Annalen der Physik. 7. Folge, Band 41, Heft 2, 1984, S. 88–92 J. A. Barth, Leipzig

Mass Scales and the Cosmological Coincidences

By P. T. LANDSBERG

Faculty of Mathematical Studies, University, Southampton, U.K.

Abstract. Theories involving the parameters $\hbar$, c, G, H (in a usual notation) are considered. A huge ratio of 10^{120} of the mass of the universe (m_u) to the smallest determinable mass m_0 in the period since the big bang occurs in such theories. Five masses are here identified and interpreted between these two limits so that one has in all seven analytical expressions for masses. They form a geometrical progression m_0, m_0R, ..., $m_0R^6 = m_u$ with $R \approx 10^{20}$. It is shown that this formulation is easily adapted to explain existing cosmological coincidences and to generate new ones.

Über die kosmische Bedingtheit einer Massenskala

Inhaltsübersicht. Es werden kosmologische Theorien diskutiert, in denen neben der Planckschen Konstante h, der Lichtgeschwindigkeit c und der Gravitationskonstante G auch noch der Hubble-Parameter H eingeht. Für solche Kosmen wird eine Massen-Scala hergeleitet, die einer geometrischen Progression, mit dem Eddingtonschen Faktor 10^{20} entspricht.

The dimensional analysis of masses which occur in a theory involving the velocity of light (c), Newton's gravitational constant (G), Planck's constant ($\hbar$) and the Hubble parameter (H) leads to the following possible masses for this theory (LANDSBERG and BISHOP 1975, LANDSBERG and EVANS 1977):

$$m(b) = \left[\frac{\hbar^3 H}{G^2}\right]^{1/5} \left[\frac{c^5}{\hbar H^2 G}\right]^{b/15}. \tag{1}$$

Here b is a parameter. Two observations (A) and (B), based on (1), were made:

(A) $m(-1) = (\hbar^2 H/Gc]^{1/3} \sim 10^{-25}$ g (pion mass)

$m(3/2) = (\hbar c/G)^{1/2} \sim 10^{-4\cdot 7}$ g (Planck mass)

$m(9) = c^3/HG \sim 10^{56\cdot 4}$ g (mass of the universe).

To this was added the smallest mass which can be measured since the big bang:

$$m(-6) = \hbar H/c^2 \sim 10^{-65\cdot 7} \text{ g}.$$

(B) Cosmological coincidences can be explained in terms of relation (1).

Dealing with (A) first, it will be pointed here that the whole range $-6 \leq b \leq 9$ contains seven specific basic masses which are in geometrical progression. The three new ones are asterisked in Table 1, below. It will be seen that they are essential to make up a geometrical progression, thus defining six distinct mass ranges. These three masses are the following.

First,

$$m\,(6\tfrac{1}{2}) \equiv \left(\frac{\hbar c^{13}}{G^5 H^4}\right)^{1/6} \sim 10^{3 \cdot 4}\ \text{g}. \tag{2}$$

To find this mass consider the lower limit on the density of matter so that it is just relativistic, $\varrho \sim m^4 c^3/\hbar^3$ with $m = m(-1)$. For such matter to be in a star of mass M whose radius is its Schwarzschild radius $2GM/c^2$, M must be of the order of equation (2) (Appendix I). Alternatively, consider again the above value of the density

$$\varrho \sim \left(\frac{H^4 c^5}{\hbar G^4}\right)^{1/3} \sim 2.65 \times 10^{12}\ \text{g cm}^{-3}.$$

This large density would go up by a factor of 2000 if the nuclear mass instead of the pion mass were used in ϱ. In any case, for such densities the velocity of sound (v) may be replaced by the velocity of light and the Jeans mass (Appendix II)

$$M_J \sim \frac{9}{4\sqrt{(2\pi)}} \frac{v^3}{G^{3/2} \varrho^{1/2}}$$

goes over into (2). Looked at in a slightly different way, the Jeans mass, referring to baryon stars, has the form (TREDER 1982, 1983)

$$[m(3/2)]^3/m^2. \tag{3}$$

If one substitutes $m(-1)$ for m in relation (3) one again finds (2).

Secondly,

$$m(4) = \left(\frac{\hbar c^4}{G^2 H}\right)^{1/3} \sim 10^{15 \cdot 7}\ \text{g}. \tag{4}$$

This is the mass of a black hole whose lifetime is the Hubble time. It has been discussed recently in a different context (SIVARAM, 1982). A black hole of mass M has a life time (see, for example LANDSBERG, 1978) $5 \times 4^5 \pi G^2 M^3/\hbar c^4 \sim H^{-1}$ whence one finds relation (4).

Thirdly,

$$m(-3\tfrac{1}{2}) = \left(\frac{\hbar^5 H^4}{c^7 G}\right)^{1/6} \sim 10^{-45 \cdot 4}\ \text{g}. \tag{5}$$

The interpretation of this relation presents some difficulty. It is of the order of magnitude originally suggested by de Broglie as an upper limit for the photon rest mass. But there was a numerical error here which, when corrected, yielded an upper limit of 10^{-39} g (GOLDHABER and NIETO 1971). Although a non-zero rest mass for the photon (baryonic or leptonic, OKUN', 1982) is still quite feasible (PRIMACK and SHER, 1980; VISSER, 1982; GEORGI et al., 1983), its upper limit has now been reduced to 10^{-16} eV or 10^{-48} g from the Pioneer-10 observations on Jupiter's magnetic field (DAVIS et al. 1975) and this is smaller than (5). Neutrino masses of order 1 eV to 100 eV (10^{-33} to 10^{-31} g) have also been considered (e.g. FREESE, KOLB and TURNER, 1983). But values down to zero are possible, so that (5) could represent the mass of a neutrino.

It is convenient in view of these findings to replace the original parameter b by

$$a \equiv \frac{2}{5}(b + 1). \tag{6}$$

The geometrical progression of masses now takes the form given in Table 1. The ratio between successive masses which define the six mass scales is

$$R \equiv (c^5/\hbar H^2 G)^{1/6} \sim 10^{20} \tag{7}$$

from relation (1), which becomes, using the notation $M(a)$ instead of $m(b)$,

$$M(a) = \left(\frac{\hbar^2 H}{cG}\right)^{1/3} \left(\frac{c^5}{\hbar H^2 G}\right)^{a/6}. \tag{8}$$

Table 1. The basic masses

a	b	mass
−2	−6	smallest mass, m_0
*−1	−7/2	{neutrino mass upper limit for the photon mass, μ
0	−1	pion mass, m_π
1	3/2	Planck mass, m_{Pl}
* 2	4	Black hole with Hubble lifetime, m_b
* 3	13/2	Jeans mass for cloud of nuclear density, m_J
4	8	mass of universe, m_u

Turning to (B), it is clear from (7) and the three new masses that many new "cosmological coincidences" result. Let us first recall perhaps the oldest of these, namely that the age of the universe (T) in atomic units satisfies (see, for example, WESSON 1978)

$$T \sim \sqrt{N}, \tag{9}$$

where N is the number of particles in the observable universe. To explain (9) in terms of (8), replace the electron mass m_e in T by $M(0)$ and, using the fine-structure constant, replace e^2 in T by $\hbar c$. Hence

$$T \equiv \frac{m_e c^3}{e^2 H} \sim R^2.$$

Also

$$N \sim \frac{m_u}{m_\pi} = \frac{M(4)}{M(0)} = R^4 \sim T^2, \tag{10}$$

as required.

Another standard result is that the ratio of the electrical to the gravitational force in a hydrogen atom behaves also as $R^2 \sim T$. By a similar argument to the above one can easily prove this result, writing m_p for the proton rest mass:

$$\frac{e^2}{G m_p m_e} \sim \frac{\hbar c}{G M(0)^2} = \left[\frac{M(1)}{M(0)}\right]^2 = R^2. \tag{11}$$

A new coincidence involving the black hole mass is

$$m_u/m_b = m_b/m_\pi \sim \sqrt{N}. \tag{12}$$

Using (8) and Table 1 it is seen that (12) is equivalent to

$$\frac{M(4)}{M(2)} = \frac{M(2)}{M(0)}.$$

Both ratios have the form $R^2 \sim \sqrt{N}$, and this proves (12). Many other illustrations of this kind can be given and will be obvious to the reader.

It may be of wider interest that there are six mass ranges defined by seven basic masses. That the ranges are equal on a logarithmic scale, i.e. that the masses are in geometric progression, is an additional regularity which was perhaps not expected. However, on the important question whether this kind of progression is of fundamental significance, and what it might be, little can be said at this point. If a deeper significance can be given to it, then of course a major advance in our understanding will be accomplished.

The author is grateful to Professor H.-J. TREDER, Einstein-Laboratorium, Caputh, for helpful correspondence.

Appendix: Derivation of Two Formulae

In this Appendix we give a simple derivation of two formulae used in the main text.

I. The Lower Limit for Relativistic Densities

In order to obtain the expression for ϱ used in connection with equation (2), note that the maximum momentum of a fermion in a degenerate gas is of order (e.g. LANDSBERG 1978, equation (15.12))

$$p_{\max}^2 = 2m\mu = \left(\frac{3\sqrt{\pi}N}{4gv}\right)^{2/3}\frac{\hbar^2}{\pi}$$

where g = spin degeneracy, $v = (4\pi/3)r_0^3$ is the volume and r_0 is the confinement radius. Thus

$$p_{\max}^2 = \frac{\hbar^2}{\pi}\,4\pi^2\cdot\pi^{1/3}\left(\frac{3N}{4g}\right)^{2/3}\left(\frac{3}{4\pi r_0^3}\right)^{2/3} = \frac{\hbar^2}{r_0^2}\left(\frac{9\pi N}{2g}\right)^{2/3}.$$

For the fermion matter to be relativistic, $p_{\max} > mc$, where m is the fermion rest mass, i.e. $\dfrac{1}{r_0}$ must be large enough. Hence the matter density is

$$\varrho \equiv \frac{m}{\left(\frac{4\pi}{3}\right)r_0^3} = \frac{3m}{4\pi r_0^3} \geq \varrho_r \equiv \frac{3m}{4\pi}\cdot\left(\frac{mc}{\hbar}\right)^3\frac{2g}{9\pi} = \frac{g}{2}\,\frac{m^4c^3}{3\pi^2\hbar^3}.$$

Thus ϱ_r is the lower limit for the density of nuclear matter to be relativistic.

Such matter within a Schwarzschild radius yields a mass

$$M = \varrho_r\cdot\frac{4\pi}{3}\left(\frac{2GM}{c^2}\right)^3.$$

Hence

$$M^2 = \frac{9\pi}{16g}\,\frac{c^3\hbar^3}{G^3m^4}.$$

Using the pion mass $\sim(\hbar^2H/Gc)^{1/3}$,

$$M = \left\{\frac{9\pi}{16g}\,\frac{c^3\hbar^3}{G^3m^4}\right\}^{1/2} = \frac{3}{4}\sqrt{\frac{\pi}{g}}\left(\frac{\hbar c^3}{G^5H^4}\right)^{1/6}$$

which is the required expression (2).

II. The Jeans Mass

Imagine a fluctuation of pressure (Δp) and matter density ($\Delta \varrho$) in a spherical volume of radius r. The resulting extra force per unit volume enclosed is $\Delta F = \Delta F_s + \Delta F_v$. Here ΔF_s is due to extra force on the boundary

$$\Delta F_s = 4\pi \Delta p r^2 \Big/ \frac{4\pi}{3} r^3 = 3\Delta p/r.$$

ΔF_v is due to the extra gravitational force

$$\Delta F_v = -\frac{G}{r^2}\left[(\varrho + \Delta\varrho)\,\frac{4\pi}{3} r^3\right]^2 \Big/ \frac{4\pi}{3} r^3 - \frac{G}{r^2}\left[\varrho\,\frac{4\pi}{3} r^3\right]^2 \Big/ \frac{4\pi}{3} r^3$$
$$= -\frac{8\pi}{3} G r \varrho \Delta\varrho.$$

The term in $(\Delta\varrho)^2$ has been neglected. Thus

$$\Delta F = \frac{a}{r} - br \qquad \left(a \equiv 3\Delta p,\; b \equiv \frac{8\pi}{3} G\varrho\Delta\varrho\right)$$

and leads to expansion for small r. The density increase thus tends to be ironed out. For large r there is a contraction and inhomogeneities tend to develop. The dividing line between these regimes occurs at the Jeans length

$$r_J \equiv \sqrt{\left(\frac{a}{b}\right)} = \frac{3v}{(8\pi\varrho G)^{1/2}}$$

where $v \equiv \sqrt{(\Delta p/\Delta\varrho)}$ is the speed of sound. The Jeans mass is obtained as

$$M_J = \frac{4\pi}{3} \varrho r_J^3 = \frac{9}{4\sqrt{(2\pi)}}\,\frac{v^3}{G^{3/2}\varrho^{1/2}}\,.$$

References

[1] Davis, L.; Goldhaber, A. S.; Nieto, M. M.: Phys. Rev. Lett. **35** (1975) 1402.
[2] Freese, K.; Kolb, E. W.; Turner, M. S.: Phys. Rev. **D 27** (1983) 1689.
[3] Georgi, H.; Ginsparg, P.; Glashow, S. L.: Nature **306** (1983) 765.
[4] Goldhaber, A. S.; Nieto, M. M.: Rev. Mod. Phys. **43** (1971) 277.
[5] Landsberg, P. T.: Thermodynamics and Statistical Mechanics, Oxford: University Press 1978, p. 296.
[6] Landsberg, P. T.; Bishop, N. T.: Phys. Lett **53 A** (1975) 109.
[7] Landsberg, P. T.; Evans, D. A.: Mathematical Cosmology: An Introduction, Oxford: University Press 1977, Appendix B.
[8] Okun', L. B.: Sov. Phys. JETP **56** (1982) 502.
[9] Primack, J. R.; Sher, M. A.: Nature **288** (1980) 680.
[10] Sivaram, C.: Am. J. Phys. **50** (1982) 279.
[11] Treder, H.-J.: Tensor **37** (1982) 93, equation (10).
[12] Treder, H.-J.: Gerlands Beitr. Geophys. **92** (1983) 99, in particular p. 103.
[13] Visser, M.: Phys. Lett B **109** (1982) 373.
[14] Wesson, P. S.: Cosmology and Geophysics. Bristol: Adam Hilger 1978.

Bei der Redaktion eingegangen am 23. Januar 1984.

Anschr. d. Verf.: Prof. P. T. Landsberg
University of Southampton
Faculty of Mathematical Studies
Southampton SO9 5NH UK