Averages:

#1	#2	#3	#4	#5	#6	total
14.1/15	15.6/20	11.7/15	15.2/20	10.6/20	6.1/10	73.4/100

median 76.5

90-100: x x x x x
80-89: x x x x x x x x x x x x x
70-79: x x x x | x x x
60-69: x x x x x x x x
50-59: x x x x x x
<50: x x x

(a)(5 points)

A topological space X is called a $\mathbf{T_3}$ space if for every point $x \in X$ and every closed set $F \subset X$ with $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.

(a)(10 points)

Let X be a compact Hausdorff space. Let $x \in X$ and let $F \subset X$ be a closed set with $x \notin F$.

For each $y \in F$, since X is Hausdorff, there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. (2 points)

The collection $\{V_y : y \in F\}$ forms an open cover of F. Since F is closed in the compact space X, F is compact. (2 points)

Therefore, there exist finitely many points $y_1, y_2, \ldots, y_n \in F$ such that $F \subset \bigcup_{i=1}^n V_{y_i}$. (2 points)

Let $U = \bigcap_{i=1}^n U_{y_i}$ (which is open, as a finite intersection of open sets) and $V = \bigcup_{i=1}^n V_{y_i}$ (which is open). (2 points)

Then $x \in U$, $F \subset V$, and $U \cap V = \varnothing$, since each U_{y_i} is disjoint from V_{y_i}. (2 points)

2.

(a)(5 points)

A topological space is called a C_2 space if it has a countable topological base.

(b)(5 points) (construction 3 points, proof 2 points)

Given a countable basis $\{B_i\}$, we select $x_i \in B_i$ for each i. For any open set A, there exists $x_i \in B_i \subseteq A$ for some i. Hence, $\{x_i\}$ is a countable dense set.

(c) (10 points) (example 2 points, separable 4 points, C_2 4 points)

$X = (\mathbb{R}, \tau)$, $\tau = \overline{\{[a, b) | a < b\}}$

separable:

$\mathbb{Q}$ is a dense subset. (2 points)

For every open set $U \subseteq \mathbb{R}$, choose $x \in U$ and $x \in [a, b) \subseteq U$.

Then, there exists $y \in \mathbb{Q}$ such that $a < y < b$.That is, $\mathbb{Q} \cap U \neq \emptyset$, $\mathbb{Q}$ is a countable dense set. (2 points for proof)

C_2:

X is not C_2. (1 points)

Assume X have countable basis $\{B_i\}$.

For every $x \in \mathbb{R}$, choose $U_x \in \{B_i\}$ such that $x \in U_x \subseteq [x, x+1)$.

Since $\inf U_x = x$, $U_x \neq U_y$ when $x \neq y$.

Then, we can construct a monomorphism $f : \mathbb{R} \to \{B_i\}; x \mapsto U_x$, an contradiction to $\{B_i\}$ is countable. (3 points for proof)

$X = (\mathbb{R}, \tau_f)$

3. （Examples 3 points ; Explanation 2 points）

(a) Examples of a Closed and Bounded Metric Space That Is Not Compact

Example 1: The Set of Natural Numbers $\mathbb{N}$ with the Discrete Metric

Let $X = \mathbb{N}$ and define the metric by

$$d(m, n) = \begin{cases} 1 & \text{if } m \neq n, \\ 0 & \text{if } m = n. \end{cases}$$

- **Closed:** $\mathbb{N}$ is trivially closed in itself.
- **Bounded:** For any $m, n \in X$, $d(m, n) \leq 1$; thus, bounded.
- **Not compact:** The open cover $\{\{n\} : n \in \mathbb{N}\}$ of singletons does not admit a finite subcover. Therefore, not compact.

Example 2: Closed Unit Ball in an Infinite-Dimensional Normed Space (ℓ^2)

Let $X = \ell^2$, the space of all square-summable sequences, with the norm

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}.$$

The closed unit ball is

$$B = \{x \in \ell^2 : \|x\| \leq 1\}.$$

- **Closed:** By definition.
- **Bounded:** Every $x \in B$ satisfies $\|x\| \leq 1$.
- **Not compact:** The standard basis vectors $e_n = (0, 0, ..., 1, 0, ...)$ (1 in the n-th entry) all lie in B and have mutual distance $\sqrt{2}$, so there is no convergent subsequence. Thus, B is not compact (by Riesz's lemma).

Example 3: $\mathbb{Q} \cap [0, 1]$ as a Subspace of $(\mathbb{Q}, |\cdot|)$

Let $X = [0, 1] \cap \mathbb{Q}$, with the metric $d(x, y) = |x - y|$.

- **Closed:** In the subspace topology of $\mathbb{Q}$, its complement is open in $\mathbb{Q}$.
- **Bounded:** $X \subset [0, 1]$, so bounded.
- **Not compact:** Since $\mathbb{Q}$ is not complete, Cauchy sequences in X may converge to irrational numbers not in X. For instance, a sequence of rationals converging to an irrational number in $[0, 1]$ does not converge in X. Thus, not compact.

Example 4: $[0,1]$ with the Discrete Metric

Let $X = [0,1]$ with the metric

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

- **Closed:** $[0,1]$ is the whole space.
- **Bounded:** All distances are at most 1.
- **Not compact:** The cover of all singletons $\{\{x\} : x \in [0,1]\}$ has no finite subcover. Thus, not compact.

Example 5: Closed Unit Ball in $C[0,1]$, the Space of Continuous Functions

Let $X = C[0,1]$, with the metric induced by the supremum norm,

$$\|f - g\|_\infty = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

The closed unit ball is

$$B = \{f \in C[0,1] : \|f\|_\infty \leq 1\}.$$

- **Closed and bounded:** Clear by construction.
- **Not compact:** The sequence $f_n(x) = x^n$ in B is not equicontinuous. By the Arzelà–Ascoli theorem, B is not compact.

(b) A pair of homeomorphic metric spaces, one complete and the other not

$\mathbb{R}$ (the set of real numbers with the usual metric) and $(0,1)$ (the open interval with the usual metric) are homeomorphic, for example via the map

$$f : \mathbb{R} \to (0,1), \quad f(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$$

This can be verified to be a continuous bijection with a continuous inverse (this is a standard homeomorphism between the real line and an open interval).

- $\mathbb{R}$ is complete: every Cauchy sequence converges.
- $(0,1)$ is not complete: for example, the sequence $x_n = \frac{1}{n}$ is Cauchy in $(0,1)$ but converges to $0 \notin (0,1)$, so it does not converge within $(0,1)$.

(c) A continuous bijection whose inverse is not continuous

Example 1: $[0,1)$ to the Unit Circle S^1

Define the function

$$f : [0,1) \to S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

by

$$f(t) = (\cos 2\pi t, \sin 2\pi t).$$

Then f is a continuous bijection.

However, the inverse $f^{-1} : S^1 \to [0, 1)$ is not continuous.

Reason: The unit circle S^1 is compact, while $[0, 1)$ is not. If the inverse were continuous, $[0, 1)$ would have to be compact as a continuous image of a compact space, which is not the case. Thus, f^{-1} is not continuous.

Example 2: The Identity Map from the Discrete to Standard Topology

Let X be an infinite set (for example, $X = \mathbb{R}$). Give X the discrete topology (all subsets are open), denoted X_d, and the usual Euclidean topology, denoted X_s.

Consider the identity map

$$\mathrm{id} : (X_d) \to (X_s), \quad \mathrm{id}(x) = x.$$

This map is a continuous bijection.

However, the inverse

$$\mathrm{id}^{-1} : (X_s) \to (X_d)$$

is not continuous, because singletons are open sets in the discrete topology but are not open in the standard topology on X.

4.

(a)(5 points)

For every open set $U \subseteq Y$ containing $f(x)$,$f^{-1}(U)$ is open in X by the continuity of f. (2 points)

Since $x \in f^{-1}(U)$ and $x_i \to x$ as $i \to \infty$,all but finitely many terms of the sequence $\{x_i\}$ lie in $f^{-1}(U)$. Consequently,all but finitely many terms of $\{f(x_i)\}$ are contained in U. (2 points)

This implies $f(x_i) \to f(x)$ as $i \to \infty$. (1 point)

(b) (5 points)

A topological space X is called C_1 if for any point $x \in X$, there exists a **countable neighborhood basis** $\{U_n\}_{n=1}^{\infty}$ such that every neighborhood of x contains some U_n.

(c) (10 points)

Suppose $f : X \to Y$ is not continuous. By condition, there exists an open set $U \subseteq Y$ such that $f^{-1}(U)$ is not open in X. (2 points)

Choose a point

$$x \in f^{-1}(U) \setminus \mathrm{int}\left(f^{-1}(U)\right).$$

Since X is a first-countable space(C_1), there exists a nested countable local basis$\{B_i\}$ at x satisfying $B_i \supseteq B_{i+1}$ for all i. (2 points)

Observe that $x \notin \mathrm{int}\left(f^{-1}(U)\right)$. Therefore, for each i , the intersection

$$B_i \cap \left(X \setminus f^{-1}(U)\right)$$

is nonempty. (2 points)

We may thus select

$$x_i \in B_i \setminus f^{-1}(U)$$

By the nested property of $\{B_i\}$, the sequence $x_i$converges to x . (1 point)

However, since $x_i \notin f^{-1}(U)$, we have $f(x_i) \notin U$ for all i. (1 point)

This contradicts the fact that $f(x) \in U$ and $f(x_i) \to f(x)$. (1 point)

Hence, f must be continuous. (1 point)

5.

(a)

For *any* $f \in \mathcal{C}(X,Y)$, show that f is in at least one $S(C,U)$:

- Take any f. Since Y is open, $f(C) \subset Y$, and $f(C) = \{f(x)\} \subset U$. So $f \in S(C,Y)$.

  OR

- Take any f. Since $\emptyset$ is compact, $f(\emptyset) = \emptyset$ is open. So $f \in S(\emptyset, \emptyset)$.

Thus, every $f \in \mathcal{C}(X,Y)$ is in at least one $S(C,U)$. So, the union of all sets in $\mathcal{S}$ is $\mathcal{C}(X,Y)$.

By definition, each $S(C,U) \subset \mathcal{C}(X,Y)$.

(b)

Step 1: $F(x)$ is continuous（5 points)

For each fixed x, the map $y \mapsto f(x,y)$ is the composition of:

- The inclusion $Y \to X \times Y \quad y \mapsto (x,y)$ (continuous)
- The map f

So $F(x)$ is the map $y \mapsto f(x,y)$, which is the composition of continuous maps, hence **continuous**.

Step 2: F is continuous（10 points)

We must show $F^{-1}(S(C,U)) \subset X$ is open for each such subbasis set.

$$F^{-1}(S(C,U)) = \{x \in X \mid F(x) \in S(C,U)\} = \{x \mid F(x)(C) \subset U\}$$

But $F(x)(C) = \{f(x,c) : c \in C\}$. So,

$$F^{-1}(S(C,U)) = \{x \mid \forall c \in C,\ f(x,c) \in U\}$$

Or,

$$F^{-1}(S(C,U)) = \{x \mid \{x\} \times C \subset f^{-1}(U)\}$$

For any $x_0 \in F^{-1}(S(C,U))$, we have $\{x_0\} \times C \subset f^{-1}(U)$. (2 points)

For each $c \in C$, the point (x_0, c) lies in $f^{-1}(U)$.

Since$f^{-1}(U)$ is an open subset of $X \times Y$(since f is continuous), by the definition of open sets in the product topology, there exist open sets $W_c \subset X$ containing x_0 and open sets $V_c \subset Y$ containing c, such that

$$W_c \times V_c \subset f^{-1}(U). \text{ (2 points)}$$

C is covered by this family of open sets $\{V_c \mid c \in C\}$. Since C is compact, there exists a finite subcover $V_{c_1}, \ldots, V_{c_n}$ covering C. (2 points)

Let

$$W = \bigcap_{i=1}^{n} W_{c_i}, \quad \text{(2 points)}$$

then W is an open neighborhood of x_0.

For any $x \in W$ and any $c \in C$, there exists some V_{c_i} such that $c \in V_{c_i}$. Hence, $(x, c) \in W_{c_i} \times V_{c_i} \subset f^{-1}(U)$, so $f(x, c) \in U$. (2 points)

Therefore, for any $x \in F^{-1}(S(C, U))$, there exists open set $W \subset F^{-1}(S(C, U))$.

6.(10 points)

Method 1:

Since f is an embedding, then $X \cong f(X)$.

Without loss of generality, we assume that $X \subseteq Y$ and $f : X \hookrightarrow Y$. (2 points)

We only need to prove that for every compact set $K \subseteq Y$, $K \cap X$ is also a compact set in X.

For any compact set $K \subseteq Y$, let $\{U_\alpha \cap X\}_{\alpha \in J}$ be a open cover of $K \cap X$ in X, where U_α is open in Y. (2 points)

Then, $\{U_\alpha\}_{\alpha \in J} \cup \{Y \setminus X\}$ form an open cover of K in Y. (1 point)

By compactness of K, there exists a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\} \cup \{Y \setminus X\}$(if needed). (2 points)

Since $K \cap X \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$ and $Y \setminus X$ is disjoint from $K \cap X$, the restricted family $\{U_{\alpha_i} \cap X\}_{i=1}^{n}$ must cover $K \cap X$. (2 points)

Hence, $K \cap X$ is a compact set in X. (1 point)

Method 2:

For any compact set $K \subseteq Y$, denote $\mathcal{A}$ be an open cover of $f^{-1}(K)$.For every point $y \in K$, since f is embedding, $f^{-1}(y)$ have at most one point.

That is, we can choose U_y such that $f^{-1}(y) \in U_y$.

Since f is an closed map, $f(X \setminus U_y)$ is closed and $y \notin f(X \setminus U_y)$.

Let $V_y = Y \setminus f(X \setminus U_y)$, then V_y is an open neighborhood of y, $\{V_y\}_{y \in K}$ is an open cover of K.

Since K is compact, we can choose a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$ of K.

Thus $f^{-1}(K)$ is covered by finitely many sets of the form $F^{-1}(V_y)$, each of which is covered by U_y, so it follows that $F^{-1}(K)$ is compact.