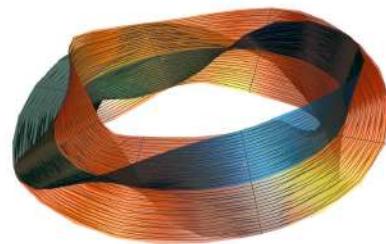


# Topology and geometry of non-Hermitian Möbius bands and their deformations

Explicit examples of rank-2 and rank-3 Higgs bundles  
in the contexts of  
quantum materials and geometric Langlands program



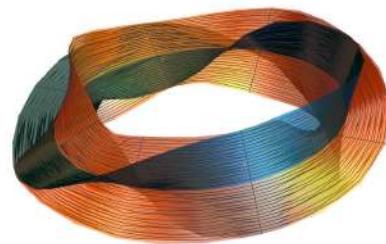
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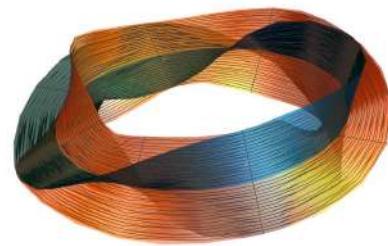
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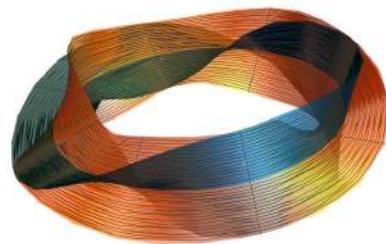
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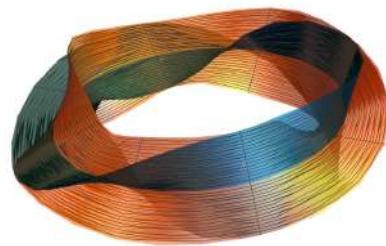
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Disclaimers: 1. Physics + math modeling, *not* mathematical physics

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## **Motivations: Quantum materials and their math modeling**

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*Holography, optical devices,  
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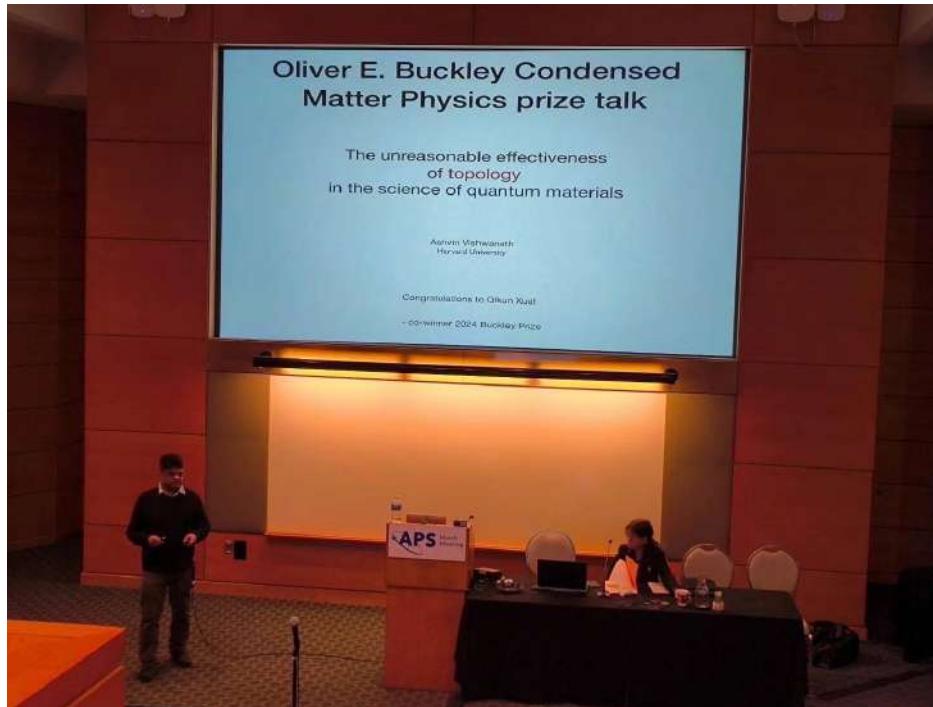
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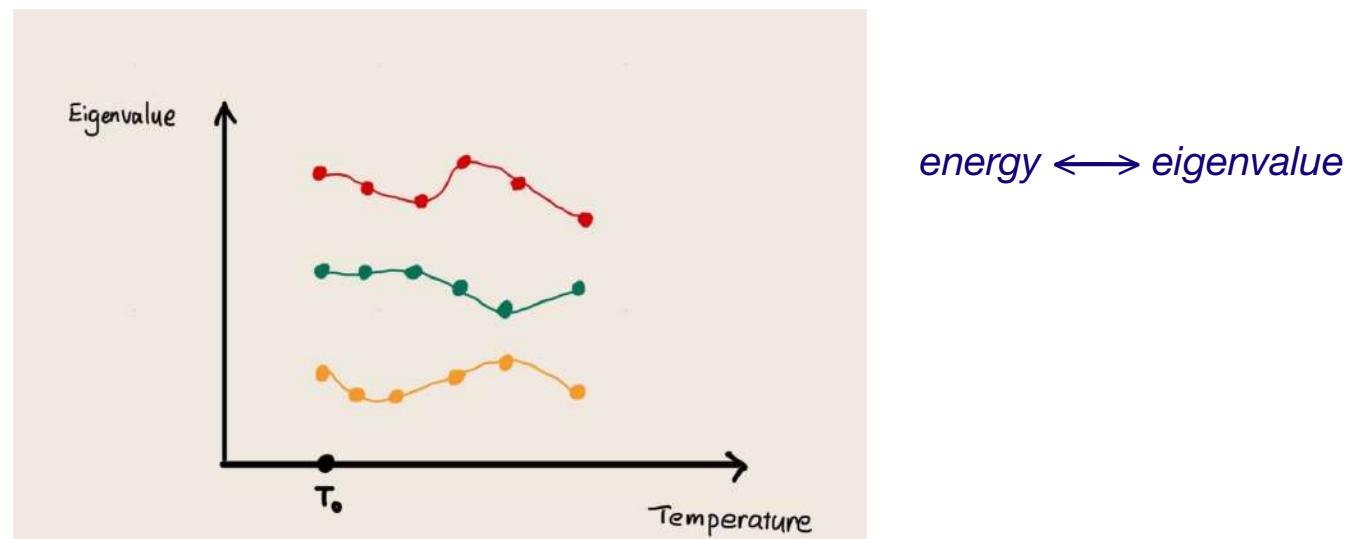
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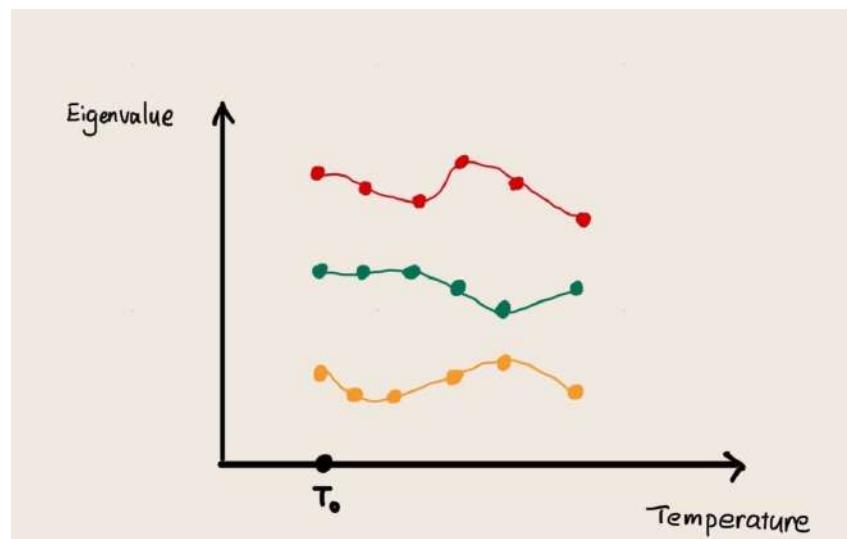
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Mathematical modeling of electronic energy *band structures* therein



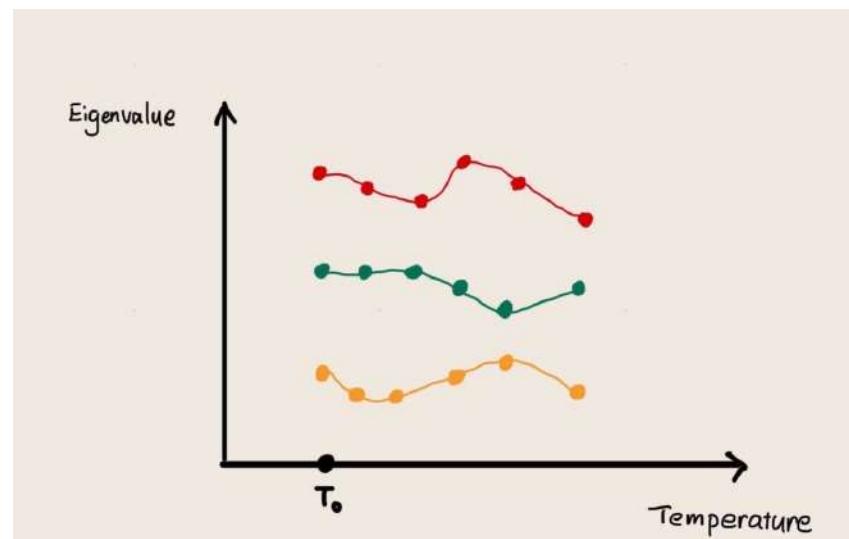
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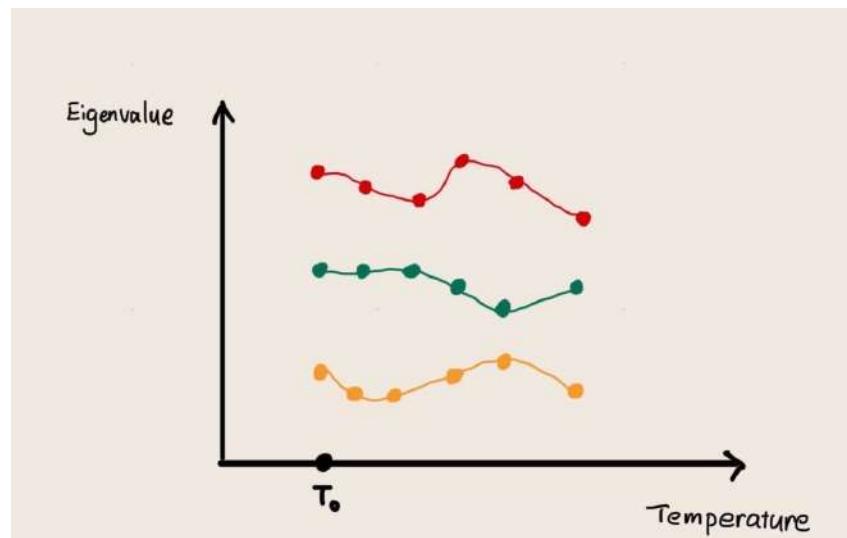
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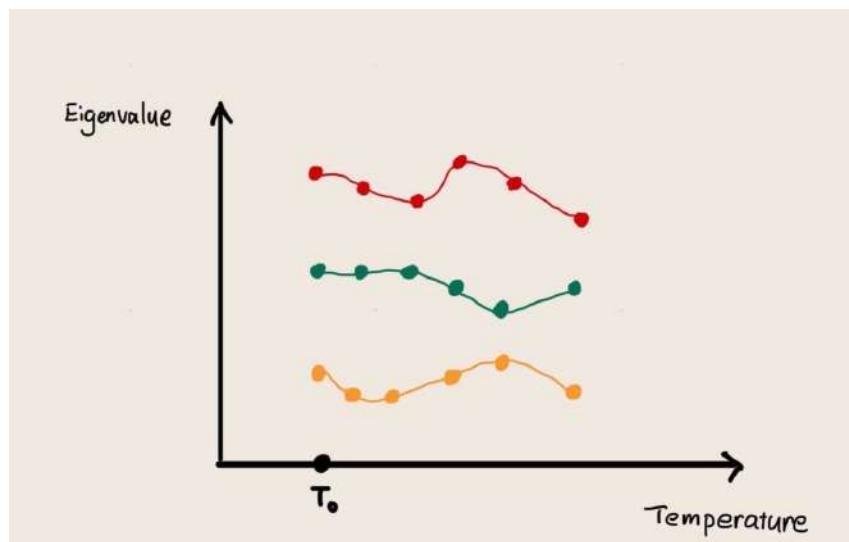
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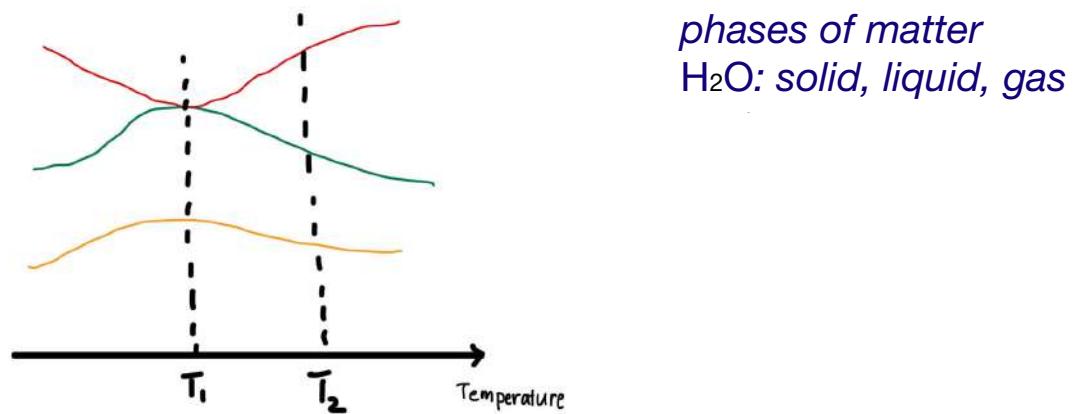


*Hermitian vs.  
non-Hermitian*

*real eigenvalues  
(observable  
energies) vs.  
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imaginary part  
(counts for  
energy exchange  
with surrounding  
environment or  
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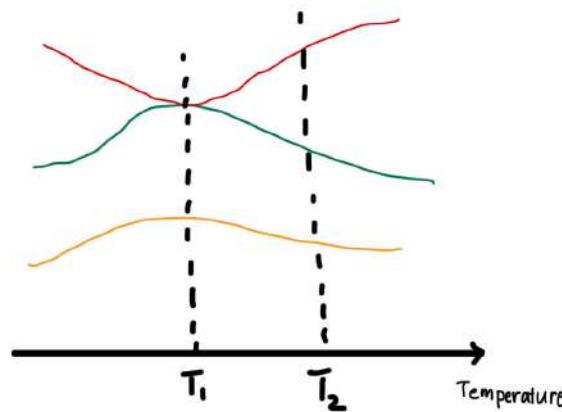
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Thanks to Hopf bundles and Higgs bundles as *eigenbundles*, we now have a **conceptually more systematic**, visibly more intuitive understanding of the topic.

## **Mathematical set-up: Eigenframe rotation of non-Hermitian systems**

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science

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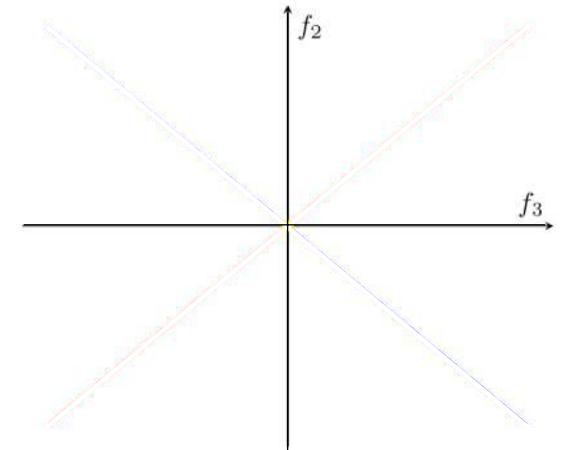
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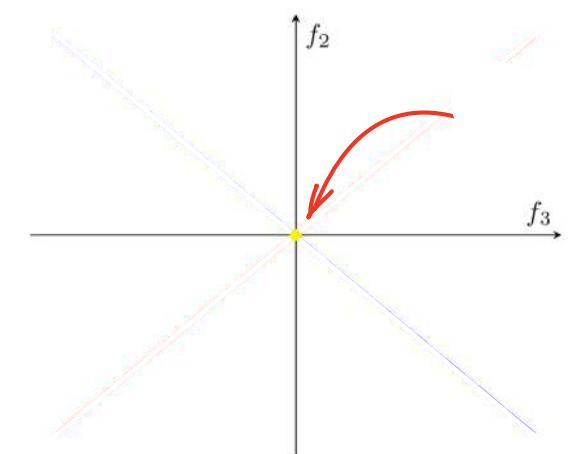
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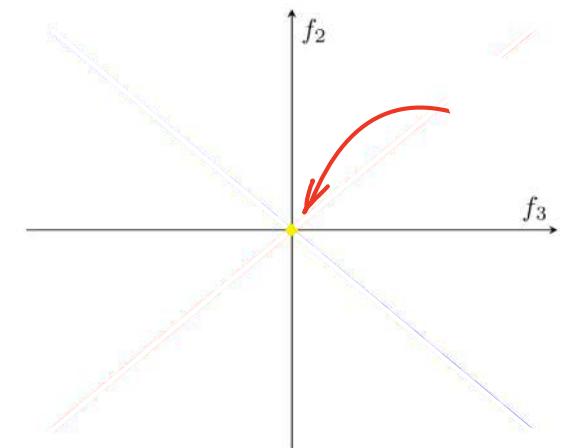
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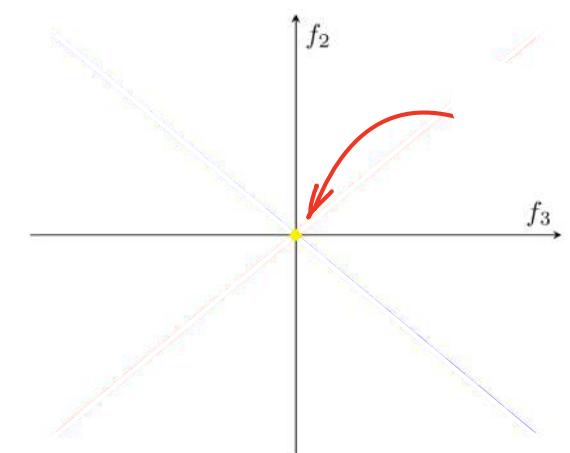
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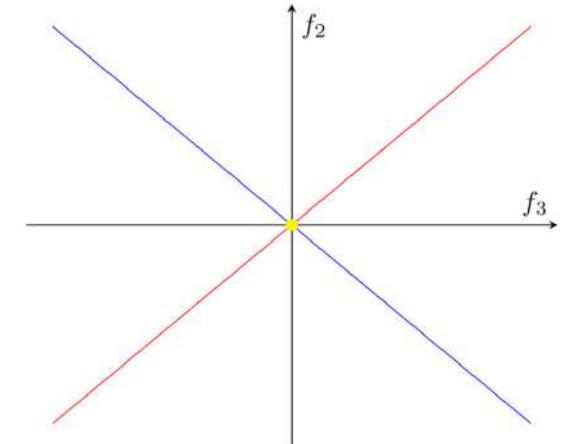
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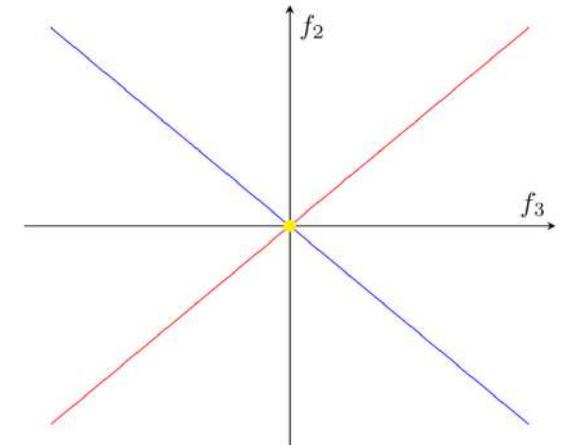
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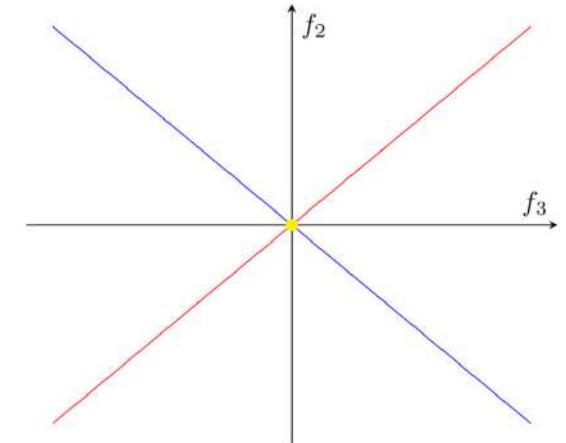
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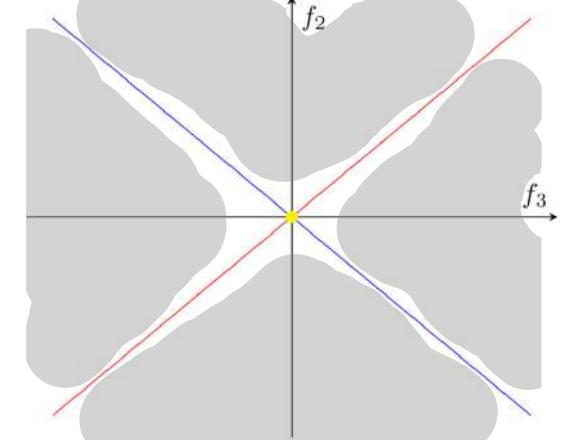
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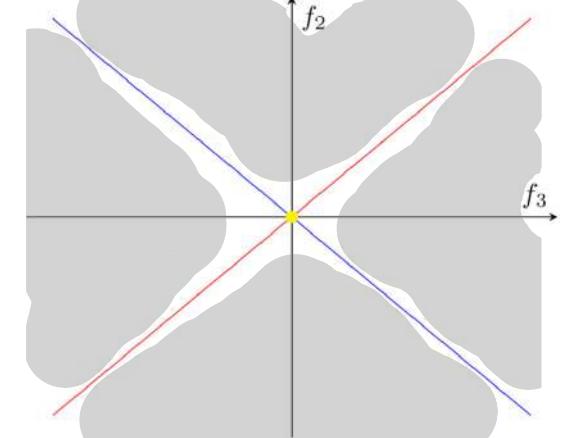
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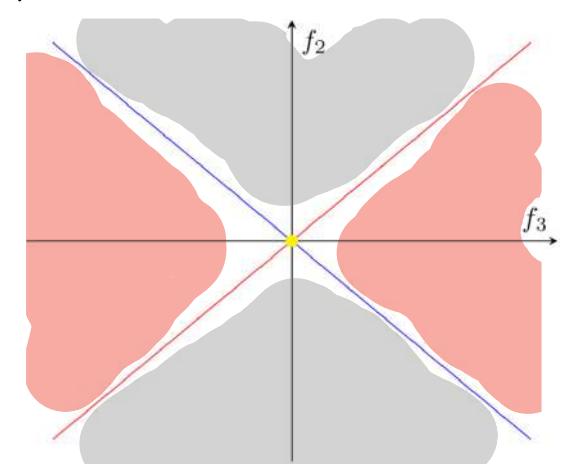
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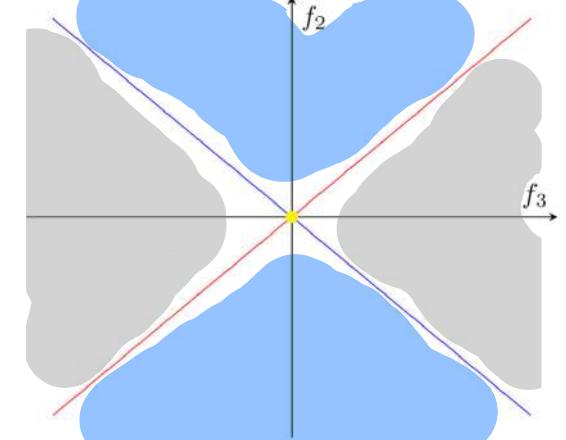
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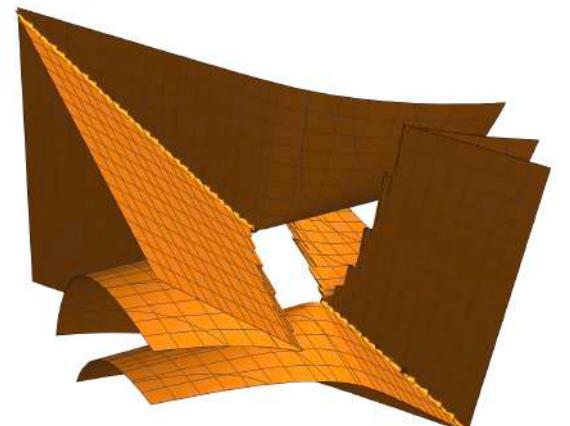
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*The equation for this surface is a non-homogeneous real polynomial in  $f_1, f_2, f_3$  of degree 6.*



Swallowtail couple sw2

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V. I. Arnold's tombstone at the Novodevichy Cemetery in Moscow

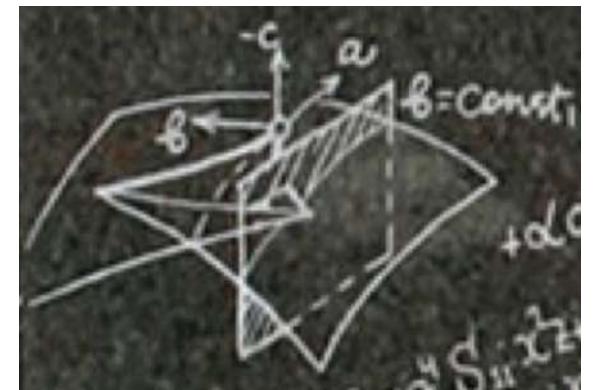
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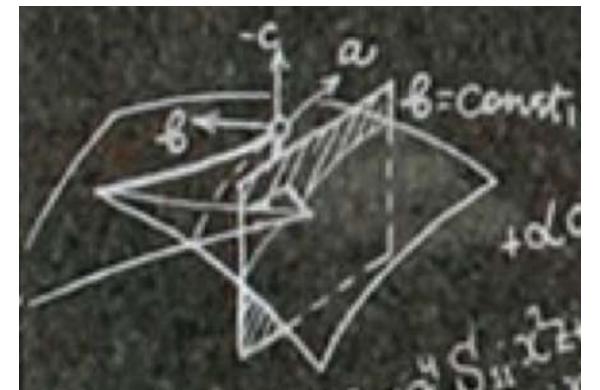
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A **local** model for moduli spaces of 3-band Hamiltonians

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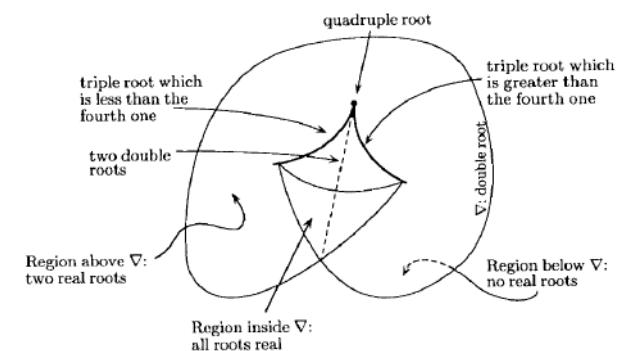
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*Arnold, Braids of algebraic functions and the cohomology of swallowtails, 1968.*

*Homological stability of braid groups*

*Portrait from Gelfand, Kapranov, Zelevinsky,  
Discriminants, resultants, and multidimensional determinants.*



The space of polynomials  $a_0 + a_1x + a_2x^2 + x^4$

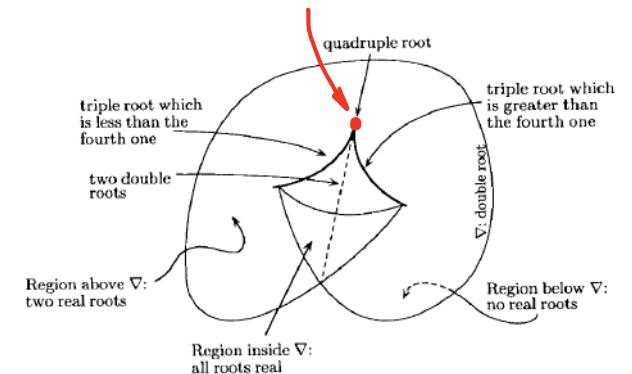
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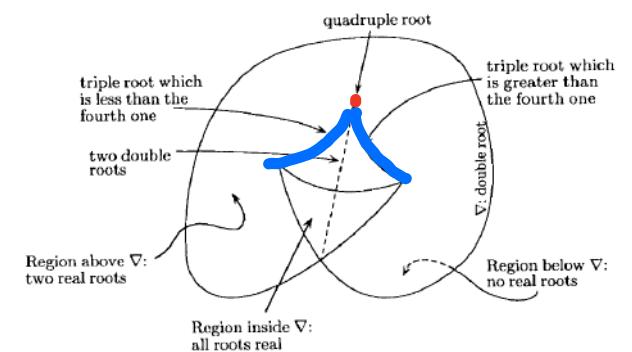
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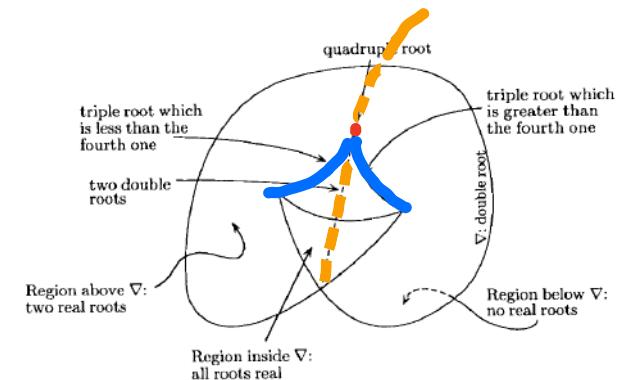
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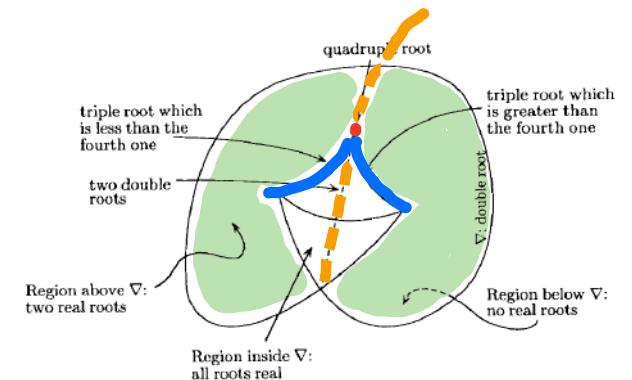
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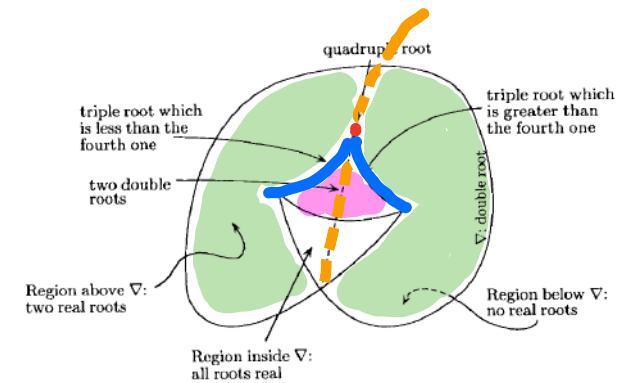
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Again, we aim to find computable algebraic invariants that systematically classify the evolutions of eigenvectors along loops in such stratified parameter spaces, including when they cross the discriminant surface resulting in degeneracies of various sorts.



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## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is **isolated**, yet has profound physical implications already known to Arnold.

*Remarks on eigenvalues and eigenvectors of Hermitian matrices,  
Berry phase, adiabatic connections and quantum Hall effect, 1995.*

*Also: Polymathematics, 2000.*

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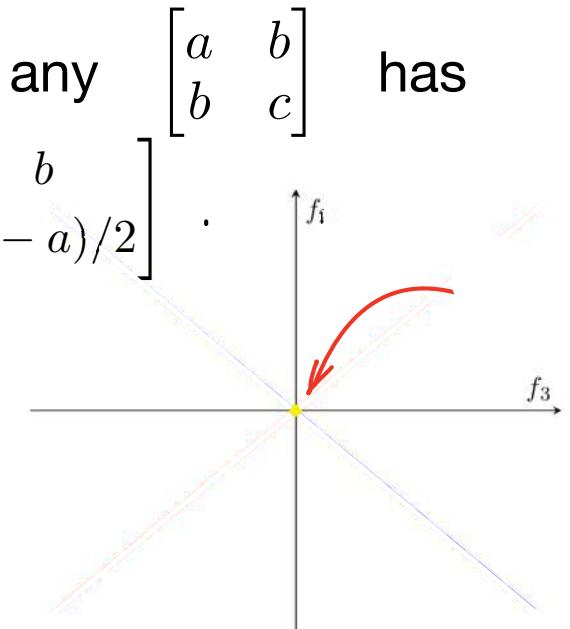
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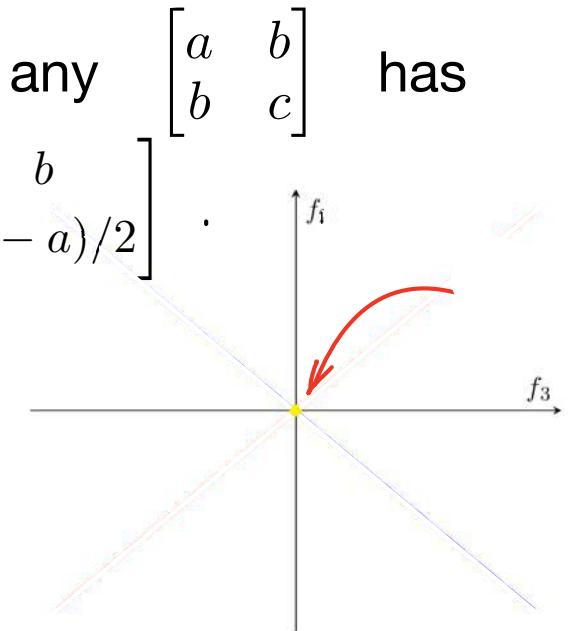
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However, their explanation for the appearance of the  $O(1)^n$ -action in constructing the moduli spaces was rather ad hoc.

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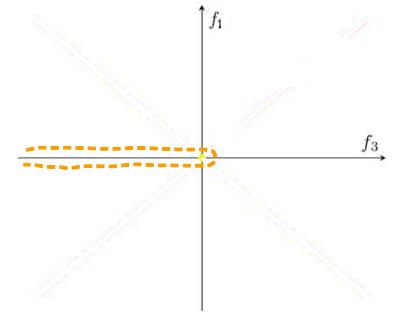
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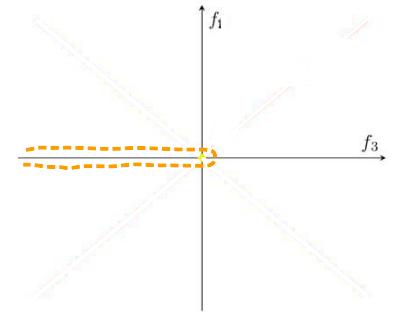
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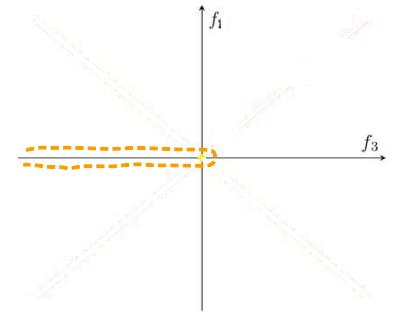
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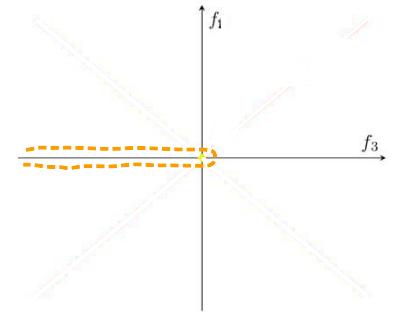
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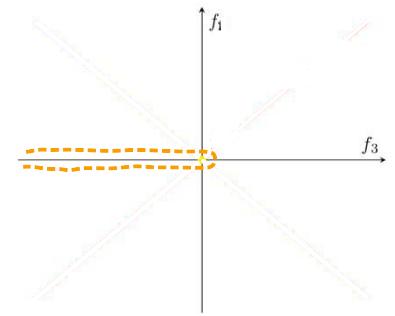
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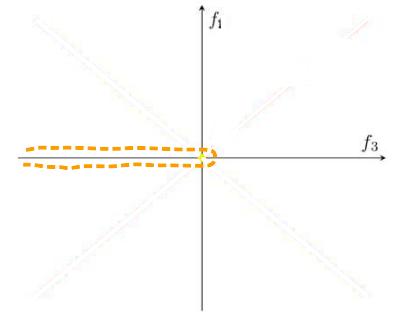
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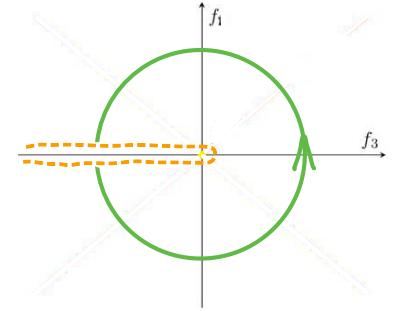
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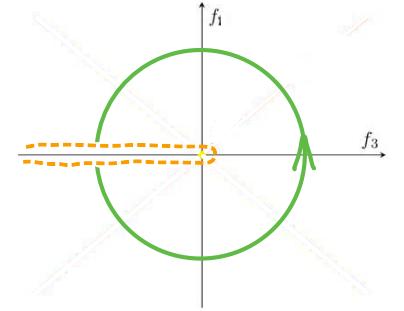
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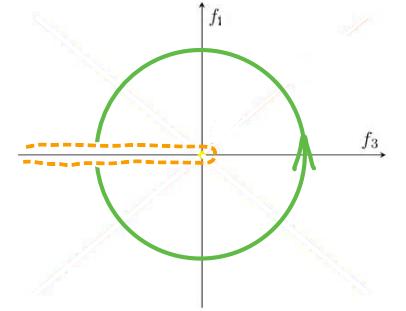
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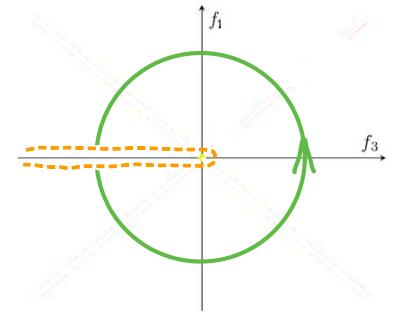


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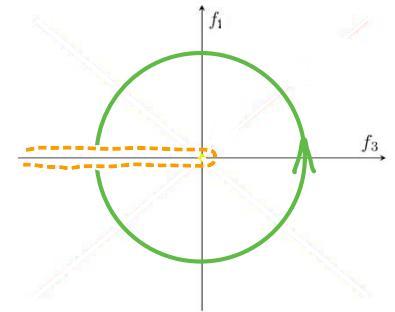
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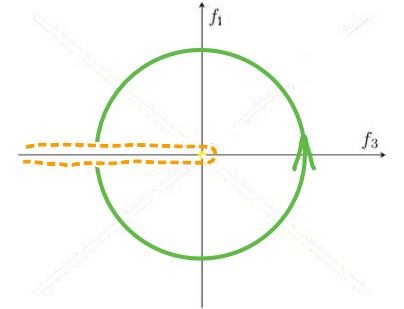
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## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{aligned} S^0 &\hookrightarrow S^1 \rightarrow S^1 & \mathbb{R} \\ S^1 &\hookrightarrow S^3 \rightarrow S^2 & \text{if the Hamiltonian is over } \mathbb{C} \\ S^3 &\hookrightarrow S^7 \rightarrow S^4 & \mathbb{H} \end{aligned}$$

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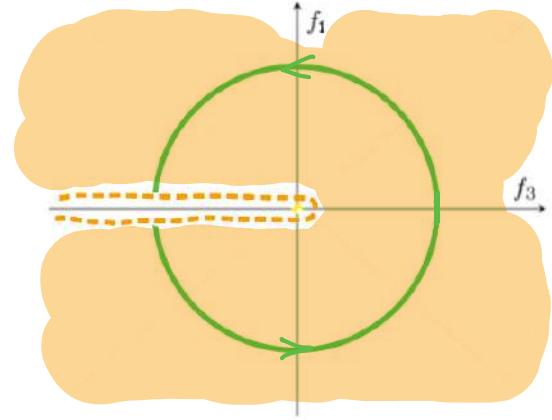
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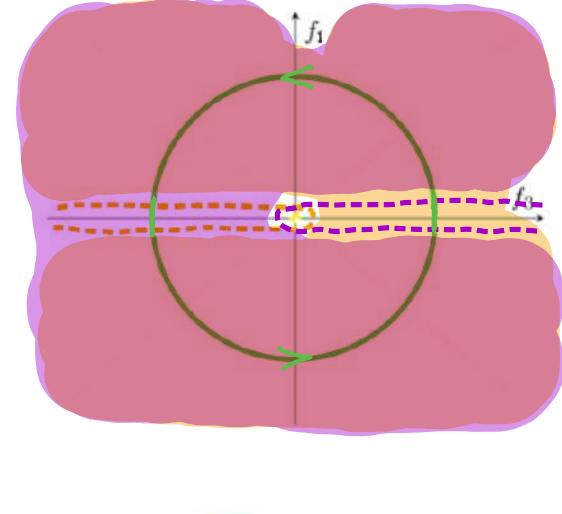
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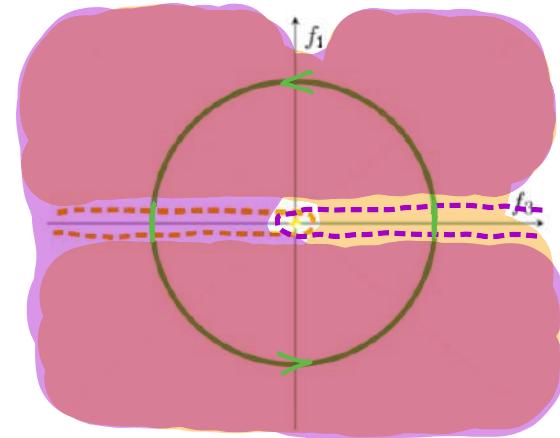
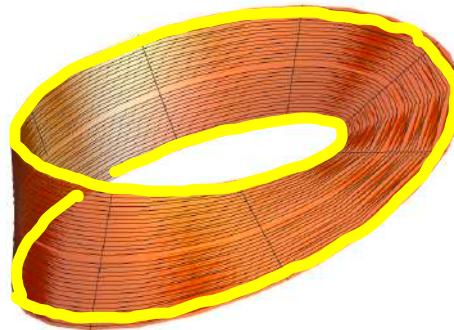
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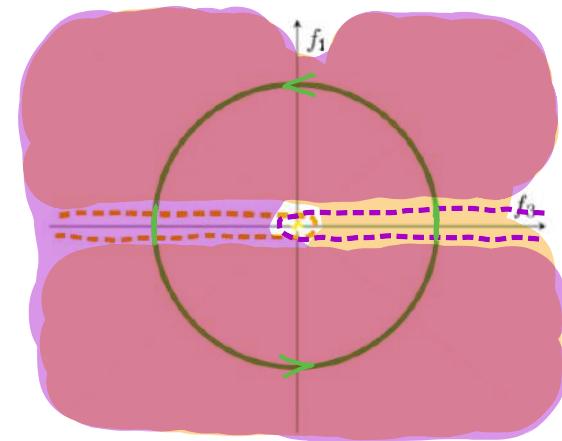
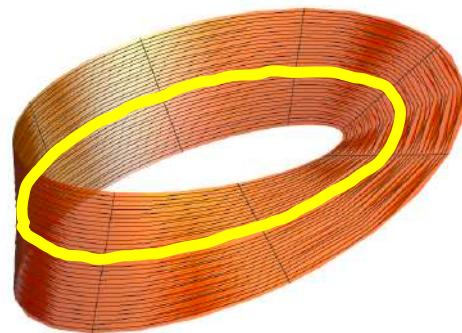
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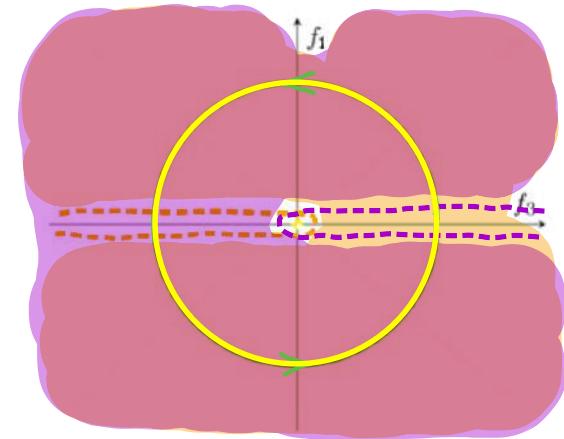
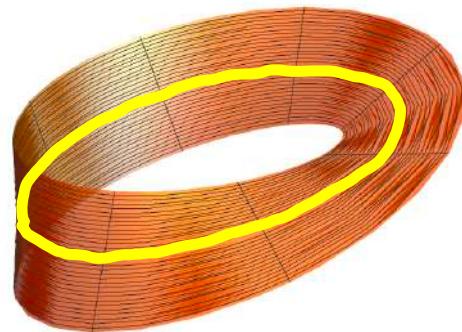
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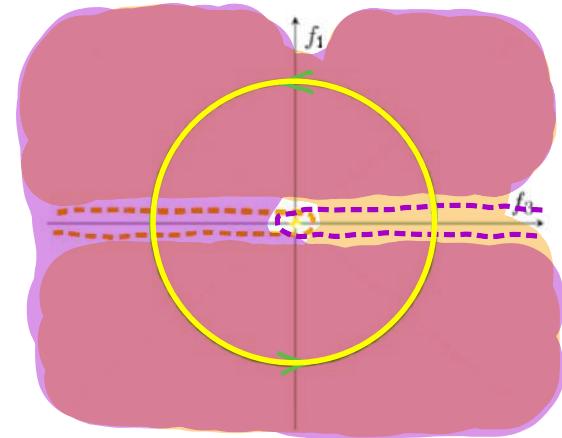
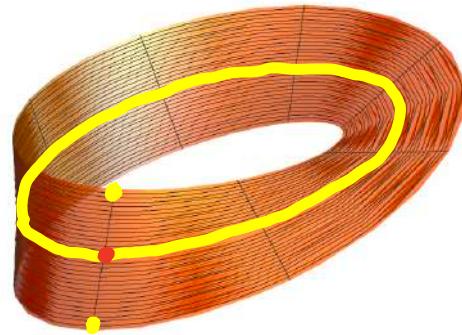
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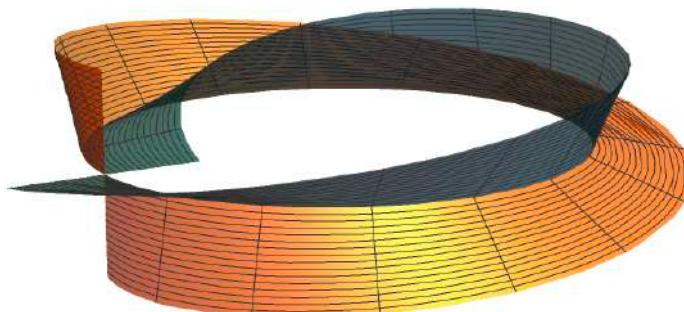
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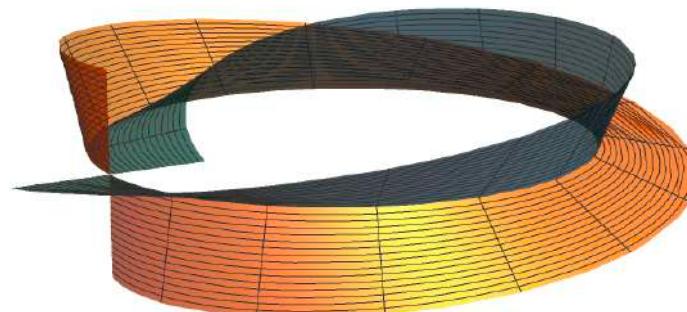


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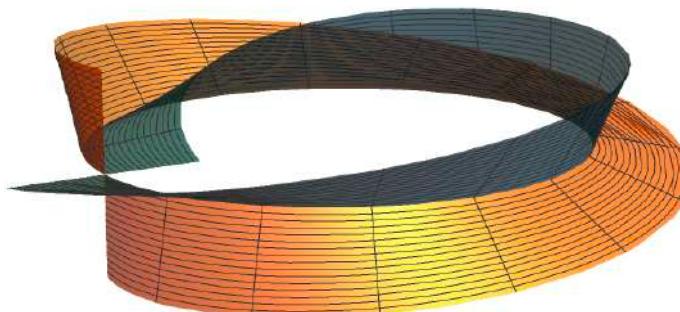


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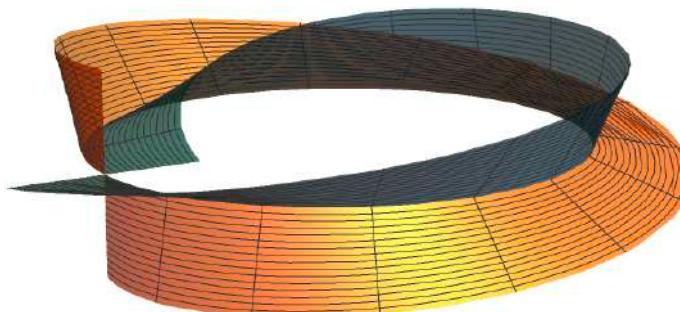


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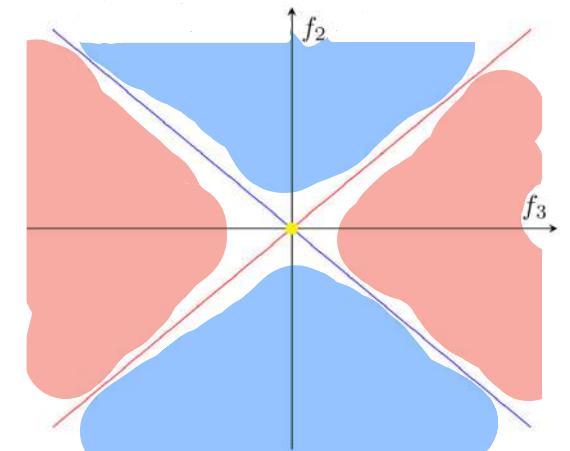
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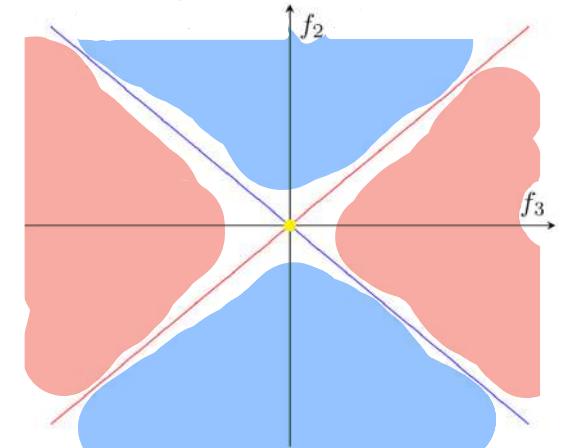


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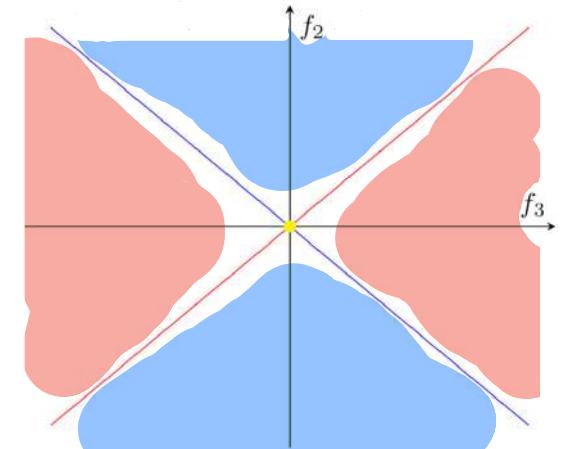


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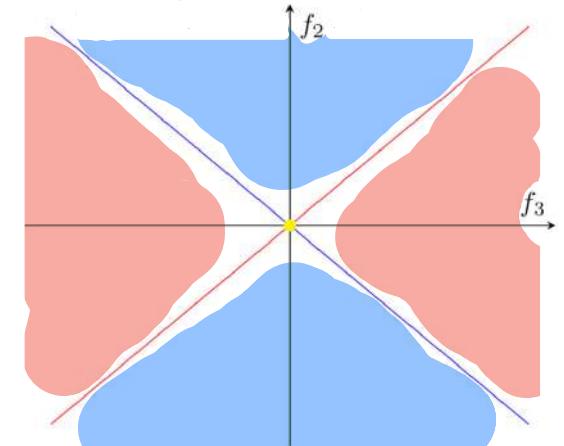


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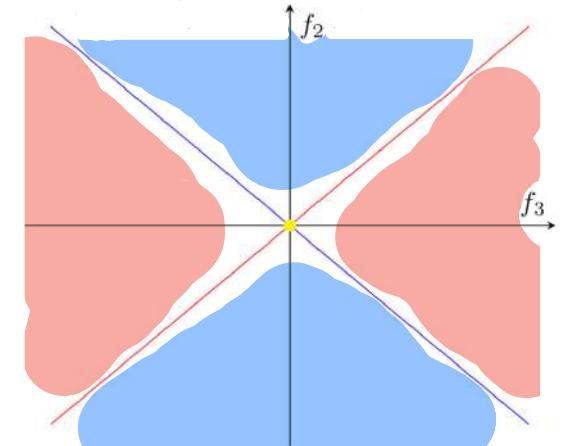


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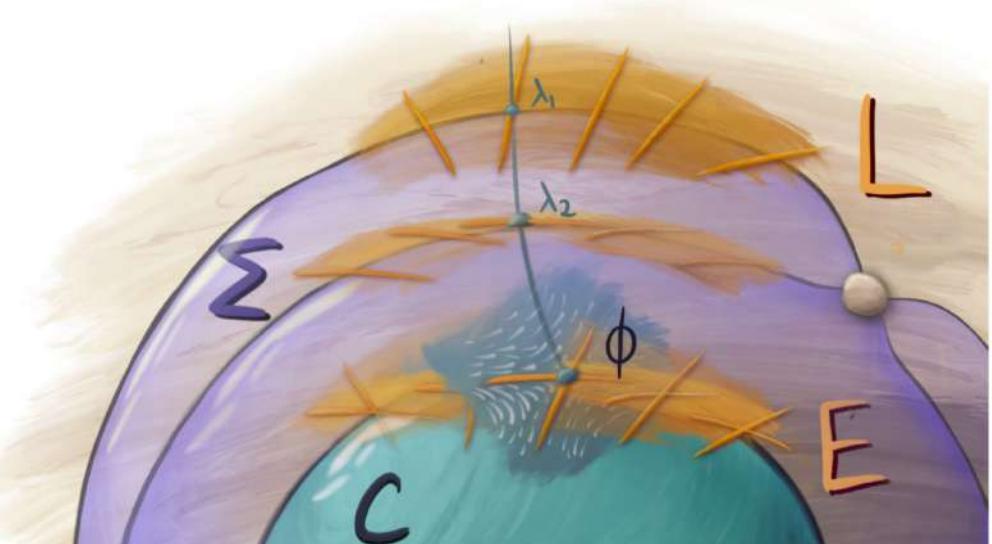
Nigel Hitchin 1987

Carlos Simpson

$C$  compact Riemann surface (or more generally Kähler manifold)

$E$  holomorphic vector bundle

$\phi$  Higgs field: a holomorphic 1-form taking values in the  
bundle of endomorphisms of  $E$  such that  $\phi \wedge \phi = 0$

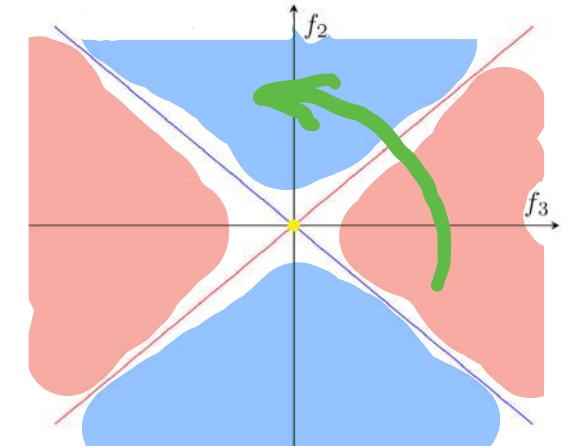


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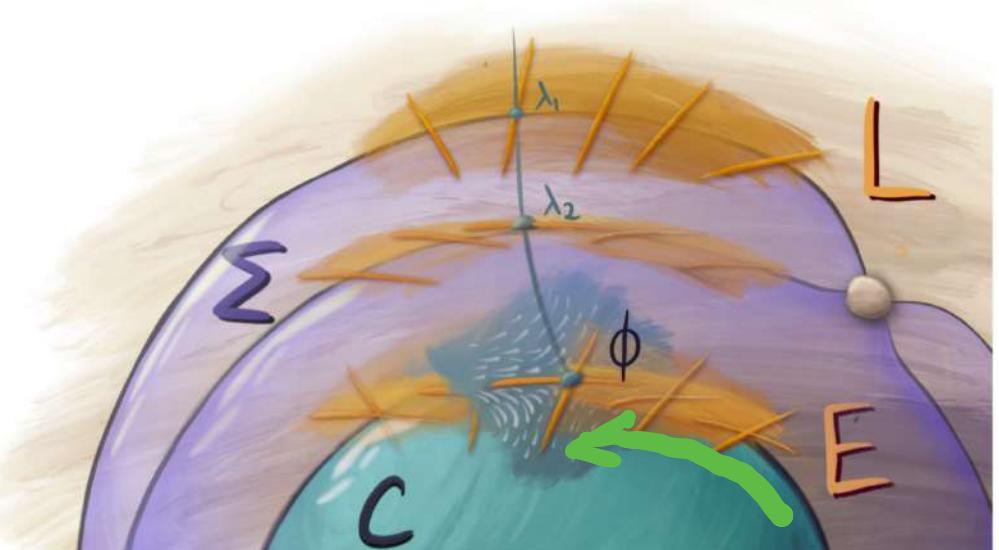
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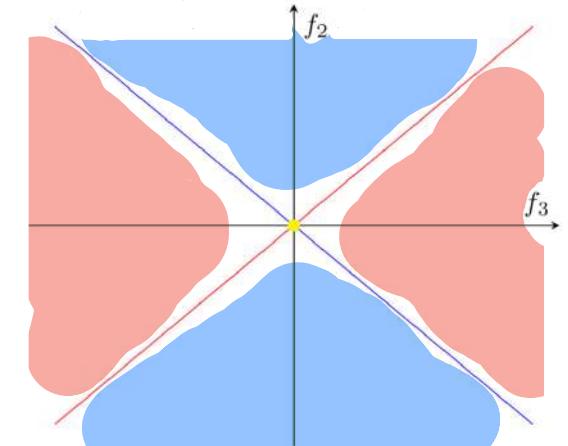


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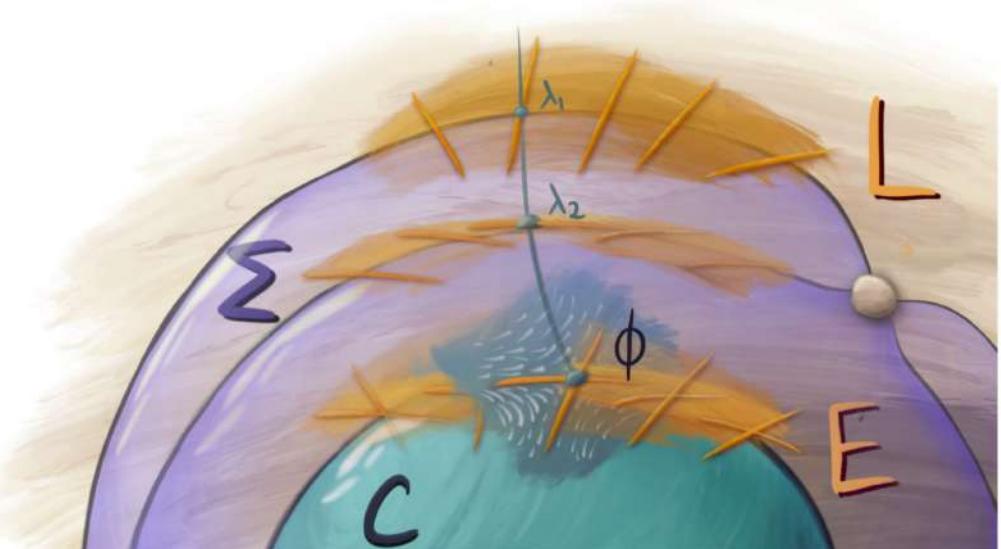
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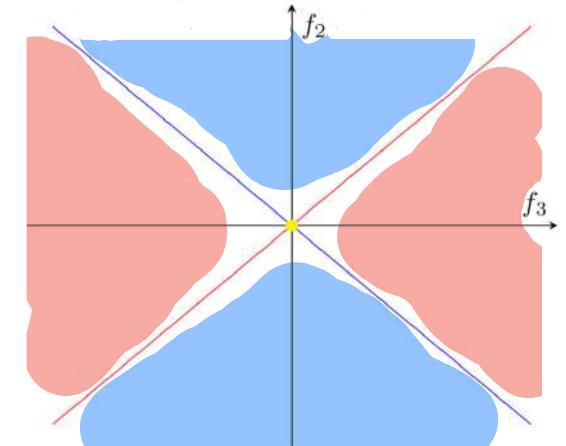


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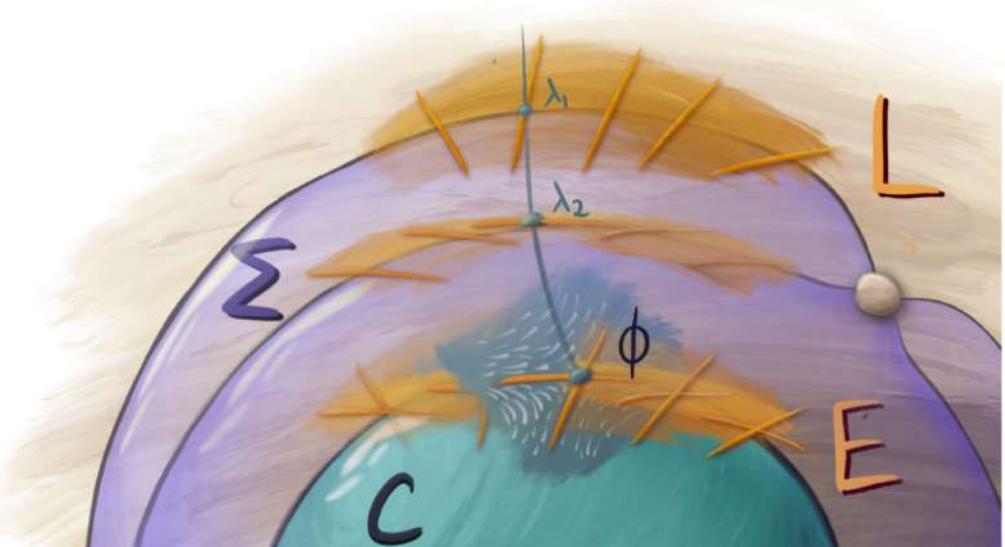
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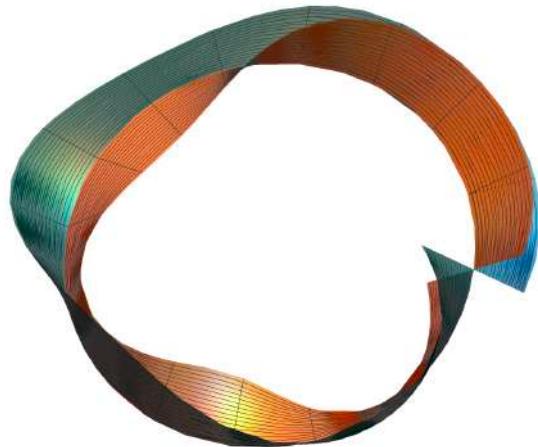
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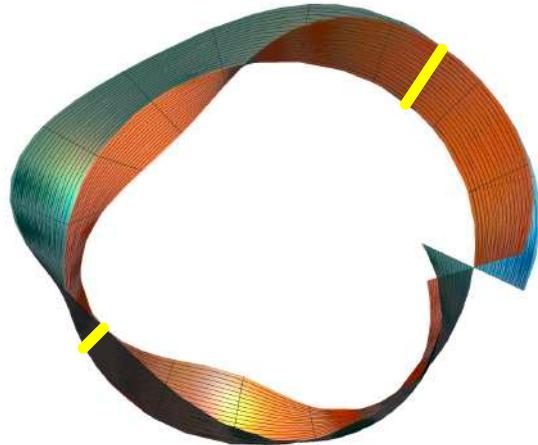
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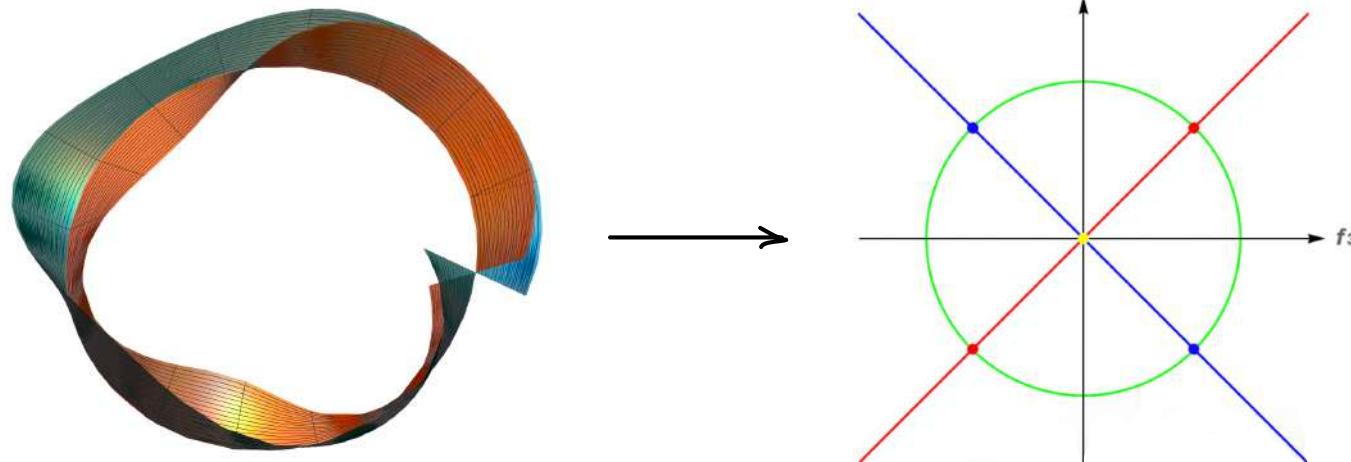
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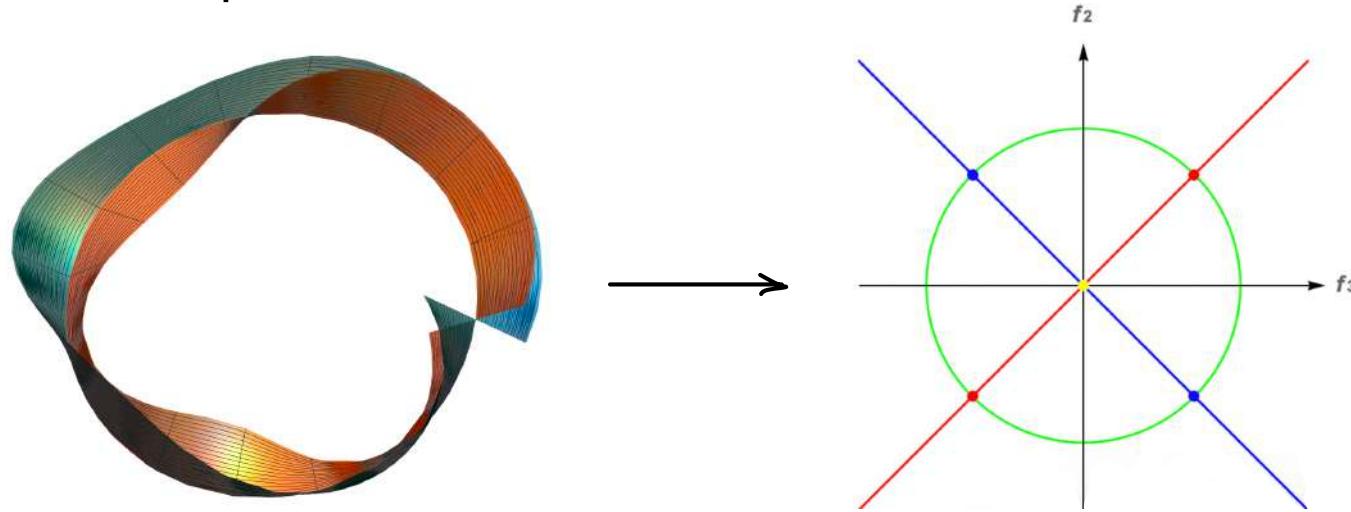
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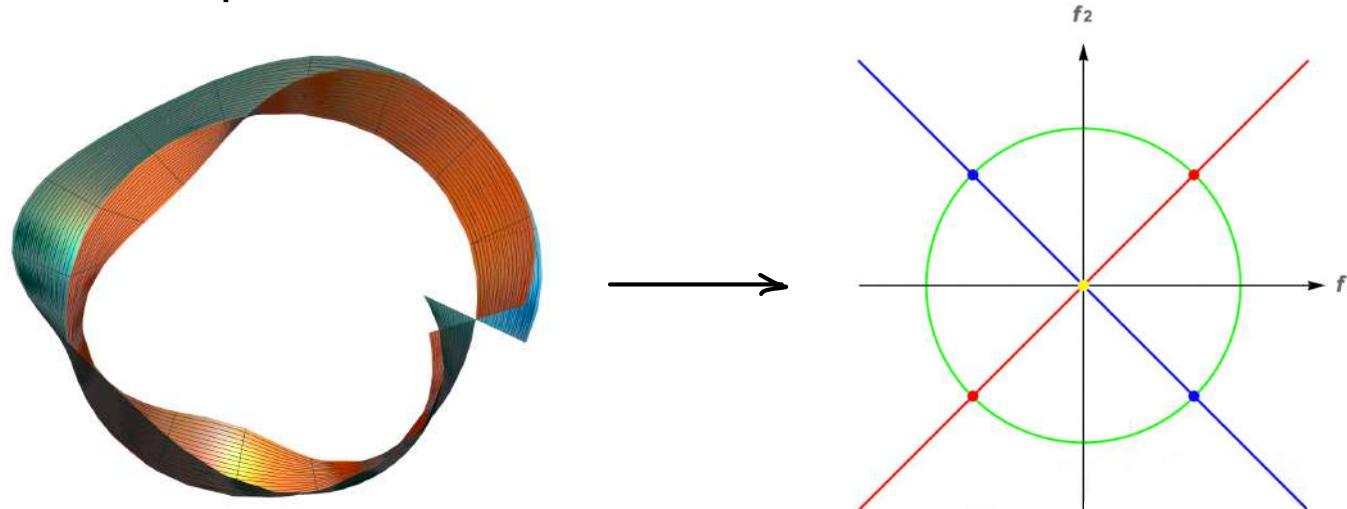
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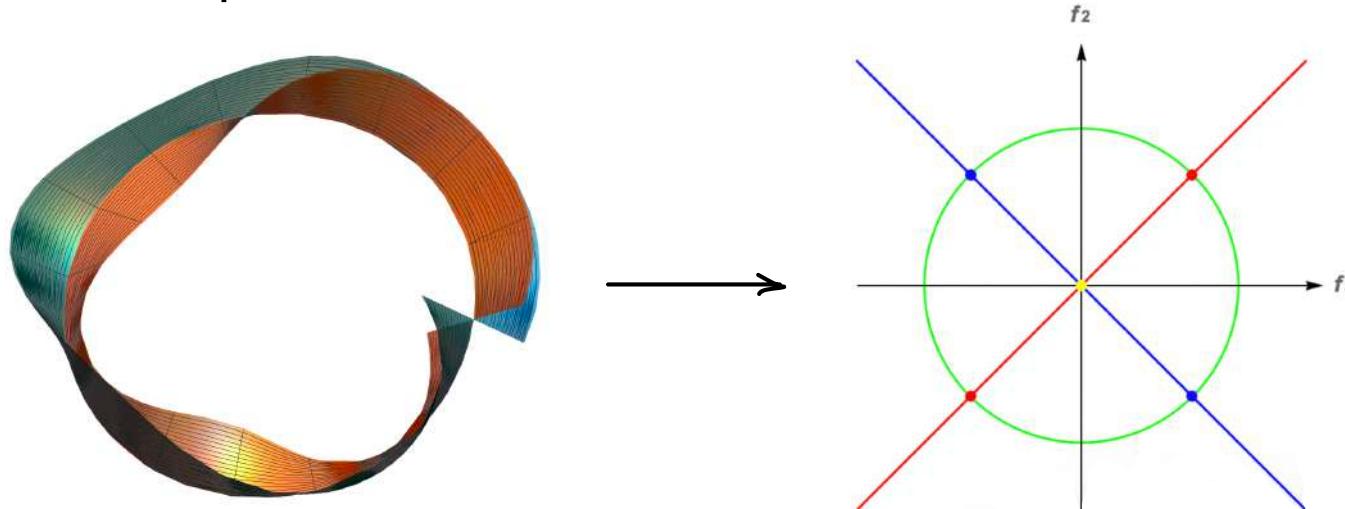
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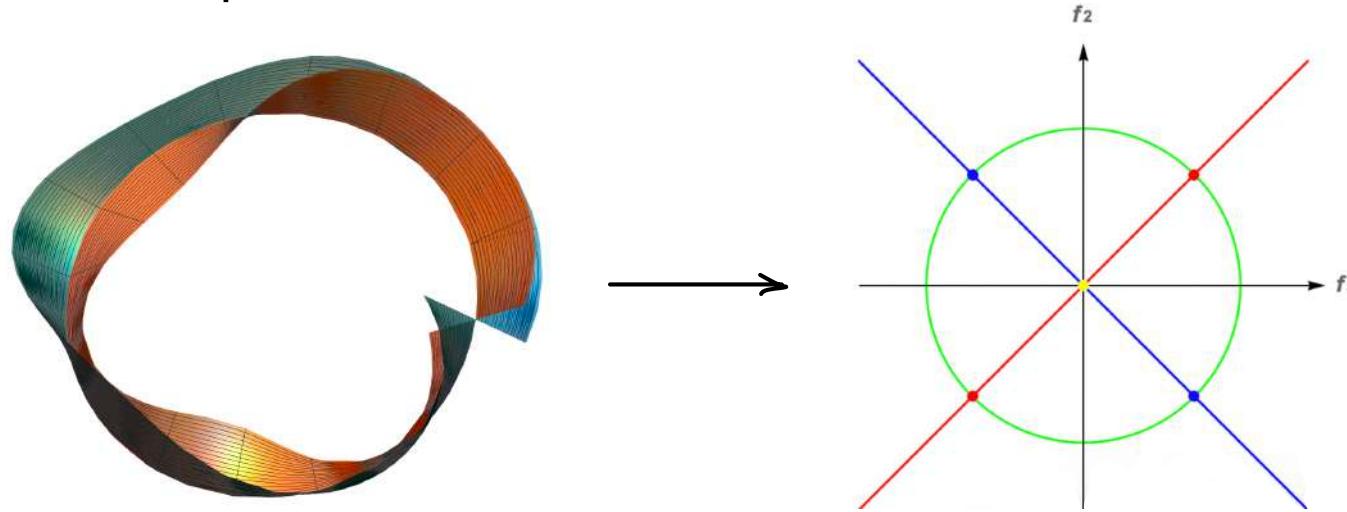


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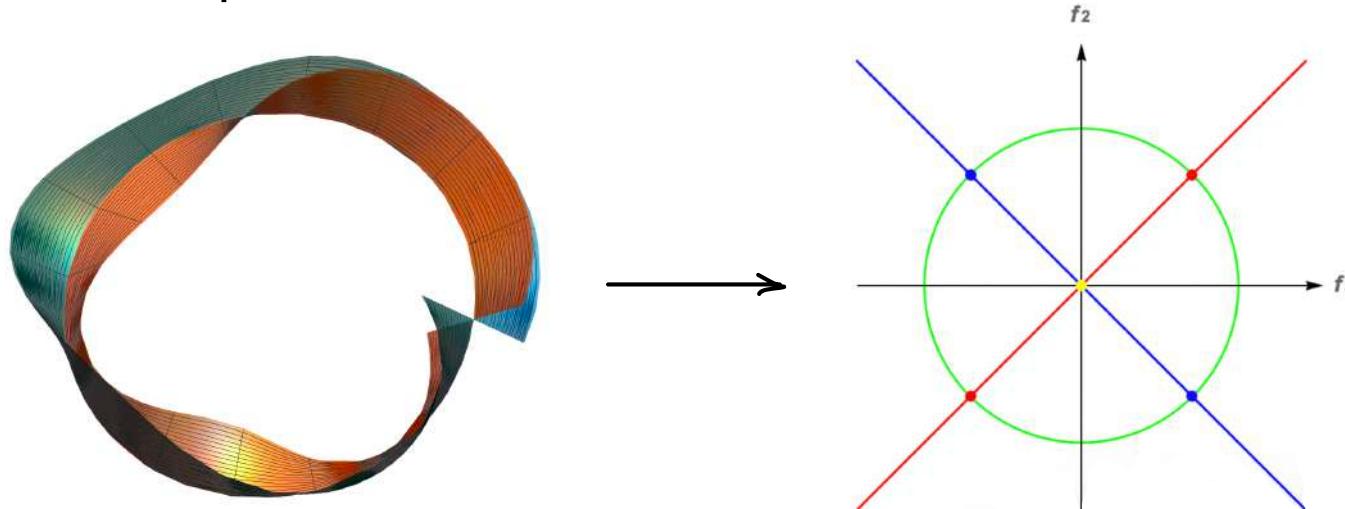


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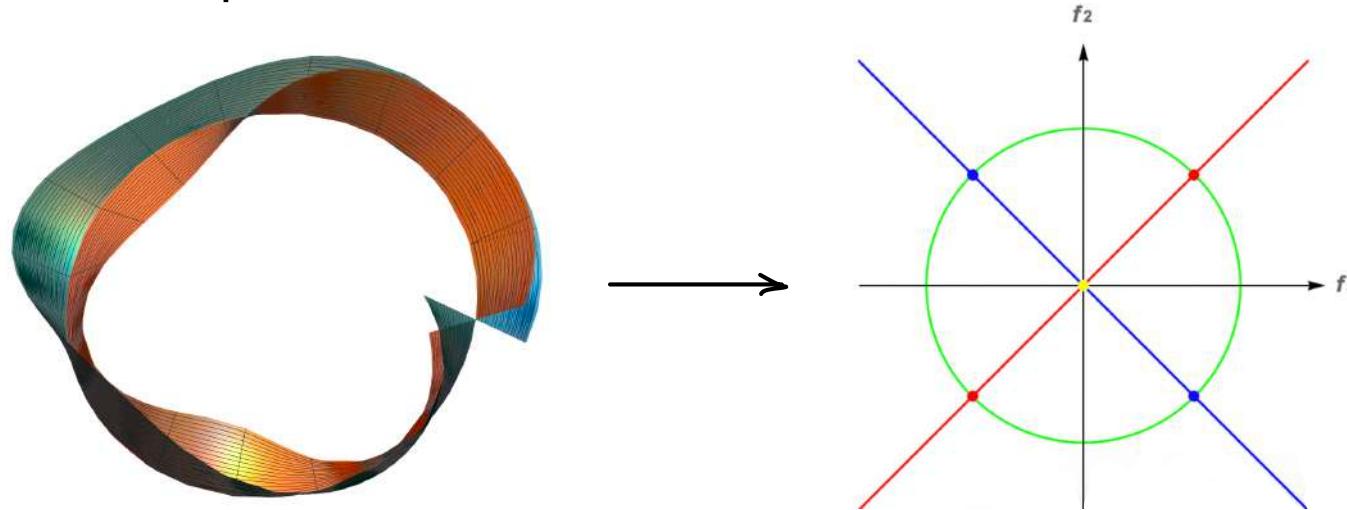
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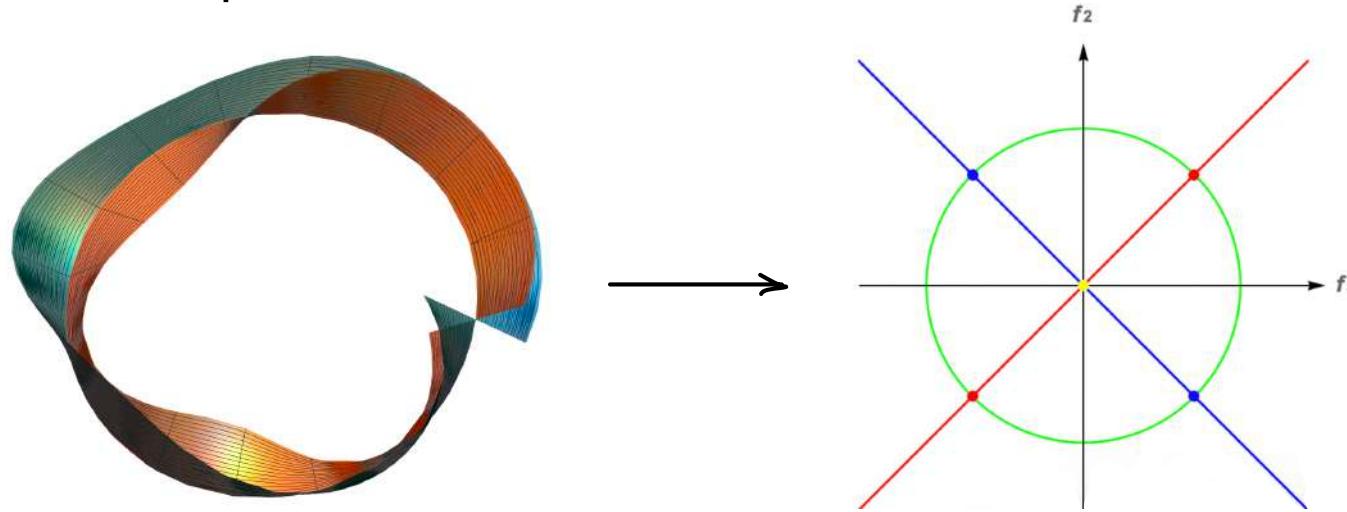
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*Gajer, The intersection Dold–Thom theorem,  
Topology, 1996. (Ph.D. student of Blaine Lawson, 1993)*

*Goresky and MacPherson, 1974.*

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Here is a video of the eigenbundle deformation: <https://yifeizhu.github.io/swallowtail/deform.mp4>

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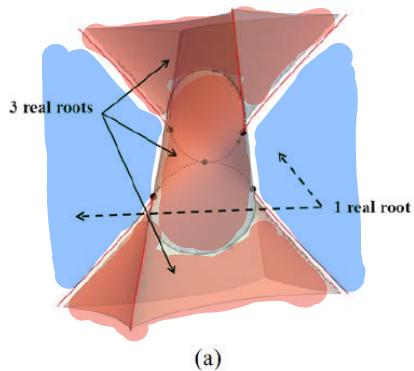
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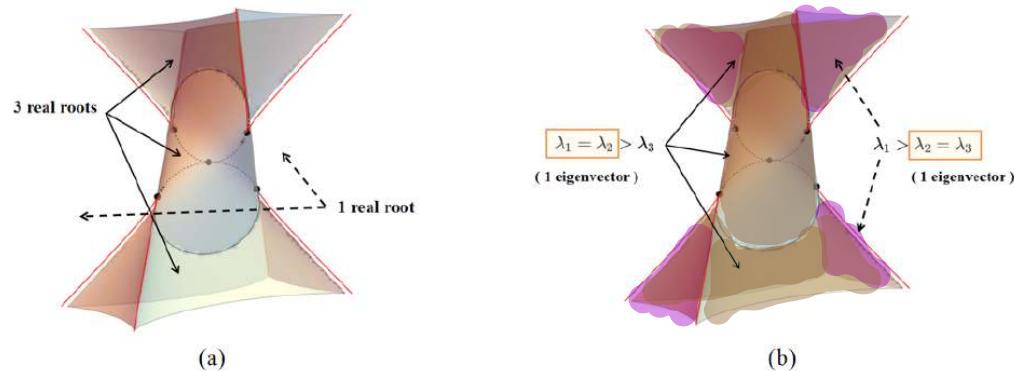
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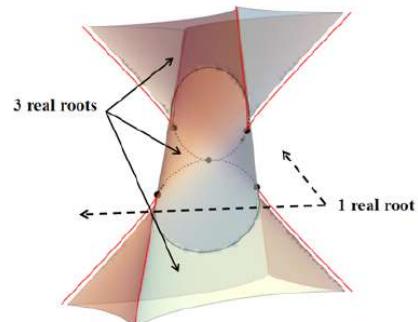
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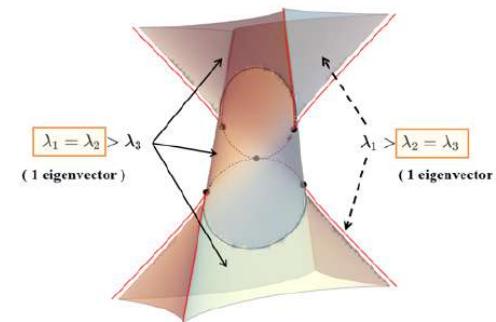
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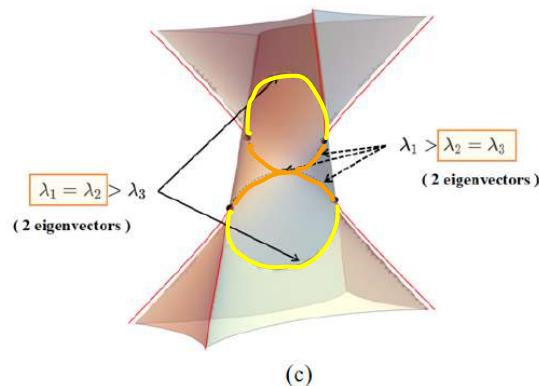
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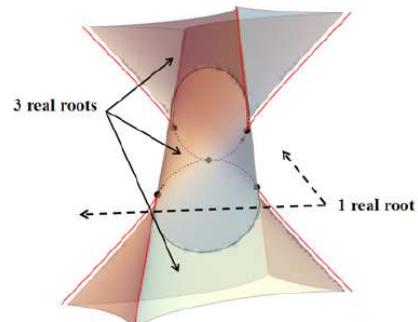
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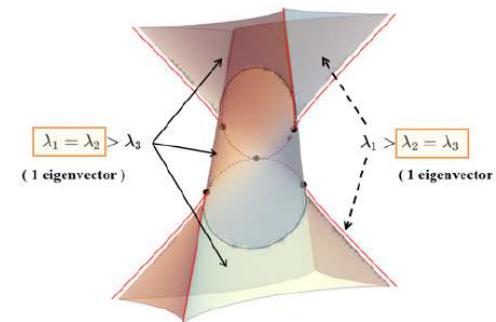
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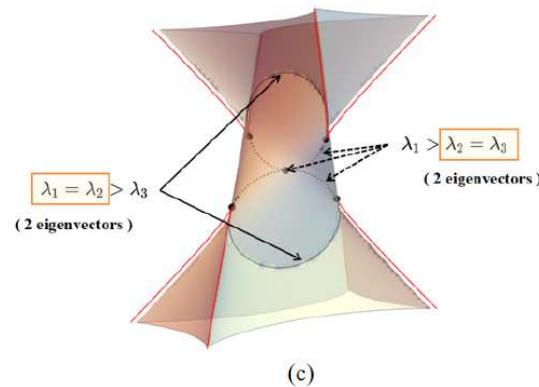
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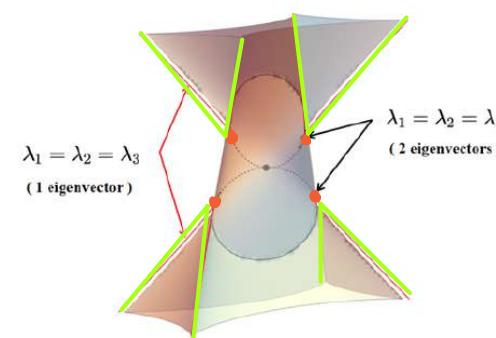
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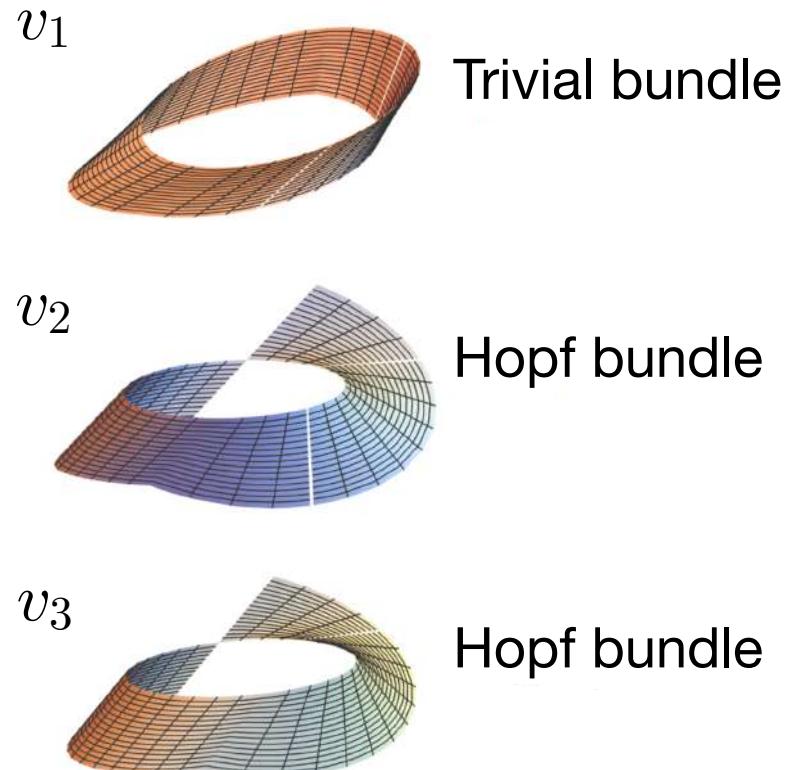
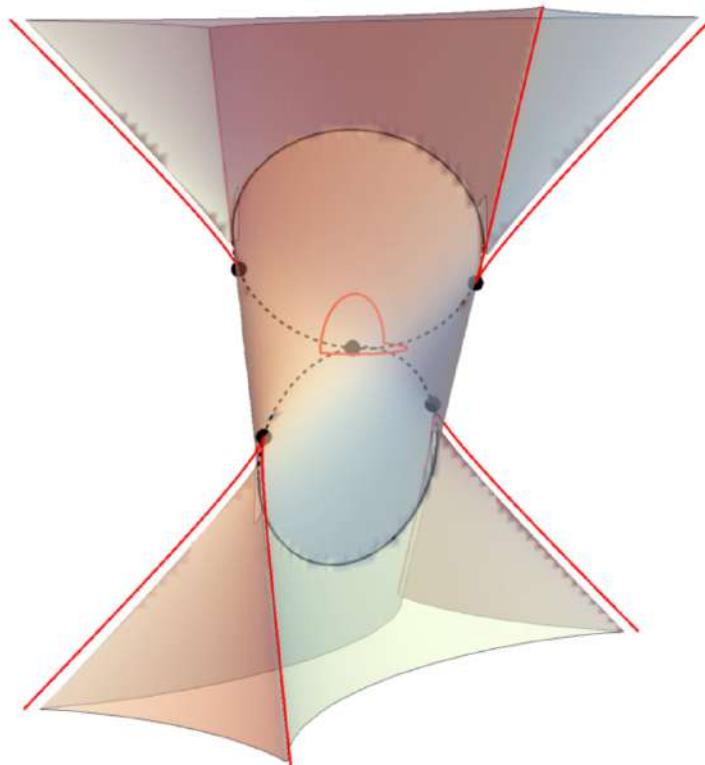
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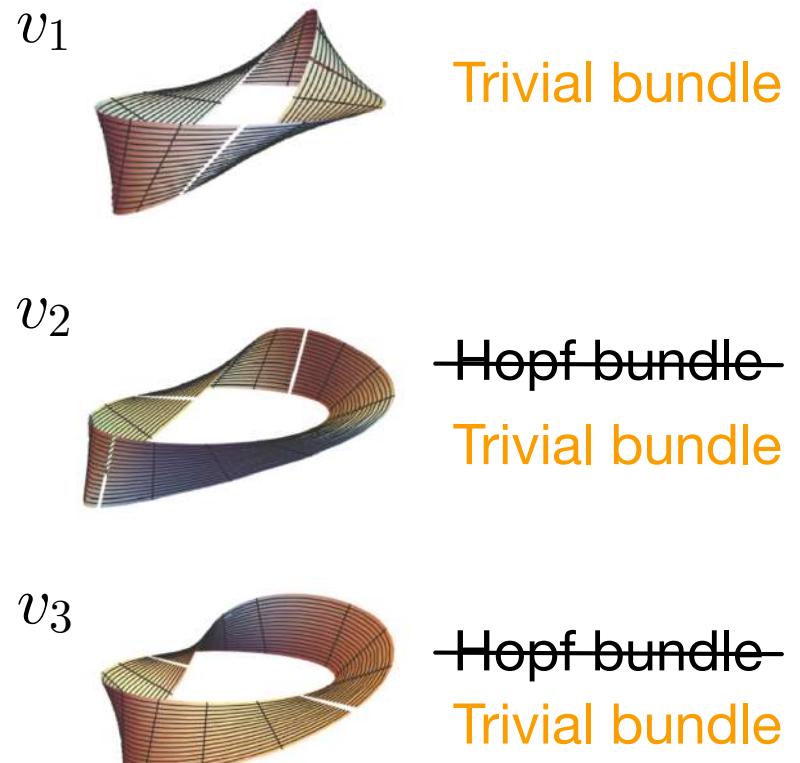
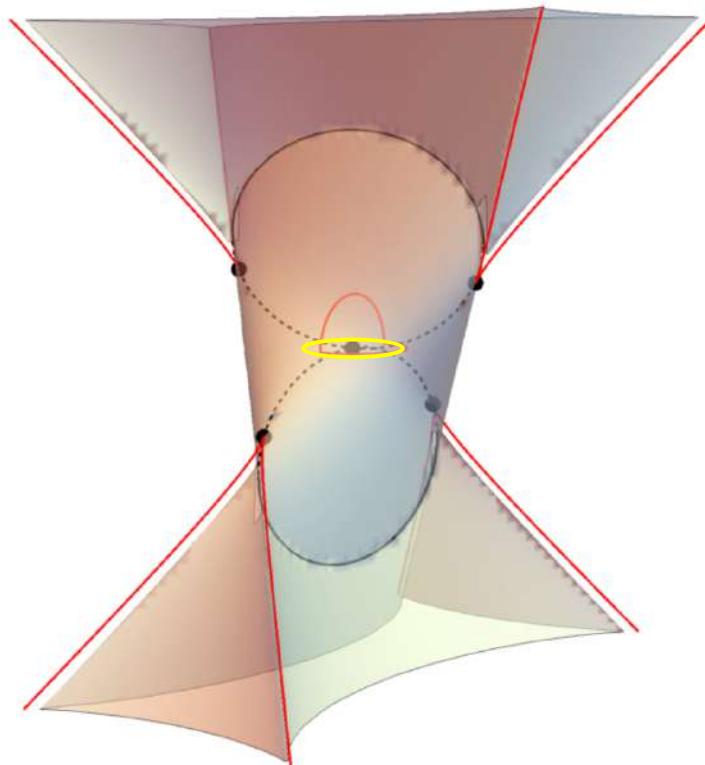


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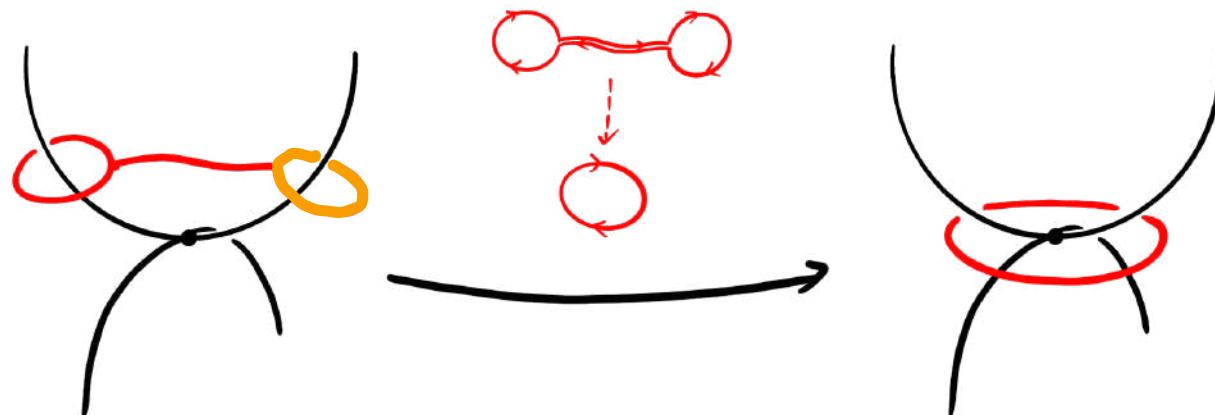


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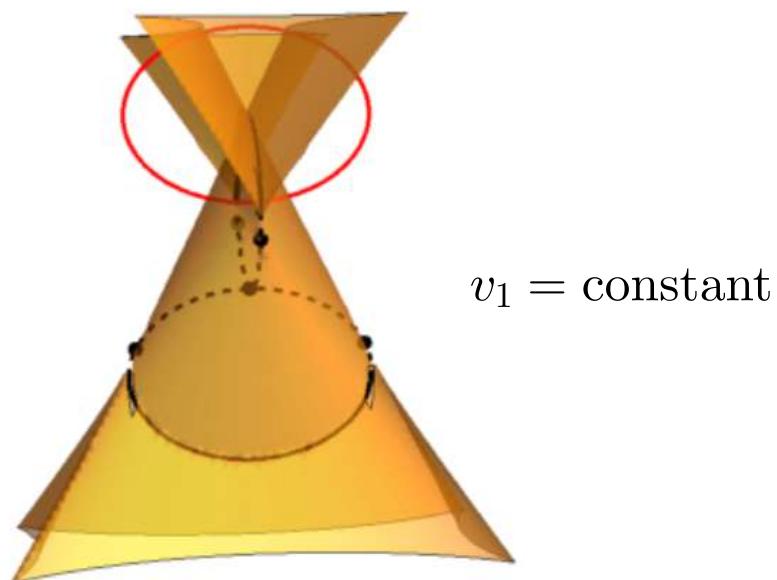


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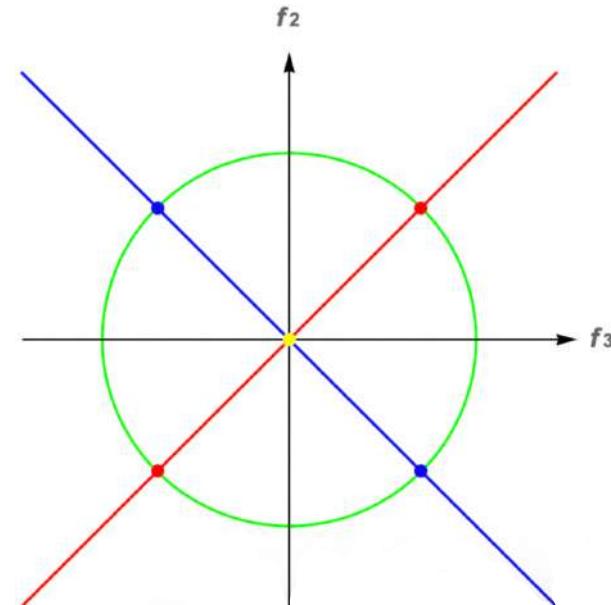
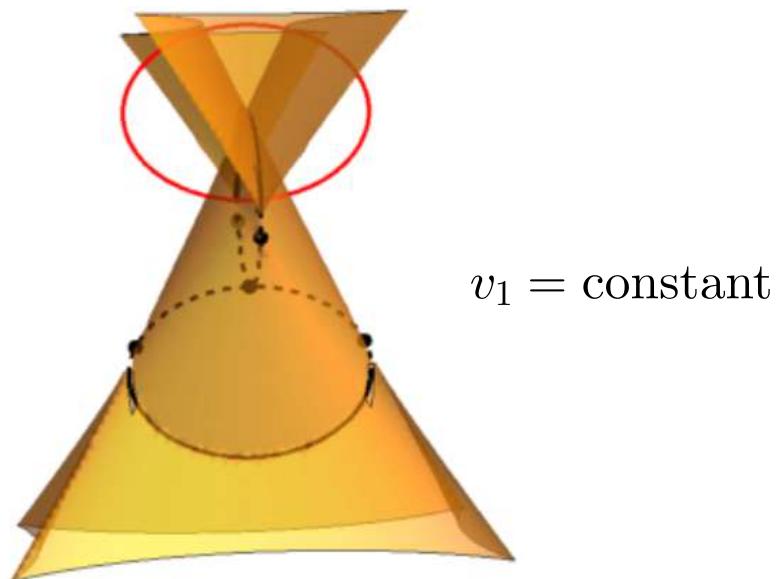


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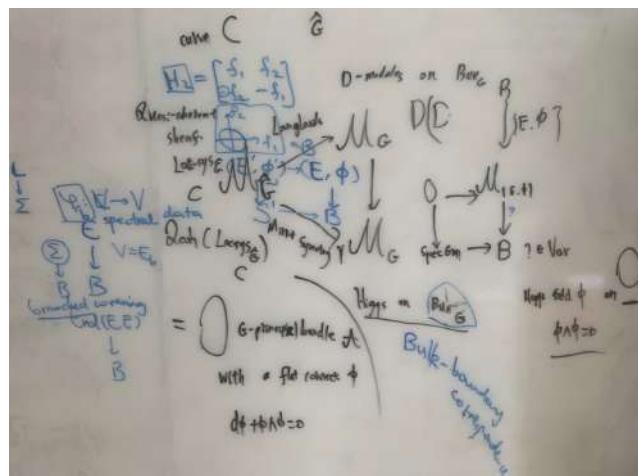
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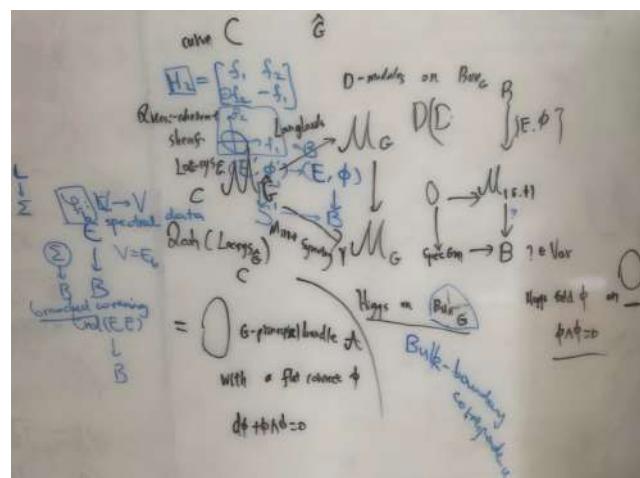


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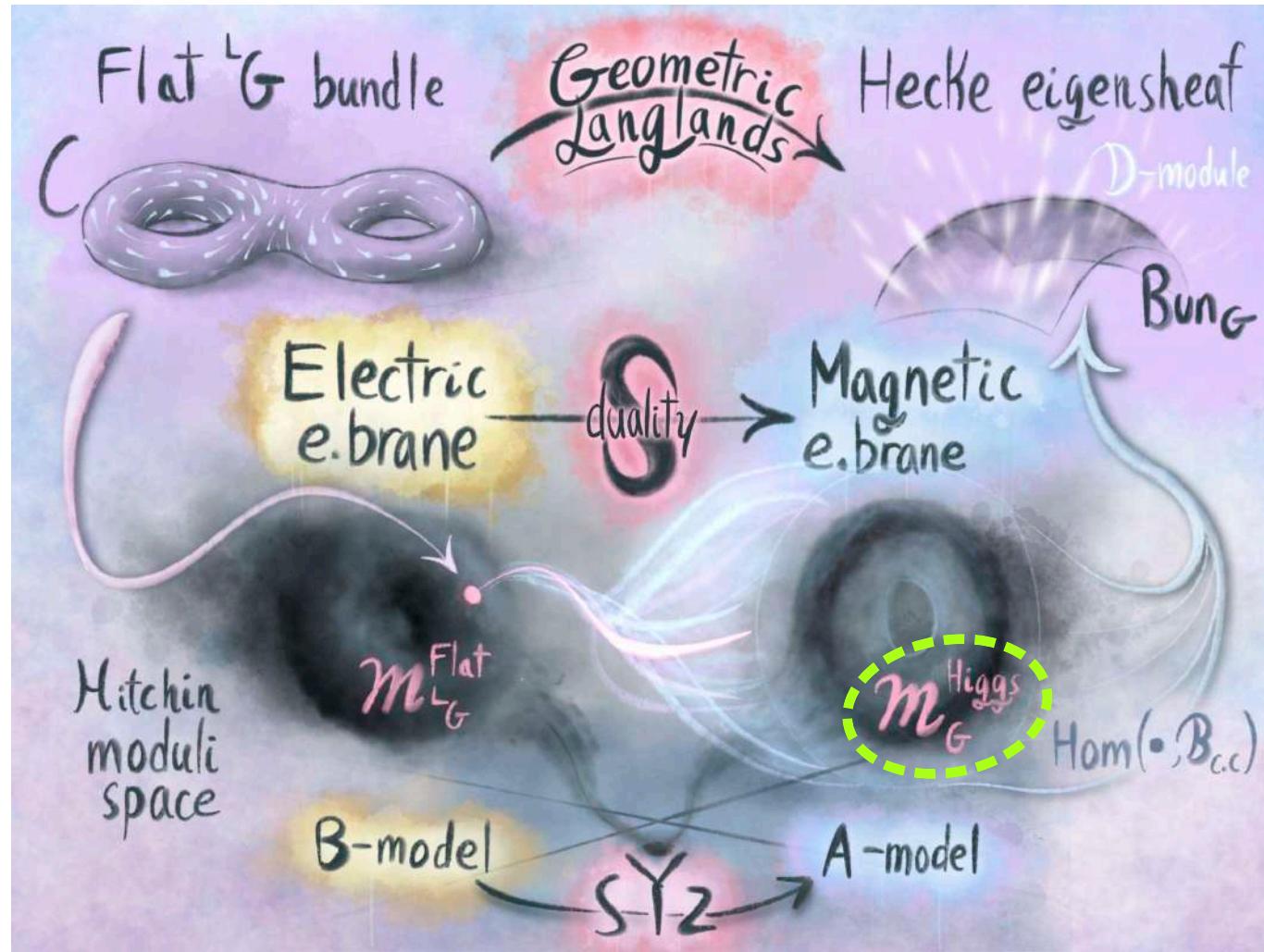
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Indeed, **Higgs bundles** sit on one side of the geometric Langlands duality! We've at least found some **testing ground**.



*Thank you.*



Portrait by Elliot Kienzle

## Illustration credits

- p. 14, Weidong Luo
- p. 21, Zhou Fang
- p. 26, Zhou Fang
- pp. 58–59, Boris Khesin and Sergei Tabachnikov. Vladimir Igorevich Arnold, 12 June 1937 — 3 June 2010. Biogr. Mem. Fell. R. Soc. 64, 7–26, 2018.
- p. 61, I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, Discriminants, resultants and multidimensional determinants. Birkhäuser, 1994.
- p. 114, Elliot Kienzle and Steven Rayan. Hyperbolic band theory through Higgs bundles. Adv. Math., 409:Paper No. 108664, 53, 2022.
- pp. 141–148, Chenlu Huang
- p. 153, Xuecai Ma
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