Derived structures in the Langlands correspondence

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ABSTRACT. We survey several recent examples of derived structures emerging in connection with the Langlands correspondence. Cases studies include derived Galois deformation rings, derived Hecke algebras, derived Hitchin stacks, and derived special cycles. We also highlight some open problems that we expect to be important for future progress.

CONTENTS

1. Motivation and overview

1.1. What do we mean by "derived structures in the Langlands correspondence"? The adjective "derived" has come to be applied to various constructions in mathematics, sometimes with very different meanings. For example, "derived functors", "derived categories", and "derived algebraic geometry" are some of the instances that we will encounter. Generally speaking, the word "derived" refers to an enhancement of mathematical constructions that incorporates homotopy theory (broadly construed). The past ten years have seen the rapid development of applications of homotopy theory to the Langlands correspondence, on several different fronts. Let us give an overview of those aspects which will be touched upon in this mini-course.

1.1.1. *Derived Hecke algebras.* Perhaps the first examples of "derived functors" that one encounters are the Tor and Ext functors, which are constructed by deriving the tensor product $\otimes$ and Hom functors. In the classical Langlands correspondence, a central role is played by the notion of Hecke operators, which form the Hecke algebra acting on automorphic forms. The Hecke algebra can be viewed as a certain space of endomorphisms; by replacing endomorphisms with "derived endomorphisms" (i.e., Ext groups) one obtains a notion of *derived Hecke algebra*. This concept has arisen in two rather different contexts: the work of Ollivier-Schneider on the p-adic Langlands correspondence [**66**], and the work of Venkatesh et. al on cohomology of arithmetic groups [**81, 34, 67**]. Like the more classical Hecke operators, the action of derived Hecke operators seems to carry deep arithmetic information.

1.1.2. *Derived moduli spaces.* Derived functors in the above sense fall under the umbrella of classical homological algebra, where one derives functors on *abelian* categories such as the category of modules over a ring. There is also a theory of derived functors on *non-abelian* categories, which goes back to Quillen [**70**]. When applied to the category of commutative rings, it enters the realm of "derived algebraic geometry", which is an enhancement of classical algebraic geometry that incorporates homotopical information.

In derived algebraic geometry one constructs "derived schemes", "derived stacks", and so on. It was recently realized that such derived enhancements provide a natural explanation of structures in the Langlands correspondence. Two different types of examples are the derived Galois deformation spaces of Galatius-Venkatesh [**34**], which were used to explain the structure of cohomology of arithmetic groups, and the derived special cycles of Feng-Yun-Zhang [**31**] and Madapusi [**57**], which were used to construct virtual fundamental cycles comprising arithmetic theta functions.

1.1.3. *Other directions, that will not be discussed.* Let us mention some further aspects of the Langlands correspondence where derived notions play a critical role, although they will *not* be elaborated upon in these notes.

- The Geometric Langlands correspondence features the moduli stack of local systems on a curve, and in particular the category of coherent sheaves on it. To obtain the "correct" version of this category, meaning the one which has a Langlands dual interpretation, one needs to understand this moduli stack in a derived way. This is discussed in the article of Raskin.
- Derived commutative algebra plays an important technical role in recent work of Bhatt-Morrow-Scholze [**6**] and Bhatt-Scholze [**8**] on integral p-adic Hodge theory.

1.2. Why derive? Given the significant technical background required for derived algebraic geometry, it is natural to ask why and how it is useful for researchers interested in the more classical aspects of the Langlands correspondence, such as the reciprocity between automorphic forms and Galois representations. (We exclude branches like Geometric Langlands theory where derived algebraic geometry is baked into the very formulations of the core problems, and therefore relevant in an obvious way.)

There are several independent reasons, some of which are touched upon in this survey. One is that higher cohomology (of locally symmetric) has become central to the Langlands correspondence, because cohomology provides natural integral structures on the spaces of automorphic forms, and integral structures are needed

to bring in p-adic methods. In many classical situations, the relevant cohomology groups are concentrated in a single degree, which allows one to ignore derived aspects to a large extent, but in general they are spread out over many degrees and it becomes necessary to work with complexes, derived functors, derived categories, etc.

So far, we have only mentioned deriving *abelian structures* (in the sense of abelian categories), which is a theory that has been developed and widely used since the 1970s. But in order to make contact with Galois representations, it becomes necessary derive the notion of a commutative ring. This is the starting point of derived algebraic geometry, and requires more sophisticated homotopical methods.

A third, independent thread treated in these notes concerns enumerative geometry related to the so-called Kudla program, which seeks to develop an incarnation of theta functions in arithmetic geometry. More broadly, one would hope to develop an incarnation of relative Langlands duality, in the sense of Venkatesh's article in these proceedings, in arithmetic geometry. This means that we seek some incarnation of automorphic forms as cycle classes in the Chow groups or higher cohomology groups of Shimura varieties and moduli of shtukas. We explain here that such cycle classes should arise from derived geometry: the cycles in classical algebraic geometry are poorly behaved in general, while their derived versions have the desired properties.

There are also other motivations which are *not* treated in these notes, including the conjectural description of cohomology of moduli of shtukas explained in Zhu's article, and the categorical versions of the local Langlands correspondence discussed in the articles of Fargues-Scholze (in the "$\ell \neq p$" situation) and of Emerton-Gee-Hellman (in the "$\ell = p$" situation). In all of these cases, one needs to incorporate derived structure in order to get the "correct" answers, e.g., the category of coherent sheaves.

1.3. The style of these notes. As the above summary is intended to convey, the past few years have witnessed a rapid and remarkably broad influx of homotopy theory into number theory, in the guise of ∞-categories, derived algebraic geometry, stable homotopy theory, etc.

The technical foundations for this theory are formidable, and complicated further by the multitude of different approaches that divide the literature. For example, the literature is split between the language of model categories and ∞-categories (the latter has acquired more popularity recently), and even within the framework of ∞-categories one finds different approaches (for example quasi-categories, DG-categories, simplicial categories). Derived algebraic geometry also comes in different flavors, being built out of commutative differential graded algebras, simplicial commutative rings, and $\mathbb{E}_\infty$-algebras. These issues are significant, but often completely orthogonal to those encountered in applying the tools to number theory.

Therefore, in view of the vast literature that already exists for homotopy theoretic background, we opted to write these notes as a kind of "user's manual" for how it is applied to number theory. We do not even attempt to give a thorough technical treatment of the foundational material. Occasionally we will point out when different approaches exist and comment on their "intuitive" differences. Mostly, we will leave things somewhat vague and informal, relying on analogies and intuitions rather than formal definitions.

1.4. Outline of this article. In §2 we provide a moral introduction to derived algebraic geometry. We do not provide any technical details – in fact, we make almost no mathematically precise statements; rather, we try to give some guide on "how to think about" certain words and phrases which come up in derived algebraic geometry. This is counterbalanced by Appendix A, which focuses purely on the basic definitions of derived algebra.

In §3 we discuss the notion of derived moduli spaces, which are our raison d'être for using derived algebraic geometry in the first place. We try to give a general pattern of heuristics which one uses in practice to recognize when derived moduli spaces should be relevant, and how to construct them. We illustrate these heuristics in several examples of relevance to the Langlands correspondence.

In §4 we survey in more detail the emergence and application of derived moduli spaces related to the Kudla program, focusing on the work of Feng-Yun-Zhang [**30, 31, 29**] on higher theta series for unitary groups over function fields.

In §5 we survey the work of Galatius-Venkatesh [**34**] on derived Galois deformation rings. Since the article of Galatius-Venkatesh is quite massive, we spend significant effort trying to give an intuitive (and emphatically informal) sketch of what is going on.

In §6, we introduce local derived Hecke algebras, discussing in some length the $\ell \neq p$ results of Venkatesh [**81**] and also briefly the $\ell = p$ results of Schneider [**72**].

In §7 we discuss the cohomology of locally symmetric spaces and the global derived Hecke algebra. Then in §8 we explain Venkatesh's motivic conjectures.

Finally, in §9 we formulate several open problems at the interface of derived algebraic geometry and the Langlands program, which should have important consequences.

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2. A guide to derived algebraic geometry

2.1. What is derived algebraic geometry? Algebraic geometry is about a theory of geometry for objects called *schemes*. These are geometric spaces built locally out of pieces called *affine schemes* (much like how manifolds are built locally out of pieces looking like Euclidean space), which are governed by commutative rings. Therefore, one could say that algebraic geometry is the study of geometry built out of commutative rings.

In this vein, *derived* algebraic geometry is about a theory of geometry built out of objects that we will informally call "derived commutative rings" for now. Derived commutative rings are some sort of enhancement of commutative rings; they govern what one might call "derived affine schemes", which form the local building blocks for *derived schemes*. Therefore, the passage from algebraic geometry to derived

algebraic geometry mostly amounts to the passage from commutative rings from derived commutative rings.

2.2. What are derived commutative rings? Let us caution at the start that there are multiple different approaches to the notion of "derived commutative ring". All of them are quite technical, and so we will not go into the formal definitions of any of them here. Instead, we will content ourselves with explaining intuitions, analogies, and how we think about things in practice.

2.2.1. *Simplicial commutative rings.* For the bulk of *these* notes, our notion of a "derived commutative ring" will be that of *simplicial commutative ring*. The adjective "simplicial" can be thought of as synonymous to "topological", so it is a reasonable first intuition to think of a simplicial commutative ring as a topological commutative ring.[1]

In particular, a simplicial commutative ring $\mathcal{R}$ has associated (abelian) homotopy groups $\pi_0(\mathcal{R}), \pi_1(\mathcal{R}), \pi_2(\mathcal{R})$, etc. These should be thought of as the homotopy groups of the underlying "topological space" of $\mathcal{R}$, but moreover the commutative ring structure on $\mathcal{R}$ equips $\pi_*(\mathcal{R})$ with the structure of a graded-commutative ring. Understanding this graded-commutative ring is an approximation to "understanding" $\mathcal{R}$, in the same sense that the understanding the usual topological homotopy groups of a topological space is an approximation to "understanding" the space.

The technical definition of a simplicial commutative ring is quite involved, and for this reason will be quarantined to Appendix A. The main body of this text is written to be comprehensible with the notion of "derived commutative rings" treated as a black box, and it might be advisable to read it as such on a first pass.

2.2.2. *Other models for derived commutative rings.* We briefly mention other possible models for "derived commutative rings" that are commonly encountered in the literature.

- *Commutative differential graded algebras* (CDGAs). These are the most concrete to define and write down "explicitly". For example, it is relatively easy to write down CDGAs by generators and relations, while any such presentation would be too huge to write down for almost any simplicial commutative ring. The crucial flaw of CDGAs from our perspective is that they only lead to a functional theory[2] of derived commutative rings over characteristic zero, while we definitely want to work integrally or in characteristic p. However, in characteristic zero CDGAs could be used just as well as simplicial commutative rings.
- $\mathbb{E}_\infty$-*algebras.* Roughly speaking, an $\mathbb{E}_\infty$-algebra is a ring with whose multiplication is commutative up to homotopy coherence. The precise definition of $\mathbb{E}_\infty$-algebras requires even more homotopy theory than simplicial commutative rings. In practice, $\mathbb{E}_\infty$-algebras are more relevant for homotopy theory than algebraic geometry. It is true that $\mathbb{E}_\infty$ lead to a functional theory in all characteristics and could be used just as well as

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[1]One reason we avoid working literally with topological commutative ring, which of course is itself a well-defined mathematical concept, is that we want to rule out "pathological" topological spaces, so a finer synonym for "simplicial" might be "topological and nice".

[2]By this, we mean that there does not exist a model category or ∞-category of CDGAs outside characteristic zero.

simplicial commutative rings in characteristic zero, but away from characteristic zero they have different behavior which seems to be less relevant for us.

- *Animated rings*, [**18**, §5.1]. In principle, this notion is exactly what we want (see §2.2.3), and even has the additional feature that it is "model-independent", but the definition is rooted firmly in ∞-category theory so we view it as less elementary. Also, the description of animated rings as coming from simplicial commutative rings clarifies certain structures that exist on them – see §A.5.1 – so it is useful to begin with simplicial commutative rings anyway.

2.2.3. *The category of derived commutative rings.* For the purposes of moduli theory, it is not the notion of a simplicial commutative ring itself which is essential, but the correct notion of its ambient category. In terms of the intuition for simplicial commutative rings as being like topological commutative rings, what we would like to do is to take the category of topological commutative rings *up to weak equivalence*; informally speaking, we want to regard "homotopic" things as being equivalent.

For this discussion we would like to make an analogy to the following objects: abelian groups, complexes of abelian groups, and the derived category of abelian groups. In our analogy a commutative ring is parallel to an abelian group. To "derive" functors on the category of abelian groups, one would consider "simplicial abelian groups", but the category of such is equivalent to the category of connective chain complexes (by the Dold-Kan equivalence, §A.4), we may just as well work with chain complexes. However, for many purposes (such as in defining derived functors), one is not really after the category of abelian groups but the *derived category of abelian groups*. Informally, one can construct this category so that the objects are chain complexes, and then the morphisms are obtained from the morphisms among chain complexes by inverting quasi-isomorphisms.

Simplicial commutative rings are parallel to chain complexes of abelian groups. The category we are after, called *the ∞-category of simplicial commutative rings*, is not the literal category of simplicial commutative rings but rather a category obtained from this by inverting weak equivalences, which are the analogues of "quasi-isomorphisms". By abuse of notation one refers to the objects of this category as simplicial commutative rings, but this obscures the fact that the ambient category is really some kind of "derived category of simplicial commutative rings". The analogy is depicted below.

Z-module	Commutative ring
Chain complex of **Z**-module	Simplicial commutative ring
Category of chain complexes of **Z**-modules	Category of simplicial commutative rings
Derived ∞-category of **Z**-modules	∞-category SCR
Animated abelian group	Object of SCR

There is an intrinsic definition of the ∞-category of interest, documented in [**18**, §5]. Objects of this category are called *animated commutative rings*, and so the ambient category is called the ∞-category of animated rings. (Their perspective would also lead to an approach to the derived ∞-category of abelian groups, which does not present the latter as a localization of the category of chain complexes.)

In summary, an "animated commutative ring" is represented by a simplicial

commutative rings, in the sense that animated commutative rings can be defined as objects of a category whose objects are simplicial commutative rings. However, the morphisms in this category are not the naive notion of maps between simplicial commutative rings, but obtained from those by inverting the analogues of quasi-isomorphisms. Other than in Appendix A, we use the phrase "simplicial commutative ring" to denote an object of the ∞-category SCR, so that it is synonymous with "animated commutative ring".
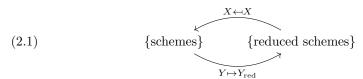
I still feel like it's confusing, maybe better to actually used "animated"…

Remark 2.1. An analogy which is sometimes used to describe the situation is that an ∞-category is like a vector space, and a choice of model category giving rise to a given ∞-category is like a choice of basis for a vector space. Under this analogy, an animated abelian group or animated ring corresponds to a vector in the vector space, while a chain complex or simplicial commutative ring corresponds to a presentation of the vector by coordinates in some chosen basis.
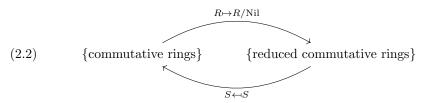
2.3. Visualizing derived schemes. In algebraic geometry one learns how to think about commutative rings geometrically. For example, in introductory textbooks on scheme theory (such as [**79**], whose §4.2 is the inspiration for the title of this subsection) one learns that non-reducedness of a ring – which at first seems "ungeometric" because it is invisible at the level of field-valued points – can be pictured geometrically as "infinitesimal fuzz". In other words, one visualizes Spec R as an infinitesimal thickening of Spec R_{red}. In this subsection we will explain how one can similarly visualize a derived scheme as a type of infinitesimal thickening of a classical scheme. The slogan that guides this is:

> The relationship between derived schemes and classical schemes
> is analogous to the relationship between (classical) schemes and
> reduced schemes.

Let us first explain the formal similarities. A scheme has an "underlying reduced" scheme, whose formation is functorial and defines a right adjoint functor to the inclusion of the category of reduced schemes into the category of all schemes.

$$(2.1) \qquad \{\text{schemes}\} \underset{Y \mapsto Y_{\mathrm{red}}}{\overset{X \hookleftarrow X}{\rightleftarrows}} \{\text{reduced schemes}\}$$

At the level of rings, the formation of underlying reduced scheme corresponds locally to quotienting a ring by its nilradical, and this is left adjoint to the forgetful functor including the category of reduced commutative rings into the category of all commutative rings.

$$(2.2) \qquad \{\text{commutative rings}\} \underset{S \hookleftarrow S}{\overset{R \mapsto R/\mathrm{Nil}}{\rightleftarrows}} \{\text{reduced commutative rings}\}$$

Now let us explain the analogous picture for simplicial commutative rings. Any commutative ring R can be viewed as a simplicial commutative ring $\underline{R}$, intuitively

by "equipping it with the discrete topology". Then $\pi_0(\underline{R}) = R$, while $\pi_i(\underline{R}) = 0$ for $i > 0$. This induces a fully faithful embedding (even at the level of ∞-categories) of the category of commutative rings into the category of simplicial commutative rings, and in particular justifies the perspective that animated rings form an "enlargement" of commutative rings.

On the other hand, for any animated ring $\mathcal{R}$, the zero-th homotopy group $\pi_0(\mathcal{R})$ has a natural ring structure. These functors fit into an adjunction analogous to (2.2):

$$\text{(2.3)} \qquad \{\text{animated commutative rings}\} \xrightarrow{\;R \mapsto \pi_0(\mathcal{R})\;} \xleftarrow{\;\underline{R} \leftmapsto R\;} \{\text{commutative rings}\}$$

This construction glues, so that any derived scheme Y has a *classical truncation* Y_{cl}, whose formation is functorial and defines a right adjoint to the functor from schemes to derived schemes:

$$\text{(2.4)} \qquad \{\text{derived schemes}\} \xleftarrow{\;X \leftmapsto X\;} \xrightarrow{\;Y \mapsto Y_{\mathrm{cl}}\;} \{\text{schemes}\}$$

Analogous to (2.1), one can visualize a derived scheme as a classical scheme plus some "derived infinitesimal fuzz". One difference in practice is that "derived fuzz" tends to carry *negative dimension*.

Example 2.2. We have not defined the étale site of a derived scheme, but just as the étale topos of a scheme is isomorphic to that of its underlying reduced scheme, it turns out that the étale site of a derived scheme is isomorphic to that of its classical truncation. This further supports the analogy between derived and non-reduced structure.

Going forward we will use calligraphic letters such as $\mathcal{A}, \mathcal{B}, \mathcal{R}$ for simplicial commutative rings, and Roman letters such as A, B, R for classical commutative rings. By the above remarks, we may regard classical commutative rings as simplicial commutative rings, but we still prefer to use this convention to emphasize that a simplicial commutative ring comes from a classical commutative ring. Similarly, we use roman letters like M or M for classical moduli spaces, and calligraphic letters like $\mathcal{M}$ for derived moduli spaces.

Are we being consistent about animated versus simplicial?

2.4. Characteristics of simplicial commutative rings. We would like to make some remarks about how to represent the datum of a simplicial commutative ring. In terms of the intuition for a simplicial commutative ring as a "topological commutative ring up to homotopy", it is natural to compare this question to that of representing the datum of a topological space up to homotopy. A topological space is a kind of amorphous object, but algebraic topology furnishes two approaches to measuring it by algebraic invariants: homotopy groups, and (singular) homology groups. We shall explain two parallel invariants for "measuring" simplicial commutative rings: homotopy groups, and the cotangent complex.

2.4.1. *Homotopy groups.* A simplicial commutative ring $\mathcal{R}$ has a sequence of homotopy groups $\pi_0(\mathcal{R}), \pi_1(\mathcal{R}), \ldots$, which are all abelian groups. Furthermore, the multiplication on $\mathcal{R}$ equips

$$\pi_*(\mathcal{R}) := \bigoplus_{i=0}^{\infty} \pi_i(\mathcal{R})$$

with the additional structure of a graded-commutative ring. The construction is detailed in A.5.1. Under our intuitive analogy of a simplicial commutative ring being like a topological commutative ring, you can think of $\pi_*(\mathcal{R})$ as being the homotopy groups of the underlying topological space, with the ring structure induced by the multiplicative structure of $\mathcal{R}$.

Example 2.3. If $\mathcal{R}$ comes from a topological commutative ring, then $\pi_*(\mathcal{R})$ agrees with the graded-commutative ring of homotopy groups (defined in the usual topological sense) of the topological commutative ring (with respect to the basepoint at the zero element).

For a topological space X, the homotopy groups of X are a good measure for "understanding" X itself. For example, if X and X' are "nice" topological spaces then a morphism $f\colon X \to X'$ is a homotopy equivalence if and only if it induces a bijection on π_i for all i, suitably interpreted to account for connected components.

Similarly, we regard $\pi_i(\mathcal{R})$ as a good approximation to "understanding" a simplicial commutative ring $\mathcal{R}$. In particular, a morphism of simplicial commutative rings $f\colon \mathcal{R} \to \mathcal{R}'$ is an isomorphism if and only if it induces a bijection on π_i for all i.

2.4.2. *Cotangent complex.* Let $f\colon A \to B$ be a morphism of classical commutative rings. Then there is a cotangent complex $\mathbf{L}_f$, defined in [**45, 46**], which was originally introduced for applications to deformation theory. For our purposes we regard $\mathbf{L}_f$ as a simplicial B-module. We do not give the definition here because it requires a significant amount of development; Appendix A gives a quick introduction to the cotangent complex and its applications.

Example 2.4. Let $f\colon A \to B$ be a morphism of commutative rings and $\widetilde{A} \to A$ a square-zero thickening with ideal I (i.e., $I^2 = 0$). Let $J = I \otimes_A B$. We consider $\widetilde{A}$-algebra deformations of B, i.e. flat $\widetilde{A}$-algebras $\widetilde{B}$ such that $\widetilde{B} \otimes_{\widetilde{A}} \widetilde{A}/I \xrightarrow{\sim} B$. It is explained in §A.16 that this deformation theory problem is "controlled" by $\mathbf{L}_f$, in the sense that:

- There is a class $\mathrm{obs}(I) \in \mathrm{Ext}_B^2(\mathbf{L}_f, J)$ which vanishes if and only if such a deformation $\widetilde{B}$ exists.
- If $\mathrm{obs}(I) = 0$, then the set of deformations $\widetilde{B}$ has the natural structure of a torsor for $\mathrm{Ext}_B^1(\mathbf{L}_f, J)$.
- The automorphism group of any deformation $\widetilde{B}$ is naturally ismorphic to $\mathrm{Ext}_B^0(\mathbf{L}_f, I) \cong \mathrm{Hom}_B(\Omega_f, I)$.

Example 2.5. If f is smooth, then $\mathbf{L}_f$ has cohomology concentrated degree 0, i.e. is represented by a B-module. In particular, smooth A-algebras always admit deformations.

In fact, the construction of the cotangent complex uses simplicial commutative rings, and was one of the earliest applications for this theory. Roughly speaking, the idea is that when B is not smooth over A then one "resolves" B by a simplicial

A-algebra, and forms the Kähler differentials of this resolution. This is reminiscent of how one defines derived functors in homological algebra, by replacing inputs with "resolutions" by certain types of chain complexes with good properties. While chain complexes only makes sense for objects of abelian categories, simplicial objects make sense in much more generality.

Example 2.6. Given a sequence of maps

$$A \xrightarrow[\quad f \quad]{\quad h \quad} B \xrightarrow{\quad g \quad} C$$

we get an exact sequence

$$\Omega_f \otimes_B C \to \Omega_h \to \Omega_g \to 0.$$

This will extend to an exact *triangle* (in the derived category of C-modules) of cotangent complexes

$$\mathbf{L}_f \overset{\mathbf{L}}{\otimes}_B C \to \mathbf{L}_h \to \mathbf{L}_g \xrightarrow{+1}$$

which in particular gives a long exact sequence in cohomology,

$$\dots \to H^{-1}(\mathbf{L}_h) \to H^{-1}(\mathbf{L}_g) \to \Omega_f \otimes_B C \to \Omega_h \to \Omega_g \to 0.$$

It was arranged by definition that a morphism of simplicial commutative rings which induces an isomorphism on homotopy groups is an isomorphism in the ∞-category of simplicial commutative rings. On the other hand, it is unclear "how much" information the cotangent complex carries. It turns out that it carries "all" the information of the derived structure. This is made precise by the following lemma of Lurie:

LEMMA 2.7 ([**53**, Corollary 3.2.17]). *Suppose a morphism of simplicial commutative rings* $f \colon \mathcal{A} \to \mathcal{B}$ *induces an isomorphism on* π_0 *and an isomorphism* $\mathbf{L}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\sim} \mathbf{L}_{\mathcal{B}}$. *Then* f *is an isomorphism.*

Remark 2.8. The cotangent complex can be globalized: for a morphism of derived stacks $\mathcal{X} \to \mathcal{Y}$, one has a relative cotangent complex $\mathbf{L}_{\mathcal{X}/\mathcal{Y}}$ in the category of simplicial quasicoherent sheaves on $\mathcal{X}$. This is obtained by gluing from the affine-local case, cf. §A.19.

We will not give the formal argument for Lemma 2.7, but we sketch the intuition. From a functor of points perspective, what we have to show is that f induces an equivalence between morphisms to an arbitrary simplicial commutative ring from $\mathcal{A}$ and from $\mathcal{B}$. An arbitrary simplicial commutative ring can be thought of as a "derived nilpotent thickening" of its classical truncation. Since morphisms to a classical ring are controlled by π_0, the morphisms to any classical ring from $\mathcal{A}$ and $\mathcal{B}$ are identified by f. Then, by (a generalization of) Example 2.4, morphisms to a derived nilpotent thickening of a classical ring are controlled by the respective cotangent complexes, which are again identified by f.

Remark 2.9. We regard the cotangent complex as a "homology theory" for simplicial commutative rings. From this perspective, Lemma 2.7 is analogous to the Hurewicz theorem, which says that for "nice" spaces, it is equivalent for a map to induce a bijection on homotopy sets and homology groups.

2.4.3. *Tangent complex.* Suppose $f\colon \mathcal{A} \to \mathcal{B}$ has cotangent complex $\mathbf{L}_f$ which is perfect as a simplicial $\mathcal{B}$-module. Then we define the *tangent complex* of f to be $\mathcal{R}\mathcal{H}om_{\mathcal{B}}(\mathbf{L}_f, \mathcal{B})$. This globalizes to the case where f is a morphism of derived schemes, or even a morphism of derived Artin stacks $\mathcal{X} \to \mathcal{Y}$.

<div style="float:right; font-style:italic;">probably elaborate more on this</div>

2.5. Examples of derived schemes.

2.5.1. *Derived fibered products.* One of the most natural ways that derived schemes arise through natural operations, even when starting out with purely classical objects as input, is through the *derived fibered product* operation.

The classical avatar of this operation is the usual fibered product of a two morphisms of schemes $X \to Z$ and $Y \to Z$. The resulting fibered product $X \times_Z Y$ is characterized by a universal property among schemes with maps to X and Y, such that the composite maps to Z agree. It is built affine locally by the tensor product operation on commutative rings.

Similarly, the derived fibered product, which we also sometimes call the *homotopy fibered product* $X \overset{h}{\times}_Z Y$, is characterized by an analogous universal property. It can be built affine locally by the *derived tensor product* operation on simplicial commutative rings (§A.10). More generally, the same considerations allow to construct the derived fibered product of derived schemes.

Given morphisms of simplicial rings $C \to A$ and $C \to B$, the derived tensor product $A \overset{h}{\otimes}_C B$ can be characterized by its universal property as a coproduct of A and B in the category of simplicial C-algebras. An explicit model can be built in terms of simplicial commutative rings, by choosing "resolutions" of A and B as C-algebras. In particular, this shows that the homotopy groups of $A \overset{h}{\otimes}_C B$ are the Tor-groups,

$$\tag{2.5} \pi_i(A \overset{h}{\otimes}_C B) \cong \mathrm{Tor}_i^C(A, B);$$

although this does not illuminate the multiplicative structure on $\pi_*(A \overset{h}{\otimes}_C B)$, and so it really only encodes information about the underlying simplicial abelian group of $A \overset{h}{\otimes}_C B$. From this, we see that the derived fibered product $X \overset{h}{\times}_Z Y$ of classical schemes is classical if and only if $\mathcal{O}_X$ and $\mathcal{O}_Y$ are Tor-independent over $\mathcal{O}_Z$. In particular, this holds if either $X \to Z$ or $Y \to Z$ are flat.

Example 2.10. The derived fibered product interacts well with formation of (co)tangent complexes, which is actually a technical advantage of working in derived algebraic geometry. More generally, a slogan is that "formation of tangent complexes commutes with homotopy limits". In particular, we have that

$$\mathbf{T}_{\mathcal{X} \times_Z \mathcal{Y}} \cong \mathrm{Fib}\left(\mathbf{T}_{\mathcal{X}}|_{\mathcal{X} \times_Z \mathcal{Y}} \oplus \mathbf{T}_{\mathcal{Y}}|_{\mathcal{X} \times_Z \mathcal{Y}} \to \mathbf{T}_Z|_{\mathcal{X} \times_Z \mathcal{Y}}\right),$$

with the fiber formed in the derived category of quasicoherent sheaves on $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$.

2.5.2. *Derived vector bundles.* Let X be a proper variety over a field k and $\mathcal{E}$ a locally free coherent sheaf on X. Consider the scheme representing the functor $S \in \mathrm{Sch}_{/k}$ to global sections of $\mathcal{E}$ on X_S; it is represented by $\underline{H^0(X, \mathcal{E})} = H^0(X, \mathcal{E}) \otimes_k \mathbf{A}_k^1$, the k-vector space $H^0(X, \mathcal{E})$ regarded as an affine space over k. It can be described as $\mathrm{Spec}\left(\mathrm{Sym}\, H^0(X, \mathcal{E})^*\right)$ where $H^0(X, \mathcal{E})^*$ is the dual of $H^0(X, \mathcal{E})$ over k.

A perspective one learns in homological algebra is to view $H^0(X, \mathcal{E})$ as the zeroth cohomology group of a complex $R\Gamma(X, \mathcal{E})$, whose higher cohomology groups

are $H^i(X, \mathcal{E})$ for $i \geq 1$. There will be a derived affine scheme

$$\underline{R\Gamma(X, \mathcal{E})} = \text{``Spec}\,(\text{Sym}\, R\Gamma(X, \mathcal{E})^*)\text{''},$$

whose classical truncation is $\underline{H^0(X, \mathcal{E})}$. Thus, we can say that derived algebraic geometry allows to construct derived schemes which extend classical schemes of global sections in the same way that derived functors extend classical global sections. Now, we have put quotations in the formulate above because we have not explained what it means to take Sym of a complex.

Remark 2.11. More generally, one can perform an analogous construction "relative to a base": given a perfect complex $\mathcal{K}$ on a (derived) stack S, one can form a "total space" $\mathcal{K}$ as a derived stack over S. Such constructions are called *derived vector bundles* in [**29**], since they generalize vector bundles, which are the total spaces arising in the special case where $\mathcal{K}$ is a locally free coherent *sheaf* (viewed as an perfect complex in degree 0).

Example 2.12. Let $S = \text{Bun}_n$, the moduli stack of rank n vector bundles on smooth projective curve C. Consider the space $\mathcal{M} \to S$, with R-points the groupoid of rank n vector bundles $\mathcal{F}$ on C plus a global section of $\mathcal{F}$. Although the fibers of $\mathcal{M} \to S$ are vector spaces, the morphism $\mathcal{M} \to S$ is not even flat, since the fibers have different dimensions. This is because $\mathcal{F} \mapsto H^0(\mathcal{F})$ behaves "discontinuously", e.g., the dimension of $H^0(\mathcal{F})$ varies discontinuously with $\mathcal{F}$. However, the perfect complex $R\Gamma(\mathcal{F})$ behaves "continuously", e.g., the Euler characteristic of $R\Gamma(\mathcal{F})$ varies continuously with $\mathcal{F}$.

We can assemble $R\Gamma(\mathcal{F})$ into a perfect complex on S, whose totalization $\mathcal{M} \to S$ has classical truncation $\text{M} \to S$ and whose derived fiber over $\mathcal{F}$ is the derived scheme $R\Gamma(\mathcal{F})$ in the above sense. This $\mathcal{M}$ is a variant of the "derived Hitchin stacks" to be discussed in §3 and §4.

2.5.3. *Derived moduli spaces.* Perhaps the most interesting examples are *derived moduli spaces*. The entirety of §3 is devoted to such examples, so let us just give a brief teaser here. The idea here is that one begins with a classical moduli space, which by definition is a type of functor defined on commutative rings. Then one finds a way to possibly reformulate the moduli problem so that it makes sense on simplicial commutative rings as well, and moreover outputs a simplicial set instead of just a set. If the functor is representable, then this gives rise to a derived moduli space.[3] We shall see later some of the benefits of promoting moduli spaces to derived moduli spaces.

2.6. Quasi-smoothness. In this subsection, we define a property of morphisms of derived schemes which is called *quasi-smoothness*, which turns out to play a very important role in derived algebraic geometry. It is the derived generalization of a *local complete intersection* morphism. For motivation, we recall what this means:

Definition 2.13. Let $f \colon A \to B$ be a finite type morphism of noetherian rings. We say that f is a *local complete intersection* (LCI) if it is Zariski-locally on A the composition of a map of the form $A \to A[x_1, \ldots, x_n]$ followed by a quotient by a regular sequence.

---

[3]The subtlety here is usually that the functor needs to carry our notion of isomorphism ♠♠♠ TONY: [leaning towards calling it quasi-isomorphism?] to isomorphisms of spaces.

THEOREM 2.14. *Let $f\colon A \to B$ be a finite type homomorphism of Noetherian rings. Then f is LCI if and only if $\mathbf{L}_f$ is represented by a complex with cohomology concentrated in degrees $[-1, 0]$.*

Definition 2.15. A morphism $f\colon \mathcal{A} \to \mathcal{B}$ of simplicial commutative rings is *quasi-smooth* if $\mathbf{L}_f$ is Zariski-locally on $\mathcal{B}$ a perfect complex lf vector bundles in degrees $-1, 0$.

More generally, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of derived stacks is *quasi-smooth* if $\mathbf{L}_f$ is a perfect complex with cohomology concentrated in degrees ≥ 1.

If f is quasi-smooth, then we define the *relative dimension of f* to be $\chi(\mathbf{L}_f)$.

Theorem 2.14 explains why we say that quasi-smoothness is a derived generalization of "LCI". This property enables certain important constructions, for example the construction of "Gysin pullbacks" in intersection theory. Moreover, there is a sense in which it is more common to encounter quasi-smooth morphisms than LCI morphisms in nature; this is actually one of the main motivations for the entrance of derived algebraic geometry into enumerative algebraic geometry.

PROPOSITION 2.16. *Quasi-smooth morphisms are preserved by compositions and derived base changes.*

Example 2.17. Any finite type affine scheme can be realized as the classical truncation of a quasi-smooth derived affine scheme, for trivial reasons. Indeed, given any choice of presentation $R = \mathbf{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, the derived fibered product of the diagram

$$
\begin{array}{c}
\operatorname{Spec} \mathbf{Z}[x_1, \ldots, x_n] \\
\downarrow {\scriptstyle (f_1, \ldots, f_m)} \\
\operatorname{Spec} \mathbf{Z} \xrightarrow{\ 0\ } \operatorname{Spec} \mathbf{Z}[y_1, \ldots, y_m]
\end{array}
$$

has classical truncation isomorphic to $\operatorname{Spec} R$.

If $f\colon \mathcal{X} \to \mathcal{Y}$ has a tangent complex $\mathbf{T}_f$, then it can be used to reformulate quasi-smoothness: f is quasi-smooth if and only if $\mathbf{T}_f$ is represented by a perfect complex with cohomology in degrees ≤ 1.

Example 2.18. By Example 2.10, a derived scheme which is Zariski-locally isomorphic to a derived fibered product of smooth schemes is quasi-smooth. The converse is also true.

3. Derived moduli spaces in the Langlands program

The relevance of derived algebraic geometry in the Langlands program is through *derived moduli spaces*. In this section we will give an overview of some examples, discuss how to think about them, and hint at why they are useful.

3.1. The hidden smoothness philosophy. Historically, the notion of derived moduli spaces was anticipated in 1980s, even before the advent of derived algebraic geometry, by Beilinson, Deligne, Drinfeld, and Kontsevich. They emphasized a principle called the *hidden smoothness philosophy*, which predicts natural moduli spaces should have derived versions which are *quasi-smooth* (and of the "expected dimension").

The backdrop for this principle is that in certain areas of classical algebraic geometry, one frequently encounters classical moduli spaces – that is to say, moduli functors represented by classical schemes or stacks – which are not LCI or have the "wrong" dimension. In such situations, the hidden smoothness philosophy predicts that these are the classical truncations of derived moduli spaces which do have the "correct" geometric properties. It can then be useful to find the definition of these derived enhancements. In this section we will discuss derived moduli spaces from this perspective: what to look for when trying to "upgrade" a classical moduli space to a derived one. Subsequent sections will (hopefully) illustrate why this might be a useful thing to do.

The motivations for the hidden smoothness philosophy are not obvious, but are rather "empirical" in a certain sense, so we will try to illustrate it through examples.

Example 3.1 (Moduli of Betti local systems). Let us begin by brainstorming informally about a somewhat older example in derived algebraic geometry, which has relevance for example in (Betti) Geometric Langlands [**4**]: the moduli space of (Betti) local systems on a smooth projective (connected) curve $C/\mathbf{C}$ of genus g. After choosing a basepoint, one could view a Betti local system as a representations of the fundamental group of C. There is a more canonical way to phrase things without choosing a basepoint, but for concreteness we make such a choice $c_0 \in C$, and then $\pi_1(C, c_0)$ has a standard presentation

$$(3.1) \qquad \pi_1(C, c_0) \approx \langle a_1, \ldots, a_g, b_1, \ldots, b_g \colon \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle.$$

This presentation suggests a reasonable guess for how to write down the moduli space of local systems of C as a fibered product

$$(3.2) \qquad
\begin{array}{c}
\mathrm{GL}_n^{2g} \\
\downarrow{\scriptstyle \prod_{i=1}^{g}[x_i, y_i]} \\
e \longrightarrow \mathrm{GL}_n
\end{array}$$

Since all the spaces in this diagram are smooth (and in particular the bottom horizontal arrow is a regular embedding), it suggests that the moduli space of local systems might be a local complete intersection, of dimension equal to $2g \dim \mathrm{GL}_n - \dim \mathrm{GL}_n$. This turns out not to be the case, however, because a regular sequence cutting out $\{e\} \hookrightarrow \mathrm{GL}_n$ does not pull back to a regular sequence in GL_n^{2g}.

However, if instead of taking the fibered product of diagram (3.2) in classical schemes, one takes the *derived* fibered product, then by Example 2.18 the resulting derived scheme is quasi-smooth and has the "expected" dimension $2g \dim \mathrm{GL}_n - \dim \mathrm{GL}_n$.

The Hidden Smoothness philosophy it justified in many examples by the fact that one can find natural (local) presentations for naturally occurring moduli spaces, similar to (3.2). More precisely, the quasi-smoothness and dimensionality conditions are suggested by deformation theory, as shall be explained below.

This section is written as a guide to solve the following problem:

> Given a classical moduli space M, find the "correct" derived moduli space $\mathcal{M}$ of which it is a classical truncation.

By definition a classical moduli space is a type of functor defined on the category of commutative rings, while a derived moduli space is a type of functor on the category of simplicial commutative rings, so the content of this problem is one of extending the domain of definition for the functor in a "good" way. (The target should also be extended, from sets or groupoids to simplicial sets.) In analogy to the notion of derived functors, we will informally call this problem "deriving the classical moduli space M".

Of course, this problem is extremely ill-defined: a given classical moduli space can be realized as the classical truncation of many different (quasi-smooth) derived moduli spaces, just as a given reduced scheme can be realized as the underlying reduced of many different schemes (see Example 2.17). Therefore, what we mean by "correct" derived moduli space is heuristic, and is dictated by the needs of the particular problem at hand. In practice, the problem may be rigidified by the desired to obtain a quasi-smooth $\mathcal{M}$, or even to obtain an $\mathcal{M}$ with a particular tangent complex.

In this section we will describe certain patterns and heuristics for how this process tends to be carried out in practice. We will illustrate it through two main examples: the *derived Galois deformation rings* of Galatius-Venkatesh [**34**], and the *derived Hitchin stacks* of Feng-Yun-Zhang [**31**].

3.2. Heuristics for derived moduli spaces. Suppose that we begin with certain classical moduli space M, which we wants to promote to a *derived* moduli space $\mathcal{M}$. The following outline describes the most standard pattern for finding the "correct" derived enhancement $\mathcal{M}$.

Recall the notion of tangent complex from §2.4.3.

(1) The tangent space to M at a point $m \in M$ can be calculated, and has some interpretation as a natural cohomology group. Furthermore, the automorphisms lie in a "previous" cohomology group, and obstructions lie in the "next" cohomology group.

(2) One then tries to construct $\mathcal{M}$ so that its tangent complex is the cohomology complex computing the cohomology groups from the previous step.

We will give some explicit examples to make this concrete. First however, we will discuss heuristics for when this process is unnecessary.

3.2.1. *Heuristics for classicality.* In this subsection we give some guiding principles for when one expects to find a derived enhancement which is not represented by a classical scheme or stack. The basic philosophy is that:

> If a moduli space is LCI of the "correct" dimension, then it
> does not need to be derived.

The notion of "correct" dimension is itself heuristic, and might arise from an intersection-theoretic setup such as in (3.2), or from calculating the Euler characteristic of the expected tangent complex.

A little more precisely, the statement that a moduli space "does not need to be derived" means that the "correct" derived moduli space should just be isomorphic to its classical truncation.

The basis for this principle is as follows. Suppose $\mathcal{M}$ is a derived scheme with classical truncation $M \hookrightarrow \mathcal{M}$. Following the Hidden Smoothness philosophy, we

suppose that $\mathcal{M}$ is quasi-smooth of the "correct" dimension, which at $m \in \mathcal{M}$ is the Euler characteristic of the tangent complex at m, $\chi(\mathbf{L}_{\mathcal{M},m})$.

LEMMA 3.2. *The canonical map $\iota\colon \mathrm{M} \to \mathcal{M}$ induces an isomorphism $H^0(\iota^*\mathbf{L}_{\mathcal{M}}) \xrightarrow{\sim} H^0(\mathbf{L}_{\mathrm{M}})$ and a surjection $H^{-1}(\iota^*\mathbf{L}_{\mathcal{M}}) \twoheadrightarrow H^{-1}(\mathbf{L}_{\mathrm{M}})$.*

We will not give a proof of Lemma 3.2, which one can find in [**34**, §7.2], but we will at least give some intuition for it. At the level of simplicial commutative rings, classical truncation is implemented by adding simplices in degrees ≥ 2 to "kill off" higher homotopy groups. In topology, the Hurewicz theorem says roughly that a map of topology spaces induces an isomorphism of low-degree homotopy groups if and only if it induces an isomorphism of low-degree homology groups. Lemma 3.2 follows from an analogous type of estimate for simplicial commutative rings.

Remark 3.3. There is a variant of Lemma 3.2 where $\mathcal{M}$ is a derived stack, which says that the map $\iota^*\mathbf{L}_{\mathcal{M}} \to \mathbf{L}_{\mathrm{M}}$ induces an isomorphisms on degrees ≥ 0 and a surjection in degree -1.

Now, suppose that M is LCI of the "correct" dimension, which we take to mean $\chi(\mathbf{L}_{\mathcal{M}})$. Then the cotangent complex of M is concentrated in degrees $0, 1$ by Theorem 2.14, and its Euler characteristic is the same as that of $\mathbf{L}_{\mathcal{M}}$. Then Lemma 3.2 forces $\iota^*\mathbf{L}_{\mathcal{M}} \to \mathbf{L}_{\mathrm{M}}$ to be an isomorphism, and therefore by Lemma 2.7 the map $\mathrm{M} \to \mathcal{M}$ is itself an isomorphism.

We have not turned the above argument into a general formal statement, since the "correct" dimension is determined heuristically in practice. The dimension used in the proof is $\chi(\mathbf{L}_{\mathcal{M}})$, but this is circular as it presupposes the existence of $\mathcal{M}$. In practice, "correct" is determined by success in applications. We caution that examples do come up "in nature" where M is LCI (or even smooth) but *not* of the "correct" dimension, such as the derived Galois deformation rings (with crystalline conditions) studied in [**34**].

Remark 3.4. In practice it can be difficult to verify that a moduli space M is LCI of the correct dimension, unless M is furthermore smooth of the correct dimension. This is because the cotangent complex of M is typically not easy to compute, unless the obstruction group vanishes (so that it is a just a sheaf, in which case M is smooth). By contrast, for a *derived* moduli space $\mathcal{M}$, the cotangent complex can usually be described concretely because it has an interpretation in terms of "higher derived dual numbers"; see §A.20 for more discussion of this.

3.2.2. *One begins with a certain classical moduli space* M, *which one wants to promote to a derived moduli space $\mathcal{M}$.* We list some examples of interest below.

Example 3.5 (Moduli of Betti local systems)**.** Let C be a smooth projective curve of genus g and $\widehat{G}$ a reductive group over $\mathbf{C}$. Motivated by Example 3.1, we define the *moduli stack space of $\widehat{G}$-Betti local systems on C* to be the stack $\mathrm{M}^{\mathrm{Betti}}_{\widehat{G}}$ obtained by taking the fibered product

$$(3.3) \qquad \begin{array}{ccc} \mathrm{M}^{\mathrm{Betti},\square}_{\widehat{G}} & \longrightarrow & \mathrm{GL}_n^{2g} \\ \downarrow & & \downarrow{\scriptstyle \prod_{i=1}^g [x_i, y_i]} \\ e & \longrightarrow & \mathrm{GL}_n \end{array}$$

and then defining $M_{\widehat{G}}^{\text{Betti}} = M_{\widehat{G}}^{\text{Betti},\square}/\widehat{G}$ to be the quotient by the conjugation $\widehat{G}$-action.

This moduli space is the subject of the "spectral side" of the Betti Geometric Langlands conjecture, which was proposed by Ben-Zvi – Nadler [**4**].

Example 3.6 (Global unrestricted Galois deformation ring)**.** Let F be a global field and $\mathcal{O}_F \subset F$ its ring of integers. Let S be a finite set of primes of $\mathcal{O}_F$ and $\mathcal{O}_F[1/S]$ be the localization of $\mathcal{O}_F$ at S. Let $\widehat{G}$ be a split reductive group and $\pi_1(\mathcal{O}_F[1/S]) \to \widehat{G}(\mathbf{F}_p)$ be a Galois representation.[4] We define the *(global unrestricted) Galois deformation functor of* $\overline{\rho}$ to be the stack $M_{\widehat{G}}^{\text{Gal}}$ on the category of Artinian $\mathbf{Z}_p$-algebras A equipped with an augmentation $A \xrightarrow{\epsilon} \mathbf{F}_p$, with A-points being the groupoid of lifts $\rho\colon \pi_1(\mathcal{O}_F[1/S]) \to \widehat{G}(A)$ which are sent to $\overline{\rho}$ by the augmentation ϵ, as in the diagram below:

$$
(3.4) \qquad
\begin{array}{ccc}
 & & \widehat{G}(A) \\
 & \nearrow & \downarrow{\scriptstyle \epsilon} \\
\pi_1(\mathcal{O}_F[1/S]) & \longrightarrow & \widehat{G}(\mathbf{F}_p)
\end{array}
$$

Such deformation functors were introduced by Mazur in [**64**], and are at the foundation of the automorphy lifting theorems pioneered by Taylor-Wiles; see the article of [CARAIANI-SHIN] for more about this. (Here "global" refers to that we are looking at representations of the fundamental group of a ring of S-integers in a *global* field, while the "unrestricted" refers to that we have not imposed any local conditions, which we would eventually want to do for applications to the Langlands correspondence.)

Remark 3.7. The need for $M_{\widehat{G}}^{\text{Gal}}$ to be derived depends subtly on F and $\widehat{G}$. We assume that $\overline{\rho}$ is absolutely irreducible and *odd* (meaning that the trace of all complex conjugations is minimal; see [**38**]). For example, if $F = \mathbf{Q}$, and $G = \text{GL}_1, \text{GL}_2$ then one does not expect any interesting derived enhancement, while if $G = \text{GL}_3, \text{GL}_4, \ldots$ then one does. If F is a number field that is not totally real, then $M_{\widehat{G}}^{\text{Gal}}$ should be derived if $\widehat{G}$ is any non-trivial semisimple group, where if F is a function field then $M_{\widehat{G}}^{\text{Gal}}$ need not be derived for any $\widehat{G}$ by [**22, 33**]. We mention finally that if $G = \text{GL}_1$, then $M_{\widehat{G}}^{\text{Gal}}$ should be derived unless $F = \mathbf{Q}$ or a quadratic imaginary field; this will be discussed more in Example 5.7.

Example 3.8 (Local unrestricted Galois deformations)**.** Below we will discuss deriving (unrestricted) Galois deformation functors of global fields. One could also ask whether (unrestricted) Galois deformation functors of local fields should also be derived. It turns out that one can prove that these are always LCI of the correct dimension, so that this is unnecessary. See [**21**] for the $\ell \neq p$ case, and [**10**] for the $\ell = p$ case. By contrast, the question of whether local deformation functors *with conditions* should be derived is unclear at present, and seems to be an important problem for the future of derived Galois deformation theory – see §**??** for more discussion.

---

[4]We suppress the choice of basepoint; for a better perspective see §5.2.

Example 3.9 (Hitchin stacks). Let C be a smooth projective curve over $\mathbf{F}_q$. For integers $m, n \geq 0$, we define the *Hitchin stack* $\mathrm{M}_{m,n}^{\mathrm{Hitch}}$ to be the stack over $\mathbf{F}_q$ with S-points being the groupoid of tuples $(\mathcal{E}, \mathcal{F}, t)$ where

- $\mathcal{E}$ is a rank m vector bundle on $C \times S$,
- $\mathcal{F}$ is a rank n vector bundle on $C \times S$ equipped with an isomorphism $Q \colon \mathcal{F} \xrightarrow{\sim} \mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \Omega_X)$ satisfying $Q^\vee = Q$ (which may be thought of as a Ω_X-valued quadratic form on $\mathcal{F}$), and
- $t \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is a morphism from $\mathcal{E}$ to $\mathcal{F}$.

This is an orthogonal variant of the Hitchin spaces M from [**30, 31**]. It was introduced in order to analyze special cycles on moduli stacks of shtukas (see the article of [YUN] in these proceedings), which feature into a function field version of arithmetic theta series. The spaces $\mathrm{M}_{m,n}^{\mathrm{Hitch}}$ are not the spaces of Higgs bundles which were originally studied by Hitchin, but are other instances of a similar constructions that has risen to importance in automorphic representation theory [**83, 28**].

3.2.3. *The tangent space to $\mathcal{M}$ at a point $m \in \mathcal{M}$ can be calculated, and has some interpretation as a natural cohomology group. Furthermore, the automorphisms lie in a "previous" cohomology group, and obstructions lie in the "next" cohomology group.* The point is that the tangent space to M at $m \in$ M has a "concrete" description in terms of first-order deformations, by virtue of the definition of M as a moduli space, and the interpretation of the tangent space in terms of dual numbers.

Example 3.10 (Moduli of Betti local systems). Continuing the notation of (3.5), let L be a (Betti) $\widehat{G}$-local system on C. We may regard L as a $\mathbf{C}$-point of $\mathrm{M}_{\widehat{G}}^{\mathrm{Betti}}$. Then the tangent space to $\mathrm{M}_{\widehat{G}}^{\mathrm{Betti}}$ at L is $H^1(\pi_1(C), \mathfrak{g}_L)$ where $\mathfrak{g}_L := L \times^{\widehat{G}} \mathfrak{g}$ is the local system on C obtained by taking the quotient of $L \times \mathfrak{g}$ by the diagonal $\widehat{G}$-action. This local system is also sometimes denoted $\mathrm{Ad}\, L$ or $\mathfrak{g}_L$. Furthermore,

- The automorphisms of L may be canonically identified with $H^0(\pi_1(C), \mathfrak{g}_L)$, and
- One can show that the obstruction to first-order deformations lie in $H^2(\pi_1(C), \mathfrak{g}_L)$.

Example 3.11 (Global unrestricted Galois deformations). Continuing the notation of Example 3.6, we may regard $\overline{\rho}$ as an $\mathbf{F}_p$-point of $\mathrm{M}_{\widehat{G}}^{\mathrm{Gal}}$. Then the tangent space to $\mathrm{M}_{\widehat{G}}^{\mathrm{Gal}}$ is $H^1(\pi_1(\mathcal{O}_F[1/S]), \mathfrak{g}_{\overline{\rho}})$, where $\mathfrak{g}_{\overline{\rho}}$ has underlying vector space $\mathfrak{g}_{\mathbf{F}_p}$ with the action of $\pi_1(\mathcal{O}_F[1/S])$ being through $\overline{\rho} \colon \pi_1(\mathcal{O}_F[1/S]) \to \widehat{G}(\mathbf{F}_p)$ and the adjoint action of $\widehat{G}(\mathbf{F}_p)$ on $\mathfrak{g}_{\mathbf{F}_p}$. Furthermore,

- The automorphisms of $\overline{\rho}$ may be canonically identified with $H^0(\pi_1(\mathcal{O}_F[1/S]), \mathfrak{g}_{\overline{\rho}})$, and
- One can show that the obstructions to first-order deformations lie in $H^2(\pi_1(\mathcal{O}_F[1/S]), \mathfrak{g}_{\overline{\rho}})$ [**64**, Proposition 2].

Example 3.12 (Hitchin stack). Continuing the notation of Example 3.9, let $(\mathcal{E}, \mathcal{F}, t) \in \mathrm{M}_{m,n}^{\mathrm{Hitch}}(\mathbf{F}_q)$. Let $\mathsf{T}(\mathcal{E}, \mathcal{F}, t)$ be the complex

$$\mathrm{End}(\mathcal{E}) \oplus \mathrm{End}(\mathcal{F}) \xrightarrow{d} \mathrm{Hom}(\mathcal{E}, \mathcal{F})$$

where $\mathrm{End}(\mathcal{E}) \oplus \mathrm{End}(\mathcal{F})$ lies in degree -1, $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ lies in degree 0, and the differential sends $(A, B) \in \mathrm{End}(\mathcal{E}) \oplus \mathrm{End}(\mathcal{F})$ to $Bt - tA \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$. Then the tangent

space to $\mathrm{M}_{m,n}^{\mathrm{Hitch}}$ at $(\mathcal{E}, \mathcal{F}, t)$ may be canonically identified with $H^1(C, \mathsf{T}(\mathcal{E}, \mathcal{F}, t))$. Furthermore, as shown in [**65**, §4.14],

- The automorphisms of $(\mathcal{E}, \mathcal{F}, t)$ may be canonically identified with $H^0(C, \mathsf{T}(\mathcal{E}, \mathcal{F}, t))$, and
- One can show that the obstruction to first-order deformations lie in $H^2(C, \mathsf{T}(\mathcal{E}, \mathcal{F}, t))$.

3.2.4. *One then tries to construct $\mathcal{M}$ so that its tangent complex is the cohomology complex computing the cohomology groups from the previous step.*

Example 3.13 (Derived moduli of Betti local systems)**.** We continue the notation of Example 3.11. In fact, there is already a subtle wrinkle here: the cohomology groups in Example 3.11 were group cohomology of $\pi_1(C)$ with coefficients in a local system $\mathfrak{g}_L$. The only dependence of this on C is through its fundamental group $\pi_1(C)$, which can be viewed as capturing the information of the Postnikov truncation $\tau_{\leq 1} C$, but turns out that one wants to incorporate the whole "homotopy type" of C. This means that we want the tangent complex to be the cohomology complex $R\Gamma_{\mathrm{Betti}}(C, \mathfrak{g}_L[1])$ rather than $R\Gamma_{\mathrm{group}}(\pi_1(C), \mathfrak{g}_L[1])$.

Let us briefly discuss the difference between these two objects. We have

$$H^0(C, \mathfrak{g}_L) \cong H^0(\pi_1(C), \mathfrak{g}_L)$$

and

$$H^1(C, \mathfrak{g}_L) \cong H^1(\pi_1(C), \mathfrak{g}_L)$$

so the difference lies in H^2. Furthermore, if C has genus $g \geq 1$ then in fact C is a $K(\pi_1(C), 1)$, so that $R\Gamma_{\mathrm{Betti}}(C, \mathfrak{g}_L[1]) \cong R\Gamma_{\mathrm{group}}(\pi_1(C), \mathfrak{g}_L[1])$. Therefore this distinction only arises (among smooth proper complex curves) in the case where $C \cong \mathbf{P}^1$. In that case $\pi_1(C)$ is trivial, so the group cohomology is trivial. However, the higher Betti cohomology of $\mathbf{P}^1$ is non-trivial (note that L is necessarily "the" trivial local system, since $\pi_1(\mathbf{P}^1) = 0$).

One way to construct the desired derived moduli stack $\mathcal{M}_{\widehat{G}}^{\mathrm{Betti}}$ is to take the *derived* fibered product of (3.2), and then quotient out by the (diagonal) conjugation action of $\widehat{G}$. To calculate the tangent complex of the resulting object, use that the formation of tangent complex respects homotopy limits (Example 2.10), and the presentation of the homotopy type of $C(\mathbf{C})$ as the homotopy pushout

(3.5)
$$\begin{array}{ccc} S^1 & \longrightarrow & \bigvee_{i=1}^{2g}(S^1) \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & C(\mathbf{C}) \end{array}$$

Note that for $g = 0$, i.e. $C \cong \mathbf{P}^1$, our presentation of $\mathcal{M}_{\widehat{G}}^{\mathrm{Betti}}$ is the *derived* self-intersection of the identity in $\widehat{G}$, quotiented by $\widehat{G}$. This has dimension $-2\dim \widehat{G}$. The classical truncation of $\mathcal{M}_{\widehat{G}}^{\mathrm{Betti}}$ is simply $\mathrm{M}_{\widehat{G}}^{\mathrm{Betti}} \cong [\mathrm{Spec}\,\mathbf{C}]/\widehat{G}$, reflecting that there is a unique local system on $\mathbf{P}^1$ (the trivial one). A more naive attempt to upgrade $\mathrm{M}_{\widehat{G}}^{\mathrm{Betti}}$ to a derived moduli space, with tangent complex $R\Gamma_{\mathrm{group}}(\pi_1(C), \mathfrak{g}_L[1])$, would have simply produced $[\mathrm{Spec}\,\mathbf{C}]/\widehat{G}$, which is still quasi-smooth but has dimension $-\dim \widehat{G}$.

Note that the description of the tangent complex of $\mathcal{M}_{\widehat{G}}^{\mathrm{Betti}}$ as $R\Gamma_{\mathrm{Betti}}(C, \mathfrak{g}_L[1])$

shows that $\mathcal{M}_{\widehat{G}}^{\mathrm{Betti}}$ is quasi-smooth, since C has Betti cohomological amplitude in $[0, 2]$.

Example 3.14 (Derived global unrestricted Galois deformations)**.** The calculations of Example 3.11 suggest that we should look for a derived stack $\mathcal{M}_{\widehat{G}}^{\mathrm{Gal}}$ whose tangent complex at $\overline{\rho}$ should be $R\Gamma(\pi_1(\mathcal{O}_F[1/S]); \mathfrak{g}_{\overline{\rho}}[1])$. Note that one might expect, based on Example 3.13, that this answer should be adjusted to $R\Gamma_{\mathrm{ét}}(\mathcal{O}_F[1/S]; \mathfrak{g}_{\overline{\rho}}[1])$. It may indeed be better to think of the latter as the "right" answer, but the ring of S-units in a global field is an "étale $K(\pi, 1)$", so that the two always coincide.

The desired $\mathcal{M}_{\widehat{G}}^{\mathrm{Gal}}$ is the *derived Galois deformation functor* of Galatius-Venkatesh. We will not give its construction right now; that requires a bit of digression into homotopy theory. Morally one wants to define the functor of points to be (3.4) but with A an "Artinian simplicial commutative ring augmented over $\mathbf{F}_q$". The resulting object should be a functor from the category of Artinian simplicial commutative rings augmented over $\mathbf{F}_q$ to the category of simplicial sets (also known as "anima", "spaces", or "∞-groupoids").

A case not quite covered by our formulation is where $F \cong \mathbf{F}_q(t)$ is the function field of $\mathbf{P}^1$ over a finite field. In this case, one could consider Galois deformations which are *everywhere unramified*. It would be then be more natural to seek a derived deformation functor with tangent complex $R\Gamma_{\mathrm{ét}}(\mathbf{P}^1_{\mathbf{F}_q}, \mathfrak{g}_{\overline{\rho}}[1])$.

The determination of the cohomology of number fields is part of class field theory and Poitou-Tate duality. The output is that if $p \neq 2$, then $R\Gamma(\pi_1(\mathcal{O}_F[1/S]); \mathfrak{g}_{\overline{\rho}})$ has cohomological amplitude in $[0, 2]$, so that $\mathcal{M}_{\widehat{G}}^{\mathrm{Gal}}$ is quasi-smooth.

However, when F is the function field of a smooth projective curve $C/\mathbf{F}_q$, then it is also natural to study *everywhere unramified* deformations of a local system on C. In this case the tangent complex is $R\Gamma_{\mathrm{ét}}(C; \mathfrak{g}_{\overline{\rho}}[1])$, and it can have nontrivial $H_{\mathrm{ét}}^3$. Therefore, the Galois deformation space is *not* quasi-smooth in this case.

Example 3.15 (Derived Hitchin stack)**.** Continuing with the notation of Example 3.12, we expect that the *derived Hitchin stack* $\mathcal{M}_{m,n}^{\mathrm{Hitch}}$ should have cotangent complex pulled back to $(\mathcal{E}, \mathcal{F}, t) \in \mathcal{M}_{m,n}^{\mathrm{Hitch}}$ is $R\Gamma(C_R, \mathsf{T}(\mathcal{E}, \mathcal{F}, t))$. This $\mathcal{M}_{m,n}^{\mathrm{Hitch}}$ is an orthogonal variant of the derived Hitchin stacks that appear in [**31**]. It may be defined as the derived vector bundle (§2.5.2) associated to the perfect complex taking $(\mathcal{E}, \mathcal{F}) \in \mathrm{Bun}_{\mathrm{GL}_m}(R) \times \mathrm{Bun}_{O_n}(R)$ to $R\mathrm{Hom}_{C_R}(\mathcal{E}, \mathcal{F})$.

There is also another, arguably more flexible approach to define the derived Hitchin stack $\mathcal{M}_{m,n}^{\mathrm{Hitch}}$, by using the notion of the "derived mapping stack" [**77**].

Remark 3.16. As we see from these examples, the justification for the hidden smoothness principle in spaces arising in number theory seems to be the low cohomological dimensions of number fields and related rings that come up in number theory, which lead to "short" tangent complexes.

Example 3.17 (Spectral Hitchin stacks and spectral periods)**.** Non-quasismooth derived moduli spaces are also of interest in the Langlands program. Several examples are studied in [**28**], where it is argued that they geometrize a spectral (i.e., Galois-theoretic) analogue of *periods* in automorphic representation theory. An example is the *spectral Hitchin stack* $\mathrm{M}_{m,n}^{\mathrm{spec}}$, which parametrizes (E, F, t) analogous to the points of $\mathcal{M}_{m,n}$ except with bundles replaced by local systems: its R-points consist of (E, F, t) where

- E is an R-family of rank m local systems on C,
- F is an R-family of rank n local systems on C, equipped with an isomorphism $Q\colon F \xrightarrow{\sim} F^\vee$ satisfying $Q^\vee = Q$ (which may be thought of as a quadratic form on F), and
- $s \in \mathrm{Hom}(E, F)$ is a flat morphism from E to F.

This has a derived enhancement $\mathcal{M}_{m,n}^{\mathrm{spec}}$ whose tangent complex is the cohomology of C with coefficients in an analogous complex $\mathsf{T}(E, F, t)$ to that of Example 3.12. However, since the étale cohomological dimension of C is 2 while the coherent cohomological dimension of C is 1, this tangent complex actually goes up to degree 2, so that $\mathcal{M}_{m,n}^{\mathrm{spec}}$ is *not* quasi-smooth. In fact, the failure of $\mathcal{M}_{m,n}^{\mathrm{spec}}$ to be quasi-smooth is actually an important facet of [**28**], which studies the relation between automorphic periods and spectral periods. One of the main points of [**28**] is that the *dualizing sheaves* of spectral Hitchin stacks, which in some sense measure the failure of smoothness, are the spectral counterpart to automorphic periods.

4. Derived special cycles

We are going to explain applications of the derived Hitchin stacks introduced in Example 3.15 towards the theory of higher arithmetic theta series developed in [**30, 31, 29**].

The context for this discussion is the Kudla program, as discussed in the article of Chao Li. This concerns the so-called *arithmetic theta functions*, which are Fourier series assembled out of special cycles. These were first studied on orthogonal or unitary Shimura varieties, but we will focus on the function-field context, as in the work of Feng-Yun-Zhang. We refer to Chao Li's article for the classical background and motivations.

4.1. Hermitian shtukas. We recall some of the definitions from [**30**, §6,7].

Let X be a smooth projective curve over $k = \mathbf{F}_q$. We regard $\mathrm{Spec}\, k$ as the base for all our geometric spaces; for example, a product means a fibered product over $\mathrm{Spec}\, k$ unless indicated otherwise. Let $\nu\colon X' \to X$ be a finite étale double cover, with the non-trivial automorphism over X denoted $\sigma\colon X' \to X'$.

For each $r \geq 0$ and $n > 0$, [**30**, §6] and [**31**, §2] construct moduli spaces of rank n unitary shtukas, denoted $\mathrm{Sht}_{U(n)}^r$. These are variants of the spaces discussed in the article of Zhiwei Yun. There is a map $\pi\colon \mathrm{Sht}_{U(n)}^r \to (X')^r$, called the "leg map". For $r = 1$, it is to be thought of as analogous to the structure map for the integral model of a unitary Shimura variety. We will now outline these constructions.

A rank n *Hermitian* bundle on $X \times S$ with respect to $\nu\colon X' \to X$ is a vector bundle $\mathcal{F}$ of rank n on $X' \times S$, equipped with an isomorphism $h\colon \mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}^\vee$ such that $\sigma^* h^\vee = h$. We refer to h as a "Hermitian structure" on $\mathcal{F}$.

We denote by $\mathrm{Bun}_{U(n)}$ the moduli stack of rank n Hermitian bundles on X, which sends a test scheme S to the groupoid of rank n Hermitian bundles on $X \times S$. The notation is justified by the following remark.

Definition 4.1. Let $r \geq 0$ be an integer. The *Hecke stack* $\mathrm{Hk}_{U(n)}^r$ has as S-points the groupoid of the following data:

    (i) $x_i' \in X'(S)$ for $i = 1, \ldots, r$, with graphs denoted by $\Gamma_{x_i'} \subset X' \times S$.

    (ii) A sequence of vector bundles $\mathcal{F}_0, \ldots, \mathcal{F}_r$ of rank n on $X' \times S$, each equipped with Hermitian structure $h_i\colon \mathcal{F}_i \xrightarrow{\sim} \sigma^*\mathcal{F}_i^\vee$.

(iii) Isomorphisms $f_i\colon \mathcal{F}_{i-1}|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}} \xrightarrow{\sim} \mathcal{F}_i|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}}$, for $1 \le i \le r$, compatible with the Hermitian structures, with the following property: there exists a rank n vector bundle $\mathcal{F}^\flat_{i-1/2}$ and a diagram of vector bundles

(4.1)
$$
\begin{array}{ccc}
 & \mathcal{F}^\flat_{i-1/2} & \\
{}^{f_i^{\leftarrow}}\swarrow & & \searrow{}^{f_i^{\rightarrow}} \\
\mathcal{F}_{i-1} & & \mathcal{F}_i
\end{array}
$$

such that $\operatorname{coker}(f_i^{\leftarrow})$ is locally free of rank 1 $\Gamma_{x_i'}$, and $\operatorname{coker}(f_i^{\rightarrow})$ is locally free of rank 1 over $\Gamma_{\sigma(x_i')}$. In particular, $f_i^{\leftarrow}$ and $f_i^{\rightarrow}$ are invertible upon restriction to $X' \times S - \Gamma_{x_i'} - \Gamma_{\sigma(x_i')}$, and the composition

$$
\mathcal{F}_{i-1}|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}} \xrightarrow{(f_i^{\leftarrow})^{-1}} \mathcal{F}^\flat_{i-1/2}|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}} \xrightarrow{f_i^{\rightarrow}} \mathcal{F}_i|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}}
$$

agrees with f_i.

For a vector bundle $\mathcal{F}$ on $X' \times S$, we abbreviate ${}^\tau\mathcal{F} := (\operatorname{Id}_X \times \operatorname{Frob}_S)^* \mathcal{F}$. If $\mathcal{F}$ has a Hermitian structure $h\colon \mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}^\vee$, then ${}^\tau\mathcal{F}$ is equipped with the Hermitian structure ${}^\tau h$; we may suppress this notation when we speak of the "Hermitian bundle" ${}^\tau\mathcal{F}$.

Definition 4.2. Let $r \ge 0$ be an integer. We define $\operatorname{Sht}^r_{U(n)}$ by the Cartesian diagram

$$
\begin{array}{ccc}
\operatorname{Sht}^r_{U(n)} & \longrightarrow & \operatorname{Hk}^r_{U(n)} \\
\downarrow & & \downarrow \\
\operatorname{Bun}_{U(n)} & \xrightarrow{(\operatorname{Id},\operatorname{Frob})} & \operatorname{Bun}_{U(n)} \times \operatorname{Bun}_{U(n)}
\end{array}
$$

Here the map $\operatorname{Frob}\colon \operatorname{Bun}_{U(n)} \to \operatorname{Bun}_{U(n)}$ is the map which on S-points is given by pre-composition with the absolute Frobenius on S. A point of $\operatorname{Sht}^r_{U(n)}$ will be called a "Hermitian shtuka (of rank n)", or "$U(n)$-shtuka".

Concretely, the S-points of $\operatorname{Sht}^r_{U(n)}$ are given by the groupoid of the following data:

(i) $x_i' \in X'(S)$ for $i = 1,\dots,r$, with graphs denoted $\Gamma_{x_i'} \subset X \times S$. These are called the *legs* of the shtuka.

(ii) A sequence of vector bundles $\mathcal{F}_0,\dots,\mathcal{F}_n$ of rank n on $X' \times S$, each equipped with a Hermitian structure $h_i\colon \mathcal{F}_i \xrightarrow{\sim} \sigma^*\mathcal{F}_i^\vee$.

(iii) Isomorphisms $f_i\colon \mathcal{F}_{i-1}|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}} \xrightarrow{\sim} \mathcal{F}_i|_{X'\times S-\Gamma_{x_i'}-\Gamma_{\sigma(x_i')}}$ compatible with the Hermitian structure, which as modifications of the underlying vector bundles on $X' \times S$ are as in the definition of the Hecke stack $\operatorname{Hk}^r_{U(n)}$.

(iv) An isomorphism $\varphi\colon \mathcal{F}_r \cong {}^\tau\mathcal{F}_0$ compatible with the Hermitian structure.

Remark 4.3. One can show that $\operatorname{Sht}^r_{U(n)}$ is a Deligne-Mumford stack locally of finite type. The map $\operatorname{Sht}^r_{U(n)} \to (X')^r$ is smooth, separated, equidimensional of relative dimension $r(n-1)$. For these and more geometric properties, see [**30**, §6.4].

4.2. Special cycles. Now we will define special cycles on $\mathrm{Sht}^r_{U(n)}$, which are analogous to the special cycles of Kudla-Rapoport on unitary Shimura varieties [**52**].

Definition 4.4. Let $\mathcal{E}$ be a rank m vector bundle on X'. We define the stack $\mathrm{Z}^r_{\mathcal{E}}$ whose S-points are the groupoid of the following data:

- A $U(n)$-shtuka with $(\{x'_1, \ldots, x'_r\}, \{\mathcal{F}_0, \ldots, \mathcal{F}_r\}, \{f_1, \ldots, f_r\}, \varphi) \in \mathrm{Sht}^r_{U(n)}(S)$.
- Maps of coherent sheaves $t_i \colon \mathcal{E} \boxtimes \mathcal{O}_S \to \mathcal{F}_i$ on $X' \times S$ such that the isomorphism $\varphi \colon \mathcal{F}_r \cong {}^\tau \mathcal{F}_0$ intertwines t_r with ${}^\tau t_0$, and the maps t_{i-1}, t_i are intertwined by the modification $f_i \colon \mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ for each $i = 1, \ldots, r$, i.e. the diagram below commutes.

$$
\begin{array}{ccccccccc}
\mathcal{E} \boxtimes \mathcal{O}_S & == & \mathcal{E} \boxtimes \mathcal{O}_S & == & \ldots & == & \mathcal{E} \boxtimes \mathcal{O}_S & \xrightarrow{\sim} & {}^\tau(\mathcal{E} \boxtimes \mathcal{O}_S) \\
\downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle t_1} & & \downarrow & & \downarrow{\scriptstyle t_r} & & \downarrow{\scriptstyle {}^\tau t_0} \\
\mathcal{F}_0 & \xdashrightarrow{f_0} & \mathcal{F}_1 & \xdashrightarrow{f_1} & \ldots & \xdashrightarrow{f_r} & \mathcal{F}_r & \xrightarrow{\ \sim\ } & {}^\tau \mathcal{F}_0
\end{array}
$$

Remark 4.5. There is an evident map $\mathrm{Z}^r_{\mathcal{E}} \to \mathrm{Sht}^r_{U(n)}$ projecting to the data in the first bullet point. This map is finite by [**30**, Proposition 7.5], hence induces a pushforward map on Chow groups.

Definition 4.6. Let $\mathrm{A}_{\mathcal{E}}(k)$ be the k-vector space of Hermitian maps $a \colon \mathcal{E} \to \sigma^* \mathcal{E}^\vee$ such that $\sigma(a)^\vee = a$.

Let $(\{x'_i\}, \{\mathcal{F}_i\}, \{f_i\}, \varphi, \{t_i\}) \in \mathcal{Z}^r_{\mathcal{E}}(S)$. By the compatibilities between the t_i in the definition of $\mathcal{Z}^r_{\mathcal{E}}$, the compositions

$$
(4.2) \qquad \mathcal{E} \boxtimes \mathcal{O}_S \xrightarrow{t_i} \mathcal{F}_i \xrightarrow{h_i} \sigma^* \mathcal{F}_i^\vee \xrightarrow{\sigma^* t_i^\vee} \sigma^* \mathcal{E}^\vee \boxtimes \mathcal{O}_S
$$

agree for each i, and (4.2) for $i = r$ also agrees with the Frobenius twist of (4.2) for $i = 0$. Hence (4.2) for every i gives the same point of $\mathrm{A}_{\mathcal{E}}(S)$, which moreover must come from $\mathrm{A}_{\mathcal{E}}(k)$. This defines a map $\mathrm{Z}^r_{\mathcal{E}} \to \mathrm{A}_{\mathcal{E}}(k)$. For $a \in \mathrm{A}_{\mathcal{E}}(k)$, we denote by $\mathrm{Z}^r_{\mathcal{E}}(a)$ the fiber of $\mathrm{Z}^r_{\mathcal{E}}$ over a. We have

$$
\mathrm{Z}^r_{\mathcal{E}} = \coprod_{a \in \mathrm{A}_{\mathcal{E}}(k)} \mathrm{Z}^r_{\mathcal{E}}(a).
$$

The $\mathrm{Z}^r_{\mathcal{E}}$ or $\mathrm{Z}^r_{\mathcal{E}}(a)$ are called *special cycles*.

4.3. Virtual fundamental classes of special cycles. Motivated by the Kudla program, we expect to attach to each special cycle $\mathrm{Z}^r_{\mathcal{E}}(a)$, $a \in \mathrm{A}_{\mathcal{E}}(k)$, a class in the Chow group $\mathrm{Ch}^*(\mathrm{Sht}^r_{U(n)})$ such that the resulting Fourier series is "automorphic". However, a cursory inspection reveals that the cycles $\mathrm{Z}^r_{\mathcal{E}}(a)$ have varying dimensions as a varies. For example, when $a = 0$ then $\mathrm{Z}^r_{\mathcal{E}}(0)$ clearly surjects onto $\mathrm{Sht}^r_{U(n)}$. On the other hand, it is not hard to see that if $r > 0$ and $a \neq 0$ then $\mathrm{Z}^r_{\mathcal{E}}(a)$ has image a proper substack of $\mathrm{Sht}^r_{U(n)}$. Hence the naive cycle classes $[\mathrm{Z}^r_{\mathcal{E}}(a)]^{\mathrm{naive}}$ would not even lie in the same Chow group as a varies.

This is a hint that we must construct *virtual* fundamental cycles $[\mathrm{Z}^r_{\mathcal{E}}(a)]^{\mathrm{vir}}$, which should lie in $\mathrm{Ch}^{mr}(\mathrm{Sht}^r_{U(n)})$ where $m = \mathrm{rank}\, \mathcal{E}$. There are various heuristics which guide such a construction. A key input is that when $m = \mathrm{rank}\, \mathcal{E} = 1$, and when a is *non-singular*, meaning that $a \colon \mathcal{E} \to \sigma^* \mathcal{E}^\vee$ is injective as a map of coherent sheaves (equivalently, an isomorphism over the generic point of X), then $\mathrm{Z}^r_{\mathcal{E}}(a)$ is LCI of the "correct" codimension, which is r. In this case, heuristics (justified

a postiori by §3.2.1) suggest that one should take $[\mathrm{Z}_{\mathcal{E}}^r(a)]^{\mathrm{vir}} = [\mathrm{Z}_{\mathcal{E}}^r(a)]^{\mathrm{naive}}$. More generally, using this observation one can define $[\mathrm{Z}_{\mathcal{E}}^r(a)]^{\mathrm{vir}}$ for any non-singular a (the point being to allow $m > 1$) using the original method of [52] by presenting $\mathrm{Z}_{\mathcal{E}}^r(a)$ as an open-closed substack of the "derived intersection" of special cycles labeled by $m = 1$ and non-singular coefficients.

The definition for singular a requires further heuristics. Over a number field, the analogous problem has a relatively simple answer. In that case a can be represented as a matrix which is of the form $a_{ns} \oplus 0$, where a_{ns} is non-singular of rank $m' < m$. Then the corresponding virtual fundamental class is the product of the virtual fundamental class for a_{ns} times the $(m - m')$th power of a the Chern class of a certain tautological line bundle. In the function field case at hand, the answer is much more complicated: it is in general a sum of infinitely many terms, each itself a product of some Chern class and a cycle obtained by a "derived intersection". To explain why the extra complication appears, we consider the case $a = 0$. Then $\mathrm{Z}_{\mathcal{E}}^r(0)$ is stratified by the kernel of the map $t_i \colon \mathcal{E} \boxtimes \mathcal{O}_S \to \mathcal{F}$, and each such stratum contributes a piece of the form described above (a Chern class times a virtual class which can be constructed via derived intersections). From this perspective, the number field situation has the special property that the governing Hermitian form is *positive definite*, which rules out all but one stratum. This is discussed in more detail in [31, §4] and [29, Introduction]. The upshot is that $[\mathrm{Z}_{\mathcal{E}}^r(a)]^{\mathrm{vir}}$ can be constructed in all situations, but the answer is extremely complicated in the function field setting.

Alternatively, it was observed in [31] that the definition of the special cycles $\mathrm{Z}_{\mathcal{E}}^r$ promote naturally, according to the pattern of §3, to *derived special cycles* $\mathcal{Z}_{\mathcal{E}}^r$ which are quasi-smooth of the correct dimension, and whose classical truncations recover $\mathrm{Z}_{\mathcal{E}}^r$. According to a construction of Khan [48], any quasi-smooth derived stack $\mathcal{Z}$ has an intrinsic notion of fundamental class $[\mathcal{Z}]^{\mathrm{naive}}$. (This rough statement had long been a folklore theorem of derived algebraic geometry, announced for example in [53], and its importance for Gromov-Witten or Donaldson-Thomas flavors of enumerative geometry were long understood. However, Khan's construction of $[\mathcal{Z}]^{\mathrm{vir}}$ is the most general to appear in the literature, and we basically use its full generality.) In [31, Theorem 6.5] it is proved that $[\mathcal{Z}_{\mathcal{E}}^r(a)]^{\mathrm{naive}}$ coincides with the previously defined (elementary) construction of $[\mathrm{Z}_{\mathcal{E}}^r(a)]^{\mathrm{vir}}$.

To sketch how $[\mathcal{Z}_{\mathcal{E}}^r]$ is defined, the key point is to realize that $\bigcup_{\mathcal{E}} \mathrm{Z}_{\mathcal{E}}^r$ admits a description as a fibered product

$$(4.3) \qquad \begin{array}{ccc} \bigcup_{\mathcal{E}} \mathrm{Z}_{\mathcal{E}}^r & \longrightarrow & \mathrm{Hk}_{\mathrm{M}}^r \\ \downarrow & & \downarrow \\ \mathrm{M} & \xrightarrow{(\mathrm{Id},\mathrm{Frob})} & \mathrm{M} \times \mathrm{M} \end{array}$$

maybe just change that earlier discusion?

for M a unitary variant of the Hitchin stack from Example 3.9, and $\mathrm{Hk}_{\mathrm{M}}^r$ a certain Hecke correspondence for M. Now, M and $\mathrm{Hk}_{\mathrm{M}}^r$ have natural enhancements to (quasismooth) derived stacks $\mathcal{M}$ and $\mathrm{Hk}_{\mathcal{M}}^r$; for example $\mathcal{M}$ is a unitary variant of Example 3.15. Actually, the important point for the quasi-smoothness of $[\mathcal{Z}_{\mathcal{E}}^r]$ is that the maps $\mathrm{Hk}_{\mathcal{M}}^r \to \mathcal{M}$ are quasismooth (whereas we do not know anything "nice" that can be said $\mathrm{Hk}_{\mathrm{M}}^r \to \mathrm{M}$). Then, we define $\mathcal{Z}_{\mathcal{E}}^r$ as a derived fibered

product

$$
\begin{array}{ccc}
\bigcup_{\mathcal{E}} \mathcal{Z}_{\mathcal{E}}^r & \longrightarrow & \mathrm{Hk}_{\mathcal{M}}^r \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{(\mathrm{Id},\mathrm{Frob})} & \mathcal{M} \times \mathcal{M}
\end{array}
$$

whose classical truncation recovers $\mathrm{Z}_{\mathcal{E}}^r$ by (4.3). Finally, the $\mathcal{Z}_{\mathcal{E}}^r(a)$ the fibers of the map $\mathcal{Z}_{\mathcal{E}}^r \to \mathcal{A}_{\mathcal{E}}(k)$.

Remark 4.7. In [**57**], Madapusi has shown that the virtual fundamental classes of special cycles on Shimura varieties can also be seen as the naive cycle classes of derived special cycles.

4.4. The modularity conjecture. Let $n \geq 1$ and $n \geq m \geq 1$. It is possible to assemble $[\mathrm{Z}_{\mathcal{E}}^r(a)]^{\mathrm{vir}}$ into a Fourier series (with Fourier parameter $a \in \mathrm{A}_{\mathcal{E}}(k)$), called a *higher theta series* in [**31**]. When $r = 0$ it recovers the classical theta series, while for $r = 1$ it recovers an arithmetic theta series in the sense of Kudla. By analogy with conjectures of Kudla, [**31**, Conjecture 4.16] predicts that these higher theta series are *modular* in the sense of automorphic forms, although the precise formulation is rather technical so we leave it to the references.

When $r = 0$, this Conjecture amounts to the modularity of classical theta functions, which has long been known. When $r = 1$, it is parallel to the conjectural modularity of arithmetic theta series on orthogonal and unitary Shimura varieties, which has seen much recent progress, as explained in the article of Chao Li. But for $r > 1$, it is a new phenomenon with no parallel in the classical theory.

We remark that the derived construction of $[\mathcal{Z}_{\mathcal{E}}^r(a)]^{\mathrm{naive}} = [\mathrm{Z}_{\mathcal{E}}^r(a)]^{\mathrm{vir}}$ already allowed in [**31**] to prove certain expected properties that were not accessible using the elementary definition. This is not an issue for the virtual classes of special cycles on Shimura varieties; the difference is because of the more complicated nature of singular terms in the function field situation, as described previously. The power of the description $[\mathcal{Z}_{\mathcal{E}}^r(a)]^{\mathrm{naive}}$ is that it is more uniform and conceptual, and "only" depends on the underlying derived stack $\mathcal{Z}_{\mathcal{E}}^r(a)$, whereas $[\mathrm{Z}_{\mathcal{E}}^r(a)]^{\mathrm{vir}}$ depends not only on $\mathrm{Z}_{\mathcal{E}}^r(a)$ but on an auxiliary presentation of it. For this reason, the authors felt at the time of writing [**31**] that the derived perspective would be more useful for proving modularity, although they did not yet see how. This view has been at least partially affirmed in [**29**], which proves that the ℓ-adic realization of higher theta series in the cohomology of the generic fiber of $\mathrm{Sht}_{U(n)}^r \to (X')^r$ is modular.

Indeed, [**29**] is heavily steeped in the formalism of derived algebraic geometry, although the derived construction of the $[\mathrm{Z}_{\mathcal{E}}^r(a)]^{\mathrm{vir}}$ is a rather small ingredient. In [**29**] modularity comes from calculations within new forms of Fourier analysis, including a so-called *arithmetic Fourier transform* on Borel-Moore homology groups that restricts to the usual finite Fourier transform in the case $r = 0$ (where it pertains to the modularity of the classical theta function), and a *derived Fourier transform* on the derived category of ℓ-adic sheaves on derived vector bundles. These situations are linked by a higher version of the trace construction, which gives a "sheaf-cycle correspondence" generalizing the classical sheaf-function correspondence. Even sketching the argument is beyond the scope of this discussion, so we refer to the [**29**, §1, §2] for an outline.

5. Derived Galois deformation rings

In this section we will give an overview of the study of derived Galois deformation rings in [**34**]. We will elide many technical details in order to give the "big picture". The reader craving more details could directly consult the paper [**34**], which is very clear and comprehensive, and contains many edifying remarks; another resource is the Oberwolfach Arbeitsgemeinschaft Report [**1**].

Readers should be familiar with the classical theory of Galois deformation rings before reading this section. We recommend [**15**] for a quick and illuminating survey on the classical Taylor-Wiles method, [**16**] for an introduction to the Calegari-Geraghty method, and [**76**] for a clear, concise account with technical details.

5.1. Overview. We begin with an overview of what will be covered in this section, to orient ourselves.

(i) First one constructs a derived global unrestricted Galois deformation ring $\mathcal{R}$ to represent the functor $\mathcal{M}_{\widehat{G}}^{\mathrm{Gal}}$ from Example 3.14. This requires some assumptions on $\overline{\rho}$, and as literally formulated in Example 3.14 can only exist under if $\widehat{G}$ is of adjoint type, although this can be generalized in a standard manner by accounting for determinants and centers.

   We remark that according to a folklore conjecture of Mazur, $\mathcal{R}$ should actually be homotopy discrete. However, this Conjecture is wide open, and treating $\mathcal{R}$ as a derived ring allows us to "circumvent" the conjecture for some purposes. Indeed, what we really need about $\mathcal{R}$ is the description of its tangent complex in terms of Galois cohomology, which we can calculate unconditionally for the derived ring $\mathcal{R}$, for the reasons explained in §A.20, while knowing the tangent complex of $\pi_0(\mathcal{R})$ is equivalent to Mazur's conjecture.

(ii) The deformation ring from the previous point does not capture local conditions which we expect to be satisfied for the Galois representations corresponding to automorphic forms. The next step is to impose local conditions, in order to obtain an object that should "match" automorphic forms. The subtlest aspect of these local conditions concerns the decomposition group for p, where the local conditions are governed by p-adic Hodge theory.

   To do this, one considers analogous derived deformation functors for local Galois groups, and then imposes local conditions by forming the *derived* fibered product against the global deformation ring. We note that the local deformation functors are usually discrete, so that it is the step of forming the derived fibered product that introduces genuine derived structure.

   For the local condition at p, we consider what may be considered the "simplest" situation from a modern perspective on integral p-adic Hodge theory: the crystalline Galois deformation functor in the Fontaine-Laffaille range. This should be regarded as a first test case; this step seems to be first obvious barrier to generalization.

   This step results in an object $\mathcal{R}^{\mathrm{crys}}$ called the *derived crystalline global Galois deformation ring*. In many situations it has dimension $-\delta$, where $\delta > 0$ is the "defect" and therefore cannot be classical.

We "know" the tangent complex of $\mathcal{R}^{\mathrm{crys}}$ in terms of Galois coho-
mology, but the structure of its homotopy groups is mysterious a priori.
It will be determined in the next step.

(iii) As a derived generalization of $R = T$ theorems, one expects that $\mathcal{R}^{\mathrm{crys}}$
acts on the cohomology of locally symmetric spaces, in a way that makes
said cohomology "free" over $\mathcal{R}^{\mathrm{crys}}$, at least generically. This action is
constructed at the level of homotopy groups in [**34**] in tandem with the
determination of $\pi_*(\mathcal{R}^{\mathrm{crys}})$, using the Calegari-Geraghty modification of
the Taylor-Wiles method.

In particular, the multiplicities in the cohomology of locally symmet-
ric space have an interesting numerical pattern which is explained by the
structure of $\pi_*(\mathcal{R}^{\mathrm{crys}})$. This numerical pattern was one of the original
motivations for [**34**] and [**81**].

In our opinion, the construction of the action at the derived level
(instead of at the level of homotopy groups) is one of the major challenges
of derived Galois deformation theory at present.

5.2. Derived local systems. Now we will circle back around to the begin-
ning, and explain how to construct the derived Galois deformation functor in step
(i) of §5.1.

The main point is that some care must be taken to formulate the moduli prob-
lem in a "homotopy-invariant" way, so that it sends homotopy equivalent simplicial
commutative rings to homotopy equivalent simplicial sets.

One example of an awkward issue that arises in the usual formulation of the
notion of representations is that issue of conjugation. If one is just interested in the
set of equivalence classes, then one can take a representation to be a homomorphism
modulo conjugacy. We prefer to take a more "canonical" perspective, which will
be explained below.

5.2.1. *Classifying spaces.* Given discrete groups Γ and H, there are *classifying
spaces* $B\Gamma$ and BH, which come with canonical basepoints. The classifying space
BH can be defined as the geometric realization of the simplicial set $N_\bullet(H)$ ob-
tained as the nerve of H viewed as a category (consisting of a single object, with
endomorphisms given by H). The simplicial set $N_\bullet(H)$ takes the form

$$(5.1) \qquad\qquad * \Leftarrow H \Lleftarrow H \times H \ldots$$

where explicit formulas for face and degeneracy maps can be found in [**61**, p.128].
In particular, BH is equipped with a distinguished basepoint pt.

More generally, the formula (5.1) can be used to extend the definition of clas-
sifying spaces to groups that are not discrete. Suppose H is a simplicial group,
which can be thought of synonymously with "topological group". Then (5.1) is a
bi-simplicial set, and we define $N_\bullet(H)$ to be the diagonal bi-simplicial set. This
generalization will be important in defining derived group representation functors,
where H will arise by evaluating an algebraic group G on a simplicial ring $\mathcal{R}$ –
it is the passage from discrete R to simplicial commutative $\mathcal{R}$ which introduces a
non-trivial simplicial structure on H.

5.2.2. *Homotopy theory of groups.* We discuss some features of the category
of simplicial groups. First, regarding simplicial groups: as explained in §A.3, the
category of simplicial groups admits a natural enrichment over simplicial sets. This

says in particular that the set of morphisms between two simplicial groups is naturally realized as the 0-simplices of a simplicial set; intuitively, it means that if Γ and H are simplicial groups, then the set of continuous homomorphisms $\Gamma \to H$ can be topologized in a natural (interesting) way. This structure is part of the ∞-category structure on simplicial groups.[5]

Below we use the word "space" as a synonym for simplicial set viewed as an object in the ∞-category of simplicial sets. By [**55**, §5.2.6.10, 5.2.6.13], the classifying space functor from simplicial groups to pointed spaces is fully faithful. Informally, this means that the space of maps of pointed spaces $(B\Gamma, \mathrm{pt}) \to (BH, \mathrm{pt})$ is equivalent to the space of continuous homomorphisms $\Gamma \to H$.

It is then natural to view the space of maps of *unpointed* spaces $B\Gamma \to BH$ as refining the set of H-valued representations of Γ. Informally speaking, it is "the space of homomorphisms from Γ to H modulo conjugacy".

5.2.3. *Homotopy theory of local systems.* Let X be a "nice" space (e.g., the geometric realization of a simplicial set) and H a discrete group. Then homotopy classes of maps from X to BH are in bijection with "H-local systems" on X. If X is connected, then after choosing a basepoint $x \in X$ these are in bijection with the set of homomorphisms $\pi_1(X, x) \to H$ modulo conjugacy.

Example 5.1. If $X = B\Gamma$, then the notion of H-local system on $B\Gamma$ recovers the notion of representation of Γ into H from §5.2.2.

Therefore, the space of maps from X to BH gives a "space of local systems of X".

More generally, if H is not discrete, then we still have a space of morphisms from X to BH, which informally speaking puts a topology on the set of H-torsors over X. If X is connected, then its underlying set is that of homomorphisms from $\pi_1(X, x)$ modulo conjugacy. Note that the underlying set depends only on $\pi_1(X, x)$, but the anima $\mathrm{Map}_{\mathsf{sSet}}(X, BH)$ depends on the full space of X (e.g., including the information of higher homotopy groups of X).

5.2.4. *Summary.* To summarize, we would like refine our point of view on what is a "representation of Γ into H". Instead of the classical notion as a homomorphism modulo conjugacy, we view it (for discrete H) as an "H-local system on Γ", or in other words a morphism $B\Gamma \to BH$.

An important feature of doing this is that we automatically get not just a *set* of morphisms $B\Gamma \to BH$ but a *space* of morphisms.

Furthermore, the same expression generalizes well to the case where H is not discrete.

5.3. Global unrestricted derived deformation ring. Let k be a finite field. For the moment let Γ be a discrete group and $\widehat{G}$ a split algebraic group over the Witt vectors $W(k)$. Let $\bar{\rho}$ be a representation of Γ in $\widehat{G}(k)$, which we regard as a map $B\Gamma \to B(\widehat{G}(k))$.

5.3.1. *Classical deformation ring.* Let $\mathrm{M}_{\widehat{G}}^{\Gamma}$ be the functor on the category of Artinian local $W(k)$-algebras A augmented over k, sending A to the set of lifts

---

[5]We remark that by [**55**, §5.2.6.10, 5.2.6.21], the ∞-category of simplicial groups is equivalent to the ∞-category of associative groups in the ∞-category of simplicial sets; the latter is the more classical context for the study of the "homotopy theory of groups" that one usually finds in the literature.
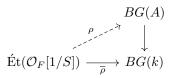
modulo conjugacy,

$$\left\{ \begin{array}{ccc} & & G(\mathrm{A}) \\ & \overset{\rho}{\nearrow} & \downarrow \\ \Gamma & \xrightarrow[\overline{\rho}]{} & G(k) \end{array} \right\} / \sim$$

If $\overline{\rho}$ is absolutely irreducible and Γ satisfies certain finite conditions on its Galois cohomology, then Schlessinger's criterion implies that $\mathcal{M}_{\widehat{G}}^{\Gamma}$ is representable. These observations, as well as the original idea to consider moduli of Galois deformations, is due to Mazur [**64**]. The story can be extended to profinite groups Γ, by treating $\mathrm{M}_{\widehat{G}}^{\Gamma}$ as the (filtered) colimit of the deformation functors obtained from the finite quotients of Γ.

MH A word about moduli stacks.

Suppose F is the ring of integers in a global field and S is a finite set of places of F, then $\Gamma = \pi_1(\mathrm{Spec}\ \mathcal{O}_F[1/S])$ satisfies the requisite finiteness conditions by class field theory. We are actually going to rename $\mathrm{M}_{\mathcal{O}_F[1/S]}^{\overline{\rho}} := \mathrm{M}_{\widehat{G}}^{\Gamma}$ in this case, because this notation emphasizes the parameters that we will focus on in this section. The pro-representing ring is the *(classical) global unrestricted Galois deformation ring*, denoted $\mathrm{R}_{\mathcal{O}_F[1/S]}^{\overline{\rho}}$.

Let us explain some terminology from [**34**]. From a scheme one X can extract, following Artin-Mazur [**3**] and Friedlander [**32**], an étale homotopy type $\mathrm{Ét}(X)$, which we regard as a pro-space, whose raison d'être is to capture the étale topology of X. For $\mathcal{O}_F[1/S]$ the ring of S-integers in a global field, it turns out that $\mathrm{Ét}(\mathcal{O}_F[1/S]) \cong B\Gamma$ where Γ is the étale fundamental group of $\mathrm{Spec}\ \mathcal{O}_F[1/S]$ with respect to some basepoint. (This is what we meant when we said earlier that $\mathrm{Spec}\ \mathcal{O}_F[1/S]$ is an "étale $K(\pi,1)$". The justification is based on class field theory.) We prefer $\mathrm{Ét}(\mathcal{O}_F[1/S])$ to $B\Gamma$, even though they are equivalent, because the former does not refer to a choice of basepoint. By the discussion in the previous section, the moduli description of $\mathrm{M}_{\widehat{G}}^{\Gamma}$ can be formulated as the set of lifts

$$\begin{array}{ccc} & & BG(A) \\ & \overset{\rho}{\nearrow} & \downarrow \\ \mathrm{Ét}(\mathcal{O}_F[1/S]) & \xrightarrow[\overline{\rho}]{} & BG(k) \end{array}$$

5.3.2. *Derived deformation functor.* Now we want to make a derived version of this construction. We begin by deriving the functor $\mathrm{M}_{\mathcal{O}_F[1/S]}^{\overline{\rho}}$. First we need to define the notion of an *Artinian local animated ring*.

Definition 5.2. An animated ring $\mathcal{A}$ is *Artinian local* if $\pi_0(\mathcal{A})$ is Artinian local and $\pi_*(\mathcal{A})$ is finitely generated as a module over $\pi_0(\mathcal{A})$.

By the discussion of §5.2, we are motivated to define $\mathcal{M}_{\mathcal{O}_F[1/S]}^{\overline{\rho}}$ to be the functor on the category of Artinian local animated ring $\mathcal{A}$ equipped with an augmented over k, sending $\mathcal{A}$ to the anima of "lifts"

$$\begin{array}{ccc} & & BG(\mathcal{A}) \\ & \overset{\rho}{\nearrow} & \downarrow \\ \mathrm{Ét}(\mathcal{O}_F[1/S]) & \xrightarrow[\overline{\rho}]{} & BG(k) \end{array}$$

Here by "lift" we mean by definition the *homotopy* fiber

$$\mathrm{Map}(\text{Ét}(\mathcal{O}_F[1/S]), BG(\mathcal{A})) \overset{h}{\times}_{\mathrm{Map}(\text{Ét}(\mathcal{O}_F[1/S]), BG(k))} \{\overline{\rho}\}.$$

Our (sketchy) formulation is not quite the same as that of Galatius-Venkatesh, who instead use the framework of model categories. While the ∞-category theory saves us some work, it is quite instructive to study the treatment of Galatius-Venkatesh, which exposes interesting homotopy-theoretic subtleties that are suppressed in ours. See [**1**, C.4] for a summary of these issues.

5.3.3. *Representability.* Lurie's thesis [**53**] establishes a derived version of Schlessinger's representability criterion. To check the criterion, one needs to compute the tangent complex; the answer was mentioned in Example 3.14 and will be revisited later.

Using Lurie's derived Schlessinger criterion, Galatius-Venkatesh show that $\mathcal{M}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$ is representable by a pro Artinian animated ring $\mathcal{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$. Checking the criterion is not the most interesting aspect of [**34**], in our opinion, and we will skip it entirely.

5.4. Desiderata. We abstract out the properties of $\mathcal{M}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$ and $\mathcal{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$ that we will actually need in the future.

5.4.1. *Tangent complex.* As already discussed in Example 3.14, the tangent complex of $\mathcal{M}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$ at $\overline{\rho}$ is $R\Gamma(\mathrm{Spec}\ \mathcal{O}_F[S^{-1}], \mathfrak{g}_{\overline{\rho}}[1])$. This is explained in [**34**, Example 4.38], although the proof there is quite concise (basically pointing out that this calculation is tautological once the definition of "local system" and "$R\Gamma$" are understood properly), so we will spell out some more discussion and context.

Example 5.3 (Tangent complex of BG). Via the cover $\pi\colon \mathrm{pt} \to BG$, we can view a complex on BG as a G-equivariant complex of k-vector spaces. In these terms, we claim that the tangent complex of BG is the adjoint representation $\mathfrak{g}$ concentrated in degree -1.

One way to see this: the map $\mathrm{pt} \to BG \to \mathrm{pt}$ induces an exact triangle

$$T_{\mathrm{pt}/BG} \to T_{\mathrm{pt}/\mathrm{pt}} \to \pi^* T_{\mathrm{pt}/BG} \overset{+1}{\longrightarrow}$$

Now, $T_{\mathrm{pt}/\mathrm{pt}}$ is obviously 0, so $\pi^* T_{\mathrm{pt}/BG}$ identifies with $T_{\mathrm{pt}/BG}[1] = \mathfrak{g}[1]$.

Example 5.4 (Tangent complexes of mapping stacks). Let X be a proper scheme and Y a scheme. The tangent complex of the derived mapping stack $\mathscr{M}\!ap(X, Y)$, if it exists, is $R\Gamma(X, f^*TY)$. See [**39**, §5.1.2].

Example 5.4 does not quite cover our case of interest, because it concerns mapping stacks out of schemes rather than pro-spaces, and therefore obtains coherent cohomology as the tangent complex, rather than the cohomology of a local system. Nevertheless, it exhibits a similar pattern, and can be proved by similar arguments.

5.4.2. *Comparison to the classical deformation functor.* We have the following compatibility with the classical theory.

As a variant of §2.3, one has a fully faithful embedding of the category of Artinian local rings A augmented over k into the category of Artinian local animated

rings augmented over k. By construction, we have a commutative diagram

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\text{Artinian local augmented}\\ \text{commutative rings } A \to k\end{array}\right\} & \xrightarrow{\mathrm{M}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}} & \mathsf{Set} \\
\downarrow & & \uparrow{\scriptstyle \pi_0} \\
\left\{\begin{array}{c}\text{Artinian local augmented simplicial}\\ \text{commutative rings } \mathcal{A} \to k\end{array}\right\} & \xrightarrow{\mathcal{M}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}} & \mathsf{sSet}
\end{array}
$$

At the level of representing rings, the commutativity of the diagram is equivalent to

$$
\pi_0(\mathcal{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}) \cong \mathrm{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}.
$$

Remark 5.5. As mentioned previously, it is actually a folklore Conjecture, attributed to Mazur, that $\mathrm{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$ is LCI of dimension $\chi(R\Gamma(\mathcal{O}_F[1/S], \mathfrak{g}_{\overline{\rho}}[1]))$. By §3.2.1, this is equivalent to the statement that $\mathcal{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$ is actually homotopy discrete, i.e., $\pi_i(\mathcal{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}) = 0$ for $i > 0$.

Why then bother with all the effort to $\mathcal{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$ at all, when we could have simply used $\mathrm{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$? The most important fact we will use going forward is the description of the tangent complex of $\mathcal{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$, which comes relatively easily. Without knowing Mazur's conjecture, we don't know anything about the tangent complex of $\mathrm{R}^{\overline{\rho}}_{\mathcal{O}_F[1/S]}$ other than its zero-th cohomology group.

5.5. Imposing local conditions. Now we begin the consideration of imposing local conditions. We will use this in two ways:

    (i) To impose conditions of p-adic Hodge-theoretic nature, which are necessary to cut down to Galois representations that could possible "match" the automorphic side.

    (ii) To implement the Taylor-Wiles method, wherein one needs to consider the effect of adding and removing ramification at auxiliary primes.

5.5.1. *Local deformation functors.* For a local field F_v and a representation $\overline{\rho}$ of $\pi_1(F_v)$, one defines the *local (unrestricted) derived deformation functor* $\mathcal{M}^{\overline{\rho}}_{F_v}$ analogously to the global case, sending

$$
\mathcal{A} \mapsto \mathrm{Map}(\mathrm{\acute{E}t}(F_v), BG(\mathcal{A})) \overset{h}{\times}_{\mathrm{Map}(\mathrm{\acute{E}t}(F_v), BG(k))} \{\overline{\rho}\}.
$$

It is also useful to have the variant $\mathcal{M}^{\overline{\rho}}_{\mathcal{O}_{F_v}}$ parametrizing *unramified* deformations, which sends

$$
\mathcal{A} \mapsto \mathrm{Map}(\mathrm{\acute{E}t}(\mathcal{O}_{F_v}), BG(\mathcal{A})) \overset{h}{\times}_{\mathrm{Map}(\mathrm{\acute{E}t}(\mathcal{O}_{F_v}), BG(k))} \{\overline{\rho}\}.
$$

When these functors come up they will typically not be representable, since $\overline{\rho}$ will typically not be irreducible (it will arise by restricting a global $\overline{\rho}$ to a "random" F_v).

5.5.2. *Cutting out local conditions.* Suppose $\overline{\rho}$ is a $\widehat{G}(k)$-local system on $\mathrm{Spec}\ \mathcal{O}_F[1/S]$ and v is a place of F. Then we have a map

$$
\mathcal{M}^{\overline{\rho}}_{\mathcal{O}_F[1/S]} \to \mathcal{M}^{\overline{\rho}}_{F_v}
$$

gotten by pulling back a local system along $\mathrm{Spec}\ F_v \to \mathrm{Spec}\ \mathcal{O}_F[1/S]$. If $v \notin S$, then the above map factors through $\mathcal{M}^{\overline{\rho}}_{\mathcal{O}_F[1/S]} \to \mathcal{M}^{\overline{\rho}}_{\mathcal{O}_{F_v}}$.

Now suppose that we are given some functor $\mathcal{M}_{F_v}^{\bar\rho,\mathrm{D}_v} \to \mathcal{M}_{F_v}^{\bar\rho}$, where the D stands for some deformation problem. Typically $\mathcal{M}_{F_v}^{\bar\rho,\mathrm{D}_v} \to \mathcal{M}_{F_v}^{\bar\rho}$ should be a closed embedding, meaning a closed embedding on classical truncations. An example could be $\mathcal{M}_{\mathcal{O}_{F_v}}^{\bar\rho} \hookrightarrow \mathcal{M}_{F_v}^{\bar\rho}$, which informally speaking cuts out the deformations which are "unramified at v". Then we can form the derived fibered product of the diagram

$$
\begin{array}{ccc}
\mathcal{M}_{\mathcal{O}_F[1/S]}^{\bar\rho,\{D_v\}} & \longrightarrow & \mathcal{M}_{\mathcal{O}_F[1/S]}^{\bar\rho} \\
\downarrow & & \downarrow \\
\mathcal{M}_{F_v}^{\bar\rho,\mathrm{D}_v} & \longrightarrow & \mathcal{M}_{F_v}^{\bar\rho}
\end{array}
$$

Example 5.6 (Adding ramification)**.** Suppose $v \notin S$, and let $S' = S \sqcup \{v\}$. The morphism $\operatorname{Spec} \mathcal{O}_F[1/S'] \to \operatorname{Spec} \mathcal{O}_F[1/S]$ induces a map $\mathcal{M}_{\mathcal{O}_F[1/S']}^{\bar\rho} \to \mathcal{M}_{\mathcal{O}_F[1/S']}^{\bar\rho}$. Informally, this morphism sends a deformation unramified outside S to the same deformation regarded as unramified outside S' (i.e., forgetting that it is unramified at v).

In the theory of classical deformation rings one has that

$$
\mathrm{M}_{\mathcal{O}_F[1/S']}^{\bar\rho} \times_{\mathrm{M}_{F_v}^{\bar\rho}} \mathrm{M}_{\mathcal{O}_{F_v}}^{\bar\rho} \cong \mathrm{M}_{\mathcal{O}_F[1/S]}^{\bar\rho}.
$$

Informally this amounts to the statement that "a deformation of local system on $\mathcal{O}_F[1/S']$ that is unramified at v comes uniquely from a deformation of a local system on $\mathcal{O}_F[1/S]$", which is immediate from Galois theory. Although this is more or less obvious, we single out the statement because of (i) it is used crucially in the Taylor-Wiles method to descend from patched deformation rings, and (ii) its derived version is non-trivial.

To follow up on the second point, it turns out that the analogous identity holds at the level of derived deformation functors:

$$
(5.2) \qquad \mathcal{M}_{\mathcal{O}_F[1/S']}^{\bar\rho} \overset{h}{\times}_{\mathcal{M}_{F_v}^{\bar\rho}} \mathcal{M}_{\mathcal{O}_{F_v}}^{\bar\rho} \cong \mathcal{M}_{\mathcal{O}_F[1/S]}^{\bar\rho}.
$$

This is proved in [**34**] by calculating the map of tangent complexes. Roughly speaking, the content of (5.2) to the statement that

$$
\begin{array}{ccc}
\operatorname{Spec} F_v & \longrightarrow & \operatorname{Spec} \mathcal{O}_F[1/S'] \\
\downarrow & & \downarrow \\
\operatorname{Spec} \mathcal{O}_{F_v} & \longrightarrow & \operatorname{Spec} \mathcal{O}_F[1/S]
\end{array}
$$

is a homotopy pushout at the level of étale homotopy types. This is similar to (3.5), where we encountered a presentation with the special property that it was not just a pushout but a homotopy pushout. To appreciate this point, it may be instructive to observe that if one uses a presentation of the fundamental group of a genus g Riemann surface which is different from that in (3.1), then one arrives at an a priori different moduli space of Betti local systems. Why is the presentation in (3.1) the "right" one?

5.5.3. *Crystalline conditions.* The next step is to cut out a derived Galois deformation ring with a p-adic Hodge theory condition imposed at p. For concreteness, we focus our attention on the "crystalline" example, which should correspond to

motives with good reduction at p, or from another perspective, automorphic forms with no level at p.

We have a map $\mathcal{M}^{\bar{\rho}}_{\mathcal{O}_F[1/S]} \to \mathcal{M}^{\bar{\rho}}_{F_p}$. (We now know that $\mathcal{M}^{\bar{\rho}}_{F_p} = \mathrm{M}^{\bar{\rho}}_{F_p}$ is classical for $\widehat{G} = \mathrm{GL}_n$ by work [**10**], and it is expected to be true in general, although we will not logically use this.) We want to define a local functor $\mathcal{M}^{\bar{\rho},\mathrm{crys}}_{F_p} \to \mathcal{M}^{\bar{\rho}}_{F_p}$ that "cuts out" the "crystalline" deformations. This is something we do not know how to do in general, since we cannot formulate a moduli-theoretic notion of crystallinity over sufficiently general rings (this will be discussed further in §9.1).

The only solution to this problem that we know of (at the present time) is to restrict ourselves to local conditions which "should" actually be classical, meaning "$\mathcal{M}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]} = \mathrm{M}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]}$" where the right hand side is some classically studied object of p-adic Hodge theory, and take as our local condition the composition

$$\mathrm{M}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]} \to \mathrm{M}^{\bar{\rho}}_{\mathcal{O}_F[1/S]} \hookrightarrow \mathcal{M}^{\bar{\rho}}_{\mathcal{O}_F[1/S]}.$$

The way that one can recognize the deformation conditions which "should" be classical is, according to §3.2.1, that they are LCI of the "correct" dimension. In practice, to analyze the resulting objects one also needs to know the tangent complexes, which is typically hard unless the deformation functor is smooth, so that the tangent complex is just the tangent space.

A general class of conditions where everything works is the *Fontaine-Laffaille range*, which is what [**34**] studies. Here the problems all go away:

- There is a complete moduli-theoretic description, even integrally.
- The obstructions vanish, so that the deformation functor is smooth and its tangent complex can be described explicitly.

For the rest of this section, we put ourselves in the same situation for simplicity. We assume we are in the Fontaine-Laffaille range, meaning that p is "large enough" compared to the Hodge-Tate weights that we are considering. This gives a closed embedding $\mathrm{M}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]} \hookrightarrow \mathcal{M}^{\bar{\rho}}_{\mathcal{O}_F[1/S]}$ as above. We then define $\mathcal{R}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]}$ to represent the derived fibered product $\mathcal{M}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]}$ of the diagram

$$
\begin{array}{ccc}
\mathcal{M}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]} & \longrightarrow & \mathcal{M}^{\bar{\rho}}_{\mathcal{O}_F[1/S]} \\
\downarrow & & \downarrow \\
\mathrm{M}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]} & \lhook\joinrel\longrightarrow & \mathrm{M}^{\bar{\rho}}_{\mathcal{O}_F[1/S]}
\end{array}
$$

A key point is that by taking the derived fibered product, one has control over the resulting tangent complex. Its Euler characteristic can be computed by using Poitou-Tate duality for Galois cohomology of global fields, and the answer is $-\delta$, where $\delta \geq 0$ is a certain quantity called the "defect" that we are not going to explicate (for our purposes, its definition can be taken to be the Euler characteristic, but it is more typical in the literature to take a different starting point). The appearance of this quantity δ is crucial; the fact that the same δ also manifests on the "automorphic side" is the lynchpin of the Calegari-Geraghty method.

5.6. Relation to the Calegari-Geraghty method. We are now going to discuss the main results of [**34**], which give a reinterpretation and elaboration of the Calegari-Geraghty method [**17**], which is itself a modification of the Taylor-Wiles

method for "$\delta > 0$" situations. The reader should be familiar with these methods at least at the level of [**16**] in order to get something out of this subsection.

5.6.1. *Conjectural picture.* We are going to abbreviate $\mathcal{R}_0 := \mathcal{R}_{\mathcal{O}_F[1/S]}^{\overline{\rho},\mathrm{crys}}$ and $\mathrm{R}_0 := \pi_0(\mathcal{R}_0)$. There is a *complex* $\mathcal{M}_0$ of "automorphic forms for G of level S", assembled out of the homology of locally symmetric spaces associated to G. We will index $\mathcal{M}_0$ to be in degrees $[-\delta, 0]$, although it might be later reindexed to be in degrees "$[-\delta - q_0, -q_0]$"♠♠♠ TONY: [compare with what Michael wrote]. The crucial point is that it has length δ – the same δ which appears as the negative of the dimension of $\mathcal{R}_0$. Finally, we write $M_0 := H^0(\mathcal{M}_0)$.

The "automorphic to Galois" direction of the Langlands correspondence, attaching Galois representations to automorphic forms, equips M_0 with the structure of an R_0-module. The expectation is (roughly):

> There is an action of $\mathcal{R}_0$ on $\mathcal{M}_0$ with the following properties.
> - It recovers the given action of R_0 on M_0 by taking homotopy/homology groups, as in the diagram below:
>
> $$\begin{array}{ccc} \mathcal{R}_0 & \curvearrowright & \mathcal{M}_0 \\ \downarrow & & \downarrow \\ \mathrm{R}_0 & \curvearrowright & M_0 \end{array}$$
>
> - It makes $\pi_*(\mathcal{M}_0)[1/p]$ generically free over $\pi_*(\mathcal{R}_0)[1/p]$.[6]

The main result of [**34**] is the construction of an action of $\pi_*(\mathcal{R}_0)$ on $\pi_*(\mathcal{M}_0)$, under certain conjectures and certain Taylor-Wiles assumptions, with the properties above.

Example 5.7 (The case of GL_1)**.** At present, the above picture can be realized completely only for $G = \mathrm{GL}_1$, and even this situation is non-trivial.[7] If $\delta = 0$, which happens when $F = \mathbf{Q}$ or a quadratic imaginary field, then all rings are classical, and the statement is a consequence of class field theory. The point is that the classical deformation ring is essentially the (completed) group ring of the abelianization of $\pi_1(\mathcal{O}_F[1/S])$. Class field theory identifies this abelianization with the class group of $\mathcal{O}_F[1/S]$, whose adelic description is the corresponding "locally symmetric space" (a finite set of points).

However, in all other cases we have $\delta > 0$, which implies:

- the derived Galois deformation ring is non-classical,
- the locally symmetric space is no longer a discrete set of points,

and the hypothesized action goes beyond classical Galois deformation theory. (See [**16**] for a discussion from the perspective of the Calegari-Geraghty method.) Nevertheless, the desired action of $\mathcal{R}_0$ on $\mathcal{M}_0$ can still be constructed, which will be explained in the forthcoming paper [**27**]. The point is that $\mathcal{R}_0$ admits a description as a (completed) "derived group ring" of the "derived abelianization" of $\pi_1(\mathcal{O}_F[1/S])$. To elaborate on what this means:

---

[6]Technically, we have defined $\mathcal{R}_0$ as a pro-ring, so that $\pi_*(\mathcal{R}_0)$ is a pro-group. Some care must be taken to "invert p" on such an object.

[7]So far we have focused on the case where $\widehat{G}$ is of adjoint type, so our definitions have to be modified slightly in order to accommodate this case.

- The abelianization of a group controls its *set* of morphisms to abelian groups. The *derived abelianization* of a group controls its *anima* of morphisms to animated abelian groups. One formulates this precisely in terms of universal properties, as a left adjoint to the forgetful functor from (animated) abelian groups to (animated) groups.
- The group ring of an abelian group can also be characterized in terms of a universal property: it is left adjoint to the functor from commutative rings to abelian groups given by extracting units. Analogously, the "derived group ring" construction can be characterized as left adjoint to the an analogous functor from animated rings to animated abelian groups.

The punchline of [**27**] is that the derived abelianization of $\pi_1(\mathcal{O}_F[1/S])$ "is" the animated abelian group obtained from the corresponding locally symmetric space for G (which has a topological group structure in the special case $G = \mathrm{GL}_1$), and its derived group ring "is" the homology chains on this animated abelian group, with its induced ring structure.

Meanwhile, $\mathcal{M}_0$ is also the homology chains on the locally symmetric space, and the sought-for action is the tautological one. In particular, $\mathcal{M}_0$ is a free module of rank one over $\mathcal{R}_0$ for $G = \mathrm{GL}_1$.

5.6.2. *Galatius-Venkatesh.* An informal statement of the main result of [**34**] is that, assuming "standard" conjectures and hypotheses related to automorphy lifting,

(i) $\pi_*(\mathcal{R}_0)$ is supported in degrees $0 \leq * \leq \delta$,
(ii) $H_*(\mathcal{M}_0)$ carries the structure of a free graded module over $\pi_*(\mathcal{R}_0)$, extending the usual structure for $* = 0$.

The conjectures alluded to above are about the existence of Galois representations attached to automorphic forms, and local-global compatibility for such representations. The hypotheses are (a stringent form of)[8] the usual "Taylor-Wiles" type conditions on the residual representation. See [**34**, §6.6, §10] for the precise formulations.

The proof of the main result is based on the Taylor-Wiles method, incorporating the modifications introduced by Calegari-Geraghty. Let us give a brief sketch here before giving a slightly less brief sketch in the remaining part of this section.

First, we orient ourselves psychologically. We believe there should be an action of $\mathcal{R}_0$ on $\mathcal{M}_0$, but here and throughout we will only be able to construct (at least at first) the "classical shadow" of this action, as depicted in the diagram below:

$$
\begin{array}{ccc}
\mathcal{R}_0 & \stackrel{?}{\curvearrowright} & \mathcal{M}_0 \\
\downarrow & & \downarrow \\
\mathrm{R}_0 & \curvearrowright & M_0
\end{array}
$$

Here $\stackrel{?}{\curvearrowright}$ stands for an action that we believe exists, but don't know how to construct.

The starting point of the Taylor-Wiles method is to consider an auxiliary family of similar situations with additional level structure added at a well-chosen collection of primes Q_n. Indeed, letting Q_n be such a set, we define

---

[8]There should be wide scope for optimization of [**34**] from a technical perspective, by incorporating Kisin's methods.

- $\mathcal{R}_n := \mathcal{R}^{\overline{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/SQ_n]}$ and $\mathrm{R}_n := \pi_0(\mathcal{R}^{\overline{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/SQ_n]})$, which by §5.4.2 is the classical crystalline deformation ring. The derived ring $\mathcal{R}_n$ has virtual dimension $-\delta$.
- $\mathcal{M}_n$ for the corresponding module of "automorphic forms for G of level SQ_n", which by assumption is a complex concentrated in degrees $[-\delta, 0]$, and $M_n := H_0(\mathcal{M}_n)$.

Then the same story holds for each $\mathcal{R}_n$: we want $\mathcal{R}_n \overset{?}{\curvearrowright} \mathcal{M}_n$, but what we can actually construct is its classical shadow $\mathrm{R}_n \curvearrowright M_n$. Therefore, we have a diagram

$$
\begin{array}{ccc}
\mathcal{R}_n & \overset{?}{\curvearrowright} & \mathcal{M}_n \\
\downarrow & & \downarrow \\
\mathrm{R}_n & \curvearrowright & M_n
\end{array}
$$

At first the picture looks the same as when $n = 0$, so it seems that no progress has been made. But the point is that by choosing Q_n artfully, one can *morally* arrange the $\mathcal{R}_n$ to "limit" to an object $\mathcal{R}_\infty$ which is actually *classical*, so that $\mathcal{R}_\infty \xrightarrow{\sim} \mathrm{R}_\infty := \pi_0(\mathcal{R}_\infty)$. Now, in reality we will not be constructing $\mathcal{R}_\infty$ in this way, but we will construct an R_∞ which will be seen to be "smooth and of the correct dimension". By the discussion in §3.2.1, this justifies setting $\mathcal{R}_\infty$ to be equal to R_∞.

Furthermore, the $\mathcal{M}_n$ "limit" to an object $\mathcal{M}_\infty$ (which really does exist), and we will see that $\mathcal{M}_\infty \xrightarrow{\sim} \mathrm{M}_\infty$. Also, one should imagine that the diagram (5.6.2) "limits" to a diagram

$$
\begin{array}{ccc}
\mathcal{R}_\infty & \overset{?}{\curvearrowright} & \mathcal{M}_\infty \\
\downarrow & & \downarrow \\
\mathrm{R}_\infty & \curvearrowright & M_\infty
\end{array}
$$

Since in this case $\mathcal{R}_\infty \twoheadrightarrow \mathrm{R}_\infty$ is actually an isomorphism of simplicial commutative rings, it automatically gives the desired action on $\mathcal{M}_\infty \xrightarrow{\sim} \mathrm{M}_\infty$.

To see how this helps with our original problem, let us contemplate the relation between $\mathcal{R}_n$ and $\mathcal{R}_0$. The difference is that $\mathcal{R}_0$ only parametrizes deformations unramified at Q_n (recall that $\overline{\rho}$ is unramified at the primes in Q_n), while $\mathcal{R}_n$ allows deformations that ramify at Q_n. As explained in Example 5.6, $\mathcal{R}_0$ can be recovered from $\mathcal{R}_n$ by forming a derived tensor product that cuts out the unramified locus from among the deformations of local Galois groups over the primes in Q_n. More precisely, one chooses Q_n so that deformations of $\overline{\rho}$ are automatically tame above Q_n. Then the maps from the tame inertia groups over Q_n to $\pi_1(\mathcal{O}_F[1/SQ_n])$ induce a ring homomorphism $\mathrm{S}_n \to \mathcal{R}_n$, where S_n is certain deformation ring for tame inertia over Q_n. There is an augmentation $\mathrm{S}_n \to \mathrm{S}_0 = W(k)$ that corresponds to the deformations which are trivial on tame inertia. By similar conditions as in Example 5.6, one has an equivalence of simplicial commutative rings

$$
(5.3) \qquad\qquad\qquad \mathcal{R}_n \overset{h}{\otimes}_{\mathrm{S}_n} \mathrm{S}_0 \xrightarrow{\sim} \mathcal{R}_0.
$$

This identity *approximately* "limits" to

$$
(5.4) \qquad\qquad\qquad \text{``}\mathcal{R}_\infty \overset{h}{\otimes}_{\mathrm{S}_\infty} \mathrm{S}_0 \xrightarrow{\sim} \mathcal{R}_0.\text{''}
$$

More generally, one has *approximately*

$$\text{(5.5)} \qquad \text{``}\mathcal{R}_\infty \overset{h}{\otimes}_{S_\infty} S_n \xrightarrow{\sim} \mathcal{R}_n.\text{''}$$

The reason we have put these statements in quotes is that it they are not exactly true, but they are true up to a certain amount of error which shrinks to 0 as $n \to \infty$, which makes them acceptable for our purposes. The source of this "error" lies in the construction of $\mathcal{R}_\infty$: it is extracted by a compactness argument from an inverse system of *finite* quotients of the $\mathcal{R}_n$.

There will also be an action of S_n on M_n by "diamond operators", for which one has

$$\text{(5.6)} \qquad \mathcal{M}_\infty \overset{h}{\otimes}_{S_n} S_0 \xrightarrow{\sim} \mathcal{M}_n,$$

Furthermore, local-global compatibility ensures that this action is compatible with the one induced by $S_n \to R_n$. Then, from the $\mathcal{R}_\infty \xrightarrow{\sim} R_\infty$ action on $M_\infty \xleftarrow{\sim} \mathcal{M}_\infty$, taking derived tensor products and using (5.4) and (5.6), we have

$$\text{``}\mathcal{R}_0 \cong \mathcal{R}_\infty \overset{h}{\otimes}_{S_\infty} S_0\text{''} \curvearrowright \mathcal{M}_\infty \overset{h}{\otimes}_{S_\infty} S_0 \cong \mathcal{M}_0.$$

As discussed above, the statement in quotes is not exactly true but can be made true on homotopy groups up to arbitrary precision. This is done by instead using (5.5) to "go from ∞ to n", which incurs arbitrarily small error as n increases, and then using (5.3) to "go from n to 0", which does not incur any error. That is enough to make some isomorphism

$$\pi_i(\mathcal{R}_0) \approx \pi_i(\mathcal{R}_\infty \overset{h}{\otimes}_{S_\infty} S_0) \cong \mathrm{Tor}_i^{S_\infty}(\mathcal{R}_\infty, S_0),$$

which we will then be able to compute explicitly (under favorable assumptions).

Having completed the informal sketch, we now proceed to the slightly more detailed sketch.

5.6.3. *Taylor-Wiles patching.* By Lemma 3.2, we have

$$\dim_k H^0(\mathbf{T}_{\mathcal{R}_0/W(k)} \otimes_{\mathcal{R}_0} k) = \dim_k H^0(\mathbf{T}_{R_0/W(k)} \otimes_{R_0} k)$$
$$\dim_k H^1(\mathbf{T}_{\mathcal{R}_0/W(k)} \otimes_{\mathcal{R}_0} k) \geq \dim_k H^1(\mathbf{T}_{R_0/W(k)} \otimes_{R_0} k)$$

We will assume that we are in the "minimally ramified" case (as in [**34**]), so that

$$\dim_k H^0(\mathbf{T}_{\mathcal{R}_0/W(k)} \otimes_{\mathcal{R}_0} k) - \dim_k H^1(\mathbf{T}_{\mathcal{R}_0/W(k)} \otimes_{\mathcal{R}_0} k) = \delta,$$

the same δ which was the amplitude of $\mathcal{M}_n$.

We will use the following fact from commutative algebra: if R is a complete local noetherian ring over A with augmentation $\epsilon \colon R \to k$, then R admits a surjection

$$W(k)[[y_1, \ldots, x_g]] \twoheadrightarrow R$$

where $g = \dim_k H^0(\mathbf{T}_{R/W(k)} \otimes_R k)$ is the dimension of the tangent space, with kernel generated by $\dim_k H^1(\mathbf{T}_{R/W(k)} \otimes_R k)$ elements.

Applying this to $R = R_0$, we have a presentation

$$R_0 \cong W(k)[[y_1, \ldots, y_g]]/(f_1, \ldots, f_{g+\delta});$$

the minimal number of relations might in fact be smaller than $g + \delta$ but we can "pad" them with trivial relations if necessary to arrange a presentation of the above sort.

The idea of Taylor-Wiles patching is that by adding level structure at well-known primes, the resulting deformation rings R_n can be arranged to have a presentation with the same number of generators and relations,

$$(5.7) \qquad\qquad R_n \cong W(k)[[y_1, \ldots, y_g]]/(f_1^{(n)}, \ldots, f_{g+\delta}^{(n)}),$$

but with the relations $f_1^{(n)}, \ldots, f_{g+\delta}^{(n)}$ going deeper and deeper into the $(p, \underline{y})$-adic topology as $n \to \infty$. Therefore, in the "limit", the relations disappear and one has a *formally* smooth object over $W(k)$ with g generators, so one would have

$$R_\infty \cong W(k)[[y_1, \ldots, y_g]].$$

In practice we will not be able to directly arrange the relations for R_n, but we will end up deducing the structure of R_∞ by a very indirect argument.

The existence of a presentation (5.7) is a cohomological condition, ensuring that the Galois cohomology groups that constitute the tangent complex of $\mathcal{R}_n$ do not change in dimension. The first step is to show that there do actually exist enough "Taylor-Wiles primes" satisfying this cohomological condition. This is where the largeness conditions on the image of $\bar\rho$ are used. We will not comment more on this.

The next step is to "patch" the deformation rings R_n along a sequence Taylor-Wiles primes, to extract a limiting object R_∞. One also patches the rings S_n and complexes $\mathcal{M}_n$ as in §5.6.2 to obtain a diagram

$$S_\infty \longrightarrow R_\infty \qquad \curvearrowright \qquad M_\infty$$

We note that because the actually places of ramification are changing, there are no "canonical" maps between the auxiliary deformation rings. One overcomes this by an elaborate construction called "Taylor-Wiles patching". This involves considering finite quotients of the auxiliary deformation rings and using pigeon-hole arguments to see that there must be many (non-canonical) maps between these finite quotients, and then extracting a limiting object by compactness. Informally, one could imagine this as saying that there exists some sequence of deformation rings whose members are "close" to having maps between each other in the $(p, \underline{y})$-adic topology; so that what one is doing geometrically is using compactness to extract some convergent subsequence among the infinitude of possible auxiliary deformation spaces.

Next, one wants to analyze the limiting object. We compare how this goes in the classical Taylor-Wiles situation where $\delta = 0$, versus the Calegari-Geraghty situation where $\delta > 0$.

5.6.4. *The classical Taylor-Wiles situation.* In the classical Taylor-Wiles method, $\delta = 0$ and one has a diagram

$$S_\infty \longrightarrow R_\infty \qquad \curvearrowright \qquad M_\infty$$

What we know is:

- M_∞ is free over S_∞.
- R_∞ is a quotient of $W(k)[[y_1, \ldots, y_g]]$.

What we want to show is:

- The action of R_∞ on M_∞ is (generically) free.
- The map $S_\infty \to R_\infty$ has large image.

We are going to parlay the given information into "upper bounds" and "lower bounds". Then it turns out that by a miracle of numerology, sometimes called the

numerical coincidence, the upper and lower bounds match, allowing us to pin down exactly what we want.

To elaborate:

(i) The "upper bound" on R_∞ comes from the fact that it admits a surjection from $\mathbf{Z}_p[[y_1, \ldots, y_g]]$, which comes from the calculation of the dimension of the tangent space, $g = \dim H^0(\mathbf{T}_{\mathcal{R}_n/W(k)} \otimes_{\mathcal{R}_n} k)$.

(ii) The "lower bound" on R_∞ comes from the fact that the action of R_∞ on M_∞ must restrict to a free action of S_∞. One can arrange $S_\infty \cong W(k)[[y_1, \ldots, y_g]]$ because $g = \dim H^1(\mathbf{T}_{\mathcal{R}_n/W(k)} \otimes_{\mathcal{R}_n} k) \geq \dim H^1(\mathbf{T}_{R_n/W(k)} \otimes_{R_n} k)$. The fact that this g is the same as from the upper bound is the so-called *numerical coincidence*. Since S_∞ acts faithfully on M_∞, and that action factors through $S_\infty \to R_\infty$, no non-trivial quotienting can occur in the map $S_\infty \to R_\infty$, which puts a "lower bound" on R_∞.

From the fact that the upper and lower bounds agree, a small commutative algebra argument shows that M_∞ is free over R_∞: we have a chain of inequalities

$$\dim S_\infty = \operatorname{depth}_{S_\infty}(M_\infty) \leq \operatorname{depth}_{R_\infty}(M_\infty) \leq \dim R_\infty$$

where the first and last terms are actually equal, so in fact all inequalities are equalities. Then by the Auslander-Buchsbaum formula [**24**, §19], M_∞ must be locally free over R_∞.

5.6.5. *The Calegari-Geraghty situation.* In the Calegari-Geraghty situation, $\delta > 0$ and we have to contend with $\mathcal{R}_n$ and $\mathcal{M}_n$. We can patch to a diagram

$$
\begin{array}{ccccc}
S_\infty & \to & \mathcal{R}_\infty & \overset{?}{\curvearrowright} & \mathcal{M}_\infty \\
& & \downarrow & & \downarrow \\
& & R_\infty & \curvearrowright & M_\infty
\end{array}
$$

We secretly believe that "$\mathcal{R}_\infty \xrightarrow{\sim} R_\infty$ is classical" (recall that this really means "$\mathcal{R}_\infty$ is smooth of the correct dimension over A", which then justifies setting $\mathcal{R}_\infty = R_\infty$) and $\mathcal{M}_\infty \xrightarrow{\sim} M_\infty$ is concentrated in one degree, but we do not know this yet. What we do know is:

- $\mathcal{M}_\infty$ is a complex of free S_∞-modules, of length δ.
- $\mathcal{R}_n$ has dimension $-\delta$.

We want to show that R_∞ is smooth of the "correct" dimension. Morally the smoothness occurs because the Taylor-Wiles limiting process is supposed to kill the relations.

Note that the $\mathcal{M}_\infty$ being a perfect complex over S_∞ does not imply that the top cohomology group M_∞ is free over S_∞. In fact it will turn out that the freeness fails. Therefore, in comparison to the classical Taylor-Wiles situation considered in §5.6.4, we will lose the "lower bound". However it turns out that one can obtain a "weaker" lower bound, and that the upper bound gets decreased by the same amount, so that they still match. To elaborate on the details:

(i) The "upper bound" on R_∞ comes from the fact that it admits a surjection from $\mathbf{Z}_p[[y_1, \ldots, y_g]]$, which comes from the calculation of the dimension of the tangent space, $g = \dim H^0(\mathbf{T}_{\mathcal{R}_n/W(k)} \otimes_{\mathcal{R}_n} k)$.

(ii) The "lower bound" comes eventually from the fact that M_∞ is the top chomology group of a complex of free S_∞-modules, of length δ.

One can arrange that $S_\infty \cong W(k)[[y_1, \ldots, y_{g+\delta}]]$ because

$$g + \delta = \dim H^1(\mathbf{T}_{\mathcal{R}_n/W(k)} \otimes_{\mathcal{R}_n} k) \geq \dim H^1(\mathbf{T}_{R_n/A} \otimes_{R_n} k).$$

The new *numerical coincidence* is that the δ appearing here coincides with the δ which is the length of the complex $\mathcal{M}_\infty$.

Indeed, the Auslander-Buchsbaum theorem implies that since M_∞ has a free S_∞-resolution of length δ (and therefore projective dimension at most δ), its depth over S_∞ is at least $\dim S_\infty - \delta$. Since the S_∞-action on M_∞ factors through $S_\infty \to R_\infty$, this puts a "lower bound" the depth of M_∞ as an R_∞-module, and thus on R_∞ itself.

The punchline is that we have again

$$g = \dim S_\infty - \delta \leq \mathrm{depth}_{S_\infty} M_\infty \leq \mathrm{depth}_{R_\infty} M_\infty \leq \dim R_\infty \leq g.$$

Therefore all inequalities are equalities, which as part of Auslander-Buchsbaum shows that $\mathcal{M}_\infty$ has cohomology in a single degree, and one concludes the rest as before.

Now we comment on the matter of calculating $\pi_*(\mathcal{R}_0)$. The argument also shows that $S_\infty \to R_\infty$ is a quotient by a regular sequence of length δ, which allows to compute $\mathrm{Tor}_i^{S_\infty}(S_0, R_\infty)$ by a standard Koszul complex calculation. From the definition of R_∞ one can ensure that $\mathrm{Tor}_i^{S_\infty}(S_n, R_\infty)$ is p-adically close to $\mathcal{R}_n$. Since $\mathcal{R}_n \overset{h}{\otimes}_{S_n} S_0 \overset{\sim}{\to} \mathcal{R}_0$, one then has that $\pi_*(\mathcal{R}_0)$ is p-adically close to $\mathrm{Tor}_i^{S_\infty}(S_0, R_\infty)$. Using a limiting argument as $n \to \infty$ one then calculates that $\pi_*(\mathcal{R}_0) \cong \mathrm{Tor}_i^{S_\infty}(S_0, R_\infty)$. The details may be found in [**34**, §14].

6. Derived Hecke algebras

6.1. The local derived Hecke algebra. ♠♠♠ TONY: [change Λ to A]

We briefly review the theory of derived Hecke algebra from [**81**]. Let G be a split reductive group over a local field K, say of residue field $\mathbf{F}_q$ having characteristic p. Let $U \subset G(K)$ be a compact open subgroup. (For our purposes, we can take $G(\mathcal{O}_K)$ to be a maximal compact subgroup.)

Let A be a commutative ring. Denoting $A[G(K)/U] = \mathrm{c\text{-}Ind}_U^{G(K)} A$ for the compact-induction of the trivial representation from U to $G(K)$, we can present the usual Hecke algebra for the pair $(G(K), U)$ as

$$H(G(K), U; \Lambda) := \mathrm{Hom}_{G(K)}(A[G(K)/U], A[G(K)/U]).$$

This presentation suggests the following generalization.

Definition 6.1. The *derived Hecke algebra* for $(G(K), U)$ with coefficients in a ring A is

$$\mathcal{H}(G(K), U; A) := \mathrm{Ext}_{G(K)}^*(A[G(K)/U], A[G(K)/U]),$$

where the Ext is formed in the category of smooth $G(K)$-representations. For $U = G(\mathcal{O}_K)$, we abbreviate $\mathcal{H}_q(G(K); A) := \mathcal{H}(G(K), G(\mathcal{O}_K); A)$.

Remark 6.2. The ring $\mathcal{H}(G(K), U; A)$ is really just a graded (associative) ring, rather any kind of derived (associative) ring. However, its description makes it clear that it arises as the cohomology groups of a differential graded algebra $\mathrm{RHom}_{G(K)}(A[G(K)/U], A[G(K)/U])$. It might be more puritanical to call the latter object the "derived Hecke algebra," but we follow [**81**] in applying this name to

the graded ring $\mathcal{H}(G(K), U; A)$. We will instead call $\mathrm{RHom}_{G(K)}(A[G(K)/U], A[G(K)/U])$ the *differential graded Hecke algebra*, and denote it $\mathrm{DGHA}(G(K), U; A)$, or sometimes simply $\mathrm{DGHA}(G(K); A)$ if $U = G(\mathcal{O}_K)$.

Example 6.3. The 0th graded cohomology group of $\mathcal{H}(G(K), U; A)$ is the classical Hecke algebra $H(G(K), U; A)$.

We next give a couple more concrete descriptions of the derived Hecke algebra, following [**81**, §2].

6.1.1. *Function-theoretic description.* Let $x, y \in G(K)/U$ and $G_{x,y} \subset G(K)$ be the stabilizer of the pair (x, y). We note that $G_{x,y}$ is a profinite open subgroup of $G(K)$. We can think of $\mathcal{H}(G(K), U; A)$ as the space of functions

$$G(K)/U \times G(K)/U \ni (x, y) \mapsto h(x, y) \in H^*(G_{x,y}; A)$$

satisfying the following constraints:

(i) The function h is "$G(K)$-invariant" on the left. More precisely, we have

$$[g]^* h(gx, gy) = h(x, y)$$

for all $g \in G(K)$, where $[g]^*: H^*(G_{gx,gy}; A) \to H^*(G_{x,y}; A)$ is pullback by $\mathrm{Ad}(g)$.

(ii) The support of h is finite modulo $G(K)$.

The multiplication is given by a convolution formula, where one uses the cup product to define multiplication on the codomain, and restriction/inflation to shift cohomology classes to the correct groups [**81**, eqn. (22)].

6.1.2. *Double coset description.* For $x \in G/U$, let $U_x = \mathrm{Stab}_U(x)$. Explicitly, if $x = g_x U$ then $U_x := U \cap \mathrm{Ad}(g_x)U$.

We can also describe $\mathcal{H}(G(\mathbf{Q}_q), U; A)$ as functions

$$x \in U\backslash G(\mathbf{Q}_q)/U \mapsto h(x) \in H^*(U_x; A)$$

which are compactly supported, i.e. supported on finitely many double cosets.

6.1.3. *The derived Hecke algebra of a torus.* Let T be a split torus over K. We can explicitly describe the derived Hecke algebra of the torus $T(K)$, using now the double coset model. Since T is abelian we simply have $T(\mathcal{O}_K)_x = T(\mathcal{O}_K)$ for all $x \in T(K)/T(\mathcal{O}_K)$.

We have $T(K)/T(\mathcal{O}_K) \cong X_*(T)$. Identify

$$(6.1) \qquad X_*(T) = T(K)/T(\mathcal{O}_K) \hookrightarrow G(K)/G(\mathcal{O}_K)$$

by the map $X_*(T) \ni \chi \mapsto \chi(\varpi) \in G(K)/G(\mathcal{O}_K)$, where ϖ is a uniformizer of K. Then $\mathcal{H}(T(K); A)$ simply consists of compactly supported functions

$$X_*(T) \to H^*(T(\mathcal{O}_K); A)$$

with the multiplication given by convolution; in other words,

$$\mathcal{H}(T(K); A) \cong A[X_*(T)] \otimes_A H^*(T(\mathcal{O}_K); A),$$

Remark 6.4. If p is invertible in A, then the structure of $H^*(T(\mathcal{O}_K); A)$ can be elucidated as follows. First of all, the reduction map $T(\mathcal{O}_K) \to T(\mathbf{F}_q)$ has pro-p kernel, which is acyclic for cohomology with coefficients in A by the assumption, and therefore induces an isomorphism $H^*(T(\mathbf{F}_q); A) \to H^*(T(\mathcal{O}_K); A)$.

Furthermore, since T is split we have $T(\mathbf{F}_q) \cong (\mathbf{F}_q^\times)^r$ where $r = \mathrm{rank}(T)$. Where are mainly interested where A is of the form $\mathcal{O}/A^n$ where $\mathcal{O}$ is the ring

of integers in a number field and A is a prime above $\ell \neq p$. In this situation, $H^*(T(\mathbf{F}_q); A)$ has non-zero higher cohomology exactly when $q \equiv 1 \pmod{\ell}$.

6.1.4. *The derived Satake isomorphism.* We henceforth assume that $q \equiv 1 \in A$. Let $T \subset G$ be a maximal torus and W the Weyl group of $T \subset G$. We consider an analogue of the classical Satake transform for the derived Hecke algebra $\mathcal{H}(G(F); A)$, which takes the form

"Derived Hecke algebra for $G \xrightarrow{\sim}$ (Derived Hecke algebra for $T)^W$."

More precisely, we define the *derived Satake transform*

$$(6.2) \qquad \qquad \mathcal{H}(G(K); A) \to \mathcal{H}(T(K); A)$$

simply by *restriction* (in the function-theoretic model §6.1.1) along the map

$$(T(K)/T(\mathcal{O}_K))^2 \to (G(K)/G(\mathcal{O}_K))^2$$

from (6.1). In more detail, let $h \in \mathcal{H}_q(G(K); A)$ be given by the function

$$(G(K)/G(\mathcal{O}_K))^2 \ni (x, y) \mapsto h(x, y) \in H^*(G_{x,y}; A).$$

Then (6.2) takes h to the composition

$$(T(K)/T(\mathcal{O}_K))^2 \longleftrightarrow (G(K)/G(\mathcal{O}_K))^2 \xrightarrow{h} H^*(G_{x,y}; A) \xrightarrow{\mathrm{res}} H^*(T_{x,y}; A)$$

Remark 6.5. It may be surprising that this is the right definition, since the analogous construction in characteristic 0, on the usual underived Hecke algebra, is far from being the usual Satake transform. It is only because of our assumptions on the relation between the characteristics (namely, that $q \equiv 1 \in A$) that this "naive" construction turns out to behave well (otherwise, it would not even be a ring homomorphism).

THEOREM 6.6 ([**81**, Theorem 3.3]). *Let W be the Weyl group of T in G. If $|W|$ is invertible in A and $q = 1 \in A$, then the map* (6.2) *induces an isomorphism*

$$\mathcal{H}(G(K); A) \xrightarrow{\sim} \mathcal{H}(T(K); A)^W.$$

6.1.5. *Commutativity?* A priori, $\mathrm{DGHA}(G(K), U; A)$ is just an associative differential graded algebra and $\mathcal{H}(G(K), U; A)$ is just an associative graded algebra. It is interesting question what more structure or properties these algebras have. Indeed, $H^0(\mathcal{H}(G(K), U; A)) \cong H(G(K), U; A)$ turns out to be commutative, although this is non-trivial to prove.

We think it is reasonable to conjecture that $\mathcal{H}(G(K); A)$ is also commutative, at least if the characteristic of A is not in "bad" position with respect to G and the residue characteristic of K. (No counterexample is known, even for small characteristics.) We list some cases in which the commutativity is known:

(i) The commutativity is known for A of characteristic $\ell \nmid |W|$ such that $q \equiv 1 \pmod{\ell}$ by Theorem 6.6 (which is due to Venkatesh).
(ii) More generally, the commutativity is known if $\#G/B(\mathbf{F}_q)$ and p are invertible in A by [**35**].[9]

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[9]This paper also investigates examples for $G = \mathrm{SL}_2$ which do not fall under these hypotheses.

To say a bit about these arguments, we recall two proofs of the commutativity of the classical spherical Hecke algebra: one via the Satake transform, and one via "Gelfand's trick". Venkatesh's argument is a generalization of the method of the Satake transform, and Gehrmann's argument is a generalization of Gelfand's trick. We note that although Gehrmann proves commutativity in more generality, Venkatesh's proof gives more refined information which is needed for global applications.

There are subtler questions that one can formulate beyond the commutativity of $\mathcal{H}(G(K); A)$, concerning the structure of the differential graded Hecke algebra $\mathrm{DGHA}(G(K); A)$. Some more speculative discussion of this problem will be given in §9.2.

6.1.6. *Extensions.* The literature on derived Hecke algebras is not as comprehensive as one would hope. It would be useful to:

- Extend the theory to non-split groups G.
- Extend the theory incorporate to non-trivial coefficients, i.e., replace $A[G(K)/U]$ with the compact-induction of a non-trivial representation of U. In a global setting, this would be related to considering non-trivial local coefficients on the corresponding locally symmetric space.

6.2. The global derived Hecke algebra. The motivation for the local derived Hecke algebra in [**81**] is to construct a "global derived Hecke algebra" acting on the cohomology of locally symmetric spaces. We will say more about this later in the subsequent sections.

6.3. Mod p derived Hecke algebras of p-adic groups. Let G be a reductive group over $\mathbf{Q}_p$ – we use the same notation for the algebraic group and its group of $\mathbf{Q}_p$-valued points – and let k be an algebraically closed field of characteristic p. Let $\mathrm{Rep}_k(G)$ denote the category of continuous representations of G on k-vector spaces, and let $\mathrm{Rep}_k^{sm}(G) \subset \mathrm{Rep}_k(G)$ denote the subcategory of smooth representations. When $G = \mathrm{GL}(2, \mathbf{Q}_p)$ or a closely related group, the irreducible objects in $\mathrm{Rep}_k^{sm}(G)$ have been classified for some time, but for every other group even an approximate classificationremains elusive, in spite of impressive efforts over the past 15 years [**13, 51, 14**].

The abelian category $\mathrm{Rep}_k^{sm}(G)$ nevertheless has a simple structure in one respect: every irreducible object admits a canonical space of surjective homomorphisms from a compact projective generator, denoted U below. This fact has been exploited by Schneider to define a derived equivalence between $\mathrm{Rep}_k^{sm}(G)$ and the derived category of dg-modules over the derived endomorphism algebra $R\,\mathrm{End}_G(U)$. The properties of the latter, a mod p analogue of the mod ℓ derived Hecke algebras introduced above, are still largely unexplored. A simpler version, called the *derived diamond algebra*, has been shown by Khare and Ronchetti to play a significant role in the theory of p-adic modular forms.

6.3.1. *Schneider's theory.* Fix an Iwahori subgroup $I \subset G$, and let $I(1) \subset I$ be its maximal pro-p normal subgroup; thus $I/I(1)$ is the group of $\mathbb{F}$-points of a torus over a finite field $\mathbb{F}$. It is easy to show that every irreducible $\pi \subset \mathrm{Rep}_k^{sm}(G)$ is generated by its subspace $\pi^{I(1)}$. This thus defines a canonical functor

$$(6.3) \qquad \tau_k : \mathrm{Rep}_k^{sm}(G) \to \mathrm{Mod}(H(G, I(1); k))$$

$$\pi \mapsto \pi^{I(1)}$$

changing to Roman for underived, calligraphic for derived, for consistency with earlier section

where $H(G, I(1); k)$ is the k-valued Hecke algebra of G relative to $I(1)$. This can be defined in terms of double cosets, as in the previous sections, but it is most convenient for us to introduce the universal module $\mathrm{U} = \text{c-Ind}_{I(1)}^G k$, and to define

$$H(G, I(1); k) = \text{End}_G(\mathrm{U})^{\text{opp}}.$$

A well-known theorem of Borel asserts that, if k is replaced by $\mathbf{C}$, then (6.3) defines an equivalence of categories between the subcategory of the left-hand side of representations generated by their $I(1)$-fixed vectors and $\text{Mod}(H(G, I(1); k))$. This is also true for k in characteristic p when $G = \text{GL}(2, \mathbf{Q}_p)$, but Ollivier proved that this is almost never the case for other G, even though every irreducible π is generated by $\pi^{I(1)}$. Schneider found the appropriate generalization of Borel's theorem by replacing $H(G, I(1); k)$ by the differential graded algebra

distinguishing notation for the differential graded Hecke alg and the derived Hecke alg – one is the cohomology ring of the other, as in prevous section

(6.4) $\text{DGHA}(G, I(1); k) = R\text{End}_G(U) := \text{RHom}_G(\mathrm{U}, \mathrm{U}).$

The replacement of the functor (6.3) is

(6.5) $\tau_k^\bullet : D(G) \to D(\text{DGHA}(G, I(1); k))$

$$\pi \mapsto \text{RHom}_{I(1)}(1, \pi) = \text{RHom}_G(\mathrm{U}, \pi)$$

Here D denotes the (unbounded) derived category, we write $D(G)$ instead of $D(\text{Rep}_k^{sm}(G))$, and the final equality is Frobenius reciprocity. Schneider's theorem is the following.

THEOREM 6.7 ([**72**], Theorem 9). *Suppose $I(1)$ is a torsion free p-adic group. Then U is a compact generator of $D(G)$ and the functor (6.5) is an equivalence of triangulated categories.*

A first description of the category $\text{Rep}_k^{sm}(G)$ is contained in [**2**]. Parabolic induction is defined as a functor from $\text{Rep}_k^{sm}(M)$ to $\text{Rep}_k^{sm}(G)$ if M is a Levi component of a parabolic subgroup P; in characteristic p this induction is not normalized and the result depends on P as well as M. The irreducible admissible representation π of G is *supercuspidal* if it does not occur as a subquotient of a representation induced from an admissible irreducible representation of a proper parabolic subgroup of G. The main theorem of [**2**] reduces the classification of $\text{Rep}_k^{sm}(G)$ to the classification of supercuspidal representations of Levi subgroups. This corresponds to a similar classification of modules for the (underived) Hecke algebra $H(G, I(1); k)$: the Hecke algebra modules corresponding to supercuspidal π are called *supersingular* and have a simple characterization in terms of $H(G, I(1); k)$ (see [**2**], p. 498]).

This suggests that the structure of $D(\text{DGHA}(G, I(1); k))$ can also be reduced to the study of triangulated subcategories of supersingular modules, but no one seems to have looked into this question.

When $I(1)$ is not torsion free, it can be replaced in the definition of U and in Theorem 6.7 by an appropriate torsion free subgroup of finite index, but the theory of the underived Hecke algebras for general open compact subgroups is much less clear.

As in the $\ell \neq p$ case, we will call the cohomology ring

$$\mathcal{H}(G, U; \Lambda) := H^*(\text{DGHA}(G, U; \Lambda))$$

the *derived Hecke algebra*; this is (a priori) just a graded associative ring. The thesis of N. Ronchetti [**71**] studies the derived *spherical* Hecke algebra $\mathcal{H}(G, G(\mathcal{O}); \Lambda)$ and

defines a Satake homomorphism

$$\mathcal{H}(G, G(\mathcal{O}); \Lambda) \to \mathcal{H}(T, T(\mathcal{O}); \Lambda) \xrightarrow{\sim} \Lambda[X_*(T)] \otimes H^*(T(\mathcal{O}); \Lambda).$$

In this case, with $T(\mathcal{O})$ p-torsion free, $H^*(T(\mathcal{O}), \Lambda)$ is an exterior algebra on H^1. This algebra is much more manageable than Schneider's DGHA$(G, I(1); k)$; however, the compact induction of the trivial representation of $G(\mathcal{O})$ does not generate $D(G)$, so Ronchetti's algebra cannot account fully for the mod p representation theory of G. However, simpler p-adic derived Hecke algebras do seem to play a role in the global theory of p-adic modular forms; see §8.4.2 below.

6.3.2. *Actions of derived Hecke algebras.* In this section G is a connected reductive group over $\mathbf{Q}_p$ and $K \subset G$ is any open compact subgroup.

Let Λ be a coefficient ring and define the universal module

$$U_\Lambda(K) = \text{c-Ind}_K^G \Lambda$$

where K acts trivially on Λ. By the *differential graded Hecke algebra (with coefficients in Λ)* we mean

$$\text{DGHA}(G, K; \Lambda) := \text{RHom}_{G(\mathbf{Q}_p)}^\bullet(U_\Lambda(K), U_\Lambda(K))$$

and by the *derived Hecke algebra (with coefficients in Λ)* or its positively graded cohomology algebra

$$\mathcal{H}(G, K; \Lambda) := H^*(\text{DGHA}(G, K; \Lambda)) = \text{Ext}_{\Lambda[G(\mathbf{Q}_p)]}^*(U_\Lambda(K), U_\Lambda(K)).$$

Fact 6.8. If $C^\bullet$ is any complex of smooth $\Lambda[G(\mathbf{Q}_p)]$-modules, we have (formally) an action of DGHA$(G, K; \Lambda)$ (resp. $\mathcal{H}(G, K; \Lambda)$) on $\text{RHom}_{\Lambda[K]}(\Lambda, C^\bullet)$ (resp. the hypercohomology $H^*(K, C^\bullet)$).

Indeed, derived endomorphisms of $U_A(K)$ act on the functor $\text{RHom}_{\Lambda[G(\mathbf{Q}_p)]}^\bullet(U_\Lambda(K), -)$. Thus the Fact follows from the Frobenius reciprocity isomorphism

$$(6.6) \qquad \text{RHom}_{\Lambda[K]}^\bullet(\Lambda, C^\bullet) \xrightarrow{\sim} \text{RHom}_{\Lambda[G(\mathbf{Q}_p)]}^\bullet(U_\Lambda(K), C^\bullet).$$

7. Cohomology of locally symmetric spaces

In this section G is a connected reductive group over $\mathbb{Q}$, with center $Z = Z_G$. We fix a subgroup $K_\infty \subset G(\mathbf{R})$ containing $Z(\mathbf{R})$ and a maximal compact subgroup of the identity component $G(\mathbf{R})^0 \subset G(\mathbf{R})$, so that $K_\infty/Z(\mathbf{R})$ is compact. Then $X = G(\mathbf{R})/K_\infty$ is a finite union of copies of the symmetric space for the derived subgroup $G(\mathbf{R})^{\text{der},0}$. Let $K_f \subset G(\mathbf{A}_f)$ be an open compact subgroup; the corresponding *congruence subgroup* of $G(\mathbf{R})$ is the subgroup $\Gamma = \Gamma_{K_f} \subset G(\mathbb{Q})$ given by the intersection $G(\mathbb{Q}) \cap K_f$ in $G(\mathbf{A}_f)$. Let $K_f' \subset K_f$ be a torsion-free normal subgroup of finite index, with quotient Q; then

$$X(\Gamma_{K_f'}) := \Gamma_{K_f'} \backslash X$$

is a C^∞ manifold, and

$$X(\Gamma_{K_f}) = \Gamma_{K_f} \backslash X = Q \backslash X(\Gamma_{K_f'})$$

is the locally symmetric space attached to K_f. For most purposes the finite group Q and the singularities of $X(\Gamma_{K_f})$ will play no role in what follows.

We will be most concerned with the adelic locally symmetric spaces

$$S_{K_f}(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbf{A}_f)/K_f$$

and with the limit
$$S(G, X) = \varprojlim_{K_f} S_{K_f}(G, X),$$
the limit taken over all open compact subgroups $K_f \subset G(\mathbf{A}_f)$. For fixed K_f, the adelic locally symmetric space can be identified with the disjoint union of a finite set of discrete arithmetic quotients of X: by reduction theory we can write $G(\mathbf{A}_f) = \coprod_j G(\mathbb{Q})\alpha_j K_f$ for a finite set of α_j, and then
$$S_{K_f}(G, X) = \coprod_j X(\Gamma_j),$$
where $\Gamma_j = G(\mathbb{Q}) \cap \alpha_j K_f \alpha_j^{-1}$.

Unless otherwise indicated, for any cohomology theory H^*, usually with twisted coefficients, the space $H^*(S(G, X))$ will be understood to be the filtered colimit
$$\varinjlim_{K_f} H^*(S_{K_f}(G, X)).$$

7.1. Review of the de Rham theory. Let $\mathcal{A}(G)$ denote the (complex) vector space of automorphic forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$, and let $\mathcal{A}_0(G) \subset \mathcal{A}(G)$ denote the subspace of cusp forms. Let $Z = Z_G$ denote the center of G, $K_\infty \subset G(\mathbf{R})$ a closed subgroup of the identity component that contains $Z(\mathbf{R})$ and whose image in the adjoint group $G(\mathbf{R})^{\mathrm{ad}}$ is compact. Depending on the circumstances, K_∞ may or may not be maximal compact modulo center. For bookkeeping purposes, we let K_∞^c denote the maximal compact subgroup of K_∞. We write $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K_\infty)$; unless otherwise indicated these are taken to be the Lie algebras of the complex points. Then we have the Cartan decomposition

(7.1) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$

The (possibly disconnected) symmetric space $X = G(\mathbf{R})/K_\infty$ has an invariant hermitian structure precisely when there is a homomorphism
$$h : \mathbf{C}^\times = R_{\mathbf{C}/\mathbf{R}}\mathbb{G}_{m,\mathbf{C}} \to G_{\mathbf{R}}$$
with image in the center of K_∞ that satisfies the axioms of a Shimura datum (see Morel's chapter in this volume). In that case, (7.1) can be refined into the $\mathfrak{k}$-invariant Harish-Chandra decomposition:

(7.2) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-,$

Here, under the map of tangent spaces
$$\mathfrak{g} = T_{G,e} \to T_{X,h},$$
where $T_{G,e}$ (resp. $T_{X,h}$) is the tangent space at the identity (resp. the tangent space at the K_∞-fixed point $h \in X$), $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) is taken to the holomorphic (resp. antiholomorphic) tangent space; they are respectively of Hodge type $(-1, 1)$ (resp. $(1, -1)$) for the Hodge structure on $\mathfrak{g}$ defined by h. We sometimes write $\mathfrak{p}^+ = \mathfrak{p}_h^+$, $\mathfrak{k} = \mathfrak{k}_h$, in order to emphasize the choice of h.

The symmetric space X has a $G(\mathbf{R})$-invariant metric, unique up to multiplication by a positive real scalar, that descends to $S(G, X)$. If G/Z is anisotropic, then $S(G, X)$ is compact and its de Rham cohomology can be computed by means of harmonic forms with respect to the invariant metric. Matsushima's formula reinterprets this computation in terms of relative Lie algebra cohomology of the space $\mathcal{A}(G)$ of automorphic forms.

Let $\rho : G \to \mathrm{Aut}(V)$ be a finite-dimensional linear representation, defined over a subfield $E \subset \mathbf{C}$. Let $\tilde{V}$ be the corresponding local system over $S(G, X)$, with coefficients in E:

$$\tilde{V} = \varprojlim_{K_f} G(\mathbb{Q}) \backslash X \times G(\mathbf{A}_f) \times V / K_f.$$

In order for $\tilde{V}$ to be a local coefficient system, we need to assume that the Zariski closure of $Z(\mathbb{Q}) \cap K_f$ in Z, for all sufficiently small K_f, is in the kernel of ρ; this assumption will be made without comment in what follows. We write $V_{\mathbf{C}} = V \otimes_E \mathbf{C}$ and define $\tilde{V}_{\mathbf{C}}$ in the same way.

By an *admissible irreducible representation* of $G(\mathbf{A})$ we always mean an irreducible $(\mathfrak{g}, K_\infty) \times G(\mathbf{A}_f)$-representation, with the chosen K_∞.

THEOREM 7.1 (Matsushima's formula). *Assume G/Z is anisotropic. Then there is a canonical isomorphism of $G(\mathbf{A}_f)$-representations*

$$H^\bullet(S(G, X), \tilde{V}_{\mathbf{C}}) \xrightarrow{\sim} \oplus_\pi m(\pi) H^\bullet(\mathfrak{g}, \mathfrak{k}; \pi_\infty \otimes V_{\mathbf{C}}) \otimes \pi_f.$$

Here π runs through admissible irreducible representations of $G(\mathbf{A})$, and there is a countable direct sum decomposition

$$(7.3) \qquad \mathcal{A}(G) \xrightarrow{\sim} \oplus_\pi m(\pi)\pi$$

where $m(\pi)$ is an (integer) multiplicity and $\pi \xrightarrow{\sim} \pi_\infty \otimes \pi_f$ is the factorization with π_∞ (resp. π_f) an irreducible $(\mathfrak{g}, K_\infty)$-module (resp. $G(\mathbf{A}_f)$-representation). The notation $H^\bullet(\mathfrak{g}, \mathfrak{k}; -)$ denotes relative Lie algebra cohomology.

The standard reference for this result is the book [**12**] of Borel and Wallach, to which we refer for definitions and proofs. The interest of Matsushima's formula is that it separates the calculation of the cohomology into a global part, corresponding to the multiplicities $m(\pi)$, and a local part that depends only on π_∞. We will only consider π for which π_∞ is tempered; in this case the cohomology $H^\bullet(\mathfrak{g}, \mathfrak{k}; \pi_\infty \otimes V_{\mathbf{C}})$ is completely calculated in [**12**], and we copy the answer in the following section.

The result is more complicated when G/Z is not anisotropic, and the proofs are considerably more difficult. In particular, there is no longer a direct sum decomposition as in (7.3). Nevertheless, it was proved by Borel and (especially) Franke that the following analogue of Theorem 7.1 holds:

$$(7.4) \qquad H^\bullet(S(G, X), \tilde{V}_{\mathbf{C}}) \xrightarrow{\sim} H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{A}(G) \otimes V_{\mathbf{C}}).$$

7.1.1. *Cuspidal cohomology.* We define the *cuspidal cohomology*

$$H_0^\bullet(S(G, X), \tilde{V}_{\mathbf{C}}) \subset H^\bullet(S(G, X), \tilde{V}_{\mathbf{C}})$$

to be the image of $H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{A}_0(G) \otimes V_{\mathbf{C}})$ in $H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{A}(G) \otimes V_{\mathbf{C}})$, with respect to the isomorphism (7.4). Borel proved that the map from the relative Lie algebra cohomology of cusp forms to cohomology is injective. Thus if we write

$$(7.5) \qquad \mathcal{A}_0(G) \xrightarrow{\sim} \oplus_\pi m_0(\pi)\pi,$$

where $m_0(\pi)$ denotes the multiplicity of π in the cuspidal spectrum, we have the analogue of Matsushima's formula:

$$(7.6) \qquad H_0^\bullet(S(G, X), \tilde{V}_{\mathbf{C}}) \xrightarrow{\sim} \oplus_\pi m_0(\pi) H^\bullet(\mathfrak{g}, \mathfrak{k}; \pi_\infty \otimes V_{\mathbf{C}}) \otimes \pi_f.$$

Remark 7.2. In (7.6) and in Theorem 7.1, the relative Lie algebra cohomology should really be replaced by $(\mathfrak{g}, K_\infty^c)$-cohomology, where K_∞^c is now a maximal compact subgroup. The version stated here is only correct when K_∞^c is connected and the center of G is finite. The discussion here can easily be adapted to handle complications introduced by disconnectedness of K_∞^c or by the center of G.

We fix a representation π_f of $G(\mathbf{A}_f)$ and let $[\pi_f]_\infty$ denote the set of irreducible representations of $G(\mathbf{R})$ such that $\pi = \pi_\infty \otimes \pi_f \subset \mathcal{A}_0(G)$. We make the hypothesis

HYPOTHESIS 7.3. *Every $\pi_\infty \in [\pi_f]_\infty$ is tempered.*

We can define

$$H_0^\bullet(S(G,X), \tilde{V}_{\mathbf{C}})[\pi_f] = \operatorname{Hom}_{G(\mathbf{A}_f)}(\pi_f, H_0^\bullet(S(G,X), \tilde{V}_{\mathbf{C}})).$$

By (7.6) we then have

$$(7.7) \qquad H_0^\bullet(S(G,X), \tilde{V}_{\mathbf{C}})[\pi_f] \xrightarrow{\sim} \oplus_{\pi_\infty \in [\pi_f]_\infty} m_0(\pi) H^\bullet(\mathfrak{g}, \mathfrak{k}; \pi_\infty \otimes V_{\mathbf{C}}).$$

7.1.2. *Relative Lie algebra cohomology in the tempered case.* The set of tempered π_∞ with non-trivial relative Lie algebra cohomology with coefficients in a given $V_{\mathbf{C}}$ is determined in [**12**, §3], especially in Theorems 3.3 and Theorem 5.1. Every such π_∞ is isomorphic to a parabolically induced representation of the form $I_{P,\sigma,0}$, in the notation of [**12**], where P is a fundamental parabolic subgroup of $G(\mathbf{R})$ with Langlands decomposition ^0MAN and σ is a discrete series representation of ^0M, whose infinitesimal character is determined by the highest weight of V:

$$\chi_\sigma = -\chi_{-s(\rho+\lambda(V))|_\mathfrak{b}}$$

(see [**12**, Theorem 3.3 (2)]) where $\lambda(V)$ is the highest weight of V, $\mathfrak{b}$ is a compact Cartan subalgebra of $\operatorname{Lie}(^0M)$, and s is a (uniquely determined) element of the Weyl group of length $\ell(s) = \dim N/2$. Moreover, the relative Lie algebra cohomology $H^\bullet(\mathfrak{g}, \mathfrak{k}; \pi_\infty \otimes V_{\mathbf{C}})$ is determined explicitly in the two theorems quoted. We let

$$q(G) = \frac{\dim(G) - \dim(K_\infty^c)}{2}; \quad \ell_0(G) = \operatorname{rank}(G) - \operatorname{rank}(K_\infty^c)$$

$$q_0(G) = q(G) - \frac{\ell_0(G)}{2}.$$

The invariant ℓ_0 coincides with $\dim A$ where A is the split part of the minimally split torus in $G(\mathbb{R})$. Then $q_0(G)$ is an integer (cf. [**12**, §4.3]) and we have

$$(7.8) \quad H^{q_0(G)+j}(\mathfrak{g}, K_\infty^c; \pi_\infty \otimes V_{\mathbf{C}}) \xrightarrow{\sim} H^{q(^0M)}(^0\mathfrak{m}, K_P; \sigma \otimes W_{s(\lambda+\rho)-\rho}) \otimes \wedge^j(\mathfrak{a}^*)$$

Here $\mathfrak{a}^*$ is the linear dual of the Lie algebra $\mathfrak{a}$ of the split component A, $W_{s(\lambda+\rho)-\rho}$ is the irreducible finite-dimensional representation of ^0M with the indicated highest weight, and K_P is the maximal compact subgroup $K_\infty^c \cap P(\mathbf{R}) = K_\infty^c \cap{}^0M(\mathbf{R})$. Moreover, ^0M is a group with discrete series and $q(^0M)$ is the dimension of its associated symmetric space.

In particular, since $\ell_0(G) = \dim A$, we have

$$(7.9) \qquad H^q(\mathfrak{g}, K_\infty^c; \pi_\infty \otimes V_{\mathbf{C}}) \neq 0 \Leftrightarrow q \in [q_0(G), q_0(G) + \ell_0(G)]$$

7.1.3. *The exterior algebra action.* A basic fact about discrete series, due to Schmid, is that the space $H^{q(^0M)}(^\circ\mathfrak{m}, K_P; \sigma \otimes W_{s(\lambda+\rho)-\rho})$ that appears in (7.8) is *one-dimensional.* It follows that

COROLLARY 7.4. $H^{q_0(G)+\bullet}(\mathfrak{g}, K_\infty^c; \pi_\infty \otimes V_\mathbf{C})$ *is naturally a free rank one differential graded module over the exterior algebra* $\wedge^\bullet(\mathfrak{a}^*)$.

Combining this with (7.7) we obtain the following global corollary, which is the starting point of Venkatesh's motivic conjectures.

COROLLARY 7.5. *Suppose* π_f *has the property that every* $\pi_\infty \in [\pi_f]_\infty$ *is tempered. Then*

$$H_0^{q_0(G)+\bullet}(S(G,X), \tilde{V}_\mathbf{C})[\pi_f]$$

is a free differential graded module over $\wedge^\bullet(\mathfrak{a}^*)$ *of rank*

$$m_0(\pi_f) = \sum_{\pi_\infty \in [\pi_f]_\infty} m_0(\pi).$$

The starting point of Venkatesh's motivic conjectures is the observation that the dimension ℓ_0 of $\mathfrak{a}^*$ coincides with a rank of a motivic cohomology group hypothetically attached to π_f. This is explained clearly in [**67, 81**], and we recall the explanation here. To a pair consisting of cohomological cuspidal automorphic representation π of G and a finite-dimensional representation τ of the Langlands L-group ^LG, the Langlands correspondence hypothetically attaches a collection of motives $M(\tau)_\pi$, obtained by composing τ with the parameter of π viewed as a homomorphism of a motivic version of the absolute Galois group of $\mathbb{Q}$ to the algebraic group ^LG. For the purposes of this and the following sections, we only need the case of the dual adjoint representation $\tau = \text{Ad}^*$; then we write $M_\pi = MAd)_\pi$. In general the motives have coefficients in a number field E_π, which we ignore temporarily.

Attached to the hypothetical motive is the group $H^1_{\text{mot}}(\mathbb{Q}, M_\pi(1))$ of motivic cohomology. This is supposed to be a direct summand of a certain K-theory group tensored with $\mathbb{Q}$, and is conjecturally a finite-dimensional $\mathbb{Q}$-vector space (more generally an E_π-vector space); we denote this space $\mathfrak{q}_\pi$ (pronounced "Va," in honor of Venkatesh). While both M_π and its motivic cohomology are inaccessible, if M_π is an actual piece of the cohomology of a smooth projective algebraic variety over $\mathbb{Q}$ cut out by correspondences, then there are regulator maps comparing $\mathfrak{q}_\pi \otimes C$ and certain spaces of cohomological realizations, where C runs through possible fields of coefficients. The key point is[10]

CONJECTURE 7.6 (Beilinson, Bloch-Kato). *(i) The space* $\mathfrak{q}_\pi$ *is of dimension* ℓ_0.
(ii) The Beilinson regulator defines an isomorphism

$$\mathfrak{q}_\pi \otimes_\mathbb{Q} \mathbf{C} \xrightarrow{\sim} H^1_\mathcal{D}(M_{\pi,HdR}, 1),$$

where $H^1_\mathcal{D}(\bullet, i)$ *is Deligne cohomology and* $M_{\pi,HdR}$ *denotes the Hodge-de Rham realization.*
(iii) For any prime p *at which* M_π *has good reduction, the* p-adic regulator *defines an isomorphism*

$$\mathfrak{q}_\pi \otimes_\mathbb{Q} \mathbb{Q}_p \xrightarrow{\sim} H^1_f(\mathbb{Q}, M_\pi(1)),$$

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[10]Normally one imposes an integrality condition on the motivic cohomology classes in $\mathfrak{q}_\pi$. In the applications Venkatesh explains how this condition can be ignored.

where H_f^1 is the Bloch-Kato Selmer group.

Each conjecture also asserts that the space on the right-hand side in the isomorphism has dimension ℓ_0. This can often be proved or deduced from other conjectures. Most importantly for our purposes, these conjectures provide rational structures on the exterior algebra $\wedge^\bullet(\mathfrak{a}^*)$ that can be compared by means of (7.7) to the rational structure on the cohomology space $H_0^\bullet(S(G,X), \tilde{V}_{\mathbf{C}})[\pi_f]$, induced by topological cohomology. Specifically, we have

Fact 7.7 ([**67**], (5.1.1))**.** Conjecture 7.6 (i) and (ii) give rise to a canonical isomorphism
$$\overline{\mathfrak{a}}_\pi \otimes_{\mathbb{Q}} \mathbf{C} \xrightarrow{\sim} \mathfrak{a}^*.$$
Thus Conjecture 7.6 endows $\mathfrak{a}^*$ with a canonical $\mathbb{Q}$-rational structure.

The conjectures of [**67, 81, 34**] all refer back to this $\mathbb{Q}$-rational structure, and in particular to the following conjecture. Since π_f is realized in cohomology, it has a model over a number field $E(\pi_f)$.[11] The π_f isotypic component of $H_0^\bullet(S(G,X), \tilde{V}_{\mathbf{C}})$ is then a direct summand of the $E(\pi_f)$-rational cohomology $H_0^\bullet(S(G,X), \tilde{V}_{E(\pi_f)})$. We state a slight generalization of the main conjecture of Prasanna and Venkatesh:

CONJECTURE 7.8 ([**67**], Conjecture 1.2.1)**.** *The action on $H_0^\bullet(S(G,X), \tilde{V}_{\mathbf{C}})[\pi_f]$ of $\wedge^\bullet \overline{\mathfrak{a}}_\pi$ preserves the $E(\pi_f)$-rational structure.*

We describe the actions of the p-adic realizations of $\overline{\mathfrak{a}}_\pi$ in greater detail in §8.

7.2. Coherent cohomology of Shimura varieties. We return to the notation of §7.1 but now we suppose that X has $G(\mathbf{R})$-invariant hermitian structure. We assume throughout that Z_G is trivial; this is certainly unnecessary but it simplifies the exposition. We define
$$\mathfrak{P} = \mathfrak{P}_h = \mathfrak{k} \oplus \mathfrak{p}^-,$$
in the notation of (7.2). This is a maximal parabolic subalgebra with unipotent radical $\mathfrak{p}^-$. The locally symmetric space $S(G,X)$ is then a Shimura variety, and has a canonical model over a number field $E(G,X)$ that is preserved under the right action of $G(\mathbf{A}_f)$. This induces a canonical model on the individual finite-level quotients $S_{K_f}(G,X)$.

Let $\tau : K_\infty \to \mathrm{GL}(W_\tau)$ be an irreducible representation, and define the holomorphic vector bundle
$$(7.10) \qquad \mathcal{E}_\tau = \varprojlim_{K_f} G(\mathbb{Q}) \backslash G(\mathbf{A}) \times W_\tau / K_\infty \times K_f$$

where now $G(\mathbb{Q})$ acts on $G(\mathbf{A})$ on the left, K_f acts on $G(\mathbf{A}_f)$ on the right, and K_∞ acts on $G(\mathbf{R})$ on the right and on W_τ on the left. If K_f is sufficiently small the bundle $\mathcal{E}_\tau$ descends to a vector bundle, also denoted $\mathcal{E}_\tau$, on $S_{K_f}(G,X)$, which has a canonical model over a finite extension $E(G,X,\tau)$ of $E(G,X)$. In practice we fix a level subgroup K_f such that $S_{K_f}(G,X)$ has a *canonical integral model*, as in the work of Kisin, Shin, and Zhu; this presupposes that $S(G,X)$ is a Shimura variety of *abelian type*, which is closely related to moduli of abelian varieties with additional structure.

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[11]If $G = GL(n)$ it follows from the multiplicity one theorem that this can identified with the field of rationality of π_f, that is the fixed field in $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ of the subgroup that fixes the isomorphism class of π_f. In general $E(\pi_f)$ may be slightly larger.

Suppose for the moment that G is anisotropic over $\mathbb{Q}$, so that $S(G, X)$ is (pro)-projective. The cohomology of $S(G, X)$ with coefficients in (the coherent sheaf attached to) the vector bundle $\mathcal{E}_\tau$ then admits an expression in terms of differential forms (Dolbeault cohomology), which in turn can be written in terms of relative Lie algebra cohomology, analogously to Matsushima's formula:

(7.11)
$$H^\bullet(S(G, X), \mathcal{E}_\tau) \overset{\sim}{\to} H^\bullet(\mathfrak{P}, \mathfrak{k}; \mathcal{A}(G) \otimes W_\tau) \overset{\sim}{\to} \oplus_\pi m(\pi) H^\bullet(\mathfrak{P}, \mathfrak{k}; \pi_\infty \otimes W_\tau) \otimes \pi_f.$$

(Again, one should replace $(\mathfrak{P}, \mathfrak{k})$-cohomology by $(\mathfrak{P}, K_h)$-cohomology, where $K_h \supset K_\infty$ is the stabilizer of h in $G(\mathbf{R})$ and is not necessarily connected.) When $S(G, X)$ is not projective, one needs to replace the left-hand side of (7.11) by $H^\bullet(S(G, X)_\Sigma, \mathcal{E}_\tau^{can})$, where $S(G, X)_\Sigma$ is a well-chosen (smooth projective) toroidal compactification and $\mathcal{E}_\tau^{can}$ is Mumford's (or Deligne's) canonical extension of $\mathcal{E}_\tau$ to a vector bundle over $S(G, X)_\Sigma$; the result is independent of the choice of Σ. The analogue of the cuspidal cohomology (7.6)

(7.12)
$$H_0^\bullet(S(G, X), \mathcal{E}_\tau) \overset{\sim}{\to} \oplus_{\pi \subset \mathcal{A}_0(G)} m(\pi) H^\bullet(\mathfrak{P}, \mathfrak{k}; \pi_\infty \otimes W_\tau) \otimes \pi_f.$$

was studied in [**42**], where in most cases the left hand side is defined algebraically in terms of two distinct extensions of $\mathcal{E}_\tau$ to $S(G, X)_\Sigma$. The analogue of Franke's theorem (7.4) holds for coherent cohomology and was proved recently in the thesis of Jun Su.

Suppose the infinitesimal character of τ, as a representation of the enveloping algebra of $\mathfrak{k}$, restricts on the enveloping algebra of $\mathfrak{g}$ to the infinitesimal character of a finite-dimensional irreducible representation. Then Schmid's theorem asserts that there is a unique π_∞ and a unique $q(\tau) \in \{0, 1, \ldots, \dim X\}$ such that $H^{q(\tau)}(\mathfrak{P}, \mathfrak{k}; \pi_\infty \otimes W_\tau) \neq 0$; moreover, the cohomology space is then of dimension 1, and π_∞ is a discrete series. This theory is described at length in [**42**]. In particular, the global coherent cohomology $H_0^q(S(G, X), \mathcal{E}_\tau)$ is non-trivial only when $q = q(\tau)$.

The version of Venkatesh's motivic conjectures for coherent cohomology, to be discussed in §8.3, concerns τ for which $H_0^\bullet(S(G, X), \mathcal{E}_\tau)$ has cohomology in several degrees, and the corresponding π_∞ belong to the *nondegenerate limit of discrete series* (NLDS). It is shown in [**9**], following work of Floyd Williams, that any given NLDS π_∞ contributes to $H^q(\mathfrak{P}, \mathfrak{k}; \pi_\infty \otimes W_\tau)$ for a unique q. Let $P(\tau)$ denote the set of NLDS σ such that $H^q(\mathfrak{P}, \mathfrak{k}; \sigma \otimes W_\tau) \neq 0$, and define

$$\Pi(\tau) = \oplus_{\sigma \in P(\tau)} \sigma.$$

For the purposes of this definition, it is most convenient to treat the different components of the restriction of any NLDS of $G(\mathbf{R})$ to its identity component as separate representations σ.

I believe this representation is parabolically induced from a discrete series representation, or possibly a sum of discrete series representations, of a Levi subgroup $L(\tau) \subset G(\mathbf{R})$.

Question 7.9. Is there an abelian Lie algebra $\mathfrak{a} \subset \mathfrak{p}$ such that

$$H^\bullet(\mathfrak{P}, \mathfrak{k}; \Pi(\tau) \otimes W_\tau)$$

is a free module over $\wedge^\bullet(\mathfrak{a}^*)$?

The answer is affirmative in the admittedly unrepresentative case where $G = \mathrm{GL}(2)$, but the question does not seem to have been explored more generally.

7.3. Action of q-adic derived Hecke algebras on cohomology. We return to the topological setting of §7.1, and specifically the Prasanna-Venkatesh Conjecture 7.7. Thus let π_f be as in that section; in particular, we assume all $\pi_\infty \in [\pi_f]_\infty$ are tempered. We assume $E(\pi_f) = \mathbb{Q}$; this hypothesis can easily be relaxed, at the cost of complicating the notation. We similarly assume throughout that the representation V is defined over $\mathbb{Q}$, and write $\tilde{V}$ for the corresponding local system with coefficients $\mathbb{Q}$.

Let p be a prime number such that the local component π_p is unramified. We fix a level subgroup $K_0 = \prod_v K_v \subset G(\mathbf{A}_f)$, with K_p hyperspecial maximal, such that $\pi_f^{K_0} \neq 0$. We will be working with $\mathbb{Q}_p$ valued cohomology of the locally symmetric space $S_{K_0}(G, X)$, with coefficients in $\tilde{V} \otimes \mathbb{Q}_p$. More precisely, we choose a $\mathbb{Z}_p$-local system $\Lambda_V \subset \tilde{V}$, and define the Hecke algebra $\mathbb{T}_{K_0}$, as in [**81**, §6.1], to be the $\mathbb{Z}_p$ algebra generated by Hecke operators at good primes, acting as endomorphisms of the chain complex of Λ_V, viewed as an object in the derived category. Then π_f determines a maximal ideal $\mathfrak{m} = \mathfrak{m}_\pi \subset \mathbb{T}_{K_0}$, which we reserve for later.

Write $S_{K_0} = S_{K_0}(G, X)$. For any $r = 1, 2, \ldots$ we let $\Lambda_{V,r} = \Lambda_V / p^r \Lambda_V$, $A_r = \mathbb{Z}/p^r\mathbb{Z}$. We consider the object

$$R\Gamma(S_{K_0}, \Lambda_{V,r})$$

as a module over $\mathbb{T}_{K_0}$. Let q be a prime such that K_q is hyperspecial maximal compact. Consider the family $\mathcal{F}_q$ of open compact subgroups $U \subset K_0$ that contain $\prod_{v \neq q} K_v$, and for each $U \in \mathcal{F}_q$, the chain complex $C^\bullet(S_U(G, X), \Lambda_{V,r})$ as an object in the derived category of A_r-modules. Then $G_q = G(\mathbb{Q}_q)$ acts on $C_q^\bullet(A_r) = \varinjlim C^\bullet(S_U(G, X), \Lambda_{V,r})$. Let U_q be the universal $A_r[G_q]$-module $\mathrm{U}_{A_r}(K_q)$. Then just as in §6.3.2, Frobenius reciprocity identifies

$$(7.13) \qquad \mathrm{RHom}_{A_r[K_q]}(A_r, C_q^\bullet(A_r)) \xrightarrow{\sim} \mathrm{RHom}_{A_r[G_q]}(\mathrm{U}_A(K_q), C_q^\bullet(A_r))$$

and thus by Fact 6.8 we have an action of the derived Hecke algebra $\mathcal{H}_{A_r}^*(G_q, K_q)$ on $H^*(K_q, C_q^\bullet(A_r))$.

By the discussion in [**43**, §7.5], we can identify the hypercohomology $H^*(K_q, C_q^\bullet(A_r))$ with $H^*(S_{K_0}, \Lambda_{V,r})$. Thus formally we have

PROPOSITION 7.10. *For any r and any q, there is a canonical action of the derived Hecke algebra $\mathcal{H}(G_q, K_q; A_r)$ on $H^*(S_{K_0}, \Lambda_{V,r})$.*

The action in Proposition 7.10 is only of interest when $\mathcal{H}_{A_r}(G_q, K_q)$ is nontrivial. Thus we assume $q \equiv 1 \pmod{p^r}$. For such q we have seen in Theorem 6.6 that
$$(7.14)$$
$$\mathcal{H}^*(G_q, K_q; A_r) \xrightarrow{\sim} \mathcal{H}^*(T(\mathbf{Q}_q), T(\mathbf{Z}_q); A_r)^W \xrightarrow{\sim} (A_r[X^*(T)] \otimes_{A_r} H^*(T_q; A_r))^W$$

where W denotes the Weyl group of G and we have adapted the notation for the algebra on the right hand side.

Remark 7.11. When $q \equiv 1 \pmod{p^r}$, the action of an individual derived Hecke operator in $\mathcal{H}_{A_r}^1(G_q, K_q)$ on cohomology modulo p^r can be made explicit in terms of the mod p^r characteristic classes of cyclic unramified topological coverings of degree p^r in the passage from Iwahori level to pro-q Iwahori level at q. They are thus seen to be derived versions of the diamond operators that play a central role in the Taylor-Wiles method. This is explained in greater detail in §8.3 in the setting

of coherent cohomology of modular curves. The topological version was developed in [**81**] and is completely analogous.

Venkatesh explains in [**81**, §2.13] how to take an increasing union of sets of q as above, while passing to the limit on r, in order to define various versions of a *global p-adic derived Hecke algebra* that acts on $H^*(S_{K_0}, \Lambda_V)$. We choose the *strict global derived Hecke algebra*, denoted $\tilde{\mathbb{T}}'$ in [**81**], and assume p is prime to the order of the Weyl group. Then $\tilde{\mathbb{T}}'$ is graded commutative and contains underived Hecke operators at good primes other than p. These generate the usual (underived) Hecke algebra $\mathbb{T}_{K_0} \subset \tilde{\mathbb{T}}'$. Moreover, the action of $\tilde{\mathbb{T}}'$ preserves the localization $H^*(S_{K_0}, \Lambda_V)_{\mathfrak{m}} = H^*(S_{K_0}, \Lambda_V)_{\mathfrak{m}_\pi}$ at the maximal ideal of $\mathbb{T}_{K_0}$ we have been saving for this moment.

The first main result of [**81**] is

THEOREM 7.12. [**81**, Theorem 7.6] *Under the assumptions listed below, $H^*(S_{K_0}, \Lambda_V)_{\mathfrak{m}}$ is generated as a graded module over $\tilde{\mathbb{T}}'$ by its minimal degree component $H^{q_0(G)}(S_{K_0}, \Lambda_V)_{\mathfrak{m}}$.*

The proof is an adaptation of the ideas of Calegari-Geraghty, in the version developed by Khare and Thorne, to the language of derived structures. It depends on a list of technical assumptions [**81**, §6.1] and one substantial assumption [**81**, §6.2]. The most significant of the technical assumptions is that the $\mathfrak{m}$-localized homology [sic!] $H_*(S_{K_0}, \Lambda_V)_{\mathfrak{m}}$ is concentrated in the interval $[q_0(G), q_0(G) + \ell_0(G)]$ that we have already seen in the calculation of relative Lie algebra cohomology of tempered representations. Related to this is the assumption that $H_*(S_{K_0}, \Lambda_V)$ is torsion-free. Finally, G is assumed to be a split $\mathbb{Q}$-group; this appears to rule out generalizations to simple groups over totally real fields, for example, though the analogous theorem must hold for such groups as well.

The more serious assumption is that π has an associated p-adic Galois parameter. This is a continuous homomorphism

$$(7.15) \qquad\qquad \rho_\pi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to G^\vee(\bar{\mathbb{Q}}_p),$$

where $G^\vee$ is the Langlands dual group to G, whose Frobenius conjugacy class at the unramified prime q corresponds to the Satake parameter of π_q, and whose restriction to a decomposition group at p is crystalline in an appropriate sense. The existence of such a ρ_π is known when G is the general linear group over a real or CM field, under our assumption that π contributes to cohomology of some S_{K_0}. Venkatesh then makes assumptions on the image of the reduction $\bar\rho_\pi$ mod p that are familiar from the Taylor-Wiles theory, in particular that the image of $\bar\rho_\pi$ is not contained in a proper parabolic subgroup of $G^\vee$ and is not too small.

Remark 7.13. Strictly speaking, Venkatesh only proves Theorem 7.12 when V is the trivial representation. Moreover, he assumes that π is congruent to no other representations at level K_0, and indeed that $\mathbb{T}_{K_0,\mathfrak{m}} \xrightarrow{\sim} \mathbf{Z}_p$. As far as we know, no one has written down a proof without these hypotheses, which presumably would require additional restrictions on p.

The second main result is the construction of an action of the p-adic realization of motivic cohomology, as in Conjecture 7.6 (iii), on $H^*(S_{K_0}, \Lambda_V)_{\mathfrak{m}}$ as endomorphisms increasing the degree by 1. This will be discussed in the following section, which is devoted to Venkatesh's main conjectures regarding this action and to generalizations in other settings.

added this

7.4. Proof sketch of Theorem 7.12. We will try to indicate the main ideas behind the proof of Theorem 7.12. It uses the Calegari-Geraghty method explained in §5.6. For a set Q_n of Taylor-Wiles primes, we denote by $S_0(Q_n)$ the covers of S_{K_0} obtained by adding $\Gamma_0(q)$ level structure for all $q \in Q_n$, and by $S_1(Q_n)$ the pro-p subcover of the cover of $S_0(Q_n)$ obtained by adding $\Gamma_1(q)$ structure for all $q \in Q_n$. The homology complex $\mathcal{M}_n$ from §5.6 is the shift by q_0 of $C_*(S_1(Q_n))_{\mathfrak{n}_n}$ where $\mathfrak{n}_n$ is any choice of maximal ideal of the Hecke algebra for $S_1(Q_n)$ lying over the maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{K_0}$. In §5.6 we patched these $\mathcal{M}_n$ into an $\mathcal{M}_\infty$, and analyzes its structure over a patched ring of diamond operators S_∞. A key point was that $\mathcal{M}_\infty \xrightarrow{\sim} M_\infty$. This yields a description of the original homology chains as

$$\mathcal{M}_0 = M_\infty \overset{h}{\otimes}_{S_\infty} S_0$$

and the original cohomology cochains as

$$\mathrm{RHom}_{S_\infty}(M_\infty, S_0).$$

Furthermore, one gets (under the Conjectures needed to make the Calegari-Geraghty method work) that M_∞ is generically a free module over its support R_∞, which is a quotient of S_∞ by a regular sequence. This gives enough information to compute the action of $\mathrm{Ext}^*_{S_\infty}(S_0, S_0)$ on $\mathrm{Ext}^*_{S_\infty}(M_\infty, S_0)$ explicitly via Koszul complexes, and one sees (under the assumptions of [81]) that this action generates $\mathrm{Ext}^*_{S_\infty}(M_\infty, S_0)$ over $\mathrm{Ext}^0_{S_\infty}(M_\infty, S_0)$.

Next, Venkatesh argues that the derived Hecke actions along the same Taylor-Wiles primes "patches" to the action of $\mathrm{Ext}^*_{S_\infty}(S_0, S_0)$ on $\mathrm{Ext}^*_{S_\infty}(M_\infty, S_0)$. Let us examine this statement more closely. The ring S_∞ is patched from rings S_{Q_n} consisting of diamond operators at primes in the Taylor-Wiles system Q_n. Informally speaking, the action of $\mathrm{Ext}^*_{S_\infty}(S_0, S_0)$ on $\mathrm{Ext}^*_{S_\infty}(M_\infty, S_0)$ is patched from the action of $\mathrm{Ext}^*_{S_{Q_n}}(S_0, S_0)$ on $\mathrm{Ext}^*_{S_{Q_n}}(M_n, S_0)$. Abstractly S_{Q_n} is the tensor product over $W(k)$ of the group rings of T_q, the maximal pro$-p$ quotient of $T(\mathbf{Z}_q)$, as $q \in Q_n$. Hence $\mathrm{Ext}^*_{S_{Q_n}}(S_0, S_0) \cong \bigotimes_{q \in Q_n} H^*(T_q, W(k))$. Now the claim is that this action on $\mathcal{M}_n$ is essentially the derived Hecke action.

We formulate the latter point precisely. For $q \in Q_n$, the ideal $\mathfrak{m}$ induces an ideal $\mathfrak{m}_q$ of $H(G(\mathbf{Q}_q), G(\mathbf{Z}_q); \mathbf{Z}/p^n\mathbf{Z})$. Part of the definition of a Taylor-Wiles prime is that q is strongly regular, which means that under the Satake isomorphism $H(G(\mathbf{Q}_q), G(\mathbf{Z}_q); \mathbf{Z}/p^n\mathbf{Z}) \xrightarrow{\sim} H(T(\mathbf{Q}_q), T(\mathbf{Z}_q); \mathbf{Z}/p^n\mathbf{Z})^W$, there are exactly $|W|$ maximal ideal of $\mathrm{Spec}\, H(T(\mathbf{Q}_q), T(\mathbf{Z}_q); \mathbf{Z}/p^n\mathbf{Z})$ lying over $\mathfrak{m}_q$, and for any choice of such maximal ideal the induced map on completions is an isomorphism. In particular, any such choice induces an isomorphism

$$\mathcal{H}^1(G(\mathbf{Q}_q), G(\mathbf{Z}_q); \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m}_q} \xrightarrow{\sim} H^1(T_q; \mathbf{Z}/p^n\mathbf{Z}).$$

Furthermore, the map $S_0(q) \to S_{K_0}$ induces $H^*(S_{K_0}) \to H^*(S_0(q))$, and then $H^*(S_{K_0})_{\mathfrak{m}} \xrightarrow{\sim} H^*(S_0(q))_{\mathfrak{n}}$ where $\mathfrak{n}$ is a maximal ideal lying over $\mathfrak{m} \in \mathrm{Spec}\,\mathbb{T}_{K_0}$ and $\mathfrak{n}_q$. Now, $H^*(S_0(q))$ has an action of the group cohomology $H^*(T_q)$ because there is a finite étale T_q-cover $S_1(q) \to S_0(q)$, and we can view $H^*(S_0(q))$ as the T_q-equivariant cohomology $H^*_{T_q}(S_0(q))$. It turns out that the action of $\mathcal{H}^1(G(\mathbf{Q}_q), G(\mathbf{Z}_q))_{\mathfrak{m}}$ on $H^*(S_{K_0})_{\mathfrak{m}}$ is intertwined with this action of $H^1(T_q)$ on $H^*(S_0(q))_{\mathfrak{n}}$.

8. Venkatesh's motivic conjectures

8.1. The action of motivic cohomology. We retain the assumptions of §7.3. Then Venkatesh's Theorem 7.12 tells us that the ℓ-adic cohomology of the locally symmetric space, localized at the maximal ideal corresponding to π, is cyclic over the strict global derived Hecke algebra (DHA) . On the other hand, the analytic theory tells us that, after tensoring with $\mathbb{Q}$, it is cyclic over the exterior algebra on a vector space of dimension $\ell_0(G)$. This invariant plays no role in the definition of the strict global DHA but is linked to motivic cohomology by the Bloch-Kato Conjecture. The relation between the two appearances of $\ell_0(G)$ is provided by a canonical map from the dual adjoint Selmer group on the right hand side of Conjecture 7.6 (iii) to the degree 1 part of the strict global DHA that is characterized by a local reciprocity law.

The local reciprocity law is nevertheless stated in a global context. We realize π and the Galois representation in (7.15) over the ring of integers $\mathcal{O}$ in a p-adic field, with residue field k. Choose a Taylor-Wiles system for π, a set $Q_r = (q_1, \ldots, q_m)$ of primes $q_i \equiv 1 \pmod{p^r}$, as in [**81**, §6.3], to which we refer for the relevant properties.[12] We write $\mathcal{H}^*_{q,A_r}$ for $\mathcal{H}^*(G_q, K_q; A_r)$. In particular π is unramified at each $q \in Q_r$ and the Satake parameters of each π_q, $q \in Q_r$, are strongly regular. This means that if $\mathfrak{m}_q \subset H(T(\mathbf{Q}_q), T(\mathbf{Z}_q); A_r)^W$ is the maximal ideal induced by the maximal ideal $\mathfrak{m}_\pi$, then the fiber over $\mathfrak{m}_q$ in

$$\mathrm{Spec}\,(H(T(\mathbf{Q}_q), T(\mathbf{Z}_q), A_r) = A_r[X^*(T^\vee)] = A_r[X_*(T)])$$

is a set of $|W|$ maximal ideals. We choose one maximal ideal in $\mathrm{Spec}\,(\mathcal{H}^0(T(\mathbf{Q}_q), T(\mathbf{Z}_q); A_r))$ for each q; this corresponds to choosing an element of the Langlands dual torus $T^\vee(k)$ that is contained in the conjugacy class of $\bar{\rho}_\pi(\mathrm{Frob}_q)$; here $\bar{\rho}$ denotes the reduction of ρ_π in $G^\vee(k)$. Since W acts on both factors in the tensor product in (7.14), this choice together with the derived Satake isomorphism allows us to define a canonical injection [**81**, §8.17]

$$\iota_{q,r} : H^1(T_q; A_r) \hookrightarrow \mathcal{H}^1_{q,A_r} \tag{8.1}$$

We can define a $\mathcal{O}$-lattice in the dual adjoint Selmer group $H^1_f(\mathbb{Q}, M_\pi(1))$. With Conjecture 7.6 in mind, we let $\overline{\mathbf{q}}^*_{\pi,p}$ denote the $\mathcal{O}$-dual of this lattice. Under the running assumptions, Venkatesh proves in [**81**, Lemma 8.9] that $\overline{\mathbf{q}}_{\pi,p}$ is free of rank $\ell_0(G)$ over $\mathcal{O}$. Note that global duality identifies $\overline{\mathbf{q}}^*_{\pi,p}$ with a lattice in (the space of obstructions) $H^2_f(\mathbb{Q}, Ad(\rho_\pi))$.

There is a second map

$$f_{q,r} : H^1(T_q, A_r) \to \overline{\mathbf{q}}^*_{\pi,p}/p^n \tag{8.2}$$

defined by local duality of Galois cohomology (see p. 78 of [**81**]).

Venkatesh's second theorem, after Theorem 7.12, is then a more precise version of the following:

THEOREM 8.1. [**81**, Theorem 8.5] *There is an action of $\overline{\mathbf{q}}^*_{\pi,p}$ on $H^*(S_{K_0}, \Lambda_V)_{\mathfrak{m}}$ by endomorphisms of degree $+1$. This action is uniquely characterized by the property that, for any $r \geq 1$ and any Taylor-Wiles prime $q \in Q_{a(r)}$ for sufficiently large*

---

[12]See also the chapters of Caraiani-Shin and Emerton-Gee-Hellmann in these proceedings. In practice we can work over a base field other than $\mathbb{Q}$, of course, and the q_i will then be prime ideals.

$a(r)$, the two actions of $H^1(T_q, A_r)$ on $H^*(S_{K_0}, \Lambda_V)_{\mathfrak{m}}$ – by derived Hecke operators (via $\iota_{q,r}$) or by the action of $\overline{\mathfrak{q}}^*_{\pi,p}$ (via $f_{q,r}$) coincide.

In particular, there is an embedding of the exterior algebra $\wedge^\bullet \overline{\mathfrak{q}}^*_{\pi,p}$ in the strict global DHA $\mathbb{T}'$ of endomorphisms of $H^*(S_{K_0}, \Lambda_V)_{\mathfrak{m}}$.

Remark 8.2 (The spectral Hecke algebra)**.** The proof of Theorem 8.1 is an intense computation in [**81**]. Another perspective is given by [**25**]. There, we construct an object which we call the *spectral Hecke algebra*; it is a derived Hecke algebra that acts on the spectral (i.e., Galois) side of the Langlands correspondence. The construction is motivated by Geometric Langlands, but we will not explain that motivation here; we refer to [**25**, Introduction] for details.

The usual spherical Hecke algebra for G is the space of functions on $G(\mathbf{Z}_q) \backslash G(\mathbf{Q}_q) / G(\mathbf{Z}_q)$ viewed as a double coset set. It is natural to view this double coset space as a *groupoid*, i.e. remembering automorphisms. Then it has a non-trivial topological structure, and taking cohomology (instead of just functions) gives the derived Hecke algebra. For $G = \mathrm{GL}_n$, the groupoid may be interpreted as the space of tuples $(\mathcal{F}_1, \mathcal{F}_2, \tau)$ where $\mathcal{F}_1, \mathcal{F}_2$ are rank n vector bundles on Spec $\mathbf{Z}_q$ and τ is an identification of their restrictions to Spec $\mathbf{Q}_q$.

To make the spectral Hecke algebra, we instead consider the space of tuples (F_1, F_2, τ) where F_1, F_2 are rank n local systems on Spec $\mathbf{Z}_q$ and τ is an identification of their restrictions to Spec $\mathbf{Q}_q$. Although the language of the two situations is parallel, the geometry is quite different. In the second case, the construction is only interesting if performed in a derived sense, because the space of rank n local systems on Spec $\mathbf{Z}_q$ embeds in the space of rank n local systems on Spec $\mathbf{Q}_q$ (so that the information of F_2 is redundant). We call this derived space the "spectral Hecke stack", and its functions are essentially the spectral Hecke algebra of [**25**].

There is a co-action of the spectral Hecke algebra on the derived Galois deformation ring, and [**25**] computes that it is dual to Venkatesh's reciprocity law for the action of the derived Hecke algebra; furthermore [**25**] explains that there is a co-multiplication on the spectral Hecke algebra making it dual to the derived Hecke algebra. If we knew the derived local-global compatibility statement that the co-action of the spectral Hecke algebra on the is compatible with the action of the derived Hecke algebra, then this would yield a conceptual explanation of Venkatesh's reciprocity law. The situation is depicted below, where we write DGHA_q for the (spherical) differential graded Hecke algebra at q and SHA_q for the spectral Hecke algebra at q, $\mathrm{SHA}_q^\vee$ for the dual in the sense of [**25**] (itself equipped with an algebra structure).

$$
\begin{array}{ccc}
\mathrm{DGHA}_q & \curvearrowright & C^*(Y)_{\mathfrak{m}} \\
{\scriptstyle\sim}\big\uparrow & {\scriptstyle?}\big\| & \big\| \\
\mathrm{SHA}_q^\vee & \curvearrowright & \mathcal{R}^{\overline{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]}
\end{array}
$$

This desired derived local-global compatibility statement can be deduced from classical local-global compatibility if one really had the action of the derived Galois deformation ring on $C^*(Y)_{\mathfrak{m}}$; but what comes out of [**34**] is only the shadow of such an action at the level of homotopy groups. Constructing the derived action is an interesting open problem which we discuss more in §9.3.

Now we tensor the motivic cohomology and the cohomology of S_{K_0} with $\mathbb{Q}_p$ and assume the Beilinson-Bloch-Kato Conjecture 7.6 (iii). We let $\overline{\mathfrak{q}}^*_{\pi,E(\pi_f)} \subset \overline{\mathfrak{q}}^*_{\pi,p} \otimes \mathbb{Q}_p$

be the space of classes whose pairing with motivic cohomology lies in $E(\pi_f)$. Let $H^*(S(G,X),\tilde{V})_\pi$ denote the π_f-isotypic subspace of $H^*(S(G,X),\tilde{V})$. The p-adic version of Venkatesh's motivic conjecture is

CONJECTURE 8.3. [**81**, Conjecture 8.8] *Under the action of* $\wedge^\bullet \overline{\mathfrak{a}}^*_{\pi,p}$ *on*

$$[H^*(S(G,X),\tilde{V})_\pi]^{K_0} \otimes \mathbb{Q}_p \subset H^*(S_{K_0},\Lambda_V)_\mathfrak{m} \otimes \mathbb{Q}_p$$

defined by Theorem 8.1, the action of $\overline{\mathfrak{a}}^*_{\pi,E(\pi_f)}$ *preserves the* $E(\pi_f)$ *rational structure induced from* $H^*(S(G,X),\Lambda_V)_\pi$.

The strongest evidence for this conjecture, apart from the results of [**20**] on a similar conjecture for coherent cohomology of modular curves (see §8.3), is the substantial evidence provided in [**67**] for the analogous conjecture for complex cohomology. In that conjecture the p-adic regular with values in the Bloch-Kato Selmer group is replaced by the Beilinson regulator with values in Deligne cohomology, as in Conjecture 7.6 (ii). Since derived structures are not required for the statement of the Prasanna-Venkatesh conjecture, we say no more about it here.

8.2. The action of the derived deformation ring. After constructing the derived deformation ring in [**34**], Galatius and Venkatesh use it to give an interpretation in derived geometry of the method of Calegari-Geraghty. The cuspidal cohomological representation π, the localized cohomology $H^*(S_{K_0},\Lambda_V)_\mathfrak{m}$, the Galois parameter ρ_π, its reduction $\bar{\rho}_\pi$ modulo p, and the dual adjoint Selmer group $H^1_f(\mathbb{Q},M_\pi(1))$ are as in §7.3 and §8.1. Let S be the set of places where π and K_0 are ramified, and assume $p \notin S$. Then we can define the crystalline derived deformation ring $\mathcal{R}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]}$, with unrestricted ramification at S. [13]

The Galatius-Venkatesh interpretation of the Calegari-Geraghty method is formulated in terms of *homology* rather than *cohomology* of S_{K_0}. To avoid complications in the duality between homology and cohomology we assume V is the trivial representation, so Λ_V is just the constant local system $\mathcal{O}$; [**34**] does not treat the more general case. The graded homology group $H_*(S_{K_0},\mathcal{O})_\mathfrak{m}$ is $H_*(\mathcal{M}_0)[q_0]$ in the notation of §5.6. The proof of the following theorem, the main result of [**34**], was sketched earlier:

THEOREM 8.4. [**34**, Theorem 14.1] *Under the hypotheses of the previous sections the localized homology* $H_*(S_{K_0},\mathcal{O})_\mathfrak{m}$ *carries the structure of a free graded module over the graded ring* $\pi_*(\mathcal{R}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]})$.

8.2.1. *Duality of the derived Hecke and derived deformation actions.* The article [**34**] does not contain any new formulations of the motivic conjectures, unlike the article [**81**] devoted to the derived Hecke action. However, when the two actions can be defined they are related canonically.

The main point is the determination of $\pi_1(\mathcal{R}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]})$ [**34**, (15.7)]: in our notation, we have

LEMMA 8.5. [**34**, Lemma 15.3] *There is an isomorphism*

$$\pi_1(\mathcal{R}^{\bar{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]}) \overset{\sim}{\longrightarrow} \overline{\mathfrak{a}}_{\pi,p}$$

---

[13]The assumptions of [**81**] already admitted in the previous section remain in force in [**34**]; these include the hypothesis that the local unrestricted ramification problems at primes in S are well-behaved.

that induces an isomorphism of graded algebras

$$\pi_*(\mathcal{R}^{\overline{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]}) \xrightarrow{\sim} \wedge^* \overline{q}_{\pi,p}$$

The relation between the two actions cannot be completely straightforward, because the derived deformation ring acts on homology while the (strict global) derived Hecke algebra acts on cohomology. Under the duality between cohomology and homology, the action of $\wedge^\bullet \overline{q}^*_{\pi,p}$ on $H^*(S_{K_0}, \mathcal{O})_\mathfrak{m}$ defined in Theorem 8.1 becomes an adjoint action on $H_*(S_{K_0}, \mathcal{O})_\mathfrak{m}$; $\overline{q}^*_{\pi,p}$ acts as operators of degree -1. The compatibility between the two actions takes the following form:

THEOREM 8.6. [**34**, Theorem 15.2] *The DHA action of $\wedge^\bullet \overline{q}^*_{\pi,p}$ and the derived deformation ring action of $\pi_*(\mathcal{R}^{\overline{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]})$ are compatible in the following sense. Under the identification*

$$\pi_*(\mathcal{R}^{\overline{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]}) \xrightarrow{\sim} \wedge^* \overline{q}_{\pi,p}$$

of Lemma 8.5, let $v \in \overline{q}_{\pi,p} \in \pi_1(\mathcal{R}^{\overline{\rho},\mathrm{crys}}_{\mathcal{O}_F[1/S]})$, $v^* \in \overline{q}^*_{\pi,p}$, $m \in H_*(S_{K_0}, \mathcal{O})_\mathfrak{m}$. Then

$$v^* \cdot v \cdot m + v \cdot v^* \cdot m = \langle v, v^* \rangle m$$

where $\langle,\rangle$ is the natural $\mathcal{O}$-valued pairing between $\overline{q}_{\pi,p}$ and $\overline{q}^*_{\pi,p}$.

PROOF SKETCH. As explained in §7.4, the derived Hecke action on cohomology is approximated by the action of $\mathrm{Ext}^*_{S_\infty}(S_0, S_0)$ on $\mathrm{Ext}^*_{S_\infty}(\mathcal{M}_\infty, S_0)$.

Meanwhile, the derived Galois deformation ring action is identified with $\mathrm{Tor}^{S_\infty}_*(R_\infty, S_0)$ on $\mathrm{Tor}_*(\mathcal{M}_\infty, S_0)$.

The assumptions of [**34**] imply that $\mathcal{M}_\infty$ is free of rank one over R_∞, and that the action of S_∞ on $\mathcal{M}_\infty$ factors through a regular quotient $S_\infty \twoheadrightarrow R_\infty$. The claimed relation then follows from calculations with Koszul complexes.

We mention also that in these terms, spectral Hecke algebra corresponds to $\mathrm{Tor}^{S_\infty}_*(S_0, S_0)$. $\square$

8.3. Coherent cohomology.

8.3.1. *The derived Hecke action.* In this section we use the classical notation for modular curves. If $d > 1$ is an integer we let $X_0(d)$ and $X_1(d)$ be the familiar modular curves over $\mathbb{Z}[\frac{1}{d}]$ with level structure of type $\Gamma_0(d)$ and $\Gamma_1(d)$ respectively. We think of $X_?(d)$, $? = 0, 1$, as the quotient

$$\Gamma_?(d)\backslash \mathfrak{H}; \ \mathfrak{H} = \mathrm{GL}(2, \mathbf{R})^0 / \mathrm{SO}(2) \cdot \mathbf{R}^\times,$$

where $\mathrm{GL}(2, \mathbf{R})^0$ is the identity component of $\mathrm{GL}(2, \mathbf{R})$. The line bundle over $X_?(d)$, $? = 0, 1$ whose sections are modular forms of weight 1 is denoted ω; the isotropy representation of $\mathrm{SO}(2)$ on the fiber of ω at its fixed point $h \in \mathfrak{H}$ is denoted ω_h.

The bundle ω is the only automorphic line bundle with non-trivial cuspidal coherent cohomology, in the sense of (7.12), in more than one degree – in other words in degrees 0 and 1. Moreover, $H^0_0(X_?(d), \omega)$ and $H^1_0(X_?(d), \omega)$ are isomorphic as modules over the classical Hecke algebra, which we denote $\mathbb{T}_{?d}$; this is the algebra generated by standard Hecke operators at all primes not dividing d. This remains true for modular curves attached to all congruence subgroups: there are two representations (Harish-Chandra modules) of the identity component $GL(2, \mathbf{R})^0$, say π_0 and π_1, such that

$$\dim H^i(\mathfrak{P}, \mathfrak{k}; \pi_i \otimes \omega_h) = 1, \ i = 0, 1$$

in the notation of (7.11). The representations π_0 and π_1 are holomorphic and anti-holomorphic limits of discrete series, respectively, and the direct sum $I(0) := \pi_0 \oplus \pi_1$ has a structure of irreducible $(\mathrm{Lie}(\mathrm{GL}(2)), \mathrm{O}(2) \cdot \mathbf{R}^\times)$-module whose restriction to $(\mathrm{Lie}(\mathrm{GL}(2)), \mathrm{SO}(2) \cdot \mathbf{R}^\times)$ is the sum of the two Harish-Chandra modules for $\mathrm{GL}(2, \mathbf{R})^0$. The notation $I(0)$ is meant to suggest (accurately) that, up to a harmless twist by a power of the absolute value of the determinant, it is obtained by parabolic induction from a unitary character of the Borel subgroup.

For any prime q we let $G_q = \mathrm{GL}(2, \mathbb{Q}_q)$, $K_q = \mathrm{GL}(2, \mathbb{Z}_q)$. Suppose q is relatively prime to d and is congruent to 1 $\pmod{p^r}$ for some $r > 0$. We let $X_?(d)_r$ denote the base change of $X_?(d)$ to $\mathrm{Spec}\,(\mathbb{Z}/p^r\mathbb{Z})$ and define ω_r analogously. One defines in [**43**] an action of the derived Hecke algebra $\mathcal{H}(G_q, K_q; \mathbf{Z}/p^r\mathbf{Z})$ on $H^*(X_?(d)_r, \omega_r)$. The action can be defined formally as in §7.3, but it also has a concrete definition in terms of cyclic coverings. We let $X_{0?}(qd)$ denote the modular curve corresponding to the congruence subgroup $\Gamma_?(d) \cap \Gamma_0(q)$, and define $X_{1?}(qd)$ similarly. As long as $q \geq 5$ the natural map $X_{1?}(qd) \to X_{0?}(qd)$ factors through an étale cover $X_\Delta(qd)/X_{0?}(qd)$ that is cyclic of degree p^r. This cover defines a class

$$(8.3) \qquad \mathfrak{S} \in H^1(X_{0?}(qd)_r; C_r)$$

that is called the *Shimura class*. Here C_r is the cyclic quotient of $(\mathbb{Z}/q\mathbb{Z})^*$ of degree p^r. This can be identified with $\mathbb{Z}/p^r\mathbb{Z}$ but not canonically; we choose an isomorphism (discrete logarithm)

$$(8.4) \qquad \log : C_r \xrightarrow{\sim} \mathbb{Z}/p^r\mathbb{Z}.$$

On the other hand, there are two maps $p_i : X_{0?}(qd) \to X_?(d)$. If we think of $X_{0?}(qd)$ as the moduli space of triples (E, C, ϕ) where E is a (generalized) elliptic curve, C is a cyclic subgroup of order q, and ϕ is some level structure at d, then π_1 is the map that forgets C and π_2 is the map that replaces E by E/C.

PROPOSITION 8.7. *(i) The composition*

$$H^0_0(X_?(d)_r, \omega_r) \xrightarrow{\pi_1^*} H^0_0(X_{0?}(d)_r, \omega_r) \xrightarrow{\cup \mathfrak{S}} H^1_0(X_{0?}(d)_r, \omega_r) \xrightarrow{\pi_{2,*}} H^1_0(X_?(d)_r, \omega_r),$$

where the middle arrow identifies $\omega_r \otimes C_r$ with ω_r by means of the discrete logarithm, coincides with the action of a (precise non-trivial) element $T_q^1 \in \mathcal{H}^1(G_q, K_q)$.

(ii) The action of $\mathcal{H}^1(G_q, K_q; \mathbf{Z}/p^r\mathbf{Z})$ on $H^*(X_?(d)_r, \omega_r)$ commutes with the action of the classical Hecke operators at primes not dividing qd.

Part (i) is equation (3-1) in [**43**], while part (ii) was observed in Mazur's Eisenstein ideal paper (and is easy to prove).

Analogous operators have been constructed and studied in the Columbia thesis of S. Atanasov for Shimura varieties attached to unitary groups of signature $(1, n-1)$. These groups also have pairs of (nondegenerate) limits of discrete series π_0, π_1 that have $(\mathfrak{P}, \mathfrak{k})$-cohomology with coefficients in the same irreducible representation of (a maximal compact) K.

8.3.2. *The conjecture.* The action of the global Hecke algebra $\mathbb{T}_{?d}$ on the weight one coherent cohomology spaces $H^i_0(X_?(d), \omega) \otimes \overline{\mathbb{Q}}_p$, $i = 0, 1$, is diagonalizable and the characters $\lambda : \mathbb{T}_{?d} \to \overline{\mathbb{Q}}_p$ that appear in the decomposition are in bijection with the cuspidal automorphic representations π of $\mathrm{GL}(2, \mathbf{A})$ whose archimedean component π_∞ is isomorphic to the irreducible representation $I(0)$ introduced above. As already mentioned, the same characters occur in H^0_0 and H^1_0, with the same multiplicity. It is also convenient to identify each π that contributes to weight one

coherent cohomology with its normalized newform g_π, which is uniquely determined by its classical q-expansion. In the statement of the conjecture we will just write $g = g_\pi$ and we always assume g to be a newform of level d.

The analogue of Venkatesh's conjecture is easier to state in this setting because the hypothetical motive M_π actually exists. By the Deligne-Serre Theorem each π (or g_π) corresponds, in the sense of the Langlands correspondence, to a 2-dimensional Galois representation

$$\rho_\pi : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{GL}(2, C)$$

for some finite extension $L/\mathbb{Q}$, where C is any algebraically closed field. Each such ρ_π is *odd*: the determinant of $\rho_\pi(c)$, for any complex conjugation $c \in \mathrm{Gal}(L/\mathbb{Q})$, equals -1. Conversely, the Artin conjecture asserts that every 2-dimensional odd representation of the Galois group of a finite extension of $\mathbb{Q}$ is of the form ρ_π; this has been known since the work of Khare and Wintenberger on Serre's Conjecture, although it had been established in most cases by Buzzard, Dickinson, Shepherd-Barron, and Taylor. Moreover, when the image of ρ_π is a dihedral group the result is essentially due to Hecke.

Thus to each π we have the 2-dimensional representation ρ_π, which can be viewed as the Galois realization of a rank 2 Artin motive, with coefficients in a ring $\mathcal{O}$ of algebraic integers, which is a direct factor of the 0-dimensional motive $R_{L/\mathbb{Q}}\,\mathrm{Spec}\,(L)$. The motive M_π is then the direct factor of $R_{L/\mathbb{Q}}\,\mathrm{Spec}\,(L)$ whose Galois realization is given by the 3-dimensional Galois representation

$$\mathrm{Ad}^0\,\rho_\pi : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{GL}(3, C),$$

the trace free summand of the 4-dimensional representation $Ad\rho_\pi$. (In general $Ad^0\rho_\pi$ factors through the Galois group of a proper subfield of L, but this is unimportant.) It is explained in [**43**, §2.7] that the motivic cohomology group $H^1_{\mathrm{mot}}(\mathbb{Q}, M_\pi(1))$, denoted ъ_π above, can be replaced in the formulation of the conjecture by a certain 1-dimensional space of *Stark units*, denoted U_π or U_g if we write $g = g_\pi$. This is defined as

$$U_g = U_\pi \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}\left[G_{L/Q}\right]} \left(\mathrm{Ad}^0\,\rho, \mathrm{U}_L \otimes \mathcal{O}\right) \tag{8.5}$$

This becomes one-dimensional upon tensoring with $E = \mathrm{Frac}(\mathcal{O})$. One introduces a natural map, denoted

$$\mathrm{red}_q : U_g \to \mathbb{F}_q^\times \otimes \mathbb{Z}/p^r\mathbb{Z}.$$

The conjecture stated in [**43**] relates the action of the derived Hecke operator T_q^1 with a rational multiple of the discrete (mod p^r) logarithm of a generator of U_g. Since rational multiplication does not sit well with reduction modulo p^r, one needs to define the inputs and the relations carefully. The Shimura class in $H^1(X_{0?}(qd)_r; C_r)$ defines a non-trivial class $\mathfrak{S}_\mathcal{O}$ in $H^1(X_{0?}(qd)_r; \mathcal{O}_r)$ (where $\mathcal{O}_r$ is here the structure sheaf of $X_{0?}(qd)_r$) through the composition of (8.4) with the inclusion of Zariski sheaves $\mathbb{Z}/p^r\mathbb{Z} \hookrightarrow \mathcal{O}_r$. A feature specific to the case of GL(2) and its inner forms is that the class $\mathfrak{S}_\mathcal{O}$ is the reduction modulo p^r of the class in $H^1(X_{0?}(qd), \mathcal{O})$ of (the complex conjugate of) an *Eisenstein cusp form* $f_E^\#$, in the sense of Mazur's Eisenstein ideal paper.[14] At least if the space of Eisenstein cusp forms is one-dimensional, this allows us to assign an invariant that measures

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[14]Atanasov has shown that for the Shimura varieties attached to unitary groups of signature $(1, n-1)$ with $n > 2$, the analogues of the Shimura classes do not lift to characteristic zero.

the action of T_q^1. Define g^* to be the weight 1 newform corresponding to the automorphic representation $\pi^\vee$ dual to π. We let $g^{*,(q)}$ be the "old form" in the space $\pi^\vee$ which in classical notation is just the analytic function $z \mapsto g^*(qz)$ for z in the upper half-plane. By Proposition 8.7, $T_q^1(g) \in H^1(X_{0?}(d)_r, \omega_r)$ has the same Hecke eigenvalues as g. Hence in the Serre duality pairing

$$\langle,\rangle : H_0^0(X_{0?}(d)_r, \omega_r) \otimes H_0^1(X_{0?}(d)_r, \omega_r) \to H_0^1(X_{0?}(d)_r, \omega_r^{\otimes 2}) \to \mathcal{O}_r,$$

where the last arrow is the trace map, the map $\bullet \mapsto \langle \bullet, T_q^1(g)\rangle$ factors through projection on the $g = g_\pi$-eigenspace of level qd. Let now $f_E^\# = a_1 \cdot f_E$ where f_E is the normalized Eisenstein cusp form (with leading coefficient 1; the constant a_1 is essentially the inverse of what is called the *Merel constant* in [43]). Then $T_q^1(g)$ is determined by the pairing $\langle gg^{*,(q)}, f_E\rangle$, which can be identified with the Petersson pairing of the weight 2 forms $gg^{*,(q)}$ and f_E; g^* has to be replaced by the old form because f_E has level q (the pairing is taking place in the second group from the right in Proposition 8.7) It can also be related to the square root of the central value of the triple product L-function $L(s, \pi \times \pi^\vee \times \pi(f_E))$, where the notation $\pi(f_E)$ is self-explanatory.

The motivic conjecture, as formulated in [43], comes down to an equality up to an integer factor:

CONJECTURE 8.8. *There exists an integer $m = m_g \geq 1$ and $u_g \in U_g$ such that, for all primes q and p as above*

$$m \cdot \langle gg^{*,(q)}, \mathfrak{S}\rangle = \log(\mathrm{red}_q(u_g)).$$

Here both sides depend linearly on the choice of log as in (8.4). The independence of m from q and p is essential in this statement, which would otherwise be vacuous.

8.3.3. *Results.* Weight 1 eigenforms $g = g_\pi$ can be classified by the images of the corresponding (odd) 2-dimensional Galois representations ρ_π, into those whose image is dihedral – induced from a character ψ_1 of a quadratic extension $K/\mathbb{Q}$ – and the "exotic" cases where the projectivized image is A_4, S_4, or A_5. In the dihedral case the Stark unit group U_π can be identified. When K is imaginary – say g or π is of CM type – U_π consists of elliptic units; when K is real – then g or π is of RM type – U_π is generated by the fundamental unit of K itself. Let D be the discriminant of K. Using the explicit determination of U_π, the article [20] proves the following theorem:

THEOREM 8.9. *If K is imaginary, assume that D is an odd prime and that ψ_1 is unramified. If K is real assume that D is odd and that ψ_1 has conductor dividing the different of $K/\mathbb{Q}$. Then Conjecture 8.8 is true for g_π.*

The restrictions on D and the conductor of ψ_1 are certainly not necessary; they have been relaxed in the Columbia thesis of R. Zhang, whose results are close to optimal. The methods of proof of Theorem 8.9 depend crucially to the realization of g_π as explicit theta functions. D. Marcil has provided striking numerical confirmation of the conjecture in some A_4 cases. As matters stand, it seems unlikely that one can prove unconditional results in the exotic case, though it's conceivable that the conjecture can be deduced somehow as a consequence of Stark's conjecture.

8.4. Venkatesh's conjectures, p-adic cohomology, and the invariant ℓ_0.
Specialists in p-adic modular forms already encountered the invariant ℓ_0 as early as
the 1990s. Hida's theory of ordinary modular forms, or automorphic cohomology
classes, for the reductive group G over $\mathbb{Q}$, constructs a p-adic analytic space whose
points parametrize p-adic modular forms, which can be viewed as the generic fiber
$\mathcal{X}_{ord,G}$ of the formal spectrum $\mathrm{Spf}\,(\mathbb{T})$, where $\mathbb{T}$ is Hida's "big Hecke algebra." The
ring $\mathbb{T}$ is an algebra over the Iwasawa algebra Λ of the (maximal compact subgroup
of the p-adic points of the) maximal torus of G. In Hida's original theory for
modular curves, and its extension to Shimura varieties, $\mathbb{T}$ is a finite flat Λ-algebra,
and the points of $\mathcal{X}_{ord}$ corresponding to eigenvalues of classical (complex) Hecke
algebras are dense in $\mathcal{E}_{ord}$. In general Hida conjectured in [**44**] that the codimension
of $\mathcal{X}_{ord}$ in the rigid generic fiber of $\mathrm{Spf}\,(\Lambda)$ is exactly ℓ_0.

Urban made the analogous conjecture in the paper [**78**] in which he constructed
p-adic families of modular forms of finite slope, parametrized by the *eigenvari-
ety* $\mathcal{X}_G$, one of several constructions of a higher-dimensional generalization of the
Coleman-Mazur eigencurve for elliptic modular forms. In the finite slope situation,
the eigenvariety is purely a rigid analytic object, Hida's Hecke algebra is replaced
by a noetherian subring, which we also denote $\mathbb{T}$, of the ring of global sections of
$\mathcal{O}_{\mathcal{X}_G}$, and the map of $\mathrm{Spf}\,(\mathbb{T})$ to $\mathrm{Spf}\,(\Lambda)$ is replaced by a morphism $p : \mathcal{X}_G \to \mathcal{W}$.
Here the *weight space* $\mathcal{W}$ is again a rigid torus of dimension equal to the absolute
rank of G, the Iwasawa algebra is replaced by the completed (noetherian) local ring
Λ_w at a point $w \in \mathcal{W}$. Fixing a point $x \in \mathcal{X}_G$, with $p(x) = w$, the map p makes the
completion $\mathbb{T}_x$ into a finite Λ_w-algebra.

Both Hida and Urban refer to their conjectures on the codimension as non-
abelian analogues of the Leopoldt Conjecture on the p-adic multiplicative indepen-
dence of global units in number fields. The Leopoldt Conjecture has for decades
been the Bermuda triangle of algebraic number theory, tempting many of the most
distinguished specialists into losing precious years in the pursuit of promising but
ultimately futile attempted proofs. Best not to try to prove the codimension con-
jectures of Hida and Urban, in other words. Nevertheless, it's natural to reinterpret
these codimension conjectures in Venkatesh's motivic framework. We briefly dis-
cuss two such interpretations. The article [**41**] of Hansen and Thorne approaches
the eigenvarieties as parameter spaces for Galois representations, whereas the un-
published article [**49**] of Khare and Ronchetti provides evidence that the derived
diamond algebras introduced in §8.4.2, acting on Hida's ordinary families, are the
wild p-adic analogues of Venkatesh's derived Hecke algebras at (tame) Taylor-Wiles
primes. In both constructions the motivic cohomology group $H^1_{\mathrm{mot}}(\mathbb{Q}, M_\pi(1))$ that
figures in Venkatesh's motivic conjecture appears through its p-adic realization as
the dual Selmer group.

8.4.1. The Hansen-Thorne construction. The article [**41**] is concerned exclu-
sively with eigenvarieties attached to the group of $G = \mathrm{GL}(n)_{\mathbb{Q}}$. The eigenvarieties
are constructed by Hansen's method in [**40**], following the topological approach of
Ash and Stevens. It has been known for about 10 years that the classical point
$x \in \mathcal{X} := \mathcal{X}_{GL(n)}$ corresponding to the cuspidal cohomological automorphic repre-
sentation π has an associated semisimple p-adic Galois representation

$$(8.6) \qquad\qquad \rho_\pi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(n, L)$$

for some finite extension $L/\mathbb{Q}_p$ over which x is defined. The point x also depends
on the secondary datum of a *refinement* t_x, or an ordering on the eigenvalues of

crystalline Frobenius, that is assumed implicitly in what follows. The representation ρ_π does not depend on the choice of t_x.

We then define the dual adjoint Selmer group $H^1_f(\mathbb{Q}, M_\pi(1)) = H^1_f(\mathbb{Q}, \mathrm{Ad} \circ \rho_\pi(1))$ as in §8.1. Note that this is a $\mathbb{Q}_p$-vector space; the constructions in [**41**], in contrast to those of [**34, 81, 43**], are carried out entirely in characteristic zero.

The main result of [**41**] is that, at least under favorable conditions, and assuming a conjecture recalled in the statement of Theorem 8.10 below, $H^1_f(\mathbb{Q}, M_\pi(1))$ controls the infinitesimal geometry of $\mathcal{E}$ at x with respect to the morphism p. Combining this with the universal coefficients spectral sequence constructed in [**40**], the authors recover the analogue of the Galatius-Venkatesh Theorem 8.4, but without introducing the derived deformation ring.

The main hypothesis required for the final results of [**41**] is a version of the isomorphism

$$(8.7) \qquad\qquad R_x \xrightarrow{\sim} \mathbb{T}_x$$

of the Taylor-Wiles method. Here R_x is an *underived* deformation ring defined using the theory of (φ, Γ)-modules over the Robba ring. In particular, R_x is defined purely by characteristic zero methods, and (pro)-represents a functor on Artinian local L-algebras for the p-adic coefficient field L of ρ_π (8.6). More precisely, it is assumed that ρ_π is absolutely irreducible; then R_x (pro)-represents *trianguline* deformations of ρ_π along with the refinement t_x. The property of being trianguline[15] is determined by the (φ, Γ)-module attached to the restriction of ρ_π to decomposition groups at p; the relevant fact for us is that the Galois representations attached to points on the eigenvariety, including ρ_π itself, necessarily have this property.

We now assume π contributes to $H^*(S_{K_0}, \tilde{V})$, with notation as in §7.3 for $G = \mathrm{GL}(n)_\mathbb{Q}$. We change notation slightly, and let $\mathfrak{m}_\pi$ denote the maximal ideal corresponding to π in $\mathbb{T}_{K_0} \otimes \mathbb{Q}_p$; $H^*(S_{K_0}, \tilde{V})_{\mathfrak{m}_\pi}$ denote the localization. Let $q_0 = q_0(\mathrm{GL}(n)_\mathbb{Q})$, $\ell_0 = \ell_0(\mathrm{GL}(n)_\mathbb{Q})$. The following theorem summarizes the main results of Theorems 4.9 and 4.13 of [**41**].

THEOREM 8.10. *Let $x \in \mathcal{X}$ be a classical point as above, with $w = p(x)$, and let $\lambda : \mathbb{T} \to L$ be the corresponding character of the Hecke algebra. Assume λ is numerically noncritical[16] Then*

(i) *$\dim \mathbb{T}_x \geq \dim \Lambda_w - \ell_0$;*

(ii) *If $\dim \mathbb{T}_x = \dim \Lambda_w - \ell_0$ let $V_x = \ker(\Lambda \to \mathbb{T}_x) \otimes_{\Lambda_w} L$, where the map from Λ_w on L is the augmentation. Then $H^*(S_{K_0}, \tilde{V})_{\mathfrak{m}_\pi}$ is free of rank 1 as graded module over the exterior algebra $\wedge^*_L V_x$ and is generated by its term in top degree $q_0 + \ell_0$.*

(iii) *Assume ρ_π is absolutely irreducible and crystalline at p, with the expected Hodge-Tate weights, and its associated Weil-Deligne representation at every prime ℓ (including p) corresponds to π_ℓ under the local Langlands correspondence. Assume also the isomorphism (8.7).*

Then V_x can be identified with $H^1_f(\mathbb{Q}, \mathrm{Ad}\,\rho_\pi(1)) = \overline{a}_{\pi,p}[\frac{1}{p}]$, and $H^*(S_{K_0}, \tilde{V})_{\mathfrak{m}_\pi}$ is free of rank 1 as graded module over the exterior algebra $\wedge^*_L \overline{a}_{\pi,p}$, by its term in top degree $q_0 + \ell_0$.

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[15]pronounced in English to rhyme with *fine*, *wine*, and *turpentine*

[16]This is a regularity condition, for which we refer to [**41**, Theorem 4.7].

Remark 8.11. It is natural to assume that the action of $\wedge_L^* \mathbf{q}_{\pi,p}$ on $H^*(S_{K_0}, \tilde{V}))_{\mathfrak{m}_\pi}$ coincides, up to duality, with the one defined in Theorem 8.4. However, the identification of V_x with $\mathbf{q}_{\pi,p}[\frac{1}{p}]$ depends on the deformation ring R_x, which in turn depends on the choice of refinement t_x. Hansen has conjectured that the action does not depend on this additional choice, but even this conjecture remains open.[17]

8.4.2. *The Khare-Ronchetti construction.* The constructions in this section are based on the paper [**49**]. Although their derived diamond algebras are defined purely locally, the construction is bound up with the action on the ordinary part of global cohomology. It is not clear to us how the representation theory of the derived diamond algebras fits in a hypothetical p-adic local Langlands program.

Let $p > 2$ be a prime number and let G be the group $\text{PGL}(n)$ over a finite extension $F/\mathbf{Q}$. For any open compact subgroup $K \subset G$ and any ring A we let $H(G, K; A)$ denote the (underived) Hecke algebra of K-biinvariant functions on G with coefficients in A. Let $B = T \cdot U$ be the upper-triangular Borel subgroup, U its unipotent radical, and T the diagonal maximal torus. We restrict attention to $\text{PGL}(n)$ in order to use explicit matrix groups, and to make use of the known construction of Galois parameters attached to cuspidal cohomology classes. The article [**49**] treats the general case and the reader should be able to make the appropriate adjustments.

Let U^- be the opposite unipotent (lower triangular) subgroup. Let v be a prime of F dividing p, with integer ring $\mathcal{O} = \mathcal{O}_v$ and uniformizer $\varpi = \varpi_v$. For $1 \le b \le c$ we let

(8.8) $I_{b,c,v} = U^-(\varpi^c \mathcal{O}) \cdot T(1 + \varpi^b \mathcal{O}) \cdot U(\mathcal{O});$

i.e. the generalization from $\text{GL}(2)$ to G of the congruence subgroup

$$I_{b,c,v} = \begin{pmatrix} 1 + \varpi^b * & * \\ \varpi^c * & 1 + \varpi^b * \end{pmatrix} \subset \text{GL}(2, \mathcal{O}).$$

We let $I_{b,c} = \prod_{v|p} I_{b,c,v}$.

Let $K = K_p \cdot K^p \subset G(\mathbf{A}_f)$, where K^p is a fixed level subgroup away from p and K_p varies over the subgroups $\prod_{v|p} I_{b,c,v}$ introduced in (8.8). We consider the cohomology of the locally symmetric spaces of dimension d

$$Y_{b,c} = G(\mathbb{Q}) \backslash G(\mathbf{A}_f)/K_\infty \cdot K^p \cdot I_{b,c}$$

where $K_\infty \subset G(\mathbf{R})$ is maximal compact and K^p is a fixed level subgroup away from p, so that $K^p \cdot I_{b,c} \subset G(\mathbf{A}_f)$ is open compact. We assume K^p is hyperspecial maximal outside a finite set of primes S.

The cohomology of $Y_{b,c}$ is a module over the unramified Hecke algebra

$$H^{un} = \prod_{q \notin S, q \neq p} H(G_q, K_q).$$

It is also a module over a derived Hecke algebra at p, which we proceed to define in the setting of Hida theory.

Following Hida, we define an operator $U_p = \prod_{\alpha \in \Sigma} U_\alpha$ acting as correspondences on each $Y(K(b,c))$, compatibly with the natural maps $Y(K(b,c)) \to Y(K(b',c'))$

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[17]We thank David Hansen for pointing this out.

for $b' < b, c' < c$. (There are really operators $U_p(b,c)$ but we don't stress this.) Roughly, with $G = \mathrm{PGL}(n)$ we have

$$U_p = \prod_{j=1}^{n-1} U_{j,p}, \; U_{j,p} = U_{\alpha_j} = I_{b,c} \prod_{v|p} \mathrm{diag}(\varpi_v \, \mathrm{Id}_j, \mathrm{Id}_{n-j}) I_{b,c}.$$

Definition 8.12. The ordinary part $H^i(Y_{b,c}, \mathbb{Z}/p^n\mathbb{Z})_{ord} \subset H^i(Y_{b,c}, \mathbb{Z}/p^n\mathbb{Z})$ is the submodule (and direct summand) where the U_p operator acts invertibly.

Fact 8.13. For all $c \geq b \geq 1$, $n \geq 1$, the pullback

$$H^i(Y_{b,b}, \mathbb{Z}/p^n\mathbb{Z})_{ord} \to H^i(Y_{b,c}, \mathbb{Z}/p^n\mathbb{Z})_{ord}$$

is an isomorphism.

PROOF. The proof in [**49**] is a calculation with double cosets acting on singular cochains. $\square$

Remark 8.14. If Fact 8.13 admits a generalization to overconvergent cohomology it will necessarily need to be formulated in characteristic zero.

The group $I_{1,c}/I_{b,c}$ is isomorphic to the abelian p-group

$$T_b := \prod_{v|p} T(1 + \varpi_v \mathcal{O}_v)/T(1 + \varpi_v^b \mathcal{O}_v).$$

Since $I_{b,c}$ is normal in $I_{1,c}$, the elements $\alpha \in I_{1,c}$ define *diamond operators*

$$\langle \alpha \rangle = I_{b,c} \alpha I_{b,c} \in H_{\mathbb{Z}/p^m\mathbb{Z}}(G, I_{b,c}).$$

for all b, c.

The action of $\langle \alpha \rangle$ on $H^i(Y_{b,c}, \mathbb{Z}/p^n\mathbb{Z})$ commutes with the maps $Y_{b,c} \to Y_{b',c'}$ for $b' < b, c' < c$. Let $\Lambda_b = \mathbb{Z}_p[T_b]$, $\Lambda = \varprojlim_b \Lambda_b$, which is the Iwasawa algebra of $T(\mathbb{Z}_p)$. We also use $\Lambda_F = \Lambda \otimes_{\mathbb{Z}_p} F$ for any finite extension $F/\mathbb{Q}_p$.

The $T_{b,c}$-covering $Y_{c,c}/Y_{b,c}$ corresponds to a morphism to the classifying space

$$\pi_{b,c} : Y_{b,c} \to BT_{b,c}.$$

Thus there is a ring homomorphism

$$\pi_{b,c}^* : H^*(BT_{b,c}, \mathbb{Z}/p^m\mathbb{Z}) = H^*(T_{b,c}, \mathbb{Z}/p^m\mathbb{Z}) \to H^*(Y_{b,c}, \mathbb{Z}/p^m\mathbb{Z})$$

and thus an action of $H^*(T_{b,c}, \mathbb{Z}/p^m\mathbb{Z})$ on $H^*(Y_{b,c}, \mathbb{Z}/p^m\mathbb{Z})$ by *derived diamond operators*. Since $I_{b,c}^{ab} = T_{b,c}$, we have an isomorphism

$$\iota : H^1(T_{b,c}, \mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\sim} H^1(I_{b,c}, \mathbb{Z}/p^m\mathbb{Z}) \subset \mathcal{H}_{\mathbb{Z}/p^m\mathbb{Z}}(G_p, I_{b,c}).$$

LEMMA 8.15 ([**49**]). *The derived diamond action of $\phi \in H^1(T_{b,c}, \mathbb{Z}/p^m\mathbb{Z})$ coincides with the derived Hecke action of $\iota(\phi)$ (obtained as in Fact 6.8).*

In particular, when $b = 1$ this action commutes with the $U_{j,p}$-operators.

The claim about commutation follows from an explicit double coset calculation. It implies (when $Y_{b,c} = Y_{1,c}$)

COROLLARY 8.16. *The derived degree 1 diamond operators act on $H^*(Y_{1,c}, \mathbb{Z}/p^m\mathbb{Z})_{ord}$, preserving all $U_{j,p}$-eigenspaces.*

Another calculation shows:

PROPOSITION 8.17. *As c varies, the actions of $H^1(T_{1,c}, \mathbb{Z}/p^m\mathbb{Z})$ on $H^*(Y_{1,1}, \mathbb{Z}/p^m\mathbb{Z})_{ord}$ are compatible, and thus give rise in the limit to an action of $H^1(T(1+p\mathbb{Z}_p)), \mathbb{Z}/p^m\mathbb{Z})$ on $H^*(Y_{1,1}, \mathbb{Z}/p^m\mathbb{Z})_{ord}$.*

Letting $m \to \infty$ we thus obtain a graded action of $H^*(T(1 + p\mathbb{Z}_p), \mathbb{Z}_p)$ on $H^*(Y_{1,1}, \mathbb{Z}_p)_{ord}$.

We return to the setting of §7.1. Let π be a cuspidal cohomological representation of G, as in that section. Thus we have, as in Corollary 7.5

$$H_0^{q_0(G)+\bullet}(Y_{1,1}, \bar{\mathbb{Q}}_p)[\pi_f]$$

is a free differential graded module over (p-adic) $\wedge^\bullet(\mathfrak{a}^*)$ of rank 0 or 1; here we are using multiplicity one, and the existence of Whittaker models, for $\mathrm{PGL}(n)$.

The following theorem is an imprecise version of the main resuilt of [**49**]. Let E be an appropriate p-adic coefficient field for π_f.

THEOREM 8.18 ([**49**]). *Under certain favorable circumstances, $H^*(Y_{1,1}, E)_\pi$ is generated over $H^{q_0}(Y_{1,1}, E)_\pi$ by the action of the derived diamond operators $H^*(T(1 + p\mathbb{Z}_p), E)$.*

Moreover, there is a canonical map from $H^1(T(1+p\mathbb{Z}_p), E)$ to the Selmer group $\overline{\mathfrak{q}}_{\pi,p}[\frac{1}{p}]^\vee$ attached to π. And under even more favorable circumstances, the action of $H^*(T(1 + p\mathbb{Z}_p), E)$ on $H^*(Y_{1,1}, E)_\pi$ factors through this Selmer group.

The "favorable circumstances" to which the theorem alludes include the isomorphism (8.7), in other words the non-abelian Leopoldt conjecture. As in Theorem 8.6, the actions of Theorem 8.18 and Theorem 8.10 should be dual to each other.

8.5. Concluding thoughts. Venkatesh and his collaborators have found striking evidence for his conjectural framework in a number of cases of low rank, and have proved the p-adic graded version of the conjectures, in the form of Theorem 8.2, under reasonably general hypotheses, by adapting the methods of modularity discussed in the chapter of Caraiani and Shin. However, especially in view of Conjecture **??**, it's natural to wonder whether the action of motivic cohomology on the cohomology of locally symmetric spaces is not a manifestation of a deeper structure.

It's probably pointless to devote too much attention to speculation about such a *derived Langlands correspondence*, when the known construction of Galois representations attached to cohomological automorphic representations is so indirect. Nevertheless, if we take as a model the relation between the geometric Langlands program and the Galois-theoretic parametrization of automorphic representations in the work of V. Lafforgue, as explained in the chapter of Cong Xue and Raskin, one can try to imagine how the action of motivic cohomology predicted by Venkatesh's conjectures might arise from a categorical version of the Langlands correspondence for number fields.

The category on the spectral side would have to be a category of sheaves on families of motives, and it's probably prudent to avoid speculating on what this might mean concretely. At this stage it would make more sense to think about families of realizations. Together with Mazur we have been looking at a model of the spectral category for p-adic realizations in the setting of Iwasawa theory. Let π and ρ_π be as in §8.2, with G a split semisimple group over $\mathbb{Q}$. Let $K_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension, let $\rho_{\pi,\infty}$ denote the restriction of ρ_π to $\mathrm{Gal}(\bar{\mathbb{Q}}/K_\infty)$, and choose a topological generator $\gamma \in \mathrm{Gal}(K_\infty/\mathbb{Q})$. Then under appropriately

restrictive hypotheses – including, for the moment, the assumption that ρ_π is ordinary at p – one can relate the derived deformation ring $\mathcal{R}_S$ of $\bar{\rho}_\pi$ can be identified with the categorical trace of the action of γ on a reasonable symmetric monoidal category of sheaves on the derived moduli stack of formal deformations of $\rho_{\pi,\infty}$.[18] This overlaps with some recent work of Tilouine and Urban on derived deformation theory in the context of Iwasawa theory, as well as with earlier work of Hida when $G = \mathrm{GL}(2)$.

maybe in the revision replace this claim with a precise theorem or delete it

It's possible to define Iwasawa-theoretic categories on the automorphic side as well, but the arguments in favor of one model or another are not particularly compelling. Of course defining the categories alone does not bring us any closer to constructing a categorical action that would recover the expected (ungraded) action of Conjecture **??**. Nor does it seem likely that we can "reverse engineer" such a categorical action from known results, such as Theorem 8.4. However, it is possible that a categorical action that recovers Theorem 8.4 at each intermediate stage of the cyclotomic tower $K_\infty/\mathbb{Q}$ is unique (up to appropriate equivalence).

At the time of writing the authors and Mazur are hoping to show that this program is not entirely fanciful by constructing an explicit categorical action when G is a torus.

9. A selection of open problems

added this

We have highlighted some open problems which seem important for future progress.

9.1. Local deformation conditions. The context for this problem is §3 and §5. In §3 we discussed examples of moduli spaces that we know how to derive. However, there are also many interesting examples, arising in connection with interesting concrete problems, for which we do not know any good derived enhancement, and for which it is unclear to authors whether such an enhancement should even exist. The examples of Galois deformation functors *with local conditions* is particularly interesting, since it would presumably have consequences for automorphy lifting.

In practice, when implementing modularity lifting arguments one *does* want to impose more general local conditions on Galois deformation functors, as explained for example in the articles of [CARAIANI-SHIN] in these proceedings. This is done by constructing Galois deformation functors for local fields with local conditions, and it turns out that these can often fail to be LCI; see [**17**, Theorem 3.19] and [**74**] for some "$\ell = p$" examples, and [**11**] for $\ell \neq p$ examples. This is an interesting phenomenon to understand, because Kisin's modification of the Taylor-Wiles method "reduces" automorphy lifting to an understanding of *local* deformation rings.

We do not know if these local deformation functors should actually be subject to the hidden smoothness philosophy, since these are typically not "honest" moduli functors in the sense that their moduli descriptions are not known "concretely". Rather, they are typically constructed from "honest" moduli spaces by processes such as formation of Zariski closure or scheme-theoretic image, which do not admit good derived versions.

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[18]It's likely that some of the restrictive hypotheses, including perhaps the ordinary hypothesis, can be relaxed if we work with characteristic zero coefficients rather than integrally over $\mathbb{Z}_p$.

For example, the local deformation functors at p are often constructed by the following sort of process (which to our knowledge was first employed in [19]): define a subset of $\overline{\mathbf{Q}}_p$-points of the unrestricted deformation functor using p-adic Hodge theory, and take their Zariski closure. (The point here is that the p-adic Hodge theory is understood most generally in the context of representations with characteristic zero coefficients.) Then the characteristic zero points admit some moduli theoretic description in terms of rational Galois representations with p-adic Hodge theoretic conditions, but it is a priori unclear how to interpret the $\mathbf{Z}_p$ or $\mathbf{F}_p$-points concretely in terms of Galois representations.

There are some cases where the theory is well-behaved with integral coefficients, for example in the Fontaine-Laffaille range. In this case, there is a full moduli-theoretic description of the local deformation functor, which even makes clear that it is smooth.

In general, the local deformation functors at p are constructed by Kisin [50], but their construction again involves processes such as formation of scheme-theoretic image or Zariski closure from the generic fiber, which we do not know how to perform in a derived way. This is why we do not know natural ways to derive the local deformation functors. However, in the past few years there have been advances in our understanding of integral p-adic Hodge theory due to Bhatt-Morrow-Scholze and Bhatt-Scholze. In particular, Bhatt-Scholze have established in [7] a new interpretation of lattices in crystalline Galois representations, in terms of prismatic F-crystals. While this itself does not suffice to give a notion of crystalline Galois representations with general coefficients, it is a promising step towards such a concept, and therefore towards a moduli-theoretic description of crystalline deformation functors, which could then be derived.

We caution, however, that even if this is all possible, it is not quite clear at present what applications this would unlock. Global methods, such as the Taylor-Wiles method, are predicated upon the premise that the structure of global deformation rings and automorphic forms can be "simplified" appropriate by adding level structure. However, this simplification occurs completely in the local-to-global aspect, and it is not clear that one can hope for the same type of simplification if local deformation rings are non-classical.

9.2. Differential graded Hecke algebra.
The context of this discussion is §6.1. In §6.1.5, we explained that commutativity of the local derived Hecke algebra $\mathcal{H}(G(K); \Lambda)$ is still unknown. We conjecture that it is true in general, and moreover should be a consequence of "higher commutativity" for the differential graded Hecke algebra $\mathrm{DGHA}(G(K); \Lambda)$. For differential graded algebras, commutativity is a structure rather than a property, and there is in fact a spectrum of structures that measures "how commutative" an algebra is, called $\mathbb{E}_n$-structures. An associative differential graded algebra has an $\mathbb{E}_1$-structure, while $\mathbb{E}_\infty$-structure is the homotopy-coherent analogue of commutative. It is clear from the construction $\mathrm{DGHA}(G(K); \Lambda)$ has an $\mathbb{E}_1$-structure. If this $\mathbb{E}_1$-structure can be promoted to an $\mathbb{E}_2$-structure, then it would follow that $\mathcal{H}(G(K); \Lambda)$ is commutative. We believe that this is the case, and we conjecture moreover that the $\mathbb{E}_1$-structure on $\mathrm{DGHA}(G(K); \Lambda)$ can be promoted to an $\mathbb{E}_3$-structure.

We explain the motivation for this conjecture. Work-in-progress of Feng-Gaitsgory [26] should imply that $\mathcal{H}(G(K); \Lambda)$ is commutative whenever Λ is of characteristic ℓ which is not too large respect to G, if K is a function field. The

strategy of [**26**] is to apply the *categorical trace of Frobenius* to the *modular derived Satake equivalence*, which has been announced by Arinkin-Bezrukavnikov. This latter result is a version of the main result of [**5**] with modular coefficients. It is interesting that this strategy is not some elaboration of a proof of commutativity for the classical Hecke algebra, but rather proceeds from a categorical equivalence. It should also clarify the "meaning" of the derived Hecke algebra in terms of the dual group, but the answer is complicated to state, so we do not formulate it here. It is plausible that the strategy will also work for p-adic local fields K.

Now, we explain why this suggests the possibility of an $\mathbb{E}_3$-structure on $\mathrm{DGHA}(G(K); A)$. Within the above strategy the differential graded Hecke algebra appears as the categorical trace of the derived Satake category. This latter object is expected to admit the structure of a "factorization category" over an algebraic curve. This "factorization" structure is an algebraic analogue of an $\mathbb{E}_3$-structure; in fact there is a precise relation between the two notions but it is rather subtle to state.

9.3. Action of the derived Galois deformation ring. The context for this discussion is §5. There we sketched the main result of [**34**], which is the construction of an action of *the homotopy groups of* a derived Galois deformation ring $\mathcal{R}_{\mathcal{O}_F[1/S]}^{\overline{\rho},\mathrm{crys}}$ on the *homology groups of* the homology of a locally symmetric space (localized at a corresponding maximal ideal of the Hecke algebra).

The above action occurs at the level of classical algebra: a graded commutative ring acts on a graded abelian group. We expect that this action can be refined to an action of the derived Galois deformation ring $\mathcal{R}_{\mathcal{O}_F[1/S]}^{\overline{\rho},\mathrm{crys}}$ on the localization of the homology chains (i.e., a simplicial commutative ring acts on a simplicial abelian group), which induces the previous action after taking homotopy groups.

The construction of this action presents an intriguing challenge. Recall that the Galatius-Venkatesh construction uses Taylor-Wiles patching, which is based on adding level structure *away* from p. Morally they find that the derived Galois deformation ring should become classical when patched, and thus deduce the derived action from the classical action, but this does not literally make sense since there is no patched derived Galois deformation ring. Instead, one could hope to add some kind of infinite level structure at p and hope that the corresponding derived Galois deformation ring is classical. Indeed, the work of Khare-Ronchetti [**49**] suggests that this should happen when ascending the ordinary (Hida) tower, if one considers ordinary deformations. On the other hand, [**49**] also points out that this approach is intimately connected with p-adic transcendence problems which are known to be extremely difficult, such as the Leopoldt Conjecture, so perhaps one should look for some other angle.

Although the construction of this refined action seems like a homotopy-theoretic problem, it would have concrete consequences for classical algebra. In fact, by the considerations sketched in the discussion of the spectral Hecke algebra in Remark 8.2, it would actually imply Venkatesh's reciprocity law [**81**, Theorem 8.5] for the action of the derived Hecke algebra. This is very desirable, since in the current formulation of the Theorem there is no effective way to determine for which primes q it applies.

9.4. Trace Conjecture for Hitchin stacks. The context for this discussion is §4. The following problem is extracted from [**29**, §8.1.1]. Let

$$\mathcal{M} \xleftarrow{\ c_0\ } \mathcal{C} \xrightarrow{\ c_1\ } \mathcal{M} \tag{9.1}$$

be a correspondence of derived stacks over $\mathbf{F}_q$. Define $\mathrm{Sht}_{\mathcal{M}}$ by the derived Cartesian square

$$
\begin{array}{ccc}
\mathrm{Sht}_{\mathcal{M}} & \xrightarrow{\ \Delta'\ } & \mathcal{C} \\
\downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle (\mathrm{Frob}\,\circ c_0, c_1)} \\
\mathcal{M} & \xrightarrow{\ \Delta\ } & \mathcal{M} \times \mathcal{M}
\end{array}
\tag{9.2}
$$

A calculation similar to those in [**31**, §5] establishes:

LEMMA 9.1. *The tangent complex* $\mathbf{T}_{\mathrm{Sht}_{\mathcal{M}}/\mathbf{F}_q}$ *is the restriction of* $\mathbf{T}_{c_1}$*. In particular, if c_1 is quasi-smooth, then* $\mathrm{Sht}_{\mathcal{M}}$ *is quasi-smooth (over* $\mathrm{Spec}\ \mathbf{F}_q$*), of dimension equals to* $d(c_1)$ *(the relative dimension of c_1).*

For an ℓ-adic sheaf $\mathcal{K}$ and $i \in \mathbf{Z}$, we write $\mathcal{K}\langle i\rangle := \mathcal{K}[2i](i)$. For a map of (derived) stacks $\mathcal{X} \to \mathcal{Y}$, we denote $\mathrm{H}_{2i}^{\mathrm{BM}}(\mathcal{X}/\mathcal{Y}) := H^{-2i}(\mathcal{X}, f^!\mathbf{Q}_{\ell,\mathcal{Y}}(i))$. When $f\colon \mathcal{X} \to \mathrm{Spec}\ k$ is the structure map to a field, we abbreviate $\mathrm{H}_{2i}^{\mathrm{BM}}(X) := \mathrm{H}_{2i}^{\mathrm{BM}}(\mathcal{X}/\mathrm{Spec}\ k)$. This is the target of the ℓ-adic cycle class map from $\mathrm{Ch}_i(X)$. If f is quasi-smooth, then we write $d(f) = \chi(\mathbf{T}_f)$ for the Euler characteristic of its tangent complex, and call it the "relative dimension" of f. By [**48**], in this situation there is a relative fundamental class $[f] \in \mathrm{H}_{2d(f)}^{\mathrm{BM}}(\mathcal{X}/\mathcal{Y})$.

Assume that in the above situation c_1 is quasi-smooth, so that $[c_1] \in \mathrm{H}_{2d(c_1)}^{\mathrm{BM}}(\mathcal{C}/\mathcal{M})$ exists. Then Lemma 9.1 implies that $\mathrm{Sht}_{\mathcal{M}}$ is quasi-smooth (over $\mathrm{Spec}\ \mathbf{F}_q$), so the derived fundamental class $[\mathrm{Sht}_{\mathcal{M}}] \in \mathrm{H}_{2d(c_1)}^{\mathrm{BM}}(\mathrm{Sht}_{\mathcal{M}})$ exists.

On the other hand, regarding $[c_1]$ as a map $c_1^*\mathbf{Q}_{\ell,\mathcal{M}} \to c_1^!\mathbf{Q}_{\ell,\mathcal{M}}\langle -d(c_1)\rangle$, we have a sequence of maps

$$(\mathrm{Frob}\circ c_0)^*\mathbf{Q}_{\ell,\mathcal{M}} = \mathbf{Q}_{\ell,\mathcal{M}} = c_1^*\mathbf{Q}_{\ell,\mathcal{M}} \xrightarrow{[c_1]} c_1^!\mathbf{Q}_{\ell,\mathcal{M}}\langle -d(c_1)\rangle,$$

whose composition we call $\mathfrak{c}_{\mathcal{M}}$. This is a *cohomological correspondence* between $\mathbf{Q}_{\ell,\mathcal{M}}$ and $\mathbf{Q}_{\ell,\mathcal{M}}\langle -d(c_1)\rangle$. We will define $\mathrm{Tr}([c_1]) \in \mathrm{H}_{2d(c_1)}^{\mathrm{BM}}(\mathrm{Sht}_{\mathcal{M}})$. Consider (9.2), and abbreviate c for the right vertical map. It is explained in §[**80**, 0.3, 0.4] that

$$\mathcal{RH}om((\mathrm{Frob}\circ c_0)^*\mathbf{Q}_{\ell,\mathcal{M}}, c_1^!\mathbf{Q}_{\ell,\mathcal{M}}) \cong c^!\mathcal{RH}om(\mathrm{pr}_0^*\mathbf{Q}_{\ell,\mathcal{M}}, \mathrm{pr}_1^!\mathbf{Q}_{\ell,\mathcal{M}})$$

$$\cong c^!(\mathbb{D}(\mathbf{Q}_{\ell,\mathcal{M}}) \boxtimes \mathbf{Q}_{\ell,\mathcal{M}}) \tag{9.3}$$

where $\mathbb{D}$ is the Verdier duality functor. The evaluation map $\mathbb{D}(\mathbf{Q}_{\ell,\mathcal{M}})\otimes\mathbf{Q}_{\ell,\mathcal{M}}\langle -d(c_1)\rangle \to \mathbb{D}_{\mathcal{M}}\langle -d(c_1)\rangle$ is adjoint to a map $\mathbb{D}(\mathbf{Q}_{\ell,\mathcal{M}}) \boxtimes \mathbf{Q}_{\ell,\mathcal{M}} \to \Delta_*\mathbb{D}_{\mathrm{Sht}_{\mathcal{M}}}\langle -d(c_1)\rangle$ where $\mathbb{D}_{\mathcal{M}}$ and $\mathbb{D}_{\mathrm{Sht}_{\mathcal{M}}}$ are the respective dualizing complexes. Composing this with (9.3) gives

$$\mathcal{RH}om((\mathrm{Frob}\circ c_0)^*\mathbf{Q}_{\ell,\mathcal{M}}, c_1^!\mathbf{Q}_{\ell,\mathcal{M}}) \to c^!\Delta_*\mathbb{D}_{\mathrm{Sht}_{\mathcal{M}}}\langle -d(c_1)\rangle. \tag{9.4}$$

Finally, using proper base change, we have isomorphisms

$$c^!\Delta_*\mathbb{D}_{\mathrm{Sht}_{\mathcal{M}}}\langle -d(c_1)\rangle \cong \Delta'_*(c')^!\mathbb{D}_{\mathcal{M}}\langle -d(c_1)\rangle \cong \Delta'_*\mathbb{D}_{\mathrm{Sht}_{\mathcal{M}}}\langle -d(c_1)\rangle. \tag{9.5}$$

We may regard $\mathfrak{c}$ as a global section of $\mathcal{RH}om(c_0^*\mathbf{Q}_{\ell,\mathcal{M}}, c_1^!\mathbf{Q}_{\ell,\mathcal{M}}\langle -d(c_1)\rangle)$. Then $\mathrm{Tr}(\alpha) \in H^0(\mathcal{C}, \Delta'_*\mathbb{D}_{\mathrm{Sht}_{\mathcal{M}}}\langle -d(c_1)\rangle) \cong \mathrm{H}_{2d(c_1)}^{\mathrm{BM}}(\mathrm{Sht}_{\mathcal{M}})$ is defined as its image under the composition of (9.4), and (9.5).

At this point, we have two natural classes in $\mathrm{H}^{\mathrm{BM}}_{2d(c_1)}(\mathrm{Sht}_{\mathcal{M}})$. One is the derived fundamental class $[\mathrm{Sht}_{\mathcal{M}}]$, and the other is $\mathrm{Tr}(\mathfrak{c}_{\mathcal{M}})$. We speculate that

$$\mathrm{Tr}(\mathfrak{c}_{\mathcal{M}}) = [\mathrm{Sht}_{\mathcal{M}}] \in \mathrm{H}^{\mathrm{BM}}_{2d(c_1)}(\mathrm{Sht}_{\mathcal{M}}).$$

We have no evidence to believe that it is true in the stated generality. However, for $\mathcal{M}$ a derived Hitchin stack of the type considered in Example 3.15, we feel that the equality is probably necessary for the Modularity Conjecture [**31**, Conjecture 4.12] to be true, so we are confident about this case. Such $\mathcal{M}$ have special properties which could conceivably be necessary for the proof; for example, they are quasi-smooth. This conditions is not relevant for the formulation of the statement, which leads us to the more general speculation. Its proof for the derived Hitchin stacks would have immediate consequences for the Modularity Conjecture.

Appendix A. A crash course on simplicial commutative rings

This appendix provides a primer on simplicial commutative rings, with the goal of working up to the definition of the cotangent complex. Its purpose is to provide more grounding for the readers to whom the "black box" approach of the main text is too vague, and to indicate references for those interested in delving further into the subject. For short introductions we like [**68, 69, 47**]; longer textbooks include [**62**] and [**37**].

In particular, we think it is edifying for typical readers to learn about simplicial commutative rings and model categories, even if their ultimate goal is work in the language of animated rings and ∞-categories. To continue the analogy of Remark 2.1: although one wants to make basis-independent arguments when possible in linear algebra, it is hard to imagine mastering the subject without knowing how to perform calculations with bases and matrices. The theory of simplicial commutative rings plays an analogous role for animated rings.

We caution that there is a discrepancy between the language used in this Appendix and in the main text. The "simplicial commutative rings" referred to in the main text were occupants of an ∞-category, while here we work within the literal categories of simplicial objects.

a case for using "animated" in the main text

Our treatment is extremely far from comprehensive. In fact, we view the sketchiness of our writeup as its main (and perhaps only) feature. Each of the subjects we touch upon is already well-documented in the literature, but simplicial homotopy theory is so vast and technical that there is real risk in getting lost in the details. In keeping with the spirit of the main text, we prefer merely to sketch the skeletal structure of the subject, contenting ourselves with slogans and intuition, and deferring details to our favorite references.

A.1. Why "simplicial"? We will introduce "simplicial sets" as a combinatorial model for topological spaces, and "simplicial commutative rings" as a combinatorial model for topological commutative rings.

Why do we say "simplicial" instead of "topological"? One answer is that the combinatorial nature of simplicial objects makes them better-behaved from a technical perspective – it rules out "pathological" topological spaces. We do not have a good philosophical answer; the reader could consult [**59**] for more musings on this question.

Furthermore, it turns out that the adjective "simplicial" is flexible enough to be applied to any object of any category, and provides a way to do abstract homotopy

theory in many examples of interest (via what is called a "model category struc-
ture"). That is, if C is a category of "widgets", we will be able to define a category
sC of "simplicial widgets".

A.2. The simplex category. The *simplex category* Δ has as its objects the
sets $[n] := \{0, 1, \ldots, n\}$ for $n \geq 0$, and $\mathrm{Hom}_\Delta([n], [m])$ consists of all maps of sets
$f\colon [n] \to [m]$ that do not reverse order, i.e., if $i \leq j$ then $f(i) \leq f(j)$.

We will name specific generator morphisms between $[n]$ and $[n+1]$. For $0 \leq
i \leq n+1$, the *coface map*

$$\delta_i\colon [n] \to [n+1]$$

is the morphism "skipping over i", or more formally

$$\delta_i(j) = \begin{cases} j & j < i, \\ j+1 & j \geq i. \end{cases}$$

For $0 \leq i \leq n$, the *codegeneracy map*

$$\sigma_i\colon [n+1] \to [n]$$

is the morphism "doubling up on i", or more formally

$$\delta_i(j) = \begin{cases} j & j \leq i, \\ j-1 & j \geq i+1. \end{cases}$$

Remark A.1. The sense in which these morphisms "generate" is that any mor-
phism in Δ can as a composition of codegeneracy maps followed by a composition
of coface maps.

A.2.1. *Simplicial sets.* A *simplicial set* is a functor $\underline{X}\colon \Delta^{\mathrm{op}} \to \mathsf{Sets}$. Concretely,
it is specified by a collection of sets $X_n := \underline{X}([n])$, for $n \geq 0$, with maps between
them indexed by the morphisms in Δ^{op}. In particular, we have *face maps*

$$d_i := \underline{X}(\delta_i)\colon X_{n+1} \to X_n \quad \text{for } 0 \leq i \leq n+1,$$

and *degeneracy maps*

$$s_i := \underline{X}(\sigma_i)\colon X_n \to X_{n+1} \quad \text{for } 0 \leq i \leq n.$$

Because of Remark A.1, to specify a simplicial set it it suffices to specify the data
of the $\{X_n\}_{n \geq 0}$ and the d_i and s_i for each n, satisfying the "simplicial identities"
[**37**, I.1]:

- $d_i d_j = d_{j-1} d_i$ for $i < j$.
- $d_i d_i = d_i d_{i+1}$.
- $d_j s_j = d_{j+1} s_j = \mathrm{Id}$.
- $d_i s_j = s_{j-1} d_i$ for $i < j$.
- $d_i s_j = s_j d_{i-1}$ for $i > j+1$.
- $s_i s_j = s_{j+1} s_i$ for $i \leq j$.

Example A.2. Each element $[n] \in \Delta$ induces a representable functor on Δ^{op},
namely $\mathrm{Hom}_\Delta(-, [n])$, and so induces a simplicial set that we shall call Δ^n. In-
tuitively, we think of it as corresponding to the topological space of the standard
n-simplex, denoted $|\Delta^n|$.

Example A.3. The definition of simplicial set is an axiomatization of the structure obtained from a topological space X by setting $\underline{X}[n]$ to be the set of continuous maps from the n-simplex $|\Delta^n|$ to X, i.e. X_n is the set of "n-simplices of X". The face maps d_i are the usual boundary maps. The degeneracy maps account for ways to view lower-dimensional simplices as "degenerate" instances of higher-dimensional simplices; these do not play an important role in the topological theory, but are technically convenient in the simplicial theory. This defines the *singular simplices* functor

$$\mathrm{Sing}\colon \mathsf{Top} \to \mathsf{sSets}.$$

Therefore, for a general simplicial set $\underline{X}$ we will refer to $\underline{X}[n]$ as the "n-simplices of $\underline{X}$".

A.2.2. *Geometric realization.* Given a simplicial set $\underline{X}$, we can build a topological space $X := |\underline{X}|$ from $\underline{X}$ by viewing $\underline{X}$ as a recipe for assembling a simplicial complex. In formulas, the geometric realization is given by

$$|\underline{X}| := \frac{\coprod_n \underline{X}[n] \times |\Delta^n|}{(d_i x, u) \sim (x, \delta_i u), (s_i x, u) \sim (x, \sigma_i u)}.$$

The topological space $|\underline{X}|$ is a CW complex [**62**, §14].

An alternative, concise way to express this is as follows: we declare $|\Delta^n|$ to be the standard n-simplex $\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \colon 0 \le x_i \le 1, \sum x_i = 1\}$ and then we define

$$|\underline{X}| = \varinjlim_{\Delta^n \to \underline{X}} \Delta^n$$

where the indexing category has as its objects the maps $\Delta^n \to \underline{X}$ for varying n, and as its morphisms the maps $\Delta^m \to \Delta^n$ respecting the given maps to $\underline{X}$. Note that the geometric realization of the simplicial set Δ^n agrees with the standard n-simplex $|\Delta^n|$, justifying the notation.

PROPOSITION A.4. *Geometric realization is left adjoint to the singular simplices functor:*

$$\mathrm{Hom}_{\mathsf{Top}}(|\underline{X}|, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(\underline{X}, \mathrm{Sing}\, Y).$$

Remark A.5. These adjoint functors define a "homotopy equivalence" of categories in a suitable sense, which we are not yet equipped to define precisely. See [**63**, §16] for the precise formulation and proof. This justifies the assertion that simplicial sets provide a combinatorial model of topological spaces. The technical statement is that the functors induce a Quillen equivalence of *model categories*, with the usual Quillen model structures on each side. The notion of model category (due to Quillen [**70**]) is an axiomatization of structure for doing abstract homotopy theory in categories. One of the structures prescribed by a model category is how to "resolve" a map by one which has "good" homotopical properties; think of this as a generalization of projective or injective resolutions. The notion of model categories will be discussed again in §A.7, although not really explained in any detail there either.

A.2.3. *Homotopy groups of a simplicial set.* A simplicial set $\underline{X}$ has *homotopy groups* $\pi_i(\underline{X})$, which could be defined as the homotopy groups (in the usual sense) of the geometric realization $|\underline{X}|$. They could also be defined directly within the category of simplicial sets, basically by using a combinatorial model for the sphere

as the simplicial set Δ^n modulo its boundary, but this involves the subtlety of "resolving" $\underline{X}$ by a *Kan complex* (see Remark A.5). We remark that for a topological space Y, $\operatorname{Sing} Y$ is a Kan complex. The point is that in "homotopy-theoretic" categories (such as the category of simplicial sets), objects may not have "enough" maps in or out of them and so must be "resolved" by ones that do.

A.2.4. *Internal Hom.* Given two simplicial sets $\underline{X}$ and $\underline{Y}$, we make a simplicial set $\underline{\operatorname{Map}}(\underline{X}, \underline{Y})$ whose set of 0-simplices is identified with $\operatorname{Hom}_\Delta(\underline{X}, \underline{Y})$. Intuitively, we are trying to put a topological structure on the set of maps from $\underline{X}$ to $\underline{Y}$. This will make the category of simplicial sets into a *simplicial category* – a category enriched over simplicial sets.

We define
$$\underline{\operatorname{Map}}(\underline{X}, \underline{Y})([n]) := \operatorname{Hom}_\Delta(\underline{X} \times \Delta^n, \underline{Y}).$$
Note that the product of presheaves is formed level-wise.

Remark A.6. A simplicial category is a possible model for the notion of ∞-category (although not the main one used in [**54**]). Intuitively, this is a type of category enriched over *spaces*, i.e., where the sets of morphisms are promoted to spaces.

A.3. Simplicial widgets. More generally, if C is any category, then we say that a *simplicial object of* C is a functor $\Delta^{\operatorname{op}} \to \mathsf{C}$. If C is the category of "widgets", then the functor category $\operatorname{Hom}(\Delta^{\operatorname{op}}, \mathsf{C})$ will be called the category of "simplicial widgets", and denoted sC.

Example A.7. The following examples will be of particular interest to us.
- A *simplicial abelian group* is a functor from $\Delta^{\operatorname{op}}$ to the category of abelian groups.
- Given a commutative ring R (for us this always means commutative and with unit), a *simplicial R-module* is a functor from $\Delta^{\operatorname{op}}$ to the category of R-modules.
- A *simplicial commutative ring* is a functor from $\Delta^{\operatorname{op}}$ to the category of commutative rings.
- Given a commutative ring R, a *simplicial R-algebra* is a functor from $\Delta^{\operatorname{op}}$ to the category of R-algebras.

The categories of such objects is, in each case, enriched over sSet. To see this, we first observe that for a simplicial set $\underline{S}$ and a simplicial widget $\underline{X}$, we have a simplicial widget $(\underline{X}^{\otimes \underline{S}})$, whose n-simplices are given by
$$(\underline{X}^{\otimes \underline{S}})[n] = \coprod_{S_n} X_n$$
with the coproduct on the RHS formed in the category of widgets. Then we define
$$\underline{\operatorname{Hom}}_{\mathsf{sC}}(\underline{X}, \underline{Y})[n] := \operatorname{Hom}_{\mathsf{sC}}(\underline{X}^{\otimes \Delta^n}, \underline{Y}).$$

Example A.8. Let $\underline{X}$ be a simplicial set. Then there is a simplicial abelian group $\mathbf{Z}\langle \underline{X} \rangle$, obtained by forming the free abelian group level-wise: define $\mathbf{Z}\langle \underline{X} \rangle[n] := \mathbf{Z}\langle \underline{X}[n] \rangle$ and let the face and degeneracy maps be induced by those of $\underline{X}$.

There is also a simplicial $\mathbf{Z}$-algebra $\mathbf{Z}[\underline{X}]$ obtained by forming the free (polynomial) $\mathbf{Z}$-algebra level-wise.

Remark A.9. It turns out that simplicial groups are *automatically* Kan complexes. Therefore, the homotopy groups of simplicial groups can be calculated as maps from simplicial spheres, in contrast to the situation for general simplicial sets as cautioned in §A.2.3. The same applies for simplicial abelian groups, simplicial commutative rings, etc. because of how the model category structures are defined in each of these cases.

A.4. The Dold-Kan correspondence. Recall that if C is an abelian category, then we can form the category of chain complexes of elements in C. This is essential for example in homological algebra. It turns out that this is closely related to the category sC.

THEOREM A.10 (Dold-Kan). *The category of simplicial R-modules is equivalent to the category of non-negatively graded chain complexes of R-modules.*

We may view Theorem A.10 as explaining that "simplicial" is a generalization of "chain complex" to non-abelian categories (where the notion "chain complex" does not exist).

We will indicate the functors used to define the equivalence of Theorem A.10, starting with the functor from simplicial R-modules to chain complexes of R-modules. A proof may be found in [**37**, §III.2]; we also like the treatment in [**58**], which we follow. Let $\underline{M}$ be a simplicial R-module. First we explain an auxiliary construction called the *Moore complex* of $\underline{M}$. From $\underline{M}$ we form a chain complex M_* by taking $M_n := \underline{M}[n]$, and taking the differential $d_n\colon M_n \to M_{n-1}$ to be the alternating sum of the face maps,

$$\partial := \sum_{i=0}^{n}(-1)^i d_i.$$

Example A.11. The singular chain complex of a topological space X is by definition the Moore complex of the simplicial abelian group $\mathbf{Z}\langle \operatorname{Sing} X\rangle$.

Remark A.12. The homology groups of the Moore complex M_* coincide with the homotopy groups of the underlying simplicial set $\underline{M}$.

Let us analyze the structure on the Moore complex. There are degeneracy maps $s_i\colon M_n \to M_{n+1}$ for $0 \le i \le n$. Topological intuition suggests that "removing" the degenerate simplices should not affect the homology. Define $DM_* \subset M_*$ so that DM_{n+1} is the span of the image of the degeneracy maps $s_0, \ldots, s_n$.

Exercise A.13. Check that DM_* is a complex of M_*, i.e. is preserved by the differential ∂. Furthermore, show that the map $M_* \to (M/DM)_*$ is a quasi-isomorphism.

Basically we want to instead consider the chain complex $(M/DM)_*$. However, it is convenient to use a different normalization of this, which we will call the *normalized Moore complex*. Define NM_n to be the kernel of *all* the face maps d_i for $i < n$ (but not $i = n$). Then $(-1)^n d_n$ defines a differential $NM_n \to NM_{n-1}$.

Exercise A.14. Check that NM_* is a chain complex.

Remark A.15. The sum map

$$NM_n \oplus DM_n \to M_n$$

is an isomorphism. Therefore, NM_* maps isomorphically to $(M/DM)_*$.

The functor from simplicial R-modules to chain complexes of R-modules is $\underline{M} \mapsto NM_*$. A key step in the proof of the Dold-Kan equivalence is to show that

$$\underline{M}[n] \cong \bigoplus_{[n] \twoheadrightarrow [k]} NM_k$$

functorially in $\underline{M}$. This tells us how to define the inverse functor: given a chain complex M_*, we will define a simplicial R-module by

$$\underline{M}[n] := \bigoplus_{[n] \twoheadrightarrow [k]} M_k.$$

The details in checking that this defines an equivalence of categories are left to the reader.

A.5. Simplicial commutative rings. A *simplicial commutative ring* is a functor $\mathcal{R}$ from Δ^{op} to the category of commutative rings. Because we shall work with these a lot, we adopt a slightly more economical notation. Simplical commutative rings will be denoted using calligraphic letters, and we abbreviate $\mathcal{R}_n := \mathcal{R}[n]$. Classical commutative rings will be denoted using Roman letters such as R. So, a simplicial commutative can be specified concretely by a collection of commutative rings $\mathcal{R}_n$, for $n \geq 0$, with maps between them indexed by the morphisms in Δ^{op}, which can be specified by degeneracy maps $d_i \colon \mathcal{R}_{n+1} \to \mathcal{R}_n$ and face maps $s_i \colon \mathcal{R}_n \to \mathcal{R}_{n+1}$ satisfying the simplicial identities. This is an axiomatization of the structure that exists on the singular simplices of a topological commutative ring. The category of commutative rings is denoted CR and the category of simplicial commutative rings is denoted sCR (this is different from the ∞-category of simplicial commutative rings, which was used in the main text and denoted SCR there).

Example A.16. Let R be a commutative ring. Then may define a simplicial commutative ring $\underline{R}$ such that $\underline{R}[n] = R$, and all face and degeneracy maps are the identity map. Intuitively, $\underline{R}$ corresponds to the topological commutative ring which is R equipped with the discrete topology.

A.5.1. *Homotopy groups.* If $\mathcal{R}$ is a simplicial commutative ring, then its homotopy groups $\pi_*(\mathcal{R})$ form a graded-commutative ring. The underlying group of $\pi_*(\mathcal{R})$ coincides with that of the underlying simplicial abelian group of $\mathcal{R}$, and can therefore by computed by the Moore complex

$$\ldots \to \mathcal{R}_n \xrightarrow{\partial_n} \mathcal{R}_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_0} \mathcal{R}_0.$$

Example A.17. For the constant simplicial commutative ring $\underline{R}$, the Moore complex reads

$$\ldots \xrightarrow{\mathrm{Id}} R \xrightarrow{0} R \xrightarrow{\mathrm{Id}} R \xrightarrow{0} R$$

Hence we see $\pi_0(\underline{R}) = R$ and $\pi_i(\underline{R}) = 0$ for $i > 0$, in accordance with the intuition expressed in Example A.16.

Remark A.18. It turns out that the augmentation ideal $\pi_{>0}(\mathcal{R})$ always has a *divided power structure* in $\pi_*(\mathcal{R})$. We will not explain what this means or why it exists, except that for a $\mathbf{Q}$-algebra a divided power structure always exists and is unique, whereas in positive characteristic it is a rather special piece of structure.

A.5.2. *Classical truncation.* The adjunction in §2.3 is induced by an adjunction

(A.1)
$$\mathsf{CR} \underset{R \mapsto \underline{R}}{\overset{\pi_0(\mathcal{R}) \leftarrow\!\shortmid \mathcal{R}}{\rightleftarrows}} \mathsf{sCR}$$

To justify this, we need to argue that

$$\mathrm{Hom}_{\mathsf{sCR}}(\mathcal{R}, \underline{R}) \cong \mathrm{Hom}_{\mathsf{CR}}(\pi_0(\mathcal{R}), R).$$

A morphism $f \in \mathrm{Hom}_{\mathsf{sCR}}(\mathcal{R}, \underline{R})$ amounts to a system of maps $f_n \colon \mathcal{R}_n \to R$ compatible with the face and degeneracy maps. Since all maps on $\underline{R}$ are constant, it must therefore be the case that $f_n(r)$ can be calculated by composing with any of the $n+1$ distinct maps $\mathcal{R}_n \to \mathcal{R}_0$ and then applying $f_0 \colon \mathcal{R}_0 \to R$. For this to be well-defined, we see that f_0 must vanish on the image of $d_1 - d_0$, and thus factor through $\pi_0(\mathcal{R}) \to R$. It remains to show that any such f_0 does induce a well-defined map of simplicial commutative rings, which is completed by the following exercise.

Exercise A.19. Show that all of the $n+1$ maps $\mathcal{R}_n \to \mathcal{R}_0$ induced by the $n+1$ maps $[0] \to [n]$ have the same composition with the quotient $\mathcal{R}_0 \twoheadrightarrow \pi_0(\mathcal{R})$.

A.5.3. *Enrichment over simplicial sets.* Because of its importance, let us explicate the enrichment of simplicial commutative rings over simplicial sets, although it is a special case of the description in §A.2.4. Given a set S and a commutative ring R, we can form the commutative ring $R^{\otimes S}$. Then for a simplicial set $\underline{S}$ and a simplicial commutative $\mathcal{R}$, there is a simplicial commutative ring $\mathcal{R}^{\underline{S}}$ wit $(\mathcal{R}^{\underline{S}})_n := (\mathcal{R}_n)^{\otimes \underline{S}[n]}$. Finally, the simplicial mapping space $\underline{\mathrm{Hom}}_{\mathsf{sCR}}(\mathcal{R}, \mathcal{R}') \in \mathsf{sSet}$ has n-simplices

$$\underline{\mathrm{Hom}}_{\mathsf{sCR}}(\mathcal{R}, \mathcal{R}')[n] := \mathrm{Hom}_{\mathsf{sCR}}(\mathcal{R}^{\otimes \Delta^n}, \mathcal{R}').$$

Exercise A.20. For commutative rings R and R', calculate the homotopy groups of $\underline{\mathrm{Hom}}_{\mathsf{sCR}}(\underline{R}, \underline{R}')$. (Answer: $\pi_0 = \mathrm{Hom}_{\mathsf{CR}}(R, R')$, and π_i vanish for $i > 0$.) This justifies thinking of CR as being embedded fully faithfully, in the sense of *simplicial categories*, in sCR.

A.6. Simplicial resolutions. We now discuss a key construction, in which a morphism of simplicial commutative rings is "resolved" by a process analogous to formation of projective or injective resolutions.

A.6.1. *Derived functors.* Let us recall the paradigm of derived functors in homological algebra, which is probably familiar to you. If F is a right-exact functor on an abelian category C, then we define the "higher derived functors" of F on $M \in \mathsf{C}$ by finding a "free" resolution

$$\ldots \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$$

and then applying F instead to the complex $(\ldots \to P_n \to P_{n-1} \to \ldots \to P_0)$, which we view as a "replacement" for M. We are being deliberately vague about what the adjective "free" should mean.

Example A.21. Let C be the category of R-modules and N an R-module, $F = N \otimes_R -$. The higher derived functors of F are $\mathrm{Tor}_i^R(N, -)$.

Similarly, if F is a left-exact functor, then one performs an analogous process using *injective* resolutions instead.

Example A.22. Let C be the category of R-modules and N an R-module, $F = \mathrm{Hom}_R(N, -)$. The higher derived functors of F are $\mathrm{Ext}^i_R(N, -)$.

Simplicial commutative rings will play the role for commutative rings that chain complexes of R-modules play for R-modules. In particular, we will see that replacing a commutative ring by a type of "free resolution" will allow us to build a theory of "derived functors" on the category of simplicial commutative rings. We will illustrate this in the particular example of the cotangent complex.

Recall that for a smooth morphism $f\colon A \to B$, we have the B-module of Kähler differentials Ω^1_f, which is "well-behaved" if f is smooth. For general f, we will "resolve" B by a "smooth" simplicial A-algebra in order to define the cotangent complex $\mathbf{L}_f$. For abelian categories this meant replacing B by a complex of objects which level-wise had good properties (e.g., "free"), but that does not apply to the category of A-algebras, which is far from abelian. Instead, this theory of homological algebra will be replaced by a theory of *homotopical algebra* developed by Quillen. The correct framework for this is that of Quillen's *model categories*, but this would involve a significant digression to explain completely, so we will adopt an ad hoc presentation and then hint at the general theory later, in §A.7.

Definition A.23. A morphism of simplicial commutative rings $f\colon \mathcal{R} \to \mathcal{R}'$ is a *weak equivalence* if it induces an isomorphism of homotopy groups $\pi_*(f)\colon \pi_*(\mathcal{R}) \xrightarrow{\sim} \pi_*(\mathcal{R}')$.

This notion is an analogue for simplicial commutative rings of the notion of *quasi-isomorphism* for chain complexes. Indeed, we could make an analogous definition for simplicial R-modules, which would transport to precisely the notion of quasi-isomorphism under the Dold-Kan Theorem.

A.6.2. *Free simplicial algebras.* Let $\mathcal{A}$ be a simplicial commutative ring. A *free simplicial $\mathcal{A}$-algebra* $\mathcal{R}$ is a simplicial $\mathcal{A}$-algebra of the following form:

- There is a system of sets X_n such that $\mathcal{R}_n = \mathcal{A}[X_n]$.
- The degeneracy maps send $s_j(X_n) \subset X_{n+1}$. (Note that there is *no* condition on face maps!)

The following Lemma illustrates that a free simplicial $\mathcal{A}$-algebra has good "mapping out" properties, similar to those of free resolutions of modules.

Lemma A.24. *Consider a commutative diagram of simplicial algebras*

$$
\begin{array}{ccc}
\mathcal{A} & \longrightarrow & \mathcal{B} \\
\downarrow & \nearrow & \sim \downarrow \Phi \\
\mathcal{R} & \longrightarrow & \mathcal{C}
\end{array}
$$

were Φ is level-wise surjective and a weak equivalence, and $\mathcal{R}$ is a free simplicial $\mathcal{A}$-algebra. Then there exists a lift $\mathcal{R} \to \mathcal{B}$ making the diagram commutative.

Proof. See [**36**, (5.4)]. For the more general statements in the framework of model categories, see [**23**, §4]. □

Proposition A.25 (Existence of resolutions). *Let $f\colon A \to B$ be a map of (classical) commutative rings. Then there exists a free simplicial A-algebra $\mathcal{B}$ and*

a diagram

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow & \downarrow{\scriptstyle\sim} \\ A & \xrightarrow{\ f\ } & B \end{array}$$

Furthermore, we can even arrange that the formation of this diagram is functorial in B.

PROOF SKETCH. We will give a "canonical" resolution, which is functorial in B. Note that for any A-algebra R, we have a map

(A.2) $$A[R] \to R$$

sending $[r]$ to r. Now, from the map $A[B] \to B$, we get two maps $A[A[B]] \to A[B]$:

- Apply (A.2) with $R = B$ to get a map $A[B] \to B$, and then take $A[-]$. For example, this map sends $[a[b]] \mapsto [ab]$.
- Apply (A.2) with $R = A[B]$. For example, this maps sends $[a[b]] \mapsto a[b]$.

We also have a map $A[B] \to A[A[B]]$ sending $[b]$ to $[[b]]$, which is a section of both maps above.

Contemplating the combinatorics of the situation, one gets a simplicial object

(A.3) $$\ldots A[A[B]] \rightrightarrows A[B]$$

where the maps defined previously are the first face and degeneracy maps. Clearly this resolution is level-wise a polynomial algebra over A, and one can check that it is free. Why is it a resolution of B?

This comes from an "extra degeneracy" argument, which is part of a more general pattern. If we have a simplicial set $\underline{X}$

$$\ldots X_1 \rightrightarrows X_0$$

with an augmentation $d_0 \colon X_0 \to X_{-1}$ and "extra degeneracies" $s_{-1} \colon X_{n-1} \to X_n$ satisfying the extensions of the simplicial identities, then the map $\underline{X} \to X_{-1}$ (regarded as a constant simplicial set) is a weak equivalence. Intuitively, $s_{-1} \colon X_{-1} \to X_0$ specifies a basepoint in each connected component, and for $n \geq 1$ the extra degeneracy $s_{-1} \colon X_{n-1} \to X_n$ specifies the homotopy from an $(n-1)$-simplex to the n-simplex spanned by it and the basepoint of the connected component.

Going back to (A.3), there are extra degeneracies $B \to A[B]$ sending $b \mapsto [b]$ and $s_{-1} \colon A[B] \mapsto A[A[B]]$ sending $a[b] \mapsto [a[b]]$, etc. This can be used to show that the map from (A.3) to B is a weak equivalence.

$\square$

Remark A.26. The construction of the proof is a special case of a more general one, and we give the general statement because we feel that it elucidates the situation. Let T be a *monad* on a category C, meaning the data of an endofunctor $T \colon \mathsf{C} \to \mathsf{C}$ plus natural transformations $\eta \colon \mathrm{Id}_\mathsf{C} \to T$ and $\mu \colon T \circ T \to T$ satisfying the coherence conditions [**56**, §VI]. The example of interest is: C is the category of sets, and T is the functor taking a set S to the underlying set of $A[S]$. Then the category of algebras over T is equivalent to the category of A-algebras.

An *algebra* over the monad T consists of the data of $x \in \mathsf{C}$ plus a morphism $a \colon Tx \to x$ satisfying coherence conditions. Given an algebra x over T, one can form a simplicial set called the *bar construction* $B(T, x)_\bullet$, which has $B(T, x)_n = T^{\circ(n+1)}x$. By formal combinatorial analysis, the bar construction is equipped with

extra degeneracies. The augmentation $B(T, x)_\bullet \to x$ will therefore be a weak equivalence in great generality.

A.6.3. *Uniqueness.* Let us discuss "how unique" these resolutions are. In the classical situation of homological algebra, for example, projective resolutions are unique up to homotopy. There is a sense in which any free simplicial resolution is unique up to homotopy. The best language for formulating such statements is model category theory, but we can say what it specializes to in this situation. There is a product morphism $\mu\colon A[X] \otimes_A A[X] \to A[X]$, where $A[X] \otimes_A A[X]$ is formed by taking tensor products levelwise. Let us call $A[X_0, X_1] := A[X] \otimes_A A[X]$. This represents the "homotopy coproduct"; in particular, given $f, g\colon A[X] \to B$ we get $f \otimes g\colon A[X_0, X_1] \to B$.

Let $A[X_0, X_1, Y]$ be a simplicial resolution of $A[X]$ over $A[X_0, X_1]$. Then we say that f and g are *homotopic* if $f \otimes g$ extends to a map

$$
\begin{array}{ccc}
A[X_0, X_1] & \longrightarrow & A[X_0, X_1, Y] \\
 & & \downarrow \\
 & \overset{f \otimes g}{\searrow} & B
\end{array}
$$

Lemma A.24 may be used to show that the lifts are unique up to homotopy, and also that any two simplicial resolutions are homotopic.

A.7. A vista of model categories. Let us hint at the more general framework underlying these procedures, which is Quillen's theory of *model categories*. Our discussion will be very brief; a favorite introductory reference to the subject is [**23**].

The purpose of a model category is to incorporate homotopy theory into category theory. The basic idea is that in practice, one often wants to introduce a notion of maps that are "weak equivalences", but not necessarily isomorphisms. For example, in the category of topological spaces there is the usual notion of homotopy equivalences, and in the category of chain complexes there is the notion of quasi-isomorphism. A model category is a framework to do homotopy-theoretic constructions in such a situation.

More formally, a model category is a category equipped with several distinguished classes of morphisms, satisfying various properties. The most important are the *weak equivalences*, which designate maps that are "homotopy equivalences". The weak equivalences are morphisms that we want to "invert". Familiar examples are:

- In the category of chain complexes of R-modules, weak equivalences should be the quasi-isomorphisms. Inverting them leads to the *derived category of R-modules*.
- In the category of simplicial commutative rings, weak equivalences should be as defined in Definition A.23.

The other data in a model category are classes of morphisms called the *cofibrations* and *fibrations*, which are roughly the generalizations of projective and injective resolutions. There are various axioms placed on these classes of morphisms: for example, they are preserved by compositions and retracts. An object is called *fibrant* if its map to the terminal object is a fibration, and an object is called *cofibrant* if its map from the initial object is a cofibration.

Example A.27. In the standard model structure on simplicial sets, the *Kan complexes* referred to in §A.2.3 are the fibrant objects. Kan complexes are characterized by lifting conditions possessed by the singular simplicial set of a topological space; in particular, $\mathrm{Sing}\, Y$ is a Kan complex for any topological space Y. This is the reason why, when defining homotopy groups of simplicial sets as maps from a sphere, it was necessary to replace the target by a Kan complex.

On the other hand, in the standard model category structure on topological space the fibrations are Serre fibrations, and in particular all topological spaces are fibrant. This is the reason why, when defining homotopy groups of a topological space, it was *not* necessary to replace the target.

The axioms of a model category imply that every object can admits a weak equivalence from a cofibrant object, or a weak equivalence to a fibrant object. We think of such a map as a "resolution" by a cofibrant object or a fibrant object. As we know, derived functors on abelian categories are defined using such resolutions. A model category structure provides a more general notion of "resolution" in categories which are not necessarily abelian, and therefore allows to construct "non-abelian derived functors".

Our particular constructions with simplicial commutative rings are examples of general constructions with model categories. In particular, our notion of "free simplicial $\mathcal{A}$-algebra $\mathcal{R}$" means that $\mathcal{R}$ is a cofibrant $\mathcal{A}$-algebra in the standard model structure; meanwhile, all simplicial commutative rings are fibrant.

A.8. The cotangent complex. We may now define the *cotangent complex* of a morphism $f\colon A \to B$, as invented by Quillen and developed by Illusie in [**45, 46**]. Let $\mathcal{B} \xrightarrow{\sim} B$ be a simplicial resolution of B as a free simplicial A-algebra. Then let $\mathbf{L}_{\mathcal{B}/A}$ be the simplicial $\mathcal{B}$-module obtained by forming Kähler differentials level-wise: $\mathbf{L}_{\mathcal{B}/A}[n] := \Omega^1_{\mathcal{B}_n/A}$. Finally we define the *cotangent complex* of f to be

$$\mathbf{L}_f := \mathbf{L}_{B/A} := \mathbf{L}_{\mathcal{B}/A} \otimes_{\mathcal{B}} B.$$

Exercise A.28. Show that if $\mathcal{B}$ and $\mathcal{B}'$ are two free simplicial resolutions of B, then $\mathbf{L}_{\mathcal{B}/A} \otimes_{\mathcal{B}} B$ is homotopic to $\mathbf{L}_{\mathcal{B}'/A} \otimes_{\mathcal{B}} B$. Therefore, $\mathbf{L}_{B/A}$ is well-defined up to homotopy. In particular, it gives a well-defined object of the derived category of B-modules.

Remark A.29. Even better, if $\mathcal{A}$ is an animated ring, and $\mathcal{B}$ is an animated $\mathcal{A}$-algebra, then $\mathbf{L}_{\mathcal{B}/\mathcal{A}}$ can be constructed as an animated $\mathcal{B}$-module – see [**53**, §3.2].

We expect $\mathbf{L}_{B/A}$ to have reasonable finiteness properties when the morphism f has good finiteness properties. However, the canonical resolutions used to build $\mathcal{B}$ in the proof of Proposition A.25 are extremely large. Therefore, we would like to know that there are "smaller" resolutions. We will examine this question next.

A.9. Economical resolutions. We will show the existence of resolutions with "good" finiteness properties, following [**47**, §4].

PROPOSITION A.30. *Let A be a noetherian commutative ring and $f\colon A \to B$ a finite type morphism. Then there exists a simplicial A-algebra resolution of B by a free simplicial A-algebra $A[X]$ as in §A.6.2, with each X_n finite.*

COROLLARY A.31. *Let A be a noetherian commutative ring and $f\colon A \to B$ a finite type morphism. Then there exists a representative of $\mathbf{L}_{B/A}$ by a complex of finite free A-modules.*

Remark A.32. Typically $\mathbf{L}_{B/A}$ cannot be represented by a perfect complex; that is, it will typically have homology groups in infinitely many degrees. For example, if A is a field and $\mathbf{L}_{B/A}$ has finite Tor-amplitude, then in fact $\mathbf{L}_{B/A}$ has Tor-amplitude at most 2, and B is LCI over A. This is a result of Avramov, resolving a Conjecture of Quillen; see [**47**, §8.16] for discussion and references.

As motivation, we recall how to construct "efficient" resolutions of a finite type A-module M by a complex of free A-modules. We can build a sequence of complexes of finite free A-modules $\{F_i\}$ whose homology approximates M in degrees up to i. For F_0, we pick any surjection from a free module $A^n \twoheadrightarrow M$. Then we pick generators for $I := \ker(F_0 \twoheadrightarrow M)$, which induces a map $A^m \xrightarrow{\partial} A^n$ with image I, whose map to M induces an isomorphism on H_0. So we may take $F_0 = [A^m \xrightarrow{\partial} A^n]$. We then inductive build F_i from F_{i-1} by picking representatives in F_{i-1} for generators of $H_i(F_{i-1})$, and then adding free summands in degree $i+2$ that bound these generators.

To perform an analogous construction for simplicial commutative rings, what we want is a way to "kill cycles" in a simplicial commutative ring. This is accomplished by the following Lemma.

LEMMA A.33. *Given a simplicial commutative ring $\mathcal{A}$, and a homotopy class $[z] \in \pi_i(\mathcal{A})$, there exists a free simplicial $\mathcal{A}$-algebra $\mathcal{A}'$ such that:*

- *$\mathcal{A}'_n$ is a free $\mathcal{A}_n$-algebra on finitely many generators for each n, and $\mathcal{A}'_n = \mathcal{A}_n$ for $n \leq i$.*
- *The map $\mathcal{A} \to \mathcal{A}'$ induce an isomorphism on π_n for $n < i$, and for $n = i$ an exact sequence*

$$0 \to \mathcal{A}_i([z]) \to \pi_i(\mathcal{A}) \to \pi_i(\mathcal{A}') \to 0$$

Indeed, supposing Lemma A.33 is true, we can make a resolution as in Proposition A.30 by picking a surjection $A[x_1, \ldots, x_d] \twoheadrightarrow B$. Then, by repeatedly applying Lemma A.33, we may build a sequence of free simplicial A-algebras with the "correct" finiteness properties and homotopy groups in all finite degrees. Taking their colimit gives the desired resolution.

Now to prove Lemma A.33, we want to "adjoin" a variable whose boundary is $[z]$. However, because of the simplicial identities, the process of "adjoining" variables is necessarily quite complicated. For this we will imitate what happens to the singular simplices when attaching cells to a topological space X to kill a homology class in degree $i - 1$. If we attach a cell in degree i, then this creates additional degenerate simplices in all higher dimensions. For this reason, the construction is considerably more complicated.

PROOF OF LEMMA A.33. We define

$$X_n := \{x_t \mid t\colon [n] \twoheadrightarrow [i+1] \in \Delta\}$$

and we take $\mathcal{A}'_n = \mathcal{A}_n[X_n]$, with faces and degeneracies defined as follows.

- We have $s_j(x_t) = x_{t \circ \sigma_j}$.
- Note that $X_{i+1} = \{x_{\mathrm{Id}}\}$. We set $d_0(x_{\mathrm{Id}}) = z$ and $d_j(x_{\mathrm{Id}}) = 0$ for $j > 0$. For $n > i+1$, we define $d_j(x_t) = x_{t \circ \delta_j}$ if $t \circ \delta_j$ is surjective, and otherwise it factors through a face map $d_{j'}\colon [i] \to [i+1]$ so we define it to make

the following diagram commute

$$
\begin{array}{ccc}
[n] & \xrightarrow{\ t\ } & [i+1] \\
{\scriptstyle \delta_j}\uparrow & & {\scriptstyle \delta_{j'}}\uparrow \\
[n-1] & \longrightarrow & [i]
\end{array}
$$

Exercise A.34. Check that this $\mathcal{A}'_n$ is a simplicial commutative ring satisfying the conclusions of Lemma A.33.

$\square$

Remark A.35. A more highbrow way to phrase this construction is as follows (following [**60**]). For a simplicial A-algebra $\mathcal{A}$, a class in $\pi_i(\mathcal{A})$ is represented by a map of simplicial sets $\partial\Delta^{i+1} \to \mathcal{A}$ (here we are using Remark A.9, that $\mathcal{A}$ is fibrant). This induces a map of simplicial commutative rings $A[\partial\Delta^{i+1}] \to \mathcal{A}$, where $A[\partial\Delta^{i+1}]$ is the free simplicial A-algebra on the simplicial set $\partial\Delta^{i+1}$. We may then form

$$
\mathcal{A}' := \mathcal{A} \otimes_{A[\partial\Delta^{i+1}]} A[\Delta^{i+1}]
$$

and we claim that it satisfies the conclusions of Lemma A.33. The first bullet point is satisfied by inspection. To compute the effect on homotopy groups, we note that $\pi_*(A) \xrightarrow{\sim} \pi_*(A[\Delta^{i+1}])$ because Δ^{i+1} is contractible, and the lowest positive-degree homotopy group of $A[\partial\Delta^{i+1}]$ is in degree i and maps to $[z]$ in $\pi_i(A)$ by construction. Then we conclude using the Tor spectral sequence [**69**, (5.2)]

$$
\mathrm{Tor}_p^{\pi_*(A[\partial\Delta^{i+1}])}(\pi_*(\mathcal{A}), \pi_*(A[\Delta^{i+1}]))_q \implies \pi_{p+q}(\mathcal{A} \otimes_{A[\partial\Delta^{i+1}]} A[\Delta^{i+1}]).
$$

A.10. Derived tensor products. Let $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{B}'$ be two maps of simplicial commutative rings. Let $\mathcal{A} \hookrightarrow \mathcal{P} \xrightarrow{\sim} \mathcal{B}$ be a free resolution of $\mathcal{B}$ as an $\mathcal{A}$-algebra and $\mathcal{A} \hookrightarrow \mathcal{P}' \xrightarrow{\sim} \mathcal{B}'$ be a free resolution of $\mathcal{B}'$ as an $\mathcal{A}$-algebra. The *derived tensor product* "$\mathcal{B} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{B}'$" is represented by the simplicial commutative ring $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}'$, with

$$
(\mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}')_n = \mathcal{P}_n \otimes_{\mathcal{A}_n} \mathcal{P}'_n.
$$

Although this is not unique since it depends on a choice of resolutions, it is unique up to weak equivalence. In fact, it suffices to resolve only one of the terms, e.g., the simplicial commutative $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{B}'$ also represents the derived tensor product. Indeed, there is an evident map

$$
\mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}' \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{B}'
$$

and this is a weak equivalence of free $\mathcal{B}'$-algebras.

If $\mathcal{A} \xrightarrow{\sim} A$ is a classical classical commutative ring and $\mathcal{P}$ is a free simplicial A-algebra, then the underlying simplicial A-module of $\mathcal{P}$ associates under the Dold-Kan correspondence to a complex of free A-modules. In particular, the Dold-Kan correspondence takes free simplicial resolutions to the familiar notion of free resolutions of chain complexes. In particular, we see that if $\mathcal{A} \xrightarrow{\sim} A$, $\mathcal{B} \xrightarrow{\sim} B$, and $\mathcal{B}' \xrightarrow{\sim} B'$ are all classical, then we have

$$
\pi_i(\mathcal{B} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{B}') \cong \mathrm{Tor}_i^A(B, B').
$$

LEMMA A.36. *Let $A \to B$ and $A \to B'$ be maps of classical commutative rings, such that $\mathrm{Tor}_A^i(B, B') = 0$ for all $i > 0$. Then $B \overset{\mathbf{L}}{\otimes}_A B' \to \pi_0(B \overset{\mathbf{L}}{\otimes}_A B') \cong B \otimes_A B'$ is a weak equivalence.*

A.11. Properties of the cotangent complex. Now we establish some properties of the cotangent complex which are familiar for the module of Kähler differentials, at least in the smooth case. As a reference we recommend [**68**].

PROPOSITION A.37. *If B, B' are A-algebras with $\mathrm{Tor}_A^i(B, B') = 0$ for $i > 0$, then*

(i) $\mathbf{L}_{B \otimes_A B'/B'} \cong \mathbf{L}_{B/A} \otimes_A B'$.

(ii) $\mathbf{L}_{B \otimes_A B'/A} \cong (\mathbf{L}_{B/A} \otimes_A B') \oplus (B \otimes_A \mathbf{L}_{B'/A})$.

Remark A.38. It is true in general that

(i) $\mathbf{L}_{\mathcal{B} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{B}'/\mathcal{B}'} \cong \mathbf{L}_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B}'$.

(ii) $\mathbf{L}_{\mathcal{B} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{B}'/\mathcal{A}} \cong (\mathbf{L}_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B}') \oplus (\mathcal{B} \otimes_{\mathcal{A}} \mathbf{L}_{\mathcal{B}'/\mathcal{A}})$.

The Tor-vanishing assumption is used to ensure that $B \overset{\mathbf{L}}{\otimes}_A B' \xrightarrow{\sim} B \otimes_A B'$.
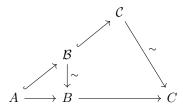
PROOF. Both arguments are completely formal from the standard properties of formation of Kähler differentials with respect to tensor products, using the observation that:

- If $\mathcal{P}$ is a free simplicial $\mathcal{A}$-algebra, and $\mathcal{B}$ is any simplicial $\mathcal{A}$-algebra, then $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{B}$ is a free simplicial $\mathcal{B}$-algebra.
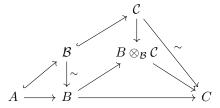
$\square$

PROPOSITION A.39. *If $A \to B \to C$ is a composition of morphisms, then there is an exact triangle in the derived category of C-modules:*

$$\mathbf{L}_{B/A} \otimes_B C \to \mathbf{L}_{C/A} \to \mathbf{L}_{C/B}$$

PROOF. Choose a free resolution $A \hookrightarrow \mathcal{B} \xrightarrow{\sim} B$. This gives C the structure of a $\mathcal{B}$-algebra, so we may then choose a free resolution $\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{\sim} C$. This gives a diagram

Then we form the tensor product

We have an exact triangle

$$\mathbf{L}_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \to \mathbf{L}_{\mathcal{C}/\mathcal{A}} \to \mathbf{L}_{\mathcal{C}/\mathcal{B}}.$$

Applying $- \otimes_{\mathcal{C}} C$, we get an exact triangle. Let us verify that the terms are as claimed.

- $\mathbf{L}_{\mathcal{B}/A} \otimes_{\mathcal{B}} \mathcal{C} \otimes_{\mathcal{C}} C \cong \mathbf{L}_{\mathcal{B}/A} \otimes_{\mathcal{B}} C \cong \mathbf{L}_{B/A} \otimes_B C$.
- $\mathbf{L}_{\mathcal{C}/A} \otimes_{\mathcal{C}} C \cong \mathbf{L}_{C/A}$.
- Let's abbreviate $C' := B \otimes_{\mathcal{B}} \mathcal{C}$. We have $\mathbf{L}_{\mathcal{C}/\mathcal{B}} \otimes_{\mathcal{B}} B \cong \mathbf{L}_{B \otimes_{\mathcal{B}} \mathcal{C}/B}$. Since this a free resolution of C as a B-algebra, tensoring with $\mathcal{C}$ gives $\mathbf{L}_{C/A}$.

$\square$

COROLLARY A.40 (Cotangent complex of localizations).

(i) If S is a multiplicative system in A, then $\mathbf{L}_{S^{-1}A/A} \cong 0$.
(ii) If S is a multiplicative system in A and T is a multiplicative system in B containing the image of S, then
$$\mathbf{L}_{T^{-1}B/S^{-1}A} \cong \mathbf{L}_{B/A} \otimes_B T^{-1}B.$$

PROOF. (i) Multiplication induces $S^{-1}A \otimes_A S^{-1}A \xrightarrow{\sim} S^{-1}A$. Putting this into Proposition A.37(ii), we get
$$\mathbf{L}_{S^{-1}A/A} \cong \mathbf{L}_{S^{-1}A \otimes_A S^{-1}A/A} \cong \mathbf{L}_{S^{-1}A/A} \oplus \mathbf{L}_{S^{-1}A/A}.$$

This forces $\mathbf{L}_{S^{-1}A/A} \cong 0$.

(ii) By Proposition A.37(i) We have $\mathbf{L}_{S^{-1}B/S^{-1}A} \cong \mathbf{L}_{B/A} \otimes_B S^{-1}B$. Apply Proposition A.39 to get an exact triangle
$$\mathbf{L}_{S^{-1}B/S^{-1}A} \otimes_{S^{-1}B} T^{-1}B \to \mathbf{L}_{T^{-1}B/S^{-1}A} \to \mathbf{L}_{T^{-1}B/S^{-1}B}.$$

The rightmost term vanishes according to (i), from which the result follows.

$\square$

A.12. Some computations of the cotangent complex. Now we will identify the cotangent complex in some special cases. Simplicial resolutions are almost always too unwieldy to compute with by hand, so we will instead need to make clever use of the formal properties explained above.

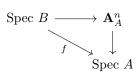PROPOSITION A.41. *Let A be a noetherian commutative ring and $f\colon A \to B$ a finite type morphism. Then*

(i) *f is étale if and only if $\mathbf{L}_f \cong 0$.*
(ii) *f is smooth if and only if $\mathbf{L}_f \cong \Omega_f^1$.*

PROOF. We will only prove the forward directions for now. The converses will be established after we have developed the connection between $\mathbf{L}_f$ and deformation theory.

Recall that $f\colon A \to B$ is weakly étale if and only if B is flat over A and $\Delta\colon \operatorname{Spec} B \to \operatorname{Spec} B \otimes_A B$ is an open immersion. If f is a finite type morphism of noetherian commutative rings, as in our situation, then weakly étale is equivalent to étale.

Suppose f is étale. Then the multiplication map $B \otimes_A B \to B$ is a localization, so Corollary A.40 implies that $\mathbf{L}_{B/B \otimes_A B} \cong 0$. By Proposition A.37 we have $\mathbf{L}_{B \otimes_A B/B} \cong \mathbf{L}_{B/A} \otimes_A B$. Then applying Proposition A.39 to $B \xrightarrow{\operatorname{Id}_B \otimes 1_B} B \otimes_A B \to B$, we find that $L_{B/A} \cong 0$.

Next suppose f is smooth. The local structure theorem for smooth maps says that locally on Spec B, f can be factored as an étale map over an affine space:

$$\begin{array}{ccc} \text{Spec } B & \longrightarrow & \mathbf{A}^n_A \\ & {\scriptstyle f} \searrow & \downarrow \\ & & \text{Spec } A \end{array}$$

At the level rings, this means we have a factorization $A \to P \to B$ where P is a polynomial ring over A and $P \to B$ is étale. Then the morphism of constant simplicial A-algebras $A \to P$ is already free, so $\mathbf{L}_{P/A} \cong \Omega_{P/A}$. From Proposition A.39 we have an exact triangle $\mathbf{L}_{P/A} \otimes_P B \to \mathbf{L}_{B/A} \to \mathbf{L}_{B/P}$ where $\mathbf{L}_{P/A} \otimes_P B \cong \Omega_{B/A}$ and $\mathbf{L}_{B/P} = 0$, which completes the proof.

$\square$

Recall that an ideal $I \subset A$ is *regular* if it has a sequence of generators $r_1, \ldots, r_n$ such that each r_i is a non-zerodivisor modulo $A/(r_1, \ldots, r_{i-1})$.

PROPOSITION A.42. *If $B = A/I$ and I is regular then*

$$\mathbf{L}_{B/A} \xrightarrow{\sim} I/I^2[1].$$

PROOF. Let us first treat the case $A = \mathbf{Z}[x]$ and $B = \mathbf{Z}$, $I = (x)$. In this case we have a section $\mathbf{Z} \to \mathbf{Z}[x]$. Apply Proposition A.39 to the sequence $\mathbf{Z} \to \mathbf{Z}[x] \to \mathbf{Z}$ to get an exact triangle

$$\mathbf{L}_{\mathbf{Z}[x]/Z} \otimes_{\mathbf{Z}[x]} \mathbf{Z} \to \mathbf{L}_{\mathbf{Z}/\mathbf{Z}} \to \mathbf{L}_{\mathbf{Z}/\mathbf{Z}[x]}.$$

This shows that $\mathbf{L}_{\mathbf{Z}/\mathbf{Z}[x]} \cong \mathbf{L}_{\mathbf{Z}[x]/\mathbf{z}}[1] \otimes_{\mathbf{Z}[x]} \mathbf{Z} \cong (x)/(x^2)$.

Now we tackle the general case. By induction, it suffices to handle the case where $I = (r)$ is principal. Note that a choice of r induces a map $\mathbf{Z}[x] \to A$ sending $x \mapsto r$, and this map induces $A/(r) \cong \mathbf{Z} \otimes_{\mathbf{Z}[x]} R$. We claim that r is a non zerodivisor if and only if $\operatorname{Tor}^i_{\mathbf{Z}[x]}(\mathbf{Z}, A) = 0$ for all $i > 0$, or in other words if and only if $\mathbf{Z} \overset{\mathbf{L}}{\otimes}_{\mathbf{Z}[x]} A \xrightarrow{\sim} A/(r)$. The claim is an elementary computation in homological algebra, using the resolution $\mathbf{Z}[x] \xrightarrow{x} \mathbf{Z}[x]$ of $\mathbf{Z}$ over $\mathbf{Z}[x]$. Then by Proposition A.37, we have

$$\mathbf{L}_{B/A} \cong \mathbf{L}_{\mathbf{Z}/\mathbf{Z}[x]} \otimes_{\mathbf{Z}[x]} A \cong (x)/(x^2) \otimes_{\mathbf{Z}[x]} A \cong (r)/(r^2).$$

$\square$

A.13. Globalization. The theory we have discussed can be globalized to schemes, and then to stacks. For example, let $Y \to X$ be a morphism of schemes. Then we define $\mathbf{L}_{Y/X}$ by finding a free replacement of $\mathcal{O}_Y$ as a simplicial $\mathcal{O}_X$-algebra, and then forming Ω-levelwise, etc. Traditionally, it was important to develop the theory this way instead of trying to "glue" the cotangent complexes from affine open subspaces, because of inadequate categorical technology: on the one hand the cotangent complex is not unique at the level of chain complexes, and on the other hand the derived category is not suitable for gluing.

However, in modern language one can indeed glue the cotangent complex affine-locally in the desired way, viewing the cotangent complex as an animated module and using that the categories of animated quasicoherent sheaves satisfy Zariski descent.

A.14. André-Quillen homology. Let $A \to B$ be a map of commutative rings. We have defined the cotangent complex $\mathbf{L}_{B/A}$ as a simplicial B-module up to homotopy.

Definition A.43. Let M be a B-module. We define the *André-Quillen homology* groups

$$D_i(B/A; M) := H_i(\mathbf{L}_{B/A} \otimes_B M)$$

and the *André-Quillen cohomology* groups

$$D^i(B/A; M) := H^i(\mathrm{Hom}_B(\mathbf{L}_{B/A}, M)).$$

As a consequence of the properties of the cotangent complex, we have various properties of André-Quillen (co)homology:

- Let B be an A-algebra and $M' \to M \to M''$ be a short exact sequence of B-modules. Then there is a long exact sequence

$$D^i(B/A; M') \to D^i(B/A; M) \to D^i(B/A; M'') \to D^{i+1}(B/A; M') \to \dots$$

- Let B, B' be A-algebras such that $\mathrm{Tor}_i^A(B, B') = 0$ for $i > 0$. Let M be a $B \otimes_A B'$-module, which we may regard as a B-module via restriction. Then

$$D^i(B \otimes_A B'/B'; M) \cong D^i(B/A; M)$$

  and

$$D^i(B \otimes_A B'/A; M) \cong D^i(B/A; M) \oplus D^i(B'/A; M).$$

- If $A \to B \to C$ is a sequence of ring homomorphisms and M is a C-module, then we get a long exact sequence

$$D^i(C/B; M) \to D^i(C/A; M) \to D^i(B/A; M) \to D^{i+1}(C/B; M) \to \dots$$

A.15. André-Quillen cohomology and commutative algebra extensions. Let $f \colon X \to S$ be a scheme and $S \hookrightarrow S'$ a square-zero thickening.

A.15.1. *Deformation theory setup.* We consider deformations of X to $X' \to S'$, as in the diagram

$$
\begin{array}{ccc}
X & \dashrightarrow & X' \\
\downarrow & & \downarrow \\
S & \longrightarrow & S'
\end{array}
$$

Then this is a matter of constructing an extension

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J} & \dashrightarrow & \mathcal{O}_{X'} & \dashrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0
\end{array}
$$

where $\mathcal{J}$ may be regarded as a sheaf on $\mathcal{O}_X$ because of the square-zero property, and as such it is isomorphic to $f^*\mathcal{I} \cong \mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{O}_X$.

A.15.2. *Square-zero algebra extensions.* Motivated by this, we consider the following problem. Let $f\colon A \to B$ a homomorphism of (classical) commutative rings and let M be an B-module. An *extension of B by M as A-algebras* is an exact sequence

$$0 \to M \to E \to B \to 0$$

which presents E as a square-zero A-algebra extension of B by M. We would like to understand the structure of all such extensions.

PROPOSITION A.44. *Let M be a B-module. Then $D^1(B/A; M)$ is the set of isomorphism classes of extensions of B by M as A-algebras.*

Example A.45. The zero element $0 \in D^1(B/A; M)$ corresponds to the trivial square-zero extension $B \oplus M$, with multiplication

$$(b_1, m_1)(b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1).$$

PROOF. Let us give maps in both directions. First suppose we have an extension $M \to E \to B$. Pick a free A-algebra resolution $A \to \mathcal{B} \xrightarrow{\sim} B$. In particular, since $\mathcal{B}_0$ is polynomial over A, we have some lifting

$$
\begin{array}{ccc}
\mathcal{B}_0 & \longrightarrow & B \\
\downarrow{\scriptstyle \phi_0} & & \| \\
M \longrightarrow E & \longrightarrow & B
\end{array}
$$

Now, we have two maps $d_0, d_1\colon \mathcal{B}_1 \rightrightarrows \mathcal{B}_0$ and we know that the image of $(d_0 - d_1)$ is an ideal in $\mathcal{B}_0$, with $\mathcal{B}_0/(d_0 - d_1) \cong \pi_0(\mathcal{B}) \cong B$. So the difference

$$\phi_0 \circ (d_0 - d_1)\colon \mathcal{B}_1 \to M$$

is a B-linear derivation from $\mathcal{B}_0$ to M, hence corresponds to an element of $\operatorname{Hom}_B(\Omega_{\mathcal{B}_1}, M)$. One checks that it is a cocycle, hence induces a class in $D^1(B/A; M)$. The different choices of ϕ_0 modify this cocycle by a coboundary, so we get a well-defined map from isomorphism classes of extensions to $D^1(B/A; M)$.

This also suggests how to construct the inverse map. Given a class in $D^1(B/A; M)$ represented by a derivation $\delta\colon \mathcal{B}_1 \to M$, define E to be the pushout of B-modules.

$$
\begin{array}{ccc}
\mathcal{B}_1 & \xrightarrow{d_0 - d_1} & \mathcal{B}_0 \\
\downarrow{\scriptstyle \delta} & & \downarrow \\
M & \longrightarrow & E
\end{array}
$$

We equip E with the multiplication induced by its structure as a quotient of $\mathcal{B}_0 \oplus M$. For this to be well-defined one checks that the image of $\mathcal{B}_1$ under $(d_1 - d_0, \delta)$ is a square-zero ideal, which is straightforward. $\qquad \square$

Example A.46. Consider $\mathbf{Z}$-algebra extensions of $\mathbf{F}_p$ by $\mathbf{F}_p$. Since we are considering arbitrary extensions of commutative rings, these are controlled by $D^1(\mathbf{F}_p/\mathbf{Z}; \mathbf{F}_p)$. By Proposition A.42, the cotangent complex of $\mathbf{Z} \to \mathbf{F}_p$ is $\mathbf{L}_{\mathbf{F}_p/\mathbf{Z}} \cong \mathbf{F}_p[1]$. So

$$D^1(\mathbf{F}_p/\mathbf{Z}; \mathbf{F}_p) \cong \operatorname{Hom}_{\mathbf{F}_p}(\mathbf{F}_p[1], \mathbf{F}_p[1]) \cong \mathbf{F}_p.$$

The 0 class corresponds to the extension $\mathbf{F}_p[\epsilon]/\epsilon^2$. The non-zero classes all have underlying ring $\mathbf{Z}/p^2\mathbf{Z}$, with the maps to $\mathbf{F}_p$ being the natural projection composed with an automorphism of $\mathbf{F}_p$.

Example A.47. Let $A \to B$ a smooth map of commutative rings and let M be any B-module. Then by Proposition A.41 we have $\mathbf{L}_{B/A} \cong \Omega^1_{B/A}$, so $D^1(B/A; M) = 0$. This says that there is a unique square-zero A-algebra extension of B by M, which is necessarily the split extension.

COMPLETION OF THE PROOF OF PROPOSITION A.41. We want to show that if $f \colon A \to B$ is a finite type morphism of Noetherian rings, and $\mathbf{L}_{B/A}$ is concentrated in degree 0, then f is smooth. Thanks to the finiteness hypotheses, it suffices to show that f is formally smooth, which means that any square-zero extension

$$
\begin{array}{ccc}
S & \longrightarrow & \mathrm{Spec}\ B \\
\downarrow & \nearrow & \downarrow \\
\widetilde{S} & \longrightarrow & \mathrm{Spec}\ A
\end{array}
$$

has a lift. It suffices to check this in the case $S = \mathrm{Spec}\ B$. Then $\mathbf{L}_{B/A}$ being concentrated in degree 0 implies that $D^1(B/A; M) = 0$ for any B-module M. But $S \to \widetilde{S}$ is a square-zero extension over A, so it must be the split extension. Therefore a lift exists. $\square$

A.16. Deformation theory of schemes. Let $\widetilde{A} \to A$ be an extension by a square-zero ideal I. We consider the problem of flat deformations

$$
\begin{array}{ccccc}
J & \longrightarrow & \widetilde{B} & \longrightarrow & B \\
\uparrow & & \uparrow & & \uparrow \\
I & \longrightarrow & \widetilde{A} & \longrightarrow & A
\end{array}
$$

or geometrically, deformations of $\mathrm{Spec}\ B \to \mathrm{Spec}\ A$ to a flat family over $\mathrm{Spec}\ \widetilde{A}$.

$$
\begin{array}{ccc}
\mathrm{Spec}\ B & \lhook\joinrel\longrightarrow & \mathrm{Spec}\ \widetilde{B} \\
\downarrow & & \downarrow \\
\mathrm{Spec}\ A & \lhook\joinrel\longrightarrow & \mathrm{Spec}\ \widetilde{A}
\end{array}
$$

Since I is square-zero, we can regard I as an A-module. Letting $J := I \otimes_{\widetilde{A}} \widetilde{B}$, we may similarly regard J as a B-module.

LEMMA A.48. *In the situation above, $\widetilde{B}$ is flat over $\widetilde{A}$ if and only if the multiplication map $I \otimes_A B \to J$ is an isomorphism.*

PROOF. If $\widetilde{B}$ is flat over $\widetilde{A}$, then applying $\widetilde{B} \otimes_{\widetilde{A}} (-)$ to the short exact sequence $I \to \widetilde{A} \to A$, and using that the $\widetilde{A}$-action on I factors through A, shows that $I \otimes_A B \xrightarrow{\sim} J$.

For the other direction, see [**75**, Tag 00MD].

$\square$

We know that $\widetilde{A}$-algebra extensions of B by $I \otimes_A B$ are classified by $D^1(B/\widetilde{A}, J)$. The composition $\widetilde{A} \to A \to B$ gives a long exact sequence

$$
\longrightarrow D^1(B/A; J) \longrightarrow D^1(B/\widetilde{A}; J) \longrightarrow D^1(A/\widetilde{A}; J)
$$

$$
\longrightarrow D^2(B/A; J) \longrightarrow \dots
$$

The equivalence class of the extension $\widetilde{A}$ may be viewed as an element $[\widetilde{A}] \in D^1(A/\widetilde{A}; J)$, which comes from $D^1(B/\widetilde{A}; J)$ if and only if its image in $D^2(B/A; J)$ vanishes. By the long exact sequence, its pre-image in $D^1(B/\widetilde{A}; J)$ forms a torsor (possibly empty) for $D^1(B/A; J)$. Hence we find:

PROPOSITION A.49. *Let the situation be as above.*

- *Attached to the extension $\widetilde{A} \to A$ is an* obstruction class $\mathrm{obs}(\widetilde{A}) \in D^2(B/A; J)$, *which vanishes if and only if an extension $\widetilde{B}$ exists.*
- *If $\mathrm{obs}(\widetilde{A}) \in D^2(B/A; J)$ vanishes, then the isomorphism classes of extensions form a non-empty torsor for $D^1(B/A; J)$.*
- *The automorphism group of any extension is $D^0(B/A; J) \cong \mathrm{Hom}_B(\Omega_{B/A}; J)$. (This is elementary; the theory of the cotangent complex is not relevant.)*

A.16.1. *Application: Witt vectors.* We will give an application of the preceding theory to the construction of *Witt vectors* for perfect $\mathbf{F}_p$-algebras.

LEMMA A.50. *If B is a perfect $\mathbf{F}_p$-algebra, then $\mathbf{L}_{B/\mathbf{F}_p} = 0$.*

PROOF. The Frobenius endomorphism on B is an isomorphism by definition, hence induces an automorphism of $\mathbf{T}_{B/\mathbf{F}_p}$. To compute the action of Frobenius on $\mathbf{T}_{B/\mathbf{F}_p}$, pick a free resolution of $\mathcal{B} \xrightarrow{\sim} B$ over $\mathbf{F}_p$, and note that Frob on B lifts to Frob on $\mathcal{B}$ (the p-power map level-wise on $\mathcal{B}$):

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\mathrm{Frob}} & \mathcal{B} \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
B & \xrightarrow{\mathrm{Frob}} & B
\end{array}
$$

Therefore the action of Frob on $\mathbf{T}_{B/\mathbf{F}_p}$ is tensored from its action on $\mathbf{T}_{\mathcal{B}/\mathbf{F}_p}$, which is zero because the Frobenius endomorphism induces 0 on Kähler differentials. So on the one hand we have that Frob induces an automorphism of $\mathbf{T}_{B/\mathbf{F}_p}$, but on the other hand it is the zero map. Therefore, it must be the case that $\mathbf{T}_{B/\mathbf{F}_p} = 0$. $\square$

LEMMA A.51. *There is a unique flat deformation of B to $\mathbf{Z}/p^n$.*

PROOF. When $n = 2$, this follows directly from Proposition A.49 with $A_1 = \mathbf{F}_p$ and $\widetilde{A} = \mathbf{Z}/p^2$. We may suppose by induction that we have a deformation B_n over $A_n := \mathbf{Z}/p^n$ and $\widetilde{A} = \mathbf{Z}/p^{n+1}$. Let $I_n \subset \widetilde{A}$ be the kernel of $\widetilde{A} \to A_n$, a module over A_n which in fact is pulled back from A_1. Let $J_n = I_n \otimes_{A_n} B_n$, which is then pulled back from B. Then the obstruction class in Proposition A.49 lies in $D^2(B_n/A_n; J_n)$. Since the action of B_n on J_n factors over $B_1 = B$, and B_n is flat over A_n, we have $D^2(B_n/A_n; J_n) \cong D^2(B/A_1; J_n) \cong 0$. $\square$

Therefore, there is a unique (up to unique isomorphism) diagram

$$
\begin{array}{ccccccc}
B & \xleftarrow{\quad} & B_2 & \xleftarrow{\quad} & B_3 & \xleftarrow{\quad} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbf{F}_p & \xleftarrow{\quad} & \mathbf{Z}/p^2\mathbf{Z} & \xleftarrow{\quad} & \mathbf{Z}/p^3\mathbf{Z} & \xleftarrow{\quad} & \cdots
\end{array}
$$

with all squares co-Cartesian and all vertical arrows flat. The inverse limit $\varprojlim_n B_n$ is called the *ring of Witt vectors of B*. This construction of the Witt vectors was pointed out by Scholze in [**73**, Remark 5.14].

A.17. Deformation theory of maps. Our next situation is motivated by the deformation theory of maps: given a square-zero extension $X \hookrightarrow \widetilde{X}$, and a map $X \to Y$, we want to understand extensions of the map

$$X \longhookrightarrow \widetilde{X}$$
$$\downarrow \quad \swarrow$$
$$Y$$

The local problem is: given an A-algebra B and a square-zero extension

$$J \to \widetilde{B} \to B,$$

equip $\widetilde{B}$ with a compatible A-algebra structure. The equivalence class of $\widetilde{B}$ as a commutative $\mathbf{Z}$-algebra extension of B by J can be viewed as an element $[\widetilde{B}] \in D^1(B/\mathbf{Z}; J)$. We want to lift this a class in $D^1(B/A; J)$. We have the long exact sequence

$$\ldots \to D^0(A/\mathbf{Z}; J) \to D^1(B/A; J) \to D^1(B/\mathbf{Z}; J) \to D^1(A/\mathbf{Z}; J) \to \ldots$$

The map $D^1(B/\mathbf{Z}; J) \to D^1(A/\mathbf{Z}; J)$ can be interpreted as sending $\widetilde{B}$ to the fibered product $A \times_B \widetilde{B}$, which is a commutative algebra extension of A by J.

PROPOSITION A.52. *Let the situation be as above.*
- *Attached to the extension $\widetilde{B} \to B$ is an* obstruction class $\mathrm{obs}(\widetilde{B}) \in D^1(A/\mathbf{Z}; J)$, *which vanishes if and only if there exists an A-algebra structure on $\widetilde{B}$ compatible with the given one on B.*
- *If $\mathrm{obs}(\widetilde{B}) \in D^1(A/\mathbf{Z}; J)$ vanishes, then the isomorphism classes of A-algebra structures form a non-empty torsor for $D^0(A/\mathbf{Z}; J) = \mathrm{Hom}_A(\Omega_A, J)$.*

A.17.1. *Application: Fontaine's map.* Let R be a p-adically complete and p-torsion-free ring. Let $R^\flat := \varprojlim_{\mathrm{Frob}_p}(R/p)$. This is a perfect ring, characterized uniquely up to unique isomorphism as "the" final perfect ring mapping to R/p.

It is a very important fact in p-adic Hodge Theory that there is a *unique* map $W(R^\flat) \to R$ that reduces mod p to the canonical map $R^\flat \to R/p$, called *Fontaine's map*, although we will not be able to explain its significance here. Here $W(R^\flat)$ are the Witt vectors, which as discussed in §A.16.1 is the inverse limit of the unique (up to unique isomorphism) family of lifts

$$R^\flat \longleftarrow W_2(R^\flat) \longleftarrow W_3(R^\flat) \longleftarrow \ldots$$
$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$
$$\mathbf{F}_p \longleftarrow \mathbf{Z}/p^2\mathbf{Z} \longleftarrow \mathbf{Z}/p^3\mathbf{Z} \longleftarrow \ldots$$

Let us build up a family of maps step-by-step, starting with $W_2(R^\flat) \to R/p^2$.

$$R/p \longleftarrow R/p^2 \longleftarrow R/p^3 \longleftarrow \ldots$$
$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$
$$R^\flat \longleftarrow W_2(R^\flat) \longleftarrow W_3(R^\flat) \longleftarrow \ldots$$
$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$
$$\mathbf{F}_p \longleftarrow \mathbf{Z}/p^2\mathbf{Z} \longleftarrow \mathbf{Z}/p^3\mathbf{Z} \longleftarrow \ldots$$

(The p-torsionfree property of R is used to see that R/p^n is flat over $\mathbf{Z}/p^n\mathbf{Z}$.) By similar arguments as above, the obstruction to such an extension is a class in

$$D^1(W_2(R^\flat)/(\mathbf{Z}/p^2\mathbf{Z}); pW_2(R^\flat)),$$

which is isomorphic to $D^1(R^\flat/\mathbf{F}_p; R^\flat)$, which vanishes because $R^\flat$ is perfect. Then, the set of such an extensions is a torsor for $D^0(R^\flat/\mathbf{F}_p; R^\flat) = 0$, so the extension is unique. A similar argument applies inductively at each step. Finally, take the inverse limit of these maps to construct Fontaine's map (here is where we use that R is p-adically complete).

A.18. Global deformations. So far we have discussed deformations of *affine* schemes. Now we begin the study of deformations of possibly non-affine schemes.

Example A.53. Let X be a *smooth* (but not necessarily affine) variety over a field k. A *first-order deformation* of X is a flat family $\widetilde{X}$ whose fiber over Spec k is X. The we claim that the set of first-order deformations of X is in bijection with $H^1(X, TX)$.

To see this, write $X = \bigcup U_i$ as a union of affines $U_i = \mathrm{Spec}\ A_i$. A first-order deformation of X is then uniquely specified by a compatible collection of first-order deformations of the U_i. Since the U_i are *smooth*, they have unique first-order deformations $\widetilde{U}_i$. The gluing data is an automorphism of $\widetilde{U}_i \cap \widetilde{U}_j := \mathrm{Spec}\ \widetilde{A}_{ij}$ lifting the given automorphism on $U_i \cap U_j := \mathrm{Spec}\ A_{ij}$, which is of the form

$$a + b\epsilon \mapsto a + (\delta a + b)\epsilon$$

where δ is a k-linear derivation from A_{ij} to itself, i.e. a section of the tangent sheaf TX over U_{ij}. Finally, there is a condition: these automorphism must satisfy the cocycle condition on triple intersections. Therefore, we see that the data of a first-order deformation is exactly that of a class in the first Cech cohomology $H^1(X, TX)$.

As an application, consider the *moduli space of (smooth, projective) genus g curves* $\mathcal{M}_g$. A k-point to this moduli space is a smooth, projective genus g curve C. The tangent space to $\mathcal{M}_g$ at the k-point corresponding to C has underlying set the set of extensions of this k-point to a $k[\epsilon]/\epsilon^2$-point, which is to say the set of first-order deformations of C/k. According to what we have said, this is $H^1(C, TC)$. Furthermore, the obstructions are valued in $H^2(C, TC)$, which vanishes because C is a curve – this tells us that $\mathcal{M}_g$ is *smooth* at the k-point corresponding to C.

A.19. Globalizing the cotangent complex. By globalizing Proposition A.54, one proves:

PROPOSITION A.54. *Let $f\colon X \to S$ be a map of schemes. Let $S \hookrightarrow \widetilde{S}$ be a square-zero thickening with ideal sheaf $\mathcal{I}$, and $\mathcal{J} = f^*\mathcal{I}$.*

- *Attached to the extension $S \hookrightarrow \widetilde{S}$ is an* obstruction class $\mathrm{obs}(\widetilde{S}) \in \mathrm{Ext}^2(\mathbf{L}_{X/S}; \mathcal{J})$, *which vanishes if and only if an extension $\widetilde{X}$ exists.*
- *If $\mathrm{obs}(\widetilde{S}) \in \mathrm{Ext}^2(\mathbf{L}_{X/S}; \mathcal{J})$ vanishes, then the isomorphism classes of extensions form a non-empty torsor for $\mathrm{Ext}^1(\mathbf{L}_{X/S}; \mathcal{J})$.*
- *The automorphism group of any extension is $\mathrm{Ext}^0(\mathbf{L}_{X/S}; \mathcal{J})$.*

Example A.55 (LCI curves)**.** Recall that a finite type morphism of Noetherian rings $A \to B$ is said to be *local complete intersection (LCI)* if it can be Zariski-locally written as a composition $A \to A[x_1, \ldots, x_n] \twoheadrightarrow B$ where the second map is

a regular quotient. More generally, a morphism of schemes is said to be LCI if it is Zariski-locally LCI. For example, a curve with at worst nodal singularities over a field is LCI. By Proposition A.42, if $Y \to X$ is LCI then $\mathbf{L}_{Y/X}$ is concentrated in degrees $-1, 0$.

Suppose C is an LCI curve over a field k, which is is smooth outside a finite number of closed points of C. Then we claim $\mathrm{Ext}^2(\mathbf{L}_{C/k}; \mathcal{J}) = 0$, which in particular shows that all infinitesimal deformations of C/k are unobstructed. By the local-global spectral sequence, it suffices to show the vanishing of

- $H^2(C, \underline{\mathrm{Ext}}^0(\mathbf{L}_{C/k}, \mathcal{J}))$,
- $H^1(C, \underline{\mathrm{Ext}}^1(\mathbf{L}_{C/k}, \mathcal{J}))$, and
- $H^0(C, \underline{\mathrm{Ext}}^2(\mathbf{L}_{C/k}, \mathcal{J}))$.

The first vanishes because C is a curve, so it has cohomological dimension 1. The third vanishes because $\underline{\mathrm{Ext}}^2(\mathbf{L}_{C/k}, \mathcal{J})$ vanishes, thanks to $\mathbf{L}_{C/k}$ being supported in degrees $[-1, 0]$. Finally, since C is smooth away from a finite collection of closed points, $\underline{\mathrm{Ext}}^1(\mathbf{L}_{C/k}, \mathcal{J})$ is a torsion sheaf by the localizing property of the cotangent complex plus its calculation in the smooth case (§A.11), so its higher cohomology vanishes.

For completeness we state the globalization of Proposition A.52:

PROPOSITION A.56. *Let $X \hookrightarrow \widetilde{X}$ be a square-zero extension with ideal sheaf $\mathcal{J}$ and $f \colon X \to Y$. Let $\mathbf{L}_Y$ be the absolute cotangent complex of Y (i.e., the cotangent complex of $Y \to \mathrm{Spec}\ \mathbf{Z}$).*

- *Attached to $\widetilde{X}$ is an obstruction class $\mathrm{obs}(\widetilde{X}) \in \mathrm{Ext}^1(\mathbf{L}_Y; \mathcal{J})$, which vanishes if f extends to $\widetilde{X} \to Y$.*
- *If $\mathrm{obs}(\widetilde{X}) \in \mathrm{Ext}^1(\mathbf{L}_Y; \mathcal{J})$ vanishes, then the isomorphism classes of extensions form a non-empty torsor for $\mathrm{Ext}^0(\mathbf{L}_Y; \mathcal{J})$.*

A.20. Geometric interpretations of André-Quillen homology. In this section we reflect on the relation between derived algebraic geometry and the cotangent complex; a more substantial exposition of this topic may be found in [**82**]. As was discussed in §3, one of the characteristic features of derived algebraic geometry in practice is its ability to construct spaces whose cotangent complexes can be "identified" in a meaningful way, for example as some natural cohomology theory. The reason this happens is because derived algebraic geometry supplies a "geometric" interpretation of the cotangent complex.

It may be helpful to consider the analogy between schemes and reduced schemes, explained in §2.3. Even if we are only interested in reduced schemes, allowing nonreduced schemes into our vocabulary allows us to give a geometric interpretation of tangent spaces as maps from dual numbers. In the context of moduli spaces, this allows to express tangent spaces as first-order deformations, and then often in practice as a cohomology group.

We could contemplate defining a "reduced moduli space" as a functor on reduced commutative rings. However, it would be a priori unclear how to describe its tangent space explicitly. If the functor can be extended to non-reduced rings, then its tangent space space could be calculated as in the previous paragraph, and if for example the moduli space turned out to be smooth then one would know a posteriori the tangent space of the underlying reduced scheme. The reader might

enjoy contemplating this thought experiment in the example of $\mathcal{M}_g$, the moduli space of smooth projective genus g curves.

Similarly, the tangent *complex* of a classical moduli space is a priori hard to describe explicitly, because the "higher" tangent spaces are not prescribed by some direct process from the moduli problem. However, in the world of derived algebraic geometry the full tangent complex does admit such a direct description, which we now explain.
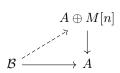
Let $A \to B$ be a morphism of commutative rings, and $\eta\colon B \to A$ an augmentation. It may be helpful to focus on the special case $A = \operatorname{Spec} k$, in which case η corresponds to a k-point of B.

Let M be a B-module. It can be turned into a simplicial B-module $M[0]$, which is characterized as the unique simplicial B-module with $\pi_0(M[0]) = M$ and $\pi_i(M[0]) = 0$ for $i > 0$. For $n \geq 0$, there is a simplicial B-module $M[n]$ with $\pi_n(M[n]) = M$ and $\pi_i(M[n]) = 0$ for $i \neq n$. (For example, $M[n]$ can be constructed using the Dold-Kan correspondence.)

Then we form the level-wise simplicial square-zero extension $A \oplus M[n]$. Its graded homotopy ring is the square-zero extension of A (in degree 0) by M (in degree n).

Example A.57. Let $A = k$ be a field and $M = k$. Then $k \oplus k[0] = k[\epsilon]/\epsilon^2$ is the ring of *dual numbers* over k. More generally, $k \oplus k[n]$ is called the ring of *derived order n dual numbers*.

Then the André-Quillen cohomology group $D^n(B/A; M)$ can be viewed as the group of homomorphisms from B to $A \oplus M[n]$ lifting η, in the homotopy category of simplicial A-algebras. The reason for this is relatively formal, once the relevant definitions are in place; for now we just give a sketch. To compute these homomorphisms, by definition one needs to resolve B by a free simplicial A-algebra $\mathcal{B}$. (This is analogous to how maps in the derived category of A-modules are computed by first resolving the source by a projective resolution.) Then, lifts

$$
\begin{array}{ccc}
 & & A \oplus M[n] \\
 & \nearrow & \downarrow \\
\mathcal{B} & \longrightarrow & A
\end{array}
$$

are computed by derivations from $\mathcal{B}$ into $M[n]$, which can be converted into a description as homomorphisms from Kähler differentials of $\mathcal{B}$ into $M[n]$, which is basically the definition of André-Quillen cohomology.

Geometrically, we can think of "$\operatorname{Spec}(A \oplus M[n])$" as being some "higher derived infinitesimal thickening" of A by M. A derived moduli problem can be evaluated on such a space, giving a geometric interpretation of André-Quillen homology groups; this makes it *easier* to understand the cotangent complex of a derived moduli space than that of a classical moduli space, in the same sense that it is "easier" to understand the tangent space of a classical moduli space than that of a "reduced moduli space".

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