# SPECTRAL MODULI PROBLEMS FOR LEVEL STRUCTURES AND AN INTEGRAL JACQUET–LANGLANDS DUAL OF MORAVA E-THEORY

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Abstract. Given an $\mathbb{E}_\infty$-ring spectrum $R$, with motivation from chromatic homotopy theory, we define relative effective Cartier divisors for a spectral Deligne–Mumford stack over $\mathrm{Sp\acute{e}t}\, R$ and prove that, as a functor from connective $R$-algebras to topological spaces, it is representable. This enables us to solve various moduli problems of level structures on spectral abelian varieties, overcoming difficulty at primes dividing the level. In particular, we obtain higher-homotopical refinement for finite levels of a Lubin–Tate tower as $\mathbb{E}_\infty$-ring spectra, which generalizes Morava, Hopkins, Miller, Goerss, and Lurie's spectral realization of the deformation ring at the ground level. Moreover, passing to the infinite level and then descending along the equivariantly isomorphic Drinfeld tower, we obtain a Jacquet–Langlands dual to the Morava E-theory spectrum, along with homotopy fixed point spectral sequences computing the homotopy groups of spheres dual to those studied by Devinatz and Hopkins. These serve as potential tools for detecting torsion classes in higher-periodic homotopy types from pro-étale cohomology of $p$-adic general linear groups.

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## 1. Introduction

The stable homotopy category is a central object of study in algebraic topology, and chromatic homotopy theory provides it with an organizing principle through the height filtration for the moduli stack of one-dimensional formal groups [Goe08, Far24]. Structured ring spectra are the most common examples therein, such as $\mathbb{E}_\infty$ and $\mathbb{H}_\infty$ structures encoding homotopy coherence [May77, BMMS86]. In [Lur09a, Lur18b], Lurie uses methods from spectral algebraic geometry, i.e., algebraic geometry over the sphere spectrum $\mathbf{S}$, to give a proof for the Goerss–Hopkins–Miller theorem of topological modular forms. Besides applications to elliptic cohomology, Lurie also proves the $\mathbb{E}_\infty$ structures on Morava E-theory spectra [Lur18b], which relies on a spectral version of deformation theory for certain $p$-divisible groups.

The earlier proof of $\mathbb{E}_\infty$ structures on E-theories was due to Goerss, Hopkins, and Miller [GH04]. They turned the problem into a moduli problem of realizations for commutative algebras in certain comodules as $\mathbb{E}_\infty$-ring spectra, and developed an obstruction theory, completing the proof by computing André–Quillen cohomology groups. Compared with their methods, Lurie's proof is more conceptual, realizing the chromatic point of view with suitable tools from higher category theory. There have been more and more applications of spectral algebraic geometry in algebraic topology, such as topological automorphic forms [BL10], Morava E-theories over any $\mathbf{F}_p$-algebra (not just perfect fields of characteristic $p$) [Lur18b], equivariant topological modular forms [GM23, BM25], and elliptic Hochschild homology [ST23], among others. (Meanwhile, the Goerss–Hopkins obstruction theory has also been generalized to higher categorical settings [PV22, MG24], but see the historical footnote 3 of [MG24].)

In classical algebraic geometry, moduli problems concerning deformations of formal groups with level structure are well studied. In particular, moduli spaces for all levels form a Lubin–Tate tower [RZ96, FGL08, SW13]. We know from the Goerss–Hopkins–Miller–Lurie theorem that the universal objects of deformations of formal groups have higher-homotopical analogues which are the Morava E-theories. A natural question is then: what can higher-homotopical analogues be for moduli problems of deformations *with level structure*? Moreover, can we find such analogues for Lubin–Tate towers? Although $\mathbb{E}_\infty$ structures on topological modular forms with level structure have been studied in [HL16], even functorially over the compactified moduli stacks through logarithmic structures, we aim to find derived stacks of spectral elliptic curves with level structure that allow the level to not be inverted therein, as opposed to the properties of étale or log-étale required by earlier methods using (logarithmic) obstruction theory. Besides, in the computation of chromatic unstable homotopy groups of spheres, after applying the EHP sequence and Bousfield–Kuhn functor, we observe that certain terms on the $E_2$-page of a homotopy fixed point spectral sequence arise from universal deformations of isogenies of formal groups. These are computed through the Morava E-theory on classifying spaces of symmetric groups [Str97, Str98]. They can be suitably interpreted as sheaves on a Lubin–Tate tower. We aim to provide a more conceptual perspective on this fact via the proposed higher-homotopical Lubin–Tate tower and its geometry.

In this paper, we address these questions by studying specific moduli problems in spectral algebraic geometry. A main ingredient of our work is the derived analogue of Artin's representability theorem established in [Lur04, TV08]. We will apply the spectral algebraic geometry version from [Lur18c]. Specifically, we define and study relative effective Cartier divisors in this context. By imposing certain conditions, we further define derived level structures on relevant geometric objects in spectral algebraic geometry. Using the spectral Artin representability theorem, we prove representability results for moduli problems that arise from our derived level structures. Moreover, we give a range of applications involving derived level structures. In particular, we consider the moduli problem of spectral deformations with derived level structures on $p$-divisible groups. We prove that these moduli problems are representable by certain formal affine spectral Deligne–Mumford stacks, and the corresponding $\mathbb{E}_\infty$-ring spectra provide us with interesting generalized cohomology theories.

*Remark* 1.1. We note that the Goerss–Hopkins–Miller–Lurie sheaf of $\mathbb{E}_\infty$-ring spectra does not directly apply to the moduli problems we consider here, due to the latter's failure of étaleness over the base moduli problems without level structure (cf. the moduli problem associated with quasimodular forms considered in [Dev23] and see also Remarks 4.23, 4.25, and 4.15 below). This is fixed by relative effective Cartier divisors analogous to Drinfeld's original approach to arithmetic moduli of (classical) elliptic curves (see [KM85, Introduction] for historical remarks). See also Remark 1.3 below for another approach.

As a concrete form of the above ramification phenomena, homotopy theorists and researchers working in related fields have been aware of what we might call the "curse of roots of unity" in $K(h)$-local stable homotopy categories, where the prime number underlying the Morava K-theory spectrum $K(h)$ divides the exponents of those roots of unity. This dates back to Schwänzl, Vogt, and Waldhausen's work on realization problems through topological Hochschild homology (implicitly in the case of $h = 1$) in the late 1990s (cf. [SVW99, Dev20, Sch21, Sal23]). We hope that the current paper further probes this delicate point towards clarity in understanding arithmetic and $K(h)$-local phenomena.

**Notation and terminology.**

- Let CAlg denote the $\infty$-category of $\mathbb{E}_\infty$-rings and $\mathrm{CAlg}^{\mathrm{cn}}$ denote the $\infty$-category of connective $\mathbb{E}_\infty$-rings.
- Let $\mathcal{S}$ denote the $\infty$-category of spaces ($\infty$-groupoids).
- Given a spectral Deligne–Mumford stack $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$, let $\tau_{\leqslant n}\mathsf{X}$ denote its $n$-truncation $(\mathcal{X}, \tau_{\leqslant n}\mathscr{O}_{\mathcal{X}})$ and $\mathsf{X}^{\heartsuit}$ denote its underlying ordinary stack $(\mathcal{X}^{\heartsuit}, \tau_{\leqslant 0}\mathscr{O}_{\mathcal{X}})$.
- By a spectral Deligne–Mumford stack $\mathsf{X}$ over an $\mathbb{E}_\infty$-ring $R$, we mean a morphism of spectral Deligne–Mumford stacks $\mathsf{X} \to \operatorname{Spét} R$. Given an $R$-algebra $S$, we sometimes write $\mathsf{X} \times_R S$ for the fiber product $\mathsf{X} \times_{\operatorname{Spét} R} \operatorname{Spét} S$.
- Let $\mathcal{M}_{\mathrm{Ell}}$ denote the spectral Deligne–Mumford stack of spectral elliptic curves, as defined in [Lur18a], and $\mathcal{M}^{\mathrm{cl}}_{\mathrm{Ell}}$ denote the (classical) Deligne–Mumford stack of (classical) elliptic curves.

**Overview and statement of results.** Throughout the paper, we work in the $\infty$-categorical setting of spectral algebraic geometry after Lurie, while referencing and comparing with classical moduli problems as treated in, e.g., [KM85, Kol96].

In Section 2, we define derived isogenies between spectral elliptic curves and prove that the kernel of a derived isogeny in some cases has properties as in the classical case. This provides evidence that our derived versions of level structures must induce classical level structures. For representability reasons, we use moduli associated with sheaves to detect higher homotopy of derived versions of level structures.

As a main tool, we define relative effective Cartier divisors. Given a spectral Deligne–Mumford stack $\mathsf{X}$ over a spectral Deligne–Mumford stack $\mathsf{S}$, a relative effective Cartier divisor is a morphism $\mathsf{D} \to \mathsf{X}$ of spectral Deligne–Mumford stacks such that it is a closed immersion, the ideal sheaf of $\mathsf{D}$ is a line bundle over $\mathsf{X}$, and the morphism is flat, proper, and locally almost of finite presentation (Definition 2.12). Denote by $\mathrm{CDiv}(\mathsf{X}/\mathsf{S})$ the space of such divisors. We use Lurie's representability theorem to prove that relative effective Cartier divisors are representable in certain cases. A major part of our proof involves computing with the associated cotangent complex.

**Theorem A** (Theorem 2.18)**.** *Suppose that $\mathsf{E}$ is a spectral algebraic space over a connective $\mathbb{E}_\infty$-ring $R$, such that $\mathsf{E} \to R$ is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor*

$$\begin{aligned} \mathrm{CDiv}_{\mathsf{E}/R} \colon \mathrm{CAlg}_R^{\mathrm{cn}} &\to \mathcal{S} \\ R' &\mapsto \mathrm{CDiv}(\mathsf{E}_{R'}/R') \end{aligned}$$

*is representable by a spectral algebraic space which is locally almost of finite presentation over $R$.*

*Remark* 1.2. Motivated by questions different from what we consider here, Gregoric introduced a related notion of *extended effective Cartier divisor* in the context of spectral algebraic geometry [Gre21, Definition 2.2.4].

In Section 3, we first define derived level structures for spectral elliptic curves. Roughly speaking, given a finite abelian group $A$, e.g., $\mathbf{Z}/N\mathbf{Z}$ and $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$, a derived level-$A$ structure on a spectral elliptic curve $\mathsf{E}$ over an $\mathbb{E}_\infty$-ring $R$ is just a relative effective Cartier divisor $\mathsf{D} \to \mathsf{E}$ whose restriction to the heart comes from a classical level-$A$ structure (see Remark 3.20 for more details concerning the *derived* nature of such structures). We let $\mathrm{Level}(A, \mathsf{E}/R)$ denote the space of derived level-$A$ structures on $\mathsf{E}/R$ and show that moduli problems associated with such structures are representable relative to $\mathsf{E}/R$.

**Theorem B** (Theorem 3.6)**.** *Suppose that $\mathsf{E}$ is a spectral elliptic curve over a connective $\mathbb{E}_\infty$-ring $R$. Then the functor*

$$\begin{aligned} \mathrm{Level}^A_{\mathsf{E}/R} \colon \mathrm{CAlg}_R^{\mathrm{cn}} &\to \mathcal{S} \\ R' &\mapsto \mathrm{Level}(A, \mathsf{E}_{R'}/R') \end{aligned}$$

*is representable by an affine spectral Deligne–Mumford stack which is locally almost of finite presentation over $R$.*

In classical algebraic geometry, closely related to elliptic curves and more general abelian varieties, we are interested in level structures on $p$-divisible groups.

They can be defined through full sets of sections of commutative finite flat group schemes. In Section 3.2, we consider derived level structures on spectral $p$-divisible groups. Let $\mathrm{Level}(r, \mathbf{G}/R)$ denote the space of derived level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structures on a height-$h$ spectral $p$-divisible group $\mathbf{G}/R$. Again, this moduli problem is relatively representable.

**Theorem C** (Theorem 3.21)**.** *Suppose that $\mathbf{G}$ is a spectral $p$-divisible group of height $h$ over a connective $\mathbb{E}_\infty$-ring $R$. Then the functor*

$$\begin{aligned}\mathrm{Level}^r_{\mathbf{G}/R} \colon \mathrm{CAlg}^{\mathrm{cn}}_R &\to \mathcal{S}\\ R' &\mapsto \mathrm{Level}(r, \mathbf{G}_{R'}/R')\end{aligned}$$

*is representable by an affine spectral Deligne–Mumford stack* $\mathrm{Sp\acute{e}t}\,\mathcal{P}^r_{\mathbf{G}/R}$.

The remaining Sections 4 and 5 give applications of derived level structures of relevance to algebraic geometry, algebraic topology, number theory, and representation theory. We first prove in Section 4.1 that the moduli problem of spectral elliptic curves with derived level-$A$ structure is (absolutely) representable by a spectral Deligne–Mumford stack, based on and generalizing [Lur18a, Theorem 2.4.1] (cf. Remark 4.7).

**Theorem D** (Theorem 4.6)**.** *Let $\mathrm{Ell}^A(R)$ denote the $\infty$-category of spectral elliptic curves with derived level-$A$ structure over a connective $\mathbb{E}_\infty$-ring $R$, with $\mathrm{Ell}^A(R)^\simeq$ the largest Kan complex contained in $\mathrm{Ell}^A(R)$. Then the functor*

$$\begin{aligned}\mathcal{M}^A_{\mathrm{Ell}} \colon \mathrm{CAlg}^{\mathrm{cn}} &\to \mathcal{S}\\ R &\mapsto \mathrm{Ell}^A(R)^\simeq\end{aligned}$$

*is representable by a spectral Deligne–Mumford stack, and this stack is locally almost of finite presentation over the sphere spectrum* $\mathbf{S}$.

*Remark* 1.3. With a motivation of constructing an arithmetically global version of modular-equivariant topological modular forms, Grossman-Naples obtained related results about ramified level structures on *oriented* (strict) elliptic curves over nonconnective bases, via derived isogenies and their moduli [GN25] (cf. [KM85, Corollary 6.8.7] and also Section 2.1 below, esp. Definition 2.5, as well as Remark 4.7). With applications to chromatic homotopy theory in mind, in Sections 4.2.2 and 4.3 for $p$-divisible groups we also consider orientations and nonconnective $\mathbb{E}_\infty$-rings.

In [Lur18b], Lurie considers spectral deformations of classical $p$-divisible groups. As we develop the concept of derived level structures, it is natural to consider the moduli of spectral deformations with derived level structures for certain $p$-divisible groups. Suppose $\mathbf{G}_0$ is a $p$-divisible group of height $h$ over a perfect $\mathbf{F}_p$-algebra $R_0$. For each nonnegative integer $r$, we consider a functor

$$\mathrm{Def}^{\mathrm{or},r}_{\mathbf{G}_0} \colon \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}, \quad R \mapsto \mathrm{Def}^{\mathrm{or}}(r, \mathbf{G}_0, R)$$

where $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ is the $\infty$-category of complete adic $\mathbb{E}_\infty$-rings, and $\mathrm{Def}^{\mathrm{or}}(r, \mathbf{G}_0, R)$ is the $\infty$-category spanned by quadruples $(\mathbf{G}, \alpha, e, \lambda)$ with

- $\mathbf{G}$ a spectral $p$-divisible group over $R$,
- $\alpha$ an equivalence class of $\mathbf{G}_0$-taggings of $\mathbf{G}$,
- $e$ an orientation of the identity component of $\mathbf{G}$, and
- $\lambda \in \mathrm{Level}(r, \mathbf{G}/R)$ a derived level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on $\mathbf{G}$ as above.

Our next main result is the following (absolute) representability.

**Theorem E** (Theorem 4.13)**.** *Let $R^{\mathrm{or}}_{\mathbf{G}_0}$ be the oriented deformation ring of $\mathbf{G}_0/R_0$ from* [Lur18b]*. Then the functor $\mathrm{Def}^{\mathrm{or},r}_{\mathbf{G}_0}$ above is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_{h,r}$, where $\mathcal{JL}_{h,r}$ is an $R^{\mathrm{or}}_{\mathbf{G}_0}$-algebra such that $\pi_0\mathcal{JL}_{h,r}$ is finite over $\pi_0 R^{\mathrm{or}}_{\mathbf{G}_0}$. In particular, $\mathcal{JL}_{h,0} \simeq R^{\mathrm{or}}_{\mathbf{G}_0}$.*

As our notation indicates, the significance of these *Jacquet–Langlands spectra* $\mathcal{JL}_{h,r}$ is that they serve as a higher-homotopical realization of the Lubin–Tate tower for $\mathbf{G}_0/R_0$, whose $\ell$-adic étale cohomology in turn realizes the Jacquet–Langlands correspondence. See Proposition 4.14 and Remark 4.15 for details.

Of particular relevance to algebraic topology, specifically chromatic power operations, we give another example in Section 4.3 of $\mathbb{E}_\infty$-ring spectra obtained by considering moduli of spectral deformations with (non-full) level structure of a $p$-power order subgroup. These spectra can be viewed as topological realizations for Strickland's deformation rings of powers of Frobenius. We note some features here in Remarks 4.23 and 4.25. Here is the main result of this application.

**Theorem F** (Theorem 4.22)**.** *Let $\widehat{\mathbf{G}}_0$ be a height-$h$ formal group over a perfect field $k$ of characteristic $p$, and let $E_h$ be the associated Morava E-theory. For each positive integer $r$, let $A_r$ be the ring from* [Str97, Str98] *classifying deformations of $\widehat{\mathbf{G}}_0/k$ with level-$\Gamma_0(p^r)$ structure, so that*

$$A_r \simeq E_h^0(B\Sigma_{p^r})/\{\textit{transfers}\}$$

*Then there exists an $\mathbb{E}_\infty$-ring $E_{h,r}$ such that $\pi_0 E_{h,r} \simeq A_r$, which depends functorially on $\widehat{\mathbf{G}}_0/k$.*

In Section 5, we construct for each classical $p$-divisible group of height $h$ an $\mathbb{E}_\infty$-ring spectrum $\mathcal{JL}_{h,\infty}$, a Jacquet–Langlands spectrum at infinite level (Definition 5.3 and Proposition 5.4). By taking homotopy fixed points, we get a *Jacquet–Langlands dual* of Morava E-theory (Definition 5.6 and Proposition 5.7).

Below is an illustration. Given a formal group $\widehat{\mathbf{G}}_0$ of height $h$ over a perfect field $k$ of characteristic $p$, there is the diagram on the left, of moduli spaces from arithmetic algebraic geometry:

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "\mathrm{GL}_h(\mathcal{O}_K)"'] \arrow[dr, "\mathcal{O}_D^\times"] & & & & \mathcal{JL}_{h,\infty} \arrow[dl, "\mathrm{GL}_h(\mathbf{Z}_p)"'] \arrow[dr, "\mathbb{G}_h"] & \\ \mathcal{LT}_K & & \mathcal{H} & & E_h & & {}^{\mathrm{L}}E_h \end{tikzcd} \tag{1.1}$$

where $K$ is the maximal unramified extension of the $p$-adic completion of $k$, $\mathcal{LT}_K$ is the Lubin–Tate moduli space of deformations of $\widehat{\mathbf{G}}_0$ over complete local rings with residue field containing $k$, $\mathcal{X}$ is the moduli space of deformations at infinite level, $D$ is the central division algebra over $K$ of invariant $1/h$, and $\mathcal{H}$ is the Drinfeld upper half-space of dimension $h-1$. The left diagram naturally lifts to the right diagram of $\mathbb{E}_\infty$-ring spectra (or rather, spectral moduli spaces, as we keep the directions of the arrows). Here, $E_h$ is the Morava E-theory spectrum associated to $\widehat{\mathbf{G}}_0/k$, $\mathbb{G}_h = \mathrm{Aut}_k(\widehat{\mathbf{G}}_0) \rtimes \mathrm{Gal}(k/\mathbf{F}_p)$ is a Morava stabilizer group, and ${}^{\mathrm{L}}E_h$ the Jacquet–Langlands dual of $E_h$ from Definition 5.6. Given this higher-homotopical realization of moduli spaces, we deduce a consequence of particular relevance to structure and calculations in chromatic homotopy theory.

**Theorem G** (Proposition 5.9)**.** *Let $\mathbf{S}_{K(h)}$ be the Bousfield localization of the sphere spectrum with respect to the Morava K-theory $K(h)$. Then there is a natural strongly convergent spectral sequence*

$$E_2^{s,t} = H_c^s\big(\mathrm{GL}_h(\mathbf{Z}_p), \pi_t{}^{\mathrm{L}}E_h\big) \implies \pi_{t-s}\mathbf{S}_{K(h)} \tag{1.2}$$

*whose $E_2^{s,t}$-term is the s'th continuous cohomology of $\mathrm{GL}_h(\mathbf{Z}_p)$ with coefficients the profinite $\mathrm{GL}_h(\mathbf{Z}_p)$-module $\pi_t{}^{\mathrm{L}}E_h$. Moreover, it fits naturally into a commutative diagram of spectral sequences*

$$\begin{tikzcd}[column sep=small]
 & H_c^s\big(\mathrm{GL}_h(\mathbb{Z}_p) \times \mathbb{G}_h, \pi_t\mathcal{JL}_{h,\infty}\big) \arrow[dl, Rightarrow, "\textit{Lubin–Tate tower}"'] \arrow[dr, Rightarrow, "\textit{Drinfeld tower}"] & \\
H_c^r(\mathbb{G}_h, \pi_{t-s}E_h) \arrow[dr, Rightarrow, "{[DH04]}"'] & & H_c^r\big(\mathrm{GL}_h(\mathbf{Z}_p), \pi_{t-s}{}^{\mathrm{L}}E_h\big) \arrow[dl, Rightarrow, "(1.2)"] \\
 & \pi_{t-s-r}\mathbf{S}_{K(h)} &
\end{tikzcd}$$

*where the colors indicate the abutments of the corresponding homotopy fixed point spectral sequences with respect to the action of a profinite group.*

We discuss further consequences and implications in Remark 5.10, as well as in Sections 5.3 and 5.4, where we list questions for investigation. These include algebraic calculations at height one for odd primes, and connections to cyclotomic extensions, to spectral algebra models of unstable chromatic homotopy theory, to representations of Hecke algebras, to categorical Langlands correspondence. A first question is the following.

**Question 1.4.** What are higher homotopy groups of the spectral moduli $\mathbb{E}_\infty$-rings from Theorems E and F above, including the finite-level and infinite-level Jacquet–Langlands spectra?

These homotopy groups should encode refined arithmetic algebro-geometric information. As a point of entry, we give some observations and calculations at chromatic height 1 in Examples 4.11, 4.26, and Section 5.3. It is relevant to compute the (co)tangent complex of the corresponding moduli problems (see, e.g., [MPR24] and cf. [BR20a]).

**Conjecture 1.5.** Hypothesis and notations as in Theorem F, there is a spectral sequence

$$E_2^{s,t} = H^s(\pi_* E_h, \pi_{*+t}L_{E_{h,r}/E_h}) \implies \pi_{t-s}E_{h,r}$$

where

$$L_{E_{h,r}/E_h} \simeq \mathrm{TAQ}^{E_h}(E_{h,r})$$

is the (topological) cotangent complex in [Lur18c] along the $\mathbb{E}_\infty$-ring map $E_h \to E_{h,r}$ as a globalization of topological André–Quillen homology (TAQ).

## 2. Effective Cartier divisors of spectral Deligne–Mumford stacks

A main construct of this paper concerns derived level structures. We begin with a derived version of isogenies and prove that, in certain cases, the kernel of a derived isogeny behaves similarly as in the classical setting. This gives evidence that our derived version of level structures must induce classical level structures. In Section 2.2, we define relative effective Cartier divisors in the setting of spectral algebraic geometry. We then use Lurie's representability theorem to prove that certain functors associated with relative effective Cartier divisors are representable by spectral Deligne–Mumford stacks. This paves the way for Section 3, where we establish specifically the representability of derived level structures for spectral elliptic curves and spectral $p$-divisible groups.

2.1. **Isogenies of spectral elliptic curves.** To define derived level structures, the first question we must address is what higher-categorical analogues of finite abelian groups are. Let us recall from [Lur17, Section 7.2.4] and [Lur18c, Section 2.7] some finiteness conditions in the context of $\mathbb{E}_\infty$-rings.

Given an $\mathbb{E}_\infty$-ring $A$ and an $A$-module $M$, we say that $M$ is

- *perfect*, if it is a compact object of the $\infty$-category $\mathrm{LMod}_A$ of left $A$-modules;
- *almost perfect*, if there exists an integer $k$ such that $M \in (\mathrm{LMod}_A)_{\geqslant k}$ and $M$ is an almost compact object of $(\mathrm{LMod}_A)_{\geqslant k}$, that is, $\tau_{\leqslant n} M$ is a compact object of $\tau_{\leqslant n}\big((\mathrm{LMod}_A)_{\geqslant k}\big)$ for all $n \geqslant 0$;
- *perfect to order $n$*, if given any filtered diagram $\{N_\alpha\}$ in $(\mathrm{LMod}_A)_{\leqslant 0}$, the canonical map
$$\varinjlim_{\alpha} \mathrm{Ext}^i_A(M, N_\alpha) \to \mathrm{Ext}^i_A(M, \varinjlim_{\alpha} N_\alpha)$$
is injective for $i = n$ and bijective for $i < n$;
- *finitely $n$-presented*, if $M$ is $n$-truncated and perfect to order $n+1$.

Next we recall finiteness conditions on algebras. We say that a morphism $\phi : A \to B$ of connective $\mathbb{E}_\infty$-rings is

- *of finite presentation*, if $B$ belongs to the smallest full subcategory of $\mathrm{CAlg}_A$ which contains $\mathrm{CAlg}_A^{\mathrm{free}}$ and is stable under finite colimits;
- *locally of finite presentation*, if $B$ is a compact object of $\mathrm{CAlg}_A$;
- *almost of finite presentation*, if $B$ is an almost compact object of $\mathrm{CAlg}_A$;
- *of finite generation to order $n$*, if the following condition holds:

  Let $\{C_\alpha\}$ be a filtered diagram of connective $\mathbb{E}_\infty$-rings over $A$ having colimit $C$. Assume that each $C_\alpha$ is $n$-truncated and that each of the transition maps $\pi_n C_\alpha \to \pi_n C_\beta$ is a monomorphism. Then the canonical map
  $$\varinjlim_{\alpha} \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$$
  is a homotopy equivalence.
- *of finite type*, if it is of finite generation to order 0.

**Proposition 2.1** ([Lur18c, Propositions 2.7.2.1 and 4.1.1.3])**.** *Let $\phi\colon A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. Then the following conditions are equivalent:*

- *The morphism $\phi$ is perfect to order 0 (resp. of finite type).*
- *The commutative ring $\pi_0 B$ is finite (resp. of finite type) over $\pi_0 A$.*

**Definition 2.2** (cf. [Lur18c, Definition 4.2.0.1])**.** Let $f\colon \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne–Mumford Stacks. We say that $f$ is *locally of finite type* (resp. *locally of finite generation to order* $n$, *locally almost of finite presentation*, *locally of finite presentation*) if the following condition holds. Given any commutative diagram

$$\begin{tikzcd} \mathrm{Sp\acute{e}t}\, B \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "f"] \\ \mathrm{Sp\acute{e}t}\, A \arrow[r] & \mathsf{Y} \end{tikzcd}$$

where the horizontal morphisms are étale, the $\mathbb{E}_\infty$-ring $B$ is of finite type (resp. of finite generation to order $n$, almost of finite presentation, locally of finite presentation) over $A$.

**Definition 2.3** ([Lur18c, Definition 5.2.0.1])**.** Let $f\colon (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}})$ be a morphism of spectral Deligne–Mumford stacks. We say that $f$ is *finite* if the following conditions hold:

(1) The morphism $f$ is affine, and
(2) The pushforward $f_*\mathscr{O}_{\mathcal{X}}$ is perfect to order 0 as a $\mathscr{O}_{\mathcal{Y}}$-module.

*Remark* 2.4. By [Lur18c, Example 4.2.0.2], a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne–Mumford stack is locally of finite type if and only if the underlying map of ordinary stacks is locally of finite type in the sense of classical algebraic geometry. Moreover, by [Lur18c, Remark 5.2.0.2], $f$ is finite if and only if the underlying map $f^\heartsuit\colon \mathsf{X}^\heartsuit \to \mathsf{Y}^\heartsuit$ is finite. In particular, when $\mathsf{X}$ and $\mathsf{Y}$ are spectral algebraic spaces, $f$ is finite if and only if $f^\heartsuit$ is finite in the classical sense.

Recall that a morphism $f\colon \mathsf{X} \to \mathsf{Y}$ of spectral Deligne–Mumford stacks is *surjective*, if for every field $k$ and any map $\mathrm{Sp\acute{e}t}\, k \to \mathsf{Y}$, the fiber product $\mathrm{Sp\acute{e}t}\, k \times_{\mathsf{Y}} \mathsf{X}$ is nonempty [Lur18c, Definition 3.5.5.5].

**Definition 2.5.** Let $R$ be a connective $\mathbb{E}_\infty$-ring and $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral abelian varieties over $R$. We call $f$ an *isogeny* if it is finite, flat, and surjective.

**Lemma 2.6.** *Let $f : \mathsf{X} \to \mathsf{Y}$ be an isogeny of spectral abelian varieties. Then $f^\heartsuit\colon \mathsf{X}^\heartsuit \to \mathsf{Y}^\heartsuit$ is an isogeny in the classical sense.*

*Proof.* For ordinary abelian varieties, $f^\heartsuit$ being an isogeny means that it is surjective and its kernel is finite. This is equivalent to $f^\heartsuit$ being finite, flat, and surjective [Mil86, Proposition 7.1]. From Definition 2.5, it is clear that $f^\heartsuit$ is finite and flat. We need only show that $f^\heartsuit$ is surjective.

By definition of surjectivity above for morphisms of spectral Deligne–Mumford stacks, we get a commutative diagram

$$\begin{tikzcd} \mathrm{Sp\acute{e}t}\, k' \arrow[r] \arrow[d] & \mathsf{X} \arrow[d] \\ \mathrm{Sp\acute{e}t}\, k \arrow[r] & \mathsf{Y} \end{tikzcd}$$

The upper horizontal morphism corresponds to a morphism $\mathrm{Sp\acute{e}t}\, k' \to \mathsf{X}^\heartsuit$ by the inclusion–truncation adjunction [Lur18c, Proposition 1.4.6.3]. On underlying topological spaces, this then corresponds to a point $|\mathrm{Sp\acute{e}t}\, k'| \to |\mathsf{X}^\heartsuit|$. It is clear that this point in $|\mathsf{X}^\heartsuit|$ is a preimage of $|\mathrm{Sp\acute{e}t}\, k|$ in $|\mathsf{Y}^\heartsuit|$. Therefore $f^\heartsuit$ is surjective. □

**Lemma 2.7.** *Let $f\colon \mathsf{X} \to \mathsf{Y}$ be an isogeny of spectral elliptic curves over a connective $\mathbb{E}_\infty$-ring $R$. Then $\mathrm{fib}(f)$ exists and is a finite and flat nonconnective spectral Deligne–Mumford stack over $R$.*

*Proof.* By [Lur18c, Proposition 1.4.11.1], finite limits of nonconnective spectral Deligne–Mumford stacks exist, so we can define $\mathrm{fib}(f)$. Let us consider the commutative diagram

$$\begin{tikzcd}
\mathrm{fib}(f) \arrow[r] \arrow[d, "f'"] & \mathsf{X} \arrow[d, "f"] \arrow[ddr] & \\
* \arrow[r] \arrow[drr, "i"] & \mathsf{Y} \arrow[dr] & \\
 & & \mathrm{Sp\acute{e}t}\, R
\end{tikzcd}$$

where the square is a pullback diagram. We find that $\mathrm{fib}(f)$ is over $\mathrm{Sp\acute{e}t}\, R$. By [Lur18c, Remark 2.8.2.6], $f' : \mathrm{fib}(f) \to *$ is flat because it is a pullback of a flat morphism. Clearly $i\colon * \to \mathrm{Sp\acute{e}t}\, R$ is flat, so by [Lur18c, Example 2.8.3.12] (being a flat morphism is a property local on the source with respect to the flat topology), $i \circ f'\colon \mathrm{fib}(f) \to \mathrm{Sp\acute{e}t}\, R$ is flat.

Next we show that $\mathrm{fib}(f)$ is finite over $R$. Since $*$, $\mathsf{X}$, and $\mathsf{Y}$ are all spectral algebraic spaces, so is $\mathrm{fib}(f)$. Moreover, $\mathrm{Sp\acute{e}t}\, R$ is a spectral algebraic space [Lur18c, Example 1.6.8.2]. By Remark 2.4, we need only prove that the underlying morphism is finite. Since the truncation functor is a right adjoint, it preserves limits. Thus we get a pullback diagram

$$\begin{tikzcd}
\mathrm{fib}(f)^\heartsuit \arrow[r] \arrow[d] & \mathsf{X}^\heartsuit \arrow[d] \\
* \arrow[r] & \mathsf{Y}^\heartsuit
\end{tikzcd}$$

So we are reduced to showing that given an isogeny $f^\heartsuit : \mathsf{X}^\heartsuit \to \mathsf{Y}^\heartsuit$ of ordinary abelian varieties over a commutative ring $R$, its kernel is finite over $R$. This is true in classical algebraic geometry [Mil86, Proposition 7.1]. $\square$

**Lemma 2.8.** *Given an integer $N \geqslant 1$, let $f_N : \mathsf{E} \to \mathsf{E}$ be an isogeny of spectral elliptic curves over a connective $\mathbb{E}_\infty$-ring $R$ such that the underlying morphism is the multiplication-by-$N$ map $[N]\colon \mathsf{E}^\heartsuit \to \mathsf{E}^\heartsuit$. Then $\mathrm{fib}(f_N)$ is finite flat of degree $N^2$ in the sense of* [Lur18c, Definition 5.2.3.1]. *Moreover, if $N$ is invertible in $\pi_0 R$, then $\mathrm{fib}(f_N)$ is an étale-locally constant sheaf.*

*Proof.* By [KM85, Theorem 2.3.1], we know that $[N]\colon \mathsf{E}^\heartsuit \to \mathsf{E}^\heartsuit$ is finite locally free of rank $N^2$ in the classical sense. When $N$ is invertible in $\pi_0 R$, its kernel is an étale-locally constant sheaf. Now, from Lemma 2.7, $\mathrm{fib}(f_N)$ is a spectral algebraic space that is finite and flat, and its underlying space $\mathrm{fib}(f_N)^\heartsuit = \ker[N]$ is locally free of rank $N^2$. We need to prove that $\mathrm{fib}(f_N) \to \mathrm{Sp\acute{e}t}\, R$ is locally free of rank $N^2$ in spectral algebraic geometry. Observe that since $\mathrm{fib}(f_N)$ is finite and flat, it is affine. We are thus reduced to proving the above for affines, i.e., $f_N|_{\mathrm{Sp\acute{e}t}\, S}\colon \mathrm{Sp\acute{e}t}\, S \to \mathrm{Sp\acute{e}t}\, R$ is locally free of rank $N^2$ for any affine substack $\mathrm{Sp\acute{e}t}\, S$ of $\mathrm{fib}(f_N)$. This is equivalent to proving that $R \to S$ is locally free of rank $N^2$ in the sense of [Lur18c, Definition 2.9.2.1]. Therefore we need to prove the following:

(1) The ring $S$ is locally free of finite rank over $R$ (by [Lur17, Proposition 7.2.4.20], this is equivalent to saying that $S$ is a flat and almost perfect $R$-module).
(2) For every $\mathbb{E}_\infty$-ring maps $R \to k$ with $k$ a field, the vector space $\pi_0(k \otimes_R S)$ is an $N^2$-dimensional $k$-vector space.

For (1), we know that $\pi_0 S$ is a projective $\pi_0 R$-module and that $S$ is a flat $R$-module, so by [Lur17, Proposition 7.2.2.18], $S$ is a projective $R$-module. By [Lur17, Corollary 7.2.2.9], since $\pi_0 S$ is a finitely generated $\pi_0 R$-module, $S$ is a retract of a finitely generated free $R$-module, and is therefore locally free of finite rank.

For (2), by [Lur17, Corollary 7.2.1.23], since $R$ and $S$ are connective, we have $\pi_0(k \otimes_R S) \simeq k \otimes_{\pi_0 R} \pi_0 S$, which is an $N^2$-dimensional $k$-vector space, as $\pi_0 S$ is a rank-$N^2$ free $\pi_0 R$-module from above.

We next show that if $N$ is invertible in $\pi_0 R$, then $\mathrm{fib}(f_N)$ is a locally constant sheaf. Since $\mathrm{fib}(f_N)$ is a spectral Deligne–Mumford stack, its associated functor of points $\mathrm{fib}(f_N)\colon \mathrm{CAlg}_R \to \mathcal{S}$ is nilcomplete and locally almost of finite presentation. By [KM85, Theorem 2.3.1], $\mathrm{fib}(f_N)|_{\mathrm{CAlg}^\heartsuit_{\pi_0 R}}$ is a locally constant sheaf. The desired result then follows from the lemma below. $\square$

**Lemma 2.9.** *Let $R$ be a connective $\mathbb{E}_\infty$-ring. Let $\mathscr{F} \in \mathcal{S}\mathrm{hv}^{\text{ét}}(\mathrm{CAlg}^{\mathrm{cn}}_R)$ be nilcomplete and locally almost of finite presentation. Suppose that $\mathscr{F}|_{(\mathrm{CAlg}^{\mathrm{cn}}_R)^\heartsuit}$ is a locally constant presheaf. Then $\mathscr{F}$ is a (homotopy) locally constant sheaf (i.e., sheafification of a homotopy-locally constant presheaf).*

*Proof.* Let us choose an étale cover $\{U_i^0\}$ of $\pi_0 R$ such that $\mathscr{F}|_{U_i^0}$ is a constant sheaf for each $i$. By [Lur17, Theorem 7.5.1.11], this corresponds to an étale cover $\{U_i\}$ of $R$ such that $\pi_0 U_i = U_i^0$. For each $i$ and $n$, we consider the diagram

$$\begin{tikzcd} \tau_{\leqslant 0} R \arrow[r] \arrow[d] & \tau_{\leqslant 0} U_i \arrow[d] \\ \tau_{\leqslant n} R \arrow[r] & \tau_{\leqslant n} U_i \end{tikzcd}$$

which is a pushout diagram, since $U_i$ is an étale $R$-algebra. This is a colimit diagram in $\tau_{\leqslant n}\mathrm{CAlg}_R$. Since $\mathscr{F}$ is a sheaf locally almost of finite presentation, we then get a pushout diagram

$$\begin{tikzcd} \mathscr{F}(\tau_{\leqslant 0} R) \arrow[r] \arrow[d] & \mathscr{F}(\tau_{\leqslant 0} U_i) \arrow[d] \\ \mathscr{F}(\tau_{\leqslant n} R) \arrow[r] & \mathscr{F}(\tau_{\leqslant n} U_i) \end{tikzcd}$$

Without loss of generality, we may assume that each $U_i$ is connective. Thus the values $\mathscr{F}(\tau_{\leqslant 0} U_i)$ is independent of $i$. This implies that $\mathscr{F}(\tau_{\leqslant n} U_i)$ are all equivalent. Since $\mathscr{F}$ is nilcomplete, we have $\mathscr{F}(U_i) \simeq \varinjlim_n \mathscr{F}(\tau_{\leqslant n} U_i)$, and so all $\mathscr{F}(U_i)$ are equivalent. $\square$

2.2. **Cartier divisors and an exercise of spectral Artin representability.** In this subsection, we define relative effective Cartier divisors in the context of spectral algebraic geometry. We then use Lurie's spectral Artin representability theorem to prove that relative effective Cartier divisors are representable in certain cases. Let us first recall this spectral analogue of Artin's representability criterion in classical algebraic geometry [Lur18c, Theorem 18.3.0.1].

**Theorem 2.10** (Lurie)**.** *Let $X\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Suppose that we have a natural transformation $f\colon X \to \operatorname{Spec} R$, where $R$ is a Noetherian $\mathbb{E}_\infty$-ring with $\pi_0 R$ a Grothendieck ring. Given $n \geqslant 0$, $X$ is representable by a spectral Deligne–Mumford $n$-stack which is locally almost of finite presentation over $R$ if and only if the following conditions are satisfied:*

1. *For every discrete commutative ring $A$, the space $X(A)$ is $n$-truncated.*
2. *The functor $X$ is a sheaf for the étale topology.*
3. *The functor $X$ is nilcomplete, infinitesimally cohesive, and integrable.*
4. *The functor $X$ admits a connective cotangent complex $L_X$.*
5. *The natural transformation $f$ is locally almost of finite presentation.*

Given a locally spectrally ringed topos $\mathsf{X} = (\mathcal{X}, \mathscr{O}_\mathcal{X})$, we can consider its functor of points

$$h_\mathsf{X}\colon \infty\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}} \to \mathcal{S}, \quad \mathsf{Y} \mapsto \mathrm{Map}_{\infty\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X})$$

In particular, by [Lur18c, Remark 3.1.1.2], a closed immersion $f : (\mathcal{Y}, \mathscr{O}_\mathcal{Y}) \to (\mathcal{X}, \mathscr{O}_\mathcal{X})$ of locally spectrally ringed topoi corresponds to a morphism $\mathscr{O}_\mathcal{X} \to f_*\mathscr{O}_\mathcal{Y}$ of sheaves over $\mathcal{X}$ of connective $\mathbb{E}_\infty$-rings such that $\pi_0\mathscr{O}_\mathcal{X} \to \pi_0 f_*\mathscr{O}_\mathcal{Y}$ is an epimorphism. We denote this epimorphism by $\alpha$. Given a closed immersion $f\colon \mathsf{D} \to \mathsf{X}$ of spectral Deligne–Mumford stacks, we let $\mathcal{I}(\mathsf{D})$ denote $\ker(\alpha)$, called the ideal sheaf of $\mathsf{D}$.

To prove relative representability for effective Cartier divisors below, we need the representability of Picard functors. Given a map $f : \mathsf{X} \to \operatorname{Spét} R$ of spectral Deligne–Mumford stacks, we can define a functor

$$\mathscr{P}\mathrm{ic}_{\mathsf{X}/R}\colon \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}, \quad R' \mapsto \mathscr{P}\mathrm{ic}(\operatorname{Spét} R' \times_{\operatorname{Spét} R} \mathsf{X})$$

If $f$ admits a section $x\colon \operatorname{Spét} R \to \mathsf{X}$, then pullback along $x$ gives a natural transformation of functors $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R} \to \mathscr{P}\mathrm{ic}_{R/R}$. We let $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}\colon \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ denote the fiber of this map.

**Theorem 2.11** ([Lur18c, Theorem 19.2.0.5])**.** *Let $f : \mathsf{X} \to \operatorname{Spét} R$ be a map of spectral algebraic spaces which is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected over an $\mathbb{E}_\infty$-ring $R$. Suppose that $x\colon \operatorname{Spét} R \to \mathsf{X}$ is a section of $f$. Then the functor $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ is representable by a spectral algebraic space which is locally of finite presentation over $R$.*

In the classical setting, schemes representing relative effective Cartier divisors are open subschemes of Hilbert schemes [Kol96, Theorem 1.13]. However, in the derived setting, the Hilbert functor is representable by a spectral algebraic space [Lur04, Theorem 8.3.3], and it is hard to establish an analogous relationship. We will directly study relative effective Cartier divisors and their spectral moduli as follows.

**Definition 2.12** (Relative effective Cartier divisor)**.** Let $\mathsf{X}$ be a spectral Deligne–Mumford stack over a spectral Deligne–Mumford stack $\mathsf{S}$. Define a *relative effective Cartier divisor of* $\mathsf{X}/\mathsf{S}$ to be a closed immersion $\mathsf{D} \to \mathsf{X}$ such that $\mathsf{D}$ is flat, proper, locally almost of finite presentation over $\mathsf{S}$, and that the associated ideal sheaf of $\mathsf{D}$ over $\mathsf{X}$ is locally free of rank 1. We let $\mathrm{CDiv}(\mathsf{X}/\mathsf{S})$ denote the $\infty$-category of such closed immersions.

*Remark* 2.13. It is not hard to see that given any spectral Deligne–Mumford stack $\mathsf{X}$ over $\mathsf{S}$, $\mathrm{CDiv}(\mathsf{X}/\mathsf{S})$ is a Kan complex, since all objects are closed immersions of $\mathsf{X}$. Let $\mathsf{D} \to \mathsf{D}'$ be a morphism. Then we have a diagram

$$\begin{tikzcd} \mathsf{D} \arrow[rr, "f"] \arrow[dr] & & \mathsf{D}' \arrow[dl] \\ & \mathsf{X} & \end{tikzcd}$$

By the definition of closed immersions, they are all equivalent to the same substack of $\mathsf{X}$, so $f$ is an isomorphism (cf. [Lur18c, Remark 3.1.1.2]).

**Example 2.14** (Meier et al.)**.** In a recent work, motivated by quantum field theories as cobordism invariants for low-dimensional manifolds as well as by physics applications, Gukov, Krushkal, Meier, and Pei constructed certain modules over the spectrum TMF of topological modular forms from *derived Looijenga line bundles*, whose sections define quantum field theories with supersymmetry [GKMP25, Sections 4–6, esp. Theorem 4.5 (Meier)]. These derived line bundles are locally trivial sheaves over the spectral Deligne–Mumford stack of spectral elliptic curves (or rather fiber products of the universal curve over it). Their 0'th homotopy recovers the classical (integral) Looijenga line bundles associated to symmetric bilinear forms, which in turn determine interesting effective Cartier divisors [Loo77, Theorem 3.4]. Cf. Remark 4.7 below and also [ST23] for related derived objects. The authors thank Lars Hesselholt for suggesting this connection.

**Lemma 2.15.** *Let $\mathsf{X}/\mathsf{S}$ be a spectral Deligne–Mumford stack as above, and $\mathsf{T} \to \mathsf{S}$ be a map of spectral Deligne–Mumford stacks. If we have a relative effective Cartier divisor $\mathsf{D} \to \mathsf{X}$, then $\mathsf{D}_\mathsf{T}$ is a relative effective Cartier divisor of $\mathsf{X}_\mathsf{T}$.*

*Proof.* We first note that $\mathsf{D}_\mathsf{T}$ is a closed immersion of $\mathsf{X}_\mathsf{T}$ [Lur18c, Corollary 3.1.2.3]. After base change, $\mathsf{D}_\mathsf{T}$ is flat, proper, and locally almost of finite presentation over $\mathsf{T}$. It remains to show that $\mathcal{I}(\mathsf{D}_\mathsf{T})$ is a line bundle over $\mathsf{X}_\mathsf{T}$. Indeed, we have a fiber sequence

$$\mathcal{I}(\mathsf{D}) \to \mathscr{O}_{\mathcal{X}} \to \mathscr{O}_{\mathcal{D}}$$

By the flatness of $\mathsf{D}$, pullback along the base change $f\colon \mathsf{T} \to \mathsf{S}$ gives another fiber sequence

$$f^*\big(\mathcal{I}(\mathsf{D})\big) \to \mathscr{O}_{\mathcal{X}_\mathsf{T}} \to \mathscr{O}_{\mathcal{D}_\mathsf{T}}$$

So we have that $\mathcal{I}(\mathsf{D}_\mathsf{T})$ is just $f^*\big(\mathcal{I}(\mathsf{D})\big)$, which is invertible. To prove that $\mathcal{I}(\mathsf{D}_\mathsf{T})$ is in fact a line bundle (cf. [Lur18c, Remark 2.9.5.3]), we note that $\mathcal{I}(\mathsf{D})$ is an invertible object in $\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ by [Lur18c, Proposition 2.9.4.2], and hence $\mathcal{I}(\mathsf{D})^{-1} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$. Since $f^*$ is a right t-exact functor, we then have $\mathcal{I}(\mathsf{D}_\mathsf{T}) = f^*\big(\mathcal{I}(\mathsf{D})\big) \in \mathrm{QCoh}(\mathsf{X}_\mathsf{T})^{\mathrm{cn}}$ and $\big(f^*\mathcal{I}(\mathsf{D})\big)^{-1} = f^*\big(\mathcal{I}(\mathsf{D})^{-1}\big) \in \mathrm{QCoh}(\mathsf{X}_\mathsf{T})^{\mathrm{cn}}$ (cf. [Lur18c, Remark 1.3.2.8, Propositions 2.1.1.1 and 2.2.5.2]). Thus $\mathcal{I}(\mathsf{D}_\mathsf{T})$ is an invertible object in $\mathrm{QCoh}(\mathsf{X}_\mathsf{T})^{\mathrm{cn}}$, hence a line bundle. $\square$

Suppose that $\mathsf{X}$ is a spectral Deligne–Mumford stack over an affine spectral Deligne–Mumford stack $\mathsf{S} = \mathrm{Sp\acute{e}t}\, R$. From Definition 2.12, we then have a functor

$$\mathrm{CDiv}_{\mathsf{X}/R}\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{CDiv}(\mathsf{X}_{R'}/R')$$

Our main goal in this section is to prove that this functor is representable when $\mathsf{X}/R$ is a spectral algebraic space satisfying certain conditions. To achieve this, we need some preparations for computing the cotangent complex of a relative effective Cartier divisor functor. The main issue has to do with square-zero extensions, for which we need the following facts about pushouts of two closed immersions.

By [Lur18c, Theorem 16.2.0.1 and Proposition 16.2.3.1], given a pushout square of spectral Deligne–Mumford stacks

$$\begin{tikzcd} \mathsf{X}_{01} \arrow[r, "i"] \arrow[d, "j"] & \mathsf{X}_0 \arrow[d, "j'"] \\ \mathsf{X}_1 \arrow[r, "i'"] & \mathsf{X} \end{tikzcd}$$

such that $i$ and $j$ are closed immersions, the induced square of $\infty$-categories

$$\begin{tikzcd} \mathrm{QCoh}(\mathsf{X}_{01}) & \mathrm{QCoh}(\mathsf{X}_0) \arrow[l] \\ \mathrm{QCoh}(\mathsf{X}_1) \arrow[u] & \mathrm{QCoh}(\mathsf{X}) \arrow[l] \arrow[u] \end{tikzcd}$$

determines an embedding $\theta\colon \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X}_0) \times_{\mathrm{QCoh}(\mathsf{X}_{01})} \mathrm{QCoh}(\mathsf{X}_1)$, which restricts to an equivalence between connective objects

$$\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{X}_0)^{\mathrm{cn}} \times_{\mathrm{QCoh}(\mathsf{X}_{01})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}_1)^{\mathrm{cn}}$$

Moreover, let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$, and set $\mathscr{F}_0 = j'^*\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_0)$ and $\mathscr{F}_1 = i'^*\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_1)$. Then $\mathscr{F}$ is $n$-connective if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are $n$-connective, and this statement is also true for the conditions of almost connective, Tor-amplitude $\leqslant n$, flat, perfect to order $n$, almost perfect, perfect, and locally free of finite rank, respectively.

Also, by [Lur18c, Theorem 16.3.0.1], we have a pullback square of $\infty$-categories

$$\begin{tikzcd} \mathrm{SpDM}_{/\mathsf{X}} \arrow[r] \arrow[d] & \mathrm{SpDM}_{/\mathsf{X}_0} \arrow[d] \\ \mathrm{SpDM}_{/\mathsf{X}_1} \arrow[r] & \mathrm{SpDM}_{/\mathsf{X}_{01}} \end{tikzcd}$$

Let $f: \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne–Mumford stacks. Let $\mathsf{Y}_0 = \mathsf{X}_0 \times_{\mathsf{X}} \mathsf{Y}$, $\mathsf{Y}_1 = \mathsf{X}_1 \times_{\mathsf{X}} \mathsf{Y}$, and let $f_0 : \mathsf{Y}_0 \to \mathsf{X}_0$ and $f_1 : \mathsf{Y}_1 \to \mathsf{X}_1$ be the projection maps. Then we have that $f$ is locally almost of finite presentation if and only if both $f_0$ and $f_1$ are locally almost of finite presentation. The statement remains true for the following individual conditions: locally of finite generation to order $n$, locally of finite presentation, étale, equivalence, open immersion, closed immersion, flat, affine, separated, and proper [Lur18c, Proposition 16.3.2.1].

Now, let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne–Mumford stack, $\mathscr{E} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ be a connective quasi-coherent sheaf, and $\eta \in \mathrm{Der}(\mathscr{O}_{\mathcal{X}}, \Sigma\mathscr{E})$ be a derivation, i.e., a morphism $\eta\colon \mathscr{O}_{\mathcal{X}} \to \mathscr{O}_{\mathcal{X}} \oplus \Sigma\mathscr{E}$. We let $\mathscr{O}_{\mathcal{X}}^{\eta}$ denote the square-zero extension of $\mathscr{O}_{\mathcal{X}}$ by $\mathscr{E}$ determined by $\eta$, so that we have a pullback diagram

$$\begin{tikzcd} \mathscr{O}_{\mathcal{X}}^{\eta} \arrow[r] \arrow[d] & \mathscr{O}_{\mathcal{X}} \arrow[d, "\eta"] \\ \mathscr{O}_{\mathcal{X}} \arrow[r, "0"] & \mathscr{O}_{\mathcal{X}} \oplus \Sigma\mathscr{E} \end{tikzcd}$$

By [Lur18c, Proposition 17.1.3.4], $(\mathcal{X}, \mathscr{O}^{\eta}_{\mathcal{X}})$ is a spectral Deligne–Mumford stack, which we will denote by $\mathsf{X}^{\eta}$. In the case of $\eta = 0$, we denote it by $\mathsf{X}^{\mathscr{E}} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}} \oplus \mathscr{E})$. We then have a pushout square of spectral Deligne–Mumford stacks

$$\begin{tikzcd}
\mathsf{X}^{\mathscr{E}} & \mathsf{X} \arrow[l] \\
\mathsf{X} \arrow[u] & \mathsf{X}^{\Sigma\mathscr{E}} \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

such that $f$ and $g$ are closed immersions. In turn, by [Lur18c, Theorem 16.2.0.1], there is a pullback diagram

$$\begin{tikzcd}
\mathrm{QCoh}(\mathsf{X}^{\mathscr{E}})^{\mathrm{acn}} \arrow[r] \arrow[d] & \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}} \arrow[d] \\
\mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}} \arrow[r] & \mathrm{QCoh}(\mathsf{X}^{\Sigma\mathscr{E}})^{\mathrm{acn}}
\end{tikzcd}$$

of categories spanned by almost connective quasi-coherent sheaves. Passing to homotopy fibers over some $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}}$, we obtain an equivalence

$$\mathrm{QCoh}(\mathsf{X}^{\mathscr{E}})^{\mathrm{acn}} \times_{\mathrm{QCoh}(\mathsf{X})} \{\mathscr{F}\} \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}\big(\mathscr{F}, \Sigma(\mathscr{E} \otimes \mathscr{F})\big)$$

as in [Lur18c, Proposition 19.2.2.2]. Similarly, by passing to the homotopy fibers over some $\mathsf{Z} \in \mathrm{SpDM}_{/\mathsf{X}}$ with $f : \mathsf{Z} \to \mathsf{X}$, we obtain the classification of first-order deformations of $\mathsf{X}$:

$$\mathrm{SpDM}_{/\mathsf{X}^{\mathscr{E}}} \times_{\mathrm{SpDM}_{/\mathsf{X}}} \{\mathsf{Z}\} \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Z})}(L_{\mathsf{Z}/\mathsf{X}}, \Sigma f^* \mathscr{E})$$

[Lur18c, Proposition 19.4.3.1].

**Lemma 2.16.** *Let $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ be a morphism of spectral Deligne–Mumford stacks, and $M$ be a connective $R$-module. Consider the $\infty$-category of Deligne–Mumford stacks $\mathsf{X}'$ equipped with a morphism $f' : \mathsf{X}' \to \mathrm{Sp\acute{e}t}\,(R \oplus M)$ that fits into the pullback diagram*

$$\begin{tikzcd}
\mathsf{X} \arrow[r] \arrow[d, "f"'] & \mathsf{X}' \arrow[d, "f'"] \\
\mathrm{Sp\acute{e}t}\, R \arrow[r] & \mathrm{Sp\acute{e}t}\,(R \oplus M)
\end{tikzcd}$$

*Then this $\infty$-category is a Kan complex, and it is canonically homotopy equivalent to the mapping space $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}, \Sigma f^* M)$. Moreover, if $f$ is flat, proper, and locally almost of finite presentation, then so is $f'$.*

*Proof.* We have a pullback square of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd}
R \oplus M \arrow[r] \arrow[d] & R \arrow[d, "(\mathrm{id},0)"] \\
R \arrow[r] & R \oplus \Sigma M
\end{tikzcd}$$

which corresponds to a pushout square of spectral Deligne–Mumford stacks

$$\begin{tikzcd} \operatorname{Spét} R \oplus M & \operatorname{Spét} R \arrow[l] \\ \operatorname{Spét} R \arrow[u] & \operatorname{Spét}(R \oplus \Sigma M) \arrow[l] \arrow[u] \end{tikzcd}$$

such that the morphisms $\operatorname{Spét}(R \oplus \Sigma M) \to \operatorname{Spét} R$ are closed immersions. This exhibits $\operatorname{Spét}(R \oplus M)$ as an "infinitesimal thickening" of $\operatorname{Spét} R$ determined by $R \xrightarrow{(\mathrm{id},0)} R \oplus \Sigma M$.

The first part of this lemma follows from the formula for first-order deformations of [Lur18c, Proposition 19.4.3.1]. The second part follows from properties of pushout of two closed immersions [Lur18c, Corollary 16.4.2.1]. □

**Lemma 2.17.** *Suppose that we are given a pushout diagram of spectral Deligne–Mumford stacks*

$$\begin{tikzcd} \mathsf{X}_{01} \arrow[r, "i"] \arrow[d, "j"] & \mathsf{X}_0 \arrow[d] \\ \mathsf{X}_1 \arrow[r] & \mathsf{X} \end{tikzcd}$$

*where $i$ and $j$ are closed immersions. Let $f\colon \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne–Mumford stacks. Let $\mathsf{Y}_0 = \mathsf{X}_0 \times_{\mathsf{X}} \mathsf{Y}$, $\mathsf{Y}_1 = \mathsf{X}_1 \times_{\mathsf{X}} \mathsf{Y}$, and let $f_0 : \mathsf{Y}_0 \to \mathsf{X}_0$ and $f_1\colon \mathsf{Y}_1 \to \mathsf{X}_1$ be the projection maps. If $f_0$ and $f_1$ are both closed immersions and determine line bundles over $\mathsf{Y}_0$ and $\mathsf{Y}_1$ respectively, then $f$ is a closed immersion and determines a line bundle over $\mathsf{Y}$.*

*Proof.* The statement concerning closed immersions follows from [Lur18c, Proposition 16.3.2.1]. For the line-bundle part, we note that by [Lur18c, Theorem 16.2.0.1 and Proposition 16.2.3.1], $f$ determines a sheaf locally free of finite rank. To show that this sheaf is a line bundle, we proceed locally. By [Lur18c, Theorem 16.2.0.2], given a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

such that $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, there is an equivalence

$$F\colon \mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{Mod}_{A_0}^{\mathrm{cn}} \times_{\mathrm{Mod}_{A_{01}}^{\mathrm{cn}}} \mathrm{Mod}_{A_1}^{\mathrm{cn}}$$

Moreover, this is a symmetric monoidal equivalence. Indeed, since

$$F(M) = (A_0 \otimes_A M, A_1 \otimes_A M, A_{01} \otimes_{A_0} A_0 \otimes_A M \simeq A_{01} \otimes_{A_1} A_1 \otimes_A M)$$

we have $F(M \otimes_A N) \simeq F(M) \otimes F(N)$. By [Lur18c, Proposition 2.9.4.2], line bundles over $A_1$, $A_{01}$, and $A_0$ determine invertible objects of $\mathrm{Mod}_{A_1}^{\mathrm{cn}}$, $\mathrm{Mod}_{A_{01}}^{\mathrm{cn}}$, and $\mathrm{Mod}_{A_1}^{\mathrm{cn}}$ respectively, which in turn determine an invertible object of $\mathrm{Mod}_A^{\mathrm{cn}}$, hence a line bundle over $A$. □

Here is the main result of this section.

**Theorem 2.18.** *Given a connective $\mathbb{E}_\infty$-ring $R$, let $\mathsf{E}/R$ be a spectral algebraic space that is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor*

$$\mathrm{CDiv}_{\mathsf{E}/R}\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{CDiv}(\mathsf{E}_{R'}/R')$$

*is representable by a spectral algebraic space which is locally almost of finite presentation over* $\mathrm{Spét}\, R$.

*Remark* 2.19. In general, given such a functor $\mathcal{M}: \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ which is representable by a spectral Deligne–Mumford stack $(\mathcal{X}, \mathscr{O}_\mathcal{X})$, a colimit preserving functor

$$\widetilde{\mathcal{M}}\colon \mathrm{SpDM}_R^{\mathrm{op}} \to \mathcal{S}$$

with $\widetilde{\mathcal{M}}\big|_{\mathrm{CAlg}_R^{\mathrm{cn}}} \simeq \mathcal{M}$ is also representable by $(\mathcal{X}, \mathscr{O}_\mathcal{X})$. Indeed, any spectral Deligne–Mumford stack $\mathcal{Y}$ over $R$ can be written as $\varinjlim_\alpha \mathcal{Y}_\alpha$ with affine $\mathcal{Y}_\alpha \simeq \mathrm{Spét}\, B_\alpha$. We thus obtain

$$\begin{aligned} \mathrm{Map}_{\mathrm{SpDM}_R}(\mathcal{Y}, \mathcal{X}) &\simeq \mathrm{Map}_{\mathrm{SpDM}_R}\big(\varinjlim_\alpha \mathrm{Spét}\, B_\alpha, \mathcal{X}\big) \\ &\simeq \varprojlim_\alpha \mathrm{Map}_{\mathrm{SpDM}_R}(\mathrm{Spét}\, B_\alpha, \mathcal{X}) \\ &\simeq \varprojlim_\alpha \mathcal{X}(B_\alpha) \\ &\simeq \varinjlim_\alpha \widetilde{\mathcal{M}}(\mathrm{Spét}\, B_\alpha) \\ &\simeq \widetilde{\mathcal{M}}(\mathcal{Y}) \end{aligned}$$

In this paper, we focus on proving representability for functors on affines, as the moduli problems we consider are all local. By the above general principle, it then follows that the extended functors on spectral Deligne–Mumford stacks are also representable.

*Proof of Theorem 2.18.* In view of the equivalence between $\infty$-categories of functors

$$\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\mathrm{Spec}\, R}$$

we apply Lurie's spectral Artin representability theorem and verify the five criteria from Theorem 2.10 one by one, in the case of $n = 0$, as follows:

(1) Lemma 2.20;
(2) Lemma 2.21;
(3) Lemmas 2.22, 2.23, 2.24;
(4) Lemma 2.26; and
(5) Lemma 2.25.

The statements of these lemmas and their proofs occupy the rest of this section. □

**Lemma 2.20.** *For every discrete commutative $R_0$, the space $\mathrm{CDiv}_{\mathsf{E}/R}(R_0)$ is 0-truncated.*

*Proof.* Recall that $\mathrm{CDiv}_{\mathsf{E}/R}(R_0)$ consists of closed immersions $\mathsf{D} \to \mathsf{E} \times_R R_0$ such that $\mathsf{D}$ is flat and proper over $R_0$. Therefore, if $R_0$ is discrete, so are the objects $\mathsf{D}$, and so $\mathrm{CDiv}_{\mathsf{E}/R}(R_0)$ is 0-truncated. □

**Lemma 2.21.** *The functor $\mathrm{CDiv}_{\mathsf{E}/R}$ is a sheaf for the étale topology.*

*Proof.* Let $\{R' \to U_i\}_{i\in I}$ be an étale cover of $\operatorname{Spét} R'$, and $U_\bullet$ be the associated Čech-simplicial object. We need to prove that the map

$$\mathrm{CDiv}_{\mathsf{E}/R}(R') \to \varprojlim_{\mathbf{\Delta}} \mathrm{CDiv}_{\mathsf{E}/R}(U_\bullet)$$

is an equivalence. Unwinding the definitions, we need only prove the following general result: Given a spectral Deligne–Mumford stack $\mathsf{X}/\mathsf{S}$ and an étale cover $\mathsf{T}_i \to \mathsf{S}$, we have a homotopy equivalence

$$\mathrm{CDiv}(\mathsf{X}/\mathsf{S}) \to \varprojlim_{\mathbf{\Delta}} \mathrm{CDiv}(\mathsf{X} \times_{\mathsf{S}} \mathsf{T}_\bullet)$$

This follows from the fact that our conditions on relative effective Cartier divisors from Definition 2.12 are local with respect to the étale topology. □

**Lemma 2.22.** *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *is nilcomplete.*

*Proof.* By [Lur18c, Definition 17.3.2.1], we need to show that the canonical map

$$\mathrm{CDiv}_{\mathsf{E}/R}(R') \to \varprojlim_{n} \mathrm{CDiv}_{\mathsf{E}/R}(\tau_{\leqslant n} R')$$

is a homotopy equivalence for every $\mathbb{E}_\infty$-ring $R'$. This can be deduced from the following: Given a flat, proper, locally almost of finite presentation spectral algebraic space $\mathsf{X}$ over a connective $\mathbb{E}_\infty$-ring $S$, we have an equivalence

$$\mathrm{CDiv}(\mathsf{X}/S) \to \varprojlim_{n} \mathrm{CDiv}(\mathsf{X} \times_S \tau_{\leqslant n} S)$$

Let us now prove this equivalence. Given a relative effective Cartier divisor $\mathsf{D} \to \mathsf{X}$, we have the following commutative diagram

$$\begin{tikzcd}
\mathsf{D} \times_S \tau_{\leqslant n} S \arrow[r] \arrow[d, dashed] \arrow[dd, bend right] & \mathsf{D} \arrow[d] \\
\mathsf{X} \times_S \tau_{\leqslant n} S \arrow[r] \arrow[d] & \mathsf{X} \arrow[d] \\
\operatorname{Spét} \tau_{\leqslant n} S \arrow[r] & \operatorname{Spét} S
\end{tikzcd}$$

where we get an induced map $\mathsf{D} \times_S \tau_{\leqslant n} S \to \mathsf{X} \times_S \tau_{\leqslant n} S$. It is not hard to prove that this map is a closed immersion [Lur18c, Corollary 3.1.2.3]. Moreover, the map $\mathsf{D} \times_S \tau_{\leqslant n} S \to \operatorname{Spét} \tau_{\leqslant n} S$ is flat, proper, and locally almost of finite presentation, since $\mathsf{D} \times_S \tau_{\leqslant n} S$ is the base change of $\mathsf{D}$ along $\operatorname{Spét} \tau_{\leqslant n} S \to \operatorname{Spét} S$. The associated ideal sheaf of $\mathsf{D} \times_S \tau_{\leqslant n} S$ remains a line bundle over $\mathsf{X} \times_S \tau_{\leqslant n} S$. Therefore $\mathsf{D} \times_S \tau_{\leqslant n} S$ is a relative effective Cartier divisor of $\mathsf{X} \times_S \tau_{\leqslant n} S$. Thus we define a functor

$$\theta \colon \mathrm{CDiv}(\mathsf{X}/S) \to \varprojlim_{n} \mathrm{CDiv}(\mathsf{X} \times_S \tau_{\leqslant n} S), \quad \mathsf{D} \mapsto \{\mathsf{D} \times_S \tau_{\leqslant n} S\}_n$$

This functor is fully faithful, since we have from [Lur18c, Proposition 19.4.1.2] an equivalence

$$\mathrm{SpDM}_{/S} \to \varprojlim_{n} \mathrm{SpDM}_{/\tau_{\leqslant n} S}$$

defined by $\mathsf{X} \mapsto \mathsf{X} \times_S \tau_{\leqslant n} S$. For $\theta$ to be an equivalence, we need only show that it is essentially surjective.

Suppose $\{\mathsf{D}_n \to \mathsf{X} \times_S \tau_{\leqslant n} S\}_n$ is an object in $\varprojlim_n \mathrm{CDiv}(\mathsf{X} \times_S \tau_{\leqslant n} S)$. It is a morphism in $\varprojlim_n \mathrm{SpDM}_{/\tau_{\leqslant n} S}$. By [Lur18c, Proposition 19.4.1.2], there is a morphism $\mathsf{D} \to \mathsf{X}$ in $\mathrm{SpDM}_{/S}$ such that $\mathsf{D} \times_S \tau_{\leqslant n} S \to \mathsf{X} \times_S \tau_{\leqslant n} S$ are equivalent to $\mathsf{D}_n \to \mathsf{X} \times_S \tau_{\leqslant n} S$.

Next, we need to show that $\mathsf{D} \to \mathsf{X}$ from above is a relative effective Cartier divisor. The conditions that $\mathsf{D} \to \mathsf{X}$ is flat, proper, and locally almost of finite presentation follow immediately from [Lur18c, Proposition 19.4.2.1]. It remains to prove that $\mathsf{D} \to \mathsf{X}$ is a closed immersion and determines a line bundle over $\mathsf{X}$.

Without loss of generality, we may assume that $\mathsf{X} = \mathrm{Spét}\, B$ is affine, so that we have closed immersions

$$\mathsf{D}_n \to (\mathrm{Spét}\, B) \times_S \tau_{\leqslant n} S \simeq \mathrm{Spét}\,(B \otimes_S \tau_{\leqslant n} S)$$

the last equivalence from [Lur18c, Proposition 1.4.11.1 (3)]. By [Lur18c, Theorem 3.1.2.1], each $\mathsf{D} \times_S \tau_{\leqslant n} S$ is equivalent to $\mathrm{Spét}\, B'_n$ for some $B'_n$ such that $\pi_0(B \otimes_S \tau_{\leqslant n} S) \to \pi_0 B'_n$ is surjective. Since $\tau_{\leqslant n+1} S \to B'_{n+1}$ is flat, we have

$$\begin{aligned} \mathrm{Spét}\, B'_n &= (\mathrm{Spét}\, B'_{n+1}) \times_{\tau_{\leqslant n+1} S} \tau_{\leqslant n} S = \mathrm{Spét}\,(B'_{n+1} \otimes_{\tau_{\leqslant n+1} S} \tau_{\leqslant n} S) \\ &\simeq \mathrm{Spét}\, \tau_{\leqslant n} B'_{n+1} \end{aligned}$$

Thus we obtain a spectrum $B'$ such that

$$\mathrm{Spét}\, \tau_{\leqslant n} B' \simeq \mathrm{Spét}\, B'_n = \mathsf{D} \times_S \tau_{\leqslant n}$$

Consequently, $\mathsf{D} = \mathrm{Spét}\, B'$ and $\pi_0 B \to \pi_0 B'$ is surjective, and so

$$\mathsf{D} = \mathrm{Spét}\, B' \to \mathrm{Spét}\, B = \mathsf{X}$$

is a closed immersion.

Finally, to prove that the associated ideal sheaf of $\mathsf{D}$ is a line bundle, we note the pullback diagrams

$$\begin{tikzcd} I_n \arrow[r] \arrow[d] & B \otimes_S \tau_{\leqslant n} S \arrow[d] \\ * \arrow[r] & B' \otimes_S \tau_{\leqslant n} S \end{tikzcd}$$

where each $I_n$ is an invertible module over $B \otimes_S \tau_{\leqslant n} S = \tau_{\leqslant n} B$. Passing to inverse limits, we obtain a pullback diagram

$$\begin{tikzcd} \varprojlim I_n \arrow[r] \arrow[d] & B \arrow[d] \\ * \arrow[r] & B' \end{tikzcd}$$

Consequently, we have $I(\mathsf{D}) \simeq \varprojlim I_n$. Now, by nilcompleteness of the Picard functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}/S}$ from [Lur18c, Proposition 19.2.4.7 (1)], $I(\mathsf{D})$ is an invertible $B$-module. Therefore the associated ideal sheaf of $\mathsf{D}$ is a line bundle over $\mathsf{X}$. □

**Lemma 2.23.** *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *is infinitesimally cohesive.*

*Proof.* This follows from Lemma 2.17 and [Lur18c, Proposition 16.3.2.1]. □

**Lemma 2.24.** *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *is integrable.*

*Proof.* Given a local Noetherian $\mathbb{E}_\infty$-ring $R'$ which is complete with respect to its maximal ideal $\mathfrak{m} \subset \pi_0 R'$, we need to prove that the inclusion functor $\operatorname{Spf} R' \hookrightarrow \operatorname{Spec} R'$ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(\operatorname{Spec} R', \mathrm{CDiv}_{\mathsf{E}/R}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(\operatorname{Spf} R', \mathrm{CDiv}_{\mathsf{E}/R})$$

This can be deduced from the following result: Given a flat, proper, and separated spectral algebraic space $\mathsf{X}$ locally almost of finite presentation over a connective local Noetherian $\mathbb{E}_\infty$-ring $S$ which is complete with respect to its maximal ideal, we have an equivalence

$$\mathrm{CDiv}(\mathsf{X}/S) \simeq \mathrm{CDiv}(\mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S)$$

Indeed, let $\mathrm{Hilb}(\mathsf{X}/S)$ denote the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ consisting of those $\mathsf{D} \to \mathsf{X}$, such that each $\mathsf{D} \to \mathsf{X}$ is a closed immersion and is flat, proper, and locally almost of finite presentation. Then by the formal GAGA theorem [Lur18c, Corollary 8.5.3.4] and the base-change properties of being flat, proper, and locally almost of finite presentation, we have $\mathrm{Hilb}(\mathsf{X}/S) \simeq \mathrm{Hilb}(\mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S)$.

To prove the above equivalence for relative effective Cartier divisors, we need to further check that $\mathsf{D} \to \mathsf{X}$ associates a line bundle over $\mathsf{X}$ if and only if $\mathsf{D} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S$ associates a line bundle over $\mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S$. Note that the morphism $f\colon \mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S \to \mathsf{X}$ is flat by [Lur18c, Corollary 7.3.6.9], and so we have

$$\mathcal{I}(\mathsf{D} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S) = \mathcal{I}(f^*\mathsf{D}) \simeq f^*\mathcal{I}(\mathsf{D})$$

over the pullback square

$$\begin{tikzcd}
\mathsf{D} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S \arrow[r] \arrow[d] & \mathsf{D} \arrow[d] \\
\mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S \arrow[r, "f"] & \mathsf{X}
\end{tikzcd}$$

By [Lur18c, proof of Proposition 19.2.4.7], we have an equivalence

$$\mathrm{QCoh}(\mathsf{X}/S)^{\mathrm{aperf,cn}} \simeq \mathrm{QCoh}(\mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S)^{\mathrm{aperf,cn}}$$

We need only restrict to the subcategories spanned by invertible objects via [Lur18c, Proposition 2.9.4.2] to complete the proof. □

**Lemma 2.25.** *The functor $\mathrm{CDiv}_{\mathsf{E}/R}$ is locally almost of finite presentation over $\operatorname{Spec} R$.*

*Proof.* By [Lur18c, Definition 17.4.1.1 (b)], we need to prove that

$$\mathrm{CDiv}_{\mathsf{E}/R}\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{CDiv}(\mathsf{E}_{R'}/R')$$

commutes with filtered colimits when restricted to each $\tau_{\leqslant n}\mathrm{CAlg}_R^{\mathrm{cn}}$. We note that $\mathrm{CDiv}(\mathsf{E}_{R'}/R')$ is a full subcategory of $\mathrm{SpDM}_{/(\mathsf{E}_{R'} \to \operatorname{Sp\acute{e}t} R')}$ and first consider instead the functor

$$\mathrm{Var}^+\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty, \quad R' \mapsto \mathrm{Var}^+_{/(\mathsf{E}_{R'} \to \operatorname{Sp\acute{e}t} R')}$$

where $\mathrm{Var}^+_{/(\mathsf{E}_{R'} \to \operatorname{Sp\acute{e}t} R')}$ consists of diagrams

$$\begin{tikzcd}
\mathsf{D} \arrow[r] \arrow[dr] & \mathsf{E}_{R'} \arrow[d] \\
& \operatorname{Sp\acute{e}t} R'
\end{tikzcd}$$

such that $\mathsf{D} \to \operatorname{Sp\acute{e}t} R'$ is flat, proper, and locally almost of finite presentation. Then by [Lur18c, Proposition 19.4.2.1], this functor commutes with filtered colimits when restricted to $\tau_{\leqslant n}\mathrm{CAlg}_R^{\mathrm{cn}}$. It remains to verify that when $\{\mathsf{D}_i \to \mathsf{E}_{i,R'}\}_{i\in I}$ are closed immersions and determine line bundles over $\{\mathsf{E}_{i,R'}\}$, $\varinjlim_{i\in I} \mathsf{D}_i \to \varinjlim_{i\in I} \mathsf{E}_{i,R'}$ are closed immersions and determine line bundles over $\varinjlim_{i\in I} \mathsf{E}_{i,R'}$. As we recalled earlier in this subsection, this follows from properties of closed immersions and the property of Picard functors that they are locally almost of finite presentation. □

**Lemma 2.26.** *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *admits a cotangent complex which is connective and almost perfect.*

*Proof.* Let $S$ be a connective $R$-algebra, $\eta \in \mathrm{CDiv}_{\mathsf{E}/R}(S)$, and $M$ be a connective $S$-module. We then have a pullback diagram

$$\begin{tikzcd} F_\eta(M) \arrow[r] \arrow[d] & \mathrm{CDiv}_{\mathsf{E}/R}(S\oplus M) \arrow[d] \\ \{\eta\} \arrow[r] & \mathrm{CDiv}_{\mathsf{E}/R}(S) \end{tikzcd}$$

From this we obtain a functor

$$F_\eta \colon \mathrm{Mod}_S \to \mathcal{S}, \quad M \mapsto F_\eta(M)$$

We first need to prove that the above functor is corepresentable. Here, $\eta$ is a morphism $\mathsf{D} \to \mathsf{E}\times_R S$, and $\mathsf{E}\times_R (S\oplus M)$ is a square-zero extension of $\mathsf{E}\times_R S$. Thus by the classification of first-order deformations [Lur18c, Proposition 19.4.3.1], the space of spectral algebraic spaces $\mathsf{D}'$ which fit into the pullback diagram

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[d, "\eta"] & \mathsf{D}' \arrow[d] \\ \mathsf{E}\times_R S \arrow[r] \arrow[d, "p"] & \mathsf{E}\times_R (S\oplus M) \arrow[d] \\ \operatorname{Sp\acute{e}t} S \arrow[r] & \operatorname{Sp\acute{e}t}(S\oplus M) \end{tikzcd}$$

is equivalent to $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{D})}\big(L_{\mathsf{D}/(\mathsf{E}\times_R S)}, \Sigma\eta^*(p^*M)\big)$. Pushing forward along $p\circ\eta$, by [Lur18c, Proposition 6.4.5.3], we then have

$$\mathrm{Map}_{\mathrm{QCoh}(\mathsf{D})}\big(L_{\mathsf{D}/(\mathsf{E}\times_R S)}, \Sigma\eta^*(p^*M)\big) \simeq \mathrm{Map}_{\mathrm{QCoh}(\operatorname{Sp\acute{e}t} S)}\big(\Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)}), M\big)$$

By Lemma 2.17, any such $\mathsf{D}' \to \mathsf{E}\times_R (S\oplus M)$ is a closed immersion and determines a line bundle over $\mathsf{E}\times_R (S\oplus M)$. Since the diagram

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[d] & \mathsf{D}' \arrow[d] \\ \operatorname{Sp\acute{e}t} S \arrow[r] & \operatorname{Sp\acute{e}t}(S\oplus M) \end{tikzcd}$$

is a pullback square, $\mathsf{D}'$ is a square-zero extension of $\mathsf{D}$. By [Lur18c, Proposition 16.3.2.1], $\mathsf{D}' \to \operatorname{Sp\acute{e}t}(S\oplus M)$ is flat, proper, and locally almost of finite presentation. Combining these facts, we find that

$$F_\eta(M) = \mathrm{Map}_{\mathrm{QCoh}(\operatorname{Sp\acute{e}t} S)}\big(\Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)}), M\big)$$

Consequently, the functor $\mathrm{CDiv}_{\mathsf{E}/R}$ satisfies condition (a) in [Lur18c, Example 17.2.4.4]. Condition (b) therein follows from the compatibility of $(p \circ \eta)_+$, as a left adjoint of the functor $(p \circ \eta)^*$, with base change (cf. [Lur18c, Construction 6.4.5.1 and Proposition 6.4.5.3]). Therefore the functor $\mathrm{CDiv}_{\mathsf{E}/R}$ admits a cotangent complex $L_{\mathrm{CDiv}_{\mathsf{E}/R}}$ satisfying

$$\eta^* L_{\mathrm{CDiv}_{\mathsf{E}/R}} = \Sigma^{-1} p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)})$$

Since the quasi-coherent sheaf $L_{\mathsf{D}/(\mathsf{E}\times_R S)}$ is connective and almost perfect [Lur18c, Proposition 17.1.5.1 (3)], the $S$-module $\Sigma^{-1} p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)})$ is $(-1)$-connective.

Next, we show that $L_{\mathrm{CDiv}_{\mathsf{E}/R}}$ is almost perfect. This follows from [Lur18c, 17.4.2.2] and Lemma 2.25.

Finally, we show that it is connective. As above, let $S$ be a connective $R$-algebra and $\eta \in \mathrm{CDiv}_{\mathsf{E}/R}(S)$. We need to prove that $M_\eta \coloneqq \eta^* L_{\mathrm{CDiv}_{\mathsf{E}/R}} \in \mathrm{Mod}_S$ is connective. We already knew that $M_\eta$ is $(-1)$-connective and almost perfect. In particular, the homotopy group $\pi_{-1} M_\eta$ is a finitely generated $\pi_0 S$-module. To prove that it in fact vanishes, by Nakayama's lemma, we note that this is equivalent to proving that

$$\pi_{-1}(\kappa \otimes_{\pi_0 S} M_\eta) \simeq \mathrm{Tor}_0^{\pi_0 S}(\kappa, \pi_{-1} M_\eta)$$

equals 0 for every residue field $\kappa$ of $\pi_0 S$. Thus we may replace $S$ by $\kappa$ and assume $\kappa$ is an algebraically closed field.

Let $A = \kappa[\epsilon]/(\epsilon^2)$. Unwinding the definitions, we find that the dual space $\mathrm{Hom}_\kappa(\pi_{-1} M_\eta, \kappa)$ can be identified with the set of automorphisms of the base change $\eta_A$ such that they restrict to be the identity of $\eta$. It remains to prove that this set is trivial. This boils down to the following assertion in classical algebraic geometry:

> Let $X/\kappa$ be a scheme, $L$ be a line bundle over $X$, and assume $L_A$ is also a line bundle over $X_A$. If $f$ is an automorphism of $L_A$ such that $f|L$ is the identity on $L$, then $f$ is the identity.

This can be proved, mutatis mutandis, as in the last part of [Lur18a, proof of Proposition 2.2.6]. □

## 3. Level structures for spectral abelian varieties

For spectral Deligne–Mumford stacks, Theorem 2.18 gives the relative representability (with respect to a fixed $\mathsf{E}/R$) of relative effective Cartier divisors (over $\mathrm{Spét}\, R$). Their analogues in classical algebraic geometry are crucial to Drinfeld's approach to arithmetic moduli of elliptic curves with level structure over $\mathbf{Z}$, as developed in [KM85], which applies nicely at primes dividing the level. In this section, we define level structures on spectral abelian varieties and related objects from effective Cartier divisors. The applications we aim at are of a similar nature to those considered by the earlier authors, i.e., incorporating ramification or regardless of failure of étaleness, which we will discuss in the next two sections.

3.1. **Level structures on elliptic curves.** Let $C$ be a one-dimensional smooth commutative group scheme over a base scheme $S$, and $A$ be a finite abelian group. Recall from [KM85, 1.5.1] that a group homomorphism

$$\phi\colon A \to C(S)$$

is said to be an *$A$-structure on $C/S$*, if the effective Cartier divisor $\sum_{a\in A}[\phi(a)]$ is a subgroup scheme of $C/S$.

The following result gives the relative representability of moduli problems of level structures.

**Proposition 3.1** ([KM85, Proposition 1.6.2]). *Let $C$ be a one-dimensional smooth commutative group scheme over $S$. Then the functor*

$$\mathrm{Level}^A_{C/S}\colon \mathrm{Sch}_S \to \mathrm{Set}, \quad T \mapsto \textit{the set of level-}A\textit{ structures on } C_T/T$$

*is representable by a closed subscheme of* $\mathrm{Hom}_{\mathrm{Grp}/S}(A, C)$.

**Definition 3.2.** Let $R$ be a connective $\mathbb{E}_\infty$-ring and $\mathsf{E}/R$ be a spectral elliptic curve. A *(derived) level-$A$ structure on* $\mathsf{E}$ is a pair $(\mathsf{D}, \phi)$, where $\mathsf{D} \to \mathsf{E}$ is a relative effective Cartier divisor, and $\phi\colon A \to \mathsf{E}^\heartsuit(\pi_0 R)$ is an $A$-structure on $\mathsf{E}^\heartsuit/\pi_0 R$ as above, such that the underlying morphism $\mathsf{D}^\heartsuit \to \mathsf{E}^\heartsuit$, necessarily a closed immersion, equals the inclusion of the associated relative effective Cartier divisor $\sum_{a\in A}[\phi(a)]$ into $\mathsf{E}^\heartsuit$. We denote by $\mathrm{Level}(A, \mathsf{E}/R)$ the $\infty$-category of level-$A$ structures on $\mathsf{E}/R$, whose objects can be viewed as relative effective Cartier divisors satisfying an extra property.

Given a spectral elliptic curve $\mathsf{E}/R$, the $\infty$-category $\mathrm{Level}(A, \mathsf{E}/R)$ is an $\infty$-groupoid, since it is a full subcategory of $\mathrm{CDiv}(\mathsf{E}/R)$, which is an $\infty$-groupoid (see Remark 2.13).

We note that derived level structures are stable under base change, as follows.

**Lemma 3.3.** *Let $\mathsf{E}/R$ be a spectral elliptic curve, $\mathsf{S} = \mathrm{Spét}\, R$, and $(\mathsf{D}, \phi)$ be a level structure. Suppose that $\mathsf{T} \to \mathsf{S}$ is a morphism of nonconnective spectral Deligne–Mumford stacks. Then the induced pair $(\mathsf{D}_\mathsf{T}, \phi_\mathsf{T})$ is a level structure on $\mathsf{E}_\mathsf{T}/\mathsf{T}$.*

*Proof.* The induced closed immersion $\mathsf{D}_\mathsf{T} \to \mathsf{E}_\mathsf{T}$ is a relative effective Cartier divisor by Lemma 2.15. It remains to check that $\phi_\mathsf{T}\colon A \to \mathsf{E}^\heartsuit(\mathsf{T}^\heartsuit) = \mathsf{E}^\heartsuit_\mathsf{T}(\mathsf{T}^\heartsuit)$ is a classical level structure, so that $\mathsf{D}^\heartsuit_\mathsf{T}$ is the associated classical relative effective Cartier divisor. This follows from the base-change property of classical level structures observed in [KM85, Section 1.5.1]. □

We next recall a result on when a divisor becomes a (finite flat) subgroup.

**Proposition 3.4** ([KM85, Corollary 1.3.7]). *Given a smooth curve $C/S$ which is a group scheme over a scheme $S$ along with a relative effective Cartier divisor $D$ of $C$, there exists a closed subscheme $Z$ of $S$ with the property that, for any $T \to S$, $D_T$ is a subgroup of $C_T$ if and only if $T \to S$ factors through $Z$. Moreover, locally on $S$, $Z$ is defined by finitely many equations.*

Here, we have an analogous incidence object for the relation "$\mathsf{D}^\heartsuit$ is a subgroup." To construct it, given an $\mathbb{E}_\infty$-ring $R$ and a finitely generated ideal $I \subset \pi_0 R$, let us first recall from [Lur18c, Definition 7.1.2.1, Notations 7.2.4.6 and 7.3.1.5] the stable $R$-linear $\infty$-categories

$$\mathrm{Mod}_R^{\mathrm{Nil}(I)}, \quad \mathrm{Mod}_R^{\mathrm{Loc}(I)}, \quad \mathrm{Mod}_R^{\mathrm{Cpl}(I)}$$

of $I$-nilpotent $R$-modules, $I$-local $R$-modules, $I$-complete $R$-modules, respectively. Their relationships can be summarized in the following diagram, which also encode geometric constructions related to quasi-coherent sheaves over $\mathrm{Spét}\, R$ supported on the closed subscheme $K \coloneqq \operatorname{Spec} \pi_0 R/I \subset \operatorname{Spec} \pi_0 R \simeq |\mathrm{Sp\acute{e}t}\, R|$ [Lur18c, Propositions 7.1.5.3 and 7.3.1.7]:

$$\begin{tikzcd}
& & \mathrm{Mod}_R^{\mathrm{Loc}(I)} \arrow[d, hook, shift left] & \\
& & \mathrm{Mod}_R \arrow[u, shift left, "L_I"] \arrow[u, phantom, "\dashv"] \arrow[dl, shift left, "\Gamma_I"] \arrow[dr, shift left, "(-)^\wedge_I"] & \\
\mathrm{QCoh}_K(\mathrm{Sp\acute{e}t}\, R) \arrow[r, shift right] \arrow[r, phantom, "\perp \simeq"] & \mathrm{Mod}_R^{\mathrm{Nil}(I)} \arrow[l, shift right] \arrow[ur, hook, shift left] \arrow[ur, phantom, "\perp"] \arrow[rr, shift left, "(-)^\wedge_I"] \arrow[rr, phantom, "\perp \simeq"] & & \mathrm{Mod}_R^{\mathrm{Cpl}(I)} \arrow[ul, hook, shift left] \arrow[ul, phantom, "\perp"] \arrow[ll, shift left, "\Gamma_I"]
\end{tikzcd}$$

**Proposition 3.5.** *Let $\mathsf{E}/\mathrm{Sp\acute{e}t}\, R$ be a spectral elliptic curve and $\mathsf{D} \to \mathsf{E}$ be a relative effective Cartier divisor. Then there exists an ideal $I \subset \pi_0 R$ such that, given any $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, the following conditions are equivalent:*

1. *The relative effective Cartier divisor $\mathsf{D}_{R'}^{\heartsuit}$ is a subgroup of $\mathsf{E}_{R'}^{\heartsuit}$.*
2. *The composite $\Gamma_I R \to R \to R'$ in $\mathrm{Mod}_R$ restricts to be a morphism $\Gamma_I R \to R'$ in $\mathrm{Mod}_R^{\mathrm{Nil}(I)}$.*
3. *The morphism $R \to R'$ factors through $R^\wedge_I$ in $\mathrm{Mod}_R$ to give a morphism $R^\wedge_I \to R'$ in $\mathrm{Mod}_R^{\mathrm{Cpl}(I)}$ by restriction.*

*Proof.* Given the elliptic curve $\mathsf{E}^{\heartsuit}/\pi_0 R$ and its relative effective Cartier divisor $\mathsf{D}^{\heartsuit}$, let $I \subset \pi_0 R$ be an ideal cutting out the closed subscheme $Z \subset \operatorname{Spec} \pi_0 R$ from the preceding proposition. In particular, $I$ is finitely generated. Let $f \colon R \to R'$ be the structure morphism of $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$. We now show that $I$ satisfies the property as stated in terms of the three equivalent conditions.

(1) $\Longrightarrow$ (2): By the proposition, if $\mathsf{D}_{R'}^{\heartsuit}/\pi_0 R'$ is a subgroup of $\mathsf{E}_{R'}^{\heartsuit}/\pi_0 R'$, the morphism $\operatorname{Spec} \pi_0 R' \to \operatorname{Spec} \pi_0 R$ must factor through $Z$. Thus, as an $R$-module, $R'$ is $I$-nilpotent by [Lur18c, Proposition 7.1.5.3]. It then follows from [Lur18c, Definition 7.1.2.1] that the diagram in $\mathrm{Mod}_R$ on the left gives rise to a diagram in $\mathrm{Mod}_R^{\mathrm{Nil}(I)}$ on the right, as desired:

$$\begin{tikzcd}
& \Gamma_I R \arrow[dl] \arrow[drr, dashed] & & \\
R \arrow[rr, "f"] & & R' & R' \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& \Gamma_I R \arrow[dl, "\mathrm{id}"'] \arrow[drr, dashed] & & \\
\Gamma_I R \arrow[rr, "\Gamma_I f"] & & \Gamma_I R' & R' \arrow[l, "\simeq"']
\end{tikzcd}$$

(1) $\Longrightarrow$ (3): Again, $\operatorname{Spec} \pi_0 R' \to \operatorname{Spec} \pi_0 R$ factors through $Z$ defined by the same $I$. In view of [Lur18c, Proposition 7.3.1.7], $R'$ is then an $I$-complete $R$-module. Thus dually [Lur18c, Notation 7.3.1.5] implies that the diagram in $\mathrm{Mod}_R^{\mathrm{Cpl}(I)}$ gives in turn a diagram in $\mathrm{Mod}_R$:

$$\begin{tikzcd}
R^\wedge_I \arrow[r, "f^\wedge_I"] \arrow[dr, "\mathrm{id}"'] & R'^\wedge_I \arrow[r, "\simeq"] & R' \\
& R^\wedge_I \arrow[ur, dashed] &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
R \arrow[r, "f"] \arrow[dr] & R' \arrow[r, "\mathrm{id}"] & R' \\
& R^\wedge_I \arrow[ur, dashed] &
\end{tikzcd}$$

(2) $\Longrightarrow$ (1): Suppose that $R'$ is $I$-nilpotent and receives a morphism in $\mathrm{Mod}_R^{\mathrm{Nil}(I)}$ from $\Gamma_I R$ through which the structure map $f: R \to R'$ of $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$ factors in $\mathrm{Mod}_R$. Then by [Lur18c, Proposition 7.1.5.3] $\mathscr{O}_{\mathrm{Sp\acute{e}t}\, R'}$ is supported on $\mathrm{Spec}\,\pi_0 R/\sqrt{I} \subset \mathrm{Spec}\,\pi_0 R \simeq |\mathrm{Sp\acute{e}t}\, R|$. Therefore, the map $\mathrm{Spec}\,\pi_0 R' \to \mathrm{Spec}\,\pi_0 R$ factors through $\mathrm{Spec}\,\pi_0 R/I = Z$, and so $\mathsf{D}^{\heartsuit}_{R'}$ is a subgroup of $\mathsf{E}^{\heartsuit}_{R'}$. (We can prove (3) $\Longrightarrow$ (1) analogously.)

(3) $\Longrightarrow$ (2): Given $R' \simeq R'^{\wedge}_I$ in $\mathrm{Mod}_R^{\mathrm{Cpl}(I)}$, $\Gamma_I R'$ is its image in $\mathrm{Mod}_R^{\mathrm{Nil}(I)}$ under the equivalence from [Lur18c, Proposition 7.3.1.7] between full subcategories of $\mathrm{Mod}_R$. Moreover, the morphism $h\colon R^{\wedge}_I \to R'$ in $\mathrm{Mod}_R^{\mathrm{Cpl}(I)}$ factoring $f: R \to R'$ corresponds to $\Gamma_I h\colon \Gamma_I(R^{\wedge}_I) \to \Gamma_I R'$. Thus they fit into the following commutative diagrams in $\mathrm{Mod}_R$, on the left and on the right, respectively:

$$\begin{tikzcd} R^{\wedge}_I \arrow[r, "h"] & R' \arrow[dd, equal] \\ & \\ R \arrow[uu] \arrow[r, "f"] & R' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \Gamma_I(R^{\wedge}_I) \arrow[r, "\Gamma_I h"] & \Gamma_I R' \arrow[d, "\wr"] \\ \Gamma_I R \arrow[u] \arrow[d] & \Gamma_I(R'^{\wedge}_I) \\ R \arrow[r, "f"] & R' \arrow[u, dashed, "\wr"'] \end{tikzcd}$$

where the dashed isomorphism demands that $R'$ also be $I$-nilpotent. It then follows that the composite $\Gamma_I R \to R \to R'$ in $\mathrm{Mod}_R$ restricts to be a morphism $\Gamma_I R \to R'$ in $\mathrm{Mod}_R^{\mathrm{Nil}(I)}$. (We can prove (2) $\Longrightarrow$ (3) analogously.) $\square$

The following is our main result in this subsection on relative representability of level structures over the spectral moduli stack of spectral elliptic curves.

**Theorem 3.6.** *Let $\mathsf{E}/R$ be a spectral elliptic curve and $A$ be a finite abelian group. Then the functor*

$$\mathrm{Level}^A_{\mathsf{E}/R}\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{Level}(A, \mathsf{E}_{R'}/R')$$

*is representable by a spectral Deligne–Mumford stack $\mathsf{S}(A)$. Moreover, $\mathsf{S}(A) = \mathrm{Sp\acute{e}t}\,\mathcal{P}_{\mathsf{E}/R}$ for some $\mathbb{E}_\infty$-ring $\mathcal{P}_{\mathsf{E}/R}$, which is locally almost of finite presentation over $R$.*

*Proof.* By definition, the functor $\mathrm{Level}^A_{\mathsf{E}/R}$ is a subfunctor of the representable functor $\mathrm{CDiv}_{\mathsf{E}/R}$ from Theorem 2.18. In view of Proposition 3.5 and Lemma 2.15, we are reduced to the affine-local case and consider a spectral Deligne–Mumford stack $\mathrm{GrpCDiv}_{\mathsf{E}/R}$ defined locally by completing $\mathrm{CDiv}_{\mathsf{E}/R}$ with respect to the incidence ideals $I$ associated to the universal object $\mathsf{D}_{\mathrm{univ}} \to \mathsf{E} \times_R \mathrm{CDiv}_{\mathsf{E}/R}$ (not the ideal sheaf defining the divisor). Cf. Remark 2.19 for the existence of this universal object over $\mathrm{CDiv}_{\mathsf{E}/R}$. We verify that $\mathrm{GrpCDiv}_{\mathsf{E}/R}$ valued on an $R$-algebra $R'$ is the space of relative effective Cartier divisors $\mathsf{D}$ of $\mathsf{E}_{R'}$ such that $\mathsf{D}^{\heartsuit}$ is a finite flat subgroup of $\mathsf{E}^{\heartsuit}_{R'}$.

Moreover, there is a clopen substack $\mathrm{GrpCDiv}^A_{\mathsf{E}/R}$ of $\mathrm{GrpCDiv}_{\mathsf{E}/R}$ whose value on an $R$-algebra $R'$ is the space of relative effective Cartier divisors $\mathsf{D}$ of $\mathsf{E}_{R'}$ such that $\mathsf{D}^{\heartsuit}$ is a subgroup of $\mathsf{E}^{\heartsuit}_{R'}$ finite locally free over $\pi_0 R'$ of rank equal to $\#A$. We then further evoke [KM85, Proposition 1.6.5] and obtain $\mathsf{S}(A)$ representing $\mathrm{Level}^A_{\mathsf{E}/R}$ as a completion of $\mathrm{GrpCDiv}^A_{\mathsf{E}/R}$ similarly as with Lemma 3.5.

To prove the remaining statement, we consider the morphism $\mathsf{S}(A) \to \operatorname{Sp\acute{e}t} R$, both being spectral algebraic spaces. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary algebraic spaces is finite in the sense of classical algebraic geometry. Thus we need only prove that $\mathsf{S}(A)^\heartsuit$ is finite over $\operatorname{Spec} \pi_0 R$. This is precisely the classical case: by definition of the derived and classical moduli problems in question, $\mathsf{S}(A)^\heartsuit$ is the representing object of the classical level-$A$ structures, which is a finite $\pi_0 R$-scheme of finite presentation according to [KM85, Corollary 1.6.3]. □

3.2. **Level structures on $p$-divisible groups.** Before we move on and introduce derived level structures for spectral $p$-divisible groups, let us first recall some classical facts needed about level structures of commutative finite flat group schemes.

3.2.1. *Classical finite flat group schemes.* Let $S$ be a scheme and $X/S$ be a finite flat $S$-scheme of finite presentation and rank $N$. It can be proved that $X/S$ is finite locally free of rank $N$. This means that for every affine scheme $\operatorname{Spec} R \to S$, the pullback scheme $X \times_S \operatorname{Spec} R$ over $\operatorname{Spec} R$ has the form $\operatorname{Spec} R'$, where $R'$ is an $R$-algebra which is locally free of rank $N$. For an element $f \in R'$ acting on $R'$ by multiplication, define an $R$-linear endomorphism of $R'$. Because $R'$ is locally free of rank $N$, multiplication by $f$ has a characteristic polynomial

$$\det(T - f) = T^N - \operatorname{trace}(f)\, T^{N-1} + \cdots + (-1)^N \operatorname{norm}(f)$$

Recall the following definition from [KM85, 1.8.2]. Let $\{P_1, \dots, P_N\}$ be a set of $N$ points not necessarily distinct in $X(S)$. We call it a *full set of sections of $X/S$* if one of the following two equivalent conditions is satisfied:

- For any $\operatorname{Spec} R \to S$ and $f \in R' = H^0(X_R, \mathscr{O})$, we have
$$\det(T - f) = \prod_{i=1}^N \bigl(T - f(P_i)\bigr)$$
- For any $\operatorname{Spec} R \to S$ and $f \in R' = H^0(X_R, \mathscr{O})$, we have
$$\operatorname{norm}(f) = \prod_{i=1}^N f(P_i)$$

Given $N$ not necessarily distinct points $P_1, \dots, P_N$ in $X(S)$, we have a morphism $\mathscr{O}_X \to \bigotimes_i (P_i)_*(\mathscr{O}_S)$ of sheaves over $X$. It is not hard to see that this morphism is surjective and defines a closed subscheme $D$ of $X$ which is flat and proper over $S$. Thus, given a finite abelian group $A$ and a map $\phi\colon A \to X(S)$ of sets, we can define a closed subscheme $D$ of $X$ by the sheaf $\bigotimes_{a \in A} \phi(a)_* \mathscr{O}_S$.

**Lemma 3.7.** *Given a finite flat $S$-scheme $Z$ of finite presentation,* $\operatorname{Hom}(A, Z)$ *is an open substack of the Hilbert stack* $\mathrm{Hilb}_{Z/S}$.

*Proof.* Let $T \to S$ be an $S$-scheme. For any $D \to Y := T \times_S Z$ in $\mathrm{Hilb}_{Z/S}(T)$, we need only verify that the set of points $t \in T$ over which $D_t \to Y_t$ comes from the closed subscheme associated to $\phi : A \to Z(T) = Y(T)$ is open in $T$, as a permutation of elements in the source $A$ of $\phi$ corresponds to an automorphism on the subscheme. Since $D$ is the closed subscheme defined by $\mathscr{O}_Y \to \mathscr{O}_D$, if $D_t$ comes from $\mathscr{O}_{Y,t} \to \bigotimes_{a\in A} \phi(a)_*\mathscr{O}_{T,t}$, then by the definition of a stalk, there exists an open subset $U$ of $T$ such that $t \in U$ and $D_U$ is defined by $\mathscr{O}_Y|_U \to \bigotimes_{a\in A} \phi(a)_*\mathscr{O}_T|_U$. □

Suppose that $G/S$ is a finite flat commutative group scheme of finite presentation and $A$ is a finite abelian group of order $N$. Let $K$ be a finite flat $S$-subgroup-scheme of $G$ locally free of rank $N$, and $\phi\colon A \to G(S)$ be a homomorphism landing in $K(S)$. Recall from [KM85, Remark 1.10.10] that the pair $(K, \phi)$ is called an *$A$-structure on $G/S$*, if the $N$ points $\phi(a), a \in A$ form a full set of sections of $K$.

**Lemma 3.8.** *Suppose that $G/S$ is a finite flat commutative group scheme of finite presentation and $K \subset G$ is a closed subscheme which is finite flat and of finite presentation. Then there exists a closed subscheme $Z \subset S$ such that given any morphism of schemes $T \to S$, $K_T$ is a subgroup scheme of $G_T$ if and only if the morphism $T \to S$ factors through $Z$.*

*Proof.* This is an analogue of [KM85, Corollary 1.3.7] for finite flat group schemes. Following the proof strategy there, we need only prove: Given finite flat closed subschemes $K_1, K_2$ of $G$, there exists a closed subscheme $Z \subset S$ such that given any morphism of schemes $T \to S$, $(K_1)_T$ is a closed subscheme of $(K_2)_T$ if and only if the morphism $T \to S$ factors through $Z$ (cf. [KM85, Lemma 1.3.4 (1)]).

Since we consider the case of finite flat group schemes of finite presentation and the question is local on $S$, we are reduced to proving:

> Let $B$ be a finite free $A$-algebra, and $\operatorname{Spec} B/I_1, \operatorname{Spec} B/I_2$ be two closed subschemes of $\operatorname{Spec} B$ such that $B/I_1$ and $B/I_2$ are also free. Then there exits a closed subscheme $\operatorname{Spec} W$ of $\operatorname{Spec} A$ such that given any $A \to A'$, $\operatorname{Spec}(B/I_1 \otimes_A A')$ is a closed subscheme of $\operatorname{Spec}(B/I_2 \otimes_A A')$ if and only if $A \to A'$ factors through $W$.

Let $\bar{1} \in B/I_2$ be the identity. Since $B/I_1$ is a free $A$-module of finite rank, the image of $\bar{1}$ under the map $B/I_2 \to B/I_2 \otimes_A B/I_1$ can be written as $\sum_{i=1}^d r_i e_i$, where $d$ is the rank and $\{e_i\}_{1\leqslant i\leqslant d}$ is an $A$-basis of $B/I_2 \otimes_A B/I_1$. It is not hard to see that $V(\{r_1, \cdots, r_d\})$ is the desired closed subscheme of $\operatorname{Spec} A$. □

**Proposition 3.9.** *Hypotheses and notations as above, the functor $A$-$\mathrm{Str}(G/S)$ on $S$-schemes defined by*

$$T \mapsto \big\{\big(K \subset G_T,\, \phi\colon A \to G(T)\big) \,\big|\, (K, \phi) \text{ is an } A\text{-structure on } G_T\big\}$$

*is representable by a finite $S$-scheme of finite presentation.*

*Proof.* This is a variant of [KM85, Lemma 1.10.11 and Proposition 1.10.13 (1)]. Let us proceed in three steps.

First, the functor

$$T \mapsto \{D_T \subset G_T \,|\, D_T \text{ is a closed subscheme finite flat over } T \text{ of rank } N\}$$

is representable by a finite $S$-scheme $\mathrm{Hilb}^N_{G/S}$ (a Grassmannian).

Second, applying the preceding lemma to the universal example over $\mathrm{Hilb}^N_{G/S}$, we obtain a finite $S$-scheme $Z$ classifying finite flat subgroup schemes of $G$ locally free of rank $N$.

Third, given such a subgroup scheme $K \subset G$, observe that the functor

$$T \mapsto \{\phi\colon A \to G(T) \,|\, (K_T, \phi) \text{ is an } A\text{-level structure on } G_T\}$$

is equivalent to the functor

$$T \mapsto \{\phi\colon A \to K(T) \mid \phi \text{ is an } A\text{-generator of } K_T\}$$

(cf. [KM85, Remark 1.10.10 and 1.10.5]). Since the latter is representable by a finite $S$-scheme of finite presentation by [KM85, Proposition 1.10.13 (1)], we further apply this representability to the universal example $K_{\mathrm{univ}} \subset G_Z$ to complete the proof. ☐

3.2.2. *Spectral finite flat group schemes.* Let $R$ be a connective $\mathbb{E}_\infty$-ring and $G$ be a commutative finite flat group scheme over $R$. By the definition of finite-flatness, we have $G = \mathrm{Sp\acute{e}t}\, B$ for a finite flat $R$-algebra $B$ [Lur18a, Definition 6.1.2]. We let $\mathrm{Hilb}(G/R)$ denote the full subcategory of $\mathrm{SpDM}_{/G}$ spanned by those $\mathsf{D} \to G$ such that $\mathsf{D} \to G$ is a closed immersion of spectral Deligne–Mumford stacks and that the composite $\mathsf{D} \to G \to \mathrm{Sp\acute{e}t}\, R$ is flat, proper, and locally almost of finite presentation. Then $\mathrm{Hilb}(G/R)$ is equivalent to the $\infty$-category of diagrams of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & S & \end{tikzcd}$$

such that $S$ is flat, proper, and locally almost of finite presentation over $R$ subject to certain additional conditions. It is not hard to see that $\mathrm{Hilb}(G/R)$ is a Kan complex (cf. Remark 2.13), so that we can define a functor

$$\mathrm{Hilb}_{G/R}\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{Hilb}(G_{R'}/R')$$

The representability of this functor is a special case of [Lur04, Theorem 8.3.3], which we record below. Like that theorem and Theorem 2.18, it can be deduced from the spectral Artin representability theorem 2.10.

**Theorem 3.10** (Lurie)**.** *Suppose that $G$ is a commutative finite flat group scheme over a connective $\mathbb{E}_\infty$-ring $R$. Then $\mathrm{Hilb}_{G/R}$ is representable by a spectral Deligne–Mumford stack which is locally almost of finite presentation over $R$.*

**Corollary 3.11.** *Hypotheses and notations as above, for each positive integer $N$, there exists a substack $\mathrm{Hilb}_{G/R}^N$ of $\mathrm{Hilb}_{G/R}$ such that given any $R'$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, the space $\mathrm{Hilb}_{G/R}^N(R') =: \mathrm{Hilb}^N(G_{R'}/R') \subset \mathrm{Hilb}(G_{R'}/R')$ consists of those $\mathsf{D} \to G_{R'}$ locally free of rank $N$ over $R'$.*

**Definition 3.12.** Let $R$ be a connective $\mathbb{E}_\infty$-ring and $G$ be a spectral commutative finite flat group scheme over $R$. Given a finite abelian group $A$ of order $N$, a *level-$A$ structure on $G$* is a pair $(\mathsf{D}, \phi)$, where $i\colon \mathsf{D} \to G$ is an object in $\mathrm{Hilb}^N(G/R)$ and $\phi\colon A \to G^\heartsuit(\pi_0 R)$ is a homomorphism, such that $(\mathsf{D}^\heartsuit, \phi)$ is an $A$-structure in the sense of [KM85, Remark 1.10.10], i.e., $\pi_0 i_* \mathscr{O}_{\mathsf{D}} = \bigotimes_{a \in A} \phi(a)_* \mathscr{O}_{\mathrm{Spec}\, \pi_0 R}$. We denote by $\mathrm{Level}(A, G/R)$ the $\infty$-category of level-$A$ structures on $G/R$.

*Remark* 3.13. Given a level-$A$ structure $(\mathsf{D}, \phi)$ on $G$, $\mathsf{D}$ is locally free of rank $N$ over $R$, since $\mathsf{D} \to G$ is a closed immersion, $\mathsf{D} \to \mathrm{Sp\acute{e}t}\, R$ is flat, and $\pi_0 i_* \mathscr{O}_{\mathsf{D}} = \bigotimes_{a \in A} \phi(a)_* \mathscr{O}_{\mathrm{Spec}\, \pi_0 R}$. The last identity also ensures the group structure on $\mathsf{D}^\heartsuit$.

*Remark* 3.14. Comparing Definition 3.12 with Definition 3.2, we see that [KM85, Proposition 1.10.6] establishes an equivalence between the two definitions in the classical case where $G^\heartsuit/\pi_0 R$ is embeddable as a closed subscheme of an elliptic curve $E^{\heartsuit}/\pi_0 R$. Thus these two definitions are compatible if the spectral group scheme $G/R$ is embeddable as a closed substack of a spectral elliptic curve $\mathsf{E}/R$.

To prove representability for the functor of level-$A$ structures, we present a second result concerning the existence of incidence spectral Deligne–Mumford stacks (cf. Proposition 3.5).

**Proposition 3.15.** *Let $G/R$ be a spectral commutative finite flat group scheme over a connective $\mathbb{E}_\infty$-ring $R$. Let $A$ be a finite abelian group of order $N$. Given an object $\mathsf{D} \to G$ in $\mathrm{Hilb}^N(G/R)$, there exists an $\mathbb{E}_\infty$-ring $W$ satisfying the following universal property:*

> *For any $R \to R'$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, $\mathsf{D}_{R'}$ affords a level-$A$ structure on $G_{R'}$ if and only if $R \to R'$ factors through $W$.*

*Proof.* Given $R'$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, it is clear that $\mathsf{D}_{R'}$ is in $\mathrm{Hilb}(G_{R'}/R')$. For $\mathsf{D}_{R'}$ to afford a level-$A$ structure as in Definition 3.12, $\operatorname{Spec} \pi_0 R' \to \operatorname{Spec} \pi_0 R$ must factor through $\mathrm{Hom}(A, G^{\heartsuit})$, which is an open substack of the classical stack $\mathrm{Hilb}_{G^{\heartsuit}/\pi_0 R}$ by Lemma 3.7. Thus $\pi_0 R \to \pi_0 R'$ factors through $B_0$ for some localization $B_0$ of $\pi_0 R$. This lifts to a factorization of $R \to R'$ through an $\mathbb{E}_\infty$-ring $B$, which is a localization of $R$ with $\pi_0 B \simeq B_0$ (see [Lur18c, Remark 1.1.4.2]).

By now, along the map $\operatorname{Sp\acute{e}t} R' \to \operatorname{Sp\acute{e}t} B$, we already have $i\colon \mathsf{D}_{R'} \to G_{R'}$ in $\mathrm{Hilb}^N(G_{R'}/R')$ and a map $\phi\colon A \to G^{\heartsuit}(\pi_0 R')$ associated with $\pi_0 i_* \mathscr{O}_{\mathsf{D}_{R'}}$ (up to automorphism). For $(\mathsf{D}_{R'}, \phi)$ to be a level-$A$ structure,

$$\bigotimes_{a \in A} \phi(a)_* \mathscr{O}_{\operatorname{Spec} \pi_0 R'} \to \pi_0 i_* \mathscr{O}_{\mathsf{D}_{R'}}$$

needs to be an isomorphism, i.e., the $N$ points $\phi(a), a \in A$ must form a full set of sections of $\mathsf{D}_{R'}^{\heartsuit}$. By [KM85, Proposition 1.9.1], $\operatorname{Spec} \pi_0 R' \to \operatorname{Spec} \pi_0 B$ must then factor through a closed subscheme of $\operatorname{Spec} \pi_0 B$ defined by finitely many equations. Thus $\pi_0 B \to \pi_0 R'$ factors through $\pi_0 B/I$ for some finitely generated ideal $I$. Analogous to the proof of (1) $\Longrightarrow$ (3) in Proposition 3.5, as a $B$-module, $R'$ is $I$-complete by [Lur18c, Propositions 7.1.5.3 and 7.3.1.7], and the morphism $B \to R'$ factors through $B_I^{\wedge}$ in $\mathrm{Mod}_R$ to give a morphism $B_I^{\wedge} \to R'$ in $\mathrm{Mod}_R^{\mathrm{Cpl}(I)}$ by restriction. In view of [Lur18c, Variant 7.3.5.6], we obtain the $\mathbb{E}_\infty$-ring $W \coloneqq B_I^{\wedge}$ as desired.

We can show the converse by a similar argument to the one in the proof of Proposition 3.5. □

*Remark* 3.16. As in Proposition 3.5, we can equivalently construct the incidence object using the local cohomology functor $\Gamma_I$ with respect to the same ideal $I \subset \pi_0 B$. However, unlike the $I$-completion functor, $\Gamma_I$ does not automatically preserve the monoidal structure (cf. [Lur18c, Section 7.3.5]).

*Remark* 3.17. For representability of the Hilbert functor, Lurie proved [Lur04, Theorem 8.3.3] with greater generality for relative spectral algebraic spaces $\mathsf{X}/\mathsf{S}$ (cf. Theorem 3.10 and also Remark 2.19). As the statement and proof of Lemma 3.15 are local on the base (see, in particular, [KM85, Proposition 1.9.1]), given $\mathsf{X}/\mathsf{S}$ and $\mathsf{D} \to \mathsf{X}$ in $\mathrm{Hilb}^N(\mathsf{X}/\mathsf{S})$, the incidence affine spectral Deligne–Mumford stacks $\operatorname{Sp\acute{e}t} B$ (resp. $\operatorname{Sp\acute{e}t} W$) from above glue into a global one $\mathsf{B}$ (resp. $\mathsf{W}$) over $\mathsf{S}$.

**Theorem 3.18.** *Suppose that $G$ is a spectral commutative finite flat group scheme over a connective $\mathbb{E}_\infty$-ring $R$ and $A$ is a finite abelian group. Then the functor*

$$\mathrm{Level}^A_{G/R}\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{Level}(A, G_{R'}/R')$$

*is representable by an affine spectral Deligne–Mumford stack* $\mathrm{Sp\acute{e}t}\,\mathcal{P}_{G/R}$.

*Proof.* We first prove the representability. By definition, the functor $\mathrm{Level}^A_{G/R}$ is a subfunctor of the representable functor $\mathrm{Hilb}^N_{G/R}$, where $N = \#A$. In view of Remark 3.17, applying Lemma 3.15 and the intermediate results in its proof to the universal object $\mathsf{D}_{\mathrm{univ}} \to G_{\mathrm{Hilb}^N_{G/R}}$ over the spectral algebraic space $\mathrm{Hilb}^N_{G/R}$, we obtain below from right to left consecutive pullbacks of universal objects, with the subscripts of $G$ for base change omitted:

$$\begin{tikzcd}
(G, \text{univ. level-}A \text{ str. on } G) \arrow[r] \arrow[d] & (G, \mathsf{D}_{\mathrm{univ}}, \phi_{\mathrm{univ}}) \arrow[r] \arrow[d] & (G, \mathsf{D}_{\mathrm{univ}}) \arrow[r] \arrow[d] & G \arrow[d] \\
\mathsf{W} \arrow[r] & \mathsf{B} \arrow[r] & \mathrm{Hilb}^N_{G/R} \arrow[r] & \mathrm{Sp\acute{e}t}\,R
\end{tikzcd}$$

It is straightforward to verify that $\mathsf{W}$ valued on an $R$-algebra $R'$ is precisely the space of level-$A$ structures on $G_{R'}$.

For the affineness property, we need to prove that $\mathsf{W}$ is finite over $R$ in the sense of spectral algebraic geometry. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary algebraic spaces is finite in the sense of classical algebraic geometry. Thus we need only prove that $\mathsf{W}^\heartsuit$ is finite over $\pi_0 R$. Again as in the proof of Theorem 3.6, given the definitions of the derived and classical moduli problems in question, this is a consequence of Proposition 3.9. □

3.2.3. *Spectral $p$-divisible groups.* Given an $\mathbb{E}_\infty$-ring $R$, let $\mathrm{FFG}(R)$ denote the $\infty$-category of spectral commutative finite flat group schemes over $R$. Let

$$X\colon (\mathrm{Ab}^p_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{FFG}(R)$$

be a spectral $p$-divisible group of height $h$ over an $\mathbb{E}_\infty$-ring $R$ (see [Lur18a, Definition 6.5.1] and cf. [Lur18b, Definition 2.0.2]). For each nonnegative integer $r$, we write $X[p^r]$ for the image of $\mathbf{Z}/p^r\mathbf{Z}$ under $X$, which is a degree-$(p^r)^h$ spectral commutative finite flat group scheme over $R$.

**Definition 3.19.** Let $\mathbf{G}$ be a spectral $p$-divisible group of height $h$ over a connective $\mathbb{E}_\infty$-ring $R$. A *level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on* $\mathbf{G}$ is a level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on $\mathbf{G}[p^r]$ as in Definition 3.12. We let $\mathrm{Level}(r, \mathbf{G}/R)$ denote the $\infty$-groupoid of level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structures on $\mathbf{G}/R$.

*Remark* 3.20. Recall that a level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on $\mathbf{G}[p^r]$ is a pair $(\mathsf{D}, \phi)$, where $\mathsf{D} \subset \mathbf{G}[p^r]$ is a finite flat closed substack of rank $(\mathbf{Z}/p^r\mathbf{Z})^h$ over $R$, and $\phi : (\mathbf{Z}/p^r\mathbf{Z})^h \to \mathbf{G}[p^r]^\heartsuit(\pi_0 R)$ is a homomorphism such that $(\mathsf{D}^\heartsuit, \phi)$ is a level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on $\mathbf{G}[p^r]^\heartsuit/\pi_0 R$. Given such a structure $(\mathsf{D}, \phi)$, since $\mathbf{G}[p^r]$ is locally free of rank $(p^r)^h$ over $R$, the rank of $\mathsf{D}^\heartsuit$ over $\pi_0 R$ equals that of $\mathbf{G}[p^r]^\heartsuit$. Since $\mathsf{D}^\heartsuit$ is a closed subscheme of $\mathbf{G}[p^r]^\heartsuit$, they must then equal. Thus $\phi$ is a $(\mathbf{Z}/p^r\mathbf{Z})^h$*-generator of* $\mathbf{G}[p^r]^\heartsuit(\pi_0 R)$ in the sense of [KM85, 1.10.5]. Note that as spectral Deligne–Mumford stacks, even though $\mathsf{D}$ has the same rank as $\mathbf{G}[p^r]$, they are not equivalent, since closed immersions in spectral algebraic geometry are not categorical monomorphisms (see [Lur18a, Warning 6.2.3]). For this reason, we do not introduce the concept of $A$-generators when discussing derived level structures, and this is where the higher homotopical information of *derived* level structures resides.

**Theorem 3.21.** *Let* $\mathbf{G}$ *be a spectral* $p$*-divisible group of height* $h$ *over a connective* $\mathbb{E}_\infty$*-ring* $R$*. Then the functor*

$$\mathrm{Level}^r_{\mathbf{G}/R}\colon \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}, \quad R' \mapsto \mathrm{Level}(r, \mathbf{G}_{R'}/R')$$

*is representable by an affine spectral Deligne–Mumford stack* $\mathrm{Sp\acute{e}t}\,\mathcal{P}^r_{\mathbf{G}/R}$*.*

*Proof.* We just notice that by the definition of a spectral $p$-divisible group, $\mathbf{G}[p^r]$ is a spectral commutative finite flat group scheme. Thus the statement follows from Theorem 3.18 above about general spectral commutative finite flat group schemes. □

*Remark* 3.22. Our derived level structure above is defined for spectral $p$-divisible groups over a *connective* $\mathbb{E}_\infty$-ring. More generally, in view of [Lur18b, Variant 2.0.6] and [Lur18a, Remarks 6.1.3 and 0.0.4], we can define such structures on $\mathbf{G}/R$ where $R$ is not necessarily connective, by setting

$$\mathrm{Level}(r, \mathbf{G}/R) \coloneqq \mathrm{Level}(r, \mathbf{G}/\tau_{\geqslant 0}R) \tag{3.1}$$

The corresponding functor $\mathrm{Level}^r_{\mathbf{G}/R} : \mathrm{CAlg}_R \to \mathcal{S}$ is representable by the fiber product

$$\mathrm{Sp\acute{e}t}\,\mathcal{P}^r_{\mathbf{G}/\tau_{\geqslant 0}R} \times_{\mathrm{Sp\acute{e}t}\,\tau_{\geqslant 0}R} \mathrm{Sp\acute{e}t}\,R \simeq \mathrm{Sp\acute{e}t}\,(\mathcal{P}^r_{\mathbf{G}/\tau_{\geqslant 0}R} \otimes_{\tau_{\geqslant 0}R} R)$$

via base change along the canonical map $\tau_{\geqslant 0}R \to R$. Indeed, with justifications given below, we have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{P}^r_{\mathbf{G}/\tau_{\geqslant 0}R} \otimes_{\tau_{\geqslant 0}R} R, R') &\simeq \mathrm{Map}_{\mathrm{CAlg}_{\tau_{\geqslant 0}R}}(\mathcal{P}^r_{\mathbf{G}/\tau_{\geqslant 0}R}, R') \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R}}(\mathcal{P}^r_{\mathbf{G}/\tau_{\geqslant 0}R}, \tau_{\geqslant 0}R') \\
&\simeq \mathrm{Level}\big(r, \mathbf{G}_{\tau_{\geqslant 0}R'}/\tau_{\geqslant 0}R'\big) \\
&= \mathrm{Level}(r, \mathbf{G}_{\tau_{\geqslant 0}R'}/R') \\
&= \mathrm{Level}(r, \mathbf{G}_{R'}/R') \\
&= \mathrm{Level}^r_{\mathbf{G}/R}(R')
\end{aligned}$$

Here, the first equivalence follows from an adjunction between extension and restriction of scalars along $\tau_{\geqslant 0}R \to R$. The second equivalence follows from an adjunction between inclusion of connective objects and taking connective covers, as $\mathcal{P}^r_{\mathbf{G}/\tau_{\geqslant 0}R}$ is connective. The third equivalence is a consequence of Theorem 3.21. Next, the first equality follows from (3.1), and the second from [Lur18b, Remark 2.0.7].

In the nonconnective case, the *relative* representability of level structures with respect to $\mathbf{G}/R$ stands in contrast to the *absolute* representability, e.g., of the functor of spectral elliptic curves as treated in [Lur18a, proof of Theorem 2.4.1], particularly its last paragraph.

This generality will be useful in Section 4.2.2 when we consider *oriented* deformations of spectral $p$-divisible groups, whose classifying $\mathbb{E}_\infty$-ring is not connective (cf. [Gre25, BDL25] for its significance). The authors thank Yuchen Wu for bringing the nonconnective case to their attention and for related discussions.

3.2.4. *Non-full level structures.* So far we have treated only full level structures on commutative finite flat group schemes. Here let us consider more general level structures, such as those relevant for power operations in Morava E-theories (see Section 4.3).

**Definition 3.23.** Suppose that $G$ is a spectral commutative finite flat group scheme over a connective $\mathbb{E}_\infty$-ring $R$. We let $\mathrm{Level}_1(r, G/R)$ denote the $\infty$-groupoid of derived level-$(\mathbf{Z}/p^r\mathbf{Z})$ structures on $G/R$. We let $\mathrm{Level}_0(r, G/R)$ denote the $\infty$-groupoid of equivalence classes $(\mathsf{D}, \phi)$ in $\mathrm{Level}_1(r, G/R)$ where two objects $(\mathsf{D}, \phi)$ and $(\mathsf{D}', \phi')$ are equivalent if the scheme-theoretic image of $\mathsf{D}^\heartsuit$ under $\phi$ and that of $(\mathsf{D}')^\heartsuit$ under $\phi'$ are equal in $G^\heartsuit/\pi_0 R$.

*Remark* 3.24. Our notations above are intended to be consistent with the standard ones $\Gamma_1(p^r)$, $\Gamma_0(p^r)$, etc. for the classical moduli problems. We drop the prime $p$ altogether for readability when the level appears in a superscript, as the results here apply to all primes.

**Proposition 3.25.** *Hypotheses and notations as above, for each $i \in \{0, 1\}$, the functor*

$$\mathrm{Level}^{i,r}_{G/R} \colon \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}, \quad R' \mapsto \mathrm{Level}_i(r, G_{R'}/R')$$

*is representable by an affine spectral Deligne–Mumford stack* $\mathrm{Sp\acute{e}t}\, \mathcal{P}^{i,r}_{G/R}$.

*Proof.* The statement for $i = 1$ is a direct consequence of the more general Theorem 3.18. For $i = 0$, we just notice that the classical level structure functor $\mathrm{Level}^{0,r}_{G^\heartsuit/\pi_0 R}$ is representable by a closed subscheme of the Grassmannian of all rank-$p^r$ quotients of $G^\heartsuit[p^r]$ (cf. [KM85, Theorem 6.6.1 and proof of Proposition 6.5.1]). By an argument analogous to that for the case of full level structures, we obtain the desired result. □

*Remark* 3.26. From the above proposition, we obtain analogous representability results for spectral $p$-divisible groups as in Section 3.2.3.

## 4. Moduli problems of derived level structures

In this section, we apply the derived level structures and their representability results from Section 3 and discuss several related spectral moduli problems.

### 4.1. Spectral elliptic curves with level structure.

In Section 3.1, given a finite abelian group $A$, we defined level-$A$ structures for spectral elliptic curves (Definition 3.2) and showed their representability relative to an object $\mathsf{E}/R$ (Theorem 3.6). Here, we consider their absolute representability (cf. [KM85, Sections 4.2–4.3]).

There exists a spectral Deligne–Mumford stack $\mathcal{M}_{\mathrm{Ell}}$ whose functor of points is

$$\mathcal{M}_{\mathrm{Ell}} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}, \quad R \mapsto \mathcal{M}_{\mathrm{Ell}}(R) = \mathrm{Ell}(R)^{\simeq}$$

where $\mathrm{Ell}(R)^{\simeq}$ is the underlying $\infty$-groupoid of the $\infty$-category of spectral elliptic curves over $R$ [Lur18a, Theorem 2.4.1].

In classical algebraic geometry, we have the Deligne–Mumford stack of (ordinary) elliptic curves, which can be viewed as a spectral Deligne–Mumford stack

$$\mathcal{M}^{\mathrm{cl}}_{\mathrm{Ell}} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}, \quad R \mapsto \mathcal{M}^{\mathrm{cl}}_{\mathrm{Ell}}(\pi_0 R)$$

where $\mathcal{M}^{\mathrm{cl}}_{\mathrm{Ell}}(\pi_0 R)$ is the groupoid of elliptic curves over the commutative ring $\pi_0 R$. Moreover, if $A$ equals $\mathbf{Z}/N\mathbf{Z}$ or $(\mathbf{Z}/N\mathbf{Z})^2$ with $N \geqslant 1$ an integer, we have the Deligne–Mumford stack of elliptic curves with level-$A$ structures, which can also be viewed as a spectral Deligne–Mumford stack

$$\mathcal{M}^{\mathrm{cl},A}_{\mathrm{Ell}} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}, \quad R \mapsto \mathcal{M}^{\mathrm{cl},A}_{\mathrm{Ell}}(\pi_0 R)$$

where $\mathcal{M}^{\mathrm{cl},A}_{\mathrm{Ell}}(\pi_0 R)$ is the groupoid of elliptic curves with level-$A$ structure over $\pi_0 R$.

In Section 3.1, for derived level-$A$ structures, the assignment $\mathsf{X} \mapsto \mathrm{Level}(A, \mathsf{X}/R)$ determines a functor $\mathrm{Ell}(R) \to \mathcal{S}$ which classifies a left fibration $\mathrm{Ell}^A(R) \to \mathrm{Ell}(R)$ of $\infty$-categories by the unstraightening construction (see [Lur09b, Definition 3.3.2.2 and Section 2.2.1]). Objects of $\mathrm{Ell}^A(R)$ are triples $(\mathsf{E}, \mathsf{D}, \phi)$ where $\mathsf{E}$ is a spectral elliptic curve over $R$ and $(\mathsf{D}, \phi)$ is a derived level-$A$ structure on $\mathsf{E}$ as in Definition 3.2. For each $R \in \mathrm{CAlg}^{\mathrm{cn}}$, consider all spectral elliptic curves over $R$ with level-$A$ structure. This moduli problem can be thought of as a functor

$$\mathcal{M}^A_{\mathrm{Ell}} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}, \quad R \mapsto \mathrm{Ell}^A(R)^{\simeq}$$

where $\mathrm{Ell}^A(R)^{\simeq}$ is the space of spectral elliptic curves $\mathsf{E}/R$ with a derived level-$A$ structure $(\mathsf{D}, \phi)$. To prove its representability, we proceed as follows.

**Lemma 4.1.** *For each discrete commutative $R_0$, the space $\mathcal{M}^A_{\mathrm{Ell}}(R_0)$ is 1-truncated.*

*Proof.* This follows from the fact that the classical moduli problem above is represented by a Deligne–Mumford 1-stack. $\square$

**Lemma 4.2.** *The functor $\mathcal{M}^A_{\mathrm{Ell}} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is an étale sheaf.*

*Proof.* Let $\{R \to U_i\}$ be an étale cover of $R$, and $U_\bullet$ be the associated Čech-simplicial object. Consider the diagram

$$\begin{tikzcd} \mathrm{Ell}^A(R)^{\simeq} \arrow[r, "f"] \arrow[d, "p"] & \varprojlim_{\mathbf{\Delta}} \mathrm{Ell}^A(U_\bullet)^{\simeq} \arrow[d, "q"] \\ \mathrm{Ell}(R)^{\simeq} \arrow[r, "g"] & \varprojlim_{\mathbf{\Delta}} \mathrm{Ell}(U_\bullet)^{\simeq} \end{tikzcd}$$

The map $p$ is a left fibration between Kan complexes, and so is a Kan fibration by [Lur09b, Lemma 2.1.3.3]. The map $q$ is a pointwise Kan fibration. By picking the projective model structure for the homotopy limit we may assume that $q$ is a Kan fibration as well. The map $g$ is an equivalence by [Lur18a, Theorem 2.4.1]. To show that $f$ is an equivalence, we need only show that for every $\mathsf{E} \in \mathrm{Ell}(R)$, the map

$$p^{-1}(\mathsf{E}) \simeq \mathrm{Level}(A, \mathsf{E}/R) \to \varprojlim_{\mathbf{\Delta}} \mathrm{Level}(A, \mathsf{E} \times_R U_\bullet/U_\bullet) \simeq q^{-1}g(\mathsf{E})$$

is an equivalence. Observe that $\mathrm{Level}(A, \mathsf{E}/R)$ is a full $\infty$-subcategory of $\mathrm{CDiv}(\mathsf{E}/R)$ and $\varprojlim_{\mathbf{\Delta}} \mathrm{Level}(A, \mathsf{E} \times_R U_\bullet/U_\bullet)$ is a full $\infty$-subcategory of $\varprojlim_{\mathbf{\Delta}} \mathrm{CDiv}(\mathsf{E} \times_R U_\bullet/U_\bullet)$. Since $\mathrm{CDiv}_{\mathsf{E}/R}$ is an étale sheaf by Lemma 2.21, the functor

$$\mathrm{Level}(A, \mathsf{E}/R) \to \varprojlim_{\mathbf{\Delta}} \mathrm{Level}(A, \mathsf{E} \times_R U_\bullet/U_\bullet)$$

is fully faithful. To show that it is an equivalence, we need only show that it is essentially surjective.

Given any $\{(\mathsf{D}_{U_\bullet}, \phi_{U_\bullet})\}$ in $\varprojlim_{\mathbf{\Delta}} \mathrm{Level}(A, \mathsf{E} \times_R U_\bullet/U_\bullet)$, clearly we can find a morphism $\mathsf{D} \to \mathsf{E}$ in $\mathrm{CDiv}(\mathsf{E}/R)$ whose image under the equivalence

$$\mathrm{CDiv}(\mathsf{E}/R) \simeq \varprojlim_{\mathbf{\Delta}} \mathrm{CDiv}(\mathsf{E} \times_R U_\bullet/U_\bullet)$$

is $\{\mathsf{D}_{U_\bullet} \to \mathsf{E} \times_R U_\bullet\}$, along with $\phi \colon A \to \mathsf{E}^{\heartsuit}(\pi_0 R)$ lifting $\{\phi_{U_\bullet}\}$. It remains to show that $(\mathsf{D}, \phi)$ is a derived level-$A$ structure. This is true because in the classical case,

$$\mathrm{Level}(A, \mathsf{E}^{\heartsuit}/\pi_0 R) \simeq \varprojlim_{\mathbf{\Delta}} \mathrm{Level}(A, \mathsf{E}^{\heartsuit} \times_{\pi_0 R} \pi_0 U_\bullet/\pi_0 U_\bullet) \qquad \square$$

**Lemma 4.3.** *The functor $\mathcal{M}^A_{\mathrm{Ell}}\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is nilcomplete, infinitesimally cohesive, and integrable.*

*Proof.* Consider the following diagram in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$:

$$\begin{tikzcd} \mathcal{M}^A_{\mathrm{Ell}} \arrow[r, "f"] \arrow[dr, "h"'] & \mathcal{M}_{\mathrm{Ell}} \arrow[d, "g"] \\ & * \end{tikzcd} \tag{4.1}$$

By [Lur18c, Remark 17.3.7.3], since $\mathcal{M}_{\mathrm{Ell}}$ is nilcomplete, infinitesimally cohesive, and integrable from [Lur18a, Theorem 2.4.1], we need only prove that $f$ is so. By [Lur18c, Proposition 17.3.8.4], $f$ has these properties if and only if each fiber of $f$ does, i.e., for each $R \in \mathrm{CAlg}^{\mathrm{cn}}$ and a point $\eta_{\mathsf{E}} \in \mathcal{M}_{\mathrm{Ell}}(R)$ which represents a spectral elliptic curve $\mathsf{E}$, the functor

$$\mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{M}^A_{\mathrm{Ell}}(R') \times_{\mathcal{M}_{\mathrm{Ell}}(R')} \{\eta_{\mathsf{E}}\}$$

is nilcomplete, infinitesimally cohesive, and integrable. This functor is precisely $\mathrm{Level}^A_{\mathsf{E}/R}$, which is so by Theorem 3.6. □

**Lemma 4.4.** *The functor $\mathcal{M}^A_{\mathrm{Ell}}$ admits a cotangent complex which is connective and almost perfect.*

*Proof.* Again, let us consider the diagram (4.1). Then [Lur18c, Proposition 17.3.9.1] reduces us to proving that $f$ admits a cotangent complex.

By [Lur18c, Proposition 17.2.4.7], a morphism $j : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ admits a cotangent complex if, for any corepresentbale $Y' \simeq \mathrm{Map}(R, -)\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and any natural transformation $Y' \to Y$, $j'$ in the following pullback diagram admits a cotangent complex:

$$\begin{tikzcd} Y' \times_Y X \arrow[r] \arrow[d, "j'"] & X \arrow[d, "j"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

Thus, to prove that $\mathcal{M}^A_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{Ell}}$ admits a cotangent complex, we need only prove that, for any $R \in \mathrm{CAlg}^{\mathrm{cn}}$ and any spectral elliptic curve $\mathsf{E}$ which corresponds to a natural transformation $\mathrm{Sp\acute{e}t}\, R \to \mathcal{M}_{\mathrm{Ell}}$, or to $\eta_{\mathsf{E}} \in \mathcal{M}_{\mathrm{Ell}}(R)$, the functor

$$\mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{M}^A_{\mathrm{Ell}}(R') \times_{\mathcal{M}_{\mathrm{Ell}}(R')} \{\eta_{\mathsf{E}}\}$$

admits a cotangent complex. Again we identify this functor as $\mathrm{Level}^A_{\mathsf{E}/R}$ and apply Theorem 3.6. Moreover, the properties of the desired cotangent complex being connective and almost perfect also follow from those associated with $\mathrm{Level}^A_{\mathsf{E}/R}$. □

**Lemma 4.5.** *The functor $\mathcal{M}^A_{\mathrm{Ell}}$ is locally almost of finite presentation.*

*Proof.* The morphism $h\colon \mathcal{M}^A_{\mathrm{Ell}} \to *$ in (4.1) is infinitesimally cohesive and admits an almost perfect cotangent complex. By [Lur18c, 17.4.2.2], it is locally almost of finite presentation. Therefore $\mathcal{M}^A_{\mathrm{Ell}}$ is locally almost of finite presentation, since $*$ is a final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. □

Here is a generalization of [Lur18a, Theorem 2.4.1].

**Theorem 4.6.** *The functor*

$$\mathcal{M}^A_{\mathrm{Ell}}\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}, \quad R \mapsto \mathcal{M}^A_{\mathrm{Ell}}(R) = \mathrm{Ell}^A(R)^{\simeq}$$

*is representable by a spectral Deligne–Mumford* 1*-stack which are locally almost of finite presentation over the sphere spectrum.*

*Proof.* We apply Theorem 2.10 and verify the set of conditions one by one through the above series of lemmas. □

*Remark* 4.7. With motivation from chromatic homotopy theory, as a variant of [Lur18a, Theorem 2.4.1], Lurie proved the representability of a functor of *oriented* elliptic curves by a nonconnective spectral Deligne–Mumford stack [Lur18b, Proposition 7.2.10] (cf. [Lur18a, Warning 0.0.5]). In particular, taking global sections of its structure sheaf, he recovered the periodic $\mathbb{E}_\infty$-ring spectrum TMF of topological modular forms previously constructed by Goerss, Hopkins, and Miller through obstruction theory [Lur18b, Definition 7.0.3]. Given his results, Theorem 4.6 is readily adaptable to oriented elliptic curves with level structure and their corresponding periodic spectra of topological modular forms with level structure, though we do not include the details here. The interested reader may compare Theorems 4.10 and 4.13 below, including their proofs. In view of [HL16], it would be interesting to discuss suitably compactified objects in this setting as well (cf. [DL25]).

4.2. **Higher-homotopical Lubin–Tate towers.** Based on Section 3.2.3, our goal in this subsection is to generalize to higher levels some of the representability results on *deformations* of $p$-divisible groups from [Lur18b, §3 and §6].

4.2.1. *Unoriented deformations.* Let $\mathbf{G}_0$ be a nonstationary $p$-divisible group over a commutative ring $R_0$ of height $h$, and $R$ be a complete adic $\mathbb{E}_\infty$-ring. Recall from [Lur18b, Definitions 3.1.4 and 3.1.1] that a *deformation of* $\mathbf{G}_0$ *over* $R$ is a spectral $p$-divisible group $\mathbf{G}$ over $R$ together with an equivalence class of $\mathbf{G}_0$-taggings of $\mathbf{G}$. A main result therein is a representability theorem with the *spectral deformation ring* $R^{\mathrm{un}}_{\mathbf{G}_0}$ corepresenting the moduli problem of such deformations [Lur18b, Theorem 3.0.11] (see also [Lur18b, Theorems 3.1.15 and 3.4.1]). In particular, $R^{\mathrm{un}}_{\mathbf{G}_0}$ is a connective $\mathbb{E}_\infty$-ring.

**Definition 4.8.** As in Section 3.2.3, we let $\mathrm{Level}(r, \mathbf{G}/R)$ denote the space of derived level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structures on $\mathbf{G}$. Consider the functor

$$\mathrm{Def}^r_{\mathbf{G}_0}\colon \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}, \quad R \mapsto \mathrm{Def}(r, \mathbf{G}_0, R)$$

Here $\mathrm{Def}(r, \mathbf{G}_0, R)$ denotes the $\infty$-category whose objects are triples $(\mathbf{G}, \alpha, \lambda)$ such that

- $\mathbf{G}$ is a spectral $p$-divisible group over $R$,
- $\alpha$ is an equivalence class of $\mathbf{G}_0$-taggings of $\mathbf{G}$, and
- $\lambda \in \mathrm{Level}(r, \mathbf{G}/R)$ as in Definition 3.19 and Remark 3.22.

We call each object $(\mathbf{G}, \alpha, \lambda)$ a *deformation with level-*$(\mathbf{Z}/p^r\mathbf{Z})^h$ *structure of* $\mathbf{G}_0$ *over* $R$.

*Remark* 4.9. Given $\alpha$ as above, a derived level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure $\lambda$ on $\mathbf{G}/R$ determines a level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on $\mathbf{G}_0/R_0$ (up to an extension of scalars) by base change along $\alpha$.

**Theorem 4.10.** *With the above hypotheses and notations, the functor* $\mathrm{Def}^r_{\mathbf{G}_0}$ *is corepresentable by an* $\mathbb{E}_\infty$*-ring whose 0'th homotopy group is finite over that of the spectral deformation ring* $R^{\mathrm{un}}_{\mathbf{G}_0}$.

*Proof.* Here we take an alternative approach from our proof of Theorem 4.6. Following [KM85, 4.3.4], we view this moduli problem as a product of a representable one and a relatively representable one, and apply [Lur18b, Theorem 3.1.15] and Theorem 3.21 consecutively.

To be specific, let $\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}$ denote the universal deformation of $\mathbf{G}_0/R_0$ from [Lur18b, Theorem 3.1.15]. Suppose that $\mathbf{G}$ is a spectral deformation of $\mathbf{G}_0$ to $R$ classified by a map of connective $\mathbb{E}_\infty$-rings $R^{\mathrm{un}}_{\mathbf{G}_0} \to R$, along which $\mathbf{G} \simeq \mathbf{G}^{\mathrm{un}}_{\mathrm{univ}} \times_{R^{\mathrm{un}}_{\mathbf{G}_0}} R$ as spectral $p$-divisible groups over $R$. We then obtain from Remark 3.22 and Theorem 3.21 equivalences

$$\begin{aligned}
\mathrm{Level}(r, \mathbf{G}/R) &\simeq \mathrm{Level}(r, \mathbf{G}^{\mathrm{un}}_{\mathrm{univ}} \times_{R^{\mathrm{un}}_{\mathbf{G}_0}} R/R)\\
&= \mathrm{Level}(r, \mathbf{G}^{\mathrm{un}}_{\mathrm{univ}} \times_{R^{\mathrm{un}}_{\mathbf{G}_0}} \tau_{\geqslant 0}R/\tau_{\geqslant 0}R)\\
&\simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{R^{\mathrm{un}}_{\mathbf{G}_0}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}}, \tau_{\geqslant 0}R)\\
&\simeq \mathrm{Map}_{\mathrm{CAlg}_{R^{\mathrm{un}}_{\mathbf{G}_0}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}}, R)
\end{aligned}$$

where $\mathcal{P}^r_{\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}}$ classifies derived level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structures on the universal deformation $\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}$ as a spectral $p$-divisible group, with $\pi_0\mathcal{P}^r_{\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}}$ finite over $\pi_0 R^{\mathrm{un}}_{\mathbf{G}_0}$ by affineness (cf. [Lur18c, Remark 5.2.0.2]).

Let us verify that the $\mathbb{E}_\infty$-ring $\mathcal{P}^r_{\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}}$ is as desired. Consider the functor

$$\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}, \quad R \mapsto \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}}, R)$$

Given $R \in \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$, $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}}, R)$ can be viewed as the $\infty$-category of pairs $(f, g)$ where $f\colon R^{\mathrm{un}}_{\mathbf{G}_0} \to R$ along the structure morphism of Spét $\mathcal{P}^r_{\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}}$ over $R^{\mathrm{un}}_{\mathbf{G}_0}$ classifies a deformation $(\mathbf{G}, \alpha)$ over $R$ of $\mathbf{G}_0/R_0$, and

$$g \in \mathrm{Map}_{\mathrm{CAlg}_{R^{\mathrm{un}}_{\mathbf{G}_0}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}}, R) \simeq \mathrm{Level}(r, \mathbf{G}/R)$$

along the restriction from $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ to CAlg (cf. claim (ii) in [Lur18b, proof of Theorem 3.4.1]) specifies a derived level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure $\lambda$ on $\mathbf{G}/R$. Thus we recover precisely the functor $\mathrm{Def}^r_{\mathbf{G}_0}$ as in Definition 4.8. □

Although we have obtained $\mathbb{E}_\infty$-rings as classifying objects associated with the above spectral moduli problems, these spectra may be complicated, and we do not know yet their homotopy groups in general. See the next example and further discussion in Section 5.4.1 below.

**Example 4.11** (Lurie)**.** At height 1, in the case where $R_0 = \mathbf{F}_p$ and $\mathbf{G}_0 = \mu_{p^\infty}$, the functor $\mathrm{Def}^0_{\mathbf{G}_0}$ is corepresented by the $p$-completed sphere spectrum [Lur18b, Warning 3.0.15 and Corollary 3.1.19] (cf. Remark 3.20).

4.2.2. *Oriented deformations.* In algebraic topology, the orientation of an $\mathbb{E}_\infty$-ring spectrum makes the $E_2$-page of its associated Atiyah–Hirzebruch spectral sequence degenerate and gives us certain information of its homotopy groups.

**Definition 4.12.** Let $\mathbf{G}_0$ be a height-$h$ $p$-divisible group over $R_0$ as above. Define the functor

$$\mathrm{Def}^{\mathrm{or},r}_{\mathbf{G}_0}\colon \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}, \quad R \mapsto \mathrm{Def}^{\mathrm{or}}(r, \mathbf{G}_0, R)$$

where $\mathrm{Def}^{\mathrm{or}}(r, \mathbf{G}_0, R)$ is the space of *oriented deformation* $(\mathbf{G}, \alpha, e, \lambda)$ *of* $\mathbf{G}_0$ *over* $R$ *with level-*$(\mathbf{Z}/p^r\mathbf{Z})^h$ *structure*, with

- $\mathbf{G}$ a spectral $p$-divisible group over $R$,
- $\alpha$ an equivalence class of $\mathbf{G}_0$-taggings of $\mathbf{G}$,
- $e\colon S^2 \to \Omega^\infty \mathbf{G}^\circ(\tau_{\geqslant 0}R)$ an orientation of the identity component of $\mathbf{G}$, and
- $\lambda$ a derived level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on $\mathbf{G}$.

**Theorem 4.13.** *Hypotheses and notations as above, for each nonnegative integer* $r$*, the functor* $\mathrm{Def}^{\mathrm{or},r}_{\mathbf{G}_0}$ *is corepresentable by an* $\mathbb{E}_\infty$*-ring, depending functorially on* $\mathbf{G}_0/R_0$*, as an algebra over the oriented deformation ring* $R^{\mathrm{or}}_{\mathbf{G}_0}$*. Moreover, its 0'th homotopy group is finite over* $\pi_0 R^{\mathrm{or}}_{\mathbf{G}_0}$*.*

*Proof.* Let $\mathrm{Def}^{\mathrm{or}}(\mathbf{G}_0, R)$ denote the $\infty$-groupoid of triples $(\mathbf{G}, \alpha, e)$ where $\mathbf{G}$ is a $p$-divisible group of over $R$, $\alpha$ is an equivalence class of $\mathbf{G}_0$-taggings of $\mathbf{G}$, and $e$ is an orientation of the identity component of $\mathbf{G}$. By [Lur18b, Remark 6.0.7], the functor

$$\mathrm{Def}^{\mathrm{or}}_{\mathbf{G}_0}\colon \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}, \quad R \mapsto \mathrm{Def}^{\mathrm{or}}(\mathbf{G}_0, R)$$

is corepresented by $R^{\mathrm{or}}_{\mathbf{G}_0}$, i.e., there is an equivalence of spaces $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}}(R^{\mathrm{or}}_{\mathbf{G}_0}, R) \simeq \mathrm{Def}^{\mathrm{or}}(\mathbf{G}_0, R)$. Note that the $\mathbb{E}_\infty$-ring $R^{\mathrm{or}}_{\mathbf{G}_0}$ is not connective. Let $\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}$ be the associated universal oriented deformation of $\mathbf{G}_0$ over $R^{\mathrm{or}}_{\mathbf{G}_0}$. Then, analogous to the unoriented case discussed above, the $\mathbb{E}_\infty$-ring $\mathcal{P}^r_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} \otimes_{\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} R^{\mathrm{or}}_{\mathbf{G}_0}$ from Theorem 3.21 and Remark 3.22 is the desired spectrum.

Indeed, let us make explicit the necessary modification regarding the nonconnective classifying ring $R^{\mathrm{or}}_{\mathbf{G}_0}$ when we adapt the proof of Theorem 4.10. Given $R \in \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$, we obtain from Remark 3.22 and Theorem 3.21 equivalences

$$\begin{aligned}
\mathrm{Level}(r, \mathbf{G}/R) &\simeq \mathrm{Level}(r, \mathbf{G}^{\mathrm{or}}_{\mathrm{univ}} \times_{R^{\mathrm{or}}_{\mathbf{G}_0}} R/R) \\
&= \mathrm{Level}(r, \mathbf{G}^{\mathrm{or}}_{\mathrm{univ}} \times_{\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} \tau_{\geqslant 0}R/\tau_{\geqslant 0}R) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}}, \tau_{\geqslant 0}R) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}_{\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}}, R) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}_{R^{\mathrm{or}}_{\mathbf{G}_0}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} \otimes_{\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} R^{\mathrm{or}}_{\mathbf{G}_0}, R)
\end{aligned}$$

To verify that the $\mathbb{E}_\infty$-ring $\mathcal{P}^r_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} \otimes_{\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} R^{\mathrm{or}}_{\mathbf{G}_0}$ is as desired, consider the functor

$$\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}, \quad R \mapsto \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} \otimes_{\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} R^{\mathrm{or}}_{\mathbf{G}_0}, R)$$

Here, $\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} \otimes_{\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} R^{\mathrm{or}}_{\mathbf{G}_0}, R)$ can again be viewed as the $\infty$-category of pairs $(f, g)$. The map $f : R^{\mathrm{or}}_{\mathbf{G}_0} \to R$ along the structure morphism of

$$\mathrm{Sp\acute{e}t}\,\mathcal{P}^r_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} \times_{\mathrm{Sp\acute{e}t}\,\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} \mathrm{Sp\acute{e}t}\,R^{\mathrm{or}}_{\mathbf{G}_0}$$

over $R^{\mathrm{or}}_{\mathbf{G}_0}$ classifies an oriented deformation $(\mathbf{G}, \alpha, e)$ over $R$ of $\mathbf{G}_0/R_0$. The map

$$g \in \mathrm{Map}_{\mathrm{CAlg}_{R^{\mathrm{or}}_{\mathbf{G}_0}}}(\mathcal{P}^r_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} \otimes_{\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}} R^{\mathrm{or}}_{\mathbf{G}_0}, R) \simeq \mathrm{Level}(r, \mathbf{G}/R)$$

along the restriction from $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ to CAlg specifies a derived level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure $\lambda$ on $\mathbf{G}/R$. The rest of the proof goes as before in Theorem 4.10. $\square$

We shall call this $\mathbb{E}_\infty$-ring from Theorem 4.13 a *Jacquet–Langlands spectrum of level* $p^r$ and denote it by $\mathcal{JL}_{h,r}$. Indeed, by functoriality, $\mathcal{JL}_{h,r}$ admits an action of $\mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z}) \times \mathrm{Aut}(\mathbf{G}_0)$. Moreover, since $(\mathbf{Z}/p^r\mathbf{Z})^h \subset (\mathbf{Z}/p^{r+1}\mathbf{Z})^h$, we obtain a tower of spectral Deligne–Mumford moduli stacks

$$\begin{tikzcd} \vdots \arrow[d] \\ \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r+1} \arrow[d] \\ \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r} \arrow[d] \\ \vdots \arrow[d] \\ \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,0} \end{tikzcd} \tag{4.2}$$

The following proposition relates this tower to a classical Lubin–Tate tower (cf. [Lur18b, Theorem 6.0.3, Corollary 3.0.13, and Remark 3.0.14] as well as [GV18, Lemma 7.1]).

**Proposition 4.14.** *The 0'th homotopy of the tower* (4.2) *recovers the finite levels of the Lubin–Tate tower of* $\mathbf{G}_0/R_0$ *in* (1.1). *More precisely, for each nonnegative integer* $r$*, the commutative ring* $\pi_0\mathcal{JL}_{h,r}$ *classifies deformations with Drinfeld (full) level-*$p^r$ *structure of* $\mathbf{G}_0$ *over complete local Noetherian rings. Moreover, taking* $\pi_0$ *of the natural connecting morphisms between adjacent levels of* (4.2) *yields the corresponding morphisms along the Lubin–Tate tower.*

*Proof.* For formal reasons, this follows from the definitions of derived level structures (cf. Definitions 4.12, 3.19, and 3.12) and the universal properties of the (co)representing objects. Indeed, from Definition 4.12 and Theorem 4.13 we have

$$\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}}(\mathcal{JL}_{h,r}, R) \simeq \mathrm{Def}^{\mathrm{or}}(r, \mathbf{G}_0, R)$$

As a special case of truncating the functor $\mathrm{Def}^{\mathrm{or}}_{\mathbf{G}_0}$, the components of the right-hand side are identified with the set of deformations with level-$p^r$ structure of $\mathbf{G}_0$ over $\pi_0 R$. Here, the structure of an orientation $e$ becomes vacuous. Moreover, by Definition 3.12, the derived level structure on $\mathbf{G}/R$ induces a classical one on $\mathbf{G}^\heartsuit/\pi_0 R$. As explained in [Wei16, Section 2.2], by a result of Drinfeld, the truncated functor is representable by a formal affine scheme $\mathcal{M}^{(0)}_{\mathbf{G}_0,r}$ as the $r$'th level of the Lubin–Tate tower. See [Wei16, Section 1.1] for notations, including the distinction between $\mathcal{M}^{(0)}_{\mathbf{G}_0,r}$ and $\mathcal{M}_{\mathbf{G}_0,r}$, and cf. [Lur18b, Definition 3.1.1]. Since the (co)representing object is uniquely determined up to equivalence, upon passage to $\pi_0$, the $r$'th level of the tower (4.2) recovers $\mathcal{M}^{(0)}_{\mathbf{G}_0,r}$, as claimed.

The statement regarding the connecting morphisms along the towers also boils down to the correspondence between derived level structures and their underlying classical ones, as both sequences of morphisms are induced by the inclusions $(\mathbf{Z}/p^r\mathbf{Z})^h \subset (\mathbf{Z}/p^{r+1}\mathbf{Z})^h$ of abelian groups. $\square$

*Remark* 4.15. In classical arithmetic algebraic geometry, the Lubin–Tate tower can be used to realize the Jacquet–Langlands correspondence [HT01, Theorem B and Chapter II]. Naturally we may ask if there is a topological (or higher-homotopical) realization of this correspondence. Such a construction has appeared recently [Sal23]. In contrast to our approach, the methods therein are based on the Goerss–Hopkins–Miller–Lurie sheaf, by considering certain degenerate level structures whose representing objects are *étale* over the Lubin–Tate space carrying universal deformations. *Integrally*, it would be interesting if our higher-categorical analogues of Lubin–Tate towers above can also lead to a topological version of the classical Jacquet–Langlands correspondence, which means that we construct representations in the category of spectra (see further Section 5.4.2 below).

*Remark* 4.16. Analogous to Proposition 4.14, Theorem 4.10 gives an alternative higher-homotopical lift for the Lubin–Tate tower of $\mathbf{G}_0/R_0$. While its construction may be more natural from the viewpoint of algebraic geometry, we expect its homotopy groups to be more complex (cf. Example 4.11 and [Lur18b, Corollary 6.0.6], and see Section 5.4.1 below). With applications to algebraic topology in mind, we will revisit (4.2) in Section 5.

On a related note, as Lurie's unoriented and oriented spectral deformation rings, the 0'th homotopy groups of both the corepresenting ring from Theorem 4.10 and that from Theorem 4.13 recover the classical rings parameterizing deformations with level structure. However, Lurie's argument for his oriented version in relation to the Lubin–Tate deformation ring is substantially more involved than our proof for Proposition 4.14. It includes showing that the higher homotopy groups of the unoriented spectral deformation ring are all $p$-torsion (see [Lur18b, Theorems 6.0.3 (a) and 6.3.1]). This difference results from the facts that orientations do not interact with derived level structures as defined here, and that Theorems 4.10 and 4.13 are established on Lurie's absolute representability results for deformations and our relative representability results for level structures.

Recall from [Lur09a, Definition 1.1 and Remark 1.4] and [Lur18b, Definition 4.1.8] that an $\mathbb{E}_\infty$-ring $A$ is said to be

- *even*, if $\pi_* A = 0$ whenever $i$ is odd,
- 2-*periodic*, if there exists $\beta \in \pi_2 A$ invertible in $\pi_* A$, so that in particular $\pi_2 A$ is a free $\pi_* A$-module of rank 1,
- *weakly* 2-*periodic*, if the natural map $\pi_n A \otimes_{\pi_* A} \pi_2 A \to \pi_{n+2} A$ is an isomorphism for all $n$, so that in particular $\pi_2 A$ is a projective $\pi_* A$-module of rank 1, and
- *complex periodic*, if it is both weakly 2-periodic and complex orientable.

There has been an evolving understanding of "evenness" in algebraic topology, (non-connective) spectral algebraic geometry, and related fields [Wil73, HH18, HRW25, Gre25, BDL25]. See also Remarks 3.22 above and 5.18 below. As a first property, we obtain the following.

**Proposition 4.17.** *The spectra $\mathcal{JL}_{h,r}$ are* 2*-periodic.*

*Proof.* This follows from the facts that each $\mathcal{JL}_{h,0} \simeq E_h$ is 2-periodic and that the structure morphism $\operatorname{Sp\acute{e}t} \mathcal{JL}_{h,r} \to \operatorname{Sp\acute{e}t} \mathcal{JL}_{h,0}$ induces a map $\mathcal{JL}_{h,0} \to \mathcal{JL}_{h,r}$ of ring spectra. □

In [Rog08], Rognes defines Galois extensions for commutative ring spectra. Suppose that $E$ is an $\mathbb{E}_\infty$-ring spectra and $F$ is an $\mathbb{E}_\infty$ $E$-algebra with an action of a finite group $G$. We say that $F$ is a *$G$-Galois extension of $E$*, if $F^{\mathrm{h}G} \simeq E$ and $F \otimes_E F \to \prod_G F$ is an equivalence. The next property will be useful in Section 5.2 when we consider Galois descent of $\mathbb{E}_\infty$-structures.

**Proposition 4.18.** *Hypotheses and notations as above, each $\mathcal{JL}_{h,r}$ is a $\mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z})$-Galois extension of $R^{\mathrm{or}}_{\mathbf{G}_0}$.*

*Proof.* By functoriality of the moduli problems, $\mathcal{JL}_{h,r}^{\mathrm{hGL}_h(\mathbf{Z}/p^r\mathbf{Z})}$ is equivalent to $R^{\mathrm{or}}_{\mathbf{G}_0}$. We need only show that

$$\mathcal{JL}_{h,r} \otimes_{R^{\mathrm{or}}_{\mathbf{G}_0}} \mathcal{JL}_{h,r} \xrightarrow{\simeq} \prod_{\mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z})} \mathcal{JL}_{h,r}$$

By [Lur18c, Proposition 1.4.11.1], this is equivalent to

$$\mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r} \times_{\mathrm{Sp\acute{e}t}\, R^{\mathrm{or}}_{\mathbf{G}_0}} \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r} \xleftarrow{\simeq} \coprod_{\mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z})} \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r}$$

for which we resort to modular interpretations. Given an $\mathbb{E}_\infty$-ring $R$, let us consider the diagram

$$\begin{tikzcd}
\coprod\limits_{\mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z})} \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r} \arrow[dr] & & \\
& \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r} \times_{\mathrm{Sp\acute{e}t}\, R^{\mathrm{or}}_{\mathbf{G}_0}} \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r} \arrow[r] \arrow[d] & \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r} \arrow[d] \\
& \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r} \arrow[r] & \mathrm{Sp\acute{e}t}\, R^{\mathrm{or}}_{\mathbf{G}_0}
\end{tikzcd}$$

where:

- The moduli space $\mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r} \times_{\mathrm{Sp\acute{e}t}\, R^{\mathrm{or}}_{\mathbf{G}_0}} \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r}$ parametrizes
$$\{(\mathbf{G}, \alpha, e, \lambda), (\mathbf{G}', \alpha', e', \lambda')\}$$
where each of the two quadruples is an oriented deformation with level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure of $\mathbf{G}_0$ over $R$ as in Definition 4.12. Since the fiber product is over $\mathrm{Sp\acute{e}t}\, R^{\mathrm{or}}_{\mathbf{G}_0}$, we have $\mathbf{G} = \mathbf{G}'$, $\alpha = \alpha'$, and $e = e'$. Thus this moduli space carries a universal example
$$(\mathbf{G}_{\mathrm{univ}}, \alpha_{\mathrm{univ}}, e_{\mathrm{univ}}, \lambda_{\mathrm{univ}}, \lambda'_{\mathrm{univ}})$$
- Over $\mathrm{Sp\acute{e}t}\, R^{\mathrm{or}}_{\mathbf{G}_0}$, the moduli space $\coprod\limits_{\mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z})} \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{h,r}$ parametrizes
$$\big\{\big(\mathbf{G}, \alpha, e, g(\lambda)\big)\big\}_{g \in \mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z})}$$
where $(\mathbf{G}, \alpha, e)$ is an oriented deformation of $\mathbf{G}_0$ over $R$, $\lambda$ is a level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure, and each $g$ acts on $\lambda$ by a change of Drinfeld basis (cf. Definition 3.19). This moduli space in turn carries a universal example
$$\big(\mathbf{G}_{\mathrm{univ}}, \alpha_{\mathrm{univ}}, e_{\mathrm{univ}}, \lambda_{\mathrm{univ}}, g_{\mathrm{univ}}(\lambda_{\mathrm{univ}})\big)$$

Since $\mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z})$ acts freely and transitively on $(\mathbf{Z}/p^r\mathbf{Z})^h$, we see that these two moduli are equivalent. $\square$

4.3. **Topological lifts of power operation rings.** Continuing Section 3.2.4, here we consider certain non-full level structures relevant to power operations in Morava E-theories. Let us first recall the classical deformation theory of one-dimensional commutative formal groups.

4.3.1. *Deformations of formal groups and of Frobenius.* Given a formal group $\widehat{\mathbf{G}}_0$ over a perfect field $k$ of characteristic $p$, a deformation of $\widehat{\mathbf{G}}_0$ over a complete local ring $R$ is a triple $(\widehat{\mathbf{G}}, i, \eta)$ such that

- $\widehat{\mathbf{G}}$ is a formal group over $R$,
- $i\colon k \to R/\mathfrak{m}$ is a ring homomorphism, with $\mathfrak{m}$ the maximal ideal of $R$, and
- $\eta\colon \pi^*\widehat{\mathbf{G}} \simeq i^*\widehat{\mathbf{G}}_0$ is an isomorphism of formal groups over $R/\mathfrak{m}$, with $\pi\colon R \to R/\mathfrak{m}$ the natural projection.

We simply write $\widehat{\mathbf{G}}$ for a deformation if $(i, \eta)$ is understood.

Recall the relative Frobenius isogeny $\mathrm{Frob}\colon \widehat{\mathbf{G}}_0 \to \sigma^*\widehat{\mathbf{G}}_0$ over $k$, with $\sigma\colon k \to k$, $x \mapsto x^p$. For each nonnegative integer $r$, a deformation of the $p^r$-power Frobenius $\mathrm{Frob}^r$ over $R$ consists of deformations $(\widehat{\mathbf{G}}, i, \eta)$ and $(\widehat{\mathbf{G}}', i', \eta')$ of $\widehat{\mathbf{G}}_0$ over $R$, together with an isogeny $\psi : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}'$ of formal groups over $R$, such that the following compatibility conditions hold (cf. [Zhu20, §5.5]):

(1) The triangle

$$\begin{tikzcd} k \arrow[r, "i'"] \arrow[d, "\sigma^r"'] & R/\mathfrak{m} \\ k \arrow[ur, "i"'] & \end{tikzcd}$$

commutes, so that $(i')^*\widehat{\mathbf{G}}_0 = i^*(\sigma^r)^*\widehat{\mathbf{G}}_0$.

(2) The rectangle

$$\begin{tikzcd} \pi^*\widehat{\mathbf{G}} \arrow[r, "\pi^*(\psi)"] \arrow[d, "\eta"'] & \pi^*\widehat{\mathbf{G}}' \arrow[d, "\eta'"] \\ i^*\widehat{\mathbf{G}}_0 \arrow[r, "i^*(\mathrm{Frob}^r)"] & i^*(\sigma^r)^*\widehat{\mathbf{G}}_0 \end{tikzcd}$$

of formal groups over $R/\mathfrak{m}$ commutes.

In particular, when $\pi^*(\psi)$ equals the $p^r$-power relative Frobenius isogeny on $\pi^*\widehat{\mathbf{G}}$, the association of $\psi$ with $\widehat{\mathbf{G}}$ is equivalent to the choice of a subgroup scheme $\widehat{\mathbf{H}} = \ker(\psi)$ of $\widehat{\mathbf{G}}$ that is *cyclic of order* $p^r$ in the sense of [KM85, 6.1] (cf. the $\mathrm{Level}_0(r, G/R)$-structure in Definition 3.23).

We say that two deformations of Frobenius

$$(\widehat{\mathbf{G}}_1, i_1, \eta_1) \to (\widehat{\mathbf{G}}_1', i_1', \eta_1') \quad \text{and} \quad (\widehat{\mathbf{G}}_2, i_2, \eta_2) \to (\widehat{\mathbf{G}}_2', i_2', \eta_2')$$

are isomorphic, if their sources are isomorphic and their target are isomorphic, both as deformations of $\widehat{\mathbf{G}}_0$ over $R$, i.e., $\psi : \widehat{\mathbf{G}}_1 \simeq \widehat{\mathbf{G}}_2$, $i_1 = i_2$, $\eta_1 = \eta_2 \circ \pi^*(\psi)$, and $\psi'\colon \widehat{\mathbf{G}}_1' \simeq \widehat{\mathbf{G}}_2'$, $i_1' = i_2'$, $\eta_1' = \eta_2' \circ \pi^*(\psi')$.

We have the following classification theorem for deformations of Frobenius from [Str97, Theorem 42 and Section 13], as reformulated in [Zhu20, Proposition 5.7 and Corollary 5.12] (cf. [Rez13, Proposition 3.5]).

**Theorem 4.19** (Strickland)**.** *Let $\widehat{\mathbf{G}}_0$ be a height-$h$ formal group over a perfect field $k$ of characteristic $p$. For each nonnegative integer $r$, there exists a complete local ring $A_r$ which carries a universal deformation of the $p^r$-power Frobenius*

$$\psi^r_{\mathrm{univ}} \colon (\widehat{\mathbf{G}}^r_s, i^r_s, \eta^r_s) \to (\widehat{\mathbf{G}}^r_t, i^r_t, \eta^r_t)$$

*Namely, given complete local rings $R$ and local homomorphisms $f : A_r \to R$, the assignment $\psi^r_{\mathrm{univ}} \mapsto f^*(\psi^r_{\mathrm{univ}})$ gives a bijection between the set of local homomorphisms $f : A_r \to R$ and the set of isomorphism classes of deformations of the $p^r$-power Frobenius over $R$. Moreover, the following hold:*

(1) *When $r = 0$, the ring $A_0$ is the Lubin–Tate deformation ring of $\widehat{\mathbf{G}}_0/k$, isomorphic to $W(k)[\![v_1, \ldots, v_{h-1}]\!]$.*
(2) *There is a local homomorphism $s^r \colon A_0 \to A_r$ classifying the source of $\psi^r_{\mathrm{univ}}$ such that*
$$(\widehat{\mathbf{G}}^r_s, i^r_s, \eta^r_s) = \big((s^r)^*\widehat{\mathbf{G}}_{\mathrm{univ}}, \mathrm{id}_k, \mathrm{id}_{\widehat{\mathbf{G}}_0}\big)$$
*with $(\widehat{\mathbf{G}}_{\mathrm{univ}}, \mathrm{id}_k, \mathrm{id}_{\widehat{\mathbf{G}}_0})$ over $A_0$ the universal deformation of $\widehat{\mathbf{G}}_0$, along which $A_r$ is finite and free as an $A_0$-module.*
(3) *There is a local homomorphism $t^r \colon A_0 \to A_r$ classifying the target of $\psi^r_{\mathrm{univ}}$ such that*
$$(\widehat{\mathbf{G}}^r_t, i^r_t, \eta^r_t) = \big((t^r)^*\widehat{\mathbf{G}}_{\mathrm{univ}}, \sigma^r, \mathrm{id}_{(\sigma^r)^*\widehat{\mathbf{G}}_0}\big)$$

These rings $A_r$ also bear topological meanings in relationship to the Morava E-theory $E$ of $\widehat{\mathbf{G}}_0/k$, as in [Str98, Theorem 1.1].

**Theorem 4.20** (Strickland)**.** *Hypotheses and notations as above, there is a natural ring isomorphism*

$$A_r \simeq E^0(B\Sigma_{p^r})/I_{\mathrm{tr}}$$

*where $I_{\mathrm{tr}}$ is the ideal generated by the images of transfer maps from proper subgroups of the symmetric group $\Sigma_{p^r}$ on $p^r$ letters.*

*Remark* 4.21. Strickland's proof relies on rational *computations* which match up the ranks of the two sides of this isomorphism as $E^0$-modules (see [Str98, Theorems 9.2 and 8.6]). From the perspective of [Lur18b, Example 0.0.6 and Theorem 0.0.8], this theorem can be viewed as a "partial" realization by $E$-cohomology of certain spaces modulo equivalences, in the setting of $\mathbb{E}_\infty$-ring spectra, of the solution to a moduli problem in classical deformation theory.

The collection $\{A_r\}_{r\geqslant 0}$ has the structure of a graded coalgebra over $A_0$, with structure maps

$$s = s^r \colon A_0 \to A_r, \quad t = t^r \colon A_0 \to A_r, \quad \mu^{m,n} \colon A_{m+n} \to A_m {}^s\!\otimes^t_{A_0} A_n$$

which classify the source, target, and composite of deformations of Frobenius, respectively. In particular, given a $K(h)$-local $E$-algebra $F$, there is a $p^r$-power operation $F^0(X) \to F^0(X \times B\Sigma_{p^r})/I_{\mathrm{tr}}$, and when $X = *$, we have

$$\pi_0 F \to E^0(B\Sigma_{p^r})/I_{\mathrm{tr}} \otimes_{E^0} \pi_0 F \simeq A_r {}^s\!\otimes_{A_0} \pi_0 F$$

These equip $\pi_0 F$ with the structure of a $\Gamma$-module, where the $\mathbf{Z}_{\geqslant 0}$-graded pieces of $\Gamma$ are $A_0$-linear duals of $A_r$ along $s^r$. For more details about power operations in Morava E-theories, the interested reader may refer to [Rez24, Rez09, Rez13]. Explicit computations of $\Gamma$ have been carried out at height 2 in [Rez08, Zhu14, Zhu19] through relevant moduli schemes of level structures. The cases at height greater than 2 are still lack of quantitative understanding.

4.3.2. *Spectral deformations.* As observed above, the assignment $\psi \mapsto \ker(\psi)$ gives a one-to-one correspondence between isomorphism classes of deformations of $\mathrm{Frob}^r$ with source $\widehat{\mathbf{G}}$ and cyclic degree-$p^r$ finite flat subgroup scheme of $\widehat{\mathbf{G}}$. Therefore, we see that $A_r$ corepresents the moduli problem

$$(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}})^{\heartsuit} \to \mathrm{Set}, \quad R \mapsto \mathrm{Def}_0(r, \widehat{\mathbf{G}}_0, R)$$

where $\mathrm{Def}_0(r, \widehat{\mathbf{G}}_0, R)$ consists of pairs $\widehat{\mathbf{H}} \subset \widehat{\mathbf{G}}$, with $\widehat{\mathbf{G}}$ a deformation of $\widehat{\mathbf{G}}_0$ over $R$ and $\widehat{\mathbf{H}}$ a cyclic subgroup of order $p^r$.

**Theorem 4.22.** *Hypothesis and notations as in Theorem 4.19, for each positive integer $r$, there exists an $\mathbb{E}_\infty$-ring spectrum $E_{h,r}$ such that $\pi_0 E_{h,r} \simeq A_r$, which depends functorially on $\widehat{\mathbf{G}}_0/k$.*

*Proof.* Given the formal group $\widehat{\mathbf{G}}_0$ over the perfect field $k$ of characteristic $p$, necessarily nonstationary [Lur18b, Example 3.0.10], we view it as a connected $p$-divisible group and consider instead the functor

$$\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}, \quad R \mapsto (\mathbf{G}, \alpha, e, \lambda)$$

where $(\mathbf{G}, \alpha)$ is a *spectral* deformation of $\widehat{\mathbf{G}}_0$ over $R$ as in Section 4.2, $e$ is an orientation of $\mathbf{G}^\circ$, and $\lambda \in \mathrm{Level}_0(r, \mathbf{G}/R)$ is a derived level structure as in Section 3.2.4, esp. Remark 3.26. This functor lifts the one corepresented by $A_r$ precisely as the case of $r = 0$ [Lur18b, Remarks 6.4.8 and 3.0.14] (cf. Proposition 4.14). Analogous to the proof of Theorem 4.13 with full level structures, we then deduce the current theorem from the representability of the spectral moduli problem $\mathrm{Level}^{0,r}_{\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0} R^{\mathrm{or}}_{\widehat{\mathbf{G}}_0}}$ by an affine spectral Deligne–Mumford stack, followed by base change along $\tau_{\geqslant 0} R^{\mathrm{or}}_{\widehat{\mathbf{G}}_0} \to R^{\mathrm{or}}_{\widehat{\mathbf{G}}_0}$. □

*Remark* 4.23. Neither the $\mathbb{E}_\infty$-ring spectra $E_{h,r}$ from Theorem 4.22 nor those from Theorem 4.13 with $r > 0$ are $K(h)$-local, in light of [Dev20, Theorems 1.3 and 3.5] (cf. [Sal23, Section 1.3] and [SVW99, Proposition 2]). In particular, they are not *finite* algebras over the Morava E-theory spectrum $E_h$, even though the morphisms $\mathrm{Sp\acute{e}t}\, E_{h,r} \to \mathrm{Sp\acute{e}t}\, E_h$ of spectral Deligne–Mumford stacks *are* finite (cf. [Lur17, Definition 7.2.2.1], [Lur18c, Proposition 2.7.2.1], and the finiteness conditions we recalled at the beginning of Section 2.1, esp. Remark 2.4). Tracing through the constructions, we find that it is at the incidence spectral Deligne–Mumford stack $\mathrm{Sp\acute{e}t}\, W$ of Proposition 3.15 where $K(h)$-locality breaks down. The authors thank Sanath Devalapurkar and Akira Tominaga for vigilant, helpful discussions on this point.

**Example 4.24** (The case where $k$ is algebraically closed)**.** Let $r > 0$. By [BSY22, Theorem G], $E_h$ admits an $\mathbb{E}_\infty$ complex orientation $\mathrm{MU} \to E_h$. The structure morphism $\mathrm{Sp\acute{e}t}\, E_{h,r} \to \mathrm{Sp\acute{e}t}\, R^{\mathrm{or}}_{\widehat{\mathbf{G}}_0}$ further gives an $\mathbb{E}_\infty$-algebra map $E_h \to E_{h,r}$. Their composite is then an $\mathbb{E}_\infty$ complex orientation for $E_{h,r}$ by [Lur18b, Remark 4.1.3]. The authors thank Niko Naumann for correcting an erroneous statement about complex orientability in an earlier version of the preceding remark.

On the other hand, suppose that $E_{h,r}$ were $K(h)$-local (cf. [Lur18b, Theorem 4.5.2]). By [BSY22, Theorem A], it would then admit an $\mathbb{E}_\infty$-algebra map to $E_h$. Analogous to Proposition 4.18, $E_{h,r}$ is Galois over $E_h$ (cf. [KM85, 7.4.3]). Thus this Galois extension would split, exhibiting $E_{h,r}$ as a finite free $E_h$-algebra. The failure of this stands in contrast to the structure of $\pi_0 E_{h,r} \simeq A_r$, which *is* finite free over $\pi_0 E_h \simeq A_0$ by [Str97, Theorem 42]. The authors thank Xiansheng Li for supplying this perspective.

*Remark* 4.25. Although we obtained spectra whose 0'th homotopy groups recover the power operation rings of Morava E-theories, we do not know yet the higher homotopy groups of these spectra concretely or explicitly, as these spectra are not even periodic in general unless $r = 0$, and as they are not étale over E-theory spectra. This non-even-periodicity with $r > 0$ should be a manifestation of the structure of a *pile*, i.e., a presheaf of categories (rather than of groupoids), as indicated in [Rez14, Section 4.3] (cf. [HL16, Question 1.3]). See further discussion in Sections 5.3 and 5.4.1 below.

**Example 4.26** (The case of height one)**.** When $h = 1$, all $A_r$ are canonically isomorphic to $A_0 \simeq \pi_0 E_h$, as there is a *unique* subgroup of the formal deformation $\mathbf{G}^\heartsuit$ that is cyclic of order $p^r$ for each $r$. Nevertheless, as explained in Remark 3.20, each subgroup may be the underlying ordinary formal group of nonequivalent spectral formal groups, which in turn give rise to nonequivalent derived level structures. Thus we do not have $E_{1,r} \simeq E_1$ in general (cf. Example 4.11).

4.4. **Strickland rings.** Let $\widehat{\mathbf{G}}_0$ be a formal group of finite height over a field of characteristic $p > 0$, and $A$ be a finite abelian group. In [Str97, Str98], with motivation from algebraic topology, Strickland comprehensively studied moduli problems related to level-$A$ structures on deformations of $\widehat{\mathbf{G}}_0$. In particular, as recalled in Theorems 4.19 and 4.20, he proved representability for the moduli problem of degree-$p^r$ subgroups of the universal deformation. Moreover, he established a relationship between its corepresenting ring $A_r$ and the Morava E-cohomology associated to $\widehat{\mathbf{G}}_0$, in terms of the latter's value in degree 0 on the symmetric group $\Sigma_{p^r}$ (cf. Remark 4.21).

Here, with suitable generalization, it is natural to propose the following definition for classes of ring spectra that corepresent analogous spectral moduli problems, towards a systematic investigation of their properties and interrelationships.

**Definition 4.27.** We call an $\mathbb{E}_\infty$-ring $S$ a *Strickland ring*, if there exist a spectral $p$-divisible group $\mathbf{G}$ over an $\mathbb{E}_\infty$-ring $R$ and a finite abelian $p$-group $A$ such that the affine spectral Deligne–Mumford stack $\operatorname{Spét} S$ represents one of the following moduli problems:

- The functor $\mathrm{Level}^A_{\mathbf{G}/R}$ of level-$A$ structures on $\mathbf{G}$ from Definition 3.12
- The analogous functor of equivalence classes $(\mathsf{D}, \phi)$ in $\mathrm{Level}(A, \mathbf{G}/R)$ where two objects $(\mathsf{D}, \phi)$ and $(\mathsf{D}', \phi')$ are equivalent if the scheme-theoretic image of $\mathsf{D}^\heartsuit$ under $\phi$ and that of $(\mathsf{D}')^\heartsuit$ under $\phi'$ are equal in $\mathbf{G}^\heartsuit/\pi_0 R$

**Example 4.28** (Spectral deformation rings with full level structure)**.** As in Theorem 4.10, let $r$ be a nonnegative integer, and $\mathbf{G}_0$ be a nonstationary $p$-divisible group over a commutative ring of height $h$. Then the corepresenting ring for the functor $\mathrm{Der}^r_{\mathbf{G}_0}$ from Definition 4.8 is a Strickland ring associated to the universal deformation $\mathbf{G}^{\mathrm{un}}_{\mathrm{univ}}/R^{\mathrm{un}}_{\mathbf{G}_0}$ and $A = (\mathbf{Z}/p^r\mathbf{Z})^h$.

**Example 4.29** (Oriented deformation rings with full level structure)**.** As in Theorem 4.13, with notations as in the previous example, the corepresenting ring $\mathcal{JL}_{h,r}$ for the functor $\mathrm{Der}^{\mathrm{or},r}_{\mathbf{G}_0}$ from Definition 4.12 is the base change of a Strickland ring associated to the universal deformation $\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0}$ and $A = (\mathbf{Z}/p^r\mathbf{Z})^h$, along the map $\tau_{\geqslant 0}R^{\mathrm{or}}_{\mathbf{G}_0} \to R^{\mathrm{or}}_{\mathbf{G}_0}$. In particular, Morava E-theory spectra $E_h \simeq \mathcal{JL}_{h,0}$ are Strickland rings up to base change.

In Section 5.3 below, we calculate the homotopy groups of $\mathcal{JL}_{1,r}$ with $p > 2$.

**Example 4.30** (Oriented deformation rings for Frobenius)**.** As in Theorem 4.22, let $\widehat{\mathbf{G}}_0$ be a height-$h$ formal group over a perfect field $k$ of characteristic $p$. Then the ring $E_{h,r}$, with $\pi_0 E_{h,r} \simeq A_r$, is the base change along the map $\tau_{\geqslant 0}R^{\mathrm{or}}_{\widehat{\mathbf{G}}_0} \to R^{\mathrm{or}}_{\widehat{\mathbf{G}}_0}$ of a Strickland ring associated to the universal deformation $\mathbf{G}^{\mathrm{or}}_{\mathrm{univ}}/\tau_{\geqslant 0}R^{\mathrm{or}}_{\widehat{\mathbf{G}}_0}$ and $A' = \mathbf{Z}/p^r\mathbf{Z}$.

In Section 5.4.1 below, we outline a strategy for computing the homotopy groups of $E_{h,r}$, which is 2-periodic as an algebra over the Morava E-theory spectrum $E_h$.

*Remark* 4.31. Notations as in the previous example, let $A$ be a perfect $k$-algebra, $E(A)$ be the Morava E-theory spectrum associated to $(\widehat{\mathbf{G}}_0)_A$, and $\mathbb{T}$ be the monad of $K(h)$-local power operation structure studied by Rezk in [Rez09]. In [BSY22, Theorem 3.50], Burklund, Schlank, and Yuan determined the algebraic structure of the homotopy groups of $E(A)$ in terms of the value on $A$ of a Witt vector functor $W_{\mathbb{T}}$ associated to $\mathbb{T}$. In particular, when $h = 1$, $\pi_0 E(A) \simeq W_{\mathbb{T}}(A) \simeq W(A)$ is the $\delta$-ring of $p$-typical Witt vectors. It would be interesting to make such structures at higher height more concrete, by leveraging the family $\{E_{h,r}\}_{r\geqslant 0}$ (cf. [Zhu20, Zha23]).

## 5. More applications

5.1. **Jacquet–Langlands spectra.** The Langlands program has arguably developed into a paradigm in contemporary mathematics and related fields which connects multiple subareas, including number theory, representation theory, and harmonic analysis, in a precise way. To be more specific, the global Langlands correspondence gives a (partly conjectural) bijection between

- $n$-dimensional complex linear representations of the absolute Galois group $\mathrm{Gal}_F$ of a given number field $F$ and
- automorphic representations of the general linear group $\mathrm{GL}_n(\mathbf{A}_F)$ with coefficients in the ring of adèles of $F$ that arise within the representations given by functions on the double coset space $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbf{A}_F)/\mathrm{GL}_n(\mathcal{O})$, with $\mathcal{O} = \prod_\nu \mathcal{O}_\nu$ the product of the rings of integers of completions at all valuations of $F$

which is compatible with certain $L$-function conditions. More generally, the group $\mathrm{GL}_n$ may be replaced by any reductive group. The Langlands correspondence has many specific examples in number theory. For the group $\mathrm{GL}_1$, this correspondence recovers global class field theory. For $\mathrm{GL}_2$ it affords the famous modularity theorem for semistable elliptic curves [Wil95, TW95].

The Langlands correspondence has a local version. Let $E$ be a local field and $G$ be a reductive group over $E$. The local Langlands correspondence predicts that given any irreducible smooth representation $\pi$ of $G(E)$, we can naturally associate an $L$-parameter, i.e., a continuous homomorphism $\phi_\pi \colon W_E \to {}^{\mathrm{L}}G(\mathbf{C})$, where $W_E$ is the Weil group of $E$, and ${}^{\mathrm{L}}G$ is the Langlands dual group of $G$.

Of particular relevance to this paper is the Jacquet–Langlands correspondence. Let $K$ be a $p$-adic field and $D$ be a central division algebra over $K$ of dimension $d^2$. We fix a positive integer $r$ and let $G = \mathrm{GL}_n(K)$, $G' = \mathrm{GL}_r(D)$, where $n = rd$.

The Jacquet–Langlands correspondence aims to relate irreducible smooth representations of $G$ to those of $G'$, while the local Langlands correspondence relates such representations of $G$ to $n$-dimensional complex representations of $W_K$.

We shall focus on the case of $r = 1$, when $D$ is a central division algebra over $K$ of dimension $n^2$ (and invariant $1/n$). There is a Jacquet–Langlands bijection between square integrable representations of $\mathrm{GL}_n(K)$ and such representations of $D^\times$. In Section 4.2, we have built a higher-homotopical realization of the Lubin–Tate tower associated to a nonstationary $p$-divisible group over a commutative ring, and discussed its relevance to a potential topological version of the Jacquet–Langlands correspondence (see Remark 4.15). On the other hand, we know the actions of certain Galois groups and automorphism groups on certain objects, such as Morava E-theories, topological Hochschild homology, and topological cyclic homology. In particular, these groups act on their homotopy groups. For example, we have the action of Morava stabilizer groups $\mathbb{G}_h$ on Morava E-theories $E_h$. It can be used to compute homotopy groups of the ($K(h)$-local) sphere spectrum via a spectral sequence

$$E_2^{s,t} = H_{\mathrm{c}}^s(\mathbb{G}_h, \pi_t E_h) \implies \pi_{t-s}\mathbf{S}_{K(h)} \tag{5.1}$$

where the $E_2$-page consists of continuous cohomology groups of $\mathbb{G}_h$ [DH04]. In general, however, this group cohomology is complicated to compute, which manifests a common phenomenon in the Langlands program, i.e., the Galois side is usually harder to understand than the automorphic side. One strategy for relevant problems is to transfer from the Galois side to the automorphic side. Let us see an example first [BSSW25].

**Theorem 5.1** (Barthel–Schlank–Stapleton–Weinstein)**.** *There is an isomorphism of graded $\mathbf{Q}$-algebras*

$$\mathbf{Q} \otimes \pi_* \mathbf{S}_{K(h)} \simeq \Lambda_{\mathbf{Q}_p}(\zeta_1, \zeta_2, \ldots \zeta_h)$$

*where the latter is the exterior $\mathbf{Q}_p$-algebra with generators $\zeta_i$ in degree $1 - 2i$.*

A key step in the proof of this theorem is to leverage the equivariantly isomorphic Lubin–Tate tower and Drinfeld tower [FGL08, SW13] between the generic fibers, *rationally* transferring the computation of cohomology of $\mathbb{G}_h$ to that for the Drinfeld symmetric space $\mathcal{H} \simeq \mathcal{D}r_K$ [BSSW25, Theorem 3.9.1]:

$$\begin{tikzcd}
& \mathcal{M}^{\mathrm{LT}}_\infty \simeq \mathcal{M}^{\mathrm{Dr}}_\infty \arrow[dl, bend right, "\mathrm{GL}_h(\mathcal{O}_K)"'] \arrow[dr, bend left, "\mathcal{O}_D^\times"] & \\
\mathcal{LT}_K & & \mathcal{D}r_K
\end{tikzcd} \tag{5.2}$$

Here, $\mathbb{G}_h \simeq \mathcal{O}_D^\times \rtimes \mathrm{Gal}(k/\mathbf{F}_p)$ with $k$ the residue field of $K$.

*Remark* 5.2. For our purposes, including the homotopy fixed point spectral sequences in Proposition 5.9 below, the extension by the Galois group $\mathrm{Gal}(k/\mathbf{F}_p)$ does not make an essential difference (cf. [BG18, Lemmas 1.32, 1.37, and Remark 1.39]).

In a sequel [BSSW24], Barthel, Schlank, Stapleton, and Weinstein compute the Picard group of $K(h)$-local spectra by using some results on the Drinfeld symmetric space from [CDN20, CDN21].

At the ground level of the Lubin–Tate tower, we know by work of Goerss, Hopkins, Miller, and Lurie that the Lubin–Tate space has a higher-homotopical refinement, namely, the Morava E-theory spectrum of the associated formal group over the residue field $k$ of $K$, as an $\mathbb{E}_\infty$-ring (or spectral Deligne–Mumford stack). It is thus a natural question how to lift the two towers to the higher-categorical setting of spectral algebraic geometry, even *integrally*, as a fully structured apparatus affording transfers such as those above and more.

Let $\mathbf{G}_0$ be a height-$h$ $p$-divisible group over $\bar{k}$. In Section 4.2, for each nonnegative integer $r$, we considered the functor

$$\mathrm{Def}^{\mathrm{or},r}_{\mathbf{G}_0}\colon \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}, \quad R \mapsto \mathrm{Def}^{\mathrm{or}}(r, \mathbf{G}_0, R)$$

and showed that it is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_{h,r}$. The duality of Lubin–Tate and Drinfeld towers then leads us to the following.

**Definition 5.3.** Define the *Jacquet–Langlands spectrum* $\mathcal{JL}_{h,\infty}$ to be the colimit of the spectra $\mathcal{JL}_{h,r}$ from Theorem 4.13, i.e., $\mathcal{JL}_{h,\infty} \coloneqq \varinjlim_r \mathcal{JL}_{h,r}$.

**Proposition 5.4.** *The spectrum $\mathcal{JL}_{h,\infty}$ is an $\mathbb{E}_\infty$-ring.*

*Proof.* This is because the $\infty$-category of $\mathbb{E}_\infty$-rings admits small colimits. See [Lur17, Corollary 3.2.3.3] for details. □

*Remark* 5.5. This spectrum is a higher-homotopical realization of $\mathcal{M}^{\mathrm{LT}}_\infty \simeq \mathcal{M}^{\mathrm{Dr}}_\infty$ in (5.2), the Lubin–Tate/Drinfeld moduli space of height $h$ for $K$ at infinite level, which has the structure of a perfectoid space [SW13] (cf. Proposition 4.14).

5.2. **The Jacquet–Langlands dual of a Morava E-theory spectrum.** By definitions of the Jacquet–Langlands spectra from Theorem 4.13 and Definition 5.3, each $\mathcal{JL}_{h,r}$ admits an action by $\mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z}) \times \mathbb{G}_h$, and thus $\mathcal{JL}_{h,\infty}$ admits an action by $\varprojlim_r \mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z}) \times \mathbb{G}_h \simeq \mathrm{GL}_h(\mathbf{Z}_p) \times \mathbb{G}_h$. Descending along the Drinfeld tower in (5.2) yields the following.

**Definition 5.6.** Given a height-$h$ formal group $\widehat{\mathbf{G}}_0$ over a perfect field of characteristic $p$, let $E_h \simeq \mathcal{JL}_{h,0}$ be the associated Morava E-theory spectrum, and $\mathcal{JL}_{h,\infty}$ be the associated Jacquet–Langlands spectrum at infinite level. Define the *Jacquet–Langlands dual of $E_h$* to be the homotopy fixed point spectrum $\mathcal{JL}^{\mathrm{h}\mathbb{G}_h}_{h,\infty}$, denoted by the symbol ${}^{\mathrm{L}}E_h$.

**Proposition 5.7.** *Given a Morava E-theory spectrum $E_h$ associated to a formal group $\widehat{\mathbf{G}}_0$ over a perfect field $k$ of characteristic $p$, its Jacquet–Langlands dual spectrum ${}^{\mathrm{L}}E_h$ is an $\mathbb{E}_\infty$-ring.*

*Proof.* By construction, we have ${}^{\mathrm{L}}E_h = \mathcal{JL}^{\mathrm{h}\mathbb{G}_h}_{h,\infty} \simeq \varinjlim_r \mathcal{JL}^{\mathrm{h}\mathbb{G}_h}_{h,r}$ (see, e.g., [Mal14] for the last equivalence). Thus it suffices to show that each $\mathcal{JL}^{\mathrm{h}\mathbb{G}_h}_{h,r}$ is an $\mathbb{E}_\infty$-ring spectrum. By [Rog08, Theorem 5.4.4 (d)] and Proposition 4.18, we have Galois extensions

$$\mathbf{S}_{K(h)} \to E_h \to \mathcal{JL}_{h,r}$$

In view of this, we notice that $\mathcal{JL}_{h,r}$ is a profinite $\mathrm{GL}_h(\mathbf{Z}/p^r\mathbf{Z}) \times \mathbb{G}_h$-spectrum in the sense of [Qui13b, Definition 5.1] (cf. [DQ16, BD10]). In particular, it is a profinite $\mathbb{G}_h$-spectrum. By [Qui13a, Proposition 3.23], we then have

$$\mathcal{JL}_{h,r}^{\mathrm{h}\mathbb{G}_h} \simeq \mathrm{Tot}\big(\mathrm{Map}(\mathbb{G}_h^{\bullet}, \mathcal{JL}_{h,r})\big)$$

As the $\infty$-category CAlg admits inverse limits (cf. [Lur17, Corollary 3.2.2.4]), $\mathcal{JL}_{h,r}^{\mathrm{h}\mathbb{G}_h}$ is an $\mathbb{E}_\infty$-ring. $\square$

*Remark* 5.8. In view of Proposition 4.14 and [SW13, Theorem 7.2.3], the generic fiber of $\pi_0{}^{\mathrm{L}}E_h$ should recover the Drinfeld symmetric space [Dri76] (see Proposition 5.21 and Remark 5.22 below for a more precise formulation at height $h = 1$). It is the rigid-analytic space $\mathcal{H} = \mathbf{P}_K^{h-1} \setminus \bigcup H$, where $\mathbf{P}_K^{h-1}$ is a rigid-analytic projective space, and $H$ runs over all $K$-rational hyperplanes in $\mathbf{P}_K^{h-1}$. It is the generic fiber of a formal scheme which parametrizes deformations of a certain formal $\mathcal{O}_D$-module related to $\widehat{\mathbf{G}}_0$. In future work, we aim to show that ${}^{\mathrm{L}}E_h$ as constructed in Definition 5.6 arises from a corresponding spectral moduli problem.

As indicated in the discussion below (5.2), our structured higher-homotopical realizations or spectral lifts of the moduli spaces in question bear specific computational tools. The following is "dual" to the $K(h)_*$-local $E_h$-Adams spectral sequence studied by Devinatz and Hopkins as a homotopy fixed point spectral sequence [DH04, Theorem 1 (iv)].

**Proposition 5.9.** *Hypothesis and notations as in Proposition 5.7, there is a natural strongly convergent spectral sequence*

$$E_2^{s,t} = H_{\mathrm{c}}^s\big(\mathrm{GL}_h(\mathbf{Z}_p), \pi_t{}^{\mathrm{L}}E_h\big) \implies \pi_{t-s}\mathbf{S}_{K(h)} \tag{5.3}$$

*whose $E_2^{s,t}$-term is the s'th continuous cohomology of $\mathrm{GL}_h(\mathbf{Z}_p)$ with coefficients the profinite $\mathrm{GL}_h(\mathbf{Z}_p)$-module $\pi_t{}^{\mathrm{L}}E_h$. Moreover, it fits naturally into a commutative diagram of spectral sequences*

$$\begin{tikzcd}
& H_{\mathrm{c}}^s\big(\mathrm{GL}_h(\mathbf{Z}_p) \times \mathbb{G}_h, \pi_t\mathcal{JL}_{h,\infty}\big) \arrow[dl, Rightarrow, "\textit{Lubin–Tate tower}"'] \arrow[dr, Rightarrow, "\textit{Drinfeld tower}"] & \\
H_{\mathrm{c}}^r(\mathbb{G}_h, \pi_{t-s}E_h) \arrow[dr, Rightarrow, "{[\mathrm{DH04}]}"'] & & H_{\mathrm{c}}^r\big(\mathrm{GL}_h(\mathbf{Z}_p), \pi_{t-s}{}^{\mathrm{L}}E_h\big) \arrow[dl, Rightarrow, "(5.3)"] \\
& \pi_{t-s-r}\mathbf{S}_{K(h)} &
\end{tikzcd} \tag{5.4}$$

*where the colors indicate the abutments of the corresponding homotopy fixed point spectral sequences with respect to the action of a profinite group.*

*Proof.* In view of (5.2), we deduce the proposition from the general result that given any profinite group $G$ and a $G$-equivariant spectrum $E$, there is a homotopy fixed point spectral sequence

$$E_2^{s,t} = H_c^s(G, \pi_t E) \implies \pi_{t-s} E^{hG}$$

See [Qui13a, Theorem 3.17] and [May96] for more details. □

*Remark* 5.10. Computationally, in general, the $E_2$-page of (5.3) is expected to be more accessible than that of (5.1) (see discussion immediately following). This connects to some of the recent progress in $p$-adic geometry, e.g., [CDN21, Bos23]. Complementary to Theorem 5.1, it opens up possibilities for systematically detecting torsion classes in the homotopy groups of the $K(h)$-local sphere spectrum $\mathbf{S}_{K(h)}$.

In the same vein, the dual spectrum ${}^{\mathrm{L}}E_h$ with $\mathrm{GL}_h(\mathbf{Z}_p)$-action potentially leads to finite resolutions of $\mathbf{S}_{K(h)}$ analogous to those pursued in [Hen07, GHMR05, Bea15, BG18, BBH25].

There is also an unstable analogue of (5.4) for computing $v_h$-periodic homotopy groups of spheres (cf. [Wan15, Zhu18, BR20b]). It is both computationally (specifically in the $h = 2$ case with meromorphic modular forms) and conceptually (in terms of geometry) related to a *filtered*, *equivariant*, *quasi-syntomic* sheaf over the Lubin–Tate tower of the formal group associated with $E_h$. Here, the filtration is compatible with an "EHP filtration" of odd-dimensional spheres, the symmetry again bears the Jacquet–Langlands duality, and the topology reflects the $p$-adic geometry of the Lubin–Tate tower across finite and infinite levels. This is from joint work in progress of Guozhen Wang and the second author.

5.3. **The case of height one at odd primes.** In this subsection, to illustrate the square (5.4) of homotopy fixed point spectral sequences from Proposition 5.9, for $h = 1$ and $p > 2$ we give explicit algebraic calculations of the actions of $\mathrm{GL}_1(\mathbf{Z}_p) \simeq \mathbf{Z}_p^\times \simeq \mathbb{G}_1$ on the homotopy groups of $\mathcal{JL}_{1,\infty}$, $E_1$, and ${}^{\mathrm{L}}E_1$, without detailed proofs (to appear elsewhere). Among these, the action of $\mathbb{G}_1$ on $\pi_* E_1$ is well understood. See, e.g., [Lur10, Lecture 35] and [BB20, Section 5.4].

The authors thank Guchuan Li, Xiansheng Li, Guozhen Wang, and Wei Yang for enlightening discussions pertinent to the content here. In relation to equivariant $E_h$-theories, Wang has also formulated the structure underlying the spectra $\{\mathcal{JL}_{h,r}\}_{0 \leqslant r \leqslant \infty}$ as that of a genuine $G$-equivariant spectrum, for the profinite group $G = \mathrm{GL}_h(\mathbf{Z}_p)$, equipped with extra compatibility properties [Wan25].

Let us now proceed in four steps consecutively along the top-left, bottom-left, top-right, and bottom-right spectral sequences in (5.4) as follows. The first step is key.

5.3.1. *Ascending along the Lubin–Tate tower.* Recall from Definition 5.3 and Proposition 4.14 that

$$\mathcal{JL}_{1,\infty} = \varinjlim_r \mathcal{JL}_{1,r}$$

with $\pi_0 \mathcal{JL}_{1,r} \simeq \mathbf{Z}_p[\zeta_{p^r}]$ and natural connecting morphisms as $r \geqslant 0$ varies, where $\zeta_{p^r}$ is a primitive $p^r$'th root of unity (cf. [Dev20, Example 3.4]). Denote $\varinjlim_r \mathbf{Z}_p[\zeta_{p^r}]$ by $\mathbf{Z}_p[\zeta_{p^\infty}]$, which is the ring of integers of the $p$-adic field $\mathbf{Q}(\zeta_{p^\infty})$ obtained from $\mathbf{Q}_p$ by adjoining all $p$-power roots of unity. Write $K_0 \coloneqq \mathbf{Q}_p$ and $K_\infty \coloneqq \mathbf{Q}_p(\zeta_{p^\infty})$

so that

$$\mathcal{O}_{K_\infty} \simeq \mathbf{Z}_p[\zeta_{p^\infty}] \simeq \pi_0 \mathcal{JL}_{1,\infty}$$

as homotopy groups of spectra commute with filtered colimits. Then $\mathrm{GL}_1(\mathbf{Z}_p)$ acts on $\pi_* \mathcal{JL}_{1,\infty}$ as $\mathrm{Gal}(K_\infty/K_0)$ via level structures. Specifically, if we write $K_r \coloneqq \mathbf{Q}_p(\zeta_{p^r})$, as studied in Iwasawa theory there is a tower of field extensions (cf. (4.2) and (5.2))

$$\begin{tikzcd}
K_\infty \arrow[d, dash] \arrow[dd, dash, bend right=50, "\mathbf{Z}_p"'] \\
K_r \arrow[d, dash, "\mathbf{Z}/p^{r-1}\mathbf{Z}"] \\
K_1 \arrow[d, dash, "\boldsymbol{\mu}_{p-1}"'] \\
K_0
\end{tikzcd} \tag{5.5}$$

where $\mathrm{Gal}(K_\infty/K_0) \simeq \mathbf{Z}_p^\times \simeq \mathbf{F}_p^\times \times (1 + p\mathbf{Z}_p) \simeq \boldsymbol{\mu}_{p-1} \times \mathbf{Z}_p$ with $\boldsymbol{\mu}_{p-1}$ the multiplicative group of $(p-1)$'st roots of unity. Here we used the assumption of $p > 2$, as $\mathbf{Z}_2^\times$ decomposes differently into $\{\pm 1\} \times \mathbf{Z}_2$.

We begin with the following instances of Proposition 4.18 along the tower (5.5).

**Example 5.11** (Galois extension of $\mathcal{JL}_{1,1}$ over $\mathcal{JL}_{1,0} \simeq E_1$)**.** Given

$$\pi_* E_1 \simeq \mathbf{Z}_p[u^{\pm 1}] \qquad \text{with } |u| = 2 \tag{5.6}$$

and $\mathrm{Gal}(K_1/K_0) \simeq \boldsymbol{\mu}_{p-1}$ as in (5.5), the homotopy group $\pi_0 \mathcal{JL}_{1,1} \simeq \mathbf{Z}_p[\zeta_p]$ as a module over the group ring $\mathbf{Z}_p[\boldsymbol{\mu}_{p-1}]$ has a basis consisting of $1, \zeta_p, \ldots, \zeta_p^{p-1}$. Let $\sigma$ be a generator of the Galois group $\boldsymbol{\mu}_{p-1} \simeq (\mathbf{Z}/p\mathbf{Z})^\times$. The action of $\boldsymbol{\mu}_{p-1}$ on $\mathbf{Z}_p[\zeta_p]$ is then given by

$$\sigma^i \cdot \zeta_p^j = \zeta_p^{j\sigma^i} \qquad \text{for } 0 \leqslant i \leqslant p-2 \text{ and } 0 \leqslant j \leqslant p-1$$

which fixes 1 and permutes the primitive $p$'th roots of unity. In particular, the element $N_\sigma \coloneqq 1 + \sigma + \cdots + \sigma^{p-2}$ acts by

$$N_\sigma \cdot \zeta_p^j = \begin{cases} p-1 & \text{if } j = 0 \\ -1 & \text{if } 1 \leqslant j \leqslant p-1 \end{cases}$$

Thus a standard calculation of the cohomology of a finite cyclic group gives

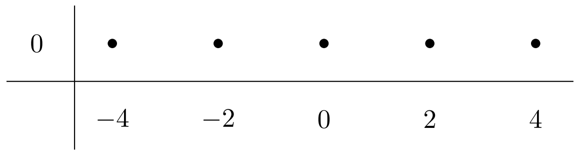
$$H^s(\boldsymbol{\mu}_{p-1}, \pi_0 \mathcal{JL}_{1,1}) \simeq \begin{cases} \mathbf{Z}_p & \text{if } s = 0 \\ 0 & \text{if } s > 0 \end{cases} \tag{5.7}$$

Since $E_1$ is 2-periodic, so is $\mathcal{JL}_{1,1}$, and $\boldsymbol{\mu}_{p-1}$ acts trivially on a periodicity generator $u = u_1$ in degree 2.

Now the $E_2$-page of the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(\boldsymbol{\mu}_{p-1}, \pi_t \mathcal{JL}_{1,1}) \implies \pi_{t-s} E_1$$

in the Adams grading $(t-s, s)$ contains classes illustrated as follows:




where each line of slope $-1$ beginning in an even degree in the row $s = 0$ corresponds to a copy of the cohomology groups (5.7), with each $\bullet$ a copy of $\mathbf{Z}_p$.

Given [Dev20, Theorem 1.3], $\mathcal{JL}_{1,1}$ must then have nonzero homotopy groups in odd degrees, so that its $K(1)$-localization does not contain $\zeta_p$ in the 0'th homotopy group. They contribute classes to odd-numbered columns of the $E_2$-page to support differentials which always move only one step to the left. Write $I_1$ for $\pi_1\mathcal{JL}_{1,1}$. The $d_r$-differentials with $r \geqslant 3$ between the cohomology groups of $I_1$ in different degrees will then eliminate all the classes eventually, save for each $\bullet$ on the row $s = 0$. We thus obtain

$$\pi_*\mathcal{JL}_{1,1} \simeq (\mathbf{Z}_p[\zeta_p] \oplus I_1[\epsilon])[u^{\pm 1}] \qquad \text{with } |\epsilon| = 1,\ \epsilon^2 = 0, \text{ and } |u| = 2 \tag{5.8}$$

for some $\boldsymbol{\mu}_{p-1}$-module $I_1$ with nontrivial group cohomology above degree 0.

**Example 5.12** (Galois extension of $\mathcal{JL}_{1,2}$ over $\mathcal{JL}_{1,1}$)**.** Given $\pi_*\mathcal{JL}_{1,1}$ as in (5.8) and $\mathrm{Gal}(K_2/K_1) \simeq \mathbf{Z}/p\mathbf{Z}$ from (5.5), the $(\mathbf{Z}[\zeta_p])[\mathbf{Z}/p\mathbf{Z}]$-module $\pi_0\mathcal{JL}_{1,2} \simeq \mathbf{Z}_p[\zeta_{p^2}]$ has a basis consisting of $1, \zeta_{p^2}, \ldots, \zeta_{p^2}^{p-1}$. Let $\tau$ be a generator of $\mathbf{Z}/p\mathbf{Z}$, acting as $1 + p$. The action of $\mathbf{Z}/p\mathbf{Z}$ on $\mathbf{Z}_p[\zeta_{p^2}]$ is then given by

$$\tau^i \cdot \zeta_{p^2}^j = (\zeta_p^j)^i \zeta_{p^2}^j \qquad \text{for } 0 \leqslant i \leqslant p-1 \text{ and } 0 \leqslant j \leqslant p-1$$

In particular, the element $N_\tau \coloneqq 1 + \tau + \cdots + \tau^{p-1}$ acts by

$$N_\tau \cdot \zeta_{p^2}^j = \begin{cases} p & \text{if } j = 0 \\ 0 & \text{if } 1 \leqslant j \leqslant p-1 \end{cases}$$

Again, a standard calculation gives

$$H^s(\mathbf{Z}/p\mathbf{Z}, \pi_0\mathcal{JL}_{1,2}) \simeq \begin{cases} \mathbf{Z}_p[\zeta_p] & & \text{if } s = 0 \\ \bigoplus_{j=1}^{p-1} \mathbf{Z}_p[\zeta_p]/(1-\zeta_p^j) & \simeq \mathbf{F}_p^{p-1} & \text{if } s > 0 \text{ is odd} \\ \mathbf{Z}_p[\zeta_p]/(p) & \simeq \mathbf{F}_p[\zeta_p] & \text{if } s > 0 \text{ is even} \end{cases} \tag{5.9}$$

Since $\mathcal{JL}_{1,1}$ is 2-periodic, so is $\mathcal{JL}_{1,2}$, and $\mathbf{Z}/p\mathbf{Z}$ acts trivially on a periodicity generator $u = u_2$ in degree 2.

This time the $E_2$-page of the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(\mathbf{Z}/p\mathbf{Z}, \pi_t\mathcal{JL}_{1,2}) \implies \pi_{t-s}\mathcal{JL}_{1,1}$$

in the Adams grading $(t - s, s)$ contains classes illustrated as follows:

$$
\begin{array}{c|ccccccccc}
4 & \square & & \square & & \square & & \square & & \square \\
3 & & \triangle & & \triangle & & \triangle & & \triangle & \\
2 & \square & & \square & & \square & & \square & & \square \\
1 & & \triangle & & \triangle & & \triangle & & \triangle & \\
0 & \blacksquare & & \blacksquare & & \blacksquare & & \blacksquare & & \blacksquare \\
\hline
 & -4 & & -2 & & 0 & & 2 & & 4
\end{array}
$$

where each line of slope $-1$ beginning in an even degree in the row $s = 0$ corresponds to a copy of the cohomology groups (5.9), denoted by $\blacksquare$, $\triangle$, and $\square$ in increasing degrees, the latter two alternating with periodicity.

Towards the expected abutment (5.8), in each even-numbered column only $\blacksquare$ survives to the $E_\infty$-page. Given the mathematical expressions for $\square$ and $\triangle$ in (5.9), $\mathcal{JL}_{1,2}$ must then additionally have nonzero homotopy groups in odd degrees, whose isomorphism class we denote by $I_2$. Moreover, besides supporting differentials which eliminate $\square$ (and possibly Galois cohomology of $I_2$ in odd degrees), classes in odd-numbered columns should eventually abut to $I_1$ from (5.8). As such, the $\mathbf{Z}/p\mathbf{Z}$-module

$$\pi_*\mathcal{JL}_{1,2} \simeq (\mathbf{Z}_p[\zeta_{p^2}] \oplus I_2[\epsilon])[u^{\pm 1}] \qquad \text{with } |\epsilon| = 1,\ \epsilon^2 = 0, \text{ and } |u| = 2$$

plays the role of an *injective hull* for $(\pi_0\mathcal{JL}_{1,2})[u^{\pm 1}]$.

**Proposition 5.13.** *Hypothesis and notations as above, for each positive integer $r$ we have*

$$\pi_*\mathcal{JL}_{1,r} \simeq (\mathbf{Z}_p[\zeta_{p^r}] \oplus I_r[\epsilon])[u^{\pm 1}] \qquad \textit{with } |\epsilon| = 1,\ \epsilon^2 = 0, \textit{ and } |u| = 2$$

*where each $I_r$ is a finitely generated abelian group with an action by $\boldsymbol{\mu}_{p-1} \times \mathbf{Z}/p^{r-1}\mathbf{Z}$, and together they form a direct system. Moreover, their corresponding group cohomology classes are compatible for convergence of homotopy fixed point spectral sequences, as explained in the last paragraphs of Examples 5.11 and 5.12.*

At infinite level, by adapting [BSSW25, proof of Theorem C], we first obtain the following cohomology groups for the action of $\mathrm{GL}_1(\mathbf{Z}_p) \simeq \mathrm{Gal}(K_\infty/K_0)$ on $\pi_0\mathcal{JL}_{1,\infty} \simeq \mathcal{O}_{K_\infty}$ as in the tower (5.5), which the authors learned from Wei Yang. Note that unlike the finite Galois groups discussed above, $\mathrm{GL}_1(\mathbf{Z}_p)$ has cohomological dimension 1, hence vanishing of its group cohomology above degree 1.

**Lemma 5.14** (Yang)**.** *Notations as above, we have*

$$H^s_{\mathrm{c}}\big(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}_{K_\infty}\big) \simeq \begin{cases} \mathbf{Z}_p & \textit{if } s = 0 \\ \mathbf{Z}_p \oplus T & \textit{if } s = 1 \\ 0 & \textit{if } s > 1 \end{cases}$$

*where $T$ is $p$-power torsion of bounded exponent.*

**Example 5.15** (Yang)**.** Let $g$ be a topological generator of $\mathrm{Gal}(K_\infty/K_0) \simeq \mathbf{F}_p^\times \times (1 + p\mathbf{Z}_p)$ and $x \coloneqq (1 - \zeta_{p^2})^{(1-p)^2} \in \mathcal{O}_{K_\infty}$ so that $g \cdot x = x^g$. Then the image of $(x^g - x)/p \in \mathcal{O}_{K_\infty}$ in the $(s = 1)$'st cohomology group above is $p$-torsion and so is a class in $T$.

*Remark* 5.16. The nonvanishing of the 1st cohomology group in Lemma 5.14 agrees with that of the odd-degree homotopy groups in (5.8), along the long exact sequence of homotopy groups associated to the fiber sequence

$$\mathcal{JL}_{1,1} \to \mathcal{JL}_{1,\infty} \to \mathcal{JL}_{1,\infty}$$

from the $\mathbf{Z}_p$-Galois extension of $\mathcal{JL}_{1,\infty}$ over $\mathcal{JL}_{1,1}$.

**Corollary 5.17.** *We have*

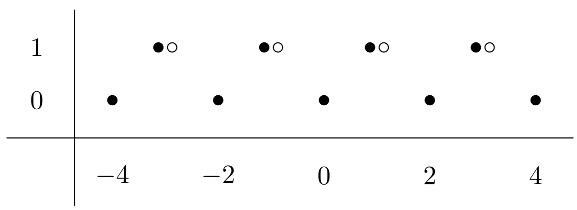
$$\pi_*\mathcal{JL}_{1,\infty} \simeq (\mathcal{O}_{K_\infty} \oplus I_\infty[\epsilon])[u^{\pm 1}] \qquad \text{with } |\epsilon| = 1,\ \epsilon^2 = 0, \text{ and } |u| = 2 \tag{5.10}$$

*where* $I_\infty = \varinjlim_r I_r$ *from Proposition 5.13. Moreover,* $I_\infty \supset \mathbf{Z}_p$*, and only a finite subgroup of* $\mathrm{GL}_1(\mathbf{Z}_p)$ *acts nontrivially on* $I_\infty$*.*

*Sketch proof.* The expression for $\pi_*\mathcal{JL}_{1,\infty}$ follows from Proposition 5.13 and Definition 5.3. We then deduce the properties of the homotopy groups $I_\infty$ in odd degrees, as stated, from the homotopy fixed point spectral sequence

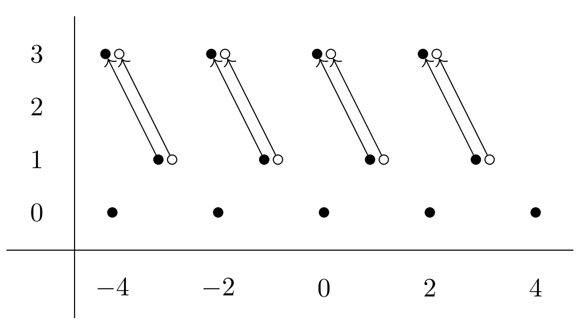
$$E_2^{s,t} = H^s_{\mathrm{c}}\big(\mathrm{GL}_1(\mathbf{Z}_p), \pi_t\mathcal{JL}_{1,\infty}\big) \implies \pi_{t-s}E_1 \tag{5.11}$$

whose $E_2$-page in the Adams grading $(t-s, s)$ contains classes depicted below:

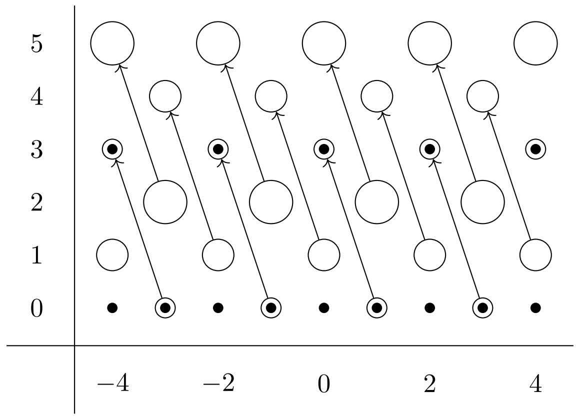



Here, each line of slope $-1$ beginning in an even degree in the row $s = 0$ corresponds to a copy of the cohomology groups from Lemma 5.14, with each $\bullet$ a copy of $\mathbf{Z}_p$ and $\circ$ a copy of $T$.

Write $\iota_s$ for the $s$'th Galois cohomology of $I_\infty$. For the spectral sequence to converge to the expected abutment (5.6) with no nonzero classes in odd-numbered columns on the $E_\infty$-page, $\iota_3$ (and possibly higher cohomology groups) must contain $\mathbf{Z}_p \oplus T$, so that $d_2$-differentials between copies at $(2m+1, 1)$ and $(2m, 3)$ eliminate them both:




Since $\mathrm{Gal}(K_\infty/K_0) \simeq \mathrm{GL}_1(\mathbf{Z}_p)$ has cohomological dimension 1, we then deduce that $\iota_s \simeq H^s(G_I, I_\infty)$ for some (nontrivial) finite subgroup $G_I \subset \mathrm{GL}_1(\mathbf{Z}_p)$, i.e., a cyclic group of order dividing $p-1$. Moreover, since $\iota_3$ contains a copy of $\mathbf{Z}_p$, so does $\iota_0 \subset I_\infty$. We can thus draw the $E_3$-page




where a small circle with a $\bullet$ inside denotes $\iota_0 \supset \mathbf{Z}_p$, a medium circle denotes $\iota_1$, a large circle denotes $\iota_2$, so that $\iota_3 \simeq \mathbf{Z}_p \oplus T \oplus \iota_0$, $\iota_4 \simeq \iota_1$, and $\iota_5 \simeq \iota_2$. The classes represented by these symbols are then eliminated in pairs by $d_3$-differentials (or possibly higher differentials for corresponding pairs). The higher nonzero cohomology groups $\iota_s$ with $s > 5$, if they exist, do not survive to the $E_\infty$-page. We obtain the abutment (5.6) as expected. □

*Remark* 5.18. As such, the spectra $\mathcal{JL}_{1,r}$ with $r \geqslant 1$ and $\mathcal{JL}_{1,\infty}$ are complex periodic $\mathbb{E}_\infty$-rings, at least in good cases (cf. Example 4.24 and Proposition 4.17). However, they are not $K(1)$-local (cf. Remark 4.23 and [Lur18b, Theorem 4.5.2]).

*Remark* 5.19. The peculiar way in which $\mathrm{GL}_1(\mathbf{Z}_p)$ acts on $\pi_*\mathcal{JL}_{1,\infty}$, in contrast to finite levels, may be a manifest of the vanishing of the cotangent complex of its corresponding classical moduli problem at infinite level [Sch12, Proposition 5.13] (cf. Section 5.4.1 below).

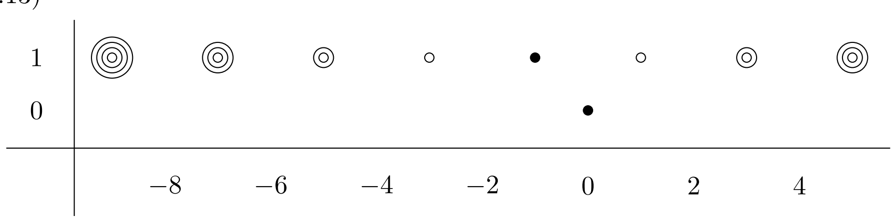
**Question 5.20.** Given [Sal23, Section 1.4], what would be the modular and arithmetic significance of the direct system $\{I_r\}_{r\geqslant 1}$ in odd degrees of $\pi_*\mathcal{JL}_{1,\infty}$ (also potentially in relation to Iwasawa theory)? Can we make it (and $G_I \subset \mathrm{Gal}(K_\infty/K_0)$ acting on $I_\infty = \varinjlim_r I_r$) more explicit?

5.3.2. *The Devinatz–Hopkins spectral sequence.* To prepare for Section 5.3.3 below, as a special case of (5.1), let us recall the homotopy fixed point spectral sequence

$$E_2^{s,t} = H_c^s(\mathbb{G}_1, \pi_t E_1) \implies \pi_{t-s}\mathbf{S}_{K(1)} \tag{5.12}$$

which collapses at the $E_2$-page (for odd primes $p$):

(5.13)




We adhere to the Adams grading $(t-s, s)$. As above, a dot denotes $\mathbf{Z}_p$. In the $(2m-1)$'st column, a symbol of $|m|$ concentric circles denotes $\mathbf{Z}_p/(1-g^m)$, again with $g$ a topological generator of $\mathbb{G}_1 \simeq \mathbf{Z}_p^\times$. Here, $\mathbb{G}_1$ acts on $\pi_* E_1$ by $g^m \cdot u := g^m u$, or equivalently, $g^m$ acts on $\pi_{2m} E_1 \simeq \mathbf{Z}_p$ by multiplying $g^m$ (see, e.g., [Lur10, Lecture 35]). We thus obtain

$$\begin{aligned}\pi_n \mathbf{S}_{K(1)} &\simeq \begin{cases} \mathbf{Z}_p & \text{if } n = 0 \\ \mathbf{Z}_p/(1-g^m) & \text{if } n = 2m-1 \\ 0 & \text{otherwise} \end{cases} \\ &\simeq \begin{cases} \mathbf{Z}_p & \text{if } n = 0 \text{ or } n = -1 \\ \mathbf{Z}/p^{k+1}\mathbf{Z} & \text{if } n = 2(p-1)p^k i - 1 \text{ with } p \nmid i \\ 0 & \text{otherwise} \end{cases}\end{aligned} \tag{5.14}$$

5.3.3. *Descending along the Drinfeld tower.* Given the analogous action of $\mathbb{G}_1$ on $\pi_* \mathcal{JL}_{1,\infty}$ from Corollary 5.17, the homotopy fixed point spectral sequence

$$E_2^{s,t} = H_c^s(\mathbb{G}_1, \pi_t \mathcal{JL}_{1,\infty}) \implies \pi_{t-s}{}^{\mathrm{L}}E_1 \tag{5.15}$$

behaves similarly as (5.12). The difference lies in the fact that, besides nonzero homotopy groups $I_\infty$ in odd degrees, $p$-power torsion therein may contribute additional cohomology classes. Thus again the spectral sequence collapses at the $E_2$-page and yields the following.

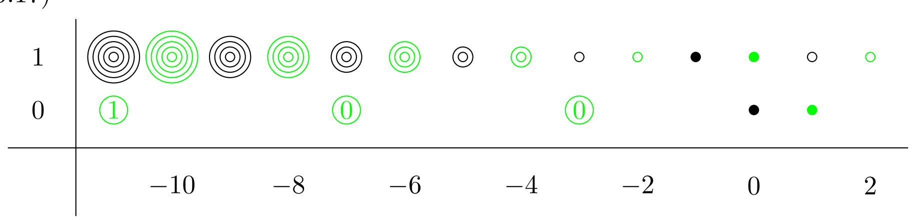
**Proposition 5.21.** *Notations as in Corollary 5.17, we have*

$$\pi_n{}^{\mathrm{L}}E_1 \simeq \begin{cases} \mathcal{O}_{K_\infty} \oplus I_\infty & \text{if } n = 0 \\ \mathcal{O}_{K_\infty} & \text{if } n = -1 \\ I_\infty & \text{if } n = 1 \\ I_\infty/(1-g^m) & \text{if } n = 2m \text{ with } m \neq 0 \\ \mathcal{O}_{K_\infty}/(p^{k+1}) & \text{if } n = 2(p-1)p^k i - 1 \text{ with } p \nmid i \\ T^{(k)} & \text{if } n = 2(p-1)p^k i + 1 \text{ with } p \nmid i \\ 0 & \text{otherwise} \end{cases} \tag{5.16}$$

*where $g$ is a topological generator of $\mathbb{G}_1$, and $T^{(k)} \subset I_\infty$ is $p$-power torsion of exponent bounded by $p^{k+1}$.*

*Proof.* Given the action of $\mathbb{G}_1$ on $u$, let us illustrate the $E_2$-page of (5.15) in the Adams grading $(t-s, s)$ as below (cf. (5.13)), with explanations following:

(5.17)




Since $\mathcal{O}_{K_\infty}$ in even degrees of $\pi_* \mathcal{JL}_{1,\infty}$ is torsion-free, its $\mathbb{G}_1$-cohomology contributes to the $E_2$-page a copy of classes in (5.13), with $\mathbf{Z}_p \simeq \mathcal{O}_{K_0}$ replaced by

$\mathcal{O}_{K_\infty}$. These classes all survive to the $E_\infty$-page and give an abutment (cf. (5.14))

$$\mathfrak{O}_n := \begin{cases} \mathcal{O}_{K_\infty} & \text{if } n = 0 \\ \mathcal{O}_{K_\infty}/(1-g^m) & \text{if } n = 2m-1 \\ 0 & \text{otherwise} \end{cases}$$
$$\simeq \begin{cases} \mathcal{O}_{K_\infty} & \text{if } n = 0 \text{ or } n = -1 \\ \mathcal{O}_{K_\infty}/(p^{k+1}) & \text{if } n = 2(p-1)p^k i - 1 \text{ with } p \nmid i \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, in odd degrees of $\pi_* \mathcal{JL}_{1,\infty}$, the $\mathbb{G}_1$-cohomology of $I_\infty$ contributes a second copy of classes in (5.13) shifted by degree 1 to the right, colored in green with $\mathbf{Z}_p$ replaced by $I_\infty$. Moreover, a $p^n$-torsion class in $I_\infty$ appears in the 0'th $\mathbb{G}_1$-cohomology whenever $p-1$ divides $m = (t-1)/2$ and $\nu_p(m) + 1 \geqslant n$. Indeed, in this case, $g^{p-1}$ is a generator for the topologically cyclic pro-$p$-group $(1+p\mathbf{Z}_p)^\times$, and $1-g^m$ is a generator for $p^{\nu_p(m)+1}\mathbf{Z}_p \subset \mathbf{Z}_p$. We depict such torsion classes in the $(2m+1)$'st column by a green circle around the value of $\nu_p(m)$, with $p = 3$ as an example. All these classes also survive to the $E_\infty$-page and give a second abutment

$$\mathfrak{I}_n := \begin{cases} I_\infty & \text{if } n = 1 \\ I_\infty/(1-g^m) & \text{if } n = 2m \\ T^{(k)} & \text{if } n = 2(p-1)p^k i + 1 \text{ with } p \nmid i \\ 0 & \text{otherwise} \end{cases}$$

where $T^{(k)}$ consists of the torsion classes above with $k = \nu_p(m)$.

We thus obtain the stated expression for $\pi_n {}^{\mathrm{L}}E_1 \simeq \mathfrak{O}_n \oplus \mathfrak{I}_n$ by overlapping the two displays above. Note that the green single circles in (5.17) actually represent 0, as neither 1 nor $-1$ is a multiple of $p-1$ for $p$ odd. $\square$

*Remark* 5.22. The summand $\mathcal{O}_{K_\infty} \subset \pi_0 {}^{\mathrm{L}}E_1$ corresponds to the Drinfeld symmetric space $\mathcal{H}$ discussed in Remark 5.8. In the case where $h = 1$, $\mathcal{H} = \mathbf{P}^0_{K_\infty}$ consists of a single point.

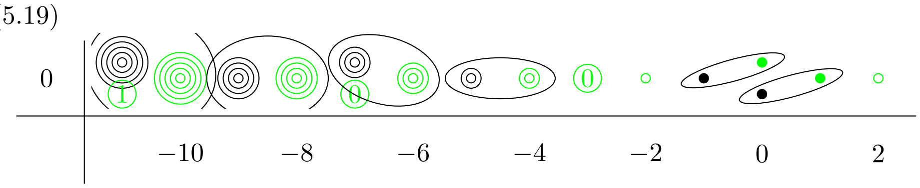
5.3.4. *Completing the square.* We now compare the action of $\mathrm{GL}_1(\mathbf{Z}_p)$ on $\pi_* \mathcal{JL}_{1,\infty}$ from Section 5.3.1 and analyze the remaining homotopy $\mathrm{GL}_1(\mathbf{Z}_p)$-fixed point spectral sequence

$$E_2^{s,t} = H^s_{\mathrm{c}}\big(\mathrm{GL}_1(\mathbf{Z}_p), \pi_t {}^{\mathrm{L}}E_1\big) \implies \pi_{t-s} \mathbf{S}_{K(1)} \tag{5.18}$$

in (5.4), whose abutment should match up with (5.14) as calculated from the $\mathbb{G}_1$-action on $\pi_* E_1$.

For convenience, without loss of generality for odd primes $p$, let us focus on the case $p = 3$. In view of (5.16) and (5.17), we then depict the line $s = 0$ on the $E_2$-page as follows:

(5.19)




where as in the proof of Proposition 5.21, we organize nonzero classes into two families. The family in black is computed from the $\mathbb{G}_1$-cohomology $\mathfrak{O}_n$ of $\mathcal{O}_{K_\infty}$

in even degrees from (5.10), while the family in green is computed from the $\mathbb{G}_1$-cohomology $\mathfrak{I}_n$ of $I_\infty$ in odd degrees.

Observe that as in the sketch proof of Corollary 5.17, here each line of slope $-1$ beginning with a group $H^0\big(\mathrm{GL}_1(\mathbf{Z}_p), \mathfrak{O}_n\big)$ in the black family is paired with one in the green family next to it on the right. We indicate such pairs by circling them in (5.19). Thus differentials between the two lines in a pair are to eliminate classes in $H^s\big(\mathrm{GL}_1(\mathbf{Z}_p), \mathfrak{O}_n\big)$ with $s > 0$ as before.

The new feature here is that $\mathfrak{O}_n$ is a quotient of $\mathcal{O}_{K_\infty}$ in general, so is $\mathfrak{I}_n$ a quotient of $I_\infty$, and taking quotient does not commute with computing $\mathrm{GL}_1(\mathbf{Z}_p)$-cohomology. In particular, by the long exact sequence of cohomology groups associated to each short exact sequence of $\mathbf{Z}_p[\mathrm{GL}_1(\mathbf{Z}_p)]$-modules

$$0 \to \mathcal{O}_{K_\infty} \xrightarrow{\cdot(1-g^m)} \mathcal{O}_{K_\infty} \to \mathcal{O}_{K_\infty}/(1-g^m) \to 0$$

the group

$$\mathbf{Z}_p/(1-g^m) \simeq H^0\big(\mathrm{GL}_1(\mathbf{Z}_p), \mathcal{O}_{K_\infty}\big)/(1-g^m)$$

is a submodule of

$$E^2_{2m-1,0} = H^0\big(\mathrm{GL}_1(\mathbf{Z}_p), \mathcal{O}_{K_\infty}/(1-g^m)\big)$$

This submodule is the only term in the $(2m-1)$'st column that survives to the $E_\infty$-page for the expected abutment (5.14).

As a remedy for this discrepancy, the remaining cohomology groups in the green family computed from $p$-power torsion in $I_\infty$ (represented by circled 3-adic valuations) come into play. Together with possibly higher differentials beyond the $E_3$-page, they clear up all classes in the green family, all classes of nonzero cohomological degree in the black family, as well as those of bidegree $(2m-1, 0)$ in the latter family but not in $H^0\big(\mathrm{GL}_1(\mathbf{Z}_p), \mathcal{O}_{K_\infty}\big)/(1-g^m)$.

A more detailed examination in future work on the differentials of this spectral sequence (5.18), besides those of (5.11), will reveal additional constraints on $H^*_{\mathrm{c}}\big(\mathrm{GL}_1(\mathbf{Z}_p), \mathcal{O}_{K_\infty}\big)$ and $H^*\big(G_I, I_\infty\big)$, and supply clues to Question 5.20.

## 5.4. **Further problems.**

5.4.1. *Homotopy groups of $E_{h,r}$ and $\mathcal{JL}_{h,r}$ for $r > 0$.* In this paper, we defined and studied level structures in the context of spectral algebraic geometry, and obtained from Theorems 4.22 and 4.13 the $\mathbb{E}_\infty$-ring spectra $E_{h,r}$ and $\mathcal{JL}_{h,r}$ as global sections of moduli spaces for these derived level structures. In particular, the level-0 cases $E_{h,0} \simeq \mathcal{JL}_{h,0} \simeq E_h$ recover the Morava E-theory spectrum. Moreover, for all $r \geqslant 0$, $\pi_0 E_{h,r} \simeq A_r$ recover Strickland's deformation rings (see Remark 4.23), which play a central role in $E_h$-power operations, and $\pi_0 \mathcal{JL}_{h,r}$ recover the finite levels of a Lubin–Tate tower (see Proposition 4.14). As explained in Remarks 4.25 and 4.23, the homotopy groups $\pi_n E_{h,r}$ and $\pi_n \mathcal{JL}_{h,r}$ for $n, r > 0$ may encode refined information of arithmetic algebraic geometry, in terms of higher-homotopical coherence of isogenies (instead of isomorphisms). It would therefore be desirable to develop strategies for quantitative investigation of these invariants.

In this direction, we propose a Serre-type spectral sequence

$$E_2^{s,t} = H^s(\pi_* E_h, \pi_{*+t} L_{E_{h,r}/E_h}) \implies \pi_{t-s} E_{h,r}$$

where

$$L_{E_{h,r}/E_h} \simeq \mathrm{TAQ}^{E_h}(E_{h,r})$$

is the (topological) cotangent complex in [Lur18c] along the $\mathbb{E}_\infty$-ring map $E_h \to E_{h,r}$ as a globalization of topological André–Quillen homology. Conjecturally, the $E_2$-page is inspired by a mapping space spectral sequence for augmented $K(h)$-local commutative $E_h$-algebras studied by Rezk [Rez13, §§2.7 and 2.13], in the context of $E_h$-cohomology of the $K(h)$-localized TAQ. It is key to compute the cotangent complex of the corresponding moduli problem (cf. [MPR24]). The abutment is based on a conceptualization of Behrens and Rezk's modular description related to a comparison between TAQ and the $K(h)$-localized Bousfield–Kuhn functor by examining $E_h$-cohomology of layers of the latter applied to the Goodwillie tower of the identity functor [BR20a, Rez12]. On a related note, in view of Remark 4.23, we aim to further investigate the nature of $K(h)$-localization in spectral algebraic geometry.

5.4.2. *Derived level structures and representations.* Galatius and Venkatesh defined and studied derived Galois deformations in [GV18], and compared the action by the homotopy groups of the derived Galois deformation ring and that by the derived Hecke algebra introduced in [Ven19] on the (co)homology of locally symmetric spaces (see, in particular, [GV18, Definition 5.4 (iii), Theorem 4.33, Section 7.3, and Theorem 15.2]). In Section 3.1, we constructed moduli stacks of spectral elliptic curves with derived level structures. We do not know yet specifically what sorts of Hecke algebras may act on their rings of functions. In [Dav24, CS24], there have been constructions in derived settings of Hecke operators on topological modular forms and Hecke eigenforms, e.g., based on Lurie's theorem [Dav24, Theorem 1.1], without involving *derived* level structures.

In view of the close relationship between level structures and representations, it would be interesting to develop a general theory of Hecke algebras in the context of spectral algebraic geometry and find a reasonable construction of derived Hecke stacks compatible with Hecke algebras for topological modular forms. A natural, related question to investigate is the *classicality* of the moduli spaces involved (see, e.g., [FH25, Section 3.2.1] and cf. [GV18, Conjecture in Section 1.3]).

5.4.3. *Relationship to categorical local Langlands correspondence.* From the recently developing categorical perspective, the Langlands correspondence (in its many facets) should be thought of as setting up equivalences between ($\infty$-)categories of certain geometric objects. See [EGH25] and the references therein. For example, for local Langlands correspondence, let $F$ be a finite extension of $\mathbf{Q}_p$, and $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbf{Q}_\ell$; then the category of smooth $\mathrm{GL}_h(F)$-representations on torsion $\mathcal{O}$-modules is conjecturally equivalent to a subcategory of quasi-coherent sheaves on a moduli stack $\mathcal{X}$ which parametrizes $h$-dimensional $\ell$-adic representations of the absolute Galois group $\mathrm{Gal}_F$. When $\ell \neq p$, such a stack $\mathcal{X}$ was introduced in [Zhu21] and [DHKM25]. In [FS24], this stack is realized by a stack of equivariant vector bundles on the Fargues–Fontaine curve for $F$ (whose étale fundamental group is isomorphic to $\mathrm{Gal}_F$). When $\ell = p$, this stack was constructed in [EG23] as a stack of $(\varphi, \Gamma)$-modules.

Our $\mathbb{E}_\infty$-ring spectrum $\mathcal{JL}_{h,\infty}$ from Section 5.1 defines a functor

$$p\text{-complete spectra} \to \big(\mathrm{GL}_h(\mathcal{O}_F) \times D^\times\big)\text{-equivariant spectra}, \quad X \mapsto \mathcal{JL}_{h,\infty}^X$$

where $D/F$ is the central division algebra of invariant $1/h$, and $\mathcal{JL}_{h,\infty}^X$ denotes the function spectrum of maps $X \to \mathcal{JL}_{h,\infty}$. Given the perspective of chromatic homotopy theory [Goe08], we may ask if this functor can be upgraded to a functor

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}}) \to \text{equivariant sheaves on } \mathcal{X}$$

for some suitably constructed moduli stack $\mathcal{X}$ on the Galois side, where $\mathcal{M}_{\mathrm{FG}}$ is the moduli stack of one-dimensional formal groups. In this case, the categorical local Langlands correspondence may take a form

$$\mathrm{Mod}_{{}^{\mathrm{L}}E_h} \simeq \mathrm{IndCoh}(\mathcal{X})$$

between the $\infty$-category of module spectra over the $\mathrm{GL}_h(\mathcal{O}_F)$-equivariant $\mathbb{E}_\infty$-ring spectrum ${}^{\mathrm{L}}E_h$ from Proposition 5.7, and the stable $\infty$-category of Ind-coherent complexes over $\mathcal{X}$.

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