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本科生毕业设计（论文）

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Homotopical Methods in The Classification of Topological Materials and Applications

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（Department of Physics　Thesis Advisor：Yifei Zhu）

[ABSTRACT]: This thesis investigates topological phases of matter, quantum anomalies, and their interplay with modern theoretical physics from a mathematical point of view. We begin by reviewing the classification of free symmetry-protected topological (SPT) phases via the Freed-Moore framework, linking symmetry groups to fermionic band topology. Building on this, the Freed-Hopkins classification is employed to unify SPT phases and t' Hooft anomalies through invertible field theories (IFTs). Anomalies are analyzed from dual perspectives: index theory (e.g., anomaly polynomials, local anomaly) and bordism invariants (e.g., η-invariants, global anomalies), which are unified under the above formalism of IFTs. These tools are applied to anomaly cancellation in string theory and we compare Freed-Hopkins' bordism methods on explicitly calculating anomalies with Tachikawa-Yamashita's approach making use of Stolz-Teichner conjecture. Synthesizing bordism, K-theory, index theory, and elliptic cohomology, this survey bridges mathematical frameworks and physical phenomena in quantum gravity, condensed matter, and high-energy theory.

[Key words]: SPT Phases, Quantum Anomaly, Functorial Field Theory, Homotopy Theory

[摘要]：本论文从数学角度研究物质拓扑相、量子反常及其与现代理论物理的相互作用。首先，我们通过弗里德-摩尔（Freed-Moore）框架回顾自由对称性保护拓扑相（SPT）的分类，将对称群与费米子能带拓扑联系起来。在此基础上，采用弗里德-霍普金斯（Freed-Hopkins）分类，通过可逆场论（IFTs）统一 SPT 相与't Hooft 反常。反常从双重视角被分析：指标理论（如反常多项式、局部反常）与配边不变量（如 η-不变量、全局反常），而这些在上述 IFT 形式体系中得以统一。这些工具被应用于超弦理论中的反常抵消问题，并比较了弗里德-霍普金斯的配边方法与塔奇科瓦-山下（Tachikawa-Yamashita）基于斯托尔兹-泰希纳（Stolz-Teichner）猜想的反常计算方法。本综述综合配边论、K-理论、椭圆上同调与指标理论，架起了数学框架与量子引力、凝聚态及高能物理理论中物理现象之间的桥梁。

[关键词]：对称保护拓扑序；量子反常；函子场论；同伦论

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1. Introduction

In this section, we briefly introduce some main ingredients in the thesis.

1.1 What is a symmetry protected topological phase?

Symmetry-protected topological (SPT) phases represent a fascinating class of quantum many-body systems that exhibit robust boundary phenomena—such as gapless edge modes or fractionalized excitations—while remaining gapped and trivial in the bulk. These phases are "protected" by global symmetries: if the symmetry is explicitly broken, the system can be adiabatically connected to a trivial phase without closing the bulk energy gap. First brought to prominence by the discovery of topological insulators and superconductors in condensed matter physics, SPT phases have since become a cornerstone of modern quantum many-body theory, bridging abstract algebraic topology with experimentally observable phenomena.

At their core, SPT phases are characterized by topological invariants —quantities that remain invariant under continuous deformations of the system' s Hamiltonian, provided the symmetry constraints are preserved. For non-interacting fermions, these invariants are elegantly captured by K-theory, which classify band structures based on their symmetry class (e.g., time-reversal, particle-hole, or chiral symmetries) and spatial dimension. For example, the classification of free-fermion SPT phases in the tenfold way[1] reveals a periodic structure across spatial dimensions and symmetry classes, mirroring the mathematics of Clifford algebras. However, interactions complicate this picture, necessitating more sophisticated tools such as group cohomology[2], cobordism theory[3-4] , or more generally invertible field theories (IFTs)[5] to classify interacting SPT phases. The general classification regime is based on the idea that the low-energy effective field theories of SPT phases are invertible topological field theories, which possess rigorous mathematical formulation in the language of functorial field theories. The main result of this thesis is due to D. Freed and J. Hopkins[5], by explicitly relating the classification of invertible field theories to homotopy theory, in particular the bordism theories and its anderson duals.

Theorem 1.1.1. *Freed-Hopkins (2016)*

$$\left\{\begin{array}{c} n+1\text{Dbosonic symmetry protected} \\ \text{topological phases with symmetry } G \end{array}\right\} = (I_{\mathbb{Z}}\Omega^{\mathrm{SO}})^{n+2}(BG)_{\mathrm{tor}},$$
$$\left\{\begin{array}{c} n+1\text{Dfermionic symmetry protected} \\ \text{topological phases with symmetry } G \end{array}\right\} = (I_{\mathbb{Z}}\Omega^{\mathrm{Spin}})^{n+2}(BG)_{\mathrm{tor}},$$

Moreover, this description of SPT phases should be complete since topological feature of SPT phases is invariant under the RG flow, whence the corresponding invertible topological field theories entirely captures phases of symmetry-protected topological materials.

1.2 What is an anomaly?

Another defining feature of SPT phases is their bulk-boundary correspondence : the topological invariant of the bulk guarantees the existence of anomalous gapless modes at the physical boundary. For instance, the edge of a two-dimensional quantum spin Hall insulator hosts helical one-dimensional modes protected by time-reversal symmetry, while the surface of a three-dimensional topological insulator supports massless Dirac fermions. These boundary modes are "anomalous" in the sense that they cannot exist independently of the bulk—they require the higher-dimensional SPT phase to cancel gauge or gravitational inconsistencies. From the quantum field theory point of view, these boundary theories alone are said to possess quantum anomalies, which are captured by invertible field theories from modern perspectives. In the language of functorial field theories, we have mathematical underpinnings of the above boundary-bulk correspondence, dated back to D. Freed[6].

Theorem 1.2.1. *Freed, Freed-Hopkins*

Quantum anomalies of a nD functorial field theory with structure ξ are given by elements in $(I_{\mathbb{Z}}\Omega^{\xi})^{n+2}$, which lies in

$$0 \longrightarrow \mathrm{Hom}(\Omega^{\xi}_{n+1}, \mathbb{C}^{\times}) \longrightarrow (I_{\mathbb{Z}}\Omega^{\xi})^{n+2} \longrightarrow \mathrm{Hom}(\Omega^{\xi}_{n+2}, \mathbb{Z}) \longrightarrow 0.$$

The last term describes the local anomaly of the quantum field theory while the first term captures the global anomaly.

The deep connection between SPT phases and quantum anomalies lies at the heart of their classification and physical relevance.

1.3 Outline

We begin by revisiting the classification of free fermionic SPT phases through the lens of the Freed-Moore formalism, which systematically associates symmetry groups with topological invariants in band theory. This framework provides a rigorous foundation for understanding how global symmetries constrain the possible phases of quantum matter. Building on this, we turn to the Freed-Hopkins classification, a groundbreaking approach that unifies SPT phases and ’t Hooft anomalies under the umbrella of invertible field theories (IFTs). By treating anomalies as obstructions to gauging symmetries, this perspective reveals a hidden unity between seemingly distinct phenomena: the topological order of materials and the consistency of quantum field theories.

A key focus of this work is the dual analysis of anomalies through two complementary mathematical paradigms: index theory and bordism invariants. Index theorems, exemplified by anomaly polynomials and local anomaly formulas, connect anomalies to the geometry of gauge and gravitational backgrounds. Meanwhile, bordism theory, through tools like η-invariants and global anomaly measures, classifies anomalies via topological obstructions in higher-dimensional manifolds. By synthesizing these viewpoints within the Freed-Hopkins framework, we demonstrate how IFTs serve as a Rosetta Stone for translating between algebraic topology, differential geometry, and quantum field theory.

The physical implications of this formalism are explored in two major contexts. First, we apply these tools to anomaly cancellation mechanisms in string theory, where the consistency of spacetime geometries hinges on the precise matching of chiral fermion content and background fields. Second, we compare the bordism-based calculations of Freed-Hopkins with the Stolz-Teichner-inspired approach[7] of Tachikawa and Yamashita[8], illuminating distinct pathways to characterize anomalies. This comparison not only clarifies the strengths and limitations of each method but also hints at deeper connections between extended topological field theories and algebraic topology.

Ultimately, this survey weaves together three mathematical pillars—bordism theory, K-theory, index theory, and elliptic cohomology—into a cohesive narrative that transcends traditional disciplinary boundaries, sheding light on open questions in quantum gravity, condensed matter, and beyond at the intersection of mathematics and theoretical physics.

The appendix serves as a concise introduction to necessary homotopical tools for the whole story in a self-contained way. Moreover, the organization and logical order of the appendix is very different from the traditional ones. All theorems and proofs are not original and most of the results in the main body are consequences of the main reference[5] by Freed and Hopkins.

2. Classification of SPT phases

The mathematical formalism of SPT phases has evolved significantly in recent years. The Freed-Moore framework systematically links symmetry groups to topological invariants by incorporating graded algebras and twisted equivariant K-theory, generalizing earlier results on the periodic table of topological materials to include spatial symmetries and crystalline materials. Meanwhile, the Freed-Hopkins classification captures SPT phases via its low-energy effective field theory, which is what we observe in the laboratory. We present these two framework in this chapter and make some comparisons which also have great implications to the mathematical theory of algebraic topology.

2.1 K-theory and free SPT phases

In this section, we present Kitaev's seminal work[1] on the classification of free fermion systems with prescribed symmetry in the ten-fold way, the so-called "Periodic Table of Topologial Phases".

The idea of the ten-fold way goes back to Altland and Zirnbauer, who discovered that substances can be divided into 10 kinds according to the symmetry class they live in. There are three symmetry types involved in their classification: time reversal symmetry $\mathcal{T}$ (anti-unitary), particle-hole symmetry $\mathcal{P}$ (anti-unitary) and chiral symmetry $\mathcal{C}$ (unitary). Furthermore, the fact that these physical operators $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ that square to 1, -1 or do not exist is closely connected to the classification of associative real super division algebras. There are exactly 10 associative real super division algebras up to Morita equivalence:

- Three of them are purely even, with no odd part: the usual division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.

- Even of them are not purely even. Of these, Six of them are Morita equivalent to the real Clifford algebras Cl_1, Cl_2, Cl_3, Cl_5, Cl_6 and Cl_7. These are the super algebras generated by 1, 2, 3, 5, 6, or 7 odd square roots of -1. The left is the complex Clifford algebra $\mathbb{C}\mathrm{l}_1$.

It turns out that the representations of these Clifford algebras 1-to-1 correspond to the Hilbert spaces within one of the prescribed symmetry class above, once we figure out the action of $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ on the Hilbert spaces and the presentation of each Clifford algebra. The combination of the above three fundamental symmetry operators $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ results in exactly ten possibilities exactly corresponds to the ten Morita-inequivalent real super division algebras. The Hamiltonian $\mathcal{H}$ of the system is commutative under the change with these operators in the prescribed symmetry class. Mathematically, it corresponds to the extension of a new generator $\mathcal{H}$ to the Clifford algebra associated to the symmetry class. i.e. the extension of Cl_k to Cl_{k+1} We have the following classical result:

Theorem 2.1.1. *(Atiyah-Bott-Shapiro)*

$$KO^{-k}(\bullet) \cong \mathrm{Mod}_{\mathrm{Cl}_k} \,/\, \mathrm{Mod}_{\mathrm{Cl}_{k+1}}$$

Therefore, the phases according to the symmetry are given by

$$KO^{n+s-2}(\bullet), \tag{1}$$

The calculation of K-theory and KO theory is mathematically known for a quite long time, given by

	0	1
$\widetilde{K}^*(\bullet)$	$\mathbb{Z}$	0

	0	1	2	3	4	5	6	7
$\widetilde{KO}^*(\bullet)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

where n is the spatial dimension and s indicates the symmetry class in terms of how many fundamental operators live in that class.

Therefore, we get Kitaev's table of topological materials

class	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$d=0$	1	2	3	4	5	6	7
A				$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
AIII	1				$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
AI			1	$\mathbb{Z}$				$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$		$\mathbb{Z}_2$
D		1		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$	
DIII	1	1	-1		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$
AII			-1	$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
CII	1	-1	-1		$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
C		-1				$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
CI	1	-1	1				$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Example 2.1.2. Some examples of topological materials lying in the periodic table are

- 1d: Majorana chain of class D $(s=0)$ represents the nontrivial phase in $KO^{-1}(\bullet)$
- 2d: Chern insulator (A), p-wave superconductor (D), topological insulator (AII).
- 3d: the strong phase of the TRS topological insulator of class AII ($s=-2$) generates $KO^{-1}(\bullet)$

One might be wondering what the AZ labels really mean, i.e. A, AIII, AI, ...These actually coincide with the labels of 10 classical infinite families of compact symmetric spaces discovered by Éllie Cartan and there are also 17 exceptional ones. Starting from a Clifford algebra, one can explicitly construct a compact symmetric space and vice versa which is compatible with the labelling in the table. The construction is as follows: We consider the unitary elements in a Clifford algebra and this gives rise to a Lie group! By the way, these groups are not the Spin groups that Clifford algebras are famously used to construct. Since each Clifford algebra sits inside the next one as an algebra, we can take quotient to get a compact symmetric space. It turns out that the π_0 of these symmetric spaces are given exactly by the KO-group above (1). The physically underpinning of these construction utilize our original notion of deformation of Hamiltonians of SPT phases.

2.2 Invertible topological field theories and SPT phases

The low energy effective field theory of SPT phases are captured by the invertible topological field theories, mathematically modelld by the functorial formalism of field theories as we have seen in the introductory part of this thesis. The invertibility, which is the mathematical incarnation of the very property of SPT phases requiring a unique gapped ground state, is a very strong constraints on the mathematical structure so that we can rigorously derive its classification result, as we will present in this section.

Since the partition function of SPT phases are in the form of

$$\mathcal{Z}(W) = \int \prod D\phi e^{-S_W(\phi)} = |\mathcal{Z}(W)| e^{iS_{\text{top}}(W)} = |\mathcal{Z}(W)| \mathcal{Z}_{\text{top}}(W),$$

where the partition function of the topological field theory valued in $\mathbb{C}^\times$, in the formalism of functorial field theory, this motivates the following definition.

Definition 2.2.1. A $n+1$ dimensional functorial field theory $\alpha : \text{Bord}_{(n,n+1)}(\mathcal{F}) \longrightarrow \text{Vect}_{\mathbb{C}}$ is invertible if, for each closed n-manifold N with fields $\mathcal{F}$, the vector space $\alpha(N)$ is one-dimensional, and for every $(n+1)$-dimensional bordism $M : N_0 \to N_1$, the linear operator $\alpha(M) : \alpha(N_0) \to \alpha(N_1)$ is invertible. Equivalently, α lies in a Picard groupoid:

$$\alpha : \text{Bord}_{(n-1,n)}(\mathcal{F}) \longrightarrow \text{Vect}^{\times}_{\mathbb{C}} = \text{Line}_{\mathbb{C}}\,.$$

Describing the topological phases via its low-energy effective field theory is conjecturally complete since the term topological reminiscent of the property that is invariant under the RG flow, and in the case of $n+1$D G-SPT phases, the IR picture should be probed as invertibel topological field theories

$$\alpha : \text{Bord}^{\xi}_{n+1} \longrightarrow \text{Line}_{\mathbb{C}},$$

together with some equivariance condition under the symmetry group G. Follow from strategy in the introduction, in order to classify SPT phases, we would like to understand

$$\mathrm{ITFT}^{n+1}_{\xi,G} := \pi_0\left(\left\{\begin{array}{c}\alpha : \mathrm{Bord}^{\xi}_{n+1} \longrightarrow \mathrm{Line}_{\mathbb{C}},\\ \text{with prescribed symmetry } G\end{array}\right\}\right).$$

The homotopy theory provides powerful tools to attack this problem which is based on the following foundational theorem saying that the bordism category is represented by the spectrum $MT\xi_{n+1}$ in the world of homotopy theory

Theorem 2.2.2 (Madsen-Tillmann[9]).
There are homotopy equivalences $|\overline{\mathrm{Bord}^{\xi}_{n+1}}| \simeq \tau_{\leqslant 1}\Sigma^n MT\xi_{n+1}$, $|\,\mathrm{sLine}^{\times}_{\mathbb{C}}\,| \simeq \tau_{\leqslant 1}\Sigma I\mathbb{Z}$ *and* $|\,\mathrm{Line}^{\times}_{\mathbb{C}}\,| \simeq \Sigma H\mathbb{Z}$*, where* $I\mathbb{Z}$ *is called the Pontrjagin dual to the sphere spectrm satisfying the universal property* $[E, \Sigma I\mathbb{Z}] \cong \mathrm{Hom}(\pi_{n+1}E, \mathbb{C}^{\times})$.

Therefore, $\mathrm{ITFT}^{n+1}_{\xi,G}$ should be an abelian group like $\mathrm{Hom}(\pi_{n+1}E, \mathbb{C}^{\times})$ as we have promised in the introduction. The homotopy groups of the spectrum $MT\xi_{n+1}$ is precisely the bordism group Ω^{ξ}_{n+1}. Therefore, with some mild efforts, we can deduce that

Theorem 2.2.3 (Freed-Hopkins[5]).
$n+1$ *D invertible topological field theories with* ξ*-structures and symmetry G, or equivalently* $n+1D$ *G-SPT phases are classified by* $(I_{\mathbb{Z}}\Omega^{\xi})^{n+2}(BG)_{\mathrm{tor}}$.

Proof. The proof relies on the stable homotopy hypothesis: Let C, D be Picard groupoids, constructing classifying spectra establishes an isomorphism of abelian groups $\mathrm{Hom}_{\otimes}(C, D) \cong [|NC|, |ND|]$. $F : C \to D$ is invertible means that we have $C \to D^{\times}$. Therefore, the theorem boils down to understand $|\overline{\mathrm{Bord}^{\xi}_{n+1}}|$ and $|\,\mathrm{sLine}^{\times}_{\mathbb{C}}\,|$ which are $MT\xi_n$ and $\tau_{\leqslant 1}\Sigma I\mathbb{Z}$ respectively according to Theorem 2.2.2, where $\overline{\mathrm{Bord}^{\xi}_{n+1}}$ is the Picard groupiod of $\mathrm{Bord}^{\xi}_{n+1}$. While with additional symmetry G, the homotopy classes of G-SPT phases form the group $(I_{\mathbb{Z}}\Omega^{\xi})^{n+2}(BG)_{\mathrm{tor}}$, where the group $(I_{\mathbb{Z}}\Omega^{\xi})^{n+2}(BG)$ lies in the short exact sequence

$$\ldots \longrightarrow \mathrm{Hom}(\Omega^{\xi}_{n+1}(BG), \mathbb{C}^{\times}) \longrightarrow (I_{\mathbb{Z}}\Omega^{\xi})^{n+2}(BG) \longrightarrow \mathrm{Hom}(\Omega^{\xi}_{n+2}(BG), \mathbb{Z}) \longrightarrow \cdots.$$

This splits and often the torsion part is given by $\mathrm{Hom}(\Omega^{\xi}_{n+1}(BG), \mathbb{C}^{\times})$. $\square$

In general, for invertible non-topological field theories, we have the theorem

Theorem 2.2.4 (Freed-Hopkins[5], Grady[10])**.**

The group of $n+1D$ invertible field theories with ξ-structures lies in the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(\Omega^{\xi}_{n+1}, \mathbb{C}^{\times}) \longrightarrow \mathrm{IFT}^{n+1}_{\xi} \longrightarrow \mathrm{Hom}(\Omega^{\xi}_{n+2}, \mathbb{Z}) \longrightarrow 0,$$

which is mathematically the group $(I_{\mathbb{Z}}\Omega^{\xi})^{n+2}(\bullet)$. The underlining spetrum is the anderson dual of the Madsen-Tillmann spectrum.

Let's look at some explicit examples.

Example 2.2.5 (Bosonic SPT phases)**.**

The tangential structure corresponding to bosonic SPT phases without any additional symmetry, the so-called invertible bosonic topological order, is the oriented structure. Therefore, it suffices to calculate oriented bordism groups at relevant dimensions

$$\Omega^{\mathrm{SO}}_4(\bullet) = \mathbb{Z}, \quad \Omega^{\mathrm{SO}}_5(\bullet) = \mathbb{Z}/2$$

- The $\mathbb{Z}$ group generated by $\mathbb{CP}^4$ in the Anderson dual of the bordism group corresponds to the free part, interpreted as invertible phases in one dimension lower. This $\mathbb{Z}$ indicates the existence of $2+1$D invertible topological phases that require no symmetry protection. The generator of this $\mathbb{Z}$ is the so-called E_8 *phase*, whose boundary can host a $1+1$D chiral CFT – the $(E_8)_1$ theory.

- The $\mathbb{Z}/2$ generator is represented by the Wu manifold SU(3)/SO(3), with the topological invariant given by the integral . The Anderson dual of the finite part of the bordism group provides the classification of invertible topological phases in this dimension. Thus, there exists a (4+1)D invertible TQFT whose partition function on the Wu manifold equals -1.

Previously, the bosonic SPT phases are classified by the group cohomology $H^{n+2}(BG, \mathbb{Z})$[2] in physical relevant dimensions. However, in higher dimensions, this is wrong in the language of topological field theories and mathematically we have a truncation map $H^{n+2}(BG, \mathbb{Z}) \to (I_{\mathbb{Z}}\Omega^{\mathrm{SO}})^{n+2}(BG)$.

Example 2.2.6 (Fermionic SPT phases)**.**

The tangential structure correponding to fermionic SPT phases without any additional symmetry, the so-called invertible fermionic topological order, is the Spin structure. The spin bordism groups at some dimensions are

$$\Omega_1^{\mathrm{Spin}}(\bullet) = \mathbb{Z}/2, \quad \Omega_2^{\mathrm{Spin}}(\bullet) = \mathbb{Z}/2, \quad \Omega_4^{\mathrm{Spin}}(\bullet) = \mathbb{Z}$$

- The first $\mathbb{Z}/2$ corresponds to the parity of fermion number in $0+1$D.
- The second $\mathbb{Z}/2$ represents the $1+1$D Majorana chain.
- The $\mathbb{Z}$ group (similar to the bosonic case) now has K3 surfaces as generators, corresponding to $2+1$D class D topological superconductors.

In general, the tangential structures corresponding to fermionic SPT phases with ten-fold-way symmetry type are twisted spin structures. For time-reversal symmetry, the twisted spin structure is the Pin$^-$ structrue. The torsion part of the corresponding bordism group is

$$\Omega_2^{\mathrm{Pin}^-}(\bullet)_{\mathrm{tor}} \cong \mathbb{Z}/8.$$

This nontrivial phase is given by the time-reversal Majorana Chain whose low energy field theory has partition function valued 1 on the manifold $\mathbb{RP}^2$.

2.3 Conclusion and outlook

Let's summarize previous two sections and analyze some mathematical implications out of the following two main results. The ten fold way classification of free fermionic SPTs with symmetry type s at spatial dimension n is given by

$$KO^{n+s-2}(\bullet)$$

Meanwhile, the low energy field theories of $n+1$D interacting fermionic SPT phases are described by reflection-positive invertible TQFTs for manifolds with twisted spin structures τ, which can be classified by the homotopy group

$$[MT\tau, \Sigma^{n+2} I_{\mathbb{Z}}]_{\text{tor}} \cong \text{Hom}(\Omega^{\tau}_{n+1}, \mathbb{C}^{\times}).$$

Therefore, it is natural to ask whether this two approaches match when we restricting our field theory classification to only weakly interacting cases. The answer is no in general. Time-reversal Majorana chain is an great counter-example of this hypothesis. The class BDI in the Kitaev's table is given by $\mathbb{Z}$ and generated by the time-reversal Majorana chain with Hamiltonian $\mathcal{H} = \frac{i}{2}\Sigma_l c_{2l}c_{2l+1}$. It can be shown that eight copies of this Hamiltonian can be adiabatically connected to the trivial topological phase, matched the bordism classification

$$\Omega_2^{\text{Pin}^-}(\bullet)_{\text{tor}} \cong \mathbb{Z}/8.$$

The reason is that when classifying free fermionic phases, we only allow Hamiltonians with quadratic terms in the path of the deformation, while in the interacting case, high order terms are permitted. Therefore, these two classification resulted from two notions of deformation, i.e. π_0 of the configuration space. The goal of this section is to mathematically capture this phenenomena by the so-called free-to-interacting map between two spectra. The kernel and cokernel are both physically interesting. The kernels are free phases that become trivialized after tunning on interaction-allowed deformations.

According to the periodic table[1], free topological phases without symmetry is given by $\text{KO}^{n-2}(\bullet)$ and we have a map of spectra

$$KO^{n-2} \longrightarrow \text{IFT}^{n+1}_{\text{Spin}} \cong (I_{\mathbb{Z}}\Omega^{\mathcal{S}})^{n+2} \tag{2}$$

The above map (2) is the Anderson dual to the Atiyah-Bott-Shapiro orientation

$$\Omega_n^{\text{Spin}} \longrightarrow KO_n \quad M \mapsto \mathcal{D}_M \tag{3}$$

sending to the Cl_n-linear Dirac operator. Here we use the anderson self-dual of the KO to get (2). Homotopically speaking, the K-Thom class is given by the following. First, let τ_V be the relative K class $[\Lambda^{\text{even}}V, \Lambda^{\text{odd}}V, (v, w) \mapsto (v, v \wedge w)]$, where V is a vector bundle on M. The relative K-theory is isomorphic to the $K^0(\mathrm{Th}(V))$ and let $\beta \in K^{-2}$ be the Bott element. Then the class

$$\beta^{-n}\tau_V \in K^{2n}(\mathrm{Th}(V))$$

is a K-Thom class of the vector bundle V, therefore according to the appendix, we have the orientation map (3).

Freed-Hopkins[5] generalized ABS map $\Omega_n^{\tau(s)} \to KO_{n+s}$ homotopically for ten twisted spin structures $\tau(s)$ and following their work, Camerana et al.[11] generalize these to symmetries beyond the ten fold way in the language of fermionic groups. The most general form of free-to-interacting map gives mathematical underpinnings of the Bott spiral when studing two kinds of deformations classes of Hamiltonians by Queiroz, Khalaf and Stern and makes possible prediction on the presence of weak topological phases that have potentially applications to topological quantum computing. Moreover, these orientations maps might play pivotal roles in the homotopy theory as well.

3. Anomalies, index theory, and invertible field theory

3.1 Introduction

Consider a quantum field theory with action $S[\psi]$

$$\mathcal{Z} = \int \mathcal{D}\psi e^{-S[\psi]}$$

Suppose Lie group G is a symmetry of the action $S[g \cdot \psi] = S[\psi]$, $\forall g \in G$, the Noether's Theorem says that there exists vector fields $j^{a\mu}$ measuring the variation of the action

$$S[g \cdot \psi] - S[\psi] = -\int_X \mathrm{d}^4x \partial_\mu j^{a\mu}(x)\theta^a(x),$$

where $\theta^a(x)$ is an infinitesimal transformation

$$\partial_\mu j^{a\mu} = 0.$$

A quantum symmetry of this theory is construed as the invariance of the partition function, i.e. under symmetry transformation of G. In the quantum theory, equation $\partial_\mu j^{a\mu} = 0$ is replaced by the statement that the correlation function of $\partial_\mu j^{a\mu}$ with any number of operators $\mathcal{O}_i(x_i)$ at points $x_i \neq x$ must vanish:

$$\left\langle \partial_\mu j^{a\mu}(x) \prod_i \mathcal{O}_i(x_i) \right\rangle = 0.$$

This is known as Ward's identity, which is the quantum version of Noether's theorem. Very roughly speaking, anomalies are the failures of lifting classical symmetries to corresponding quantum symmetries.

3.1.1 ABJ anomaly

Let M^d be a closed even Riemannian spin manifold. Consider massless Dirac fermions under $U(1)$ background fields A on M. The partition function is given by

$$\mathcal{Z}[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathrm{e}^{-\int_M \bar{\psi}\mathfrak{D}_M \psi}$$

where $\mathfrak{D}_M$ is the Dirac operator on M as discussed in the appendix. The action term $S = \mathrm{e}^{-\int_M \bar{\psi}\mathfrak{D}_M \psi}$ is invariant under axial $U(1)$ transformation $\psi(x) \to \mathrm{e}^{i\gamma_5 \theta(x)}\psi(x)$, $\bar{\psi}(x) \to \bar{\psi}(x)\mathrm{e}^{i\gamma_5\theta(x)}$, denote the Noether current by j_A^μ.

Heuristically, since $\mathfrak{D}_X$ is a self-adjoint operator, we have the spectral decompostion $\mathfrak{D}_X \psi_n = \lambda_n \psi_n$, where $\{\psi_n\}$ is an orthonormal basis for the Dirac operator. If we can expand fields ψ in terms of this orthonormal basis as

$$\psi = \sum a_n \psi_n, \quad \bar{\psi} = \sum \bar{b}_n \bar{\psi}_n,$$

Therefore, the path integral becomes

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathrm{e}^{-\int_X \bar{\psi}\mathfrak{D}_X\psi} = \prod_n \mathrm{d}\bar{b}_n \mathrm{d}a_n \mathrm{e}^{\sum_n a_n\bar{b}_n\lambda_n} = \det \mathfrak{D}_X.$$

Since γ_5 anti-commutes with the Dirac operator $\{\gamma_5, \mathfrak{D}\} = 0$, γ_5 only changes the sign of each eigenvalue. Hence, after applying infinitesimal chiral transformation, only zero modes attribute:

$$\mathcal{D}\psi D\bar{\psi} \to \exp(-(n_+ - n_-))\mathcal{D}\psi D\bar{\psi},$$

where we denote the subspace of zero modes as $E_0 = E^+ \otimes E^-$, where $E^{\pm}$ is the subspace of $\gamma_5 = \pm 1$ with dimension $n_{\pm}$, and we refer $(n_+ - n_-)$ as the chiral anomaly, since after modification of the Ward identity, the chiral anomaly measures the non-conservative of $\partial_\mu j^\mu_A$ Then the chiral anomaly is essentially

$$\mathrm{ind}_a(\mathfrak{D}) = \dim\ker\mathfrak{D} - \dim\ker\mathfrak{D}^\dagger$$

By Atiyah-Singer index theorem, the chiral anomaly can be computed as

$$\mathrm{ind}_t(\mathfrak{D}) = \int_X \hat{A}(R)\,\mathrm{ch}(F) \tag{4}$$

If instead the symmetry is the gauge symmetry of the quantum system, the presence of anomaly will render the theory inconsistent, since the partition fucntion $\det\mathfrak{D}_X$ above is not a “function” rather a section of the determinant line bundle associated to the Dirac operator $\mathfrak{D}_X$, there is always ambiguity in the definition of the partition function. Roughly speaking, the topological index is the characteristics class of this determinant line bundle.

3.1.2 What is an anomaly?

Roughly speaking, a theory has an anomaly if a symmetry that is present at the classical level is absent at the quantum level. This could happen for both global and gauge symmetries and in fact they are of very different nature. While the existence of an anomalous global symmetry may be useful, the existence of an anomalous gauge symmetry is fatal because it spoils the gauge invariance, which manifests as the redundant degrees of freedom in the physical descriptions of our theories, and hence renders the theory inconsistent.

As the first kind of quantum anomaly to ever be discovered, anomalous global symmetry is in fact very amiable and obliging, for example, it helps us to explain the puzzle in the calculation of the decay rate of neutral pions into two photons in its original discovery, therefore anomalies of global symmetry are sometimes called ABJ anomalies. While anomalies in global symmetries are physically interesting, gauge anomalies kill all the physics completely: they render the theory mathematically inconsistent! This is because "gauge symmetries" are not really symmetries at all, but redundancies in our description of the theory. It we wish to build a consistent theory, we must ensure that all gauge anomalies vanish.It also has many other phenomenological applications ranging form the computation of quantum numbers in the Skyrme model of hadrons to the mechanisms for baryogenesis in the standard model. We can also distinguish between several possible types of gauge anomalies . Local anomalies, the ones that are most frequently referred as anomalies in the mathematics literature, are defined to be the absence of invariance of continuous gauge transformations that are local at the quantum level, here local means that they can be continously connected to the identity transformation, gauge transformations in electrodynamics are local. Global anomalies are therefore referred to those related to global gauge transformations. For example, we can view general relativity as a gauge theory with gauge transformations being diffeomorphisms of the ambient spacetime and in this case, they are usually dubbed global gravitational anomalies[12].

3.1.3 Gauge anomaly

For similar reasons in section 3.1.1, the path integral of massless chiral fermions is also ill-defined, which is characterized by the Pfaffian line bundle. We have already seen that certain theory of massless fermions living on the boundary of a SPT is perfectly fine. Now suppose our gauge theory live in X which is the boundary of Y in which massive Dirac fermions live. By Dai-Freed theorem and later by Witten[13], the partition function of the whole theory on Y is

$$\mathcal{Z}_Y = |\det \mathfrak{D}_X| \exp(-i\pi\eta(\mathfrak{D}_Y)/2)$$

which is always a well-defined function on the moduli space of connections $\mathcal{M}$. Therefore, this might be our definition of the partition function. However, this depends on the choices of Y which might results in different answers for different Y, except all such choices are compatible with each other, i.e.

$$\exp\left(i\pi\eta\left(\mathfrak{D}_{Y_1\sqcup\overline{Y_2}}\right)/2\right) = 1,$$

for any choices of Y_1 and Y_2. Here we have used the gluing property of η-invariant. This means that for all closed manifolds Y, the η-invariant vanishes. Using this picture, we can not only recover our previous derivation of perturbative anomalies but also that of non-perturbative anomaly (a.k.a. global anomaly) as we will explained below.

For g continuously connected to the identity, one can write $Y = \partial Z$, where $Z = X \times \mathbb{D}^2$ is a $(d+2)$-dimensional manifold, since the gauge bundle can be extended to Z without problem. In this case, we can use the Atiyah-Patodi-Singer index theorem for manifolds with boundary which relates

$$\mathrm{ind}_a(\mathfrak{D}_Z) = \int_{X\times\mathbb{D}^2} \hat{A}(R)\,\mathrm{ch}(F) + \frac{\eta(\mathfrak{D}_Y)}{2},$$

Since $\mathrm{ind}_a(\mathfrak{D}_Z)$ is always an integer, we get

$$\exp(-i\pi\eta(\mathfrak{D}_Y)/2) = \exp\left(i\pi\int_{X\times\mathbb{D}^2} \hat{A}(R)\,\mathrm{ch}(F)\right).$$

Moreover, for g not continuously connected to identity, the global anomaly is $\exp\big(-i\pi\eta(\mathfrak{D}_{X_g})/2\big)$, the η-invariant of the mapping torus. For a singlet $\mathrm{SU}(2)$ Weyl fermion, this invariant coincides with the mod 2 Dirac operator on the mapping torus.

Example 3.1.1. ($SU(2)$ global anomaly[14])

Let us examine a single Weyl fermion interacting with a background SU(2) gauge field in four-dimensional Euclidean spacetime. A gauge transformation g that decays at spatial infinity falls under the homotopy group $\pi_4(SU(2)) = \mathbb{Z}_2$, indicating that transformations corresponding to non-trivial elements of this homotopy group cannot be smoothly linked to the identity. The partition function is expressed through

$$\mathcal{Z}[A] = \int \mathcal{D}\psi D\bar{\psi} \mathrm{e}^{-\int \mathrm{d}^4 x \bar{\psi} i \mathfrak{D} \psi} = \mathrm{Pf}[i\mathfrak{D}]$$

The famous mapping torus argument is the following: consider a loop in the moduli space of connections: $A(s) = sA' + (1-s)A$ and by assumption $[A] = [A'] \in \mathcal{M}$. If $Z[A] \neq Z[A']$, we say there is a gauge anomaly. In this case the partition function is not a function on the moduli space, rather a section. The gauge anomaly is given by the non-trivial holonomy around loops. From another point of view, $A(s)$ determines an extension of bundle P on X to $X \times S^1$ which is usually dubbed mapping torus. By the theorem in the appendix:

$$\mathrm{hol}_\varphi \, \mathrm{DetD}_{X/S}(V) = \lim_{\varepsilon \to 0} e^{-2\pi i \xi_{X_{\varphi(\varepsilon)}}(V)},$$

we see that the η-invariant is precisely this holonomy.

All in all, we get the statement that once there is no local anomaly, the exponentiated η-invariant, which measures global anomaly in this case is bordism invariant since it is zero whenever Y is null-bordant by the APS index theorem. Therefore, whenever the anomaly polynomial vanishes, the exponentiated η-invariant is a homomorphism from appropriate bordism group to $U(1)$.

To study local and global anomalies, we can follow these two steps to verify the absence of gauge anomalies:

- Compute $\int_{X \times \mathbb{D}^2} \hat{A}(R) \, \mathrm{ch}(F)$, in general this is replaced by the anomaly polynomial depending on the fermion contents of the theory. If it vanishes, there is no local anomaly.
- Compute $\Omega^{\xi}_{d+1}(U(1))$, in general this is replaced by $\Omega^{\xi}_{d+1}(BG)$. If it vanishes, there can be no global anomaly. If $\Omega^{\xi}_{d+1}(BG) \neq 0$, compute η-invariant explicitly on gen-

erators of the bordism group.

3.1.4 Anomaly inflow and invertible field theory

An 't Hooft anomaly is a mild violation of quantum gauge symmetry when turning on background gauge field A. It gives a lot of information about the dynamics of the theory.

$$Z[A^g] = Z[A] \exp\left(-2\pi i \int_X \alpha(g, A)\right).$$

It is convenient to describe anomalies using a classical, local action for the gauge fields A in $(d+1)$-spacetime dimensions. Such actions are also referred to as invertible field theories. In this presentation the d-dimensional manifold X supporting the dynamical field theory is viewed as the boundary of a $(d+1)$-manifold Y , and we extend the classical gauge field sources A to the manifold Y. On Y there is a local, classical Lagrangian $-2\pi i\omega(A)$ with the property that

$$\exp\left(2\pi i \int_Y \omega(A^g) - 2\pi i \int_Y \omega(A)\right) = \exp\left(2\pi i \int_X \alpha(g, A)\right),$$

where $\omega(A)$ is the Lagrangian of the anomaly theory. Introduce a modified partition function

$$\tilde{Z}[A] := Z[A] \exp\left(2\pi i \int_Y \omega(A)\right),$$

which is gauge invariant under the transformation $A \mapsto A^g$. The partition function of the anomaly theory is defined as

$$\mathcal{A}[A] = \exp\left(2\pi i \int_Y \omega(A)\right),$$

which is an invertible field theory, this is exactly a field theory charactering a non-trivial SPT phase. Therefore in this scheme, the 't Hooft anomaly of the boundary theory is provided by inflow from the nontrivial bulk Y. This is known as the anomaly inflow.

Example 3.1.2. A prototypical example of this phenomena is the integer quantum Hall effect. The $1+1$D boundary possesses gapless chiral edge modes, i.e. massless chiral fermions which cannot exist alone in $1+1$D as in the case of $3+1$D in the previous section. The effective field theory of the bulk is the renowned Chern-Simons field theory

$$S_{CS}[A]=\frac{\nu}{4\pi}\int A\wedge F,$$

where ν here gives the integer characterizing the quantized Hall conductance $\sigma_H=\nu\frac{e^2}{h}$, called level. The Chern-Simons theory also suffers form the gauge inconsistencies $\mathcal{Z}_{\mathrm{CS}}[A^g]\neq\mathcal{Z}_{\mathrm{CS}}[A]$ when living on manifolds with boundaries. The gauge variation is

$$\delta S_{\mathrm{CS}}\sim\delta\left(\int_{M^3}A\wedge F\right)\sim\int_{N^2}F\wedge F,$$

which is exactly the anomaly polynomial as in (4) at $1+1$D.

From our previous discussions, we have seen that directly starting from studying invertible field theories smoothly connects the analysis of various kinds of anomalies.

Remark 3.1.3. Whenever $\exp(-i\pi\eta(\mathfrak{D}_Y)/2)$ can be expressed in terms of a characteristic class $\int_Y\Phi$, we can define the partition function according to[13], independent of the extension Y, even if the exponentiated η-invariant does not vanish for closed manifold $\bar{Y}$.

$$Z_X=|\operatorname{Pf}\mathfrak{D}_X|\exp\left(-\frac{i\pi}{2}\eta_Y\right)\exp\left(i\int_X I^0_{d+1}\right)$$

3.2 Anomalies and index theory

It is widely recognized that local anomalies arising in a d-dimensional quantum field theory are captured by a $(d+2)$-dimensional characteristic form, with chiral fermions contributing a degree-$(d+2)$ index density linked to a specific Dirac operator. Both global and local anomalies may be derived from a $(d+1)$-dimensional geometric invariant I. In this section, we give general formalism of this slogan and explicitize some examples and calculations.

3.2.1 Local anomaly and anomaly polynomials

Local anomalies on a d-dimensional curved spacetime are determined by an characteristic class in $d+2$ dimensions called anomaly polynomial I_{d+2}, as its name suggests, is written as a polynomial in traces of powers of the spacetime curvature trR^k and the gauge field strength $\mathrm{tr}_r\, F^k$, here r indicates a representation of the gauge group, which varies from theories to theories

$$I_{d+2} = P(\mathrm{tr}\,\mathcal{R}^k, \mathrm{tr}_r\,\mathcal{F}^k)$$

In general, the anomaly polynomial I_{d+2} is written as the summation of contributions from all of the fields in the theory, each are given by an characteristic class in $d+2$ dimensions served as certain index density. By the index-type theorem, the integration of anomaly polynomial over the compact manifoldequals the index of certain elliptic operator in $d+2$ dimensions, which in most cases we consider is a Dirac-type operator. Also, by the theory of Chern-Weil, the anomaly polynomial I_{2n} is closed. Therefore, it is locally exact $I_{d+2} = \mathrm{d}I^0_{d+1}$, where I^0_{d+1} is the Chern-Simons form. This $(2n-1)$-form can be integrated over M^{d+1}, whose boundary ∂M^{d+1} is identified with the physical spacetime.

$$\Gamma_{\text{eff}} = 2\pi i \int_{M^{d+1}} I^0_{d+1},$$

One can showcase that the gauge variation of the Chern-Simons form is also closed, $\mathrm{d}\delta I^0_{d+1} = 0$, therefore locally we have $\delta I^0_{d+1} = \mathrm{d}I^1_d$. The anomaly is the variation of the effective action under gauge transformation

$$\delta\Gamma_{\text{eff}} = 2\pi i \int_{(\partial M)^d} I^1_d.$$

Conversely, the precise relationship between local anomalies and the anomaly polynomial is as follows. The anomalous variation of the quantum effective action $\delta\Gamma_{\text{eff}}$ can be related to the $(d+2)$-dimensional anomaly polynomial through what is called the Wess-Zumino descent procedure:

$$\delta\Gamma_{\text{eff}} = \int_{M^d} I_d^1 = \int_{W^{d+1}} \mathrm{d}I_{d+1}^0 = \delta_\Lambda \left[\int_{W^{d+1}} I_{d+1}^0\right] \equiv \delta[\alpha(W^{d+1})]$$

where $\partial W^{d+1} = M^d$ and $\alpha(W^{d+1})$ is called the anomaly theory. The anomaly of certain gauge theory is computed by the integral of I_d^1 over the spacetime manifold M^d.

Proposition 3.2.1. *Here are anomaly polynomials for some fermions of some types.*

i. Weyl fermion

$$\mathcal{I}_{1/2} = [\hat{A}(R)\,\mathrm{tr}_r\,\mathrm{e}^{iF/2\pi}]_{d+2},$$

in particular, fermion singlet $\mathcal{I}_{\text{Dirac}} = [\hat{A}(R)]_{d+2}$.

ii. Left-handed Weyl gravitino :

$$\mathcal{I}_{3/2} = [\hat{A}(R)(\mathrm{tr}\,\mathrm{e}^{iR/2\pi})\,\mathrm{tr}_r\,e^{iF/2\pi}]_{d+2}.$$

iii. Self-dual tensor field:

$$\mathcal{I}_{\text{SD}} = \left[-\frac{1}{8}L(R)\right]_{d+2}$$

3.2.2 Examples

Example 3.2.2. Non-linear Sigma model

A non-linear sigma model has bosonic fields $\varphi \in \mathcal{F} = \{\varphi : X \to M\}$. The manifold M is called the target space, X^d is the d-dimensional worldsheet. The bosonic action functional is given by

$$S_b = \int_X \langle d\varphi, d\varphi\rangle.$$

With supersymmetry, we get fermionic fields on the worldsheet X which are sections of spinor bundles $S^{\pm}$ tensored with $\varphi^*(TM)$, then the total action is given by

$$S = \int_X \langle\bar{\psi},\, \not{D}_\varphi\psi\rangle + \langle d\varphi, d\varphi\rangle.$$

Anomalous gauge symmetries compromise the consistency of the standard Fadeev-Popov gauge-fixing approach, resulting in unphysical negative-norm states and/or ultraviolet-divergent theories. The quantization procedure encounters the problem that the regularized fermion determinant is no well-defined function, but the section of a line bundle L which is characterized by

$$m = \int_X c_1(L)$$

We can now relate it to the anomaly

$$m = \int_X c_1(L) = \mathrm{ind} D_{2n+2} = \int_X \mathrm{ch}_{n+1}(\varphi^*(TM))$$

By naturality, we see that the theory is anomaly free if and only if the $n+1$-th chern character of the tangent space vanishes. For $n = 1$, $\varphi^* \mathrm{ch}_2(TM) = \frac{1}{2} c_1(X) c_1(M)$. Therefore, for 2d sigma models to be physically available, one must either impose generic base manifolds alongside the trivialization of the target space's first Chern class or constrain the base manifold to trivial configurations. This establishes the celebrated conclusion that the non-linear sigma model is anomaly-free precisely when the target manifold's first Chern class vanishes—that is, when the target space qualifies as a Calabi-Yau manifold.

Example 3.2.3. The standard model

The standard model of particle physics is the jewlery of modern theoretical physics. It has been verififed to be utterly precise through enourmous amount of experiments. Therefore, theoretically, is should be anomaly-free. Let's look at a generation of fermions including leptons and quarks.

	Q_L	$\bar{u}_R$	$\bar{d}R$	l_L	$\bar{e}R$
$SU(3)$	3	$\bar{3}$	3	1	1
$SU(2)$	2	1	1	2	1
$U(1)$	$1/6$	$-2/3$	$1/3$	$-1/2$	1

The anomaly of any spin-$1/2$ fermion in this generation is given by

$$\begin{aligned} I_{1/2}[\mathcal{F}, \mathcal{R}] = & \int_{M^4 \times \mathbb{D}^2} [\hat{A}(R)\, \mathrm{ch}(F)]_6, \\ = & \frac{1}{48\pi^3} \left(\mathrm{tr}_f\, \mathcal{F}^3 - \frac{1}{8} \mathrm{tr}\, \mathcal{R}^2\, \mathrm{tr}_f\, \mathcal{F} \right), \end{aligned} \tag{5}$$

where f indicates the representation of the gauge group that this type of fermion lives in. The total anomaly is then the summation of (5) over the representation of the gauge group f of all particles in the table, which is given by

$$\begin{aligned} 3\cdot 2\cdot(\tfrac{1}{6})^3+3\cdot(-\tfrac{2}{3})^3+3\cdot(\tfrac{1}{3})^3+2\cdot(-\tfrac{1}{2})^3+1 = {} & 0, \\ 3\cdot 2\cdot(\tfrac{1}{6})+3\cdot(-\tfrac{2}{3})+3\cdot(\tfrac{1}{3})+2\cdot(-\tfrac{1}{2})+1 = {} & 0, \end{aligned}$$

up to some constant depending on the base manifold. Therefore, the standard model is free of local anomalies. The global anomaly of the standard model also vanishes[13], by computing the bordism group of the $SU(5)$ Grand Unified Theory $\Omega_4^{SU(5)}$ and evaluate the η-invariant on its generator, which results in zero.

3.3 Anomalies and Invertible Field Theory

In this section, we first mathematically indicate how invertible field theories characterize the quantum anomalies in the language of relative field theories. Then we derive from the statement that anomalies are invertible field theories in one higher dimension to recover our previous discussions on anomalies using topological index, η-invariant and bordism groups.

3.3.1 Relative field theory

Suppose the quantum field theory Z has a symmetry G; in order for the partition function to respect this symmetry, it must descend to a map $Z' : \mathcal{F}(M)/G \to \mathbb{C}$. In general, there is no reason for this to be possible. To understand the obstruction, let us view $Z : \mathcal{F}(M) \to \mathbb{C}$ as a section of the trivial complex line bundle $\epsilon_{\mathcal{F}(M)}$ over $\mathcal{F}(M)$. If the G-symmetry is anomalous, then we will find that $Z(g\cdot\Phi) = P(g,\Phi)Z(\Phi)$ for some coefficient $P(g,\Phi)$ which depends on g and Φ. This says that Z can be understood as the section of the line bundle $\mathcal{L}_M := (\mathbb{C}\times\mathcal{F}(M))/G$ over $\mathcal{F}(M)/G$, where G acts on $\mathbb{C}\times\mathcal{F}(M)$ by the formula

$$g : (\lambda, \Phi) \mapsto (P(g,\Phi)\lambda, g\cdot\Phi)$$

Therefore, the G-symmetry being anomalous is equivalent to the failure of the line bundle $\mathcal{L}_M$ to be trivial over $\mathcal{F}(M)/G$, i.e., the non-vanishing of $c_1(\mathcal{L}_M) \in \mathrm{H}^2(\mathcal{F}(M)/G;\mathbb{Z})$.

Similarly, suppose N is a closed $(n-1)$-dimensional manifold. Then Z can be viewed as a function from $\mathcal{F}(N)$ to finite-dimensional vector spaces. Running a similar argument as above, one finds that there is an obstruction to descending Z from $\mathcal{F}(N)$ to $\mathcal{F}(N)/G$, and it is given by the failure of an invertible gerbe $\mathcal{G}_N$ to be trivializable. One can think of this obstruction as a class in $H^3(\mathcal{F}(N)/G)$. Despite the fact that for a closed n-dimensional manifold M, the map $Z : \mathcal{F}(M) \to \mathbb{C}$ may not descend to a map $Z' : \mathcal{F}(M)/G \to \mathbb{C}$, we see that there is a canonically-defined line bundle $\mathcal{L}_M$ over $\mathcal{F}'(M) := \mathcal{F}(M)/G$. We might therefore wish to consider a functor $\alpha : \mathrm{Bord}_{[n-1,n]}(\mathcal{F}') \to \mathrm{Cat}_{\mathbb{C}}$ which assigns to a closed n-manifold M the line bundle $\mathcal{L}_M$ over $\mathcal{F}'(M)$, and to a closed $(n-1)$-manifold N the invertible gerbe $\mathcal{G}_N$ over $\mathcal{F}'(N) := \mathcal{F}(N)/G$. Since α assigns a vector space (really, vector bundle) to an n-manifold, α is begging to be viewed as an $(n+1)$-dimensional field theory, extended to dimension $n-1$. α can indeed be viewed as an $(n+1)$-dimensional field theory, i.e. a functor $\mathrm{Bord}_{[n-1,n+1]}(\mathcal{F}') \to \mathrm{Mod}_{\mathrm{Mod}_{\mathbb{C}}}$, where $\mathrm{Mod}_{\mathrm{Mod}_{\mathbb{C}}}$ is a target $(\infty, 2)$-category whose objects are $\mathbb{C}$-linear categories, whose morphisms are $\mathbb{C}$-vector spaces, and whose 2-morphisms are complex matrices. Moreover, notice that since $\mathcal{L}_M$ and $\mathcal{G}_N$ are invertible, the functor α can be viewed as an invertible field theory.

To recover $\mathcal{Z} : \mathrm{Bord}_{[n-1,n]}(\mathcal{F}) \to \mathrm{Vect}_{\mathbb{C}}$ from $\alpha : \mathrm{Bord}_{[n-1,n+1]}(\mathcal{F}') \to \mathcal{C}$, note that the partition function can be viewed as a section of $\mathcal{L}_M$, i.e., as a bundle map $\epsilon_M \to \mathcal{L}_M$ from the trivial line bundle over $\mathcal{F}'(M)$ to $\mathcal{L}_M$. Similarly, for a closed $(n-1)$-manifold N, we can view Z on $\mathcal{F}'(N)$ as a "section" of $\mathcal{G}_N$, i.e., as a gerve map $\epsilon_N \to \mathcal{G}_N$ from the trivial gerbe over $\mathcal{F}'(N)$ to $\mathcal{G}_N$. Motivated by this, we make the following definition. Let $\mathcal{D}$ be a symmetric monoidal ∞-category. Define the "trivial" invertible field theory $1 : \mathrm{Bord}_{[n-1,n+1]}(\mathcal{F}') \to \mathcal{D}$ to be the tensor unit in the category of field theories, i.e., the one which assigns to every closed ($n-1$)-manifold the tensor unit in $\mathcal{C}$, and to every closed n-manifold the tensor unit in $\mathrm{End}\mathcal{C}(\mathbf{1}_{\mathcal{C}})$. Then, Z can be viewed as a natural transformation $\mathbf{1} \to \tau_{\leq n}\alpha$, where $\tau_{\leq n}\alpha$ is the restriction of α to $(n-1)$ - and n-dimensional manifolds. In this generalized setup, α is the anomaly theory, and the anomaly is trivializable if it is equipped with an isomorphism $\mathbf{1} \xrightarrow{\sim} \alpha$ of (invertible) $(n+1)$-dimensional field theories. This motivates the following definition.

Definition 3.3.1. A d-dimensional relative quantum field theory is therefore defined as a (2-)natural transformation $\mathcal{R} : \mathcal{A} \to \mathbf{1}^{d+1}$, where $\mathcal{A} : \mathcal{B}^{d+1} \to \mathcal{H}$ is a $(d+1)$-dimensional field theory functor.

Now suppose $\mathcal{Z} : \mathrm{Bord}_n(\mathcal{F}) \to \mathcal{C}$ is an n-dimensional extended QFT. Then an anomaly theory for Z is an $(n+1)$-dimensional invertible extended QFT $\alpha : \mathrm{Bord}_{n+1}(\mathcal{F}') \to \mathfrak{C}'$ such that Z is a natural transformation $\mathbf{1} \to \tau_{\leq n}\alpha$, where $\mathfrak{C}'$ is a symmetric monoidal $(\infty, n+1)$-category such that $\mathcal{C} = \mathrm{End}_{\mathcal{C}'}(\mathbf{1}_{\mathcal{C}'}) = \Omega\mathcal{C}'$. Given this observation, consider invertible TQFTs $\alpha : \mathrm{Bord}_{n+1}(\mathcal{F}') \to \mathrm{Mod}_{\mathbb{C}}^{(n+1)}$, where $\mathrm{Mod}_{\mathbb{C}}^{(n+1)}$ is the $(\infty, n+1)$-category defined inductively by $\mathrm{Mod}_{\mathbb{C}}^{(1)} := \mathrm{Mod}_{\mathbb{C}}$ and $\mathrm{Mod}_{\mathbb{C}}^{(n+1)} := \mathrm{Mod}_{\mathrm{Mod}_{\mathbb{C}}^{(n)}}$. Let $|\mathrm{Bord}_{n+1}(\mathcal{F}')|$ be the geometric realization of $\mathrm{Bord}_{n+1}(\mathcal{F}')$, so that it is an infinite loop space, the associated connective spectrum by $\mathrm{MT}(\mathcal{F}')$ of Madsen-Tillmann as we have seen in the previous chapter, we also saw that such invertible TQFTs were classified by maps of infinite loop spaces from $|\mathrm{Bord}_{n+1}(\mathcal{F}')|$ to the Picard groupoid of $\mathrm{Mod}_{\mathbb{C}}^{(n+1)}$. But this Picard space is $K(\mathbb{C}^\times, n+1) = \Omega^\infty\Sigma^{n+1}H\mathbb{C}^\times$, so we see that invertible TQFTs $\alpha : \mathrm{Bord}_{n+1}(\mathcal{F}') \to \mathrm{Mod}_{\mathbb{C}}^{(n+1)}$ are classified by elements of

$$\mathrm{Map}_{\text{inf. loop}}\left(|\mathrm{Bord}_{n+1}(\mathcal{F}')|, K(\mathbb{C}^\times, n+1)\right) \simeq \mathrm{Map}_{\mathrm{Sp}}(\mathrm{MT}(\mathcal{F}'), \Sigma^{n+1}H\mathbb{C}^\times)$$

whose π_0 is $H^{n+1}(\mathrm{MT}(\mathcal{F}'); \mathbb{C}^\times)$. If $\mathrm{MT}(\mathcal{F}')$ is the Thom spectrum of a map $\boldsymbol{BF} : \mathcal{F}' \to \mathrm{BO} \times \mathbb{Z}$ from some space BF' behaving like the moduli space of tangential structures, then the Thom isomorphism gives $H^{n+1}(\mathrm{MT}(\mathcal{F}'); \mathbb{C}^\times) \simeq H^{n+1}(\mathrm{BF}'; \mathbb{C}^\times)$. Initially, this was the group that was assumed to classify deformation classes of anomalies of n-dimensional QFTs; for example, if BF' is the classifying space of some group $\mathcal{G}$, then this is the group cohomology $H^{n+1}(BG; \mathbb{C}^\times)$.

The target category of n-dimensional QFTs should be $\Omega^\infty\Sigma^{n+1}I_{\mathbb{C}^\times}$, where $I_{\mathbb{C}^\times}$ is the Brown-Comenetz dualizing spectrum. Following the above analysis, one posits that deformation classes of anomalies of n-dimensional QFTs are classified by $I^{n+1}_{\mathbb{C}^\times}(\mathrm{MT}(\mathcal{F}'))$. This is closely related to $H^{n+1}(\mathrm{MT}(\mathcal{F}');\mathbb{C}^\times)$: the connective cover of $I^\times_{\mathbb{Z}}$ is $H\mathbb{C}^\times$, so we obtain a canonical map $H^{n+1}(\mathrm{MT}(\mathcal{F}');\mathbb{C}^\times) \to I^{n+1}_{\mathbb{C}^\times}(\mathrm{MT}(\mathcal{F}'))$. If α is trivializable, then there is an isomorphism $1 \xrightarrow{\sim} \alpha$. The space of all isomorphisms is precisely π_1 of the space of invertible field theories, which is $\pi_1(\mathrm{Map}_{\mathrm{Sp}}(\mathrm{MT}(\mathcal{F}'), \Sigma^{n+1} I_{\mathbb{C}^\times}) = I^n_{\mathbb{C}^\times}(\mathrm{MT}(\mathcal{F}'))$

3.3.2 Recovering the classical picture

Hamiltonian anomaly Assuming A invertible, $\mathcal{A}(M^d) \simeq \mathcal{H}$ and G acts on through the tensor product of lines:

$$g \cdot V = L_g \otimes V$$

After the identification $L_g \cong \mathbb{C}$ we get an endomorphism ϕ_g of H for each g such that

$$\phi_{g_1 g_2} \circ \phi_{g_2^{-1}} \circ \phi_{g_1^{-1}} = \alpha_{g_1,g_2} 1_H$$

which means ϕ is a projective representation of G on $\mathcal{H}$, i.e. Hamiltonian anomalies. The Lie algebra cocycle condition is the so-called the Wess-Zumino consistency conditions. Let's unravel it in detail. In the BRST formalism, let's denote $\mathcal{G} : X \to G$ local gauge group and $\mathfrak{g}_{\mathrm{loc}}$ be its corresponding Lie algebra. s is the BRST charge, ω represents the ghost field. $W[\omega, A] = sS_{\mathrm{eff}}[A]$ denotes the anomaly polynomial. $\mathcal{F}_{\mathrm{loc}}[\mathcal{A}]$ denotes all the local functionals on the space of connections. Wess-Zumino consistency condition writes

$$sW = 0,$$

which means W is an element in the BRST cohomology $H^!_{\mathrm{BRST}}(\mathfrak{g}_{\mathrm{loc}}; \mathcal{F}_{\mathrm{loc}}[\mathcal{A}])$. If W is BRST exact, then it is null-homologous, indicating that there is no perturbative anomaly, since in this case, S_{eff} is a local functional which can be cancelled by local counterterms.

The anomaly descent equation is basically deriving anomaly through a gauge-invariant closed $n+2$-form $P[A]$. In the BRST formalism, the derivation is as follows. Since P is closed, locally $P = \mathrm{d}Q^0_{n+1}$, where subscript indicates the de Rham degree and superscript indicates ghost number. By gauge invariance $sP = 0 = \mathrm{d}(sQ^0_{n+1})$, then locally $sQ^0_{n+1} = \mathrm{d}Q^1_n$. Therefore $\int_X Q^1_n$ gives an element in BRST cohomology of degree one, which is the anomaly.

A short calculation shows that the generators obey the algebra

$$[\mathcal{J}^a(x), \mathcal{J}^b(y)] = if^{abc}\delta^4(x-y)\mathcal{J}^c(x)$$

and this, along with the definition of the anomaly $\mathcal{A}^a$ as the covariant divergence of the current implies the Wess-Zumino condition

$$\mathcal{J}^a(x)\mathcal{A}^b[y,A] - \mathcal{J}^b(x)\mathcal{A}^a[x,A] = if^{abc}\delta^4(x-y)\mathcal{A}^c[y,A].$$

This equation is non-linear in the gauge potential implying that the anomaly can be determined completely once the leading order piece is known.

Local anomaly For all established anomaly field theories $\mathcal{A}$, the anomaly polynomial can be reconstructed from the value.

$$\frac{1}{2\pi i}\ln \mathcal{A}(M\times S^1) \quad \mathrm{mod}\, \mathbb{Z}.$$

For instance, in the case of complex chiral fermions $\frac{1}{2\pi i}\ln \mathcal{A}(M\times S^1)$ is the modified eta invariant $\xi_V(M\times S^1)$, which according to the Atiyah-Patodi-Singer theorem can be written as

$$\xi_V(M\times S^1) = \int_{M\times D^2} I_V - \mathrm{ind}(D_{V,M\times D^2}),$$

where I_V is its index density. Since the index is an integer, we find that the anomaly polynomial of the complex chiral fermion is given by the degree $d+2$ component of I_V.

Global anomaly and bordism Recall the theorem on the classification of invertible field theories which says

Theorem 3.3.2. *(Freed-Hopkins[5], Grady[10])*

Let IFT_{ξ}^{d+1} *denote the abelian group of (d+1)-dimensional reflection-positive IFTs on manifolds with* ξ*-structure. Then we have a short exact sequence*

$$0 \longrightarrow \mathrm{Hom}(\Omega_{d+1}^{\xi}, \mathbb{C}^{\times}) \longrightarrow \mathrm{IFT}_{\xi}^{d+1} \longrightarrow \mathrm{Hom}(\Omega_{d+2}^{\xi}, \mathbb{Z}) \longrightarrow 0$$

According to the appendix $\mathrm{Hom}(\Omega_{d+2}^{\xi}, \mathbb{Z})$ is basically a characteristic classes which is the anomaly polynomial in the current framework. Therefore, if we assume the vanishing of the local anomaly, according to[15], the global anomalies are then bordism invariants characterized by torsion homomorphisms $\Omega_{n+1}^{\xi} \longrightarrow \mathbb{C}^{\times}$.

4. Stolz-Teichner program and Anomaly Cancellation

4.1 Introduction

The Stolz-Teichner conjecture[7] stands as a profound bridge between algebraic topology and quantum field theory, proposing a geometric realization of topological modular forms (TMF) —a generalized cohomology theory—through supersymmetric Euclidean field theories (SEFTs). This conjecture asserts that the space of sufficiently well-behaved supersymmetric QFTs, when organized into a moduli space, geometrically encodes the spectrum of TMF. In doing so, it elevates the interplay between supersymmetry, modular invariance, and bordism invariants to a foundational principle, unifying insights from string theory, condensed matter physics, and homotopy theory.

To motivate the conjecture, consider the archetypal role of supersymmetry in quantum systems: in 0+1D dimensions, supersymmetric quantum mechanics (SQM) provides a physical model for K-theory, where the Witten index—a supersymmetric analog of the Euler characteristic—classifies ground-state degeneracies protected by symmetry. Extending this to $1+1$D, supersymmetric sigma models with Calabi-Yau targets exhibit modular invariance, with their partition functions valued in the ring of weakly holomorphic modular forms. These observations suggest a hierarchy: low-dimensional SUSY QFTs naturally probe generalized cohomology theories, with TMF emerging as the 2+1D analog of K-theory.

Stolz and Teichner formalized this intuition by defining Euclidean field theories—a variant of Atiyah-Segal topological quantum field theories (TQFTs)—that incorporate Riemannian metrics and supersymmetry. A d-dimensional SEFT assigns to every closed $(d-1)$-manifold a Hilbert space and to every d-dimensional bordism a linear operator, subject to compatibility conditions ensuring supersymmetry and modular covariance. Crucially, these theories are "Euclidean" in the sense that their partition functions depend only on the conformal structure of spacetime, mirroring the worldsheet geometry of strings. The conjecture posits that the space of 2-dimensional SEFTs (with $\mathcal{N} = (1,1)$ supersymmetry) realizes the spectrum TMF, with homotopy groups π_* TMF encoding deformation classes of such theories.

The Stolz-Teichner program also intersects with the Freed-Hopkins classification of invertible field theoreis, which we have encountered in the last two chapters. For instance, anomalies in 2-dimensional SCFTs—such as those arising from chiral fermions—can be canceled by coupling to 3-dimensional invertible bulk theories, mirroring the bulk-boundary correspondence of SPT phases. This duality reveals a hidden unity: both anomalies and supersymmetric genera are governed by Thom spectra (e.g., MString for TMF), with now SEFTs acting as "generators" for TMF.

Despite its elegance, the conjecture remains open in full generality. However, partial results—such as the construction of TMF via 2-dimensional gauged sigma models and the classification of 1-dimensional SQM theories via K-theory—support its validity. These developments underscore the power of supersymmetry as a tool for probing topology, while challenging physicists to uncover deeper physical interpretations of TMF, such as its role in quantum gravity or the classification of interacting topological phases.

4.2 Anomaly cancellation through TMF

Given a modular invariant 2D worldsheet CFT $T_{w.s.}$ with $(c_L, c_R) = (16, 0)$ we can produce 10D heterotic string theory on target space X^{10} equipped with a 3-form H with $\mathrm{d}H = p_1(X)/2$ in order to be non-anomalous. According to bordism picture discussed in the previous chapter, each worldsheet chiral fermion has the anomaly

$$\mathcal{A}_{w.s.}[X] = \exp\left(-2\pi i \int_{Y^{11}} \frac{f^*(p_1(X))}{2}\right),$$

which can be cancelled by introducing a 3-form H with the coupling $\exp\left(2\pi i \int_X f^*(H)\right)$. The condition on the existence of such 3-form H is called string structure on the manifold X^{10} since it can also be treated as a trivialization of the characteristic classes $p_1/2$. There will be N 10d fermions, where N is the number of states that have eigenvalue 1 under L_0. From the last chapter we know that the anomaly polynomial, which measures local anomalies, of a 10D gravitino is

$$[\mathcal{I}_{3/2}]_{12} = \frac{p_1^3}{3780} - \frac{13p_1p_2}{756} + \frac{31p_3}{3780} = \frac{31p_3}{3780},$$

while that of chiral fermions is

$$[\mathcal{I}_{1/2}]_{12} = -\frac{31p_1^3}{967680} + \frac{11p_1p_2}{241920} - \frac{p_3}{60480} = -\frac{p_3}{16\cdot 3780}.$$

Note that we have a 3-form field H satisfying $\mathrm{d}H = p_1/2$. We see that there is no mixed gravitational-gauge anomaly if and only if $N = 31\cdot 16 = 496$. There are only two such 2d CFTs, with $E_8 \times E_8$ or SO(32) Lie group as the gauge group.

Due to the vanishement of local anomaly, recall that, the global anomaly is measured by a bordism invariant

$$\mathcal{A}_{\text{het}} : \Omega_{11}^{\text{string}} \to U(1).$$

Surprisingly, the string bordism group at 11 is zero, implying that the global anomaly automatically vanishes. Now we switch our gear to consider lower dimensional compactifications of heterotic string theory. A compactifications to D dimensions produces a heterotic string theory on D dimensional spacetime X^D and replaces our worldsheet CFT $T_{w.s.}$ with a 2d $\mathcal{N} = (0,1)$ SCFT with $(c_L, c_R) = (36 - D, 3(10 - D)/2)$.

Example 4.2.1. $D = 4$ and $T_{w.s.}$ is give by 2D $\mathcal{N} = (0,1)$ SCFT with $(c_L, c_R) = (22, 9)$. The R-sector states of $T_{w.s.}$ with $(L_0, \bar{L}_0) = (1, 0)$ produce 4D chiral fermions. In some cases we have $\mathrm{SU}(2)$ symmetry. The $\mathrm{SU}(2)$ symmetry and chirality might yield potential anomalies. This is directly related to our previous discussions on Witten's $\mathrm{SU}(2)$ global anomaly that is given by the exponentiated η-invariant and equivalently mod 2 index of the Dirac operator in this case. Therefore this 4d theory is anomaly-free iff the number of SU(2) doublet fermions in the the R-sector states is even.

Example 4.2.2. $D = 2$ and $T_{w.s.}$ is a 2D $\mathcal{N} = (0, 1)$ SCFT with $(c_L, c_R) = (24, 12)$. The R-sector states give 2D chiral fermions as before and the anomaly polynomial is $p_1/48$, which is automatically absent since $\mathrm{d}H = p_1/2$. Therefore the exponentiated η-invariant gives the global anomaly. We know that the generator of $\Omega_3^{\text{string}} = \mathbb{Z}/24$ is a 3-sphere with $\int_{S^3} H = 1$, then the anomaly is characterized by

$$\mathcal{A}_{\text{het}} : \Omega_3^{\text{string}} = \mathbb{Z}/24 \to U(1), \quad 1 \mapsto \exp\left(2\pi i \frac{N}{24}\right).$$

As a result, the theory has the $\mathbb{Z}/24$ global anomaly unless $T_{w.s.}$ has net number of chiral fermions equal to 24.

The above two constraints can all be derived from TMF calculation based on the Segal-Stolz-Teichner conjecture, in fact we could do more: the global anomalis of all heterotic string theories vanish. The second examples will be treated in the third section. Now we sketch what the Stolz-Teichner conjecture is and how one can use TMF spectra to justify the absence of global anomalies.

Similar to Freed-Hopkins conjecture, the Segal-Stolz-Teichner conjecture says the deformation classes of $2\mathrm{D}\mathcal{N} = (0, 1)$ supersymmetric QFTs are given by the TMF spectra.

$$\mathrm{TMF}_n = \pi_0\left(\left\{\begin{array}{c} 2\mathrm{D}\mathcal{N} = (0,1)\,\text{supersymmetric QFTs} \\ \text{of degree}\, n = 2(c_R - c_L) \end{array}\right\}\right)$$

An 2D $\mathcal{N} = (0, 1)$ SCFT T with $2(c_R - c_L) = n$ should then determine an element $[T] \in \mathrm{TMF}_n$. As we have seen in the appendix, there is a map from TMF to the ring of modular forms of weight $n/2$ with integer coefficients

$$\phi_W : \mathrm{TMF}_n \to \mathbb{Z}((q)),$$

which resembles the elliptic genus of a SQFT. Moreover, we also have the string orientation map from string bordism to TMF

$$\sigma : \Omega_n^{\text{string}} \to \mathrm{TMF}_n,$$

where the image is given by $\mathcal{N} = (0,1)$ sigma model. This map manifests as the quantization map under the conjecture. Now the composition of the above two maps physically gives elliptic genera of $\mathcal{N} = (0,1)$ sigma models $\sigma(M, H)$ and in fact this coincide with the comutations done in the mathematical side.

The Witten's SU(2) global anomaly in 4d is related to the nontrivial loops in the space of gauge transformations. Heuristically, in the first example $D = 4$ we described above, the global anomaly is similarily associated to nontrivial loops in configurations space of 2d $\mathcal{N} = (0,1)$ SCFTs with $(c_L, c_R) = (26, 15)$, where $D = 0$ in the above formula. So, the global anomaly is given by

$$\pi_1\left(\left\{\begin{array}{c} 2\mathrm{D}\mathcal{N} = (0,1)\ \mathrm{SCFTs} \\ \mathrm{with}(c_L, c_R) = (26, 15) \end{array}\right\}\right),$$

which is TMF_{-21} according to Stolz-Teichner conjecture and in fact this group was calculated long before to be zero! Therefore there is no global anomaly. The heterotic compactification is a construction sending 2D $\mathcal{N} = (0,1)$ SCFTs with $(c_L, c_R) = (36 - D, \frac{3(10-D)}{2})$ with flavor symmetry G to D-dimensional heterotic string theories with gauge symmetry G. We can calculate the anomaly of this theory and according to the Freed-Hopkins' conjecture, this is given by $(I_{\mathbb{Z}}\Omega^{\text{string}})^{D+2}(\mathrm{BG})$. Therefore we get a homomorphism

$$\mathcal{A} : \mathrm{TMF}^{D+22}(BG) \to (I_{\mathbb{Z}}\Omega^{\text{string}})^{D+2}(BG),$$

characterizing local and global anomalies of heterotic string theories.

This map, first of all, should be a map of spectra, i.e.

$$\mathcal{A} \in [\text{TMF}, \Sigma^{-20} I_{\mathbb{Z}} \Omega^{\text{string}}]$$

Also, D dimensional heterotic string theory with $T_{w.s.}$ can be further compactified on a d-dimensional string manifold (M, H), which is a $D-d$ dimensional heterotic string theory with $T_{w.s.} \times \sigma(M, H)$. This corresponds to a multiplication map mathematically

$$\star : \text{TMF} \wedge \Omega^{\text{string}} \to \text{TMF}, \quad [T_{w.s.}] \times [(M, H)] \mapsto [T_{w.s.} \times \sigma(M, H)]$$

which should be compatible with the anomaly map in the following sense: $\mathcal{A}$ can be identified with an element in $[\text{TMF} \wedge \Omega^{\text{string}}, I_{\mathbb{Z}}]$ and this element should factor through the multiplication map $\star$. Therefore we get another map of spectra $\mathcal{B}$: TMF $\longrightarrow \Sigma^{-20} I\mathbb{Z}$ which is in $(I_{\mathbb{Z}}\,\text{TMF})^{-20}(\bullet)$,which lies in

$$\ldots \longrightarrow \text{Ext}_{\mathbb{Z}}(\text{TMF}_{-21}(\bullet), \mathbb{Z}) \longrightarrow (I_{\mathbb{Z}}\,\text{TMF})^{-20}(\bullet) \longrightarrow \text{Hom}_{\mathbb{Z}}(\text{TMF}_{-20}(\bullet), \mathbb{Z}) \longrightarrow \cdots$$

As we have explained in the previous chapter, the global anomaly part is in the torsion subgroup, i.e. $\text{Ext}_{\mathbb{Z}}(\text{TMF}_{-21}(\bullet), \mathbb{Z})$. However, we have $\text{TMF}_{-21}(\bullet) = 0$. Therefore, there is no global anomaly. As we will see in the third section, the bordism calculation to prove the vanishment of global anomalies is fairly complicated, though more general.

4.3 Anomaly cancellation using bordism

Local anomalies can somtimes be cancelled by introducing new terms, now known as the Green-Schwarz Mechanism[16]. In relevant dimensions, the anomaly polynomial factors as

$$I_{12} = X_4 X_8.$$

We can kill the anomaly by adding the term:

$$-\int B_2 \wedge X_8,$$

to the action functional, where B_2 is a 2-form called B-field and it relates to X_4 through its gauge-invariant curvature $H_3 \equiv \mathrm{d}B_2 - \omega_{\mathrm{CS}}$. Using the Bianchi identity and $P_{d+2} = \mathrm{d}I_{d+1}$, we see that this adds a term $-X_4X_8$ to the anomaly polynomial, so the total local anomaly vanishes.

In the seminal work of Freed and Hopkins[17], they calculate the global anomalies from the non-vanishing bordism groups by evaluating the Green-Schwarz term on all generators of the bordism group and see whether it vanishes or not. This method is transparent in that on the one hand, local anomaly can be cancelled by the Green-Schwarz term, and on top of that, on the other hand, the global anomaly is an bordism invariant from this term which can be expressed as integral of characteristic classes, utilizing the power of algebraic topology. The local expression of anomaly field theories, which are invertible, are then given by the Green-Schwarz term.

$$\alpha_{\mathrm{GS}}(Y_{11}) = \int_{Y_{11}} H \wedge X_8.$$

On a manifold which is itself a boundary, $\tilde{Y}_{11} = \partial Z_{12}$ which has a twisted string structures,

$$\alpha_{\mathrm{GS}}(\tilde{Y}_{11}) = \int_{Z_{12}} \mathrm{d}H \wedge X_8 = \int_{Z_{12}} X_4 \wedge X_8,$$

which is used to cancel local anomaly. In general

Example 4.3.1. ($E_8 \times E_8$ heterotic string theory)

Our spacetime in $E_8 \times E_8$ heterotic string theory is a spin manifold M and field are sections of two principal E_8-bundles $V, U \to M$. The Green-Schwarz data, which is basically the B-field, given by a trivialization of

$$\frac{1}{2}p(M) - c(V) - c(U) \in H^4(M;\mathbb{Z})$$

where c is the canonical generator of $H^4(BE_8;\mathbb{Z})$ that defines a universal characteristic class for E_8-bundles. This is called a twisted string structure. One can show that the corresponding bordism group is trivial. However, $E_8 \times E_8$ heterotic string theory also possesses a $\mathbb{Z}/2$

symmetry which switches the two E_8-bundles. In this case, the corresponding cobordism group $\Omega^{\xi}_{n+1} \neq 0$. To rescue this, we find a nice set of generators for Ω^{ξ}_{n+1}, then calculate the anomaly field theory using Green-Schwarz mechanism, eventually evaluate the anomaly on the set to prove the vanishment.

Theorem 4.3.2. *(Basile-Debray-Delgado-Montero[18])*

Let ξ denote the tangential structure for the $E_8 \times E_8$ heterotic string with its $\mathbb{Z}/2$ symmetry.

i. *Ω^{ξ}_{11} is of order 64 (isomorphic to either $\mathbb{Z}/8 \oplus \mathbb{Z}/8$, $\mathbb{Z}/16 \oplus \mathbb{Z}/4$, $\mathbb{Z}/32 \oplus \mathbb{Z}/2$, or $\mathbb{Z}/64$. And a generating set is* Bott $\times \mathbb{RP}^3$ *and (possibly) a certain ($S^4 \times S^4$)-bundle over $\mathbb{RP}^3$.*

ii. *The anomaly theory vanishes.*

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Appendix

A Modern homotopy theory

A.1 Introduction to homotopy theory through the lens of characteristic classes

In this introductory section, we illustrate an important concept heavily used, but in several different ways, during the main body of the thesis - characteristic classes, together with several closely-related generalized cohomology theories which are also frequently referred to in the thesis, such as K-theory, cobordism, elliptic cohomology and differential cohomology. The structure presented here is quite unique in order to elaborate approaches to characteristic classes as well as concepts in modern homotopy theory for readers with minimal backgrounds on algebraic topology in a concise manner, though we assume the reader familiar with basic theory of ordinary cohomology and differential geometry.

A.1.1 Four approaches to characteristic classes

First approach: axiomatic approach

Definition A.1.1. The i^{th} Chern class of a complex vector bundle $V \to M$ is defined to be a $2i$-degree class $c_i(V) \in H^{2i}(M;\mathbb{Z})$ satisfying the following axioms:

i. $c_0(V) = 1$.

ii. (Naturality) $c_i(f^*V) = f^*(c_i(V))$

iii. (Whitney sum formula) Let $U \to M$ be another complex vector bundle, we have

$$c_k(V \oplus U) = \sum_{i+j=k} c_i(V)c_j(U).$$

If we define the total Chern class $c(V) := c_0(V) + c_1(V) + \cdots$, we have $c(V \oplus U) = c(V)c(U)$

iv. (Normalization) $c_i(M) := c_i(TM)$. We have $c(\mathbb{CP}^n) = (1+x)^{n+1}$ where x be the generator of $H^2(\mathbb{CP}^n) \cong \mathbb{Z}$.

Grothendieck proves the existence and uniqueness of the Chern classes in the context of algebraic geometry. Here are some properties that can be deduced from these axioms:

i. (Additivity) $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

ii. (Stability) $c\left(V \oplus \underline{\mathbb{C}}\right) = c(V)$.

iii. $c_i(V) = 0$ if $i > \text{rank}(V)$.

Second approach: classifying space Let G be a compact Lie group. Let $\text{Bun}_G(M)$ denote the set of principal G-bundles over M up to bundle isomorphism. We can construct a universal principal G-bundle $EG \to BG$ in the following sense.

Theorem A.1.2. *To each map $f : M \to BG$ we can assign the pullback bundle $f^*(EG) \to M$. This becomes a natural bijection if we consider the homotopy classes of maps $f : M \to BG$, i.e.*

$$[M, BG] \overset{\cong}{\to} \text{Bun}_G M, \quad f \mapsto f^*(EG).$$

The general principle of building characteristic classes is as follows: to construct characteristic classes of principal G-bundles, in particular vector bundles as principle GL_n-bundles, it suffices to choose one in the cohomology of the corresponding classifying space. Then we can build characteristic classes for all bundles through pulling back which is a map

$$c : \text{Bun}_G(M) \to H^*(M; \mathbb{Z}) \quad [f : M \to BG] \mapsto f^*c,$$

the naturalness of these classes is for free. In other words, a characteristic class c for principle G-bundles is determined by its value on the classifying space $c \in H^*(BG)$.

Example A.1.3. (Chern classes)

Let's consider a rank n complex vector bundles $V \to M$, we get a map $f_V : M \to B\text{U}_n$. If we select a class $c \in H^*(B\text{U}_n)$, which satisfies the axioms in Definition A.1.1 for its universal bundles, then let $c(E) := f_E^* c$, this is what we want. We could do this construction for all rank at the same time by considering the the classifying space for complex vector bundles

$$B\mathrm{U} := \mathrm{colim}_{n\to\infty} B\mathrm{U}_n.$$

Therefore, it suffices to consider the cohomology of this space. We have the following theorem:

Theorem A.1.4. *$H^*(B\mathrm{U}) \cong \mathbb{Z}[c_1, c_2, \ldots]$, with $|c_k| = 2k$, where c_k is the k^{th} Chern class.*

Third approach: the splitting principle In fact, it suffices to only construct the first Chern class, as we will see in later section that, the first Chern class also gives higher Chern classes in generalized cohomology theories. This follows from the splitting principle which asserts that every vector bundles can be decomposed to direct sum of line bundles after pulling back. Since the Chern classes are natural and higher Chern classes of line bundles vanishes, we only need to consider the first Chern class of the line bundle in related problem. Here we present how to construct higher Chern classes from first Chern class to elaborate the splitting principle.

Theorem A.1.5. *(Splitting principle)*

After pulling back along the flag bundle $p : \mathrm{F}\ell(V) \to X$, a complex vector bundle $V \to X$ can be decomposed as a direct sum of line bundles $p^*V = L_1 \oplus \ldots \oplus L_n$. Moreover, the map $p^* : H^*(X;\mathbb{Z}) \to H^*(\mathrm{F}\ell(V);\mathbb{Z})$ is injective.

Then the total Chern class $c(V) = 1 + c_1(V) + \cdots + c_n(V)$, with $c_k(V) \in H^{2k}(X;\mathbb{Z})$, equals to

$$\begin{aligned} 1 + c_1(V) + \cdots + c_n(V) = c(L_1 \oplus \ldots \oplus L_n) = &\ (1 + x_1) \cdots (1 + x_n) \\ = &\ 1 + \sigma_1(x_1, \ldots, x_n) + \cdots + \sigma_n(x_1, \ldots, x_n) \end{aligned}$$

with Chern roots $x_i = c_1(L_i) \in H^2(X;\mathbb{Z})$, $i = 1, 2, \ldots, n$. Then by degree reason $c_k(E)$ is the k^{th} symmetric polynomial σ_k in these roots. Reverse the above procedure, we can define Chern classes for all complex vector bundles once we know how to define the total Chern class of line bundles $c(L)$. We can identify $p^*c(E)$ with $c(E)$ since by Theorem A.1.5, p^* is injective.

The above story can also be generalized to the case of principle G-bundles. However, we can no longer work over $\mathbb{Z}$ anymore, we have to work over $\mathbb{Q}$ and the decomposition analogue involves the maximal torus T of the structure group G. Let G be a compact, connected Lie group, then T is isomorphic to $\mathbb{T}^n$ for some n called rank of G. We define a G/T-bundle $p : Z \to M$ similar to the flag bundle above as the pullback along $f_P : M \to BG$ and $i : T \hookrightarrow G$

$$Z = M \times_{BG} BT.$$

The generalized splitting principle says that after pulling back to Z, the bundle p^*P can be viewed as a T-bundle. Furthermore, The map $q^* : H^*(X;\mathbb{Q}) \to H^*(Z;\mathbb{Q})$ is also an injection. Therefore, to define a characteristic class c for principle G-bundles, it suffices to consider its values $c(Q)$ on principle T-bundles Q. Since we have the isomorphism $T \cong \mathbb{T}^n$, $c(Q)$ decomposes as a product

$$\prod_{i=1}^{n}(1 + x_i)$$

Definition A.1.6. Choose a connection ∇ on V. The total Chern class $c(V)$ is

$$c(V) := \det\left(I - \frac{F_\nabla}{2\pi i}\right) \in H^{2*}_{\mathrm{dR}}(M)$$

The renowned Chern-Weil theorem says it is independent of the choice of connection. Moreover, this lives in $H^{2k}_{\mathrm{dR}}(M;\mathbb{Z}) \subset H^{2k}_{\mathrm{dR}}(M) \otimes \mathbb{Q}$ a priori only in $H^{2k}_{\mathrm{dR}}(M) \otimes \mathbb{Q}$. More general Chern-Weil theory and the phenomenon of integral lifts of de Rham classes are the genesis of differential cohomology, which we will elaborate at the end of our introductory part in section A.1.9.

A.1.2 Characteristic classes as obstructions to tangential structures

Theorem A.1.7. *As graded rings, $H^*(BO;\mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, w_3, \ldots]$, with $|w_k| = k$.*

Therefore, characteristic classes for real vector bundles in $H\mathbb{F}_2$ is a polynomial in w_k.

These w_k's are the Stiefel-Whitney classes. Set $w_0 = 1$, the total Stiefel-Whitney class is $w(V) := 1 + w_1(V) + w_2(V) + \cdots$. The Stiefel-Whitney classes also satisfy the axioms in Definition A.1.1 as well as the properties like stability and rank condition as the Chern classes do. It turns out that Stiefel-Whitney classes are obstruction to several structures ubiquitous in geometry and topology.

Here are some examples of some common topological structure. Notice that we say M has a H-structure if its frame bundle $\mathcal{F}(M)$, which is a $\mathrm{GL}_n(\mathbb{R})$-bundle, has one.

Example A.1.8. Using the more geometric language of reduction, we unravel some familiar concepts:

i. Consider $\rho : \mathrm{O}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$. A lift of $\mathcal{F}(M)$ to a O_n-bundle is equivalent to a Riemannian metric, i.e. smoothly varing inner product on T_xM.

ii. Consider $\rho : \mathrm{SO}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$. A lift of $\mathcal{F}$(M) to a SO_n-bundle is equivalent to an orientation. Two lifts are homotopic iff they define the same orientation.

iii. Consider $\rho : \mathrm{U}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$. A lift of $\mathcal{F}$(M) to U_n-bundle is equivalent to an almost complex structure.

Definition A.1.9. (Spin and $\mathrm{Pin}^{\pm}$-structrues)

A spin structure on M is a Spin_n-structure along the map $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n \hookrightarrow \mathrm{GL}_n(\mathbb{R})$.

Spin_n is the double cover of SO_n. Moreover, there are two groups Pin_n^+ and Pin_n^- double covering the O_n and they both possess the connected component of the identity as Spin_n. Similarily, we can speak of $\mathrm{Pin}^{\pm}$-structures.

Theorem A.1.10. *Let M be a manifold.*

- *M is orientable if and only if $w_1(M) = 0$.*
- *M is spin if and only if $w_1(M) = 0$ and $w_2(M) = 0$.*
- *M has a Pin^+(Pin^-)-structures if and only if $w_2(M) = 0$. $(w_2 + w_1^2(M) = 0 \quad \text{resp.})$*

Characteristic classes and cobordism The theorem is a culminating result in modern differential topology and is of enormous importance to the problem of immersions and embedding of manifolds.

A.1.3 Digression: cohomology operations and Wu classes

Definition A.1.11. A cohomology operation is a natural transformation of contravariant functors $\mathcal{O} : H^p(-;A) \to H^q(-;B)$. We say $\mathcal{O}$ is stable if it also commutes with suspension isomorphism Σ.

An example of cohomology operation is $x \mapsto x^2$, which exists in any degree. Though this is in some sense universal, it's not stable. They are genesis of a large class of stable operations.

Let's consider the case $p = 2$.

Definition A.1.12. All stable cohomology operations $H^*(-;\mathbb{F}_2) \to H^*(-;\mathbb{F}_2)$ over $\mathbb{F}_2$ constitute a graded $\mathbb{F}_2$-algebra called the Steenrod algebra $\mathscr{A}$, which is generated by classes $\mathrm{Sq}^n \in \mathscr{A}_n$, called Steenrod squares, such that:

i. $\mathrm{Sq}^n : H^k(-;\mathbb{F}_2) \to H^{k+n}(-;\mathbb{F}_2)$.

ii. $\mathrm{Sq}^0 = \mathrm{id}$, $\mathrm{Sq}^1 = \beta_4$.

iii. Restricted degree n, Sq^n is the squaring map $x \mapsto x^2$ and if $n > |x|$, then $\mathrm{Sq}^n x$ vanishes.

iv. The total Steenrod square $\mathrm{Sq} := 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots$ is a ring homomorphism, or equivalently,

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y), \tag{6}$$

which is called the Cartan formula.

v. The Steenrod squares have the Adem relations

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{\lfloor i/2\rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$$

plays an important role in the theory of characterisitc classes.

Recall that the classical theorem of Poincaré duality says for any closed n-manifold M,

$$H^k(M;\mathbb{Z}/2) \otimes H^{n-k}(M;\mathbb{Z}/2) \xrightarrow{\smile} H^n(M;\mathbb{Z}/2) \xrightarrow[\sim]{\frown[M]} \mathbb{Z}/2$$

is a nondegenerate pairing. In particular, $H^k(M;\mathbb{Z}/2) \cong (H^{n-k}(M;\mathbb{Z}/2))^*$ and $\mathrm{Sq}^k : H^{n-k}(M;\mathbb{Z}/2) \to H^n(M;\mathbb{Z}/2)$ is such a linear functional. Therefore, we can use Poincaré duality to turn the Steenrod squares into characteristic classes, the *Wu* classes.

Definition A.1.13. (Wu classes)

The k^{th} Wu class v_k of M is the Poincaré dual of k^{th} Steenrod square, i.e. $v_k \in H^k(M;\mathbb{Z}/2)$: $v_k \smile x = \mathrm{Sq}^k(x)$. Similarly, the total Wu class is $v := 1 + v_1 + v_2 + \cdots$. The total Wu class satisfies

$$\langle v \smile x, [M]\rangle = \langle \mathrm{Sq}x, [M]\rangle$$

for all $x \in H^*(M;\mathbb{Z}/2)$.

Theorem A.1.14. *(Wu formula)*

$$\mathrm{Sq}(v) = w. \tag{7}$$

Using the Ádem relations, we get

$$\mathrm{Sq}^i w_k = \sum_{j=0}^{i} \binom{k+j-i-1}{j} w_{i-j} w_{k+j}.$$

The Wu classes are very useful in computation. Here are some examples

Proposition A.1.15. *If M is a closed 2- or 3-manifold, $w_1(M)^2 = w_2(M)$.*

Proof. Looking at homogeneous terms and using Wu formula (7),

$$w_1 = \mathrm{Sq}^1 v_0 + \mathrm{Sq}^0 v_1 = v_1$$
$$w_2 = \mathrm{Sq}^2 v_0 + \mathrm{Sq}^1 v_1 + \mathrm{Sq}^0 v_2 = v_1^2 + v_2 = w_1^2$$

Moreover, we deduce that orientable manifolds are also spin in low dimensions. □

Example A.1.16. (Wu manifold)

The *Wu manifold* $W := \mathrm{SU}_3/\mathrm{SO}_3$ is a five-dimensional manifold with mod 2 cohomology being $H^*(W;\mathbb{Z}/2) \cong \mathbb{F}_2[z_2, z_3]/(z_2^2, z_3^2)$, and the $\mathscr{A}$-action being $\mathrm{Sq}^1 z_2 = z_3$ and $\mathrm{Sq}^2 z_3 = z^5$. Hence $v(W) = 1 + v_2$ and only w_2 and w_3 can be nonzero,

$$w_2(W) = \mathrm{Sq}^2 v_0 + \mathrm{Sq}^1 v_1 + \mathrm{Sq}^0 v_2 = v_2 = z_2$$
$$w_3(W) = \mathrm{Sq}^3 v_0 + \mathrm{Sq}^2 v_1 + \mathrm{Sq}^1 v_2 + v_3 = \mathrm{Sq}^1 z_2 = z_3$$

so $w(W) = 1 + z_2 + z_3$ and Stiefel-Whitney number $w_{2,3} = \langle w_2(W) w_3(W), [W] \rangle = 1$. Thus, $\Omega_5^O \cong \mathbb{Z}/2$ has W as a generator.

Definition A.1.17. The k^{th} integral Stiefel-Whitney class $W_n(V)$ of V is defined as

$$\beta_0 w_{n-1}(V) \in H^n(M;\mathbb{Z}).$$

The Lie group Spin_n^c is the quotient

$$\mathrm{Spin}_n^c := (\mathrm{Spin}_n \times \mathrm{U}_1)/(\mathbb{Z}/2) \tag{8}$$

where $\mathbb{Z}/2$ acts as -1 on both components. Therefore, just as the Spin-structures, we could talk about Spin^c-structures and Pin^c-structures by replacing Spin_n with Pin_n^+ or Pin_n^- in (8). Like Theorem A.1.10, these structures are also obstructed by certain characteristic classes.

Proposition A.1.18. *Let M be a manifold.*

- *M has a Spin^c-structure if and only if $w_1(M) = 0$ and $W_3(M) = 0$.*
- *M has a Pin^c-structure if and only if $W_3(M) = 0$.*

A.1.4 Chern classes and Pontrjagin classes

A natural question to ask is whether Chern classes have the similar connection to cobordism rings and as obstructions to topological structures. We can generally define Chern classes and Chern numbers similarly for stably almost complex manifolds, which means we have complex structure on $TM \oplus \underline{\mathbb{R}^k}$ for some k.

We can define it for stably almost complex manifolds, which is called the complex cobordism ring and denoted as Ω_*^{U}. As we have seen many times, it is a general phenomenon that the cohomology rings of classifying space encode characteristic classes which classify manifolds up to cobordism through characteristic numbers, determining the structure of the cobordism rings.

Theorem A.1.19 (Milnor, Novikov)**.** *As graded rings,*

$$\Omega_*^{\mathrm{U}} \cong \mathbb{Z}[x_1, x_2, \ldots]$$

where $|x_k| = 2k$*. Rationally,* $\Omega_*^U \otimes \mathbb{Q}$ *is generated by the complex projective spaces* $\mathbb{CP}^i$ *for* $i \geq 1$*.*

Moreover, two stably almost complex manifolds are cobordant if and only if they have the same Chern numbers.

A necessary condition for possessing stably almost complex structures is the vanishment of w_{2k+1} and W_{2k+1} for all k.

We have already introduced the Stiefel-Whitney class related to the real vector bundles and orthogonal group O_n and the Chern classes related to complex vector bundles and the unitary groups U_n, what about oriented real bundles? They should be related to special orthogonal groups SO_n according to our previous discussions. The corresponding characteristic classes are called Pontrjagin classes.

The maximal torus $\mathbb{T}^n$ of SO_n consists of the diagonal matrices in $\mathrm{U}_{[n/2]} \subset \mathrm{SO}_n$. According to the splitting principle, it suffices to define it on line bundles $L \to X$. We define the total Pontrjagin class $p(L) = 1 + p_1(L)$ and $p_1(L) = c_1(L)^2 \in H^4(X;\mathbb{Z})$.If V is an oriented real vector bundle, then after pulling back.

Theorem A.1.20 (Thom, Wall)**.** *i. As graded rings,*

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[x_1, x_2, \dots,]$$

where $|x_k| = 4k$*, and* $x_k = [\mathbb{CP}^{2k}]$.

ii. Two oriented n-manifolds are oriented cobordant if and only if they have the same Pontrjagin and Stiefel-Whitney numbers.

Since $\mathrm{Spin}_n \to \mathrm{SO}_n$ is a double cover, the forgetful map $\Omega_*^{\mathrm{Spin}} \to \Omega_*^{\mathrm{SO}}$ is an isomorphism rationally. In particular, $\Omega_*^{\mathrm{Spin}} \otimes \mathbb{Q} \cong \mathbb{Q}[\tilde{x}_1, \tilde{x}_2, \dots]$ with $|\tilde{x}_k| = 4k$ but they are not represented by $\mathbb{CP}^{4k}$ anymore. The characteristic numbers that classify spin cobordism classes are KO-characteristic classes, the KO-theory is a generalized cohomology theory.

A.1.5 Characteristic classes for generalized cohomology theory

$$\begin{aligned}\tilde{E}_n(M) &= \lim_{k\to\infty} \pi_{n+k}(M \wedge E_k),\\ \tilde{E}^n(M) &= \lim_{k\to\infty} [\Sigma^k M, E_{n+k}].\end{aligned}$$

In this subsection, we require our spectrum E to possess a graded commutative ring structure $E^i(X) \times E^j(X) \to E^{i+j}(X)$, dubbed a ring spectrum. Theories represented by ring spectra are named multiplicative cohomology theories.

Example A.1.21 (K-theory)**.**

Theorem A.1.22. *(Bott periodicity).* $K^0(\Sigma^2 X) \cong K^0(X)$.

We can generalize construction of the cobordism group to any stable tangential structure.

Cobordism between closed n-manifolds with ξ-structure is an equivalence relation, and the equivalence classes in dimension n form an abelian group denoted ${}^{\xi}_{n}$, under disjoint union. As before, we can define abelian groups ${}^{\xi}_{n}(X)$. The functors Ω_*^{ξ} is also an generalized homology theory, and the Pontryagin-Thom construction shows they are represented by the Thom spectra whose n-th space is the Thom space M^V for some vector bundle $V \to M$ which is defined as the unit disc bundle modulo the unit sphere bundle $D(V)/S(V)$.. Let $\tau_k \to B\mathrm{O}_k$ denote the tautological bundle, and $V_k \to \xi_k$ denote its pullback across $\xi_k \to B\mathrm{O}_k$.

Theorem A.1.23 (Pontrjagin-Thom)**.** *Pontrjagin-Thom collapse map defines a bijection:*

$$\Omega_n^{\xi} \xrightarrow{\cong} [S^m, \xi_{m-n}^{V_{m-n}}] = \pi_m(\xi_{m-n}^{V_{m-n}})$$

Define the Thom spectrum $M\xi$ to have n^{th} space $\xi_n^{V_n}$ whose structrue map is

$$\Sigma\xi_n^{V_n} \cong \xi_n^{V_n \oplus \mathbb{R}} \to \xi_{n+1}^{V_{n+1}}$$

then we get an isomorphism between ξ-bordism group and $\pi_n(M\xi)$.

Proof. For large m, the classifying map of the normal bundle $\phi : \nu_m \to V_{n-m}$ extends to a map $S^m \to \xi_{m-n}^{V_{m-n}}$ via Pontrjagin-Thom collapse as follows.Choose a basepoint on $S^m \backslash \nu$, then we can acquire a map $S^m \to \xi_{m-n}^{V_{m-n}}$. Conversely, given a map $S^m \to D(V_{m-n})$ sending the basepoint to $S(V_{m-n})$, by transversality, the preimage of a regular value is what we want. This is an isomorphism up to homotopy by transversality. □

Moreover, we get the oriented cobordism $M\,\mathrm{SO}$, spin cobordism $M\,\mathrm{Spin}$, and complex cobordism MU by the stable tangential G-structure where ξ's could also be $B\,\mathrm{Spin}^c$, $B\,\mathrm{Pin}^{\pm}$.

Remark A.1.24. Not all tangential G-structures are stable, and in general, we do NOT always get a ring structure on Ω_*^{ξ}. For example, this happens for Pin^+ and Pin^- structure. Their spectra are modules over $M\,\mathrm{Spin}$. Moreover, the Pontrjagin-Thom theorem only works for stable normal structures, which again in most cases are equivalent to stable tangential structures, except for those module spectra, like $M\,\mathrm{Pin}^{\pm}$.

Generalized orientations and characteristic classes for generalized cohomology

Definition A.1.25. Let E be a multiplicative cohomology theory. A characteristic classes c for generalized cohomology simply takes values in $E^*(X)$ that is natural in the sense of Definition A.1.1.

Definition A.1.26. Let $V \to X$ be a topological vector bundle of rank k. Then an E-orientation or E-Thom class on V is an element of degree k $u_V \in \tilde{E}^k(\mathrm{Th}(V))$ in the reduced E-cohomology of the Thom space of V, such that for every point $x \in X$ there exist a map $\varphi_x : \mathbb{R}^n \to V_x$ makes the composite

$$\tilde{E}^k(\mathrm{Th}(V)) \to \tilde{E}^k(\mathrm{Th}(V_x)) \xrightarrow{\varphi_x^*} \tilde{E}^k(S^k) \cong E^0(\bullet)$$

map the element u_V to the unit in $E^0(\bullet)$.

Theorem A.1.27. *Let $\pi : V \to X$ be a E-oriented vector bundle of rank k. We have the Thom isomorphisms*

$$E^*(X) \xrightarrow{\smile u_V} \tilde{E}^{*+k}(Th(V)).$$

The pushforward map $\pi_! : E^*(V) \to E^{*-k}(X)$ is given by the composite

$$E^*(V) \underset{\cong}{\xrightarrow{\smile u}} \tilde{E}^{*+n-k}(Th(\nu_V^n)) \xrightarrow{\mathrm{PT}^*} \tilde{E}^{*+n-k}(\Sigma^n X_+) \cong \tilde{E}^{*-k}(X)$$

where ν_E^n is the normal bundle of $V \hookrightarrow \mathbb{R}^n$, u is the Thom class for this bundle. PT stands for Pontrjagin-Thom collapse map.

A.1.6 Characteristic series: complex orientation, formal group law and genera

Definition A.1.28. A complex orientation of E is an E-orientation for each rank-n universal complex vector bundle $\mathcal{O}(n) \to \mathrm{BU}_n$:

$$u_n \in \tilde{E}^k(\mathrm{Th}(\mathcal{O}(n)))$$

such that these are compatible in that

i. for any n, we have $u_n = \varphi_n^* u_{n+1}$, where

$$u_n \in \tilde{E}^k(\mathrm{Th}(\mathcal{O}(n))) \cong \tilde{E}^{k+1}(\mathrm{Th}(\mathbb{R} \oplus \mathcal{O}(n)))$$

and where $\varphi_n : \mathbb{R} \oplus \mathcal{O}(n) \to \mathcal{O}(n+1)$.

ii. for any n_1, n_2, we have $u_{n_1} \cdot u_{n_2} = u_{n_1+n_2}$.

Constructed from pulling back along its classifying map. Moreover, this collection is compatible with pullback (between complex vector bundles) and direct sum (multiplicativity).

From our previous discussions on bordism theory, we know that the sequence $\{\mathrm{Th}(\mathcal{O}(n))\}$ is basically the Thom spectra which represents the complex cobordism theory MU and a complex orientation basically assigns a natural collection of classes in $\tilde{E}^*(MU)$, such assignment together with the conditions above actually give what is called a map between ring spectra $MU \to E$.

Remark A.1.29. The above definition can be further generalized to any tangential G-structure, by simply replacing $O(n)$ with rank n the universal vector bundle with G-structure. When $G = U$, we recover our original definition.

The generalization bears the name a universal E-orientation for vector bundles with G-structure and it is a big theorem that it is equivalent to a homomorphism of $\mathbb{E}_\infty$-ring spectra

$$u : MG \longrightarrow E$$

Quillen's theorem implies there is a one-to-one correspondence between formal group laws over R and a ring homomorphism $MU^* \to R$. Over a $\mathbb{Q}$-algebra, any formal group law is isomorphic to the additive formal group law F_+.

$$F \underset{\exp_F}{\overset{\log_F}{\rightleftarrows}} F_+ \tag{9}$$

The isomorphism $\log_F$ is given by

$$\log_F(x) = f(x) = \int_0^x \frac{dt}{\frac{\partial F}{\partial x_2}(t,0)}. \tag{10}$$

Example A.1.30. Rationally, the universal formal group law F_{MU} has the logarithm using (10) as

$$\log_{F_{MU}}(x) = \sum_{n\geq 0} \frac{[\mathbb{CP}^n]}{n+1} x^n$$

Remark A.1.31. Over $\mathbb{F}_p$, there is exactly one formal group law for each $n \in \mathbb{N}$, called height. This is really the pillar of modern chromatic homotopy theory.

Definition A.1.32. (Genus)

An genus is a ring homomorphism

$$\varphi : MSO_* \to R,$$

where R is a $\mathbb{Q}$-algebra. And a complex genus is a ring homomorphism

$$\varphi : MU_* \to R.$$

Therefore, any complex genus $\varphi : MU^* \to R$ has a logarithm

$$\log_\varphi(x) = \sum_{n\geq 0} \frac{\varphi([\mathbb{CP}^n])}{n+1} x^n$$

Genera can be completely described by its charactersitic series according to the splitting principle. We can associate the characteristic class for any complex vector bundles to a genus $\varphi : MU_* \to R$

$$q_\varphi(V \to M) \in H^*(M; R),$$

which is given entirely in terms of the characteristic series

$$q_\varphi(x) = q_\varphi(\mathcal{O}(1) \to \mathbb{CP}^\infty) \in H^*(\mathbb{CP}^\infty; R) \cong R[x], \tag{11}$$

where $x = c_1(\mathcal{O}(1))$ the usual first Chern class. Conversely, starting from the characteristic series, we can define a genus for any stably almost complex manifold M,

$$\varphi(M) = \langle q_\varphi(TM), [M] \rangle.$$

Proposition A.1.33 (Hirzebruch)**.** *(Hirzebruch)*

For R a $\mathbb{Q}$-algebra, there are bijections

$$\begin{aligned} \{q(x) = 1 + a_1 x + a_2 x^2 + \cdots \in R[\![x]\!]\} &\longleftrightarrow \{\varphi : MU^* \to R\} \\ \{q(x) = 1 + a_2 x^2 + a_4 x^4 + \cdots \in R[\![x]\!] \mid a_{odd} = 0\} &\longleftrightarrow \{\varphi : M\,\mathrm{SO}^* \to R\} \end{aligned}$$

Proof. Dissecting formula (11), the isomorphism on the right hand side is really induced by the usual first Chern class c_1, and according to Theorem **??** this comes from a complex orientation yielding the additive formal group law F_+ over R. What really associated to the complex genus $\varphi : MU^* \to R$ really is another formal group law F over R and the universal first Chern class

$$c_1^\varphi \in H^*(BU(1); R) \cong R[c_1^\varphi],$$

where the isomorphism is induced by it. The characteristic class $q_\varphi(V \to X)$ is constructed from this first Chern class according to the splitting principle. Therefore, the characteristic series is basically an automorphism of $H^*(BU(1); R)$ which maps c_1 to c_1^φ.

According to (9) and since $c_1^\varphi = F(x, 0)$ we have

$$\begin{aligned} q_\varphi(x) = F(x,0) = &\quad \exp_\varphi F_+(\log_\varphi(x), \log_\varphi(1)) \\ = &\quad \tfrac{x}{\exp_\varphi(x)}, \end{aligned}$$

where $\exp_\varphi(x)$ is the inverse to $\log_\varphi(x)$.

□

Example A.1.34. From (10), the logarithm for $F_\times(x_1, x_2) = x_1 + x_2 - x_1 x_2$ is

$$\log_\times(x) = \int_0^x \frac{dt}{1-t} = -\log(1-x)$$

Therefore,

$$\exp_\times(x) = 1 - e^{-x}$$

$$q_\times(x) = \frac{x}{\exp_\times(x)} = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots \in \mathbb{Q}[\![x]\!]$$

generates the Todd genus td.

Example A.1.35. Here are some important characteristic classes in topology given in terms of their characteristic seris:

- Chern character $\mathrm{ch}(V)$ corresponds to $q(x) = e^x$. The Chern character has this special properties as being a ring homomorphism

$$\mathrm{ch} : K^0(M) \longrightarrow H^{\text{even}}(M;\mathbb{Q}), \quad [V] \mapsto \mathrm{ch}(V)$$

  which is not possessed by other genera.

- A roof genus $\hat{A}(V)$ corresponds to $q(x) = \frac{x/2}{\sinh(x/2)}$=$1 - \frac{x^2}{24} + \frac{7x^4}{5760} + \cdots,$ we could expand it:

$$\hat{A}(X) = 1 - \frac{1}{24}p_1(X) + \frac{7p_1(X)^2 - 4p_2(X)}{5760} + \cdots$$

- L-genus $L(V)$ corresponds to $q(x) = \frac{x}{\tanh(x)}$.

Genera have a geometric (or analytic) interpretation and lift to integral invariants, whose explanation for the integrality is given by the index theory.

Theorem A.1.36. *(Hirzebruch-Riemann-Roch)*

Let M be a projective complex manifold and let $V \to M$ be a holomorphic vector bundle. Then

$$\chi(M, V) = \langle \mathrm{td}(M)\mathrm{ch}(V), [M] \rangle \in \mathbb{Z}. \tag{12}$$

The left hand side of (12) also equals to the index of the operator $\partial + \bar{\partial}$ which is defined to be an integer. Hirzebruch's signature theorem, a step in the proof of Theorem A.1.36 is another example.

Theorem A.1.37. *(Hirzebruch Signature Theorem)*

The signature of a closed oriented smooth manifold X is

$$\sigma(M) = \langle L(M), [M] \rangle, \tag{13}$$

where $\sigma(M)$ is the signature of the intersection form, which is a prior an integer.

In fact, the signature can also be interpreted as an Fredholm index of certain operator, i.e. $\sigma(M) = \mathrm{ind}(\mathrm{d} + \mathrm{d}^*)$.

A.1.7 Geometry of characteristic classes: index theory

The index theory relates, on one side, analytic index of Fredholm operators between vector bundles on M and, on the other side, some topological invariants of M typically involving characteristic classes defined above, usually dubbed the topological index.

Let's set up the index problem on a closed oriented n-manifold M. Let $V, U \to M$ be two vector bundles over M, and suppose $P : \Gamma(V) \to \Gamma(U)$ is a linear differential operator of order m. Locally, P can be written as:

$$Pu(x) = \sum_{|\alpha| \leq m} a_\alpha(x) (\frac{\partial}{\partial x^1})^{\alpha_1} \cdots (\frac{\partial}{\partial x^n})^{\alpha_n} u(x)$$

where u is a smooth section of V, a_α is a smooth bundle map between V and U. The symbol of this differential operator $\sigma(P)$ is the highest order term, which gives a bundle map

$$\sigma(P) : \mathrm{Sym}^m(T^*X) \otimes V \to U$$

The differential operator P is elliptic if its symbol is invertible. It follows from elliptic theory that P has finite dimensional kernel and cokernel. The Fredholm index of P is

$$\mathrm{ind}\, P = \dim \ker P - \dim \mathrm{coker} P.$$

In fact, the Fredholm index is a complete deformation invariant in the space of Fredholm operators. The principal symbol is a bundle map and hence define a class $\sigma(D) \in K^0(T^*M, T^*M_0)$. From elliptic theory, we know that the index of the operator depends only on the relative K-

class $[\sigma(D)]$. Therefore, the Fredholm index can be interpretted as the so-called analytic index

$$\mathrm{ind}_a : K^0(T^*M, T^*M_0) \longrightarrow \mathbb{Z}, \quad [\sigma(P)] \mapsto \mathrm{ind}\, P$$

The Atiyah-Singer index theorem

Theorem A.1.38. *(Atiyah-Singer)*

Let M be a closed oriented manifold, and $P : \Gamma(V) \to \Gamma(U)$ is an elliptic operator, then

$$\mathrm{ind}\, P = \langle \mathrm{td}(TM_{\mathbb{C}}) \cdot \phi^{-1}\, \mathrm{ch}(\sigma(P)), [M] \rangle$$

where ϕ is the Thom isomorphism in ordinary cohomology theory. Moreover, we can interpret the right hand side as an topological index $\mathrm{ind}_t : K^0(T^*M, T^*M_0) \to \mathbb{Z}$ *using Thom isomorphism ϕ and cobordism theory purely topologically. Therefore the Atiyah-Singer index theorem basically says*

$$\mathrm{ind}_a = \mathrm{ind}_t,$$

as maps between $K^0(T^*M, T^*M_0)$ and $\mathbb{Z}$.

When M is a even dimensional closed spin manifold, one has the Thom class in K-theory since a spin manifold has a canonical K-orientation $\tau \in K^0(T^*M, T^*M_0)$ gives rise to the isomorphism

$$K^0(M) \xrightarrow{\cong} K^0(T^*M, T^*M_0), \quad [V] \mapsto \tau \cdot \pi^* V \tag{14}$$

The corresponding operator D of the class τ is the so-called Dirac operator $D : \Gamma(S_+) \to \Gamma(S_-)$, where $S_\pm$ are spinor bundles associated to the spin structure on M. General classes on the right hand side of (14) corresponds to the twisted Dirac operators D^V. In this case, according Theorem A.1.27 , we have topological pushforwards in K-theory along $p : M \to \bullet$, and composite with the inverse of Thom isomorphism (14) we get

$$\mathrm{ind}_t : K^0(T^*M, T^*M_0) \xrightarrow{\cong} K^0(M) \xrightarrow{p_!} K^0(\bullet) \cong \mathbb{Z}$$

This turns out to be equal to Thom's construction on topological index in this special case.

Theorem A.1.39. *(The "famous" Atiyah-Singer Index Theorem)*

Let M be an even dimensional closed spin manifold and $V \to M$ a complexvector bundle, then

$$\operatorname{ind} D^V = \langle \hat{A}(M)\,\mathrm{ch}(V), [M] \rangle$$

An application of Atiyah-Singer index theorem is the explanation for the integrality of $\hat{A}$-genus on spin manifolds. This is evident from the special case where we take V to be trivial in Theorem A.1.39. Moreover, along the same line of thoughts, one can prove variants of index theorems like above, for example, the equivariant index theorem considers manifolds with G-action, when G is abelian, ind_G lives in the equivariant K-theory $K_G(\bullet)$ which is exactly the character of the group G. Another example if the family index theorem, which generalize to a family of spin manifolds $M \to S$, we can then define the family index $\hat{A}(M) \in KO^{-n}(S)$. In the language of homotopy theory, all of these stories are encoded in a map of ring spectra called the ABS orientation MSpin $\to$ KO. We can generalize to the cases of E-oriented manifolds where the family genus defines a class $\phi(M) \in E^*(S)$, the key is to define certain analytic index which also lives in this group, where it is the map of ring spectra $\phi : MG \to E$ relates two sides of the story.

A.1.8 Elliptic genera and elliptic cohomology

Definition A.1.40. (Elliptic genus)

An genus $\varphi : \Omega_*^{SO} \longrightarrow R$ (R is a $\mathbb{Q}$-algebra.) is elliptic if its logarithm

$$g(x) = \sum_{n \geq 0} \frac{\varphi(\mathbb{CP}^{2n})}{2n+1} x^{2n+1}$$

is an elliptic integral

$$g(x) = \int_0^x \frac{dt}{\sqrt{R(t)}}$$

with $R(t) = 1 - 2\delta t^2 + \epsilon t^4 \quad (\delta, \epsilon \in R)$.

The formal group law of elliptic genus comes from the group structure of the Jacobi quartic elliptic curve. Assume $\delta, \epsilon \in \mathbb{C}$ and the discriminant $\Delta = \epsilon(\delta^2 - \epsilon)^2 \neq 0$. We define

$$\log_J(x) := \int_0^x \frac{dt}{\sqrt{1 - 2\delta t^2 + \epsilon t^4}} = \int_0^x \frac{dt}{\sqrt{R(t)}}$$

Theorem A.1.41.

$$F_J(x_1, x_2) = \frac{x_1\sqrt{R(x_2)} + x_2\sqrt{R(x_1)}}{1 - \epsilon x_1^2 x_2^2}.$$

Genesis of elliptic genera vaishes. Moreover, As graded rings,

$$\Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[x_4, x_8, x_{12}, x_{16}, \cdots].$$

with generators $x_4 = [\mathbb{CP}^2]$, $x_8 = [\mathbb{HP}^2]$, $x_{4i} = [\mathbb{CP}^{2i}], i > 2$. Obviously $(x_{12}, x_{16}, \ldots) \subset I_*$.

Conjecture A.1.42. $I_* = (x_{12}, x_{16}, \ldots)$ under the above isomorphism.

One has

$$\varphi(\mathbb{P}(V^{2m})) = 0,$$

Since $\varphi(\mathbb{CP}^2) = \delta$ and $\varphi(\mathbb{HP}^2) = \epsilon$ for any elliptic genus, it follows that $I_* = (x_{12}, x_{16}, \ldots)$. In fact, an equivalent definition of elliptic genus is the requirements on vanishment of any $\mathbb{CP}(V^{2m})$.

For an oriented manifold M with a complex vector bundle $V \to M$, build

$$S_q(V) = \sum_{n \geq 0} S^n(V) q^n \in K(X)[\![t]\!], \quad \Lambda_q(V) = \sum_{n \geq 0} \Lambda^n(V) q^n \in K(X)[\![t]\!].$$

from the symmetric and exterior powers of T, and then write

$$R(V) := \bigotimes_{l=1}^{\infty} S_{q^l}(V) \otimes \bigotimes_{l=1}^{\infty} \Lambda_{q^l}(V) =: \sum_{k \geq 0} R_k(V) q^k.$$

Then one finds that

$$\begin{array}{ll} R_0 = 1 & R_2 = \Lambda^2 T + T \\ R_1 = -T & R_3 = -(\Lambda^3 T + T \otimes T) \end{array}$$

In fact, the characteristics appeared earlier that vanish on $\mathbb{CP}(V^{2m})$ are exactly

$$\hat{A}(M^n)\mathrm{ch}(R_k(TM^{\mathbb{C}} - \mathbb{C}^n)) =: \rho_k(M^n).$$

Indeed, we get the so-called universal elliptic genus

$$\varphi_{LS} : \Omega_*^{SO} \longrightarrow \mathbb{Q}[\![q]\!], \quad [M] \mapsto \sum_{k\geq 0} \langle \rho_k(M), [M] \rangle q^k.$$

It bears the name since the its formal group law is the one given by Euler in Theorem A.1.41.

Similarily, we can define the following genus which is not elliptic, now known as the Witten genus

$$\varphi_W(M) := \left\langle \hat{A}(M)\mathrm{ch}\left(\bigotimes_{l=1}^{\infty} S_{q^l}(TM^{\mathbb{C}} - \mathbb{C}^n)\right), [M^n] \right\rangle \in \mathbb{Q}[\![q]\!].$$

Integrality, modularity and index theory

Theorem A.1.43. *(Chudnovsky, Zagier)*

The universal elliptic genus φ_{LS} maps to modular form for

$$\Gamma_0(2) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2\,\mathbb{Z} \middle| \; c\mathit{even} \right\}$$

When M is a spin manifold, then $\varphi_{LS}(M)$ and $\varphi_W(M)$ both have integer coefficients. This can be deduced from their definitions as power series with coefficients equal to the indices of twisted Dirac operators. But, where does the modularity come from? Index theory again, but this time as equivariant index theorem. When M is a spin manifold, Witten formally defined the signature operator on the free loop space $\mathcal{L}M$, and he showed its S^1-equivariant index equals φ_{LS}. The modularity of it can be physically deduced as being the index of signature operator.

Theorem A.1.44. *When M is a string manifold, φ_{LS} maps to integral modular form, i.e.*

$$\varphi_W(M) \in \mathrm{MF}_*\,.$$

in general, $\varphi_W(M)$ lives in the ring of almost modular forms

$$\widehat{\mathrm{MF}_*} = \mathbb{Q}[G_2, G_4, G_6] \supset \mathrm{MF}_* \,.$$

The characteristic series associated to the Witten genus

$$q_W(x,\tau) = \exp\left(2\sum_{k\geq 2} G_{2k}(\tau)\frac{x^{2k}}{(2k)!}\right)$$

This is also $q_W(x) = x/\sigma(x)$, where $\sigma(x,\tau)$ is the Weierstrass σ-function. The full modularity of Witten genus of string manifolds and almost modularity in the general case can be explained via characteristic series. Let $q = e^{2\pi i\tau}$ and $u = e^x$, the characteristic series has the product expansion

$$q_W(u,q) = \frac{x/2}{\sinh(x/2)} \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n u)(1-q^n u^{-1})} e^{-G_2(\tau)x^2}.$$

The last term contributes nothing exactly when $(p_1/2)(M) = 0$ and the rest is exactly the characteristic series of a twisted $\hat{A}$-genus that have integer coefficients according to the index theorem.

Elliptic cohomology and orientation As we have already seen, the integrality of $\hat{A}$-genus of spin manifolds through Atiyah-Singer index theorem can be encoded in the orientation map $MU \to K$ between spectra. It is the natural to ask whether there is also an orientation map in the case of elliptic genera, in particular φ_{LS}, and the Witten genus φ_W? The answer is yes, the target spectra are elliptic cohomology and topological modular form respectively. Let's first focus on elliptic genera and elliptic cohomologies.

Theorem A.1.45. *(Conner-Floyd)*

$$K^*(M) \cong MU^*(M) \otimes_{MU^*} \mathbb{Z}[\![\beta, \beta^{-1}]\!].$$

A.1.9 Chern-Weil theory and differential cohomology

In this section, we introduce the general Chern-Weil theory. Let's now consider a principal G-bundle $\pi : P \to M$ with connection ∇ and curvature 2-form $F_\nabla \in \Omega^2_M(\mathfrak{g})$. Choose a $\mathfrak{g}$-invariant polynomial g of degree k: $g \in \mathrm{Sym}^k(\mathfrak{g}^\vee)^G$. We can construct a $2k$-form out of this polynomial as follows. First, we pullback the curvature 2-form to the total bundle P, then apply g on its k-power, i.e. $g((\pi^* F_\nabla)^{\wedge k}) \in \Omega^{2k}_P(\mathbb{R})$. Since g is an invariant polynomial, it descends to a $2k$-form $g(F_\nabla) \in \Omega^{2k}_M(\mathbb{R})$ on the base space which we call the Chern-Weil form. Similar to the trace powers, it is a closed form according to the Bianchi identity. In fact, more is true.

Theorem A.1.46. *(Chern-Weil)*

- $g(F_\nabla)$ *is closed and its de Rham class* $[g(F_\nabla)] \in H^{2k}_{\mathrm{dR}}(M)$ *is independent of* ∇*,*
- *The Chern-Weil map cw:* $\mathrm{Sym}^k(\mathfrak{g}^*)^G \to H^{2k}_{\mathrm{dR}}(M)$ *is a functorial ring homomorphism.*

Example A.1.47. (Chern classes)

For complex vector bundles, we have previously seen that the total Chern class is given by

$$c(F) = \det\left(1 - \frac{iF}{2\pi}\right) \in H^{2*}(M;\mathbb{Z}).$$

Expand this we get the following

$$\begin{aligned} c_1(F) = &\ \tfrac{\mathrm{tr}(F)}{2\pi i}, \\ c_2(F) = &\ -\tfrac{1}{8\pi^2}(\mathrm{tr}\, F \wedge \mathrm{tr}\, F - \mathrm{tr}(F \wedge F)), \\ c_n(F) = &\ \left(\tfrac{1}{2\pi i}\right)^k \det F. \end{aligned}$$

Example A.1.48. (Pontrjagin classes)

For real vector bundles $V \to M$, the total Pontrjagin class is the total Chern class of its complexification, equivalently, we can define

$$p(F) = \det\left(1 - \frac{F}{2\pi}\right) \in H^{4*}(M;\mathbb{Z}).$$

Some lower degree examples are

$$\begin{aligned} p_1(F) = & -\tfrac{1}{8\pi^2}\operatorname{tr}(F \wedge F), \\ p_2(F) = & \tfrac{1}{128\pi^4}(\operatorname{tr}(F \wedge F)^2 - 2\operatorname{tr}(F^4)). \end{aligned}$$

Example A.1.49. (Chern character)

The chern character is also defined by

$$\operatorname{ch}(F) = \operatorname{tr}\exp\left(\frac{F}{2\pi i}\right) = \sum_k \frac{1}{k!}\operatorname{tr}\left(\frac{F}{2\pi i}\right)^k \in H^{2*}(M;\mathbb{Q}).$$

The k-th Chern character $\operatorname{ch}_k(F)$ is the degree $2k$ component $\frac{1}{k!}\operatorname{tr}\left(\frac{F}{2\pi i}\right)^k$.

Chern-Simons invariant

Theorem A.1.50. *(Gauss-Bonnet-Chern)*

Let M^{2n} be a closed Riemannian manifold with curvature F. Then we have

$$\int_M p_\chi(F, \dots, F) = \chi(M),$$

where p_χ is the Pfaffian polynomial.

Proof. Choose an arbitrary unit vector field v with isolated zeros on M. We can naturally construct a connection 1-form A locally which satisfies

$$\textstyle\int_{\{\mathbb{S}_{m_i}\}} \frac{1}{2\pi} A \quad = \operatorname{ind}_{m_i}(v),$$

where $\mathbb{S}_{m_i}$ is the unit sphere surrounding a zero m_i. Here the index is defined as the degree of the map $u : \partial\mathbb{D} \to \mathbb{S}^{n-1}$ given by $u(z) = v(z)/\|v(z)\|$. It is a classical theorem of Poincar□ and Hopf that

$$\sum_i \operatorname{ind}_{m_i}(v) = \chi(M).$$

Moreover, from unit vector field v, we have the associated sphere bundle $\pi : S(M) \to M$ with fibers $\mathbb{S}^{2n-1}$. One can show that there is a $(2n-1)$-form $Tp_\chi(F, A)$ on $S(M)$ such that

- $\pi^* p_\chi(F, \dots, F) = \mathrm{d}Tp_\chi(F, A)$,

- $\int_{\mathbb{S}^{2n-1}} Tp_\chi(F, A) = 1$.

$Tp_\chi(F, A)$ is called the trangression of p_χ. We can then easily deduce the theorem:

$$\begin{aligned} \int_M p_\chi(F) = \lim_{t\to 0} \int_{M-\{\mathbb{D}^t_{m_i}\}} p_\chi(F) &= \lim_{t\to 0} \int_{v^*(M-\{\mathbb{D}^t_{m_i}\})} \pi^* p_\chi(F) \\ &= \lim_{t\to 0} \int_{v^*(M-\{\mathbb{D}^t_{m_i}\})} \mathrm{d}Tp_\chi(F, A) \\ &= \lim_{t\to 0} \int_{\Sigma \deg(m_i)\mathbb{S}^t_{m_i}} Tp_\chi(F, A) \\ &= \sum_i \mathrm{ind}_{m_i}(v) \end{aligned}$$

and by Poincar□-Hopf Theorem we get desired formula. □

Trangression can generally be obtained by considering tautological pullback, since $\pi^* P \to P$ is trivial

$$\pi^*(g(F_A)) = \mathrm{d}Tg(F, A).$$

The closed $(2k-1)$ -form on P is given by

$$Tg(A) = \int_{[0,1]} g(A, \varphi_t^{k-1})\mathrm{d}t,$$

where $\varphi_t = tF_A + (t^2 - t)A \wedge A$.

Example A.1.51. (Chern-Simons invariant)

Taking $G = \mathrm{SO}(3)$, M^3 oriented, and $g = p_1$. Then $p_1(F_A) = 0$, i.e. $\mathrm{d}Tp_1(A) = 0$. Consider the frame bundle $F(M) \to M$ which is trivializable, take a global section s, the integral

$$\mathrm{CS}(M) := \int_M s^* Tp_1(A) \mod \mathbb{Z}$$

is a conformal invariant of the manifold called the *Chern-Simons* invariant. Moreover, the necessary condition for M^3 being able to be conformally immersed into $\mathbb{R}^4$ is

$$\mathrm{CS}(M) = 0.$$

Chern-Simons invariant is a case of secondary characteristic class in the sense that it distinguishes manifolds even if the primary invariant, i.e. p_1 vanishes.

When not working on the total bundle, such global formula does not exist anymore.

In the spirit of the trangression form, we have a generally the Chern-Simons form which is defined only locally. Consider two connections A_1 and A_2 on P. The Chern-Simons form is

$$\mathrm{CS}_g(A_1, A_2) = \int_{[0,1]} g(A, F_t^{k-1})\mathrm{d}t,$$

where $F_t = \mathrm{d}A_t + A_t \wedge A_t$, $A_t = tA_1 + (1-t)A_2$. We get the trangression formula:

$$g(F_{A_1}) - g(F_{A_2}) = \mathrm{dCS}_g(A_1, A_2).$$

Different choice of the conections only differs by an exact term. Therefore, we take its class in $\Omega^{2k-1}(M)/\operatorname{im}(\mathrm{d})$ and abuse the same notation $\mathrm{CS}_g(A_1, A_2)$ also referring to this class. In particular, we can take (A_1, A_2) to be $(A, 0)$ and denote it by $\mathrm{CS}_g(A)$.

Example A.1.52. Let's take g to be the k-th Chern character ch_k and denote its Chern-Simons form as CS_{2k-1}. Then we have

$$\begin{aligned}
\mathrm{CS}_1(A) = &\ \tfrac{i}{2\pi}\operatorname{tr} A, \\
\mathrm{CS}_3(A) = &\ -\tfrac{1}{8\pi^2}\operatorname{tr}\left(A\mathrm{d}A + \tfrac{2}{3}A^3\right), \\
\mathrm{CS}_5(A) = &\ -\tfrac{i}{48\pi^3}\operatorname{tr}\left(A(\mathrm{d}A)^2 + \tfrac{3}{2}A^3\mathrm{d}A + \tfrac{3}{5}A^5\right).
\end{aligned}$$

The second one is the most familiar Chern-Simons form in the literature.

Differential cohomology So far, we have seen many examples of the Chern-Weil form coming from integral descent of certain charactersitic classes, is there a general story? Recall that a cohomology class $c \in H^{2k}(BG; \mathbb{Z})$ determines a characteristic class. Therefore, we try to do the Chern-Weil construction on the classifying space, we get an isomorphism

$$\mathrm{Sym}^k(\mathfrak{g}^*)^G \cong H^{2k}(BG; \mathbb{R}),$$

compatible with the graded ring structrue on $H^*(BG; \mathbb{R})$. However, the Chern-Weil construction involves the choice of a connection ∇. This suggest we need to consider the classifying stack $B_\nabla G$ of principle G-bundles with connection.

Now let $c_\mathbb{R}$ be the image of c in $H^{2k}(BG; \mathbb{R})$, which is isomorphic to say $f : M \to B_\nabla G$

gives our principle G-bundle P with connection ∇, then Chern-Weil theorem says

$$f^* c_{\mathbb{R}} = g(F_\nabla).$$

This suggests a more refined cohomology which is not $H^{2k}(B_\nabla G;\mathbb{Z}) \cong H^{2k}(BG;\mathbb{Z})$ as the pullback. This is the differential cohomology. And the full statement goes like this: there is a unique class $\hat{c} \in \hat{H}^{2k}(B_\nabla G;\mathbb{Z})$ that gives $f^*c \in H^{2k}(M;\mathbb{Z})$ which lifts the Chern-Weil form $g(F_\nabla) \in \Omega^{2k}_{\text{cl}}(M)$ and complement the Chern-Simons form $\mathrm{CS}_g(A) \in \Omega^{2k-1}(M)/\operatorname{im}(\mathrm{d})$, i.e. $[f^*c]_{\mathbb{R}} = [g(F_\nabla)] \in H^{2k}(M;\mathbb{R})$. In the case of $g = p_1$, we also have secondary characteristic classes - Chern-Simons invariant $\mathrm{CS}(M) \in H^{2k-1}(M;\mathbb{R}/\mathbb{Z})$ which is more refined than the usual characteristic classes discussed in previous sections. All of these indicates we are seeking for $\hat{H}^{2k}(M;\mathbb{Z})$ such that

$$\begin{tikzcd}
 & H^{2k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "Bock"] \arrow[dr] & & H^{2k}(M;\mathbb{Z}) \arrow[dr, "\otimes\mathbb{R}"] & \\
H^{2k-1}(M;\mathbb{Z}) \arrow[ur] \arrow[dr] \arrow[rr, phantom, ">"] & & H^{2k}(M;\mathbb{Z}) \arrow[ur, "I_{\mathrm{CS}}"] \arrow[dr, "R_{\mathrm{CS}}"'] \arrow[rr, phantom, ">"] & & H^2k(M;\mathbb{R}) \\
 & \Omega^{2k-1}(X)/Im(d) \arrow[ur, "a_{\mathrm{CS}}"] \arrow[rr, "d"] & & \Omega^{2k}_{\text{cl}}(X)_{\mathbb{Z}} \arrow[ur, "\mathrm{dR}"'] &
\end{tikzcd}$$

the digram is commutative and diagonal sequences are exact. We call $\hat{H}^{2k}(M;\mathbb{Z})$ satisfies these a differential cohomology. Here is a model of ordinary differential cohomology.

Definition A.1.53. Differential Characters (Cheeger-Simons)

For $n \geqslant 0$, the group of *differential characters* $\hat{H}^n_{\text{CS}}(X;\mathbb{Z})$ on X is the abelian group consisting of pairs (ω, k), where

- A closed differential form $\omega \in \Omega^n_{\text{clo}}(X)$,
- A group homomorphism $\varphi : Z_{\infty,n-1}(X;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$,

such that for any $c \in C_{\infty,n}(X;\mathbb{Z})$ we have

$$\varphi(\partial c) = \int_c \omega \mod \mathbb{Z}$$

Here $C_{\infty,n}(X;\mathbb{Z})$ and $Z_{\infty,n-1}(X;\mathbb{Z})$ denote the smooth singular chains and cycles with integer coefficients and automatically implies $\omega \in \Omega^n_{\text{cl}}(X)_{\mathbb{Z}}$.

Theorem A.1.54. *We have homomorphisms*

$$\begin{array}{l}
R_{\text{CS}} : \hat{H}^n_{\text{CS}}(X;\mathbb{Z}) \to \Omega^n_{\text{cl}}(X), \quad (\omega,\varphi) \mapsto \omega, \\
a_{\text{CS}} : \Omega^{n-1}(X)/\text{Im}(d) \to \hat{H}^n_{\text{CS}}(X;\mathbb{Z}), \quad \alpha \mapsto (d\alpha, \int \alpha \mod \mathbb{Z}), \\
I_{\text{CS}} : \hat{H}^n_{\text{CS}}(X;\mathbb{Z}) \to \hat{H}^n_{\text{CS}}(X;\mathbb{Z})/\text{Im}(a_{\text{CS}}) \simeq H^n(X;\mathbb{Z}), \quad (\omega,\varphi) \mapsto [\omega - \varphi_{\mathbb{R}} \circ \partial].
\end{array}$$

such that the the following digram is commutative and diagonal sequences are exact.

$$\begin{tikzcd}
 & H^{n-1}(X;\mathbb{R}/\mathbb{Z}) \arrow[rr, "Bock"] \arrow[dr] & & H^n(X;\mathbb{Z}) \arrow[dr, "\otimes\mathbb{R}"] & \\
H^{n-1}(X;\mathbb{Z}) \arrow[ur] \arrow[dr] & & \hat{H}^n(X;\mathbb{Z}) \arrow[ur, "I_{\text{CS}}"] \arrow[dr, "R_{\text{CS}}"'] & & H^n(X;\mathbb{R}) \\
 & \Omega^{n-1}(X)/Im(d) \arrow[rr, "d"] \arrow[ur, "a_{\text{CS}}"] & & \Omega^n_{\text{cl}}(X)_{\mathbb{Z}} \arrow[ur, "\text{dR}"'] &
\end{tikzcd}$$

Example A.1.55. An example of differential character is constructed from Chern-Simons invariants. The basic setting is let $(E,\nabla) \to X$ be a hermitian vector bundle with connection. For $f : M^3 \to X$ with M closed oriented, set

$$\begin{array}{rl}
\text{CS}(E,\nabla)(f : M \to X) := & \text{CS}(f^*E, f^*\nabla) \\
= & \int_M f^* \,\text{Tr}(\text{d}A \wedge A + \frac{2}{3} A \wedge A \wedge A)(\text{modZ}).
\end{array}$$

The second Chern character form is

$$\text{ch}_2(F_A) = \text{Tr}((dA \wedge A + A \wedge A)^2) \in \Omega^4_{\text{cl}}(X).$$

We get

$$(\text{ch}_2(F_A), \text{CS}(E,A)) \in \hat{H}^4(X;\mathbb{Z}).$$

Actually, the definition of the Chern-Simons invariants uses $\widehat{H\mathbb{Z}}^*$s

Let M^{n-1} closed oriented $(n-1)$-manifold. We have differential extension integration maps denoted by $\int_M$,

$$\int_M : \hat{H}^n(M;\mathbb{Z}) \to \hat{H}^1(M;\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}, \quad (\omega,\varphi) \mapsto \varphi(1_M)$$

Generally, for fiber bundle $p : N \to X$ with oriented fibers, we have

$$\int_{N/X} : \hat{H}^n(N;\mathbb{Z}) \to \hat{H}^{n-r}(X;\mathbb{Z}),$$

where $r = \dim N - \dim X$. Differential integration is a refinement of integrations in $H\mathbb{Z}^*$ and Ω^*.

Proposition A.1.56. *Suppose $(W^{2n}, \partial W)$ is a compact oriented manifold, for any $\hat{x} \in \hat{H}^n(W;\mathbb{Z})$, we have*

$$\int_{\partial W} \hat{x} \mid_{\partial W} \equiv \int_W R(\hat{x}) \mod \mathbb{Z}$$

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