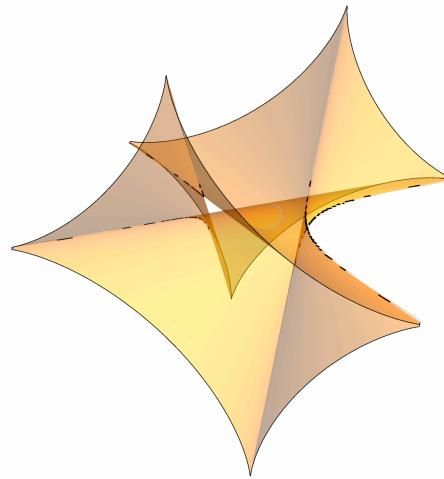


# **Explicit examples of Higgs bundles in the contexts of quantum materials and geometric Langlands correspondence**



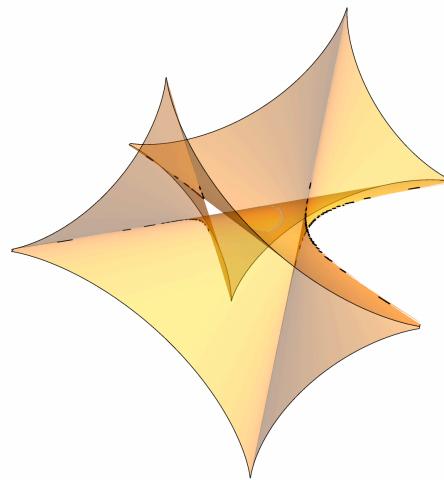
Yifei Zhu

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2025.1.11

Physics + mathematical modeling

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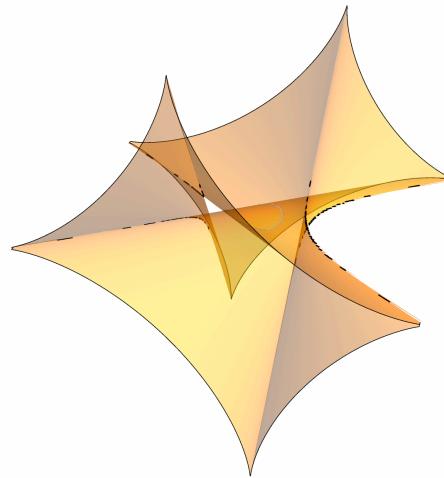
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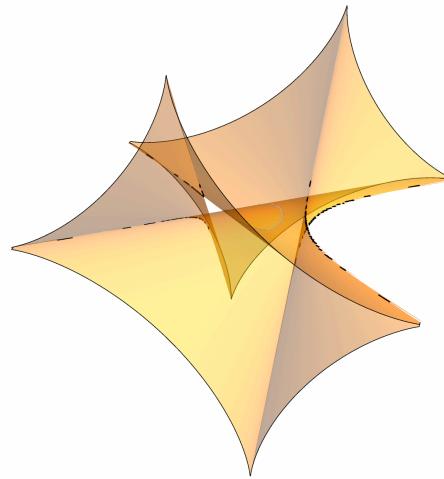
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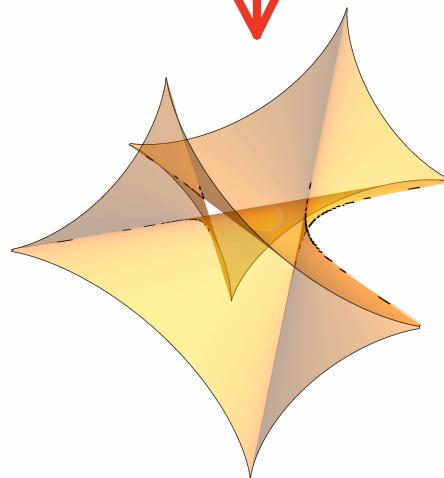
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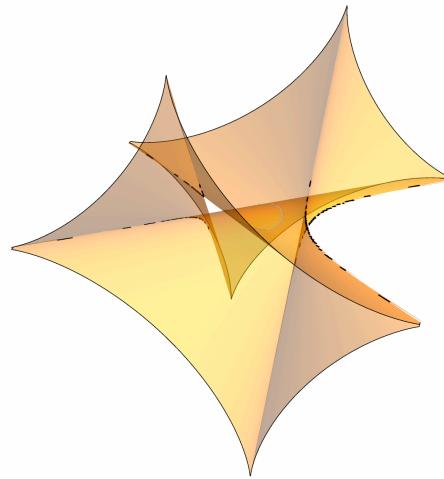
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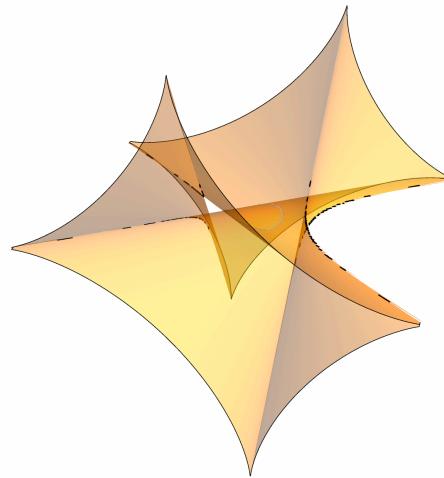
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*Holography, optical devices,  
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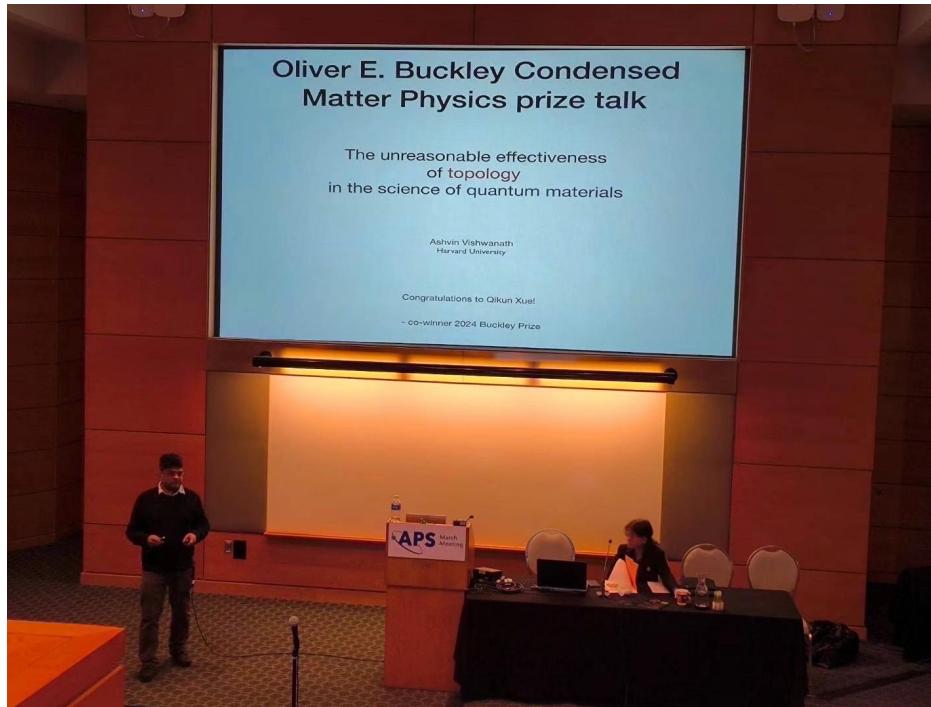
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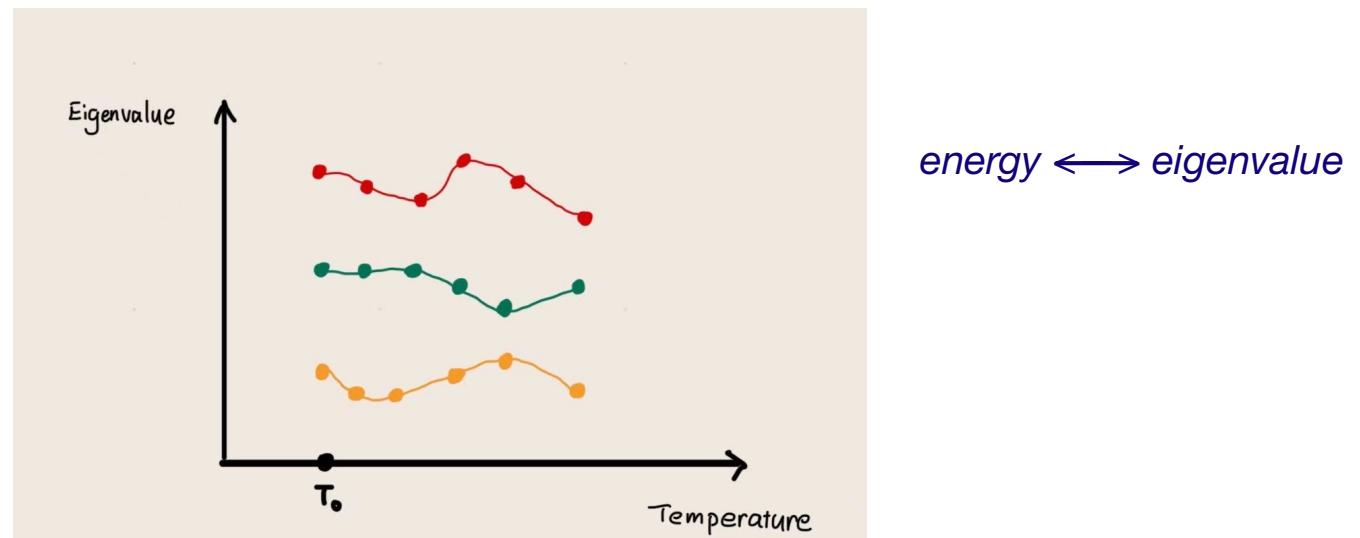
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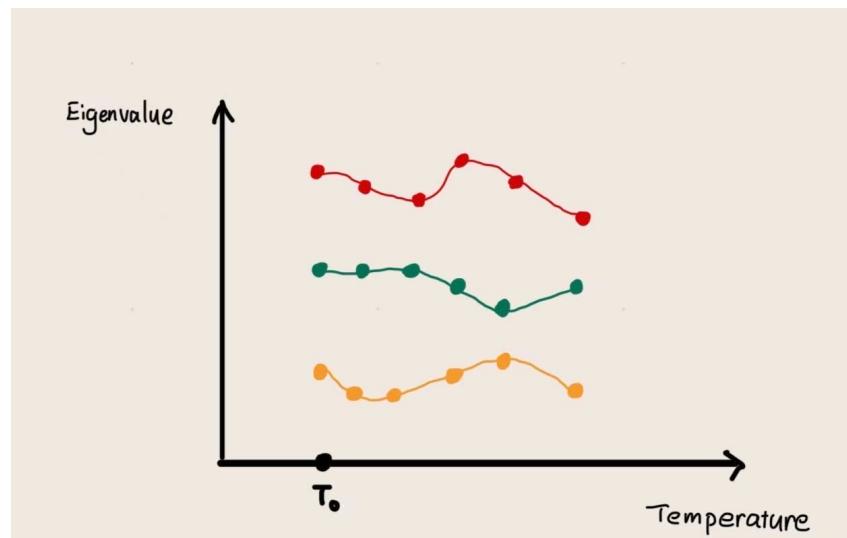
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Mathematical modeling of electronic energy *band structures* therein



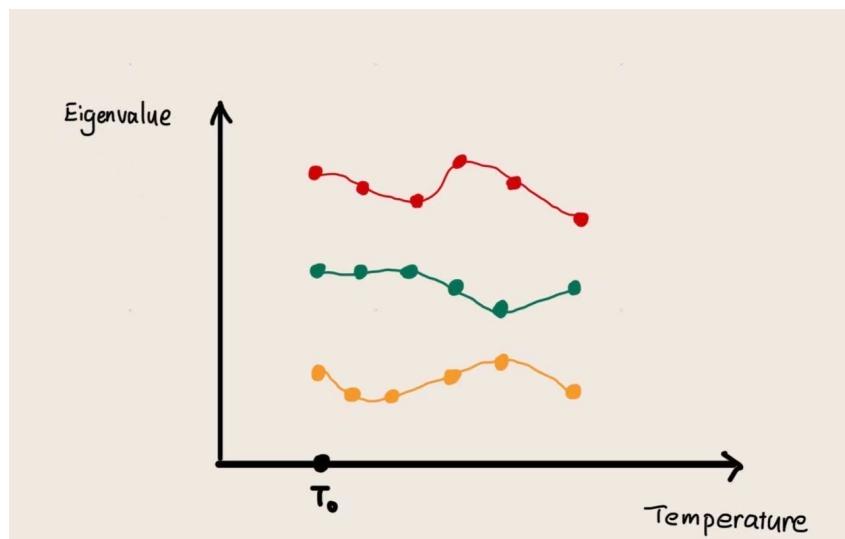
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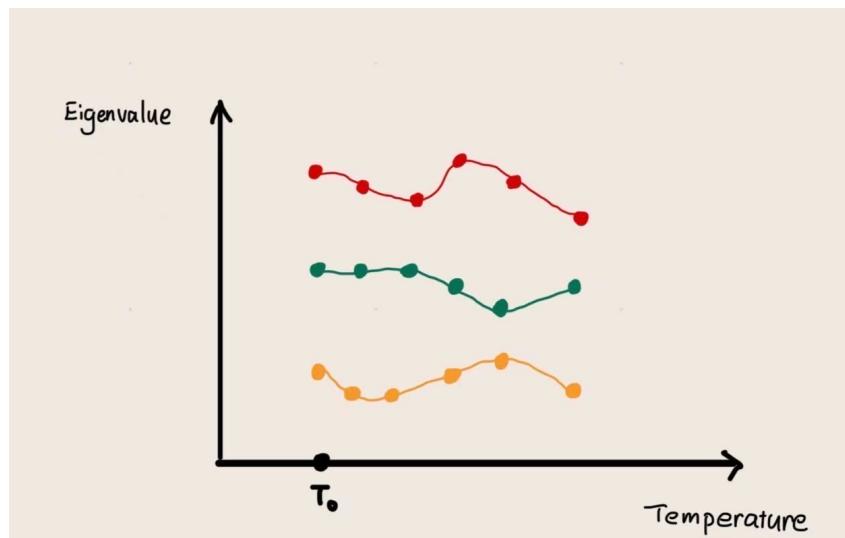
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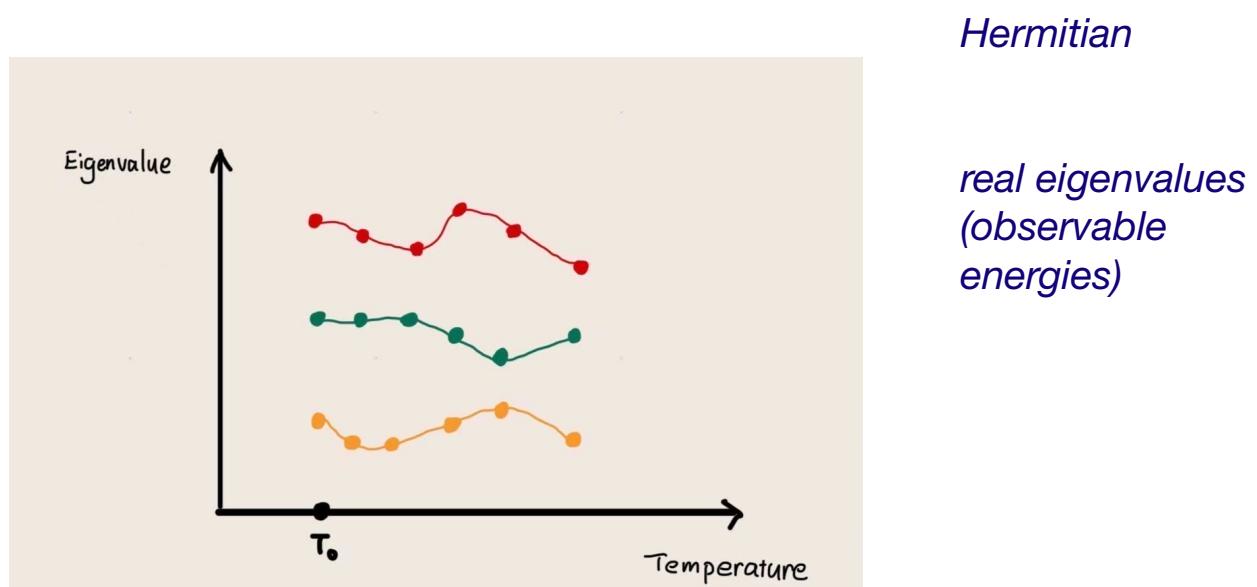
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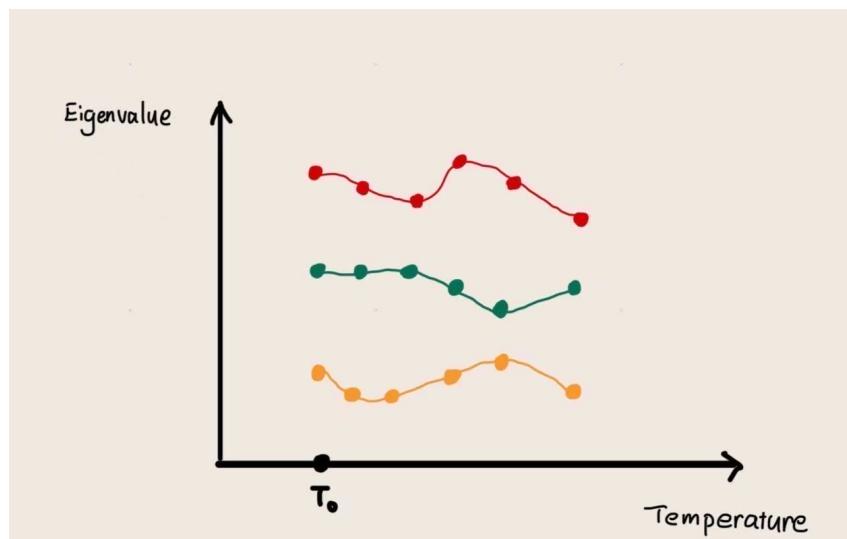
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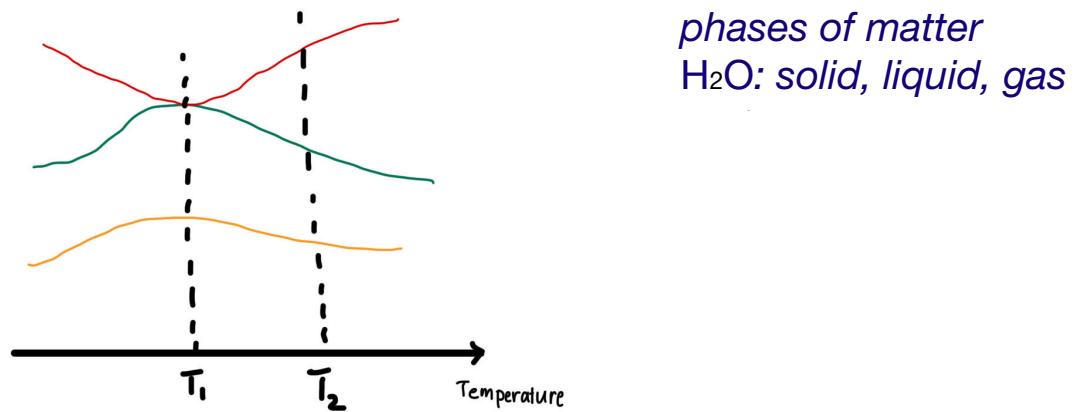


*Hermitian vs.  
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*real eigenvalues  
(observable  
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imaginary part  
(counts for  
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environment or  
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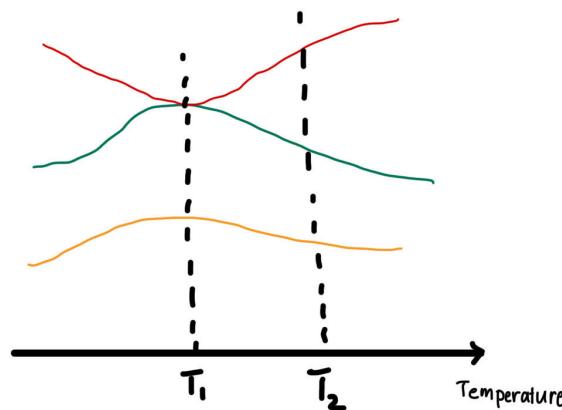
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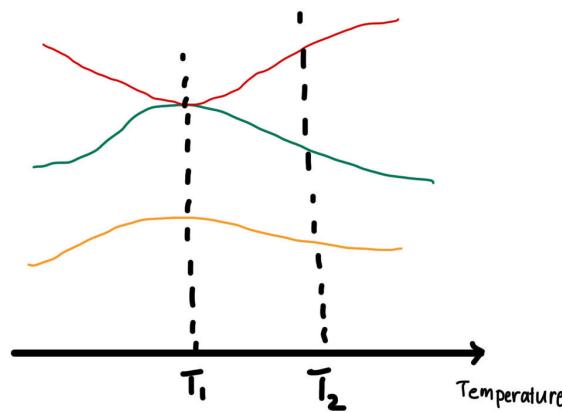
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*It is indeed “pointless” and is better understood as a **functor**!*

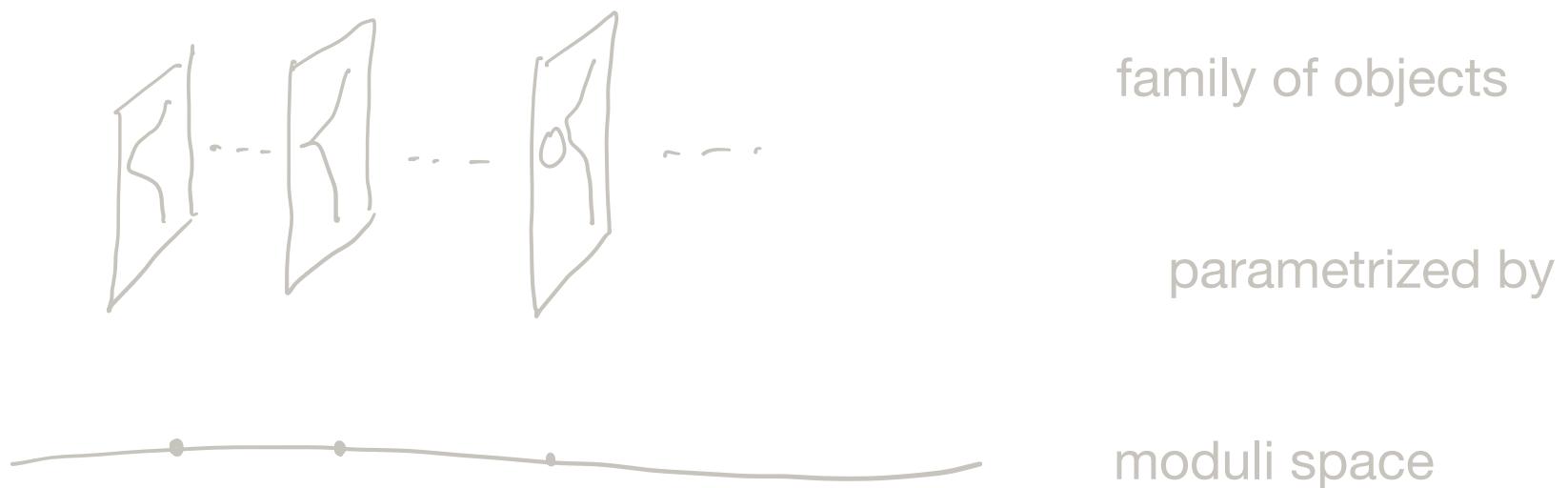
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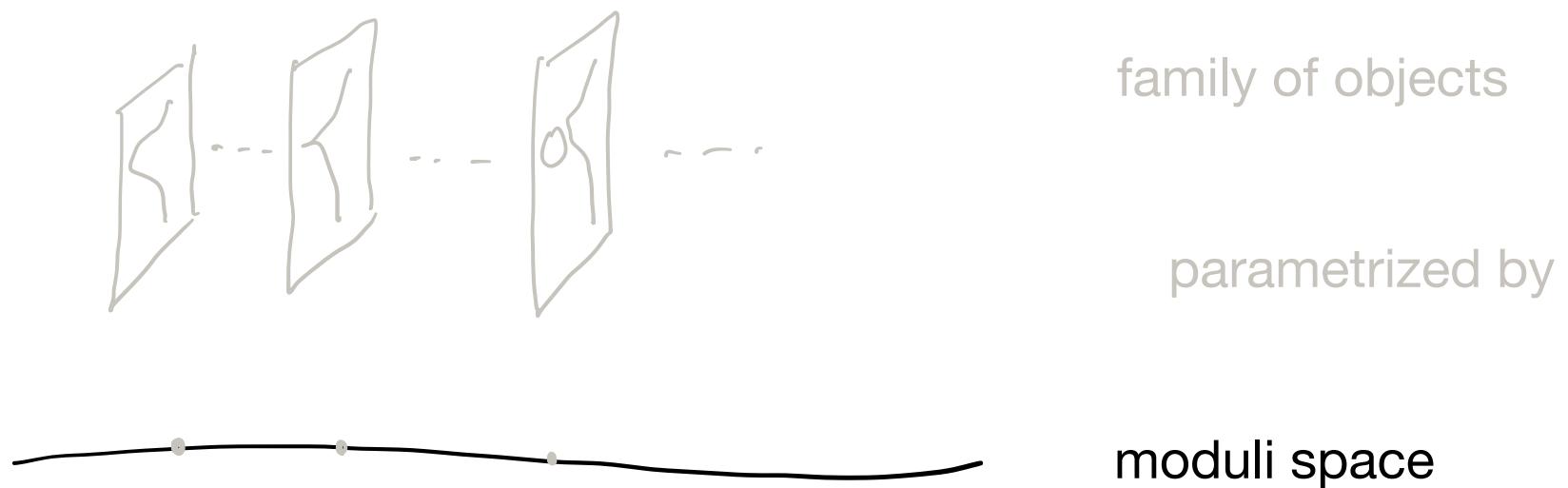
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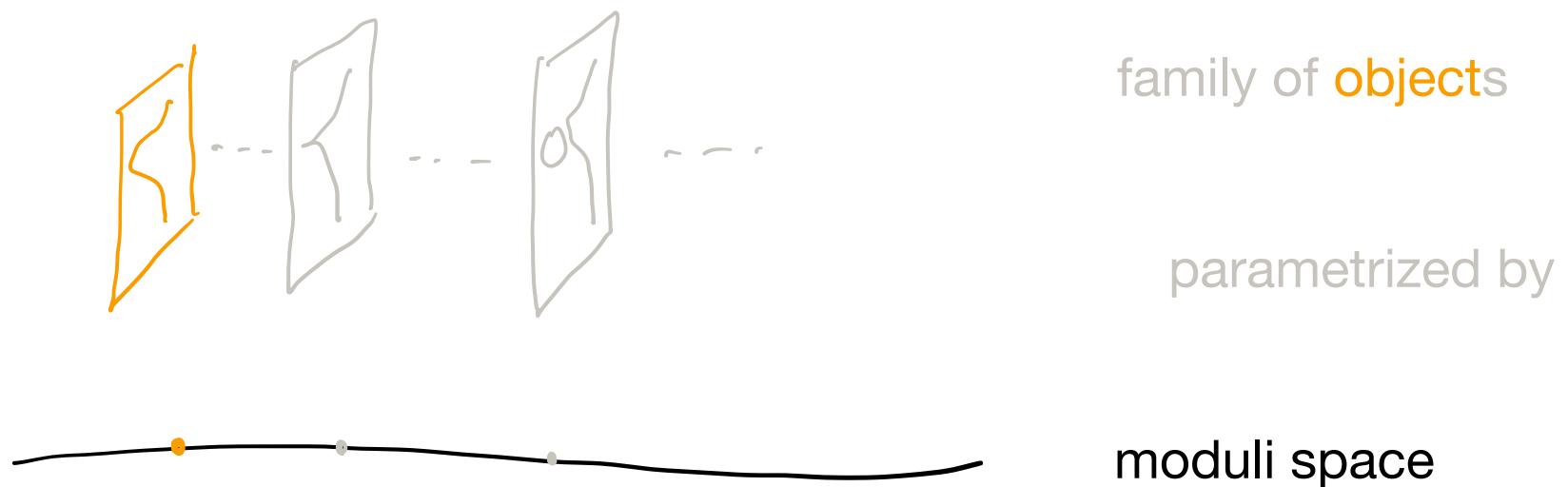
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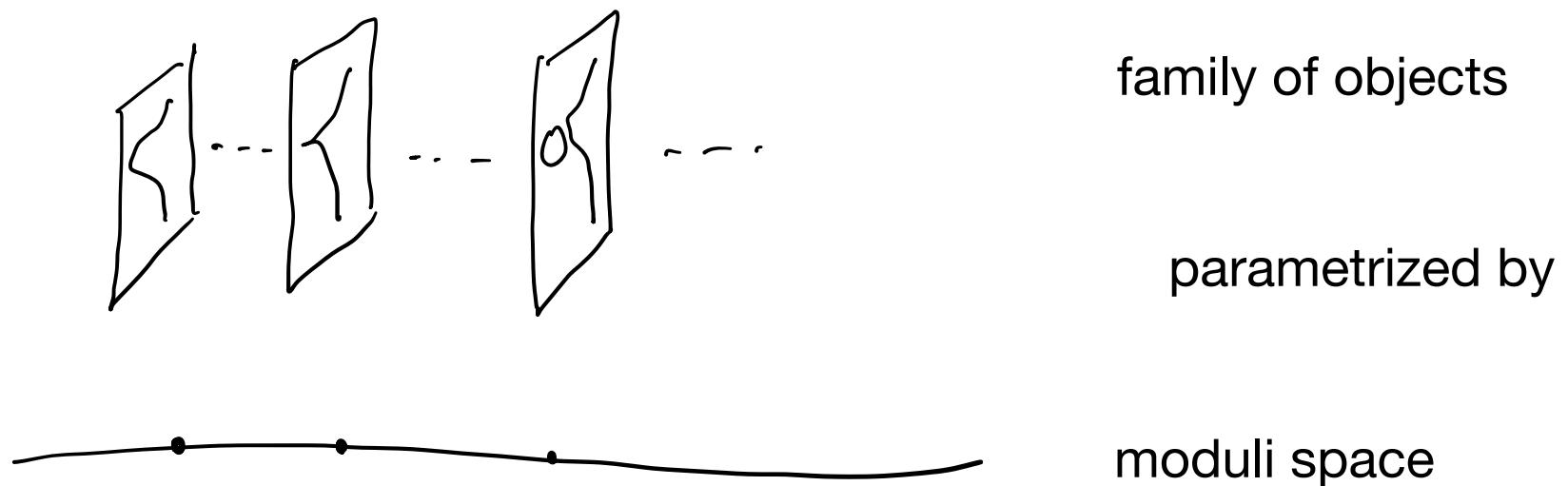
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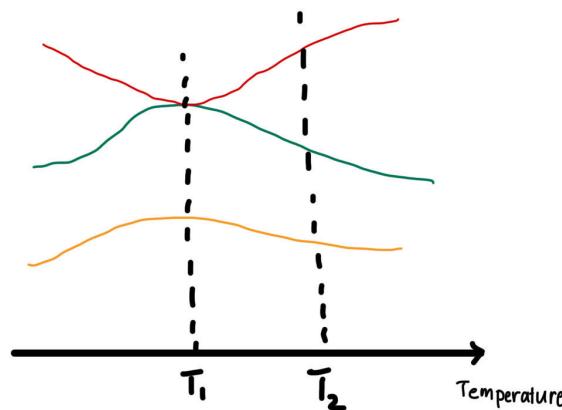
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- In this sense, studying moduli spaces is of the **second-order** nature.

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*Guzman et al., Model-free characterization of topological edge and corner states in mechanical networks, PNAS 2024.*

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### Preprints

- H. Jia, J. Hu, R.-Y. Zhang, Y. Xiao, D. Wang, M. Wang, S. Ma, X. Ouyang, **Y. Zhu**, and C. T. Chan. *Anomalous bulk-edge correspondence intrinsically beyond line-gap topology in non-Hermitian swallowtail gapless phase.*
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## **Mathematical set-up: Eigenframe evolution of non-Hermitian systems**

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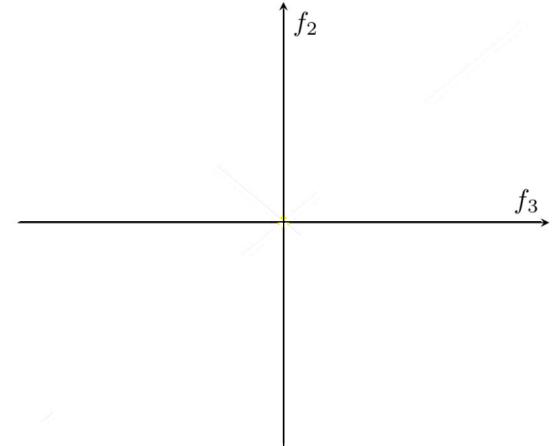
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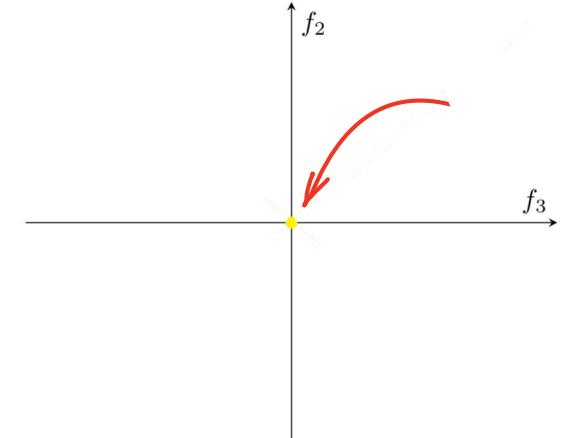
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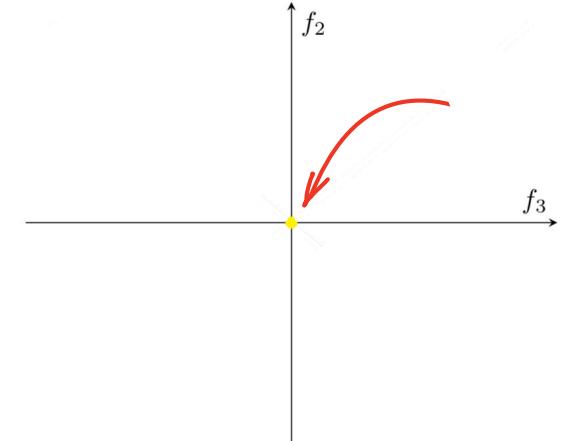
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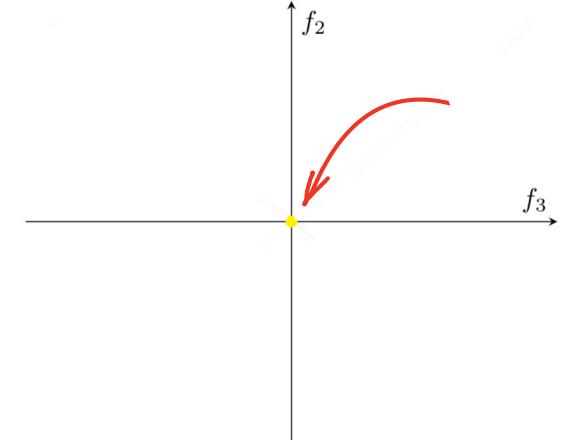
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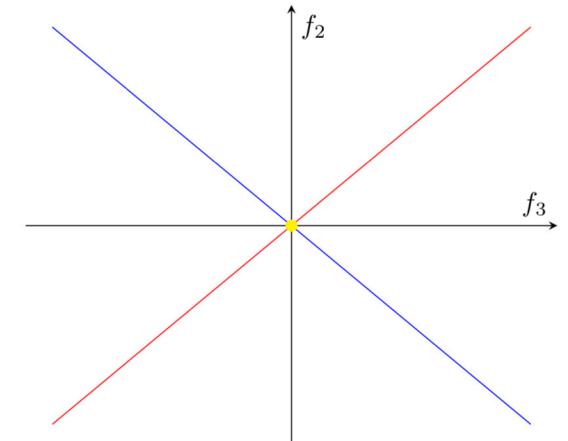
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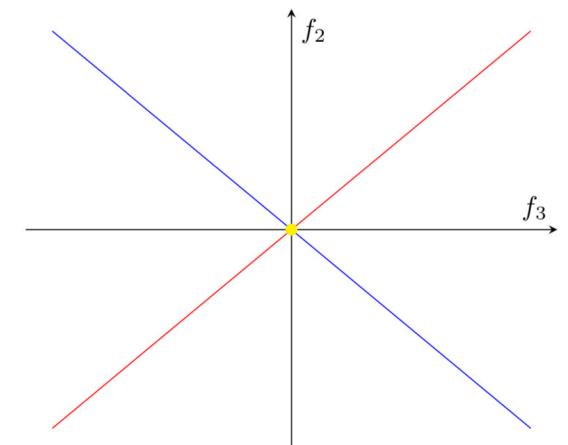
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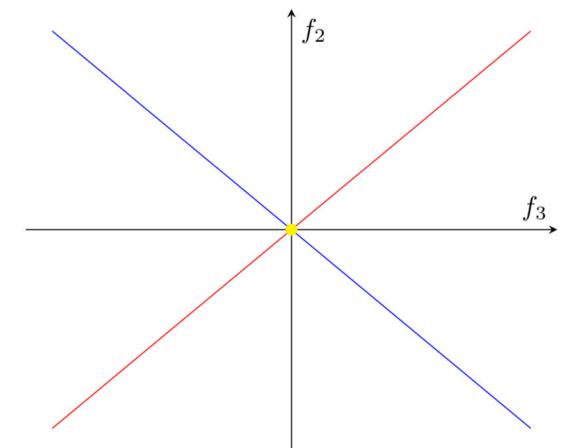
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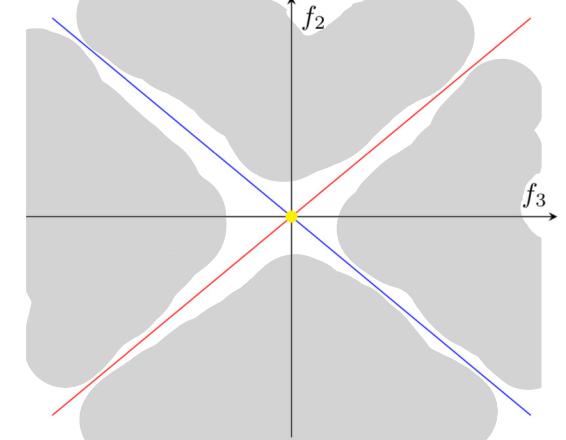
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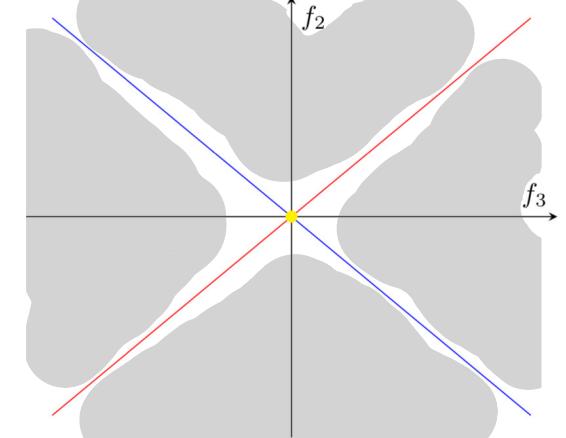
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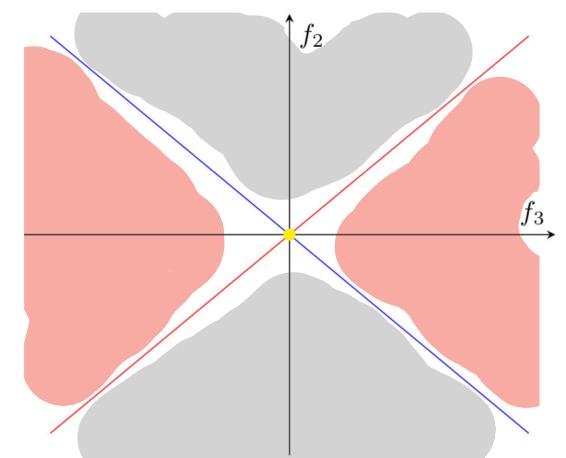
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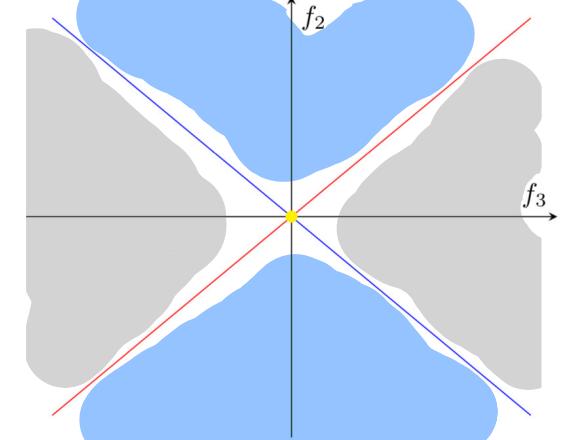
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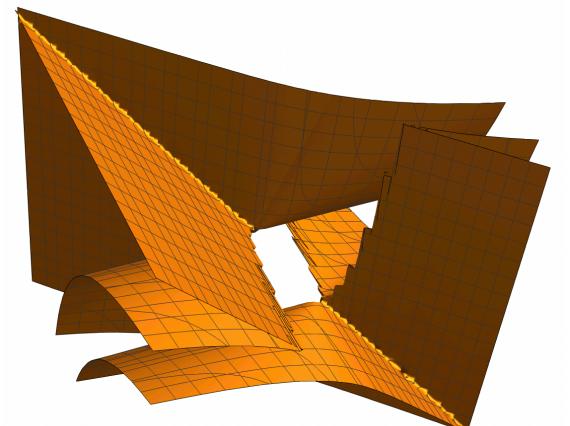
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*The equation for this surface is a non-homogeneous real polynomial in  $f_1, f_2, f_3$  of degree 6.*



Swallowtail couple sw2

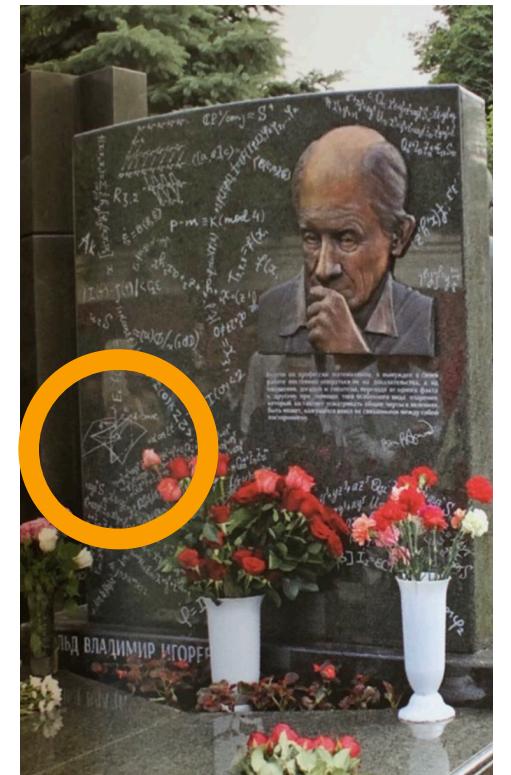
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V. I. Arnold's tombstone at the Novodevichy Cemetery in Moscow

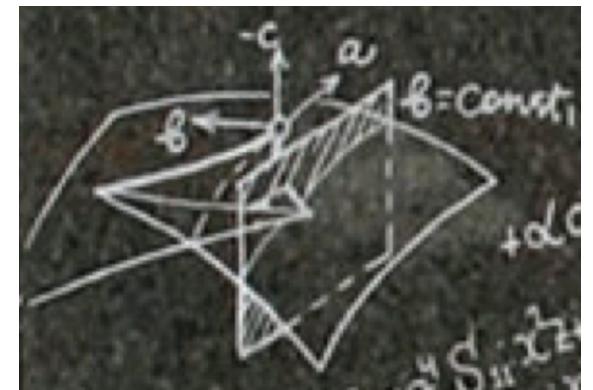
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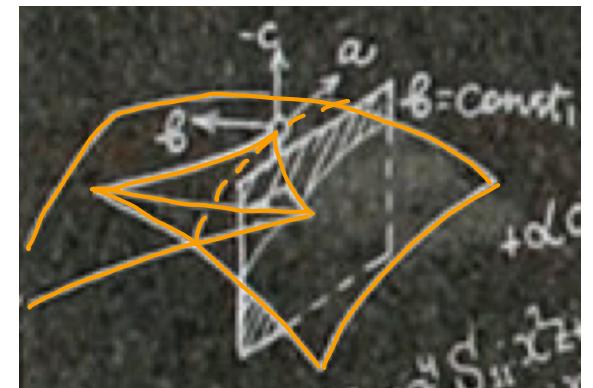
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A **local** model for moduli spaces of 3-band Hamiltonians

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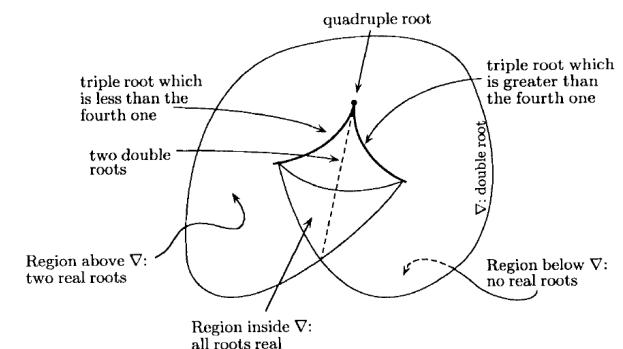
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*Arnold, Braids of algebraic functions and the cohomology of swallowtails, 1968.*

*Homological stability of braid groups*

*Portrait from Gelfand, Kapranov, Zelevinsky,  
Discriminants, resultants, and multidimensional determinants.*



The space of polynomials  $x^4 + ax^2 + bx + c$

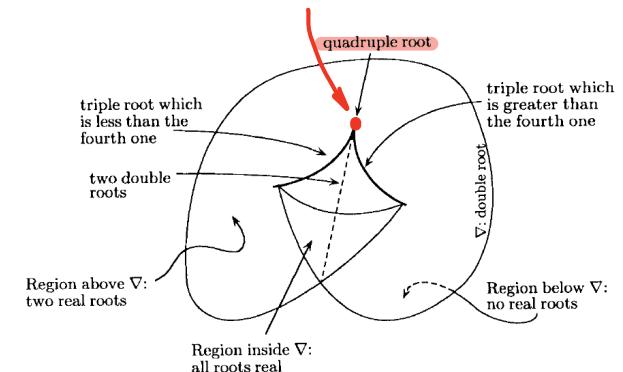
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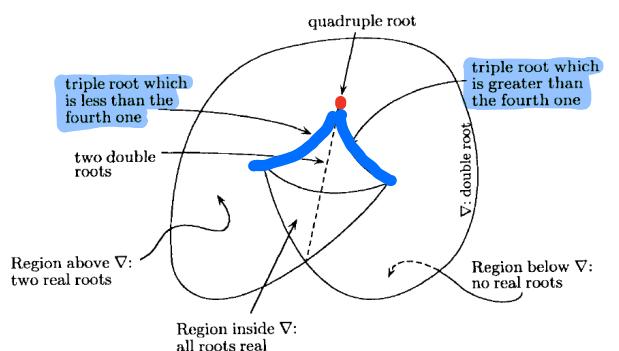
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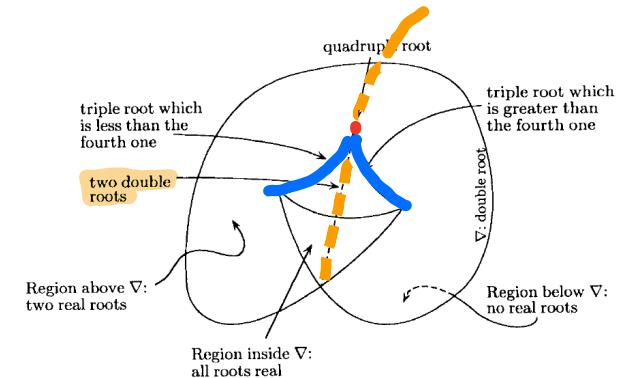
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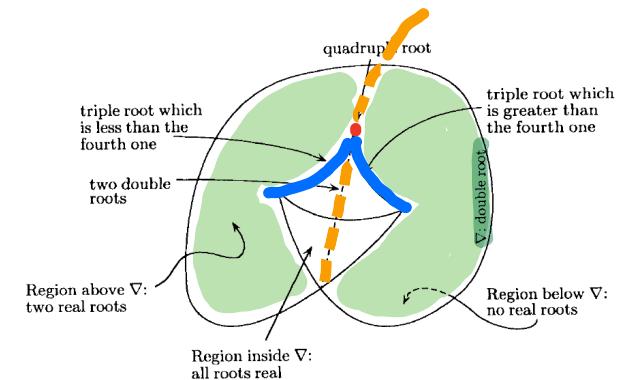
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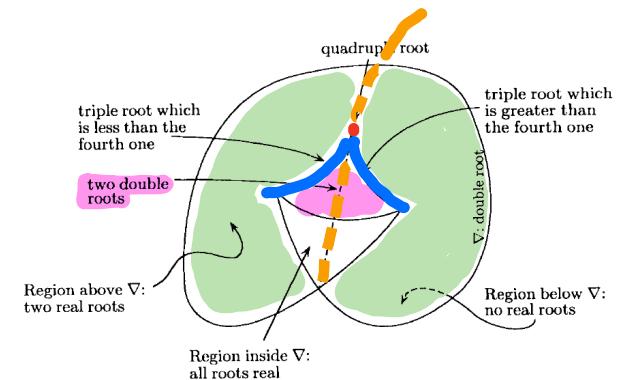
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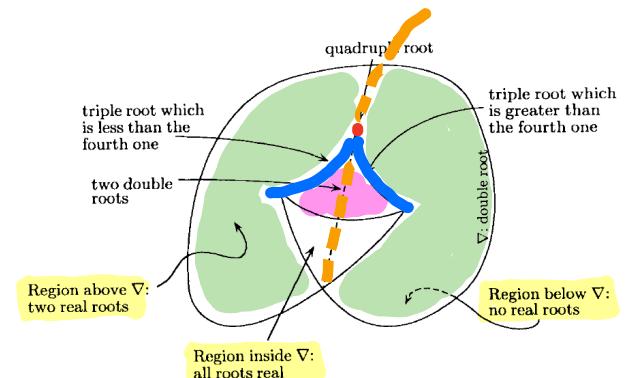
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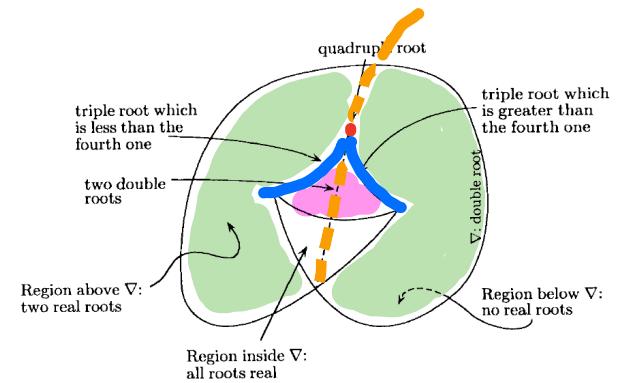
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Again, we aim to find computable algebraic invariants that systematically classify the evolutions of eigenvectors along loops in such stratified parameter spaces, including when they cross the discriminant surface resulting in degeneracies of various sorts.



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*Remarks on eigenvalues and eigenvectors of Hermitian matrices,  
Berry phase, adiabatic connections and quantum Hall effect, 1995.*

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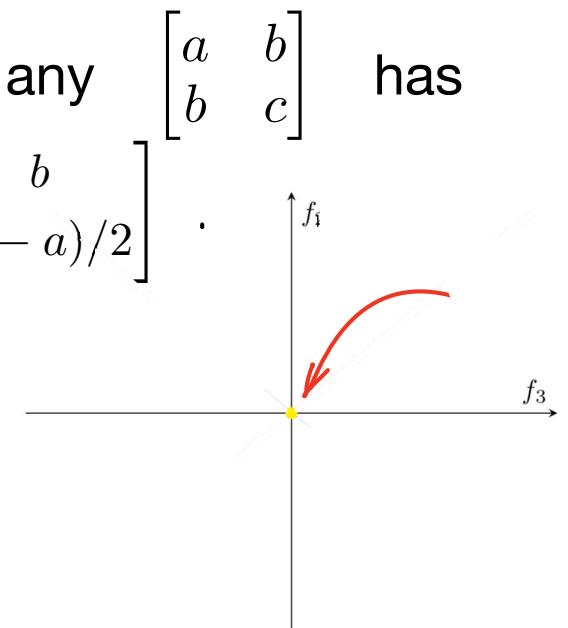
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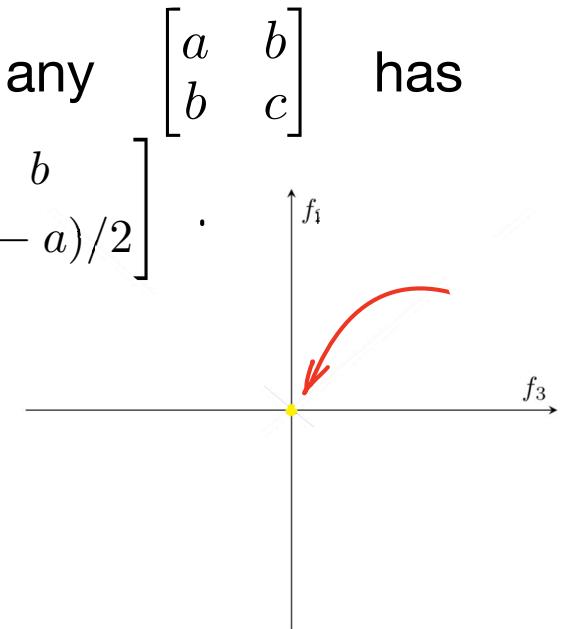
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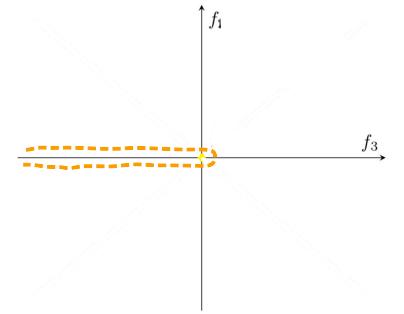
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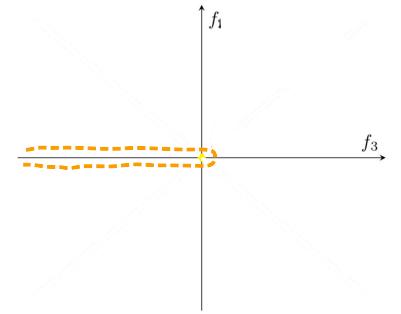
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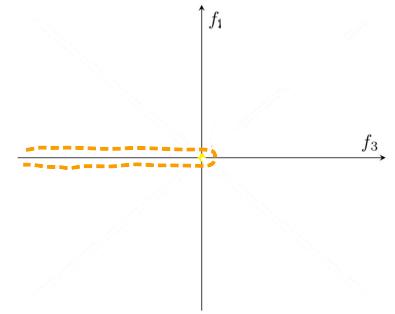


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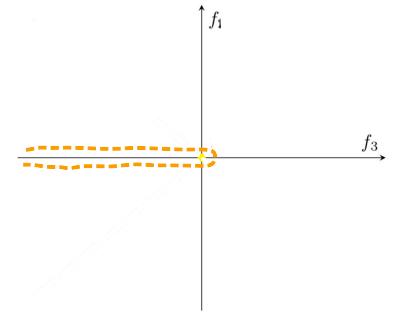
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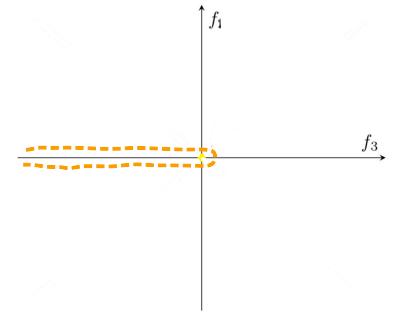
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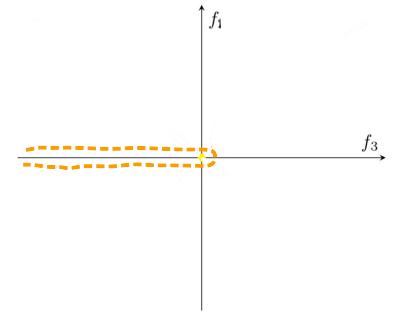
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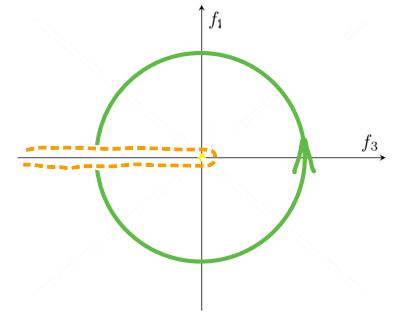
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## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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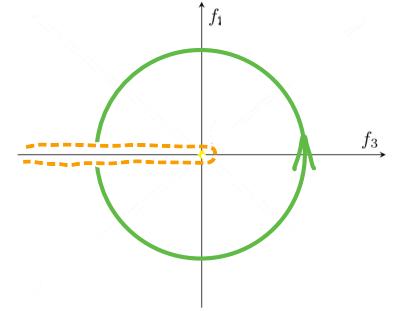
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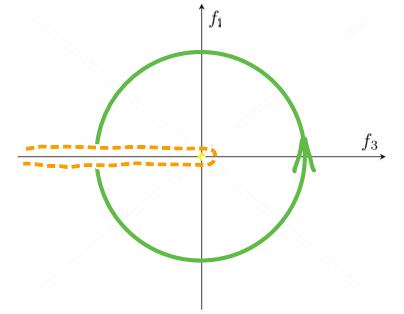


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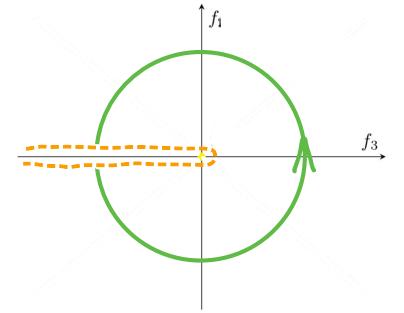
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We compute that

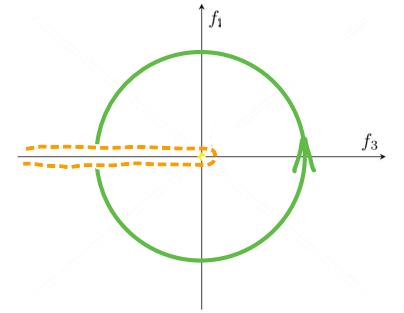
$$\lim_{\theta \rightarrow (-\pi)_+} \frac{v_+}{|v_+|} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

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## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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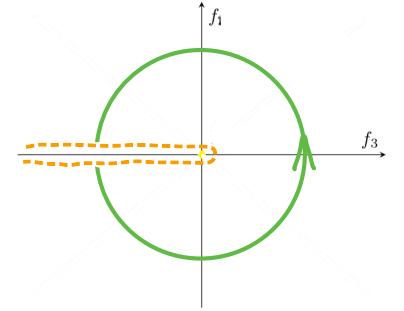
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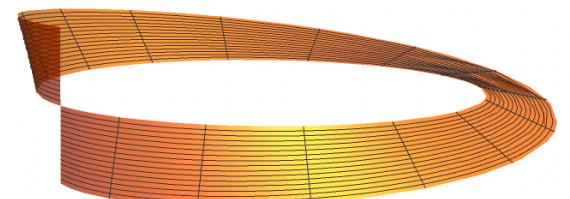
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## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

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## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

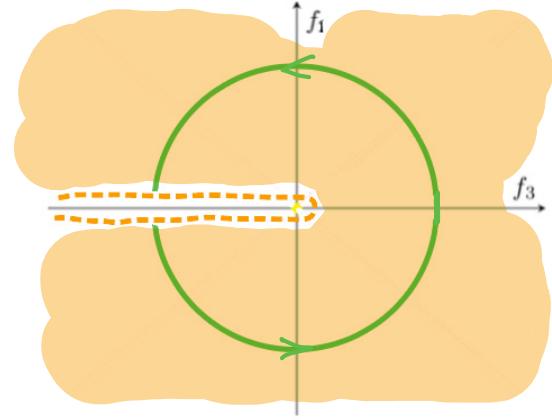
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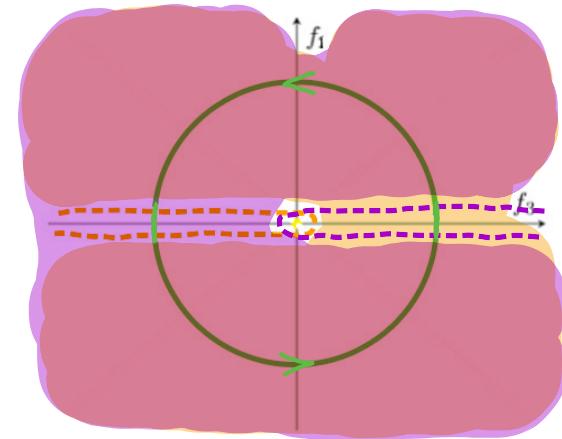
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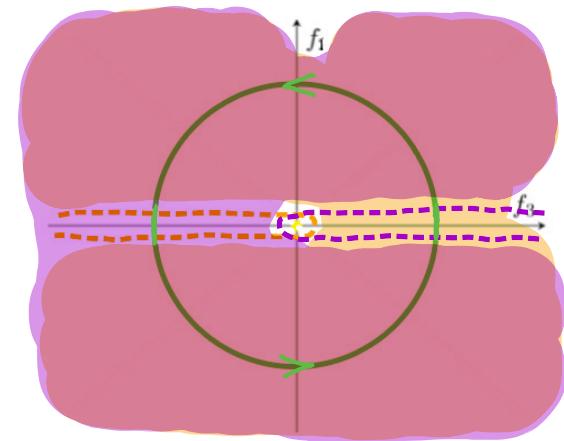
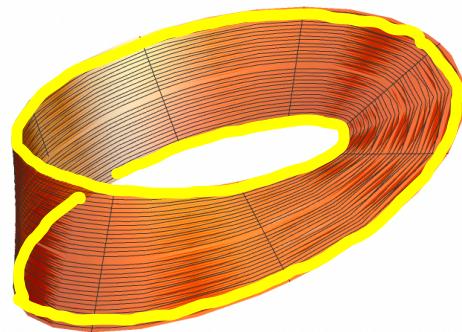
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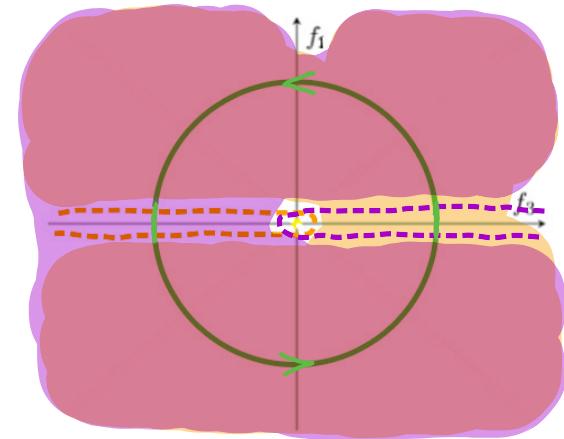
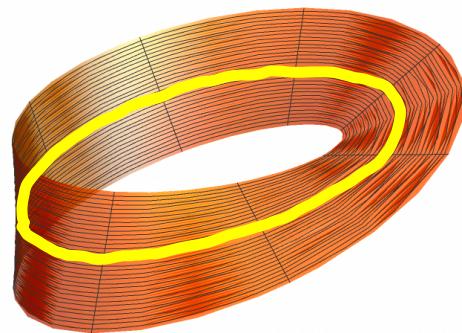
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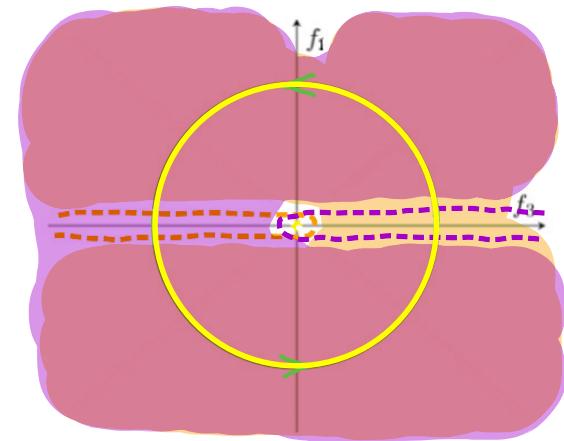
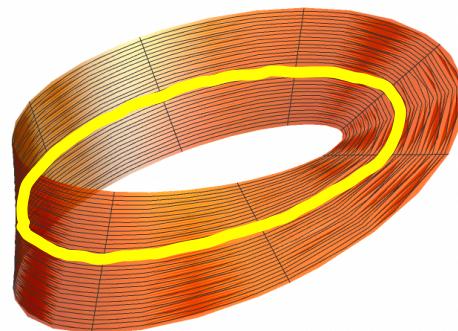
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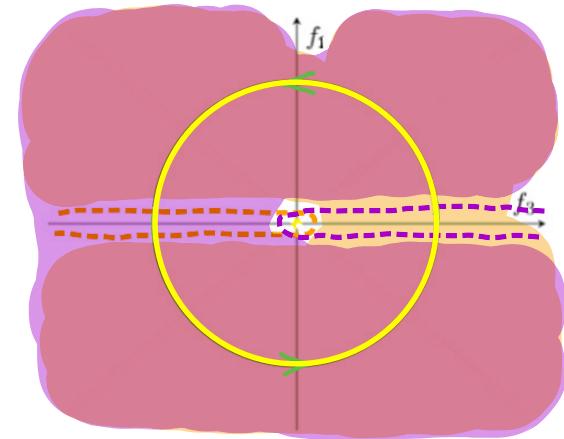
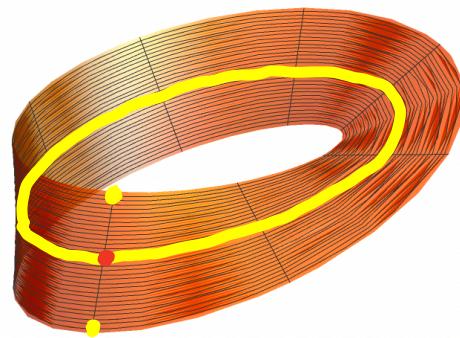
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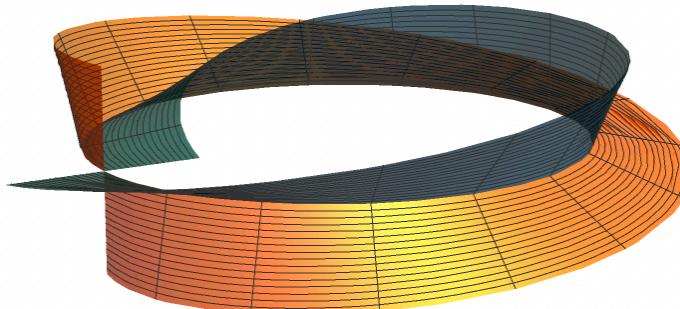
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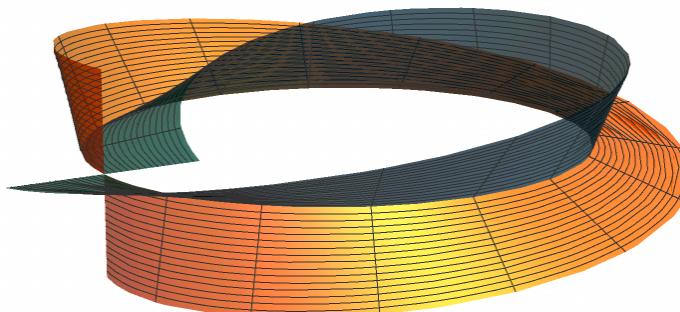


## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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**Corollary.** The universal eigenbundle for real Hermitian 2-band systems is given by a pair of orthogonally intersecting half Möbius bands over the **unit circle** in the punctured parameter plane.

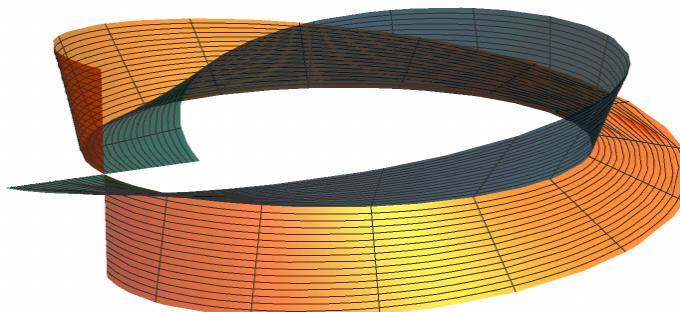


## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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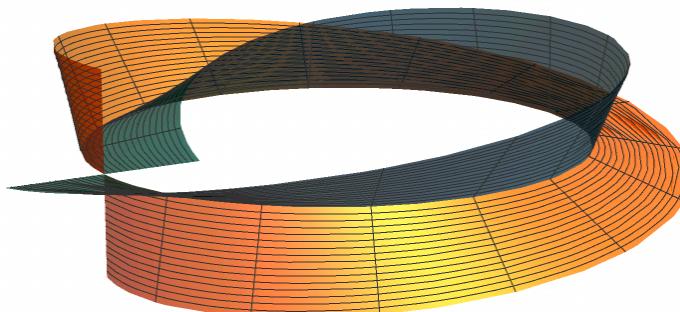


## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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# Eigenframe evolution as Higgs bundles: The non-Hermitian case



## Eigenframe evolution as Higgs bundles: The non-Hermitian case

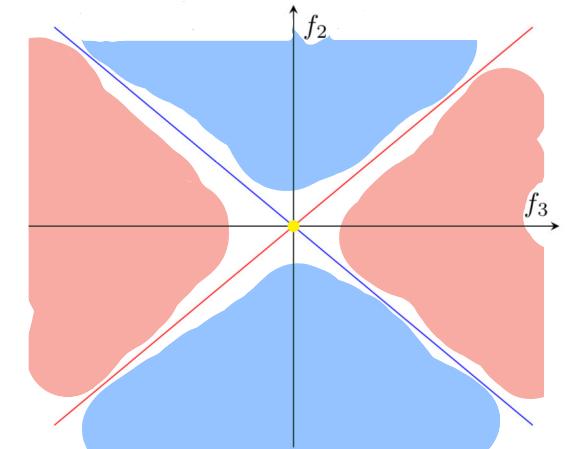
Recall that non-Hermitian 2-band systems

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

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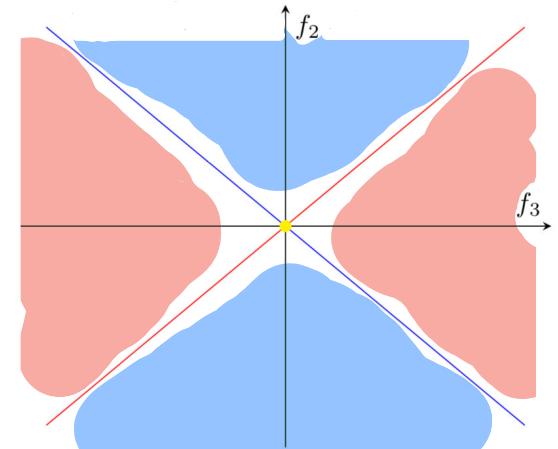


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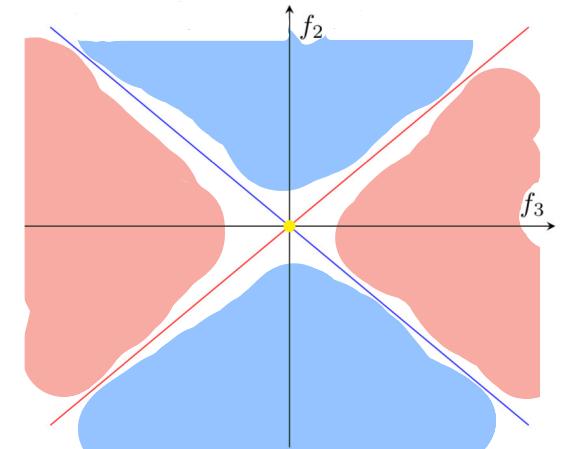


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1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.

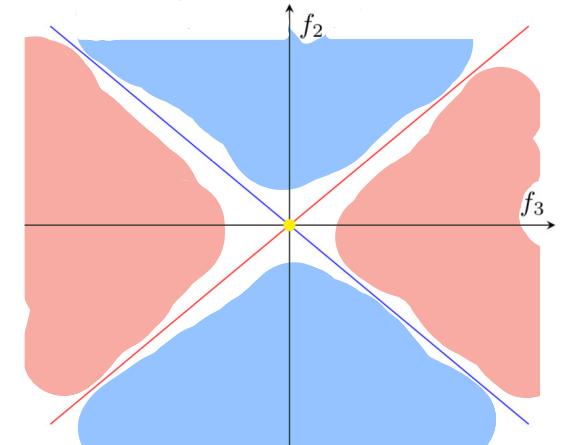


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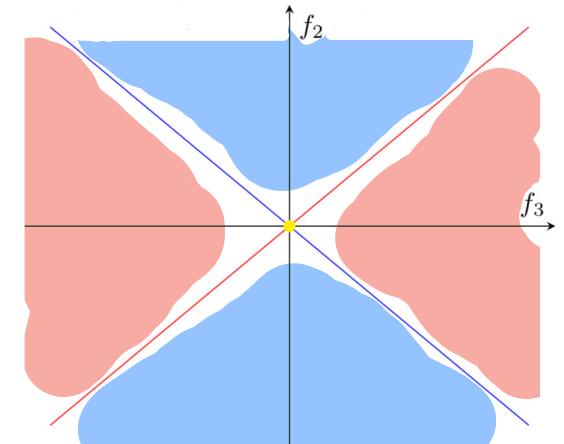


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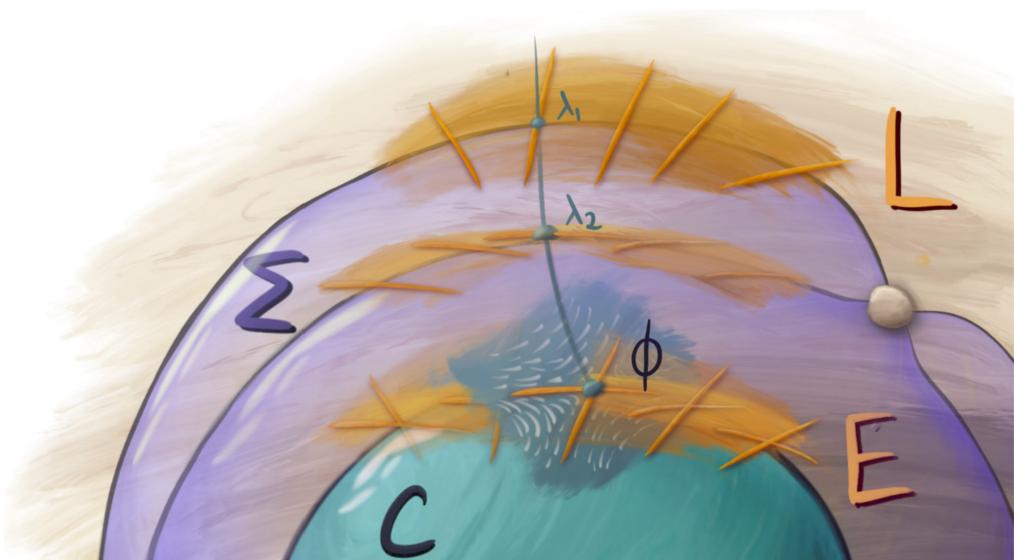
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A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

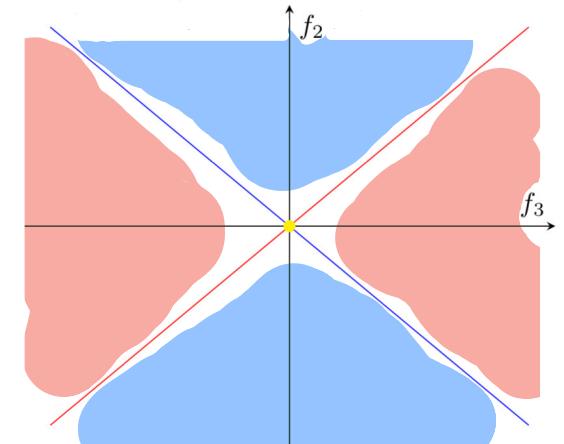


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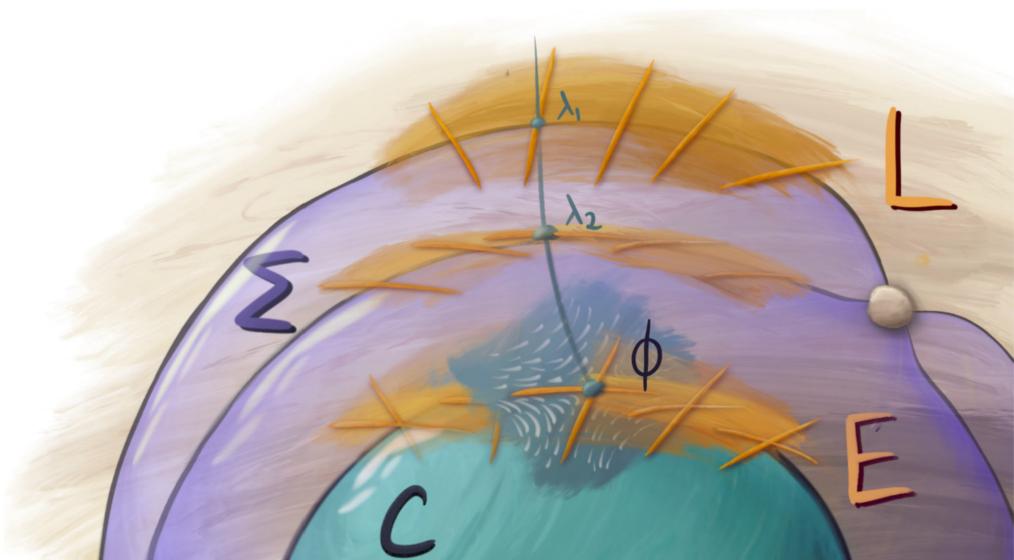
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Peter Higgs (bosons)

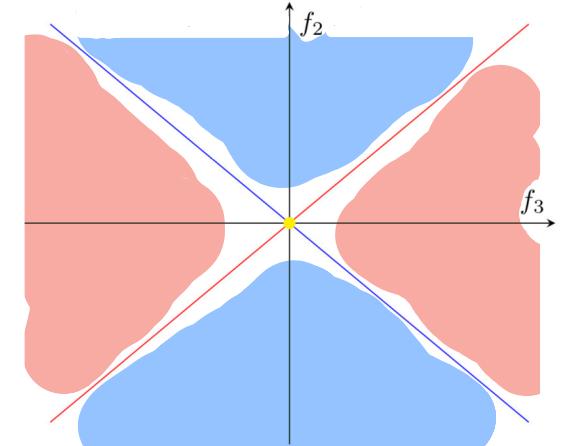


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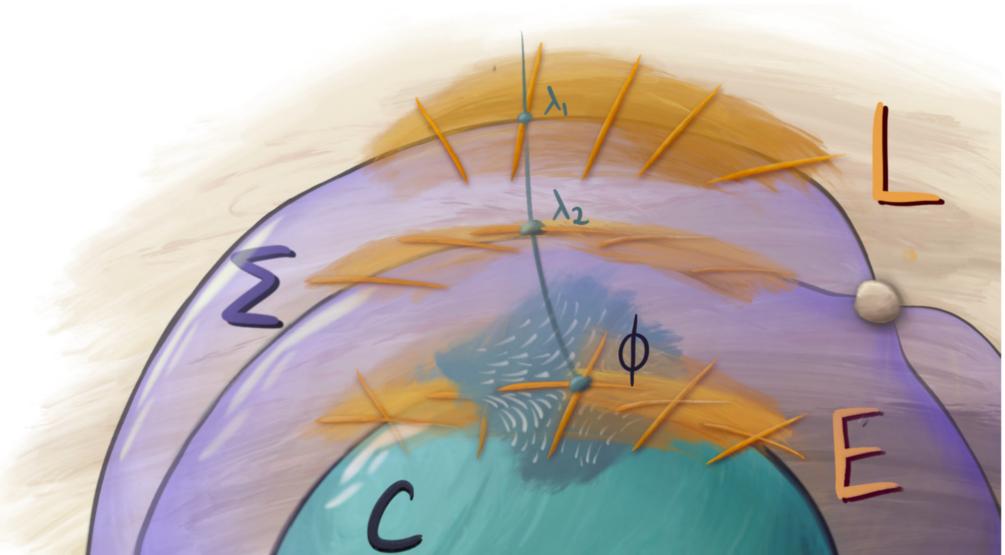


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Peter Higgs (bosons)



1929–2024

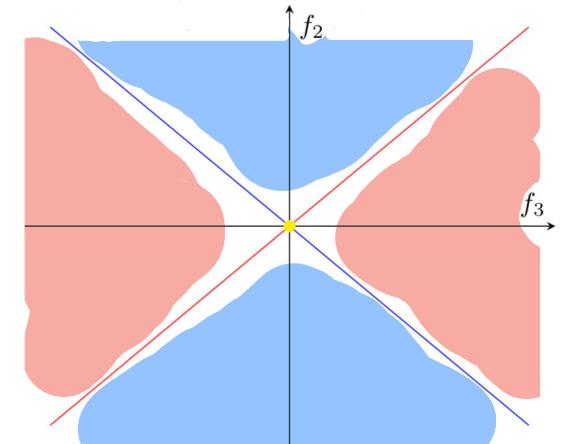


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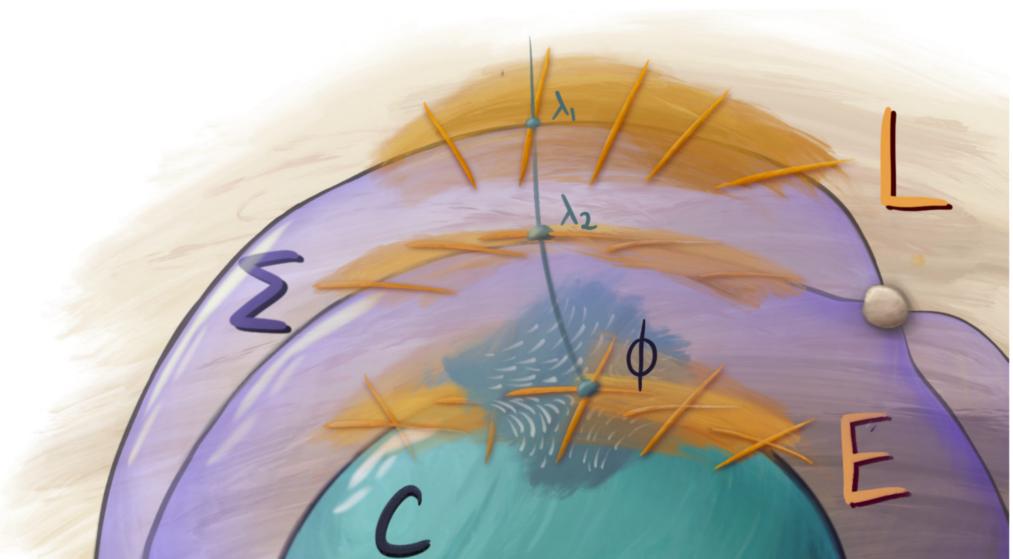
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Peter Higgs (bosons)

Nigel Hitchin 1987

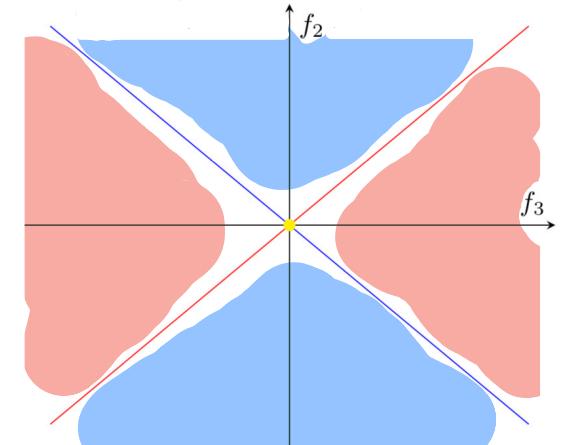


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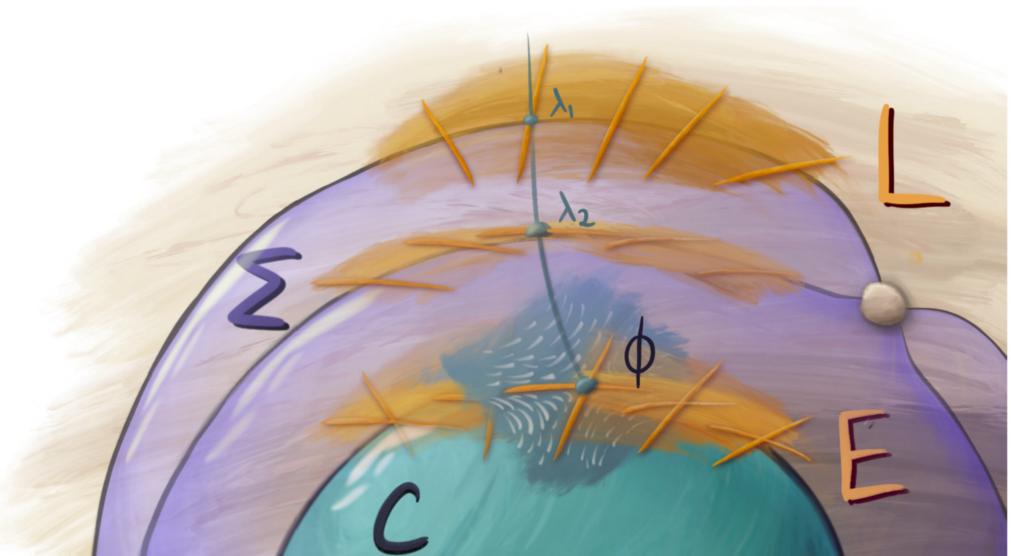


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Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface

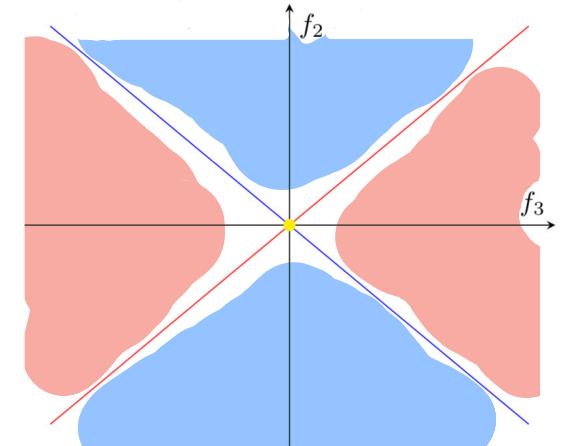


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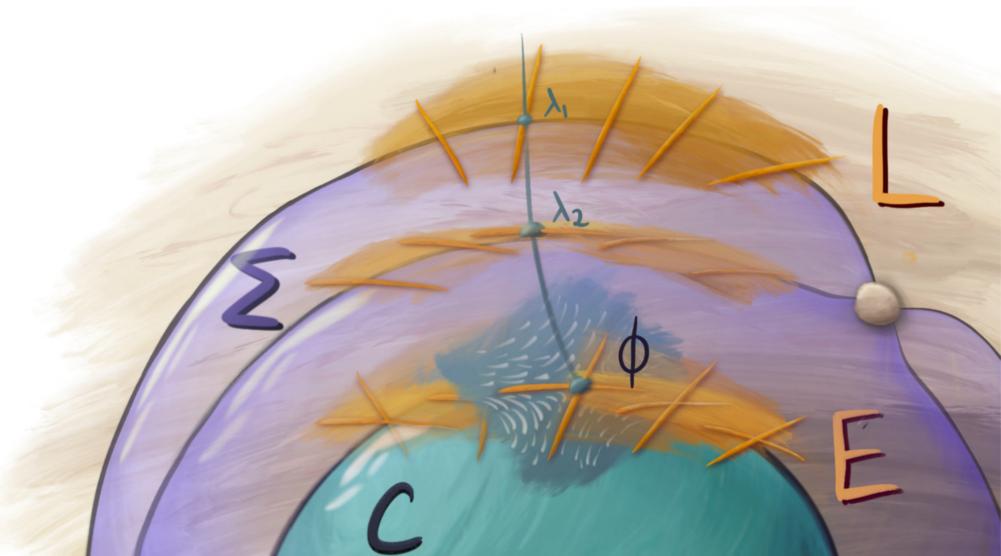


A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface  
E holomorphic vector bundle

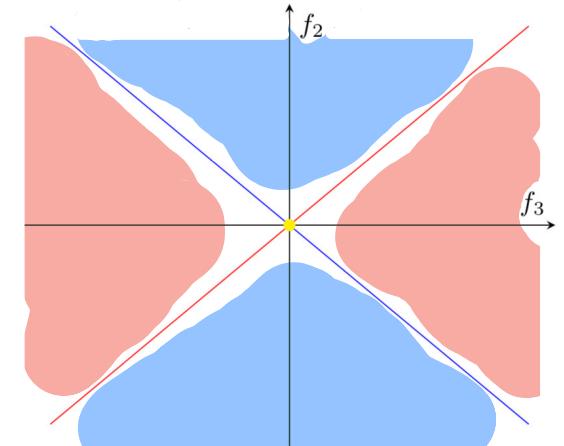


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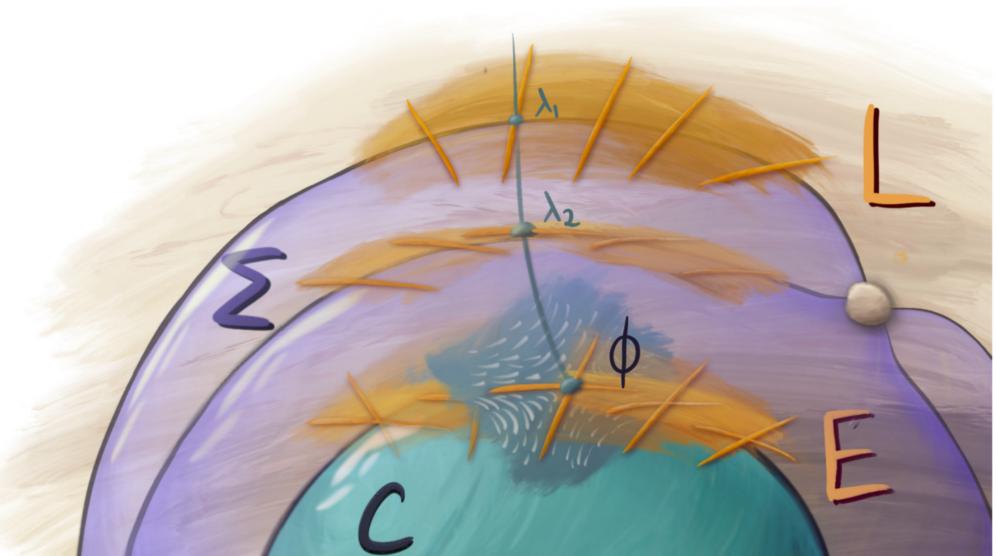
Peter Higgs (bosons)

Nigel Hitchin 1987

$C$  compact Riemann surface

$E$  holomorphic vector bundle

$\phi$  Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of  $E$

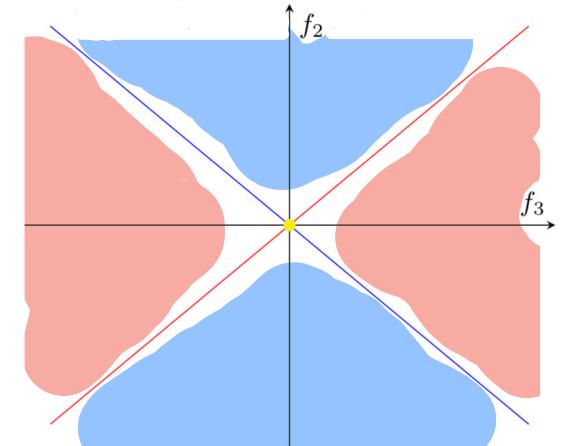


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A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of **matrices**

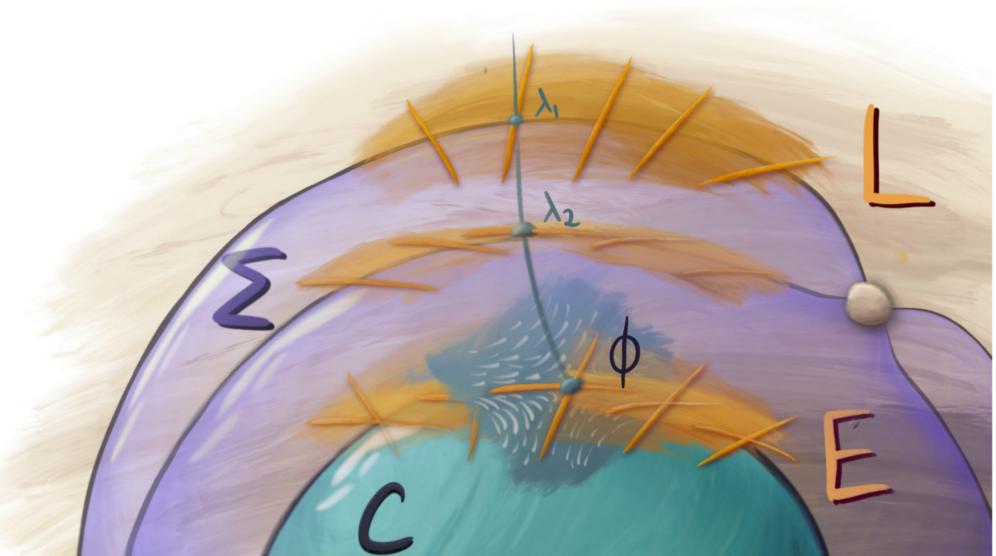
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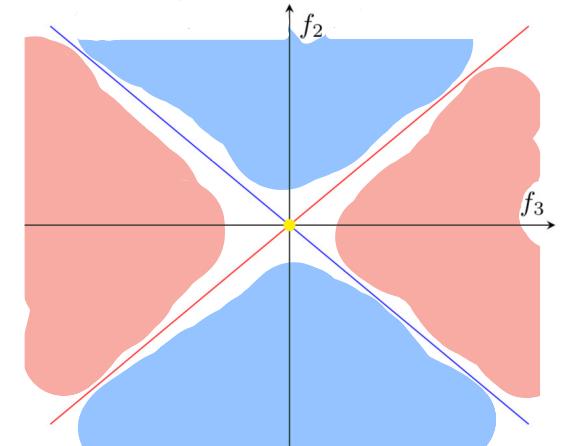


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A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

Peter Higgs (bosons)

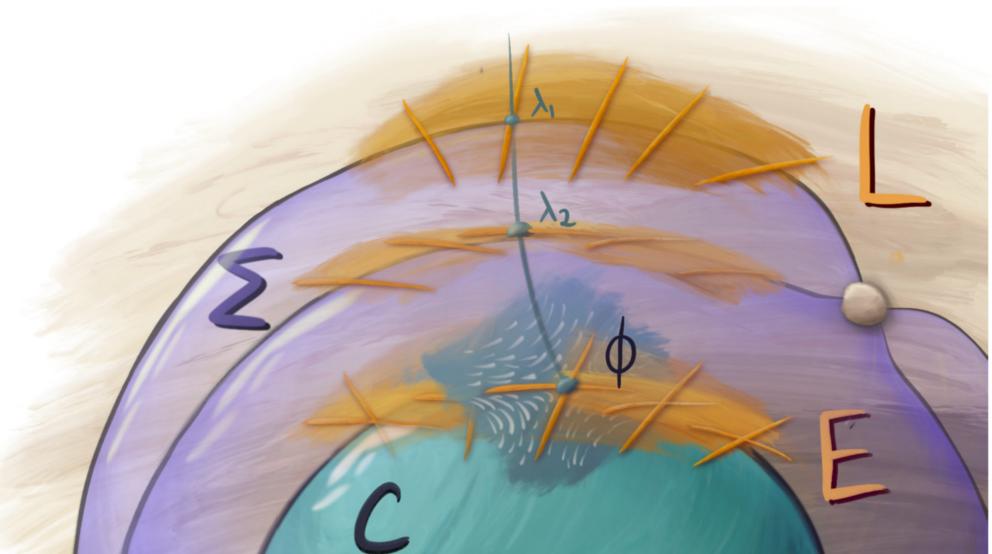
Nigel Hitchin 1987

Carlos Simpson

$C$  compact Riemann surface (or more generally Kähler manifold)

$E$  holomorphic vector bundle

$\phi$  Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of  $E$  such that  $\phi \wedge \phi = 0$

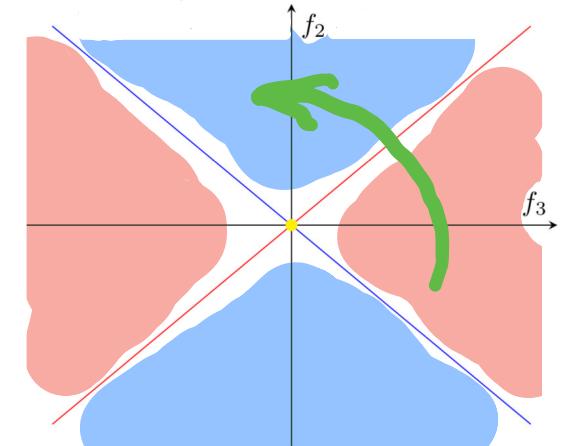


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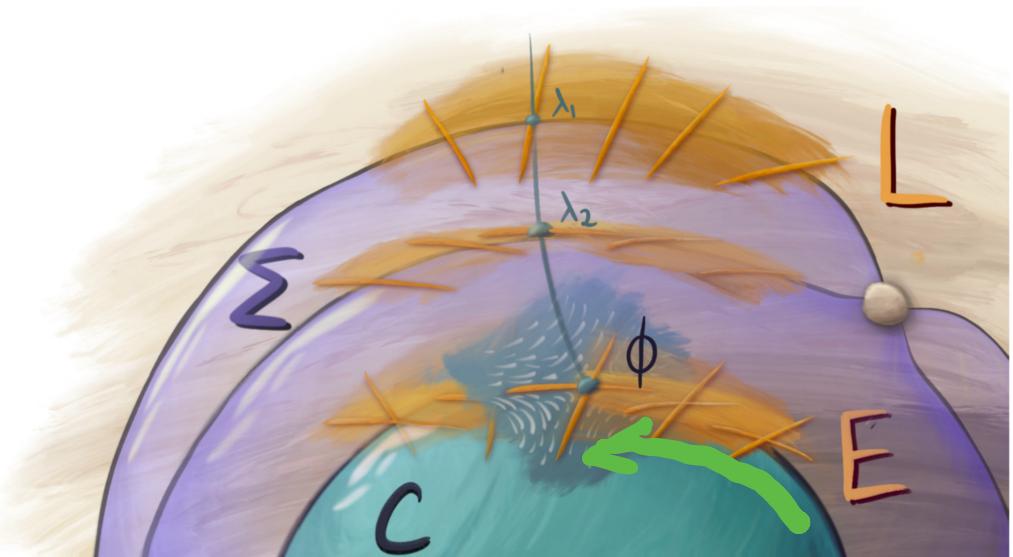
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$$\phi_x \in \text{End}(E_x), x \in C$$

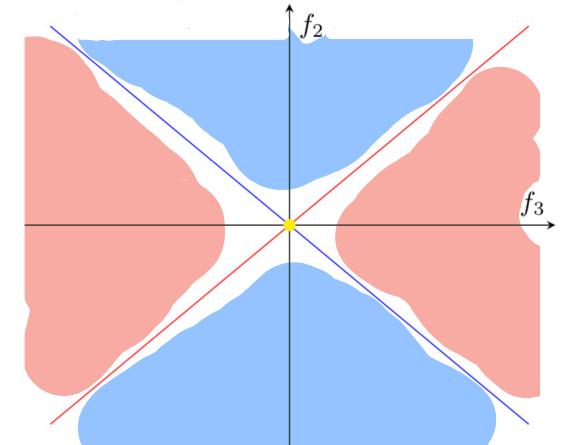


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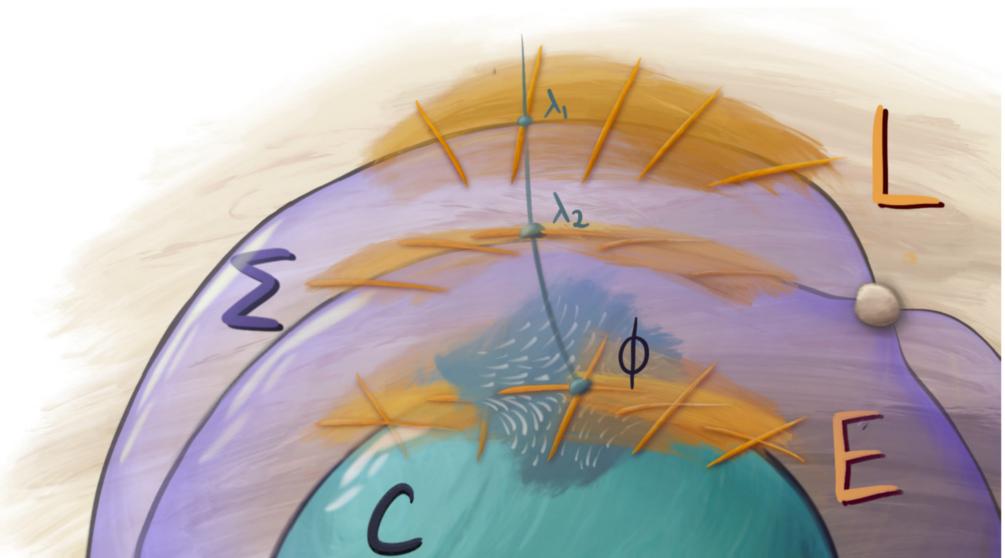
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Portrait from Kienzle and Rayan,  
*Hyperbolic band theory through Higgs bundles*, **Adv. Math.**, 2022.

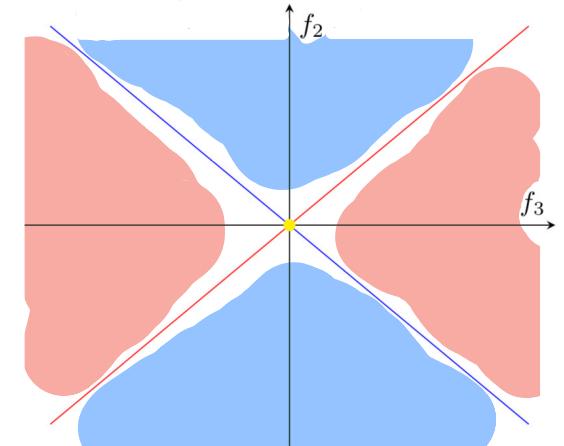


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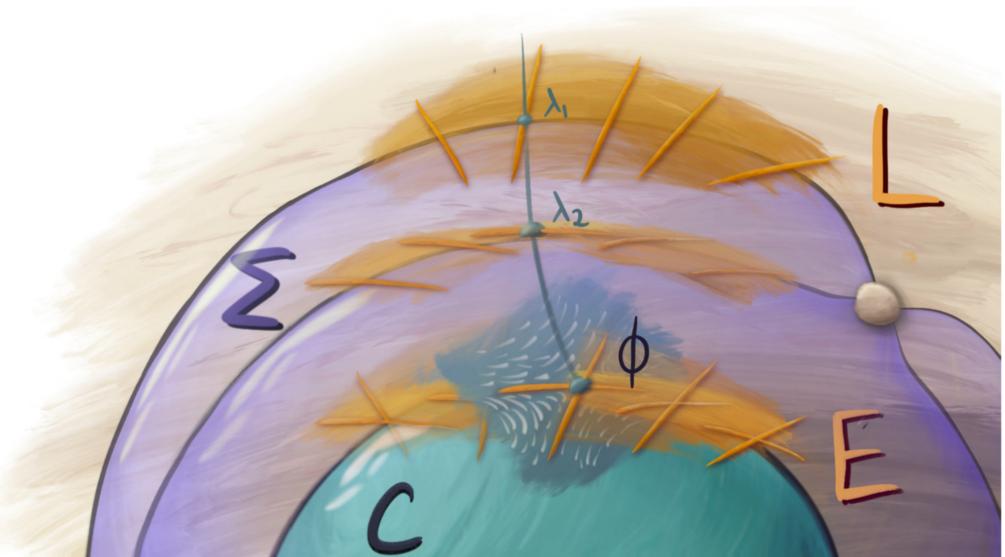
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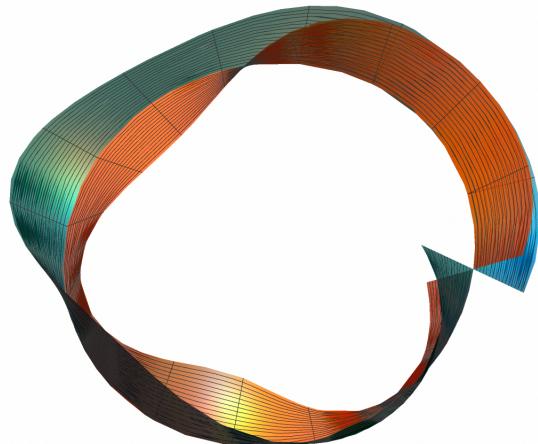
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Hyperbolic metric on the base  $C$ . Kollár et al., *Nature*, 2019.



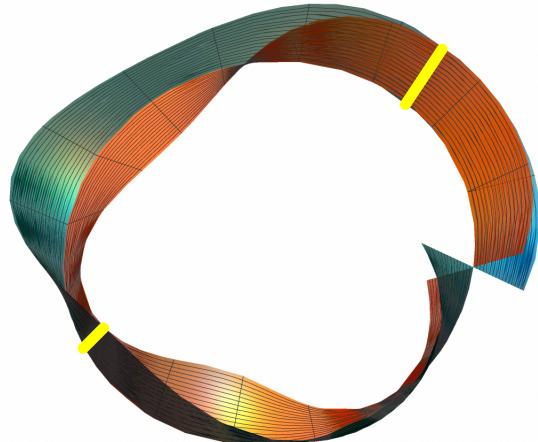
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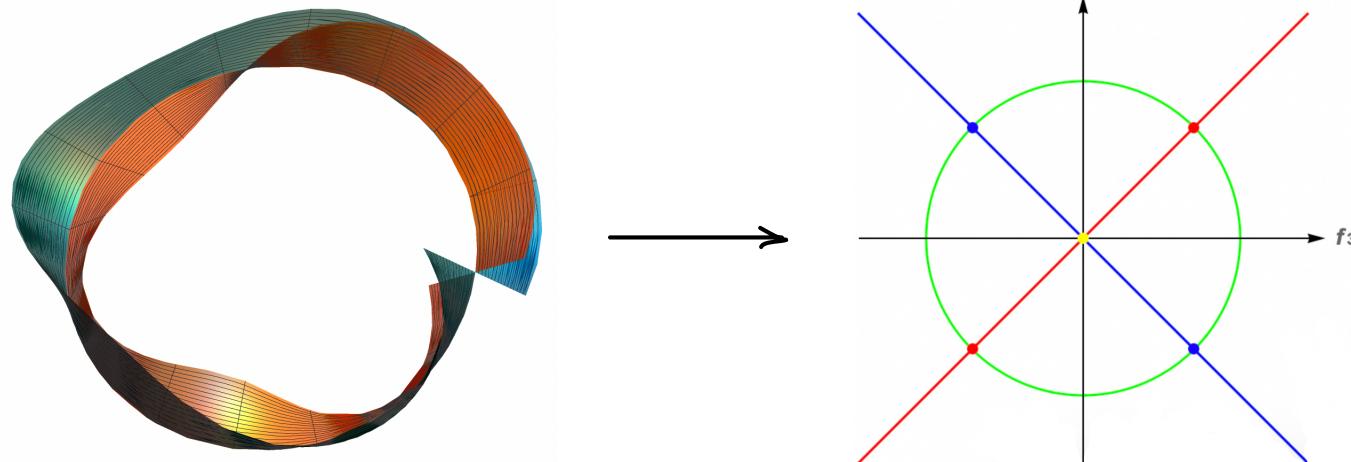
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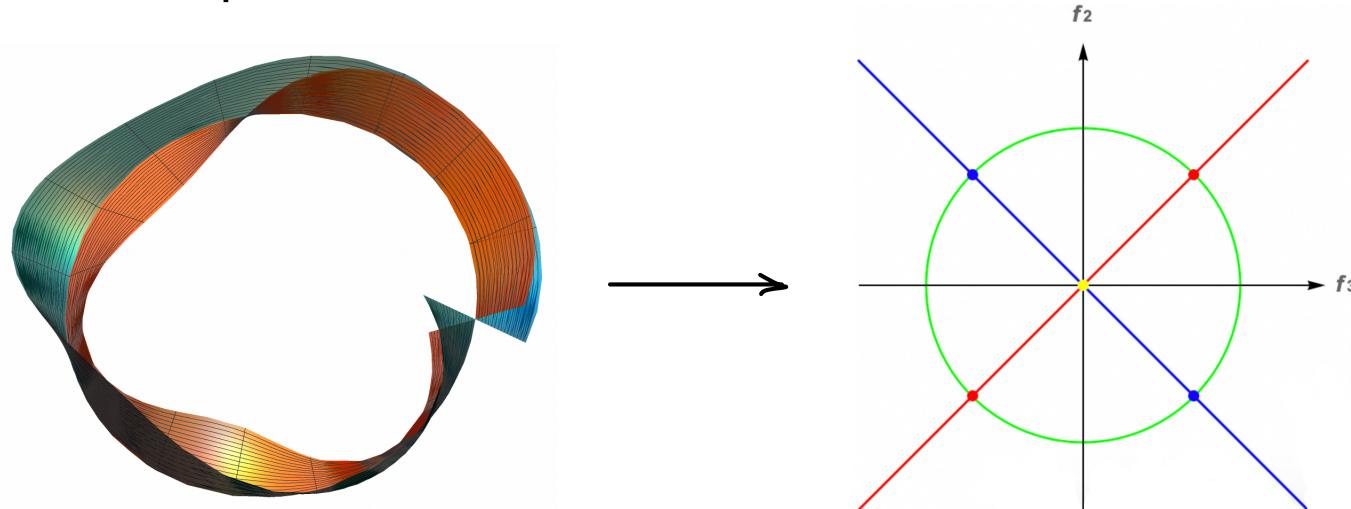
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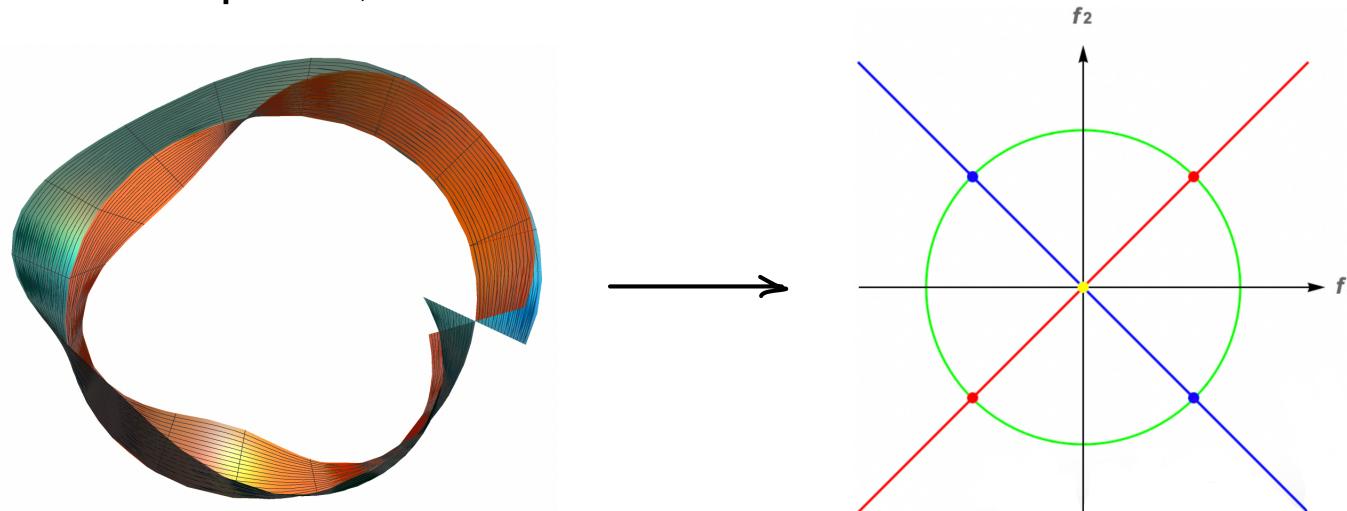
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## Eigenframe evolution as Higgs bundles: The non-Hermitian case

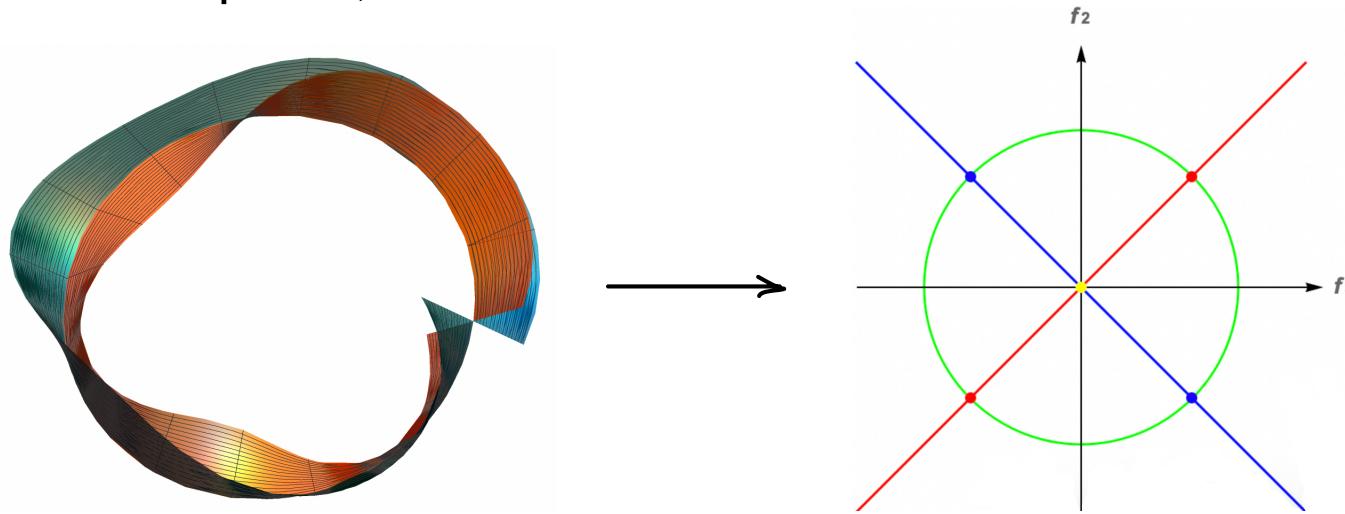
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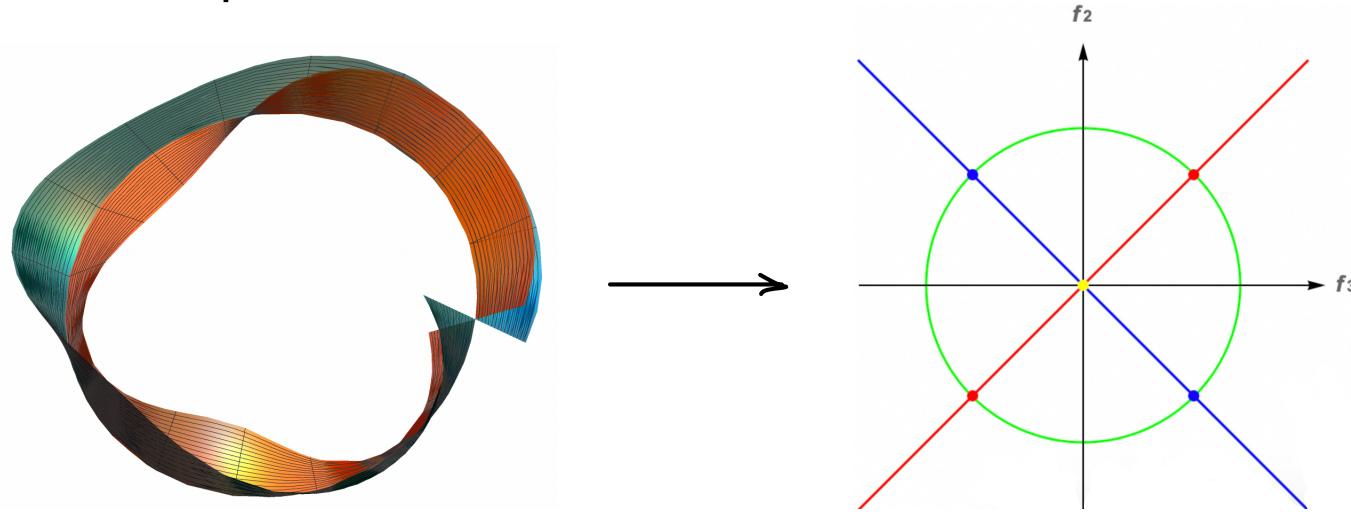


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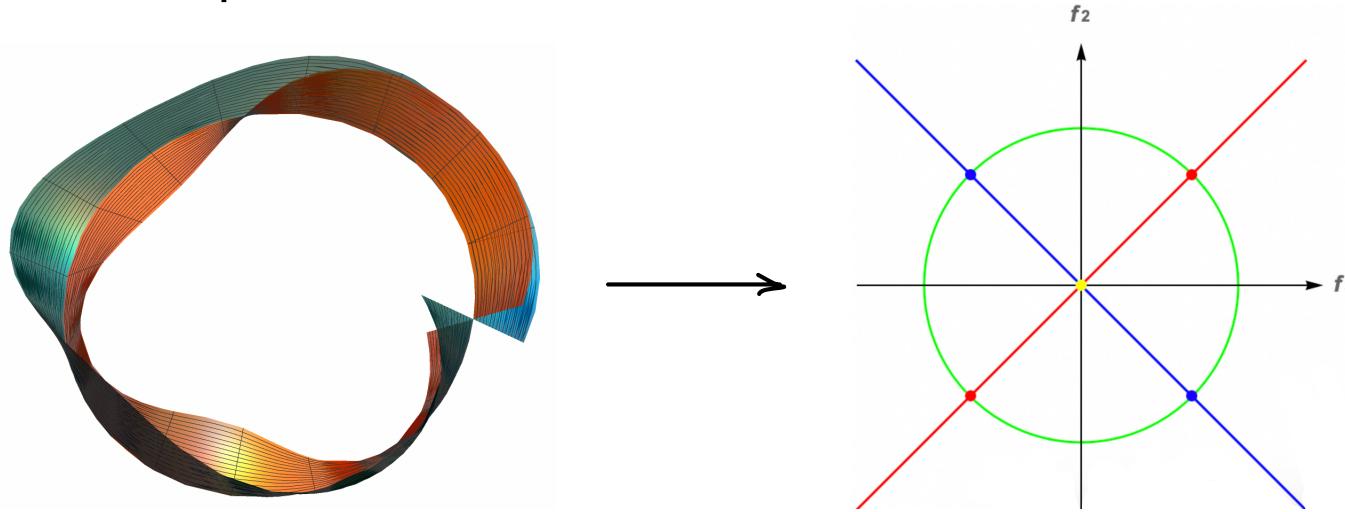


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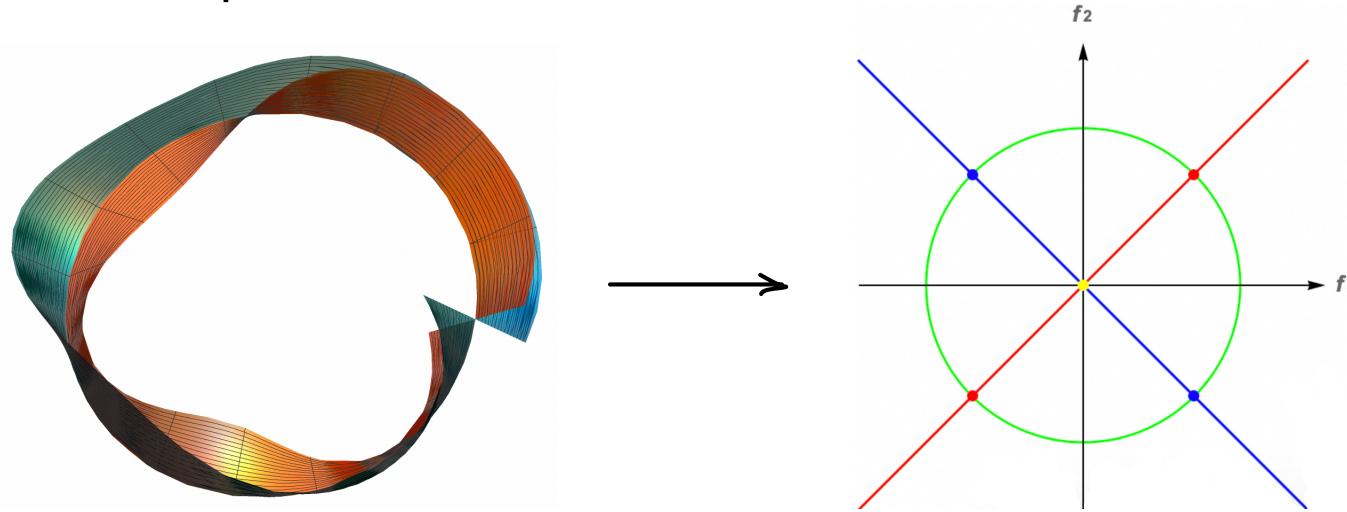
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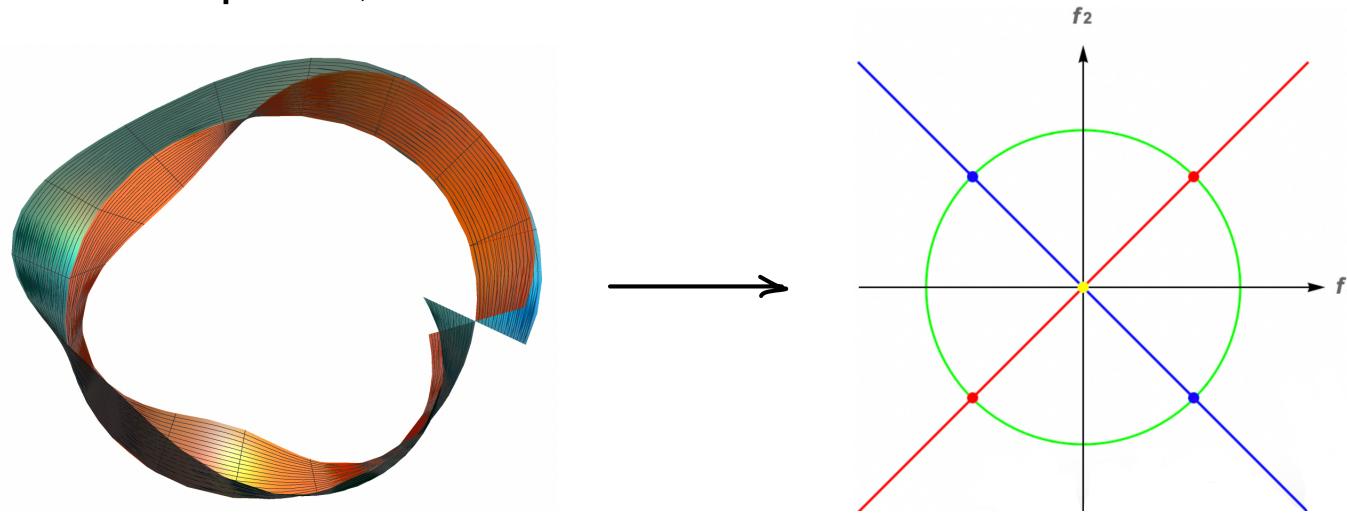
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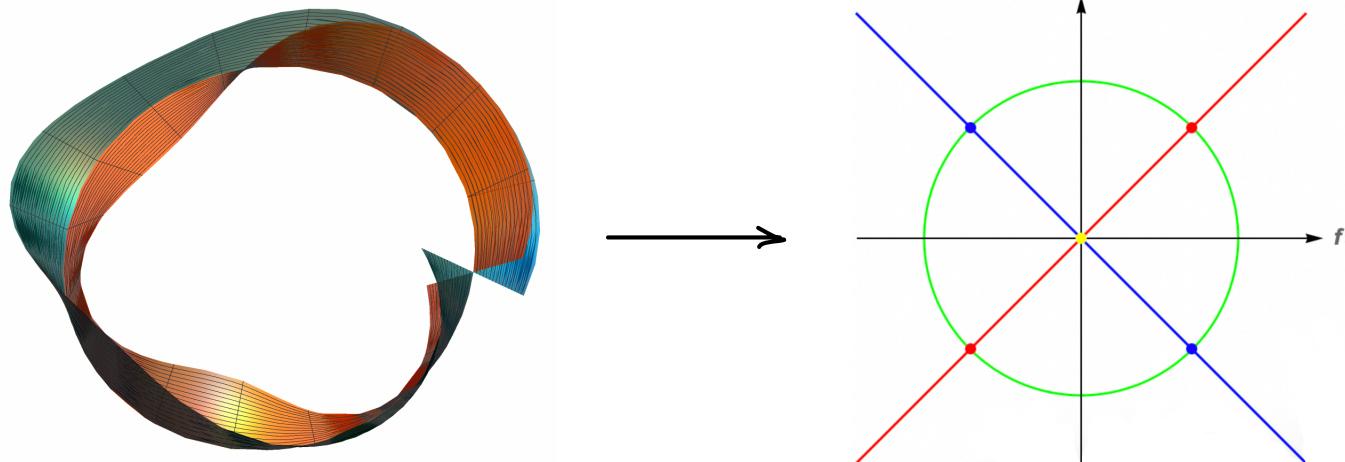
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**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.



Here is a video showing the eigenframe evolution: <https://yifeizhu.github.io/swallowtail/rotate.mp4>

**Note.** In the non-Hermitian case, since the eigenvectors are in  $\mathbb{C}^2$ , we have adopted (a variant of) the *Hermitian angle* to properly characterize the eigenframe rotation and deformation:

$$\frac{\langle v_+, v_- \rangle_{\mathbb{C}}}{|v_+| |v_-|} = \rho e^{i\psi}, \quad \cos(v_+, v_-)_{\text{Herm}} := \rho$$

*The intrinsic geometry should be independent of real/complex coordination, though.*

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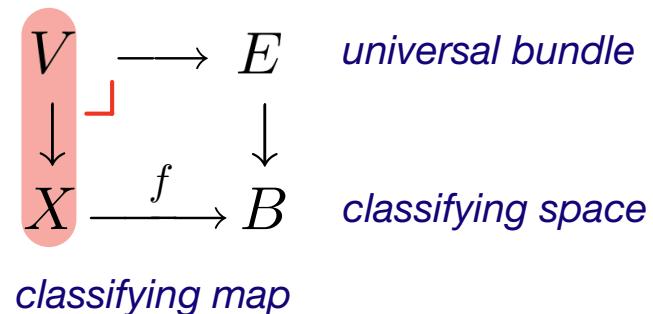
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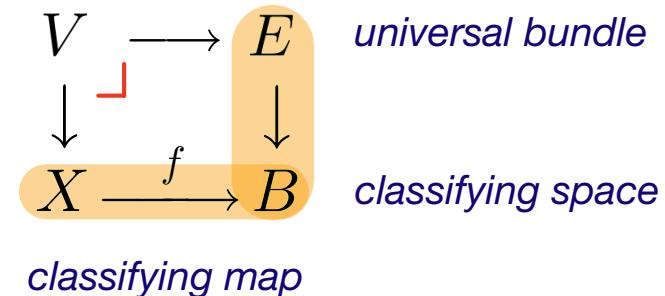
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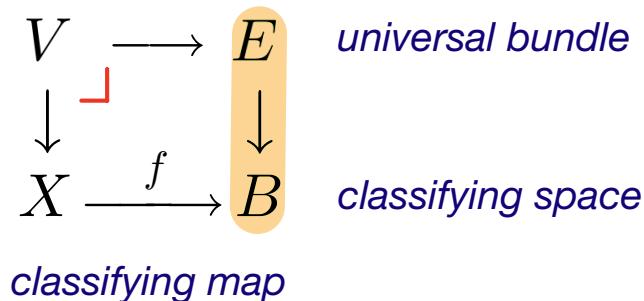
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This breaks the classification problem into two parts:

- Describe the universal bundle
- Find **computable** and **effective algebraic invariants** (topological charge) for the classifying/moduli space

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**In progress:** Need to compute the *intersection fundamental group* of the **stratified** moduli space.

*Gajer, The intersection Dold–Thom theorem,  
Topology, 1996. (Ph.D. student of Blaine Lawson, 1993)*

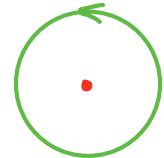
*Goresky and MacPherson, 1974.*

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$$\begin{array}{c|c|c|c} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \xrightarrow{\bar{p}}$$

Intersection homology of  $\mathbb{R}^2$  with one singular point: from top to bottom are  $I^{\bar{p}}H_0, I^{\bar{p}}H_1, I^{\bar{p}}H_2$ , where  $\bar{p}$  is the perversity function.

From blue to red regions, they detect the singular point.

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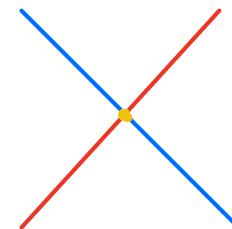
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$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
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0	0	0	0
0	0	0	0
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0
0	0	0	0
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
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Intersection homology of  $\mathbb{R}^2$  with a pair of intersecting singular lines:  
from top to bottom are  $I^{\bar{p}}H_*$  with  $* = 0, 1, 2$ .

From green to blue regions, they detect the singular lines.

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*May need to work at the chain level.*

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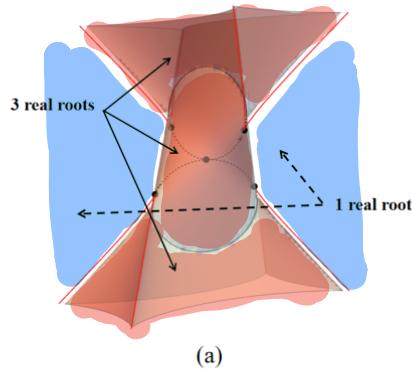
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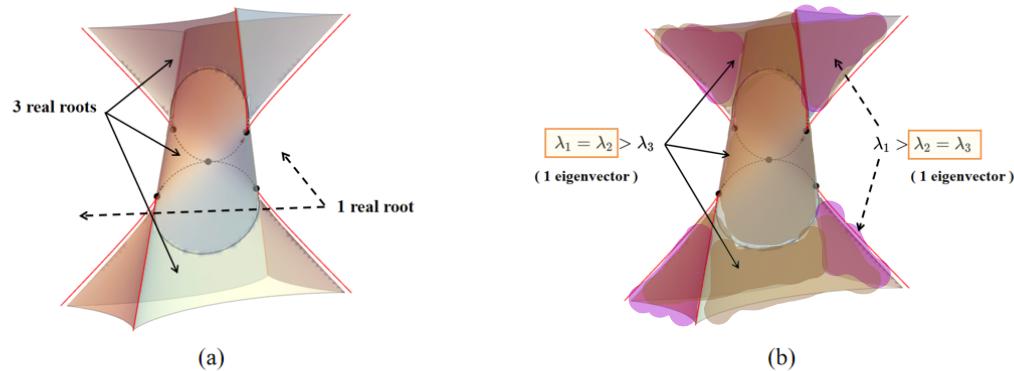
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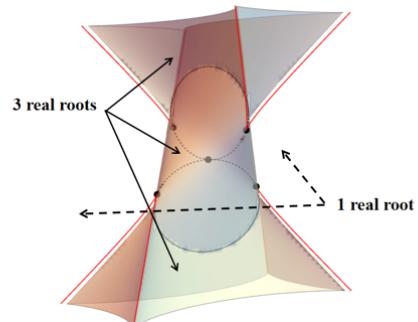
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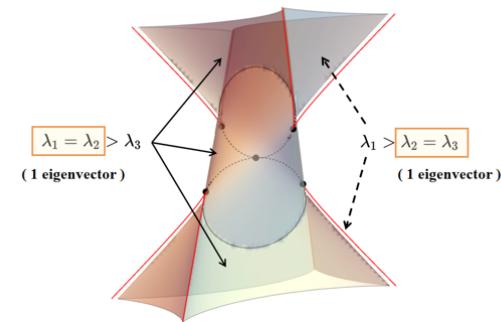
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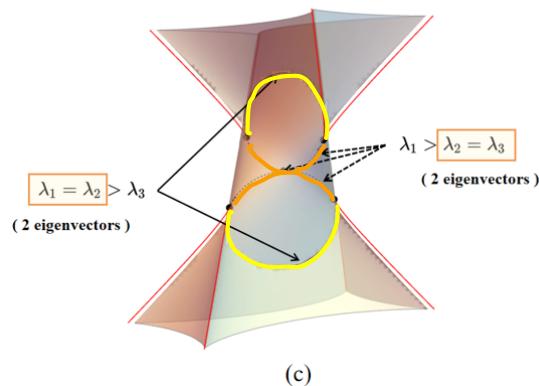
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(a)



(b)



(c)

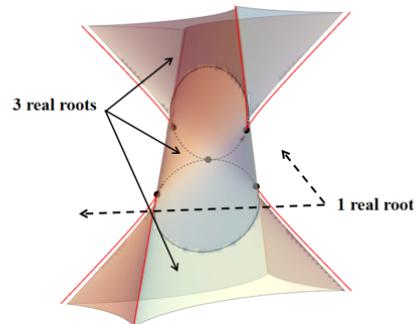
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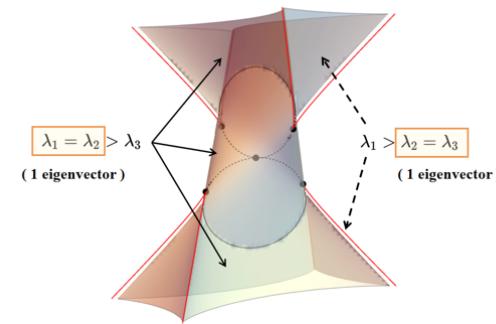
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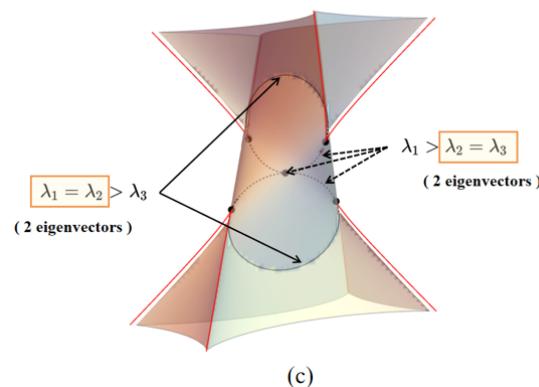
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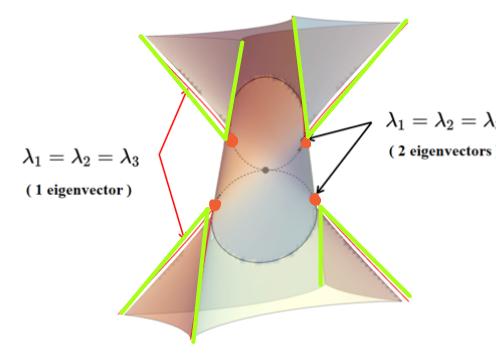
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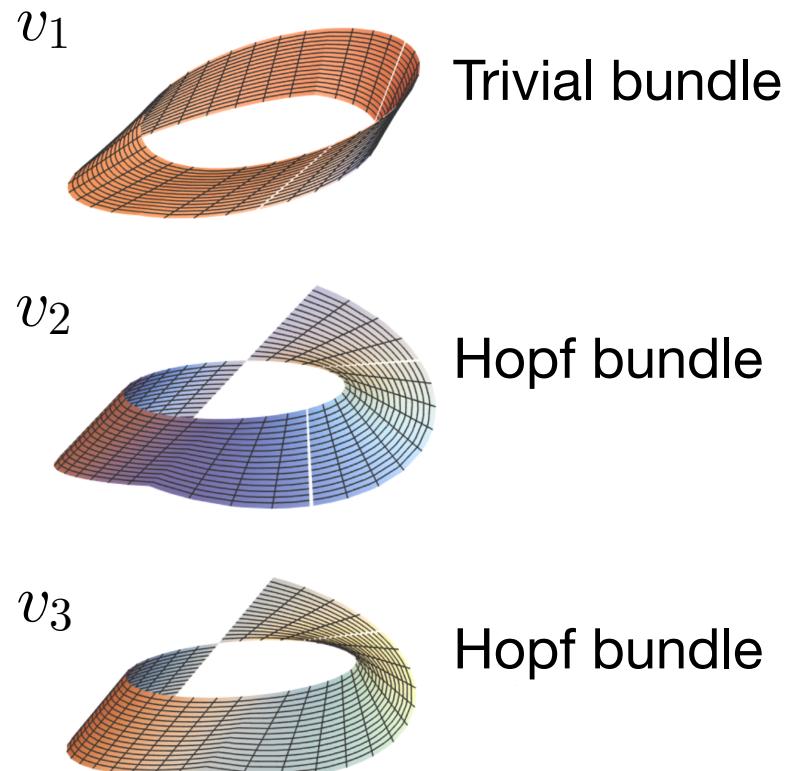
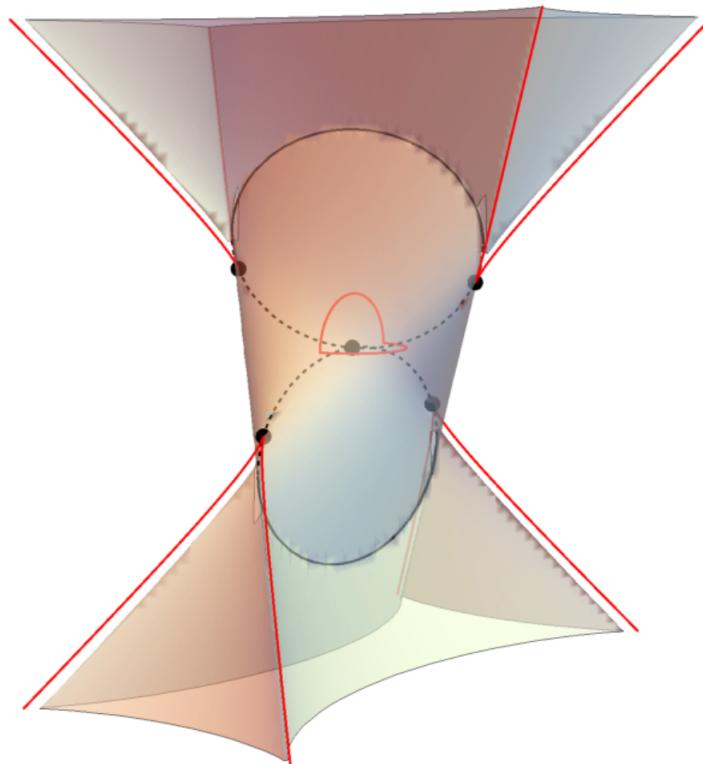
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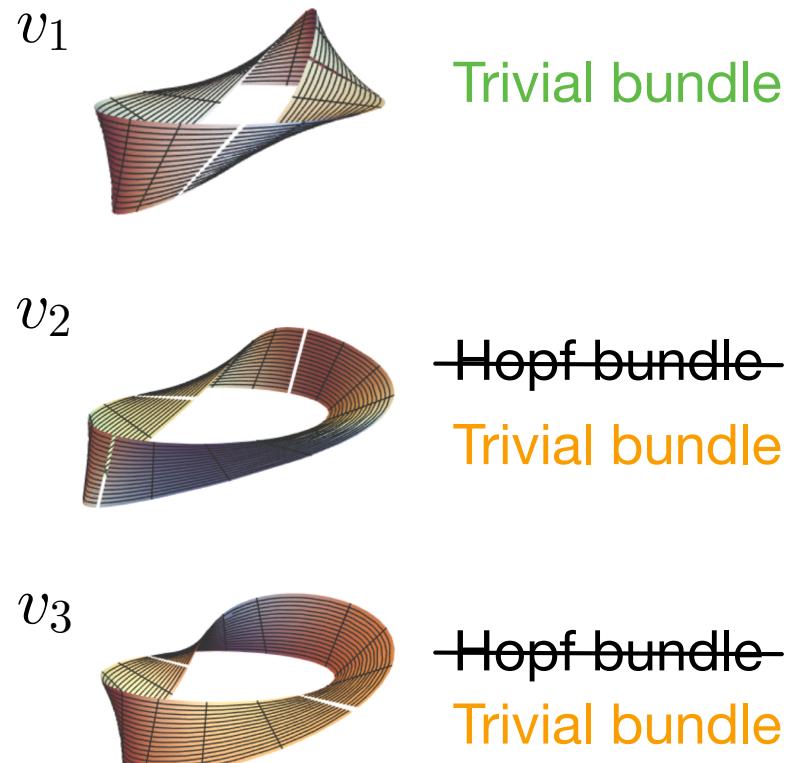
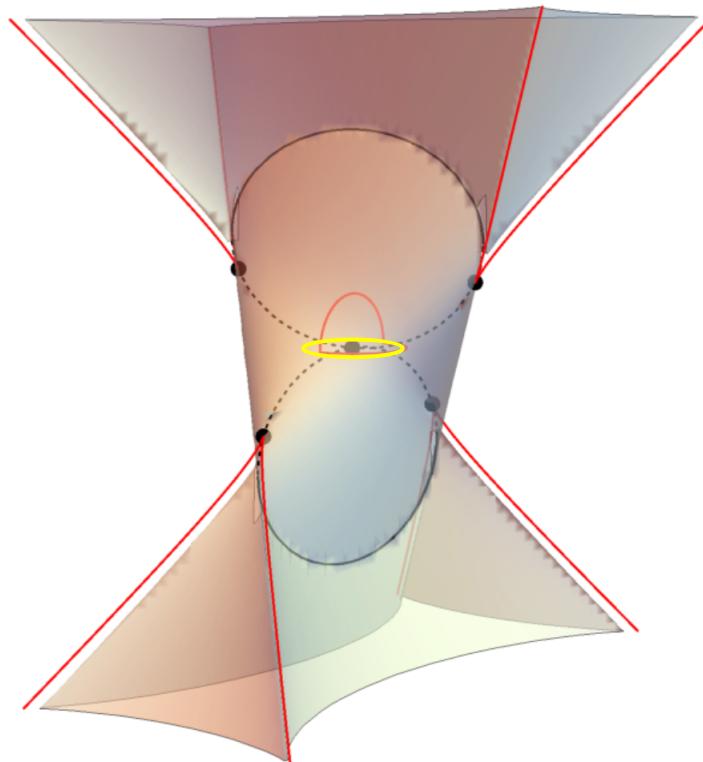


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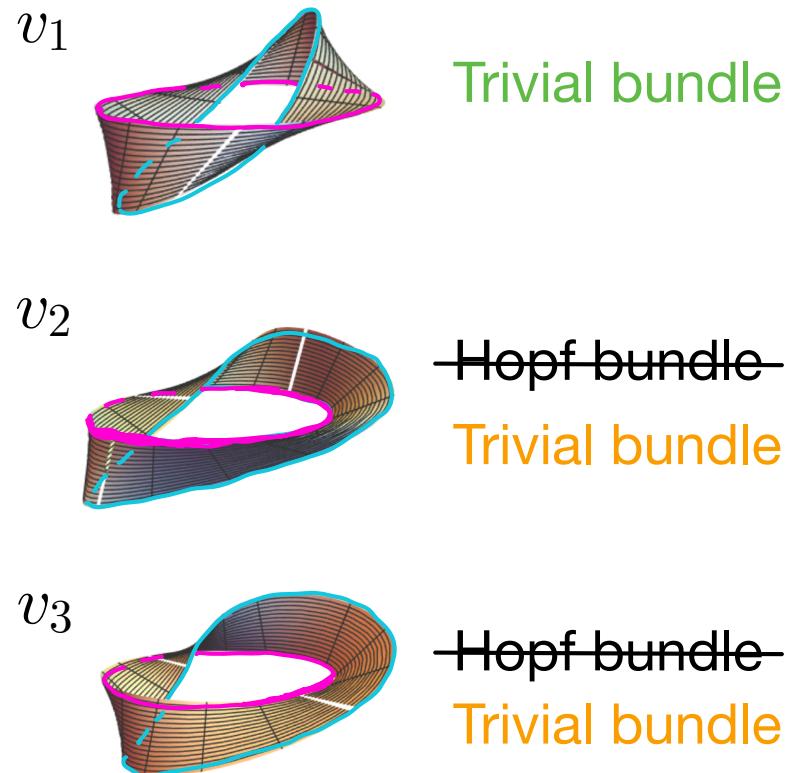
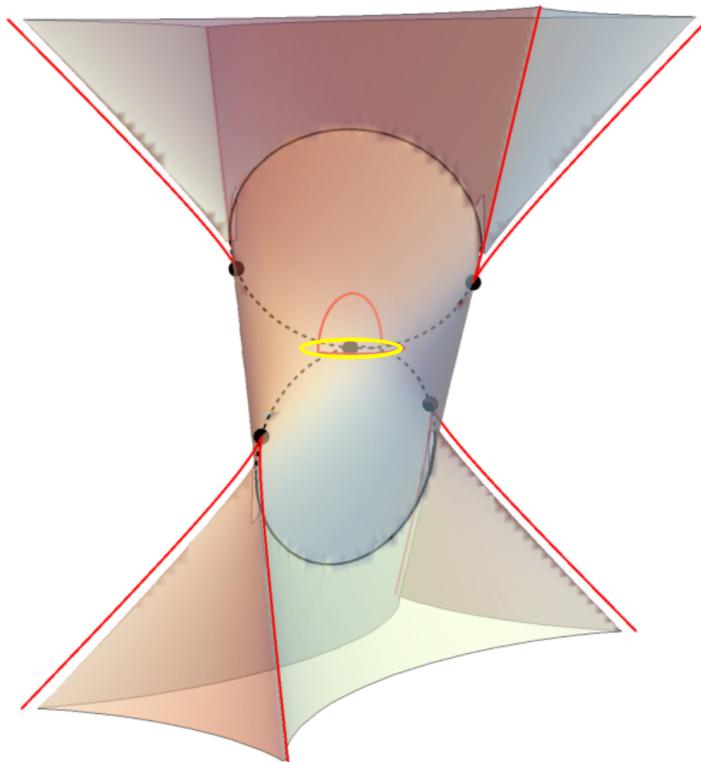


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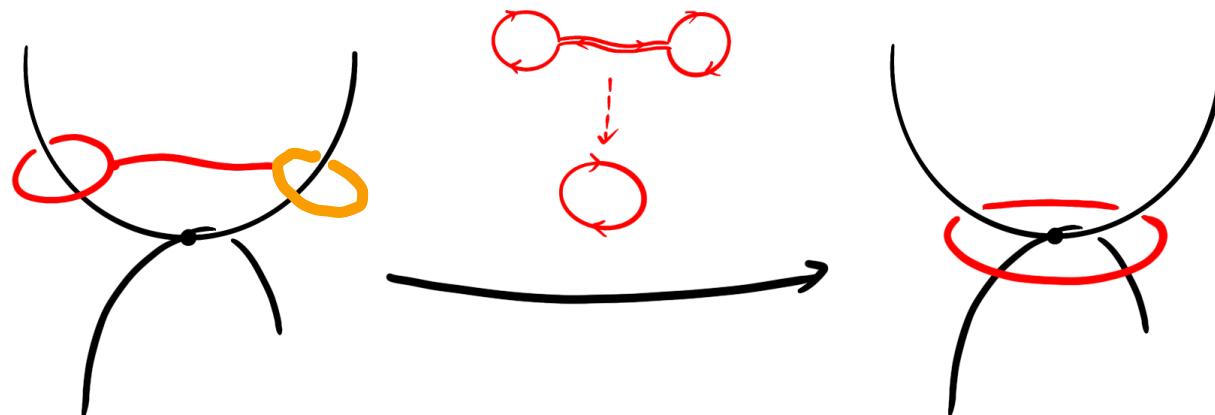
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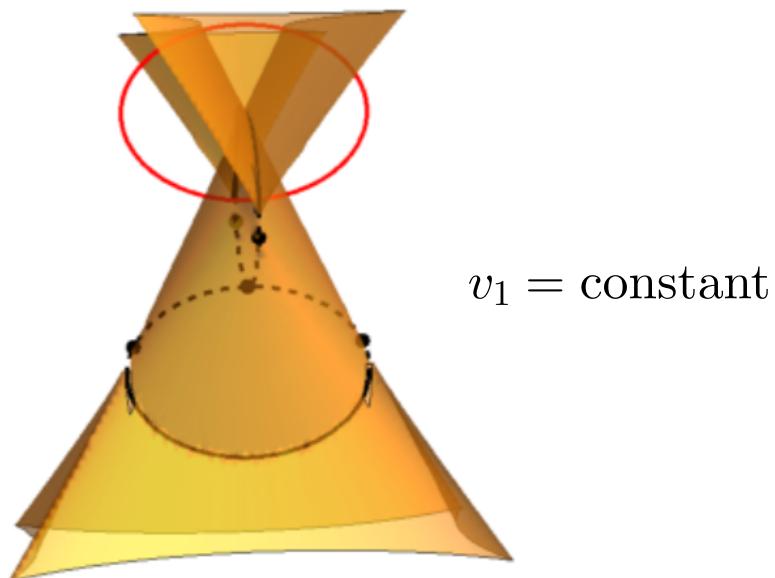


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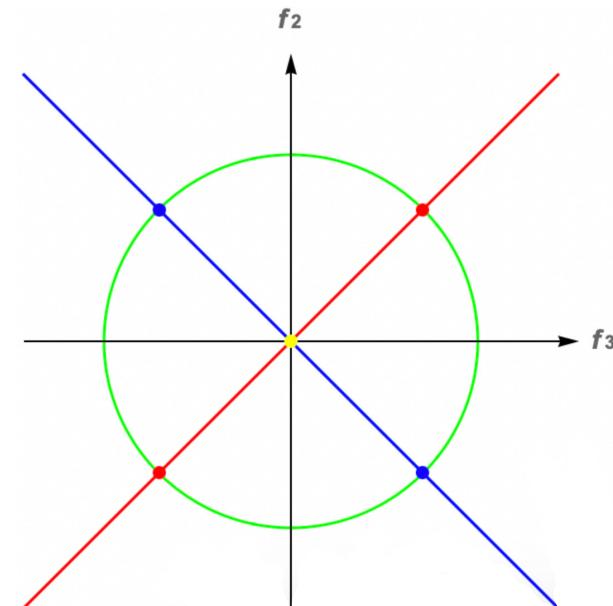
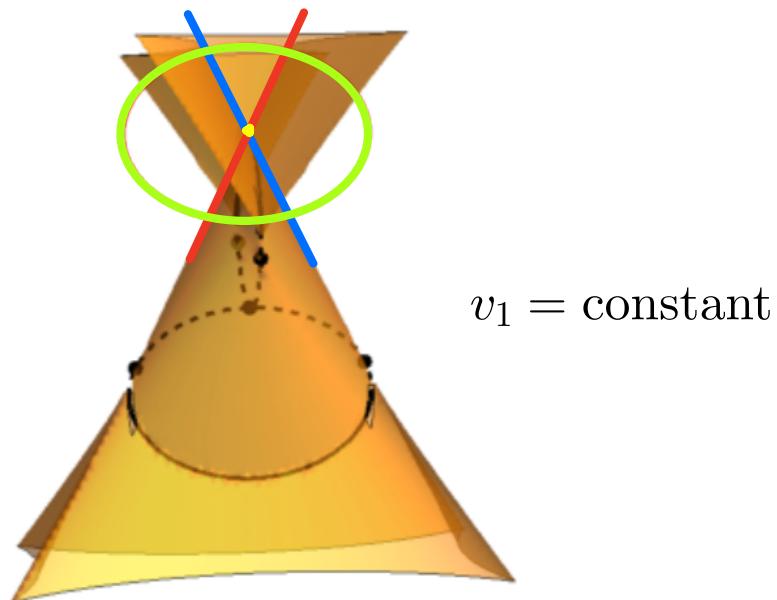


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**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}, \text{ where each } g_i \text{ is a linear function of the parameters } f_j.$$

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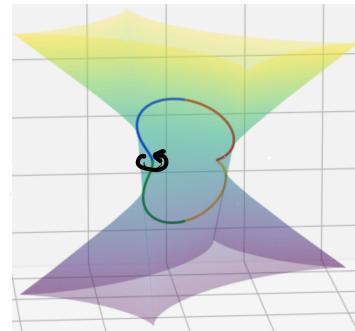
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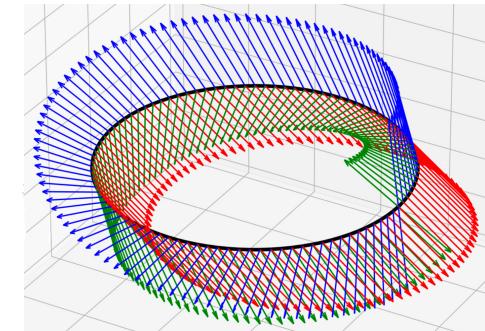
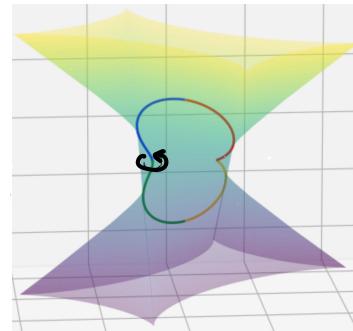
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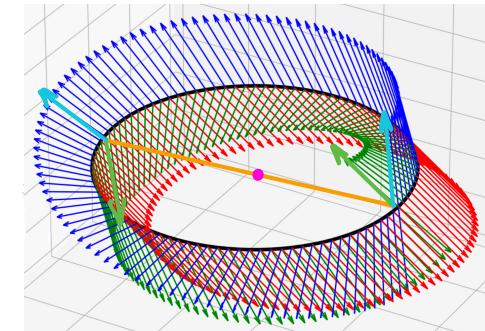
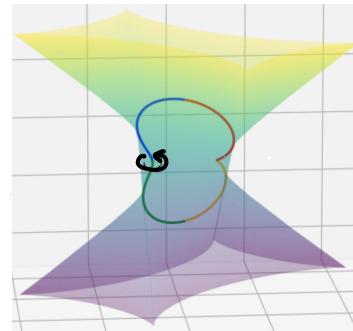
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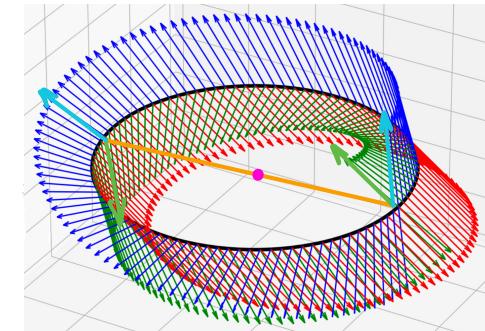
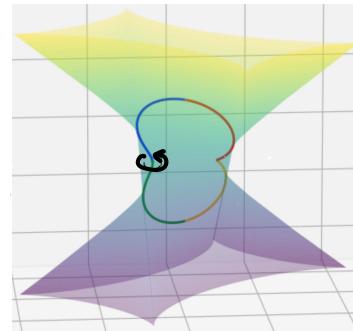
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Visualizing at <https://www.wolframcloud.com/env/zhuwf0/Presentation.nb>

- Opening of 2 tunnels and a new “big” loop around, along which the rank-3 eigenbundle is trivial
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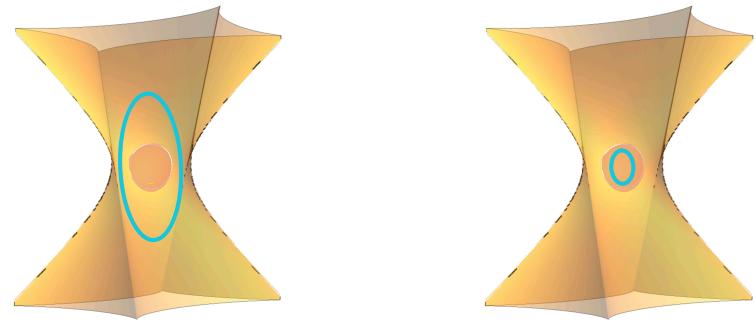
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*Shrinking this loop into the **enclosed region**, we find the eigenbundle along it remains trivial.*

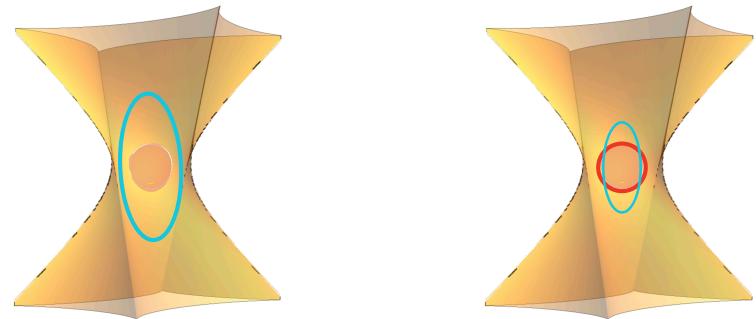
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*What about loops transversing the **nodal intersection lines**? Band inversion again?*

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*tangent developable, along the **cuspidal lines***

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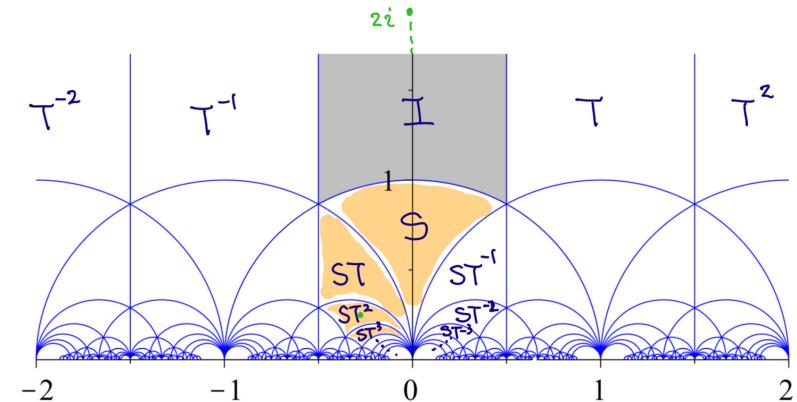
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A prototypical 2D hyperbolic lattice with a *straight-line* boundary

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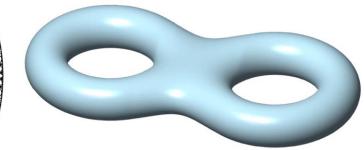
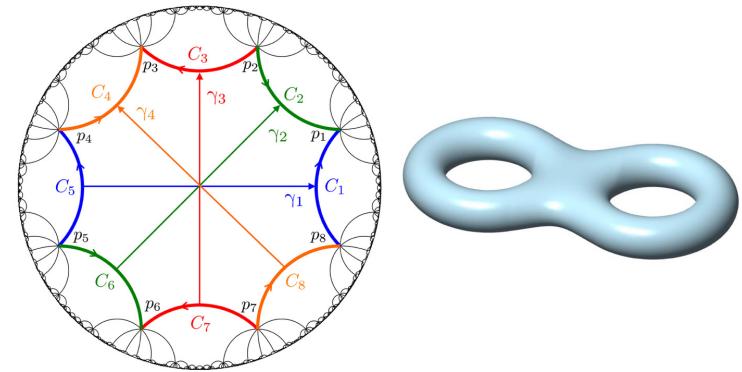
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*Another basic example of a hyperbolic lattice associated to a genus-2 surface*  
(from Maciejko and Rayan, *Hyperbolic band theory*, **Sci. Adv.**, 2021)

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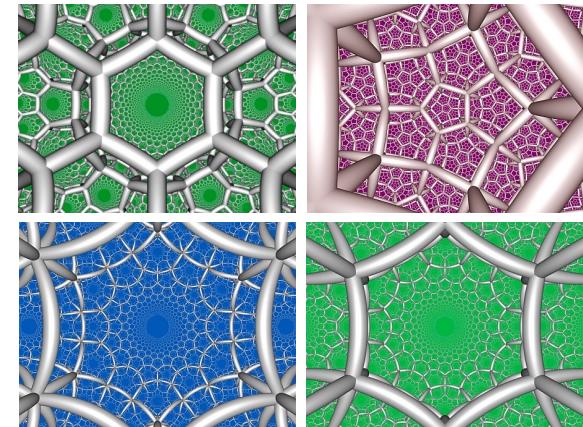
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Four 3D hyperbolic lattices tiling up the hyperbolic 3-space  $\mathbb{H}^3$  (from John Baez's blog)

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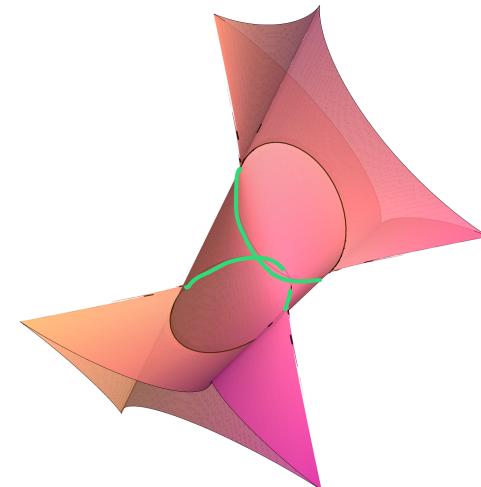
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Existence of **nodal curves** inside also gives evidence, supporting nontrivial loops around (generating a free group on 3 letters) acting on a 3D hyperbolic lattice.

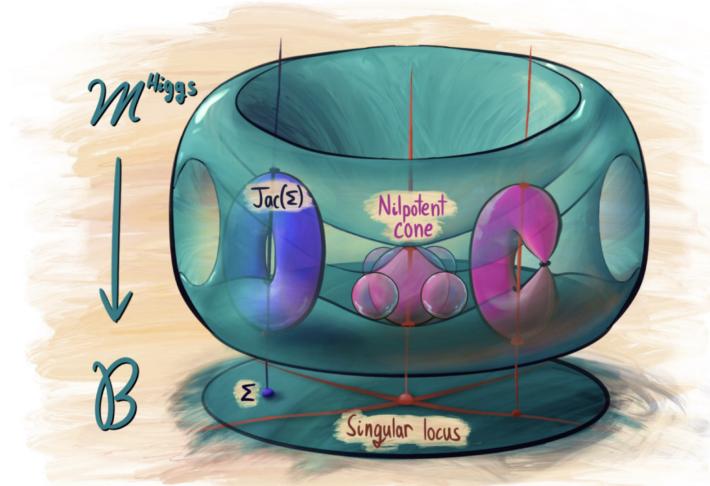


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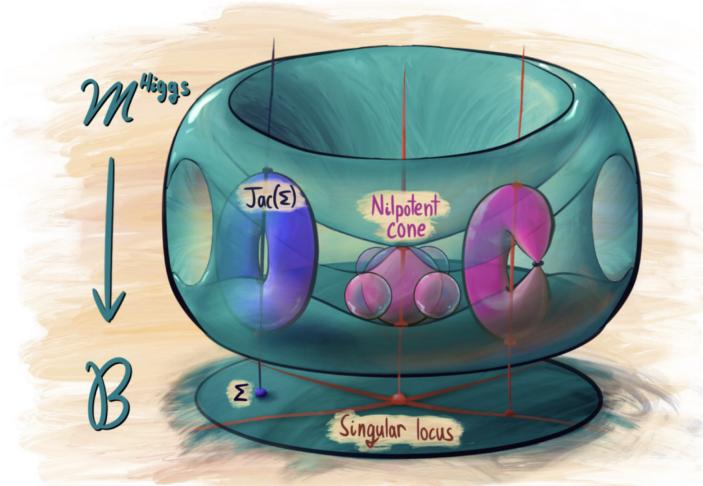


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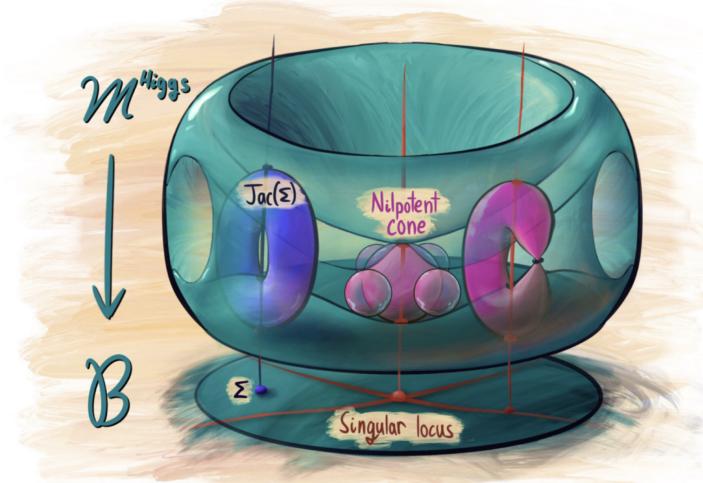


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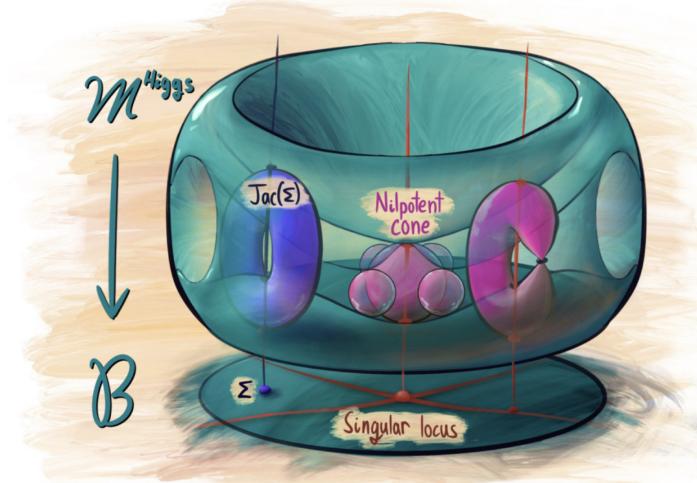


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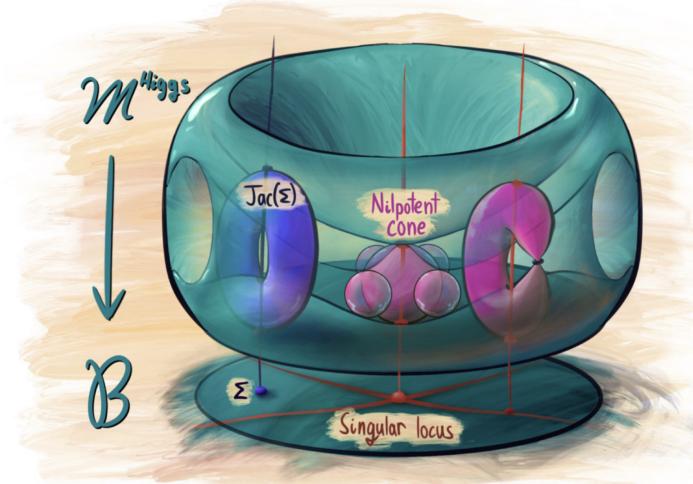
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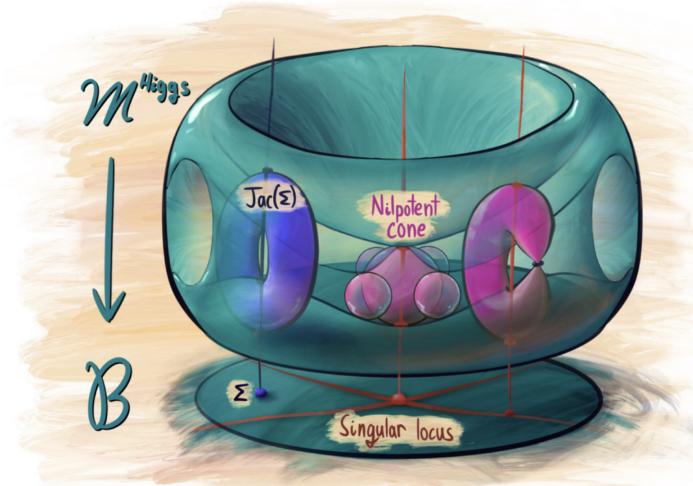
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- Nilpotent cone: The most degeneration occurs over  $0 \in \mathcal{B}$ . The fiber  $h^{-1}(0)$  is called the **nilpotent cone**.

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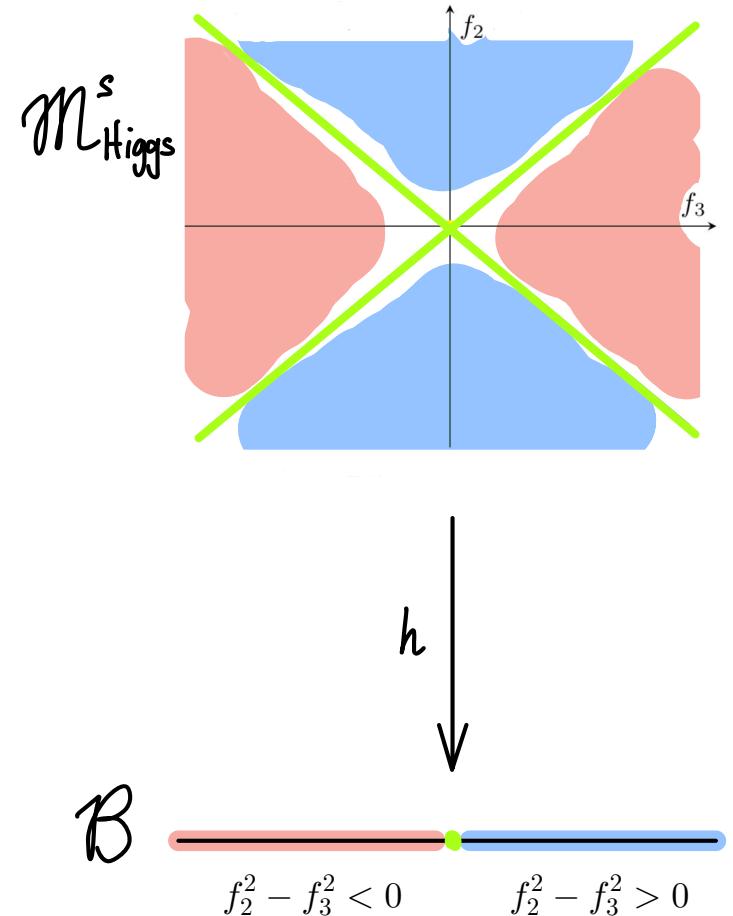
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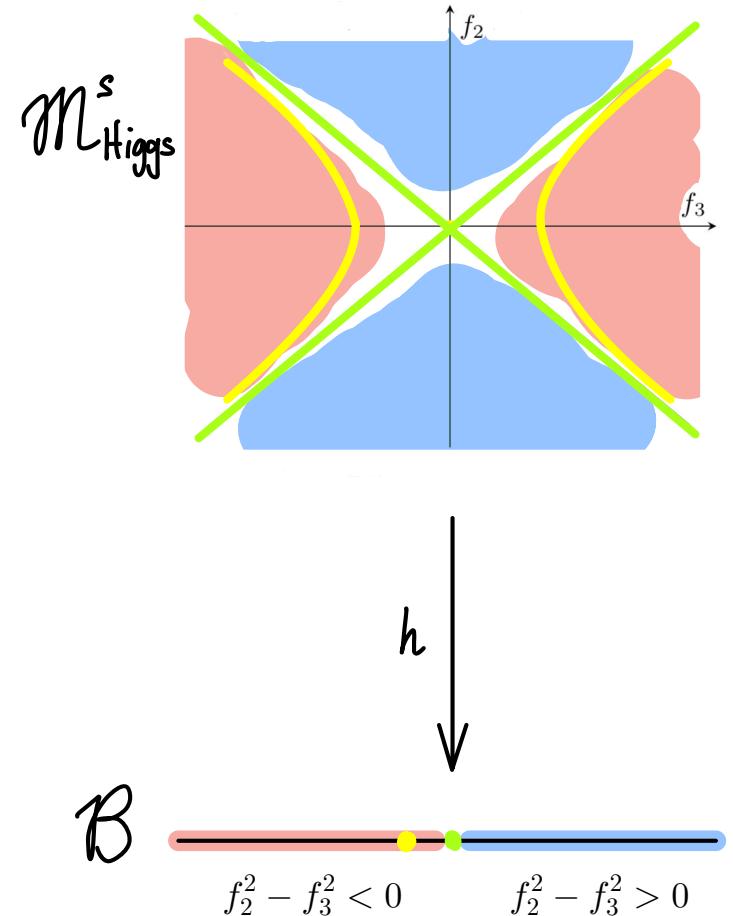
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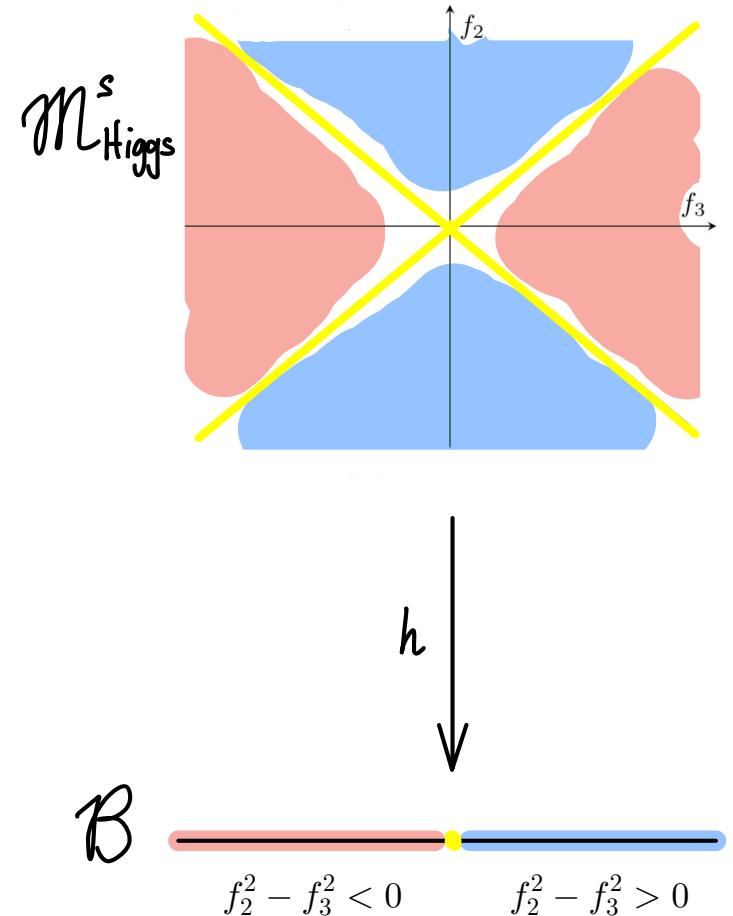
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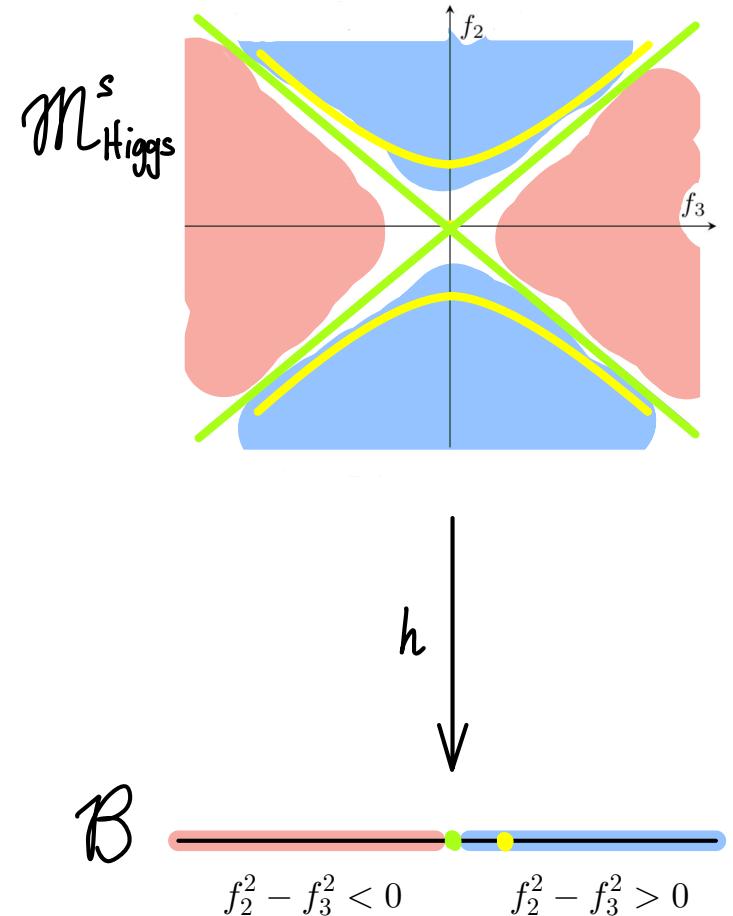
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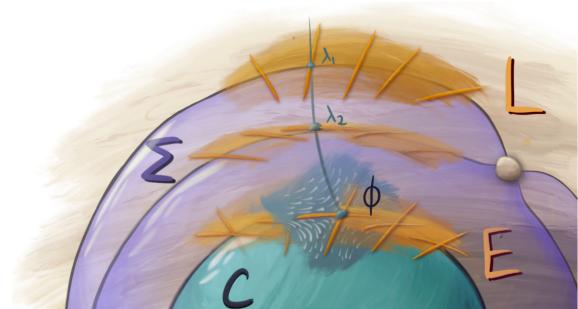
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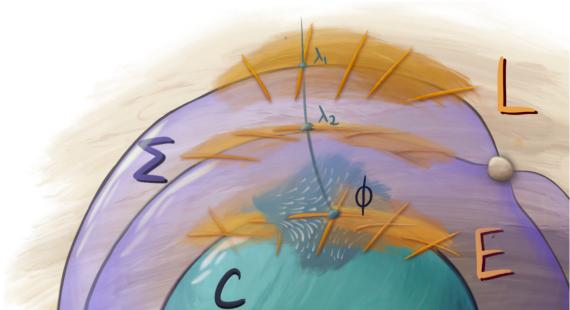
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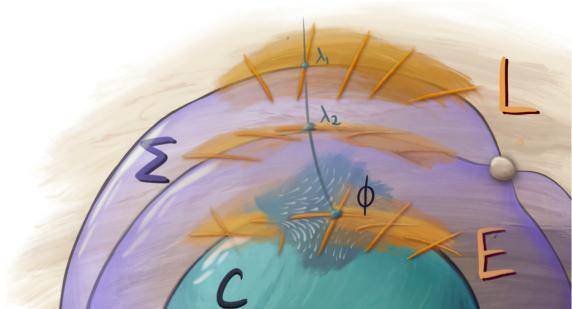
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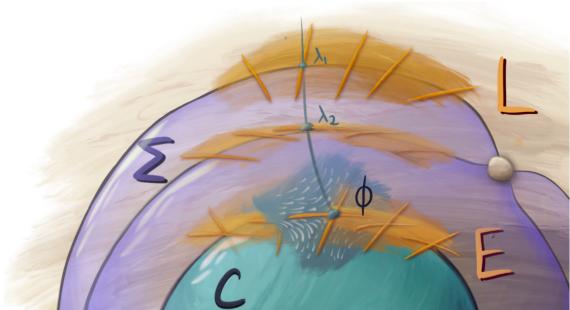
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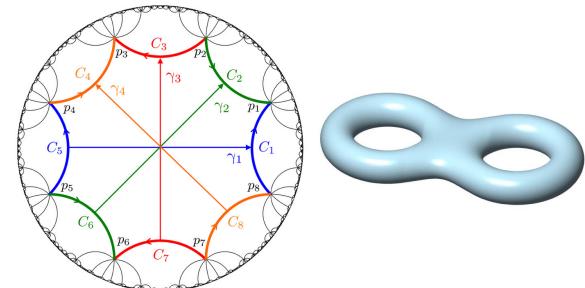
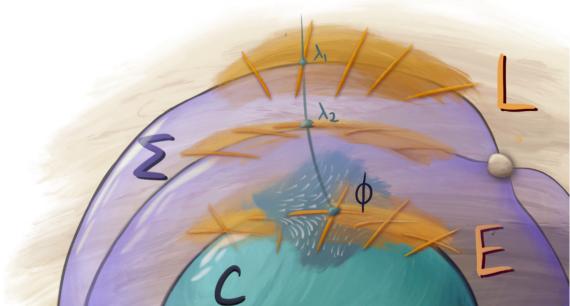
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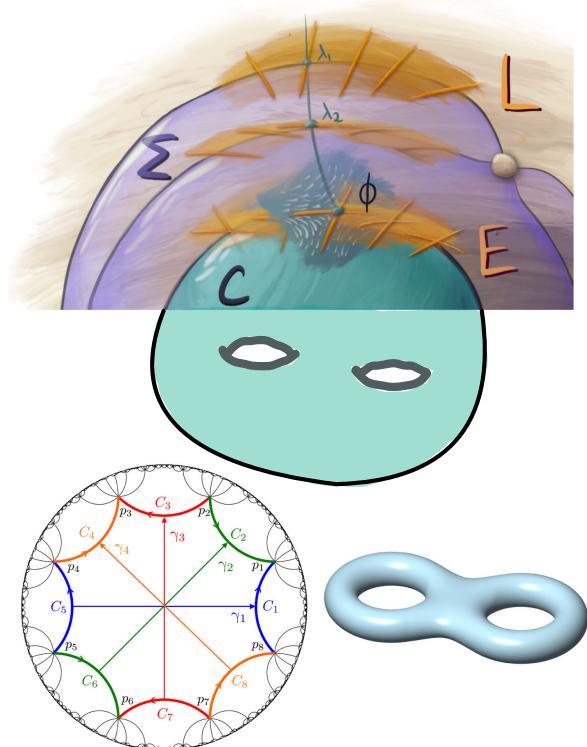
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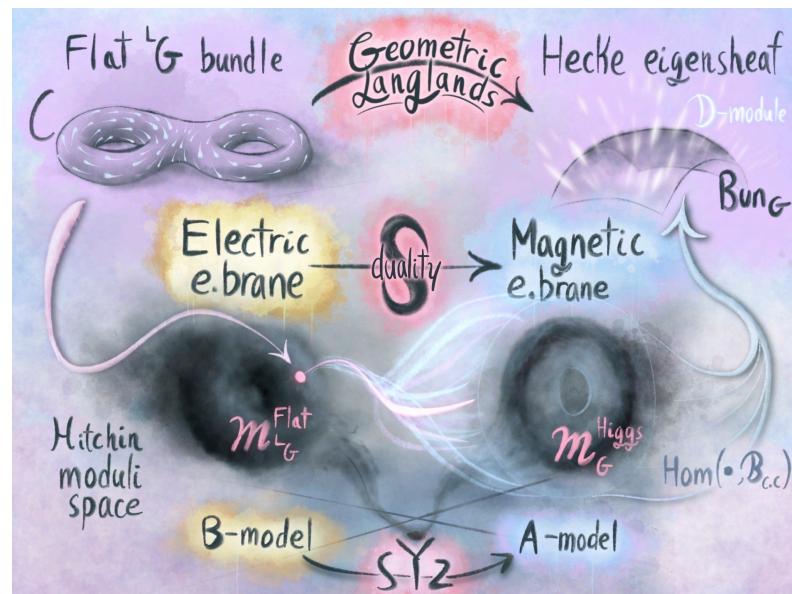
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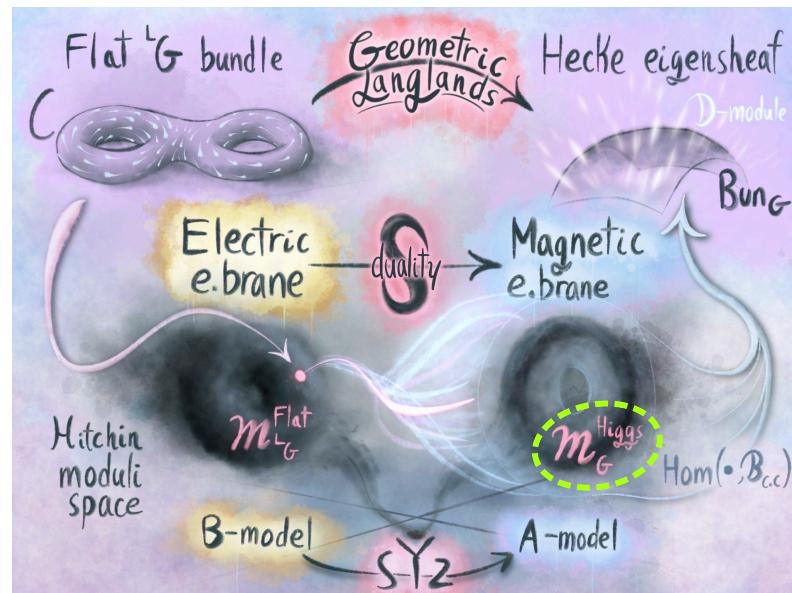
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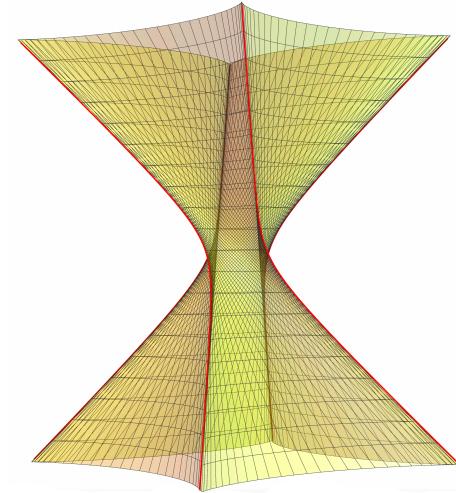
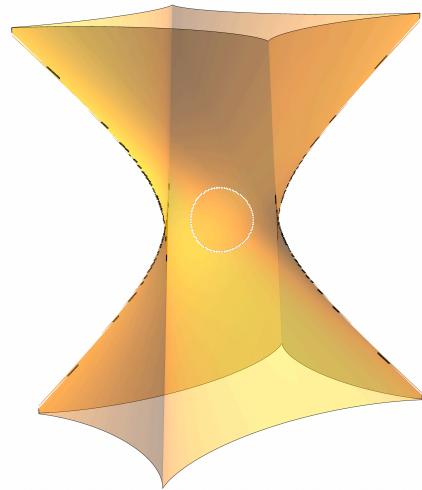
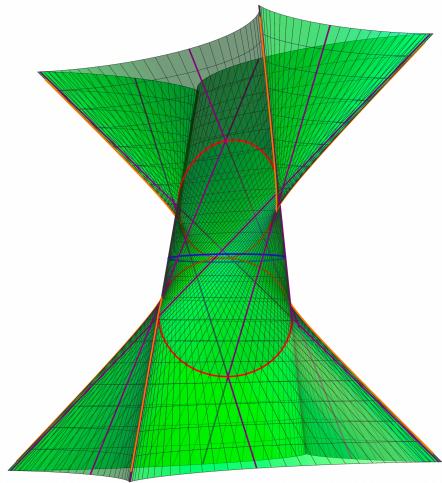


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*Thank you.*

