MAT8021, Algebraic Topology

Assignment 6

Due in-class on Friday, May 9

1. Find all $(2,3)$-shuffles α and give formulas for the associated shuffle maps $f_\alpha\colon \Delta[5] \to \Delta[2] \times \Delta[3]$.
2. Find recursive formulas for $\dim_{\mathbb{Z}/2} H_k\big((\mathbb{RP}^2)^n; \mathbb{Z}/2\big)$ in terms of k and n.
3. Find a pair of chain complexes C_* and D_* such that the tensor product chain complex $C_* \otimes D_*$ does not satisfy the Künneth formula, i.e., there is some n such that
$$H_n(C_* \otimes D_*) \ncong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \;\oplus \bigoplus_{p+q=n-1} \operatorname{Tor}\big(H_p(C_*), H_q(D_*)\big)$$
4. Find the homology of the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(3,5)$.
5. There is a continuous map from one Grassmannian $\mathrm{Gr}(k,n)$ to the next $\mathrm{Gr}(k,n+1)$ by sending a plane $V \subset \mathbb{R}^n$ to the plane
$$\{(0, x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in V\}$$
Show that the image consists of a union of Schubert cells, and find the dimension of the smallest cell not in the image.

The following is a series of additional exercises/examples and will not be collected for grading.

1. A *differential graded algebra* is a chain complex A_* with associative multiplication maps $\cdot\colon A_p \times A_q \to A_{p+q}$ satisfying the Leibniz rule
$$\partial(x \cdot y) = (\partial x) \cdot y + (-1)^p x \cdot (\partial y)$$
for $x \in A_p, y \in A_q$.

   Show that given elements $[x] \in H_p(A)$ and $[y] \in H_q(A)$, we get a well-defined element $[x] \cdot [y]$ in $H_{p+q}(A)$. Show that this makes $H_*(A)$ into a graded ring.
2. Recall that a *topological group* G is a space with continuous maps
$$\begin{array}{ll} \mu\colon G \times G \to G & \text{multiplication} \\ \nu\colon G \to G & \text{inverse} \\ \iota\colon \{*\} \to G & \text{identity} \end{array}$$

so that on the underlying set, we get a group with $g \cdot h = \mu(g,h)$, $g^{-1} = \nu(g)$, and $e = \iota(*)$.

(a) Show that $H_*(G)$ is a ring by defining a multiplication on $C_*(G)$. This is called a *Pontryagin ring* structure on $H_*(G)$.

(b) If G is abelian, show that $C_*(G)$ (and hence $H_*(G)$) is *graded commutative*, i.e., $x \cdot y = (-1)^{|x||y|} y \cdot x$ for any $x, y \in C_*(G)$, where $|x|$ and $|y|$ denote the degrees of x and y respectively.

3. (a) Let $G = \mathbb{R}$. What is the Pontryagin ring structure on $H_*(\mathbb{R})$?

   (b) Show that $H_*(S^1)$ is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^2)$ with $|\alpha| = 1$.

   (c) More generally, it turns out that

$$H_*(S^1 \times S^1) \cong \mathbb{Z}[\alpha,\beta]/(\alpha^2, \beta^2, \alpha\beta + \beta\alpha) =: \Lambda[\alpha,\beta]$$

   is an *exterior algebra* on α, β with $|\alpha| = |\beta| = 1$. Similarly $H_*(S^1 \times S^1 \times S^1) \cong \Lambda[\alpha,\beta,\gamma]$, etc. In contrast, if $G = S^3$ regarded as the unit quaternions, what is the Pontryagin ring structure on $H_*(S^3)$?

4. Let $G = \mathrm{SO}(3)$ be the 3×3 matrices over $\mathbb{R}$ with determinant 1.

   (a) Viewing it as the group of rotations in $\mathbb{R}^3$, describe a homeomorphism $\mathrm{SO}(3) \cong \mathbb{RP}^3$ by defining a map $D^3 \to \mathrm{SO}(3)$ that factors through $\mathbb{RP}^3$.

   (b) Give a presentation for $H_*\big(\mathrm{SO}(3)\big)$ as a ring.

   (c) What about $H_*\big(\mathrm{SO}(3); \mathbb{Z}/2\big)$? In particular, show that the square of the generator of $H_1\big(\mathrm{SO}(3); \mathbb{Z}/2\big)$ equals zero.

5. Suppose G is a topological group and X is a topological space with a continuous map $G \times X \to X$ which is an action of G. Show that $H_*(X)$ becomes a left module over the Pontryagin ring $H_*(G)$.

The following is a series of exercises on intersection homology,[1] not to be collected either (Greg Friedman's book is recommended for further reading).

Let X be a simplicial complex.

- A *filtration* of X is a sequence of subcomplexes of X:

$$X = X^n \supseteq X^{n-1} \supseteq \cdots \supseteq X^2 \supseteq X^1 \supseteq X^0 \supseteq X^{-1} = \emptyset$$

  Each connected component of $X^i - X^{i-1}$ is called a *stratum*.

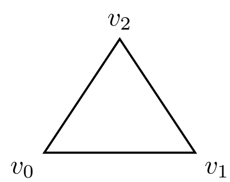
- Let $\mathcal{F}$ be the set of strata of X. A *perversity* on X is a function $\bar{p} \colon \mathcal{F} \to \mathbb{Z}$ such that $\bar{p}(S) = 0$ if $S \subset X^n - X^{n-1}$.

- An i-simplex σ is said to be $\bar{p}$-*allowable* if $\dim(\sigma \cap S) \le \dim(\sigma) + \dim(S) - n + \bar{p}(S)$ for every stratum S of X.

[1] Thanks to Zhou Fang for supplying it.

- An i-chain ζ is said to be *$\bar{p}$-allowable* if every simplex of ζ and of $\partial\zeta$ is $\bar{p}$-allowable.
- Define the group $I^{\bar{p}}C_i(X)$ to be the subset of $C_i(X)$ consisting of $\bar{p}$-allowable i-chains. It can be shown that the chain complex $\big(C_*(X), \partial\big)$ restricts to a chain complex $\big(I^{\bar{p}}C_*(X), \partial\big)$. The *Goresky–MacPherson intersection homology groups* are defined to be $I^{\bar{p}}H_i(X) \coloneqq H_i\big(I^{\bar{p}}C_*(X)\big)$.

Now, let X be the boundary of the simplex $[v_0, v_1, v_2]$. Suppose that X is filtered as $\{v_2\} = X^0 \subset X^1 = X$.



1. Compute the intersection homology of this stratified space. (Hint: Consider the three cases of $\bar{p}(\{v_2\}) > 0$, $\bar{p}(\{v_2\}) = 0$, and $\bar{p}(\{v_2\}) < 0$.)
2. Compute the intersection homology of $S^1 \vee S^1$ with a stratification by its singular point similar to the above.
3. Is the intersection homology defined above independent of choice of a filtration? Give a proof or a counterexample.