



The Sigma Orientation Is an H_{∞} Map

Author(s): Matthew Ando, Michael J. Hopkins, Neil P. Strickland

Reviewed work(s):

Source: *American Journal of Mathematics*, Vol. 126, No. 2 (Apr., 2004), pp. 247-334

Published by: [The Johns Hopkins University Press](#)

Stable URL: <http://www.jstor.org/stable/40067875>

Accessed: 02/11/2011 12:34

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at

<http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



The Johns Hopkins University Press is collaborating with JSTOR to digitize, preserve and extend access to
American Journal of Mathematics.

<http://www.jstor.org>

THE SIGMA ORIENTATION IS AN H_∞ MAP

By MATTHEW ANDO, MICHAEL J. HOPKINS, and NEIL P. STRICKLAND

Abstract. In an earlier paper, the authors constructed a natural map, called the *sigma orientation*, from the Thom spectrum $MU\langle 6 \rangle$ to any elliptic spectrum. $MU\langle 6 \rangle$ is an H_∞ ring spectrum, and in this paper we show that if (E, C, t) is the elliptic spectrum associated to the universal deformation of a supersingular elliptic curve over a perfect field of characteristic $p > 0$, then the sigma orientation is a map of H_∞ ring spectra.

1. Introduction. In [Hop95, AHS01], we introduced the notion of an *elliptic spectrum* and showed that any elliptic spectrum (E, C, t) admits a *canonical* $MU\langle 6 \rangle$ orientation

$$MU\langle 6 \rangle \xrightarrow{\sigma(E, C, t)} E$$

called the *sigma orientation* (see also §16). We conjectured that the spectrum TMF of “topological modular forms” of Hopkins and Miller admits an $MO\langle 8 \rangle$ orientation, such that for any elliptic spectrum (E, C, t) the diagram

$$\begin{array}{ccc} MO\langle 8 \rangle & \longrightarrow & TMF \\ \uparrow & & \downarrow \\ MU\langle 6 \rangle & \xrightarrow{\sigma(E, C, t)} & E \end{array}$$

commutes. For more about the conjecture, see [Hop95] and the introduction to [AHS01].

The conjecture seems now to be within reach, although that is the subject of another paper in preparation. The proof depends on the following feature of the sigma orientation, which was not proved in [AHS01]. Let C_0 be a supersingular elliptic curve over a perfect field k of characteristic $p > 0$, and let E be the even periodic ring spectrum associated to the universal deformation of the formal group of C_0 (so it is a form of E_2). The Serre-Tate theorem endows E with the structure

Manuscript received April 3, 2002.

Research supported in part by the NSF; research of the first author supported in part by NSF Grant DMS-0071482.

American Journal of Mathematics 126 (2004), 247–334.

of an elliptic spectrum (see §15.3), and so a map of ring spectra

$$(1.1) \quad \sigma: MU\langle 6 \rangle \rightarrow \mathbf{E}.$$

Goerss and Hopkins, building on work of Hopkins and Miller, have shown that \mathbf{E} is an E_∞ ring spectrum [GH02]; it is classical that $MU\langle 6 \rangle$ is. We need to know that the map (1.1) is an H_∞ map; we prove that in this paper. (The paper of Goerss and Hopkins has not yet been published. Our result depends only on the existence of the H_∞ structure, and so a cautious statement of the our result is that if \mathbf{E} is an H_∞ ring spectrum, then the map (1.1) is H_∞ . See Remark 12.13.)

In Part 1, we study the general problem of showing that an orientation

$$MU\langle 2k \rangle \xrightarrow{g} \mathbf{E}$$

is H_∞ , i.e. that for each n the diagram

$$(1.2) \quad \begin{array}{ccc} D_n MU\langle 2k \rangle & \xrightarrow{D_n g} & D_n \mathbf{E} \\ \downarrow & & \downarrow \\ MU\langle 2k \rangle & \xrightarrow{g} & \mathbf{E} \end{array}$$

commutes up to homotopy. Our analysis is based on [And95], which treats the case of $MU\langle 0 \rangle$, the Thom spectrum associated to $BU\langle 0 \rangle = \mathbb{Z} \times BU$. We review that case in §4, in a form which generalizes to $MU\langle 6 \rangle$.

Briefly, suppose that \mathbf{E} is a homotopy commutative ring spectrum with the property that $\pi_{\text{odd}} \mathbf{E} = 0$ and $\pi_2 \mathbf{E}$ contains a unit, so $\pi_0 \mathbf{E}^{\mathbb{C}P^\infty_+}$ is the ring of functions on a formal group $G = G_E$ over $S = \pi_0 \mathbf{E}$. If

$$MU\langle 0 \rangle \xrightarrow{g} \mathbf{E}$$

is an orientation, then the composition

$$(\mathbb{C}P^\infty)^L \longrightarrow MU\langle 0 \rangle \xrightarrow{g} \mathbf{E}$$

represents a trivialization s_g of the ideal sheaf $\mathcal{I}_G(0)$ of functions on G which vanish at the identity, that is, a coordinate on the formal group G . The association

$$g \mapsto s_g$$

gives a bijection between $MU\langle 0 \rangle$ -orientations on \mathbf{E} and coordinates on G .

Suppose in addition that $\pi_0 \mathbf{E}$ is a complete local ring with perfect residue field of characteristic $p > 0$, and the height of the formal group G is finite. In

§3, following [And95], we show that an H_∞ structure on \mathbf{E} adds the following structure to the formal group G . Given a map (of complete local rings) $i: S \rightarrow R$, a finite abelian group A , and a level structure (in the sense of [Dri74]; see §9)

$$(1.3) \quad \ell: A \rightarrow i^*G(R),$$

there is a map $\psi_\ell: S \rightarrow R$, and an isogeny $f_\ell: i^*G \rightarrow \psi_\ell^*G$ with kernel A . (The behavior of this structure with respect to variation in A gives *descent data for level structures* as described in Definition 3.1 or Proposition 11.14.)

If s is the coordinate on G associated to an orientation g , then the H_∞ structure gives *two* coordinates on ψ_ℓ^*G : one (ψ_ℓ^*s) comes from pulling back along ψ_ℓ ; the other ($N_\ell i^*s$) is obtained from the invariant function

$$(1.4) \quad \prod_{a \in A} T_a^* i^* s$$

on i^*G by descent along the isogeny f_ℓ (see Proposition 10.14). In §4.3, we show that these two coordinates arise from the two ways of navigating the diagram

$$\begin{array}{ccccc} (BA^* \times \mathbb{C}P^\infty)^{V_{\text{reg}} \otimes L} & \longrightarrow & D_n MU\langle 0 \rangle & \xrightarrow{D_n g} & D_n \mathbf{E} \\ \downarrow & & \downarrow & & \downarrow \\ MU\langle 0 \rangle & \xrightarrow{g} & \mathbf{E}, & & \end{array}$$

where $|A| = n$ and V_{reg} denotes the regular representation of A^* (a key point is that (1.4) is the Euler class of the bundle $V_{\text{reg}} \otimes L$ associated to the orientation g). It follows that if g is an H_∞ map, then

$$(1.5) \quad \psi_\ell^* s = N_\ell i^* s.$$

This condition is equivalent to the condition in [And95]; see Remark 4.16.

In §5 we modify the discussion of §4 to handle $MU\langle 6 \rangle$ -orientations. If

$$MU\langle 6 \rangle \xrightarrow{g} \mathbf{E}$$

is an orientation, then the composition

$$((\mathbb{C}P^\infty)^3)^{\prod_i (1-L_i)} \longrightarrow MU\langle 6 \rangle \xrightarrow{g} \mathbf{E}$$

represents a *cubical structure* s_g on the line bundle $\mathcal{I}_G(0)$ (Definition 13.5); in [AHS01], we showed that the assignment $g \mapsto s_g$ is a bijection between the set of $MU\langle 6 \rangle$ -orientations of \mathbf{E} and the set $C^3(G; \mathcal{I}_G(0))$ of cubical structures.

As before, a cubical structure s on $\mathcal{I}_G(0)$ gives rise to *two* cubical structures ψ_ℓ^*s and $\tilde{N}_\ell s$ on $\psi_\ell^*\mathcal{I}_G(0)$. If $s = s_g$ is the cubical structure associated to an $MU\langle 6 \rangle$ -orientation g , then these two cubical structures correspond to the two ways of navigating the diagram (1.2). If g is an H_∞ orientation, then the cubical structure s must satisfy the equation

$$(1.6) \quad \psi_\ell^*s = \tilde{N}_\ell i^*s.$$

In Proposition 6.1 we show that the necessary conditions (1.5) and (1.6) are sufficient if we suppose in addition that p is not a zero divisor in \mathbf{E} . We have given a direct proof, but our argument amounts to showing that for $k \leq 3$, the character map of [HKR00] for $\mathbf{E}^0(D_p BU\langle 2k \rangle_+)$ is injective.

Thus we have reduced the problem of checking whether the orientation (1.1) is H_∞ to the problem of checking the equation (1.6). That problem is mostly a matter of recalling the construction of the sigma orientation; we do that in Part 4. Here are the main points.

Definition 1.7. An *elliptic spectrum* consists of

- (1) an even, periodic, homotopy commutative ring spectrum \mathbf{E} ;
- (2) an elliptic curve C over $\text{spec } \pi_0 \mathbf{E}$; and
- (3) an isomorphism of formal groups

$$t: G_{\mathbf{E}} \cong \widehat{C}$$

over $\text{spec } \pi_0 \mathbf{E}$.

The Theorem of the Cube (or Abel's Theorem, for that matter) shows that if C is an elliptic curve, then $\mathcal{I}_C(0)$ has a *unique* cubical structure $s(C/S)$. If C is the elliptic curve associated to an elliptic spectrum (\mathbf{E}, C, t) , then

$$(t^3)^* \mathfrak{s}(C/S)$$

is a cubical structure on $\mathcal{I}_{G_{\mathbf{E}}}(0)$; the associated $MU\langle 6 \rangle$ -orientation is the sigma orientation. (The name comes from the fact that if C is a complex elliptic curve, then there is a simple formula for $s(C/S)$ in terms of the Weierstrass σ -function, which shows that the sigma orientation for the Tate curve is the Witten genus. See [AHS01].)

Now suppose that (\mathbf{E}, C, t) is an elliptic spectrum, that \mathbf{E} is an H_∞ spectrum, and that $\pi_0 \mathbf{E}$ is a complete local ring with perfect residue field of characteristic $p > 0$. Suppose that for each level structure

$$A \xrightarrow{\ell} i^* G(R)$$

we are given an isogeny of elliptic curves

$$h_\ell: i^* C \longrightarrow \psi_\ell^* C$$

with kernel $[\ell(A)]$, such that

$$t^* \hat{h}_\ell = f_\ell$$

(such structure, with compatibility with variation in A , is called an H_∞ elliptic spectrum in Definition 16.4). The uniqueness of the cubical structure $s(C/S)$ implies that

$$\psi_\ell^* s(C/S) = N_\ell t^* s(C/S),$$

which implies equation (1.6). Thus we have the following:

PROPOSITION 1.8. (16.5) *If (E, C, t) is an H_∞ elliptic spectrum, and p is regular in $\pi_0 E$, then the sigma orientation*

$$MU\langle 6 \rangle \xrightarrow{\sigma(E, C, t)} E$$

is an H_∞ map.

The Serre-Tate Theorem together with the result of Goerss and Hopkins implies that the elliptic spectrum associated to the universal deformation of a supersingular elliptic curve over a perfect field of characteristic p is an H_∞ elliptic spectrum (Corollary 15.13), and so the Proposition implies our result.

COROLLARY 1.9. (16.6) *If (E, C, t) is the elliptic spectrum associated to the universal deformation of a supersingular elliptic curve over a perfect field of characteristic $p > 0$, then the sigma orientation*

$$MU\langle 6 \rangle \xrightarrow{\sigma(E, C, t)} E$$

is H_∞ .

We have analyzed H_∞ ring spectra using the algebraic geometry of group schemes and in particular level structures, and we have analyzed orientations (i.e. Thom isomorphisms) using the algebraic geometry of line bundles. Part 2 describes the relationship to topology. Section 7 discusses the relationship between level structures and the cohomology of abelian groups; this is a variation of [HKR00]. Section 8 expresses some familiar results about the even-periodic cohomology of Thom complexes in the language of line bundles.

The construction of the homomorphism ψ_ℓ and the isogeny f_ℓ and the proof of the sufficiency of the equations (1.5) and (1.6) depend on two technical results

(Propositions 9.15 and 9.24) about level structures. We prove those results in Part 3. In order to make the discussion more self-contained, we also recall there some results about level structures, primarily from [Dri74, KM85, Str97].

2. Notation.

2.1. Groups. If X is an object in some category with products, and $J \subseteq I$ is an inclusion of sets, the projection map $X^I \rightarrow X^J$ will be denoted π_J , while $\hat{\pi}_J$ will denote the projection map $X^I \rightarrow X^{(I \setminus J)}$. The set J will often be indicated by the sequence of its elements. For example, π_{23} will denote projection to product of the 2nd and 3rd factors, while $\hat{\pi}_1$ will denote projection away from the first factor. If $\sigma: I \rightarrow I$ is an automorphism, the symbol π_σ refers to the induced automorphism of X^I .

If X is a commutative group object, then the symbol μ_J will denote the map $X^J \rightarrow X$ obtained by composing π_J with the iterated multiplication, while $\hat{\mu}_J$ will denote the map $\mu_J \times \hat{\pi}_J$. In punctual notation,

$$\begin{aligned}\mu_{23}(a_1, a_2, a_3, \dots) &= a_2 a_3 \\ \hat{\mu}_{23}(a_1, a_2, a_3, \dots) &= (a_1, a_2 a_3, \dots)\end{aligned}$$

and so forth.

If X is a commutative group in a category of objects over a base S , then the symbol $0: S \rightarrow X$ will stand for the identity section, and we shall generally abbreviate to π the symbol for the structural map

$$\pi_0: X \rightarrow S.$$

2.2. Formal schemes and formal groups. As in [AHS01], we view affine schemes as representable functors from rings to sets, and define a *formal scheme* to be a filtered colimit of affine schemes; the value of the colimit is the colimit of the values

$$(2.1) \quad (\operatorname{colim}_\alpha X_\alpha)(R) = \operatorname{colim}_\alpha X_\alpha(R).$$

In this paper we make one important modification to the notation (2.1). Recall from [Gro60, 0, 7.1.2] that a *preadmissible* ring is a linearly topologized ring which contains an *ideal of definition*: an open ideal I such that, for all open neighborhoods V of zero, $I^n \subset V$ for some $n > 0$. An *admissible* ring is preadmissible ring which is complete and separated. If R is an admissible ring, then the ideals of definition form a fundamental system of neighborhoods of 0. The

formal spectrum of R is the formal scheme [Gro60, I, 10.1.2, 10.6]

$$\mathrm{spf} R \stackrel{\mathrm{def}}{=} \mathrm{colim}_J \mathrm{spec} R/J,$$

where the colimit is over the poset of ideals of definition. In fact we shall only need the case that R is a local ring; a local ring is admissible if it is complete and separated in its adic topology. If R is an admissible ring and if X is a formal scheme, then we define $X(R)$ to be the set of *natural transformations*

$$\mathrm{spf} R \rightarrow X.$$

Thus if R is admissible then $\hat{\mathbb{A}}^1(R)$ is the set of *topologically* nilpotent elements of R , rather than just the set of nilpotent elements. Similarly

$$(\mathrm{spf} R')(R)$$

is the set of continuous ring homomorphisms from R' to R .

A *local scheme* (of residue characteristic p) is a scheme of the form $\mathrm{spec} R$, where R is a local ring (of residue characteristic p). An *local formal scheme* (of residue characteristic p) is a formal scheme of the form $\mathrm{spf} R$, where R is an admissible local ring (of residue characteristic p).

Let S be a formal scheme. A *formal group scheme* over S is a commutative group in the category of formal schemes over S . If A is a finite abelian group, then A_S will denote the constant formal group scheme over S given by A . A *formal group* over S is a formal group scheme which is locally isomorphic to $S \times \hat{\mathbb{A}}^1$ as a pointed formal scheme over S . If R is complete local ring, then a formal group over R means a formal group over $\mathrm{spf} R$. If G is a formal group over R and

$$j: R \rightarrow R'$$

is a map of complete local rings, then with the pull-back diagram

$$\begin{array}{ccc} j^*G & \longrightarrow & G \\ \downarrow & & \downarrow \\ \mathrm{spf} R' & \xrightarrow{j} & \mathrm{spf} R \end{array}$$

in mind, we write j^*G for the resulting formal group over R' .

We shall be primarily interested the case of a formal group G of finite height over a local formal scheme S whose closed point S_0 is the spectrum of a perfect field of characteristic $p > 0$. Let G_0 be the fiber of G over S_0 , i.e. the pull-back

in the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{i} & G \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S. \end{array}$$

By construction, $(G/S, i, \text{id}_{S_0})$ is a deformation of G_0 in the sense of Lubin and Tate (see Definition 12.1).

This deformation is classified by a pull-back diagram

$$\begin{array}{ccc} G & \longrightarrow & G' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S', \end{array}$$

where $(G'/S', f_{\text{univ}}, j_{\text{univ}})$ is the universal deformation of Lubin and Tate ([LT66]; see §12). Various facts about G/S then follow from facts about G'/S' by change of base.

2.3. Ideal sheaves associated to divisors. If G is a formal group or elliptic curve over a local formal scheme S , then an (*effective*) divisor on G is a closed subscheme D of G such that the ideal sheaf $\mathcal{I}(D)$ is invertible, and D is finite, free, and of finite presentation over S . The sheaf $\mathcal{I}(D)$ is the inverse of the sheaf which is usually denoted $\mathcal{O}(D)$. For example, if $w \in G(R)$ then $\mathcal{I}(w)$ is the ideal of functions on G which vanish at w . More generally, if W is a finite set and

$$\ell: W \rightarrow G(R)$$

is a map of sets, then we will write $\mathcal{I}(\ell)$ for the ideal associated to the divisor

$$[\ell(W)] \stackrel{\text{def}}{=} \sum_{w \in W} [\ell(w)],$$

so

$$\mathcal{I}(\ell) \cong \bigotimes_{w \in W} \mathcal{I}(\ell(w)).$$

2.4. Spectra. The category of spectra over a universe U will be denoted \mathbb{S}_U . The category \mathbb{S}_U is enriched over the category Spaces_+ of pointed topological spaces, and our notation will reflect this. Thus, the object $\mathbb{S}_U(E, F)$ will refer to a pointed topological space, and for a pointed space X , E^X is the function object (spectrum). What would in category theory denoted $E \otimes X$ will in this case be

denoted $E \wedge X$. There are natural homeomorphisms of pointed spaces

$$(2.2) \quad \text{Spaces}_+(X, \mathbb{S}_U(E, F)) \cong \mathbb{S}_U(E \wedge X, F) \cong \mathbb{S}_U(E, F^X).$$

If V is a vector bundle over a space X , then X^V will refer to the pointed space which is the Thom complex of V . When V is a virtual bundle, then X^V will refer to the Thom *spectrum* of V , arranged so that the “bottom cell” is in the virtual (real) dimension of V . With this convention, the Thom spectrum of an “honest” vector bundle is the suspension spectrum of the Thom complex, so no real problem should come up when regarding an actual vector bundle as a virtual.

We write V_{std} for (the vector bundle over $B\Sigma_n$ associated to) the standard complex representation of Σ_n , and if A is an abelian group, then V_{reg} will denote (the vector bundle over BA associated to) the complex regular representation of A .

2.5. Even periodic ring spectra. A (homotopy commutative) ring spectrum \mathbf{E} will be called *even* if $\pi_{\text{odd}}\mathbf{E} = 0$, and *periodic* if $\pi_2\mathbf{E}$ contains a unit. A ring spectrum \mathbf{E} will be called *homogeneous* if it is a homotopy commutative algebra spectrum over an even periodic ring spectrum. We will be particularly interested in homogeneous spectra \mathbf{E} in which the ring $\pi_0\mathbf{E}$ is preadmissible in some natural topology (possibly discrete). If \mathbf{E} is a such a spectrum, then we write $\hat{\pi}_0\mathbf{E}$ for the separated completion of $\pi_0\mathbf{E}$, and we define

$$S_{\mathbf{E}} \stackrel{\text{def}}{=} \text{spf}(\hat{\pi}_0\mathbf{E})$$

for the formal scheme defined by $\hat{\pi}_0\mathbf{E}$.

Let \mathbf{E} be such a spectrum, and let X be a space. If $\{X_\alpha\}$ is the set of compact subsets of X and $\{I_\beta\}$ is the set of ideals of definition of $\pi_0\mathbf{E}$, then $\pi_0\mathbf{E}^{X_+}$ is preadmissible in the topology defined by the kernels of the maps

$$\pi_0\mathbf{E}^{X_+} \rightarrow (\pi_0\mathbf{E}^{(X_\alpha)_+})/I_\beta,$$

and we define $X_{\mathbf{E}}$ to be the formal scheme

$$X_{\mathbf{E}} = \text{spf } \hat{\pi}_0\mathbf{E}^{X_+};$$

this gives a covariant functor from spaces to formal schemes over $S_{\mathbf{E}}$. If $\mathbf{F} = \mathbf{E}^{X_+}$ then

$$S_{\mathbf{F}} = X_{\mathbf{E}},$$

and we shall use these notations interchangeably.

The most important example of these constructions is that $\mathbf{E} \cong E_n$ is the spectrum associated to the universal deformation of a formal group of height n over a perfect field k of characteristic $p > 0$, so

$$\pi_0 \mathbf{E} \cong \mathbb{W}k[[u_1, \dots, u_{n-1}]],$$

and X is a space with the property that $H_*(X, \mathbb{Z})$ is concentrated in even degrees. In that case, the natural map of rings

$$\pi_0 \mathbf{E}^{X_+} \rightarrow \mathcal{O}(X_{\mathbf{E}}) = \hat{\pi}_0 \mathbf{E}^{X_+}$$

is an isomorphism, but in general all we have is a surjective map.

If \mathbf{E} is a homogeneous ring spectrum, then it is complex orientable, and

$$G_{\mathbf{E}} = (\mathbb{C}P^\infty)_{\mathbf{E}}$$

is a formal group over $S_{\mathbf{E}}$.

Part 1. H_∞ orientations.

3. Algebraic geometry of even H_∞ ring spectra.

3.1. Descent data for level structures. Let \mathbf{E} be a homogeneous ring spectrum. In this section we investigate the additional structure which adheres to $G_{\mathbf{E}} = (\mathbb{C}P^\infty)_{\mathbf{E}}/S_{\mathbf{E}}$ when \mathbf{E} is an H_∞ spectrum (see §A). In order to make precise statements, it is convenient to suppose that $\pi_0 \mathbf{E}$ is a complete local ring with perfect residue field of characteristic $p > 0$, and that $G_{\mathbf{E}}$ is a formal group of finite height. In that case, we shall show that an H_∞ structure on \mathbf{E} determines “descent data for level structures” on $G_{\mathbf{E}}$. In §11 we shall give a definition of this notion in the usual language of descent; the definition we give there is equivalent to the following:

Definition 3.1. Let G be a formal group over a formal scheme S . *Descent data for level structures on G* assign to every map of formal schemes $i: T = \text{spf } R \rightarrow S$, finite abelian group A , and level structure (Definition 9.9)

$$(3.2) \quad \ell: A_T \rightarrow i^* G,$$

a map of formal schemes $\psi_\ell: T \rightarrow S$ and an isogeny $f_\ell: i^* G \rightarrow \psi_\ell^* G$ with kernel

$$[\ell(A)] \stackrel{\text{def}}{=} \sum_{a \in A} [\ell(a)],$$

satisfying the following.

(1) If $j: T \rightarrow T'$ is a map of formal schemes and

$$j^*\ell: A_{T'} \rightarrow j^*i^*G$$

is the resulting level structure, then

$$\psi_{j^*\ell} = j \circ \psi_\ell,$$

and

$$f_{j^*\ell} = j^*f_\ell.$$

(2) If $B \subseteq A$, then with the notation

$$(3.3) \quad \begin{array}{ccccc} B & \longrightarrow & A & \longrightarrow & A/B \\ \ell' \downarrow & & \ell \downarrow & & \downarrow \ell'' \\ i^*G & \xlongequal{\quad} & i^*G & \xrightarrow{f_{\ell'}} & \psi_{\ell'}^*G, \end{array}$$

we have

$$(3.4) \quad \begin{aligned} \psi_{\ell''} &= \psi_\ell: T \rightarrow S \\ f_\ell &= f_{\ell''} \circ f_{\ell'}: i^*G \rightarrow \psi_{\ell'}^*G = \psi_{\ell''}^*G. \end{aligned}$$

(3) If ℓ is the inclusion of the trivial subgroup, then f_ℓ and ψ_ℓ are the identity maps. Among other things this implies that if ℓ and ℓ' differ by an automorphism of A , then $f_\ell = f_{\ell'}$.

We shall write (ψ, f) for such descent data. Formal groups with descent data for level structures form a category: if G/S and G'/S' are two formal groups with descent data for level structures, then a *map* from G'/S' to G/S is a pull-back

$$\begin{array}{ccc} G' & \longrightarrow & G \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

in the category of formal schemes, such that the induced isomorphism

$$G' \rightarrow S' \times_S G$$

is a group homomorphism, and such that the descent data for G pull back to the descent data for G' .

Let \mathcal{C} be the category whose objects are homogeneous ring spectra \mathbf{E} with the property that $\pi_0 \mathbf{E}$ is a complete local ring with perfect residue field of positive characteristic and $G_{\mathbf{E}}$ is a formal group of finite height, and whose morphisms are maps $f: \mathbf{E} \rightarrow \mathbf{F}$ of ring spectra with the property that $\pi_0 f$ is a map of local rings. Let $H_{\infty} \mathcal{C}$ be the subcategory of \mathcal{C} consisting of H_{∞} ring spectra and H_{∞} maps. We shall construct the dotted arrow in the diagram

$$(3.5) \quad \begin{array}{ccc} H_{\infty} \mathcal{C} & \dashrightarrow & (\text{formal groups with descent data}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\quad} & (\text{formal groups}). \end{array}$$

The main result is Theorem 3.25.

3.2. Descent data from H_{∞} ring spectra. The basic operation on the homotopy groups of an H_{∞} ring spectrum is the transformation

$$D_n: \pi_0 \mathbf{E} \rightarrow \pi_0 \mathbb{S}_U(D_n S^0, \mathbf{E}) = \pi_0 \mathbf{E}^{B\Sigma_{n+}}.$$

This map is multiplicative in the sense that $D_n(fg) = D_n(f)D_n(g)$, but it is not quite additive. In fact, it follows from Proposition A.5 that

$$(3.6) \quad D_n(f + g) = \sum_{i+j=n} \text{Tr}_{ij} D_i(f)D_j(g),$$

where Tr_{ij} is the transfer map associated to the inclusion $\Sigma_i \times \Sigma_j \subseteq \Sigma_n$.

If \mathbf{E} is a spectrum such that $\pi_0 \mathbf{E}$ is a complete local ring with perfect residue field of characteristic $p > 0$, and the formal group $G_{\mathbf{E}}$ is of finite height, then there is a slightly more convenient operation to work with. Suppose that A is a finite abelian group, and let A^* be its group of complex characters. With these hypotheses, Proposition 7.3 says that the natural map (7.2)

$$(3.7) \quad (BA^*)_{\mathbf{E}} \rightarrow \underline{\text{hom}}(A, G_{\mathbf{E}})$$

is an isomorphism. Define a functor $D_A: \mathbb{S}_U \rightarrow \mathbb{S}_U$ by

$$(3.8) \quad D_A(X) = \mathcal{L}(U^{A^*}, U) \underset{A^*}{\wedge} X^{(A^*)},$$

where $X^{(A^*)}$ denotes the external smash product

$$\bigwedge_{\alpha \in A^*} X \in \text{Ob } \mathbb{S}_{U^{A^*}}$$

(we will also have use for the functor on pointed spaces given by the the analogue of (3.8)).

Definition 3.9. Given a complete local ring R and a level structure (9.9)

$$A_{\text{spf } R} \xrightarrow{\ell} i^* G,$$

we define $\psi_\ell^E: \pi_0 E \rightarrow R$ to be the map given by the composition

$$\pi_0 E \xrightarrow{D_A} \pi_0 S_U(D_A S^0, E) = \pi_0 E^{BA_*^+} \rightarrow \mathcal{O}((BA^*)_E) \xrightarrow{\chi_\ell} R,$$

where χ_ℓ is the map classifying the homomorphism ℓ as in (7.4).

LEMMA 3.10. [And95] *The map ψ_ℓ^E is a continuous ring homomorphism.*

Proof. ψ_ℓ^E is certainly multiplicative. It's additive because equation (3.6) and the double coset formula imply that $\psi_\ell^E(x + y) - \psi_\ell^E(x) - \psi_\ell^E(y)$ is a sum of elements in the image of the transfer map from proper subgroups of A^* . The result therefore follows from Proposition 7.5.

To see that ψ_ℓ^E is continuous, note that

$$(3.11) \quad \mathcal{O}((BA^*)_E) \cong \mathcal{O}(\underline{\text{hom}}(A, G_E))$$

is a local ring by Proposition 9.8. It suffices to show that for y in the maximal ideal of $\pi_0 E$, $D_A y$ is in the maximal ideal of $\mathcal{O}((BA^*)_E)$. Since, modulo the augmentation ideal of $\pi_0 E^{BA_*^+}$ we have

$$D_A y = y^{|A|},$$

it follows that ψ_ℓ^E is continuous. \square

Remark 3.12. It is not necessary to use an abelian group A . The initial ring R over which we may define a *ring homomorphism*

$$\psi: \pi_0 E \longrightarrow \pi_0 E^{B(\Sigma_k)_+} \longrightarrow R$$

as above is the quotient of $\pi_0 E^{B(\Sigma_k)_+}$ by the ideal generated by the images of transfers from proper subgroups of Σ_k . Strickland [Str98] shows that

$$\text{spf}(\pi_0 E_n^{B(\Sigma_k)_+} / \text{proper transfers})$$

is the scheme of “subgroups of order k of G_{E_n} ”. The analysis of this paper can be carried through in that setting.

The operation ψ_ℓ is clearly natural in the sense that given a map $f: \mathbf{E} \rightarrow \mathbf{F}$ of H_∞ spectra of the indicated kind, with the property that $\pi_0 f$ is continuous, the level structure ℓ gives a level structure

$$A_{\text{spf } R'} \xrightarrow{\ell} j^* G_{\mathbf{F}},$$

where R' and j are defined by

$$\pi_0 F \xrightarrow{j} R' = R \hat{\otimes}_{i, \hat{\pi}_0 \mathbf{E}, \hat{\pi}_0 f} \pi_0 \mathbf{F},$$

and the diagram

$$\begin{array}{ccc} R' & \xleftarrow{\psi_\ell^F} & \pi_0 \mathbf{F} \\ \uparrow & & \uparrow \pi_0 f \\ R & \xleftarrow{\psi_\ell^E} & \pi_0 \mathbf{E} \end{array}$$

commutes.

In the language of algebraic geometry, let

$$T = \text{spf } R \xrightarrow{i} S_{\mathbf{E}}.$$

The map ψ_ℓ^E is a map of formal schemes

$$\psi_\ell^E: T \rightarrow S_{\mathbf{E}},$$

and the naturality is expressed in terms of the commutative diagram

$$\begin{array}{ccc} T \times_{i, S_{\mathbf{E}}, S_f} S_{\mathbf{F}} & \xrightarrow{\psi_\ell^F} & S_{\mathbf{F}} \\ \downarrow & & \downarrow s_f \\ T & \xrightarrow{\psi_\ell^E} & S_{\mathbf{E}}. \end{array}$$

Making use of the isomorphism

$$T \times_{i, S_{\mathbf{E}}, S_f} S_{\mathbf{F}} \cong i^* S_{\mathbf{F}},$$

we find that the map ψ_ℓ^F can be factored through a *relative* map

$$(3.13) \quad \psi_\ell^{F/\mathbf{E}}: i^* S_{\mathbf{F}} \rightarrow \psi_\ell^{E*} S_{\mathbf{F}}$$

as in

(3.14)

$$\begin{array}{ccccc}
 i^* S_{\mathbf{F}} & & & & \\
 \downarrow \psi_{\ell}^{\mathbf{F}/\mathbf{E}} & \nearrow \psi_{\ell}^{\mathbf{F}} & & & \\
 & (\psi_{\ell}^{\mathbf{E}})^* S_{\mathbf{F}} & \xrightarrow{S_f^* \psi_{\ell}^{\mathbf{E}}} & S_{\mathbf{F}} & \\
 \downarrow & & & & \downarrow S_f \\
 T & \xrightarrow{\psi_{\ell}^{\mathbf{E}}} & S_{\mathbf{E}}. & &
 \end{array}$$

For example, let $G = G_{\mathbf{E}}$, and take $\mathbf{F} = \mathbf{E}^{\mathbb{C}P_+^\infty}$ so that $G = S_{\mathbf{F}}$. There results a map

$$(3.15) \quad \psi_{\ell}^{G/\mathbf{E}}: i^* G \rightarrow (\psi_{\ell}^{\mathbf{E}})^* G.$$

This map turns out to be a homomorphism of groups, as one can see by considering the (H_∞) map

$$\mathbf{E}^{\mathbb{C}P_+^\infty} \rightarrow \mathbf{E}^{(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+}$$

coming from $\mu: \mathbb{C}P^\infty{}^2 \rightarrow \mathbb{C}P^\infty$. We shall eventually show (Proposition 3.21) that $\psi_{\ell}^{G/\mathbf{E}}$ is an isogeny, with kernel $\ell: A \rightarrow i^* G$. In order to give the proof, it is essential to understand the effect of the operation $\psi_{\ell}^{\mathbf{E}}$ on the cohomology of Thom complexes.

Suppose that V is a virtual bundle over a space X , and write

$$\mathbf{F} = \mathbf{E}^{X_+}.$$

If we start with an element $m \in \pi_0 \mathbb{S}_U((X)^V, \mathbf{E})$ and follow the construction of the map $\psi_{\ell}^{\mathbf{E}}$, we wind up with an element $\psi_{\ell}^V(m)$ in

$$(3.16) \quad R \underset{\chi_{\ell}, \pi_0 \mathbf{E}^{BA_+^*}}{\widehat{\otimes}} \hat{\pi}_0 \mathbb{S}_U((BA^* \times X)^{V_{\text{reg}} \otimes V}, \mathbf{E}),$$

where V_{reg} denotes the regular representation of A^* . As before, this map is additive, and in fact $\psi_{\ell}^{\mathbf{F}}$ -linear:

$$(3.17) \quad \psi_{\ell}^V(xm) = \psi_{\ell}^{\mathbf{F}}(x)\psi_{\ell}^V(m).$$

Let $T = \text{spf } R$; then we have a commutative diagram

$$\begin{array}{ccccc} i^* S_{\mathbf{F}} & \xrightarrow{\chi_{\ell}} & \underline{\text{hom}}(A, G_{\mathbf{F}}) & \longrightarrow & S_{\mathbf{F}} \\ \downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{\chi_{\ell}} & \underline{\text{hom}}(A, G_{\mathbf{E}}) & \longrightarrow & S_{\mathbf{E}} \\ & & \curvearrowright_i & & \end{array}$$

in which all the squares are pull-backs. In the language of §8, the element m is a section of the line bundle $\mathbb{L}(V)$ over $S_{\mathbf{F}}$. Elements of (3.16) are sections of

$$\chi_{\ell}^* \mathbb{L}(V_{\text{reg}} \otimes V)$$

over $i^* S_{\mathbf{F}}$. Taking into account the linearity (3.17) we find that the map ψ_{ℓ}^V can be interpreted as a map

$$(3.18) \quad \psi_{\ell}^V: (\psi_{\ell}^{\mathbf{F}})^* \mathbb{L}(V) \rightarrow \chi_{\ell}^* \mathbb{L}(V_{\text{reg}} \otimes V)$$

of line bundles over $i^* S_{\mathbf{F}}$.

LEMMA 3.19. *The map ψ_{ℓ}^V has the following properties*

(1) *If m is a trivialization of $\mathbb{L}(V)$, then $\psi_{\ell}^V(m)$ is a trivialization of $\chi_{\ell}^* \mathbb{L}(V_{\text{reg}} \otimes V)$.*

(2) *With the obvious identifications*

$$\psi_{\ell}^{V_1 \oplus V_2} = \psi_{\ell}^{V_1} \otimes \psi_{\ell}^{V_2}.$$

(3) *If $f: Y \rightarrow X$ is a map, then*

$$\psi_{\ell}^{f^* V} = f^* \psi_{\ell}^V.$$

The most important example is the case $X = \mathbb{C}P^{\infty}$ and $V = L$, so $S_{\mathbf{F}} = G = G_{\mathbf{E}}$ and $\mathbb{L}(L) = \mathcal{I}_G(0)$ (8.4). Then (8.11) gives an isomorphism

$$\chi_{\ell}^* \mathbb{L}(V_{\text{reg}} \otimes L) \cong \mathcal{I}_{i^* G}(\ell),$$

and so we may think of ψ_{ℓ}^L as a map of line bundles over $i^* G$

$$(\psi_{\ell}^{\mathbf{F}})^* \mathcal{I}_G(0) \rightarrow \mathcal{I}_{i^* G}(\ell),$$

or on sections a $\psi_{\ell}^{\mathbf{F}}$ -linear map

$$(3.20) \quad \Gamma(\mathcal{I}_G(0)) \rightarrow \Gamma(\mathcal{I}_{i^* G}(\ell)).$$

PROPOSITION 3.21. [And95] (1) *The map*

$$\psi_\ell^{G/\mathbf{E}}: i^*G \rightarrow (\psi_\ell^{\mathbf{E}})^*G$$

of (3.15) is an isogeny with kernel $[\ell(A)]$.

(2) If the isogeny $\psi_\ell^{G/\mathbf{E}}$ is used to identify

$$(\psi_\ell^{G/\mathbf{E}})^*I_{(\psi_\ell^{\mathbf{E}})^*G}(0) \cong I_{i^*G}(\ell)$$

as in (10.15), then the map ψ_ℓ^L (3.20) sends a coordinate x on G to the trivialization $(\psi_\ell^{G/\mathbf{E}})^*(\psi_\ell^{\mathbf{E}})^*x$ of $I_{i^*G}(\ell)$.

Proof. First, observe that $\psi_\ell^{G/\mathbf{E}}$ is an isogeny of degree $|A|$. This follows from the Weierstrass Preparation Theorem [Lan78, pp. 129–131], because, after reducing modulo the maximal ideal in R , the ring homomorphism

$$(\psi_\ell^{G/\mathbf{E}})^*: R \hat{\otimes}_{\psi_\ell^{\mathbf{E}}} \pi_0 E^{\mathbb{C}P^\infty+} \rightarrow R \hat{\otimes}_i \pi_0 E^{\mathbb{C}P^\infty+}$$

sends a coordinate x to $x^{|A|}$. To see this, note that the composition

$$\pi_0 \mathbf{F} \xrightarrow{D_A} \pi_0 \mathbf{F}^{BA_*^*} \xrightarrow{\pi_0 \mathbf{F}^{S^0 \rightarrow BA_*^*}} \pi_0 \mathbf{F}$$

is the map $x \mapsto x^{|A|}$.

To prove the first part of the Proposition, it remains to show that the $[\ell(A)]$ is contained in the kernel, i.e. that $\psi_\ell^{G/\mathbf{E}} \ell = 0$, i.e. that if x is a coordinate on $G_{\mathbf{E}}$, then $(\psi_\ell^{G/\mathbf{E}})^*(\psi_\ell^{\mathbf{E}})^*x$ vanishes on $[\ell(A)]$. A coordinate on G is a generator the ideal $\mathcal{I}(0)$, which the zero section identifies with $\mathbb{L}(L)$. The commutativity of the diagram

$$(3.22) \quad \begin{array}{ccccc} & & \pi_0 \mathbf{E}^{\mathbb{C}P^\infty} & & \\ & \swarrow \psi_\ell^{G/\mathbf{E}} & & \searrow \psi_\ell^L & \\ \pi_0 \mathbf{E}^{BA_*^* \times \mathbb{C}P^\infty} & \xleftarrow{\pi_0 \mathbf{E}^\zeta} & & \xleftarrow{\pi_0 \mathbf{E}^{(BA_*^* \times \mathbb{C}P^\infty)^{V_{\text{reg}} \otimes L}}} & \pi_0 \mathbf{E}^{\mathbb{C}P^\infty L} \\ \downarrow D_A & & & & \downarrow D_A \\ & & \pi_0 \mathbf{E}^{BA_*^* \times \mathbb{C}P^\infty} & & \\ & \uparrow \pi_0 \mathbf{E}^\zeta & & & \\ R \hat{\otimes}_{\pi_0} \mathbf{E}^{BA_*^* \times \mathbb{C}P^\infty} & \xleftarrow{} & & \xleftarrow{} & R \hat{\otimes}_{\pi_0} \mathbf{E}^{(BA_*^* \times \mathbb{C}P^\infty)^{V_{\text{reg}} \otimes L}} \end{array}$$

(in which the tensor products are taken over the ring $\hat{\pi}_0 \mathbf{E}^{BA_*^*}$) shows that $\psi_\ell^{G/\mathbf{E}}$ takes a section of $\mathcal{I}(0)$ to a section of $\mathcal{I}_{i^*G}(\ell)$. The claim about $\psi_\ell^L x$ also follows from inspection of the diagram (3.22). \square

In fact Proposition 3.21 gives a simple description of the map ψ_ℓ^V for a general virtual bundle V , and in particular, shows that it is determined by maps which have already been constructed. We shall express the answer in the language of §8, where line bundles of the form $\mathbb{L}(V)$ are computed in terms of divisors. As in §8, it is illuminating to work at the outset with $V \otimes L$ over $X \times \mathbb{C}P^\infty$ and then pull back along the identity section of G_F .

With this in mind, let $\mathbf{F} = \mathbf{E}^{X_+}$, let

$$\mathbf{G} = \mathbf{F}^{\mathbb{C}P_+^\infty} = \mathbf{E}^{(\mathbb{C}P^\infty \times X)_+},$$

and let $G = G_F = S_{\mathbf{G}}$. Let $D = D_V$ be the divisor on G corresponding to V as in Proposition 8.12, so that there is an isomorphism

$$t_{V \otimes L}: \mathbb{L}(V \otimes L) \cong \mathcal{I}(D^{-1}).$$

Thus we may replace the domain of

$$\psi_\ell^{V \otimes L}: (\psi_\ell^{\mathbf{G}})^* \mathbb{L}(V \otimes L) \rightarrow \chi_\ell^* \mathbb{L}(V_{\text{reg}} \otimes V \otimes L).$$

with

$$(\psi_\ell^{\mathbf{G}})^* \mathcal{I}(D^{-1}) \cong (\psi_\ell^{G/F})^* (\psi_\ell^{\mathbf{F}})^* \mathcal{I}(D^{-1}).$$

Using the analogous isomorphism (8.9)

$$\chi_\ell^* \mathbb{L}(V_{\text{reg}} \otimes V \otimes L) \cong \mathcal{I} \left(\sum_a T_a^* D^{-1} \right)$$

to interpret the range, we may think of $\psi_\ell^{V \otimes L}$ as a map

$$(3.23) \quad \psi_\ell^{V \otimes L}: (\psi_\ell^{G/F})^* (\psi_\ell^{\mathbf{F}})^* \mathcal{I}(D^{-1}) \rightarrow \mathcal{I} \left(\sum_a T_a^* D^{-1} \right).$$

PROPOSITION 3.24. *In the guise of (3.23), the map $\psi_\ell^{V \otimes L}$ is given by*

$$f \mapsto (\psi_\ell^{G/F})^* (\psi_\ell^{\mathbf{F}})^* f.$$

Proof. There are actually two assertions. One is that

$$\langle f \rangle \geq D^{-1} \Rightarrow \langle ((\psi_\ell^{\mathbf{F}})^* f) \circ \psi_\ell^{G/F} \rangle \geq \sum_a T_a^* D^{-1}.$$

The other is that this gives the map $\psi_\ell^{V \otimes L}$. The verification of both assertions follows the lines of the proof of Proposition 3.21. Indeed, everything involved

takes Whitney sums in V to tensor products, and commutes with base change in V . It suffices then to verify the case when X is a single point, and V has dimension 1. In this case, the isomorphism t_L is given by the inclusion of the zero section $\mathbb{C}P^\infty \rightarrow \mathbb{C}P^{\infty L}$, and the result follows from naturality of the maps ψ_ℓ , as in the diagram (3.22). \square

The results of this section assemble to give the following:

THEOREM 3.25. *Let \mathbf{E} be a homogeneous H_∞ ring spectrum. Suppose that $\pi_0 \mathbf{E}$ is a local ring with perfect residue field of characteristic $p > 0$, and the formal group $G = G_{\mathbf{E}}$ is of finite height. The rule which associates to a level structure*

$$(3.26) \quad \ell: A_{\text{spf } R} \rightarrow i^* G$$

the map of formal schemes $\psi_\ell^{\mathbf{E}}: \text{spf } R \rightarrow S_{\mathbf{E}}$ and the isogeny

$$\psi_\ell^{G/\mathbf{E}}: i^* G \rightarrow \psi_\ell^* G$$

is descent data for level structures on the formal group $G/S_{\mathbf{E}}$, and gives the dotted arrow in the diagram (3.5).

Proof. Lemma 3.10 and Proposition 3.21 show that $\psi_\ell^{\mathbf{E}}$ is a ring homomorphism and $\psi_\ell^{G/\mathbf{E}}$ is an isogeny with kernel $[\ell(A)]$. The compatibility of $\psi_\ell^{\mathbf{E}}$ and $\psi_\ell^{G/\mathbf{E}}$ with variation in A as described in Definition 3.1 follows from the commutativity of the diagrams (A.2); a proof is given in Appendix B. \square

4. A necessary condition for an $MU\langle 0 \rangle$ -orientation to be H_∞ . Let $MU\langle 0 \rangle$ be the Thom spectrum of the tautological bundle over $\mathbb{Z} \times BU$, and let \mathbf{E} be a homogeneous ring spectrum. In §4.2 we recall that to give a map of (homotopy commutative) ring spectra

$$(4.1) \quad g: MU\langle 0 \rangle \rightarrow \mathbf{E}.$$

is to give a coordinate s on $G = G_{\mathbf{E}}$.

In §4.3 we give a necessary condition for the map g to be a map of H_∞ spectra, in the case that \mathbf{E} is an H_∞ ring spectrum, that $\pi_0 \mathbf{E}$ is a complete local ring with perfect residue field of characteristic $p > 0$, and that the formal group G is of finite height. The result may be described as follows.

Let $s = s_g$ be the coordinate on G associated to the orientation (4.1). In §3 we showed that the hypotheses on \mathbf{E} give descent data for level structures on G . Given a level structure (9.9)

$$(4.2) \quad A_T \xrightarrow{\ell} i^* G,$$

we get *two* coordinates on the formal group $(\psi_\ell^E)^*G$: one is just $(\psi_\ell^E)^*s$, the other is the norm $N_\ell i^*$ s of the coordinate i^*s with respect to the isogeny

$$i^*G \xrightarrow{\psi_\ell^{G/E}} \psi_\ell^*G$$

as in Proposition 10.14. We show that these two coordinates correspond to the two ways of going around the diagram

$$\begin{array}{ccc} D_A MU\langle 0 \rangle & \longrightarrow & D_A E \\ \downarrow & & \downarrow \\ MU\langle 0 \rangle & \longrightarrow & E; \end{array}$$

the main result is Proposition 4.13.

4.1. H_∞ structures on Thom spectra of infinite loop spaces. Suppose that $B \rightarrow \mathbb{Z} \times BO$ is a homotopy multiplicative map, and let M be the associated Thom spectrum. The spectrum M has a natural multiplication. If $W: X \rightarrow B$ is a vector bundle over X with a B -structure, then the Thom complex X^W comes equipped with a canonical M -Thom class

$$\Phi_M(W): X^W \rightarrow M.$$

LEMMA 4.3. *The Thom class $\Phi_M(W)$ has the following properties.*

- (i) *It is multiplicative:* $\Phi_M(W \oplus W') = \Phi_M(W)\Phi_M(W')$.
- (ii) *It is preserved under base change:* given $f: X \rightarrow Y$,

$$\Phi_M(f^*W) = f^*\Phi_M(W).$$

An infinite loop map

$$B \rightarrow \mathbb{Z} \times BO$$

gives, for every vector bundle $W: X \rightarrow B$ with a B -structure, a B -structure to the vector bundle $D_n W$ over $D_n X$, and so also to its restriction $V_{\text{reg}} \otimes W$ to $B\Sigma_n \times X$. The Thom spectrum spectrum M is then an E_∞ ring spectrum, whose underlying H_∞ -structure is such that if $u_W: X^W \rightarrow M$ is the M -Thom class of the B -bundle W , then the composition

$$(B\Sigma_n \times X)^{V_{\text{reg}} \otimes W} \rightarrow D_n M \rightarrow M$$

is the M -Thom class of $V_{\text{reg}} \otimes W$.

4.2. The spectrum $MU\langle 0 \rangle$. A $BU\langle 0 \rangle = \mathbb{Z} \times BU$ bundle over a space X is just a virtual complex vector bundle W , with rank given by the locally constant function

$$X \xrightarrow{W} \mathbb{Z} \times BU \rightarrow \mathbb{Z}.$$

The tautological line bundle L over $\mathbb{C}P^\infty$ gives rise to a natural map

$$\Phi_{MU\langle 0 \rangle}(L): \mathbb{C}P^{\infty L} \rightarrow MU\langle 0 \rangle.$$

If \mathbf{E} is an even periodic ring spectrum with formal group $G = G_E$ and

$$g: MU\langle 0 \rangle \rightarrow \mathbf{E}$$

is a homotopy multiplicative map, then by Proposition 8.14 the composition

$$(\mathbb{C}P^\infty)^L \xrightarrow{\Phi_{MU\langle 0 \rangle}(L)} MU\langle 0 \rangle \xrightarrow{g} \mathbf{E}$$

is a trivialization s_g of the ideal sheaf $\mathbb{L}(L) \cong \mathcal{I}(0)$ over G , that is, a coordinate on G . The standard result about $MU\langle 0 \rangle$ -orientations is

LEMMA 4.4. *The assignment $g \mapsto s_g$ is a bijection between the set of maps of homotopy commutative ring spectra $MU\langle 0 \rangle \rightarrow \mathbf{E}$ and coordinates on G_E .*

Proof. For $MU = MU\langle 2 \rangle$ instead of $MU\langle 0 \rangle$ the standard reference is [Ada74]. The minor modifications for $MU\langle 0 \rangle$ may be found in [AHS01]. \square

It is customary to express Lemma 4.4 in terms of formal group laws. A formal group law is the same thing as a formal group together with a coordinate: the equivalence sends a formal group G over R with multiplication

$$G \times G \xrightarrow{m} G,$$

and coordinate $s \in \mathcal{O}(G)$ to the power series

$$m^*s \in \mathcal{O}(G \times G) \cong R[[s, t]].$$

We shall write (G, s) for this group law.

For example, the tautological map

$$(\mathbb{C}P^\infty)^L \rightarrow MU\langle 0 \rangle$$

gives a coordinate $s_{MU\langle 0 \rangle}$ on $G_{MU\langle 0 \rangle}$. (Quillen's Theorem [Qui69] is that $(G_{MU\langle 0 \rangle}, s_{MU\langle 0 \rangle})$ is the *universal* formal group law.)

The commutative diagram

$$\begin{array}{ccc} \mathrm{spf} \hat{\pi}_0 E^{\mathbb{C}P_+^\infty} & \xrightarrow{\mathrm{spf} \hat{\pi}_0 g^{\mathbb{C}P_+^\infty}} & \mathrm{spf} \hat{\pi}_0 MU\langle 0 \rangle^{\mathbb{C}P_+^\infty} \\ \downarrow & & \downarrow \\ \mathrm{spf} \hat{\pi}_0 E & \xrightarrow{\mathrm{spf} \hat{\pi}_0 g} & \mathrm{spec} \pi_0 MU\langle 0 \rangle \end{array}$$

gives a *relative* map

$$\bar{g}: G_E = \mathrm{spf} \hat{\pi}_0 E^{\mathbb{C}P_+^\infty} \rightarrow (\mathrm{spf} \hat{\pi}_0 g)^* G_{MU\langle 0 \rangle}.$$

Naturality together with the analogous diagram for

$$(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+ \rightarrow \mathbb{C}P_+^\infty$$

shows that \bar{g} is a homomorphism of formal groups over S_E . The construction of s_g shows that \bar{g} is an isomorphism of formal groups, and

$$\bar{g}^* s_{MU\langle 0 \rangle} = s_g.$$

In particular, we have the following.

LEMMA 4.5. *If*

$$g: MU\langle 0 \rangle \rightarrow E$$

is a homotopy multiplicative map, then

$$\pi_0 g: \pi_0 MU\langle 0 \rangle \rightarrow \pi_0 E$$

classifies the group law (G_E, s_g) .

To understand the Thom class associated to a general virtual complex vector bundle W over a pointed space X , it is convenient as in §8 to work first with the bundle $W \otimes L$ over $X \times \mathbb{C}P^\infty$, and then pull back along the identity section. So let $F = E^{X_+}$, and let $f: E = E^{S^0} \rightarrow F$ be the map associated to the map $X_+ \rightarrow S^0$.

The map

$$(X \times \mathbb{C}P^\infty)^{W \otimes L} \rightarrow MU\langle 0 \rangle \xrightarrow{g} E$$

represents a trivialization s_W of the line bundle $\mathbb{L}(W \otimes L)$ over $G_F = (\mathrm{spec} \pi_0 f)^* G_E$.

LEMMA 4.6. *Suppose that W is a line bundle, and let $b \in G(\pi_0 \mathbf{F})$ be the corresponding point. Under the isomorphism (8.7)*

$$\mathbb{L}(W \otimes L) \cong T_b^* \mathcal{I}(0),$$

s_W is the section

$$s_W = T_b^*(\pi_0 f)^* s_g$$

Proof. The map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$ which classifies the tensor product of line bundles is responsible for the group structure of G . \square

Now take $X = BA^*$ so that $S_{\mathbf{F}} = (BA^*)_{\mathbf{E}}$, and take $W = V_{\text{reg}}$. Let

$$\epsilon: S_{\mathbf{F}} \rightarrow S_{\mathbf{E}}$$

be the structural map. We have a homomorphism

$$A \longrightarrow G_{\mathbf{F}}$$

as in (7.1) and an isomorphism of line bundles over $S_{\mathbf{F}}$

$$(4.7) \quad \mathbb{L}(V_{\text{reg}} \otimes L) \cong \bigotimes_{a \in A} T_a^* \epsilon^* \mathcal{I}(0),$$

as in (8.10). Lemma 4.6 implies the following.

PROPOSITION 4.8. *Under the isomorphism (4.7), we have*

$$s_{V_{\text{reg}}} = \bigotimes_{a \in A} T_a^* \epsilon^* s_g.$$

4.3. Comparing the H_∞ structures. We continue to suppose that \mathbf{E} is a homogeneous ring spectrum, and that

$$g: MU\langle 0 \rangle \rightarrow \mathbf{E}$$

is a map of homotopy commutative ring spectra. Now suppose in addition that $\pi_0 \mathbf{E}$ is a complete local ring with perfect residue field of characteristic $p > 0$, and that the formal group $G = G_{\mathbf{E}}$ is of finite height. Let A be a finite abelian group. Proposition 7.3 implies that, with our hypotheses on \mathbf{E} , the natural map (7.2)

$$(BA^*)_{\mathbf{E}} \rightarrow \underline{\text{hom}}(A, G)$$

is an isomorphism. If

$$A_T \xrightarrow{\ell} i^*G$$

is a level structure (9.9) with cokernel

$$i^*G \xrightarrow{q} G',$$

then the homomorphism ℓ is classified by a map χ_ℓ making the diagram

$$\begin{array}{ccc} T & \xrightarrow{\chi_\ell} & \underline{\text{hom}}(A, G) \\ & \searrow i & \downarrow \epsilon \\ & & S_E \end{array}$$

commute. After changing base along $\chi_\ell : T \times G \rightarrow \underline{\text{hom}}(A, G) \times G$, the isomorphism (4.7) becomes

$$(4.9) \quad \chi_\ell^* \mathbb{L}(V_{\text{reg}} \otimes L) \cong \bigotimes_{a \in A} T_a^* i^* \mathcal{I}(0),$$

and then Proposition 10.14 gives an isomorphism

$$(4.10) \quad \chi_\ell^* \mathbb{L}(V_{\text{reg}} \otimes L) \cong q^* N_q i^* \mathcal{I}_G(0) \cong q^* \mathcal{I}_{G'}(0).$$

Proposition 4.8 and Proposition 10.14 have the following

COROLLARY 4.11. *With respect to the isomorphism (4.10), we have*

$$\chi_\ell^* s_{V_{\text{reg}}} = q^* N_q i^* s_g.$$

Now suppose in addition that E is an H_∞ ring spectrum. Using the isogeny (3.21)

$$\psi_\ell^{G/E} : i^*G \rightarrow (\psi_\ell^E)^*G,$$

equation (4.10) becomes

$$(4.12) \quad \chi_\ell^* \mathbb{L}(V_{\text{reg}} \otimes L) \cong (\psi_\ell^{G/E})^* \mathcal{I}_{(\psi_\ell^E)^* G}(0).$$

The two ways of going around the diagram

$$\begin{array}{ccccc} (BA^* \times \mathbb{C}P^\infty)^{V_{\text{reg}} \otimes L} & \longrightarrow & D_A MU\langle 0 \rangle & \xrightarrow{D_A g} & D_A \mathbf{E} \\ & & \downarrow & & \downarrow \\ & & MU\langle 0 \rangle & \xrightarrow{g} & \mathbf{E} \end{array}$$

give two different trivializations s_{cl} and s_{cc} of $\mathbb{L}(V_{\text{reg}} \otimes L)$ over

$$(BA^* \times \mathbb{C}P^\infty)_{\mathbf{E}} = \underline{\text{hom}}(A, G) \times G,$$

and Corollary 4.11 shows that

$$\chi_\ell^* s_{\text{cc}} = \chi_\ell^* s_{V_{\text{reg}}} = (\psi_\ell^{G/\mathbf{E}})^* N_{\psi_\ell^{G/\mathbf{E}}} s_g.$$

By definition,

$$\chi_\ell^* s_{\text{cl}} = \psi_\ell^L(s_g),$$

and so with respect to the isomorphism (4.12), Proposition 3.21 gives

$$\chi_\ell^* s_{\text{cl}} = (\psi_\ell^{G/\mathbf{E}})^* (\psi_\ell^{\mathbf{E}})^* s_g.$$

Thus we have the following:

PROPOSITION 4.13. *Let $g: MU\langle 0 \rangle \rightarrow \mathbf{E}$ be a homotopy multiplicative map, and let $s = s_g$ be corresponding trivialization of $\mathcal{I}_G(0)$. If the map g is H_∞ , then for any level structure*

$$A \xrightarrow{\ell} i^* G,$$

the section s satisfies the identity

$$(4.14) \quad N_{\psi_\ell^{G/\mathbf{E}}} i^* s = (\psi_\ell^{\mathbf{E}})^* s,$$

in which the isogeny $\psi_\ell^{G/\mathbf{E}}$ has been used make the identification

$$N_{\psi_\ell^{G/\mathbf{E}}} i^* \mathcal{I}_G(0) \cong \mathcal{I}_{(\psi_\ell^{\mathbf{E}})^* G}(0).$$

Remark 4.15. The proposition can be stated in terms of formal group laws along the lines of Lemma 4.5. Given an orientation g and a level structure ℓ as

in the Proposition, we get two ring homomorphisms

$$\alpha, \beta: \pi_0 MU\langle 0 \rangle \rightarrow \mathbf{E},$$

namely

$$\alpha: \pi_0 MU\langle 0 \rangle \xrightarrow{P_A} \pi_0 MU\langle 0 \rangle^{BA_+^*} \xrightarrow{\pi_0 g_+^{BA_+^*}} \pi_0 \mathbf{E}^{BA_+^*} \xrightarrow{\chi_\ell} R$$

and

$$\beta: \pi_0 MU\langle 0 \rangle \xrightarrow{\pi_0 g} \pi_0 \mathbf{E} \xrightarrow{\psi_\ell^\mathbf{E}} R.$$

The proposition implies that α classifies the group law $((\psi_\ell^\mathbf{E})^* G, N_\ell i^* s_g)$, while β classifies the group law $((\psi_\ell^\mathbf{E})^* G, (\psi_\ell^\mathbf{E})^* s_g)$.

Remark 4.16. The necessary condition of Proposition 4.13 was introduced in [And95], in the case that \mathbf{E} is the spectrum associated to the universal deformation of the Honda formal group of height n . In that case, if one has a level structure

$$(\mathbb{Z}/p)^n \xrightarrow{\ell} i^* G_{\mathbf{E}},$$

one finds that

$$\begin{aligned} \psi_\ell^\mathbf{E} &= i \\ \psi_\ell^{G/\mathbf{E}} &= p: G_{\mathbf{E}} \rightarrow G_{\mathbf{E}}, \end{aligned}$$

so that equation (4.14) becomes

$$N_p i^* s = i^* s,$$

or after pulling back along p ,

$$\prod_{a \in (\mathbb{Z}/p)^n} T_a^* s = p^* s.$$

5. A necessary condition for an $MU\langle 2k \rangle$ -orientation to be H_∞ . In this section we describe the modifications to Proposition 4.13 needed in the case that $k \geq 1$ and

$$g: MU\langle 2k \rangle \rightarrow \mathbf{E}$$

is a homotopy multiplicative map from the Thom spectrum of $BU\langle 2k \rangle$ to \mathbf{E} .

Let bu denote connective K -theory. We recall that

$$[X, BU\langle 2k \rangle] \cong bu^{2k}(X).$$

This makes it clear that if V is a $BU\langle 2k \rangle$ -bundle over X and W is any (virtual) vector bundle, then $W \otimes V$ has a canonical $BU\langle 2k \rangle$ -structure. Also, as the map

$$\mathbb{C}P^\infty \rightarrow BU = BU\langle 2 \rangle$$

classifying the reduced tautological bundle $1 - L$ may be viewed as an element of $bu^2(\mathbb{C}P^\infty)$, it follows that the bundle

$$V = (1 - L_1) \otimes \cdots \otimes (1 - L_k)$$

over $(\mathbb{C}P^\infty)^k$ has a $BU\langle 2k \rangle$ -structure.

Suppose that \mathbf{E} is an even periodic ring spectrum and let $G = G_{\mathbf{E}}$.

LEMMA 5.1. (1) *Proposition 8.12 gives an isomorphism*

$$t_V: \mathbb{L}(V) \cong \Theta^k(\mathcal{I}_G(0)).$$

(2) *For the bundle $L \otimes V$ over $\mathbb{C}P^{\infty k+1}$, Proposition 8.12 gives an isomorphism*

$$t_{L \otimes V}: \mathbb{L}(L \otimes V) \cong \frac{\hat{\mu}_{12}^* \mathbb{L}(V)}{\hat{\pi}_2^* \mathbb{L}(V)}$$

of line bundles over $G_{\mathbf{E}}^{k+1}$.

(3) *In the notation of Lemma 4.3, the $MU\langle 2k \rangle$ -Thom class of the bundle $L \otimes V$ over $(\mathbb{C}P^\infty)^{k+1}$ is given by*

$$\Phi_{MU\langle 2k \rangle}(L \otimes V) = \frac{\hat{\mu}_{12}^* \Phi_{MU\langle 2k \rangle}(V)}{\hat{\pi}_2^* \Phi_{MU\langle 2k \rangle}(V)}.$$

Proof. The first part follows from the discussion of the line bundles $\mathbb{L}(V)$ in §8. For the second two parts, simply write

$$L \otimes V = (1 - LL_1) \otimes (1 - L_2) \otimes \cdots \otimes (1 - L_k) - (1 - L) \otimes (1 - L_2) \otimes \cdots \otimes (1 - L_k),$$

and use Lemma 4.3. □

The lemma implies that if $g: MU\langle 2k \rangle \rightarrow \mathbf{E}$ is a homotopy multiplicative map, then the composition

$$(\mathbb{C}P^{\infty k})^V \rightarrow MU\langle 2k \rangle \xrightarrow{g} \mathbf{E}$$

represents a trivialization $s = s_g$ of $\Theta^k(\mathcal{I}_G(0))$ (In fact it is easily seen to be a Θ^k -structure on $\mathcal{I}_G(0)$ in the sense of (13.5)). If \mathbf{E} is an H_∞ ring spectrum, and A is a finite abelian group, then the two ways of going around the diagram

$$(5.2) \quad \begin{array}{ccc} (BA^* \times (\mathbb{C}P^\infty)^k)^{V_{\text{reg}} \otimes V} & \longrightarrow & D_A MU\langle 2k \rangle \xrightarrow{D_A g} D_A \mathbf{E} \\ \downarrow & & \downarrow \\ MU\langle 2k \rangle & \xrightarrow{g} & \mathbf{E} \end{array}$$

give two different trivializations s_{cl} and s_{cc} of $\mathbb{L}(V_{\text{reg}} \otimes V)$ over $\text{spf } \hat{\pi}_0 \mathbf{E}^{(BA^* \times (\mathbb{C}P^\infty)^k)_+} = \underline{\text{hom}}(A, G) \times G^k$. The second part of Lemma 5.1 implies that

$$(5.3) \quad \mathbb{L}(V_{\text{reg}} \otimes V) \cong \bigotimes_{a \in A} \tilde{T}_a \mathbb{L}(V),$$

where \tilde{T}_a is translation operation introduced in (14.1).

If

$$A_T \xrightarrow{\ell} i^* G$$

is a level structure on G (9.9), then after changing base along the map

$$T \times G^k \xrightarrow{\chi_\ell \times G^k} \underline{\text{hom}}(A, G) \times G^k$$

and using the isogeny

$$\psi_\ell^{G/\mathbf{E}}: i^* G \longrightarrow (\psi_\ell^{\mathbf{E}})^* G$$

of Proposition 3.21, we have isomorphisms

$$(5.4) \quad \begin{aligned} \chi_\ell^* \mathbb{L}(V_{\text{reg}} \otimes V) &\cong (\psi_\ell^{G/\mathbf{E}})^* \tilde{N}_{\psi_\ell^{G/\mathbf{E}}} i^* \Theta^k(\mathcal{I}_G(0)) \\ &\cong (\psi_\ell^{G/\mathbf{E}})^* \Theta^k(\mathcal{I}_{(\psi_\ell^{\mathbf{E}})^* G}(0)); \end{aligned}$$

from (5.3) and (14.9). The third part of Lemma 5.1 and Definition 14.7 imply that with respect to this isomorphism we have

$$\chi_\ell^* s_{\text{cc}} = (\psi_\ell^{G/\mathbf{E}})^* \tilde{N}_{\psi_\ell^{G/\mathbf{E}}} i^* s_g.$$

By definition, we have

$$\chi_\ell^* s_{\text{cl}} = \psi_\ell^V(s_g),$$

and with respect to the isomorphism (5.4), Proposition 3.24 gives the equation

$$\chi_\ell^* s_{\text{cl}} = (\psi_\ell^{G/\mathbf{E}})^*(\psi_\ell^\mathbf{E})^* s_g.$$

The analogue of Proposition 4.13 is:

PROPOSITION 5.5. *Let $g: MU\langle 2k \rangle \rightarrow \mathbf{E}$ be a homotopy multiplicative map, and s corresponding section of $\Theta^k(\mathcal{I}_G(0))$. If the map f is H_∞ , then for each level structure*

$$A \xrightarrow{\ell} i^* G$$

the section s satisfies the identity

$$\tilde{N}_{\psi_\ell^{G/\mathbf{E}}} i^* s = (\psi_\ell^\mathbf{E})^* s,$$

in which the map $\psi_\ell^{G/\mathbf{E}}$ has been used make the identification

$$\tilde{N}_{\psi_\ell^{G/\mathbf{E}}} \Theta^k(\mathcal{I}_{i^* G}(0)) \cong \Theta^k(\mathcal{I}_{(\psi_\ell^\mathbf{E})^* G}(0))$$

as in (5.4).

6. The necessary condition is sufficient for $k \leq 3$. Suppose that \mathbf{E} is an even periodic H_∞ spectrum. Suppose that $\pi_0 \mathbf{E}$ is a p -regular admissible local ring with perfect residue field of characteristic p , and the formal group $G = G_{\mathbf{E}}$ is of finite height. Suppose that $k \leq 3$, and let $g: MU\langle 2k \rangle \rightarrow \mathbf{E}$ be a homotopy multiplicative map. Let $s = s_g$ be the section of $\Theta^k(\mathcal{I}_G(0))$ as in §5.

PROPOSITION 6.1. *The map g is H_∞ if and only if for each level structure*

$$A \xrightarrow{\ell} i^* G,$$

the section s satisfies the identity

$$\tilde{N}_{\psi_\ell^{G/\mathbf{E}}} s = (\psi_\ell^\mathbf{E})^* i^* s,$$

in which, as in Proposition 5.5, the isogeny $\psi_\ell^{G/\mathbf{E}}$ has been used make the identification

$$\tilde{N}_{\psi_\ell^{G/\mathbf{E}}} \Theta^k(\mathcal{I}_{i^* G}(0)) \cong \Theta^k(\mathcal{I}_{(\psi_\ell^\mathbf{E})^* G}(0)).$$

Proof. We must show that, for all n , the diagram

$$\begin{array}{ccc} D_n MU\langle 2k \rangle & \xrightarrow{D_ng} & D_n \mathbf{E} \\ \downarrow & \cdot & \downarrow \\ MU\langle 2k \rangle & \xrightarrow{g} & \mathbf{E} \end{array}$$

commutes. The hypotheses on $\pi_0 \mathbf{E}$ and the algebra of the D_n 's together with the Sylow structure of the symmetric groups reduce us immediately to checking that the diagram

$$\begin{array}{ccc} D_A MU\langle 2k \rangle & \xrightarrow{D_A g} & D_A \mathbf{E} \\ \downarrow & & \downarrow \\ MU\langle 2k \rangle & \xrightarrow{g} & \mathbf{E} \end{array}$$

commutes when A is a Sylow subgroup of Σ_p [McC86, §7].

Let g_{cl} and g_{cc} be the two ways of navigating this diagram. Each is a generator of $\pi_0 \mathbb{S}_U(D_A MU\langle 2k \rangle, \mathbf{E})$, so by the Thom isomorphism their ratio is a generator of

$$\pi_0 \mathbb{S}_U(D_A BU\langle 2k \rangle_+, \mathbf{E}).$$

For $k \leq 3$, the natural map

$$\pi_0 \mathbb{S}_U(D_A BU\langle 2k \rangle_+, \mathbf{E}) \longrightarrow \Delta^* \pi_0 \mathbb{S}_U((BA^* \times BU\langle 2k \rangle)_+, \mathbf{E})$$

is injective (see e.g. [McC86, 7.3]). Let $\mathbf{F} = \mathbf{E}^{BU\langle 2k \rangle_+}$. Our hypotheses on \mathbf{E} and the fact that, for $k \leq 3$, $H_*(BU\langle 2k \rangle, \mathbb{Z})$ is concentrated in even degrees (for $B\mathbf{U}\langle 6 \rangle$ see [Sin68] or [AHS01]) imply that the natural maps induce isomorphisms

$$\begin{aligned} \pi_0 \mathbf{E} &\cong \mathcal{O}(S_{\mathbf{E}}) \\ \pi_0 \mathbf{F} &\cong \mathcal{O}(S_{\mathbf{F}}) \\ \pi_0 \mathbb{S}_U((BA^* \times BU\langle 2k \rangle)_+, \mathbf{E}) &\cong \mathcal{O}(\underline{\text{hom}}(A, G_{\mathbf{F}})). \end{aligned}$$

By Proposition 9.24, it suffices to show that $g_{\text{cl}}/g_{\text{cc}} = 1$ after changing base along the two maps

$$\begin{aligned} \underline{\text{level}}(A, G_{\mathbf{F}}) &\rightarrow \underline{\text{hom}}(A, G_{\mathbf{F}}) \\ S_{\mathbf{F}} &\rightarrow \underline{\text{hom}}(A, G_{\mathbf{F}}) \end{aligned}$$

classifying respectively the level structure and the zero homomorphism.

After changing base to $\underline{\text{level}}(A, G)$, $g_{\text{cl}}/g_{\text{cc}}$ becomes

$$((\psi_\ell^F)^* s) / (\tilde{N}_{\psi_\ell^{G/F} i^*} s),$$

as in Proposition 5.5. The base change $S_F \rightarrow \underline{\text{hom}}(A, G)$ corresponds to the augmentation

$$\pi_0 F^{BA_+^*} \rightarrow \pi_0 F,$$

under which each g restricts to the Thom class

$$BU\langle 2k \rangle^{V^p} \xrightarrow{\Phi_{MU\langle 2k \rangle}(V^p)} MU\langle 2k \rangle \xrightarrow{g} E$$

of V^p , where V is the standard bundle over $BU\langle 2k \rangle$. \square

Part 2. Even periodic cohomology of abelian groups and Thom complexes.

7. Even cohomology of abelian groups. Suppose that E is a homogeneous ring spectrum, with formal group $G = G_E$, and let A be a finite abelian group. Let A^* be the character group $A^* = \text{hom}(A, \mathbb{C}^\times)$. An element a of A may be viewed as a character of A^* , giving a line bundle V_a over BA^* and so a map $(BA^*)_E \rightarrow (\mathbb{C}P^\infty)_E = G$, i.e. a $(BA^*)_E$ -valued “point” of G . As a varies we get a map of sets

$$(7.1) \quad A \xrightarrow{\chi} G((BA^*)_E).$$

Since

$$V_{a+b} = V_a \otimes V_b,$$

and since the group structure of G comes from the map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$$

which classifies the tensor product of line bundles, the map χ is a group homomorphism, and so it is classified by a map of formal schemes

$$(7.2) \quad (BA^*)_E \xrightarrow{\bar{\chi}} \underline{\text{hom}}(A, G).$$

This map is often an isomorphism. For example, we have the

PROPOSITION 7.3. *If $\pi_0 E$ is a complete local ring of residue characteristic $p > 0$, and if the height of the formal group G_E is finite, then the map $\bar{\chi}$ is an isomorphism of formal schemes over S_E .*

Proof. This formulation of the \mathbf{E} -cohomology of abelian groups appeared in [HKR00]. \square

Suppose that $\tilde{\chi}$ is an isomorphism, and suppose that we have a level structure

$$A_T \xrightarrow{\ell} i^* G$$

over a formal scheme T . The homomorphism ℓ is classified by a map χ_ℓ making the diagram

$$(7.4) \quad \begin{array}{ccc} T & \xrightarrow{\chi_\ell} & (BA^*)_{\mathbf{E}} \\ & \searrow i & \downarrow \\ & & S_{\mathbf{E}} \end{array}$$

commute.

PROPOSITION 7.5. *If $\pi_0 \mathbf{E}$ is a complete local ring, and G is of finite height, and if $A'^* \subset A^*$ is a proper subgroup, then the composite map of $\pi_0 \mathbf{E}$ -modules*

$$\pi_0 \mathbf{E}^{BA'^*_+} \xrightarrow{\text{transfer}} \pi_0 \mathbf{E}^{BA^*_+} \xrightarrow{\chi_\ell} \mathcal{O}(T)$$

is zero.

Proof. It suffices to consider the case that

$$\ell: A \rightarrow i^* G$$

is the tautological level structure over $\underline{\text{level}}(A, G)$.

If A is not a p -group then $\underline{\text{level}}(A, G)$ is empty and the result is trivial.

Suppose that $A' = 0$ and $A = \mathbb{Z}/p$. Let $t \in \pi_0 \mathbf{E}^{\mathbb{C}P^\infty_+}$ be a coordinate, and let F be the resulting group law. Then

$$\pi_0 \mathbf{E}^{BA^*_+} \cong \pi_0 \mathbf{E}[[t]]/[p]_F(t)$$

and $\tau: \pi_0 \mathbf{E}^{BA'^*_+} = \pi_0 \mathbf{E} \rightarrow \pi_0 \mathbf{E}^{BA^*}$ is given by

$$\tau(1) = \langle p \rangle(t)$$

(see e.g. [Qui71]), where $\langle p \rangle(t)$ is the power series such that

$$t \langle p \rangle(t) = [p]_F(t).$$

The result follows from the isomorphism (9.23)

$$\mathcal{O}(\underline{\text{level}}(\mathbb{Z}/p, G_{\mathbf{E}})) \cong \pi_0 \mathbf{E}[[t]] / \langle p \rangle(t).$$

For the general case, we may suppose that $A'^* \subsetneq A^*$ is maximal, and so we have a pull-back diagram

$$\begin{array}{ccc} BA'^* & \xrightarrow{i'} & B0 \\ j' \downarrow & & \downarrow j \\ BA^* & \xrightarrow{i} & BC^* \end{array}$$

where $C \subsetneq A$ is cyclic of order p . The commutativity of the diagram

$$\begin{array}{ccccc} \pi_0 \mathbf{E}^{BA'^*} & \xleftarrow{i'^*} & \pi_0 \mathbf{E}^{B0} & & \\ \tau \downarrow & & \downarrow \tau & & \\ \pi_0 \mathbf{E}^{BA^*} & \xleftarrow{i^*} & \pi_0 \mathbf{E}^{BC^*} & & \\ \pi \downarrow & & \downarrow & & \\ \mathcal{O}(\underline{\text{level}}(A, G)) & \longleftarrow & \mathcal{O}(\underline{\text{level}}(C, G)) & & 0 \end{array}$$

implies that

$$\pi(\tau(1)) = 0.$$

The result follows, since $\pi_0 \mathbf{E}^{BA'^*}$ is a cyclic $\pi_0 \mathbf{E}^{BA^*}$ -module via j'^* , and τ is a map of $\pi_0 \mathbf{E}^{BA^*}$ -modules. \square

8. Algebraic geometry of the Thom isomorphism. Suppose that X is a space, and that V is a complex vector bundle over X . The $\pi_0 \mathbf{E}^{X^+}$ -module $\pi_0 \mathcal{S}_U(X^V, \mathbf{E})$ is free of rank one (since \mathbf{E} is complex orientable) and so can be interpreted as the module of sections of a line bundle $\mathbb{L}(V)$ over $X_{\mathbf{E}}$. The fact that the Thom complex of an external Whitney sum is the smash product of the Thom complexes gives rise to a canonical isomorphism

$$(8.1) \quad \mathbb{L}(V \oplus W) \cong \mathbb{L}(V) \otimes \mathbb{L}(W).$$

This property can then be used to extend the definition of $\mathbb{L}(V)$ to virtual bundles; we define

$$(8.2) \quad \mathbb{L}(V - W) = \mathbb{L}(V) \otimes \mathbb{L}(W)^{-1}.$$

If $f: X \rightarrow Y$ is a map, and V is a virtual bundle over Y , then there is an isomorphism

$$\pi_0 \mathbb{S}_U(X^{f^* V}, \mathbf{E}) \cong (\pi_0 \mathbf{E}^{X_+}) \otimes \pi_0 \mathbb{S}_U(Y^V, \mathbf{E}).$$

In terms of algebraic geometry, this means that there is a natural isomorphism

$$(8.3) \quad \mathbb{L}(f^* V) \cong (f_{\mathbf{E}})^* \mathbb{L}(V).$$

Here is a series of examples which lead to a fairly complete understanding of the functor \mathbb{L} :

(1) If L denotes the canonical line bundle over $\mathbb{C}P^\infty$, then the zero section identifies $\pi_0 \mathbb{S}_U(\mathbb{C}P^\infty L, \mathbf{E})$ with the augmentation ideal in $\pi_0 \mathbf{E}^{\mathbb{C}P^\infty}$, and so we have an isomorphism

$$(8.4) \quad \mathbb{L}(L) \cong \mathcal{I}(0).$$

(2) Suppose that V is a line bundle over X , classified by a map $b: X \rightarrow \mathbb{C}P^\infty$. In terms of algebraic geometry, the map b defines an $X_{\mathbf{E}}$ -value point $b = b_{\mathbf{E}}$ of G . It follows from (8.3) that

$$(8.5) \quad \mathbb{L}(V) \cong b^* \mathcal{I}(0) \cong 0^* \mathcal{I}(-b).$$

(3) Taking X to be a point and V to be the trivial complex line bundle in (2), we have

$$(8.6) \quad \mathbb{L}(V) \cong 0^* \mathcal{I}(0).$$

Now $\mathbb{L}(V)$ is the sheaf associated to $\pi_2 \mathbf{E}$, while $0^* \mathcal{I}(0)$ is the sheaf of cotangent vectors at the origin of G , isomorphic to the sheaf ω_G of invariant differentials on G .

(4) If V is the trivial bundle of dimension k , then by (8.6) and (8.1), $\mathbb{L}(V)$ is just ω_G^k . If $f: \mathbf{E} \rightarrow \mathbf{F}$ is an \mathbf{E} -algebra (e.g. $\mathbf{F} = \mathbf{E}^{X_+}$), this gives an interpretation of the homotopy group $\pi_{2k} \mathbf{F}$ as the sections of $f^* \omega_G^k$.

(5) If $V = (1 - L)$ is the reduced canonical line bundle over $\mathbb{C}P^\infty$, then using (8.2), (8.4), and (8.6) we have

$$\mathbb{L}(V) \cong \pi^* 0^* \mathcal{I}(0) \otimes \mathcal{I}(0)^{-1} = \Theta^1(\mathcal{I}(0)),$$

where $\pi: G_{\mathbf{E}} \rightarrow S_{\mathbf{E}}$ is the structural map and Θ^1 is defined in Definition 13.2.

(6) With the notation of example (2) consider the line bundle $V \otimes L$ over $X \times \mathbb{C}P^\infty$. Then $\mathbb{L}(V \otimes L)$ is pulled back from $\mathcal{I}_G(0)$ along the map

$$X_{\mathbf{E}} \times G \xrightarrow{b \times 1} G \times G \xrightarrow{\mu} G.$$

It follows that

$$(8.7) \quad \mathbb{L}(V \otimes L) \cong T_b^* \mathcal{I}(0) = \mathcal{I}_{X_E \times G}(-b).$$

(7) More generally, suppose that $V = \sum n_i L_i$ is a virtual sum of line bundles over X . The line bundles L_i define points b_i of G over X_E , and the bundle V determines the divisor $D = \sum n_i [b_i]$. It follows using (8.1) that

$$(8.8) \quad \mathbb{L}(V \otimes L) = \mathcal{I}_{X_E \times G}(D^{-1}),$$

where $D^{-1} = \sum n_i [b_i^{-1}]$.

(8) In fact, by the splitting principle, the line bundle $\mathbb{L}(V \otimes L)$ can be computed in this manner even when V is not a virtual sum of line bundles. Indeed, by the splitting principle, there is a map $f: F \rightarrow X$ with the properties that f_E is finite and faithfully flat, and $f^* V$ is a virtual sum of line bundles. The line bundle $\mathbb{L}(f^*(V) \otimes L)$ can then be computed as $\mathcal{O}(D^{-1})$ as above. But the divisor D descends to $X_E \times G$, even if none of its points do.

(9) Let A be a finite abelian group. An element $a \in A$ can be regarded as a character of A^* . Let V_a be the associated line bundle over BA^* . Recall (7.1) that this construction defines a group homomorphism

$$\chi: A \rightarrow G(BA_E^*).$$

The line bundle $\mathbb{L}(V_a \otimes V \otimes L)$ over $BA_E^* \times X_E \times G$ is

$$\mathbb{L}(V_a \otimes V \otimes L) \cong T_a^* \mathcal{I}(D^{-1});$$

taking V to be the trivial line bundle over a point gives

$$\mathbb{L}(V_a \otimes L) \cong T_a^* \mathcal{I}(0) = I(a^{-1}).$$

(10) Now let

$$V_{\text{reg}} = \bigoplus_{a \in A} V_a$$

be the regular representation of A^* . Over the scheme $(BA^*)_E \times G$, the line bundle associated to the Thom complex of $V_{\text{reg}} \otimes V \otimes L$ is

$$(8.9) \quad \mathbb{L}(V_{\text{reg}} \otimes V \otimes L) \cong \bigotimes_{a \in A} T_a^* \mathcal{I}(D^{-1}) \cong \mathcal{I} \left(\sum_a T_a^* D^{-1} \right).$$

In particular,

$$(8.10) \quad \mathbb{L}(V_{\text{reg}} \otimes L) \cong \bigotimes_{a \in A} T_a^* \mathcal{I}(0) \cong \mathcal{I}(\chi).$$

(11) Suppose that the map

$$\tilde{\chi}: (BA^*)_{\mathbf{E}} \rightarrow \underline{\text{hom}}(A, G)$$

of (7.2) is an isomorphism. If

$$A_T \xrightarrow{\ell} i^* G \xrightarrow{q} G'$$

is a level structure with cokernel q over T , then changing base in (8.10) along

$$T \times G \xrightarrow{\chi_\ell} \underline{\text{hom}}(A, G) \times G$$

(where χ_ℓ is the map classifying the homomorphism ℓ ; see (7.4)) and using (10.15) gives

$$(8.11) \quad \chi_\ell^* \mathbb{L}(V_{\text{reg}} \otimes L) \cong q^* N_q \mathcal{I}_G(0) \cong q^* \mathcal{I}_{G'}(0) \cong \mathcal{I}_G(\ell).$$

(12) Restricting the above example to BA^* we find that

$$\begin{aligned} \chi_\ell^* \mathbb{L}(V_{\text{reg}}) &= 0_G^* q^* \mathcal{I}_{G'}(0) = 0_{G'}^* \mathcal{I}_{G'}(0) \\ &= \omega_{G'}. \end{aligned}$$

This series of examples establishes the following results:

PROPOSITION 8.12. *For a pointed topological space X , let \mathbf{F} be the spectrum \mathbf{E}^{X_+} , and let*

$$G = G_{\mathbf{F}} = (\mathbb{C}P^\infty)_{\mathbf{F}}$$

be the associated formal group. Attached to each (virtual) complex vector bundle V over X is a divisor $D = D_V$ on G , and an isomorphism

$$(8.13) \quad t_V: \mathbb{L}(V \otimes L) \cong \mathcal{I}_G(D^{-1}).$$

The map t_V restricts to an isomorphism

$$t_V: \mathbb{L}(V) \cong 0^* \mathcal{I}(D^{-1}).$$

PROPOSITION 8.14. *The correspondence $V \mapsto D_V$ and the isomorphism (8.13) are determined by the following properties:*

(i) *If $V = V_1 \oplus V_2$, then $D_V = D_{V_1} + D_{V_2}$, and with the identifications*

$$\begin{aligned}\mathbb{L}(V_1) \otimes \mathbb{L}(V_2) &\cong \mathbb{L}(V) \\ \mathcal{I}(D_{V_1}^{-1}) \otimes \mathcal{I}(D_{V_2}^{-1}) &\cong \mathcal{I}(D_V^{-1}),\end{aligned}$$

there is an equality

$$t_V = t_{V_1} \otimes t_{V_2}.$$

(ii) *If $f: Y \rightarrow X$ is a map of pointed spaces, and if $W = f^*V$, then $D_W = f^*D_V$, and $t_W = f^*t_V$.*

(iii) *If X is a point, and V has dimension 1, then $D = [0]$, and the isomorphism*

$$(8.15) \quad t_L: \mathbb{L}(L) \cong \mathcal{I}(0)$$

is given by applying $\pi_0 \mathbf{E}^{(-)}$ to the zero section $\mathbb{C}P_+^\infty \rightarrow \mathbb{C}P^{\infty L}$.

Part 3. Level structures and isogenies of formal groups.

9. Level structures.

9.1. Homomorphisms. Suppose that A is a finite abelian group and G is a formal group over a formal scheme S .

Definition 9.1. We write $\underline{\text{hom}}(A, G)$ for the functor from formal schemes to groups defined by the formula

$$\underline{\text{hom}}(A, G)(T) = \{\text{pairs } (u, \ell) \mid u: T \rightarrow S, \ell \in \text{hom}(A, u^*G(T))\}.$$

Remark 9.2. We shall use the notation

$$A_T \xrightarrow{\ell} u^*G$$

to indicate that T is a formal scheme and $(u, \ell) \in \underline{\text{hom}}(A, G)(T)$.

Example 9.3. Let G be a formal group over R , and suppose that x is a coordinate on G . Let F be the resulting group law. The “ n -series” of F is the power series $[n](t) \in R[[t]]$ defined by the formula

$$[n](x) = n^*x,$$

where the right-hand-side refers to the pull-back of functions along the homomorphism $n: G \rightarrow G$. To give a homomorphism

$$\ell: \mathbb{Z}/n \rightarrow G(T)$$

is to give a topologically nilpotent element $x(\ell(1))$ of $\mathcal{O}(T)$, with the property that

$$[n](x(\ell(1))) = 0;$$

the homomorphism ℓ is then given by

$$x(\ell(j)) = [j](x(\ell(1))).$$

It follows that

$$\underline{\text{hom}}(\mathbb{Z}/n, G) = \text{spf}(R[[x(\ell(1))]] / ([n](x(\ell(1))))).$$

It is clear from the definition that if $B \subseteq A$ then there is a restriction map

$$\underline{\text{hom}}(A, G) \rightarrow \underline{\text{hom}}(B, G),$$

and if $A = B \times C$ then the resulting map

$$(9.4) \quad \underline{\text{hom}}(A, G) \rightarrow \underline{\text{hom}}(B, G) \times_S \underline{\text{hom}}(C, G)$$

is an isomorphism. Also from the definition we see that if $j: S' \rightarrow S$ is a map of formal schemes, then the natural map

$$\underline{\text{hom}}(A, j^*G) \rightarrow j^*\underline{\text{hom}}(A, G)$$

is an isomorphism. Combining these observations with Example 9.3 and the structure of finite abelian groups gives the following.

LEMMA 9.5. *The functor $\underline{\text{hom}}(A, G)$ is represented by an affine formal scheme over S . If $j: S' \rightarrow S$ is a map of formal schemes, then the natural map*

$$\underline{\text{hom}}(A, j^*G) \rightarrow j^*\underline{\text{hom}}(A, G)$$

is an isomorphism of formal schemes over S' .

For formal groups over p -local rings, only the p -groups give anything interesting.

Example 9.6. Returning to Example 9.3, the n -series is easily seen to be of the form

$$[n](t) = nt + o(2).$$

If n is a unit in R , then

$$R[[x]]/([n](x)) \cong R$$

so $\underline{\text{hom}}(\mathbb{Z}/n, G)$ is the trivial group scheme over R .

Example 9.7. If R is a complete local ring of residue characteristic p , then there is an h with $1 \leq h \leq \infty$ such that

$$[p^m](t) \equiv \epsilon t^{p^{mh}} + o(t^{p^h} + 1) \pmod{\mathfrak{m}_R}.$$

This h is called the *height* of G . If h is finite, then the Weierstrass Preparation Theorem [Lan78, pp. 129–131] implies that there are monic polynomials $g_m(t)$ of degree p^{mh} such that

$$[p^m](t) = g_m(t) \cdot \epsilon,$$

where ϵ is a unit of $R[[t]]$. It follows that $\mathcal{O}(\underline{\text{hom}}(\mathbb{Z}/p^m, G))$ is finite and free of rank p^{hm} over R .

These examples generalize to give the following:

PROPOSITION 9.8. *Let G be a formal group of finite height over a local formal scheme S . Then $\underline{\text{hom}}(A, G)$ is a local formal scheme over S . For $B \subseteq A$, the forgetful map*

$$\underline{\text{hom}}(A, G) \rightarrow \underline{\text{hom}}(B, G)$$

is a map of formal schemes, finite and free of rank d^h , where d is the order of the p -torsion subgroup of A/B .

9.2. Level structures. The scheme $\underline{\text{hom}}(A, G)$ has an important closed subscheme $\underline{\text{level}}(A, G)$, which was introduced by Drinfel'd [Dri74]. Suppose that G is a formal group over a formal scheme S . For simplicity, we suppose that S is a local formal scheme of residue characteristic $p > 0$.

Definition 9.9. Let T be a formal scheme. A T -valued point

$$A_T \xrightarrow{\ell} i^*G$$

of $\underline{\text{hom}}(A, G)$ is a *level structure* if for each prime q dividing $|A|$, the subgroup $i^*G[q] = \ker(q : i^*G \rightarrow i^*G)$ is a divisor on G/T , and there is an inequality of divisors

$$\sum_{\substack{a \in A \\ qa=0}} [\ell(a)] \leq i^*G[q]$$

in i^*G . The subfunctor of $\underline{\text{hom}}(A, G)$ consisting of level structures will be denoted $\underline{\text{level}}(A, G)$.

Remark 9.10. If we say that

$$A_T \xrightarrow{\ell} i^*G$$

“is a level structure,” we mean that T is a formal scheme, and (i, ℓ) is a T -valued point of $\underline{\text{level}}(A, G)$. We may omit one of T and i if it is clear from the context.

Here are some examples to give a feel for level structures. First of all, only p -groups of small rank can produce level structures.

LEMMA 9.11. *If $|A|$ is not a power of p , then*

$$\underline{\text{level}}(A, G) = \emptyset.$$

If the height of G is h and the p -rank of A is greater than h , then again $\underline{\text{level}}(A, G) = \emptyset$.

Proof. If $|A|$ is not a power of p , then there is a prime $q \neq p$ such that the divisor

$$\sum_{qa=0} [\ell(a)]$$

has degree greater than 1. However, $q : G \rightarrow G$ is an isomorphism, so $G[q] = [0]$ has degree 1. Similarly, if the height of G is h then the degree of $G[p]$ is p^h . \square

A level structure is trying to be a monomorphism; for example if R is a domain in which $|A| \neq 0$, then a homomorphism

$$\ell : A \rightarrow G(R)$$

is a level structure if and only if it is a monomorphism (Corollary 9.21). However, naive monomorphisms from A to G can't in general be a representable functor.

Example 9.12. Let $\widehat{\mathbb{G}}_m$ be the formal multiplicative group with coordinate x so that the group law is

$$x \underset{F}{+} y = x + y - xy.$$

The p -series is

$$[p](x) = 1 - (1 - x)^p.$$

The monomorphism

$$\mathbb{Z}/p \longrightarrow \widehat{\mathbb{G}}_m(\mathbb{Z}[[y]]/[p](y))$$

given by $j \mapsto [j](y)$ becomes the zero map under the base change

$$\begin{aligned} \mathbb{Z}[[y]]/([p](y)) &\rightarrow \mathbb{Z}/p \\ y &\mapsto 0. \end{aligned}$$

On the other hand, the functor level(A, G) is representable.

LEMMA 9.13. *Let G be a formal group of finite height over a local formal scheme S , and let A be a finite abelian group. The functor level(A, G) is a closed formal subscheme of hom(A, G).*

Proof. See Katz and Mazur [KM85, 1.3.4] or [Str97]. □

It is clear from the definition that if $j: S' \rightarrow S$ is a map of formal schemes, then the natural map

$$\underline{\text{level}}(A, j^*G) \rightarrow j^*\underline{\text{level}}(A, G)$$

is an isomorphism of formal schemes over S' .

PROPOSITION 9.14. *Suppose that G is a formal group of finite height h over a local formal scheme S with perfect residue field of characteristic $p > 0$, and suppose that A is a finite abelian p -group with $|A[p]| \leq p^h$. Then we have the following:*

(i) *The functor level(A, G) is represented by a local formal scheme which is finite and flat over S : indeed $\mathcal{O}(\underline{\text{level}}(A, G))$ is a finite free $\mathcal{O}(S)$ -module.*

(ii) *If G is the universal deformation of a formal group over a perfect field (see §12) then level(A, G) is the formal spectrum of a Noetherian complete local domain which is regular of dimension h .*

Proof. With our hypotheses, we may suppose that G/S is the universal deformation of a formal group of height h over a perfect field k of characteristic p ; the general case follows by change of base.

If $A = A[p]$, then the result is precisely the Lemma of [Dri74, p. 572, in proof of Prop. 4.3]. The proof in the general case follows similar lines and is given in [Str97].

The proof of (i) for a general A can be given easily: by definition of $\underline{\text{level}}(A, G)$, the diagram

$$\begin{array}{ccc} \underline{\text{level}}(A, G) & \xrightarrow{j} & \underline{\text{hom}}(A, G) \\ i \downarrow & & \downarrow k \\ \underline{\text{level}}(A[p], G) & \xrightarrow{l} & \underline{\text{hom}}(A[p], G) \end{array}$$

is a pull-back. Proposition 9.8 implies that k is finite and free, and so i is too. \square

9.3. Level structures over p -regular schemes. In this section, we suppose that G is a formal group of finite height over a complete local ring E of residue characteristic $p > 0$. The following description of the subscheme $\underline{\text{level}}(A, G)$ was found by Hopkins in the course of his work on [HKR00].

PROPOSITION 9.15. *Suppose that p is not a zero divisor in E . Let x be a coordinate on G . The scheme $\underline{\text{level}}(A, G)$ is the closed subscheme of $\underline{\text{hom}}(A, G)$ defined by the ideal of annihilators of $x(\ell(a))$, where a ranges over the non-zero elements of $A[p]$.*

The proof will be given at the end of this section. Note that the ideal in the proposition is independent of the coordinate used to describe it.

For $n \geq 1$ let $A[n]$ denote the n -torsion in A . Let R be a complete local E -algebra, and consider the following conditions on a homomorphism

$$\ell: A \rightarrow G(R).$$

Again, they are phrased in terms of a choice of a coordinate x on G , but they are easily seen to be independent of that choice.

- (A) If $0 \neq a \in A[p]$ then $x(\ell(a))$ is regular (i.e. not a divisor of zero).
- (B) If $0 \neq a \in A[p]$ then $x(\ell(a))$ divides p .
- (C) $\prod_{a \in A[p]} (x - x(\ell(a)))$ divides $[p](x)$.
- (D) The natural map

$$(9.16) \quad R[[x]] \left/ \left(\prod_{a \in A[p]} (x - x(\ell(a))) \right) \right. \rightarrow \prod_{a \in A[p]} (R[[x]]/(x - x(\ell(a))))$$

is a monomorphism. Condition (C) says precisely that there is an inequality of Cartier divisors

$$\sum_{pa=0} [\ell(a)] \leq G[p].$$

Thus condition (C) is that ℓ is a level structure.

PROPOSITION 9.17. *If R is p -torsion free, then these conditions are equivalent.*

First we prove the following result. It will be convenient to use the symbol ϵ to denote a generic unit. Its value may change from line to line.

LEMMA 9.18. *Let $n = |A[p^m]|$. The discriminant of the set*

$$\{x(\ell(a)) \mid a \in A[p^m]\}$$

is

$$\Delta = \epsilon \prod_{0 \neq a \in A[p^m]} x(\ell(a))^n.$$

Proof. Let F be the group law associated to a coordinate on G . The formula

$$x - y = (x - y)\epsilon(x, y),$$

where $\epsilon(x, y) \in E[[x, y]]^\times$, gives

$$\begin{aligned} \Delta &= \prod_{a \neq b \in A[p^m]} (x(\ell(a)) - x(\ell(b))) \\ &= \epsilon \prod_F (x(\ell(a)) - x(\ell(b))) \\ &= \epsilon \prod x(\ell(a) - \ell(b)) \\ &= \epsilon \prod_{c \neq 0} \prod_{a-b=c} x(\ell(c)) \\ &= \epsilon \prod_{c \neq 0} x(\ell(c))^n. \end{aligned} \quad \square$$

Proof of Proposition 9.17. Under the hypothesis that p is regular in R , it is clear that (B) implies (A). Let's check that (A) implies (B). Note that

$$[p](x) = x(p + xe(x))$$

for some $e(x) \in E[[x]]$. For $a \in A[p]$ we have

$$0 = [p](x(\ell(a))) = x(\ell(a))(p + x(\ell(a))e(x(\ell(a)))).$$

If $x(\ell(a))$ is not a zero-divisor in R , then we must have

$$p = -x(\ell(a))e(x(\ell(a))).$$

Next, let check that (C) implies (B). If (C) holds, then there is a power series $e(x) \in E[[x]]$ such that

$$e(x) \prod_{a \in A[p]} (x - x(\ell(a))) = [p](x) = px + o(x^2)$$

The coefficient of x on the left is (up to a sign)

$$e(0) \prod_{0 \neq a \in A[p]} x(\ell(a))$$

so (B) holds.

Next let's check that (A) implies (D). With respect to the basis of powers of x in the domain and the obvious basis in the range, the matrix of (9.16) is the Vandermonde matrix on the set $x(\ell(A[p]))$. Condition (A) and Lemma 9.18 together imply that (9.16) is a monomorphism.

Finally, let's check that (D) implies (C). Each $x(\ell(a))$ is a root of $[p](x)$, so the image of $[p](x)$ in the range of (9.16) is zero. If (D) holds then $[p](x)$ is zero in the domain, which implies (C). \square

LEMMA 9.19. *Condition (A) holds if and only if, for all non-zero $a \in A$, $x(\ell(a))$ is a regular element of R . Condition (B) holds if and only if, for all non-zero $a \in A$, $x(\ell(a))$ divides a power of p . If p is regular in R , then (C) holds if and only if, for each m , $\prod_{a \in A[p^m]} (x - x(\ell(a)))$ divides $[p^m](x)$, and (D) holds if and only if for each m , the natural map*

$$(9.20) \quad R[[x]] \left/ \left(\prod_{a \in A[p^m]} (x - x(\ell(a))) \right) \right. \rightarrow \prod_{a \in A[p^m]} (R[[x]]/(x - x(\ell(a))))$$

is a monomorphism.

Proof. Recall that the p -series $[p](x)$ is divisible by x : let $\langle p \rangle(x)$ be the power series such that

$$[p](x) = x \langle p \rangle(x).$$

Thus

$$x(\ell(pa)) = x(\ell(a)) \langle p \rangle(x(\ell(a))).$$

So if $x(\ell(pa))$ divides zero (resp. a power of p), then so does $x(\ell(a))$; this proves the statement about (A) and (B). For the statement about (D), suppose that p is not a zero divisor in R , and condition (D) holds; by Proposition 9.17, condition (A) holds. With respect to the basis of powers of x in the domain and the obvious

basis in the range, the matrix of (9.20) is the Vandermonde matrix on the set $x(\ell(A[p^m]))$. Lemma 9.18 and the statement about (A) prove the statement about (D). For the statement about (C), suppose that p is not a zero divisor in R and condition (C) holds; by Proposition 9.17, condition (D) holds. It follows that for each m , the natural map (9.20) is a monomorphism. If $p^m a = 0$, then $x(\ell(a))$ is a root of $[p^m](x)$, so the image of $[p^m](x)$ in the range of (9.20) is zero, which implies the statement about (C). \square

COROLLARY 9.21. *If R is a domain of characteristic 0, then the conditions (A)–(C) hold if and only if $\ell: A \rightarrow G(R)$ is a monomorphism.*

Proof of Proposition 9.15. By Proposition 9.14, $R = \mathcal{O}(\underline{\text{level}}(A, G))$ is a finite free E -module. It follows that p is not a zero divisor in R . Proposition 9.17 implies that R is initial among complete local E -algebras satisfying (A). \square

Example 9.22. Let G be a formal group of finite height over a p -regular complete local ring R of residue characteristic p . Suppose that x is a coordinate on G . Let $\langle p \rangle(t) \in R[[t]]$ be the power series such that

$$t\langle p \rangle(t) = [p](t).$$

Proposition 9.15 implies that

$$(9.23) \quad \underline{\text{level}}(\mathbb{Z}/p, G) \cong \text{spf}(R[[x(\ell(1))]]/\langle p \rangle(x(\ell(1)))).$$

This calculation occurs as part of the proof of the Lemma in the proof of Proposition 4.3 of [Dri74].

9.4. Calculations in $\underline{\text{hom}}(\mathbb{Z}/p, G)$ via level structures. Let G be a formal group of finite height over a p -regular complete local ring R with perfect residue field of characteristic $p > 0$. Let A be a finite abelian group. By construction there is a natural map

$$\mathcal{O}(\underline{\text{hom}}(A, G)) \rightarrow \mathcal{O}(\underline{\text{level}}(A, G)).$$

There is also a ring homomorphism

$$\mathcal{O}(\underline{\text{hom}}(A, G)) \rightarrow R$$

classifying the zero homomorphism. The proof of Proposition 6.1 uses the following result:

PROPOSITION 9.24. *The natural map*

$$(9.25) \quad \mathcal{O}(\underline{\text{hom}}(\mathbb{Z}/p, G)) \rightarrow R \times \mathcal{O}(\underline{\text{level}}(\mathbb{Z}/p, G))$$

is injective.

Remark 9.26. This result is equivalent to the injectivity for the group \mathbb{Z}/p of the character map of Hopkins-Kuhn-Ravenel, and as such is proved in [HKR00].

Proof. Let h be the height of G . Let $\Lambda = (\mathbb{Z}/p)^h$. Let $g(x)$ be the monic polynomial of degree p^h such that

$$[p](x) = g(x)\epsilon$$

where $\epsilon \in R[[x]]^\times$. Then

$$(9.27) \quad \underline{\text{hom}}(\mathbb{Z}/p, G) = \text{spf } R[[x]]/[p](x) \cong \text{spf } R[[x]]/g(x),$$

and Proposition 9.15 (see Example 9.22) implies that

$$\mathcal{O}(\underline{\text{level}}(\mathbb{Z}/p, G)) = R[[x]]/\langle p \rangle(x).$$

Let

$$D = \mathcal{O}(\underline{\text{level}}(\Lambda, G))$$

and let

$$\ell: \Lambda \rightarrow G(D)$$

be the tautological homomorphism. By definition,

$$(9.28) \quad D = \mathcal{O}(\underline{\text{hom}}(\Lambda, G))/J,$$

where J is the ideal obtained by equating coefficients in

$$(9.29) \quad \prod_{a \in \Lambda} (x - x(\ell(a))) = g(x).$$

By Proposition 9.14, D is finite and free over R . Therefore, letting u denote the map (9.25), it suffices to show that $D \hat{\otimes} u$ is injective.

Each nonzero $a \in \Lambda$ gives a monomorphism $\mathbb{Z}/p \hookrightarrow \Lambda$ and so a homomorphism

$$\begin{aligned} \mathcal{O}(\underline{\text{level}}(\mathbb{Z}/p, G)) &\xrightarrow{a} D \\ x &\mapsto x(\ell(a)). \end{aligned}$$

We may view these all together as a ring homomorphism

$$D \hat{\otimes} \mathcal{O}(\underline{\text{level}}(\mathbb{Z}/p, G)) \xrightarrow{M} \prod_{0 \neq a \in \Lambda} D[[x]]/(x - x(\ell(a))).$$

Note that the identity map of D may be written as

$$D \xrightarrow{F} D[[x]]/(x - x(\ell(0))) = D.$$

With this notation, the diagram

$$\begin{array}{ccc} D \hat{\otimes} \mathcal{O}(\underline{\text{hom}}(\mathbb{Z}/p, G)) & \xrightarrow{D \hat{\otimes} u} & D \hat{\otimes} (R \times \mathcal{O}(\underline{\text{level}}(\mathbb{Z}/p, G))) \\ \cong \downarrow & & \downarrow F \times M \\ D[[x]]/(\prod_{a \in \Lambda} (x - x(\ell(a)))) & \longrightarrow & \prod_{a \in \Lambda} D[[x]]/(x - x(\ell(a))) \end{array}$$

commutes, where the map across the bottom is the evident map (9.16). It is a monomorphism by Proposition 9.17. \square

10. Isogenies. Throughout this section, G is a formal group of finite height over a local formal scheme S with perfect residue field of characteristic $p > 0$. If

$$A_S \xrightarrow{\ell} G$$

is a level structure, then the Cartier divisor

$$[\ell(A)] \stackrel{\text{def}}{=} \sum_{a \in A} [\ell(a)]$$

is a subgroup scheme of G , and the quotient $G/[\ell(A)]$ is a formal group. A finite free map of formal groups $G \rightarrow G'$ is called an *isogeny*. In this section we recall the construction of the isogeny $G \rightarrow G/[\ell(A)]$.

10.1. The norm. An important ingredient in the construction of the quotient is the following (see for example [Mum70, III §12], [DG70, III §2 no. 3], [Str97]). Let $X \xrightarrow{\pi} Y$ be a finite free map of local formal schemes. Multiplication by a section $f \in \mathcal{O}_X$ defines an \mathcal{O}_Y -linear endomorphism f^\cdot of \mathcal{O}_X .

Definition 10.1. The *norm* of f is the determinant

$$\mathbf{N}_\pi f = \text{Det}(f^\cdot) \in \mathcal{O}_Y.$$

This is a multiplicative (but not additive) map $\mathcal{O}_X \rightarrow \mathcal{O}_Y$.

Definition 10.2. A list $P_j: Y \rightarrow X$, $j = 1, \dots, k$ of not-necessarily distinct sections of π is called a *full set of sections* if, for all $f \in \mathcal{O}_X$,

$$\mathbf{N}_\pi f = \prod_j f(P_j),$$

where we have viewed P_j as a Y -valued point of X and so written $f(P_j)$ for $P_j^* f$.

Definition 10.3. Let \mathcal{L} be an invertible sheaf of ideals on X . The *norm* of \mathcal{L} is the ideal sheaf $\mathbf{N}_\pi \mathcal{L}$ on Y generated by $\mathbf{N}_\pi t$, where t is a generator of \mathcal{L} .

The multiplicativity of \mathbf{N}_π implies that $\mathbf{N}_\pi \mathcal{L}$ is independent of the choice of generator t , and, if s is a section of \mathcal{L} , then its norm $\mathbf{N}_\pi s$ is a section of $\mathbf{N}_\pi \mathcal{L}$. The norm is not additive, but it is multiplicative in the sense that

$$\mathbf{N}_\pi(fs) = \mathbf{N}_\pi f \cdot \mathbf{N}_\pi s$$

if f is a section of \mathcal{O}_X and s is a section of \mathcal{L} .

If $\{P_1, \dots, P_k\}$ is a full set of sections of π , then the map

$$\bigotimes_j f_j \otimes s_j \mapsto \prod_j f_j s_j(P_j)$$

defines an isomorphism of \mathcal{O}_Y -modules

$$\bigotimes_j P_j^* \mathcal{L} \cong \mathbf{N}_\pi \mathcal{L}.$$

In particular, $\mathbf{N}_\pi \mathcal{L}$ is a line bundle, and the norm extends to a multiplicative map from the group of invertible fractional ideals on X to the the group of invertible fractional ideals on Y .

10.2. Quotients by finite subgroups.

Definition 10.4. A *finite subgroup* of G is a divisor K on G which is also a subgroup scheme.

Let us write π and μ respectively for the projection and multiplication maps

$$\pi, \mu: G \times K \rightarrow G,$$

and let $\mathcal{O}_{G/K}$ be the equalizer

$$(10.5) \quad \mathcal{O}_{G/K} \longrightarrow \mathcal{O}_G \xrightarrow{\begin{array}{c} \pi^* \\ \mu^* \end{array}} \mathcal{O}_G \otimes \mathcal{O}_K.$$

PROPOSITION 10.6. (i) For $f \in \mathcal{O}_G$, we have $\mathbf{N}_\pi \mu^* f \in \mathcal{O}_{G/K}$.
(ii) If x is a coordinate on G and $y = \mathbf{N}_\pi \mu^* x$, then

$$\mathcal{O}_{G/K} = \mathcal{O}_S[[y]].$$

In particular, $\mathcal{O}_{G/K}$ is the ring of functions on a formal scheme G/K over S .

(iii) G/K has naturally the structure of a formal group, and as such is the categorical cokernel of the inclusion

$$K \rightarrow G.$$

(iv) For any map $f: T \rightarrow S$ of local formal schemes, there is a canonical isomorphism of formal groups

$$f^* G / f^* K \cong f^*(G/K).$$

Proof. The construction of quotients of formal groups by finite subgroups goes back to Lubin [Lub67]. For the construction of the quotient in the generality considered here, see [Dri74] and [Str97]. \square

Definition 10.7. An *isogeny* is a finite free homomorphism $q: G \rightarrow G'$ of formal groups, i.e. a homomorphism of formal groups such that $\ker q$ is a finite subgroup of G .

Proposition 10.6 implies that isogenies are epimorphisms.

LEMMA 10.8. If $f: G \rightarrow G'$ is an isogeny, then there is a unique isomorphism $G/\ker f \cong G'$ making the diagram

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ G/\ker f & \xrightarrow{\cong} & G' \end{array}$$

commute. If $g: G \rightarrow G''$ is another isogeny, such that

$$\ker f \subseteq \ker g,$$

then there is a unique isogeny

$$h: G' \rightarrow G''$$

such that $g = hf$.

Let $q: G \rightarrow G'$ be an isogeny, and let $K = \ker q$. By Proposition 10.6, the norm map

$$\mathbf{N}_\pi \mu^*: \mathcal{O}_G \rightarrow \mathcal{O}_G$$

takes values in $\mathcal{O}_{G/K} \subseteq \mathcal{O}_G$.

Definition 10.9. We write N_q for the norm

$$N_q: \mathcal{O}_G \rightarrow \mathcal{O}_{G'}$$

induced by $\mathbf{N}_\pi \mu^*$ and the isomorphism $G/K \cong G'$.

10.3. Level structures, quotients, and the norm of an ideal. Let A be a finite abelian group, and let

$$A_S \xrightarrow{\ell} G$$

be a level structure.

PROPOSITION 10.10. (i) *The Cartier divisor*

$$[\ell(A)] = \sum_{a \in A} [\ell(a)]$$

is a subgroup scheme of G .

(ii) *The sections*

$$\ell(a): S \rightarrow [\ell(A)]$$

for $a \in A$ are a full set of sections of $[\ell(A)]$.

Proof. It suffices to prove the first part in the case that G is the universal deformation a formal group of height h over a field of characteristic p , and $R = \mathcal{O}(\underline{\text{level}}(A, G))$. In that case the result is essentially Proposition 4.4 of [Dri74]: Drinfel'd actually considers a level structure of the form

$$\Lambda = (\mathbb{Z}/p^n)^h \xrightarrow{\ell} G(R)$$

and a subgroup $A \subseteq \Lambda$, but his argument uses only the A -structure and the fact that $R_\Lambda(G)$ is a Noetherian p -regular complete local domain. Strickland [Str97] gives a complete proof in the generality considered here. Katz and Mazur prove

the second part as [KM85, Thm. 1.10.1], in the case that G is smooth curve over a scheme S ; their proof proceeds by reducing to and proving the local case considered here. \square

It follows from Proposition 10.10 that, if

$$G \xrightarrow{q} G'$$

is the cokernel of the inclusion $[\ell(A)] \rightarrow G$, then the norm of Definition 10.9 is given by the formula

$$q^*N_q f = \prod_{a \in A} T_a^* f.$$

In particular, if x is a coordinate on G , then

$$(10.11) \quad y = \prod_{a \in A} T_a^* x$$

is the coordinate on the quotient G' given in Proposition 10.6. Indeed this is the coordinate discovered by Lubin [Lub67]. A coordinate on G is a trivialization of the line bundle $\mathcal{I}_G(0)$, and it is useful to interpret the norm in terms of line bundles.

Let \mathcal{L} be an invertible sheaf of ideals in \mathcal{O}_G (or, more generally, an invertible fractional ideal on G). Proposition 10.6 implies that there is a line bundle $N\mathcal{L} = N_q \mathcal{L}$ on G' , characterized by the formula

$$q^*N\mathcal{L} = \mathbf{N}_\pi \mu^* \mathcal{L};$$

and if t is a trivialization of \mathcal{L} , then Nt is a trivialization of $N\mathcal{L}$. The map

$$(10.12) \quad \bigotimes_{a \in A} s_a \mapsto \prod_{a \in A} s_a$$

defines an isomorphism

$$(10.13) \quad \bigotimes_{a \in A} T_a^* \mathcal{L} \cong q^*N\mathcal{L}.$$

If s is a section of \mathcal{L} , then under this isomorphism

$$\bigotimes_{a \in A} T_a^* s = q^*Ns.$$

Thus in the presence of the level structure, the first two parts of Proposition 10.6 take the following form:

PROPOSITION 10.14. *There are canonical natural isomorphisms*

$$(10.15) \quad \begin{aligned} N\mathcal{O}_G &\cong \mathcal{O}_{G'} \\ N\mathcal{I}_G(0) &\cong \mathcal{I}_{G'}(0) \\ q^*\mathcal{I}_{G'}(0) &\cong \bigotimes_{a \in A} T_a^* I_G(0) \cong I_G(\ell). \end{aligned}$$

If s is a coordinate on G , then Ns is a coordinate on G' , and under the isomorphism (10.15),

$$q^*Ns = \bigotimes_{a \in A} T_a^* s.$$

11. Descent for level structures. In Definition 3.1 we described “descent data for level structures” as they appear on the formal group of an H_∞ ring spectrum. In this section, we give an equivalent description (see Proposition 11.14) which displays the relationship to the usual notion of descent data. In addition to justifying the terminology, the new formulation simplifies the task of showing that the Lubin-Tate formal groups have canonical descent data for level structures (Proposition 12.9).

11.1. Composition of isogenies: the simplicial functor $\underline{\text{level}}_*$. Let FGps be the functor from admissible local rings R to sets whose value on R is the set of formal groups $G/\text{spf } R$. If $f: R \rightarrow R'$ is a map of admissible local rings, then $\text{FGps}(f)$ sends $G/\text{spf } R$ to $f^*G/\text{spf } R'$.

Let

$$\underline{\text{level}}(A) \longrightarrow \text{FGps}$$

be the functor over FGps whose value on R is the set of formal groups $G/\text{spf } R$ equipped with a level structure

$$A_{\text{spf } R} \longrightarrow G.$$

We define

$$\underline{\text{level}}_1 \stackrel{\text{def}}{=} \coprod_{A_0} \underline{\text{level}}(A_0);$$

the coproduct is over all finite abelian groups. We have adorned the $\underline{\text{level}}$ and the A with subscripts so that we can make the more general definition

$$\underline{\text{level}}_n = \coprod_{0=A_n \subseteq A_{n-1} \cdots \subseteq A_0} \underline{\text{level}}(A_0).$$

The coproduct is over all sequences of *inclusions of finite abelian groups with $A_n = 0$* . With this convention we also have

$$\mathrm{FGps} = \underline{\mathrm{level}}_0.$$

We write

$$(11.1) \quad d_0: \underline{\mathrm{level}}_1 \rightarrow \mathrm{FGps}$$

for the structural map.

Over $\underline{\mathrm{level}}_1(A)$ we have a level structure

$$A \xrightarrow{\ell} d_0^* G$$

and an isogeny

$$d_0^* G \xrightarrow{q_A} G/\ell(A)$$

with kernel A . These assemble to give a group G/ℓ and an isogeny

$$d_0^* G \xrightarrow{q} G/\ell$$

over $\underline{\mathrm{level}}_1$. We write

$$(11.2) \quad d_1: \underline{\mathrm{level}}_1 \longrightarrow \mathrm{FGps}$$

for the map classifying G/ℓ .

LEMMA 11.3. *Let*

$$A \xrightarrow{\ell} G$$

be a level structure. If $B \subseteq A$, then the induced map

$$B \xrightarrow{\ell|_B} G$$

is a level structure. If $q: G \rightarrow G'$ is an isogeny with kernel $\ell|_B$, then the induced map

$$\ell': A/B \rightarrow G'$$

is a level structure.

Proof. The first part is clear from the definition of a level structure (9.9). For the second part, consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ell} & G \\ \downarrow & & \downarrow q \\ A/B & \xrightarrow{\ell'} & G'. \end{array}$$

Let D be the divisor

$$D = \sum_{\substack{a \in A \\ pa=0}} [\ell(a)]$$

on G ; by hypothesis we have an inequality of Cartier divisors

$$D \leq G[p].$$

It follows that

$$\sum_{b \in B} T_b^* D \leq \sum_{b \in B} T_b^* G[p].$$

The formula (10.11) for the coordinate on the quotient G' shows that the left side descends to the divisor

$$\sum_{\substack{c \in (A/B) \\ pc=0}} [\ell'(c)],$$

while the right side descends to the divisor $G'[p]$. □

The lemma gives maps

$$d_j: \underline{\text{level}}_n \rightarrow \underline{\text{level}}_{n-1}$$

for $0 \leq j \leq n$ as follows. For $0 \leq j \leq n-1$, the map d_j sends a point

$$(11.4) \quad 0 = A_n \subseteq \cdots \subseteq A_j \subseteq \cdots \subseteq A_0 \longrightarrow G$$

of $\underline{\text{level}}_n$ to the point

$$0 = A_n \subseteq \cdots \widehat{A_j} \cdots \subseteq A_0 \longrightarrow G$$

of $\underline{\text{level}}_{n-1}$ obtained by omitting A_j . The map d_n sends (11.4) to

$$0 = A_{n-1}/A_{n-1} \subseteq \cdots \subseteq A_0/A_{n-1} \longrightarrow G/\ell(A_{n-1}).$$

In the case $n = 1$ these are just the maps (11.1) and (11.2). We also have for $0 \leq j \leq n$ a map

$$s_j: \underline{\text{level}}_n \rightarrow \underline{\text{level}}_{n+1}$$

which sends the sequence (11.4) to the sequence

$$A_n \subseteq \cdots \subseteq A_j \subseteq A_j \subseteq \cdots \subseteq A_0 \longrightarrow G$$

obtained by repeating A_j . It is easy to check that

LEMMA 11.5. $(\underline{\text{level}}_*, d_*, s_*)$ is a simplicial functor.

11.2. Descent data for functors over formal groups. Now suppose that

$$\mathcal{P} \longrightarrow \text{FGps}$$

is a functor over FGps , and if $x \in \mathcal{P}(R)$ is an R -valued point, let's write G_x for the resulting formal group over $\text{spf } R$. As in the previous section, we define

$$\begin{aligned}\underline{\text{level}}(A, \mathcal{P}) &= \underline{\text{level}}(A) \times_{\text{FGps}} \mathcal{P} \\ \underline{\text{level}}_n(\mathcal{P}) &= \underline{\text{level}}_n \times_{\text{FGps}} \mathcal{P}\end{aligned}$$

and so on. A point $(\ell, x) \in \underline{\text{level}}(A, \mathcal{P})(R)$ is a point x of $\mathcal{P}(R)$ and a level structure

$$A \xrightarrow{\ell} G_x.$$

We write

$$(11.6) \quad d_0: \underline{\text{level}}_1(\mathcal{P}) \rightarrow \underline{\text{level}}_0(\mathcal{P}) = \mathcal{P}$$

for the forgetful map

$$d_0(\ell, x) = x.$$

We also always have degeneracies

$$s_j: \underline{\text{level}}_n(\mathcal{P}) \rightarrow \underline{\text{level}}_{n+1}(\mathcal{P})$$

for $0 \leq j \leq n$.

If (ℓ, x) is an R -valued point of $\underline{\text{level}}(\mathcal{P})$, then we get an isogeny

$$G_x \longrightarrow G_x/\ell.$$

Suppose that we have a natural transformation

$$(11.7) \quad d_1: \underline{\text{level}}_1(\mathcal{P}) \longrightarrow \mathcal{P}$$

such that

$$(11.8) \quad G_{d_1(\ell, x)} = G_x/\ell,$$

or equivalently that the diagram

$$(11.9) \quad \begin{array}{ccc} \underline{\text{level}}_1(\mathcal{P}) & \longrightarrow & \underline{\text{level}}_1 \\ d_1 \downarrow & & \downarrow d_1 \\ \mathcal{P} & \longrightarrow & \text{FGps} \end{array}$$

commutes. Lemma 11.3 then gives maps

$$d_j: \underline{\text{level}}_n(\mathcal{P}) \rightarrow \underline{\text{level}}_{n-1}(\mathcal{P})$$

for $0 \leq j \leq n$.

Definition 11.10. Descent data for level structures on the functor \mathcal{P} consist of a natural transformation (11.7) such that:

- (1) the diagram (11.9) commutes, and
- (2) $(\underline{\text{level}}_*(\mathcal{P}), d_*, s_*)$ is a simplicial functor.

Remark 11.11. It is equivalent to ask for natural transformations

$$d_j: \underline{\text{level}}_n(\mathcal{P}) \rightarrow \underline{\text{level}}_{n-1}(\mathcal{P})$$

for $n \geq 1$ and $0 \leq j \leq n$, such that $(\underline{\text{level}}_*(\mathcal{P}), d_*, s_*)$ is a simplicial functor, and the levelwise natural transformation

$$\underline{\text{level}}_*(\mathcal{P}) \longrightarrow \underline{\text{level}}_*$$

is a map of simplicial functors.

For example, let G be a formal group of finite height over a p -local formal scheme S . The formal scheme S has the structure of a functor over FGps: if

$x: \text{spf } R \rightarrow S$ is a point of S , then

$$G_x = x^* G;$$

we briefly write G/S for S , considered as a functor over FGps in this way. The functor level($A, G/S$) is just the functor called level(A, G) in § 9; in particular it is represented by the S -scheme level(A, G) of Lemma 9.13. To give maps ψ_ℓ and f_ℓ which satisfy condition (3.1) of Definition 3.1 amounts to giving a map

$$d_1: \underline{\text{level}}_1(G/S) \rightarrow S$$

and an isogeny

$$d_0^* G \xrightarrow{q} d_1^* G$$

whose kernel on level($A, G/S$) is A . Lemma 11.3 gives maps

$$d_j: \underline{\text{level}}_n(G/S) \longrightarrow \underline{\text{level}}_{n-1}(G/S)$$

for $0 \leq j \leq n$ as explained above. With these definitions, parts (2) and (3) of Definition 3.1 are equivalent to asserting that

$$(\underline{\text{level}}_*(G/S), d_*, s_*)$$

is a simplicial functor, and over level₂(G) the diagram

(11.12)

$$\begin{array}{ccc} & d_0^* d_0^* G & \\ & \swarrow \quad \searrow & \\ d_1^* d_0^* G & & d_0^* d_1^* G \\ \downarrow d_1^* q & & \parallel \\ d_1^* d_1^* G & & d_2^* d_0^* G \\ & \swarrow \quad \searrow & \\ & d_2^* d_1^* G & \end{array}$$

commutes.

A more convenient formulation of Definition 3.1 is the following. Let $\underline{G/S}$ to be the functor over FGps whose value on R is the set of pull-back diagrams

$$\begin{array}{ccc} G' & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ \mathrm{spf} R & \xrightarrow{i} & S \end{array}$$

such that the map

$$G' \rightarrow i^*G$$

induced by f is a homomorphism (hence isomorphism) of formal groups over $\mathrm{spf} R$. For a finite abelian group A , $\underline{\mathrm{level}}(A, \underline{G/S})(R)$ is the set of diagrams

$$\begin{array}{ccccc} A_{\mathrm{spf} R} & \xrightarrow{\ell} & G' & \xrightarrow{f} & G \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{spf} R & \xrightarrow{i} & S, \end{array}$$

where the square part is a point of $\underline{G/S}(R)$ and ℓ is a level structure. To give a map of functors

$$(11.13) \quad \underline{\mathrm{level}}_1(\underline{G/S}) \xrightarrow{d_1} \underline{G/S}$$

making the diagram

$$\begin{array}{ccc} \underline{\mathrm{level}}_1(\underline{G/S}) & \xrightarrow{d_1} & \underline{G/S} \\ \downarrow & & \downarrow \\ \underline{\mathrm{level}}_1 & \xrightarrow{d_1} & \mathrm{FGps} \end{array}$$

commute is to give a pull-back diagram

$$\begin{array}{ccc} G/\ell & \longrightarrow & G \\ \downarrow & & \downarrow \\ \underline{\mathrm{level}}_1(\underline{G/S}) & \longrightarrow & S; \end{array}$$

it is equivalent to give a map of formal schemes

$$d_1: \underline{\mathrm{level}}_1(\underline{G/S}) \rightarrow S$$

and an isogeny

$$d_0^*G \xrightarrow{q} d_1^*G$$

whose kernel on $\underline{\text{level}}(A, G/S)$ is A .

PROPOSITION 11.14. *Let G be a formal group over an admissible local ring R , and let $S = \text{spf } R$. Descent data for level structures on the group G/S are equivalent to descent data for level structures on the functor $\underline{G/S}$.*

Proof. One checks that the commutativity of the diagram (11.12) has been incorporated in the structure of the functor $\underline{G/S}$. \square

11.3. Noetherian rings and Artin rings. Suppose that \mathcal{D} is a subcategory of the category of admissible local rings. If \mathcal{P} is a functor from complete local rings to sets, let $\mathcal{P}^\mathcal{D}$ denote its restriction to \mathcal{D} .

Definition 11.15. *Descent data for level structures on $\mathcal{P}^\mathcal{D}$ consists of a natural transformation*

$$d_1: \underline{\text{level}}_1(\mathcal{P})^\mathcal{D} \xrightarrow{d_1} \mathcal{P}^\mathcal{D},$$

such that the restriction to \mathcal{D} of the diagram (11.9) commutes, and such that the $(\underline{\text{level}}_*(\mathcal{P}))^\mathcal{D}, d_*, s_*$ is a simplicial functor.

For example, let \mathcal{N} be the category of Noetherian complete local rings, and let \mathcal{A} be the category of Artin local rings. If S and T are Noetherian local formal schemes, then the natural maps

$$(11.16) \quad (\text{formal schemes})(S, T) \rightarrow (\text{functors})(S^\mathcal{N}, T^\mathcal{N}) \rightarrow (\text{functors})(S^\mathcal{A}, T^\mathcal{A})$$

are isomorphisms.

PROPOSITION 11.17. *If G is a formal group over a Noetherian local formal scheme S with perfect residue field of characteristic $p > 0$, then the forgetful maps, from the set of descent data for level structures on $\underline{G/S}$ to the set of descent data for level structures on $\underline{G/S}^\mathcal{N}$ and on $\underline{G/S}^\mathcal{A}$, are isomorphisms.*

Proof. If G is a formal group over a Noetherian local formal scheme, then by Proposition 9.14, $\underline{\text{level}}(A, G)$ is also a Noetherian local formal scheme. The result follows easily from the isomorphism (11.16). \square

12. Lubin-Tate groups. Let k be a perfect field of characteristic $p > 0$, and let Γ be a formal group of finite height over k . In this section we shall prove that the universal deformation of Γ has descent for level structures.

12.1. Frobenius. Let k be a perfect field of characteristic $p > 0$, and let Γ be a formal group of finite height over k . The Frobenius map ϕ gives rise to a relative Frobenius F map as in the diagram

$$\begin{array}{ccccc} & & \phi_{\Gamma} & & \\ & \nearrow F & \downarrow & \searrow & \\ \Gamma & \xrightarrow{\phi_k^* \Gamma} & \Gamma & & \\ \downarrow & & \downarrow & & \\ \text{spec } k & \xrightarrow{\phi_k} & \text{spec } k. & & \end{array}$$

The Frobenius map F is an isogeny of degree p , with kernel the divisor $p[0]$.

12.2. Deformations. If T is a local formal scheme, then we write T_0 for its closed point.

Definition 12.1. Let T be a local formal scheme. A *deformation of Γ to T* is a triple $(H/T, f, j)$ consisting of a formal group H over T and a pull-back diagram

$$\begin{array}{ccc} H_{T_0} & \xrightarrow{f} & \Gamma \\ \downarrow & & \downarrow \\ T_0 & \xrightarrow{j} & \text{spec } k, \end{array}$$

such that the induced map $H_{T_0} \rightarrow j^*\Gamma$ is a homomorphism (and so isomorphism) of formal groups over T_0 . The functor from complete local rings to sets which assigns to R the set of deformations of Γ to $\text{spf } R$ will be denoted $\text{Def}(\Gamma)$.

From the definition it is clear that if $(H/T, f, j)$ is a deformation of Γ , then there is a natural transformation

$$\underline{H/T} \rightarrow \text{Def}(\Gamma).$$

Lubin and Tate [LT66] construct a deformation $(G/S, f_{\text{univ}}, j_{\text{univ}})$ with an isomorphism

$$(12.2) \quad S \cong \text{spf } \mathbb{W}k[[u_1, \dots, u_{h-1}]]$$

inducing j_{univ} : $S_0 \cong \text{spec } k$ such that the natural transformation

$$(12.3) \quad \underline{G/S} \rightarrow \text{Def}(\Gamma)$$

is an isomorphism of functors over FGps.

Remark 12.4. Lubin and Tate claim only that the restriction to Noetherian complete local rings

$$\underline{G/S^N} \rightarrow \underline{\text{Def}(\Gamma)^N}$$

is an isomorphism. In fact, their argument proves the stronger statement. The main point is that, if Φ is a formal group law over a field k of characteristic $p > 0$ and if M is *any* k -vector space (not necessarily finite-dimensional), then the natural map

$$H_k^2(\Phi) \otimes M \rightarrow H_M^2(\Phi)$$

is an isomorphism, a result which Lubin and Tate assert at the beginning of 2.3 for M finite dimensional. Indeed, the argument of their Proposition 2.6 may be applied to give this calculation of $H_M^2(\Phi)$.

12.3. Descent for level structures on deformations. We continue to fix a formal group Γ of finite height over a perfect field k of characteristic $p > 0$.

Let A be a finite abelian group. If R is a complete local ring, then a point of $\underline{\text{level}}(A, \text{Def}(\Gamma))$ is a commutative diagram

$$(12.5) \quad \begin{array}{ccccccc} A & \xrightarrow{\ell} & H & \longleftarrow & H_{T_0} & \xrightarrow{f} & \Gamma \\ & & \downarrow & & \downarrow & & \downarrow \\ & & T & \longleftarrow & T_0 & \xrightarrow{j} & \text{spec } k, \end{array}$$

consisting of a deformation (H, f, j) of Γ to R and a level structure ℓ on H . The level structure in (12.5) of $\text{Def}(A, \Gamma)(R)$ has a cokernel

$$H \xrightarrow{q} H'$$

If x is a coordinate on H , then $x(\ell(a))$ is topologically nilpotent in $\mathcal{O}(T)$. It follows that $x(\ell(a)) = 0$ on T_0 , and so there is a canonical isomorphism \bar{q} making the diagram

$$\begin{array}{ccccccc} H_{T_0} & \xrightarrow{F^r} & (\phi^r)^* H_{T_0} & \xrightarrow{\text{can}} & H_{T_0} & \xrightarrow{f} & \Gamma \\ q_{T_0} \downarrow & \nearrow \bar{q} & \downarrow & & \downarrow & & \downarrow \\ H'_0 & \xrightarrow{\quad} & T_0 & \xrightarrow{\phi^r} & T_0 & \xrightarrow{j} & \text{spec } k \end{array}$$

commute. In other words $(H', f \circ \text{can} \bar{q}, j \circ \phi^r)$ is a point of $\text{Def}(\Gamma)(T)$, and we have

constructed a natural transformation

$$(12.6) \quad \underline{\text{level}}_1(\text{Def}(\Gamma)) \xrightarrow{d_1} \text{Def}(\Gamma).$$

satisfying (11.8). The fact that $\phi^r \phi^s = \phi^{r+s}$ then implies

LEMMA 12.7. *The map d_1 is descent data for level structures on the functor $\text{Def}(\Gamma)$.*

Now let $(G/S, f_{\text{univ}}, j_{\text{univ}})$ be Lubin and Tate's universal deformation of $\Gamma / \text{spec } k$. Using the isomorphism (12.3) and Proposition 11.17 we may trade $\underline{G/S}$ for $\text{Def}(\Gamma)$ in (12.6) to get a map

$$(12.8) \quad d_1: \underline{\text{level}}_1(\underline{G/S}) \longrightarrow \underline{G/S}$$

such that

PROPOSITION 12.9. *The natural transformation d_1 is descent data for level structures on the functor $\underline{G/S}$, and so gives descent data for level structures on the formal group G/S .*

12.4. Comparison to the descent data coming from the E_∞ structure of Goerss and Hopkins. The construction of the descent data in Proposition 12.9 uses the equality

$$(12.10) \quad p^r[0] = \ker F^r: \Gamma \rightarrow (\phi^r)^*\Gamma$$

of divisors on Γ and the equation

$$(12.11) \quad F^{r+s} = ((\phi^r)^*F^s)F^r: \Gamma \rightarrow (\phi^{r+s})^*\Gamma.$$

More generally, to give descent data for level structures on $\underline{G/S}$ is equivalent to giving a collection of isogenies

$$F_r: \Gamma \rightarrow (\phi^r)^*\Gamma$$

for $r \geq 1$ satisfying the analogues of (12.10) and (12.11). The descent data in the proposition are uniquely determined by the choice $F_r = F^r$.

Now let \mathbf{E} be the homogeneous ring spectrum such that $G = G_{\mathbf{E}}$ is Lubin and Tate's universal deformation of Γ , so

$$S_{\mathbf{E}} = S = \text{spf } \mathbb{W}k[[u_1, \dots, u_{h-1}]].$$

In work in preparation, Goerss and Hopkins [GH02] have shown that \mathbf{E} is an E_∞ ring spectrum; by Theorem 3.25 it follows that there is a map

$$\underline{\text{level}}_1(\underline{G}/\underline{S}) \xrightarrow{d_1^{\text{gh}}} \underline{G}/\underline{S}$$

giving descent data for level structures on G/S .

Let A be a finite group of order p^r , and let

$$A_{\text{spf} R} \xrightarrow{\ell} i^* G$$

be a level structure on G . Reducing modulo the maximal ideal in the construction of $\psi_\ell^{\mathbf{E}}$ (3.9), one sees that

$$\psi_\ell^{\mathbf{E}} = \phi^r: S_{\mathbf{E}} \rightarrow S_{\mathbf{E}}.$$

Examination of the construction (3.14) of $\psi_\ell^{G/\mathbf{E}}$ shows that

$$(\psi_\ell^{G/\mathbf{E}})_{S_0} = F^r: G_{S_0} \rightarrow (\phi^r)^* G_{S_0}.$$

Thus we have the following result:

PROPOSITION 12.12. *If \mathbf{E} is the spectrum associated to the universal deformation of a formal group Γ of finite height over a perfect field k , then the descent data for level structures on $G_{\mathbf{E}}$ provided by the E_∞ structure of Goerss and Hopkins coincide with the descent data in Proposition 12.9.*

Remark 12.13. At the time of the writing of this paper, the result of Goerss and Hopkins is not published. The arguments of this section do not depend on their result beyond the existence of the H_∞ structure, so a cautious statement of the proposition is that the descent data for level structures on $G_{\mathbf{E}}$ provided by any H_∞ structure on \mathbf{E} coincide with the descent data in Proposition 12.9.

Part 4. The sigma orientation.

13. Θ^k -structures.

13.1. The functors Θ^k . Suppose that G is a formal group over a formal scheme S , and suppose that \mathcal{L} is a line bundle over G .

Definition 13.1. A *rigid* line bundle over G is a line bundle \mathcal{L} equipped with a specified trivialization of $O^* \mathcal{L}$. A *rigid section* of such a line bundle is a section s which extends the specified section at the identity. A *rigid isomorphism* between two rigid line bundles is an isomorphism which preserves the specified trivializations.

Definition 13.2. Suppose that $k \geq 1$. We define the line bundle $\Theta^k(\mathcal{L})$ over G^k by the formula

$$(13.3) \quad \Theta^k(\mathcal{L}) \stackrel{\text{def}}{=} \bigotimes_{I \subseteq \{1, \dots, k\}} (\mu_I^* \mathcal{L})^{(-1)^{|I|}}.$$

If s is a section of \mathcal{L} , then we write $\Theta^k s$ for the section

$$\Theta^k s = \bigotimes_{I \subseteq \{1, \dots, k\}} (\mu_I^* s)^{(-1)^{|I|}}.$$

of $\Theta^k(\mathcal{L})$. We define $\Theta^0(\mathcal{L}) = \mathcal{L}$ and $\Theta^0 s = s$.

For example we have

$$\begin{aligned} \Theta^1(\mathcal{L}) &= \frac{\pi^* 0^* \mathcal{L}}{\mathcal{L}} \\ \Theta^1(\mathcal{L})_a &= \frac{\mathcal{L}_0}{\mathcal{L}_a} \\ \Theta^2(\mathcal{L})_{a,b} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b}}{\mathcal{L}_a \otimes \mathcal{L}_b} \\ \Theta^3(\mathcal{L})_{a,b,c} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b} \otimes \mathcal{L}_{a+c} \otimes \mathcal{L}_{b+c}}{\mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c \otimes \mathcal{L}_{a+b+c}}. \end{aligned}$$

We observe three facts about these bundles.

- (1) $\Theta^k(\mathcal{L})$ has a natural rigid structure for $k > 0$.
- (2) For each permutation $\sigma \in \Sigma_k$, there is a canonical isomorphism

$$\xi_\sigma: \pi_\sigma^* \Theta^k(\mathcal{L}) \cong \Theta^k(\mathcal{L}).$$

Moreover, these isomorphisms compose in the obvious way.

- (3) There is a canonical identification (of rigid line bundles over X^{k+1})

$$(13.4) \quad \Theta^k(\mathcal{L})_{a_1, a_2, \dots} \otimes \Theta^k(\mathcal{L})_{a_0+a_1, a_2, \dots}^{-1} \otimes \Theta^k(\mathcal{L})_{a_0, a_1+a_2, \dots} \otimes \Theta^k(\mathcal{L})_{a_0, a_1, \dots}^{-1} \cong 1.$$

Definition 13.5. A Θ^k -structure on a line bundle \mathcal{L} over G is a trivialization s of the line bundle $\Theta^k(\mathcal{L})$ such that:

- (1) for $k > 0$, s is a rigid section;
- (2) s is symmetric in the sense that for each $\sigma \in \Sigma_k$, we have $\xi_\sigma \pi_\sigma^* s = s$;

(3) we have

$$(13.6) \quad s(a_1, a_2, \dots) \otimes s(a_0 + a_1, a_2, \dots)^{-1} \otimes s(a_0, a_1 + a_2, \dots) \\ \otimes s(a_0, a_1, \dots)^{-1} = 1$$

under the isomorphism (13.4).

A Θ^3 -structure is known as a *cubical structure* [Bre83]. We write $C^k(G; \mathcal{L})$ for the set of Θ^k -structures on \mathcal{L} over G . Note that $C^0(G; \mathcal{L})$ is just the set of trivializations of \mathcal{L} , and $C^1(G; \mathcal{L})$ is the set of rigid trivializations of $\Theta^1(\mathcal{L})$. We also define a functor from rings to sets by

$$\underline{C}^k(G; \mathcal{L})(R) = \{(u, f) \mid u: \text{spec}(R) \longrightarrow S, f \in C_{\text{spec}(R)}^k(u^*G; u^*\mathcal{L})\},$$

and we recall the following:

PROPOSITION 13.7. [AHS01] Let G be a formal group over a scheme S , and let \mathcal{L} be a line bundle over G . The functor $\underline{C}^k(G; \mathcal{L})$ is represented by an affine scheme over S , and for $j: S' \rightarrow S$, the natural map

$$\underline{C}^k(j^*G; j^*\mathcal{L}) \rightarrow j^*\underline{C}^k(G; \mathcal{L})$$

is an isomorphism.

13.2. Relations among the Θ^k : the functor Δ .

Definition 13.8. If \mathcal{M} is a line bundle over G^n , then we define $\Delta\mathcal{M}$ to be the rigid line bundle over G^{n+1} given fiberwise by the formula

$$\Delta\mathcal{M}_{a_1, a_2, \dots, a_{n+1}} = \frac{\mathcal{M}_{a_1, a_3, \dots, a_{n+1}} \otimes \mathcal{M}_{a_2, \dots, a_{n+1}}}{\mathcal{M}_{a_1+a_2, a_3, \dots, a_{n+1}} \otimes \mathcal{M}_{0, a_3, \dots, a_{n+1}}}.$$

If s is a section of \mathcal{M} then we write Δs for the rigid section

$$\Delta s(a_1, \dots, a_{n+1}) = \frac{s(a_1, \dots) \otimes s(a_2, \dots)}{s(a_1 + a_2, \dots) \otimes s(0, a_3, \dots)}$$

of $\Delta\mathcal{M}$.

The following can be checked directly from the definitions.

LEMMA 13.9. (i) Δ is multiplicative: if \mathcal{M}_1 and \mathcal{M}_2 are line bundles over G^n then there is a canonical isomorphism of rigid line bundles

$$(13.10) \quad \Delta(\mathcal{M}_1 \otimes \mathcal{M}_2) \cong \Delta(\mathcal{M}_1) \otimes \Delta(\mathcal{M}_2).$$

(ii) Under the identification (13.10), one has

$$\Delta(s_1 \otimes s_2) = \Delta(s_1) \otimes \Delta(s_2).$$

(iii) If \mathcal{L} is a line bundle over G then for $k \geq 2$ there is a canonical isomorphism of rigid line bundles

$$(13.11) \quad \Theta^k \mathcal{L} \cong \Delta \Theta^{k-1} \mathcal{L}.$$

(iv) If s is a section of \mathcal{L} then under the isomorphism (13.11), one has

$$\Theta^k s = \Delta \Theta^{k-1} s$$

14. The norm map for Θ^k -structures. Suppose that A is a finite group, and let

$$A_S \xrightarrow{\ell} G$$

be a level structure on a formal group G of finite height over a local formal scheme with perfect residue field of characteristic $p > 0$. Let

$$K = [\ell(A)],$$

and let

$$G \xrightarrow{q} G'$$

be the quotient of G by K .

Fix $k \geq 1$, and, for $1 \leq i \leq k$, let

$$\mu_i, w_i: G^k \times K \rightarrow G^k$$

be the maps given in punctual notation by

$$\begin{aligned} \mu_i(g_1, \dots, g_k, a) &= (g_1, \dots, g_i + a, \dots, g_k) \\ w_i(g_1, \dots, g_k, a) &= (g_1, \dots, a, \dots, g_k); \end{aligned}$$

that is, w_i replaces g_i with a . Let

$$\pi: G^k \times K \rightarrow G^k$$

the projection onto the first k factors.

Let

$$A_S \xrightarrow{\ell_i} G^k$$

be the homomorphism to the i factor, let

$$G_i = G^{i-1} \times G' \times G^{k-i},$$

and let

$$q_i: G^k \rightarrow G_i$$

be the projection.

If \mathcal{M} is an invertible fractional ideal on G^k , then by Proposition 10.6, we may define $N_i\mathcal{M}$ to be the invertible fractional ideal on G_i such that

$$q_i^* N_i \mathcal{M} = \mathbf{N}_\pi(\mu_i^* \mathcal{M}) \otimes \mathbf{N}_\pi(w_i^* \mathcal{M})^{-1}.$$

For $a \in A$, let $\tilde{T}_a \mathcal{M}$ be the line bundle whose fiber over (g_1, \dots, g_k) is

$$(14.1) \quad \tilde{T}_a \mathcal{M}_{g_1, \dots, g_k} \stackrel{\text{def}}{=} \frac{\mathcal{M}_{(a+g_1, g_2, \dots, g_k)}}{\mathcal{M}_{(a, g_2, \dots)}}.$$

If s is a section of \mathcal{M} , define $\tilde{T}_a s$ by

$$\tilde{T}_a s(g_1, \dots, g_k) = \frac{s(a + g_1, \dots, g_k)}{s(a, g_2, \dots, g_k)}.$$

Proposition 10.10 implies that there is a canonical isomorphism

$$(14.2) \quad q_1^* N_1 \mathcal{M} \cong \bigotimes_{a \in A} \tilde{T}_a \mathcal{M},$$

which after base change along ℓ_1 is equivariant with respect to the evident action of A on the right.

By construction, if s is a section of \mathcal{M} then we get a section $N_i \mathcal{M}$ of $N_i \mathcal{M}$ by the formula

$$q_i^* N_i s = \mathbf{N}_\pi(\mu_i^* s) \otimes \mathbf{N}_\pi(w_i^* s)^{-1}.$$

Under the isomorphism (14.2), we have

$$q_1^* N_1 s = \bigotimes_{a \in A} \tilde{T}_a s.$$

Now suppose that \mathcal{L} is an invertible sheaf of ideals on G .

LEMMA 14.3. *There is a canonical isomorphism of rigid line bundles*

$$(14.4) \quad q_i^* N_i \Theta^k \mathcal{L} \cong (q^k)^* \Theta^k N_q \mathcal{L}.$$

Proof. We prove the statement for $i = 1$. One checks directly that there are natural isomorphisms of rigid line bundles

$$\begin{aligned}\tilde{T}_a\Theta^1\mathcal{L} &\cong \Theta^1\tilde{T}_a\mathcal{L} \cong \Theta^1T_a^*\mathcal{L} \\ \Delta\tilde{T}_a\mathcal{M} &\cong \tilde{T}_a\Delta\mathcal{M}.\end{aligned}$$

The proof follows by induction, using the Lemma 13.9 and the isomorphism (14.2). \square

In view of the lemma, it is convenient to write $\tilde{N}\Theta^k\mathcal{L}$ for the line bundle $N\Theta^k\mathcal{L}$, considered as a line bundle on $(G')^k$; of course there is a canonical isomorphism of rigid line bundles

$$(14.5) \quad \tilde{N}\Theta^k\mathcal{L} \cong \Theta^kN\mathcal{L}.$$

PROPOSITION 14.6. *If s is a Θ^k -structure on \mathcal{L} , then*

$$q_i^*N_is = q_j^*N_js,$$

and under the isomorphism (14.4), N_is descends to a Θ^k -structure on $N\mathcal{L}$.

Proof. The first point is that N_is is independent of i . If $\mathcal{M} = \Theta^k\mathcal{L}$, then there is a canonical isomorphism

$$\frac{\mu_i^*\mathcal{M}}{w_i^*\mathcal{M}} \cong \frac{\mu_j^*\mathcal{M}}{w_j^*\mathcal{M}}.$$

The cocycle condition (13.6) for s implies that, under this isomorphism,

$$\frac{\mu_i^*s}{w_i^*s} \cong \frac{\mu_j^*s}{w_j^*s},$$

so $q_i^*N_is = q_j^*N_js$ as required. It follows from Proposition 10.6 that N_is is invariant under the action of the i -factor of K^k on G^k . The question of whether N_is descends to a section of $\Theta^kN\mathcal{L}$ amounts to the question of whether a ratio of sections of \mathcal{O}_{G^k} in the equalizer of

$$\mathcal{O}_{G^k} \xrightarrow[\mu^*]{\pi^*} \mathcal{O}_{G^k \times K^k},$$

and it suffices to check that it is in the equalizer of

$$\mathcal{O}_{G^k} \xrightarrow{\begin{array}{c} \pi^* \\ \mu_i^* \end{array}} \mathcal{O}_{G^k \times K},$$

for each i . It follows that $N_i s$ descends to a section of $\Theta^k N\mathcal{L}$. It is then straightforward if tedious to check that $N_i s$ is a Θ^k -structure. \square

Lemma 14.3 and Proposition 14.6 permit us to make the following:

Definition 14.7. If s is a Θ^k -structure on \mathcal{L} , then let $\tilde{N}s$ be the Θ^k -structure on $N\mathcal{L}$ such that

$$(14.8) \quad (q^k)^* \tilde{N}s = q_1^* N_1 s = \bigotimes_{a \in A} \tilde{T}_a s$$

under the isomorphisms (14.2) and (14.4):

$$(14.9) \quad (q^k)^* \Theta^k N\mathcal{L} \cong (q^k)^* \tilde{N}\Theta^k \mathcal{L} \cong q_1^* N_1 \Theta^k \mathcal{L} \cong \bigotimes_{a \in A} \tilde{T}_a \Theta^k \mathcal{L}.$$

15. Elliptic curves.

Definition 15.1. An *elliptic curve* is a pointed proper smooth curve

$$\begin{array}{ccc} & 0 & \\ C & \xrightarrow{\hspace{2cm}} & S \end{array}$$

whose geometric fibers are connected and of genus 1.

Much of the theory of level structures, isogenies, Θ^k -structures, which we have described in detail in this paper for formal groups, is well known in the case of elliptic curves. In this section we briefly recall some results which we need. Details may be found in [DR73, KM85, Mum70, Sil99].

15.1. Abel's Theorem. Note that the discussion of the line bundles $\Theta^k \mathcal{L}$ in §13 applies to abelian groups in any category where the notion of line bundle makes sense. The first result about elliptic curves is that they are group schemes.

THEOREM 15.2. (Abel) *An elliptic curve C/S has a unique structure of abelian group scheme such that the rigid line bundle $\Theta^3(\mathcal{I}_C(0))$ is trivial. The (necessarily unique) rigid trivialization $s(C/S)$ of $\Theta^3(\mathcal{I}(0))$ is a cubical structure.*

Proof. See for example [KM85, p. 63] or [DR73]. \square

Remark 15.3. The *theorem of the cube* says that any line bundle over an abelian variety has a unique cubical structure. A general enough statement of the theorem of the cube, together with the group structure on elliptic curves, implies Theorem 15.2. We have stated Theorem 15.2 to emphasize that the group structure on an elliptic curve is *constructed* to trivialize $\Theta^3(\mathcal{I}(0))$, so that by the time you get around to applying the theorem of the cube, you already know the conclusion for $\mathcal{I}(0)$.

15.2. Level structures on elliptic curves The study of level structures on elliptic curves is due to Katz and Mazur [KM85]. Let C be an elliptic curve over a scheme S , and let A be an abelian group.

Definition 15.4. A homomorphism

$$\ell: A_S \longrightarrow C$$

is a *level structure* if the Cartier divisor

$$[\ell(A)] = \sum_{a \in A} [\ell(a)]$$

is a sub-group scheme.

LEMMA 15.5. *Let C be an elliptic curve over a local formal scheme S with perfect residue field of characteristic $p > 0$. If*

$$\ell: A_S \rightarrow \widehat{C}$$

is a level structure on the formal group of C , then

$$A_S \rightarrow \widehat{C} \rightarrow C$$

is a level structure on C .

Proof. This follows from the definition and Proposition 10.10. □

Let

$$A \xrightarrow{\ell} C$$

be a level structure. The inclusion

$$[\ell(A)] \rightarrow C$$

has a cokernel $C/[\ell(A)]$ which is an elliptic curve. If

$$q: C \rightarrow C' = C/[\ell(A)]$$

denotes the projection, then q^* identifies $\mathcal{O}_{C'}$ with the equalizer

$$\mathcal{O}_{C'} \longrightarrow \mathcal{O}_C \xrightarrow{\begin{array}{c} \pi^* \\ \mu^* \end{array}} \mathcal{O}_C \otimes \mathcal{O}_{[\ell(A)]}.$$

If $f \in \mathcal{O}_C$, then $N\pi\mu^*f$ is in this equalizer, and we write

$$N = N_q: \mathcal{O}_C \rightarrow \mathcal{O}_{C'}$$

for the resulting map; explicitly,

$$q^*Nf = \prod_{a \in A} T_a^*f.$$

It is easy to check that, if t is a coordinate on C , then Nt is a coordinate on C' ; in particular, if

$$\ell: A_S \rightarrow \widehat{C}$$

is a level structure, then the natural map of formal groups

$$\widehat{C}/[\ell(A)] \rightarrow \widehat{C}/[\ell(A)]$$

is an isomorphism.

As in the case of a formal group, we have

$$N\mathcal{I}_C(0) = \mathcal{I}_{C'}(0).$$

After pulling back along the level structure ℓ , we have

$$q^*N\mathcal{L} \cong \bigotimes_{a \in A} T_a^*\mathcal{L},$$

and this isomorphism is equivariant with respect to the standard action of A on the right.

The discussion of the reduced norm \tilde{N} of §14 applies to elliptic curves as well. The main point is that, if \mathcal{L} is a fractional ideal on the elliptic curve C , then the isogeny q gives isomorphisms of rigid line bundles

$$\tilde{N}\Theta^k\mathcal{L} \cong \Theta^kN\mathcal{L}$$

over $(C')^k$ as in (14.5), and if s is a Θ^k structure on \mathcal{L} , then $\tilde{N}s$ is a Θ^k -structure on $\tilde{N}\mathcal{L}$, as in Proposition 14.6 and Definition 14.7.

15.3. The Serre-Tate theorem. Let C_0 be an elliptic curve over a field k of characteristic $p > 0$.

Definition 15.6. A *deformation* of C_0 is a triple $(D/T, f, j)$ consisting of an elliptic curve D over a local formal scheme T of residue characteristic $p > 0$ and a pull-back diagram

$$\begin{array}{ccc} D_{T_0} & \xrightarrow{f} & C_0 \\ \downarrow & & \downarrow \\ T_0 & \xrightarrow{j} & \text{spec } k \end{array}$$

of elliptic curves. A *map* deformations

$$(\alpha, \beta): (D, f, j) \rightarrow (D', f', j')$$

is a pull-back square

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D' \\ \downarrow & & \downarrow \\ T & \xrightarrow{\beta} & T' \end{array}$$

such that the diagram

$$\begin{array}{ccccc} & & f & & \\ & \swarrow & \curvearrowright & \searrow & \\ D_{T_0} & \xrightarrow{\alpha_{T_0}} & D'_{T'_0} & \xrightarrow{f'} & C_0 \\ \downarrow & & \downarrow & & \downarrow \\ T_0 & \xrightarrow{\beta_0} & T'_0 & \xrightarrow{j'} & \text{spec } k \\ & & \searrow & \swarrow & \\ & & j & & \end{array}$$

commutes.

Let C_0 be a supersingular elliptic curve over a perfect field k of characteristic $p > 0$.

THEOREM 15.7. (*Serre-Tate*) *The natural transformation*

$$\text{Def}(C_0)^N \rightarrow \text{Def}(\widehat{C}_0)^N$$

is an isomorphism of functors over FGps^N . Let G/S be the universal deformation of the formal group \widehat{C}_0 . Then there is a deformation $(C/S, f_{\text{univ}}, j_{\text{univ}})$ of C_0 to S such that the natural maps

$$\underline{C}/\underline{S}^N \rightarrow \text{Def}(C_0)^N \rightarrow \text{Def}(\widehat{C}_0)^N \leftarrow \underline{G}/\underline{S}^N$$

are isomorphism of functors over FGps^N .

Proof. The Serre-Tate Theorem as stated in [Kat81] proves that the forgetful natural transformation induces an isomorphism

$$(15.8) \quad \text{Def}(C_0)^A \rightarrow \text{Def}(\widehat{C}_0)^A$$

of functors of Artin local rings. On the other hand, the functor $\text{Def}(C_0)$ is *effectively pro-representable*: there is deformation $(C'/S', f', j')$ with $S' \cong \text{spf } \mathbb{W}k[[u]]$, such that the natural map

$$(15.9) \quad \underline{(C'/S')}^A \rightarrow \text{Def}(C_0)^A$$

is an isomorphism (see for example [DR73]). It follows that

$$\underline{(C'/S')}^N \cong \text{Def}(C_0)^N.$$

Combining the isomorphisms (15.8) and (15.9) with the isomorphism

$$\text{Def}(\widehat{C}_0)^N \cong \underline{G}/\underline{S}^N$$

gives an isomorphism of formal schemes

$$S \xrightarrow{\cong} S'$$

and, if C is the elliptic curve over S obtained from C'/S' by pull-back, an isomorphism

$$\underline{C}/\underline{S}^N \rightarrow \underline{G}/\underline{S}^N.$$

□

Example 15.10. In characteristic 2 the elliptic curve C_0 given by the Weierstrass equation

$$y^2 + y = x^3$$

is supersingular (e.g. [Sil99]). The universal deformation of its formal group is a formal group G over $S \cong \text{spf } \mathbb{Z}_2[[u_1]]$. It is well-known (e.g. by the Exact Functor Theorem [Lan76]) that there is a spectrum \mathbf{E} with

$$G_{\mathbf{E}}/S_{\mathbf{E}} = G/S :$$

it is a form of E_2 . The Serre-Tate Theorem endows \mathbf{E} with the structure of an elliptic spectrum: if C/S is the universal deformation of C_0 to S , then there is a canonical isomorphism

$$G_{\mathbf{E}} = G \cong \widehat{C}$$

of formal groups over $S_{\mathbf{E}}$.

15.4. Descent for level structures on a Serre-Tate curve. We suppose that C_0 is a supersingular elliptic curve over a perfect field k of characteristic $p > 0$, and that C/S is the universal deformation of C_0 provided by Theorem 15.7. Let G/S be the universal deformation of the formal group \widehat{C}_0 . Since \underline{C}/S is a functor over FGps, Definition 11.10 provides a notion of descent data for level structures on \underline{C}/S .

Theorem 15.7 gives an isomorphism of formal groups $\widehat{C} \cong G$, and so the descent data (12.6) for the Lubin-Tate formal groups give descent data

$$d_1: \underline{\text{level}}_1(\underline{C}/S) \rightarrow \underline{C}/S.$$

Explicitly, suppose that

$$A_T \xrightarrow{\ell} i^* \widehat{C}$$

is a level structure over a Noetherian local formal scheme T . The descent data provide an isogeny of formal groups

$$(15.11) \quad i^* \widehat{C} \xrightarrow{f_\ell} \psi_\ell^* \widehat{C}$$

over T with kernel $[\ell(A)]$.

It is natural to ask for an isogeny of elliptic curves

$$i^* C \xrightarrow{g_\ell} \psi_\ell^* C$$

extending f_ℓ . This corresponds, in the language of §11, to replacing the functor FGps with the functor Ell whose value on a ring R is the set of elliptic curves $C/\text{spec } R$. Thus we shall refer to descent data for level structures on $\underline{C/S}$ together with isogenies g_ℓ extending f_ℓ as *descent data for level structures on $\underline{C/S}$ over Ell*.

Over T we have an isogeny of elliptic curves

$$i^*C \xrightarrow{q} C'$$

with kernel $[\ell(A)]$, and a canonical isomorphism

$$(15.12) \quad \widehat{C}' \cong \psi_\ell^* \widehat{C}$$

of formal groups over T , as in §15.2. Theorem 15.7 implies that there is a unique isomorphism of elliptic curves

$$C' \cong \psi_\ell^* C$$

extending (15.12); put another way, we have the following.

COROLLARY 15.13. *The functor $\underline{C/S}$ has descent data for level structures over Ell, whose restriction to $\widehat{C}/S = G/S$ are the descent data given by Proposition 12.9. In particular, for each level structure*

$$A_T \xrightarrow{\ell} i^* \widehat{C},$$

there is a canonical isogeny of elliptic curves g_ℓ making the diagram

$$\begin{array}{ccc} i^* \widehat{C} & \longrightarrow & i^* C \\ f_\ell \downarrow & & \downarrow g_\ell \\ \psi_\ell^* \widehat{C} & \longrightarrow & \psi_\ell^* C \end{array}$$

commute.

15.5. The cubical structure of an elliptic curve is compatible with descent. The uniqueness of the cubical structure in Theorem 15.2 implies the following.

PROPOSITION 15.14. *Let C be an elliptic curve, and let $s(C/S)$ be the cubical structure of Theorem 15.2. If $i^* C \rightarrow \psi^* C$ is an isogeny, then*

$$\psi^* s(C/S) = \tilde{N} i^* s(C/S).$$

16. The sigma orientation. Suppose that \mathbf{E} is a homogeneous ring spectrum and let $G = G_{\mathbf{E}}$. Let V be the bundle

$$V = \prod_{j=1}^k (1 - L_i)$$

over $(\mathbb{C}P^\infty)^k$. In Lemma 5.1 we observed that Proposition 8.12 specializes to give an isomorphism

$$t_V: \mathbb{L}(V) \cong \Theta^k(\mathcal{I}_{G_{\mathbf{E}}}(0)),$$

and if

$$g: MU\langle 2k \rangle \rightarrow \mathbf{E}$$

is an orientation, then the composition

$$(\mathbb{C}P^{\infty k})^V \rightarrow MU\langle 2k \rangle \xrightarrow{g} \mathbf{E}$$

represents a rigid section s of $\Theta^k(\mathcal{I}_G(0))$. In fact it is easily seen to be a Θ^k -structure, that is a $\pi_0 \mathbf{E}$ -valued point of $\underline{C}^k(G; \mathcal{I}_G(0))$. Similarly, if $g: BU\langle 2k \rangle_+ \rightarrow \mathbf{E}$ is a homotopy multiplicative map, then the composite

$$\mathbb{C}P^{\infty k} \rightarrow BU\langle 2k \rangle \rightarrow \mathbf{E}$$

represents a Θ^k -structure on the trivial line bundle \mathcal{O}_G , and so a point of $\underline{C}^k(G; \mathcal{O}_G)$. In [AHS01] we proved the following:

THEOREM 16.1. *If \mathbf{E} is a homogeneous spectrum and $k \leq 3$, then these correspondences induce isomorphisms*

$$(16.2) \quad \text{RingSpectra}(MU\langle 2k \rangle, \mathbf{E}) \rightarrow \underline{C}^k(G; \mathcal{I}_G(0))(\pi_0 \mathbf{E})$$

and

$$\text{RingSpectra}(BU\langle 2k \rangle_+, \mathbf{E}) \rightarrow \underline{C}^k(G; \mathcal{O}_G)(\pi_0 \mathbf{E}).$$

Now suppose that (\mathbf{E}, C, t) is an *elliptic spectrum*: that is, \mathbf{E} is a homogeneous ring spectrum, C is an elliptic curve over $S_{\mathbf{E}}$, and t is an isomorphism

$$t: G_{\mathbf{E}} \cong \widehat{C}$$

of formal groups over $S_{\mathbf{E}}$. Abel's Theorem 15.2 gives a cubical structure $s(C/S)$ on C , which gives a cubical structure $t^* \widehat{s}(C/S)$ on $G_{\mathbf{E}}$.

Definition 16.3. [AHS01] The *sigma orientation* for (\mathbf{E}, C, t) is the map of ring spectra

$$\sigma(\mathbf{E}, C, t): MU\langle 6 \rangle \rightarrow \mathbf{E}$$

which corresponds to $t^* \widehat{s}(C/S)$ under the isomorphism (16.2).

Now suppose that \mathbf{E} is a homogeneous H_∞ spectrum, with the property that $\pi_0 \mathbf{E}$ is an admissible local ring with perfect residue field of characteristic $p > 0$. Let $S = S_{\mathbf{E}}$. Suppose that (\mathbf{E}, C, t) is an elliptic spectrum. In particular, the $G = G_{\mathbf{E}}$ is of finite height. By Theorem 3.25, the H_∞ structure on \mathbf{E} gives descent data for level structures on G .

Definition 16.4. An H_∞ *elliptic spectrum* is an elliptic spectrum (\mathbf{E}, C, t) whose underlying spectrum \mathbf{E} is a homogeneous H_∞ spectrum \mathbf{E} as above, together with descent data for level structures on C/S , considered as a functor over Ell as in §15.4, such that the diagram of functors over FGps

$$\begin{array}{ccc} \underline{\text{level}}_1(C/S) & \xrightarrow{t} & \underline{\text{level}}_1(G/S) \\ d_1 \downarrow & & \downarrow d_1 \\ \underline{C/S} & \xrightarrow{t} & \underline{G/S} \end{array}$$

commutes.

PROPOSITION 16.5. Let (\mathbf{E}, C, t) be an H_∞ elliptic spectrum, and suppose in addition that p is regular in $\pi_0 \mathbf{E}$. Then the sigma orientation

$$MU\langle 6 \rangle \xrightarrow{\sigma_{(\mathbf{E}, C, t)}} \mathbf{E}$$

is an H_∞ map.

Proof. By Proposition 6.1, it suffices to show that, for each level structure

$$A_T \xrightarrow{\ell} i^* \widehat{C},$$

we have

$$\tilde{N}_{g_\ell} s(C/S_{\mathbf{E}}) = (\psi_\ell^{\mathbf{E}})^* s(C/S_{\mathbf{E}}),$$

where g_ℓ is the isogeny of elliptic curves making the diagram

$$\begin{array}{ccc} i^*G_{\mathbf{E}} & \longrightarrow & i^*C \\ \psi_\ell^{G/\mathbf{E}} \downarrow & & \downarrow g_\ell \\ (\psi_\ell^{\mathbf{E}})^*G_{\mathbf{E}} & \longrightarrow & (\psi_\ell^{\mathbf{E}})^*C \end{array}$$

commute. Proposition 15.14 gives the result. \square

Now let (\mathbf{E}, C, t) be the elliptic spectrum associated to the universal deformation of a supersingular elliptic curve C_0 over a perfect field k of characteristic $p > 0$. For example, we may take C_0 to be the Weierstrass curve

$$y^2 + y = x^3$$

over \mathbb{F}_2 (Example 15.10). Applying the Proposition, Corollary 15.13, and Proposition 12.12 gives the

COROLLARY 16.6. *The orientation*

$$MU\langle 6 \rangle \xrightarrow{\sigma(\mathbf{E}, C, t)} \mathbf{E}$$

is an H_∞ map.

Appendix A. H_∞ -ring spectra. Given an integer $n \geq 0$, let $D_n: \mathbb{S}_U \rightarrow \mathbb{S}_U$ be the functor

$$(A.1) \quad \mathbf{E} \mapsto \mathcal{L}(n) \wedge_{\Sigma_n} \mathbf{E}^{(n)},$$

where $\mathcal{L}(n) = \mathcal{L}(U^n, U)$ is the space of linear isometric embeddings from U^n to U .

An E_∞ ring spectrum is a spectrum with maps

$$D_n(\mathbf{E}) \rightarrow \mathbf{E}, \quad n \geq 0,$$

making the following diagrams commute:

$$(A.2) \quad \begin{array}{ccccc} \{1_U\} \rtimes \mathbf{E} & \longrightarrow & D_1 \mathbf{E} & \longrightarrow & D_{n+m} \mathbf{E} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{E} & \xlongequal{\quad} & \mathbf{E} & \longrightarrow & \mathbf{E}. \end{array}$$

An H_∞ ring spectrum is a spectrum \mathbf{E} together with maps $D_n \mathbf{E} \rightarrow \mathbf{E}$ such that the diagrams (A.2) commute up to homotopy.

The category of E_∞ -ring spectra is naturally enriched over topological spaces. The *space* of E_∞ -maps from \mathbf{E} to \mathbf{F} is the subspace of all maps consisting of those which make the diagrams

$$\begin{array}{ccc} D_n\mathbf{E} & \longrightarrow & D_n\mathbf{F} \\ \downarrow & & \downarrow \\ \mathbf{E} & \longrightarrow & \mathbf{F} \end{array}$$

commute. For a topological space X , the spectrum which underlies the “function object” is simply the spectrum \mathbf{E}^{X+} . The spectrum which underlies $\mathbf{E} \otimes X$ is more difficult to describe. If \mathbf{E} is only an H_∞ -ring spectrum, the spectrum \mathbf{E}^{X+} is still H_∞ .

These remarks actually depend very little on the construction of the functor D_n and are mostly matters of pure category theory. Indeed, the map $D_n\mathbf{E} \rightarrow \mathbf{E}$ can be regarded as a natural transformation of functors

$$\mathbb{S}_U(\mathbf{F}, \mathbf{E}) \rightarrow \mathbb{S}_U(D_n\mathbf{F}, \mathbf{E}).$$

Given a topological space X , we can use (2.2) to define a transformation

$$(A.3) \quad \mathbb{S}_U(\mathbf{F}, \mathbf{E}^{X+}) \rightarrow \mathbb{S}_U(D_n\mathbf{F}, \mathbf{E}^{X+}).$$

as the composite

$$\begin{aligned} \mathbb{S}_U(\mathbf{F}, \mathbf{E}^{X+}) &\cong \text{Spaces}(X, \mathbb{S}_U(\mathbf{F}, \mathbf{E})) \\ &\rightarrow \text{Spaces}(X, \mathbb{S}_U(D_n\mathbf{F}, \mathbf{E})) \\ &\cong \mathbb{S}_U(D_n\mathbf{F}, \mathbf{E}^{X+}). \end{aligned}$$

A more subtle property is that the transformation (A.3) is also given by

$$(A.4) \quad \begin{aligned} \mathbb{S}_U(\mathbf{F}, \mathbf{E}^{X+}) \cong \mathbb{S}_U(\mathbf{F} \wedge X_+, \mathbf{E}) &\xrightarrow{\quad} \mathbb{S}_U(D_n(\mathbf{F} \wedge X_+), \mathbf{E}) \\ &\xrightarrow{\text{diag}} \mathbb{S}_U(D_n(\mathbf{F}) \wedge X_+, \mathbf{E}) \\ &\cong \mathbb{S}_U(D_n\mathbf{F}, \mathbf{E}^{X+}). \end{aligned}$$

An important property of the functors is summarized in the following result of [BMMS86].

PROPOSITION A.5. *There is a natural weak equivalence*

$$\bigvee_{i+j=n} \mathcal{L}(2) \wedge D_i(\mathbf{E}) \wedge D_j(\mathbf{F}) \rightarrow D_n(\mathbf{E} \vee \mathbf{F}).$$

Furthermore, the ij -component of

$$D_n(\mathbf{E}) \xrightarrow{D_n(\nabla)} D_n(\mathbf{E} \vee \mathbf{E}) \rightarrow \bigvee_{i+j=n} \mathcal{L}(2) \wedge D_i(\mathbf{E}) \wedge D_j(\mathbf{E}) \rightarrow \prod_{i+j=n} \mathcal{L}(2) \wedge D_i(\mathbf{E}) \wedge D_j(\mathbf{E})$$

is the transfer map Tr_{ij} with respect to the inclusion $\Sigma_i \times \Sigma_j \subset \Sigma_n$.

Note also that if W is a virtual bundle of dimension 0 over a space X , then $D_A(X^W)$ is the Thom spectrum the virtual bundle $V_{\text{reg}} \otimes W$ over $D_A(X)$, where V_{reg} is the regular representation of A^* .

Appendix B. Composition of operations. Let \mathbf{E} be a homogeneous H_∞ ring spectrum, such that $\pi_0 \mathbf{E}$ is a local ring with perfect residue field of characteristic $p > 0$, and the formal group $G = G_{\mathbf{E}}$ is of finite height. In §3 we associate to each map of formal schemes

$$i: T = \text{spf } R \rightarrow S_{\mathbf{E}},$$

finite abelian group A , and level structure

$$\ell: A_T \rightarrow i^* G,$$

a map of formal schemes $\psi_{\ell}^{\mathbf{E}}: T \rightarrow S_{\mathbf{E}}$ and an isogeny $\psi_{\ell}^{G/\mathbf{E}}: i^* G \rightarrow \psi_{\ell}^* G$ with kernel $[\ell(A)]$. Theorem 3.25 asserts that these constitute descent data for level structures on G . Properties (1) and (3) of Definition 3.1 follow immediately from the construction. In this section we give a proof of (2): if

$$(B.1) \quad B \rightarrow A \xrightarrow{\pi} C$$

is a short exact sequence of finite abelian groups, then with the notation

$$(B.2) \quad \begin{array}{ccccccc} B & \longrightarrow & A & \longrightarrow & C \\ \ell' \downarrow & & \ell \downarrow & & \downarrow \ell'' \\ i^* G & \xlongequal{\quad} & i^* G & \xrightarrow{\psi_{\ell'}^{G/\mathbf{E}}} & \psi_{\ell'}^* G, \end{array}$$

we have

$$(B.3) \quad \psi_{\ell''} = \psi_{\ell}: T \rightarrow S$$

$$(B.4) \quad \psi_{\ell}^{G/\mathbf{E}} = \psi_{\ell'}^{G/\mathbf{E}} \circ \psi_{\ell''}^{G/\mathbf{E}}: i^* G \rightarrow \psi_{\ell'}^* G = \psi_{\ell''}^* G.$$

In fact we shall give a proof of (B.3); the proof of (B.4) is similar.

Elsewhere in this paper we use the notation D_n for either the functor (A.1) or for the natural transformation

$$\mathbb{S}_U(\mathbf{F}, \mathbf{E}) \rightarrow \mathbb{S}_U(D_n \mathbf{F}, \mathbf{E})$$

associated to an H_∞ ring spectrum \mathbf{E} (and similarly for D_A). In this section only, it is convenient to write P_n or P_A for the natural transformation, reserving D_n and D_A to refer to the functors (A.1) and (3.8). Thus for

$$f: \mathbf{F} \rightarrow \mathbf{E},$$

the diagram

$$\begin{array}{ccc} D_n \mathbf{F} & \xrightarrow{P_n f} & \mathbf{E} \\ D_n f \downarrow & \nearrow & \\ D_n \mathbf{E} & & \end{array}$$

commutes; and if R is a complete local ring and

$$A_{\text{spf } R} \xrightarrow{\ell} i^* G$$

is a level structure, then by Definition 3.9, $\psi_\ell^\mathbf{E}: \pi_0 \mathbf{E} \rightarrow R$ is the composition

$$\pi_0 \mathbf{E} \xrightarrow{P_A} \pi_0 \mathbb{S}_U(D_A S^0, \mathbf{E}) = \pi_0 \mathbf{E}^{BA_*^*} \rightarrow \mathcal{O}((BA^*)_{\mathbf{E}}) \cong \mathcal{O}(\underline{\text{hom}}(A, G)) \xrightarrow{\chi_\ell} R.$$

If Z is a set, let Σ_Z be the group of automorphisms of the underlying set. We let Z act on itself by left multiplication, and so we consider Z to be a subgroup of Σ_Z . If $T \subseteq \Sigma_Z$ is a subgroup, and if X is a group, then we write $T \wr X$ for the wreath product

$$T \wr X = T \ltimes X^Z.$$

Suppose that

$$(B.5) \quad X \rightarrow Y \xrightarrow{\pi} Z$$

is a short exact sequence of abelian groups, and Z is finite. A splitting

$$s: Z \rightarrow Y$$

of (B.5) as a sequence of sets determines a homomorphism

$$g: Y \rightarrow Z \wr X$$

by the formula

$$g(y) = (\pi(y), f_y),$$

where

$$f_y: Z \rightarrow X$$

is the map of sets given by the formula

$$f_y(z) = y + s(z) - s(\pi(y) + z).$$

If g' is another such homomorphism defined using a section s' , then g and g' differ by conjugation by $(0, s - s') \in Z \wr X$, and so just the extension (B.5) determines a homotopy class of maps

$$BY \xrightarrow{Bg} B(Z \wr X).$$

If moreover X is finite, then s determines a homomorphism

$$h: \Sigma_Z \wr \Sigma_X \rightarrow \Sigma_Y$$

by the formula

$$h(\sigma, \tau)(s(z) + x) = s(\sigma(z)) + \tau(x)$$

for $\sigma \in \Sigma_Z$, $\tau \in \Sigma_X$, $x \in X$, and $z \in Z$. Once again the resulting map

$$B(\Sigma_Z \wr \Sigma_X) \rightarrow B\Sigma_Y$$

is independent of the choice of section. The maps g and h have been defined so that the diagram

$$\begin{array}{ccc} Z \wr X & \longrightarrow & \Sigma_Z \wr \Sigma_X \\ g \uparrow & & \downarrow h \\ Y & \longrightarrow & \Sigma_Y \end{array}$$

commutes.

Applying these observations to the dual of a short exact sequence

$$B \rightarrow A \rightarrow C$$

gives maps

$$Bg_+: D_A S^0 = BA_+^* \rightarrow B(B^* \cap BC_+^*)_+ = D_B BC_+^* = D_B D_C S^0$$

and

$$Bh_+: D_{|B|} D_{|C|} S^0 \rightarrow D_{|A|} S^0.$$

LEMMA B.6. *In this situation, the diagram*

$$\begin{array}{ccc} \pi_0 \mathbf{E} & \xrightarrow{P_A} & \pi_0 \mathbf{E}^{BA_+^*} \\ P_C \downarrow & & \uparrow \pi_0 \mathbf{E}^{Bg_+} \\ \pi_0 \mathbf{E}^{BC_+^*} & \xrightarrow{P_B} & \pi_0 \mathbf{E}^{D_{B^*} BC_+^*} \end{array}$$

commutes.

Proof. For $f: T \rightarrow E$, $P_A f$ is the composition

$$D_A T \rightarrow D_{|A|} T \xrightarrow{D_{|A|} f} D_{|A|} \mathbf{E} \rightarrow \mathbf{E}.$$

Now let $f: S^0 \rightarrow \mathbf{E}$, and consider the diagram

$$\begin{array}{ccccc} D_B D_C S^0 & \xrightarrow{\quad} & & & \\ \searrow & & & & \downarrow P_B P_C f \\ D_{|B|} D_C S^0 & \xrightarrow{D_{|B|} P_C f} & D_{|B|} \mathbf{E} & & \\ \downarrow & & \uparrow & & \downarrow \\ D_{|B|} D_{|C|} S^0 & \xrightarrow{D_{|B|} D_{|C|} f} & D_{|B|} D_{|C|} \mathbf{E} & \xrightarrow{\quad} & \mathbf{E} \\ \downarrow Bh_+ & & \downarrow & & \uparrow P_A f \\ D_{|A|} S^0 & \xrightarrow{D_{|A|} f} & D_{|A|} \mathbf{E} & & \end{array}$$

The trapezoid on the left commutes by Lemma B.6. The top inner rectangle is obtained by applying $D_{|B|}$ to the definition of $P_C f$, and so the top outer composition is the definition of $P_B P_C f$. The lower inner rectangle commutes by the naturality of D , and the lower outer composition is the definition of P_A . The right

inner triangle is a case of the left diagram of (A.2), and so commutes because \mathbf{E} is an H_∞ spectrum. \square

Now we turn to the situation of the diagram (B.2). Lemma B.6 implies that the top square in the diagram

$$(B.7) \quad \begin{array}{ccccc} \pi_0 \mathbf{E} & \xrightarrow{P_A} & \pi_0 \mathbf{E}^{BA^*_+} & & \\ \downarrow P_C & & \uparrow \pi_0 \mathbf{E}^{Bg_+} & \searrow \chi_\ell & \downarrow \psi_\ell^\mathbf{E} \\ \pi_0 \mathbf{E}^{BC^*_+} & \xrightarrow{P_B} & \pi_0 \mathbf{E}_+^{D_{B^*} BC^*} & & R \\ \uparrow & & \uparrow & & \uparrow \psi_{\ell'}^\mathbf{E} \\ \pi_0 \mathbf{E} & \xrightarrow{P_B} & \pi_0 \mathbf{E}^{BB^*_+} & \xrightarrow{\chi_{\ell'}} & R \end{array}$$

commutes. The commutativity of the bottom left square is obvious; the two right-hand corners commute by the definition of $\psi_\ell^\mathbf{E}$ (3.9).

LEMMA B.8. *If*

$$\chi_{\ell''}: \pi_0 \mathbf{E}^{BC^*_+} \rightarrow R$$

is the homomorphism classifying the homomorphism

$$C \xrightarrow{\ell''} (\psi_{\ell'}^\mathbf{E})^* G,$$

then

$$(B.9) \quad \chi_{\ell''} = \chi_\ell \circ \pi_0 \mathbf{E}^{Bg_+} \circ P_B.$$

Proof. By definition, $\chi_{\ell''}$ classifies the homomorphism

$$C \xrightarrow{\ell} \psi_{\ell'}^* G.$$

Let us temporarily write

$$u = \chi_\ell \circ \pi_0 \mathbf{E}^{Bg_+} \circ P_B.$$

As in the proof of Lemma 3.10, the axioms of an H_∞ structure together with Proposition 7.5 imply that u is a continuous ring homomorphism. The commuta-

tivity of the diagram (B.7) shows that the diagram

$$\begin{array}{ccc} \pi_0 \mathbf{E} & & \\ \downarrow & \searrow \psi_{\ell'}^{\mathbf{E}} & \\ \pi_0 \mathbf{E}^{BC_*^*} & \xrightarrow{u} & R \end{array}$$

commutes. In view of the isomorphism

$$(BC^*)_{\mathbf{E}} \cong \underline{\text{hom}}(C, G)$$

of Proposition 7.3, u classifies *some* homomorphism

$$C \xrightarrow{w} (\psi_{\ell'}^{\mathbf{E}})^* G,$$

and it remains to show that $w = \ell''$. To show that, it suffices to show that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ell} & i^* G \\ \downarrow & & \downarrow \psi_{\ell}^{G/\mathbf{E}} \\ C & \xrightarrow{w} & (\psi_{\ell'}^{\mathbf{E}})^* G \end{array}$$

commutes.

Suppose that $a \in A$ with image $c \in C$, and consider the diagram

$$\begin{array}{ccccccc} \pi_0 \mathbf{E}^{\mathbb{C}P^\infty_+} & \xrightarrow{P_B} & \pi_0 \mathbf{E}^{D_B \mathbb{C}P^\infty_+} & \xrightarrow{\pi_0 \mathbf{E}^{B\Delta_+}} & \pi_0 \mathbf{E}^{(BB^* \times \mathbb{C}P^\infty)_+} & & \\ \pi_0 \mathbf{E}^{Bc_+} \downarrow & & \downarrow \pi_0 \mathbf{E}^{D_B c_+} & & \downarrow \pi_0 \mathbf{E}^{B(\pi \times a)_+} & & \\ \pi_0 \mathbf{E}^{BC_*^*} & \xrightarrow{P_B} & \pi_0 \mathbf{E}^{D_B BC_*^*} & \xrightarrow{\pi_0 \mathbf{E}^{Bg_+}} & \pi_0 \mathbf{E}^{BA_*^*} & \xrightarrow{\chi_{\ell}} & R. \end{array}$$

The outer clockwise composition is the map of rings corresponding to the point

$$\text{spf } R \xrightarrow{\psi_{\ell}^{G/\mathbf{E}}(\ell(a))} G,$$

while the outer counterclockwise composition corresponds to the point

$$\text{spf } R \xrightarrow{w(c)} G.$$

It is clear that the left square commutes. To see that the right square commutes,

observe that we have a commutative diagram

$$\begin{array}{ccccc} C^* & \longrightarrow & A^* & \longrightarrow & B^* \\ c \downarrow & & \downarrow \pi \times a & & \downarrow = \\ \mathbb{C}^\times & \longrightarrow & B^* \times \mathbb{C}^\times & \longrightarrow & B^*. \end{array}$$

A choice of section

$$s: B^* \rightarrow A^*$$

gives the map $Bg: BA^* \rightarrow D_B BC^*$. It also gives a section

$$s: B^* \rightarrow B^* \times \mathbb{C}^\times,$$

and so a map

$$Bg: BB^* \times \mathbb{C}P^\infty \rightarrow D_B \mathbb{C}P^\infty$$

such that the diagram

$$\begin{array}{ccc} BA^* & \xrightarrow{Bg} & D_B BC^* \\ B(\pi \times a) \downarrow & & \downarrow D_B Bc \\ BB^* \times \mathbb{C}P^\infty & \xrightarrow{Bg} & D_B \mathbb{C}P^\infty \end{array}$$

commutes. But the homotopy class of the map Bg is independent of the choice of section, so the diagram

$$\begin{array}{ccc} BA^* & \xrightarrow{Bg} & D_B BC^* \\ B(\pi \times a) \downarrow & & \downarrow D_B Bc \\ BB^* \times \mathbb{C}P^\infty & \xrightarrow{B\Delta} & D_B \mathbb{C}P^\infty \end{array}$$

commutes. □

The commutativity of the diagram (B.7) and Lemma B.8 together imply that

$$\begin{aligned} \psi_\ell^E &= \chi_\ell \circ P_A \\ &= \chi_\ell \circ \pi_0 E^{Bg+} \circ P_B \circ P_C \\ &= \chi_{\ell''} \circ P_C \\ &= \psi_{\ell''}^E \end{aligned}$$

as required.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN,
URBANA IL 61801
E-mail: mando@math.uiuc.edu

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MA 02139-4307
E-mail: mjh@math.mit.edu

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF SHEFFIELD, SHEFFIELD S3 7RH,
GREAT BRITAIN
E-mail: N.P.Strickland@sheffield.ac.uk

REFERENCES

- [Ada74] J. Frank Adams, *Stable Homotopy and Generalised Homology*, Univ. of Chicago Press, 1974.
- [And95] M. Ando, Isogenies of formal group laws and power operations in the cohomology theories E_n , *Duke Math. J.* **79** (1995), 423–485.
- [AHS01] M. Ando, M. J. Hopkins, and N. P. Strickland, Elliptic spectra, the Witten genus, and the theorem of the cube, *Invent. Math.* **146** (2001), 595–687.
- [Bre83] L. Breen, *Fonctions thêta et théorème du cube*, *Lecture Notes in Math.*, vol. 980, Springer-Verlag, Berlin, 1983.
- [BMMS86] R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ Ring Spectra, Lecture Notes in Math.*, vol. 1176, Springer-Verlag, Berlin, 1986.
- [DR73] P. Deligne and M. Rapoport, Les schémas de modules de courbes elliptiques, *Modular Functions of One Variable II*, *Lecture Notes in Math.*, vol. 349, Springer-Verlag, Berlin, 1973, pp. 143–316.
- [DG70] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970.
- [Dri74] V. G. Drinfeld, Elliptic modules, *Math. USSR-Sb.* **23** (1974), 561–592.
- [GH02] P. Goerss and M. J. Hopkins, Realizing commutative ring spectra as E_∞ ring spectra, preprint, 2002.
- [Gro60] A. Grothendieck, Éléments de géométrie algébrique. I. Le langage des schémas, *Inst. Hautes Études Sci. Publ. Math.* **228** (1960).
- [Hop95] M. J. Hopkins, Topological modular forms, the Witten genus, and the theorem of the cube, *Proceedings of the International Congress of Mathematicians*, vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 554–565.
- [HKR00] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel, Generalized group characters and complex oriented cohomology theories, *J. Amer. Math. Soc.* **13** (2000), 553–594 (electronic).
- [Kat81] N. M. Katz, Serre-Tate local moduli, *Surfaces algébriques: séminaire de géométrie algébrique d'Orsay 1976–1978*, *Lecture Notes in Math.*, vol. 868, Springer-Verlag, Berlin, 1981, pp. 138–202.
- [KM85] N. M. Katz and B. Mazur, *Arithmetic Moduli of Elliptic Curves*, Princeton University Press, Princeton, NJ, 1985.
- [Lan76] P. S. Landweber, Homological properties of comodules over MU_*MU and BP_*BP , *Amer. J. Math.* **98** (1976), 591–610.
- [Lan78] S. Lang, *Cyclotomic Fields*, *Grad. Texts in Math.*, vol. 59, Springer-Verlag, New York, 1978.
- [Lub67] J. Lubin, Finite subgroups and isogenies of one-parameter formal Lie groups, *Ann. of Math. (2)* **85** (1967), 296–302.
- [LT66] J. Lubin and J. Tate, Formal moduli for one-parameter formal lie groups, *Bull. Soc. Math. France* **94** (1966), 49–60.

- [McC86] J. E. McClure, Power operations in H_∞^d ring theories, *H_∞ Ring Spectra and Their Applications, Lecture Notes in Math.*, vol. 1176, Springer-Verlag, Berlin, 1986.
- [Mum70] D. Mumford, *Abelian Varieties*, published for the Tata Institute of Fundamental Research, Bombay, 1970.
- [Qui69] D. Quillen, On the formal group laws of unoriented and complex cobordism theory, *Bull. Amer. Math. Soc.* **75** (1969), 1293–1298.
- [Qui71] —————, Elementary proofs of some results of cobordism theory using Steenrod operations, *Adv. Math.* **7** (1971), 29–56.
- [Sil99] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Springer-Verlag, New York, 1997 (corrected reprint of the 1986 original).
- [Sin68] W. M. Singer, Connective fiberings over BU and U , *Topology* **7** (1968), 271–303.
- [Str97] N. P. Strickland, Finite subgroups of formal groups, *J. Pure Appl. Algebra* **121** (1997), 161–208.
- [Str98] —————, Morava E -theory of symmetric groups, *Topology* **37** (1998), 757–779.