *be the Dyer–Lashof map. Then*

$$D^* y_{2^l-1} = [1, y_{2^l-1}] + \sum_{1 \leq m \leq l-1} e^{2^l - 2^{m+1} + 1} \otimes y_{2^m-1} \otimes y_{2^m-1}.$$

Proof. Since $y_{2^l-1} = \Omega v_{2^l}$ it follows that $\Psi(y_{2^l-1}) = y_{2^l-1} \otimes 1 + 1 \otimes y_{2^l-1}$. This accounts for the term $[1, y_{2^l-1}]$. The rest of the expression is accounted for by the description of D^* in terms of the operations Q_*^r, the observation that $y_{2^l-1} = a_{2^l-1}$ and (4.12) and (4.13).

Recall the map g_k defined in (4.4). Define a cohomology class $z_{k,l} \in H^{2^l-1}E_2^kS^7$ by

$$z_{k,l} = g_k^* y_{2^l-1}. \quad (4.15)$$

By definition g_0: $S^7 \to SO$ is just a generator of $KO^{-1}S^7$. Therefore, if we define $\sigma \in H^7S^7$ to be the non-zero element, we have

$$\begin{aligned} z_{0,l} &= 0 \qquad \text{if } l \neq 3 \\ z_{0,3} &= \sigma \qquad \text{in } H^7S^7. \end{aligned} \quad (4.16)$$

We describe the classes $z_{k,l}$ in $H^{2^l-1}E_2{}^kS^7$ inductively on k. To formulate this result some remarks on the cohomology of $E_2{}^kS^7$ are required. Observe that $E_2{}^kS^7 = E_2(E_2{}^{k-1}S^7)$. We always write $H^*E_2{}^kS^7$ as H^*E_2X where $X = E_2{}^{k-1}S^7$.

4.17 Lemma. *In* $H^{2^l-1}E_2{}^kS^7 = H^{2^l-1}E_2(E_2{}^{k-1}S^7)$

$$z_{k,l} = [1, z_{k-1,l}] + \sum_{m=3}^{l-1} e^{2^l-2^{m+1}+1} \otimes z_{k-1,m} \otimes z_{k-1,m}.$$

Proof. When $k = 0$ this is just (4.16). The general case follows from the definition of $z_{k,l}$ as $g_k^* y_{2^l-1}$, the factorisation of g_k as $D \cdot E_2 g_{k-1}$, Lemma (4.14) and the formulae (4.7).

Now let $\bar{\gamma}$: $Y \times_{G_k} (S^7)^{2^k} \to E_2{}^kS^7$ be the first map in (4.5). Then, of course,

$$h(g)^* y_{2^l-1} = \bar{\gamma}^* g_k^* y_{2^l-1} = \bar{\gamma}^* z_{k,l}.$$

The proof (4.3) is completed by choosing classes, for $l \geq 4$, $a_{k,l} \in H^{2^l-2}E_2{}^kS^7$ and $b_{k,l} \in H^{2^l-5}E_2{}^kS^7$ such that

$$z_{k,l} = Sq^1 a_{k,l} + Sq^4 b_{k,l}.$$

The class $a_{k,l} \in H^{2^l-2}E_2(E_2{}^{k-1}S^7)$ for $l \geq 4$ is defined inductively on k, by the following formulae;

$$\begin{aligned} &a_{0,l} = 0 \qquad \text{for all } l \geq 4 \qquad \text{in } H^{2^l-2}S^7. \\ &a_{k,4} = [1, a_{k-1,4}] + 1 \otimes z_{k-1,3} \otimes z_{k-1,3} \qquad \text{in } H^{14}E_2(E_2{}^{k-1}S^7) \\ &a_{k,l} = [1, a_{k-1,l}] + \sum_{m=4}^{l-1} e^{2^l-2^{m+1}} \otimes z_{k-1,m} \otimes z_{k-1,m} \\ &\text{for } l \geq 5, \text{ in } H^{2^l-2}E_2(E_2{}^{k-1}S^7). \end{aligned} \tag{4.18}$$

The class $b_{k,l} \in H^{2^l-5}E_2(E_2{}^{k-1}S^7)$ for $l \geq 4$ is defined inductively, on k, by the formulae;

$$\begin{aligned} &b_{0,l} = 0 \qquad \text{for all } l \geq 4 \qquad \text{in } H^{2^l-5}S^7 \\ &b_{k,4} = 0 \qquad \text{in } H^{11}E_2(E_2{}^{k-1}S^7) \\ &b_{k,l} = [1, b_{k-1,l}] + e^{2^l-19} \otimes z_{k-1,3} \otimes z_{k-1,3} \qquad \text{for } l \geq 5 \qquad \text{in } H^{2^l-5}E_2(E_2{}^{k-1}S^7). \end{aligned} \tag{4.19}$$

Before showing that $a_{k,l}$ and $b_{k,l}$ have the required property we need a preliminary lemma.

4.20 Lemma. $Sq^1 z_{k,l} = Sq^2 z_{k,l} = 0$, *where* $z_{k,l}$ *is the class in* $H^{2^l-1}E_2{}^kS^7$ *defined in* (4.15).

Proof. The proof is typical of several calculations we will do. We proceed by induction on k using (4.16) and (4.17) and the formulae (4.6)–(4.10). When $k = 0$ the result is obvious from (4.16). In general we compute $Sq^1 z_{k,l}$ using (4.17) and (4.10).

$$Sq^1 z_{k,l} = [1, Sq^1 z_{k-1,l}] + \sum_{m=3}^{l-1} \binom{2^l - 2^m}{1} e^{2^l-2^{m+1}+2} \otimes z_{k-1,m} \otimes z_{k-1,m}.$$

By induction $Sq^1 z_{k-1,l} = 0$. Further $\binom{2^l - 2^m}{1} \equiv 0 \bmod 2$ and so $Sq^1 z_{k,l} = 0$.

We now compute $Sq^2 z_{k,l}$ using (4.17) and (4.10).

$$Sq^2 z_{k,l} = [1, Sq^2 z_{k-1,l}] + \sum_{m=3}^{l-1} \left\{ \binom{2^l - 2^m}{2} e^{2^l-2^{m+1}+3} \otimes z_{k-1,m} \otimes z_{k-1,m} + \binom{2^l - 2^m - 1}{0} e^{2^l-2^{m+1}+1} \otimes Sq^1 z_{k-1,m} \otimes Sq^1 z_{k-1,m} \right\}.$$

By induction $Sq^2 z_{k-1,l} = 0$ and $Sq^1 z_{k-1,m} = 0$. Further $\binom{2^l - 2^m}{2} \equiv 0 \bmod 2$ and so $Sq^2 z_{k,l} = 0$.

4.21 LEMMA. *With the above definitions*, $z_{k,l} = Sq^1 a_{k,l} + Sq^4 b_{k,l}$ *in* $H^{2^l-1}E_2^k S^7$, *for* $l \geq 4$.

Proof. The proof is by induction on k. When $k = 0$ the result is trivial for referring to (4.16) we see that for $l \geq 4$, $z_{0,l} = 0$ and referring to (4.18) and (4.19) we see that for $l \geq 4$, $a_{0,l} = 0$ and $b_{0,l} = 0$.

We now do the case $l = 4$. Referring to (4.18) for the definition of $a_{k,4}$ and using (4.10) we see that

$$\begin{aligned} Sq^1 a_{k,4} &= [1, Sq^1 a_{k-1,4}] + \binom{7}{1} e \otimes z_{k-1,3} \otimes z_{k-1,3} + [Sq^1 z_{k-1,3}, z_{k-1,3}] \\ &= [1, Sq^1 a_{k-1,4}] + e \otimes z_{k-1,3} \otimes z_{k-1,3} \end{aligned}$$

since $\binom{7}{1} \equiv 1 \bmod 2$ and by (4.20), $Sq^1 z_{k-1,3} = 0$.

Referring to (4.19) for the definition of $b_{k,4}$ we see $b_{k,4} = 0$.

Now using (4.17) in the case $l = 4$ we see that

$$z_{k,4} = [1, z_{k-1,4}] + e \otimes z_{k-1,3} \otimes z_{k-1,3} = Sq^1 a_{k,4}.$$

This proves the lemma when $l = 4$.

For $l \geq 5$ we compute $Sq^1 a_{k,l}$ and $Sq^4 b_{k,l}$ using (4.18), (4.19) and (4.10):

$$\begin{aligned} Sq^1 a_{k,l} = [1, Sq^1 a_{k-1,l}] &+ \sum_{m=4}^{l-1} \binom{2^l - 2^m - 1}{1} e^{2^l - 2^{m+1} + 1} \otimes z_{k-1,m} \otimes z_{k-1,m} \\ &+ [Sq^1 z_{k-1,l-1}, z_{k-1,l-1}]. \end{aligned}$$

(The term $[Sq^1 z_{k-1,l-1}, z_{k-1,l-1}]$ comes from $Sq^1(1 \otimes z_{k-1,l-1} \otimes z_{k-1,l-1})$.) However the binomial coefficient $\binom{2^l - 2^m - 1}{1} \equiv 1 \bmod 2$ and $Sq^1 a_{k-1,l-1} = 0$ by (4.20) and so

$$Sq^1 a_{k,l} = [1, Sq^1 a_{k-1,l}] + \sum_{m=4}^{l-1} e^{2^l - 2^{m+1} + 1} \otimes z_{k-1,m} \otimes z_{k-1,m}.$$

Next we compute $Sq^4 b_{k,l}$ for $l \geq 5$

$$\begin{aligned} Sq^4 b_{k,l} = [1, Sq^4 b_{k-1,l}] &+ \binom{2^l - 12}{4} e^{2^l - 15} \otimes z_{k-1,3} \otimes z_{k-1,3} \\ &+ \binom{2^l - 13}{2} e^{2^l - 17} \otimes Sq^1 z_{k-1,3} \otimes Sq^1 z_{k-1,3} \\ &+ \binom{2^l - 14}{0} e^{2^l - 19} \otimes Sq^2 z_{k-1,3} \otimes Sq^2 z_{k-1,3}. \end{aligned}$$

Now $\binom{2^l - 12}{4} \equiv 1 \bmod 2$ and by (4.20), $Sq^1 z_{k-1,3} = Sq^2 z_{k-1,3} = 0$, therefore

$$Sq^4 b_{k,l} = [1, Sq^4 b_{k-1,l}] + e^{2^l - 15} \otimes z_{k-1,3} \otimes z_{k-1,3}.$$

Adding $Sq^1 a_{k,l}$ and $Sq^4 b_{k,l}$ and using the inductive hypothesis to show that

$$\begin{aligned} [1, Sq^1 a_{k-1,l}] + [1, Sq^4 b_{k-1,l}] &= [1, Sq^1 a_{k-1,l} + Sq^4 b_{k-1,l}] \\ &= [1, z_{k-1,l}] \end{aligned}$$

gives the following equation

$$Sq^1 a_{k,l} + Sq^4 b_{k,l} = [1, z_{k-1,l}] + \sum_{m=3}^{l-1} e^{2^l - 2^{m+1} + 1} \otimes z_{k-1,m} \otimes z_{k-1,m}.$$

The left hand side of this equation agrees with the expression for $z_{k,l}$ given in (4.17). This completes the proof of (4.21).

We have now completed the proof of (4.3) for writing $\bar{\gamma}: Y_{G_k}(S^7) \to E_2^k S^7$ for the first map occurring in (4.5), we see, referring to (4.4) and (4.5), that

$$h(g)^* y_{2^l-1} = \bar{\gamma}^* g_1^* y_{2^l-1} = \bar{\gamma}^*(z_{k,l}) = \bar{\gamma}^*(Sq^1 a_{k,l} + Sq^4 b_{k,l}).$$

Now take the classes u and v of (4.3) to be

$$u = \bar{\gamma}^* a_{k,l}, \qquad v = \bar{\gamma}^* b_{k,l}.$$

§5. THE PROOF OF THEOREM C

We begin by constructing the surface X with its free $G = \Sigma_2 \int \Sigma_2$ action, and showing that $X_G(S^7)$ can be framed. We then use the calculations of §4, in the case $l = 4$, to show that $X_G(S^7)$ can be framed to have Kervaire invariant one and to identify this framing. Throughout this section $G = \Sigma_2 \int \Sigma_2 \subset \Sigma_4$.

Note that $BG = E_2(B\Sigma_2)$. Recall $H^*B\Sigma_2 = Z/2[e]$ where $e \in H^1 B\Sigma_2$ is the non-zero class. We use the notation of (4.6)–(4.10) to describe H^*BG.

Let # stand for connected sum and define the surface $\bar{X}$ by $\bar{X} = P \# T$ where P is the projective plane and T is the torus $S^1 \times S^1$. We define a map $\bar{X} \to BG$. First define $a\colon P \to BG$ to be the composition.

$$P \subset B\Sigma_2 \xrightarrow[i_1]{} B\Sigma_2 \times B\Sigma_2 \xrightarrow[j]{} E_2 B\Sigma_2 = BG. \tag{5.1}$$

where i_1 is the inclusion of the first factor and j is the inclusion defined in (4.8).

Let $\Delta\colon B\Sigma_2 \to B\Sigma_2 \times B\Sigma_2$ be the diagonal. Note that the map $1 \times \Delta\colon S^\infty \times B\Sigma_2 \to S^\infty \times B\Sigma_2 \times B\Sigma_2$ is Σ_2 equivariant where Σ_2 acts on $S^\infty \times B\Sigma_2$ by the product of the usual action on S^∞ and the trivial one on $B\Sigma_2$, and Σ_2 acts on $S^\infty \times B\Sigma_2 \times B\Sigma_2$ by $T(x, y, z) = (-x, z, y)$ where T is the non-trivial element of Σ_2. Dividing out by Σ_2 gives the map $1 \times_{\Sigma_2} \Delta\colon B\Sigma_2 \times B\Sigma_2 \to BG$.

Now define $b\colon T \to BG$ to be the following composition:

$$T \subset B\Sigma_2 \times B\Sigma_2 \xrightarrow[1 \times_{\Sigma_2} \Delta]{} BG. \tag{5.2}$$

Finally define $c\colon \bar{X} \to BG$ to be the composite

$$\bar{X} \to P \vee T \to BG \tag{5.3}$$

where the first map is the collapsing map.

Let X be the total space of the principal G covering induced by c. Then it may be checked that X has genus 5. Clearly X has a free G action and $X/G = \bar{X}$.

Before proceeding any further we need to establish some notation for $H^*\bar{X}$,

$$H^1\bar{X} \cong H^1 P + H^1 T \cong Z/2 + Z/2 + Z/2.$$

Let $u \in H^1\bar{X}$ be the generator coming from $H^1 P$ and x_1 and x_2 those coming from $H^1 T$. Then

$$H^2\bar{X} = Z/2$$

and

$$u^2 = x_1 x_2 \neq 0, \qquad u x_1 = u x_2 = 0.$$

5.4 THEOREM. *The manifold $X_G(S^7)$ can be framed.*

Proof. With the notation of Theorem A we need to show that $\tau\bar{X} + 7\xi$ is stably trivial. Stable bundles over a surface are classified by their Stiefel–Whitney classes. We will prove the theorem by showing $W(\tau\bar{X}) \cdot (W(\xi))^7 = 1$ where W stands for the total Stiefel–Whitney class. Note that $W(\tau\bar{X}) = 1 + u + u^2$.

Let ρ be the bundle over BG classified by Br where $r\colon G \to O(4)$ is the permutation representation of $G \subset \Sigma_4$. Then ξ is just $c^*\rho$. We compute $a^*\rho$ and $b^*\rho$ where a and b are defined in (5.1) and (5.2).

By definition G is the subgroup of Σ_4 generated by the permutations

$$(12), \qquad (34) \qquad \text{and} \qquad (13)(24).$$

Let $i_1\colon \Sigma_2 \times \Sigma_2 \to G$ be the inclusion of the subgroup generated by (12) and (34). Then

$j = Bi_1\colon B\Sigma_2 \times B\Sigma_2 \to BG$ where j is the map occuring in (5.1). Let $i_2\colon \Sigma_2 \times \Sigma_2 \to G$ be the inclusion of the subgroup generated by (12)(34) and (13)(24). Then $1 \times_{\Sigma_2} \Delta = Bi_2\colon B\Sigma_2 \times B\Sigma_2 \to BG$ where $1 \times_{\Sigma_2}\Delta$ is the map occurring in (5.2).

Let $s\colon \Sigma_2 \to O(2)$ be the permutation representation. Then $ri_1\colon \Sigma_2 \times \Sigma_2 \to O(4)$ is just the direct product $s \times s$. Thus $j^*\rho$ is classified by $Bs \times Bs$. However Bs classifies the bundle $H + \epsilon^1$ where H is the Hopf line bundle over $B\Sigma_2$. Thus $j^*\rho = (H + \epsilon^1) \times (H + \epsilon^1)$. Therefore

$$a^*\rho = H_2 + \epsilon^3$$

where H_2 is the Hopf line bundle over P^2.

The representation $ri_2\colon \Sigma_2 \times \Sigma_2 \to O(4)$ is equivalent to the external tensor product $s \otimes s$. Now $(1 \times_{\Sigma_2} \Delta)^*\rho$ is classified by $B(ri_2)$ and therefore $(1 \times_{\Sigma_2} \Delta)^*\rho = (H + \epsilon^1)$ where $\otimes$ stands for the external tensor product. Therefore

$$b^*\rho = (H_1 + \epsilon^1) \otimes (H_1 + \epsilon^1) = H_1 \otimes H_1 + H_1 \otimes \epsilon^1 + \epsilon^1 \otimes H_1 + \epsilon^1 \otimes \epsilon^1$$

where H_1 is the Hopf line bundle over S^1.

A straightforward calculation now gives

$$W(a^*\rho) = 1 + u, \qquad W(b^*\rho) = 1 + x_1x_2.$$

Therefore we deduce

$$W(\xi) = W(c^*\rho) = 1 + u + x_1x_2 = 1 + u + u^2.$$

Finally $W(\tau\bar{X})$. $W(\xi)^7 = (1 + u + u^2)^8 = 1$ and the result follows.

Note. The above argument actually shows that $\tau\bar{X} + k\xi$ is stably trivial if and only if $k \equiv 3 \bmod 4$. Therefore $X_G(N)$ can be framed if and only if $\dim N \equiv 3 \bmod 4$, in particular $X_G(S^3)$ can be framed but $X_G(S^1)$ cannot be framed.

We now summarize the information required from §4. The notation is that described at the beginning of §4. Note that the Cayley number framing H is just gF, and so, as in (4.2)

$$K(X_G(S^7), \alpha_G(H)) = q_B(h(g)^*y_{15}).$$

Referring to (4.5) we see that $h(g) = g_2\bar{\gamma}$ where $\bar{\gamma}\colon X_G(S^7) \to E_2E_2S^7$, and g_2 is defined in (4.4). Therefore referring to (4.15)

$$h(g)^*y_{15} = \bar{\gamma}^*z_{2,4}$$

where $z_{2,4} \in H^{14}E_2E_2S^7$.

Recall that $\sigma \in H^7S^7$ is the non-zero element, we need the following cohomology classes

$a_{1,4} = 1 \otimes \sigma \otimes \sigma$ in $H^{14}E_2S^7$ see (4.18)

$z_{1,3} = [1, \sigma]$ in $H^7E_2S^7$ see (4.16) and (4.17)

$a_{2,4} = [1, a_{1,4}] + 1 \otimes z_{1,3} \otimes z_{1,3}$ in $H^{14}E_2E_2S^7$ see (4.18)

$b_{2,4} = 0$ in $H^{11}E_2E_2S^7$ see (4.19).

According to (4.21), $z_{2,4} = Sq^1a_{2,4}$. (Note this is easily checked in this case). Therefore, according to (1.2),

$$q_B(\bar{\gamma}^*z_{2,4}) = \bar{\gamma}^*(a_{2,4} \cdot Sq^2a_{2,4}).$$

The proof of Theorem C is completed by proving the following result.

5.5 Theorem. *With the above notation*

$$\bar{\gamma}^*(a_{2,4} \cdot Sq^2a_{2,4}) \neq 0.$$

Proof. First we expand $a_{2,4} \cdot Sq^2a_{2,4}$ using (4.6) and (4.10). From (4.10)

$$Sq^2a_{2,4} = [1, Sq^2a_{1,4}] + e^2 \otimes z_{1,3} \otimes z_{1,3}.$$

Using (4.6) we deduce that $z_{1,3} \cdot z_{1,3} = 0$. A further application of (4.6) gives

$$a_{2,4} \cdot Sq^2 a_{2,4} = [a_{1,4}, Sq^2 a_{1,4}] + [1, a_{1,4} \cdot Sq^2 a_{1,4}] + [z_{1,3} \cdot a_{1,4}, z_{1,3} \cdot Sq^2 a_{1,4}].$$

Next using (4.10), we compute $Sq^2 a_{1,4}$

$$Sq^2 a_{1,4} = e^2 \otimes \sigma \otimes \sigma \qquad \text{in } H^{16} E_2 S^7,$$

and, therefore, since $\sigma^2 = 0$ in $H^{14} S^7$, we deduce from (4.6) that

$$a_{1,4} \cdot Sq^2 a_{1,4} = 0 \qquad \text{in } H^{30} E_2 S^7$$
$$z_{1,3} \cdot a_{1,4} = 0 \qquad \text{in } H^{21} E_2 S^7.$$

Therefore we have shown

$$a_{2,4} \cdot Sq^2 a_{2,4} = [a_{1,4}, Sq^2 a_{1,4}]. \tag{5.6}$$

For ease of notation write $a = a_{1,4} = 1 \otimes \sigma \otimes \sigma$ in $H^{14} E_2 S^7$. Also write $f \in H^1 E_2 S^7$ for the class $f = e \otimes 1 \otimes 1$. A further application of (4.6) gives

$$[a, Sq^2 a] = (1 \otimes a \otimes a) \cdot [1, f]^2 \qquad \text{in } H^{30} E_2 E_2 S^7. \tag{5.7}$$

We will have proved the theorem if we can show that

$$\bar{\gamma}^*(1 \otimes a \otimes a) \cdot (\bar{\gamma}^*[1, f])^2 \neq 0. \tag{5.8}$$

We first show that $(\bar{\gamma}^*[1, f])^2$ is non-zero. Consider the commutative diagram

$$\begin{array}{ccc} X_G(S^7) & \xrightarrow{\bar{\gamma}} & E_2 E_2 S^7 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ \bar{X} & \xrightarrow[c]{} & E_2 B\Sigma_2 = BG \end{array}$$

where π and p are the projections and c is defined in (5.3). Observe that $p = E_2(q)$ where $q: E_2 S^7 \to B\Sigma_2$ is the projection. It follows from (4.6) and (4.7) that

$$p^*[1, e]^2 = [1, f]^2.$$

From the definition of c we get $c^*[1, e] = u$. Therefore

$$\bar{\gamma}^*[1, f]^2 = \bar{\gamma}^* p^*[1, e]^2 = \pi^* c^*[1, e]^2 = \pi^* u^2.$$

However u^2 is non-zero in $H^2 \bar{X}$ and $\pi^*: H^2 \bar{X} \to H^2 X_G(S^7)$ is obviously an isomorphism. Thus

$$\bar{\gamma}^*[1, f]^2 \neq 0.$$

To show $\bar{\gamma}^*(1 \otimes a \otimes a)$ is non-zero consider the commutative diagram

$$\begin{array}{ccc} (S^7)^4 & = & (S^7)^4 \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle k_2} \\ X_G(S^7) & \xrightarrow[\bar{\gamma}]{} & E_2 E_2 S^7 \end{array}$$

where k_1 and k_2 are the inclusions. The map k_2 can be factorised as follows

$$(S^7 \times S^7) \times (S^7 \times S^7) \xrightarrow[j_1 \times j_1]{} E_2 S^7 \times E_2 S^7 \xrightarrow[j_2]{} E_2(E_2 S^7)$$

where j_1 is the inclusion of $S^7 \times S^7$ in $E_2 S^7$ and j_2 is the inclusion of $E_2 S^7 \times E_2 S^7$ in $E_2 E_2 S^7$. Using (4.6) and (4.7) we see that

$$k_1^* \bar{\gamma}^*(1 \otimes a \otimes a) = k_2^*(1 \otimes a \otimes a) = j_1^* a \otimes j_1^* a = \sigma \otimes \sigma \otimes \sigma \otimes \sigma,$$

which is the non-trivial element of $H^{28}(S^7)^4$. Therefore

$$\bar{\gamma}^*(1 \otimes a \otimes a) \neq 0.$$

To complete the proof of (5.5) note that $\pi^*: H^2\bar{X} \to H^2X_G(S^7)$ is an isomorphism and so $H^2X_G(S^7) = Z/2$. By the Poincaré duality theorem $H^{28}X_G(S^7) = Z/2$. As we've just shown $\bar{\gamma}^*[1, f]^2$ and $\bar{\gamma}^*(1 \otimes a \otimes a)$ are the non-zero elements in their respective cohomology groups and so, again by the Poincaré duality theorem, the product $\bar{\gamma}^*[1, f]^2 \cdot \bar{\gamma}^*(1 \otimes a \otimes a)$ must be non-zero. By (5.8) this completes the proof of (5.5) and therefore completes the proof of Theorem C.

Referring to note (iii) after Theorem C it is clear that the above proof with only the obvious alterations shows that the 14-dimensional manifold $X_G(S^3)$ can be framed to have Kervaire invariant one. As pointed out after the proof of (5.4), the 6-dimensional manifold $X_G(S^1)$ cannot be framed.

Referring to the discussion before Lemma 2.7 and that lemma, it can be shown that $\Omega_2(BG; k\rho) = Z/2$ if $k \equiv 3 \bmod 4$, where $G = \Sigma_2 \int \Sigma_2$. Therefore the framed bordism class of $X_G(S^7)$ with framing induced by the Cayley number framing of S^7 has order 2.

§6. THE PROOF OF THEOREM D

The notation used is that established at the beginning of §4. We begin by summarising the information required from §4.

From (4.2), the $2^{l+1}-2$ dimensional manifold $Y_{G_k}(S^7)$ equipped with a framing induced by a framing of S^7 has Kervaire invariant one if and only if $q_B(h(g)^*y_{2^l-1}) = 1$. now from (4.5), (4.15) and (4.21)

$$h(g)^*y_{2^l-1} = \bar{\gamma}^*g_k^*y_{2^l-1} = \bar{\gamma}^*z_{k,l} = \bar{\gamma}^*(Sq^1a_{k,l} + Sq^4b_{k,l}),$$

where $a_{k,l}$ is defined in (4.18), $b_{k,l}$ is defined in (4.19) and $\bar{\gamma}: Y_{G_k}(S^7) \to E_2{}^kS^7$ is the first map occurring in (4.5).

Using (1.2) we see that

$$q_B(h(g)^*y_{2^l-1}) = \bar{\gamma}^*A_{k,l}$$

where $A_{k,l} \in H^{2^{l+1}-2}E_2{}^kS^7$ is the class

$$A_{k,l} = a_{k,l} \cdot Sq^2a_{k,l} + b_{k,l} \cdot Sq^8b_{k,l} + Sq^1b_{k,l} \cdot Sq^7b_{k,l} + Sq^2b_{k,l} \cdot Sq^6b_{k,l} + Sq^3b_{k,l} \cdot Sq^5b_{k,l} + Sq^1a_{k,l} \cdot Sq^4b_{k,l}. \quad (6.1)$$

The main result of this section is the following theorem.

6.2 THEOREM. *If $2^{l+1} - 2 - 7 \cdot 2^k > 2$ then the class $A_{k,l}$ is decomposable over the Steenrod algebra. That is $A_{k,l} = \Sigma\alpha_ix_i$ where the α_i's are stable primary cohomology operations with strictly positive degree.*

Proof of Theorem D assuming (6.2). $Y_{G_k}(S^7)$ is a framed $2^{l+1}-2$ dimensional manifold and so all stable primary cohomology operations with strictly positive degree taking values in $H^{2^{l+1}-2}Y_{G_k}(S^7)$ are zero. Therefore $\bar{\gamma}^*A_{k,l} = \Sigma\alpha_i\bar{\gamma}^*x_i = 0$.

We now prove (6.2). We will use the notation implicit in (6.1) without further comment. The proof is a long but straightforward calculation using (4.6)–(4.10).

6.3 LEMMA.

$$Sq^1(b_{k,l} \cdot Sq^7b_{k,l}) = Sq^1b_{k,l} \cdot Sq^7b_{k,l}$$
$$Sq^1(Sq^2b_{k,l} \cdot Sq^5b_{k,l}) = Sq^3b_{k,l} \cdot Sq^5b_{k,l}.$$

Proof. Immediate from the Cartan formula and the Adem relations for the Steenrod squares.

6.4 LEMMA.

$$Sq^1(a_{k,l} \cdot Sq^4b_{k,l}) = Sq^1a_{k,l} \cdot Sq^4b_{k,l}.$$

Proof. By (4.21), $Sq^4b_{k,l} = z_{k,l} + Sq^1a_{k,l}$. Now (4.20) and the Adem relation $Sq^1Sq^1 = 0$ show that $Sq^1(Sq^4b_{k,l}) = 0$. The result follows from the Cartan formula.

6.5 LEMMA.

$$Sq^2 b_{k,l} = Sq^8 b_{k,l} = 0.$$

Proof. If $l = 4$ then $b_{k,4} = 0$, see (4.19), and so the result is trivial. Therefore assume $l \geq 5$. If $k = 0$ the result is trivial since referring to (4.19) we see that $b_{0,l} = 0$. We proceed by induction, using the definition of $b_{k,l}$, (4.19), and (4.10):

$$Sq^2 b_{k,l} = [1, Sq^2 b_{k-1,l}] + \binom{2^l - 12}{2} e^{2^l - 17} \otimes z_{k-1,3} \otimes z_{k-1,3} + \binom{2^l - 13}{0} e^{2^l - 19} \otimes Sq^1 z_{k-1,3} \otimes Sq^1 z_{k-1,3}.$$

By the inductive hypothesis $Sq^2 b_{k-1,l} = 0$ and by (4.20) $Sq^1 z_{k-1,3} = 0$. Finally $\binom{2^l - 12}{2} \equiv 0 \bmod 2$ and so

$$Sq^2 b_{k,l} = 0.$$

That $Sq^8 b_{k,l} = 0$ follows from a similar sort of induction using (4.10), (4.20) and the extra facts that $Sq^2 z_{k-1,3} = Sq^4 z_{k-1,3} = 0$.

Lemmas 6.3–6.5 show that

$$A_{k,l} = a_{k,l} \cdot Sq^2 a_{k,l} + B_{k,l} \tag{6.6}$$

where $B_{k,l}$ is decomposable over the Steenrod algebra. We first deal with the case $l = 4$, and since $2^{l+1} - 2 - 7 \cdot 2^k > 2$, we must therefore have $k = 1$. The argument given at the beginning of (5.5) shows that $a_{1,4} \cdot Sq^2 a_{1,4} = 0$. Therefore (6.6) shows that $A_{1,4}$ is decomposable over the Steenrod algebra and so (6.2) is verified in the case $l = 4$.

For the rest of this section assume $l \geq 5$.

6.7 LEMMA.

$$a_{k,l} \cdot Sq^2 a_{k,l} = [a_{k-1,l}, Sq^2 a_{k-1,l}] + [a_{k-1,l} \cdot Sq^2 a_{k-1,l}, 1] + [z_{k-1,l-1} \cdot Sq^2 a_{k-1,l}, z_{k-1,l-1}] + C_{k,l}$$

where $C_{k,l}$ is decomposable over the Steenrod algebra.

Proof. From the definition of $a_{k,l}$ and (4.10) one checks that

$$Sq^2 a_{k,l} = [1, Sq^2 a_{k-1,l}] + \sum_{m=4}^{l-1} e^{2^l - 2^{m+1} + 2} \otimes z_{k-1,m} \otimes z_{k-1,m}$$

(recall $l \geq 5$). The rest of the calculation is a straightforward exercise in the use of (4.6). It turns out that

$$C_{k,l} = \left(\sum_{m=4}^{l-1} e^{2^l - 2^{m+1} + 1} \otimes z_{k-1,m} \otimes z_{k-1,m} \right)^2$$

which is decomposable over the Steenrod algebra.

6.8 LEMMA.

$$Sq^2 [z_{k-1,l-1} \cdot a_{k-1,l}, z_{k-1,l-1}] = [z_{k-1,l-1} \cdot Sq^2 a_{k-1,l}, z_{k-1,l-1}].$$

Proof. This is left as an exercise using (4.10) and (4.20).

Lemmas 6.3–6.8 show that in $H^{2^{l+1}-2} E_2^k S^7$, for $l \geq 5$,

$$A_{k,l} = a_{k,l} \cdot Sq^2 a_{k,l} = [a_{k-1,l}, Sq^2 a_{k-1,l}] + [a_{k-1,l} \cdot Sq^2 a_{k-1,l}, 1] \tag{6.9}$$

modulo decomposables over the Steenrod algebra.

We now prove the following result.

6.10 PROPOSITION. *If $l \geq 5$ then $[a_{k-1,l}, Sq^2 a_{k-1,l}]$ is decomposable over the Steenrod algebra.*

Proof of (6.2) *assuming* (6.10). The case $l = 4$ has been dealt with after (6.6) so assume $l \geq 5$. Then (6.9) shows that $A_{k,l}$ is decomposable over the Steenrod algebra if and only if $a_{k,l} \cdot Sq^2 a_{k,l}$ is decomposable over the Steenrod algebra. However (6.9) and (6.10) together show that $a_{k,l} \cdot Sq^2 a_{k,l}$ is decomposable over the Steenrod algebra if $a_{k-1,l} \cdot Sq^2 a_{k-1,l}$ is decomposable over the Steenrod algebra. Now an obvious induction starting from the fact that $a_{0,l} = 0$ for $l \geq 4$ (see 4.18) completes the proof.

To prove 6.10 we need to introduce some more cohomology classes. For $l \geq 4$ and $k \geq 1$ define $u_{k,l} \in H^{2^l-2}E_2^kS^7$ inductively, on k, by the following formulae:

$$u_{1,l} = e^{2^l-15} \otimes \sigma \otimes \sigma \qquad \text{in } H^{2^l-2}E_2S^7$$
$$u_{k,l} = [1, u_{k-1,l}] + \sum_{m=3}^{l-1} e^{2^l-2^{m+1}} \otimes z_{k-1,m} \otimes z_{k-1,m} \qquad \text{in } H^{2^l-2}E_2^kS^7. \tag{6.11}$$

6.12 LEMMA. $Sq^1 u_{k,l} = z_{k,l}$.

Proof. This is the usual sort of induction on k using (4.10) and (4.20) to compute $Sq^1 u_{k,l}$ and the inductive hypothesis and (4.17) to identify it with $z_{k,l}$.

Next for $l \geq 5$ and $k \geq 1$ define $v_{k,l} \in H^{2^l-3}E_2^kS^7$ inductively, on k, by the following formulae:

$$v_{1,l} = 0 \qquad \text{in } H^{2^l-3}E_2S^7.$$
$$v_{k,l} = [1, v_{k-1,l}] + \sum_{m=4}^{l-1} e^{2^l-2^{m+1}-1} \otimes u_{k-1,m} \otimes u_{k-1,m} \qquad \text{in } H^{2^l-3}E_2^kS^7. \tag{6.13}$$

Here $u_{k,l}$ is defined in (6.11).

6.14 LEMMA. *For $k \geq 1$ and $l \geq 5$, $Sq^2 a_{k,l} = Sq^2 Sq^1 v_{k,l}$.*

Proof. This is the usual sort of induction on k. Lemma 6.12 is crucial.

Proof of (6.10). If $k = 1$ then $a_{k-1,l} = 0$ and the result is trivial. If $k \geq 2$ then the following formula holds:

$$[a_{k-1,l}, Sq^2 a_{k-1,l}] = Sq^3[v_{k-1,l}, Sq^1 v_{k-1,l}] + Sq^2[Sq^1 v_{k-1,l}, a_{k-1,l}] + Sq^3Sq^1(1 \otimes v_{k-1,l} \otimes v_{k-1,l}) + Sq^2(1 \otimes Sq^1_{v_{k-1,l}} \otimes Sq^1_{v_{k-1,l}}).$$

The verification of this formula is a straightforward but tedious exercise using (4.10).

This completes the proof of (6.2) and therefore proves Theorem D.

In view of Lemma 6.12 and Theorem 1.2

$$q_B(h(g)^* y_{2^l-1}) = q_B(\bar{\gamma}^* z_{k,l}) = \bar{\gamma}^*(u_{k,l} \cdot Sq^2 u_{k,l}).$$

Therefore one might hope to be able to show that $\bar{\gamma}^*(u_{k,l} \cdot Sq^2 u_{k,l}) = 0$ and possibly simplify the proof of Theorem D. I was unable to do this, indeed the reason for introducing the equation $z_{k,l} = Sq^1 a_{k,l} + Sq^4 b_{k,l}$ was to get around the difficulty arising from this line of proof.

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