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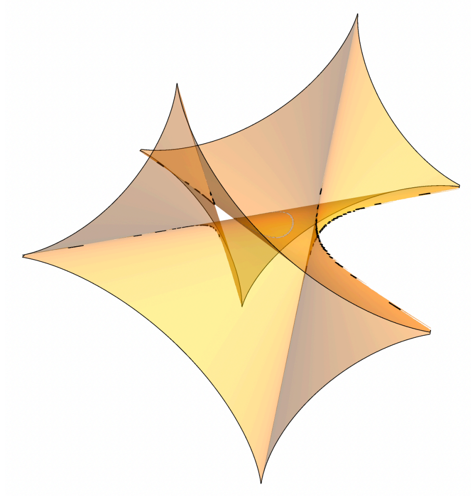

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2025.1.11

Physics + mathematical modeling

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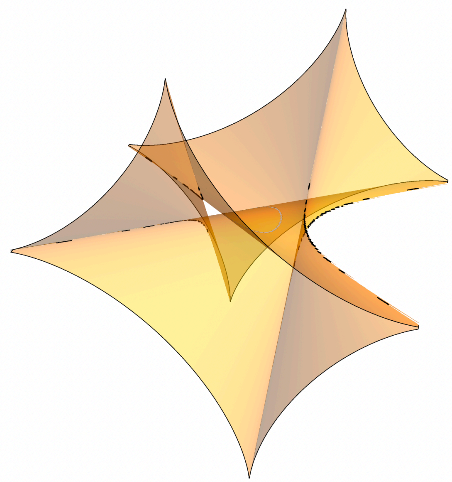

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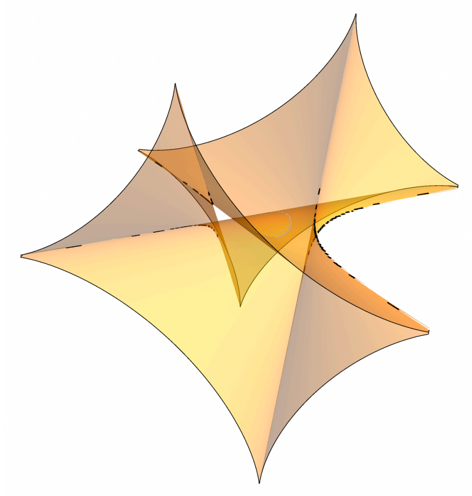

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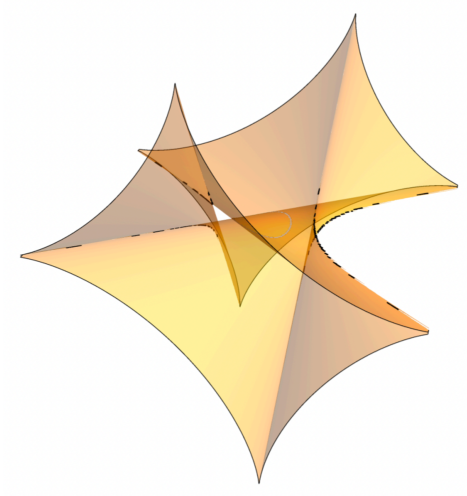

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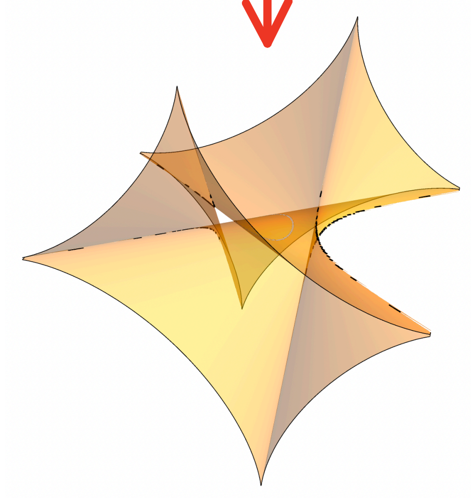

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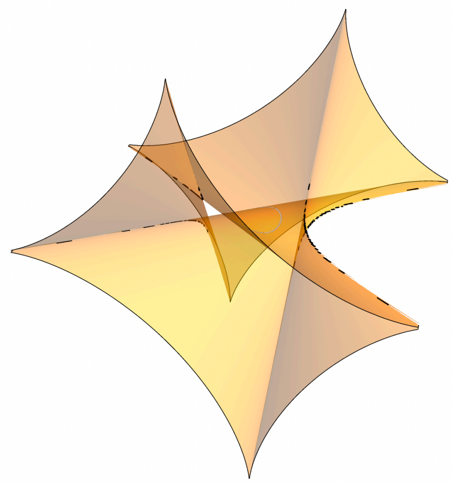

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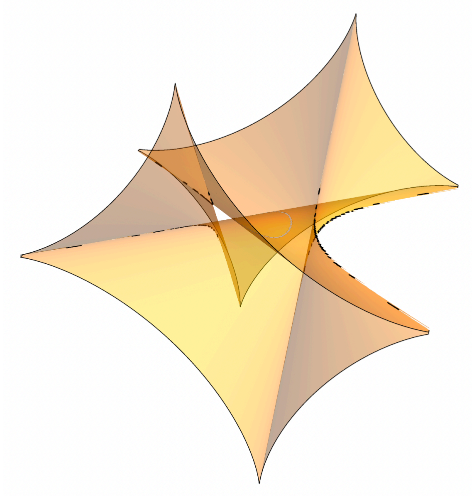

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Holography, optical devices, absorption devices, …

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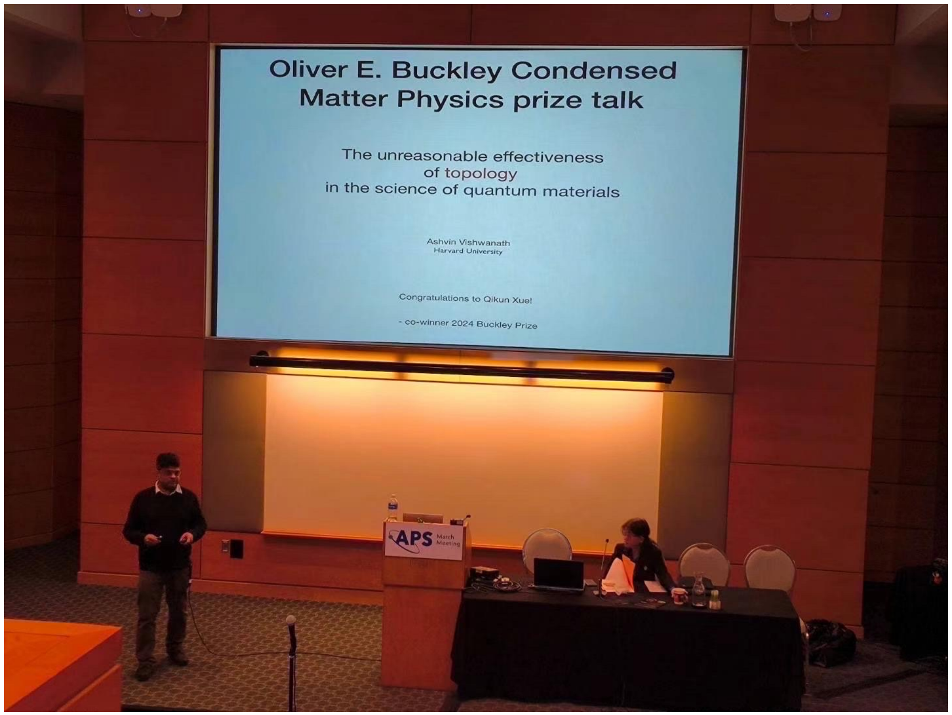

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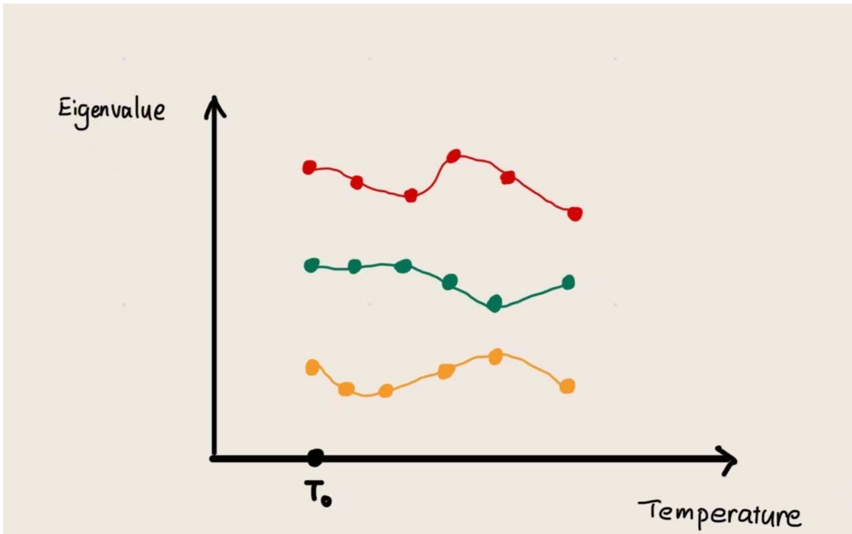
Mathematical modeling of electronic energy *band structures* therein



energy $\longleftrightarrow$ *eigenvalue*

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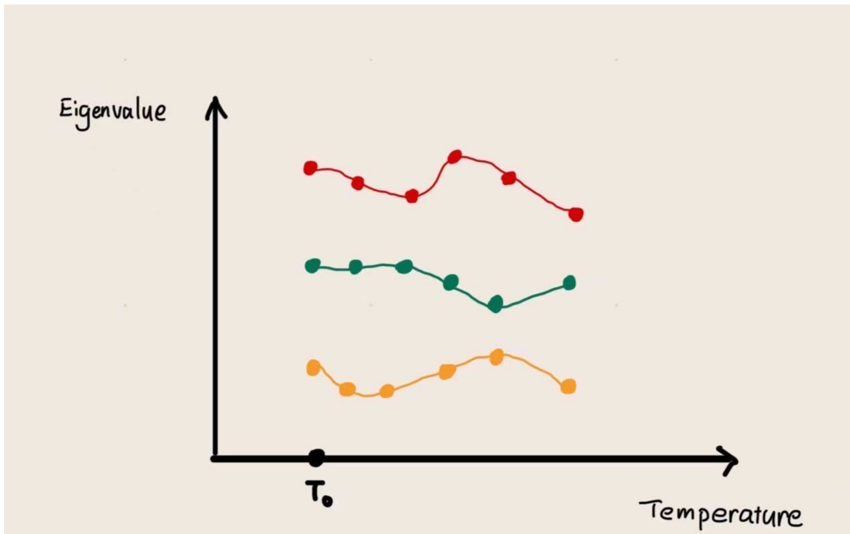
Mathematical modeling of electronic energy *band structures* therein concerns topological/homotopical classification of *Hamiltonians*



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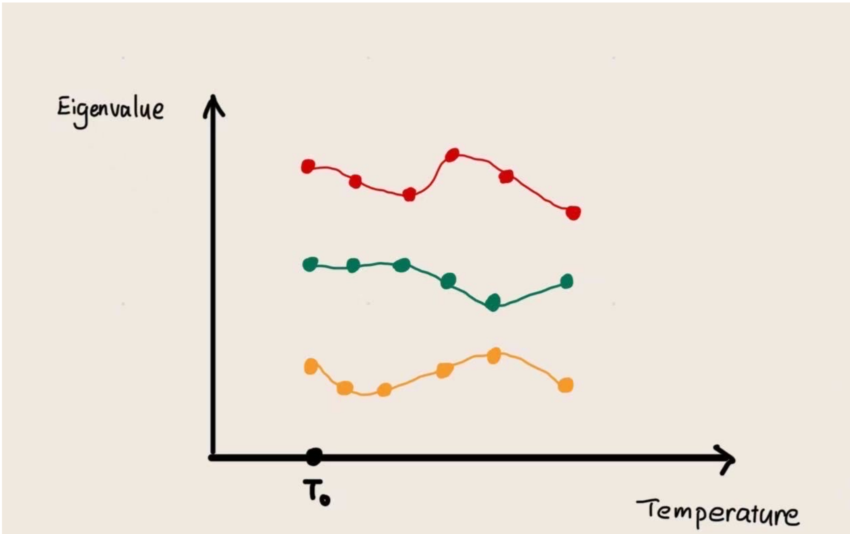

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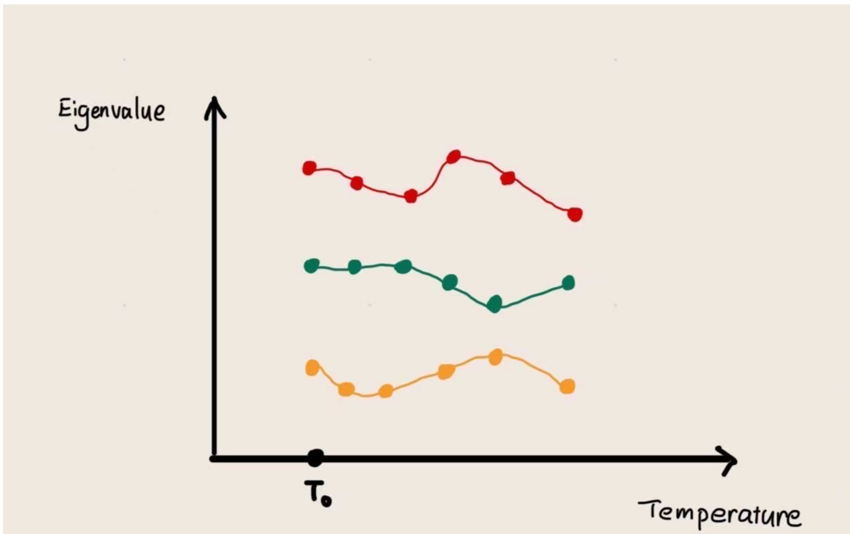

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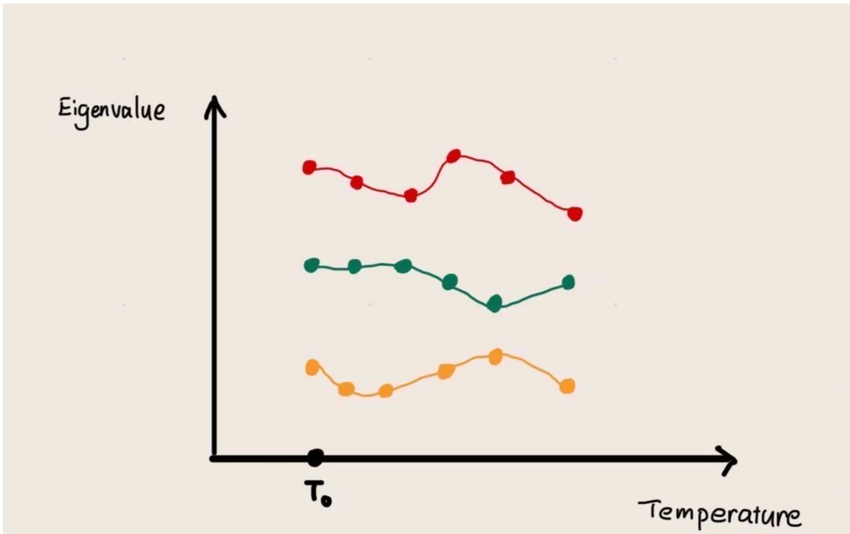

Hermitian

real eigenvalues (observable energies)

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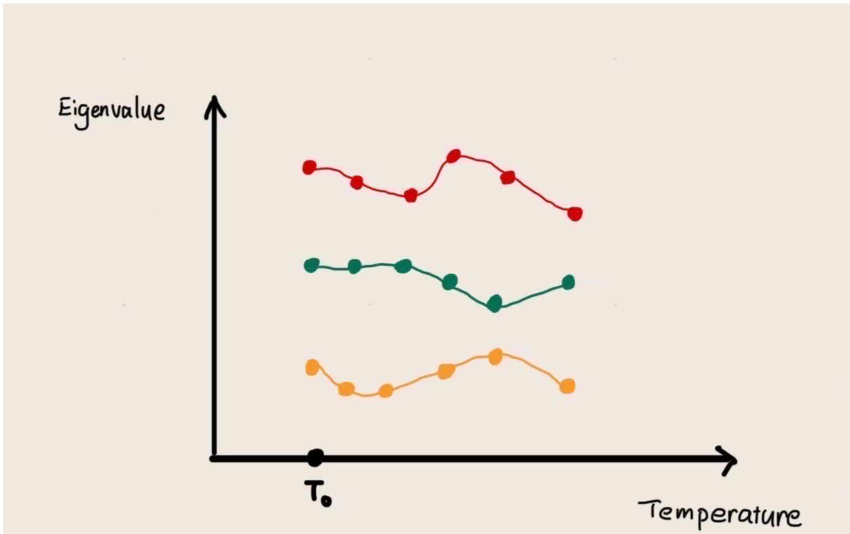

Hermitian vs.
non-Hermitian

real eigenvalues (observable energies) vs.
eigenvalues with imaginary part (counts for energy exchange with surrounding environment or other systems)

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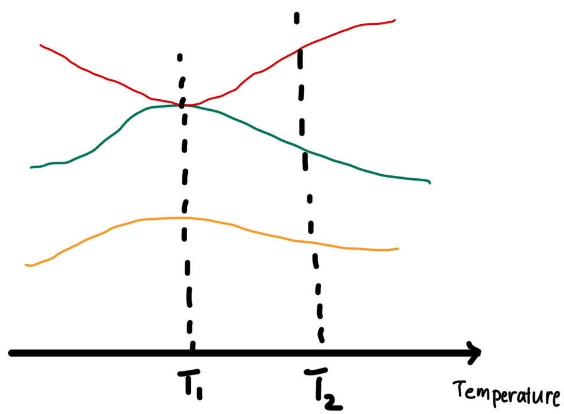

phases of matter
H_2O*: solid, liquid, gas*

T_1 : singular points (points where eigenvalues degenerate)
$H(T_1)$: gapless Hamiltonian
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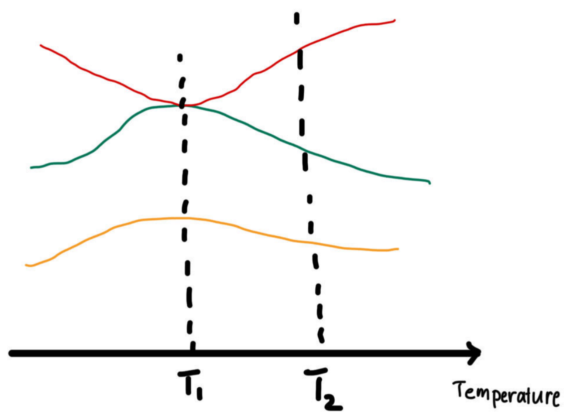

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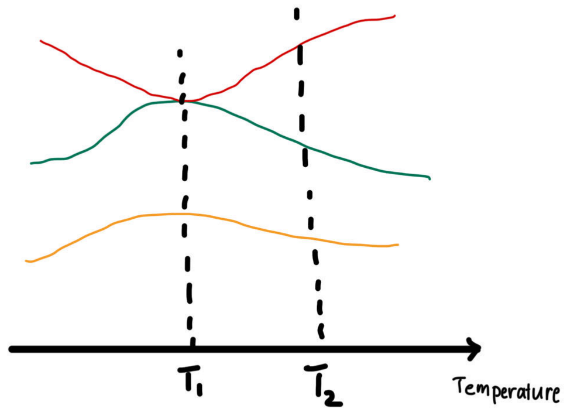

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It is indeed "pointless" and is better understood as a functor!

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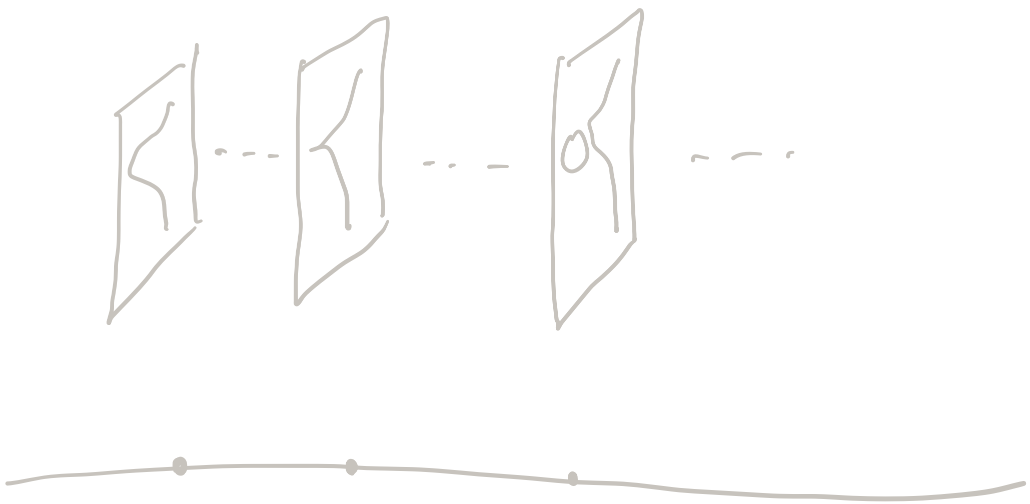

family of objects

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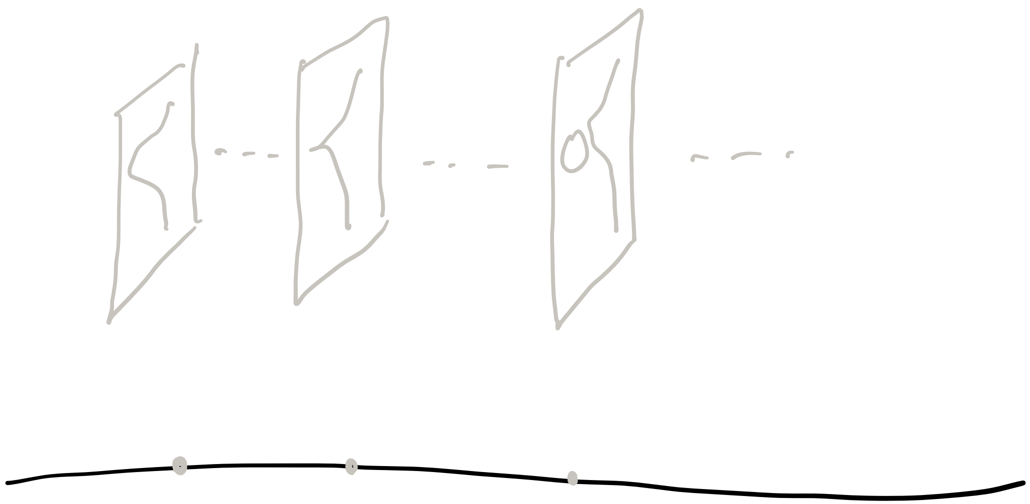

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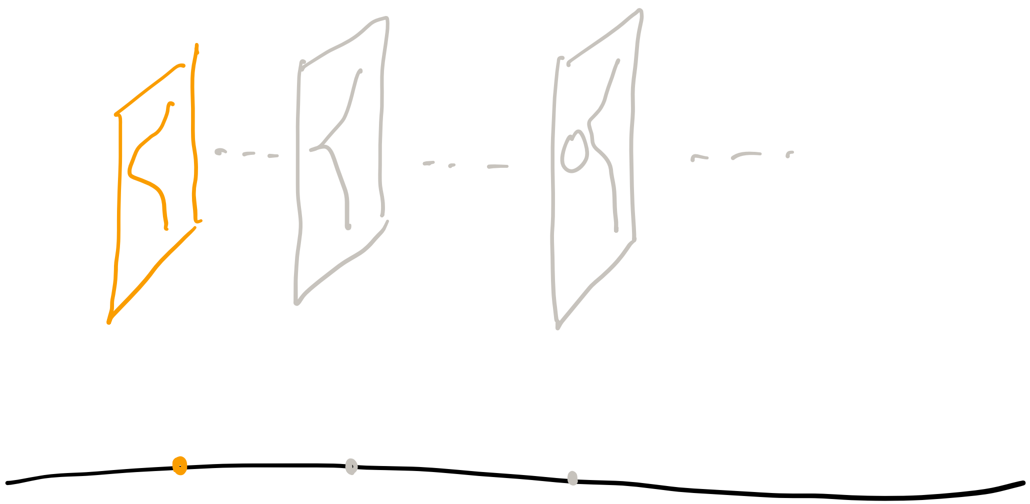

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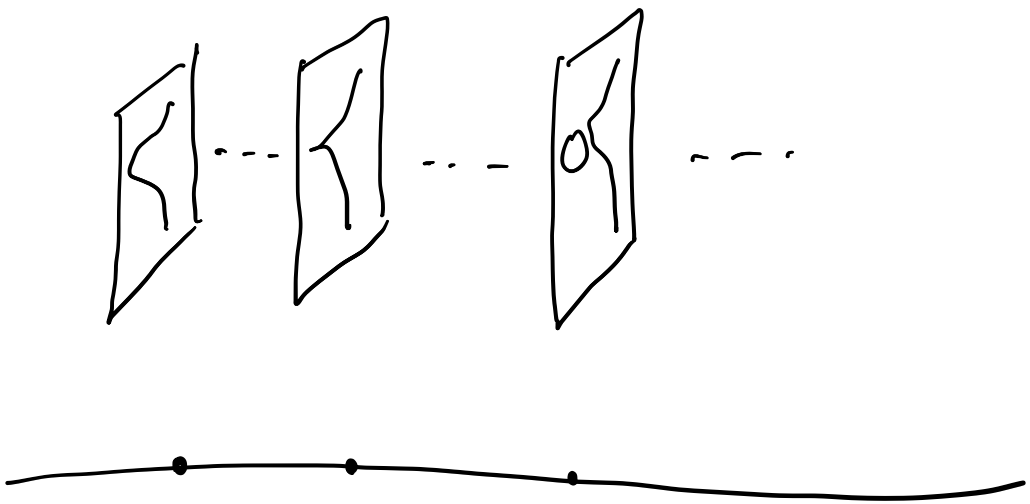

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- In this sense, studying moduli spaces is of the second-order nature.

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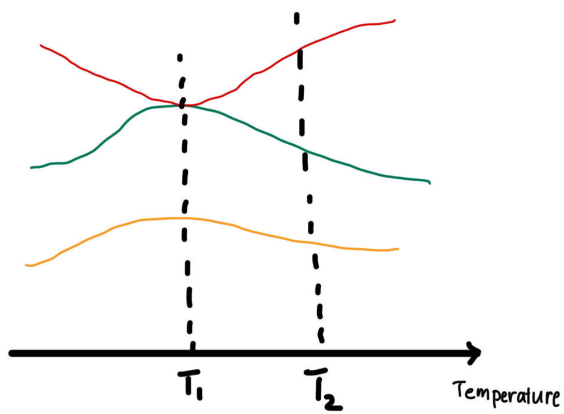

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Guzman et al., Model-free characterization of topological edge and corner states in mechanical networks, ***PNAS*** *2024.*

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- J. Hu, R.-Y. Zhang, Y. Wang, X. Ouyang, **Y. Zhu**, H. Jia, and C. T. Chan. *Non-Hermitian swallowtail catastrophe revealing transitions among diverse topological singularities*. ***Nature Physics***, 19:1098–1103, 2023.

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- H. Jia, R.-Y. Zhang, J. Hu, Y. Xiao, S. Zhang, **Y. Zhu**, and C. T. Chan. *Topological classification for intersection singularities of exceptional surfaces in pseudo-Hermitian systems*. ***Communication Physics***, 6:293, 2023.

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Preprints

- H. Jia, J. Hu, R.-Y. Zhang, Y. Xiao, D. Wang, M. Wang, S. Ma, X. Ouyang, **Y. Zhu**, and C. T. Chan. *Anomalous bulk–edge correspondence intrinsically beyond line-gap topology in non-Hermitian swallowtail gapless phase*.
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- J. Hu, R.-Y. Zhang, M. Wang, D. Wang, S. Ma, X. Ouyang, **Y. Zhu**, H. Jia, and C. T. Chan. *Unconventional bulk-Fermi-arc linking paired exceptional points of order three and their splitting from a defective triple point*.

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Thanks to Hopf bundles and Higgs bundles as *eigenbundles*, we now have a conceptually more systematic, visibly more intuitive understanding of the topic. The structure of Higgs bundles also hints at certain deeper aspects of mathematics as well as physics.

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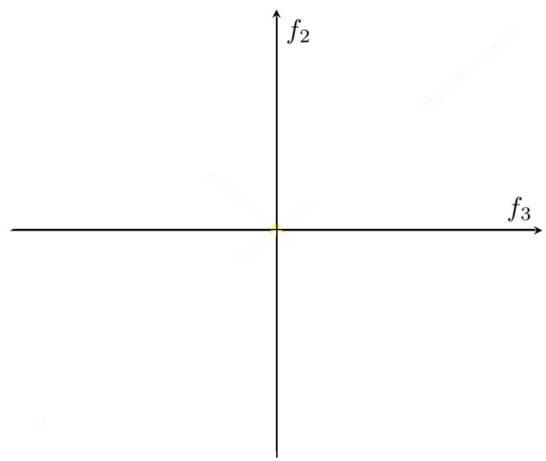

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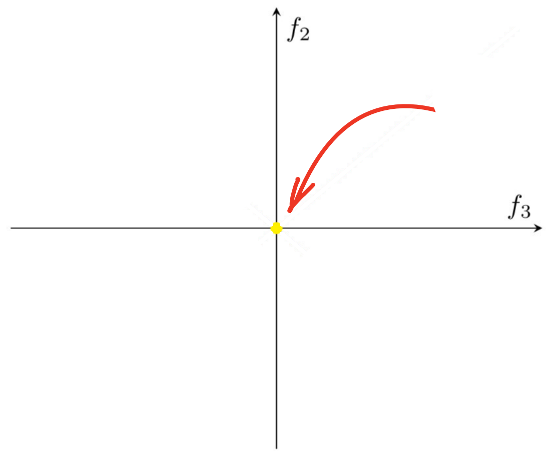

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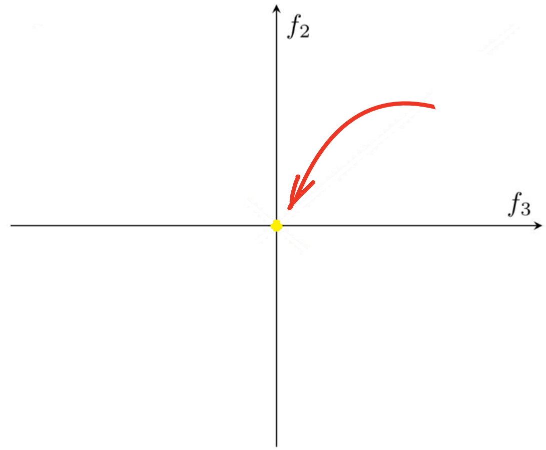

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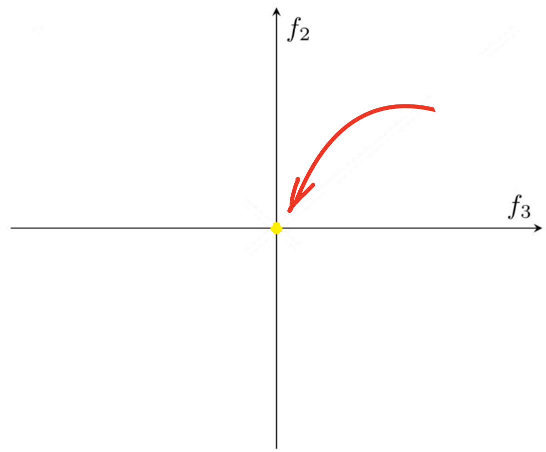

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0. Over {(0, 0)}, *H* has a double eigenvalue, whose eigenspace is 2-dimensional.

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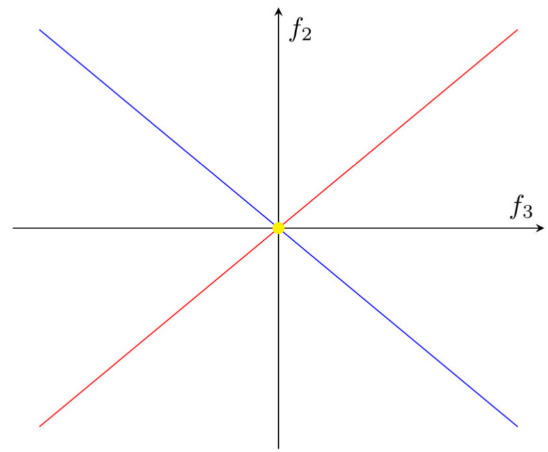

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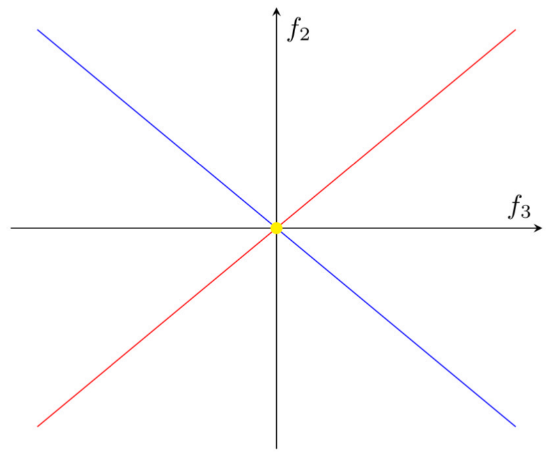

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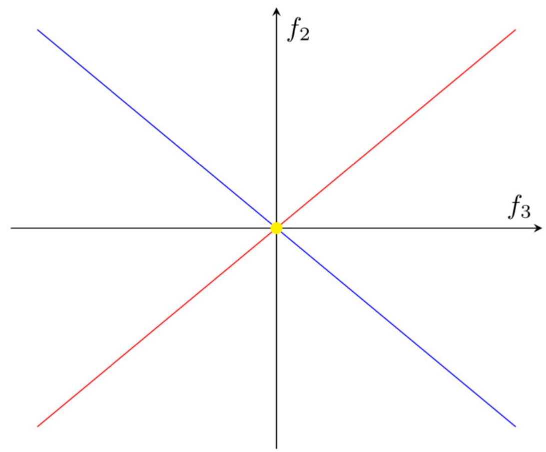

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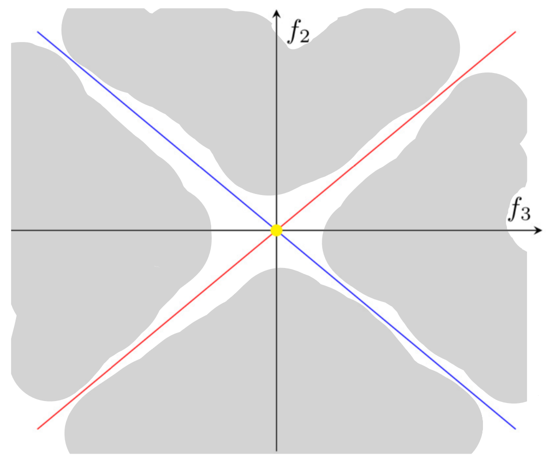

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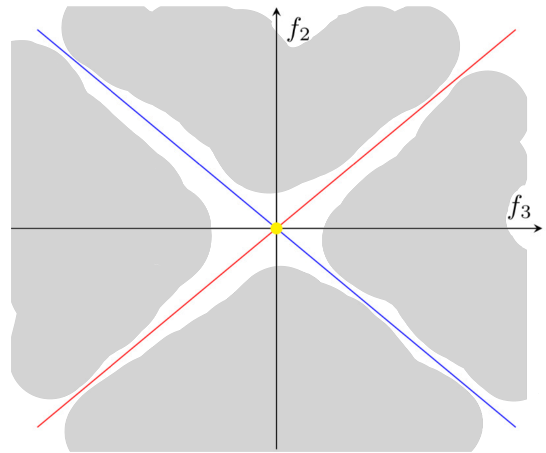

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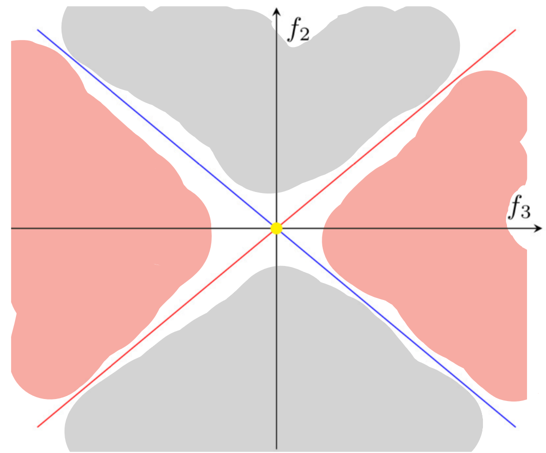

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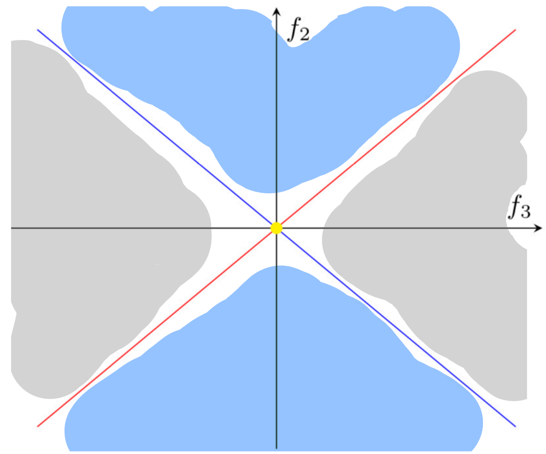

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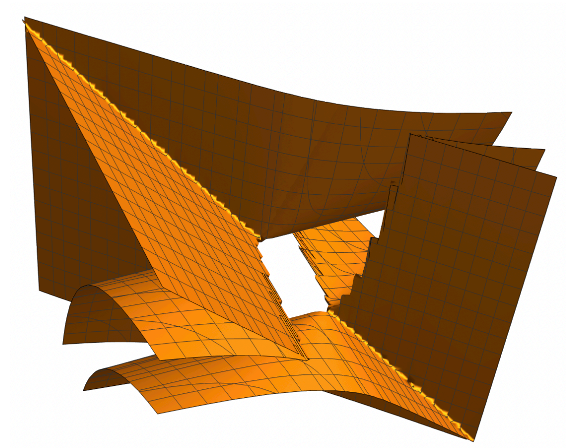
Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the $f_1 f_2 f_3$-space:

The equation for this surface is a non-homogeneous real polynomial in f_1, f_2, f_3 of degree 6.



Swallowtail couple $\mathrm{sw}2$

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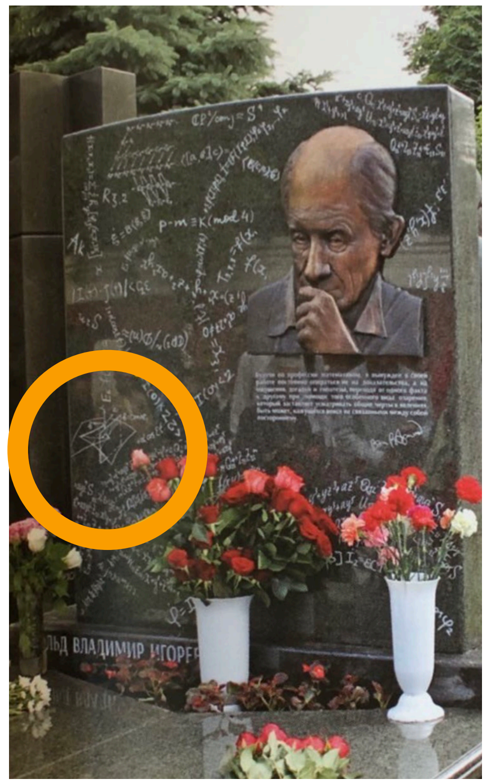

V. I. Arnold's tombstone at the Novodevichy Cemetery in Moscow

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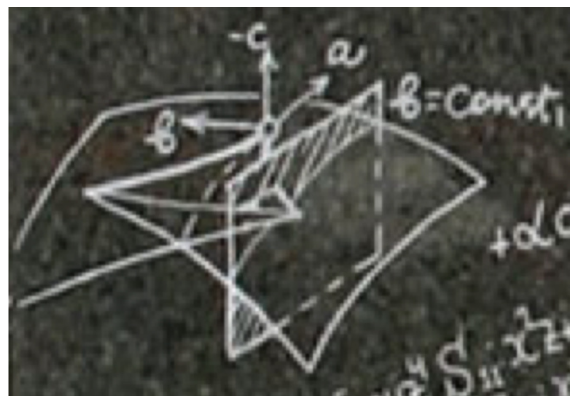

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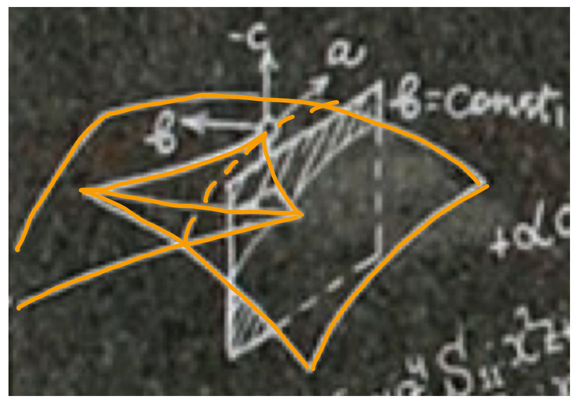

A local model for moduli spaces of 3-band Hamiltonians

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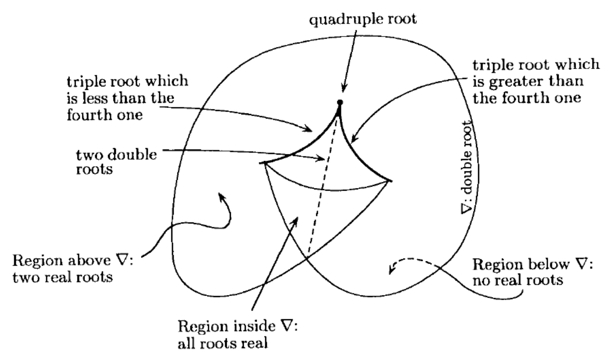

The space of polynomials $x^4 + ax^2 + bx + c$

Arnold, Braids of algebraic functions and the cohomology of swallowtails, 1968.

Homological stability of braid groups

Portrait from Gelfand, Kapranov, Zelevinsky,
Discriminants, resultants, and multidimensional determinants.

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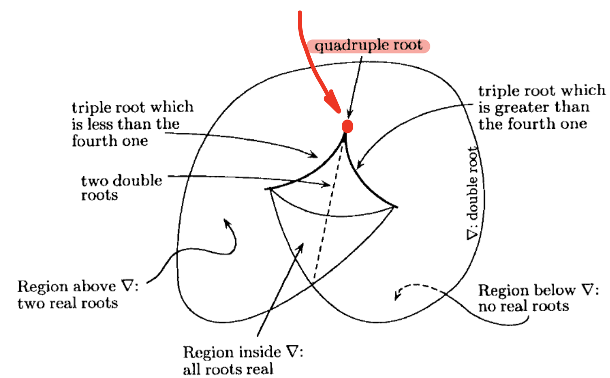

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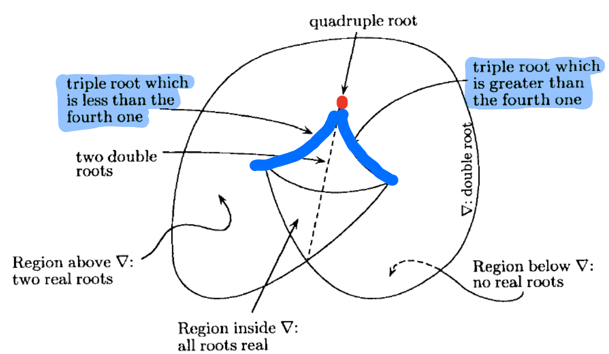

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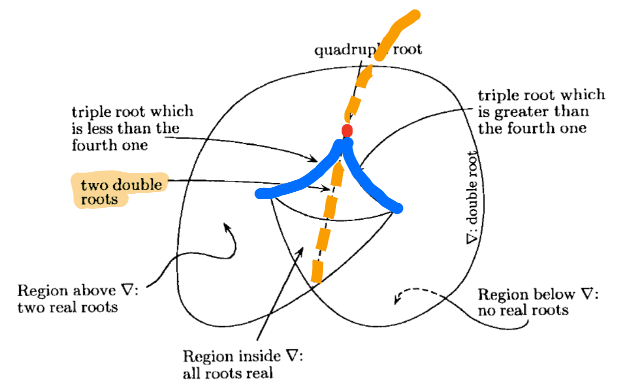

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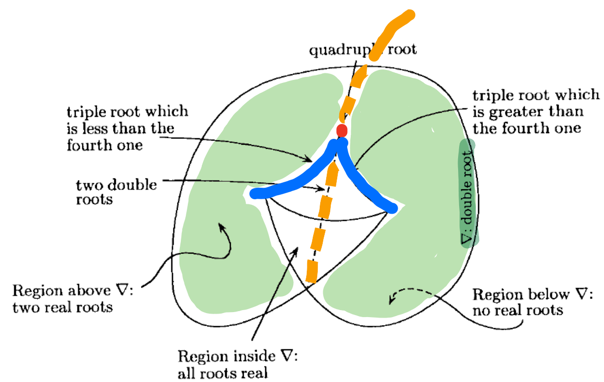

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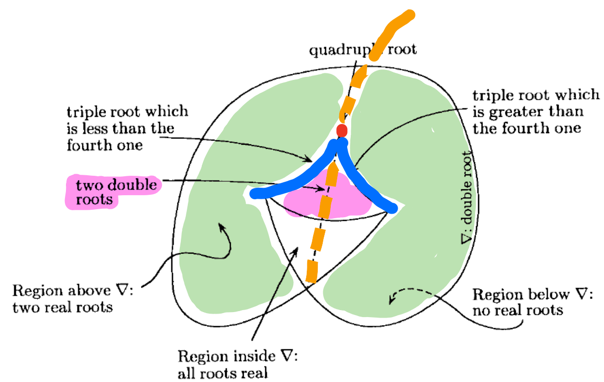

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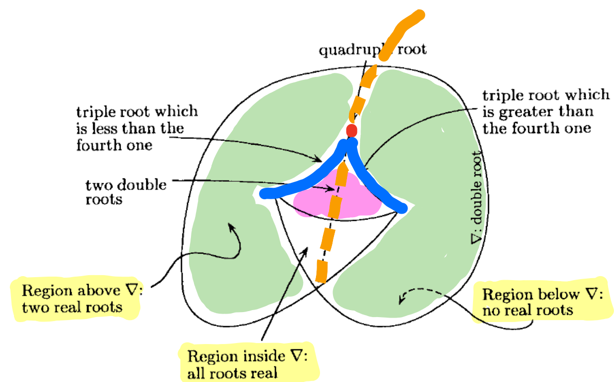

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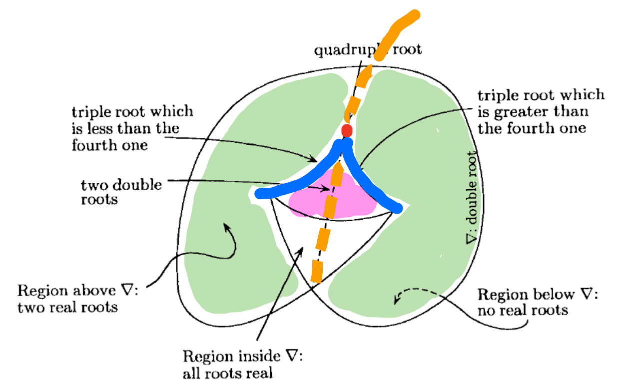

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Again, we aim to find computable algebraic invariants that systematically classify the evolutions of eigenvectors along loops in such stratified parameter spaces, including when they cross the discriminant surface resulting in degeneracies of various sorts.

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Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connections and quantum Hall effect, 1995.

Also: Polymathematics, 2000.

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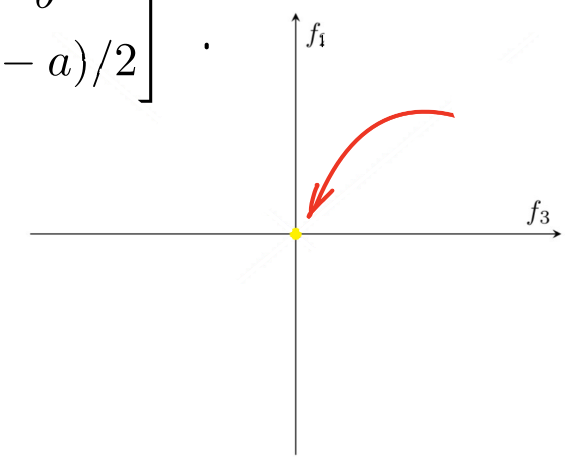

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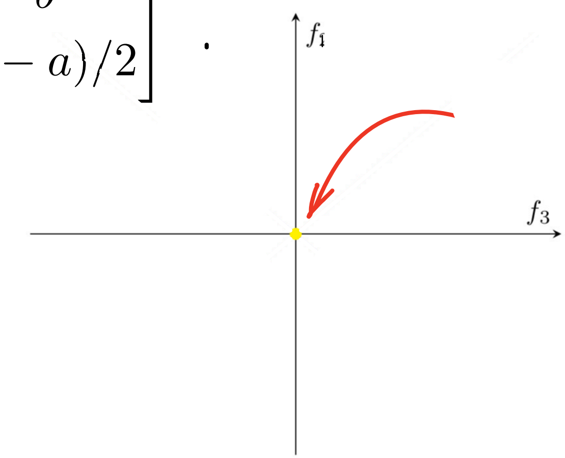

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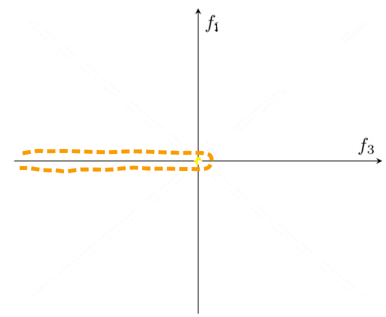

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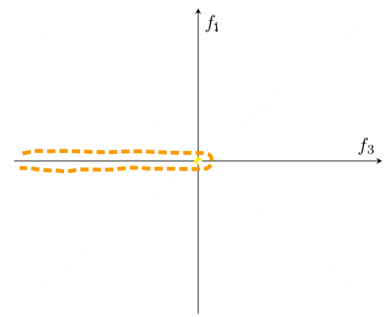

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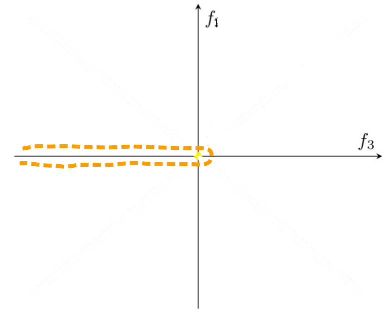

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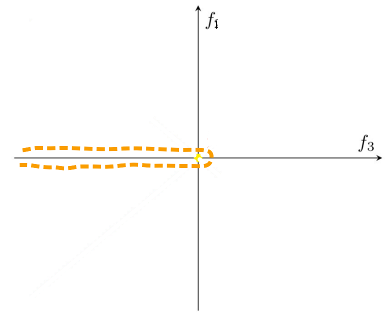

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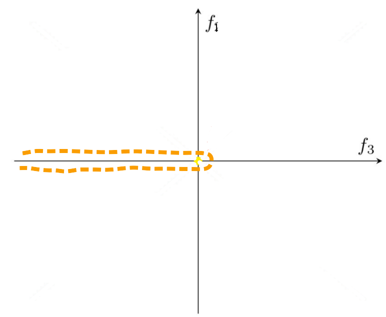

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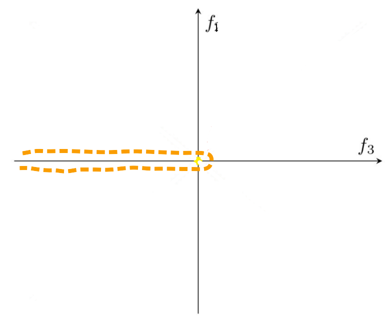

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2+f_3^2} \neq 0 \iff \cancel{f_1 = 0,\ f_3 \leq 0}}$$

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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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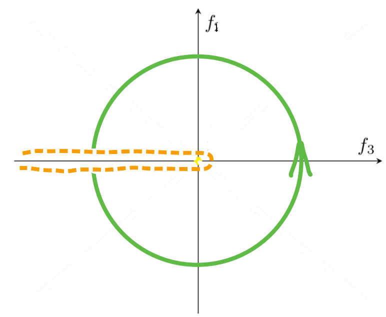

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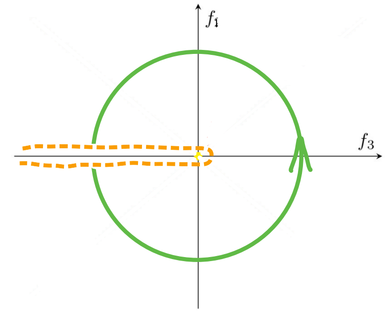

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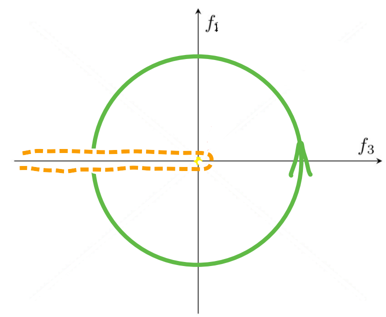

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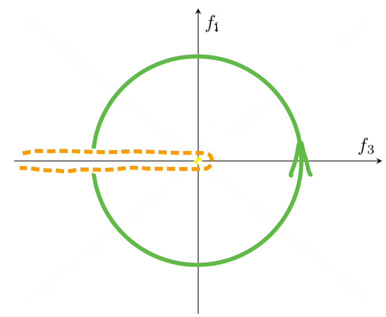

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We compute that

$$\lim_{\theta\to(-\pi)_+}\frac{v_+}{|v_+|}=\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

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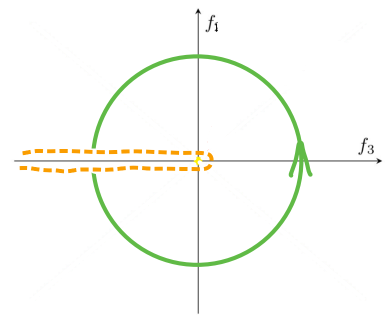

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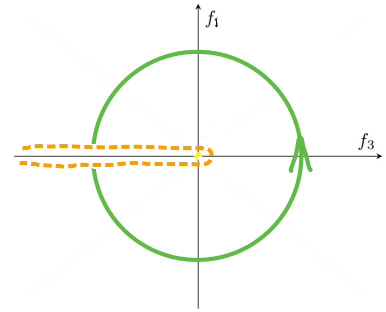

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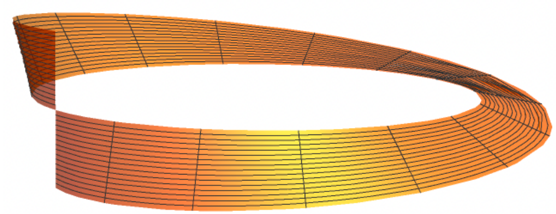

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

Lemma. The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{array}{lcc} S^0 \hookrightarrow S^1 \to S^1 & & \mathbb{R} \\ S^1 \hookrightarrow S^3 \to S^2 & \text{if the Hamiltonian is over} & \mathbb{C} \\ S^3 \hookrightarrow S^7 \to S^4 & & \mathbb{H} \end{array}$$

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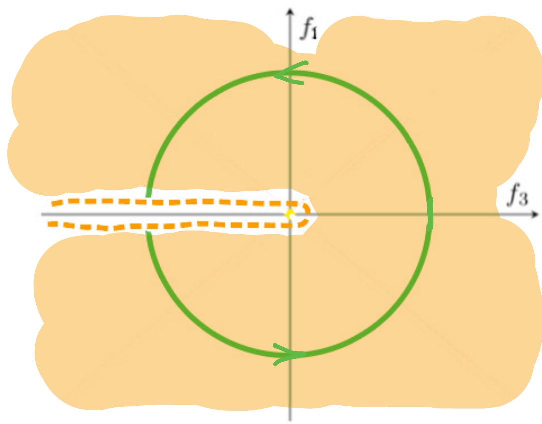

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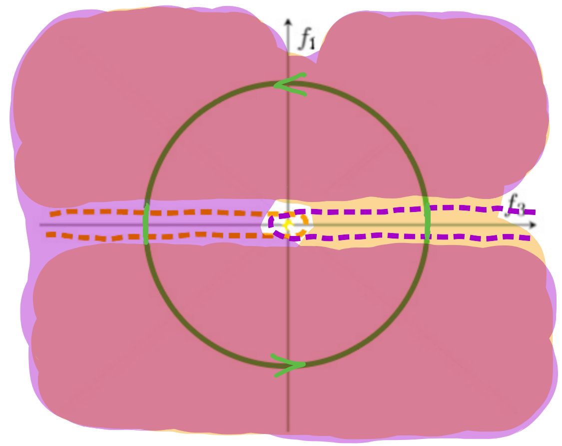

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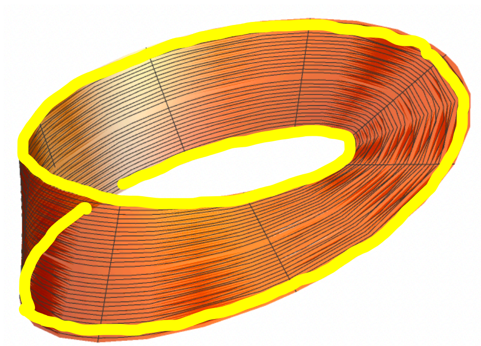

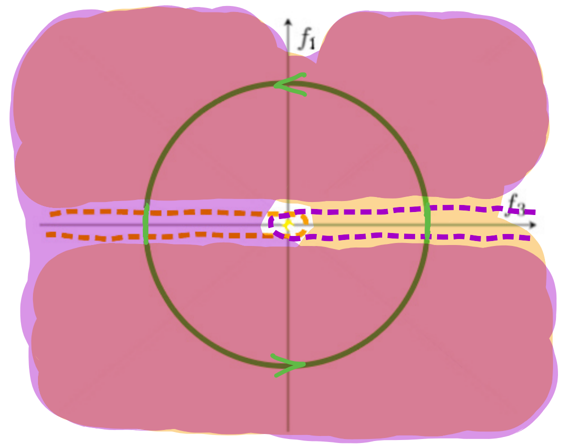

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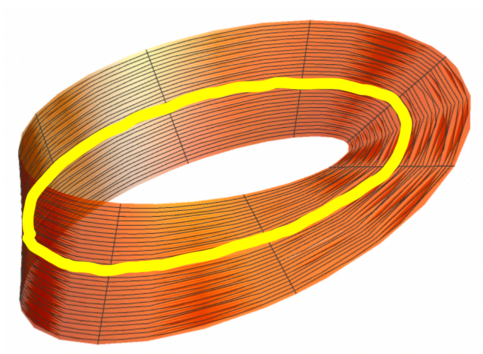

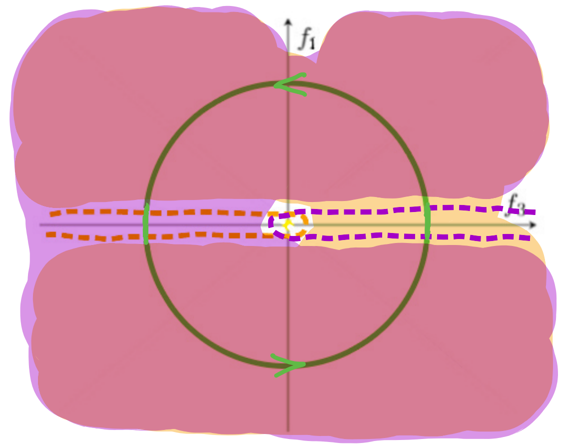

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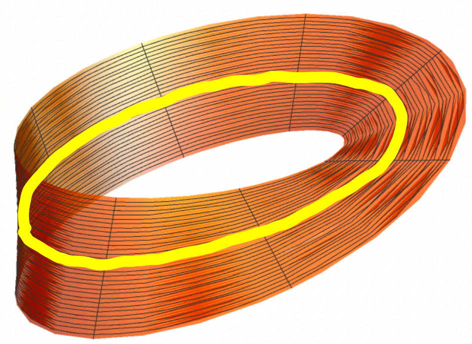

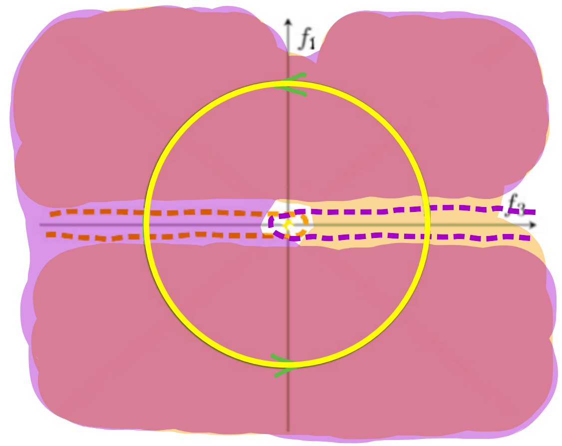

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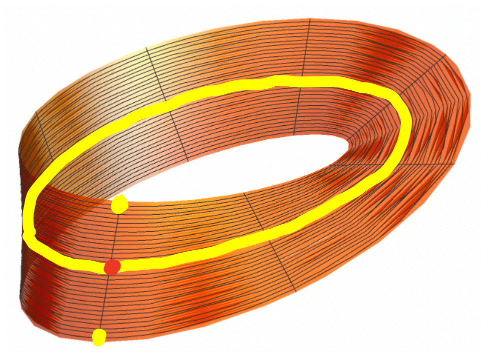

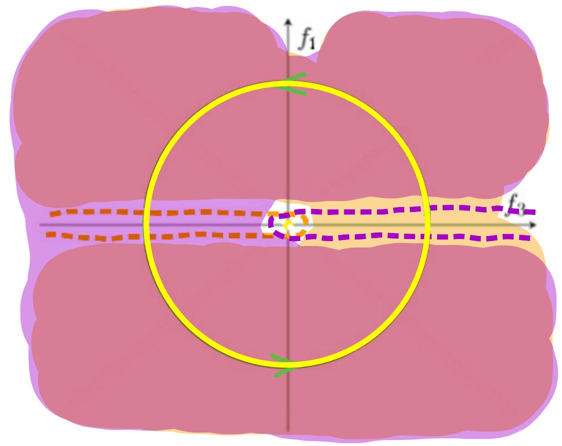

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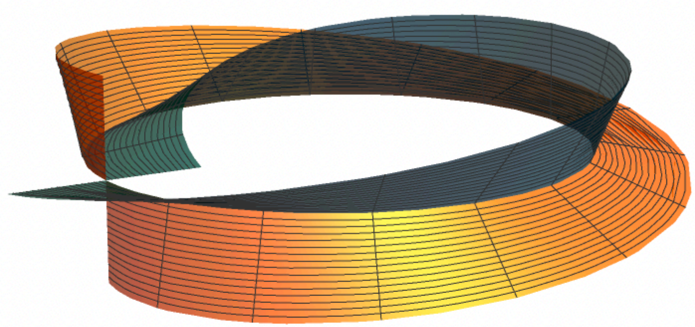

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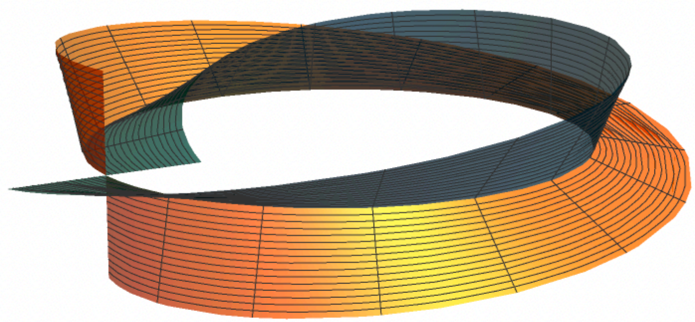

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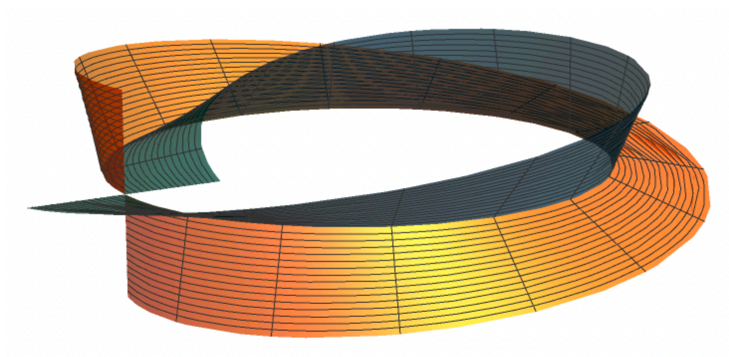

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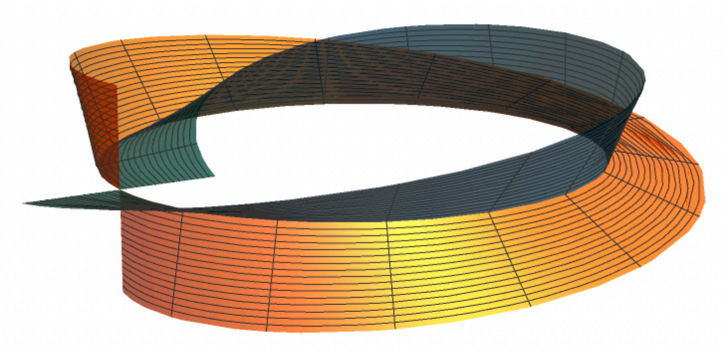

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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Recall that non-Hermitian 2-band systems

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

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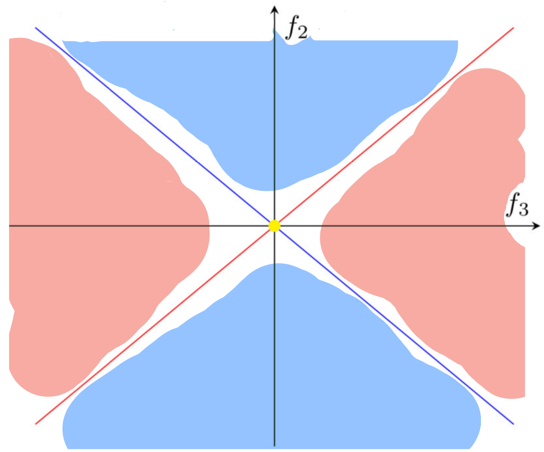

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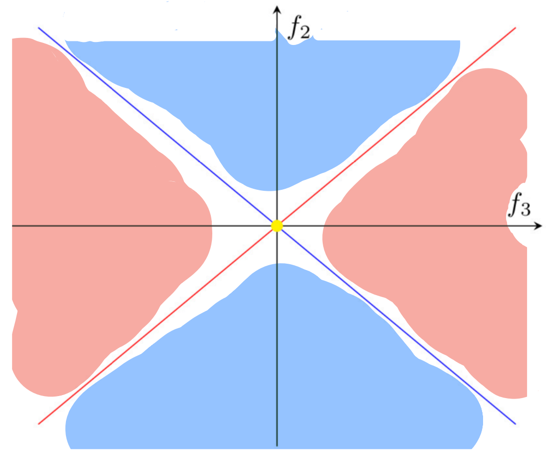

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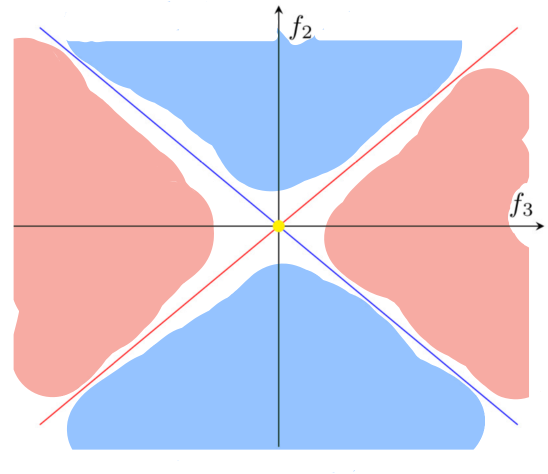

0. Over $\{(0, 0)\}$, H has a double eigenvalue, whose eigenspace is 2-dimensional.
1. Over $\{f_2 = \pm f_3\} - \{(0, 0)\}$, again H has a double eigenvalue, but its eigenspace is of dimension 1.

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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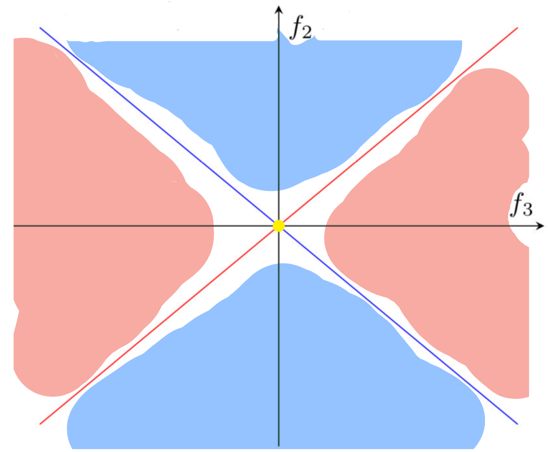

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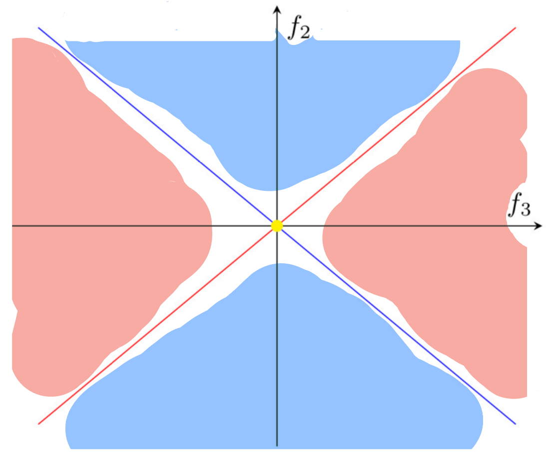

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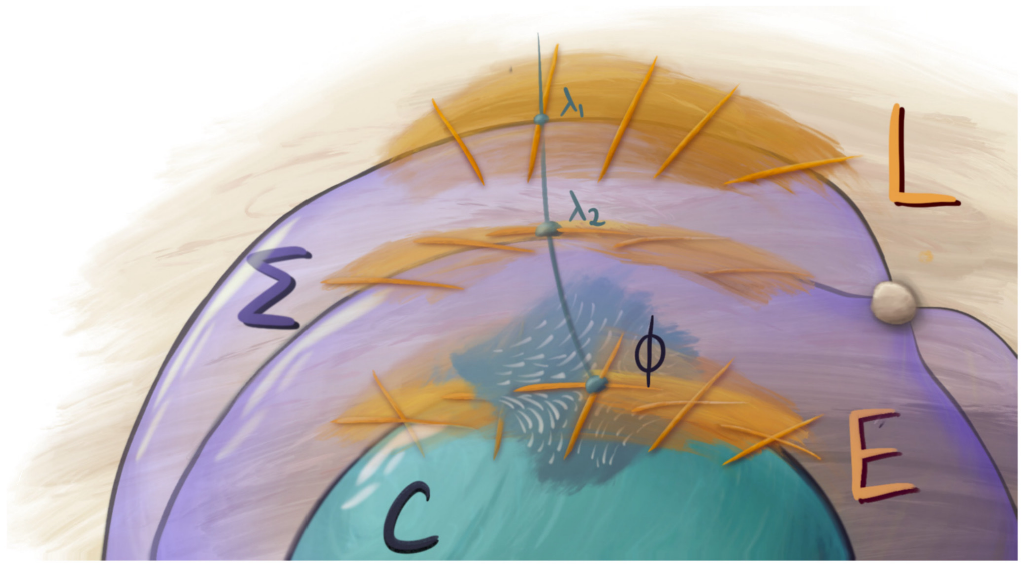
A *Higgs bundle* $(E, \phi) \to C$ is essentially a family of matrices



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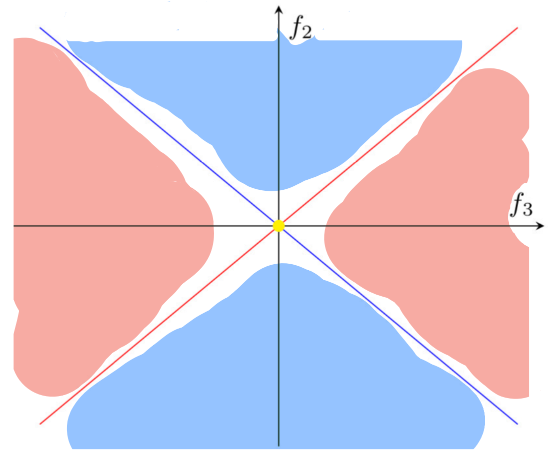

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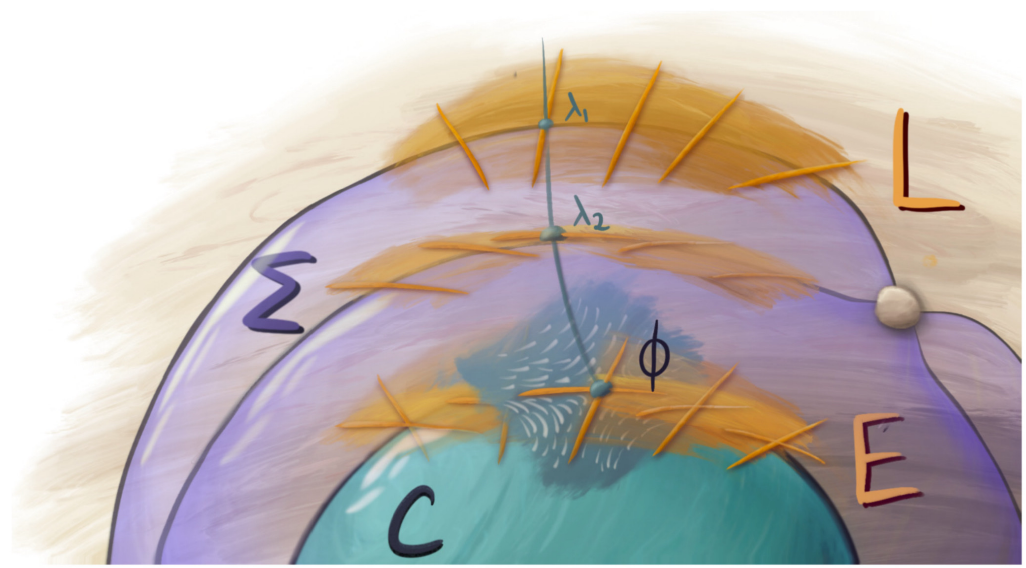
Peter Higgs (bosons)



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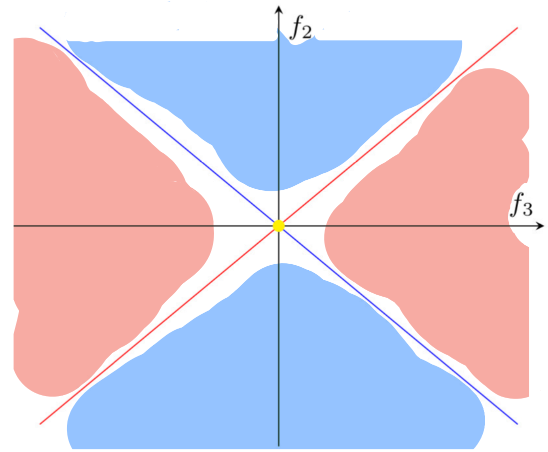

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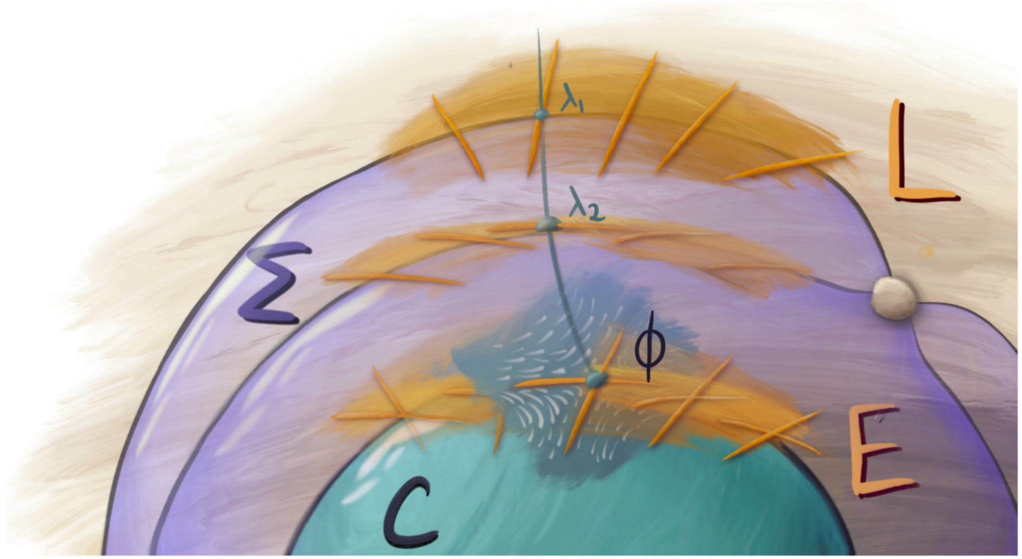

Peter Higgs (bosons)

1929–2024

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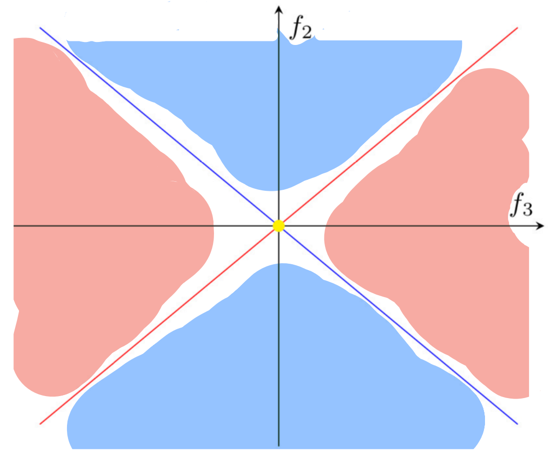

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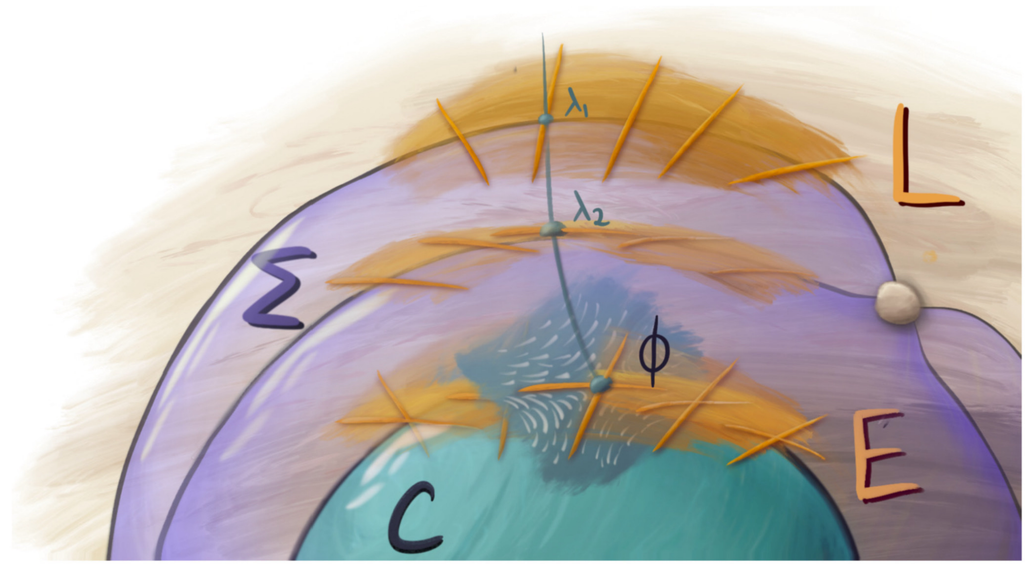
Nigel Hitchin 1987



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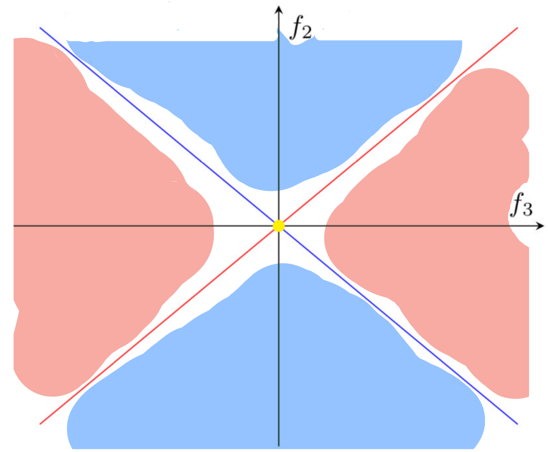

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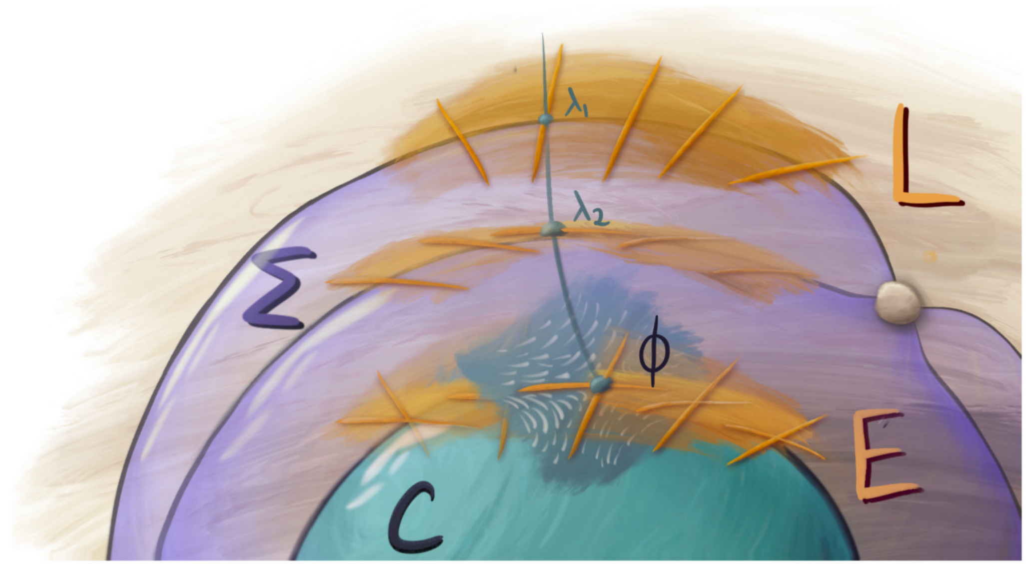
Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface



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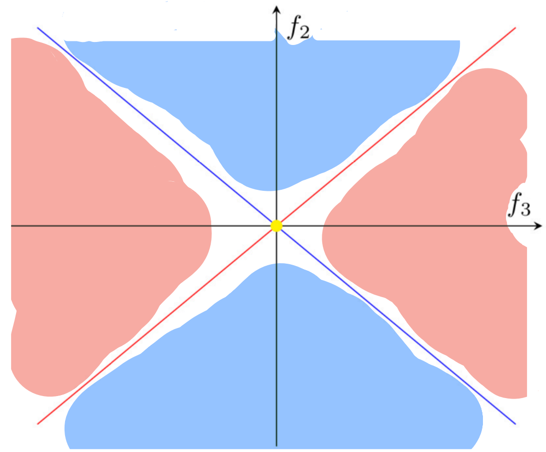

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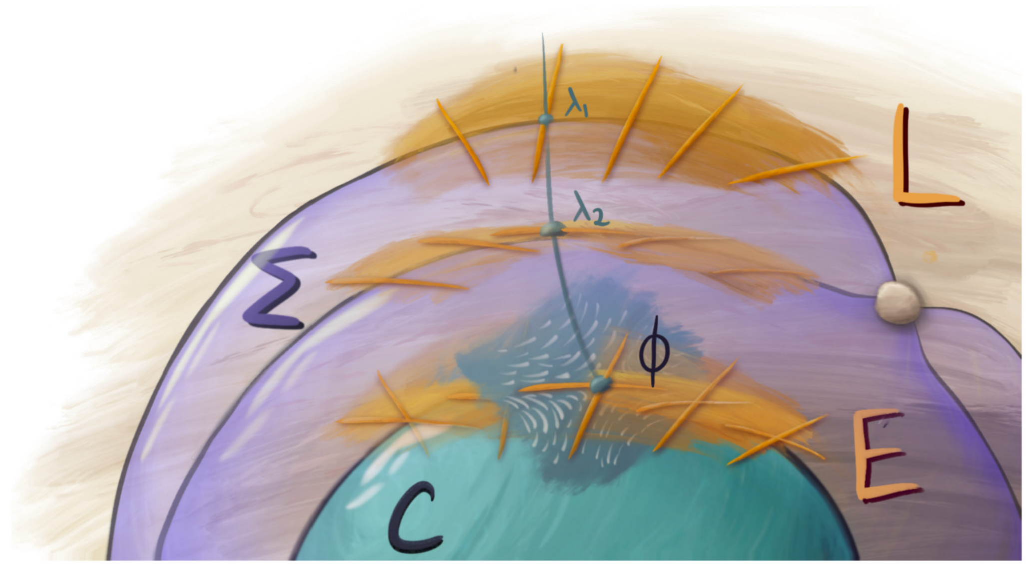
Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface
E holomorphic vector bundle



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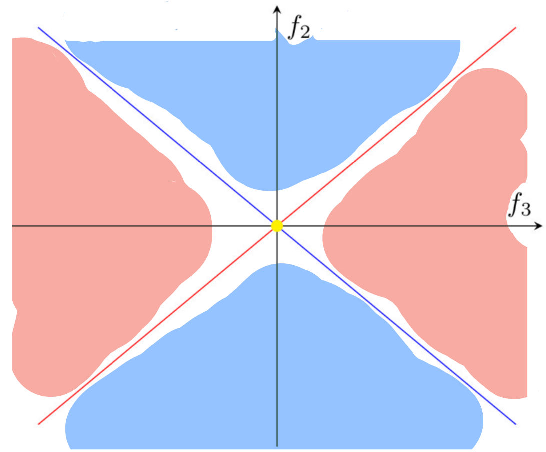

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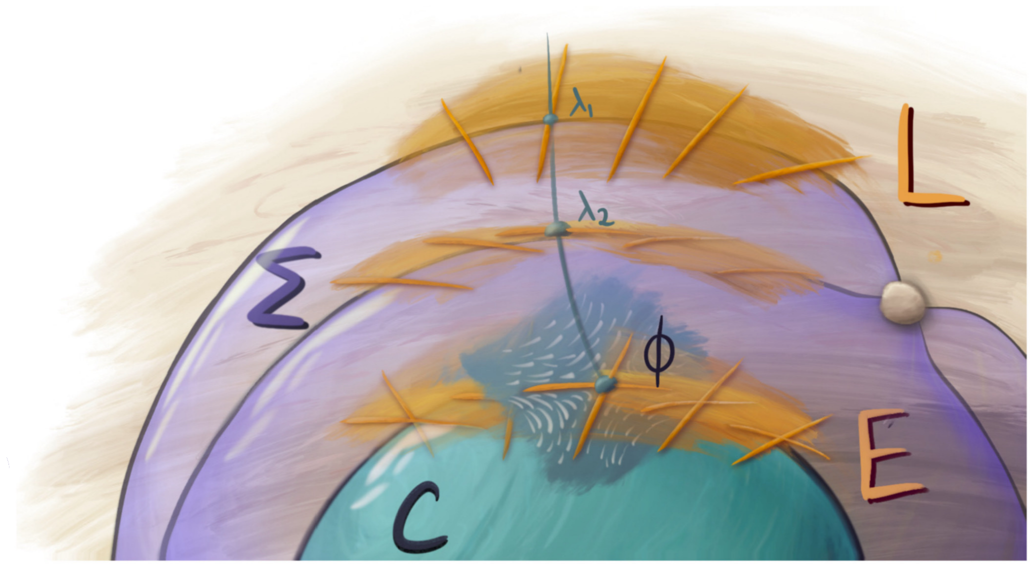
Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface
E holomorphic vector bundle
ϕ Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of E



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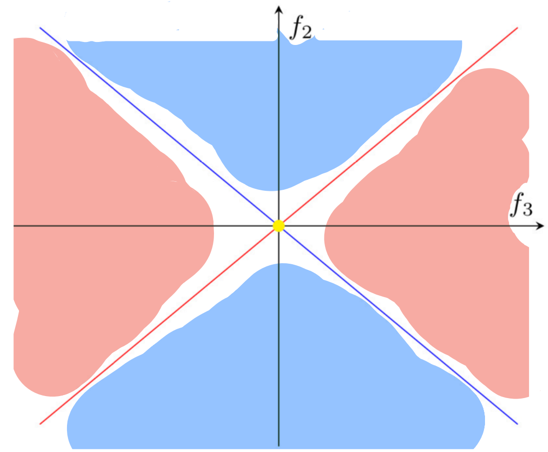

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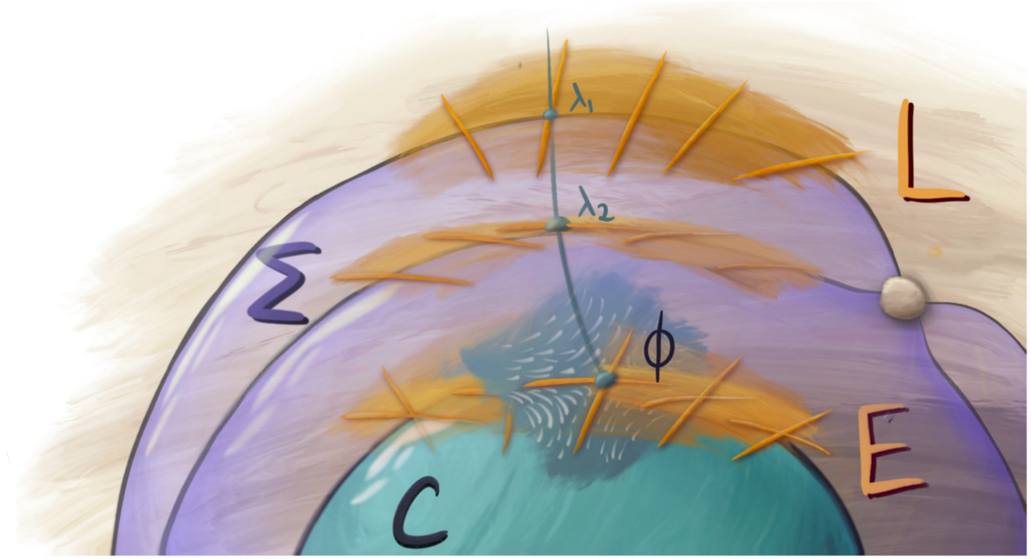
Nigel Hitchin 1987



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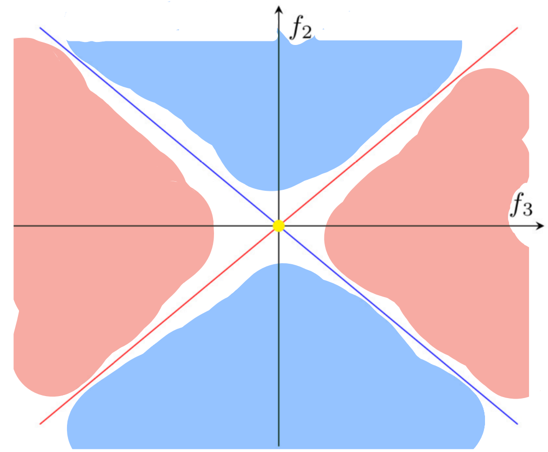

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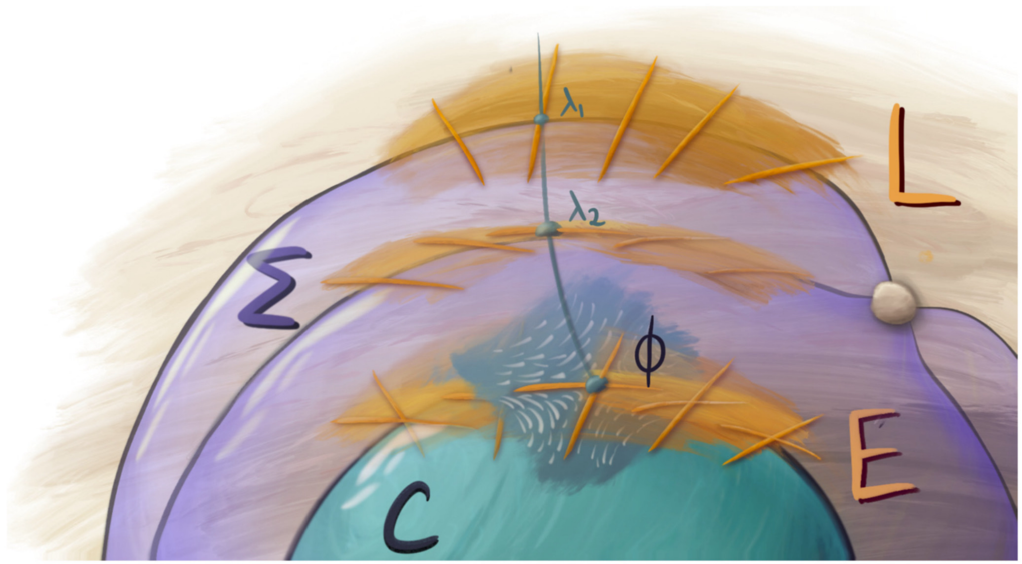
A *Higgs bundle* $(E, \phi) \to C$ is essentially a family of matrices

Peter Higgs (bosons)

Nigel Hitchin 1987
Carlos Simpson

C compact Riemann surface (or more generally Kähler manifold)
E holomorphic vector bundle
ϕ Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of E such that $\phi \wedge \phi = 0$



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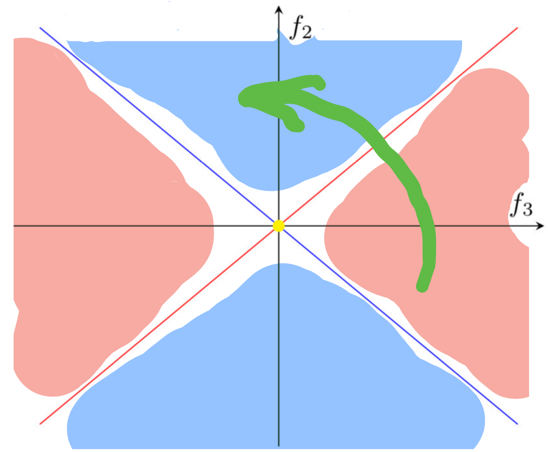

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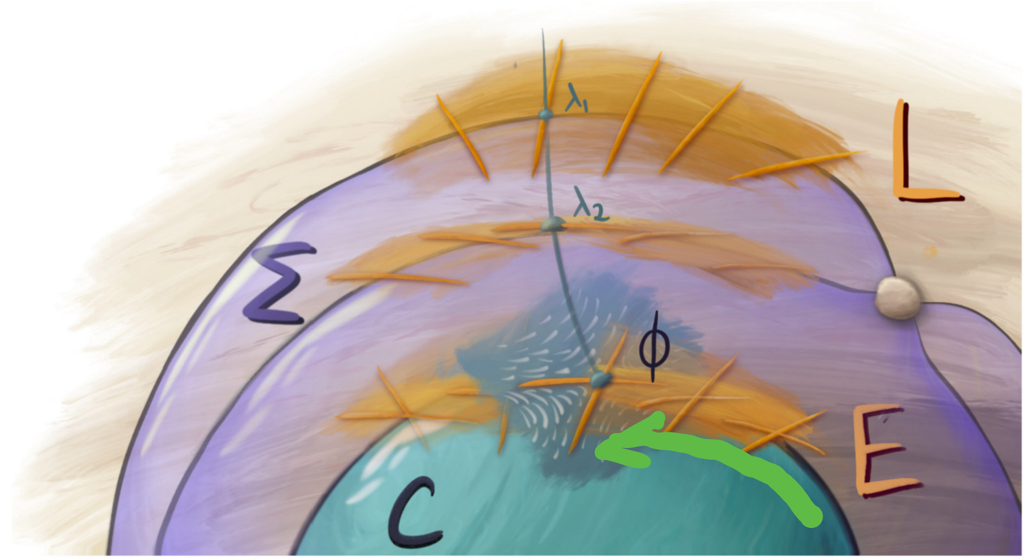
$$\phi_x \in \mathrm{End}(E_x), x \in C$$



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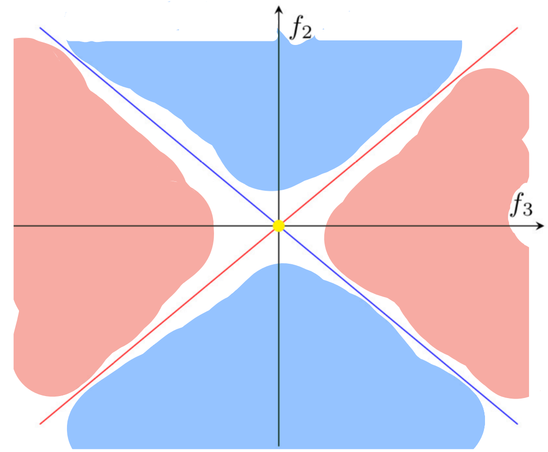

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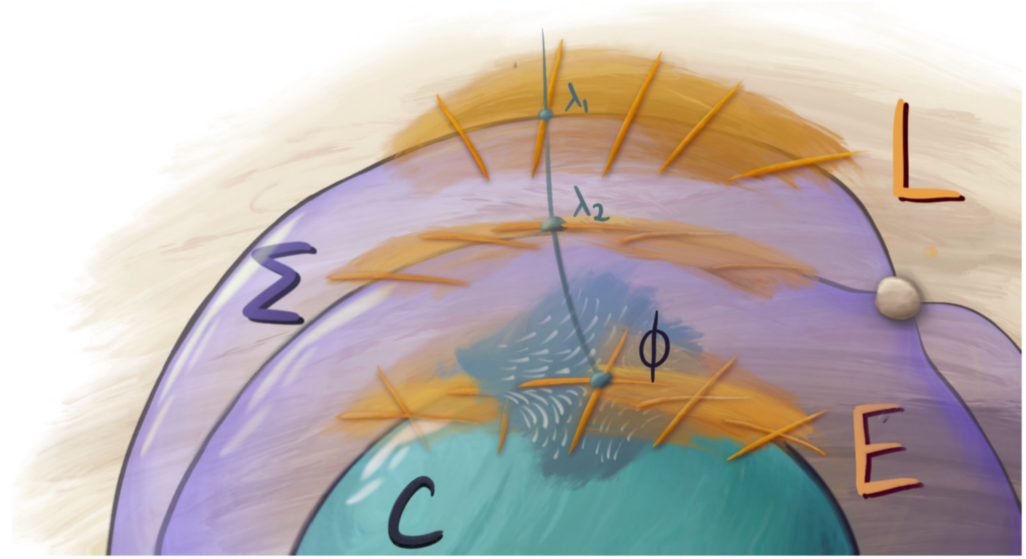
Portrait from Kienzle and Rayan, *Hyperbolic band theory through Higgs bundles*, ***Adv. Math.***, 2022.



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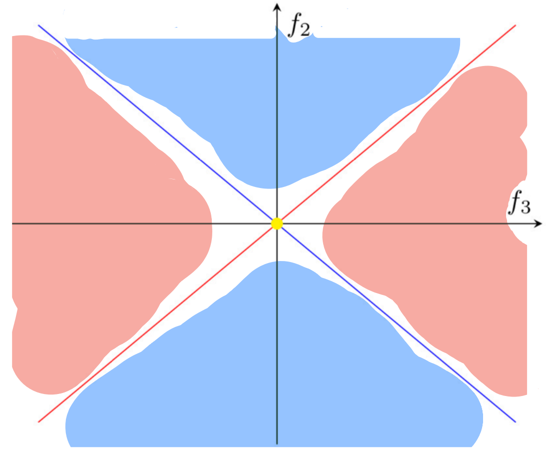

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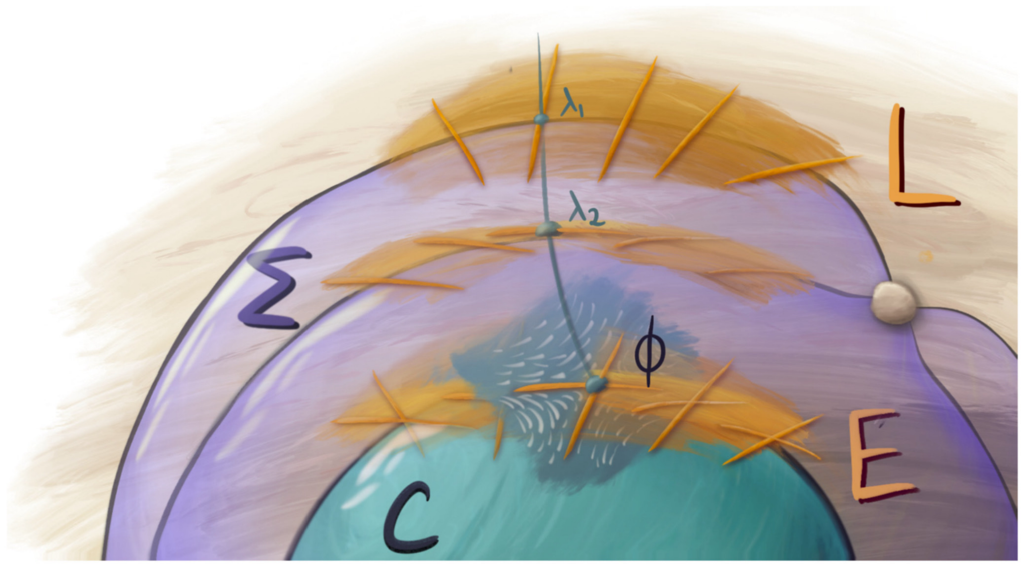

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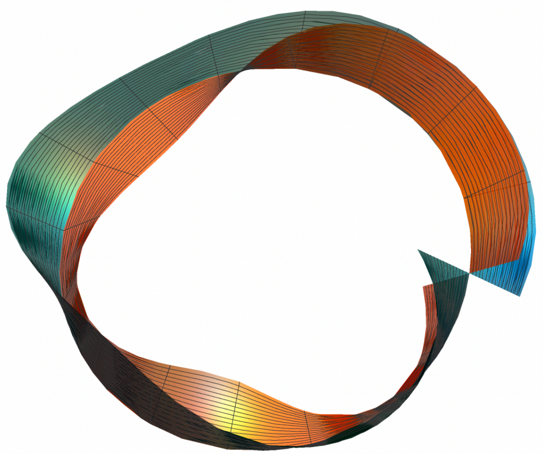
*Hyperbolic metric on the base C. Kollár et al., **Nature**, 2019.*

Eigenframe evolution as Higgs bundles: The non-Hermitian case

Proposition. The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands



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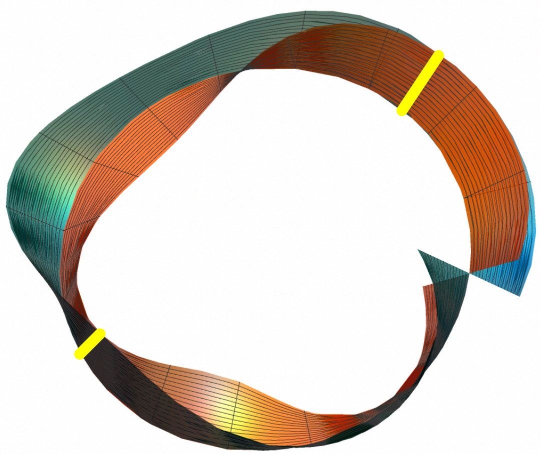

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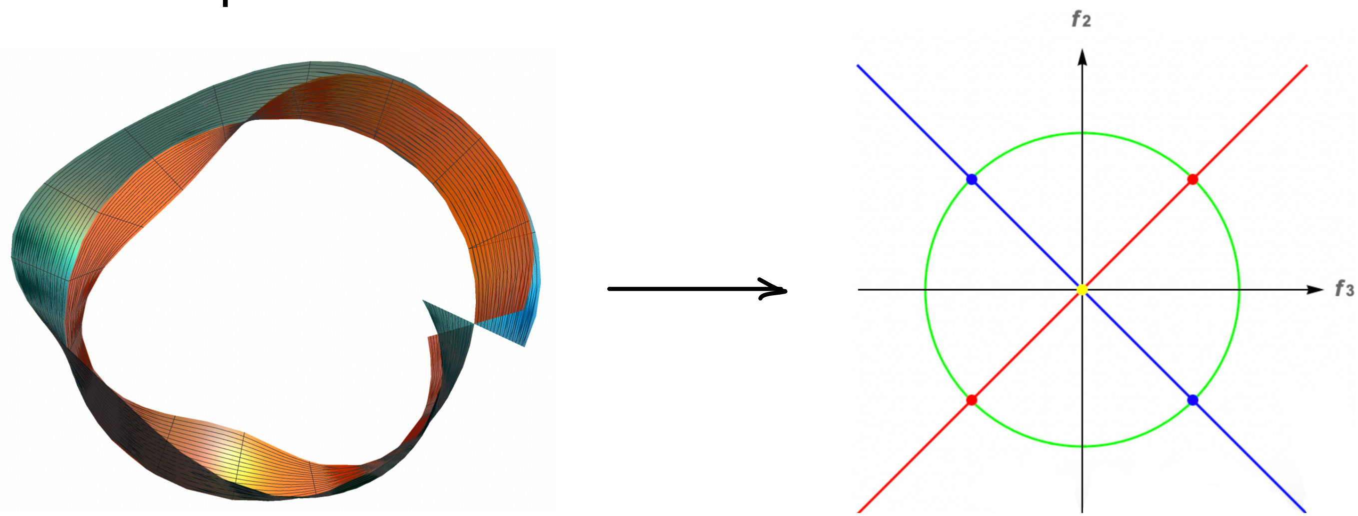

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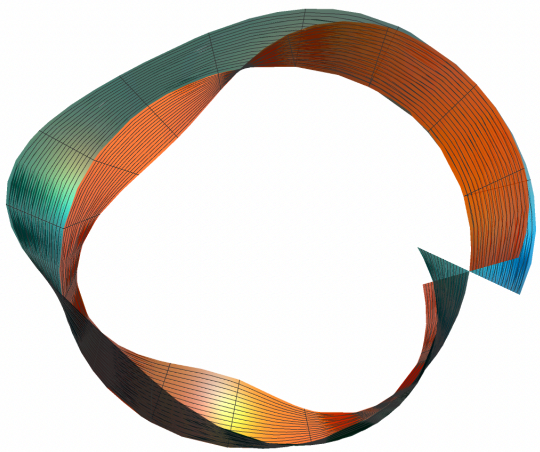

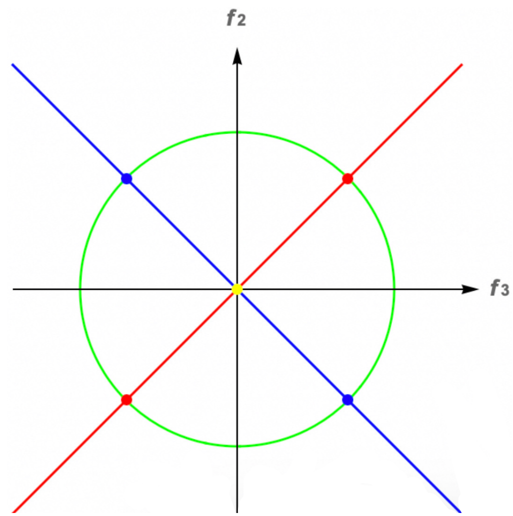

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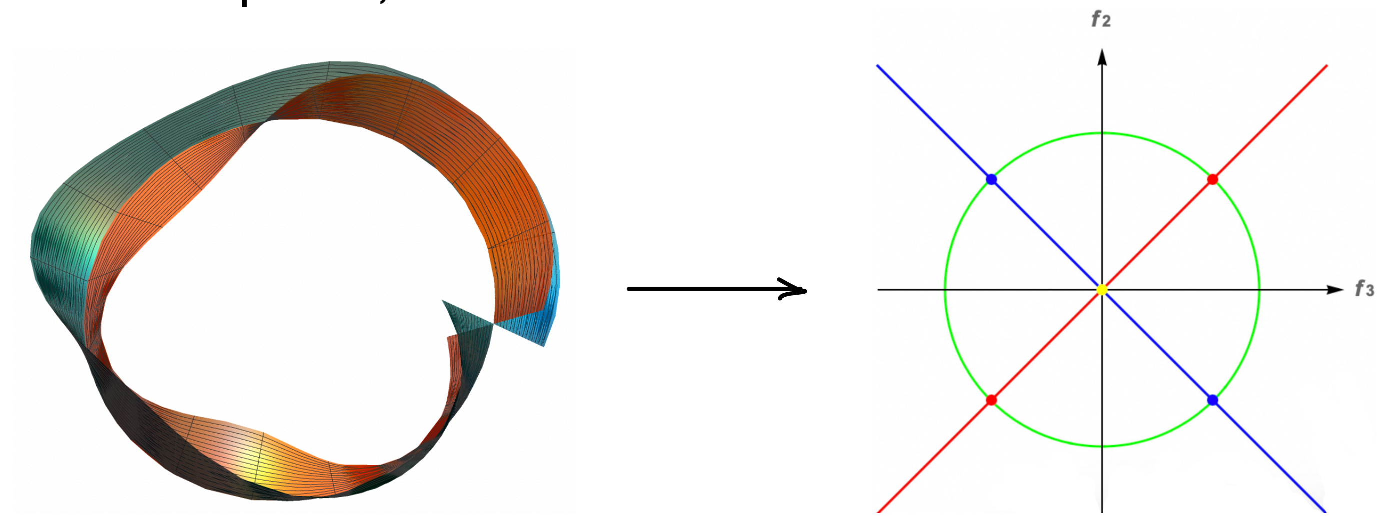

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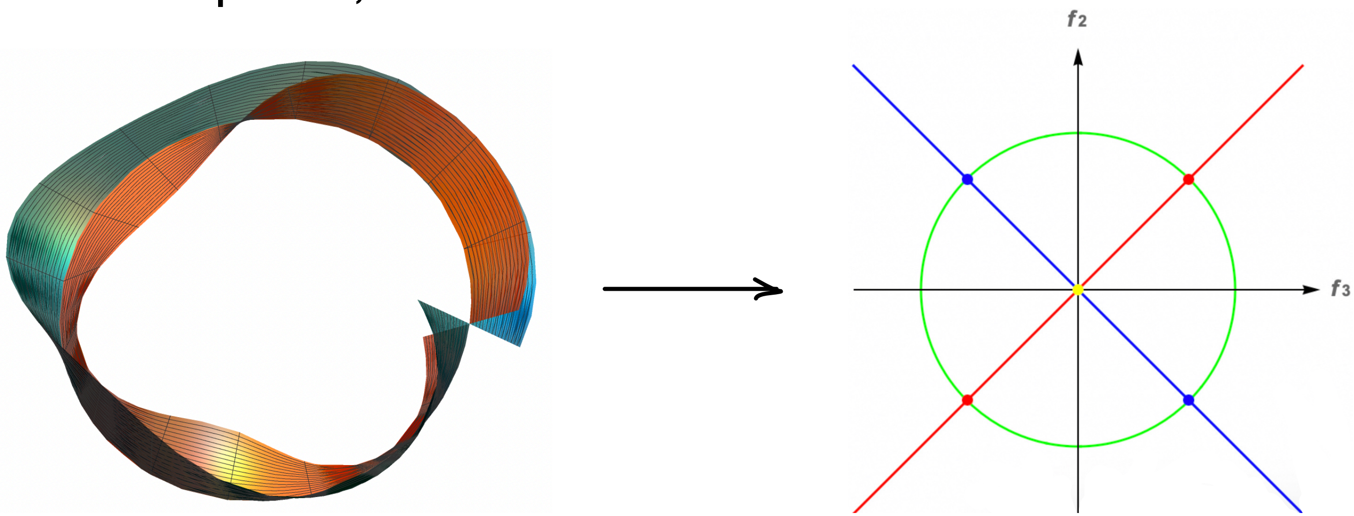

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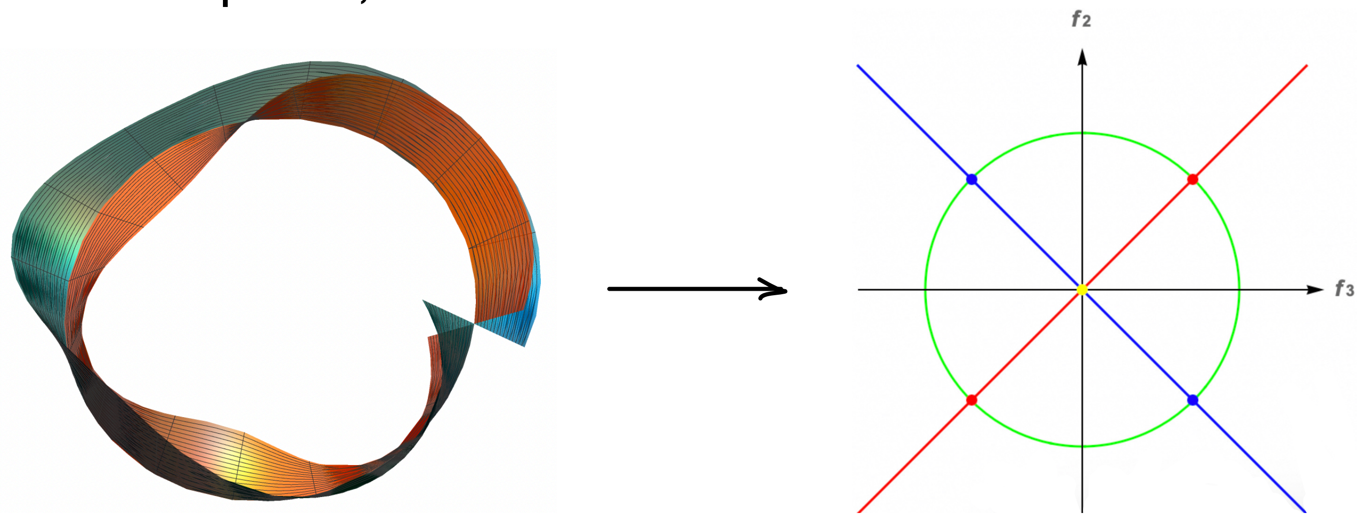

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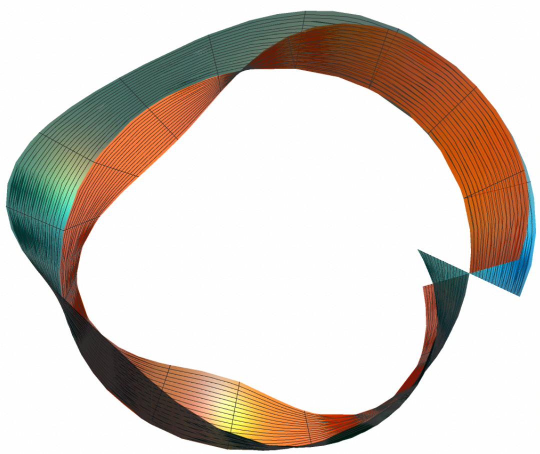

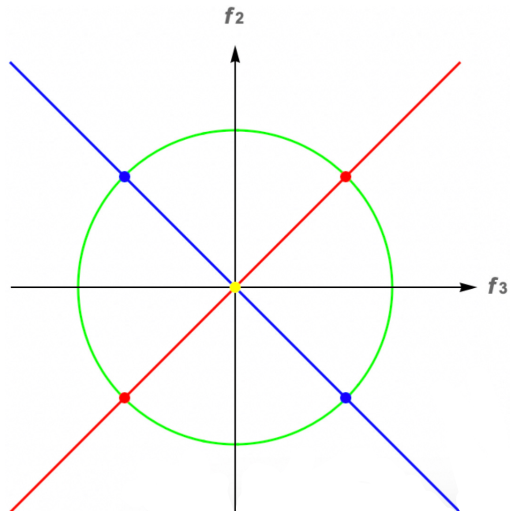

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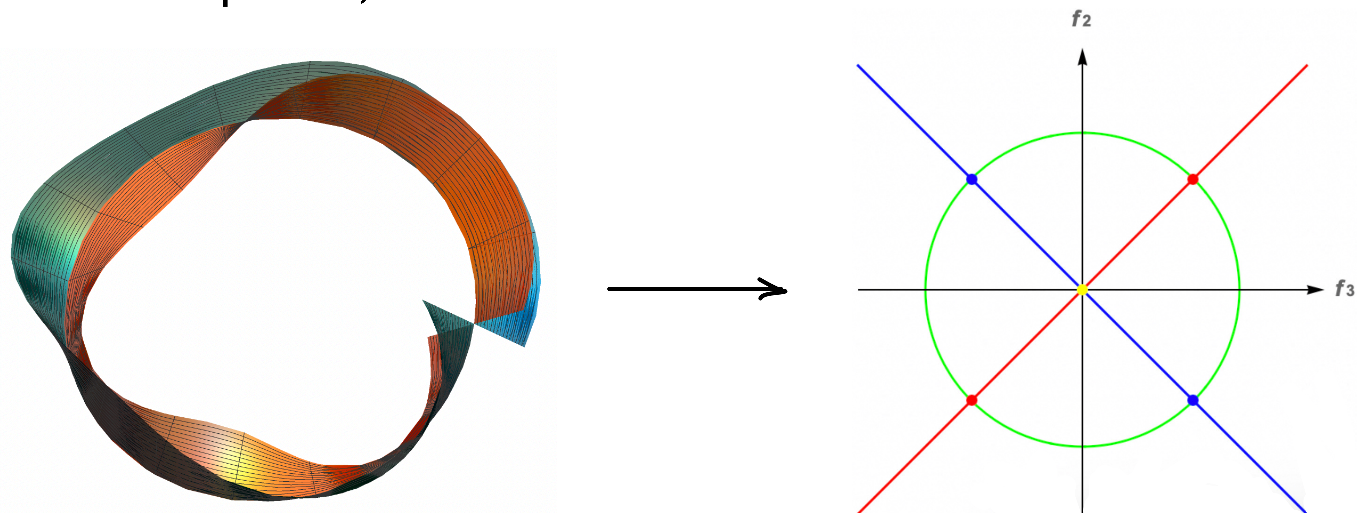

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Eigenframe evolution as Higgs bundles: The non-Hermitian case

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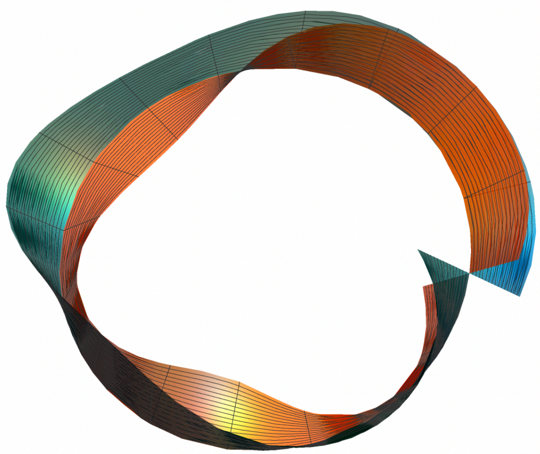

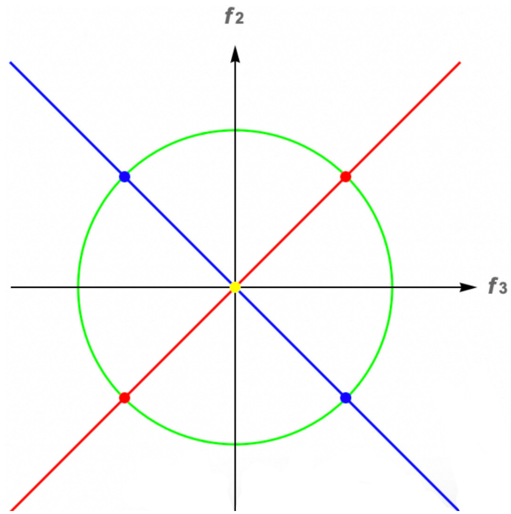

Here is a video showing the eigenframe evolution: https://yifeizhu.github.io/swallowtail/rotate.mp4

Note. In the non-Hermitian case, since the eigenvectors are in $\mathbb{C}^2$, we have adopted (a variant of) the *Hermitian angle* to properly characterize the eigenframe rotation and deformation:

$$\frac{\langle v_+, v_- \rangle_{\mathbb{C}}}{|v_+|\,|v_-|} = \rho e^{i\psi}, \quad \cos(v_+, v_-)_{\mathrm{Herm}} := \rho$$

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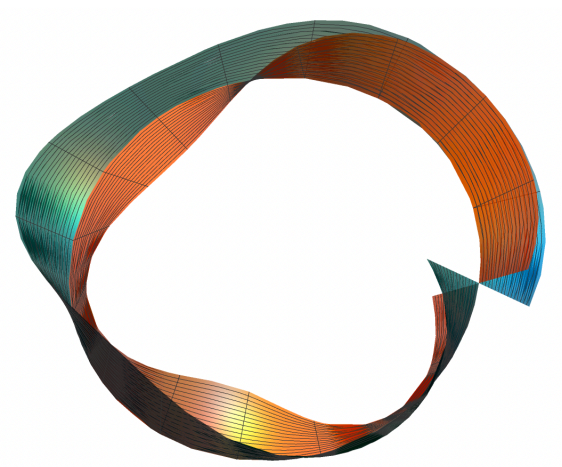

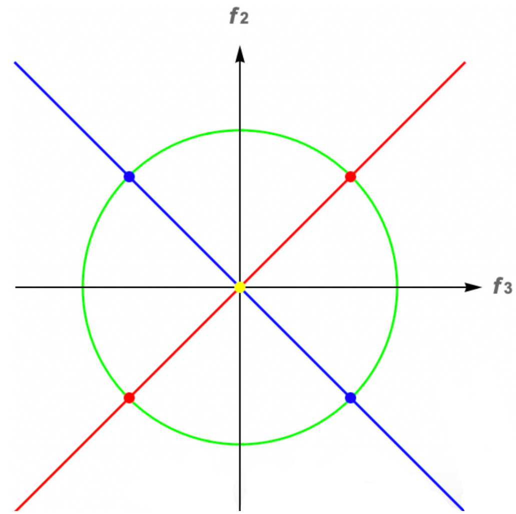

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Note. In the non-Hermitian case, since the eigenvectors are in $\mathbb{C}^2$, we have adopted (a variant of) the *Hermitian angle* to properly characterize the eigenframe rotation and deformation:

The intrinsic geometry should be independent of real/complex coordination, though.

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Mathematical interlude: Classification of bundles

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\begin{array}{ccc}
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\end{array}
$$

universal bundle

classifying space

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- Find computable and effective algebraic invariants (topological charge) for the classifying/moduli space

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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In progress: Need to compute the *intersection fundamental group* of the stratified moduli space.

Gajer, The intersection Dold–Thom theorem, ***Topology****, 1996. (Ph.D. student of Blaine Lawson, 1993)*

Goresky and MacPherson, 1974.

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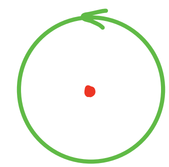

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$$\begin{array}{c|c|c|c}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}$$

$$\quad {}_0 \qquad {}_1 \qquad {}_2 \qquad \longrightarrow \bar{p}$$

Intersection homology of $\mathbb{R}^2$ with one singular point: from top to bottom are $I^{\bar{p}}H_0, I^{\bar{p}}H_1, I^{\bar{p}}H_2,$ where $\bar{p}$ is the perversity function.

From blue to red regions, they detect the singular point.

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Tolerance of ill-behaved cycles

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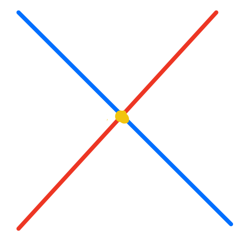

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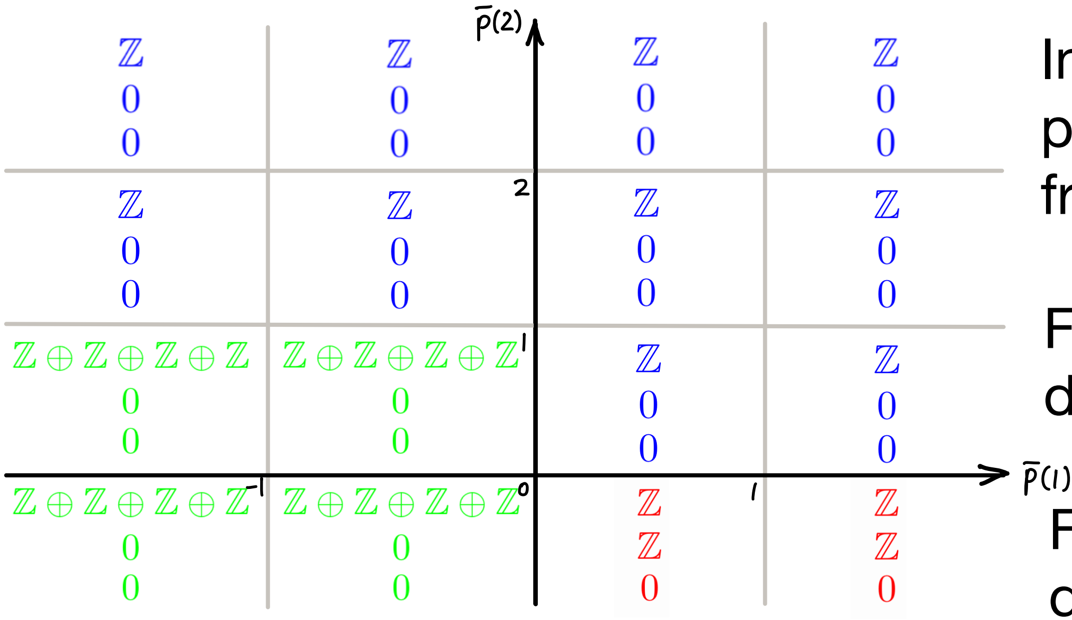

Intersection homology of $\mathbb{R}^2$ with a pair of intersecting singular lines: from top to bottom are $I^{\bar{p}}H_*$ with $* = 0, 1, 2.$

From green to blue regions, they detect the singular lines.

From blue to red regions, they detect the intersection point.

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- 1st intersection homology group recovers the Hermitian 2-band charge of $\mathbb{Z}$.
- Still need compatibility with our earlier ad hoc non-Hermitian classification:

$$\pi_1\left(S^1 \vee S^1 \vee S^1\right) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

May need to work at the chain level.

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$$\begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$$

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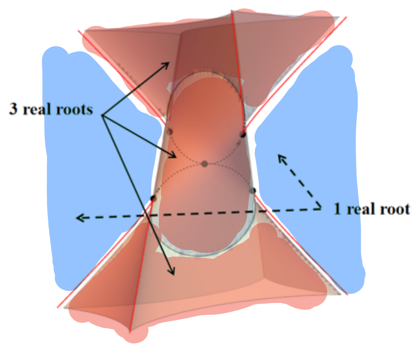

(a)

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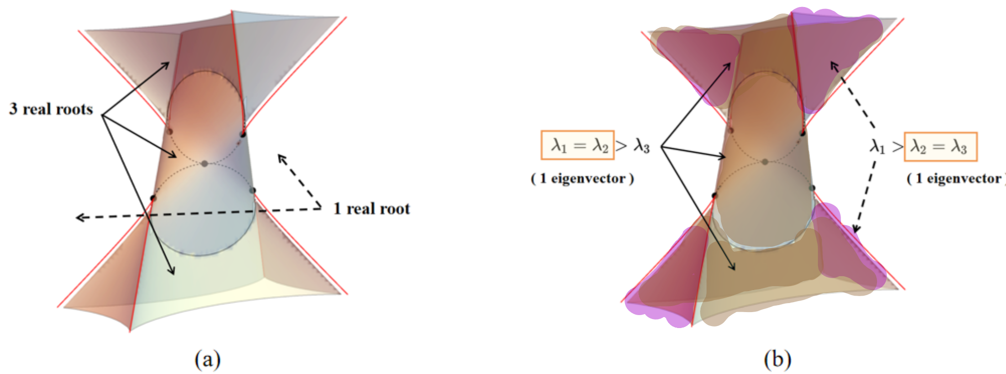

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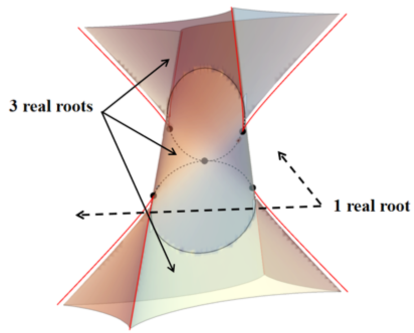

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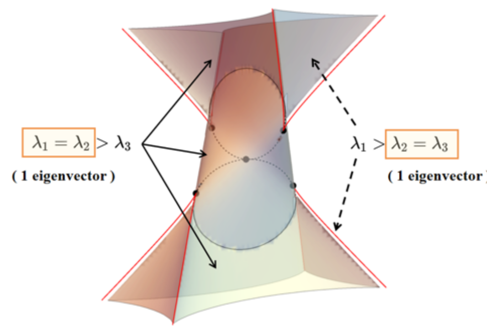

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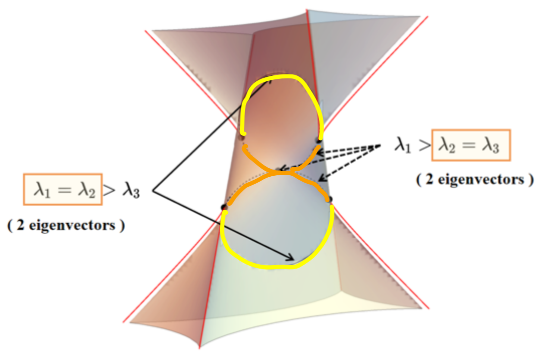

(c)

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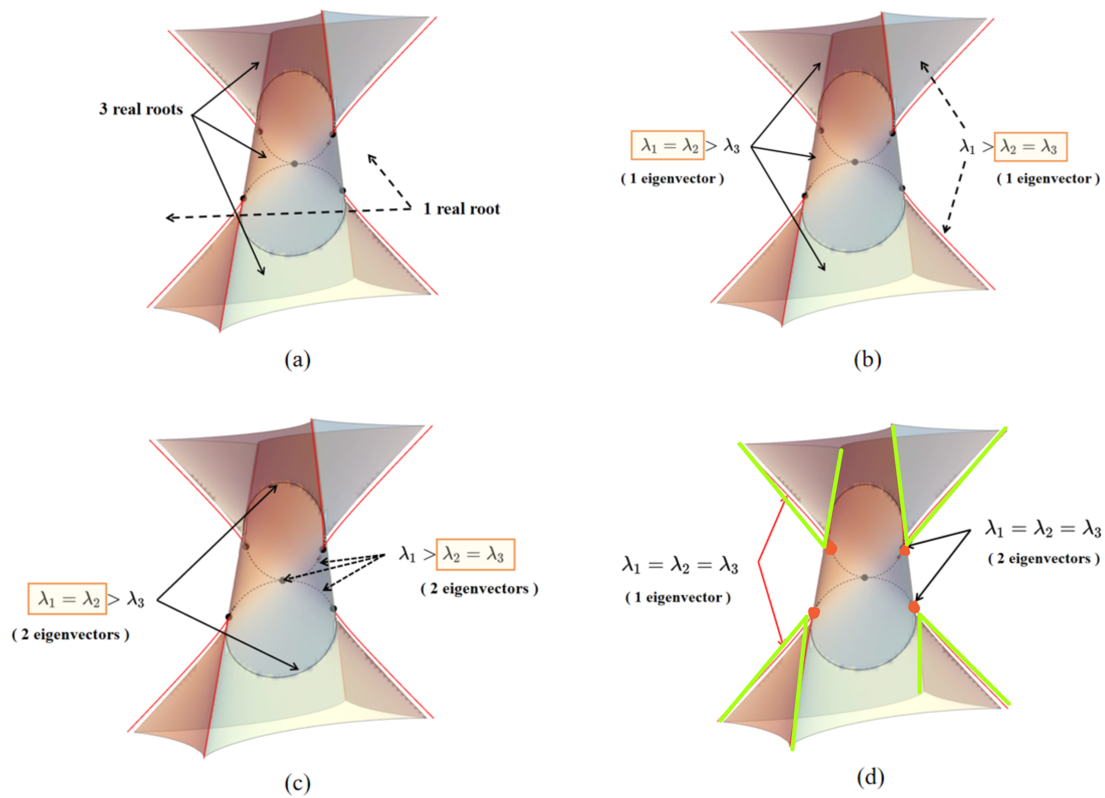

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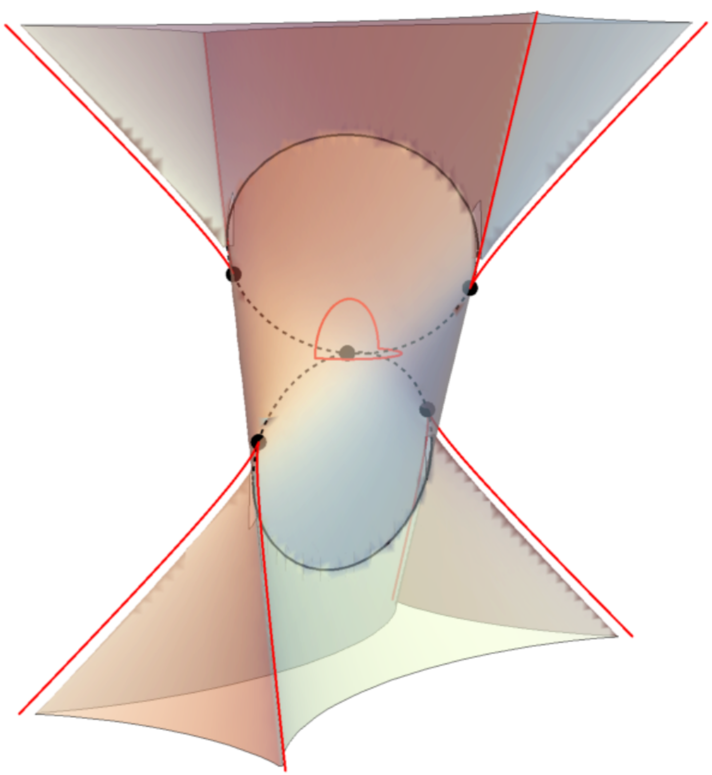

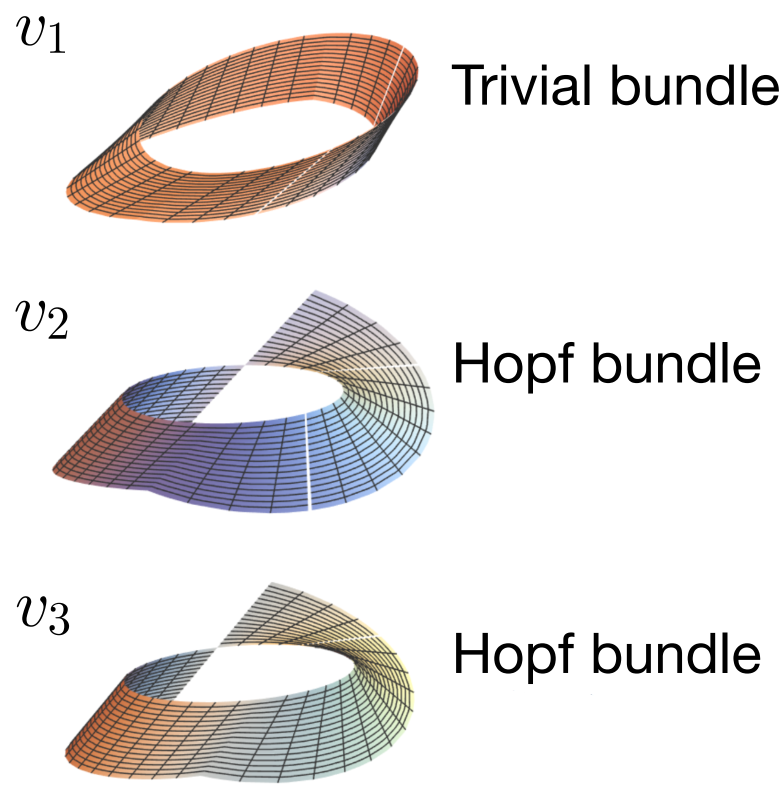

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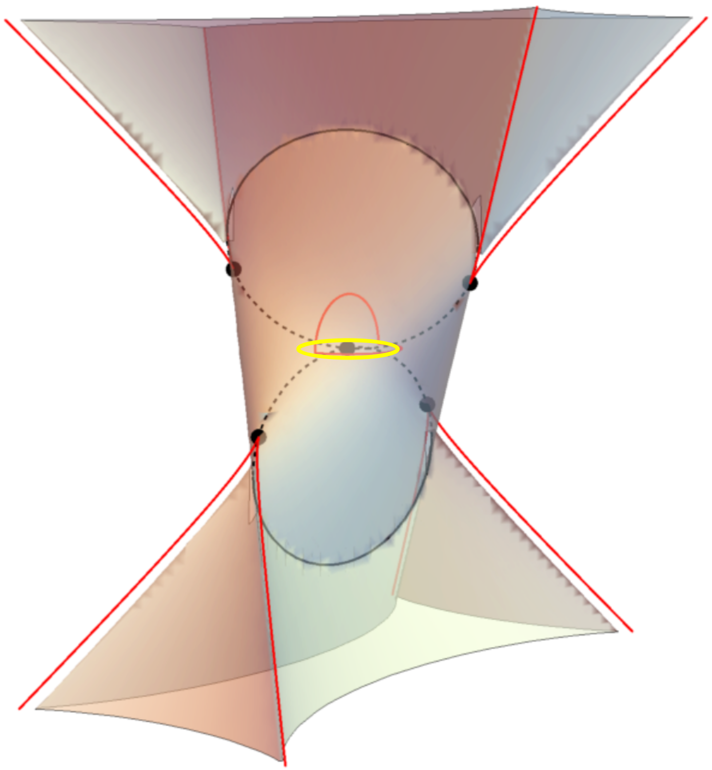
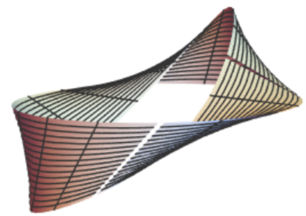
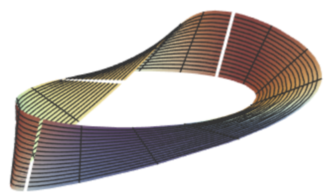


v_1



Trivial bundle

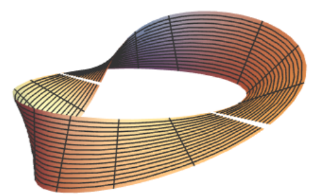
v_2



~~Hopf bundle~~

Trivial bundle

v_3



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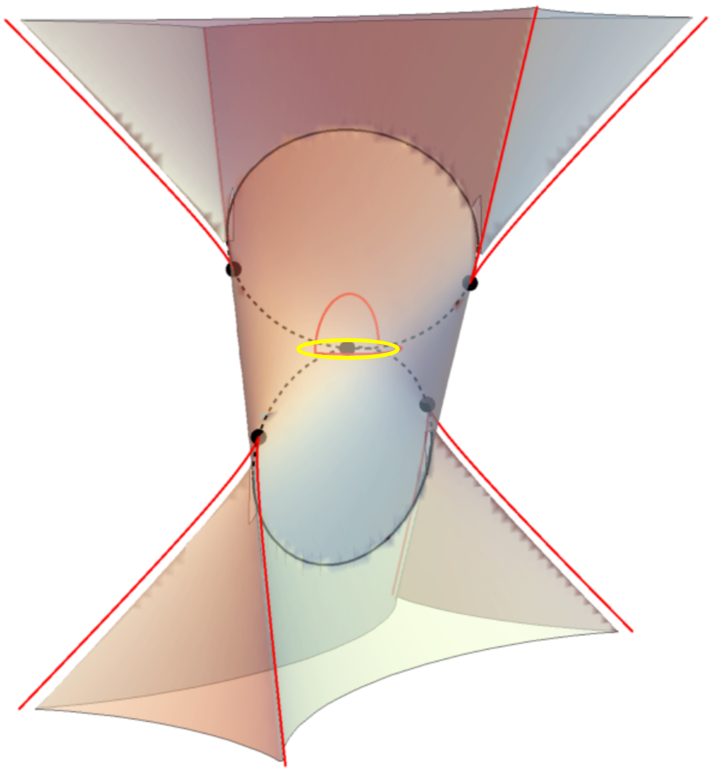

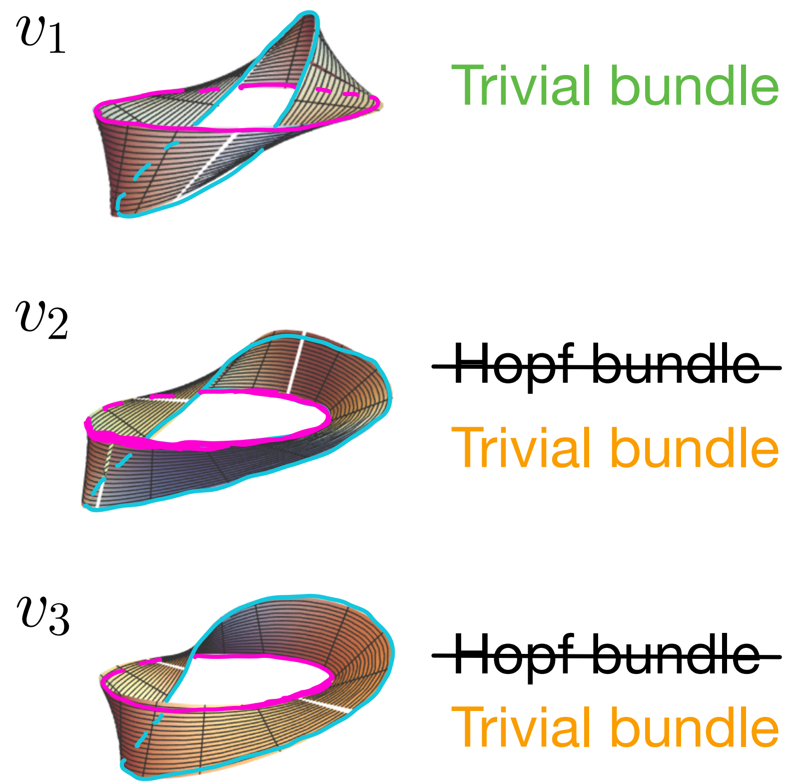

Linking number is "over-sensitive" when it involves a loop in the base space and a loop in the total space.

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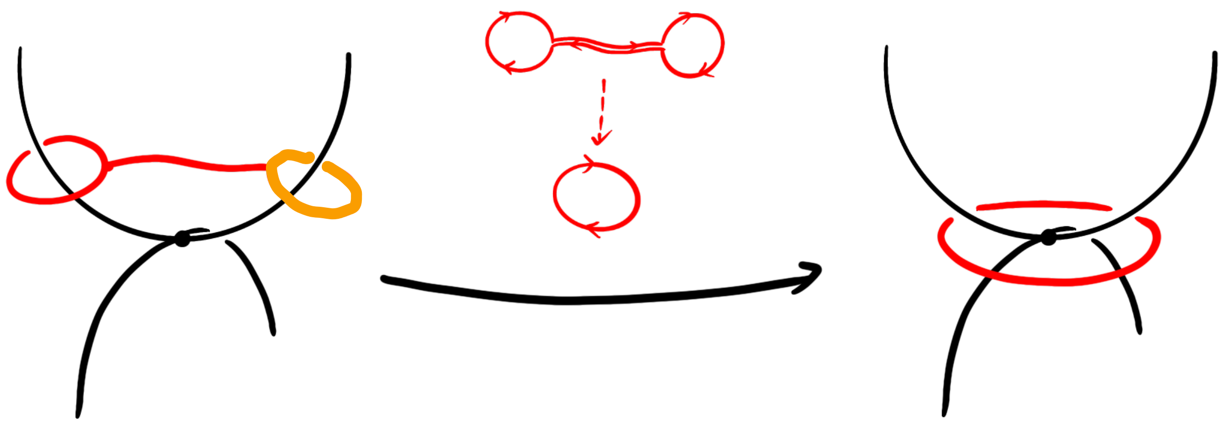

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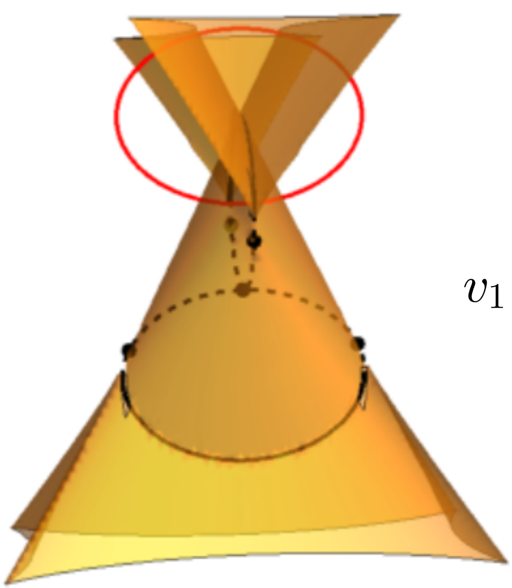

$v_1 = \text{constant}$

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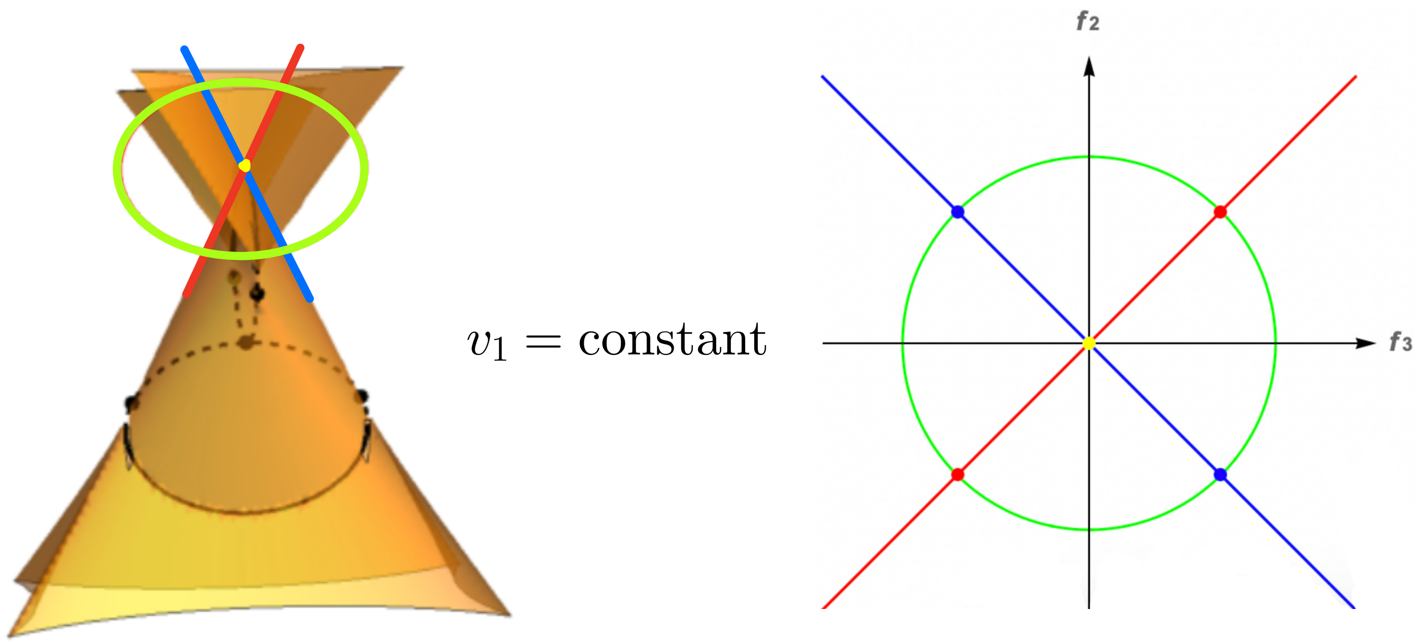

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In progress: Moreover, we have investigated the 3D moduli spaces *as a family*

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*a "**family** of families"*

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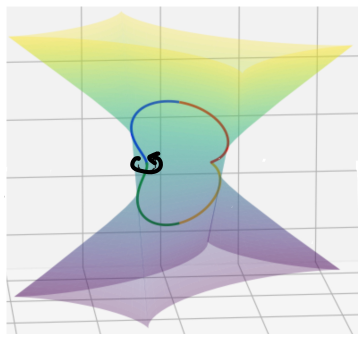

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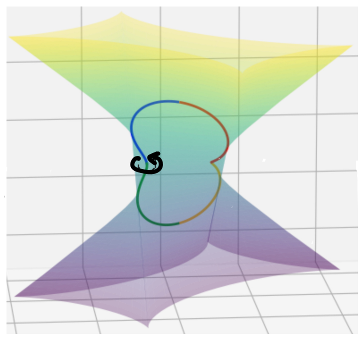

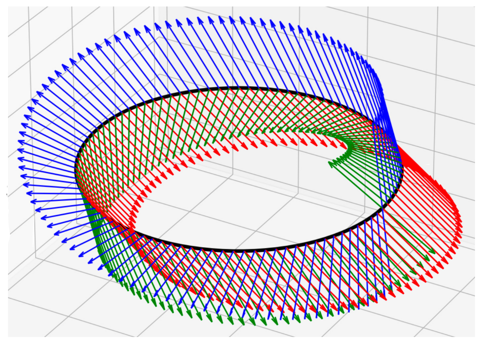

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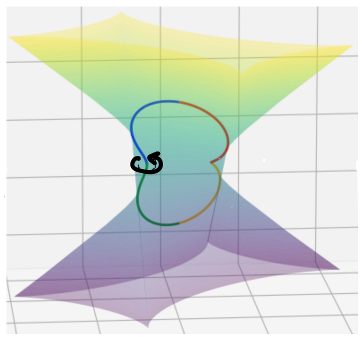

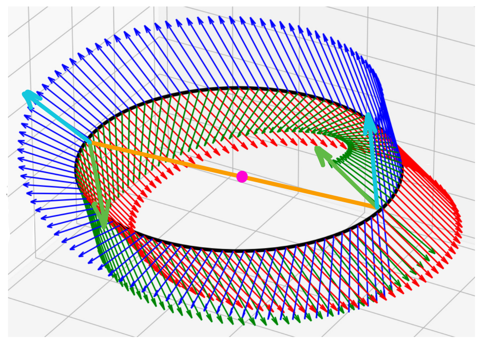

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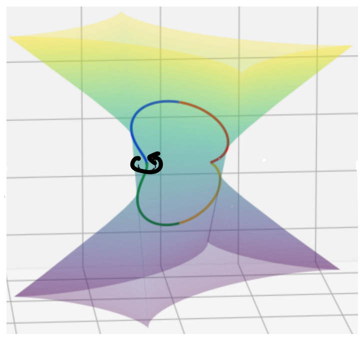

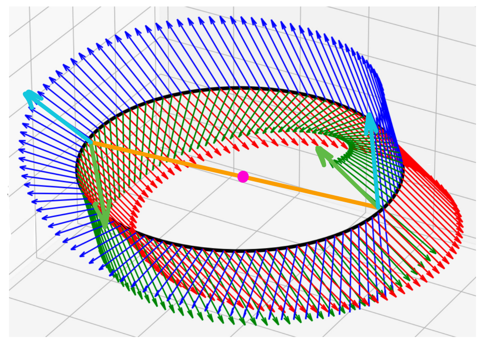

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Across the center (nodal line), the blue and green eigenstates swap — "band inversion."

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Visualizing at https://www.wolframcloud.com/env/zhuyf0/Presentation.nb

- Opening of 2 tunnels and a new “big” loop around, along which the rank-3 eigenbundle is trivial
- Merging of 8 cuspidal lines into 4
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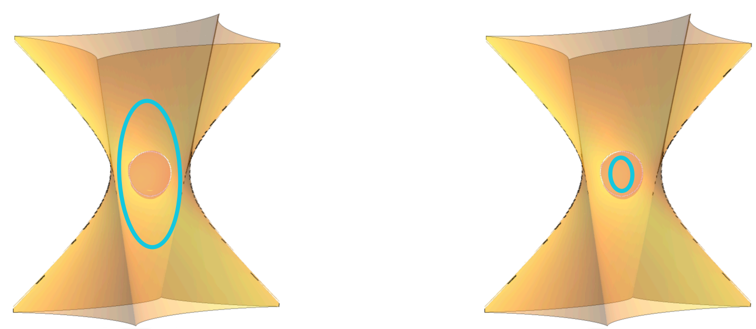

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Shrinking this loop into the enclosed region, we find the eigenbundle along it remains trivial.

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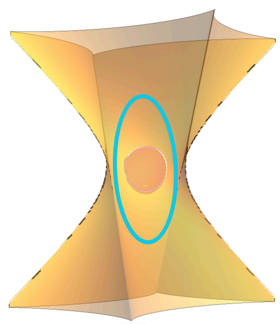

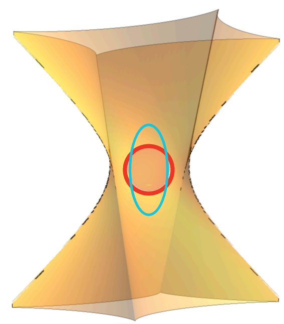

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What about loops transversing the nodal intersection lines? Band inversion again?

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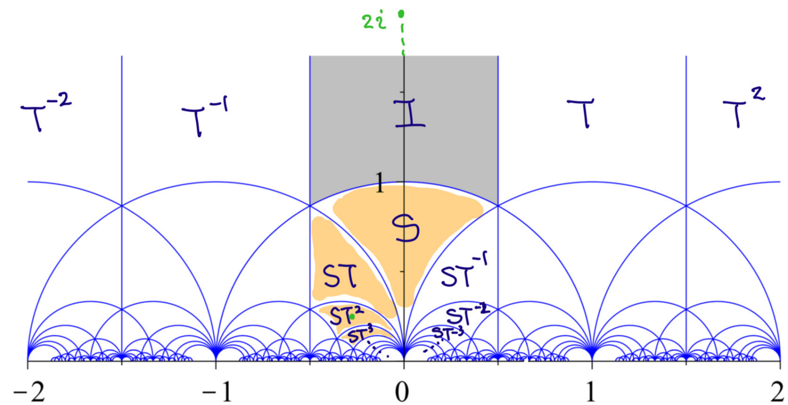

A prototypical 2D hyperbolic lattice with a straight-line boundary

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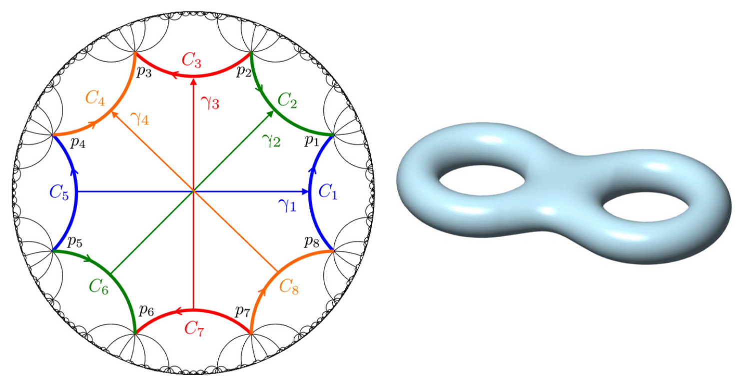

Another basic example of a hyperbolic lattice associated to a genus-2 surface
(from Maciejko and Rayan, *Hyperbolic band theory*, ***Sci. Adv.***, 2021)

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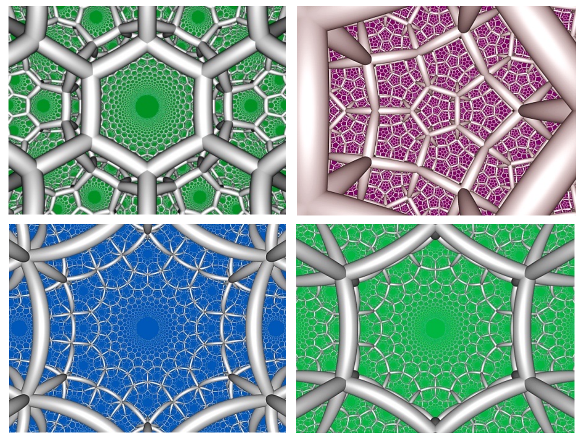

Four 3D hyperbolic lattices tiling up the hyperbolic 3-space $\mathbb{H}^3$ (from John Baez's blog)

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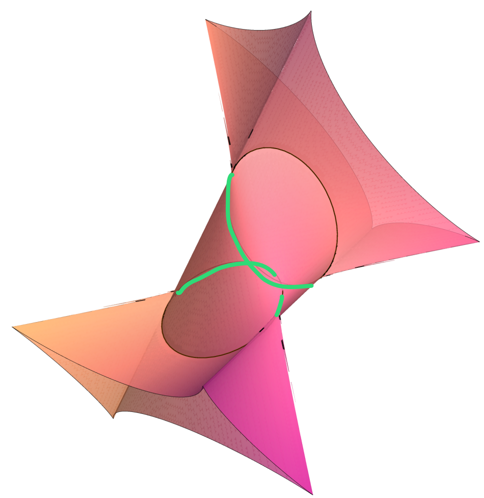

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  Existence of nodal curves inside also gives evidence, supporting nontrivial loops around (generating a free group on 3 letters) acting on a 3D hyperbolic lattice.

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There are moduli spaces for Higgs bundles of various sorts.

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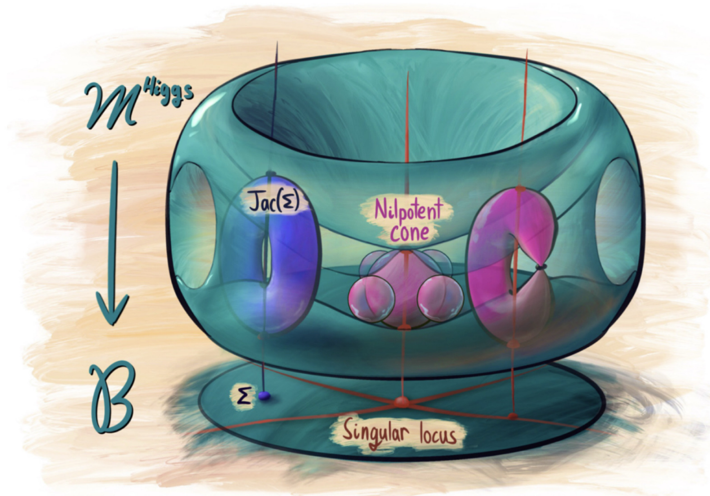

Portrait by Elliot Kienzle

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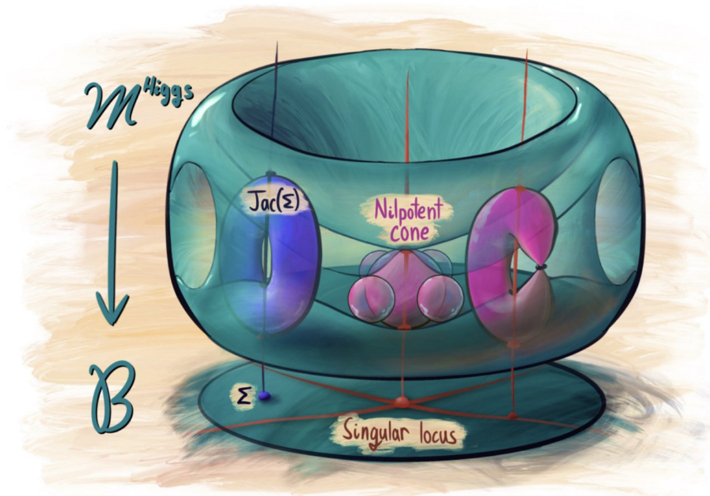
$$h\colon \mathscr{M}^{\mathrm{s}}_{\mathrm{Higgs}}\big(\mathrm{SL}_n(\mathbb{C})\big) \longrightarrow \bigoplus_{i=2}^{n} H^0(\Sigma; K^i) =: \mathscr{B}$$

$$(E,\phi) \;\mapsto \big(p_2(\phi),\ldots,p_n(\phi)\big)$$

where each p_i is a homogeneous polynomial of degree i.



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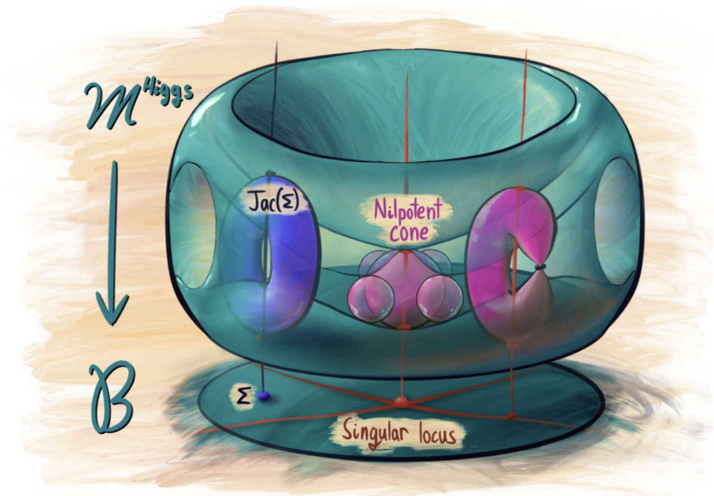

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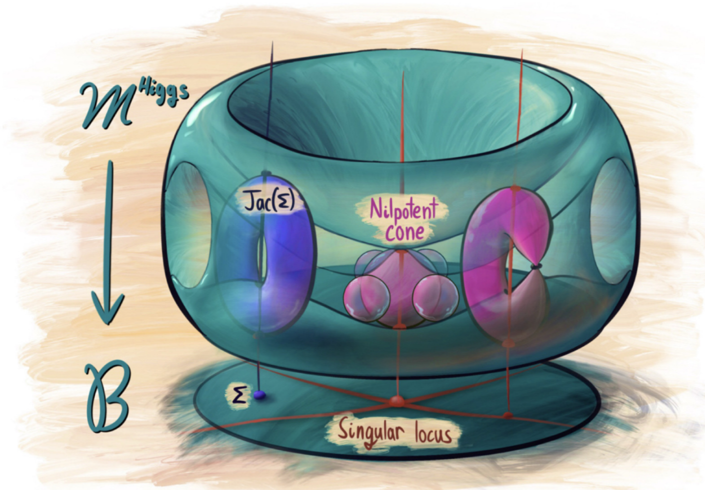

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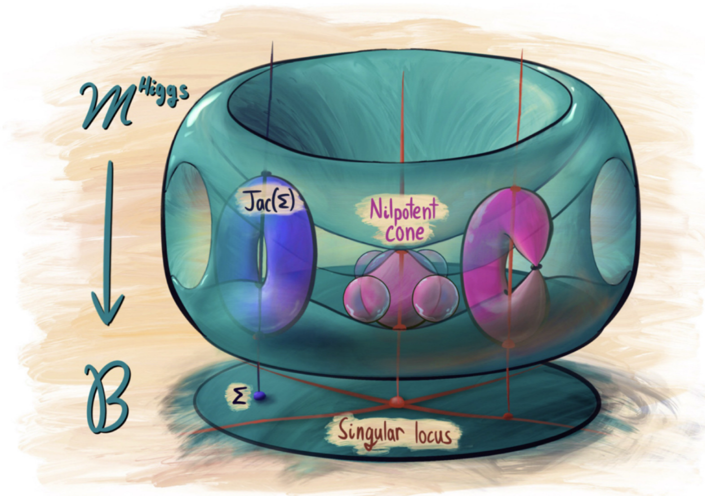

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- Singular locus: Consists of points b such that σ_b is singular. Over the singular locus, the spectral curve is singular, and the fiber can degenerate.

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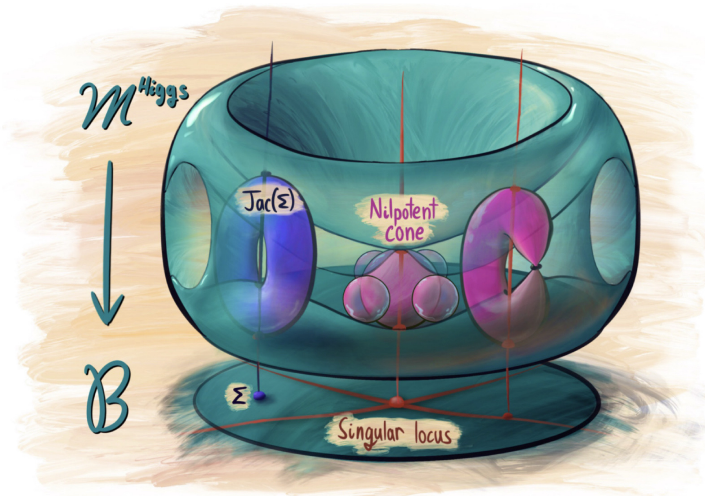

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where each p_i is a homogeneous polynomial of degree i. A point $b = \big(p_2(\phi),\dots,p_n(\phi)\big) \in \mathscr{B}$ determines a spectral curve $\sigma_b\colon \omega^n + p_2(\phi)\omega^{n-2} + \ldots + p_n(\phi) = \det(\omega I - \phi)$.

- Regular locus: Consists of points b such that σ_b is regular.
- Singular locus: Consists of points b such that σ_b is singular. Over the singular locus, the spectral curve is singular, and the fiber can degenerate.
- Nilpotent cone: The most degeneration occurs over $0 \in \mathscr{B}$. The fiber $h^{-1}(0)$ is called the nilpotent cone.

Portrait by Elliot Kienzle

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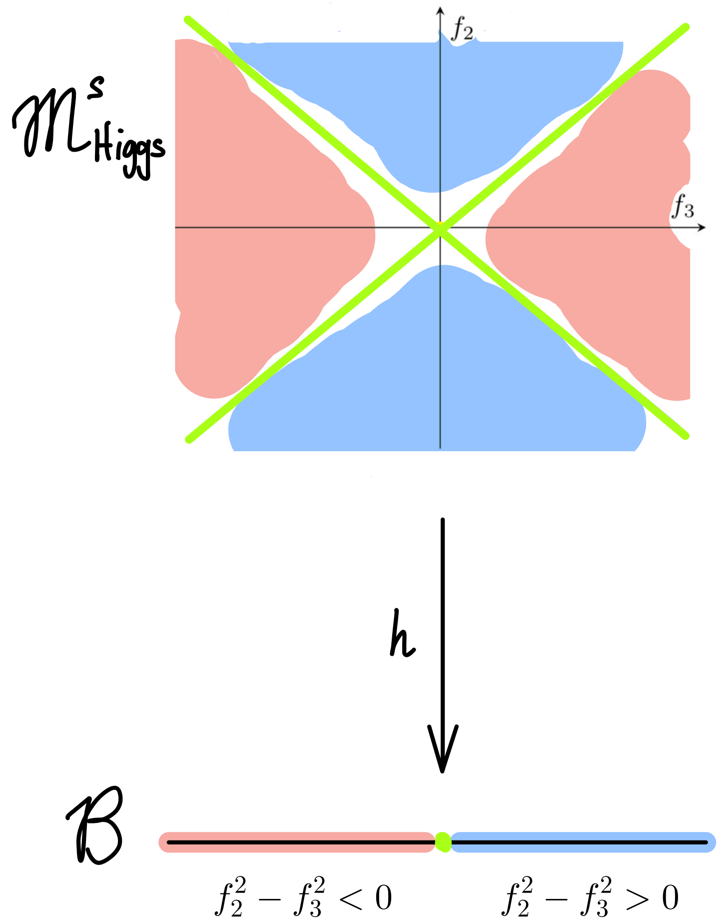

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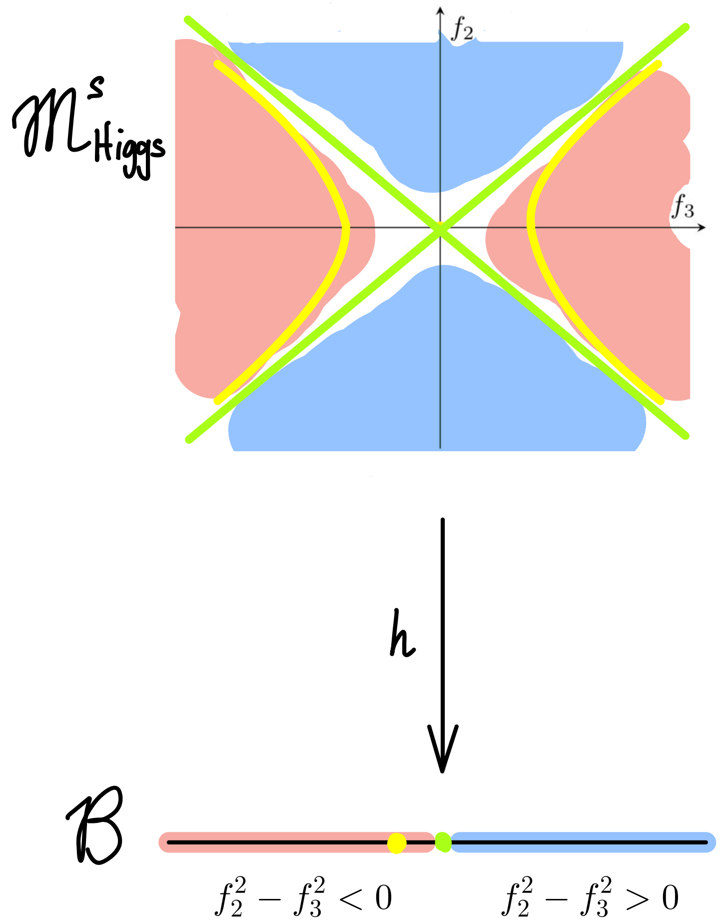

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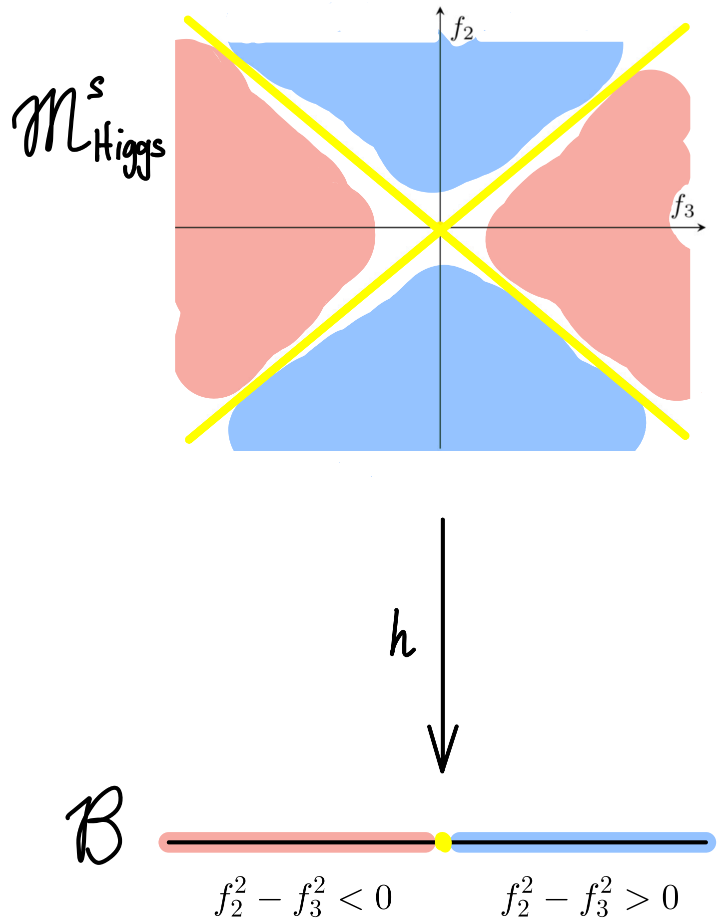

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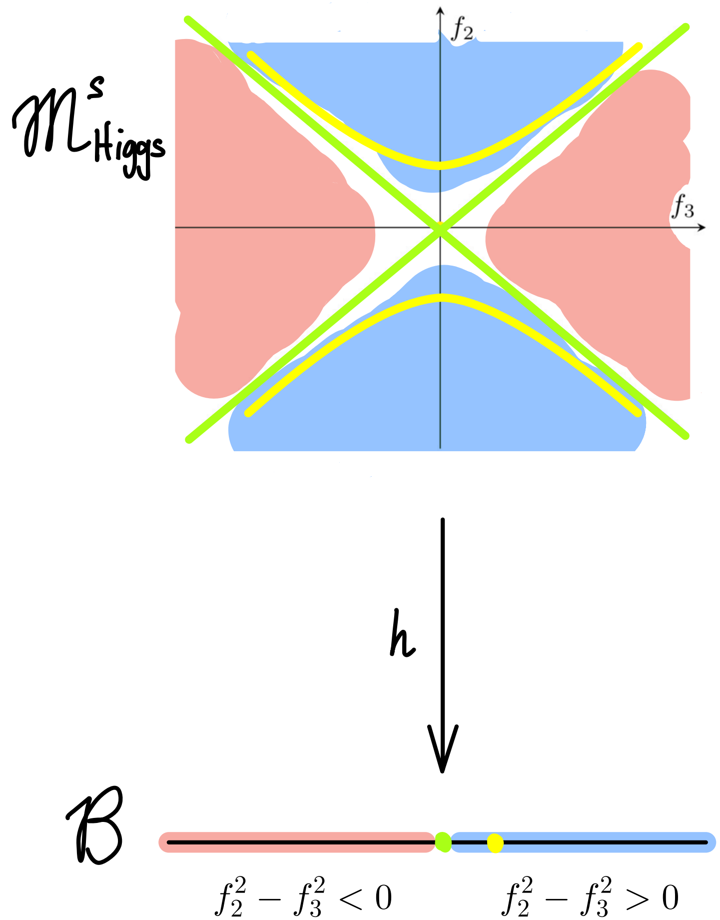

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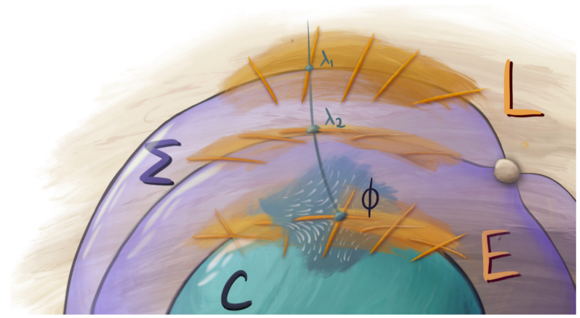

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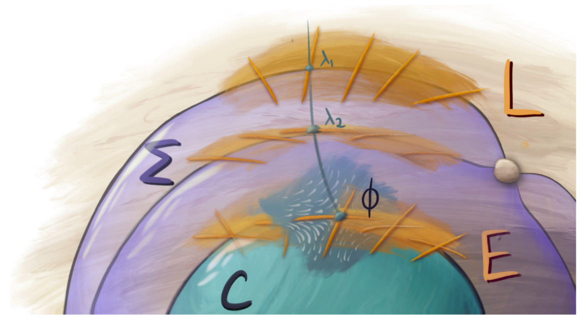

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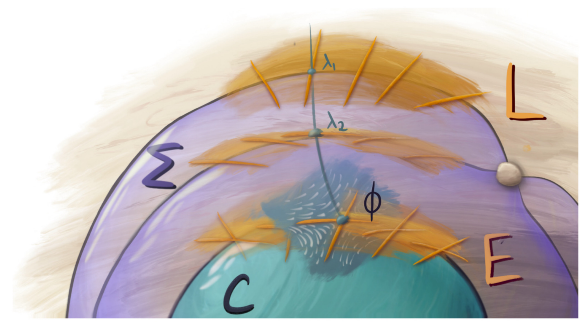

Thus, given a Higgs bundle (E, ϕ), we get a harmonic map f: $\widetilde{C} \to \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$.

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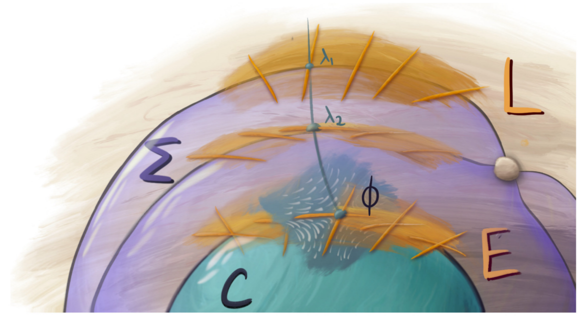

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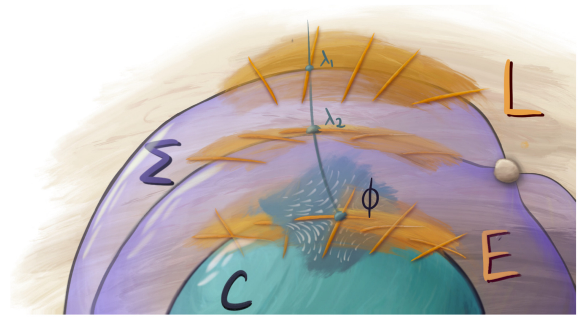

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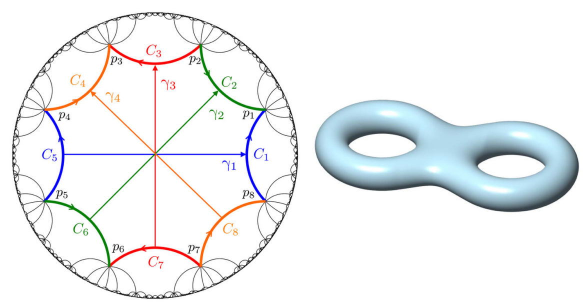

Maciejko–Rayan '21

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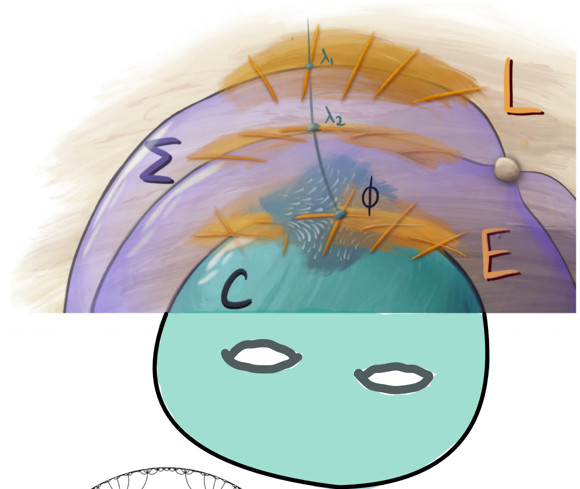

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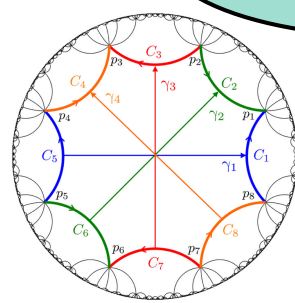

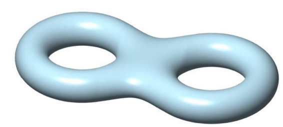

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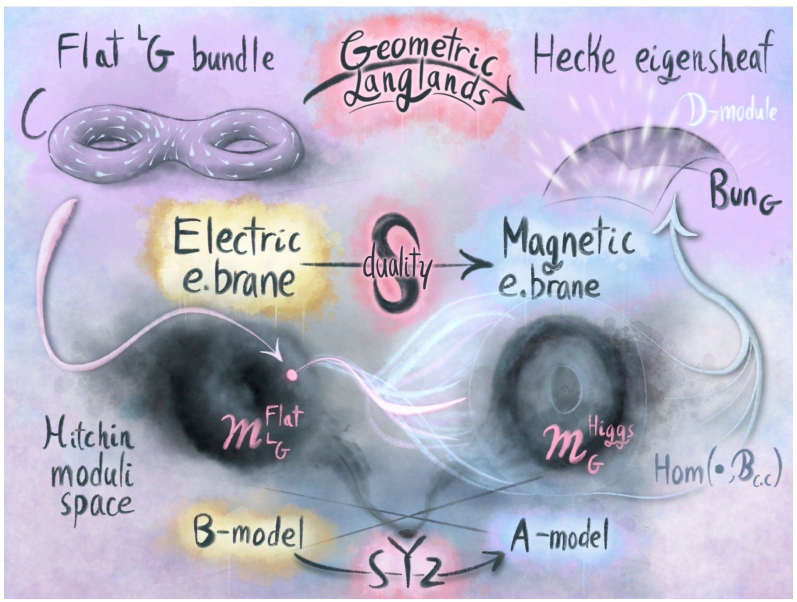

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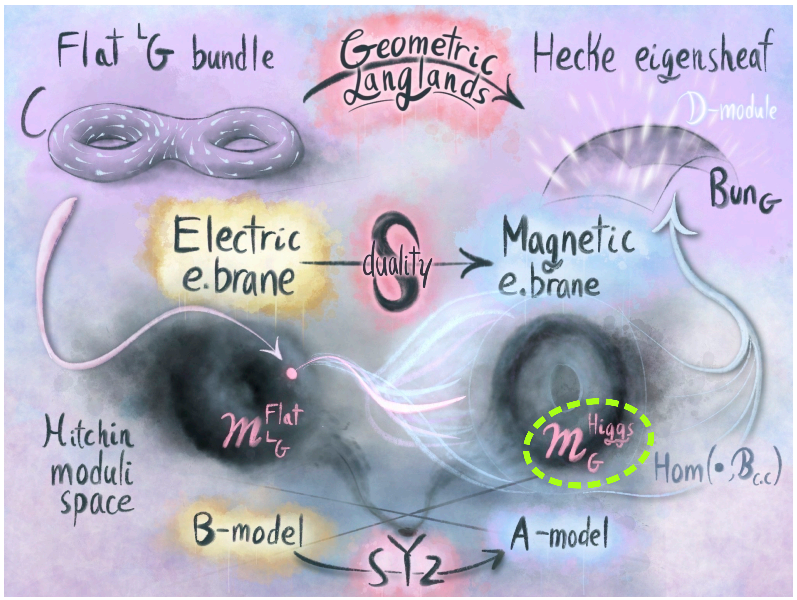

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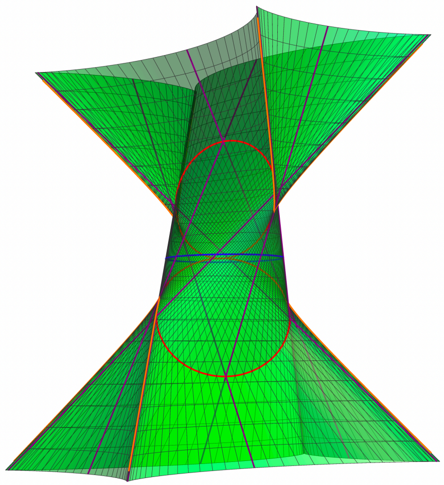
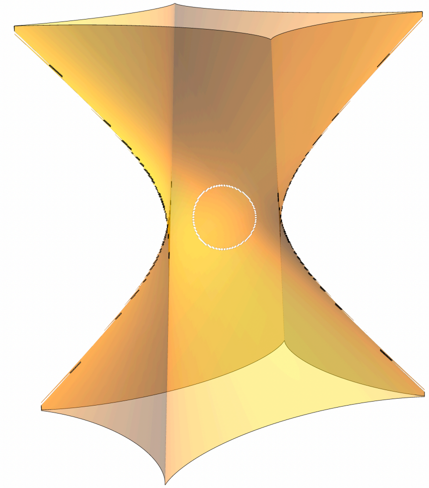
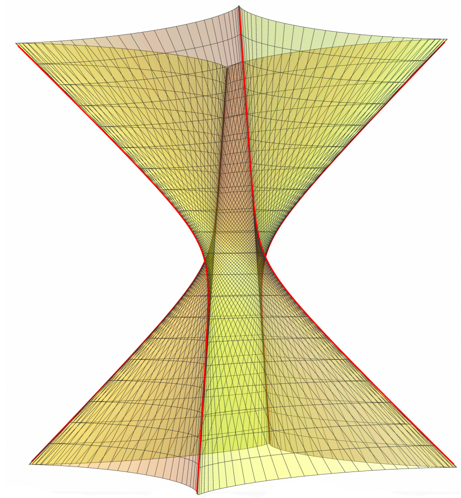

Thank you.