Assignment 9

1. [Y] Sec. 3.3 #2.
2. [Y] Sec. 3.4 #2.

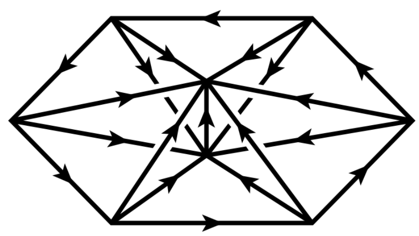
Analogous to the projective plane, the 3-dimensional real projective space $\mathbb{RP}^3$ can be obtained by identifying antipodal points of the 3-sphere S^3, or by identifying antipodal points on the boundary S^2 of the solid 3-ball D^3, or by equipping the set of 1-dimensional subspaces of $\mathbb{R}^4$ with a suitable topology.

3. This last description gives $\mathbb{RP}^3$ the structure of a 3-manifold as follows. Each point of $\mathbb{RP}^3$ has a *homogeneous coordinate* $(x_0 : x_1 : x_2 : x_3)$, i.e., the points (x_0, x_1, x_2, x_3) and $(\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3)$ of $\mathbb{R}^4 - \{(0,0,0,0)\}$ are identified as the same point in $\mathbb{RP}^3$ for any nonzero real number λ.

   Based on this, specify the local charts that cover $\mathbb{RP}^3$, each of which is homeomorphic to $\mathbb{E}^3$, and write down the transition functions on their pairwise overlaps.

In the following, let us give two more descriptions for $\mathbb{RP}^3$, first as a *lens space* (introduced by Tietze) with the structure of a simplicial complex (see, e.g., Section 3.2 of the reference [B]), second as a quotient space obtained by a *Dehn surgery*.[1]

4. Construct a 3-dimensional simplicial complex from n tetrahedra (i.e., 3-simplices) $T_1, \ldots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i.



   This simplicial complex, or its polytope (geometric realization), is an example of a *lens space*, denoted by $L(n, 1)$.

   (a) Show that $L(2, 1)$ is homeomorphic to $\mathbb{RP}^3$.

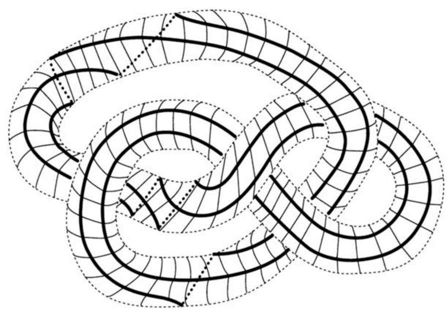
   (b) Calculate the Euler characteristic of $\mathbb{RP}^3$ by carefully enumerating the simplices of $L(2, 1)$.

[1] The figures are copied from Allen Hatcher's *Algebraic topology* and John Luecke's *Dehn surgery on knots in the 3-sphere*. The descriptions below are adapted in addition from Joshua Evan Greene's *Heegaard Floer homology*.

5. More generally, viewing $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ and given positive integers p, q with $(p, q) = 1$, we can construct the lens space $L(p, q)$ from the periodic homeomorphism $f : S^3 \to S^3, (z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2)$ as the quotient space $S^3/\sim_f$, where $x \sim_f x'$ if and only if the k-fold composite $f^k(x) = x'$ for some k.

   Show that this construction of $L(p, 1)$ gives the same space as Question 4.[2]

In 1910, Dehn devised a general method called *surgery* for constructing 3-manifolds, which can also be carried out in two steps as follows. Dehn's construction begins with a knot $K \subset S^3$, i.e., an embedded $S^1 \hookrightarrow S^3$. A closed tubular neighborhood of K is homeomorphic to a solid torus $S^1 \times D^2$. First we excise its interior from S^3 to produce the knot exterior X_K, a compact manifold with torus boundary. We then obtain a 3-manifold by regluing a solid torus $S^1 \times D^2$ to X_K along their boundaries, in such a way that a curve $\{\theta\} \times \partial D^2$ (i.e., a line of longitude) glues to a curve that wraps p times longitudinally and q times meridionally around K.



The homeomorphism type of the result depends only on K and the slope p/q, and we denote it $K(p/q)$.

6. Let $K = \bigcirc$ be the unknot. Show that $\bigcirc(p/1) \cong L(p, 1)$ and so Dehn's surgery gives yet another way of constructing $\mathbb{RP}^3$, when $p = 2$.

[2]To visualize S^3 from D^3, consider the analogue of S^2 as obtained from D^2 by folding a dumpling, i.e., by identifying pairs of points on the boundary $\partial D^2 = S^1$ that are symmetric along a diameter D^1. It is also helpful to think of S^3 as the union of a pair of linked solid tori, by drilling off a solid cylinder through the north and south poles of D^3.