

# Supplementary material: Observation of swallowtail catastrophe singularity in non-Hermitian bands and its topological origin

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## 1. Materials and methods

### 1.1 INIC for implementing non-reciprocal hoppings

To implement non-reciprocal hoppings, we use a negative impedance converter through current inversion (INIC) [1-4], which incorporates a unity-gain stable operational amplifier (OpAmp) with two resistors  $R_a$  and  $R_b$  that are in the positive and negative feedback circuit, respectively (see Fig. S1). When the OpAmp is operated stably in a negative feedback configuration, the input  $I_1$  and output  $I_2$  currents (depending on the node voltages  $V_1$  and  $V_2$ ) can be calculated by assuming that the negative potential is ideally equal to the positive input potential  $V_1$

$$I_1 = -\frac{R_b}{R_a} \cdot i\omega C_i \cdot (V_1 - V_2) \quad (\text{S1})$$

$$I_2 = i\omega C_i \cdot (V_1 - V_2) \quad (\text{S2})$$

The OpAmp, being an active circuit elements, helps break the circuit reciprocity. To experimentally implement the anti-symmetric parts of the Hamiltonian,  $R_b=R_a$  is required, which results in

$$I_1 = -I_2 \quad (\text{S3})$$

It is shown that the INIC makes the admittances (i.e. hopping parameters in tight binding model) from the node 1 to 2 ( $G_{12}$ ) and from 2 to 1 ( $G_{21}$ ) opposite

$$G_{12} = -i\omega C_i, G_{21} = i\omega C_i \quad (\text{S4})$$

Viewing from node 2, the capacitance is positive (i.e.  $C_i$ ), but from node 1, it becomes a negative capacitance (i.e.  $-C_i$ ).

In the sample, the unity-gain stable OpAmps of model ADA4625-1ARDZ-R7 were employed. To ensure normal operations, dual supply of  $\pm 5\text{V}$  was provided and the frequency of the input AC current of the circuit was set as low as  $f=1\text{kHz}$ .

### 1.2 Topoelectric circuit design

For a grounded circuit, the Laplacian formalism of admittance matrix is given by [4-6]

$$J = D + W - C, \quad (\text{S5})$$

where  $D$  donates the total node conductance,  $W$  represents the ground matrix and  $C$  is the adjacency matrix. In our designed topoelectric circuit with three nodes (in Fig. 1B), the total node conductance is a diagonal matrix

$$D = i\omega \begin{bmatrix} -C_1 - C_2 & 0 & 0 \\ 0 & C_1 + C_3 & 0 \\ 0 & 0 & C_2 + C_3 \end{bmatrix} \quad (\text{S6})$$

Each diagonal element involves the sum of all components connected to the corresponding node.

The adjacency matrix  $C$  is characterized by

$$C = i\omega \begin{bmatrix} 0 & C_1 & C_2 \\ -C_1 & 0 & C_3 \\ -C_2 & C_3 & 0 \end{bmatrix}, \quad (\text{S7})$$

where the elements in the matrix represent the hoppings via capacitances between each pair of nodes. In detail, the element  $C_1$  ( $C_2$ ) represents the hopping from node b to a (c to a), while  $-C_1$  ( $-C_2$ ) is the hopping in the opposite direction—from node a to b (a to c). The hoppings from b to c and c to b are the same, which is represented by the two symmetric elements  $C_3$  in the matrix. The ground matrix  $W$  reads

$$\begin{aligned} W = i\omega & \begin{bmatrix} C_g + 1/(i\omega R_g) + C_{g0} & 0 & 0 \\ 0 & C_g + 1/(i\omega R_g) & 0 \\ 0 & 0 & C_g + 1/(i\omega R_g) \end{bmatrix} \\ & = (i\omega C_g + 1/R_g)I + i\omega \begin{bmatrix} C_{g0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (\text{S8})$$

which is also a diagonal matrix and each element denotes the contributions from grounded capacitors ( $C_g$  and  $C_{g0}$ ) and resistors ( $R_g$ ) to each node. Inserting Eqs. (S6-8) into Eq. (S5) yields

$$J = (i\omega C_g + 1/R_g)I + i\omega \begin{bmatrix} -C_1 - C_2 + C_{g0} & -C_1 & -C_2 \\ C_1 & C_1 + C_3 & -C_3 \\ C_2 & -C_3 & C_2 + C_3 \end{bmatrix} \quad (\text{S9})$$

By setting  $C_1 = f_1 C_0$ ,  $C_2 = f_2 C_0$ ,  $C_3 = f_3 C_0$ ,  $C_{g0} = C_0$ , we finally arrive at

$$\begin{aligned} J &= \underbrace{(i\omega C_g + 1/R_g)I}_{\varepsilon_0(\omega)} + i\omega C_0 \underbrace{\begin{bmatrix} -f_1 - f_2 + 1 & -f_1 & -f_2 \\ f_1 & f_1 + f_3 & -f_3 \\ f_2 & -f_3 & f_2 + f_3 \end{bmatrix}}_{\bar{J}} \\ &= \varepsilon_0(\omega) + i\omega \bar{J} \end{aligned} \quad (\text{S10})$$

where the effective Laplacian  $\bar{J}$  is exactly the effective Hamiltonian in Eq.(1) with a common divisor  $C_0$ . Here  $C_0$  is set to be 10nF, in consideration of the magnitude of the Hamiltonian parameters and the reasonable capacitance range in the circuits. The AC driving frequency of the system is an external parameter and was generally chosen to be 1kHz in the experiments, leaving  $\omega = 2\pi f$  a constant. Therefore, the parameters  $f_1$ ,  $f_2$  and  $f_3$  can be precisely tuned by changing the capacitances  $C_1$ ,  $C_2$  and

$C_3$ , respectively. The rest part  $\varepsilon_0(\omega)$ , resulting from the equally grounded resistance  $R_g = 1M\Omega$  and capacitance  $C_g = 1nF$  to each node, is simply ignored, because it only contributes to a complex shift of eigenvalues and does not have impact on the eigenstates.

### 1.3 Experimental operation and observation

The experimental sample is basically made up of surface mounted device (SMD) capacitors, resistors and OpAmps on a printed circuit board (PCB). As shown in Fig. 1C, multiple capacitors are parallelly connected between adjacent nodes, and each capacitor is connected by a serial toggle switch, so that it can be individually controlled. Hence, the capacitance  $C_1$  (as well as  $C_2$  and  $C_3$ ) becomes a combination of the parallelly connected capacitors. The corresponding  $f_1$ ,  $f_2$  and  $f_3$  values in the Hamiltonian are thus tunable by controlling the switches. The required capacitors (with tolerance of 1%) in experiments are listed in Table S1, and the three columns are used for tuning  $C_1$ ,  $C_2$  and  $C_3$ , respectively. Taking an exceptional point at parameter  $(f_1, f_2, f_3) = (0.2283, 0.1083, 0.3)$  as an example, it can be achieved by setting  $C_1=2.283nF$ ,  $C_2=1.083nF$  and  $C_3=3nF$ , which are decomposed as the following

$$C_1=43pF+560pF+680pF+1nF;$$

$$C_2=43pF+150pF+430pF+560pF;$$

$$C_3=1.5nF+1.5nF.$$

To observe this point, we simply turn on the switches of the corresponding capacitors with the above values, and leaving other switches off.

In the experiments, a DC power supply (GPC-3030) served as the dual voltages of  $\pm 5V$  for the OpAmps to operate normally. A waveform generator (Keysight: M3201A) was used to excite the system and a sinusoidal voltage with constant amplitude (generally  $1V \sim 2V$ ) and frequency of  $1kHz$  was set to feed into each node individually. A matching oscilloscope (RS PRO IDS1074B) was employed to measure the voltage response of all nodes in the system. The input current can be acquired by connecting a shunt resistor of  $R=10\Omega$  from the input node to the voltage source.

With the measured voltage response to the input current vector, one can directly obtain the Green's function matrix  $G$ , which is inverse to the admittance matrix  $J$  [1-8]. The admittance eigenvalues and eigenstates are thus easily retrieved from the Green's function  $G$ .

## 2. Riemannian geometry

Here we demonstrate that the adiabatic transformation process of eigenstates is Riemannian geometry. The metric operator in pseudo-Hermiticity plays a similar role to the space-time metric in general relativity [9]. The time varying problem is governed by the Schrödinger equation

$$H|\varphi_m\rangle=i\partial_t|\varphi_m\rangle \quad (\text{S11})$$

and the completeness of eigenstates shows that a field can be expanded as

$$\phi_n(\lambda(t))=\sum_m[U(\lambda(t))]^{-1}{}_n{}^m\varphi_m(\lambda(t)) \quad (\text{S12})$$

Applying the partial derivative with respect to  $t$ , one obtains

$$\begin{aligned} i \frac{\partial}{\partial t} \phi_n(\lambda(t)) &= H[U(\lambda(t))]^{-1} \phi_m(\lambda(t)) \\ &= i \frac{\partial[U(\lambda(t))]^{-1}}{\partial t} \phi_m(\lambda(t)) + i[U(\lambda(t))]^{-1} \frac{\partial \phi_m(\lambda(t))}{\partial t} \end{aligned} \quad (\text{S13})$$

The adiabatic approximation allows the instantaneous eigenvalue problem

$$H(\lambda(t))\phi_m(\lambda(t)) = E_m \phi_m(\lambda(t)) \quad (\text{S14})$$

and applying a scalar product  $\langle \phi_l |$  from the left of Eq. S13 yields

$$-iE_l[U(\lambda(t))]^{-1} = \frac{\partial[U(\lambda(t))]^{-1}}{\partial t} + \langle \phi_l | \eta \frac{\partial |\phi_m(\lambda(t))\rangle}{\partial t} [U(\lambda(t))]^{-1} \quad (\text{S15})$$

The partial derivative with respect to  $t$  can be expanded as

$$\frac{\partial |\phi_m(\lambda(t))\rangle}{\partial t} = \sum_k \frac{\partial |\phi_m(\lambda(t))\rangle}{\partial \lambda^k} \frac{\partial \lambda^k}{\partial t} \quad (\text{S16})$$

We define the affine connection

$$A_k^n = -\langle \phi_n | \eta \frac{\partial |\phi_m(\lambda(t))\rangle}{\partial \lambda^k} = -\langle \phi_n | \eta \frac{\partial}{\partial \lambda^k} |\phi_m\rangle \quad (\text{S17})$$

and the solution to  $U^{-1}$  is thus obtained as

$$U^{-1} = \exp\left[\int_0^t ds \frac{\partial \lambda^k}{\partial s} A_k - i \int_0^t ds \omega(\lambda(s))\right] = \exp\left(\int_{\lambda(0)}^{\lambda(t)} d\lambda^k A_k\right) \times \exp\left[i \int_0^t ds E(\lambda(s))\right] \quad (\text{S18})$$

Ignoring the dynamical phase, the geometric phase is simply

$$U^{-1} = P \exp\left(\int_{\lambda(0)}^{\lambda(t)} d\lambda^k A_k\right) \quad (\text{S19})$$

where  $P$  denotes path ordering operator, which is important here, because the affine connection  $A$  is a matrix. Considering the non-commutative nature of matrix product,  $A$  is a non-Abelian parallel transport gauge [10], and the integration of  $A$  on closed loops depends on the path circulating singularities. Here we define a local metric  $g$  with its elements being

$$g_{mn} = \langle \phi_m | \eta \phi_n \rangle \quad (\text{S20})$$

which has explicit relations with the affine connection. The normalization of eigenstates can make  $g$  a constant and thus the partial derivative with respect to time vanishes

$$0 = \partial_t g_{mn} = \partial_t \langle \phi_m | \eta \phi_n \rangle \quad (\text{S21})$$

Inserting the identity operator  $I = \sum_l \eta |\phi_l\rangle \langle \phi_l|$ , one obtains

$$\partial_{\lambda_k} \langle \phi_m | \eta \phi_n \rangle = \sum_l \langle \partial_{\lambda_k} \phi_m | \eta | \phi_l \rangle \langle \phi_l | \eta \phi_n \rangle + \sum_l \langle \phi_m | \eta | \phi_l \rangle \langle \phi_l | \eta \partial_{\lambda_k} \phi_n \rangle \quad (\text{S22})$$

We note that

$$\langle \partial_{\lambda_k} \varphi_m | \eta | \varphi_l \rangle = \langle \varphi_l | \eta | \partial_{\lambda_k} \varphi_m \rangle^* \quad (\text{S23})$$

and we have

$$0 = A_k^{*l} g_{ln} + g_{ml} A_k^l \quad (\text{S24})$$

This important relation between the local metric  $g$  and the affine connection  $A$  reveals the Riemannian geometry of the adiabatic transformation process of eigenstates. In the next section, we will use this equation to predict the emergence of exceptional surfaces (ESs) and nodal line (NL).

### 3. Predicting the emergence of ESs and NL in view of Riemannian geometry

The metric operator in determining the pseudo-Hermitian symmetry of the considered system is the Riemannian metric. In regions with  $PT$ -exact phases, there are three real eigenstates, due to the fact that the system preserves  $PT$  symmetry and the Hamiltonian has been gauged to be real. These eigenstates are orthogonal under an indefinite inner product

$$\langle \varphi_m | \eta \varphi_n \rangle = \begin{cases} 0 & m \neq n \\ \neq 0 & m = n \end{cases} \quad (\text{S25})$$

and thus the metric  $g$  is diagonal and real. Within the three diagonal elements in  $g$ , two of which are positive, and the other is negative, i.e. the following three forms

$$g_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{S26})$$

In the swallowtail catastrophe, the parameter space is partitioned into three regions by the ESs, as shown in Fig. S2A. Here Regs I and II are the  $PT$ -exact phases, and Reg III is the  $PT$ -broken phase.

We take Reg I as an example to show that the Riemannian geometry can be used to predict the emergence of ES and NL. The local metric  $g$  of Reg I can be calculated as  $g_1$ . Since the eigenstates are real, the definition (Eq. S17) shows that the affine connection is also real. Hence, Eq. S24 reduces to

$$0 = A_k^l g_{ln} + g_{ml} A_k^l \quad (\text{S27})$$

The metric is diagonal, and one can easily demonstrate that the connection is a linear combination of the following elementary matrices

$$T_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{S28})$$

which are exactly the Lie algebraic generators of  $\text{SO}(2,1)$  group. It is shown that  $T_2$  and  $T_3$  characterize Lorentz boost between  $\varphi_1$  and  $\varphi_3$ , and between  $\varphi_2$  and  $\varphi_3$ , respectively. While  $T_3$  characterizes the rotation of  $\varphi_1$  and  $\varphi_2$ . The  $\pi$  rotation of  $\varphi_1$  and  $\varphi_2$  is expressed by the operation

$$[\varphi_1, \varphi_2, \varphi_3] \exp(\pi T_1) = [-\varphi_1, -\varphi_2, \varphi_3] \quad (\text{S29})$$

bringing them to their opposite directions, which are still the eigenstate at the same point in parameter space. Consider a closed loop on which the eigenstates adiabatically evolve and accumulate a matrix form geometric phase ( $\pi T_1$  in Eq. S29). The loop simply circulates the nodal line (NL), and the polarizations of  $\varphi_1$  and  $\varphi_2$  rotate  $\pi$  as indicated by Fig. 2C2. This is a way for predicting the existence of the NL formed by the first and the second bands. The ES can also be predicted. The Lorentz boost between the second and the third bands can be characterized by

$$[\varphi_1, \varphi_2, \varphi_3] \exp(\beta T_3) \quad (\text{S30})$$

and the matrix exponential is

$$\exp(\beta T_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \beta & \sinh \beta \\ 0 & \sinh \beta & \cosh \beta \end{bmatrix} \quad (\text{S31})$$

As  $\beta$  approaches  $+\infty$ , we have the relation  $\cosh \beta \approx \sinh \beta = \rho$ , resulting in two parallel vectors in the same directions

$$\begin{aligned} \varphi_1 \cosh \beta + \varphi_2 \sinh \beta &= \rho(\varphi_1 + \varphi_2) \\ \varphi_1 \sinh \beta + \varphi_2 \cosh \beta &= \rho(\varphi_1 + \varphi_2) \end{aligned} \quad (\text{S32})$$

Similarly, as  $\beta$  approaches  $-\infty$ ,  $\cosh \beta \approx -\sinh \beta = \rho$ , resulting in another pair of parallel vectors in opposite directions

$$\begin{aligned} \varphi_1 \cosh \beta + \varphi_2 \sinh \beta &= \rho(\varphi_1 - \varphi_2) \\ \varphi_1 \sinh \beta + \varphi_2 \cosh \beta &= -\rho(\varphi_1 - \varphi_2) \end{aligned} \quad (\text{S33})$$

In band gaps,  $\beta$  cannot approach infinity by an integration on a path, and an infinitely large  $\beta$  can only be provided by approaching the exceptional surfaces. The eigenstates that coalesce are  $\varphi_1$  and  $\varphi_2$ , and thus the ES, being the boundary of Reg I, can be formed by the second and third bands. We note that the  $\varphi_1$  and  $\varphi_3$  also experience Lorentz boost, but the second band blocks the formation of ES between the first and the third bands. However, if we consider Reg III with metric  $g_2$ , the generators will be

$$T'_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T'_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad T'_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{S34})$$

meaning that ESs can be formed by the first and the second bands, or by the second and the third bands (due to Lorentz boost characterized by  $T_1$  and  $T_3$ ). We also note that there is another rotation of  $\varphi_1$  and  $\varphi_3$  (i.e.  $\exp(\pi T'_2)$ ), but the second band blocks the formation of NL by the first and the third bands. Regions having the local metric  $g_3$  is also possible, but such regions are not present in the considered parameter ranges of the Hamiltonian (Eq. 1 in the maintext). We will not give more analysis. For exact phases, regions with different local metrics are not connected, and they are separated by broken phase.

The broken phases have very different forms of local metrics compared with the exact phases. In broken phases, there will always be a pair of eigenstates that are conjugate to each other, and the other eigenstate is real. The local metric in broken phase is not diagonal, but is a Hermitian matrix instead. Possible forms can be (note that the eigenstates are properly normalized)

$$g_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{S35})$$

The regions with different metrics in broken phase are path connected, contrary to exact phases. The boundary between regions with different metrics is a surface with real parts of all three eigenvalues being degenerate. The surface is not exactly a degeneracy because the imaginary parts of eigenvalues are not degenerate.

#### 4. ADE classification of swallowtail: A4 singularity

Before introducing ADE classification, we firstly need to introduce another mathematical concept, which is the orbifold. An orbifold is much like a smooth manifold but possibly with singularities of the form of fixed points of finite group actions. A smooth manifold is a space locally modelled on Cartesian space/Euclidean spaces  $\mathbf{R}^n$ , an orbifold is, more generally, a space that is locally modelled on smooth action groupoids (homotopy quotients)  $\mathbf{R}^n // G$  of a finite group  $G$  on a Cartesian space.

An  $n$ -dimensional orbifold is a Hausdorff topological space  $X$ , called the underlying space, with a covering by a collection of open subsets  $U_i$ , closed under finite intersections. For each  $U_i$ , there is:

1. an open subset  $V_i$  of  $\mathbf{R}^n$ , invariant under a faithful linear action of finite group  $\Gamma_i$ ;
2. a continuous map  $\varphi_i$  of  $V_i$  onto  $U_i$  invariant under  $\Gamma_i$ , called an orbifold chart, which defines a homeomorphism between  $V_i/\Gamma_i$  and  $U_i$ .

The collection of orbifold charts is called an orbifold atlas if the following properties are satisfied:

1. for each inclusion  $U_i \subset U_j$  there is an injective group homeomorphism  $f_{ij}: \Gamma_i \rightarrow \Gamma_j$
2. for each inclusion  $U_i \subset U_j$  there is a  $\Gamma_i$ -equivalent homeomorphism  $\psi_{ij}$ , called a gluing map, of  $V_i$  onto an open subset of  $V_j$
3. the gluing maps are compatible with the charts, i.e.  $\varphi_j \psi_{ij} = \varphi_i$
4. the gluing maps are unique up to composition with group elements, i.e. any other possible gluing map from  $V_i$  to  $V_j$  has the form  $g \cdot \psi_{ij}$  for a unique  $g$  in  $\Gamma_j$

The orbifold atlas defines the orbifold structure completely: two orbifold atlases of  $X$  give the same orbifold structure if they can be consistently combined to give a larger orbifold atlas. Note that the orbifold structure determines the isotropy of any point of the orbifold up to isomorphism: it can be computed as the stabilizer of the point in any orbifold chart. If  $U_i \subset U_j \subset U_k$ , then there is a unique transition element  $g_{ijk}$  in  $\Gamma_k$  such that  $g_{ijk}\psi_{ik} = \psi_{jk}\psi_{ij}$ . These transition elements satisfy  $(\text{Ad } g_{ijk})f_{ik} = f_{jk}f_{ij}$ , as well as the cocycle relation  $f_{km}(g_{ijk})g_{ikm} = g_{ijm}g_{jkm}$ .

An ADE singularity is an orbifold fixed point locally of the form  $\mathbf{C}^n // \Gamma^*$  with  $\Gamma^* \hookrightarrow SU(2)$  a finite subgroup of  $SU(2)$  given by the ADE classification (and  $SU(2)$  is understood with its defining

linear action on the complex vector space  $\mathbf{C}^2$ ). As is known, the finite subgroups of  $SO(3)$  are exhausted by the following list:

1. the cyclic group  $\mathbf{Z}_n$ ;
2. the dihedral group  $D_{2n}$ , isomorphic to the semidirect product of  $\mathbf{Z}_n$  and  $\mathbf{Z}_2$ ;
3. the groups of motions of the tetrahedron,  $T_{12}$ , of the octahedron,  $O_{24}$ , and of the icosahedron,  $I_{60}$ .

Let  $\Gamma$  be a discrete subgroup of  $SO(3)$ . Consider its preimage  $\Gamma^* \hookrightarrow SU(2)$  under the two sheeted covering map  $SU(2) \rightarrow SO(3)$ . The group  $\Gamma^*$  is called the binary group of the corresponding polyhedron and acts on  $\mathbf{C}^2$  as a subgroup of  $SU(2)$ . Consider the algebra of polynomial invariants of this action of  $\Gamma^*$ . As it turns out, this algebra is generated by three invariants  $x, y$  and  $z$ , which satisfy a single relation. This relation defines a hypersurface  $V$  in the space  $\mathbf{C}^2$  with coordinates  $x, y$  and  $z$ .  $V$  is naturally isomorphic to the orbifold of the action of  $\Gamma^*$  and has an isolated singular point at the origin.

For a suitable choice of generators in the algebra of invariants, the relations for the binary groups of polyhedra are as Table S2. Thus, the orbifold  $V$  of the action of a binary polyhedral group on  $\mathbf{C}^2$  is isomorphic to the zero level set of the corresponding singularity.

The preimage of the singular point on  $V$  is a connected union of projective lines:

$$\pi^{-1}(0) = C_1 \cup C_2 \cup \dots \cup C_\mu, \quad C_i \cong \mathbf{CP}^1 \quad (\text{S36})$$

The self-intersection index of each component  $C_i$  is equal to -2. Pairwise intersections are described by a graph in which a vertex is assigned to each component  $C_i$ , and two vertices are or are not connected by an edge depending on whether the intersection index of the corresponding components is 1 or 0. In this manner one obtains the Dynkin diagrams (see Fig. S3).

The orbit of a point  $x \in \mathbf{C}^{n+1}$  such that  $x_1 + \dots + x_\mu = 0$  under the action of the group  $A_\mu$  is described by the unordered set of  $\mu$  points (counting multiplicities)  $x_1, \dots, x_\mu$  on  $\mathbf{C}$  and is given by the polynomial.

$$\prod_{i=1}^{\mu-1} t(t - x_i) = t^{\mu+1} + \lambda_\mu t^\mu + \dots + \lambda_2 t^2 + \lambda_1 t \quad (\text{S37})$$

With the coefficients  $\lambda_i(x)$  corresponding to the parameter space (e.g. the parameter space  $f_1, f_2$  and  $f_3$  in the Hamiltonian Eq. 1 in the maintext). The swallowtail is the set of zeros of the discriminant of the polynomial, and is the  $A_4$  classification. Hence, the parameter space has three degrees of freedom  $\lambda_1, \lambda_2$  and  $\lambda_3$ .

It can be identified that the ADE classification (A4 type) cannot exactly explain the formation of the swallowtail singularity in our considered system. Because the characteristic polynomial of the Hamiltonian (Eq. 1 in the maintext) only has three eigenvalues (not four or five in Eq. S37). The discriminant of the characteristic polynomial is more complicated than that of Eq. S37. The formation of the swallowtail in band structures can only be explained by the adiabatic transformation of eigenstates.

## 5. Loops, hypersurface singularities, and fractional topological invariants

In non-Hermitian systems, it is possible that a topological invariant is not an integer, but a fraction instead. In this section, we will discuss some possible singularities that carry fractional invariants. Complete intersections (NILs), cusps (EL3s) will be included in the discussion.

Before discussing the fractional invariant, we should provide the definition of fractional topology. In topological physics, adding a minus sign to a state is treated to be the same state. Hence, if eigenstates of a Hamiltonian evolve to its negative  $\varphi \rightarrow -\varphi$  via adiabatic transformation on a closed loop, the eigenstates are considered to have evolved to the initial states. Such a loop carries an integer topological invariant, and representative examples are loops circulating 2D Dirac points [12] or 3D nodal lines [13] in Hermitian systems. However, multiplying the state by an imaginary unit is not regarded as the same state, and thus the loops bringing  $\varphi$  to  $i\varphi$  is not considered to have evolved to the initial state. Such loops carry fractional topological invariants. To obtain an integer invariant, one need to find the complementary singularity, or observe the adiabatic transformation process of more cycles of on the loop. In some cases, one eigenstate can evolve to another eigenstate on a closed loop, which is also a representative example. Next, we will discuss these situations in detail.

Firstly, let us consider a simple example, which is a two-band model and the Hamiltonian takes the form

$$H(\mathbf{k}) = (\sigma_1 + i\sigma_2) + k_x\sigma_1 + k_y\sigma_2 \quad (\text{S38})$$

The band structure is shown in Fig. S4A, which has two isolated exceptional points (EPs) in parameter space. It can be easily found that the two EPs comes from the splitting of a Hermitian Dirac point by introducing non-Hermitian perturbations (the term  $i\sigma_2$  in Eq. S38). The topology of the two EPs can be investigated by closed loops  $p_1$  and  $p_2$  circulating them ( $p_1$  and  $p_2$  have the same basepoint, the red point in Fig. S4A). By observing the evolution of eigenvalues on a closed loop circulating the EP, the two eigenvalues braid, resulting an exchange  $\omega_1 \leftrightarrow \omega_2$  (simultaneously  $\varphi_1 \leftrightarrow \varphi_2$ , see Fig. S4B-C). It is obvious that the eigenstates cannot evolve to the initial states after one cycle on the loops, and the topological invariant carried by the loop is a fraction. However, if one consider the composite loop  $p_1p_2$ , the braiding of the eigenvalues will cancel, because the eigenvalues are braiding in opposite directions (see the red arrows in Fig. S4B-C). The composite loop carries an integer topological invariant, on which the eigenstates will accumulate a  $\pi$  geometric phase, equivalent to the topology a 2D Dirac point. Two cycles on the loop ( $p_1$  or  $p_2$ ) will result in the braiding of eigenvalues twice ( $\omega_1 \rightarrow \omega_2 \rightarrow \omega_1$ ,  $\omega_2 \rightarrow \omega_1 \rightarrow \omega_2$ ), and eigenstates can evolve to their negatives ( $\varphi_1 \rightarrow -\varphi_1$ ,  $\varphi_2 \rightarrow -\varphi_2$ ), accumulating  $\pi$  geometric phase. Hence, the winding number of the exceptional point is a half integer (i.e. 1/2). It is notable that the integration of Berry connection on one cycle of the loop is not gauge independent, due to the fact that the eigenstates cannot evolve to the initial states. To avoid such problems, some researchers would like to use energy vorticity to define a gauge independent topological invariant [11]. If one would like to employ the adiabatic transformation of eigenstates, the geometric phase will be accumulated on  $n$  cycles on the closed loops [13], so that the eigenstates can evolve to the initial states to ensure gauge independence. If the total geometric phase is  $m\pi$ , the topological invariant is simply the fraction  $m/n$ .

Next, we consider the complete intersections and cusps in our Hamiltonian. The NIL is an intersection of ESs, and the corresponding Hamiltonian is a non-defective matrix and can be diagonalized. While the Hamiltonian on ES is defective (lacking one eigenstate) and cannot be diagonalized. It can be easily identified that the eigenstates at the NIL come from the contributions of the intersecting ESs. The topology of such singularities are characterized by the loops  $l_2$  and  $l_4$  in Fig. 2A and Fig. 3A in the maintext [15], whose base points are on NIL. In view of the adiabatic transformations, the eigenstates on one ES evolve to the eigenstates on the other ES. *An equivalent statement is that one eigenstate of the NIL adiabatically evolves to the other eigenstate of the NIL.* It is obvious that such process cannot bring the eigenstates to the initial states, and thus carry fractional topological invariants. To obtain a gauge independent topological invariant, we need to consider the

loops  $l_2^2$  and  $l_7^2$  in Fig. 2A and Fig. 3A in the maintext. Since eigenstates  $\varphi_2$  and  $\varphi_3$  rotate  $\mp\pi$  ( $\varphi_{2,3} \rightarrow -\varphi_{2,3}$ , see Fig. 2D and Fig. 3C in the maintext), the eigenstates evolve to the initial state, which carry integer topological invariants. Correspondingly, the winding numbers on  $l_2$  and  $l_7$  are both half integers (i.e. 1/2).

The EL3s are a little different from the NIL. In catastrophe theory, the cusps are formed due to the folding of curves in higher dimensions [16]. In band structures, the EL3s are formed in the same way. As can be obviously indicated from Fig. 2-3 in the maintext that the ES is folded at the cusps. Hence, on loop  $l_3$  (Fig. 2A), the eigenvalues experience order exchange twice ( $\omega_1 \rightarrow \omega_{2,3} \rightarrow \omega_1$  and  $\omega_{2,3} \rightarrow \omega_1 \rightarrow \omega_{2,3}$ ) at each of the cusps, due to the folding of ES. The eigenstates  $\varphi_1 \sim \varphi_3$  experience  $(-\pi, 0, \pi)$  rotations on the loop, meaning that the eigenstates return to the initial states, and thus  $l_3$  carries an integer topological invariant. However, the loop  $l_3$  can be decomposed into two sub-loops  $l_8$  and  $l_9$  (see Fig. S5A), and each loop circulates one EL3. The eigenvalues on the sub-loop  $l_8$  will experience order exchange once, and thus the final states on the loop are different from the initial states (see Fig. S5B). One may consider if the cusp is similar to the complete intersection, carrying a half integer winding number. In fact, the current case is a little different, because in cusps, all three eigenstates are involved (the NIL only involves two eigenstates). As shown in Fig. S5B-C, the eigenvalues experience the following processes of order change on  $l_8$ : starting from the ES, in the first cycle, band 2 and band 3 bifurcate in the gap, and then band 1 and band 2 coalesce on the other ES. After the parameters traverse the EL3, the coalesced bands (1 and 2) exchange the order with band 3, and finally the band sequence is rearranged as (2,3,1). Such process will be repeated on the second and third cycles, and the sequences are rearranged as (3,1,2) and (1,2,3). Next, we turn our attention to the eigenstates. The adiabatic transformation process of eigenstates on loops  $l_8$  of three cycles is shown in Fig. S5D. **The eigenstates evolve to the initial states, and the accumulated geometric phase can be obtained as  $2\pi$ . Hence, the winding number on  $l_8$  (also  $l_9$ ) is 2/3 instead of 1/2. (require discussions)** Now we consider the loop  $l_4$ , which encloses the double EL3s in the broken phase region (see Fig. 3A in the maintext). Similar to  $l_3$ ,  $l_4$  also traverses the cusps twice, leading to the order exchange of eigenvalues simultaneously. However, in broken phase regions, there exist a surface, on which the real part of all three eigenvalues are degenerate (we call it quasi-degeneracy), and whenever the loop  $l_5$  traverses surface (at least twice, or multiples of two times), the eigenvalues will also experience order exchange twice. Hence, the decomposed sub-loops  $l_{10}$  and  $l_{11}$  of  $l_5$  (see Fig. S5A), will be very different from  $l_8$  and  $l_9$ . Let us take  $l_{10}$  as an example, the loop traverses the quasi-degeneracy and EL3, experiencing the order exchange process of eigenvalues  $\omega_{2,3} \rightarrow \omega_1 \rightarrow \omega_{2,3}$  (see Fig. S5E). As for eigenstates,  $\varphi_2$  and  $\varphi_3$  rotate  $\pm\pi/2$  for one cycle on the loop, meaning that the eigenstates cannot evolve to the initial states. The eigenstates can evolve to the initial states for two cycles (see Fig. S5F), bringing  $\varphi_2$  and  $\varphi_3$  to their negatives (i.e.  $\pm\pi$  for  $\varphi_2$  and  $\varphi_3$ ) (Fig. S5G-H). This adiabatic transformation process shows that  $l_{10}$  and  $l_{11}$  both carry half integer (i.e. 1/2) winding numbers.

The above discussions provides a conclusion that singularities with fractional topological are often present in pairs, because it is always obtained from a splitting of a singularity carrying integer topological invariant. On a closed loop circulating a singularity with an integer topological invariant, both the eigenvalues and eigenstates can evolve to the initial state, which is totally opposite to the singularity with fractional invariant. The non-Hermiticity introduces a degree of freedom that eigenvalues can exchange orders (braiding or band folding) or coalesce (exceptional surface), which can break a loop (with integer invariant) into two sub-loops (with fractional invariants). This is the intrinsic factor for obtaining a singularity with fractional topological invariant. We also note that the winding numbers cannot exactly explain the formation of swallowtail, and thus the winding numbers are not discussed in detail in the maintext.

## 6. Sequence of eigenstates and hybrid topological invariant

Hypersurface singularities may have hybrid topological invariants. In [14], the hybrid invariant is illustrated by extending the parameters into complex space. However, in the considered system, extending the parameters into complex space will break the  $PT$  symmetry and  $\eta$ -pseudo-Hermitian symmetry. The swallowtail catastrophe is not expected to emerge if the symmetries are broken. Actually, in the original parameter space (parameters being real) without extending the dimensions, the hybrid invariant can also be illustrated [15]. Next, we will discuss the intrinsic property in determining the hybrid invariant of hypersurface singularities.

For an isolated singularity, a common approach in characterizing the topology is simply observing the adiabatic transformation process on a closed loop circulating it. However, if one considers a hypersurface singularity, the situation will be a little different. For example, a closed loop circulating the intersection NIL will inevitably be decomposed into a composite of four elementary loops (see Fig. S6). In the maintext, we have shown that on loops  $l_2$  and  $l_4$ , the eigenstates will bifurcate and rotate in opposite directions as the parameters depart from ES, which shows that if the order of  $\varphi_2$  and  $\varphi_3$ , one simply obtains the inverse of the topological invariant on the loops [15]. Now we denote  $\alpha_2$  and  $\alpha_7$  as the topological invariants on loops  $l_2$  and  $l_7$ , respectively. If a loop traverses an ES, the sequence of the two eigenstates forming the ES will have two different options. Hence, for a loop circulating the NIL, there will be  $2^4=16$  invariants (as listed in Table S3). All the 16 invariants are integers because eigenstates evolve to the initial states. The loop circulating an EL3 will be composite of  $l_{10}$  and  $l_8$  (with invariants  $\alpha_{10}$  and  $\alpha_8$ ) and thus have  $2^2=4$  invariants, as listed in Table S4. Different from the NIL, the invariants on such loops circulating an EL3 are all fractions. **The adiabatic transformation of eigenstates for two invariants  $\alpha_{10}\alpha_8^{-1}$  and  $\alpha_{10}\alpha_8$  are shown in Fig. S6, showing that the winding numbers are 1/2 and 2/3, respectively. (require discussions)**

The above analysis shows that it is important to specify the sequence of the eigenstates (or eigenvalues) before discussing the topology. The loop starts from the ES, at which two eigenvalues (and eigenstates) coalesce, and the two eigenstates bifurcate in opposite directions when the parameters are at gaps. Hence, different sequences of the two eigenstates will result in opposite invariants. For some composite loops, opening up the basepoint will set a constraint on the sequence of eigenstates. Let us take the loop  $l_6$  (see Fig. S7) as an example. The loop  $l_6$  is a composite of  $l_2$  and  $l_1$  in case that the sequence of eigenstates on  $l_2$  is defined by the eigenvalues (from small to large). On the contrary, if we exchange the order of  $\varphi_2$  and  $\varphi_3$ , the basepoint cannot be open. This can be easily understood, because band 2 cannot continuously evolves to band 3 unless they coalesce (note that the basepoint is on NIL).

## 7. Other equivalent loops

In the maintext, we have shown some equivalent loops circulating degeneracy lines in the swallowtail, and explained the transition within different types of lines via the MT in view of the topological charge conservation. Here, we provide more equivalent loops as supplements, which also protects the emergence of the swallowtail.

We have demonstrated that  $l_4^{-1}l_6l_4 = l_7^{-1}$ , and thus it can be easily derived that  $l_4^{-1} = l_7^{-1}l_4^{-1}l_6^{-1}$ . Since  $l_6$ ,  $l'_6$  and  $l''_6$  are equivalent, the basepoint of the composite  $l_7^{-1}l_4^{-1}l_6^{-1}$  (shown in Fig. S8A) can be open, which is obviously equivalent to  $l_7^{-1}$ . Hence, the loop  $l_7^{-1}$  can be continuously deformed to

$l_6^{-1}l_7^{-1}l_5^{-1}$ , as shown in Fig. S8A. The deformation processes do not encounter other degeneracy lines, and thus the topological charge are conserved. Besides, a closed loop circulating the NIL can be continuously deformed to the loop circulating both the double EL3s and the NL (see Fig. S8B). The demonstration of the equivalence requires the commutation relations  $l_2l_1 = l_1l_2$  and  $l_7l_1 = l_1l_7$ . This can be understood as the following. As shown in Fig. S8C, the loops  $l_{12}$  and  $l'_{12}$  circulate the singularity  $S_1$  via different paths, and it is not difficult to discover that  $l'_{12} = l_{13}^{-1}l_{12}l_{13}$ , where the loop  $l_{13}$  encloses the singularity  $S_2$ . Generally the relation  $l'_{12} = l_{12}$  does not hold [13,17], because  $l'_{12}$  cannot be continuously deformed to  $l_{12}$ , prevented by the isolated singularity  $S_2$ . However, the situation will be a little different if  $S_2$  is a hypersurface singularity (see Fig. S8D). **This is because the hypersurface singularity is on the loop, instead of being surrounded by a loop totally.** Hence, it is not difficult to imagine that the loop  $l_1$  can be continuously deformed to  $l'_1$ , and no isolated singularity can prevent the deformation process (i.e.  $l_2l_1 = l_1l_2$ ). Similarly, the other commutation relation  $l_4l_1 = l_1l_4$  also holds. (require discussions) With the commutation relations, one can easily obtain  $l_4^{-1}l_2l_4l_2^{-1} = l_3^{-1}l_1^{-1}$ . The composite loop on the left hand side encloses the NIL, and since  $l_5$  is trivial, the right hand side  $l_3^{-1}l_1^{-1}$  is equivalent to  $l_3^{-1}l_5l_1^{-1}$ , which encloses the NL and the double EL3. Hence,  $l_4^{-1}l_2l_4l_2^{-1}$  can be continuously deformed to  $l_3^{-1}l_5l_1^{-1}$ , and the double EL3 and the NL can transit to the NIL via the MT.

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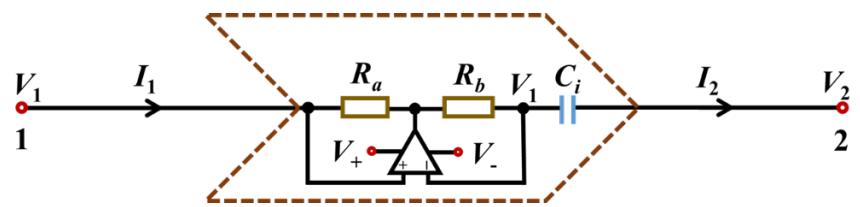


Fig. S1 Circuit diagram of a negative impedance converter structure through current inversion.

	$C_1$ (Num)	$C_2$ (Num)	$C_3$ (Num)
1	43 pF (1)	43 pF (1)	75 pF (1)
2	75 pF (3)	75 pF (3)	100 pF (3)
3	91 pF (1)	91 pF (1)	150 pF (1)
4	100 pF (3)	100 pF (3)	300 pF (3)
5	150 pF (4)	150 pF (4)	1 nF (2)
6	300 pF (7)	300 pF (7)	1.5 nF (3)
7	430 pF (1)	430 pF (1)	
8	470 pF (1)	470 pF (1)	
9	560 pF (1)	560 pF (1)	
10	680 pF (2)	680 pF (2)	
11	750 pF (1)	750 pF (1)	
12	820 pF (3)	820 pF (3)	
13	910 pF (1)	910 pF (1)	
14	1 nF (5)	1 nF (5)	
15	1.2 nF (2)	1.2 nF (2)	
16	1.5 nF (1)	1.5 nF (1)	
17	1.8 nF (1)	1.8 nF (1)	

Table S1. The capacitances mounted on the circuit for tuning  $C_1$ ,  $C_2$  and  $C_3$ .

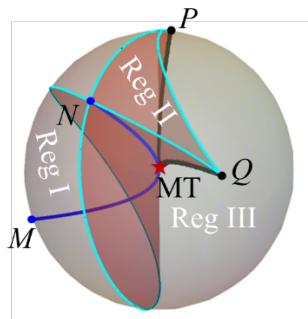


Fig. S2. Different regions partitioned by ESs. Regs I and II:  $PT$ -exact phases. Reg III:  $PT$ -broken phase.

$\mathbb{Z}_n$	$x^n + yz = 0$	$A_n$
$\mathbb{D}_{2n}^*$	$xy^2 - x^{n+1} + z^2 = 0$	$D_{n+2}$
$\mathbb{T}^*$	$x^4 + y^3 + z^2 = 0$	$E_6$
$\mathbb{O}^*$	$x^3 + xy^3 + z^2 = 0$	$E_7$
$\mathbb{I}^*$	$x^5 + y^3 + z^2 = 0$	$E_8$

Table S2. Binary groups of polyhedral.

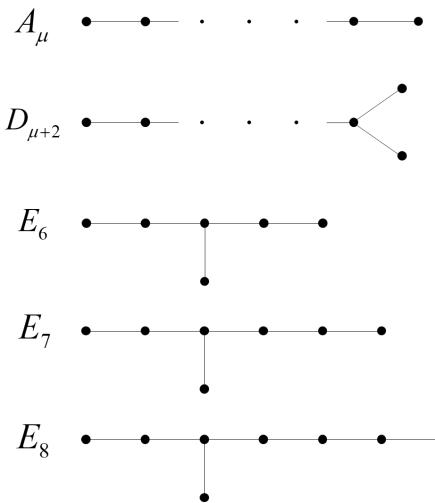


Fig. S3. Dynkin diagram.

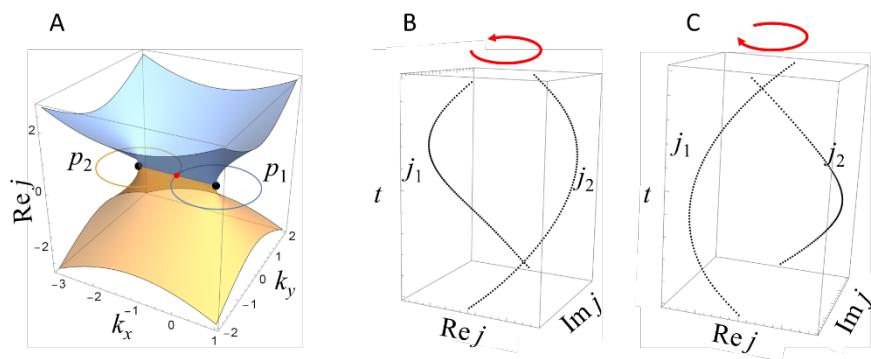


Fig. S4. Fractional winding number of isolated exceptional point induced by band braiding.

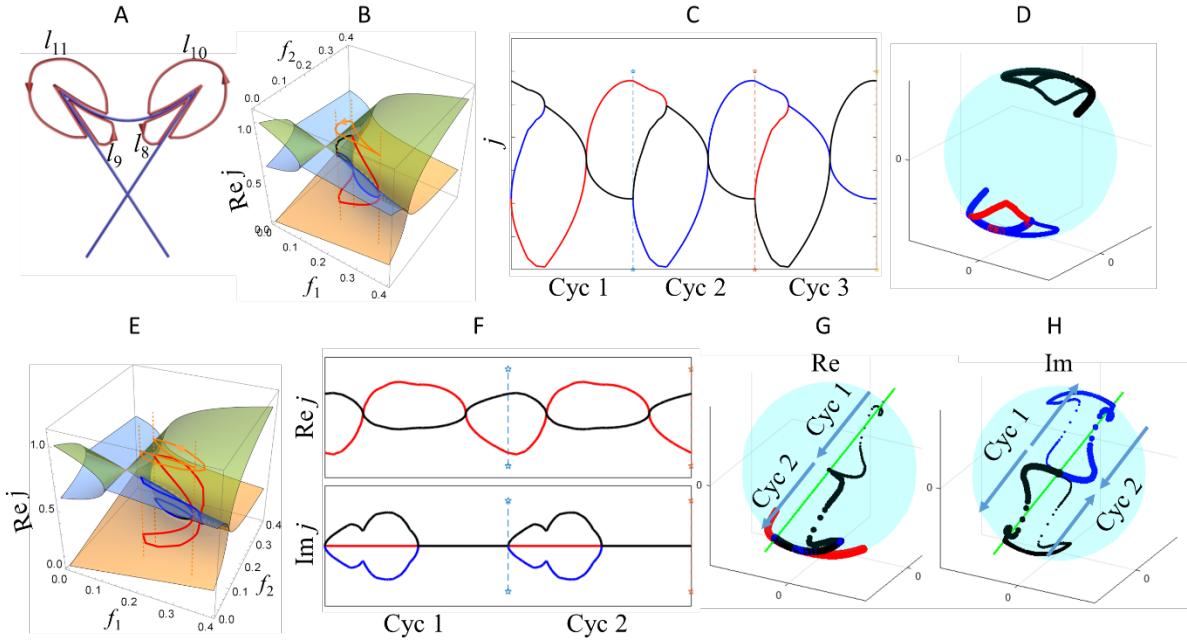


Fig. S5. Band folding induced fractional winding number at EL3s. A. sub-loops enclosing EL3s in exact phases ( $l_8$  and  $l_9$ ) and broken phases ( $l_{10}$  and  $l_{11}$ ). B. Eigenvalues at loop  $l_8$ , band 2 (blue, the corresponding eigenstate  $\varphi_2$ ) and band 3 (black, the corresponding eigenstate  $\varphi_3$ ) coalesce at ES, and exchange order with band 1 (red, the corresponding eigenstate  $\varphi_1$ ) at EL3. C. Evolution of eigenvalues on loop  $l_8$  for three cycles. D. Adiabatic transformation of eigenstates on  $l_8$  for three cycles (red,  $\varphi_1$ ; blue,  $\varphi_2$ ; black,  $\varphi_3$ ). E. Eigenvalues at loop  $l_{10}$ . F. Evolution of eigenvalues on loop  $l_{10}$  for two cycles. G-H. Adiabatic transformation of eigenstates on  $l_8$  for two cycles. G. real parts of eigenstates. H. Imaginary parts of eigenstates.

$\alpha_2\alpha_7\alpha_2\alpha_7$	$\alpha_2\alpha_7\alpha_2\alpha_7^{-1}$	$\alpha_2^{-1}\alpha_7^{-1}\alpha_2^{-1}\alpha_7$	$\alpha_2^{-1}\alpha_7^{-1}\alpha_2^{-1}\alpha_7^{-1}$
$\alpha_2\alpha_7\alpha_2^{-1}\alpha_7^{-1}$	$\alpha_2\alpha_7^{-1}\alpha_2^{-1}\alpha_7$	$\alpha_2^{-1}\alpha_7\alpha_2\alpha_7^{-1}$	$\alpha_2^{-1}\alpha_7^{-1}\alpha_2\alpha_7$
$\alpha_2^{-1}\alpha_7\alpha_2^{-1}\alpha_7$	$\alpha_2\alpha_7^{-1}\alpha_2^{-1}\alpha_7^{-1}$	$\alpha_2^{-1}\alpha_7\alpha_2\alpha_7$	$\alpha_2\alpha_7^{-1}\alpha_2\alpha_7^{-1}$
$\alpha_2^{-1}\alpha_7\alpha_2^{-1}\alpha_7^{-1}$	$\alpha_2^{-1}\alpha_7^{-1}\alpha_2\alpha_7^{-1}$	$\alpha_2\alpha_7\alpha_2^{-1}\alpha_7$	$\alpha_2\alpha_7^{-1}\alpha_2\alpha_7$

Table S3. Hybrid invariant on the closed loop circulating NIL.

$\alpha_{10}\alpha_8$	$\alpha_{10}\alpha_8^{-1}$
$\alpha_{10}^{-1}\alpha_8$	$\alpha_{10}^{-1}\alpha_8^{-1}$

Table S4. Hybrid invariant on the closed loop circulating EL3.

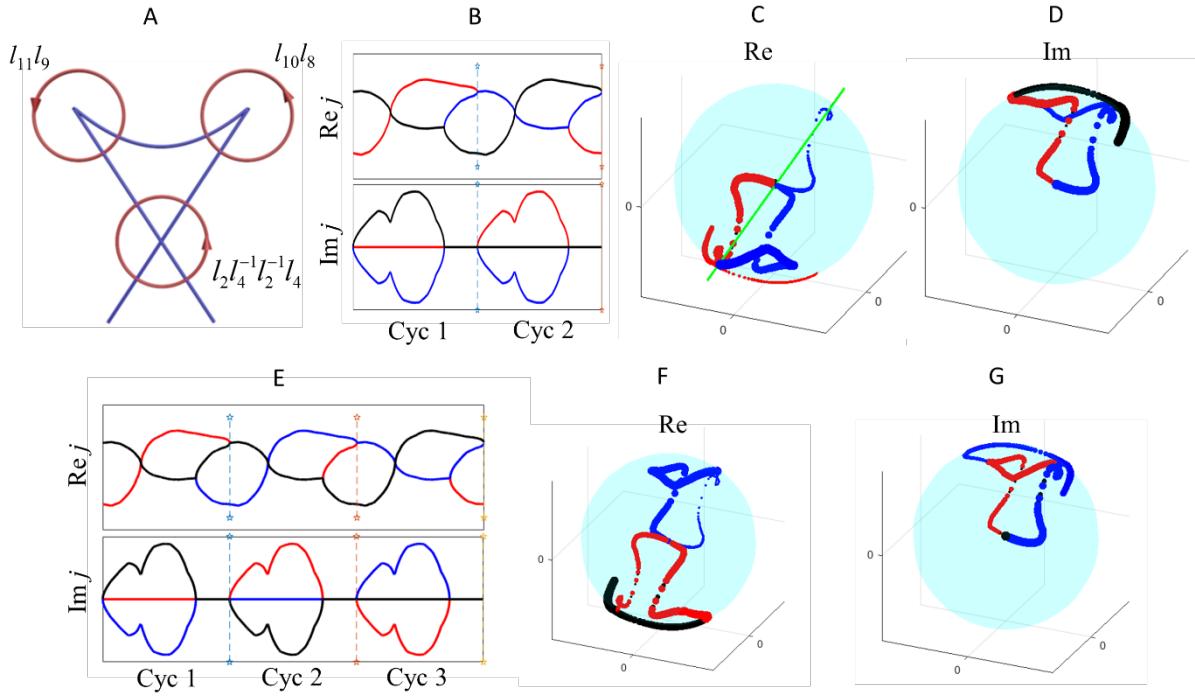


Fig. S6. Composite loops carrying hybrid topological invariant. A. Composite loops circulating EL3s ( $l_8l_{10}$  and  $l_9l_{11}$ ) and NIL ( $l_2l_4^{-1}l_2^{-1}l_4$ ). B and E. Evolution of eigenvalues on composite loop  $l_8l_{10}$  for invariants  $\alpha_{10}\alpha_8^{-1}$  (for two cycles) and  $\alpha_{10}\alpha_8$  (for three cycles), respectively. (require discussions) C-D and F-G. Adiabatic transformation of eigenstates for invariants  $\alpha_{10}\alpha_8^{-1}$  (for two cycles) and  $\alpha_{10}\alpha_8$  (for three cycles), respectively. C and F, real parts of eigenstates. D and G, imaginary parts of eigenstates.

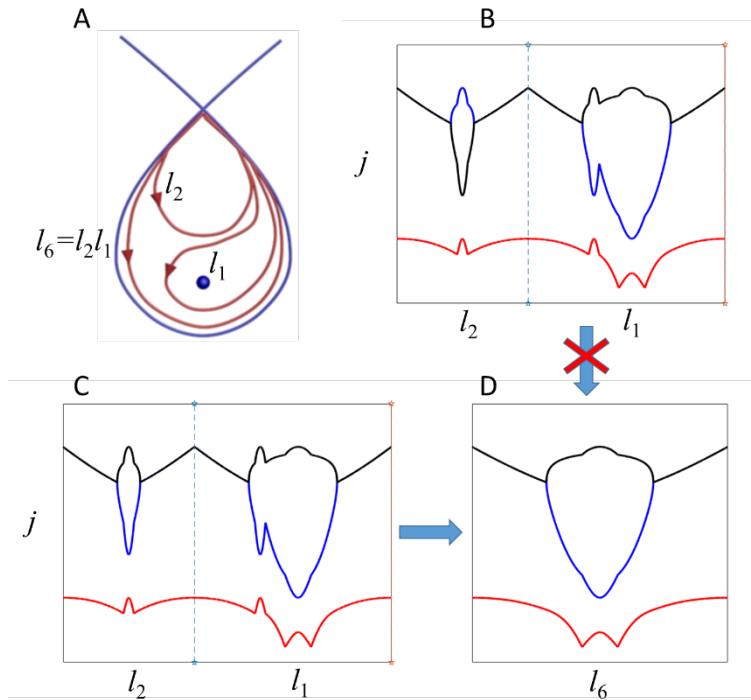


Fig. S7. Impact of the sequence of eigenvalues on opening basepoint. A.  $l_5$  is the composite of  $l_1$  and  $l_2$ . B. The common basepoint of  $l_1$  and  $l_2$  cannot be open if the sequences of eigenvalues (blue and black) are different on the two loops. On the contrary, the basepoint can be open if the sequence are identical on  $l_2$  and  $l_1$  (in C) and is equivalent to  $l_6$  (in D).

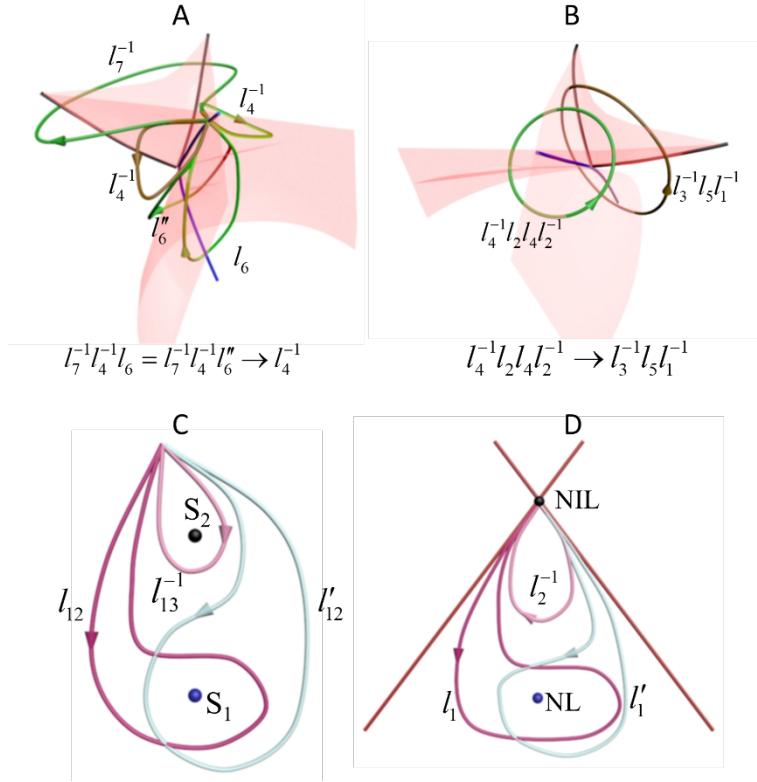


Fig. S8. Equivalent loops and commutation relations. A. The composite loop  $l_6^{-1}l_7^{-1}l_5^{-1}$  is equivalent to  $l_7^{-1}$  and can be continuously deformed to each other. B. The NIL enclosed by  $l_7^{-1}l_2l_7l_2^{-1}$  can transit to the double EL3s and NL circulated by  $l_3^{-1}l_1^{-1}$  via the MT, due to topological charge conservation. C. Non-commutation relation of the topology of two isolated singularities. The inequality  $l_{12}' \neq l_{12}$  holds because the singularity  $S_2$  blocks the deformation process from  $l_{12}$  to  $l_{12}'$ . D. Commutation relation of the topology of an isolated singularity and a hypersurface singularity. The equality  $l_1' = l_1$  holds because the NIL is on the loop  $l_2$ , which cannot prevent the deformation from  $l_1$  to  $l_1'$ .