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南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

本科生毕业设计（论文）

题　　目： **Markov-Rényi 映射的快速李雅普诺夫指数重分形分析**

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Markov-Rényi 映射的
快速李雅普诺夫指数重分形分析

王郅超

（数学系　指导教师：张一威）

[ABSTRACT]: The concept of the Lyapunov exponent was first introduced to quantify the rate at which the orbits diverge in a hyperbolic dynamical system, and the systems with countably (infinite) many branches and infinite topological entropy draw much attention. In this case, it was shown that the Lyapunov spectrum (i.e. the dimension function of the level sets of Lyapunov exponent) has a non-compact support and a horizontal asymptotic. This leads to the finer study of the behavior of Lyapunov exponent at infinity. In this paper, we will determine the (upper and lower) fast Lyapunov spectrum independent of the thermodynamic formalism for the Rényi map, which in particular has a neutral fixed point, and it is closely related to the backward continued fractions. We also prove that the Lyapunov spectrum is continuous at infinity, and calculate the set of number whose partial quotient tends to infinity. The main technique established to prove the results above is the existence of calculable cantor-like subsets in the level sets of the fast Lyapunov exponent.

[Key words]: Fast Lyapunov spectrum, Hausdorff dimension, Backward continued fractions

目录

1. Introduction

1.1 Backgrounds

Dynamical systems exhibiting strong hyperbolicity inherently produce a significant degree of mixing. In such scenarios, the orbit structure becomes highly intricate, making it particularly important to quantify the rate at which orbits diverge.

Lyapunov exponents serve as an important tool for this purpose, describing the exponential rate at which infinitesimally close orbits of a dynamical system diverge. For a piecewise differentiable interval map $T : I \to I$, where I is an at most countably many union of closed intervals, the *Lyapunov exponent* of the map T at the point $x \in I$ is defined as

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)|, \tag{1.1}$$

whenever the limit exists.

According to Birkhoff's ergodic theorem, for any ergodic T-invariant measure μ such that $\int \log |T'| d\mu$ is finite, the Lyapunov exponent $\lambda(x)$ equals $\int \log |T'| d\mu$ for μ-almost every $x \in I$. However, the Lyapunov exponent can attain a continuous range of values, forming an entire interval. This observation naturally motivates the investigation of the complexity of the level sets of the Lyapunov exponent.

For any $\alpha \in \mathbb{R} \cup \{\pm\infty\}$, we define the *level set* of the Lyapunov exponent $\lambda(x)$ as

$$J(\alpha) := \{x \in I : \lambda(x) = \alpha\}. \tag{1.2}$$

The *Lyapunov spectrum* is the function that describes how the Hausdorff dimension of $J(\alpha)$ varies with α, namely,

$$L(\alpha) := \dim_{\mathrm{H}} J(\alpha), \tag{1.3}$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension (see Section 2.3 for definition and[1] for more information).

The Lyapunov spectrum has been extensively studied for several important classes of piecewise differentiable maps. The pioneering work is due to Weiss[2]. He relates the Lyapunov exponent with the pointwise dimension of a Gibbs measure and based on the discussion on the multifractal analysis of the pointwise dimension from Pesin and Weiss[3], he proved that for conformal expanding maps with finitely many branches [1], the Lyapunov spectrum has a bounded domain, is real analytic, and is concave in the domain.

Inspired by Weiss' result, two directions of generalizations have been performed in the context of non-uniformly hyperbolic piecewise differentiable interval maps, and different phenomenon on the Lyapunov spectrum has been observed.

On the one hand, Pollicott and Weiss[4], and Nakaish[5] (see also the works of Takens and Verbitskiy[6], and Pfister and Sullivan[7]) studied the Lyapunov spectrum in the case of the Manneville–Pomeau map, which is an interval map with two branches and a parabolic fixed point at zero. In this case, the Lyapunov spectrum has a bounded domain, but it can have points where it is not analytic. Later, Gelfert and Rams[8] considered a broader class of such systems and described the Lyapunov spectrum.

On the other hand, Pollicott and Weiss[4] (see also the work of Kesseböhmer and Stratmann[9]) studied for the Lyapunov spectrum for the Gauss map, which is an expanding map with countably many branches, and infinite topological entropy. They showed that the Lyapunov spectrum (see Figure 1) is real analytic, but it has an unbounded domain $[0, \infty)$ and no longer concave on the domain. In particular $\lim_{\alpha\to\infty} L(\alpha) = 1/2$.

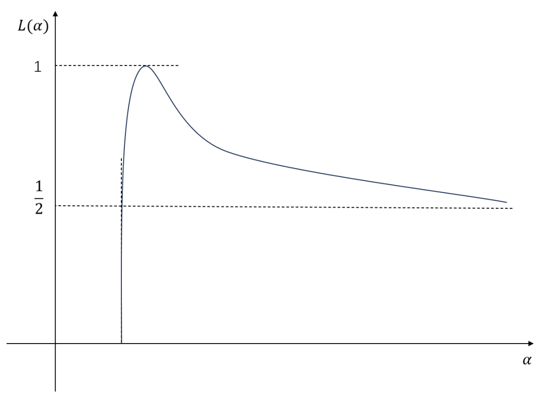

Figure 1

[1] Weiss' results are also valid for Axiom A surface diffeomorphisms.

In 2010, Iommi[10] build upon the results mentioned above, and considered a model so called *Markov-Rényi map*, which is a map with both a parabolic fixed point at zero and infinite topological entropy. Such map might have no absolutely continuous invariant probability measure with respect to Lebesgue measure, and it is closely related to the backward (or regular) continued fractions, and is also related to the geodesic flow on the modular surface. It turns out to that the Lyapunov spectrum (see Figure 2) has an unbounded domain $[0, \infty)$ and there might exist non-differentiable point in the domain. Moreover, as in the case of Gauss map, $L(\alpha)$ has a horizontal asymptote.

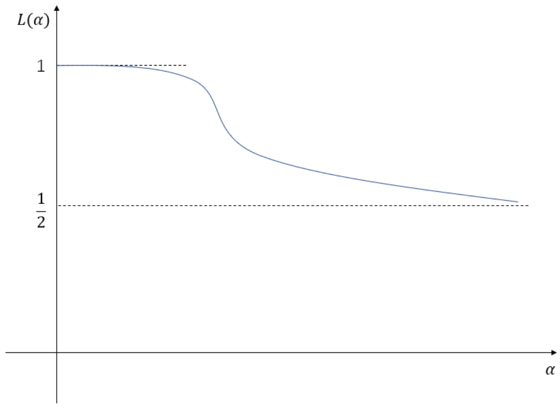

Figure 2

Based on above discussions, two natural questions arise regarding to the Lyapunov spectrum at point '∞'.

(Q1) What is the value of $L(\infty)$? Whether the Lyapunov spectrum L is continuous or not at ∞?

(Q2) Can we have further refined the spectrum on the Lyapunov exponent at ∞? If so, what is the differences between this spectrum and the Lyapunov spectrum?

In the current paper, we are aiming to answer these two questions by studying the so called *fast Lyapunov spectrum* in the context of the Rényi map (see (1.4)).

Our main results in the present paper will develop a unified approach (independent of the thermodynamic formalism) to estimate both (upper and lower) fast Lyapunov spectrum for the Rényi map. It is worth to remark that the fast Lyapunov spectrum was previously studied by Fan, Liao, Wang and Wu[11-12], in the setting of the Gauss map, [2] which is a model with absence of any parabolic fixed point. On the other hand, to the best of our knowledge, the upper and lower fast Lyapunov spectrum results are still missing other than the Gauss map.

1.2 The Rényi map

As a typical interval map with countably many branches having parabolic fixed points, the Rényi map (or backward continued fractions map) has received much attention. Let $R : [0,1) \to [0,1)$ be the Rényi map defined by

$$R(x) := \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor, \forall x \in [0,1), \tag{1.4}$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a number. The ergodic properties of the Rényi map have been investigated by Adler and Flatto[14], and Rényi[15]. The Rényi map is also closely related to the backward continued fractions algorithm[16-18]. Actually, every $x \in [0,1)$ admits a backward continued fractions expansion (BCF) of the form

$$x = 1 - \cfrac{1}{b_1(x) - \cfrac{1}{b_2(x) - \cfrac{1}{b_3(x) - \ddots}}}, \tag{1.5}$$

where the (nth) *partial quotient* $b_n(x) \in \mathbb{N}_{\geq 2}$ is given by

$$b_n(x) = \left\lfloor \frac{1}{1 - R^{n-1}(x)} \right\rfloor + 1, \forall n \geq 1. \tag{1.6}$$

1.3 Statement of the main theorems

Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a function satisfying $\psi(n)/n \to \infty$ as $n \to \infty$. Analogous to (1.1), (1.2) and (1.3), we define the *fast Lyapunov exponent* of R (with respect to ψ) at point

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[2] Fan, Liao, Wang and Wu[11-12] also studied the so called fast Khintchine spectrum. Later, the upper and lower Khintchine spectrum was studied by Liao and Rams[13].

$x \in (0,1)\backslash\mathbb{Q}$ as

$$\lambda_\psi(x) := \lim_{n\to\infty} \frac{1}{\psi(n)} \log |(R^n)'(x)|, \tag{1.7}$$

whenever the limit exists. For any $\alpha \in \mathbb{R}_+ \cup \{\infty\}$, let $J_\psi(\alpha) := \{x \in (0,1)\backslash\mathbb{Q} : \lambda_\psi(x) = \alpha\}$, and define the *fast Lyapunov spectrum* (with respect to ψ) as

$$F_\psi(\alpha) := \dim_H J_\psi(\alpha). \tag{1.8}$$

Let $\beta := \beta_\psi$ and $B := B_\psi$ be given by

$$\beta = \limsup_{n\to\infty} \frac{\psi(n+1)}{\psi(n)} \quad \text{and} \quad B = \limsup_{n\to\infty} \sqrt[n]{\psi(n)}, \tag{1.9}$$

respectively.

We say that two functions $f, g : \mathbb{N} \to \mathbb{R}_+$ are *equivalent* if $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$. The fast Lyapunov spectrum is described as follows.

Theorem A (Fast Lyapunov spectrum)**.** Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a function satisfying $\psi(n)/n \to \infty$ as $n \to \infty$. For any $0 < \alpha \leq \infty$, the level set $J_\psi(\alpha)$ is nonempty if and only if ψ is equivalent to an increasing function. Moreover, if ψ is equivalent to an increasing function, then, for $0 < \alpha < \infty$,

$$F_\psi(\alpha) = \frac{1}{\beta + 1}, \tag{1.10}$$

and

$$F_\psi(0) = 1,\ F_\psi(\infty) = \frac{1}{B+1}. \tag{1.11}$$

Next, we study the upper and lower fast Lyapunov spectra of R. For any $x \in (0,1)\backslash\mathbb{Q}$, let

$$\overline{\lambda}_\psi(x) := \limsup_{n\to\infty} \frac{1}{\psi(n)} \log |(R^n)'(x)|, \tag{1.12}$$

and let $\underline{\lambda}_\psi(x)$ be defined analogously by replacing the limit superior in (1.12) with the limit inferior.

For any $\alpha \in \mathbb{R}_+ \cup \{\pm\infty\}$, we call

$$\overline{F}_\psi(\alpha) := \dim_H \left\{x \in (0,1)\backslash\mathbb{Q} : \overline{\lambda}_\psi(x) = \alpha\right\} \quad \text{and} \quad \underline{F}_\psi(\alpha) := \dim_H \left\{x \in (0,1)\backslash\mathbb{Q} : \underline{\lambda}_\psi(x) = \alpha\right\} \tag{1.13}$$

the *upper* and *lower fast Lyapunov spectra*, respectively.

Let $b := b_\psi$ be given by

$$b = \liminf_{n\to\infty} \sqrt[n]{\psi(n)}. \tag{1.14}$$

Theorem B (Upper and lower fast Lyapunov spectra)**.** Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a function satisfying $\psi(n)/n \to \infty$ as $n \to \infty$. For any $0 < \alpha \le \infty$, we have

$$\overline{F}_\psi(\alpha) = \frac{1}{b+1} \quad \text{and} \quad \underline{F}_\psi(\alpha) = \frac{1}{B+1}. \tag{1.15}$$

In the view of (1.11) and (1.15), $\overline{F}_\psi(\alpha)$ (resp.$\underline{F}_\psi(\alpha)$) are discontinuous at $\alpha = 0$, whenever $b \neq 0$ (resp. $B \neq 0$). Additionally, it also follows from (1.15) that $\overline{F}_\psi(\alpha)$ and $\underline{F}_\psi(\alpha)$ are continuous at infinity and that

$$F_\psi(\infty) = \underline{F}_\psi(\infty) = \frac{1}{B+1}.$$

2. Preliminary

2.1 Notation

We follow the following conventions:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers (or positive integers);
- $\mathbb{Q}$ represents the set of rational numbers;
- $\mathbb{R}$ signifies the set of real numbers;
- $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ indicates the set of positive real numbers.

We use $\lfloor \cdot \rfloor$ to denote the integer part of a real number.

For a subset $A \subset \mathbb{R}$, we denote:

- $|A|$ or diam A as the diameter of A;

- $\#A$ as the cardinality of A;
- $\mathrm{int}(A)$ as the interior of A;
- $\overline{A}$ or $\mathrm{cl}(A)$ as the closure of A.

For a function $f : A \to \mathbb{R}$, a subset B of A, an element $x \in A$, and $n \in \mathbb{N}$, we denote:

- $f|_B$ as the restriction of f to B;
- $f'(x)$ as the derivative of f at $x \in A$;
- $f^0(x) \coloneqq x$ as the identity function;
- $f^n(x) \coloneqq \underbrace{f(f(\cdots f}_{n}(x))$ as the n-th iterate of f.

2.2 BCF and Cylinder Sets

For $n \geq 1$, denote

$$[\![b_1, b_2, \cdots, b_n]\!] \coloneqq 1 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \ddots - \frac{1}{b_n}}} \tag{2.1}$$

the (nth) *convergent*. Now use the notation

$$[\![b_1, b_2, \cdots]\!] = 1 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \ddots}} \tag{2.2}$$

for

$$\lim_{n\to\infty} [\![b_1, b_2, \cdots, b_n]\!], \tag{2.3}$$

a limit which always exists. Call $[\![b_1, b_2, \cdots]\!]$ the (infinite) *backward continued fraction (BCF) expansion*, and $[\![b_1, b_2, \cdots, b_n]\!]$ the (nth) *finite backward continued fraction (BCF) expansion* with respect to $\{b_n\}_{n\geq 1}$. Let $x \in [0, 1)$ be a real number, $b_n = b_n(x)$ defined in (1.6) be the partial quotient, and $[\![b_1, b_2, \cdots, b_n]\!]$ defined in (2.1) be the convergent. It follows from (2.1) that

$$[\![b_1, b_2, \cdots, b_{n+1}]\!] = 1 - \frac{1}{b_{n+1} - 1 + [\![b_1, b_2, \cdots, b_n]\!]}. \tag{2.4}$$

The formula above gives a inductive relation of convergent. Define two natural numbers p_n, q_n with $(p_n, q_n) = 1$ by

$$\frac{p_n}{q_n} := [\![b_1, b_2, \cdots, b_n]\!], \text{ for every } n \geq 1. \tag{2.5}$$

These two numbers p_n, q_n are obtained recursively from the following relation

$$\begin{pmatrix} p_n & -p_{n-1} \\ q_n & -q_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_n & -1 \\ 1 & 0 \end{pmatrix}, \text{ for every } n \geq 1. \tag{2.6}$$

In other words

$$p_{n+1} = b_{n+1}p_n - p_{n-1}, \;\; q_{n+1} = b_{n+1}q_n - q_{n-1}, \text{ for every } n \geq 1 \tag{2.7}$$

with a convention

$$p_0 = 0, q_0 = 1, p_1 = 1, q_1 = b_1. \tag{2.8}$$

By (1.6), (2.7) and induction, $\{q_n\}_{n\geq 1}$ is strictly increasing: $q_1 = b_1 \geq 2 > 1 = q_0$ and

$$q_{n+1} \geq 2q_n - q_{n-1} > 2q_n - q_n = q_n, \text{ for every } n \geq 1. \tag{2.9}$$

Taking determinant of the matrices on both side of (2.6) yields:

$$p_n q_{n-1} - p_{n-1} q_n = -1, \; \frac{p_n}{q_n} = \frac{p_{n-1}}{q_{n-1}} - \frac{1}{q_{n-1}q_n}, \text{ for every } n \geq 1. \tag{2.10}$$

For $x \in [0, 1)$, x is rational if and only if x admits a BCF expansion

$$x = [\![b_1(x), b_2(x), \cdots, b_n(x), 2, 2, \cdots]\!] \tag{2.11}$$

for some $n \in \mathbb{N}$. The values of two BCF expansions can be ordered in the sense of

$$[\![b_1, b_2, \cdots]\!] < [\![c_1, c_2, \cdots]\!] \text{ if and only if } b_m < c_m, \; m = \min\{j \in \mathbb{N} : b_j \neq c_j\}. \tag{2.12}$$

Next define the cylinder sets of BCF. Given a natural number n and the ordered indices

$(\sigma_1, \cdots, \sigma_n) \in \mathbb{N}_{\geq 2}^n$, call

$$I_n(\sigma_1, \cdots, \sigma_n) := \{x \in [0, 1) : b_i(x) = \sigma_i, \text{for all } i \leq n\} \tag{2.13}$$

a *cylinder set* of order n (associated with $(\sigma_1, \cdots, \sigma_n)$), or equivalently

$$I_n(\sigma_1, \cdots, \sigma_n) = \bigcap_{j=1}^{n} R^{-j+1} I_{\sigma_j}, \tag{2.14}$$

where $I_k = [1 - 1/(k-1), 1 - 1/k)$ for $k \geq 2$. Each cylinder set $I_n(\sigma_1, \cdots, \sigma_n)$ is a (left closed and right open) interval. Namely,

$$I_n(\sigma_1, \cdots, \sigma_n) = \Big[[\![\sigma_1, \sigma_2, \cdots, \sigma_n - 1]\!], [\![\sigma_1, \sigma_2, \cdots, \sigma_n]\!] \Big) = \left[\frac{\widehat{p}_n - \widehat{p}_{n-1}}{\widehat{q}_n - \widehat{q}_{n-1}}, \frac{\widehat{p}_n}{\widehat{q}_n} \right), \tag{2.15}$$

where $\widehat{p}_n, \widehat{q}_n \in \mathbb{N}$ with $(\widehat{p}_n, \widehat{q}_n) = 1$ are given by

$$\frac{\widehat{p}_n}{\widehat{q}_n} = [\![\sigma_1, \sigma_2, \cdots, \sigma_n]\!] = 1 - \cfrac{1}{\sigma_1 - \cfrac{1}{\sigma_2 - \cdots - \cfrac{1}{\sigma_n}}}. \tag{2.16}$$

Based on (2.10) and (2.15), it follows that

$$|I_n(\sigma_1, \cdots, \sigma_n)| = \frac{1}{\widehat{q}_n(\widehat{q}_n - \widehat{q}_{n-1})}. \tag{2.17}$$

The main proposition in this subsection is about the approximation of the length of cylinder sets and is stated as follows. As we didn't find the literature, we provide the details for the convenience of the reader.

Lemma 2.1. *Let $\{b_n\}_{n\geq 1}$ be a sequence with each $b_n \in \mathbb{N}_{\geq 2}$. Define $\{q_n\}_{n\geq 0}$ inductively by* (2.7) *and* (2.8)*. For each integer $n \geq 1$, the following assertions hold.*

i). $q_n \geq b_n q_{n-1}/2$.

ii). $q_n \geq (1 + 1/n) q_{n-1}$.

iii). $\prod_{k=1}^n b_k / 2^n \leq q_n \leq \prod_{k=1}^n b_k$.

iv).

$$\frac{1}{(b_1b_2\cdots b_n)^2}\le\frac{1}{q_n^2}\le\frac{1}{q_n(q_n-q_{n-1})}\le\frac{n+1}{q_n^2}\le\frac{(n+1)2^{2n}}{(b_1b_2\cdots b_n)^2}\le\frac{2^{3n}}{(b_1b_2\cdots b_n)^2}.$$

As a corollary of Lemma 2.1, we have

Proposition 2.2. *For any natural number* n *and all* $(\sigma_1,\cdots,\sigma_n)\in\mathbb{N}^n_{\ge 2}$

$$\frac{1}{(\sigma_1\sigma_2\cdots\sigma_n)^2}\le|I_n(\sigma_1,\cdots,\sigma_n)|\le\frac{2^{3n}}{(\sigma_1\sigma_2\cdots\sigma_n)^2}.$$

Proof of Lemma 2.1. We proceed the proof of Lemma 2.1 item by item.

For item $i)$, when $n=1$, it directly follows from (2.8). For the case $n\ge 2$, it follows from (2.7), (2.9) and (1.6) that

$$q_n=b_nq_{n-1}-q_{n-2}\ge b_nq_{n-1}-q_{n-1}\ge\frac{b_n}{2}q_{n-1},$$

as we want.

For item $ii)$, note that for each $n\in\mathbb{N}$,

$$[\![\underbrace{2,2,\cdots,2}_{n}]\!]=\frac{1}{n+1}. \tag{2.18}$$

The proof of this equality is by induction on n. For the case $n=1$, it follows from

$$[\![2]\!]=1-\frac{1}{2}.$$

Suppose the case for n is true, it follows from (2.4) that

$$[\![\underbrace{2,2,\cdots,2}_{n+1}]\!]=1-\frac{1}{2-1+[\![\underbrace{2,2,\cdots,2}_{n}]\!]}=1-\frac{1}{2-1+\frac{1}{n+1}}=\frac{1}{n+2}.$$

This yields the case $n+1$, and the induction is completed. Also, note that

$$1-\frac{q_{n-1}}{q_n}=[\![b_n,b_{n-1},\cdots,b_1]\!] \tag{2.19}$$

The proof of this equality is also by induction on n. For the case $n = 1$, it follows from

$$1 - \frac{q_0}{q_1} = 1 - \frac{1}{b_1} = [\![b_1]\!].$$

Suppose the case for n is true, then

$$\begin{aligned}
&1 - \frac{q_n}{q_{n+1}} \\
=&1 - \frac{q_n}{b_{n+1}q_n - q_{n-1}} && \text{(by (2.7))} \\
=&1 - \frac{1}{b_{n+1} - \frac{q_{n-1}}{q_n}} \\
=&1 - \frac{1}{b_{n+1} - 1 + [\![b_n, b_{n-1}, \cdots, b_1]\!]} && \text{(by induction hypothesis)} \\
=&[\![b_{n+1}, b_n, \cdots, b_1]\!]. && \text{(by (2.4))}
\end{aligned}$$

Thus the case for $n+1$ is also true. It follows from (2.19), (2.12) and (2.18) that

$$1 - \frac{q_{n-1}}{q_n} = [\![b_n, b_{n-1}, \cdots, b_1]\!] \geq [\![\underbrace{2, 2, \cdots, 2}_{n}]\!] = \frac{1}{n+1}.$$

In other words

$$q_n \geq (1 + \frac{1}{n})q_{n-1}.$$

as we want.

For item $iii)$, use item $i)$ and (2.7):

$$\frac{b_k}{2} q_{k-1} \leq q_k \leq b_k q_{k-1} \text{ for } k = 1, \cdots, n. \tag{2.20}$$

Note that by (2.8) $q_0 = 1$, taking product of each term in (2.20) from $k = 1$ to n yields

$$\frac{\prod_{k=1}^{n} b_k}{2^n} = \prod_{k=1}^{n} \frac{b_k}{2} q_{k-1} \leq \prod_{k=1}^{n} q_k \leq \prod_{k=1}^{n} b_k q_{k-1} = \prod_{k=1}^{n} b_k, \tag{2.21}$$

as we want.

For item $iv)$, the inequalities are directly derived from item $ii)$ and item $iii)$. Therefore, the proof of all assertions is completed. $\square$

2.3 Hausdorff dimension

First we recall some useful properties of Hausdorff dimension, then we introduce the main propositions calculating Hausdorff dimension for the lower bound: Proposition 3.1.

For a subset E of $\mathbb{R}$, denote

$$\mathcal{H}^\alpha_\delta(E) := \inf\{\sum_k (\operatorname{diam} F_k)^\alpha : E \subset \bigcup_k F_k,\ F_k \subset \mathbb{R},\ \operatorname{diam} F_k \leq \delta \text{ for all } k\}, \tag{2.22}$$

and such a cover $\bigcup_k F_k$ is called a *δ-cover* of E. Denote the *exterior α-dimensional Hausdorff measure* of E by

$$\mathcal{H}^\alpha(E) := \lim_{\delta\to 0} \mathcal{H}^\alpha_\delta(E). \tag{2.23}$$

Call

$$\dim_{\mathrm{H}}(E) := \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\}, \tag{2.24}$$

the *Hausdorff dimension* of E.

Hausdorff dimension satisfies monotonicity

$$A \subseteq B \text{ implies } \dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}(B), \tag{2.25}$$

and countable stability

$$\dim_{\mathrm{H}}(\bigcup_{n\geq 1} E_n) = \sup_{n\geq 1}\{\dim_{\mathrm{H}}(E_n)\}. \tag{2.26}$$

Upper bound of Hausdorff dimension can be detected by in the sense that $\mathcal{H}^\alpha(E)$

$$\dim_{\mathrm{H}}(E) \leq \alpha \text{ if and only if } \mathcal{H}^\alpha(E) < \infty. \tag{2.27}$$

Let us recall two useful lemmas from[1] calculating lower bound and upper bound of Hausdorff dimension.

Lemma 2.3 ([[1], Example 4.6: Lower bound]). *Let $[0,1] = \mathbb{E}_0 \supset \mathbb{E}_1 \supset \cdots$ be a decreasing sequence of sets and $\mathbb{E} = \bigcap_{n\geq 0} \mathbb{E}_n$. Assume that each $\mathbb{E}_n$ is a union of finite number of disjoint closed intervals (called basic intervals of order n) and each basic interval in $\mathbb{E}_{n-1}$*

contains m_n intervals of $\mathbb{E}_n$ which are seperated by gaps of length at least ε_n. If $m_n \geq 2$ and $\varepsilon_{n-1} > \varepsilon_n > 0$, then

$$\dim_{\mathrm{H}} \mathbb{E} \geq \liminf_{n\to\infty} \frac{\log(m_1 m_2 \cdots m_{n-1})}{-\log(m_n \varepsilon_n)}.$$

Lemma 2.4 ([[1], Proposition 4.1: Upper bound])**.** *Suppose $\mathbb{F}$ can be coverd by $\mathcal{N}_n$ sets of diameter at most δ_n with $\delta_n \to 0$ as $n \to \infty$. Then*

$$\dim_{\mathrm{H}} \mathbb{F} \leq \liminf_{n\to\infty} \frac{\log \mathcal{N}_n}{-\log \delta_n}.$$

3. The distribution of the digits

Now we introduce the main Proposition calculating lower bound of Hausdorff dimension. Let $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ be two sequences of reals numbers with $s_n, t_n \geq 2$ for every $n \geq 1$. Moreover, throughout this section, assume

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log s_k}{n} = \infty, \tag{3.1}$$

$$\inf_{n\geq 1} \frac{s_n}{t_n} > c > 0. \tag{3.2}$$

Write

$$\mathbb{E}(\{s_n\}\{t_n\}) \coloneqq \{x \in [0,1) : s_n < b_n(x) \leq s_n + t_n, \forall n \geq 1\}. \tag{3.3}$$

Proposition 3.1 (Hausdorff dimension of $\mathbb{E}(\{s_n\}\{t_n\})$)**.** *Assume that $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ satisfies* (3.1) *and* (3.2) *above. Then*

$$\dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\}) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \log t_k}{2\sum_{k=1}^{n+1} \log s_k - \log t_{n+1}}.$$

Remark. Proposition 3.1 will be used repeatedly since $\mathbb{E}(\{s_n\}\{t_n\})$ is a very common type of subset of several sets which we are interested in.

The proof of Proposition 3.1 is divided into two parts: the lower bound and the upper bound of $\dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\})$. With Lemma 2.3 and Lemma 2.4, we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let

$$\mathcal{C}_n := \{(\sigma_1, \cdots, \sigma_n) \in \mathbb{N}_{\geq 2}^n : s_j < \sigma_j \leq s_j + t_j, \forall 1 \leq j \leq n\}. \tag{3.4}$$

For $(\sigma_1, \cdots, \sigma_n) \in \mathcal{C}_n$, define the basic interval of order n as

$$J_n(\sigma_1, \cdots, \sigma_n) := \bigcup_{s_{n+1} < k \leq s_{n+1} + t_{n+1}} cl(I(\sigma_1, \cdots, \sigma_n, k)), \tag{3.5}$$

where $cl(\cdot)$ denotes the closure of a set. Write

$$\mathbb{F}_n := \bigcup_{(\sigma_1, \cdots, \sigma_n) \in \mathcal{C}_n} J_n(\sigma_1, \cdots, \sigma_n). \tag{3.6}$$

Then $\mathbb{E}(\{s_n\}\{t_n\}) = \bigcap_{n=1}^{\infty} \mathbb{F}_n$.

For the lower bound of $\dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\})$, by the structure of basic intervals in (3.5), we deduce that each basic interval of order $n-1$ contains

$$\frac{t_n}{2} < \lfloor t_n \rfloor \leq m_n := \lfloor s_n + t_n \rfloor - \lfloor s_n \rfloor < t_n + 1 < 2t_n \tag{3.7}$$

basic intervals of order n. Note that $J_n(\sigma_1, \cdots, \sigma_n)$ is a refinement of $I_n(\sigma_1, \cdots, \sigma_n)$ so that two different basic intervals has a non-trivial gap. We make it explicit by estimating the gaps between two basic intervals with the same order. Assume that $J_n(\sigma_1, \cdots, \sigma_n)$ and $J_n(\sigma_1^*, \cdots, \sigma_n^*)$ are two basic intervals in $\mathbb{F}_n$. Then they are separated by the cylinder of order $n+1$:

$$I_{n+1}(\sigma_1, \cdots, \sigma_n, 2) \quad \text{or} \quad I_{n+1}(\sigma_1^*, \cdots, \sigma_n^*, 2)$$

by (2.12). We may assume $J_n(\sigma_1, \cdots, \sigma_n)$ and $J_n(\sigma_1^*, \cdots, \sigma_n^*)$ is seperated by $I_{n+1}(\sigma_1, \cdots, \sigma_n, 2)$. The gap between these two basic intervals is at least

$$\begin{aligned} &|I_{n+1}(\sigma_1, \cdots, \sigma_n, 2)| \\ \geq &\frac{1}{(\sigma_1 \cdots \sigma_n 2)^2} && \big(\text{by Proposition 2.2}\big) \\ \geq &\frac{1}{4((1+\frac{1}{c})s_1 \cdot (1+\frac{1}{c})s_2 \cdots (1+\frac{1}{c})s_n)^2} && \big(\text{by (3.4) and (3.2)}\big) \end{aligned}$$

$$=\frac{1}{4(1+\frac{1}{c})^{2n}(s_1s_2\cdots s_n)^2} =: \varepsilon_n. \tag{3.8}$$

It follows that

$$\begin{aligned}
&\dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\}) \\
\geq &\liminf_{n\to\infty} \frac{\log(m_1m_2\cdots m_n)}{-\log(m_{n+1}\varepsilon_{n+1})} && \big(\text{by Lemma (2.3)}\big)\\
\geq &\liminf_{n\to\infty} \frac{\log(t_1t_2\cdots t_n) - n\log 2}{2\log(s_1s_2\cdots s_ns_{n+1}) - \log t_{n+1} + 3\log 2 + 2(n+1)\log(1+\frac{1}{c})} && \big(\text{by (3.7) and (3.8)}\big)\\
= &\liminf_{n\to\infty} \frac{\log(t_1t_2\cdots t_n)}{2\log(s_1s_2\cdots s_ns_{n+1}) - \log t_{n+1}}. && \big(\text{by (3.1)}\big)
\end{aligned}$$

For the upper bound of $\dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\})$, we see that for each $n \in \mathbb{N}$, $\dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\})$ is covered by $\mathbb{F}_n$, i.e., $\mathcal{N}_n \coloneqq \text{card}\, \mathcal{C}_n$ basic intervals of order n. Note that

$$\mathcal{N}_n < 2t_1 \cdot 2t_2 \cdots 2t_n = 2^n t_1t_2\cdots t_n \tag{3.9}$$

and

$$\begin{aligned}
&|J_n(\sigma_1,\cdots,\sigma_n)| \\
\leq &\frac{2^{3(n+1)}}{(\sigma_1\cdots\sigma_n)^2} \sum_{s_{n+1}<k\leq s_{n+1}+t_{n+1}} \frac{1}{k^2} && \big(\text{by Proposition 2.2 and (3.5)}\big)\\
\leq &\frac{2^{3(n+1)}}{(\sigma_1\cdots\sigma_n)^2} \sum_{s_{n+1}<k\leq s_{n+1}+t_{n+1}} \frac{1}{k(k-1)} \\
= &\frac{2^{3(n+1)}}{(\sigma_1\cdots\sigma_n)^2} \Big(\frac{1}{s_{n+1}} - \frac{1}{s_{n+1}+t_{n+1}}\Big) \\
< &\frac{2^{3(n+1)}}{(s_1\cdots s_n)^2} \Big(\frac{1}{s_{n+1}} - \frac{1}{s_{n+1}+t_{n+1}}\Big) && \big(\text{by (3.4)}\big)\\
\leq &\frac{2^{3(n+1)}t_{n+1}}{(s_1\cdots s_ns_{n+1})^2} =: \delta_n && (3.10)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\}) \\
\leq &\liminf_{n\to\infty} \frac{\log \mathcal{N}_n}{-\log \delta_n} && \big(\text{by Lemma } 2.4\big)\\
\leq &\liminf_{n\to\infty} \frac{n\log 2 + \log(t_1t_2\cdots t_n)}{2\log(s_1s_2\cdots s_ns_{n+1}) - \log t_{n+1} - (3n+3)\log 2} && \big(\text{by (3.9) and (3.10)}\big)
\end{aligned}$$

$$\leq \liminf_{n\to\infty} \frac{\log(t_1t_2\cdots t_n)}{2\log(s_1s_2\cdots s_ns_{n+1}) - \log t_{n+1}}. \qquad \big(\text{by (3.1)}\big)$$

And the proof of Proposition 3.1 is completed. □

Write

$$\mathbb{E}(\{e^n\},\{e^n\}) = \{x\in\Lambda : e^n < a_n(x) \leq 2e^n, \forall n\geq 1\}. \tag{3.11}$$

As a consequence of Proposition 3.1, we are able to determine its Hausdorff dimension.

Corollary 3.2.

$$\dim_{\mathrm{H}} \mathbb{E}(\{e^n\},\{e^n\}) = \frac{1}{2}.$$

We end this section by providing the Hausdorff dimension of the following set

Lemma 3.3. *Write*

$$\Pi_\infty := \{x\in[0,1) : \limsup_{n\to\infty} \frac{\log b_1(x)+\cdots+\log b_n(x)}{n} = \infty\}.$$

Then

$$\dim_{\mathrm{H}} \Pi_\infty = \frac{1}{2}.$$

Proof of Lemma 3.3. For the lower bound, since $\mathbb{E}(\{e^n\},\{e^n\})\subset\Pi_\infty$ (defined in (3.11)

$$\dim_{\mathrm{H}} \Pi_\infty \geq \dim_{\mathrm{H}} \mathbb{E}(\{e^n\},\{e^n\}) = \frac{1}{2}.$$

For the upper bound, let $0<\epsilon<\frac{1}{2}$ and $s:=\frac{1}{2}+\epsilon$. Choosing a sufficiently large number $K>1$ such that

$$K^\epsilon > 16J_\epsilon \text{ and } K>4 \tag{3.12}$$

where

$$J_\epsilon := \sum_{n=2}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty. \tag{3.13}$$

Observe that $\forall N\in\mathbb{N}$, Π_∞ is covered by

$$\bigcup_{n=N}^{\infty} \bigcup_{(\sigma_1,\cdots,\sigma_n)\in C_n(K)} I_n(\sigma_1,\cdots,\sigma_n), \tag{3.14}$$

where $C_n(K)$ is given by

$$C_n(K) \coloneqq \{(\sigma_1, \cdots, \sigma_n) \in \mathbb{N}^n : \sigma_1 \cdots \sigma_n \geq K^n\} \tag{3.15}$$

For any $\delta > 0$, by Proposition 2.2 and (3.12), there exists $M = \lfloor \log_2 \frac{1}{\delta} \rfloor + 1 > 0$, such that for all $n > M$

$$|I_n(\sigma_1, \cdots \sigma_n)| \leq \frac{2^{3n}}{(\sigma_1 \sigma_2 \cdots \sigma_n)^2} = \frac{2^{3n}}{K^{2n}} \leq (\frac{1}{2})^n \leq (\frac{1}{2})^M \leq \delta, \tag{3.16}$$

so when $N > M$, (3.14) is a δ-cover of Π_∞, and we estimate:

$$\begin{aligned}
&\mathcal{H}^s_\delta(\Pi_\infty) \\
\leq &\liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{(\sigma_1,\cdots,\sigma_n)\in C_n(K)} |I_n(\sigma_1, \cdots \sigma_n)|^s && \big(\text{by (3.16)}\big) \\
\leq &\liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{(\sigma_1,\cdots,\sigma_n)\in C_n(K)} \frac{2^{3n}}{(\sigma_1 \cdots \sigma_n)^{1+2\epsilon}} && \big(\text{by Proposition 2.2}\big) \\
\leq &\liminf_{N\to\infty} \sum_{n=N}^{\infty} \frac{2^{3n}}{K^{\epsilon n}} \sum_{(\sigma_1,\cdots,\sigma_n)\in C_n(K)} \frac{1}{(\sigma_1 \cdots \sigma_n)^{1+\epsilon}} && \big(\text{by (3.15)}\big) \\
\leq &\liminf_{N\to\infty} \sum_{n=N}^{\infty} (\frac{8J_\epsilon}{K^\epsilon})^n && \big(\text{by (3.13)}\big) \\
\leq &\liminf_{N\to\infty} \sum_{n=N}^{\infty} \frac{1}{2^n} = 0. && \big(\text{by (3.12)}\big)
\end{aligned}$$

Let $\delta \to 0^+$, there is $\mathcal{H}^s(E) = 0$, it follows from (2.27) that $\dim_{\mathrm{H}}(\Pi_\infty) \leq s$. Since ϵ is arbitrary, we obtain $\dim_{\mathrm{H}}(\Pi_\infty) \leq \frac{1}{2}$. The proof is completed. □

4. Transformation

Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a function satisfies $\psi(n)/n \to \infty$ as $n \to \infty$, let α be a real number, recall

$$J_\psi(\alpha) \coloneqq \{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} \frac{\log |(R^n)'(x)|}{\psi(n)} = \alpha\},$$

To prove Theorem 2.2, first reformulate $J_\psi(\alpha)$. Set

$$E_\psi(\alpha) = \{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} \frac{\sum_{k=1}^n \log b_k(x)}{\psi(n)} = \frac{\alpha}{2}\} \tag{4.1}$$

Denote

$$J := \bigcup_{n=1}^{\infty}\{x \in [0,1) : (R^n)'(x) \text{ does not exist}\} \tag{4.2}$$

then $x \in [0,1)\backslash J$ if and only if for any $n \geq 1$, $(R^n)'(x)$ exists.

Lemma 4.1. *$J = \mathbb{Q} \cap (0,1)$ and for $x \in [0,1)\backslash J$,*

$$0 \leq 2\sum_{k=1}^{n} \log b_k(x) - \log |(R^n)'(x)| \leq 2n \log 2, \tag{4.3}$$

Proof of Lemma 4.1. First note that $R'(x)$ dose not exist if and only if $x \in \{k/(k+1)\}_{k\geq 1}$. Let n be a natural number, by chain rule

$$(R^n)'(x) = R'(R^{n-1}(x))R'(R^{n-2}(x)) \cdots R'(x), \tag{4.4}$$

it follows that $(R^n)'(x)$ does not exists if and only if

$$\{x, R(x), \cdots, R^{n-1}(x)\} \bigcap \Big\{\frac{k}{k+1}\Big\}_{k\geq 1} \neq \emptyset. \tag{4.5}$$

Since $R([0,1)\backslash\mathbb{Q}) \subset [0,1)\backslash\mathbb{Q}$, it follows from (4.5) that $J \subset \mathbb{Q} \cap (0,1)$. On the other hand,

$$\Big\{\frac{k}{k+1}\Big\}_{k\geq 1} \subset J,$$

and

$$\Big\{\frac{2k-1}{2k+1}\Big\}_{k\geq 1} \subset R^{-1}(\frac{1}{2}) \subset \{x \in [0,1) : (R)'(x) \text{ does not exist}\} \subset J$$

So similarly

$$\Big\{\frac{3k-2}{3k+1}\Big\}_{k\geq 1} \subset R^{-1}(\frac{1}{3}) \subset R^{-2}(\frac{1}{2}) \subset \{x \in [0,1) : (R^2)'(x) \text{ does not exist}\} \subset J$$

It follows from

$$\mathbb{Q} \cap (0,1) = \bigcup \Big\{\frac{k}{k+1}\Big\}_{k\geq 1} \bigcup \Big\{\frac{2k-1}{2k+1}\Big\}_{k\geq 1} \bigcup \cdots \tag{4.6}$$

that $\mathbb{Q} \cap (0,1) \subset J$, so $J = \mathbb{Q} \cap (0,1)$.

For $x \in [0,1)\backslash J$ and $k \geq 1$,

$$\frac{1}{1-R^{k-1}(x)} < b_k(x) = \lfloor \frac{1}{1-R^{k-1}(x)} \rfloor + 1 \leq \frac{1}{1-R^{k-1}(x)} + 1 \leq \frac{2}{1-R^{k-1}(x)}. \quad (4.7)$$

Taking logarithm of each term in (4.7) yields

$$\log b_k(x) - \log 2 \leq -\log(1-R^{k-1}(x)) \leq \log b_k(x). \quad (4.8)$$

Calculation yields

$$R'(x) = \frac{1}{(1-x)^2},\ R'(R^k(x)) = \frac{1}{(1-R^k(x))^2}, \quad (4.9)$$

and then by (4.4)

$$\log |(R^n)'(x)| = -2\sum_{k=1}^{n} \log(1-R^{k-1}(x)). \quad (4.10)$$

By (4.8)

$$0 \leq 2\sum_{k=1}^{n} \log b_k(x) - \log |(R^n)'(x)| \leq 2n\log 2, \quad (4.11)$$

and the proof is completed. □

Corollary 4.2. *For all $\alpha \in \mathbb{R}$, $J_\psi(\alpha) = E_\psi(\alpha)$, and if $\alpha < 0$ $E_\psi(\alpha)$ and $J_\psi(\alpha)$ are both empty.*

Proof of Corollary 4.2. By (4.11), for $x \in [0,1)\backslash J$

$$2\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log b_k(x)}{\psi(n)} = \lim_{n\to\infty} \frac{\log |(R^n)'(x)|}{\psi(n)} \quad (4.12)$$

Then $J_\psi(\alpha) = E_\psi(\alpha)$.

Since for every $n \in \mathbb{N}$, $b_n \geq 2$, then for every $x \in [0,1)$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log b_k(x)}{\psi(n)} \geq 0.$$

□

Lemma 4.3. *Assume $\alpha > 0$, then $E_\psi(\alpha) = F_\psi(\alpha)$ is non-empty if and only if ψ is equivalent* *to an increasing function.*

Proof of Lemma 4.3. For the "only if" part, we assume that $E_\psi(\alpha)$ is non-empty. Then we can take $x \in E_\psi(\alpha)$ and define $\varphi : \mathbb{N} \to \mathbb{R}_+$ as $\varphi(n) := \frac{\log b_1(x)+\cdots+\log b_n(x)}{\alpha}$ Hence $\varphi(n+1) > \varphi(n)$ and

$$\lim_{n\to\infty} \frac{\varphi(x)}{\psi(n)} = \lim_{n\to\infty} \frac{\log b_1(x) + \cdots + \log b_n(x)}{\alpha\psi(n)} = 1$$

which means that ψ is equivalent to the increasing function φ.

For the "if" part, we suppose that ψ is equivalent to an increasing function $\widehat{\varphi}$. Then

$$\lim_{n\to\infty} \frac{\widehat{\varphi}(n)}{n} = \lim_{n\to\infty} \frac{\psi(n)}{n} \lim_{n\to\infty} \frac{\widehat{\varphi}(n)}{\psi(n)} = \infty$$

Put

$$\widehat{x} = [\![\widehat{b_1}, \widehat{b_2}, \cdots]\!] = 1 - \cfrac{1}{\widehat{b_1} - \cfrac{1}{\widehat{b_2} - \cdots}}$$

where $\widehat{b_n} = b_n(\widehat{x}) := \lfloor e^{\alpha(\widehat{\varphi}(n)-\widehat{\varphi}(n-1))+1} \rfloor$. We deduce that $\widehat{b_n} \geq 2$ and

$$e^{\alpha(\widehat{\varphi}(n)-\widehat{\varphi}(n-1))} \leq \lfloor e^{\alpha(\widehat{\varphi}(n)-\widehat{\varphi}(n-1))+1} \rfloor = b_n(\widehat{x}) \leq e^{\alpha(\widehat{\varphi}(n)-\widehat{\varphi}(n-1))+1} \tag{4.13}$$

for all $n \geq 1$, since $\alpha \geq 0$. Hence by (4.13)

$$\frac{\widehat{\varphi}(n) - \widehat{\varphi}(0)}{\widehat{\varphi}(n)}\alpha \leq \frac{\log b_1(\widehat{x}) + \cdots + \log b_n(\widehat{x})}{\widehat{\varphi}(n)} \leq \frac{\widehat{\varphi}(n) - \widehat{\varphi}(0) + n}{\widehat{\varphi}(n)}\alpha,$$

and

$$\lim_{n\to\infty} \frac{\log b_1(\widehat{x}) + \cdots + \log b_n(\widehat{x})}{\psi(n)} = \lim_{n\to\infty} \frac{\log b_1(\widehat{x}) + \cdots + \log b_n(\widehat{x})}{\widehat{\varphi}(n)} = \alpha$$

which implies that $\widehat{x} \in E_\psi(\alpha)$ and $E_\psi(\alpha)$ is non-empty. □

We also have the following corollary for regular Lyapunov spectrum at ∞.

Corollary 4.4.

$$J(\infty) = \Big\{ x \in [0,1) : \lim_{n\to\infty} \frac{\sum_{k=1}^n \log b_k(x)}{n} = \infty \Big\},$$

with $L(\infty) = 1/2$, so $L(\alpha)$ is continuous at ∞.

Proof of Corollary 4.4. By Lemma 4.1, for $x \in [0,1)\backslash J$, there is

$$0 \le \frac{2\sum_{k=1}^{n} \log b_k(x) - \log |(R^n)'(x)|}{n} \le 2\log 2.$$

By (2.11), for every $x \in J$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log b_k(x)}{n} \neq \infty.$$

Thus

$$J(\infty) = \Big\{x \in [0,1) : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log b_k(x)}{n} = \infty\Big\}.$$

Since $\mathbb{E}(\{e^n\},\{e^n\}) \subset J(\infty) \subset \Pi_\infty$, by Lemma 3.3,

$$\frac{1}{2} \le \dim_{\mathrm{H}} \mathbb{E}(\{e^n\},\{e^n\}) \le L(\infty) \le \dim_{\mathrm{H}} \Pi_\infty \le \frac{1}{2}.$$

Then $L(\infty) = 1/2$. □

5. Proof of Theorem A

For the case $\alpha = 0$, since

$$\{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log b_k(x)}{n} = \log 2\} \subset \{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log b_k(x)}{\psi(n)} = 0\}, \tag{5.1}$$

we deduce from[19] that $E_\psi(0)$ is of full Lebesgue and by Corollary 4.2 $F_\psi(0) = 1$.

Deducing from Corollary 4.2 and Lemma 4.3, since the case $\alpha = 0$ is clear, without loss of generality from now on, we assume that $\alpha > 0$ and ψ is increasing.

5.1 Lower bound

For $\alpha \in (0,\infty)$, the control of lower bound of $\dim_{\mathrm{H}} E_\psi(\alpha)$ is an application of Proposition 3.1. Let $s_1 = t_1 = e^{\alpha\psi(1)+1}$ and $s_n = t_n = e^{\alpha(\psi(n)-\psi(n-1))+1}$ for every $n \ge 2$. Since ψ is increasing, we see that for all $n \ge 1$, $s_n = t_n \ge 2$, the limit in (3.1) and (3.2) holds and

$$\frac{\widehat{\varphi}(n) + n}{\widehat{\varphi}(n)}\alpha \le \frac{\log b_1(\widehat{x}) + \cdots + \log b_n(\widehat{x})}{\widehat{\varphi}(n)} \le \frac{\widehat{\varphi}(n) + n(1+\log 2)}{\widehat{\varphi}(n)}\alpha,$$

then $\mathbb{E}(\{s_n\}\{t_n\})$ is a subset of $E_\psi(\alpha)$, and

$$
\begin{aligned}
&\dim_{\mathrm{H}} E_\psi(\alpha)\\
\geq &\dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\})\\
= &\liminf_{n\to\infty} \frac{\sum_{k=1}^{n}\log t_k}{2\sum_{k=1}^{n+1}\log s_k - \log t_{n+1}} && \big(\text{by Proposition } 3.1\big)\\
= &\liminf_{n\to\infty} \frac{n+\alpha\psi(n)}{\alpha(\psi(n+1)+\psi(n))+2n+1}\\
= &\frac{1}{\limsup_{n\to\infty}\frac{\psi(n+1)}{\psi(n)}+1}\\
= &\frac{1}{\beta+1} && \big(\text{by (3.5)}\big) && (5.2)
\end{aligned}
$$

5.2 Upper bound

First enlarge $E_\psi(\alpha)$ to a set whose Hausdorff dimension can be estimated easier.

For $x \in E_\psi(\alpha)$

$$
\limsup_{n\to\infty} \frac{\log b_1(x)+\cdots+\log b_n(x)+\log b_{n+1}(x)}{\log b_1(x)+\cdots+\log b_n(x)} = \limsup_{n\to\infty}\frac{\psi(n+1)}{\psi(n)} = \beta,
$$

which is equivalent to

$$
\tau(x) := \limsup_{n\to\infty} \frac{\log b_{n+1}(x)}{\log b_1(x)+\cdots+\log b_n(x)} = \beta - 1 = \gamma. \tag{5.3}
$$

Besides, we also deduce that

$$
\mathrm{k}(x) = \lim_{n\to\infty} \frac{\log b_1(x)+\cdots+\log b_n(x)}{n} = \infty \tag{5.4}
$$

since $\lim_{n\to\infty}\frac{\psi(n)}{n} = \infty$.

In fact, it follows that the Hausdorff dimension of

$$
\{x\in(0,1)\backslash\mathbb{Q} : \tau(x)=\beta-1\}\bigcap\{x\in(0,1)\backslash\mathbb{Q} : \mathrm{k}(x)=\infty\}
$$

approximates the desired upper bound of $\dim_{\mathrm{H}} E_\psi(\alpha)$.

Lemma 5.1. *For* $0 \le \gamma \le \infty$*, let*

$$\Gamma_\infty(\gamma) \coloneqq \{x \in (0,1)\backslash\mathbb{Q} : \tau(x) = \gamma, \mathrm{k}(x) = \infty\}. \tag{5.5}$$

and

$$\widehat{\Gamma}_\infty(\gamma) \coloneqq \{x \in (0,1)\backslash\mathbb{Q} : \tau(x) \ge \gamma, \mathrm{k}(x) = \infty\}. \tag{5.6}$$

Then

$$\dim_{\mathrm{H}} \Gamma_\infty(\gamma) = \dim_{\mathrm{H}} \widehat{\Gamma}_\infty(\gamma) = \frac{1}{\gamma+2}.$$

Proof of Lemma 5.1. We only need to approximate the lower bound of $\dim_{\mathrm{H}} \Gamma_\infty(\gamma)$ and the upper bound for $\dim_{\mathrm{H}} \widehat{\Gamma}_\infty(\gamma)$ since $\Gamma_\infty(\gamma) \subset \widehat{\Gamma}_\infty(\gamma)$. The proof is divided into three cases: $\gamma = 0, 0 < \gamma < \infty$ and $\gamma = \infty$.

$Case$ I : $\gamma = 0$. In this case, for the upper bound it follows from Lemma 3.3 that

$$\dim_{\mathrm{H}} \widehat{\Gamma}_\infty(\gamma) \le \dim_{\mathrm{H}}\{x \in (0,1)\backslash\mathbb{Q} : \limsup_{n\to\infty} \frac{\log b_1(x) + \cdots + \log b_n(x)}{n} = \infty\} = \dim_{\mathrm{H}} \Pi_\infty = \frac{1}{2}. \tag{5.7}$$

For the lower bound, by Proposition 3.1, we see

$$\dim_{\mathrm{H}} \Gamma_\infty(\gamma) \ge \dim_{\mathrm{H}}\{x \in (0,1)\backslash\mathbb{Q} : e^n < b_n(x) < 2e^n, \forall n \ge 1\} = \frac{1}{2}.$$

as the limits in (3.1) (3.2) (5.3) and (5.4) holds.

$Case$ II : $0 < \gamma < \infty$. We use Proposition 3.1 again to control the lower bound.

$$\dim_{\mathrm{H}} \Gamma_\infty(\gamma) \ge \dim_{\mathrm{H}}\{x \in (0,1)\backslash\mathbb{Q} : e^{(\gamma+1)^n} < b_n(x) < 2e^{(\gamma+1)^n}, \forall n \ge 1\} = \frac{1}{\gamma+2}.$$

For the upper bound of $\dim_{\mathrm{H}} \widehat{\Gamma}_\infty(\gamma)$, we put the following covering argument. Let $0 < \epsilon < \frac{\gamma}{2}$ and $s \coloneqq \frac{1+\epsilon}{\gamma-2\epsilon+2}$. Choose $M \ge 2$ sufficiently large such that

$$J_\epsilon(\frac{64}{M^\epsilon})^s \le \frac{1}{2}, \text{ and } M^\epsilon > 64, \tag{5.8}$$

where $J_\epsilon := \sum_{n=2}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty$. Observe that

$$\widehat{\Gamma}_\infty(\gamma) \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_n(\epsilon, M), \tag{5.9}$$

where the set $B_n(\epsilon, M)$ is given by

$$B_n(\epsilon, M) := \Big\{x \in (0,1)\backslash\mathbb{Q} : d_{n+1} > \big(\prod_{k=1}^{n} d_k(x)\big)^{\gamma-\epsilon}, \prod_{k=1}^{n} d_k(x) \geq m\Big\}. \tag{5.10}$$

Let $\mathcal{D}_n(M) := \{(\sigma_1, \cdots, \sigma_n) \in \mathrm{N}_{\geq 2}^n : \sigma_1 \cdots \sigma_n \geq M^n\}$. For $(\sigma_1, \cdots, \sigma_n) \in \mathcal{D}_n(M)$, put

$$J_n(\sigma_1, \cdots, \sigma_n) := \bigcup_{k > (\sigma_1 \cdots \sigma_n)^{\gamma-\epsilon}} I_{n+1}(\sigma_1, \cdots, \sigma_n, k). \tag{5.11}$$

Then $B_n(\epsilon, M)$ can be rewritten as

$$B_n(\epsilon, M) = \bigcup_{(\sigma_1, \cdots, \sigma_n) \in \mathcal{D}_n(M)} J_n(\sigma_1, \cdots, \sigma_n). \tag{5.12}$$

Combining this with (5.9), we see that for any $N \geq 1$, the set $\widehat{\Gamma}_\infty(\gamma)$ is covered by

$$\{J_n(\sigma_1, \cdots, \sigma_n) : n \geq N, (\sigma_1, \cdots, \sigma_n) \in \mathcal{D}_n(M)\}.$$

By Proposition 2.2, we deduce that

$$\begin{aligned}
&|J_n(\sigma_1, \cdots, \sigma_n)| \\
=&\sum_{k > (\sigma_1 \cdots \sigma_n)^{\gamma-\epsilon}} I_{n+1}(\sigma_1, \cdots, \sigma_n, k) = \frac{2^{3(n+1)}}{(\sigma_1 \cdots \sigma_n)^2} \sum_{k > (\sigma_1 \cdots \sigma_n)^{\gamma-\epsilon}} \frac{1}{k^2} \\
\leq&\frac{2^{6n}}{(\sigma_1 \cdots \sigma_n)^{\gamma-\epsilon+2}} \leq \big(\frac{64}{M^\epsilon}\big)^n \frac{1}{(\sigma_1 \cdots \sigma_n)^{\gamma-2\epsilon+2}}.
\end{aligned}$$

Now for any $\delta > 0$, there exists d such that for all $n > d$

$$|J_n(\sigma_1, \cdots, \sigma_n)| \leq \big(\frac{64}{M^\epsilon}\big)^n \frac{1}{(\sigma_1 \cdots \sigma_n)^{\gamma-2\epsilon+2}} < \frac{1}{(\sigma_1 \cdots \sigma_n)^{\gamma-2\epsilon+2}} < \delta.$$

by (5.8). We conclude that

$$\mathcal{H}_\delta^s(\widehat{\Gamma}_\infty(\gamma))$$

$$
\begin{aligned}
&\le \liminf_{N\to\infty}\sum_{n=N}^{\infty}\sum_{(\sigma_1,\cdots,\sigma_n)\in\mathcal{D}_n(M)}|J_n(\sigma_1,\cdots,\sigma_n)|^s\\
&\le \liminf_{N\to\infty}\sum_{n=N}^{\infty}\sum_{(\sigma_1,\cdots,\sigma_n)\in\mathcal{D}_n(M)}(\frac{64}{M^\epsilon})^{ns}\frac{1}{(\sigma_1\cdots\sigma_n)^{s(\gamma-2\epsilon+2)}}\\
&\le \liminf_{N\to\infty}\sum_{n=N}^{\infty}(\frac{64}{M^\epsilon})^{ns}\sum_{(\sigma_1,\cdots,\sigma_n)\in\mathbb{N}_{\ge 2}^n}\frac{1}{(\sigma_1\cdots\sigma_n)^{s(\gamma-2\epsilon+2)}}\\
&= \liminf_{N\to\infty}\sum_{n=N}^{\infty}\big(J_\epsilon(\frac{64}{M^\epsilon})^s\big)^n\\
&\le \liminf_{N\to\infty}\sum_{n=N}^{\infty}\frac{1}{2^n}=0,
\end{aligned}
$$

which yields that

$$\dim_{\mathrm{H}}\widehat{\Gamma}_\infty(\gamma)\le s=\frac{1+\epsilon}{\gamma-2\epsilon+2}.$$

Letting $\epsilon\to 0^+$, we obtain that $\dim_{\mathrm{H}}\widehat{\Gamma}_\infty(\gamma)\le s=\frac{1}{\gamma+2}$, as we want.

Case III : $\gamma=\infty$. In this case, we will show that $\dim_{\mathrm{H}}\widehat{\Gamma}_\infty(\infty)=0$. In fact, for $0<\epsilon<1$, let $B>1$ large enough such that

$$P_\epsilon := \sum_{j=2}^{\infty}\frac{1}{j^{(B+2)\epsilon}}<\frac{1}{2^{7\epsilon+1}}. \tag{5.13}$$

Note that

$$\widehat{\Gamma}_\infty(\infty)\subset\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{(\sigma_1,\cdots,\sigma_n)\in\mathbb{N}_{\ge 2}^n}J_n'(\sigma_1,\cdots,\sigma_n), \tag{5.14}$$

where $J_n'(\sigma_1,\cdots,\sigma_n)$ is defined as

$$J_n'(\sigma_1,\cdots,\sigma_n):=\bigcup_{k>(\sigma_1\cdots\sigma_n)^B}I_{n+1}(\sigma_1,\cdots,\sigma_n,k). \tag{5.15}$$

Then

$$|J_n'(\sigma_1,\cdots,\sigma_n)|=\sum_{k>(\sigma_1\cdots\sigma_n)^B}|I_n(\sigma_1,\cdots,\sigma_n,k)|\le\frac{2^{3n+4}}{(\sigma_1\cdots\sigma_n)^{B+2}},$$

and also

$$|J_n'(\sigma_1,\cdots,\sigma_n)|\le\frac{1}{2^{n(B-1)-4}}.$$

By (5.14), we see that for any $\delta > 0$

$$
\begin{aligned}
&\mathcal{H}^{\epsilon}_{\delta}(\widehat{\Gamma}_{\infty}(\infty)) \\
&\leq \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{(\sigma_1,\cdots,\sigma_n)\in\mathbb{N}^n_{\geq 2}} |J'_n(\sigma_1,\cdots,\sigma_n)|^{\epsilon} \\
&\leq \liminf_{N\to\infty} \sum_{n=N}^{\infty} 2^{7n\epsilon} \sum_{(\sigma_1,\cdots,\sigma_n)\in\mathbb{N}^n_{\geq 2}} \frac{1}{(\sigma_1\cdots\sigma_n)^{(B+2)\epsilon}} \\
&= \liminf_{N\to\infty} \sum_{n=N}^{\infty} (2^{7\epsilon} P_{\epsilon})^n \\
&\leq \liminf_{N\to\infty} \sum_{n=N}^{\infty} \frac{1}{2^n} = 0,
\end{aligned}
$$

which implies that $\dim_{\mathrm{H}} \widehat{\Gamma}_{\infty}(\infty) \leq \epsilon$. Due to arbitrariness of ϵ, $\dim_{\mathrm{H}} \widehat{\Gamma}_{\infty}(\infty) = \dim_{\mathrm{H}} \Gamma_{\infty}(\infty) = 0$.

□

Lemma 5.1 has the following corollary. Consider the set of all real numbers x in $[0, 1)$ whose partial quotients $\{b_n(x)\}_{n\geq 1}$ tends to infinity as n tends to infinity:

$$
\{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} b_n(x) = \infty\}. \tag{5.16}
$$

Corollary 5.2.

$$
\dim_{\mathrm{H}}\{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} b_n(x) = \infty\} = \frac{1}{2}.
$$

Proof of Corollary 5.2. By setting $\gamma = 0$ in Lemma 5.1, it follows that

$$
\dim_{\mathrm{H}}\{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} b_n(x) = \infty\} \leq \dim_{\mathrm{H}} \Gamma_{\infty}(\gamma) = \frac{1}{2}.
$$

Consider the set $\mathbb{E}(\{e^n\}, \{e^n\})$ in (3.11) then

$$
\dim_{\mathrm{H}}\{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} b_n(x) = \infty\} \geq \dim_{\mathrm{H}} \mathbb{E}(\{e^n\}, \{e^n\}) = \frac{1}{2}.
$$

□

By Lemma 5.1, we obtain

$$\dim_{\mathrm{H}} E_{\psi}(\alpha) \le \dim_{\mathrm{H}} \Gamma_{\infty}(\beta - 1) = \frac{1}{\beta + 1}. \tag{5.17}$$

With the desired lower bounded in (5.2) and (5.17), the proof of Theorem A is completed.

We have the following corollary as a consequence of Theorem A, which gives a full description for the growth rate of digits of the BCF expansion.

Corollary 5.3 (Growth of digits in BCF expansion)**.** *Let $\phi : \mathbb{N} \to \mathbb{R}_+$ be a function such that* $\lim_{n\to\infty} \phi(n) = \infty$*. Then*

$$\dim_{\mathrm{H}}\{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} \frac{\log b_n}{\phi(n)} = 1\} = \frac{1}{2+\xi},$$

where ξ is defined as

$$\xi := \limsup_{n\to\infty} \frac{\phi(n+1)}{\phi(1) + \cdots + \phi(n)}.$$

Proof of Corollary 5.3. The lower bound is obtained by letting $s_n = t_n = 2e^{\phi(n)}$ in Proposition 3.1. For the upper bound, let $\widehat{\psi}(n) := \sum_{k=1}^{n} \phi(k)$. Then $\widehat{\psi}$ is increasing and $\widehat{\psi}(n)/n \to \infty$ as $n \to \infty$. Moreover,

$$\lim_{n\to\infty} \frac{\log b_n(x)}{\phi(n)} = 1 \Longrightarrow \lim_{n\to\infty} \frac{\log b_1(x) + \cdots + \log b_n(x)}{\widehat{\psi}(n)} = 1.$$

Hence

$$\dim_{\mathrm{H}}\left\{x \in (0,1)\backslash\mathbb{Q} : \lim_{n\to\infty} \frac{\log b_n(x)}{\phi(n)} = 1\right\} \le \dim_{\mathrm{H}} E_{\widehat{\psi}}(1) = \frac{1}{\widehat{\beta}+1},$$

where $\widehat{\beta}$ is given by

$$\widehat{\beta} = \limsup_{n\to\infty} \frac{\widehat{\psi}(n+1)}{\widehat{\psi}(n)} = 1 + \limsup_{n\to\infty} \frac{\phi(n+1)}{\phi(1) + \cdots \phi(n)} = 1 + \xi.$$

Therefore, the desired upper bound follows.

$\square$

6. Proof of Theorem B

Before proving Theorem B, we first give several useful lemmas. Write $\Pi_n(x) := b_1(x)\cdots b_n(x)$, let $a, c \in (1,\infty)$. Set

$$\overline{D}(a,c) := \{x \in [0,1) : \Pi_n(x) \geq a^{c^n} \text{ for i.m. } n \geq 1\}$$

and

$$\underline{D}(a,c) := \{x \in [0,1) : \Pi_n(x) \geq a^{c^n} \text{ for all } n \gg 1\},$$

where “i.m.” denotes “infinitely many”.

Lemma 6.1. *For any $d \in (1,c)$, if $x \in \overline{D}(a,c)$, then*

$$\Pi_{n+1}(x) > \max\{(\Pi_n(x))^d, a^{d^{n+1}}\} \text{ for i.m. } n \geq 1. \tag{6.1}$$

Proof of Lemma 6.1. Given any $m \in \mathbb{N}$, since $d < c$ and $x \in \overline{D}(a,c)$, we can find $k > m$ such that

$$\Pi_m(x) < a^{c^k d^{m-k}} \text{ and } \Pi_k(x) > a^{c^k}.$$

Define $f(n) := a^{c^k d^{n-k}}$, then

$$\Pi_m(x) < f(m),\ \Pi_k(X) > a^{c^k} = f(k).$$

Choose the largest n such that $m \leq n < k$ and $\Pi_n(x) < f(n)$. There are two cases: $n = k-1$ if $\Pi_i(x) < f(i)$, for all i with $m \leq i < k$, or $n < k-1$ otherwise. In either case, we have

$$\Pi_{n+1}(x) \geq f(n+1) = (f(n))^d > \max\{(\Pi_n(x))^d, a^{d^{n+1}}\}.$$

□

Lemma 6.2.

$$\dim_{\mathrm{H}} \underline{D}(a,c) = \dim_{\mathrm{H}} \overline{D}(a,c) = \frac{1}{c+1}.$$

Proof of Lemma 6.2. The lower bound of $\dim_{\mathrm{H}} \underline{D}(a,c)$ is obtain by setting $s_n = t_n = a^{c^n}$ in

Proposition 3.1:

$$\dim_{\mathrm{H}} \underline{D}(a,c) \geq \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \log t_k}{2\sum_{k=1}^{n+1} \log s_k - \log t_{n+1}} = \frac{1}{c+1}.$$

As for the upper bound, we put here an covering argument. Fix $d \in (1,c)$ and $s \in (0,1)$, for any $x \in \overline{D}(a,c)$, and for i.m. $n \geq 1$, by Lemma 6.1

$$\Pi_{n+1}(x) \geq \max\{(\Pi_n(x))^d, a^{d^{n+1}}\} \geq (\Pi_n(x))^{sd} c^{(1-s)d^{n+1}}.$$

Then $\overline{D}(a,c)$ is covered by the limsup set

$$\overline{D}(a,c) \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ x \in [0,1) : b_{n+1}(x) \geq (\Pi_n(x))^{sd-1} c^{(1-s)d^{n+1}} \right\} \tag{6.2}$$

Write

$$J_n(\sigma_1, \cdots, \sigma_n) = \bigcup_{j \geq (\sigma_1 \cdots \sigma_n)^{sd-1} c^{(1-s)d^{n+1}}} I_{n+1}(\sigma_1, \cdots, \sigma_n, j)$$

Then

$$\overline{D}(a,c) \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(\sigma_1,\cdots,\sigma_n)\in\mathbb{N}^n}^{\infty} J_n(\sigma_1, \cdots, \sigma_n)$$

which means that for every $N \geq 1$, $\overline{D}(a,c)$ is covered by

$$\{J_n(\sigma_1, \cdots, \sigma_n) : (\sigma_1, \cdots, \sigma_n) \in \mathbb{N}^n, n \geq N\}.$$

Note that

$$\begin{aligned}
&|J_n(\sigma_1, \cdots, \sigma_n)| \\
=& \sum_{j \geq (\sigma_1 \cdots \sigma_n)^{sd-1} c^{(1-s)d^{n+1}}} |I_{n+1}(\sigma_1, \cdots, \sigma_n, j)| \\
\leq& \sum_{j \geq (\sigma_1 \cdots \sigma_n)^{sd-1} c^{(1-s)d^{n+1}}} \frac{2^{3(n+1)}}{(\sigma_1 \cdots \sigma_n)^2} \frac{1}{j^2} \\
\leq& \frac{2^{3(n+1)}}{(\sigma_1 \cdots \sigma_n)^2} \frac{2}{(\sigma_1 \cdots \sigma_n)^{sd-1} c^{(1-s)d^{n+1}}} = \frac{2^{3n+4}}{(\sigma_1 \cdots \sigma_n)^{sd+1} c^{(1-s)d^{n+1}}}.
\end{aligned}$$

Now for any $\epsilon < 1$ sufficiently small, take $t = \frac{1+\epsilon}{sd+1} > 0$, then for any $\delta > 0$, there exists

$M \geq 1$ such that

$$|J_n(\sigma_1, \cdots, \sigma_n)| < \delta \text{ for all } n > M \text{ and } (\sigma_1, \cdots, \sigma_n) \in \mathbb{N}^n.$$

We estimate:

$$\begin{aligned}
&\mathcal{H}^t_\delta(\overline{D}(a,c)) \\
\leq &\liminf_{n\to\infty} \sum_{n=N}^{\infty} \sum_{(\sigma_1,\cdots,\sigma_n)\in\mathbb{N}^n, n\geq N} |J_n(\sigma_1,\cdots,\sigma_n)|^t \\
\leq &\liminf_{n\to\infty} \sum_{n=N}^{\infty} \sum_{(\sigma_1,\cdots,\sigma_n)\in\mathbb{N}^n, n\geq N} \frac{2^{(3n+4)t}}{(\sigma_1\cdots\sigma_n)^{(sd+1)t} c^{(1-s)td^{n+1}}} \\
= &\liminf_{n\to\infty} \sum_{n=N}^{\infty} \frac{J_\epsilon^n 2^{(3n+4)t}}{c^{(1-s)td^{n+1}}} = 0
\end{aligned}$$

Let $\delta \to 0^+$, it follows that $\mathcal{H}^t(\overline{D}(a,c)) = 0$, then $\dim_{\mathrm{H}} \overline{D}(a,c) \leq t = \frac{1+\epsilon}{sd+1}$. Next let $\epsilon \to 0^+$, $s \to 1^-$, then $\dim_{\mathrm{H}} \overline{D}(a,c) \leq \frac{1}{d+1}$. Finally let $d \to c^-$, then $\dim_{\mathrm{H}} \overline{D}(a,c) \leq \frac{1}{c+1}$. □

Remark. Together with Lemma 6.2, since

$$\{x \in [0,1) : b_n(x) \geq a^{c^n} \text{ for i.m. } n \geq 1\} \subset \{x \in [0,1) : \Pi_n(x) \geq a^{c^n} \text{ for i.m. } n \geq 1\},$$

we also deduced:

$$\dim_{\mathrm{H}}\{x \in [0,1) : b_n(x) \geq a^{c^n} \text{ for all } n \geq 1\} = \dim_{\mathrm{H}}\{x \in [0,1) : b_n(x) \geq a^{c^n} \text{ for i.m. } n \geq 1\} = \frac{1}{c+1}.$$

Now let us prove Theorem B. When $\alpha = 0$, since

$$E_\psi(\alpha) \subset \underline{E}_\psi(\alpha) \subset \overline{E}_\psi(\alpha),$$

and (5.1), it follows that

$$\overline{F}_\psi(0) = \underline{F}_\psi(0) = 1.$$

Now assume $0 < \alpha \leq \infty$. Also since $\log b = \liminf_{n\to\infty} \log \psi(n)/n \geq 0$, we may assume $b \geq 1$.

6.1 Hausdorff dimension of $\overline{F}_\psi(\alpha)$

6.1.1 Upper bound

For $x \in \overline{E}_{\psi}(\alpha)$, when $\alpha \in (0, \infty)$, we see that $\Pi_n(x) \geq e^{\alpha\psi(n)/3}$ holds for infinitely many n; when $\alpha = \infty$, we see that $\Pi_n(x) \geq e^{\alpha\psi(n)}$ holds for infinitely many n. So

$$\overline{E}_{\psi}(\alpha) \subset \{x \in [0, 1) : \Pi_n(x) \geq A^{\psi(n)} \text{ for i.m. } n \geq 1\} \tag{6.3}$$

for some $A > 1$. This leads to study the Hausdorff dimension of the limsup set.

Lemma 6.3. *Let* $A \in (1, \infty)$. *Write*

$$\overline{F}(\psi) := \{x \in [0, 1) : \Pi_n(x) \geq A^{\psi(n)} \text{ for i.m. } n \geq 1\}. \tag{6.4}$$

Then

$$\dim_{\mathrm{H}} \overline{F}(\psi) = \frac{1}{b+1},$$

where $b \in [1, \infty]$ *is defined as in Theorem B.*

Proof of Lemma 6.3. The proof is divided into three parts: $b = 1, 1 < b < \infty, b = \infty$.

For the case $b = 1$, since $\psi(n)/n \to \infty$ as $n \to \infty$, we get that $\overline{F}(\psi)$ is a subset of Π_{∞}. By Lemma 3.3,

$$\dim_{\mathrm{H}} \overline{F}(\psi) \leq \dim_{\mathrm{H}} \Pi_{\infty} = \frac{1}{2} = \frac{1}{b+1}.$$

For any $\epsilon > 0$, by definition of b, we obtain $\psi(n) \leq (1+\epsilon)^n$ for infinitely many n, and so

$$\overline{D}(A, 1+\epsilon) \subset \overline{F}(\psi).$$

It follows from Lemma 6.2 that $\dim_{\mathrm{H}} \overline{F}(\psi) \geq \frac{1}{2+\epsilon}$. Letting $\epsilon \to 0^+$, we get the desired lower bound.

For the case $1 < b < \infty$, let $0 < \epsilon < b - 1$. By definition of b, we have:

i). $\psi(n) \leq (b+\epsilon)^n$ for infinitely many n,

ii). $\psi(n) \geq (b-\epsilon)^n$ for sufficiently large n.

Then

$$\underline{D}(A, b+\epsilon) \subset \overline{F}(\psi) \subset \overline{D}(A, b-\epsilon).$$

Applying Lemma 6.2, we see that

$$\frac{1}{b+\epsilon+1} \leq \dim_{\mathrm{H}} \overline{F}(\psi) \leq \frac{1}{b-\epsilon+1}.$$

Since ϵ is arbitrary, we obtain $\dim_{\mathrm{H}} \overline{F}(\psi) = \frac{1}{1+b}$.

For the case $b = \infty$, let $C > 1$ be large, we have $\psi(n) > C^n$ for sufficiently large n, and so

$$\overline{F}(\psi) \subset \overline{D}(A, C).$$

It follows from Lemma 6.2 that $\dim_{\mathrm{H}} \overline{F}(\psi) \leq \frac{1}{1+C}$. Letting $C \to \infty$, we get that $\dim_{\mathrm{H}} \overline{F}(\psi) = 0$. □

Combining (6.3) and Lemma 6.3, we deduce that

$$\dim_{\mathrm{H}} \overline{E}_{\psi}(\alpha) \leq \frac{1}{b+1}.$$

6.1.2 Lower bound

For the lower bound, when $\alpha < \infty$, we construct a subset $\mathbb{E}(\{s_n\}\{t_n\})$ of $\overline{E}_{\psi}(\alpha)$, and use Proposition 3.1. More precisely, we need to construct for each $\epsilon > 0$ a sequence $\{s_n\}_{n\geq 1}$ satisfying the following conditions:

a).

$$\limsup_{n\to\infty} \frac{\log s_1 + \cdots + \log s_n}{\psi(n)} = \alpha \tag{6.5}$$

b).

$$\lim_{n\to\infty} \frac{\log s_1 + \cdots + \log s_n}{n} = \infty \tag{6.6}$$

c).

$$\limsup_{n\to\infty} \frac{\log s_{n+1}}{\log s_1 + \cdots + \log s_n} \leq b+\epsilon-1 \tag{6.7}$$

Then set $t_n = s_n$ for each $n \geq 1$, it follows that

$$\begin{aligned}\dim_{\mathrm{H}} \overline{E}_{\psi}(\alpha) \geq& \dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\}) \\ =& \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \log s_k}{2\sum_{k=1}^{n+1} \log s_k - \log s_{n+1}} \\ =& (2 + \limsup_{n\to\infty} \frac{\log s_{n+1}}{\log s_1 + \cdots + \log s_n})^{-1} \geq \frac{1}{B+1+\epsilon}.\end{aligned}$$

To show this, first set

$$c_{j,k} := \begin{cases} e^{\alpha\psi(k)(b+\epsilon)^{j-k}} & 1 \leq k \leq j \\ e^{\alpha\psi(k)} & k \geq j+1 \end{cases} \tag{6.8}$$

for each $j, k \in \mathbb{N}$. Set

$$b_j := \inf_{k\geq 1}\{c_{j,k}\} = \inf\{e^{\alpha\psi(1)(b+\epsilon)^{j-1}}, e^{\alpha\psi(2)(b+\epsilon)^{j-2}}, \cdots, e^{\alpha\psi(j-1)(b+\epsilon)}, e^{\alpha\psi(j)}, e^{\alpha\psi(j+1)}, \cdots\}. \tag{6.9}$$

Since $\frac{\psi(k)}{k} \to \infty$, $\psi(k) \to \infty$ and $e^{\alpha\psi(k)} \to \infty$ as $k \to \infty$, the infimum in (6.9) is obtained, and denote t_j the smallest index the infimum in $\inf_{k\geq 1}\{c_{j,k}\}$ is obtained:

$$t_j := \min\{k \geq 1 : c_{j,k} = b_j\}. \tag{6.10}$$

Claim. For all $j, k \geq 1$,

i). $c_{j,k} \leq c_{j+1,k} \leq c_{j,k}^{b+\epsilon}$ and $b_j \leq b_{j+1} \leq b_j^{b+\epsilon}$,

ii). $t_{j+1} \geq t_j$ and $t_j \to \infty$ as $j \to \infty$,

iii). $b_j \leq e^{\alpha\psi(j)}$ and $b_{t_j} = e^{\alpha\psi(t_j)}$,

iv). $\frac{\log b_j}{j} \to \infty$ as $j \to \infty$.

Now we prove the claim item by item. For i), fix $j \geq 1$, when $1 \leq k \leq j$

$$e^{\alpha\psi(k)(b+\epsilon)^{j-k}} \leq e^{\alpha\psi(k)(b+\epsilon)^{j-k+1}} \leq e^{\alpha\psi(k)(b+\epsilon)^{j-k+1}},$$

and when $k \geq j+1$

$$e^{\alpha\psi(k)} \leq e^{\alpha\psi(k+1)} \leq e^{\alpha\psi(k+1)(b+\epsilon)}.$$

The quantities in the inequalities correspond the expression of $c_{j,k}$, so $c_{j,k} \leq c_{j+1,k} \leq c_{j,k}^{b+\epsilon}$, and their infimums satisfy the same relation: $b_j \leq b_{j+1} \leq b_j^{b+\epsilon}$, as we want.

For ii), fix $j \geq 1$, when $t_j = 1$, $t_{j+1} \geq 1 = t_j$. When $2 \leq t_j \leq j$, note that $i = t_j$ if and only if

$$c_{j,i} < c_{j,k}\ \forall k < t_j \text{ and } c_{j,i} \leq c_{j,k}\ \forall k > t_j. \tag{6.11}$$

So

$$e^{\alpha\psi(k)(b+\epsilon)^{j-k}} < e^{\alpha\psi(t_j)(b+\epsilon)^{j-t_j}}\ \forall k < t_j.$$

implies

$$e^{\alpha\psi(k)(b+\epsilon)^{j+1-k}} = (e^{\alpha\psi(k)(b+\epsilon)^{j-k}})^{b+\epsilon} < (e^{\alpha\psi(t_j)(b+\epsilon)^{j-t_j}})^{b+\epsilon} = e^{\alpha\psi(t_j)(b+\epsilon)^{j+1-t_j}}\ \forall k < t_j$$

since $b+\epsilon \geq 1+\epsilon > 1$. It follows that $t_{j+1} \geq t_j$. When $t_j \geq j+1$, $\forall k < t_j$

$$c_{j+1,t_j} = c_{j,t_j} < c_{j,k} \leq c_{j+1,k}.$$

there is $t_{j+1} \geq t_j$. If $\lim_{j\to\infty} t_j \neq \infty$, since $\{t_j\}_{j\geq 1}$ is increasing, there is $t_j = N, \forall j \gg 1$, for some $N \in \mathbb{N}$. Then when $j > N$

$$e^{\alpha\psi(j)} = c_{j,j} \geq c_{j,t_j} = c_{j,N} = e^{\alpha\psi(N)(b+\epsilon)^{j-N}}.$$

It follows that for all $j > N$

$$\frac{\log\psi(N) - N\log(b+\epsilon)}{j} + \log(b+\epsilon) \leq \frac{\log\psi(j)}{j}. \tag{6.12}$$

Letting $j \to \infty$ in (6.12) yields $\liminf_{j\to\infty} \frac{\log\psi(j)}{j} \geq \log(b+\epsilon) > \log b$, a contradiction to the definition of b. So $t_j \to \infty$ as $j \to \infty$, as we want.

For iii), for all $j \geq 1$, there is $b_j = \inf_{k\geq 1}\{c_{j,k}\} =\leq c_{j,j} = e^{\alpha\psi(j)}$. Note that $c_{j,k} \geq 1, \forall j,k \geq 1$. If $t_j < j$, for all $1 \leq k < t_j$

$$c_{t_j,k} = e^{\alpha\psi(k)(b+\epsilon)^{t_j-k}} = (e^{\alpha\psi(k)(b+\epsilon)^{j-k}})^{(b+\epsilon)^{t_j-j}} = (c_{j,k})^{(b+\epsilon)^{t_j-j}} > (c_{j,t_j})^{(b+\epsilon)^{t_j-j}} = c_{t_j,t_j},$$

for all $t_j < k \le j$

$$c_{t_j,k} = (c_{j,k})^{(b+\epsilon)^{k-j}} \ge (c_{j,t_j})^{(b+\epsilon)^{k-j}} = (c_{t_j,t_j})^{(b+\epsilon)^{k-t_j}} > c_{t_j,t_j},$$

and for all $k > j$

$$c_{t_j,k} = c_{j,k} \ge c_{j,t_j} = (c_{t_j,t_j})^{(b+\epsilon)^{j-t_j}} > c_{t_j,t_j}.$$

It follows from (6.11) that $t_{t_j} = t_j$ and $b_{t_j} = e^{\alpha\psi(t_j)}$, and in fact:

$$t_{t_j} = t_{t_j+1} = \cdots = t_j.$$

If $t_j = j$, then

$$b_{t_j} = b_j = c_{j,t_j} = c_{j,j} = e^{\alpha\psi(j)} = e^{\alpha\psi(t_j)}.$$

If $t_j > j$, then for all $1 \le k \le j$

$$c_{t_j,k} = e^{\alpha\psi(k)(b+\epsilon)^{t_j-k}} \ge e^{\alpha\psi(k)(b+\epsilon)^{j-k}} = c_{j,k} > c_{j,t_j} = c_{t_j,t_j},$$

for all $j < k < t_j$

$$c_{t_j,k} = e^{\alpha\psi(k)(b+\epsilon)^{t_j-k}} \ge e^{\alpha\psi(k)} = c_{j,k} > c_{j,t_j} = c_{t_j,t_j},$$

and for all $k > t_j$

$$c_{t_j,k} = e^{\alpha\psi(k)} = c_{j,k} \ge c_{j,t_j} = c_{t_j,t_j}.$$

Thus $b_{t_j} = e^{\alpha\psi(t_j)}$, and in fact

$$t_j = t_{j+1} = \cdots = t_{t_j},$$

as we want.

For iv), use the fact

$$\text{If } \frac{a_j}{j} \to \infty, \frac{b_j}{j} \to \infty, \text{ then } c_j := \min_{1\le k\le j}\{\frac{a_k b_{j-k}}{j}\} \to \infty, \tag{6.13}$$

as $j \to \infty$. Note that

$$\frac{\log b_j}{j} = \inf_{k \geq 1}\{\frac{\log c_{j,k}}{j}\} = \alpha \inf\{\frac{\psi(1)(b+\epsilon)^{j-1}}{j}, \cdots, \frac{\psi(j-1)(b+\epsilon)}{j}, \frac{\psi(j)}{j}, \frac{\psi(j+1)}{j}, \cdots\}.$$

tends to infinity if $\frac{\log c_{j,k}}{j} \rightrightarrows \infty$ $\forall k \geq 1$ uniformly, as $j \to \infty$. Set $a_j = \psi(j), b_j = (b+\epsilon)^j$ in (6.13),there is

$$\frac{\log b_j}{j} = \alpha \inf\{c_j, \frac{\psi(j)}{j}, \frac{\psi(j+1)}{j}, \cdots\} \geq \alpha \inf\{c_j, \frac{\psi(j)}{j}, \frac{\psi(j+1)}{j+1}, \cdots\}$$

Given $M > 0$, by (6.13) and $\frac{\psi(j)}{j} \to \infty$ as $j \to \infty$, can choose $N \in \mathbb{N}$ such that for all $j > N$, there is $c_j > M$ and $\frac{\psi(j)}{j} > M$, also since $j + k > j > N$ $\forall k \geq 1$,

$$\frac{\psi(j+1)}{j+1} > M, \frac{\psi(j+2)}{j+2} > M, \cdots, \tag{6.14}$$

and thus $\frac{\log b_j}{j} \geq \frac{M}{\alpha}$, So $\frac{\log b_j}{j} \to \infty$, as $j \to \infty$, and the proof of the claim is completed.

Finally, set $s_n = \frac{b_n}{b_{n-1}}$ for $n \geq 2$ and $s_1 = b_1$. Claim ii) and iii) imply condition a) (6.5), Claim iv) implies condition b) (6.6) and Claim i) implies condition c) (6.7). It follows that

$$\dim_{\mathrm{H}} \overline{E}_\psi(\alpha) \geq \dim_{\mathrm{H}} \mathbb{E}(\{s_n\}\{t_n\}) \geq \frac{1}{B+1+\epsilon},$$

and since ϵ is arbitrary,

$$\dim_{\mathrm{H}} \overline{E}_\psi(\alpha) \geq \frac{1}{B+1},$$

the desired lower bound.

6.2 Hausdorff dimension of $\underline{E}_\psi(\alpha)$

For $x \in \underline{E}_\psi(\alpha)$, when $\alpha \in (0, \infty)$, we see that $\Pi_n(x) \geq e^{\frac{\alpha\psi(n)}{3}}$ holds for n sufficiently large; when $\alpha = \infty$, we see that $\Pi_n(x) \geq e^{\alpha\psi(n)}$ holds for n sufficiently large. So

$$\underline{E}_\psi(\alpha) \subset \{x \in [0,1) : \Pi_n(x) \geq A^{\psi(n)} \text{ for all } n \gg 1\} \tag{6.15}$$

for some $A > 1$. This leads to study the Hausdorff dimension of the liminf set.

Lemma 6.4. *Let $A \in (1, \infty)$. Write*

$$\underline{F}(\psi) := \{x \in [0,1) : \Pi_n(x) \geq A^{\psi(n)} \text{ for all } n \gg 1\}. \tag{6.16}$$

Then

$$\dim_{\mathrm{H}} \underline{F}(\psi) = \frac{1}{B+1},$$

where $B \in [1, \infty]$ is defined as in Theorem B.

Proof of Lemma 6.4. The proof is divided into three parts: $B = 1, 1 < B < \infty, B = \infty$.

For the case $B = 1$, since $\psi(n)/n \to \infty$ as $n \to \infty$, we get that $\underline{F}(\psi)$ is a subset of Π_∞. By Lemma 3.3,

$$\dim_{\mathrm{H}} \underline{F}(\psi) \leq \dim_{\mathrm{H}} \Pi_\infty = \frac{1}{2} = \frac{1}{B+1}.$$

For any $\epsilon > 0$, by definition of B, we obtain $\psi(n) \leq (1+\epsilon)^n$ for all $n \gg 1$, and so

$$\underline{D}(A, 1+\epsilon) \subset \underline{F}(\psi).$$

It follows from Lemma 6.2 that $\dim_{\mathrm{H}} \underline{F}(\psi) \geq \frac{1}{2+\epsilon}$. Letting $\epsilon \to 0^+$, we get the desired lower bound.

For the case $1 < B < \infty$, let $0 < \epsilon < B - 1$. By definition of B, we have:

i). $\psi(n) \leq (B+\epsilon)^n$ for sufficiently large n,

ii). $\psi(n) \geq (B-\epsilon)^n$ for infinitely many n.

Then

$$\underline{D}(A, B+\epsilon) \subset \underline{F}(\psi) \subset \overline{D}(A, B-\epsilon).$$

Applying Lemma 6.2, we see that

$$\frac{1}{B+\epsilon+1} \leq \dim_{\mathrm{H}} \underline{F}(\psi) \leq \frac{1}{B-\epsilon+1}.$$

Since ϵ is arbitrary, we obtain $\dim_{\mathrm{H}} \underline{F}(\psi) = \frac{1}{1+B}$.

For the case $B = \infty$, let $C > 1$ be large, we have $\psi(n) > C^n$ for infinitely many n, and so

$$\underline{F}(\psi) \subset \overline{D}(A, C).$$

It follows from Lemma 6.2 that $\dim_{\mathrm{H}} \underline{F}(\psi) \le \frac{1}{1+C}$. Letting $C \to \infty$, we get that $\dim_{\mathrm{H}} \underline{F}(\psi) = 0$. $\square$

Combining (6.15) and Lemma 6.4, we deduce that

$$\dim_{\mathrm{H}} \underline{E}_{\psi}(\alpha) \le \frac{1}{B+1}.$$

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Appendix

For a function $\psi : \mathbb{N} \to \mathbb{R}_{>0}$ that meets the following criteria:

$$\lim_{n\to\infty} \frac{\psi(n)}{n} = \infty \tag{.17}$$

and

$$\liminf_{n\to\infty} \frac{\log \psi(n)}{n} = \log b \in [0, \infty] \tag{.18}$$

and so $b \in [1, \infty]$. We aim to construct a function that fulfills conditions iii)', which is stronger version of iii) in Section 6.1.2, and is of independent interest.

Proposition .5. *Assume $\psi : \mathbb{N} \to \mathbb{R}_{>0}$ satisfies the above conditions* (.17) *and* (.18) *with $b \in [1, \infty]$, then there exists a non-decreasing function $g_\psi : \mathbb{N} \to \mathbb{R}_{>0}$ that simultaneously satisfies the following three properties:*

i).

$$\limsup_{n\to\infty} \frac{g_\psi(n)}{\psi(n)} = 1, \tag{.19}$$

ii).

$$\lim_{n\to\infty} \frac{g_\psi(n)}{n} = \infty, \tag{.20}$$

iii)'.

$$\lim_{n\to\infty} \frac{g_\psi(n+1)}{g_\psi(n)} = b. \tag{.21}$$

Remark. In fact, the function $g_\psi(n)$ we construct in the proof of Proposition .5 satisfies $g_\psi(n) \leq \psi(n)$, for all $n \geq 1$.

Before we proceed with the proof of Proposition .5, let us state two useful lemmas.

Lemma .6. *Suppose $\psi : \mathbb{N} \to \mathbb{R}_{>0}$ satisfies the conditions* (.17) *and* (.18) *with $1 < b < \infty$, then for every $M \in \mathbb{N}$, $0 < \epsilon < b - 1$, there exists an integer $m^* = m^*(M, \epsilon) > M$ that simultaneously satisfies the following two properties:*

i).

$$\psi(m^* + n) > (b - \epsilon)^n \psi(m^*), \textit{ for all } n \geq 1; \tag{.22}$$

ii).

$$\psi(m^* - n) > (b + \epsilon)^{-n} \psi(m^*), \textit{ for all } 1 \leq n \leq m^*. \tag{.23}$$

Proof of Lemma .6. Set a real number $c_\psi := \min_{n \geq 1}\{\psi(n)\}$, then it follows follows from (.17) that c_ψ is well-defined. Since $\psi(n) > 0$ for all $n \geq 1$, we have $c_\psi > 0$. Fix $M \in \mathbb{N}, 0 < \epsilon < b - 1$ as in the hypothesis, by (.18), we can choose an integer $m_0 > M$ such that the following three properties are satisfied:

$$\psi(m_0) < c_\psi (b + \epsilon/2)^{m_0}, \tag{.24}$$

and

$$(1 + \frac{\epsilon}{2(b + \epsilon/2)})^{m_0 - M} > (b + \epsilon/2)^M, \tag{.25}$$

(in other words, (.25) means $(b - \epsilon)^{M - m_0} < (b + \epsilon/2)^{-m_0}$), and also for all $n > m_0$

$$\psi(n) > (b - \epsilon/2)^n. \tag{.26}$$

Meanwhile, we further take an integer $N > m_0$ with

$$\psi(m_0)(b - \epsilon)^{N - m_0} < (b - \epsilon/2)^N \tag{.27}$$

holds.

Let us now proceed with the proof of Lemma .6 by contradiction. Suppose, in contrast, that there is no such $m^* > M$ satisfying the properties (.22) and (.23) simultaneously, then in the rest of the proof, we recursively construct a sequence of integers $\{m_j\}_{j \geq 0}$ with $m_j > M$ for every $j \geq 0$.

Starting from m_0 and assume by induction for each $j \geq 0$, integers $m_0 > M, \cdots, m_j > M$ are already defined, the integer m_{j+1} is defined by:

i). If there exists some $\hat{n} \geq 1$ such that $\psi(m_j + \hat{n}) \leq (b-\epsilon)^{\hat{n}}\psi(m_j)$, then define $m_{j+1} \coloneqq m_j + n_j$ where $n_j \geq 1$ is the minimal positive integer such that

$$\psi(m_j + n_j) \leq (b-\epsilon)^{n_j}\psi(m_j). \tag{.28}$$

ii). Else if for all $n \geq 1$, $\psi(m_j + n) > (b-\epsilon)^{n}\psi(m_j)$. Due to our assumption, (.22) and (.23) can not satisfy simultaneously for m_j, but m_j already satisfies (.22), so m_j dissatisfies (.23). Then define $m_{j+1} \coloneqq m_j + n_j$, where $1 \leq -n_j \leq m_j$ is the minimal positive integer such that

$$\psi(m_j + n_j) \leq (b+\epsilon)^{n_j}\psi(m_j). \tag{.29}$$

Claim (A). Let m_{j+1} be the integer defined in either item i) or item ii) above, then

$$M < m_{j+1} < N, \tag{.30}$$

recall N was defined in (.27).

Once Claim (A) is proved, one can apply it recursively so that the sequence $\{m_j\}_{j\geq 0}$ is defined, and $M < m_j < N$ for every $j \geq 0$.

We proceed the proof by contradiction. Suppose on the contrary that $m_{j+1} \leq M$, by our construction $m_0 > M, \cdots, m_j > M$. Set

$$R \coloneqq \sum_{i<j, n_i>0} n_i \text{ and } S \coloneqq -\sum_{i<j, n_i<0} n_i, \tag{.31}$$

then $m_{j+1} = m_0 + R - S \leq M$ with convention that $R = 0$ if $\{i \geq 0 : i < j, n_i > 0\}$ is empty (similar for S). It follows that

$$\begin{aligned}
\psi(m_{j+1}) &\leq \psi(m_0)(b-\epsilon)^R(b+\epsilon)^{-S} && \big(\text{by (.28), (.29) and (.31)}\big) \\
&\leq \psi(m_0)(b-\epsilon)^{R-S} && \big(\text{since } b-\epsilon > 1\big) \\
&= \psi(m_0)(b-\epsilon)^{m_{j+1}-m_0}.
\end{aligned} \tag{.32}$$

And

$$
\begin{aligned}
&\psi(m_0)(b-\epsilon)^{m_{j+1}-m_0} \\
&\leq \psi(m_0)(b-\epsilon)^{M-m_0} \\
&< \psi(m_0)(b+\epsilon/2)^{-m_0} && \big(\text{by } (.25)\big) \\
&< c_\psi. && \big(\text{by } (.24)\big)
\end{aligned}
$$

This is obvious a contradiction to the minimality of c_ψ.

Next we proceed by contradiction the proof of the other half of Claim (A): $m_{j+1} < N$. Suppose otherwise $m_{j+1} \geq N$, then it follows that

$$
\begin{aligned}
\psi(m_{j+1}) &\leq \psi(m_0)(b-\epsilon)^{m_{j+1}-m_0} && \big(\text{by } (.32)\big) \\
&= \psi(m_0)(b-\epsilon)^{m_{j+1}-N}(b-\epsilon)^{N-m_0} \\
&\leq \psi(m_0)(b-\epsilon)^{m_{j+1}-N}\frac{(b-\epsilon/2)^N}{\psi(m_0)} && \big(\text{by } (.27)\big) \\
&\leq (b-\epsilon/2)^{m_{j+1}}, && \big(\text{since } m_{j+1} \geq N\big)
\end{aligned}
$$

This is a obvious contradiction to (.26) and we thus obtain (.30), and completes the proof of Claim (A).

Due to (.30) in Claim (A), there is $M < m_j < N$ for every $j \geq 0$ and $\{m_j\}_{j\geq 1}$ is a bounded infinite sequence taking value in $\mathbb{N}$, then by the pigeonhole principle, the sequence $\{m_j\}_{j\geq 1}$ must take repeated value for different indices, say there exist $0 \leq i^* < j^*$ such that $m_i^* = m_j^*$. Set

$$
\widetilde{R} := \sum_{i^* \leq t < j^*, n_t > 0} n_t \text{ and } \widetilde{S} := -\sum_{i^* \leq t < j^*, n_t < 0} n_t, \tag{.33}
$$

then $m_{i^*} = m_{j^*} = m_{i^*} + \widetilde{R} - \widetilde{S}$ and thus $\widetilde{R} = \widetilde{S} > 0$. However it follows that

$$
\begin{aligned}
\psi(m_{i^*}) &= \psi(m_{j^*}) \\
&\leq \psi(m_{i^*})(b-\epsilon)^{\widetilde{R}}(b+\epsilon)^{-\widetilde{S}} && \big(\text{by } (.28), (.29) \text{ and } (.33)\big) \\
&= \psi(m_{i^*})(\frac{b-\epsilon}{b+\epsilon})^{\widetilde{R}} && \big(\text{since } \widetilde{R} = \widetilde{S}\big) \\
&< \psi(m_{i^*}). && \big(\text{since } \widetilde{R} > 0\big)
\end{aligned}
$$

This is absurd, consequently there is some integer $m^* = m^*(M, \epsilon)$ satisfying the properties (.22) and (.23) simultaneously, as we want. $\square$

Analogous to Lemma .6

Lemma .7. *Suppose* $\psi : \mathbb{N} \to \mathbb{R}_{>0}$ *satisfies* (.17) *and* (.18) *with* $b = 1$*, then for every* $M \in \mathbb{N}$*,* $\epsilon > 0$*, there exists an integer* $m^* = m^*(M, \epsilon) > M$ *satisfying the following two properties simultaneously:*

i).

$$\psi(m^* + n) > \frac{m^* + n}{m}\psi(m^*) \text{ for all } n \geq 1; \tag{.34}$$

ii).

$$\psi(m^* - n) > (1 + \epsilon)^{-n}\psi(m^*) \text{ for all } 1 \leq n \leq m^*. \tag{.35}$$

Proof of Lemma .7. Set $c_\psi := \min_{n\geq1}\{\psi(n)\} > 0$. Fix $\epsilon > 0$ as in the hypothesis and without loss of generality we can fix $M \in \mathbb{N}$, with $M > 2/\epsilon$. By (.18), we can choose an integer $m_0 > M$ such that the following two properties are satisfied

$$(1 + \frac{\epsilon}{2(1 + \epsilon/2)})^{m_0 - M} > (1 + \epsilon/2)^M \tag{.36}$$

and

$$\psi(m_0) < c_\psi(1 + \epsilon/2)^{m_0}. \tag{.37}$$

Meanwhile, by (.17), one can further take an integer $N > m_0$ such that

$$\frac{\psi(n)}{n} > \frac{\psi(m_0)}{m_0} \text{ for all } n > N. \tag{.38}$$

Let us now proceed the proof of Lemma .7 by contradiction. Suppose in contrast that there is no such $m^* > M$ satisfying the properties (.34) and (.35) simultaneously, then we will recursively construct a sequence of integers $\{m_j\}_{j\geq0}$. The sequence of integers $\{m_j\}_{j\geq0}$ we construct satisfies the following 3 properties:

Claim (B). For each $j \geq 0$,

1). If $m_j \leq m_0$, then

$$\psi(m_j) \leq (1 + \epsilon)^{m_j - m_0}\psi(m_0); \tag{.39}$$

2). If $m_j \geq m_0$, then

$$\psi(m_j) \leq \frac{m_j}{m_0}\psi(m_0); \tag{.40}$$

3).

$$M < m_j < N. \tag{.41}$$

Recall that N was defined in (.38).

Let us begin the construction $\{m_j\}_{j\geq 0}$: we start from the integer m_0 defined above, then properties (.39) and (.40) in Claim (B) are trivially true for m_0, and by our construction of m_0, (.41) is also true. In particular, $m_0 > M$, and we construct m_1 using this property. If there exists some $\hat{n} \geq 1$ such that (.42) holds for $m_j = m_0$, we use item i) below to define m_1, otherwise if no such $\hat{n}$ exists, since $m_0 > M$, then by our assumption, there is no $m^* > M$ satisfying the properties (.34) and (.35) simultaneously, we can define m_1 by item ii) below.

Next, we prove Claim (B) for m_1 using the fact Claim (B) is true for m_0 (the details in this step will be given in the following paragraphs). In particular, $m_1 > M$, then again in the same manner we can define m_2, and prove Claim (B) for m_2 using the fact Claim (B) is true for m_1. We repeat this process to construct m_j for each $j \geq 0$.

To be more precise, assume that we have already defined integers $m_0, \cdots, m_j$, and we have also verified Claim (B) for $m_0, \cdots, m_j$, in particular $m_j > M$. Then we define m_{j+1}, and verify Claim (B) for m_{j+1}.

i). If there exists some $\hat{n} \geq 1$ such that $\psi(m_j + \hat{n}) \leq \frac{m_j+\hat{n}}{m_j}\psi(m_j)$, then define $m_{j+1} \coloneqq m_j + n_j$ where $n_j \geq 1$ is the minimal positive integer such that

$$\psi(m_j + n_j) \leq \frac{m_j + n_j}{m_j}\psi(m_j). \tag{.42}$$

ii). Else if for all $n \geq 1$, $\psi(m_j + n) > \frac{m_j+n}{m_j}\psi(m_j)$. Due to our assumption, (.34) and (.35) do not satisfy simultaneously for m_j, but m_j already satisfies (.34), so m_j dissatisfies (.35). Then define $m_{j+1} \coloneqq m_j + n_j$, where $1 \leq -n_j \leq m_j$ is the minimal positive integer such that

$$\psi(m_j + n_j) \le (1+\epsilon)^{n_j}\psi(m_j). \tag{.43}$$

We use item i) or item ii) defines m_{j+1}. Let us proceed the proof of Claim (B) for m_{j+1} item by item, using the fact Claim (B) is true for m_j.

Step I We prove (.39) for m_{j+1}. One key observation used in the following proof is when $m_j \le m_0$, since Claim (B) is true for m_j, in particular, (.39) holds for m_j. Also as $m_{j+1} \le m_0$ and $n_j \ne 0$, we distinguish the following 3 cases:

Case I.1: ($m_j \le m_0$ and $n_j < 0$). In this case, $m_{j+1} = m_j + n_j < m_j \le m_0$, since (.39) in Claim (B) holds for m_j, it follows that

$$\begin{aligned}
\psi(m_{j+1}) &\le (1+\epsilon)^{n_j}\psi(m_j) && \big(\text{since } n_j < 0\text{, and using (.43)}\big)\\
&\le (1+\epsilon)^{n_j}(1+\epsilon)^{m_j-m_0}\psi(m_0) && \big(\text{since (.39) holds for } m_j\big)\\
&= (1+\epsilon)^{m_{j+1}-m_0}\psi(m_0),
\end{aligned}$$

so item 1) (.39) in Claim (B) holds for m_{j+1}, as we want.

Case I.2: ($m_j \le m_0$ and $0 < n_j \le m_0 - m_j$). In this case, $m_{j+1} = m_j + n_j \le m_j + m_0 - m_j = m_0$, since (.39) and (.41) holds for m_j, in particular $m_j > M$, it follows from Bernoulli's inequality[3]

$$\begin{aligned}
\psi(m_{j+1}) &\le \frac{m_j+n_j}{m_j}\psi(m_j) && \big(\text{since } n_j > 0\text{, and using (.42)}\big)\\
&\le (1+1/m_j)^{n_j}\psi(m_j) && \big(\text{using Bernoulli's inequality}\big)\\
&\le (1+\epsilon)^{n_j}\psi(m_j) && \big(\text{since } m_j > M > 2/\epsilon\big)\\
&\le (1+\epsilon)^{n_j}(1+\epsilon)^{m_j-m_0}\psi(m_0) && \big(\text{since (.39) holds for } m_j\big)\\
&= (1+\epsilon)^{m_{j+1}-m_0}\psi(m_0),
\end{aligned}$$

so item 1) (.39) in Claim (B) holds for m_{j+1}, as we want.

Case I.3: ($m_j > m_0$ and $n_j < m_0 - m_j < 0$). In this case, $m_{j+1} = m_j + n_j <$

[3] Bernoulli's inequality: for real numbers $x > -1, a \ge 1$

$$(1+x)^a \ge 1 + ax. \tag{.44}$$

$m_j + m_0 - m_j = m_0$, since (.40) holds for m_j, it follows that

$$\begin{aligned}
\psi(m_{j+1}) &\le (1+\epsilon)^{n_j}\psi(m_j) && \big(\text{since } n_j < 0, \text{ and using (.43)}\big)\\
&\le (1+\epsilon)^{n_j+m_j-m_0}(1+\epsilon/2)^{m_0-m_j}\psi(m_j) && \big(\text{since } m_0 - m_j < 0\big)\\
&\le (1+\epsilon)^{n_j+m_j-m_0}(1+1/m_0)^{-(m_j-m_0)}\psi(m_j) && \big(\text{since } m_0 > M > 2/\epsilon\big)\\
&\le (1+\epsilon)^{n_j+m_j-m_0}\frac{m_0}{m_j}\psi(m_j) && \big(\text{using Bernoulli's inequality}\big)\\
&\le (1+\epsilon)^{n_j+m_j-m_0}\psi(m_0), && \big(\text{since (.40) holds for } m_j\big)
\end{aligned}$$

so item 1) (.39) in Claim (B) holds for m_{j+1}, as we want. Thus **Step I**: the proof of item i) in (.39) for m_{j+1} is completed.

Step II Next we prove item 2) (.40) for m_{j+1}. By hypothesis, we always have $m_{j+1} \ge m_0$ when proving item 2). As $n_j \neq 0$, we also distinguish the following 3 cases:

Case II.1: ($m_j \ge m_0$ and $n_j > 0$). In this case, $m_{j+1} = m_j + n_j > m_j \ge m_0$, since item 2) (.40) in Claim (B) holds for m_j, it follows that

$$\begin{aligned}
\psi(m_{j+1}) &\le \frac{m_j+n_j}{m_j}\psi(m_j) && \big(\text{since } n_j > 0, \text{ and using (.42)}\big)\\
&\le \frac{m_j+n_j}{m_j}\frac{m_j}{m_0}\psi(m_0) && \big(\text{since (.40) holds for } m_j\big)\\
&= \frac{m_{j+1}}{m_0}\psi(m_0),
\end{aligned}$$

so item 2) (.40) in Claim (B) holds for m_{j+1}, as we want.

Case II.2: ($m_j \ge m_0$ and $m_0 - m_j \le n_j < 0$). In this case, $m_{j+1} = m_j + n_j \ge m_j + m_0 - m_j = m_0$, since item 2) (.40) in Claim (B) holds for m_j, it follows that

$$\begin{aligned}
\psi(m_{j+1}) &\le (1+\epsilon)^{n_j}\psi(m_j) && \big(\text{since } n_j < 0, \text{ and using (.43)}\big)\\
&\le (1+\epsilon/2)^{n_j}\psi(m_j) && \big(\text{since } n_j < 0\big)\\
&\le (1+1/m_{j+1})^{n_j}\psi(m_j) && \big(\text{since } m_{j+1} \ge m_0 > M > 2/\epsilon\big)\\
&\le (1-\frac{n_j}{m_{j+1}})^{-1}\psi(m_j) && \big(\text{using Bernoulli's inequality}\big)\\
&\le \frac{m_{j+1}}{m_0}\psi(m_0). && \big(\text{since (.40) holds for } m_j\big)
\end{aligned}$$

so item 2) (.40) in Claim (B) holds for m_{j+1}, as we want.

Case III.3: ($m_j \le m_0$ and $n_j > m_0 - m_j$). In this case, $m_{j+1} = m_j + n_j > m_j + m_0 -$

$m_j = m_0$, since item 1) (.39) and 3) (.41) in Claim (B) holds for m_j, in particular $m_j > M$, it follows that

$$
\begin{aligned}
\psi(m_{j+1}) &\le \frac{m_j+n_j}{m_j}\psi(m_j) && \big(\text{since } n_j > 0\text{, and using (.42)}\big)\\
&\le \frac{m_j+n_j}{m_0}(1+1/m_j)^{m_0-m_j}\psi(m_j) && \big(\text{using Bernoulli's inequality}\big)\\
&\le \frac{m_j+n_j}{m_0}(1+\epsilon)^{m_0-m_j}\psi(m_j) && \big(\text{since } m_j > M > 2/\epsilon\big)\\
&\le \frac{m_j+n_j}{m_0}(1+\epsilon)^{m_0-m_j}(1+\epsilon)^{m_j-m_0}\psi(m_0) && \big(\text{since (.39) holds for } m_j\big)\\
&= \frac{m_{j+1}}{m_0}\psi(m_0),
\end{aligned}
$$

so item 2) (.40) in Claim (B) holds for m_{j+1}, as we want. Thus **Step II**: the proof of item 2) in (.40) for m_{j+1} is completed.

Step III We prove item 3) (.39) for m_{j+1}, using what we have just proven in **Step I** and **Step II** , that is, item 1) (.39) and item 2) (.40) holds for m_{j+1}.

First we show $m_{j+1} > M$, and the proof is by contradiction: suppose otherwise if $m_{j+1} \le M < m_0$, it follows that

$$
\begin{aligned}
\psi(m_{j+1}) &\le (1+\epsilon)^{m_{j+1}-m_0}\psi(m_0) && \big(\text{since } m_{j+1} < m_0 \text{ and by (.39) for } m_{j+1}\big)\\
&\le (1+\epsilon)^{M-m_0}\psi(m_0) && \big(\text{since } m_{j+1} \le M\big)\\
&< (1+\epsilon/2)^{M-m_0}(1+\epsilon/2)^{-M}\psi(m_0) && \big(\text{by (.36)}\big)\\
&= (1+\epsilon/2)^{-m_0}\psi(m_0)\\
&< c_\psi, && \big(\text{by (.37)}\big)
\end{aligned}
$$

a obvious contradiction to the minimality of c_ψ.

Finally we show $m_{j+1} < N$, and the proof is also by contradiction: suppose otherwise if $m_{j+1} \ge N > m_0$, it follows from (.40) for m_{j+1} that

$$
\psi(m_{j+1}) \le \frac{m_{j+1}}{m_0}\psi(m_0),
$$

this is a obvious contradiction to $\psi(n)/n > \psi(m_0)/m_0$ for all $n \ge N$, thus $m_{j+1} < N$. And we have completed the proof of Claim (B) for m_{j+1}.

Therefore by (.41), the sequence $\{m_j\}_{j\ge1}$ is bounded with infinite many terms and takes value in $\mathbb{N}$, then by the pigeonhole principle, the sequence $\{m_j\}_{j\geq 1}$ must take repeated value for different indices. Say there exists $0 \leq i^* < j^*$ such $m_i^* = m_j^*$. Set

$$R := \sum_{i^* \leq t < j^*, n_t > 0} n_t \text{ and } S := -\sum_{i^* \leq t < j^*, n_t < 0} n_t, \tag{.45}$$

and it follows from $m_i^* = m_j^* = m_i^* + R - S$ that $R = S > 0$. In fact for index $t \geq 0$, when $n_t > 0$, it follows from (.42), Bernoulli's inequality and the fact $m_t > M$ that

$$\begin{aligned} \psi(m_t + n_t) &\leq \frac{m_t + n_t}{m_t}\psi(m_t) \qquad \big(\text{by (.42)}\big) \\ &< (1 + 1/m_t)^{n_t}\psi(m_t)\big(\text{Bernoulli's inequality}\big) \\ &< (1 + \epsilon/2)^{n_t}\psi(m_t). \ \big(\text{since } m_t > M > 2/\epsilon\big) \end{aligned} \tag{.46}$$

Then analogous to Lemma .6, it follows from (.43), (.46) and (.45) that

$$\begin{aligned} \psi(m_i^*) &= \psi(m_j^*) \\ &\leq \psi(m_i^*)(1 + \epsilon/2)^R(1 + \epsilon)^{-S}\big(\text{by (.43), (.46) and (.45)}\big) \\ &= \psi(m_i^*)\Big(\frac{1 + \epsilon/2}{1 + \epsilon}\Big)^R \qquad \big(\text{since } R = S\big) \\ &< \psi(m_i^*), \qquad \big(\text{since } R > 0\big) \end{aligned}$$

which is absurd. So there is an integer $m^* = m^*(M, \epsilon) > M$ satisfying the properties (.34) and (.35) simultaneously, and proof of Lemma .7 is completed. □

Based on Lemma .6 and .7, we are now ready to prove Proposition .5.

Proof of Proposition .5. The main part of the proof is to construct a non-decreasing function $g : \mathbb{N} \to \mathbb{R}$. We will divide its construction into three different cases, namely, $b = \infty, 1 < b < \infty$ and $b = 1$. We verify each case individually. To achieve this, we divide the construction into three cases: $b = \infty, 1 < b < \infty$ and $b = 1$. The first case is easy, and the rest two are more difficult but shares a common pattern.

Case I($b = \infty$): In this case there is

$$\lim_{n\to\infty} \frac{\log \psi(n)}{n} = \infty. \tag{.47}$$

We first construct interleave sequences of indices $\{\widetilde{n}_j\}_{j\geq 1}, \{n_j\}_{j\geq 1}$ inductively satisfying $n_{j+1} \geq \widetilde{n}_{j+1} > n_j \geq \widetilde{n}_j$ for all $j \geq 1$. More precisely, $\{n_j\}_{j\geq 1}$ is a modification of $\{\widetilde{n}_j\}_{j\geq 1}$. By (.47), choose $\widetilde{n}_1 \in \mathbb{N}$ such that $\log \psi(n)/n > 1$ for all $n > \widetilde{n}_1$, then we further take n_1 such that

$$\frac{\log \psi(n_1)}{n_1} = \min_{n\geq \widetilde{n}_1} \left\{ \frac{\log \psi(n)}{n} \right\} := \alpha_1 \geq 1,$$

Next assume by induction that for $j > 1, \widetilde{n}_1, n_1, \cdots, \widetilde{n}_{j-1}, n_{j-1}$ are already defined (but $\widetilde{n}_j$ and n_j are not), then $\widetilde{n}_j$ and n_j are defined as follows: by (.47) choose $\widetilde{n}_j > n_{j-1}$ such that for all $n \geq \widetilde{n}_j$, we have $\log \psi(n)/n \geq j$. Choose $n_j \geq \widetilde{n}_j$ such that

$$\frac{\log \psi(n_j)}{n_j} = \min_{n\geq \widetilde{n}_j} \left\{ \frac{\log \psi(n)}{n} \right\} := \alpha_j \geq j.$$

Take

$$g_\psi(n) := e^{\alpha_j n} \tag{.48}$$

for $n_j \leq n < n_{j+1}$. If $n_1 = 1$, we have defined $g_\psi(n)$ for all $n \geq 1$. If $n_1 > 1$, we choose appropriate value for $g_\psi(n)$ such that $g_\psi(n) < \psi(n)$ and $g_\psi(n)$ is non-decreasing when $1 \leq n < n_1$. For example, we can take

$$g_\psi(n) = \min_{1\leq j<n_1} \{\psi(j)\},$$

for $1 \leq n < n_1$. This completes the definition of g_ψ.

Let us verify that g constructed in (.48) satisfies the properties (.19), (.20) and (.21) in Proposition .5. The sequence $\{\widetilde{n}_j\}_{j\geq 1}$ is non-decreasing, so $\{\alpha_j\}_{j\geq 1}$ is non-decreasing and thus g_ψ is non-decreasing. For any $n \geq 1$, if $n < n_1$, then $g_\psi(n) = \min_{1\leq j<n_1}\{\psi(j)\} \leq \psi(n)$; otherwise $n \geq n_1$, then there exists some $j \geq 1$, such that $n_j \leq n < n_{j+1}$, then it follows from the definition of g_ψ (.48) that

$$g_\psi(n) = e^{\alpha_j n} \leq e^{n\frac{\psi(n)}{n}} = \psi(n).$$

Therefore for all $n \geq 1$, $g_\psi(n) \leq \psi(n)$. Also it directly follows from the definition of α_j

that $g(n_j) = \psi(n_j)$. We conclude that (.19) in Proposition .5 is verified.

For $j \geq 1$ and $n_j \leq n < n_{j+1}$, there is

$$\frac{g_\psi(n+1)}{g_\psi(n)} \geq e^{\alpha_j} \geq e^j,$$

and consequently (.21) is true. For all $n \geq n_1$, $g_\psi(n) \geq e^n$, which proves (.20). Therefore, we obtain Proposition .5 in **Case I**.

Case II($1 < b < \infty$): Let us briefly state the outline of the proof. The construction of g consists of two parts and corresponds two types of indices that need to to be determined.

In **Part 1**, we determine the boundary indices n_j: we use Lemma .6 to construct recursively a strictly increasing sequence of integers $\{n_j\}_{j\geq 0}$ and interpolate $g(n_j) = \psi(n_j)$ for all $j \geq 1$.

In **Part 2**, we fix a positive integer j and then fill the remaining values of $g_\psi(n)$ inside $[n_j, n_{j+1}]$. Determine the first internal index r_1 by (.53) as a minimizer: $n = r_1$ minimize

$$\Big(\frac{\psi(n_j+n)}{\psi(n_j)}\Big)^{1/n} \tag{.49}$$

when n varies in $[n_j, n_{j+1}]$. Then define g by (.54), a geometric progression starting from $g(n_j)$ with common ratio (.49). If $n_j + r_1$ reaches n_{j+1}, we stop, otherwise we can similarly find $r_2, \cdots$, finitely many internal indices and define g in each interval determine by adjacent inter indices as a geometric progression with proper common ratio. Let us proceed the proof of the construction.

Part 1 In this part, we define the boundary indices $\{n_j\}_{j\geq 1}$. Let $n_0 \geq 1$ be a positive integer such that $\psi(n_0) = \min_{n\geq 1}\{\psi(n)\}$ and choose a constant $A > 1/(b-1)$. Starting from n_0, assume by induction that for $j \geq 1$, $n_0, \cdots, n_{j-1}$ are already defined, then n_j is defined as follows: take $M_j = n_{j-1}$ and $\epsilon_j = 1/jA$ in Lemma .6, define $n_j = m^*(M_j, \epsilon_j) = m^*(n_{j-1}, 1/Aj)$. Since $n_j = m_j > M_j = n_{j-1}$, we deduce that $\{n_j\}_{j\geq 0}$ is strictly increasing. The construction of boundary indices is completed.

Define $g_\psi(n) = \psi(n_0)$ for $1 \leq n < n_1$ and $g_\psi(n_j) = \psi(n_j)$ for all $j \geq 1$. Up till now, we still need to define $g_\psi(n)$ for $n_j < n < n_{j+1}$, for all $j \geq 1$.

Part 2 To this end, we further define the internal indices $r_1, \cdots$ and the remaining values of g_ψ in this part. They are defined as follows: fix each integer j, note that properties (.22) with $m^* = n_j$ and (.23) with $m^* = n_{j+1}$ in Lemma .6 are satisfied, more explicitly:

i). For all $n \geq 1$,

$$\psi(n_j + n) > \Big(b - \frac{1}{jA}\Big)^n \psi(n_j); \tag{.50}$$

ii). For all $1 \leq n \leq n_{j+1}$,

$$\psi(n_{j+1} - n) > \Big(b + \frac{1}{(j+1)A}\Big)^{-n} \psi(n_{j+1}). \tag{.51}$$

We use these two properties (.50) and (.51) to construct s_1, and define the value of g_ψ for n between $n_j + 1$ and $n_j + s_1$ as follows. It follows from (.50) and the mean value theorem that for each $0 \leq n \leq n_{j+1} - n_j$, there exists a unique real number $t_1(n) \geq -1$ such that

$$\psi(n_j + n) = \Big(b + \frac{t_1(n)}{jA}\Big)^n \psi(n_j), \tag{.52}$$

and by (.51) $t_1(n_{j+1} - n_j) \leq 1$. Choose

$$t_1 := \min_{1 \leq n \leq n_{j+1} - n_j} \{t_1(n)\} \in [-1, 1] \tag{.53}$$

with

$$\psi(n_j + r_1) = \Big(b + \frac{t_1}{jA}\Big)^{r_1} \psi(n_j)$$

for some $1 \leq r_1 \leq n_{j+1} - n_j$. Define

$$g_\psi(n) := \Big(b + \frac{t_1}{jA}\Big)^{n - n_j} \psi(n_j) \tag{.54}$$

for $n_j < n \leq n_j + r_1$. If $n_j + r_1 = n_{j+1}$, we stop, otherwise we continue the construction. In the latter case, by (.52) and (.53)

$$\psi(n_j + r_1 + n) = \Big(b + \frac{t_1(r_1 + n)}{jA}\Big)^{r_1 + n} \psi(n_j)$$

$$
\begin{aligned}
&\geq \Big(b + \frac{t_1}{jA}\Big)^{r_1+n}\psi(n_j) \\
&= \Big(b + \frac{t_1}{jA}\Big)^{n}\psi(n_j + r_1)
\end{aligned}
$$

for all $0 \leq n \leq n_{j+1} - n_j - r_1$, and thus for such n, there exists a unique real number $t_2(n) \geq t_1$ such that

$$
\psi(n_j + r_1 + n) = \Big(b + \frac{t_2(n)}{jA}\Big)^{n}\psi(n_j + r_1),
$$

and by (.51) $t_2(n_{j+1} - n_j + r_1) \leq 1$. Choose again

$$
t_2 := \min_{1 \leq n \leq n_{j+1} - n_j + r_1}\{t_2(n)\} \in [t_1, 1]
$$

with

$$
\psi(n_j + r_1 + r_2) = \Big(b + \frac{t_2}{jA}\Big)^{r_2}\psi(n_j + r_1)
$$

for some $1 \leq r_2 \leq n_{j+1} - n_j + r_1$. Define

$$
g_\psi(n) := \Big(b + \frac{t_2}{jA}\Big)^{n-n_j-r_1}\psi(n_j + r_1)
$$

for $n_j + r_1 < n \leq n_j + r_1 + r_2$.

We repeat this process until $n_{j+1} = n_j + r_1 + \cdots + r_s$ for some $s \geq 1$, and the process ends in finite step since for each $k \geq 1$, $r_k \geq 1$. The construction is completed with $g(n_{j+1})$ in the end coinciding the original definition $g(n_{j+1}) = \psi(n_{j+1})$.

Now we verify $\{g_\psi(n)\}_{n\geq 1}$ satisfies the properties in Proposition .5: $g_\psi(n)$ is non-decreasing as $b - 1/A > 1$ and $t_r \geq \cdots \geq t_1 \geq -1$. Set $r_0 = 0$ for simplicity, then for every $n_j \leq n \leq n_{j+1}$, there is some $1 \leq k \leq s$ such that $n_j + r_0 + \cdots r_{k-1} \leq n < n_j + r_0 + \cdots + r_k$, then by minimality of t_k

$$
\begin{aligned}
g_\psi(n) &= \Big(b + \frac{t_k}{jA}\Big)^{n-n_j-r_1-\cdots-r_{k-1}}\psi(n_j + r_1 + \cdots + r_{k-1}) \\
&\leq \Big(b + \frac{t_k(n - n_j + r_1 + \cdots + r_{k-1})}{jA}\Big)^{n-n_j-r_1-\cdots-r_{k-1}}\psi(n_j + r_1 + \cdots + r_{k-1}) \\
&= \psi(n).
\end{aligned}
$$

Since $\{n_j\}_{j\geq 1}$ is strictly increasing and $g(n_j) = \psi(n_j)$ for all $j \geq 1$, property (.19) in Proposition .5 is proved. It follows from $-1 \leq t_1 \leq \cdots \leq t_r \leq 1$ that

$$b - \frac{1}{jA} \leq \frac{g_\psi(n+1)}{g_\psi(n)} \leq b + \frac{1}{jA}$$

for all $n_j \leq n < n_{j+1}$ and $j \geq 1$. So $g_\psi(n+1)/g_\psi(n) \to b$, and $g_\psi(n)$ eventually grows exponentially fast so $g_\psi(n)/n \to \infty$, which proves properties (.20) and (.21).

Case III($b = 1$): The idea of the construction of g is analogous to the case $1 < b < \infty$. The reason we divide **Case II** and **Case III** into different cases is that when $b = 1$, no matter how small ϵ is, $b - \epsilon$ is no longer larger than 1, in which case the construction of g in **Case II** is invalid since g in (.54) is decreasing. We make a correction to g by multiplying a linear term n/n_j in the expression of g in (.59). The rest of the construction is similar to **Case II**. Let us proceed the proof of the construction.

Part 1 In this part, we define the boundary indices $\{n_j\}_{j\geq 1}$. Let $n_0 \geq 1$ be such that $\psi(n_0) = \min_{n\geq 1}\{\psi(n)\}$. Starting from n_0, assume by induction that for $j \geq 1, n_0, \cdots, n_{j-1}$ are already defined, then n_j is defined as follows: take $M_j = n_{j-1}$ and $\epsilon_j = 1/j$ in Lemma .7, define $n_j = m^*(M_j, \epsilon_j) = m^*(n_{j-1}, 1/j)$. since $n_j = m_j > M_j = n_{j-1}$, we deduce that $\{n_j\}_{j\geq 0}$ is strictly increasing.

Define $g_\psi(n) = \psi(n_0)$ for $1 \leq n < n_1$ and $g_\psi(n_j) = \psi(n_j)$ for all $j \geq 1$, and we still need to define $g_\psi(n)$ for $n_j < n < n_{j+1}$, for all $j \geq 1$.

Part 2 To this end, we further define the internal indices $r_1, \cdots$ and the remaining values of g_ψ in this part. Fix j, properties (.34) with $m^* = n_j$ and (.35) with $m^* = n_{j+1}$ in Lemma .7 are satisfied:

i). For all $n \geq 1$,

$$\psi(n_j + n) > \frac{n_j + n}{n_j}\psi(n_j); \tag{.55}$$

ii). for $1 \leq n \leq n_{j+1}$,

$$\psi(n_{j+1} - n) > \left(1 + \frac{1}{j+1}\right)^{-n}\psi(n_{j+1}). \tag{.56}$$

It follows from (.55) that for each $0 \leq n \leq n_{j+1} - n_j$, there exists a unique real number $t_1(n) \geq 0$ such that

$$\psi(n_j + n) = \Big(1 + \frac{t_1(n)}{j+1}\Big)^n \frac{n_j + n}{n_j} \psi(n_j), \tag{.57}$$

and by (.56) $t_1(n_{j+1} - n_j) \leq 1$. Choose

$$t_1 \coloneqq \min_{1 \leq n \leq n_{j+1} - n_j} \{t_1(n)\} \in [0, 1] \tag{.58}$$

with

$$\psi(n_j + r_1) = \Big(1 + \frac{t_1}{j+1}\Big)^{r_1} \frac{n_j + r_1}{n_j} \psi(n_j),$$

for some $1 \leq r_1 \leq n_{j+1} - n_j$. Define

$$g_\psi(n) \coloneqq \Big(1 + \frac{t_1}{j+1}\Big)^{n - n_j} \frac{n}{n_j} \psi(n_j) \tag{.59}$$

for $n_j < n \leq n_j + r_1$. If $n_j + r_1 = n_{j+1}$, we stop, otherwise we continue the construction. In the latter case by (.57) and (.58)

$$\begin{aligned}
\psi(n_j + r_1 + n) &= \Big(1 + \frac{t_1(r_1 + n)}{j+1}\Big)^{r_1 + n} \frac{n_j + r_1 + n}{n_j} \psi(n_j) \\
&\geq \Big(1 + \frac{t_1}{j+1}\Big)^{r_1 + n} \frac{n_j + r_1 + n}{n_j} \psi(n_j) \\
&= \Big(1 + \frac{t_1}{j+1}\Big)^{n} \frac{n_j + r_1 + n}{n_j + r_1} \psi(n_j + r_1),
\end{aligned}$$

for all $0 \leq n \leq n_{j+1} - n_j - r_1$, and thus for such n, there exists a unique real number $t_2(n) \geq t_1$ such that

$$\psi(n_j + r_1 + n) = \Big(1 + \frac{t_2(n)}{j+1}\Big)^{n} \frac{n_j + r_1 + n}{n_j + r_1} \psi(n_j + r_1),$$

and by (.56) $t_2(n_{j+1} - n_j + r_1) \leq 1$. Choose again

$$t_2 \coloneqq \min_{1 \leq n \leq n_{j+1} - n_j + r_1} \{t_2(n)\} \in [t_1, 1],$$

with

$$\psi(n_j + r_1 + r_2) = \Big(1 + \frac{t_2}{j+1}\Big)^{r_2} \frac{n_j + r_1 + r_2}{n_j + r_1} \psi(n_j + r_1)$$

for some $1 \leq r_2 \leq n_{j+1} - n_j + r_1$. Define

$$g_\psi(n) := \Big(1 + \frac{t_2}{j+1}\Big)^{n - n_j - r_1} \frac{n}{n_j + r_1} \psi(n_j + r_1)$$

for $n_j + r_1 < n \leq n_j + r_1 + r_2$.

We repeat this process until $n_{j+1} = n_j + r_1 + \cdots + r_s$ for some $s \geq 1$, and the process ends in finite steps since for each $1 \leq k \leq r$, $r_k \geq 1$. The construction is completed with $g(n_{j+1})$ coinciding the original definition $g(n_{j+1}) = \psi(n_{j+1})$.

Now we verify $\{g_\psi(n)\}_{n\geq 1}$ satisfies the properties in Proposition .5: By minimality of $\{t_j\}_{j\geq 1}$, $g_\psi(n) \leq \psi(n)$ for all $n \geq 1$, and it follows from $g(n_j) = \psi(n_j)$ for all $j \geq 1$ that property (.19) in Proposition .5 is true. By construction $g_\psi(n)/n \geq \psi(n_j)/n_j$ for all $n_j \leq n < n_{j+1}$, and thus property (.20) is true. Finally it follows from for all $n \geq 1$ and for some $j \geq 0, k \geq 1$ and $t_k \geq 0$

$$1 \leq \Big(1 + \frac{t_k}{j+1}\Big) \frac{n+1}{n} = \frac{g_\psi(n+1)}{g_\psi(n)} \leq \Big(1 + \frac{1}{j+1}\Big) \frac{n+1}{n}$$

that $g_\psi(n)$ is non-decreasing and property (.21) is true, and the proof of Proposition .5 is completed. $\square$

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