

## Higher-periodic homotopy types through Lubin–Tate towers

In their recent work, Barthel, Schlank, Stapleton, and Weinstein determined the periodic stable homotopy groups of the sphere spectrum rationally. They made an essential use of the structure afforded by perfectoid spaces for computing relevant group cohomology in the framework of condensed mathematics. These spaces appear in an equivariant isomorphism between two towers: (1) the Lubin–Tate tower that parametrizes deformations of a formal group of fixed height with level structures and (2) the Drinfeld tower that parametrizes those for shtukas. I’m obliged to introduce this exciting mathematical landscape to the greater “perfection” community, and appeal for further insights and collaborations. This also includes: (a) my ongoing joint work with Guozhen Wang which computes unstable higher-periodic homotopy types integrally, (b) Xuecai Ma’s spectral realization of finite levels of the Lubin–Tate tower as non-even commutative ring spectra, which generalize Morava, Hopkins, Miller, Goerss, and Lurie’s spectra at the ground level, and (c) Hongxiang Zhao’s work which connects Ando’s norm through homotopical descent of level structures along the tower, to Coleman’s norm in the context of Lubin and Tate’s explicit local class field theory.

<https://bicmr.pku.edu.cn/content/show/17-3377.html>

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Yifei Zhu

Southern University of Science and Technology

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## Context and motivations

**$p$ -adic geometry**



**topology**

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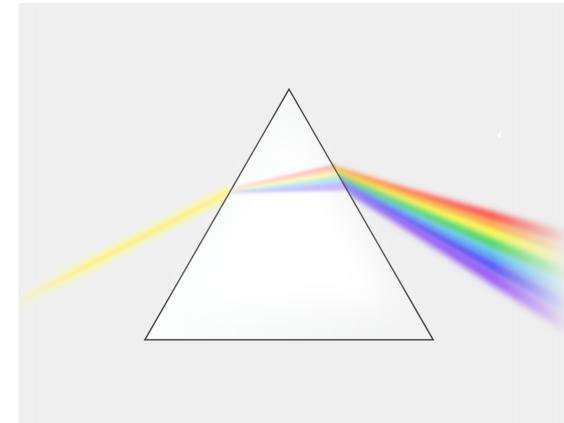


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e.g., Tate diagonal  
cyclotomic spectra à la Nikolaus and Scholze  
Liu–Wang, Jingbang Guo

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In this talk, we focus on some of the recent progresses in the **forward direction**, but the ***p*-adic geometry** involved, as well as the relevant **tools**, may be of independent interest beyond topology.

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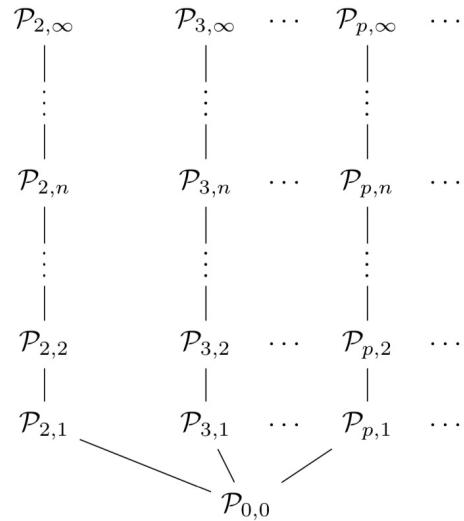
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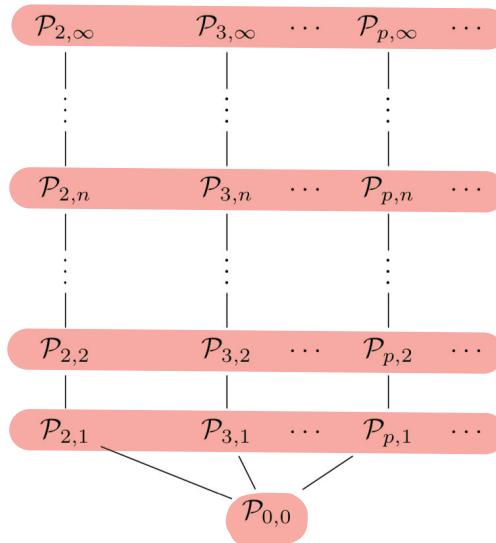
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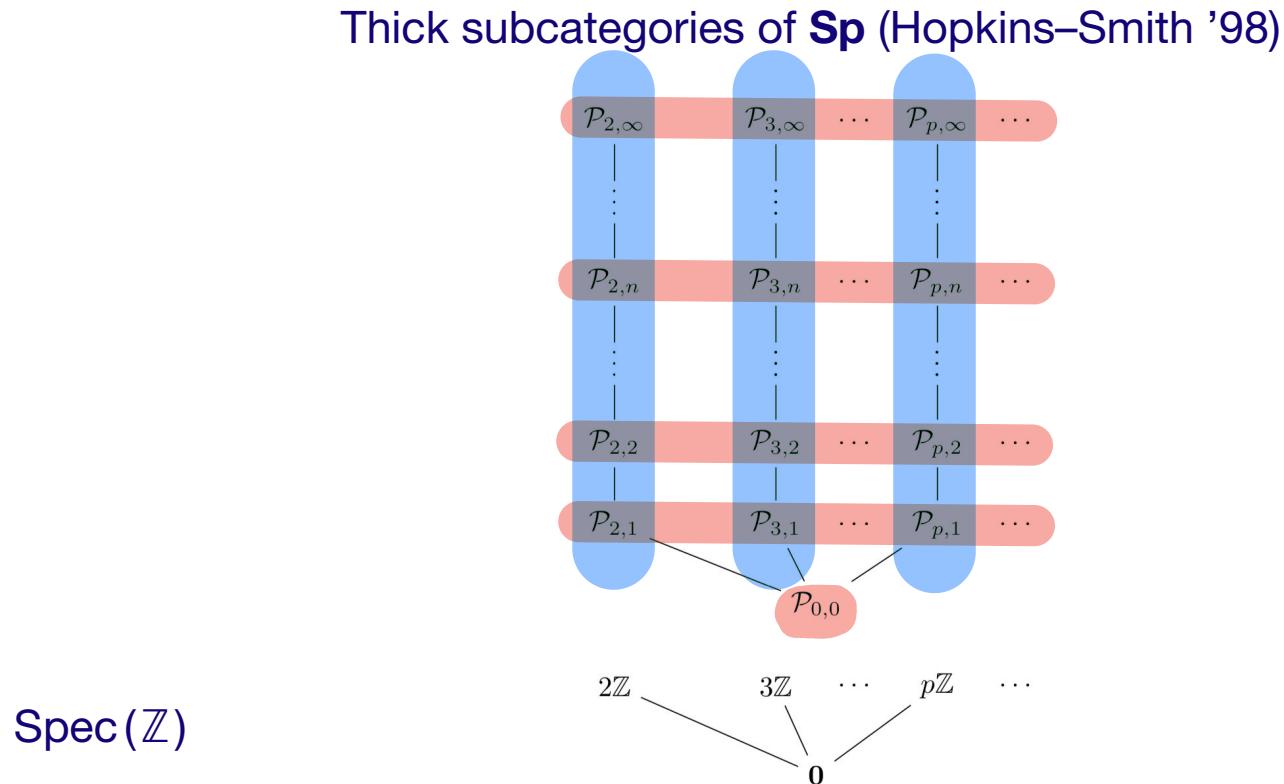
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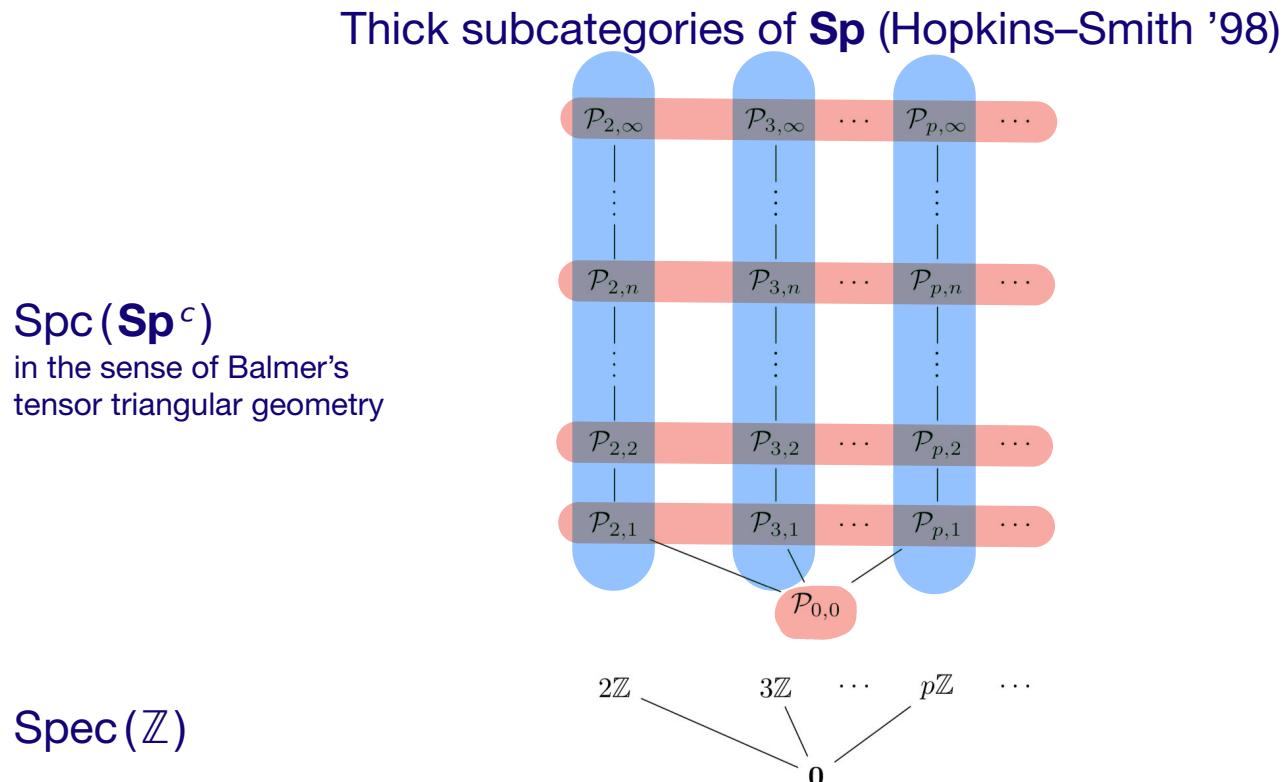
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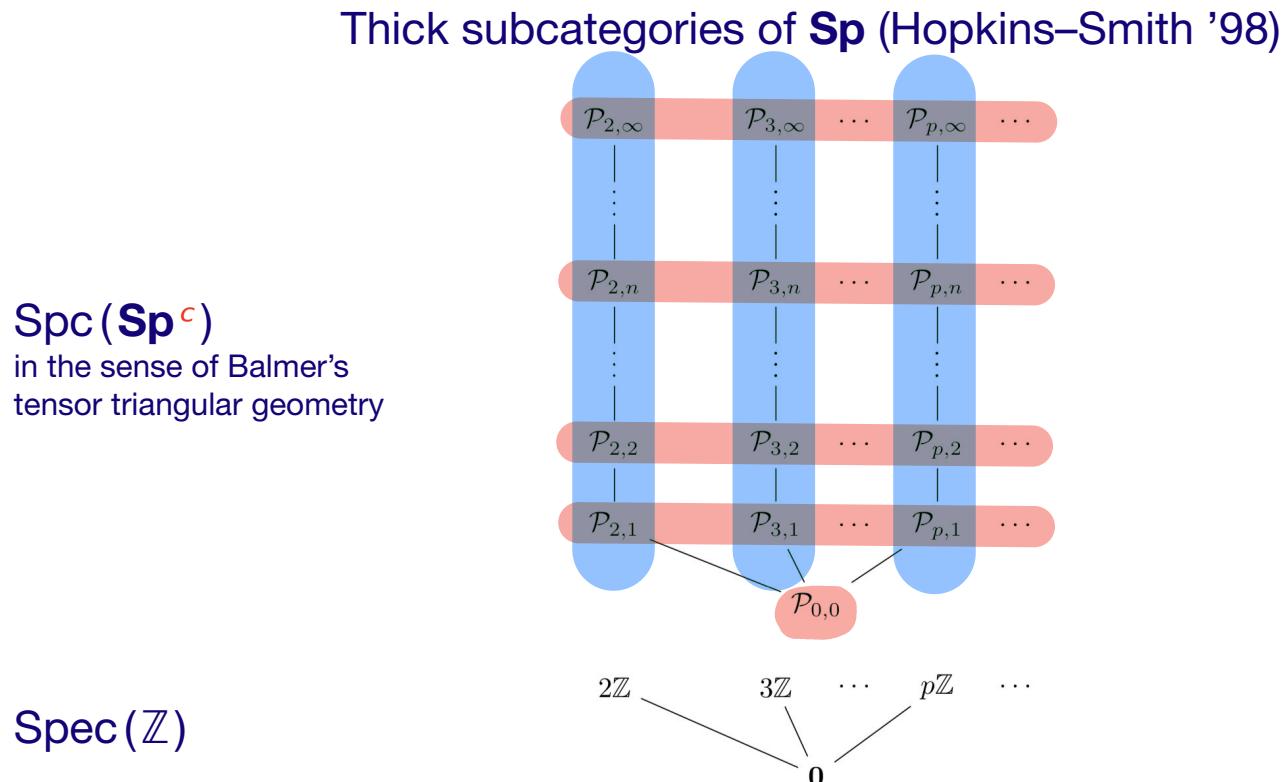
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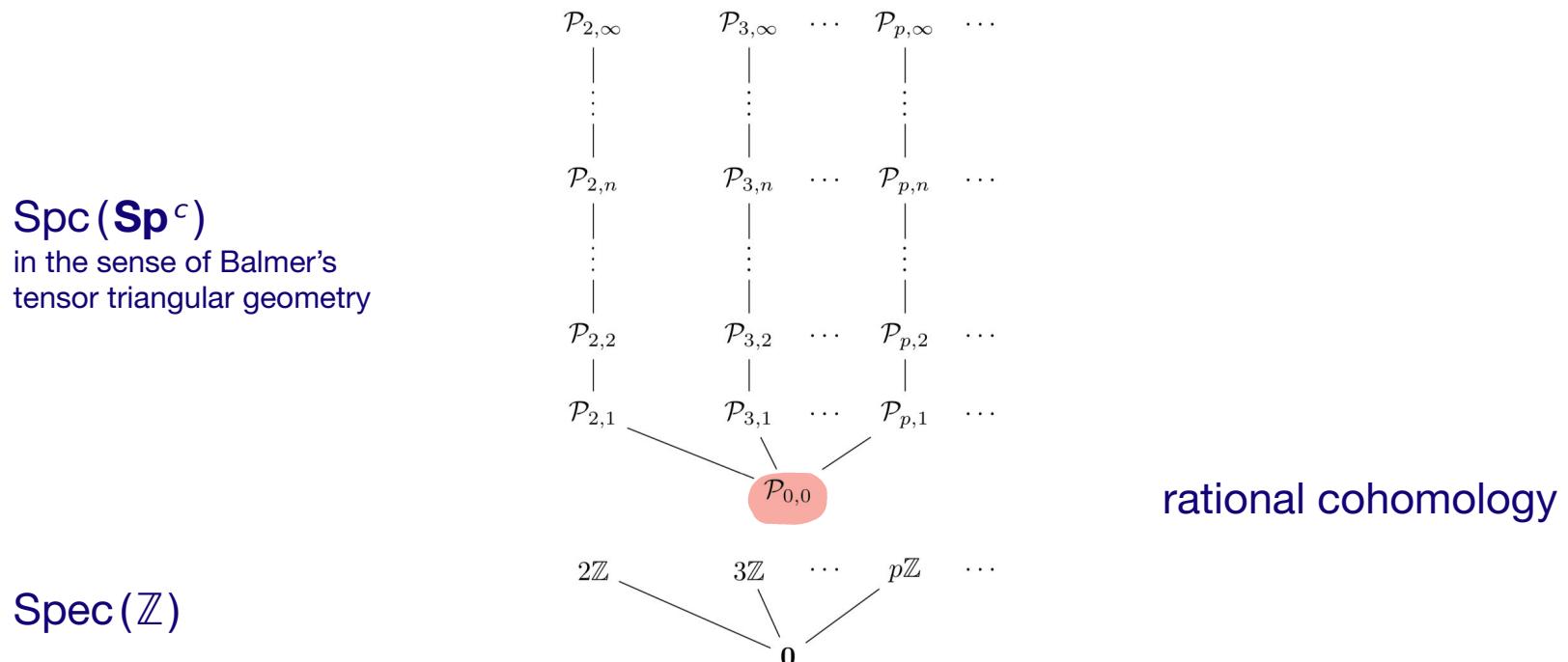


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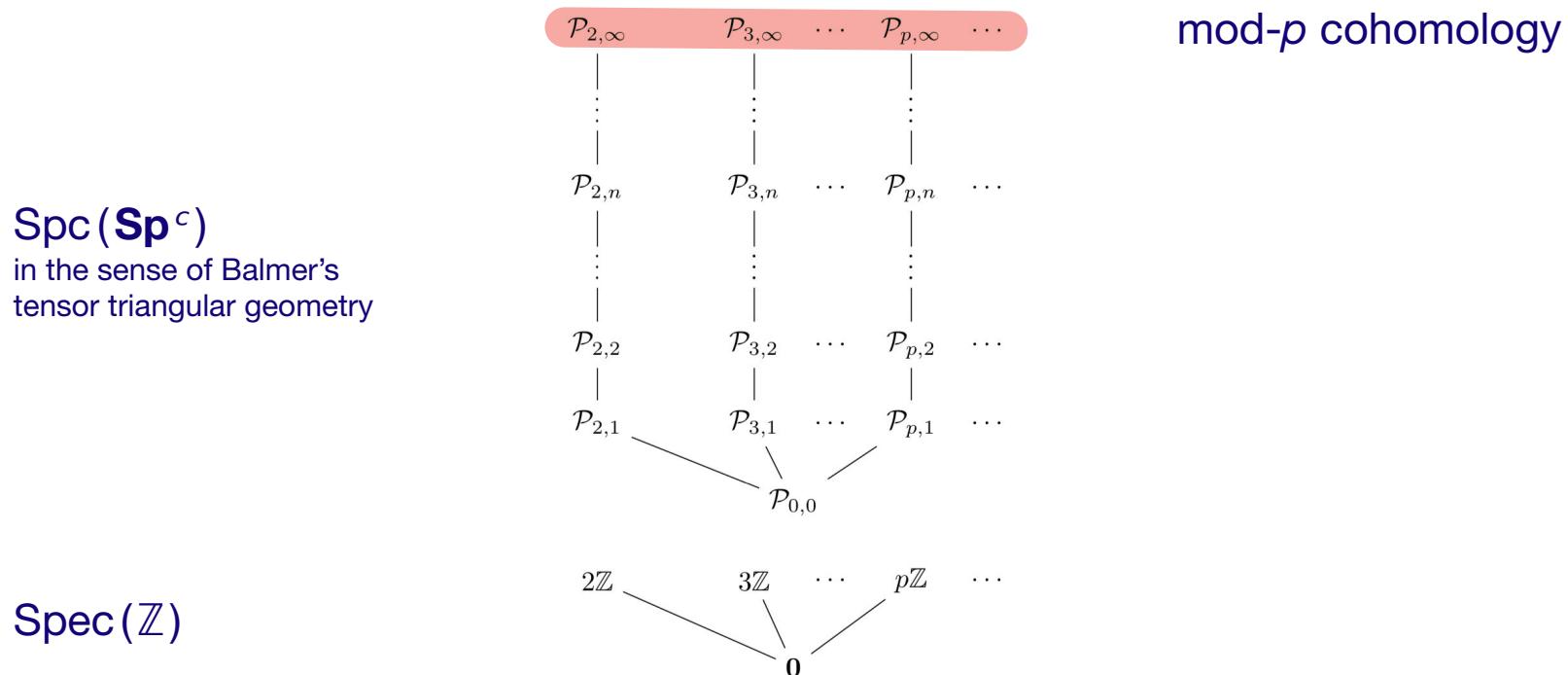


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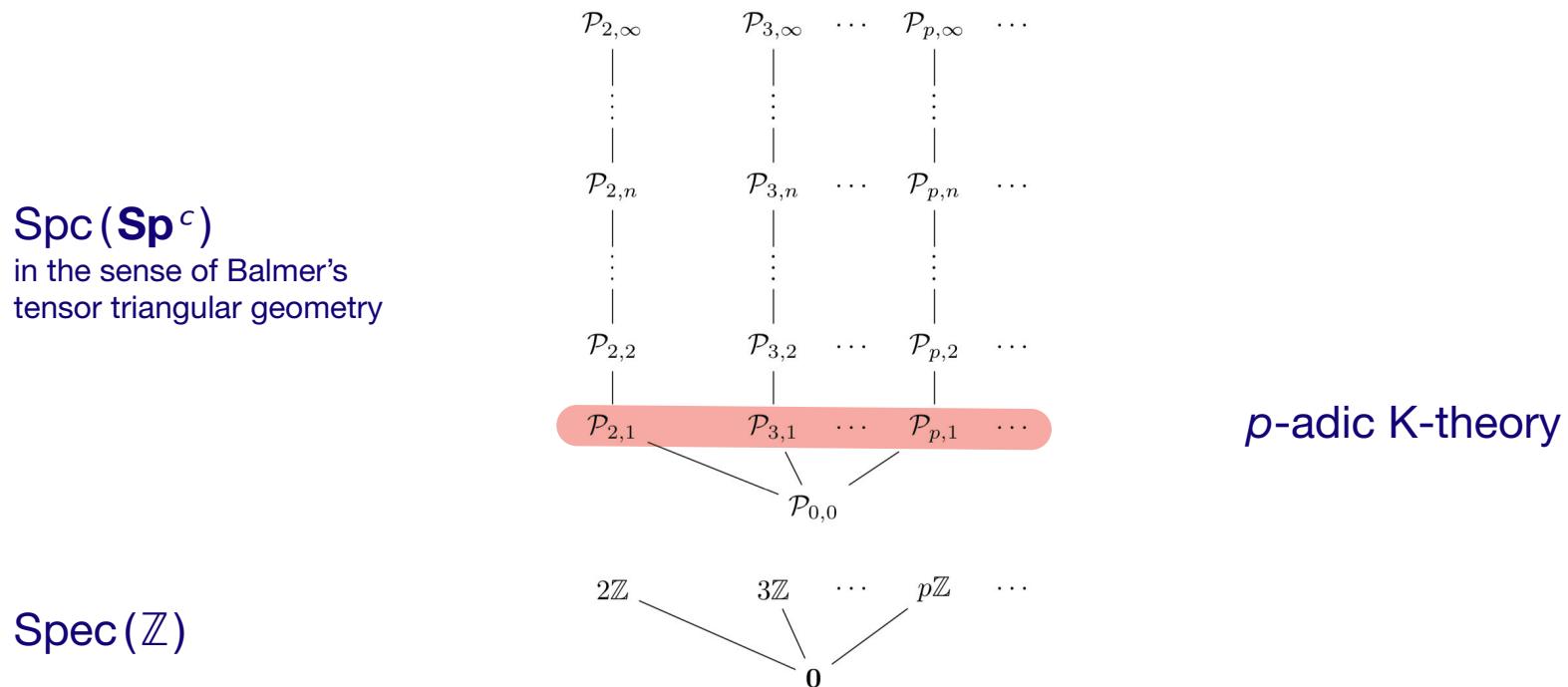


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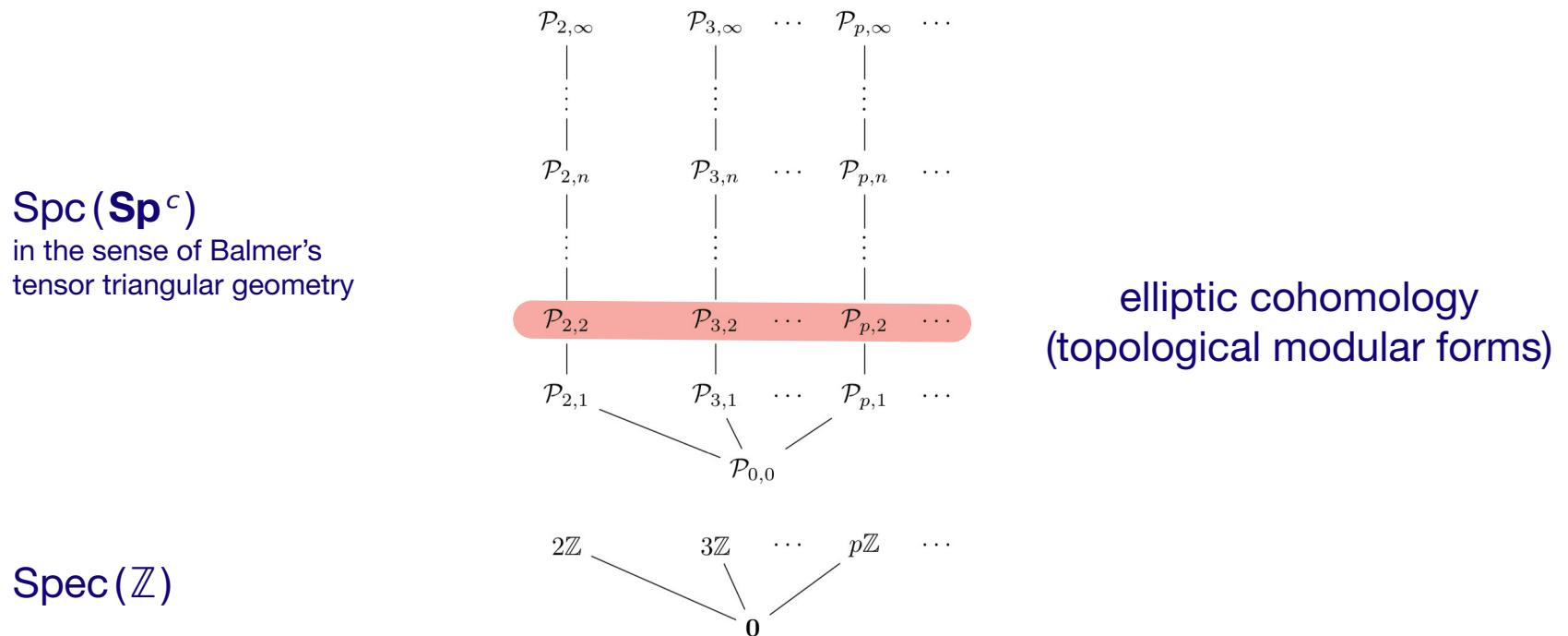


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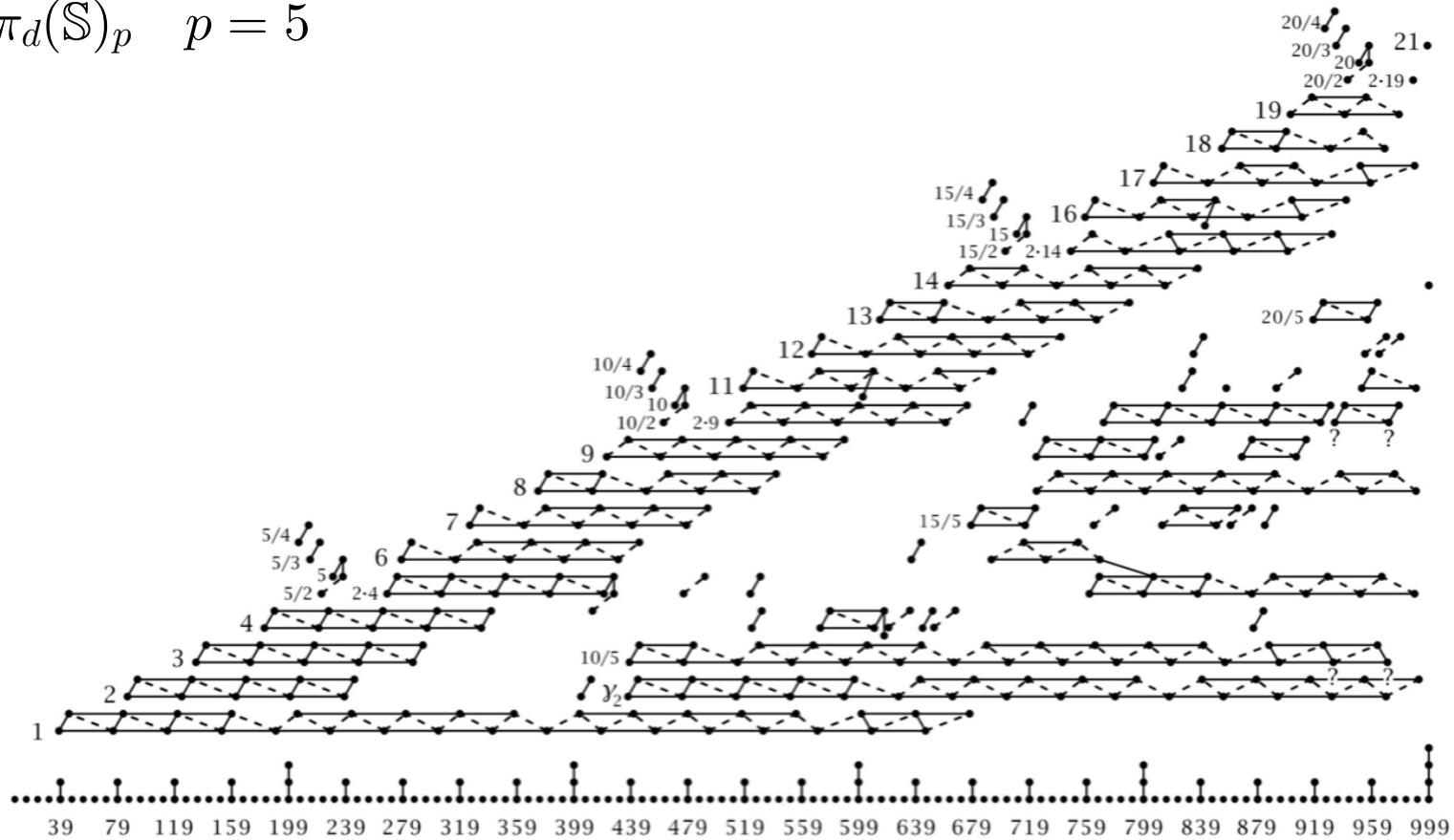
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**Spec( $\mathbb{Z}$ )**

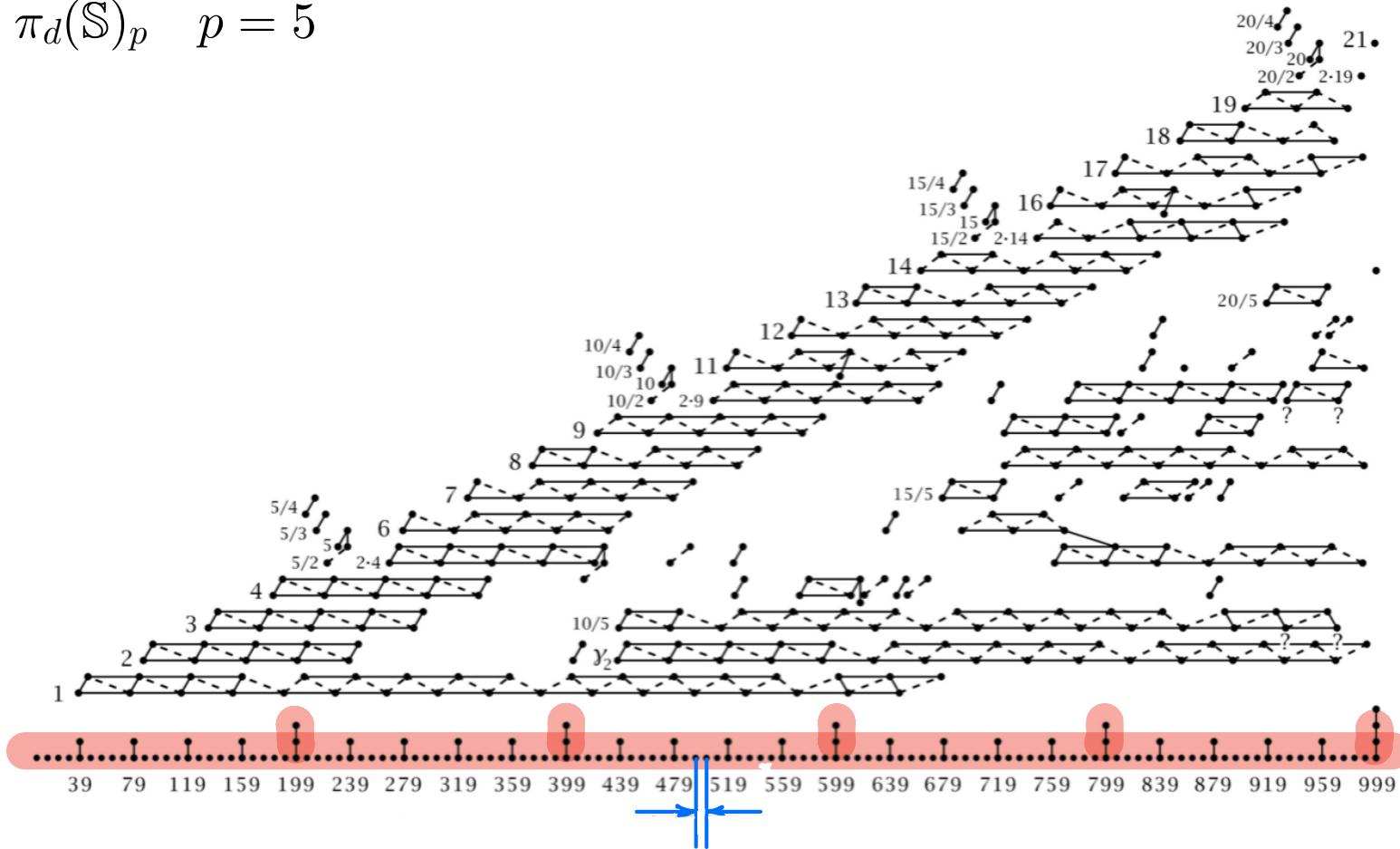
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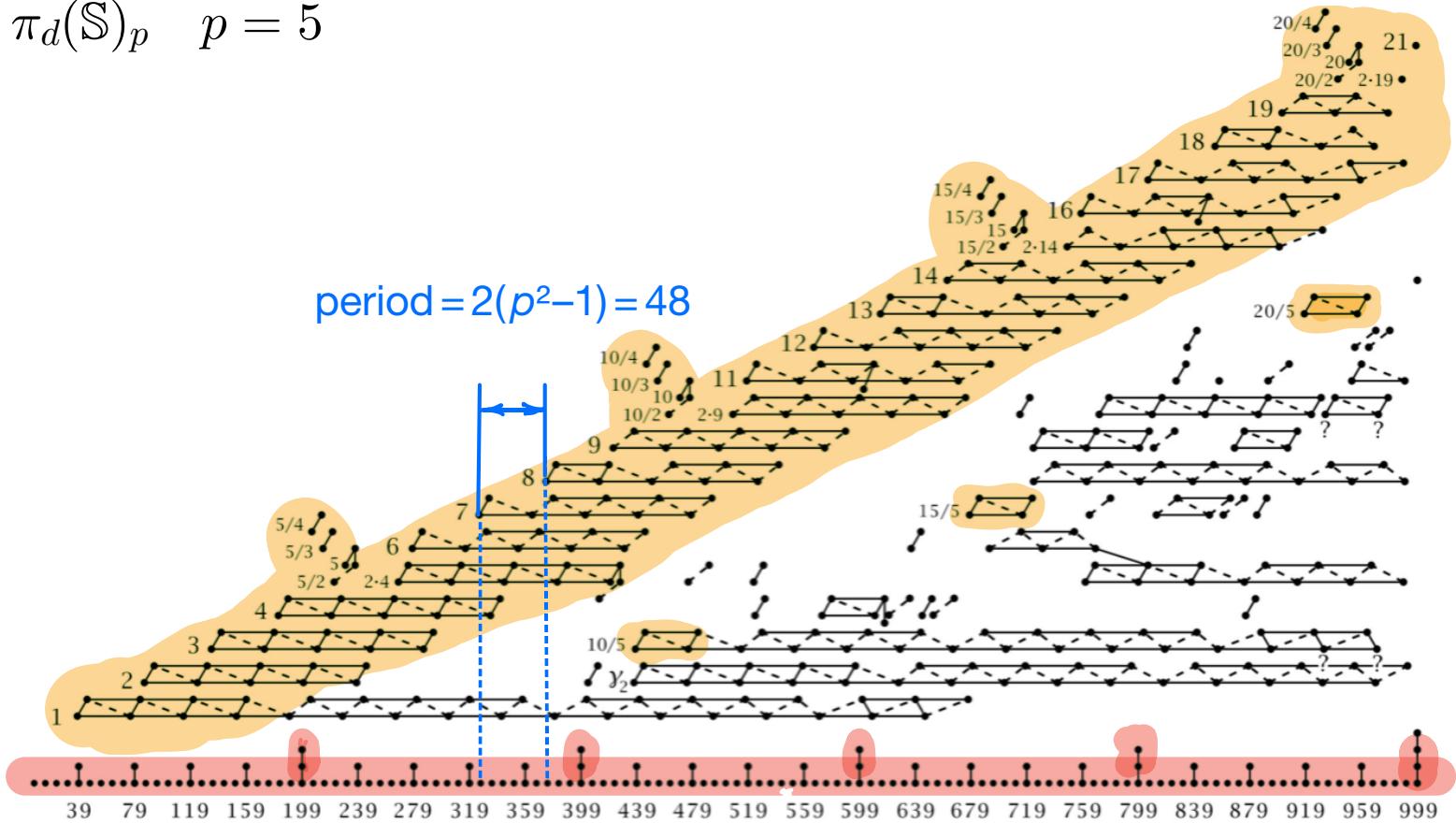


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$$\text{period} = 2(p-1) = 8$$

$v_1$ -periodic elements

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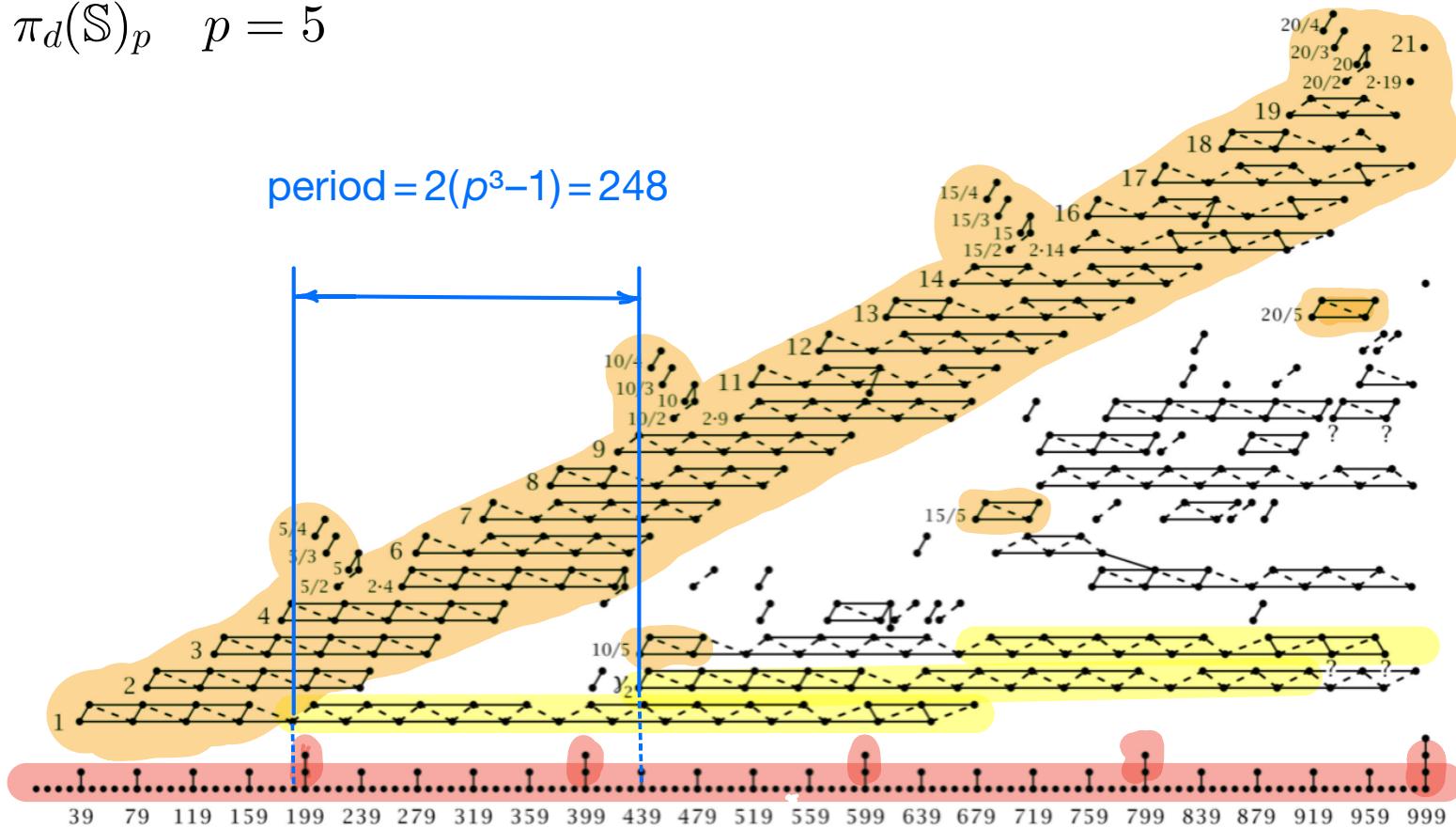


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$v_2$ -periodic elements

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period =  $2(p^3 - 1) = 248$

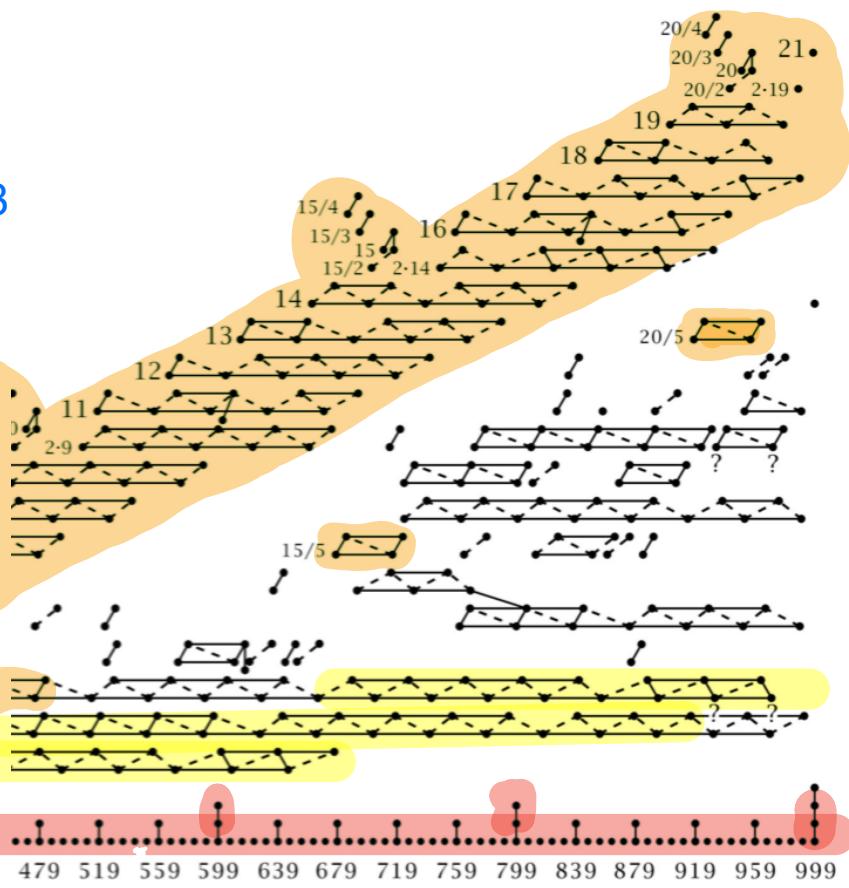


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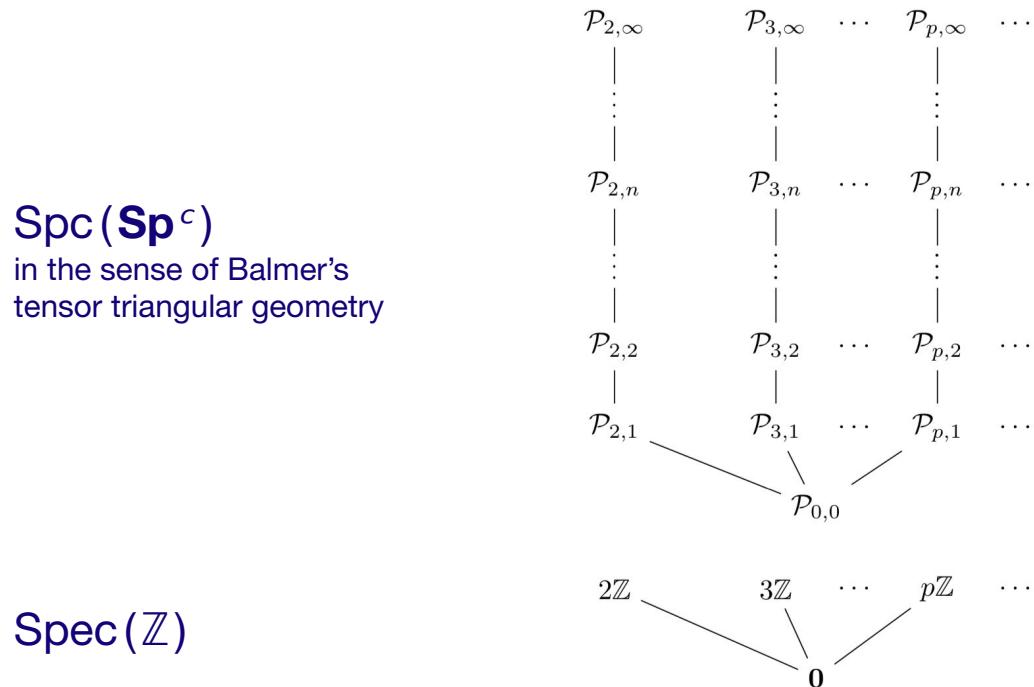


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Key ingredients in their proof

- The Devinatz–Hopkins homotopy-fixed-point spectral sequence (**descent spectral sequence**) computing the homotopy groups of the  $K(n)$ -local sphere

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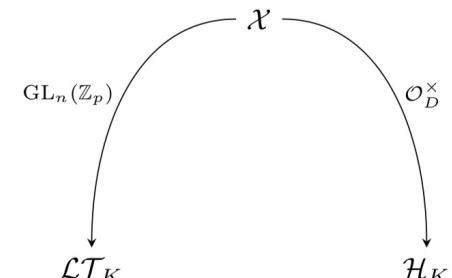
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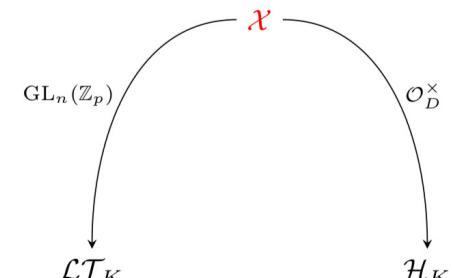
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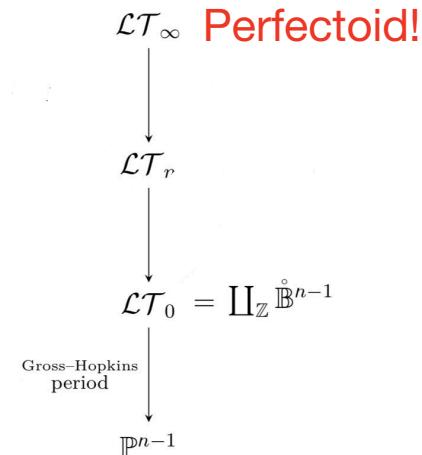
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- [Lubin–Tate ’66] for  $r = 0$ , by an affine formal scheme  $\mathrm{Spf} A_0$  of dimension  $n - 1$  over the Witt ring  $\mathcal{O}_F$  of  $k$  (with  $F = \max.$  unr. ext. of  $p$ -adic comp. of  $k$ ).
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[Goerss–Hopkins–Miller, Strickland ’98] These moduli spaces are (partially) spectrally realized by the Morava E-theory spectrum  $E = E_n(G/k)$ , i.e.,

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[Fargues ’08, Weinstein ’15] As a family of spaces indexed by  $r$ , they stack into a tower with all levels literally:



# Analytic geometry and homotopy groups of the $K(n)$ -local sphere

Recall moduli spaces for deformations of a formal group with level structures.

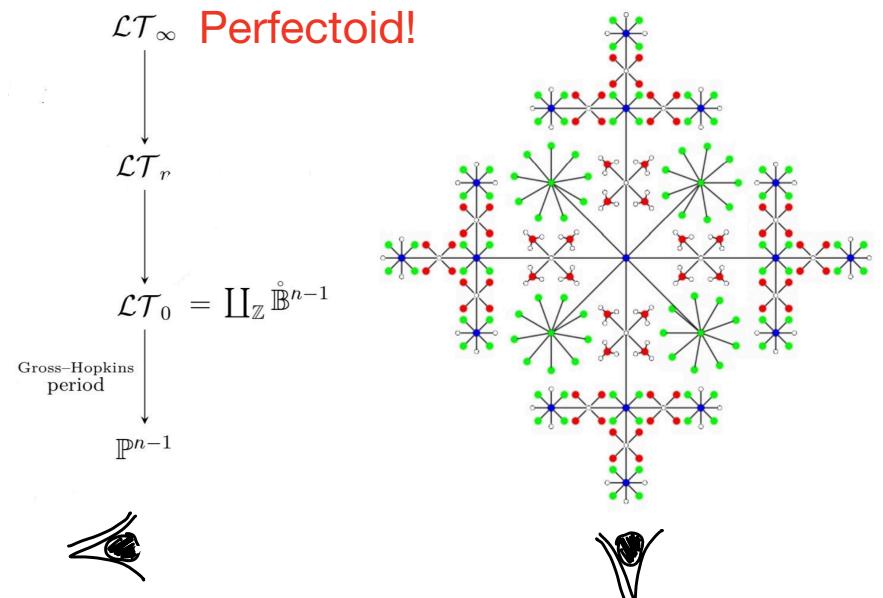
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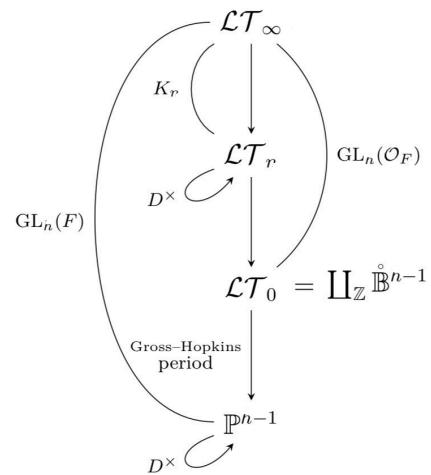
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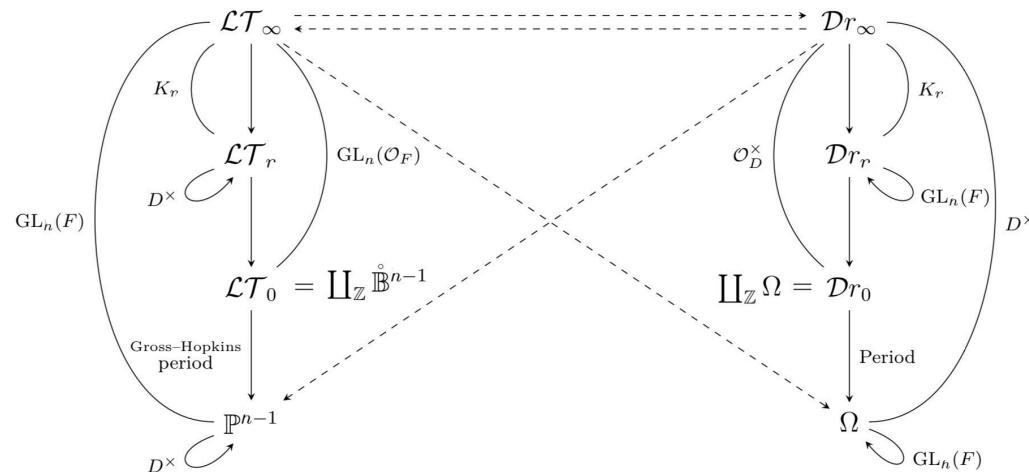
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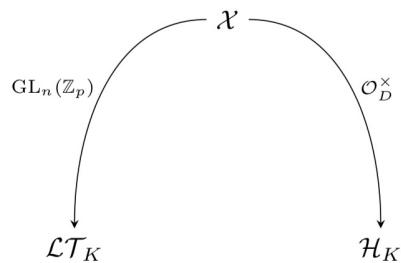
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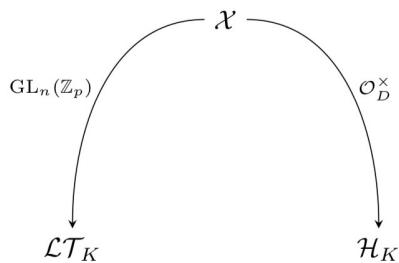
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This diagram induces an isomorphism in  $D(\text{Solid})$ :

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In general, given a pro-étale  $G$  torsor  $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$ , we have  $R\Gamma(Y, \widehat{\mathcal{O}}_{\text{cond}}^+)^{hG} \cong R\Gamma(X, \widehat{\mathcal{O}}_{\text{cond}}^+)$ .

Here taking homotopy fixed points of  $R\Gamma(\mathcal{X}_K, \widehat{\mathcal{O}}_{\text{cond}}^+)^{h(\text{GL}_n(\mathbb{Z}_p) \times \mathcal{O}_D^\times)}$  in two orders yields the above isomorphism.

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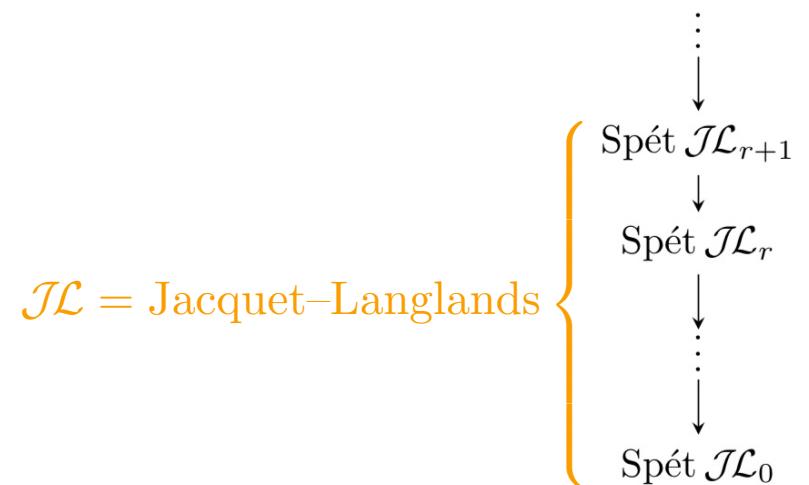
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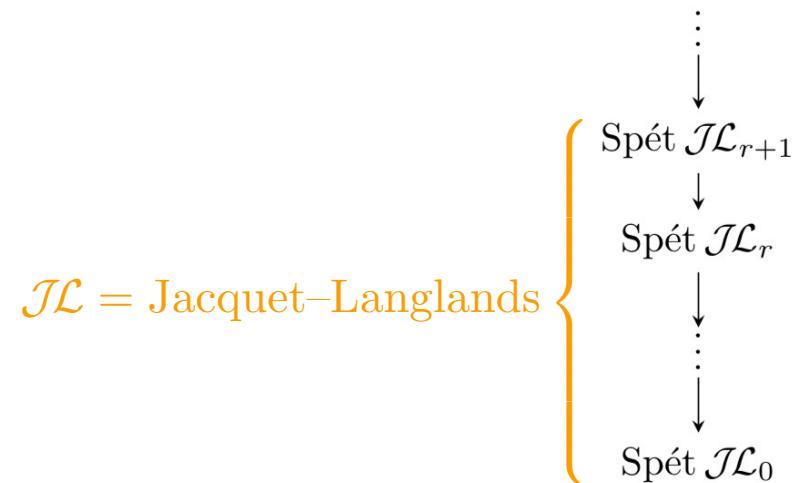
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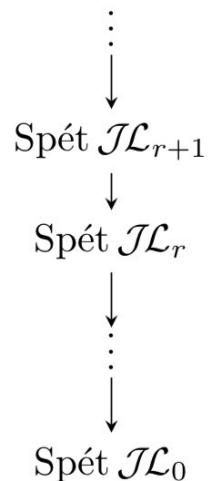
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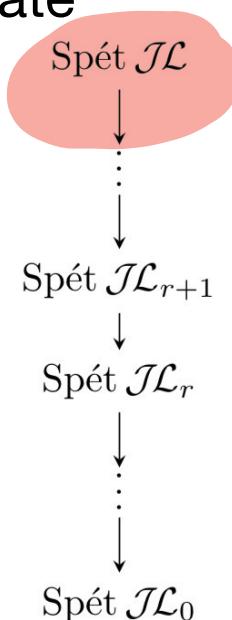


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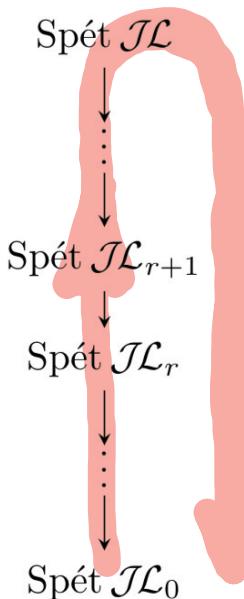


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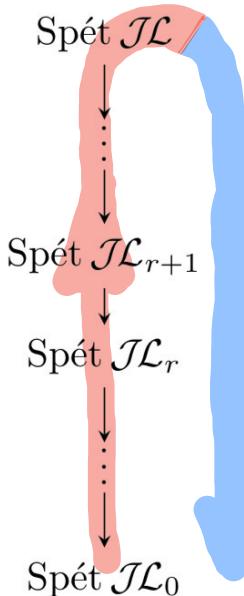


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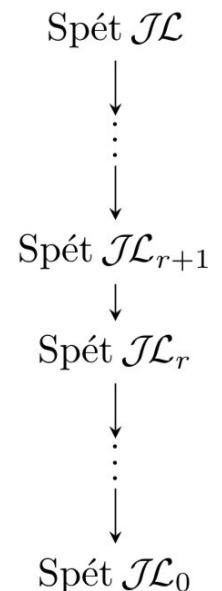
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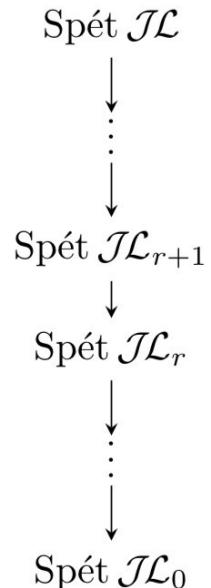
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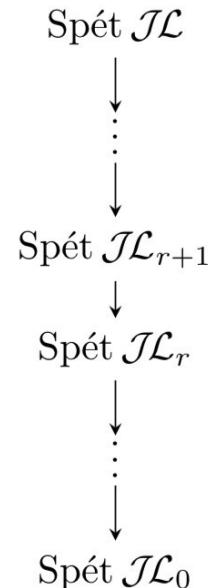
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[Hongxiang Zhao '23] Zhao gave explicit evidence that, along Lubin–Tate towers, certain norm operators from **local class field theory** have spectral realizations.

*Thank you.*



## Afterthoughts

Mathematics has become so developed and specialized, that the heavy notations of individual subfields make it feel as if each small, distinguished group of people set a rule, a roadmap for themselves, their friends and descendants to play within. That said, we must not give up the effort to increase the understandability of mathematics across subfields, convey the essential ideas, learn and get inspirations from each other, so as to make the better of it for the future.