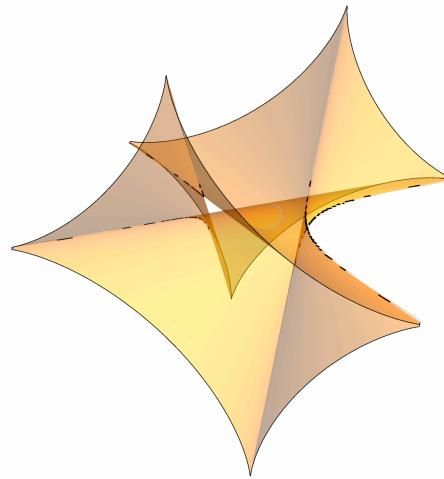


Explicit examples of Higgs bundles in the contexts of quantum materials and geometric Langlands correspondence



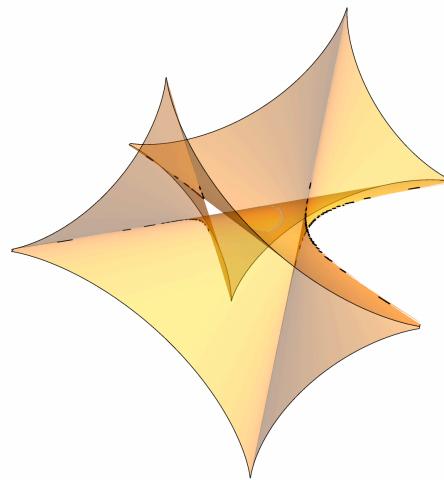
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2025.1.11

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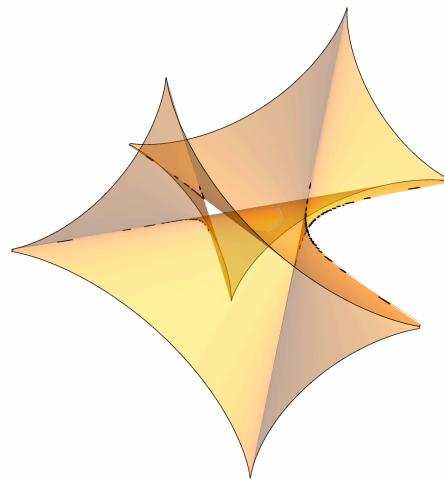
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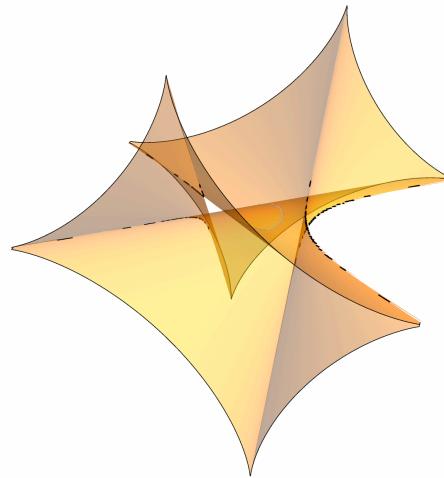
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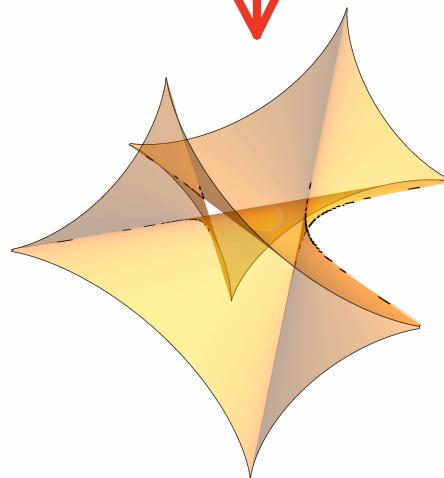
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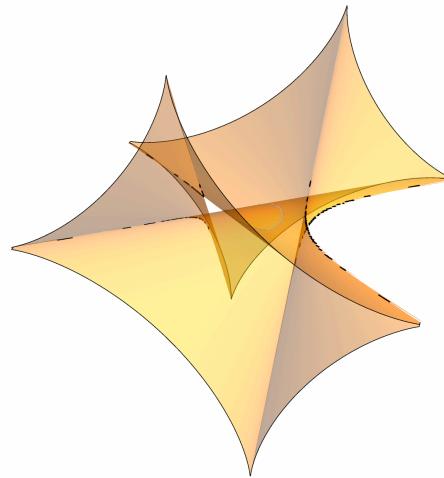
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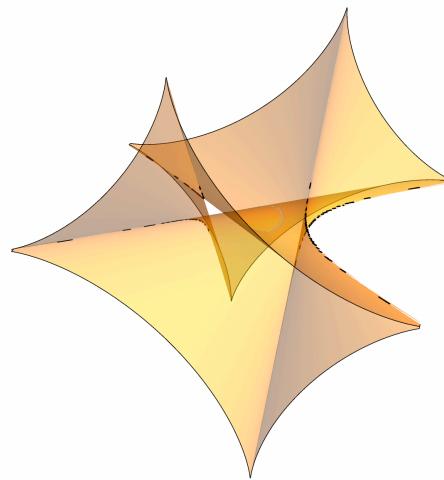
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Our goal is to explain:

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*Holography, optical devices,
absorption devices, ...*

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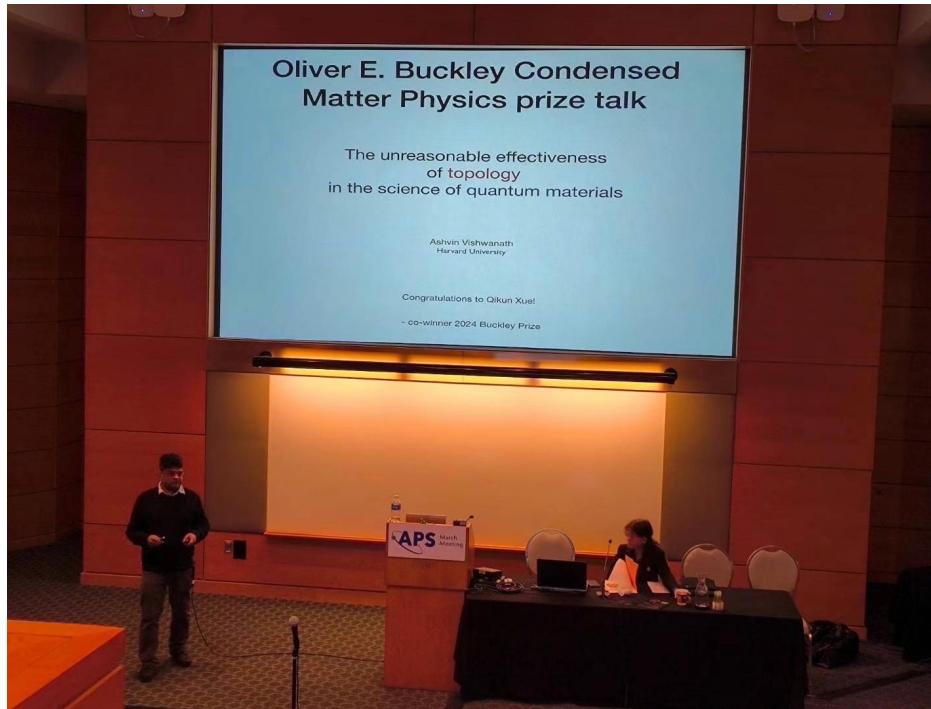
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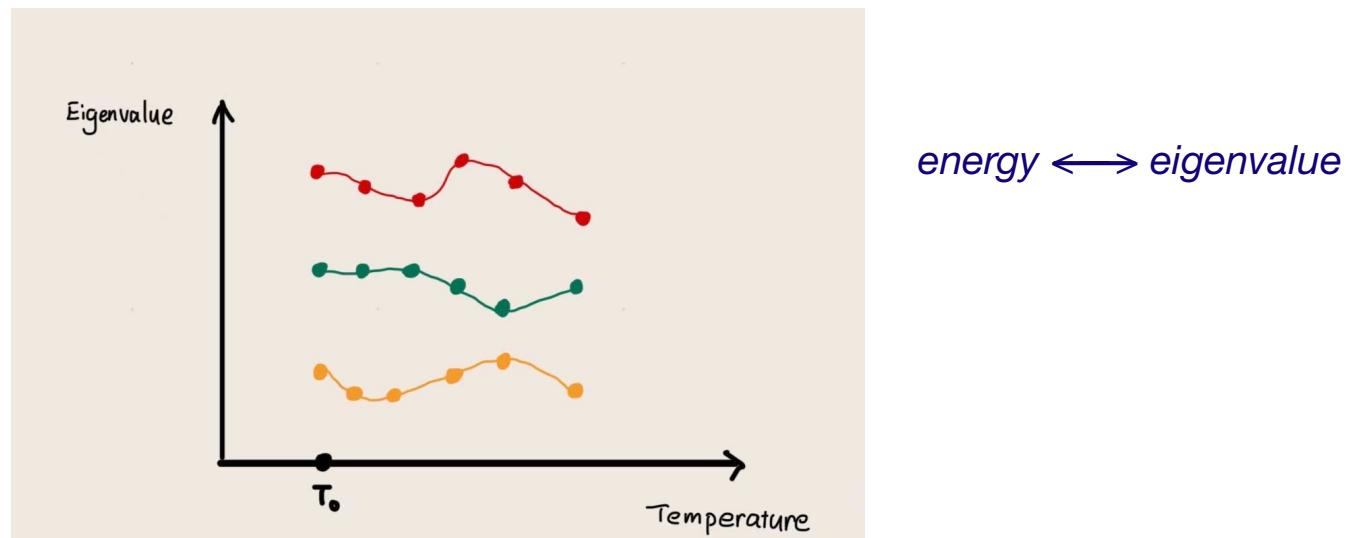
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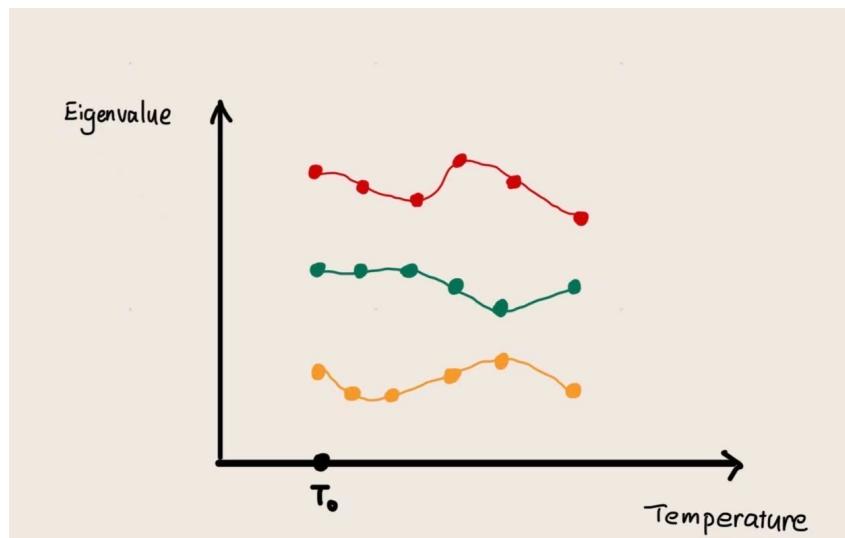
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Mathematical modeling of electronic energy *band structures* therein



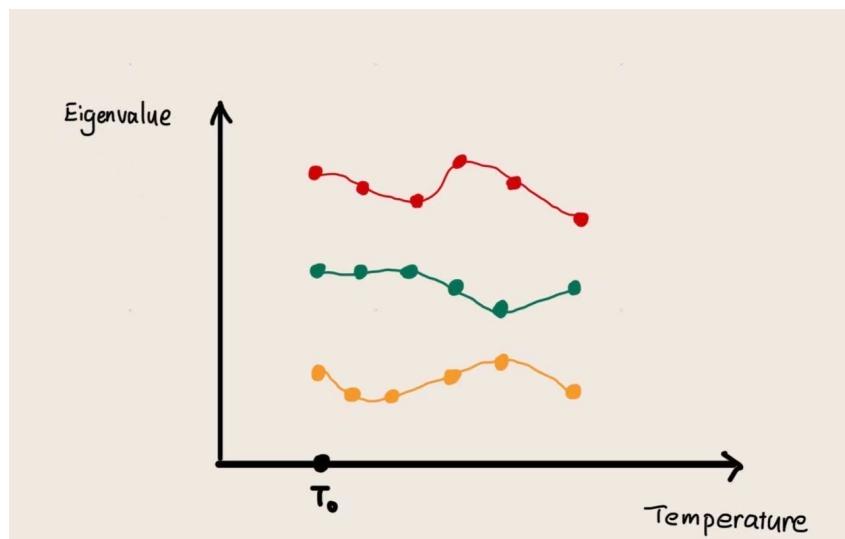
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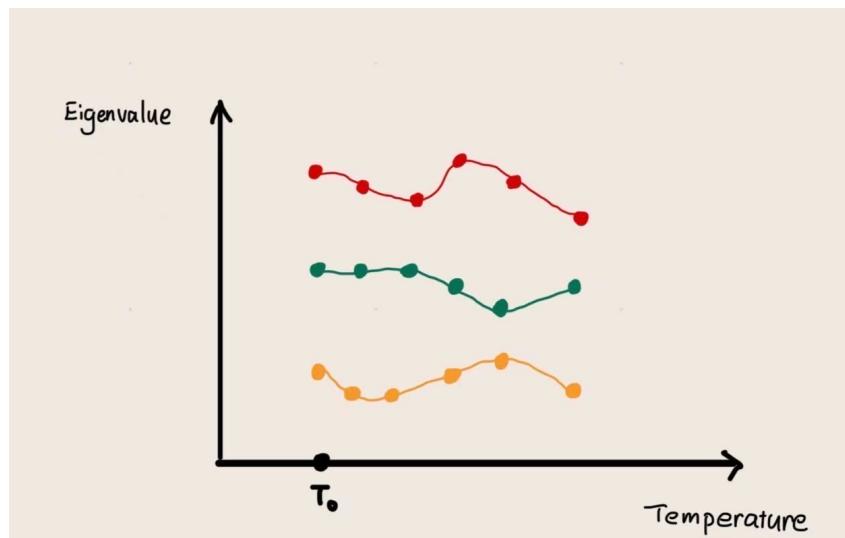
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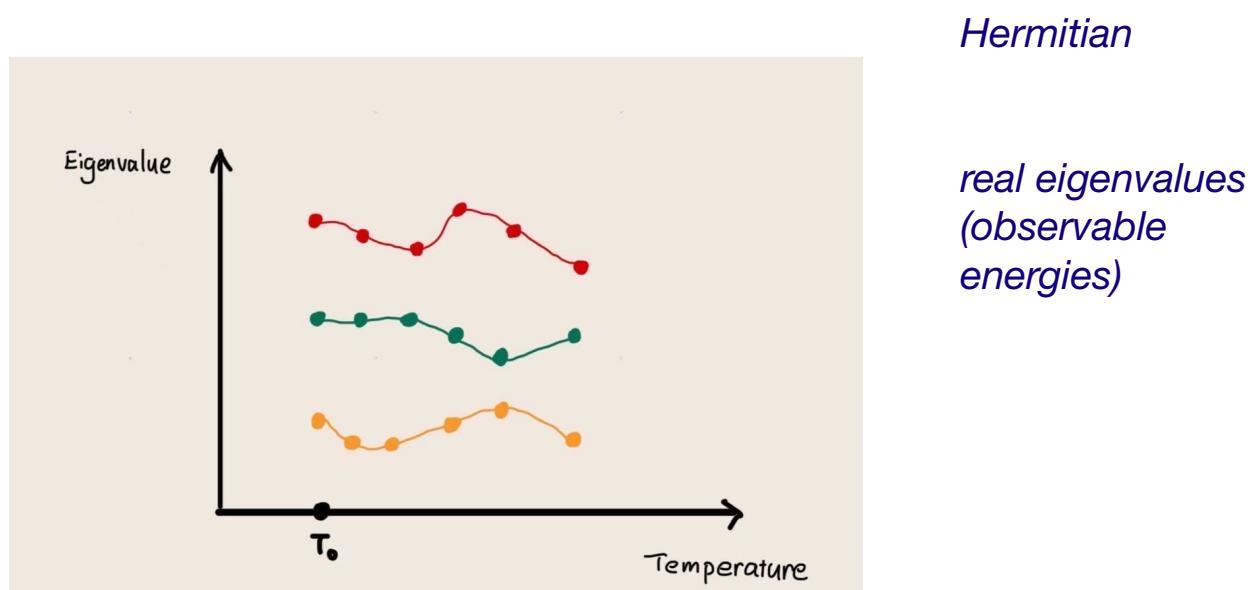
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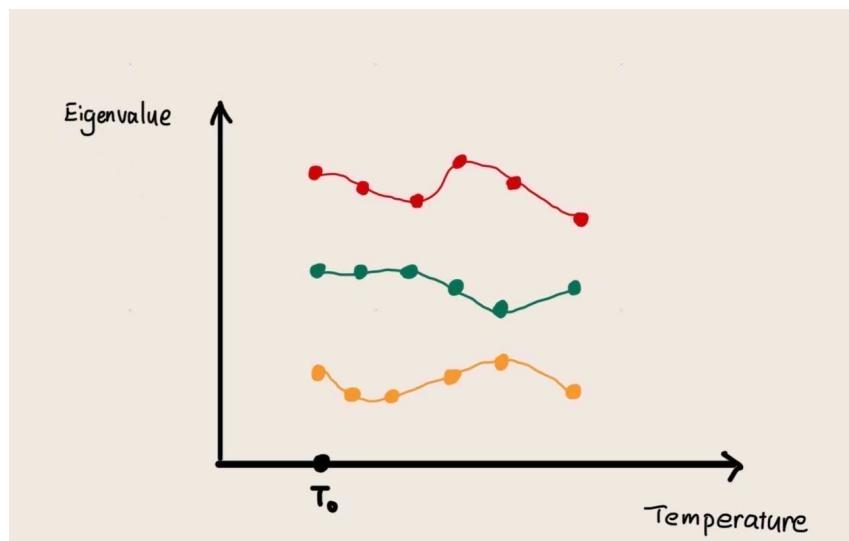
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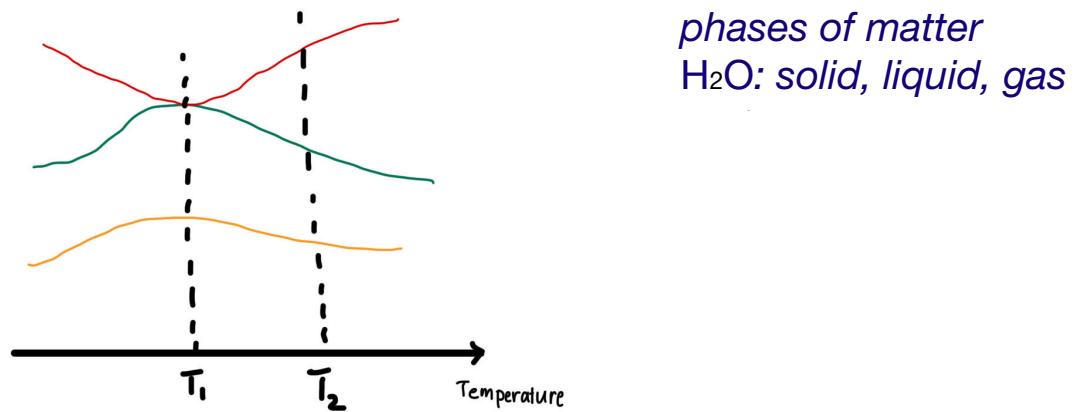


*Hermitian vs.
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*real eigenvalues
(observable
energies) vs.
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imaginary part
(counts for
energy exchange
with surrounding
environment or
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Motivations: Quantum materials and their math modeling

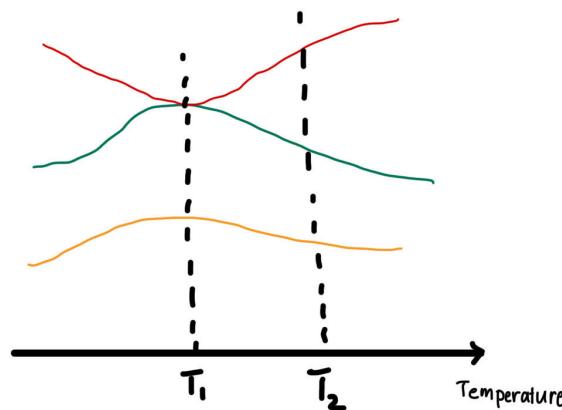
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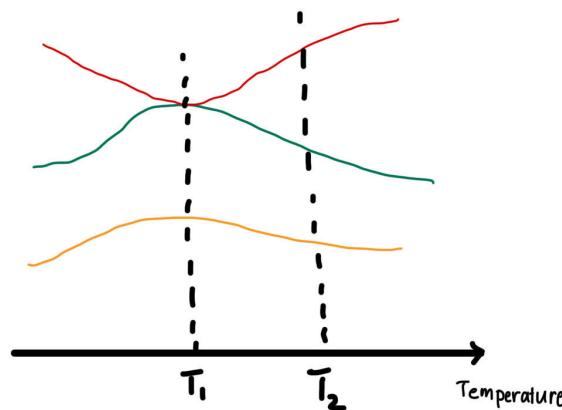
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*It is indeed “pointless” and is better understood as a **functor**!*

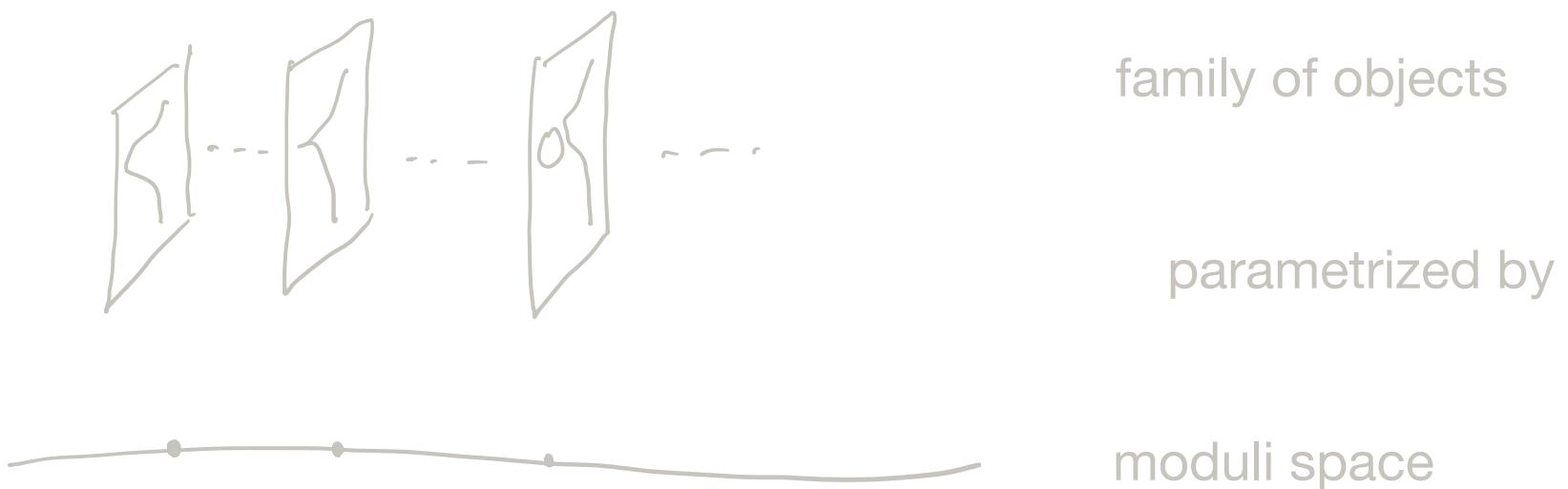
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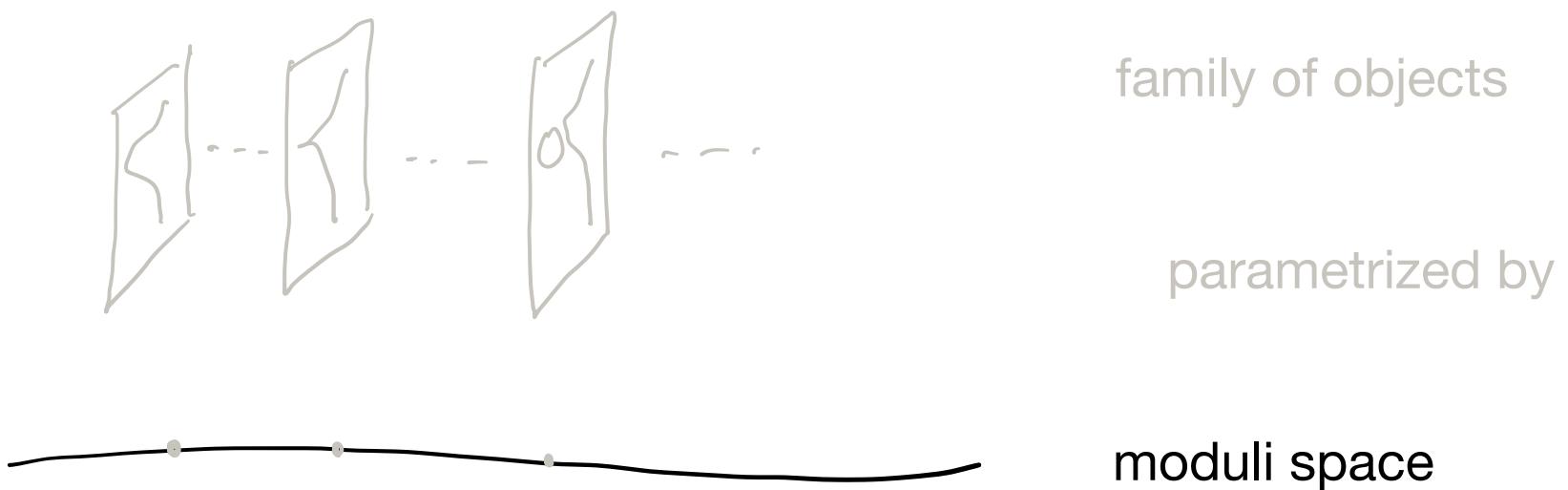
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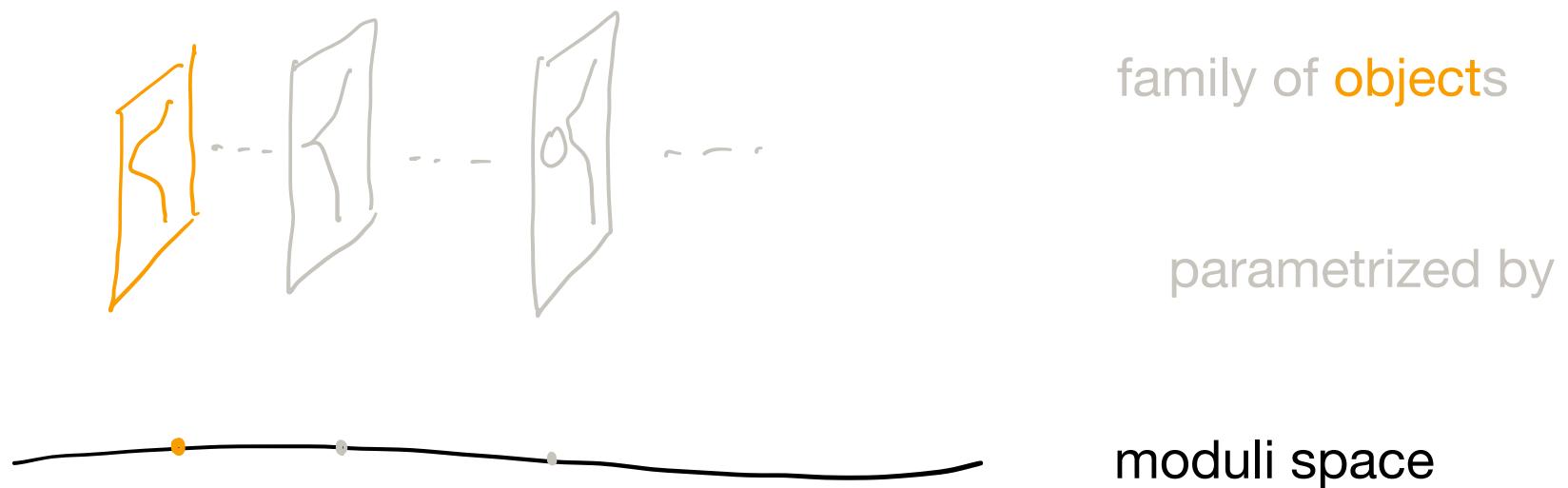
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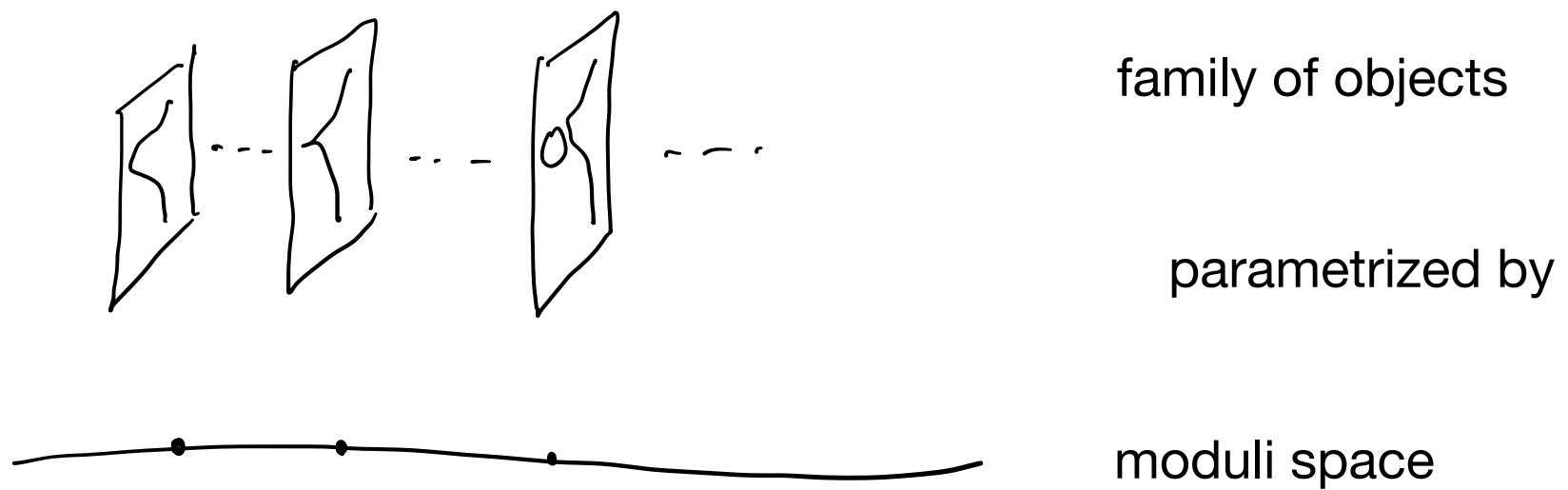
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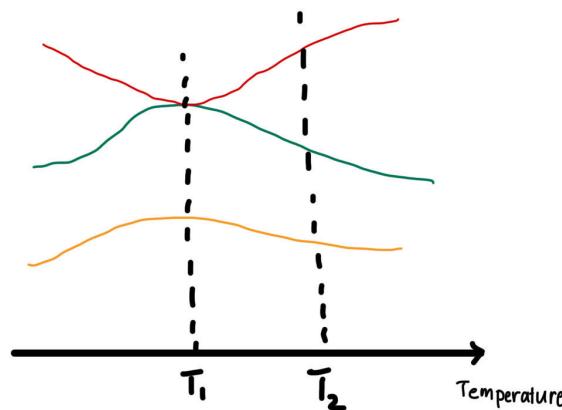
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Guzman et al., Model-free characterization of topological edge and corner states in mechanical networks, PNAS 2024.

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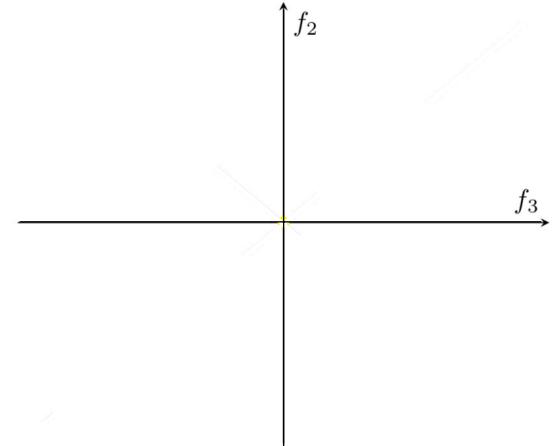
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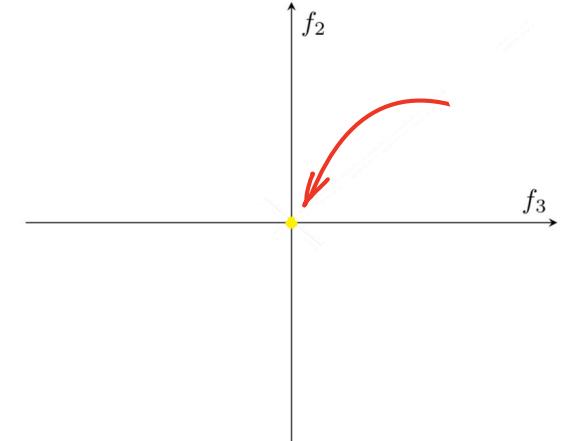
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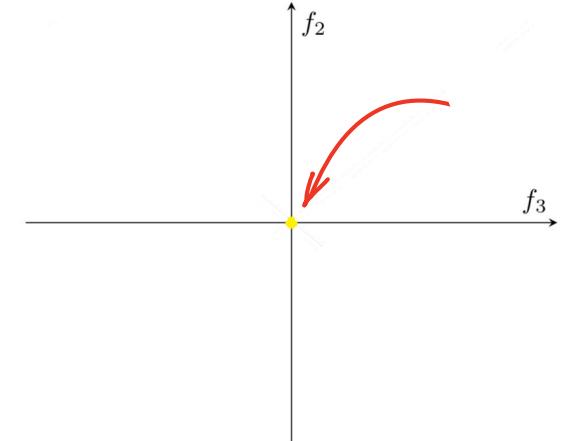
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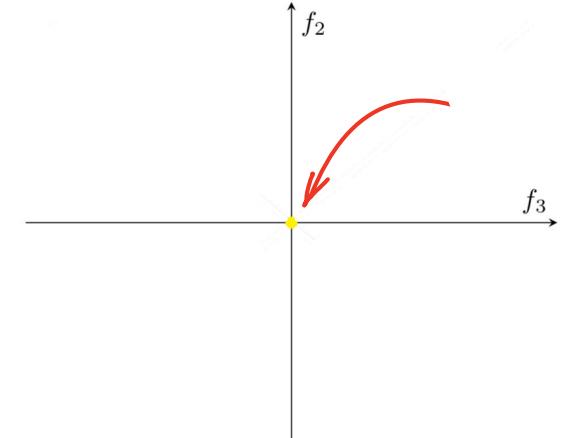
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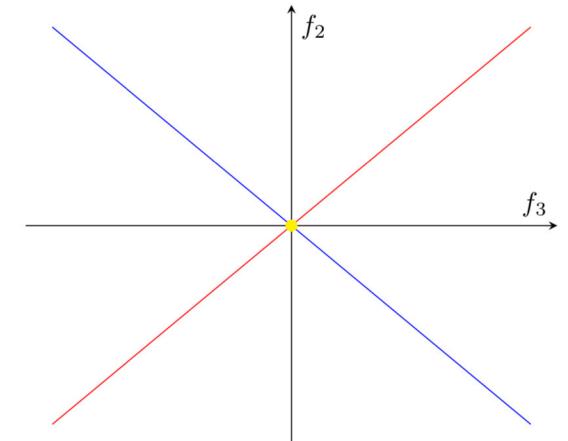
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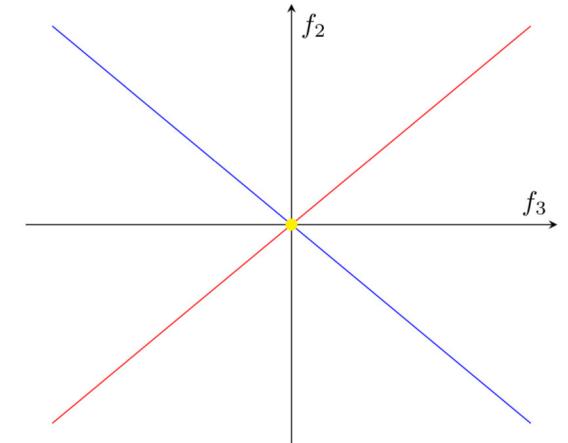
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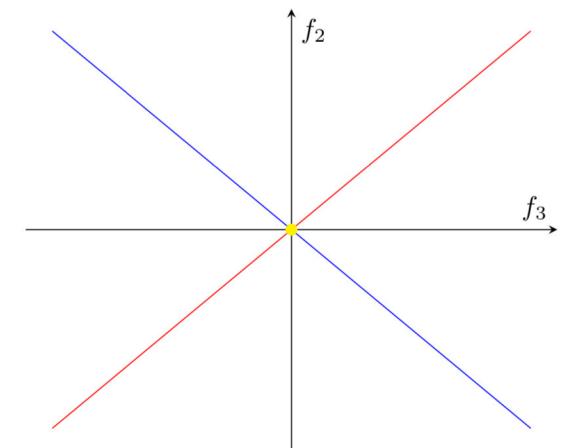
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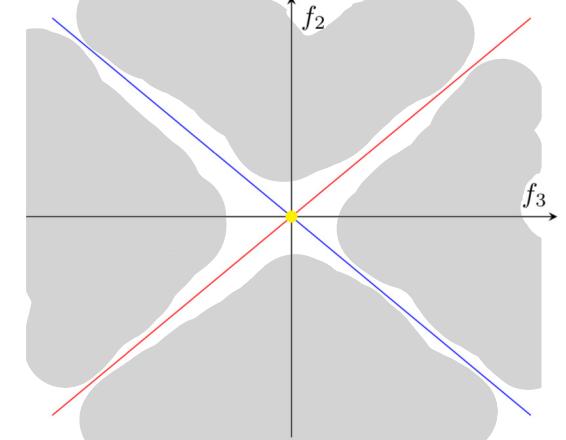
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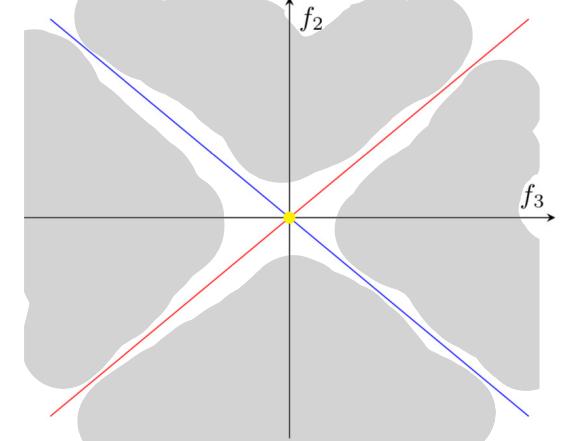
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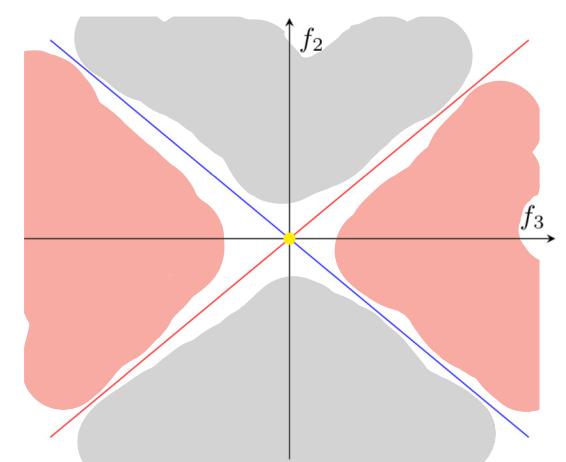
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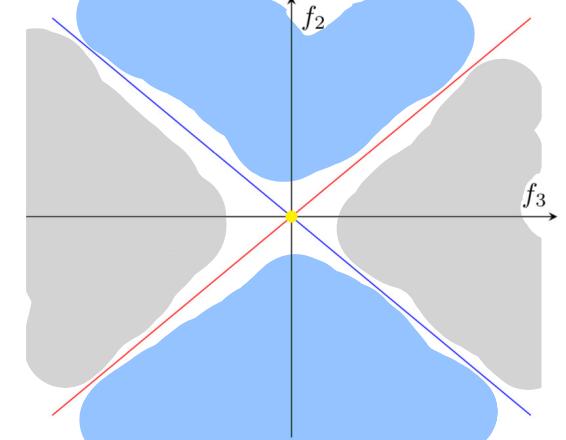
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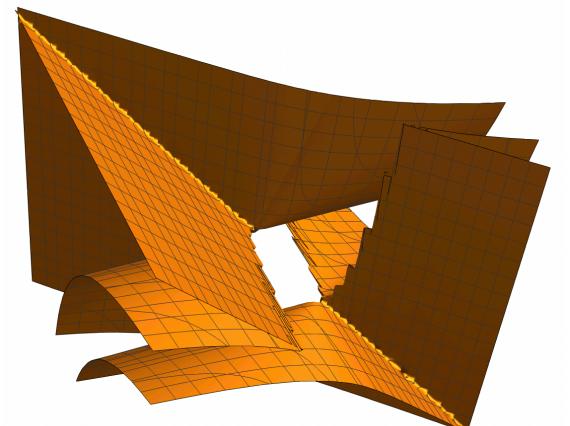
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The equation for this surface is a non-homogeneous real polynomial in f_1, f_2, f_3 of degree 6.



Swallowtail couple sw2

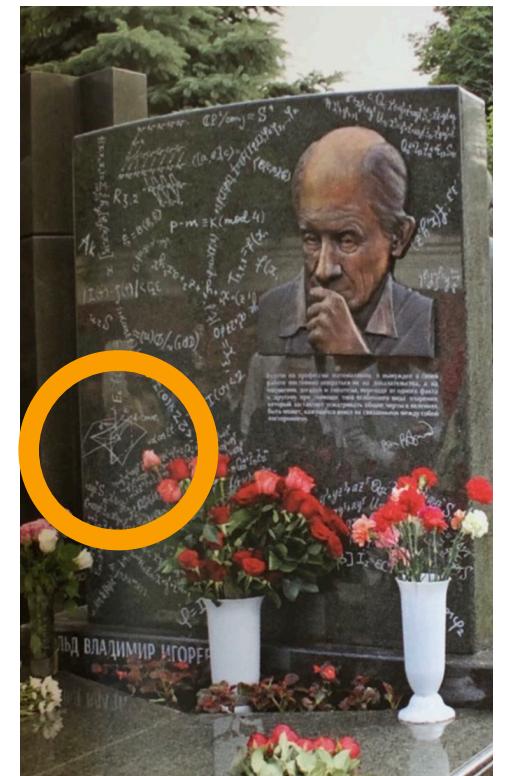
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V. I. Arnold's tombstone at the Novodevichy Cemetery in Moscow

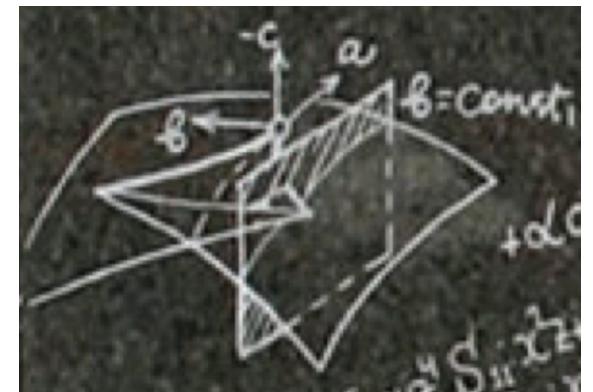
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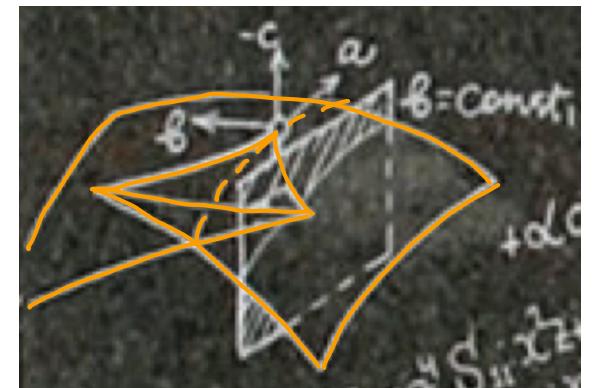
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A **local** model for moduli spaces of 3-band Hamiltonians

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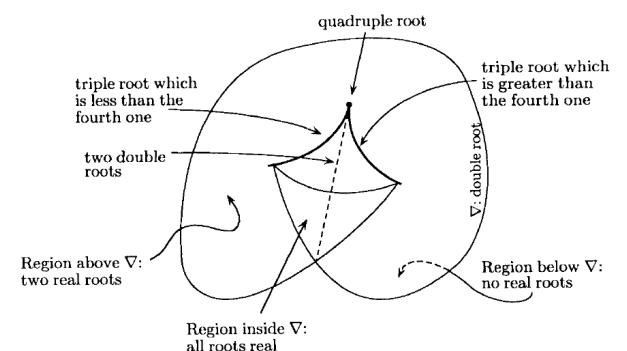
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Arnold, Braids of algebraic functions and the cohomology of swallowtails, 1968.

Homological stability of braid groups

*Portrait from Gelfand, Kapranov, Zelevinsky,
Discriminants, resultants, and multidimensional determinants.*



The space of polynomials $x^4 + ax^2 + bx + c$

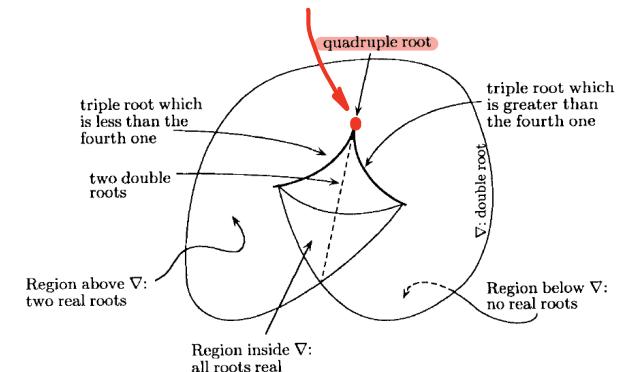
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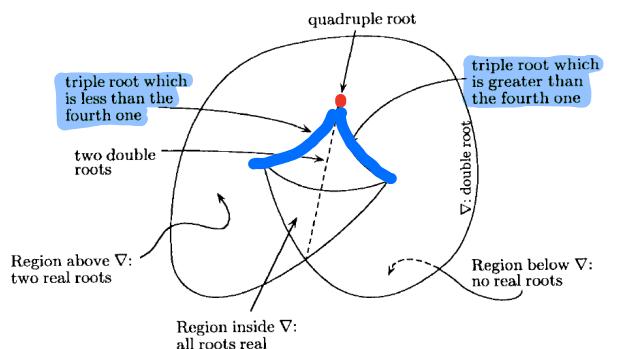
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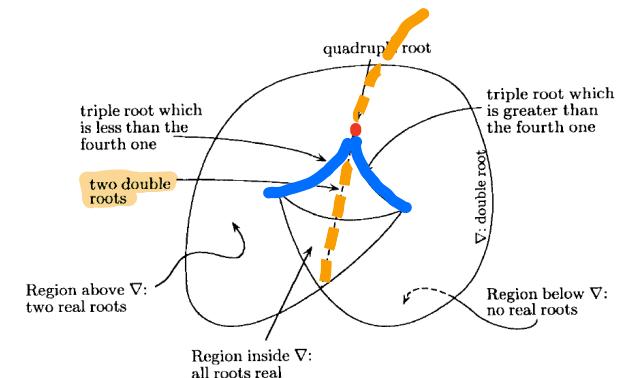
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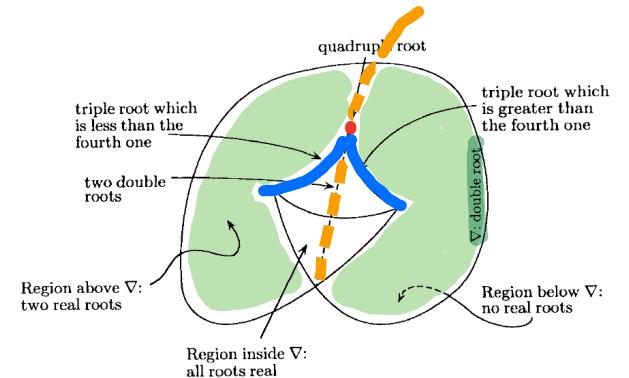
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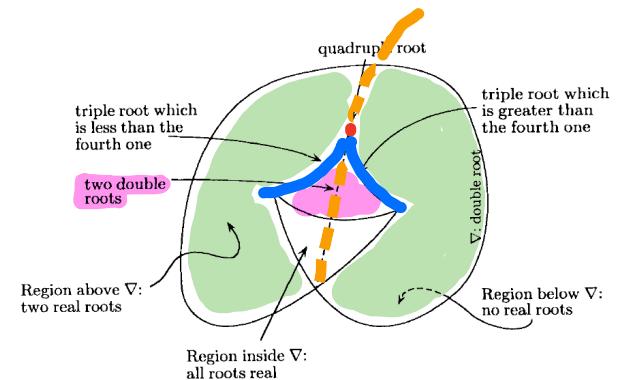
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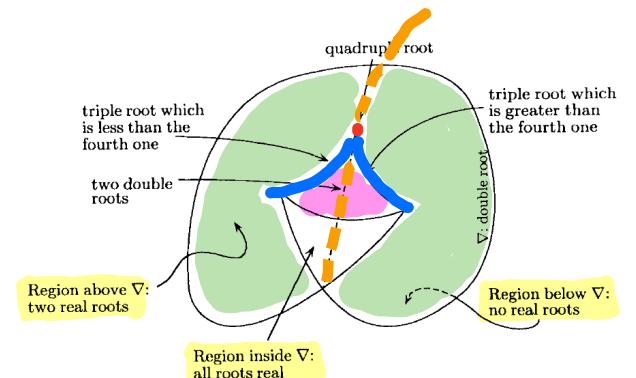
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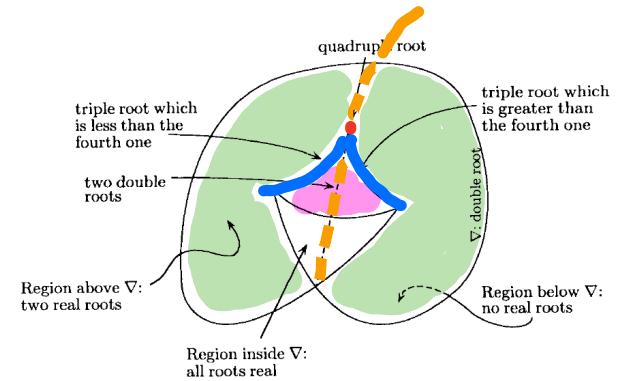
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Again, we aim to find computable algebraic invariants that systematically classify the evolutions of eigenvectors along loops in such stratified parameter spaces, including when they cross the discriminant surface resulting in degeneracies of various sorts.



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*Remarks on eigenvalues and eigenvectors of Hermitian matrices,
Berry phase, adiabatic connections and quantum Hall effect, 1995.*

Also: Polymathematics, 2000.

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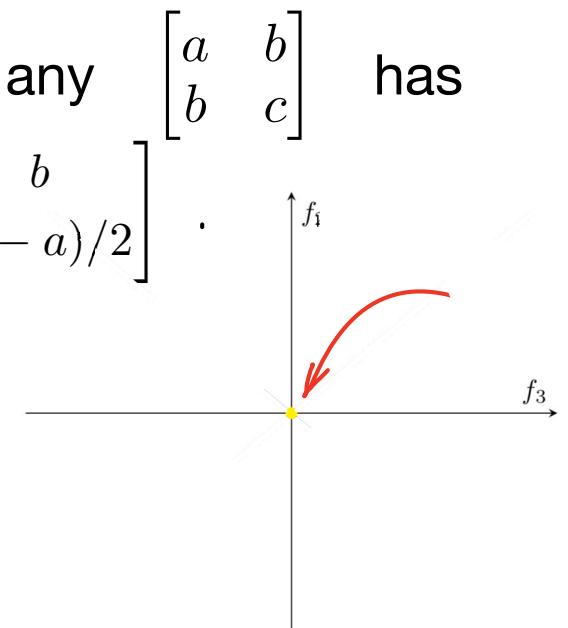
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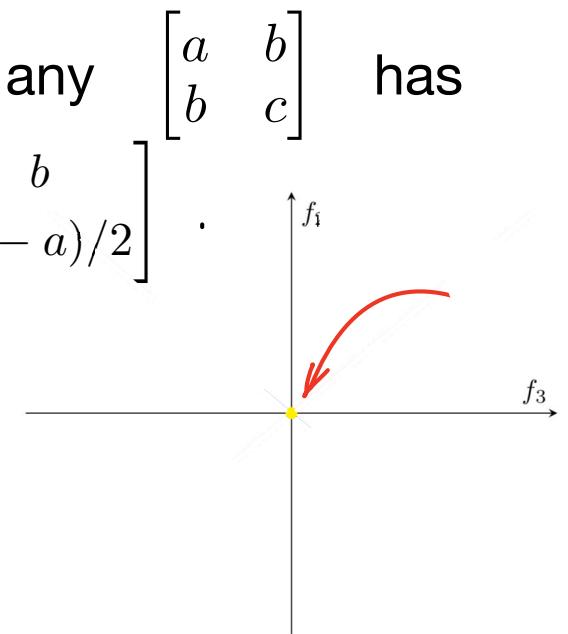
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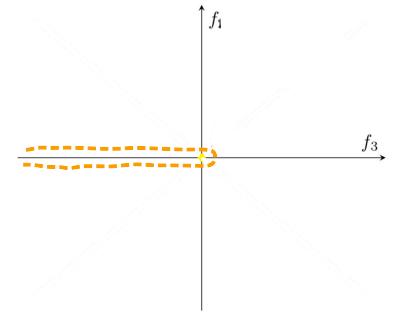
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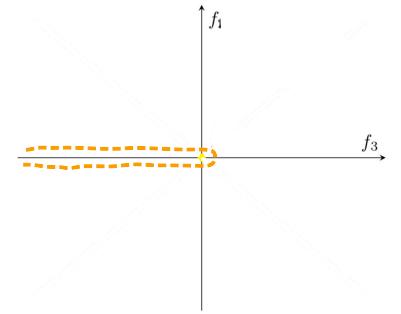
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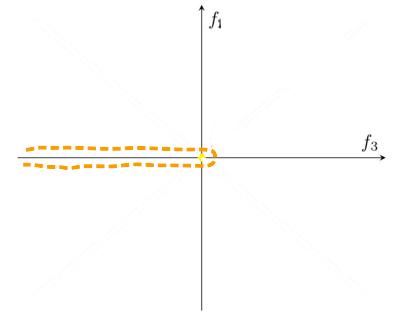


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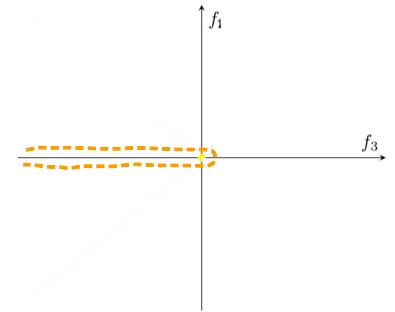
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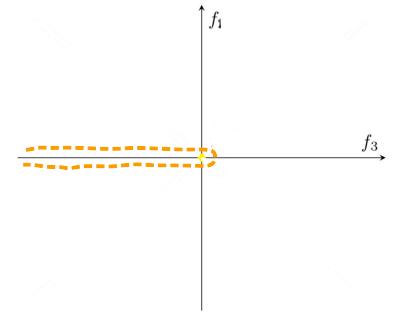
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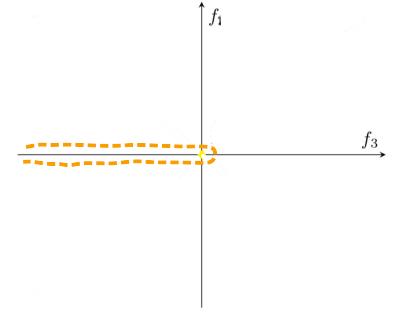
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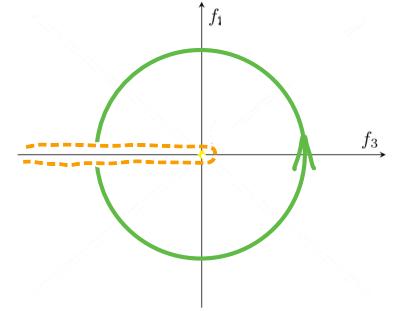
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$$\rightarrow \begin{bmatrix} 0 & 0 \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \Rightarrow v_+ = \begin{bmatrix} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{bmatrix} \xrightarrow{\begin{cases} f_3 = \cos \theta \\ f_1 = \sin \theta \end{cases} \quad -\pi < \theta < \pi} \begin{bmatrix} \cos \theta + 1 \\ \sin \theta \end{bmatrix}$$

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors v_+ corresponding to ω_+ , perform Gaussian elimination through elementary row operations:

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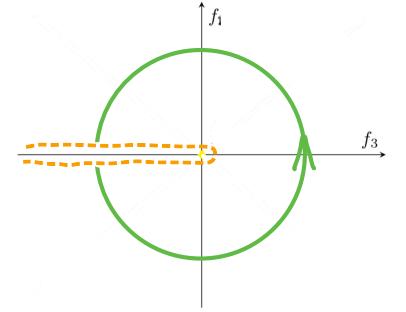
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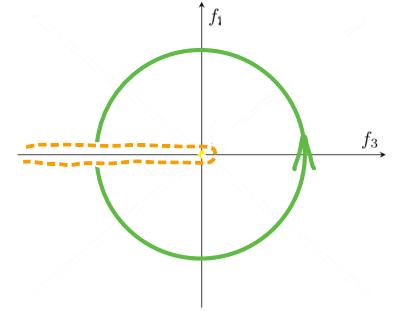


Observe that when $\theta \rightarrow (-\pi)_+$, we have $\cos \theta + 1 \rightarrow 0_+$ and $\sin \theta \rightarrow 0_-$,

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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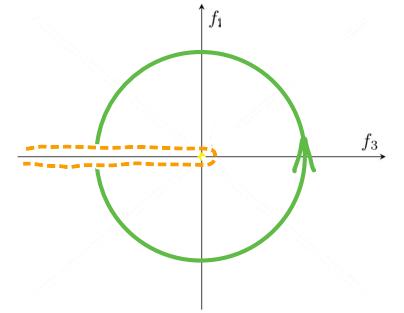


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We compute that

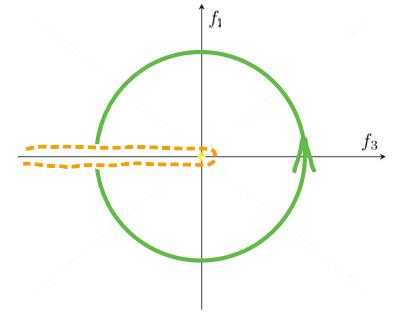
$$\lim_{\theta \rightarrow (-\pi)_+} \frac{v_+}{|v_+|} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

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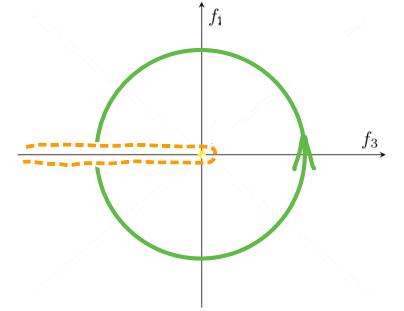
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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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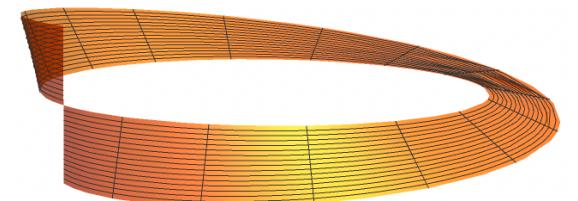
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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

Lemma. The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

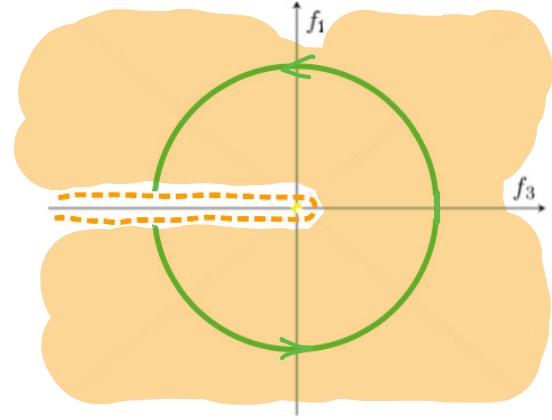
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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

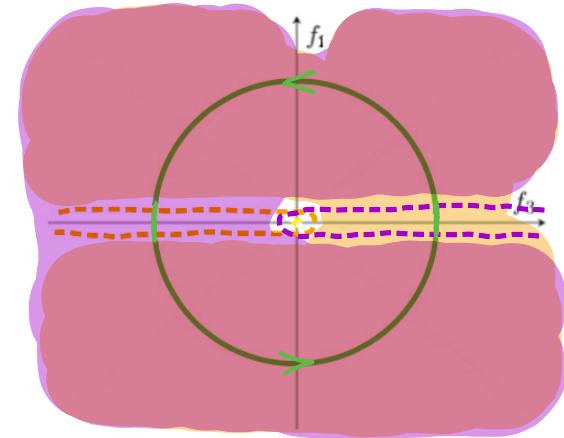
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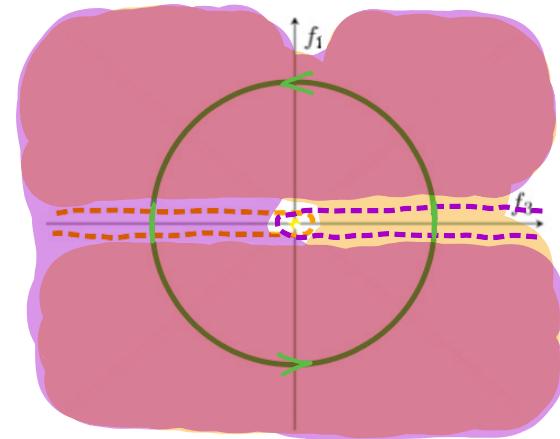
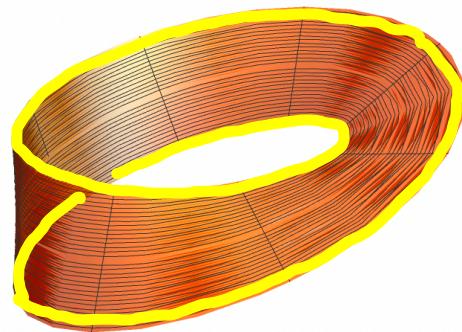
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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

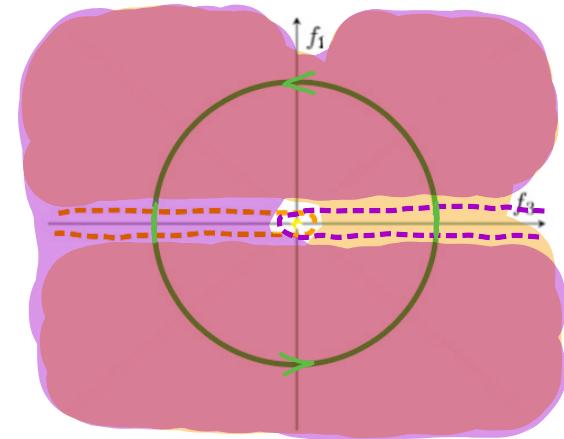
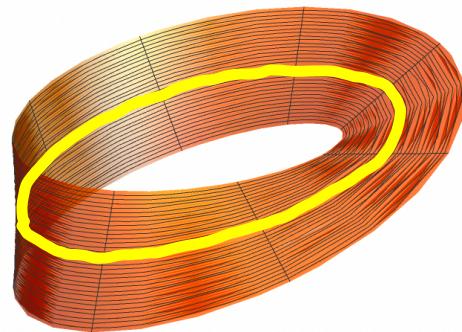
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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

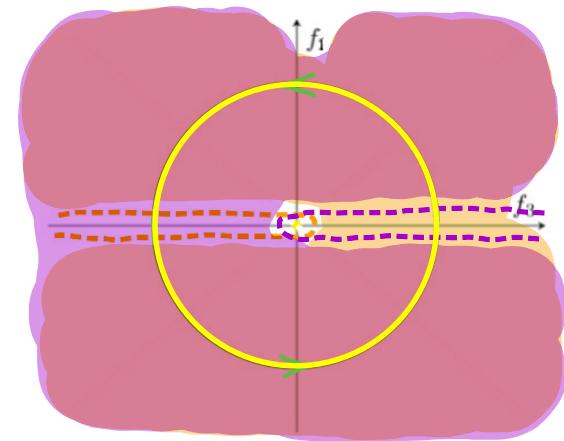
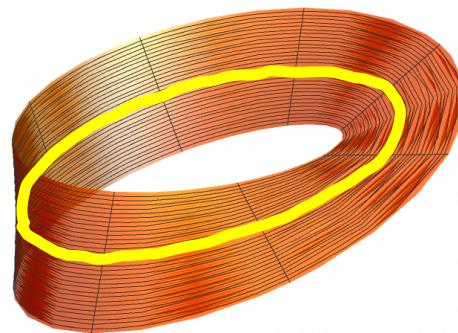
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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

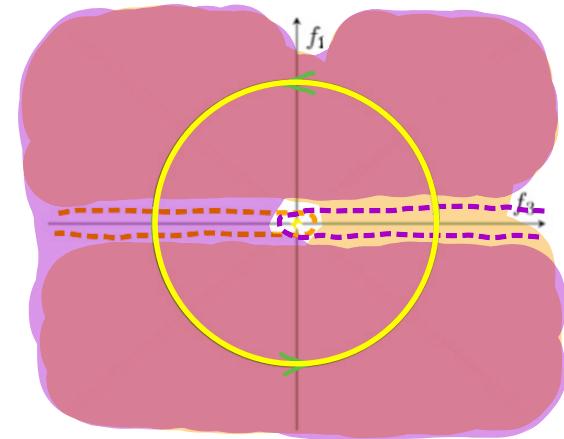
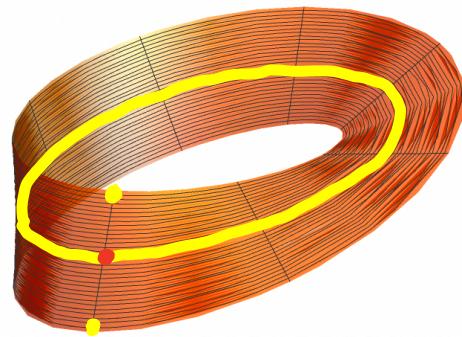
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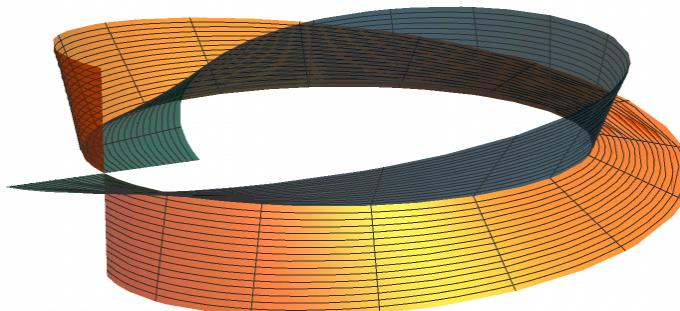
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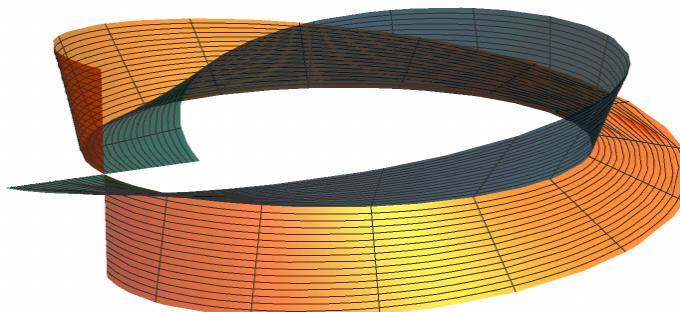


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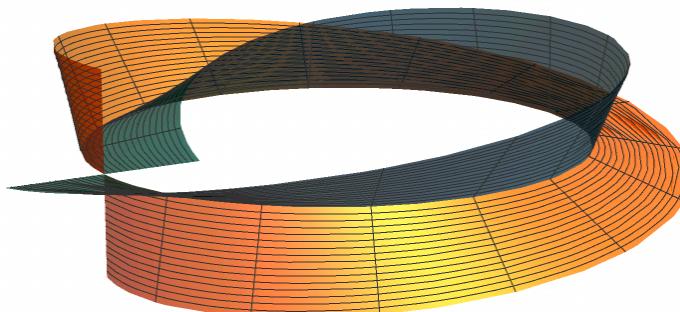


Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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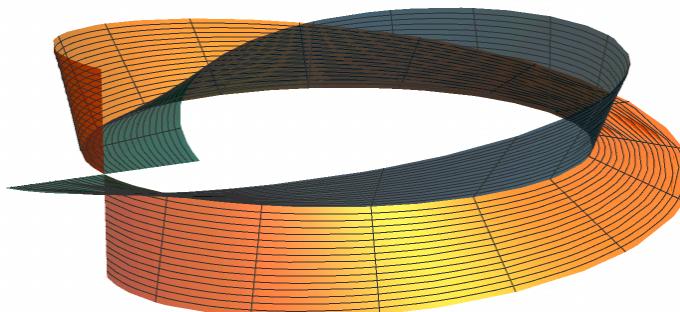


Eigenframe rotation as vector bundles: Revisiting the Hermitian case

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Eigenframe evolution as Higgs bundles: The non-Hermitian case



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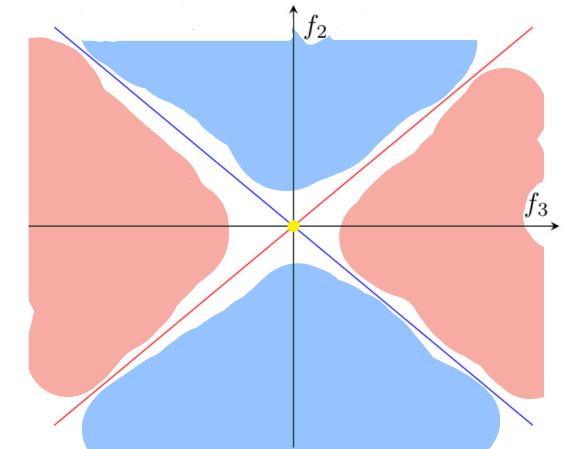
Recall that non-Hermitian 2-band systems

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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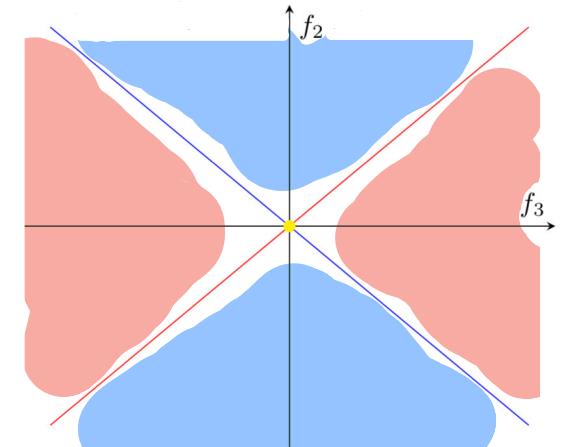


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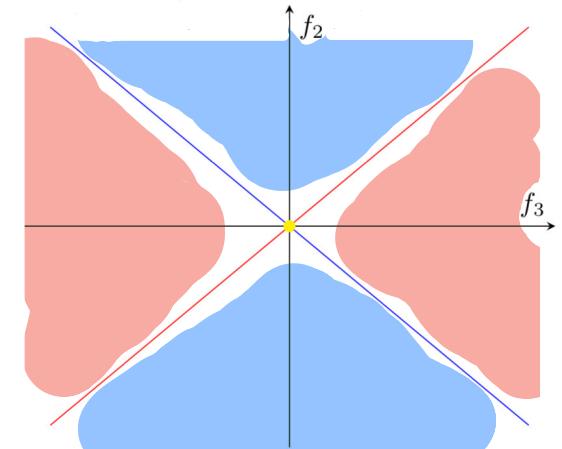


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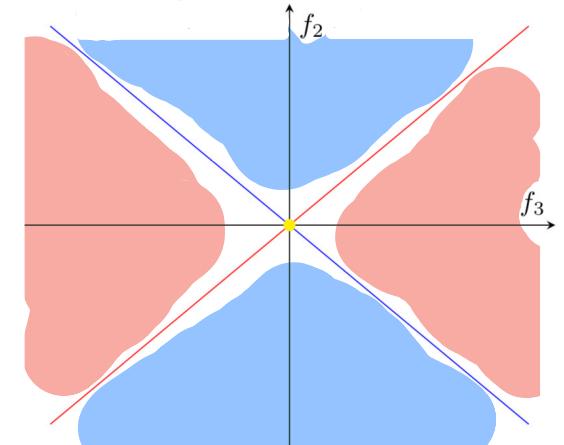


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2. Over $\{f_2 \neq \pm f_3\}$, H has 2 distinct eigenvalues. When $|f_2| < |f_3|$, the eigenvectors are **real**. When $|f_2| > |f_3|$, the eigenvectors are **not real**.

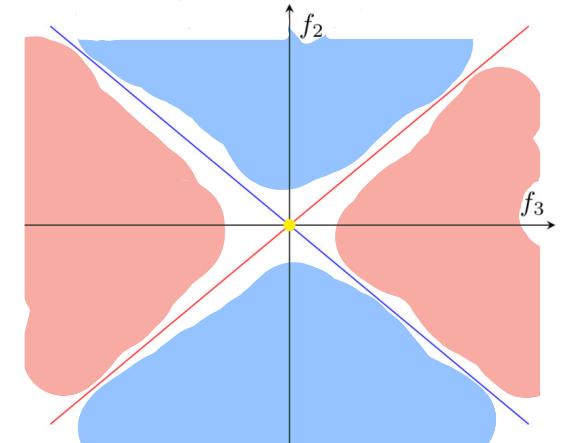


Eigenframe evolution as Higgs bundles: The non-Hermitian case

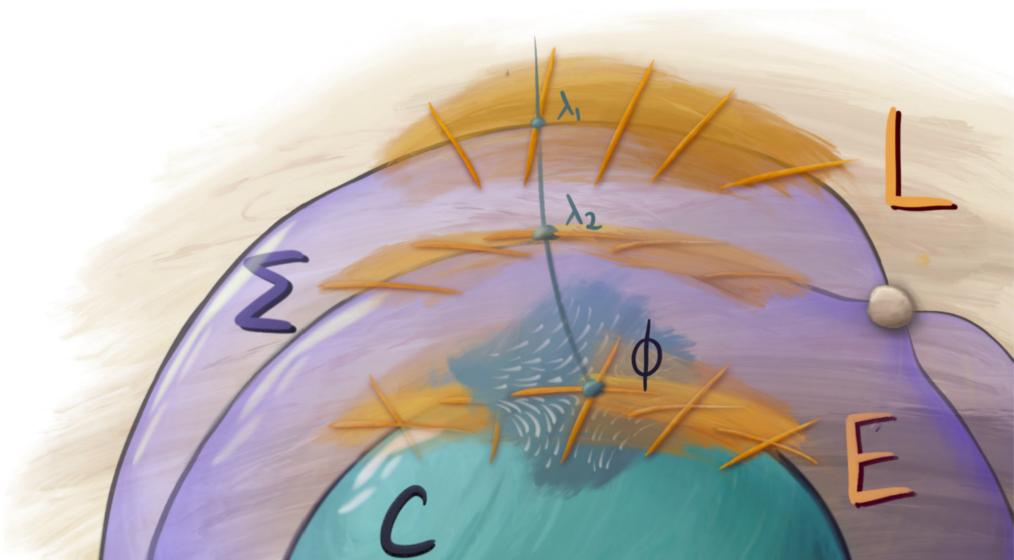
Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

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A *Higgs bundle* $(E, \phi) \rightarrow C$ is essentially a family of matrices

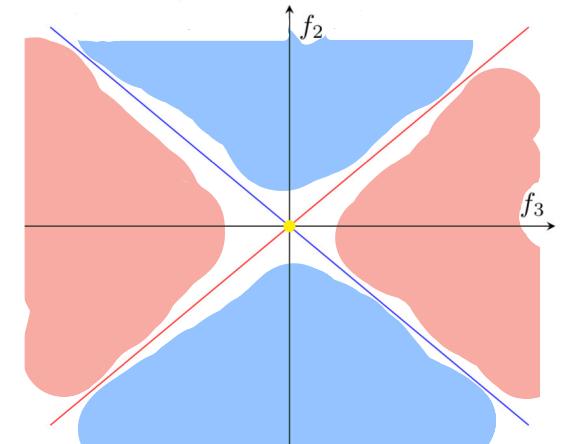


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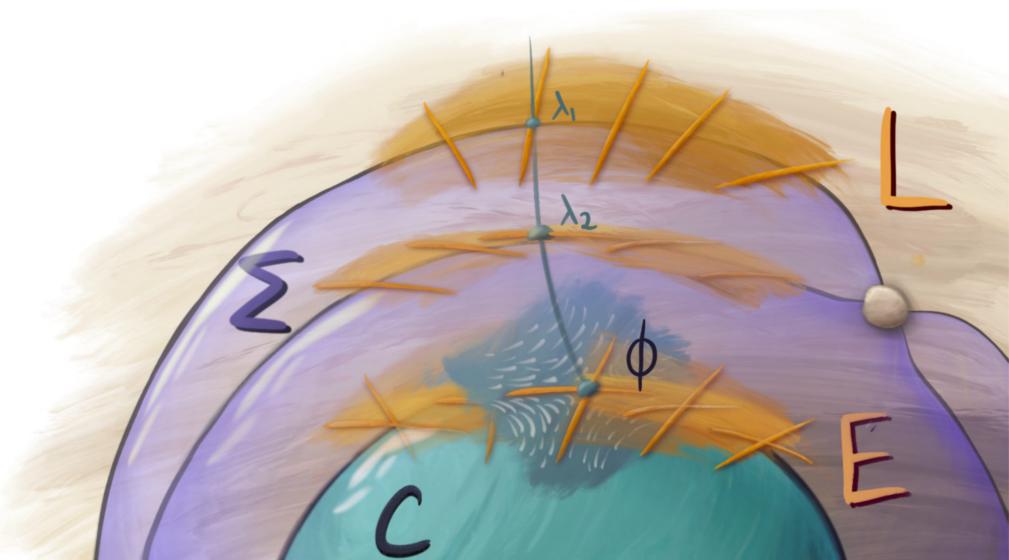
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Peter Higgs (bosons)

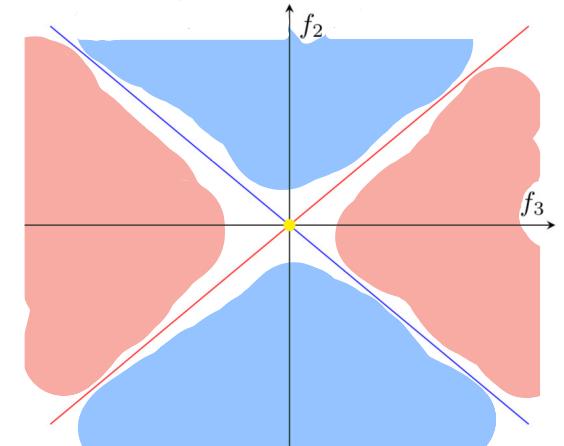


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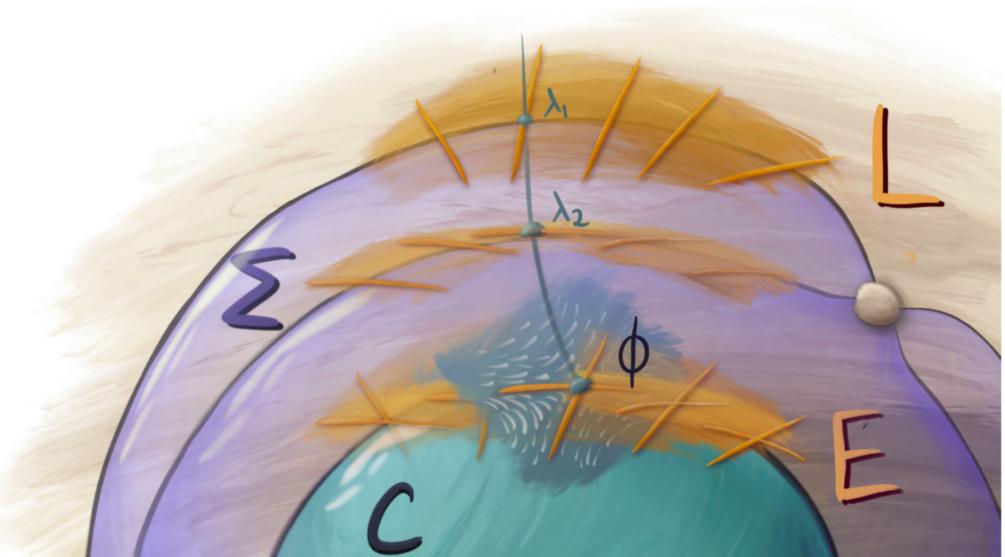


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Peter Higgs (bosons)



1929–2024

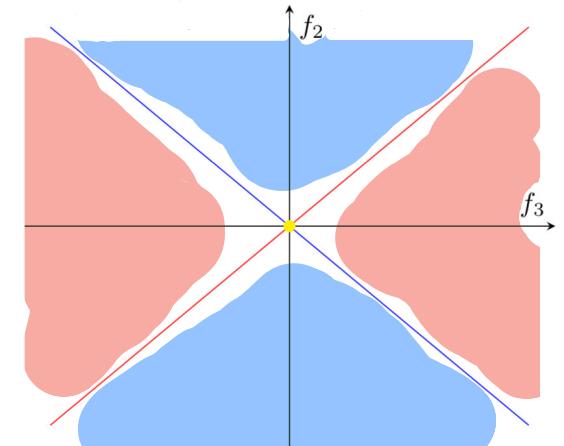


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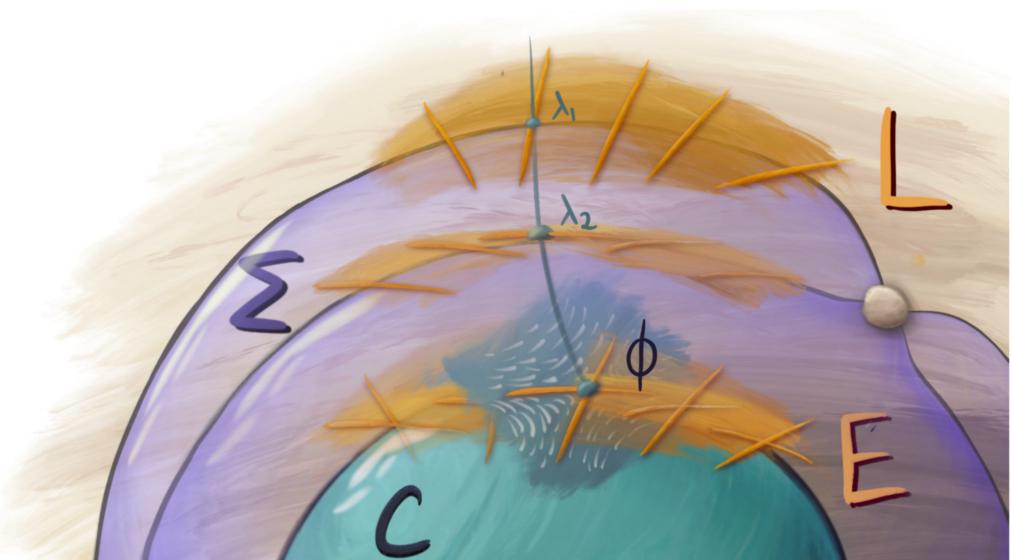
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Nigel Hitchin 1987

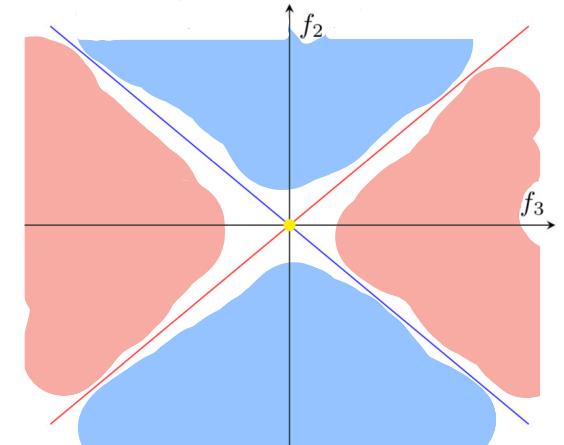


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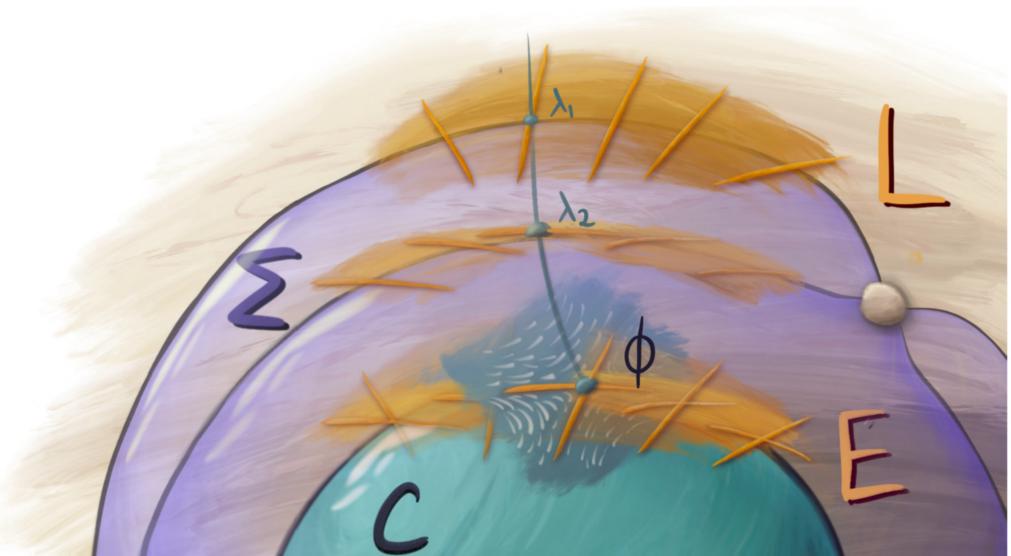


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Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface

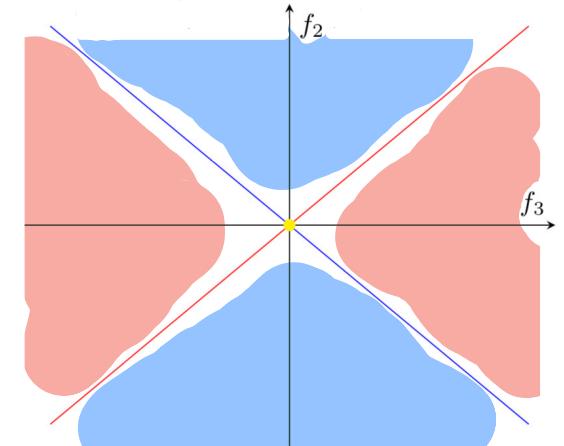


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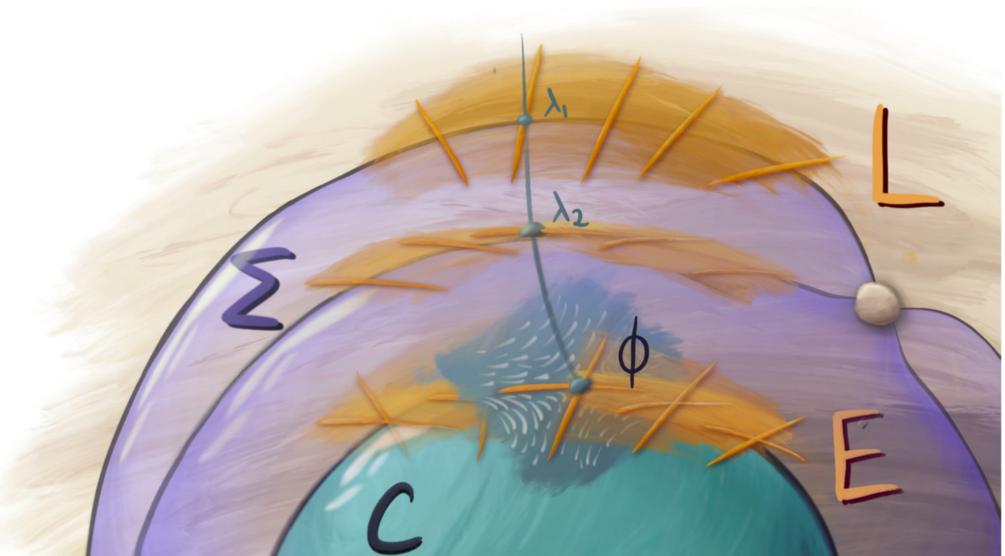


A *Higgs bundle* $(E, \phi) \rightarrow C$ is essentially a family of matrices

Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface
E holomorphic vector bundle

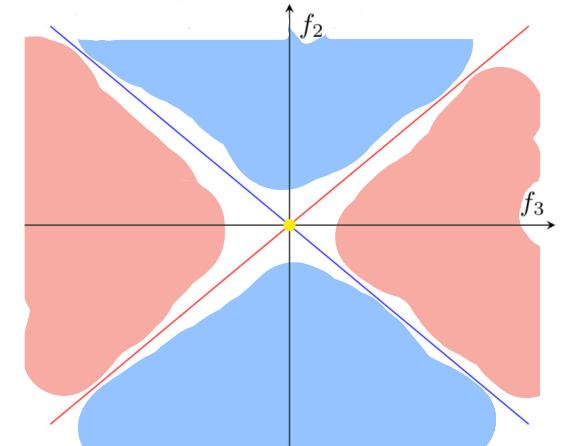


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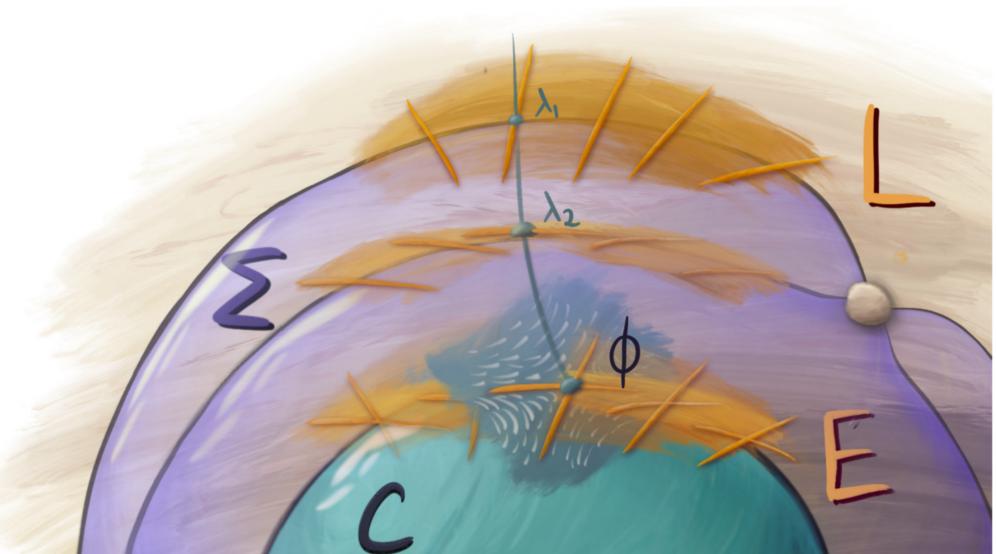
Peter Higgs (bosons)

Nigel Hitchin 1987

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ϕ Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of E

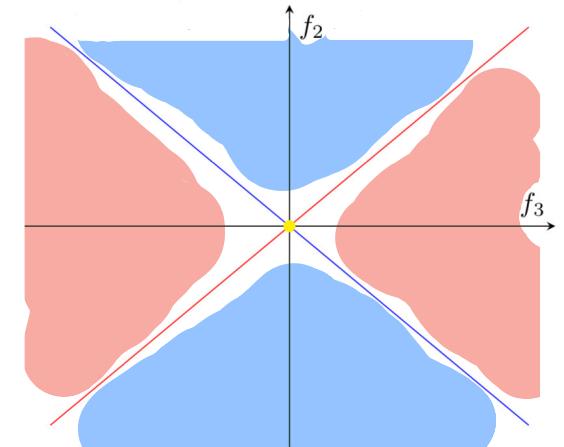


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A *Higgs bundle* $(E, \phi) \rightarrow C$ is essentially a family of **matrices**

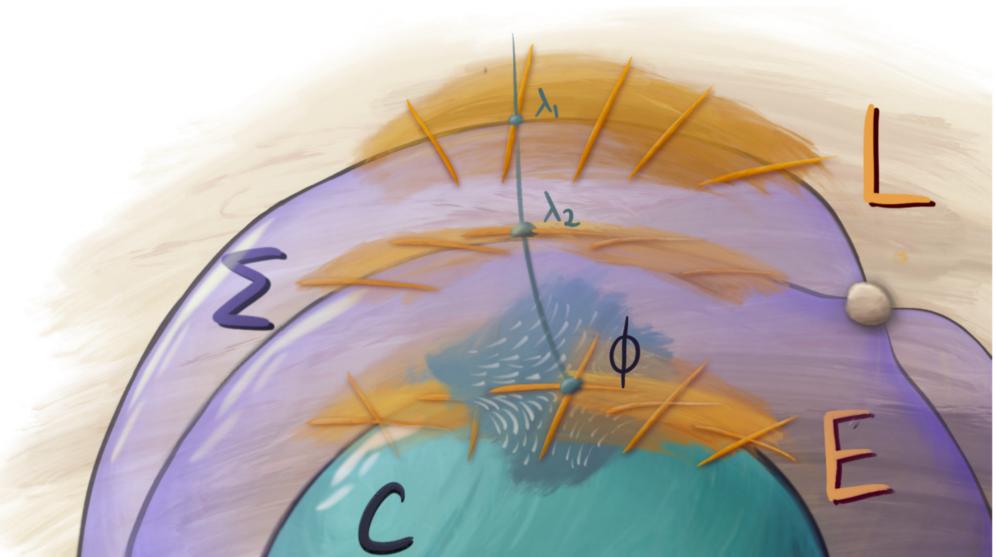
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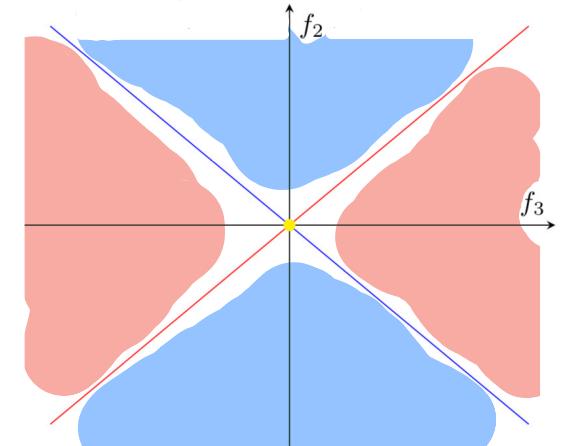


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Peter Higgs (bosons)

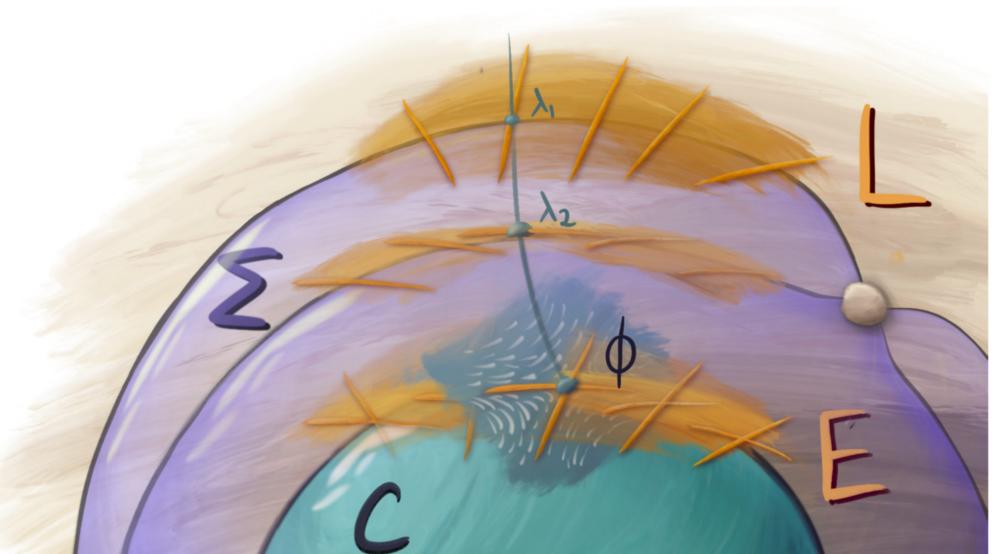
Nigel Hitchin 1987

Carlos Simpson

C compact Riemann surface (or more generally Kähler manifold)

E holomorphic vector bundle

ϕ Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of E such that $\phi \wedge \phi = 0$

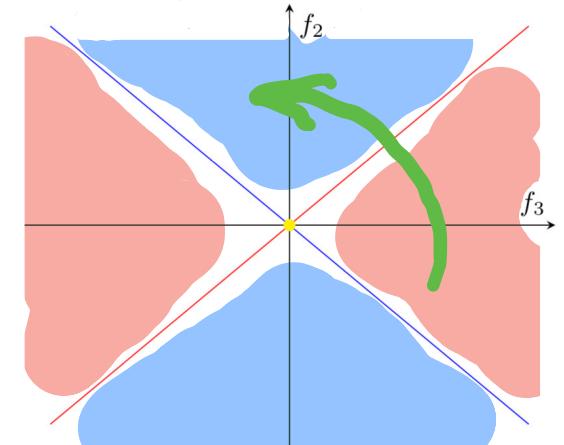


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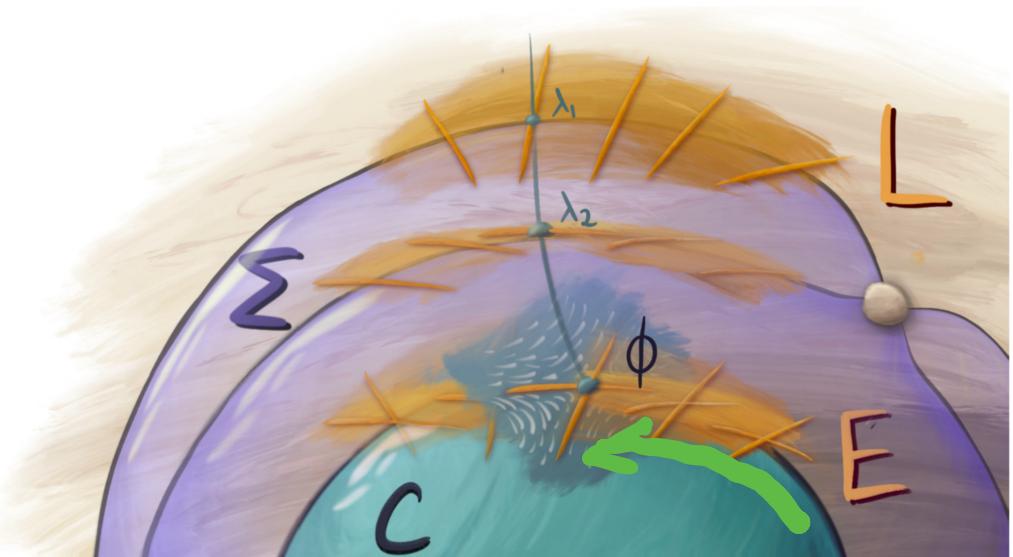
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$$\phi_x \in \text{End}(E_x), x \in C$$

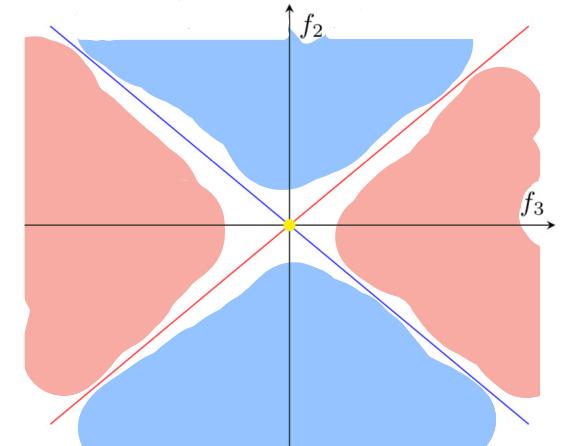


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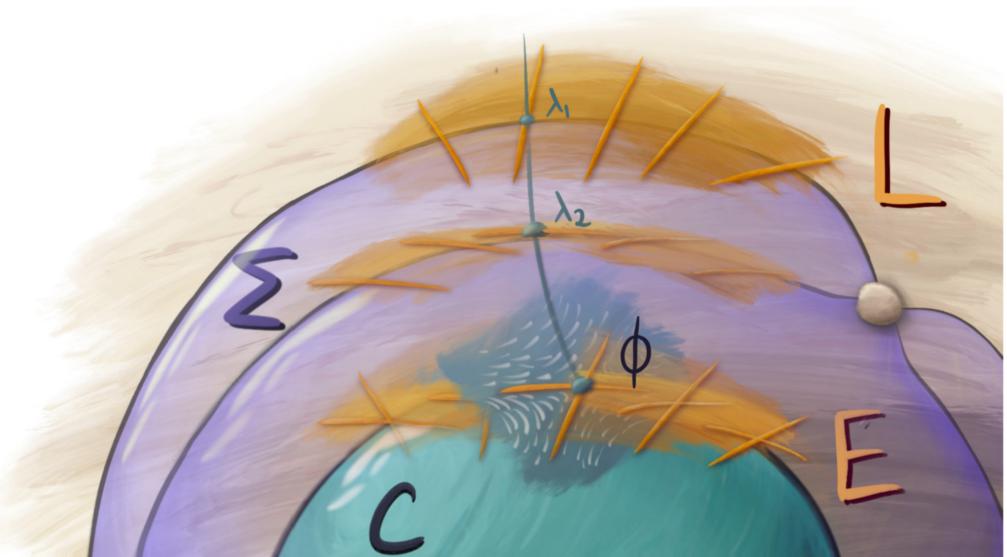
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Portrait from Kienzle and Rayan,
Hyperbolic band theory through Higgs bundles, **Adv. Math.**, 2022.

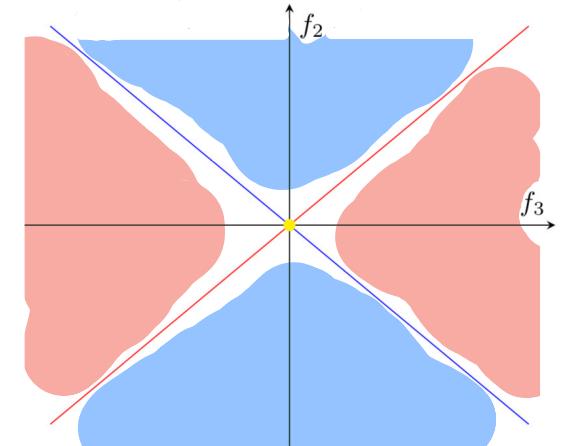


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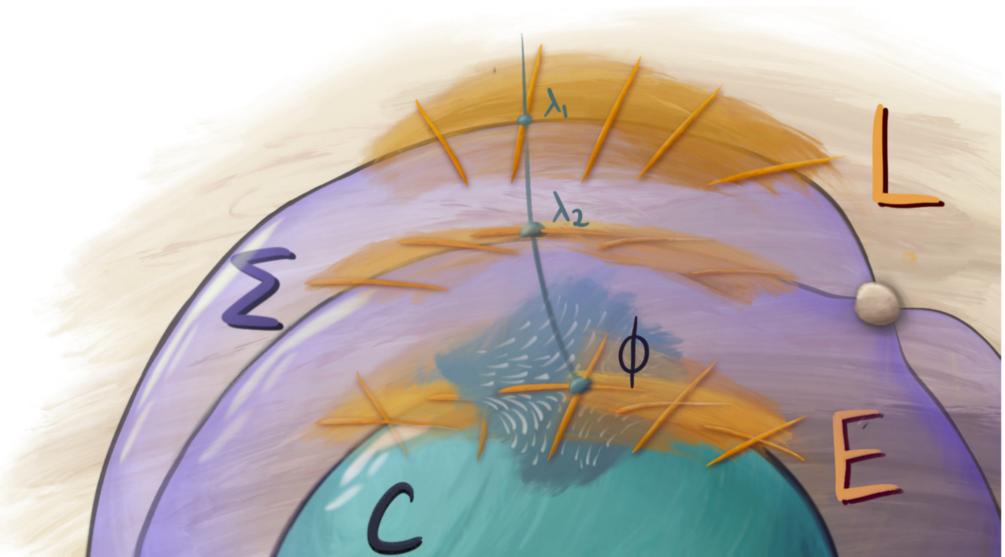
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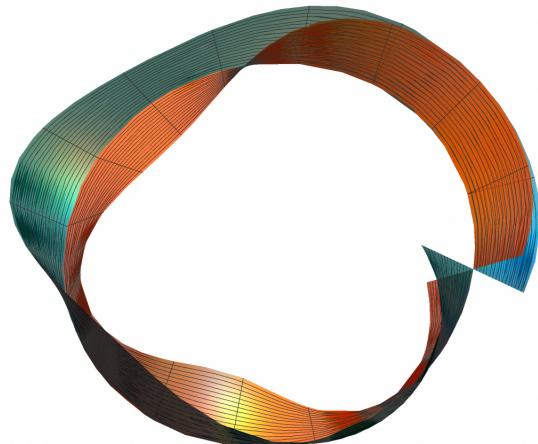
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Hyperbolic metric on the base C . Kollár et al., *Nature*, 2019.



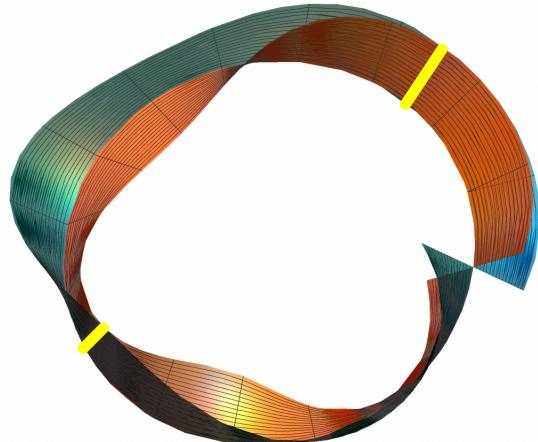
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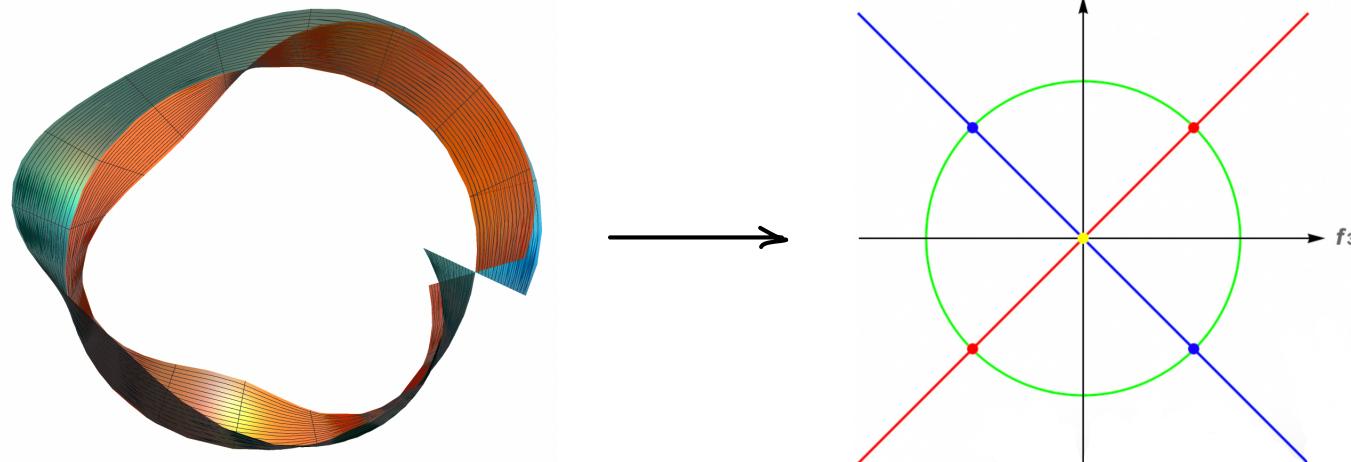
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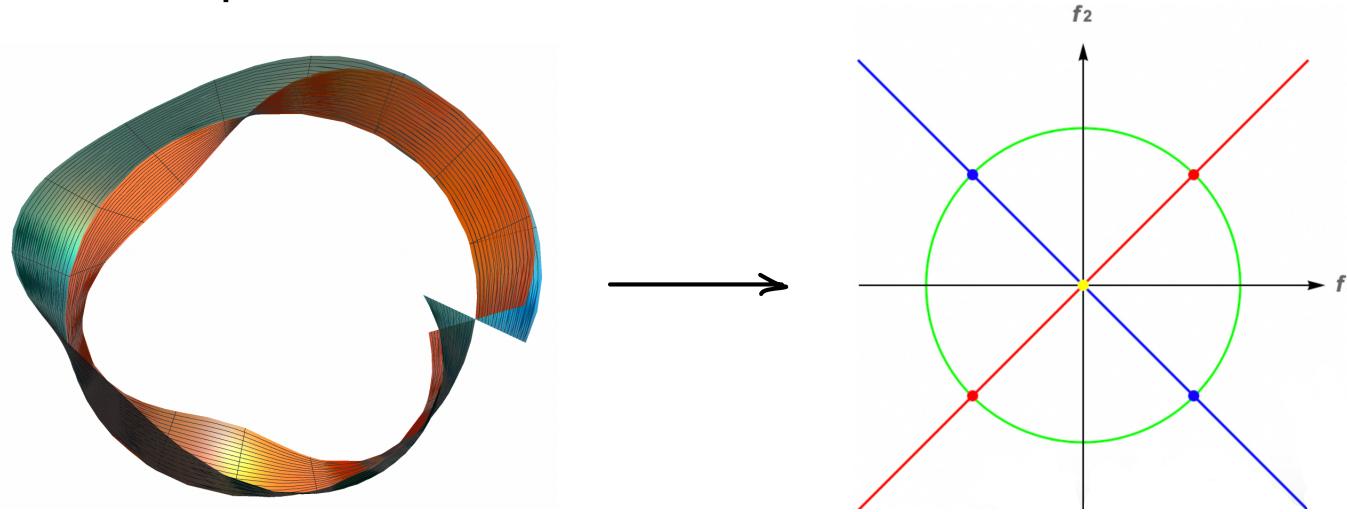
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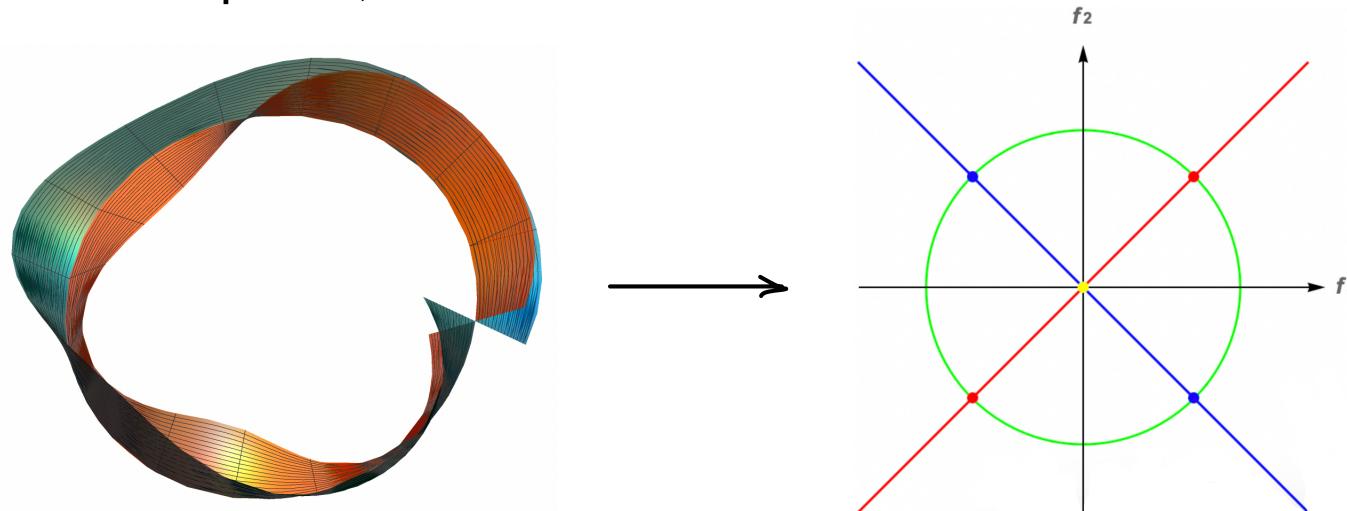
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Eigenframe evolution as Higgs bundles: The non-Hermitian case

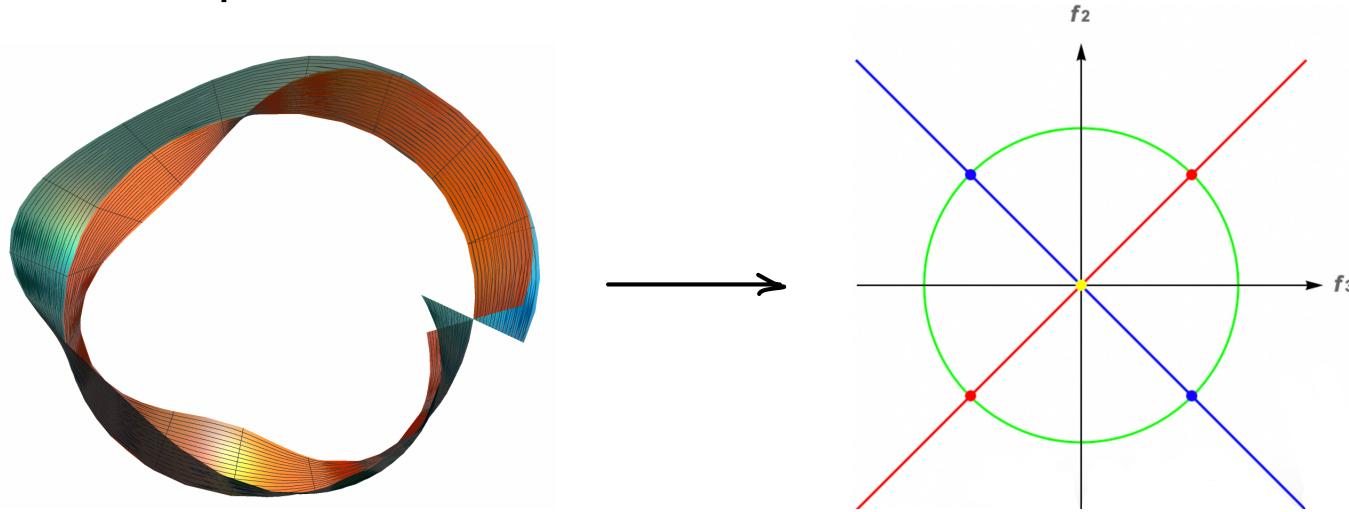
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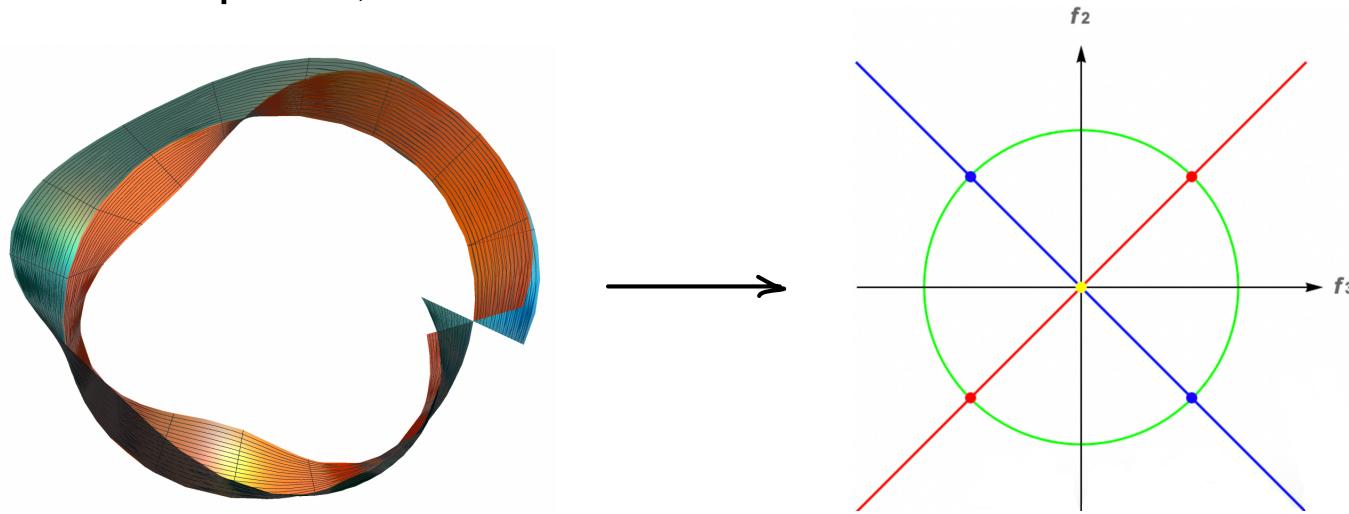


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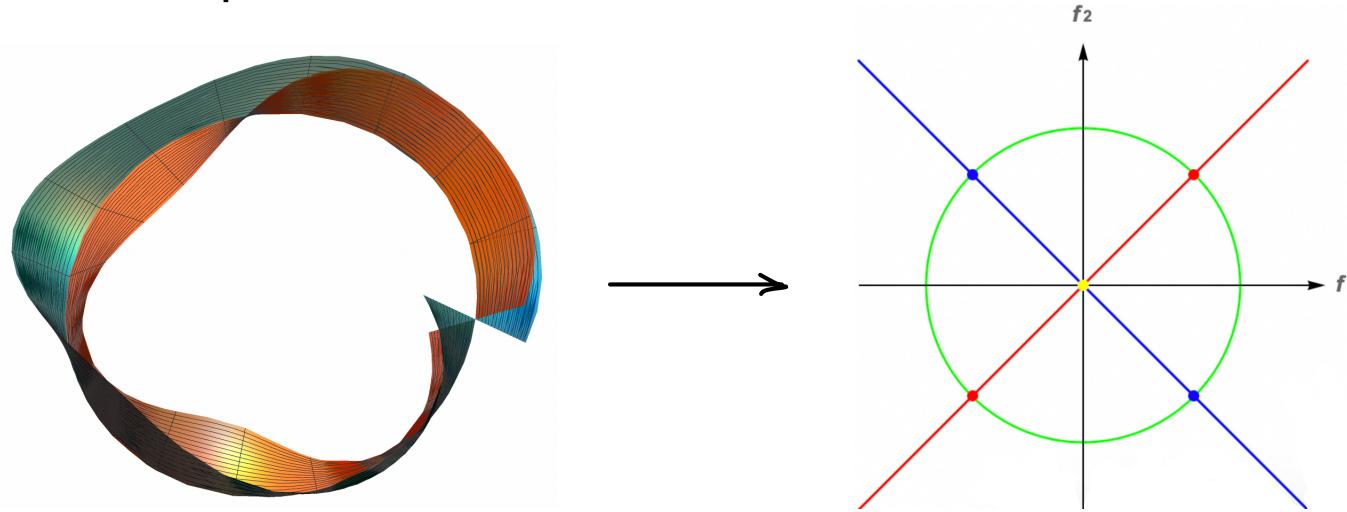


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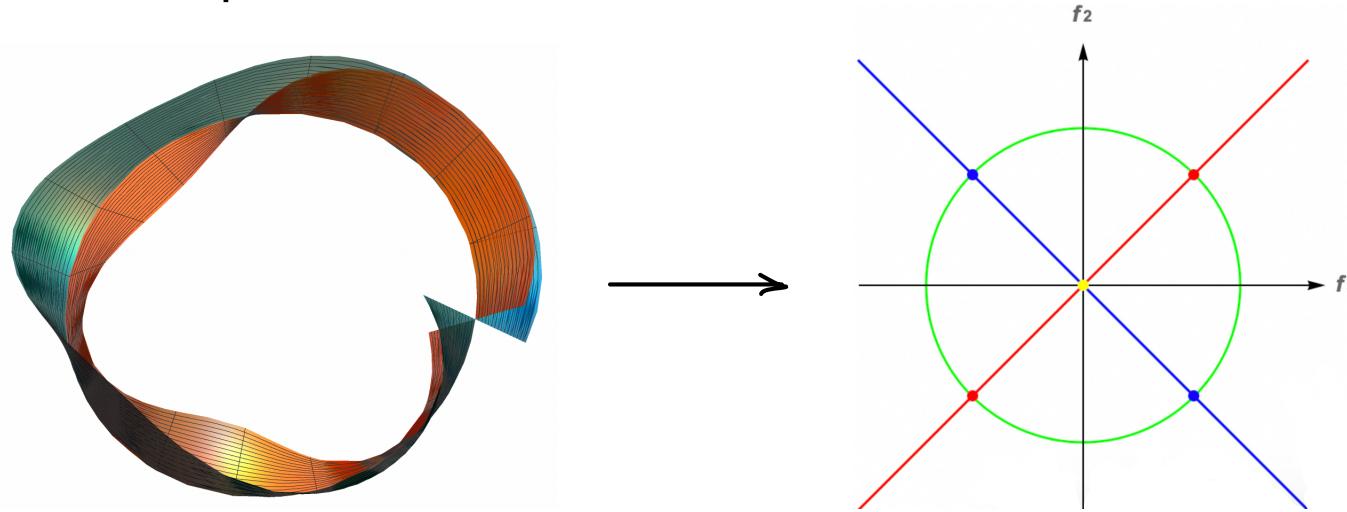
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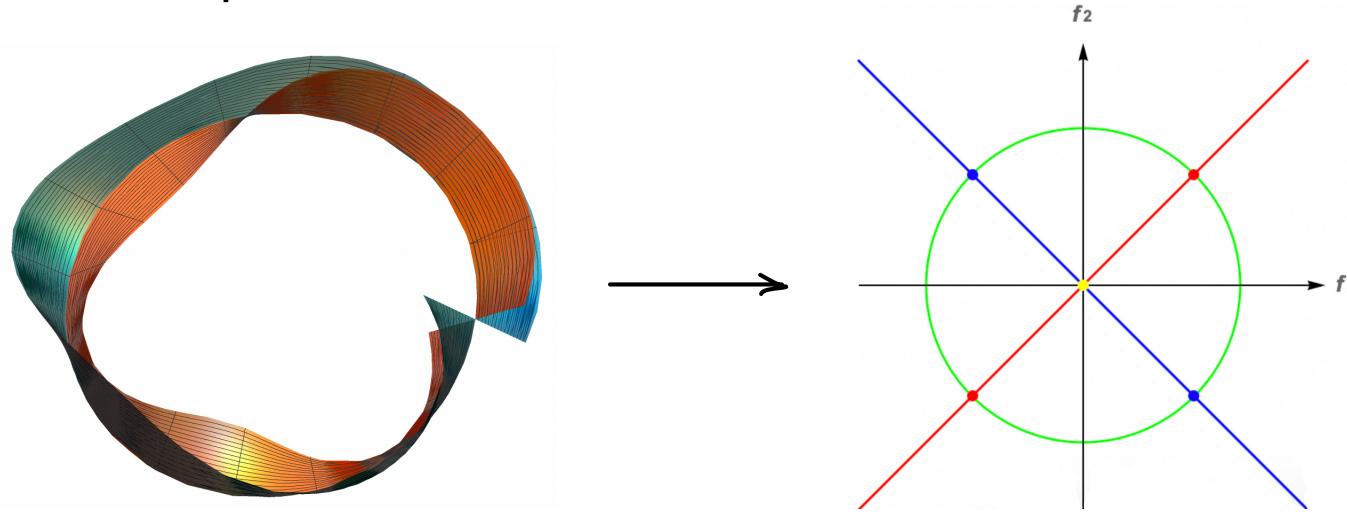
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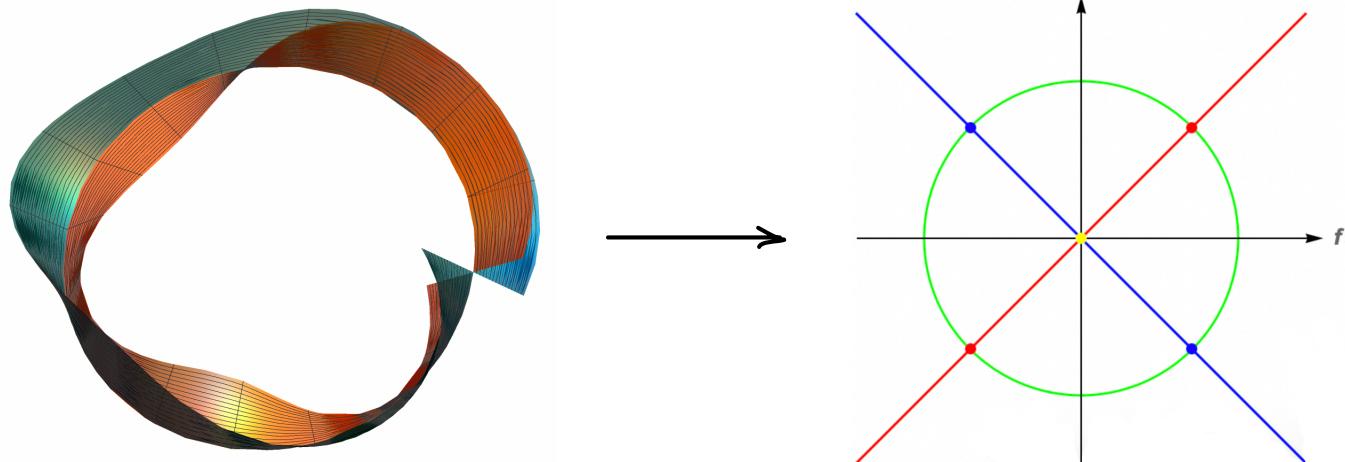
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The intrinsic geometry should be independent of real/complex coordination, though.

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Question. How to compute the topological charge?

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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Mathematical interlude: Classification of bundles

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$$\begin{array}{ccc} V \\ \downarrow \\ X \end{array}$$

Eigenframe evolution as Higgs bundles: The non-Hermitian case

Question. How to compute the **topological charge**?

Mathematical interlude: Classification of bundles

$$\begin{array}{ccc} V & E & \textit{universal bundle} \\ \downarrow & \downarrow & \\ X & B & \textit{classifying space} \end{array}$$

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$$\begin{array}{ccc} V & E & \text{universal bundle} \\ \downarrow & \downarrow & \\ X & \xrightarrow{f} & B \\ & & \text{classifying space} \\ & & \text{classifying map} \end{array}$$

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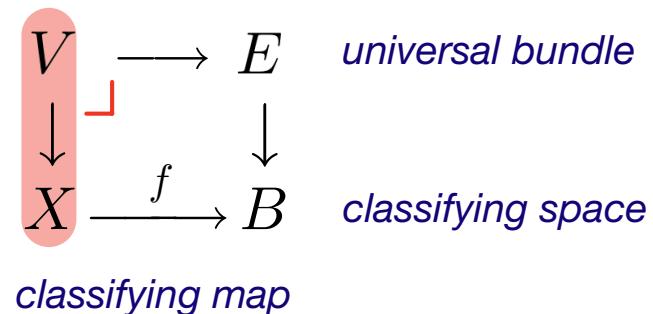
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Eigenframe evolution as Higgs bundles: The non-Hermitian case

Question. How to compute the **topological charge**?

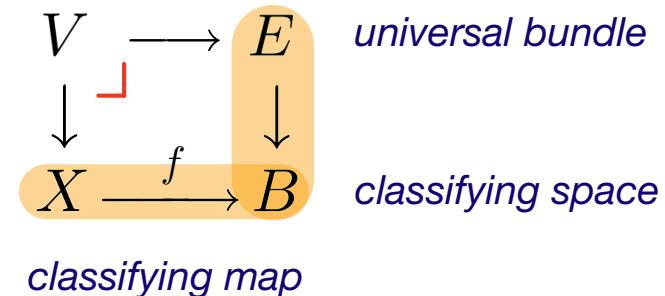
Mathematical interlude: Classification of bundles



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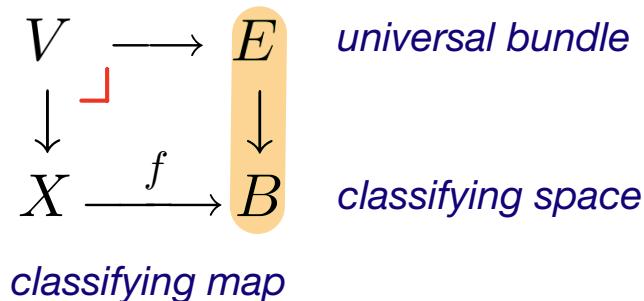
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- Describe the **universal bundle**

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This breaks the classification problem into two parts:

- Describe the universal bundle
- Find **computable** and **effective algebraic invariants** (topological charge) for the classifying/moduli space

Eigenframe evolution as Higgs bundles: The non-Hermitian case

Question. How to compute the topological charge?

In progress: Need to compute the *intersection fundamental group* of the **stratified** moduli space.

*Gajer, The intersection Dold–Thom theorem,
Topology, 1996. (Ph.D. student of Blaine Lawson, 1993)*

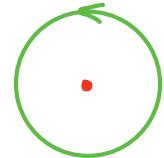
Goresky and MacPherson, 1974.

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
\mathbb{Z}	0	0	0
0	0	0	0

$\xrightarrow{\bar{p}}$

Intersection homology of \mathbb{R}^2 with one singular point: from top to bottom are $I^{\bar{p}}H_0, I^{\bar{p}}H_1, I^{\bar{p}}H_2$, where \bar{p} is the perversity function.

From blue to red regions, they detect the singular point.

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Tolerance of ill-behaved cycles

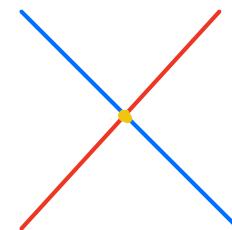
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0	0	0	0
—	—	—	—
\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
0	0	0	0
0	0	0	0
—	—	—	—
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}	\mathbb{Z}
0	0	0	0
0	0	0	0
—	—	—	—
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}	\mathbb{Z}
0	0	0	0
0	0	0	0

Intersection homology of \mathbb{R}^2 with a pair of intersecting singular lines:
from top to bottom are $I^{\bar{p}}H_*$ with $* = 0, 1, 2$.

From green to blue regions, they detect the singular lines.

From blue to red regions, they detect the intersection point.

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- Still need compatibility with our earlier ad hoc non-Hermitian classification:

$$\pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

May need to work at the chain level.

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Here is a video of the eigenbundle deformation: <https://yifeizhu.github.io/swallowtail/deform.mp4>

Eigenframe evolution as Higgs bundles: The non-Hermitian case

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Example (Swallowtail quadruple sw4).

$$\begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$$

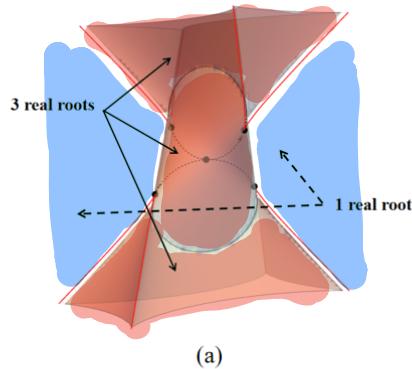
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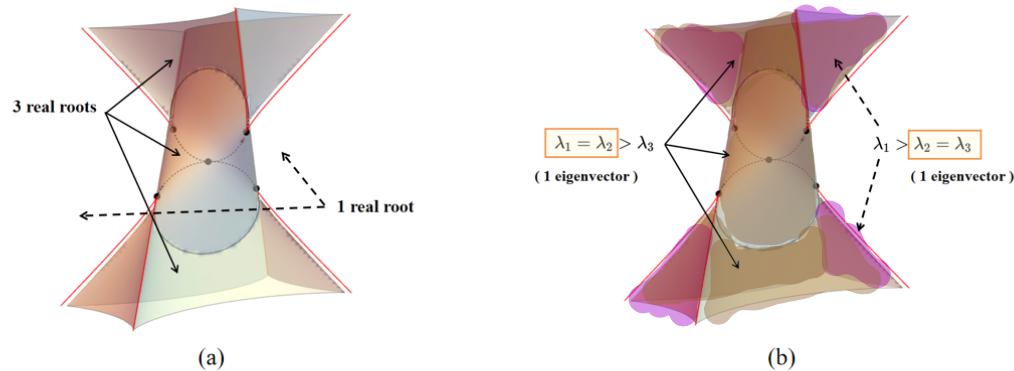
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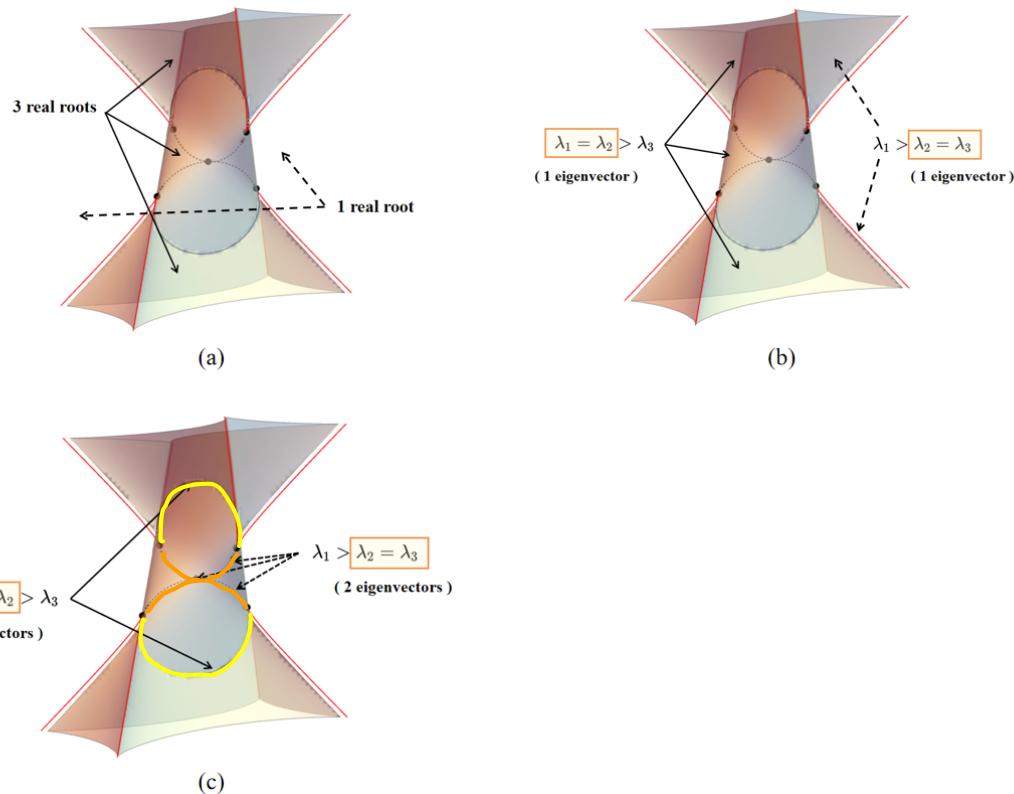
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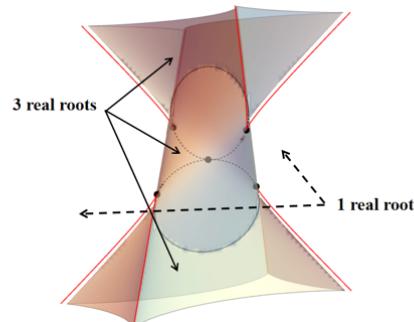
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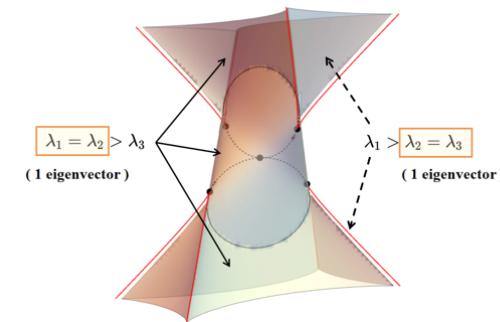
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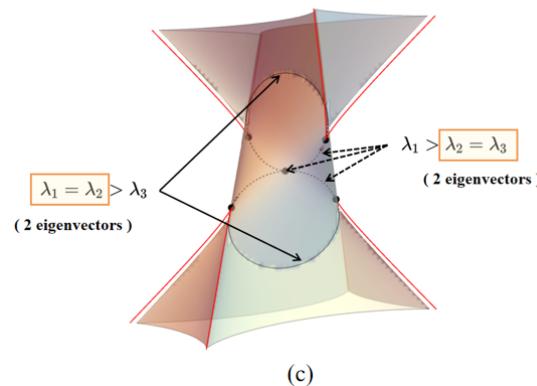
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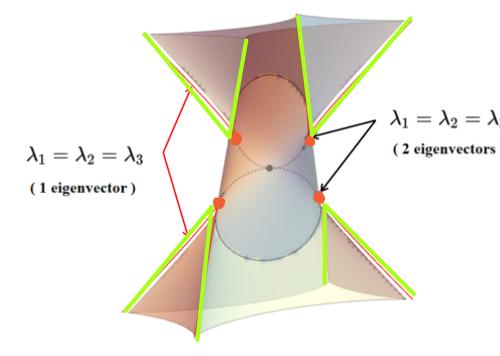
(a)



(b)



(c)



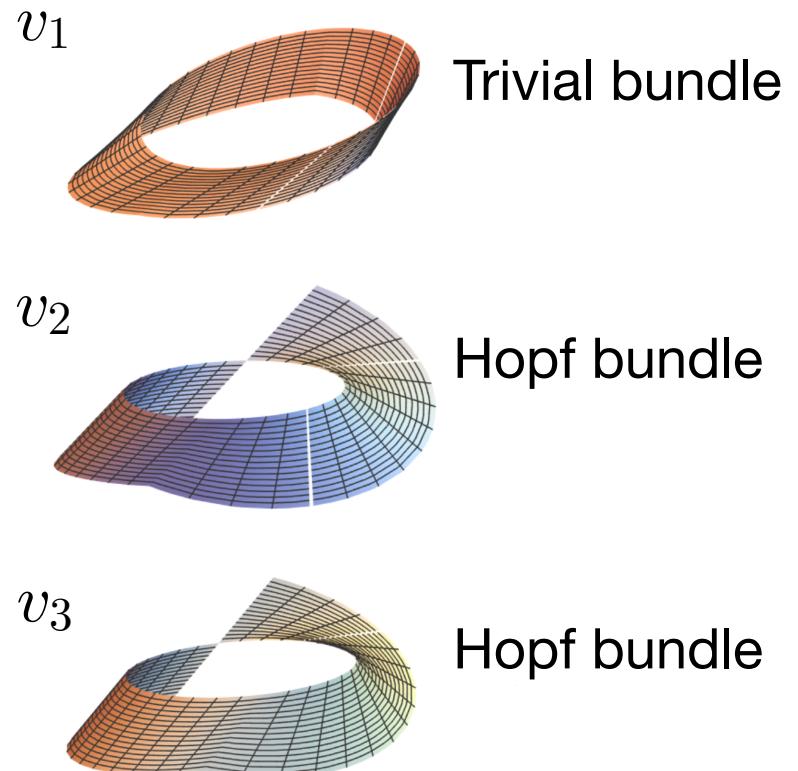
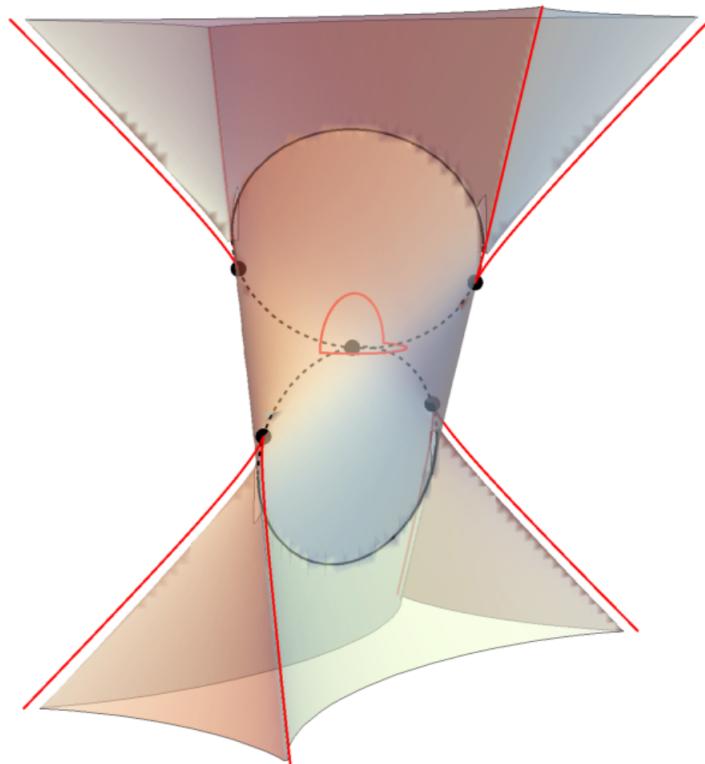
(d)

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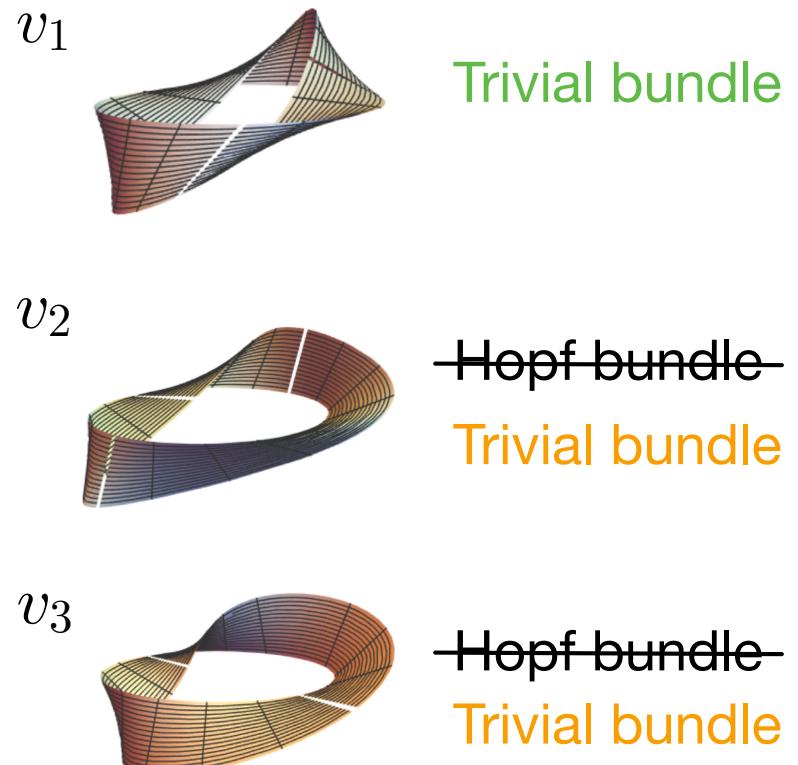
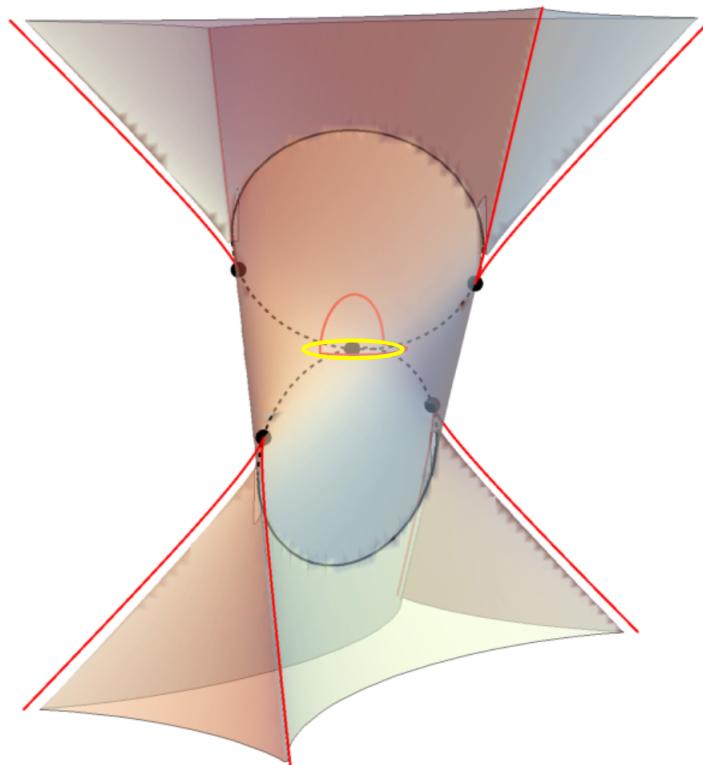


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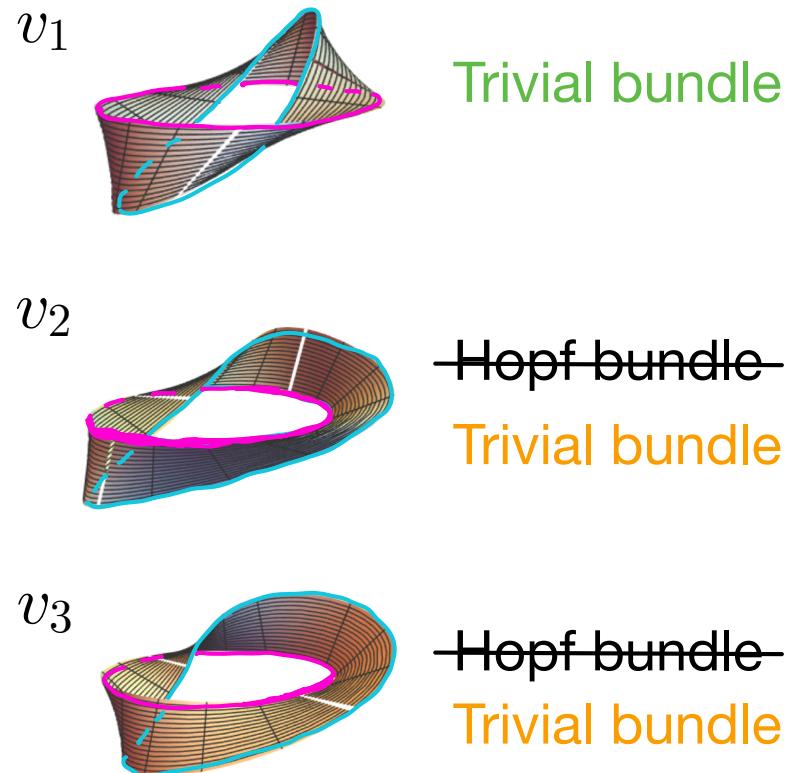
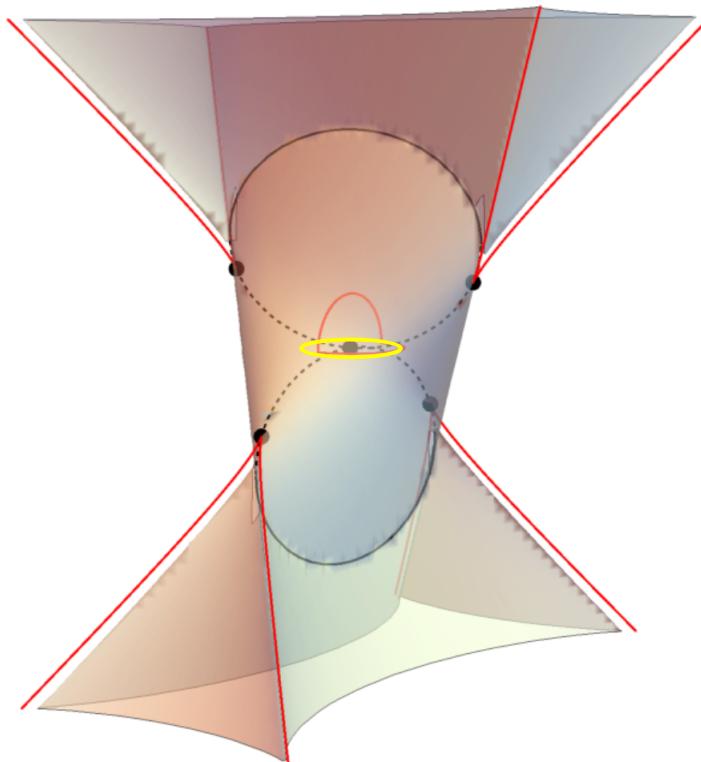


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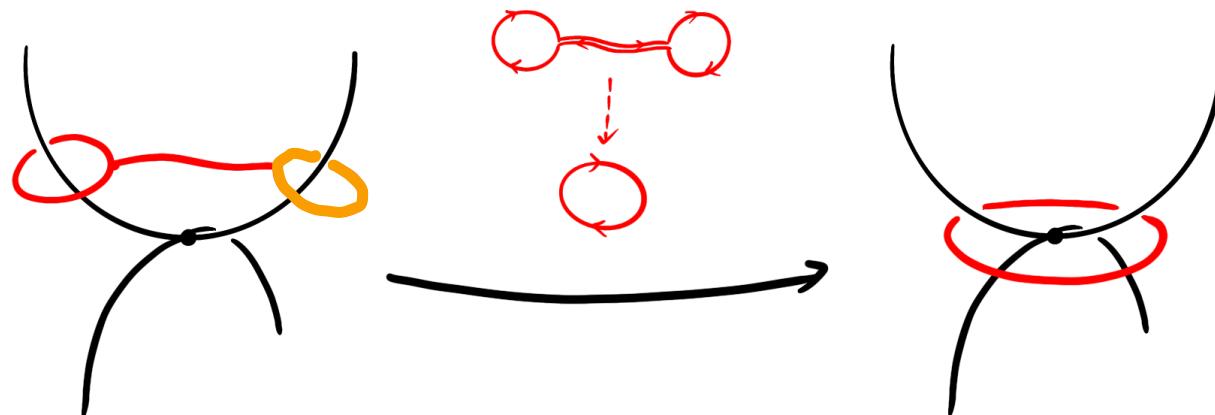
*Linking number is “over-sensitive” when it involves a loop in the **base space** and a loop in the **total space**.*

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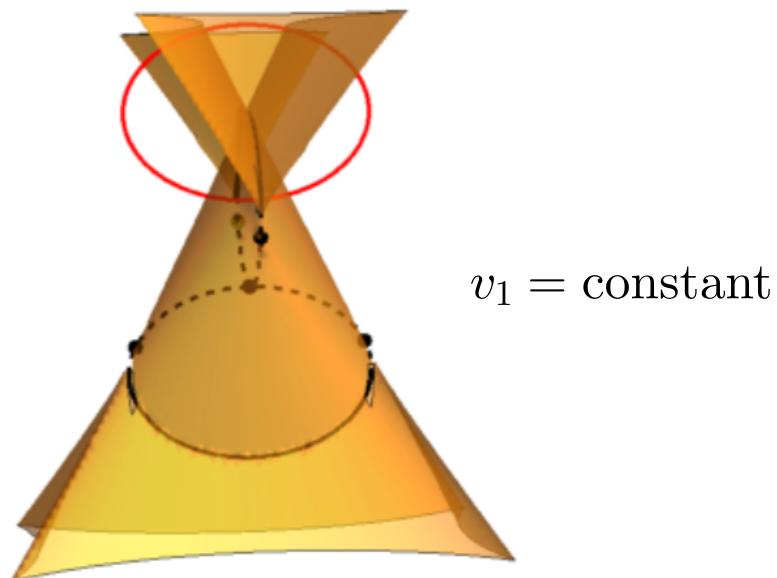


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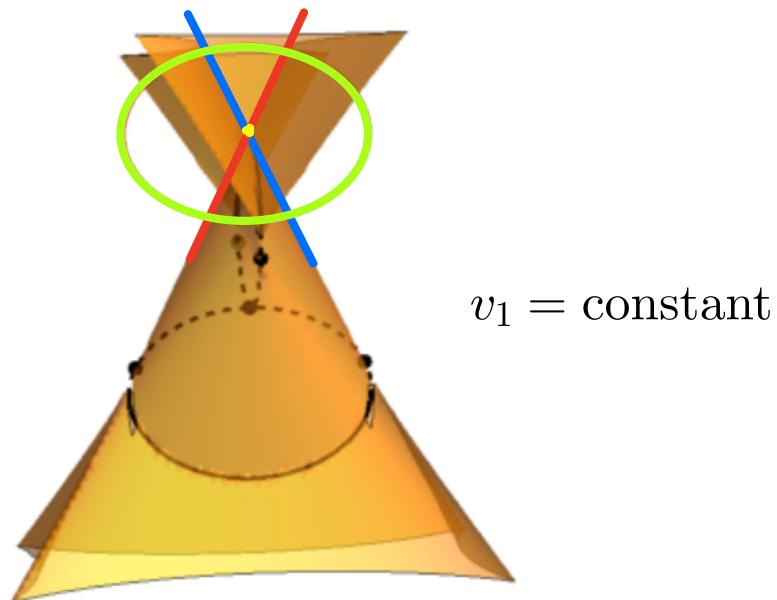


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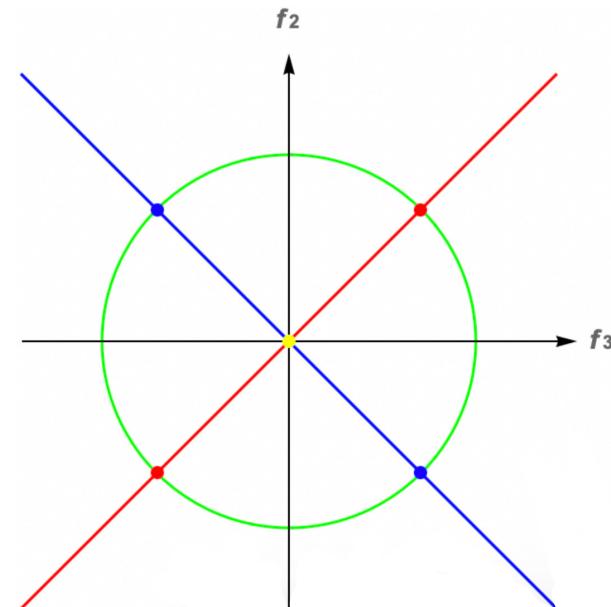
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$$v_1 = \text{constant}$$



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*a “**family of families**”*

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Example (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}, \text{ where each } g_i \text{ is a linear function of the parameters } f_j.$$

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Here is visualization from 3 angles of deforming sw4, with nodal lines degenerating:

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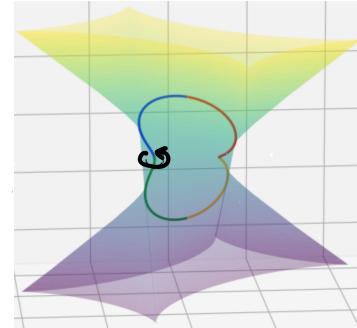
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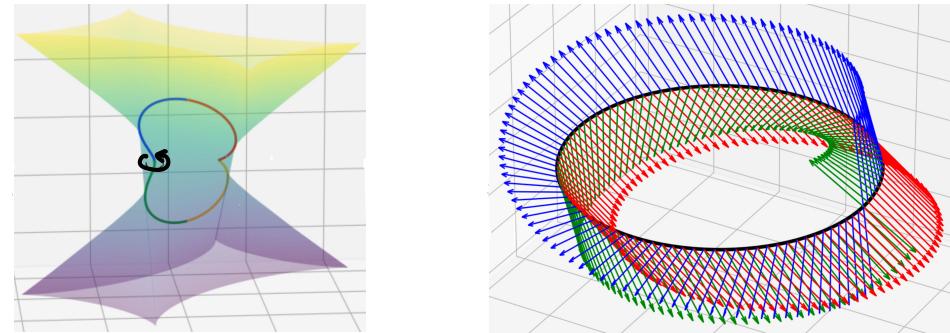
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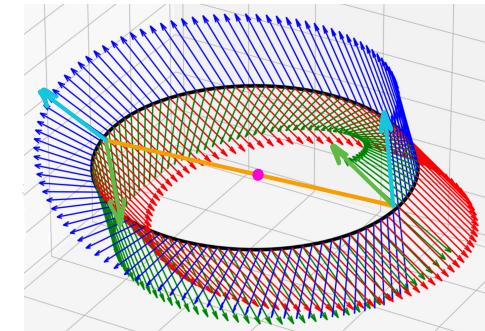
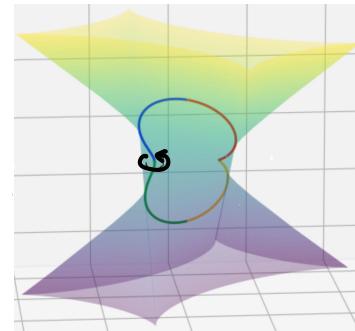
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Across the *center* (nodal line),
the *blue* and *green* eigenstates
swap

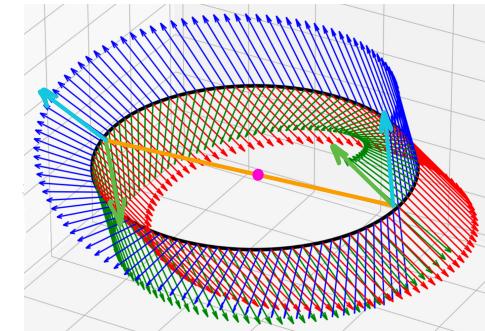
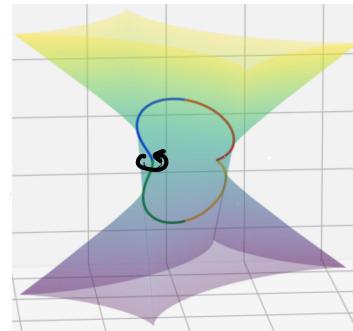
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- Opening of 2 tunnels and a new “big” loop around, along which the rank-3 eigenbundle is trivial
- Merging of 8 cuspidal lines into 4
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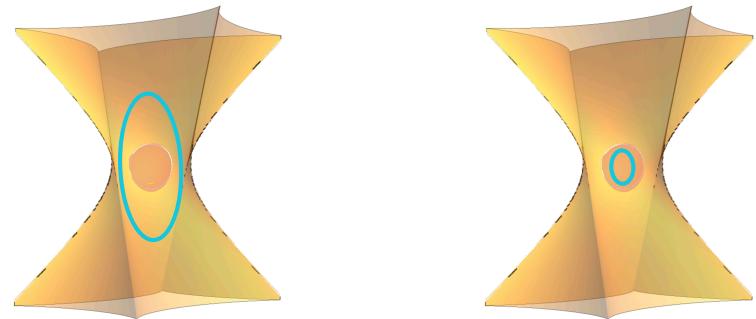
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*Shrinking this loop into the **enclosed region**, we find the eigenbundle along it remains trivial.*

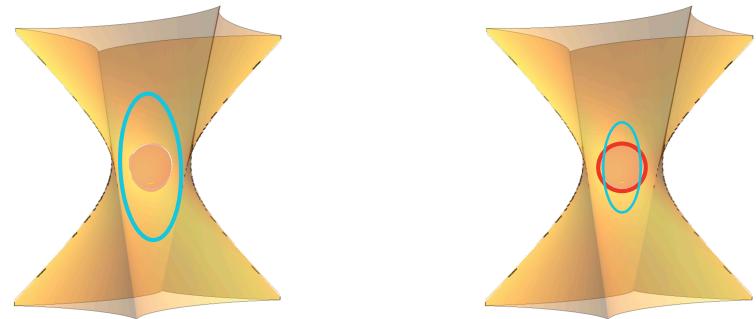
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*What about loops transversing the **nodal intersection lines**? Band inversion again?*

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*tangent developable, along the **cuspidal lines***

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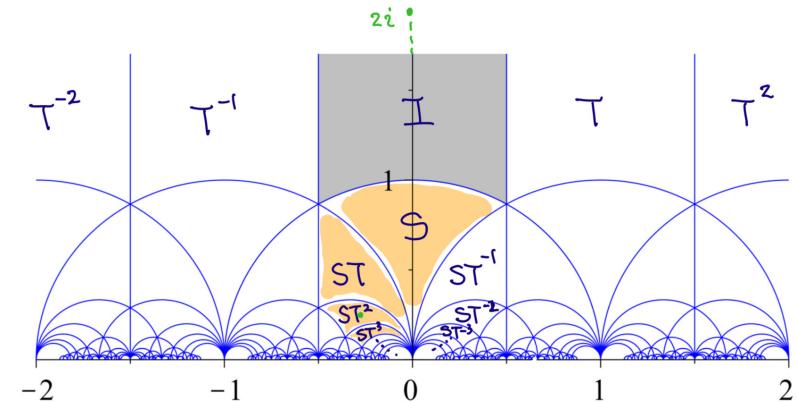
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A prototypical 2D hyperbolic lattice
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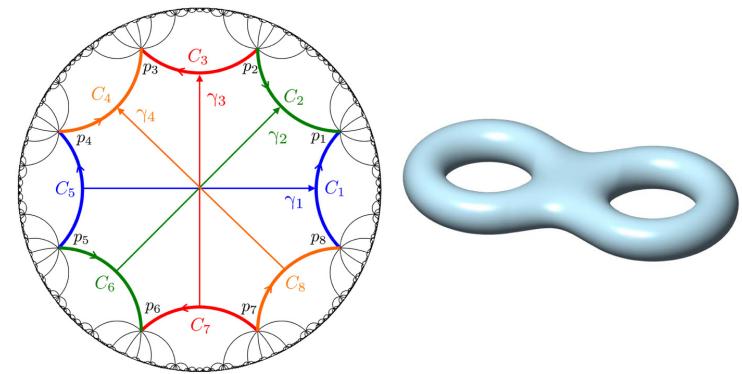
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Another basic example of a hyperbolic lattice associated to a genus-2 surface
(from Maciejko and Rayan, *Hyperbolic band theory*, **Sci. Adv.**, 2021)

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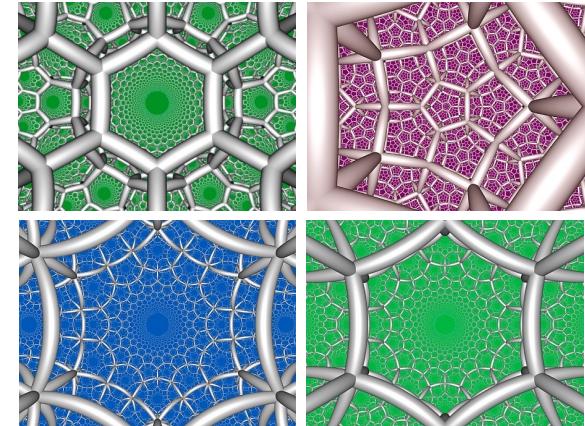
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Four 3D hyperbolic lattices tiling up the hyperbolic 3-space \mathbb{H}^3 (from John Baez's blog)

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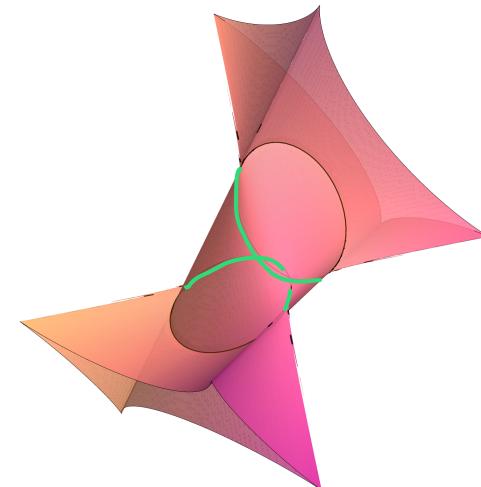
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Existence of **nodal curves** inside also gives evidence, supporting nontrivial loops around (generating a free group on 3 letters) acting on a 3D hyperbolic lattice.

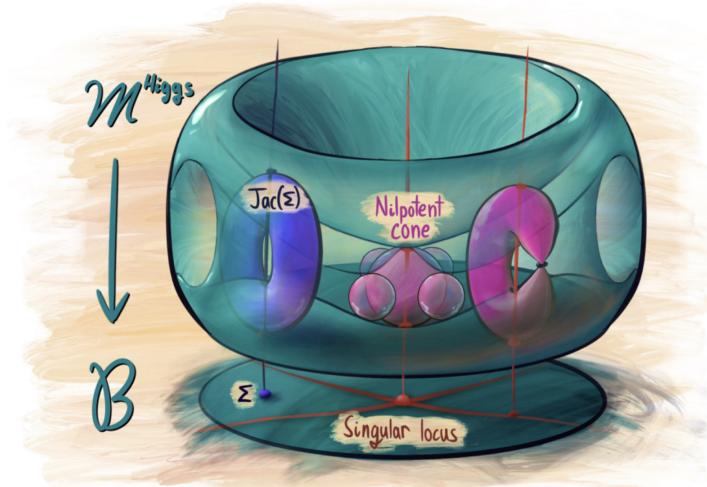


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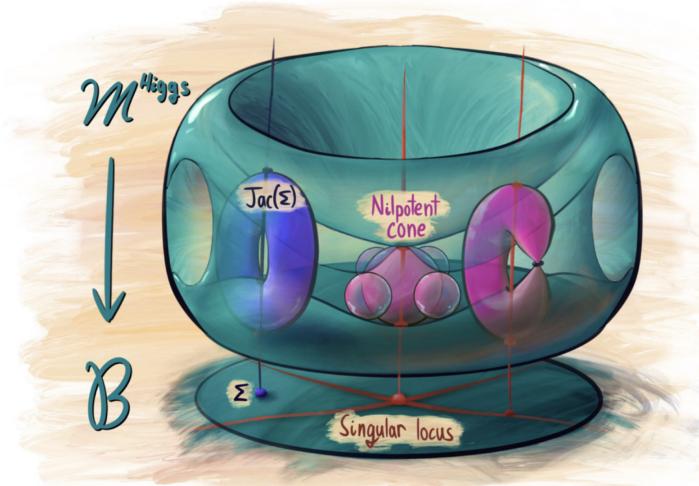


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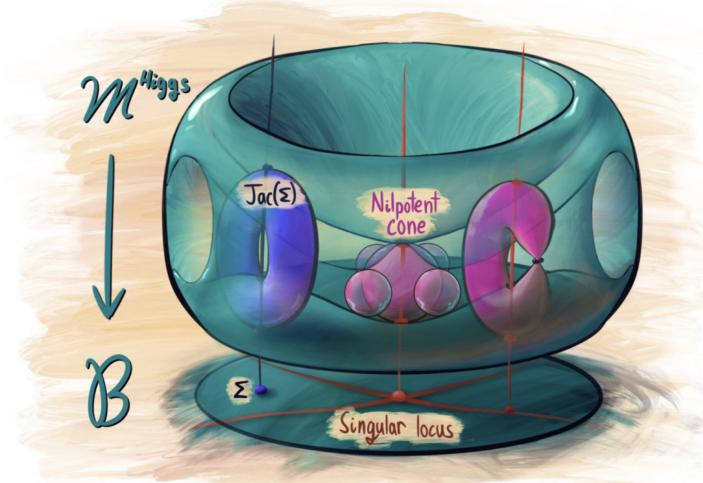


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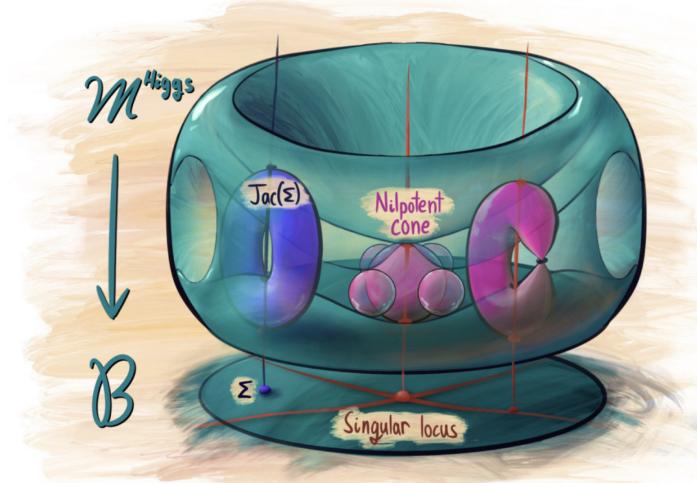


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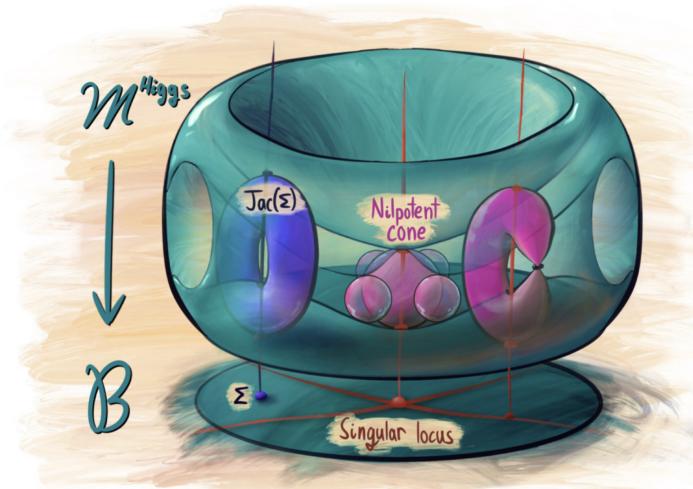
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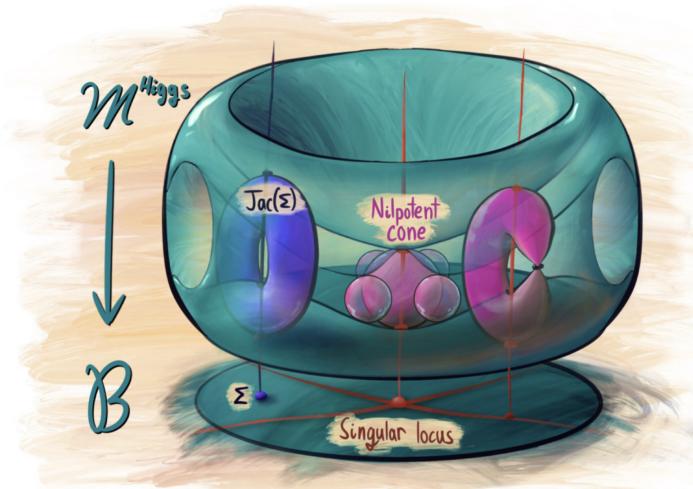
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- Nilpotent cone: The most degeneration occurs over $0 \in \mathcal{B}$. The fiber $h^{-1}(0)$ is called the **nilpotent cone**.

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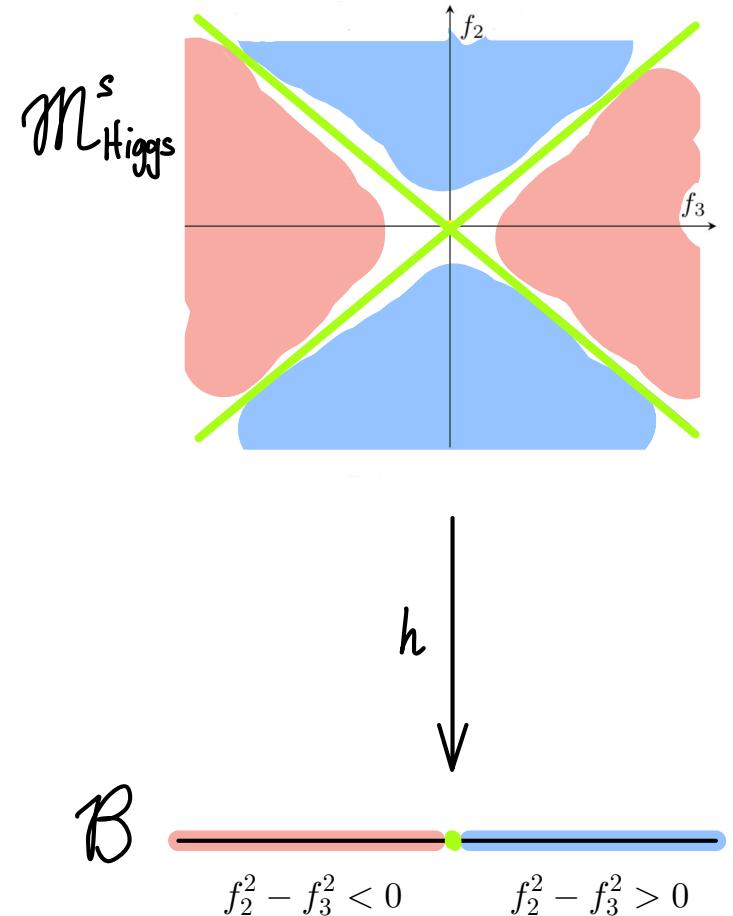
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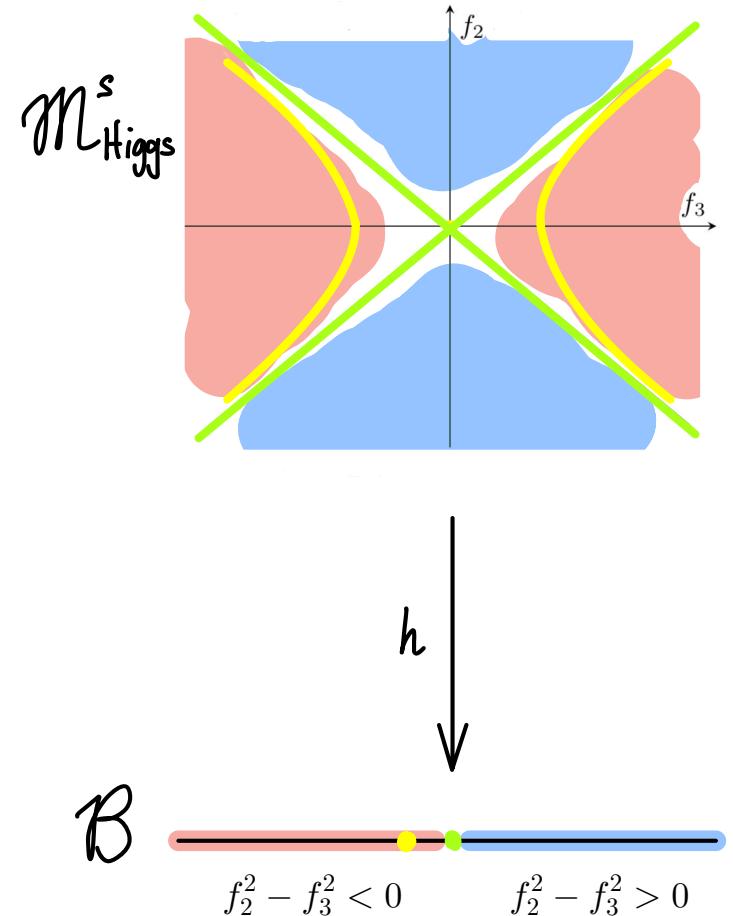
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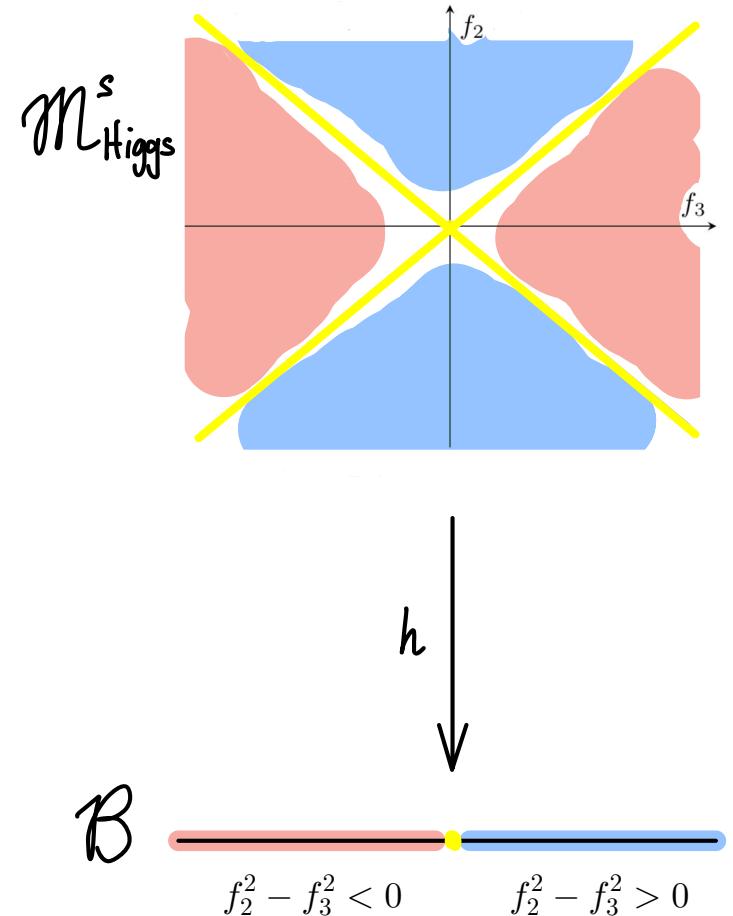
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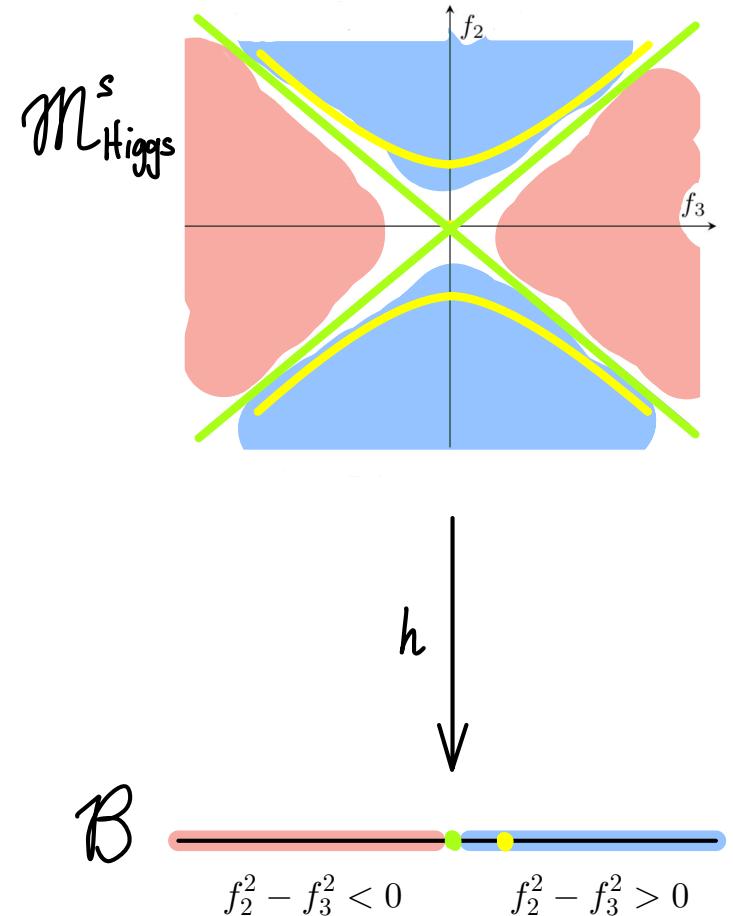
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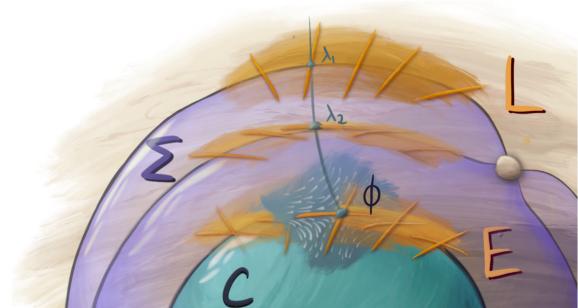
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In fact, the **non-Abelian Hodge correspondence** gives analytic isomorphisms

$$\mathcal{M}_{\text{Higgs}}^{\text{s}}(\text{SL}_n(\mathbb{C})) \cong \text{Rep}(\pi_1(C), \text{SL}_n(\mathbb{C})) \cong \mathcal{H}$$

where \mathcal{H} is the space of equivariant harmonic maps from the universal cover \tilde{C} to $\text{SL}_n(\mathbb{C})/\text{SU}(n)$, modulo isometries.



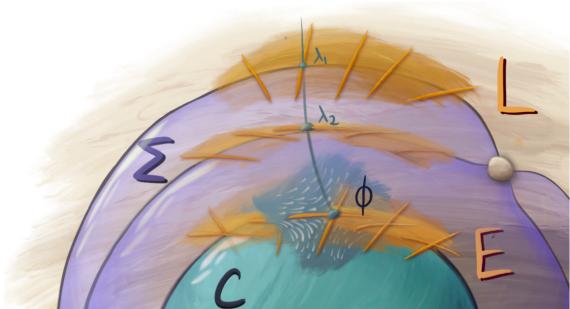
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Moduli of Higgs bundles

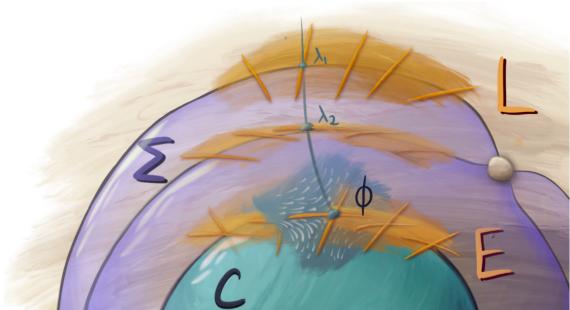
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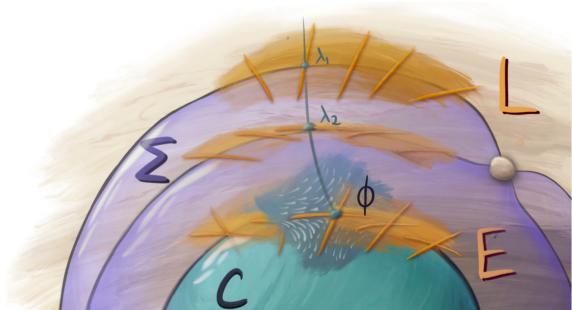
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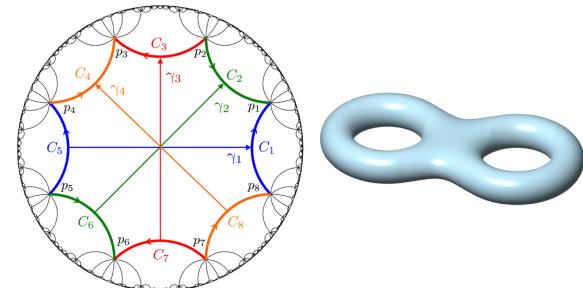
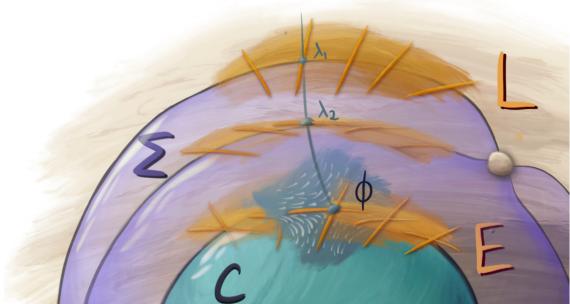
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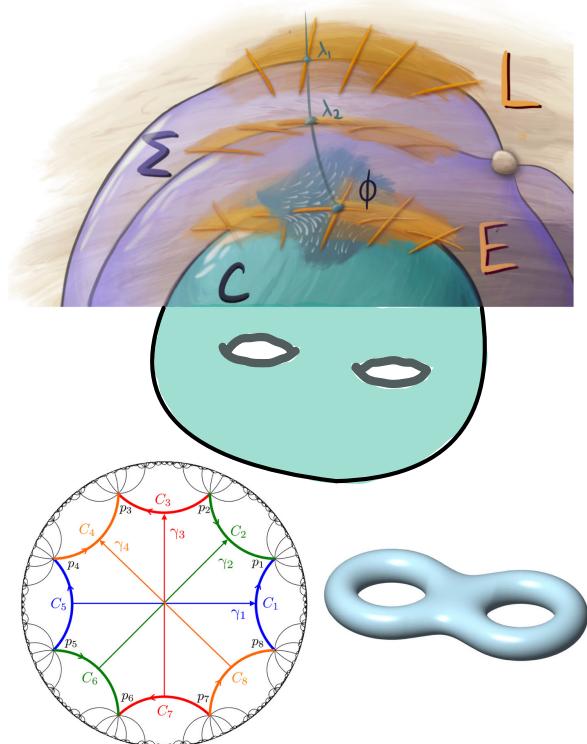
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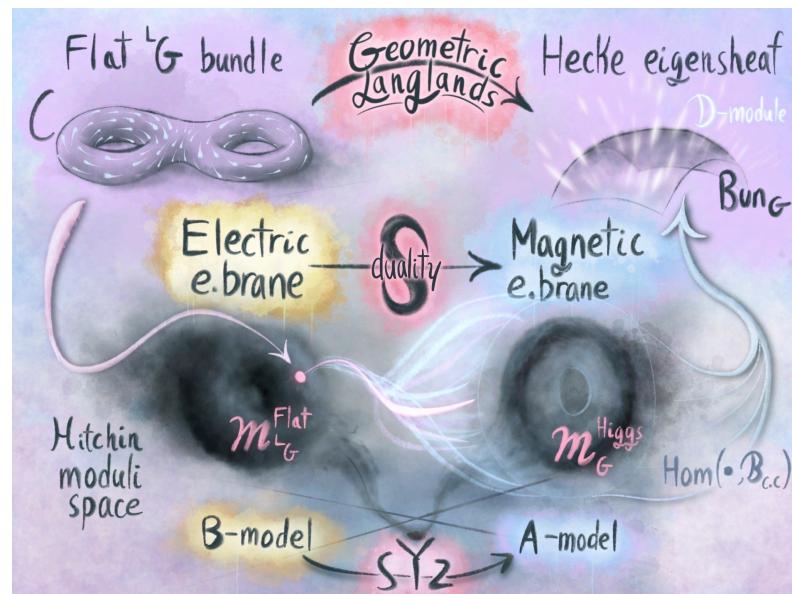
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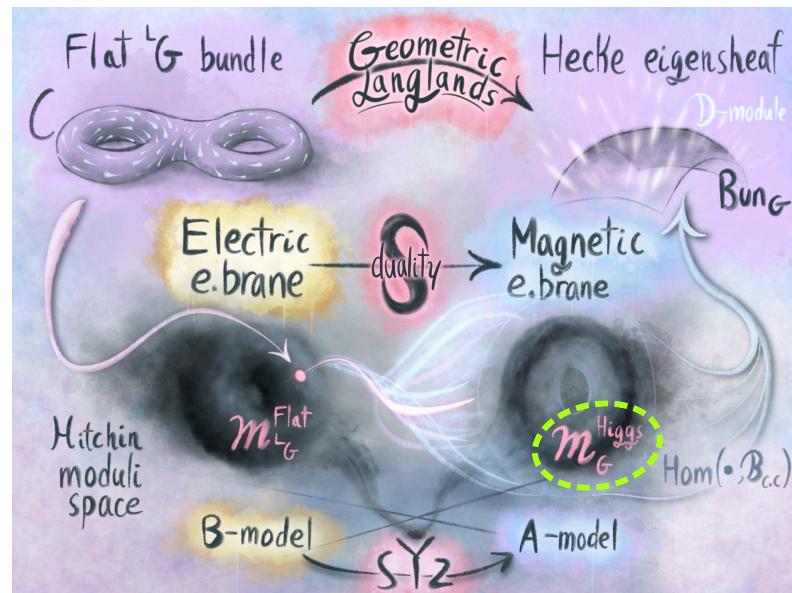
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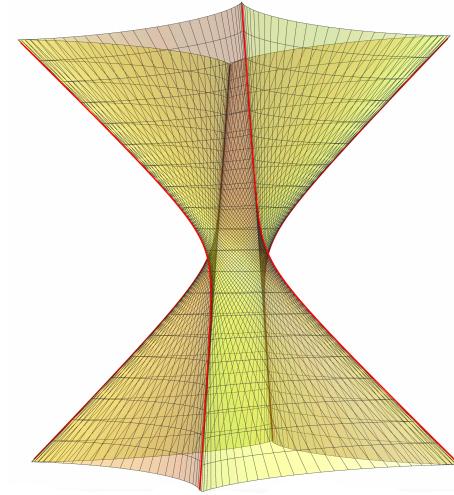
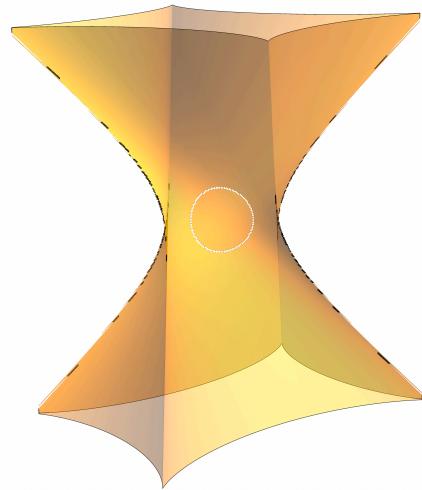
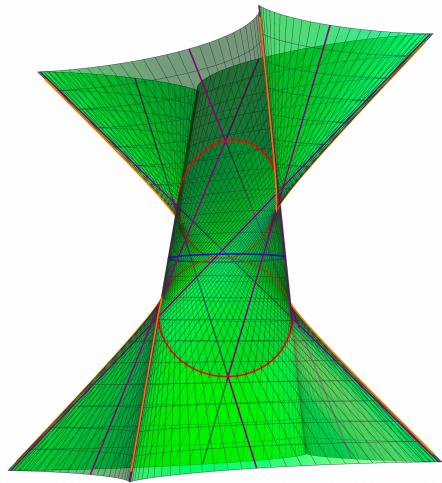


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Thank you.

