MAT8021, Algebraic Topology

Assignment 3

Due in-class on Friday, March 21

Numbered exercises are from Hatcher's "Algebraic Topology."

1. In class, we defined subdivision maps $s_n^i : \Delta[n+1] \to \Delta[n] \times [0,1]$ for $0 \le i \le n$ by
$$s_n^i(t_1, \ldots, t_{n+1}) = \big((t_1, \ldots, \widehat{t_{i+1}}, \ldots, t_{n+1}), t_{i+1}\big)$$
Show that these satisfy the relations
   - $s_n^i d_{n+1}^j = \begin{cases} (d_n^{j-1}, \mathrm{id}) \circ s_{n-1}^i & \text{if } i < j-1 \\ (d_n^j, \mathrm{id}) \circ s_{n-1}^{i-1} & \text{if } i > j \end{cases}$
   - $s_n^0 d_{n+1}^0 = i_0$
   - $s_n^n d_{n+1}^{n+1} = i_1$
   - $s_n^{i-1} d_{n+1}^i = s_n^i d_{n+1}^i$ for $1 \le i < n+1$

   Use this to show that the operator $h\colon C_n(X) \to C_{n+1}(X \times [0,1])$ given by
$$h\left(\sum a_\sigma \sigma\right) = \sum a_\sigma \sum_{i=0}^{n} (-1)^i (\sigma, \mathrm{id}) \circ s_n^i$$
satisfies $\partial\, h(x) + h\, \partial(x) = \tilde{i}_0(x) - \tilde{i}_1(x)$, where $\tilde{i}_k = (\sigma, \mathrm{id}) \circ i_k$.

2. Let C_* be the chain complex with
$$C_n = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$
Let D_* be the chain complex with
$$D_n = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$
such that the boundary map $\partial\colon D_1 \to D_0$ sends m to $2m$.

   Show that the natural projection $\pi\colon D_* \to C_*$ is a map of chain complexes and it induces the zero map $H_*(D_*) \to H_*(C_*)$.

3. With notations as in Question 1, show that there is no chain homotopy h such that $\partial h + h\partial = \pi$ (from π to zero).

4. For $Z \subset Y \subset X$ spaces, show that there is a short exact sequence of singular chain complexes

$$0 \to C_*(Y, Z) \to C_*(X, Z) \to C_*(X, Y) \to 0$$

What does the resulting long exact sequence of homology groups look like?

5. Hatcher, Exercise 12 on page 132.