International Day of Mathematics lecture, SUSTech, 2022

# Weird surfaces: Möbius band, Klein bottle, and swallowtail

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twist it

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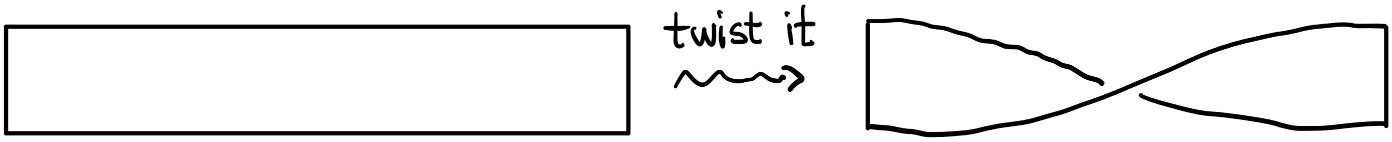

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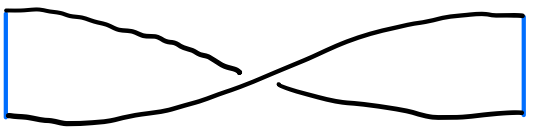

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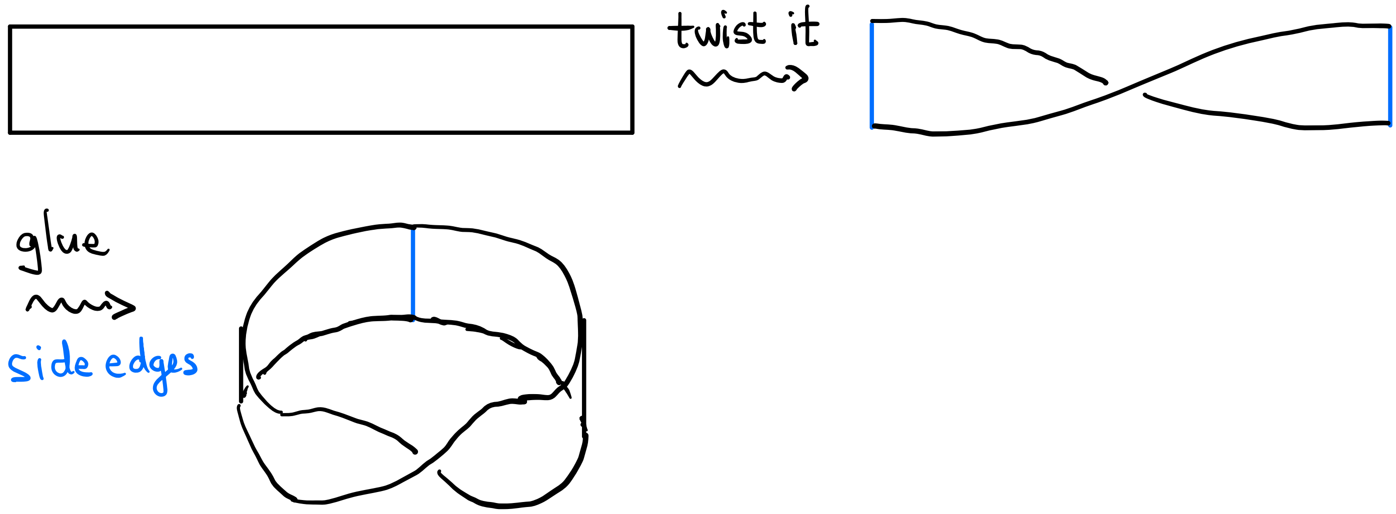

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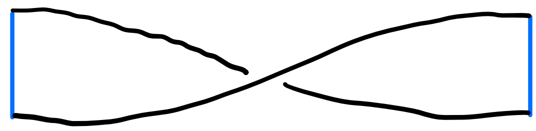

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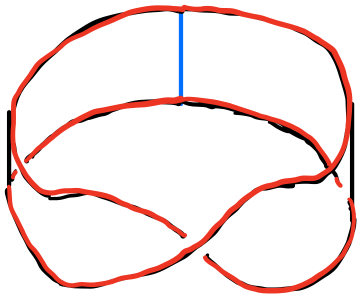

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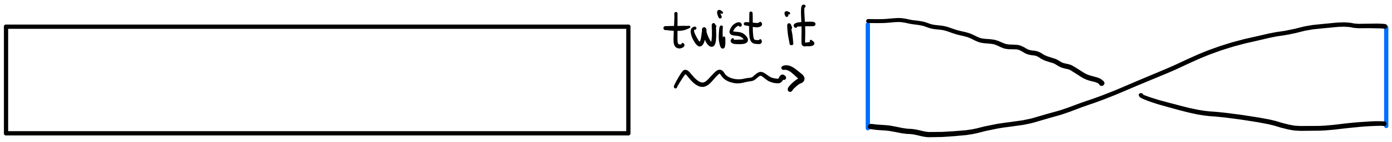


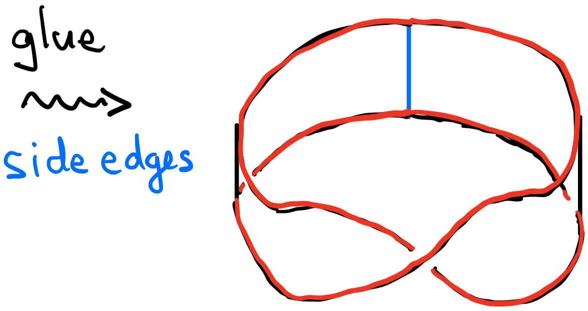


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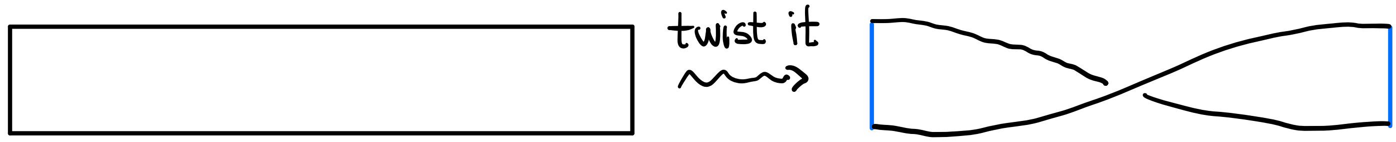


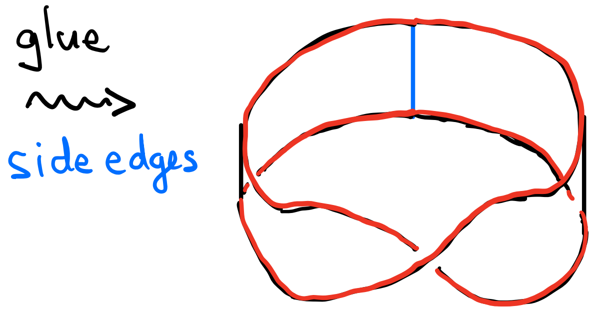


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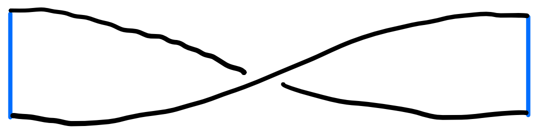

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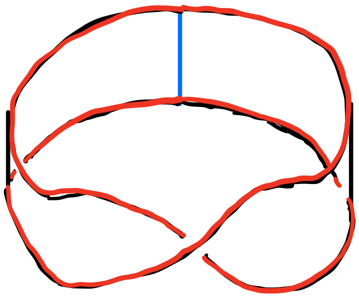

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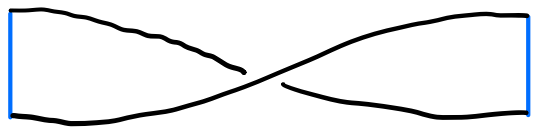

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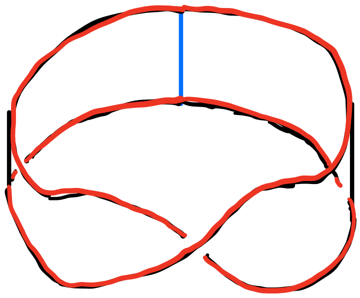

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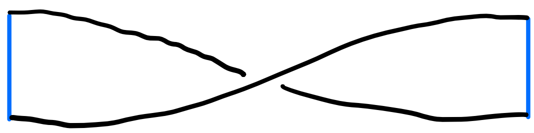

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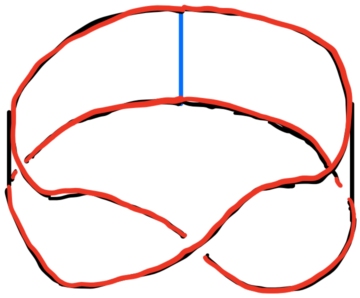

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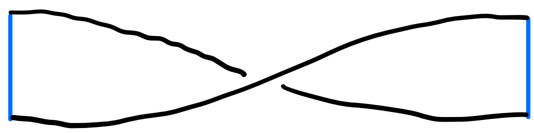

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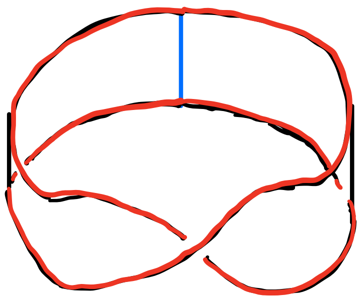

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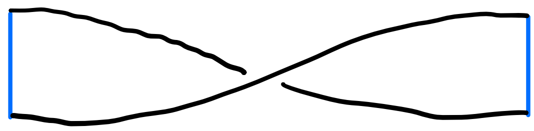

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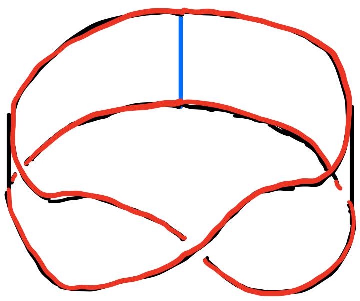

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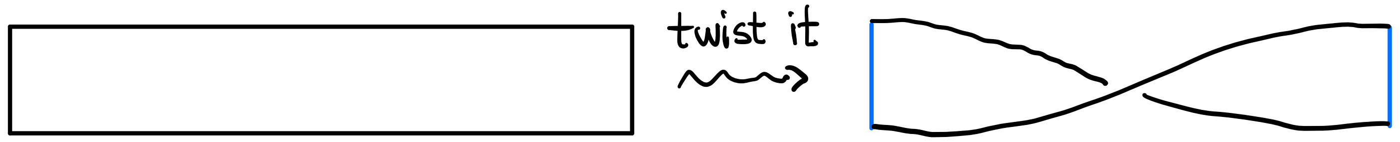


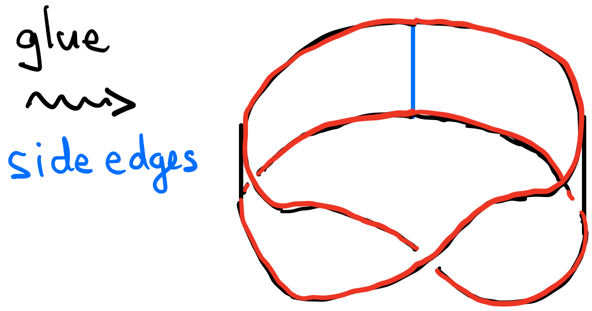


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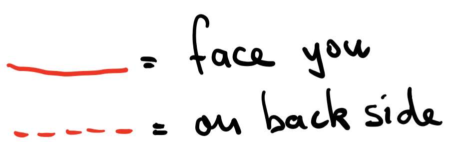


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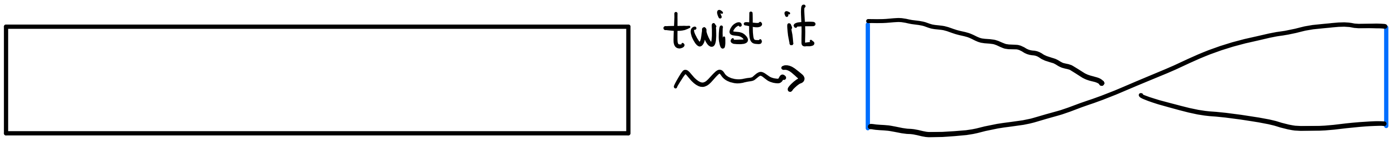


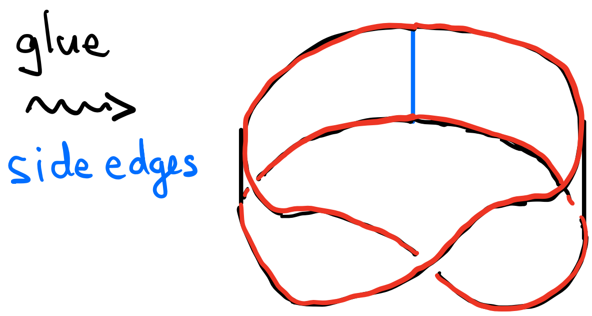


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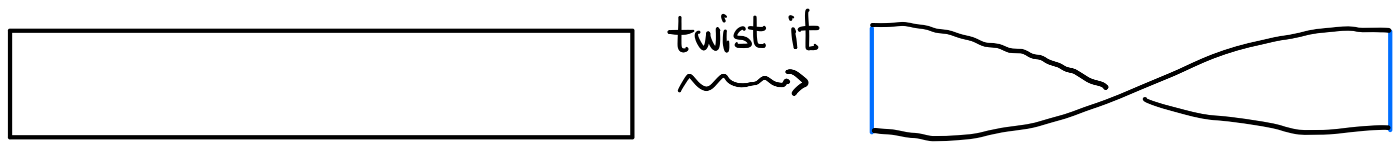


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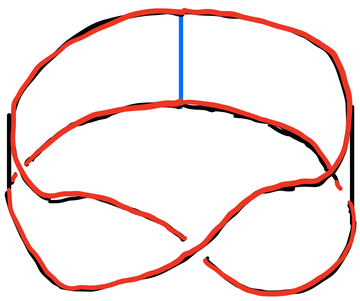

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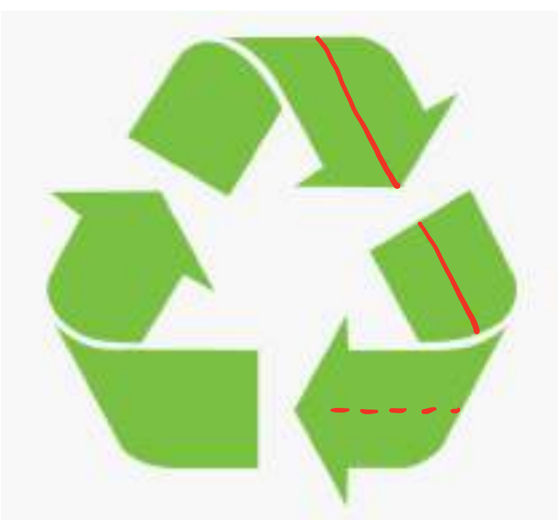

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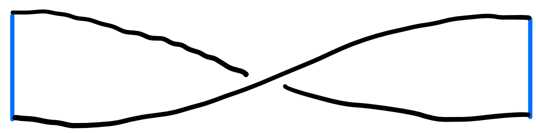

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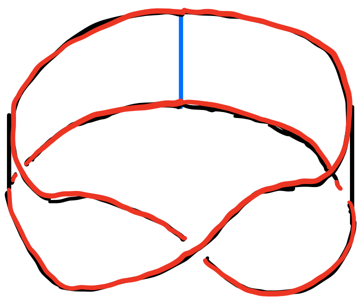

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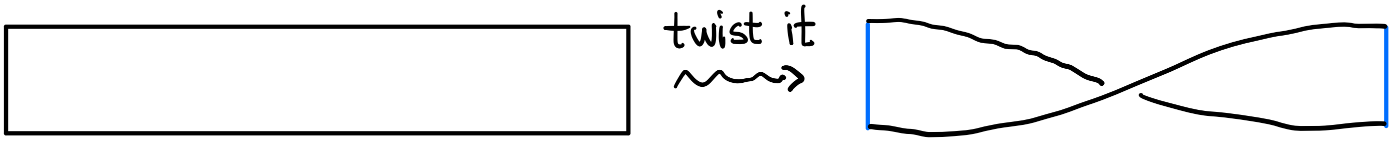


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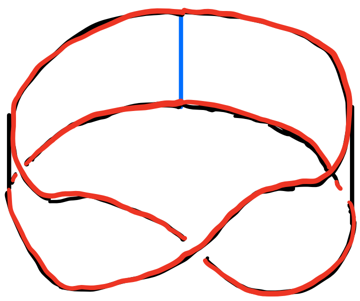

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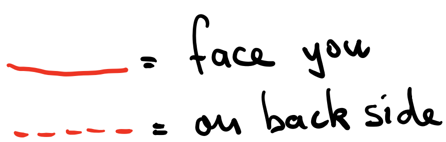


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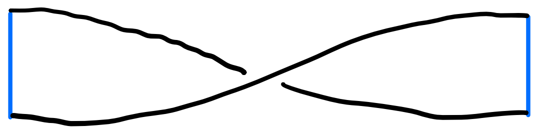

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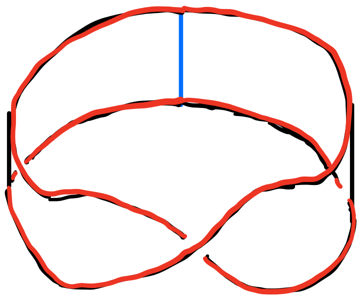

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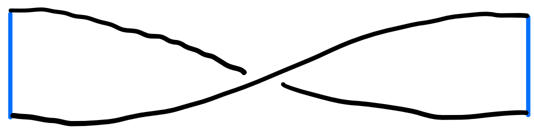

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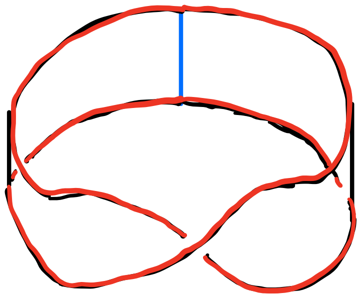

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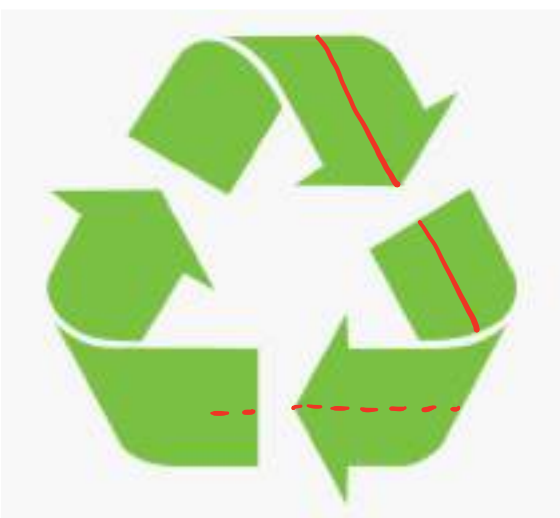

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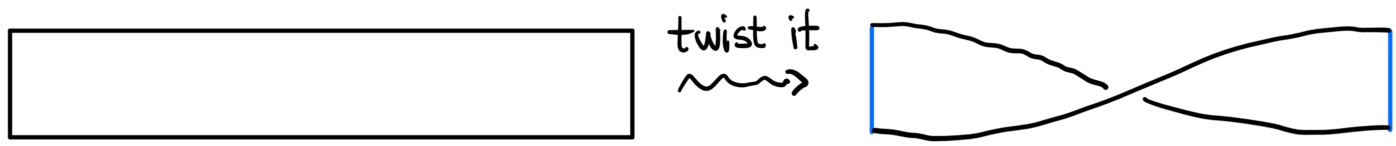


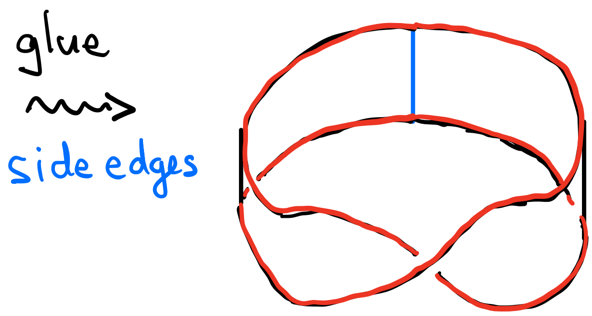


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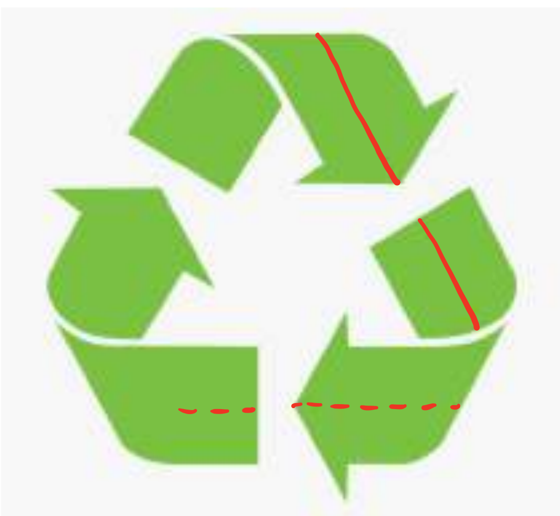

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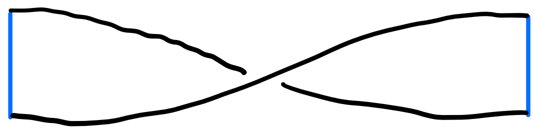

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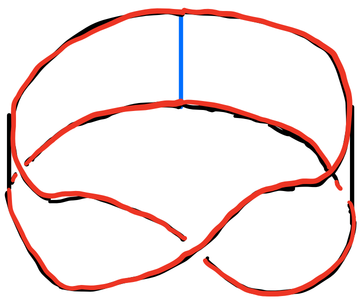

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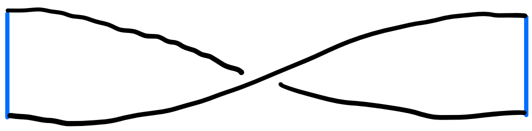

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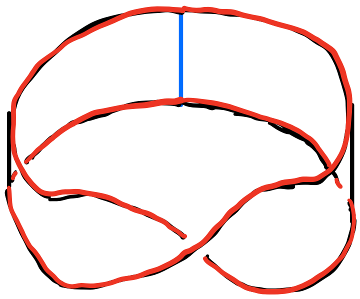

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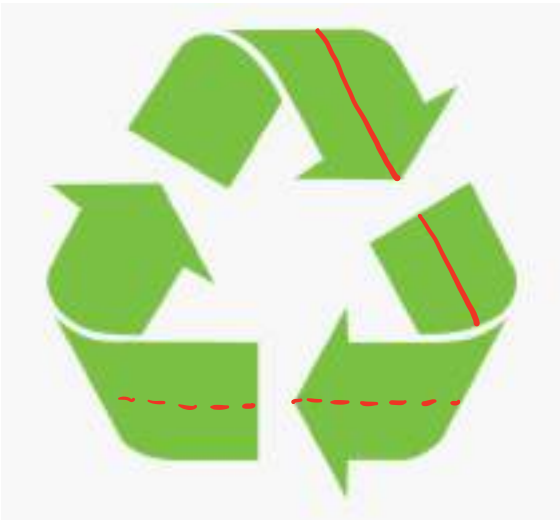

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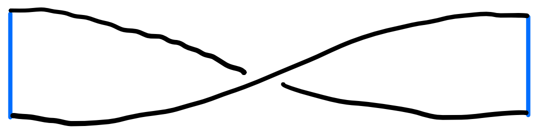

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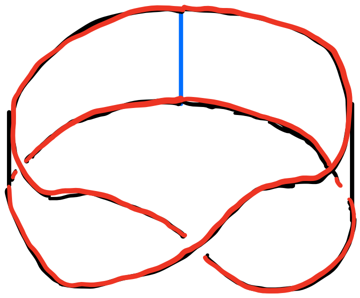

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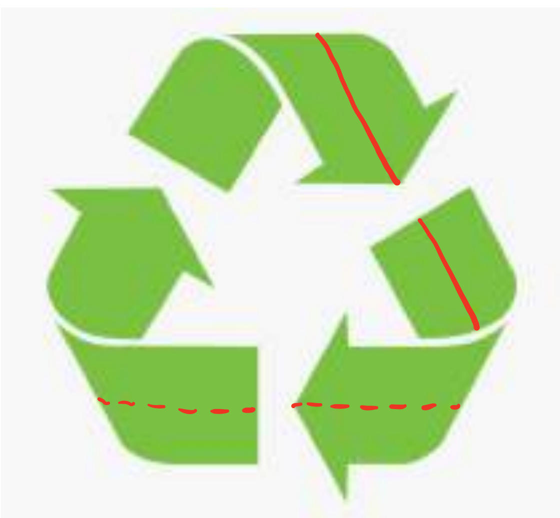

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International Day of Mathematics lecture, SUSTech, 2022

# Weird surfaces: Möbius band, Klein bottle, and swallowtail

Yifei Zhu

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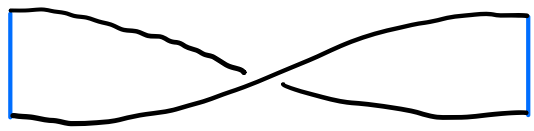
twist it



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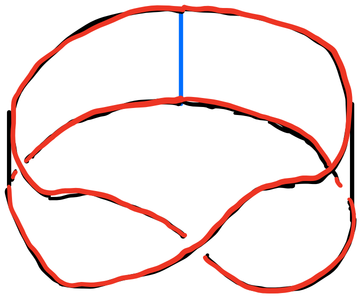

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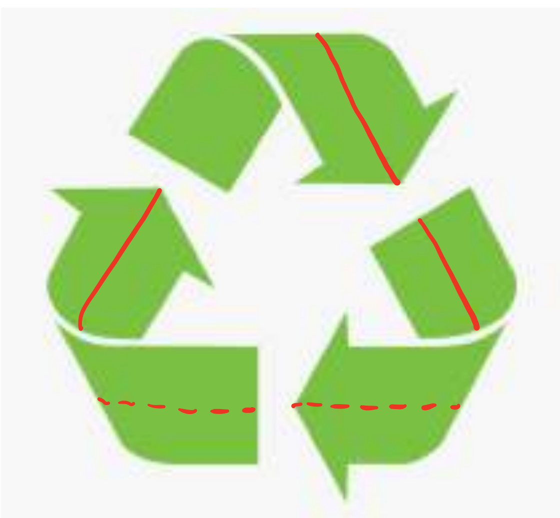

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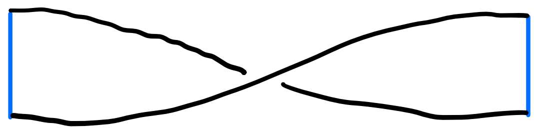

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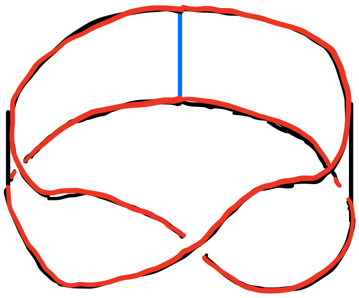

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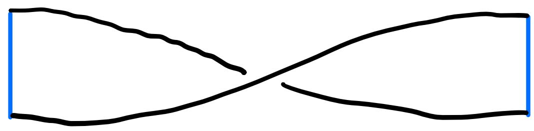

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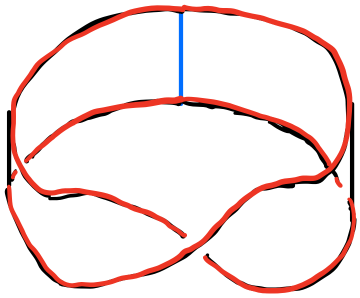

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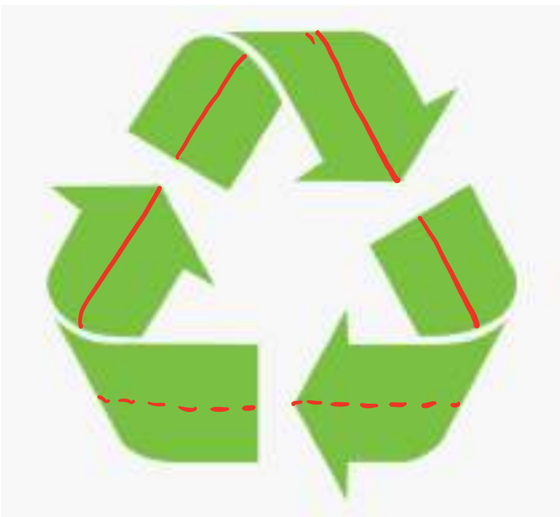

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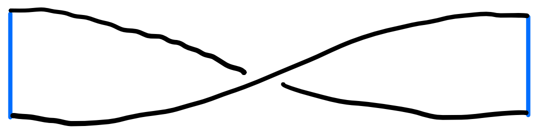

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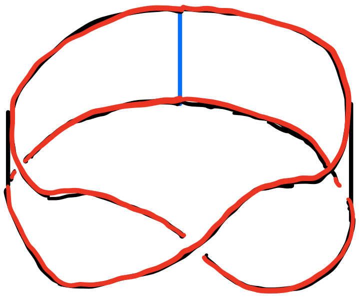

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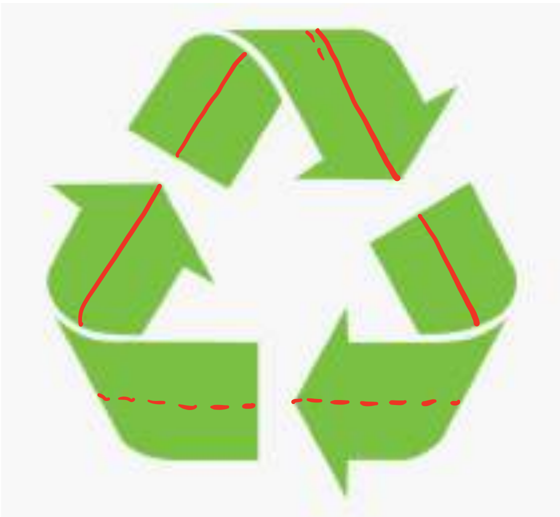

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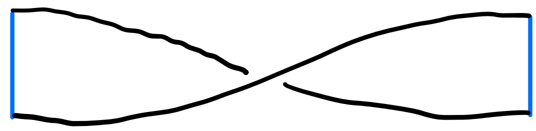

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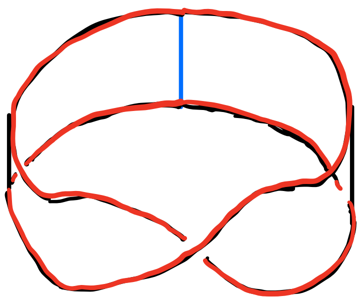

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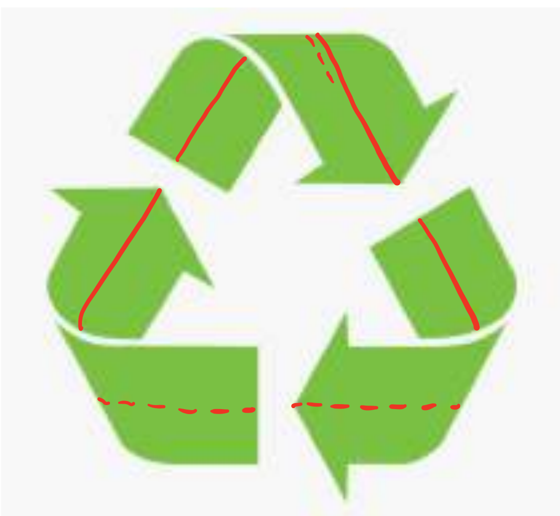

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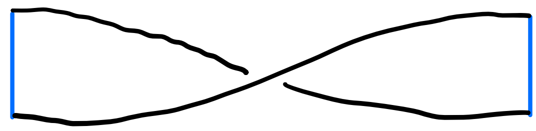

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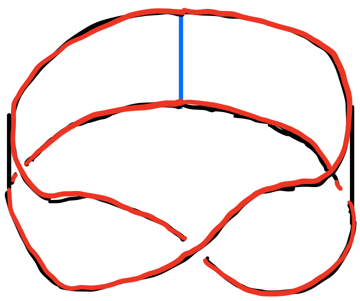

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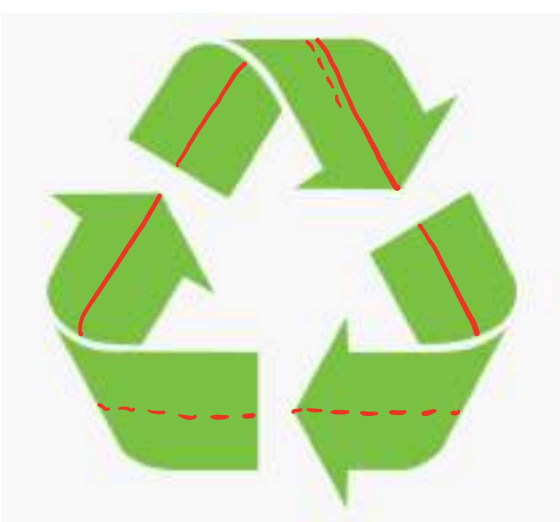

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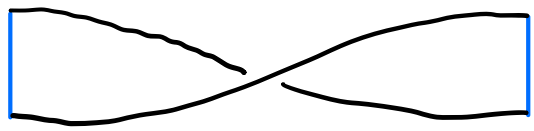

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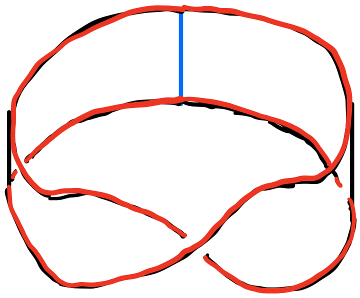

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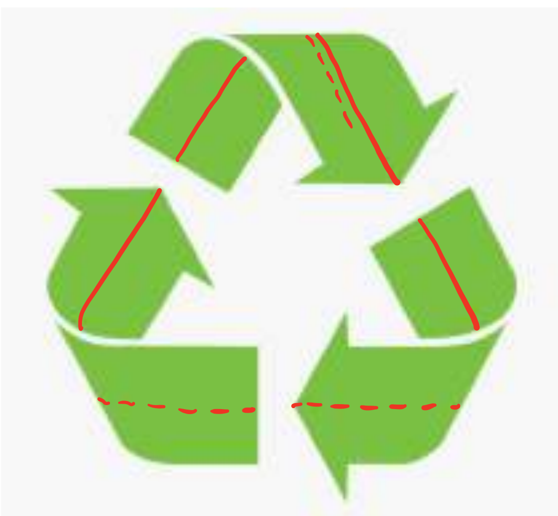

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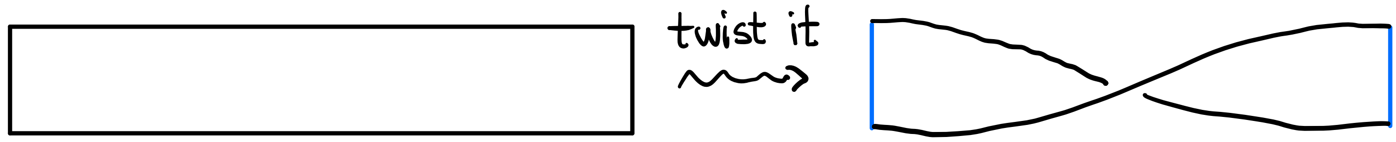


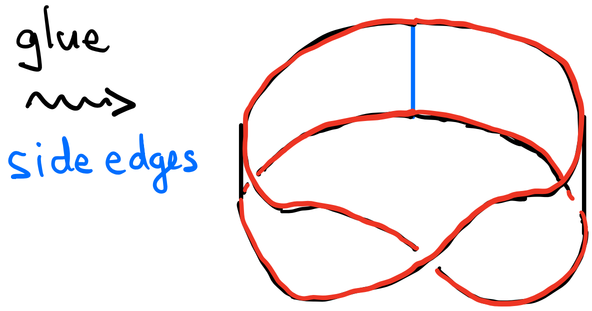


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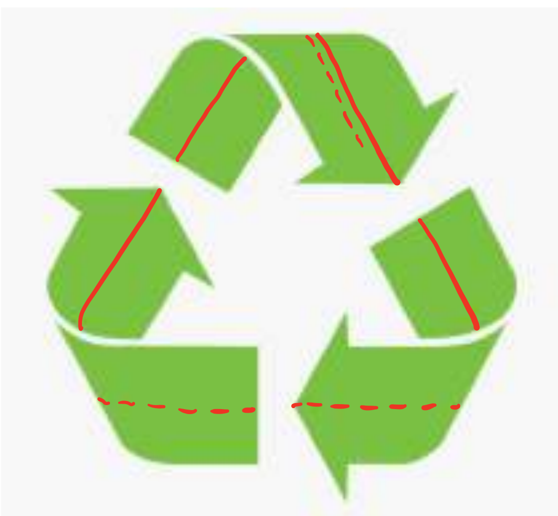

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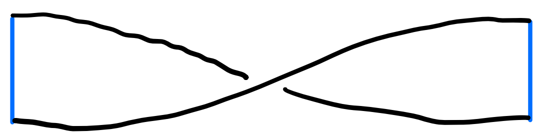

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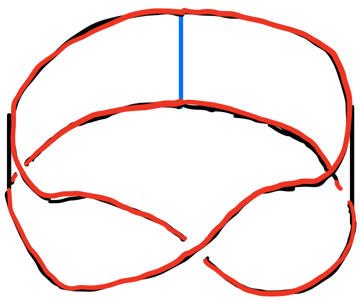

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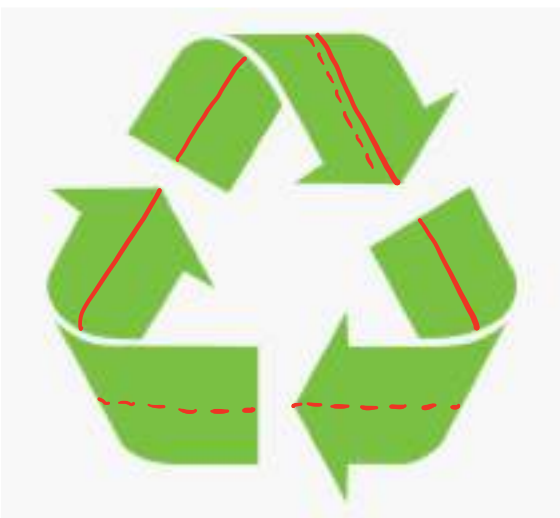

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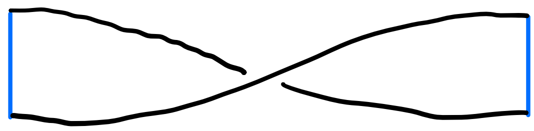

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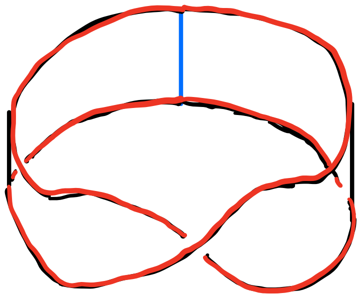

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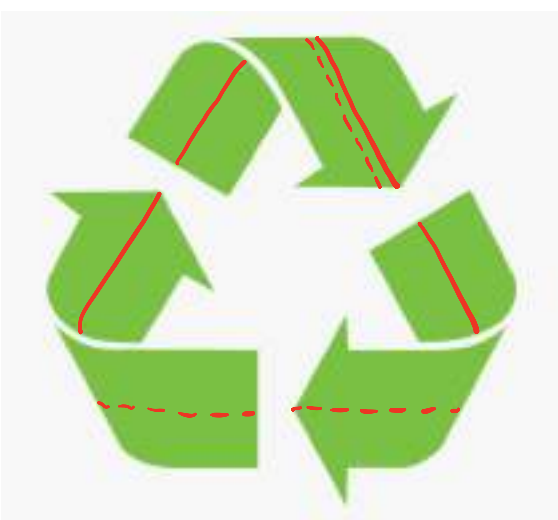

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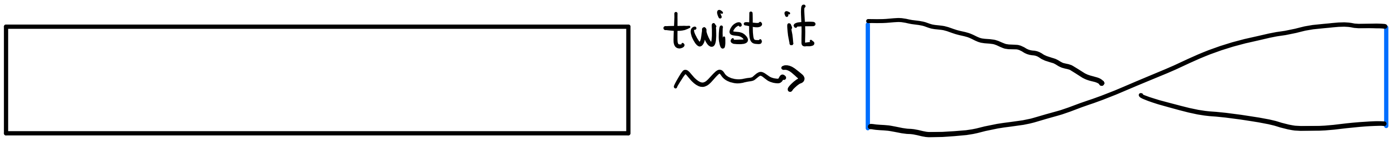


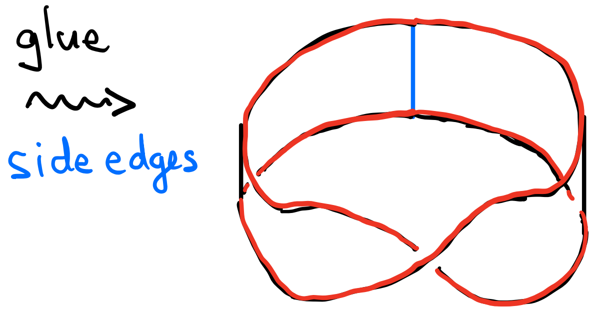


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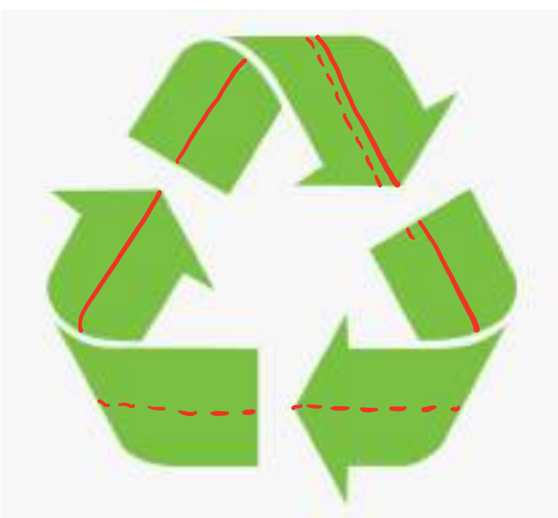

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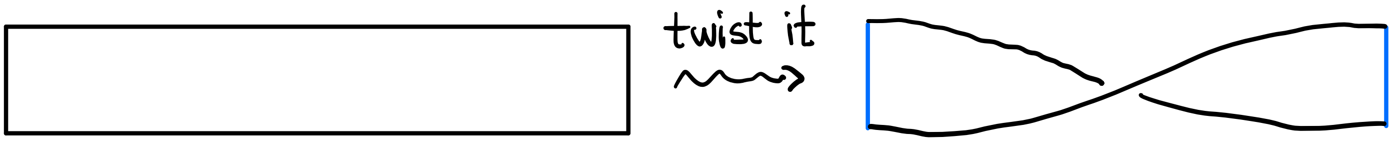


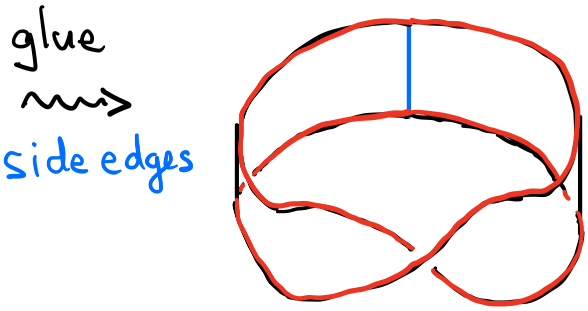


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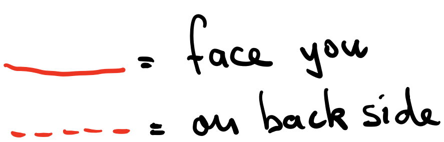


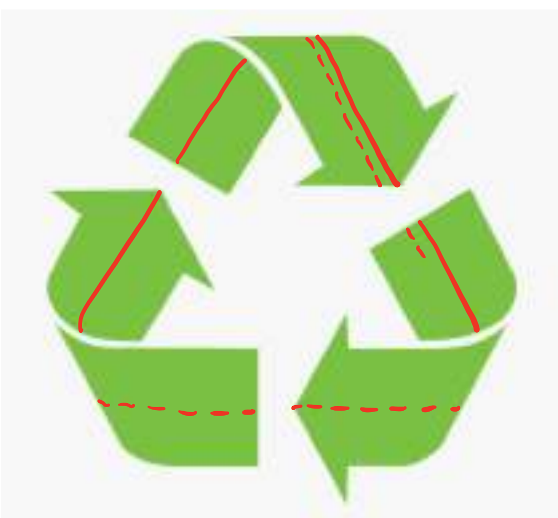

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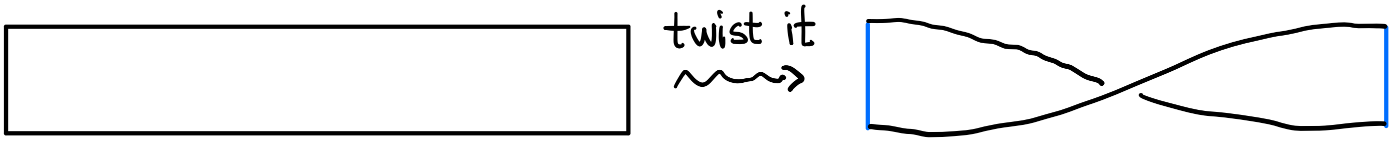


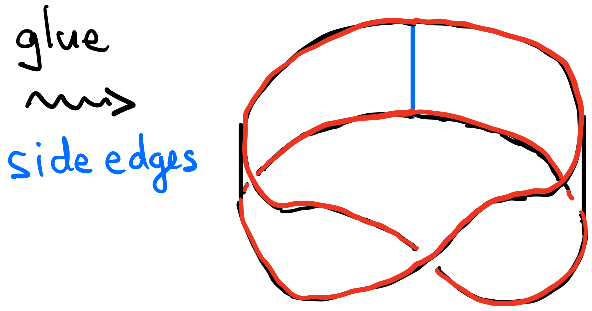


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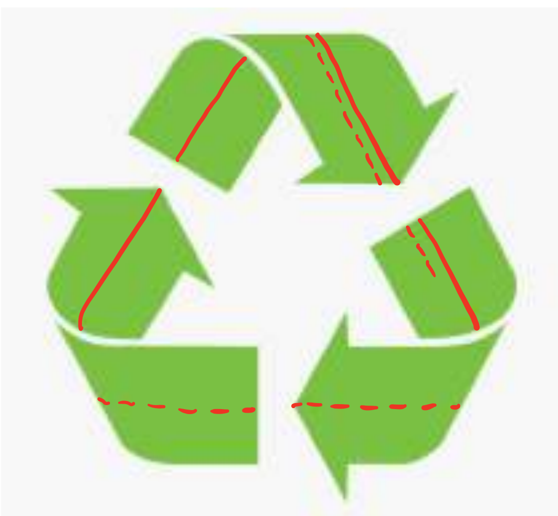

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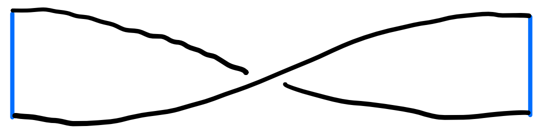

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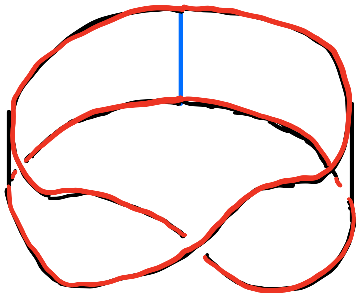

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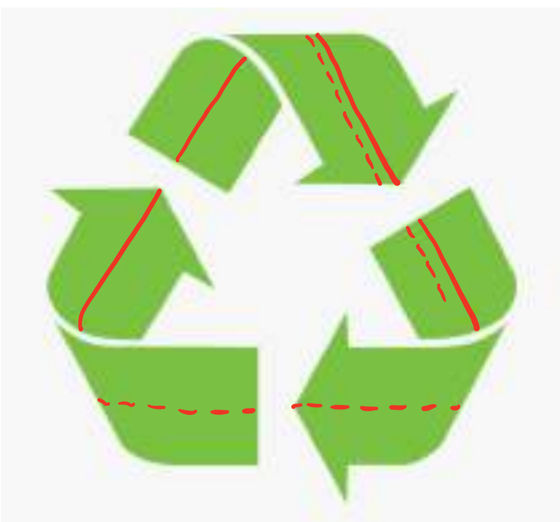

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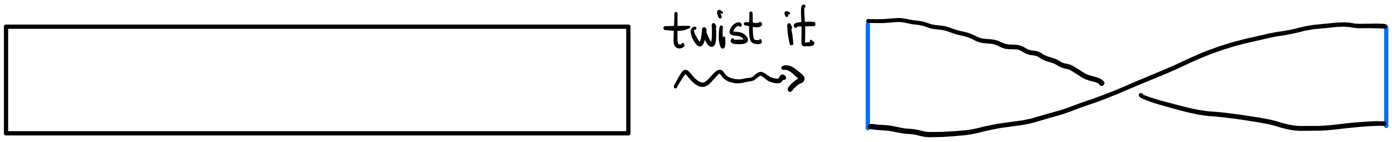


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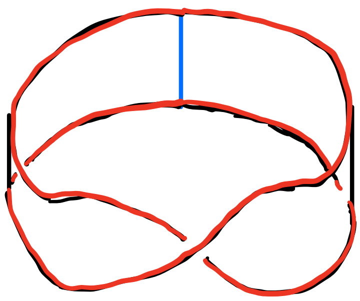

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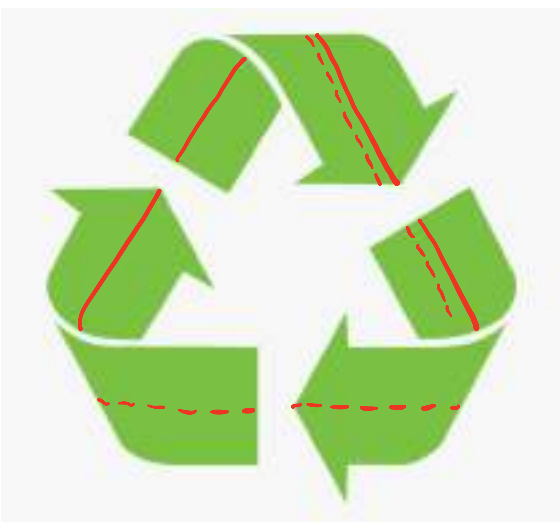

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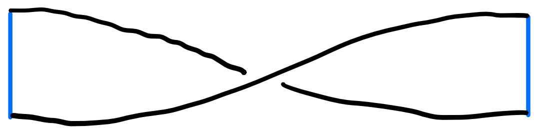

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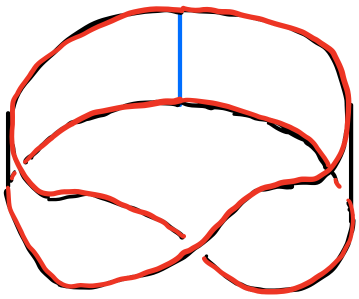

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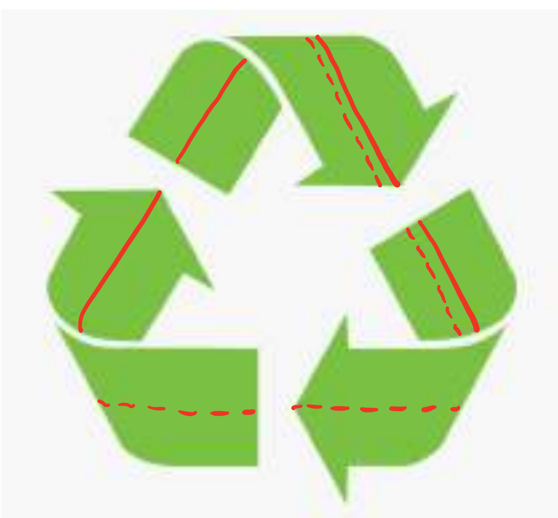

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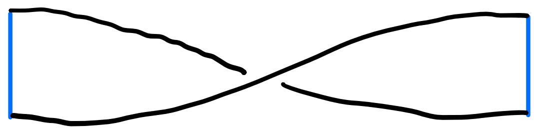

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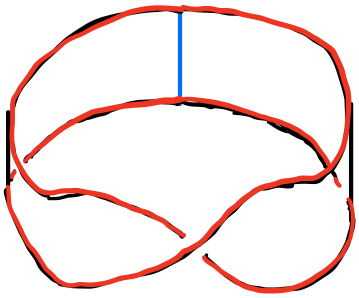

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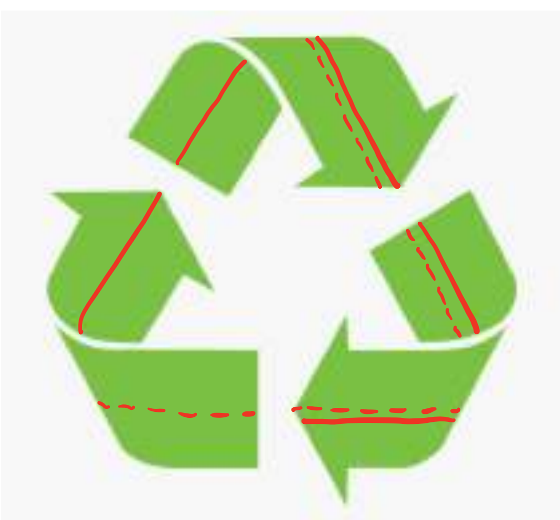

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International Day of Mathematics lecture, SUSTech, 2022

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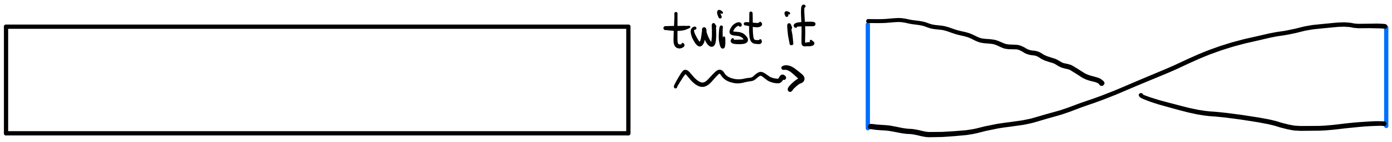


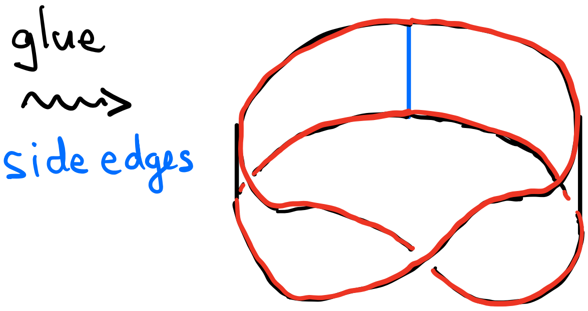


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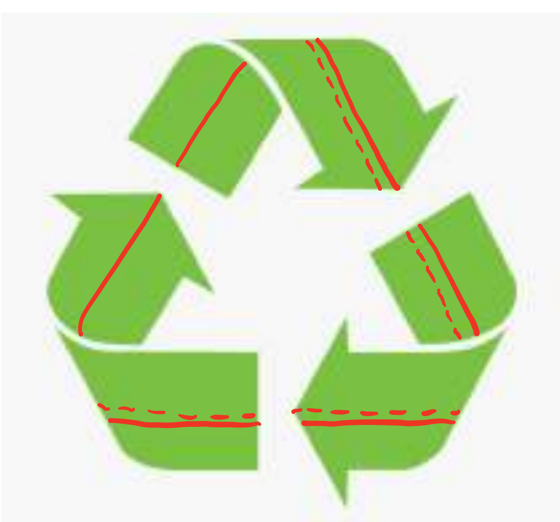

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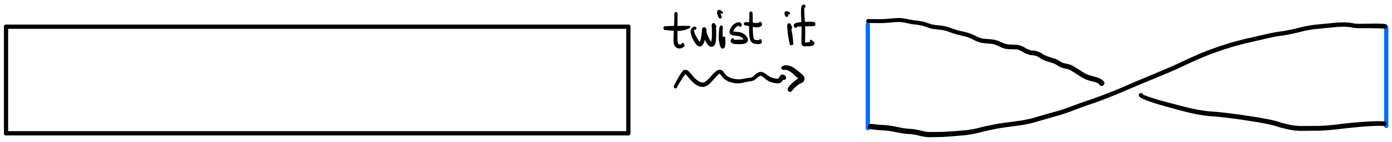


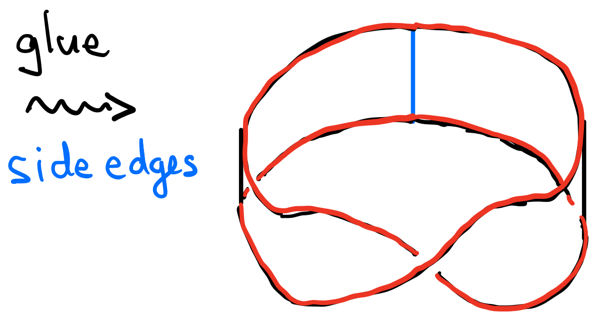


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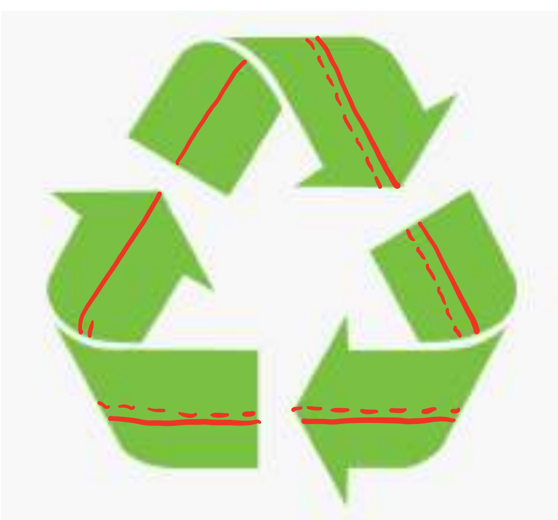

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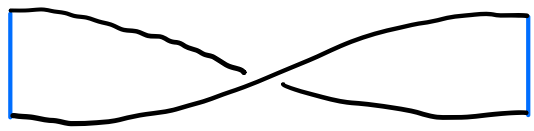

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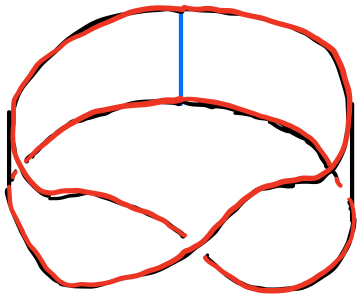

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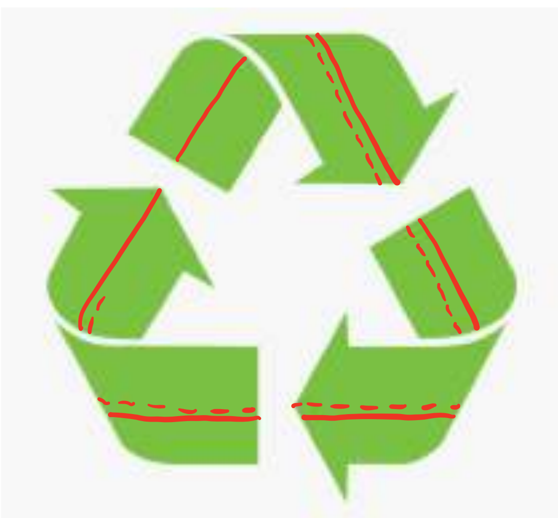

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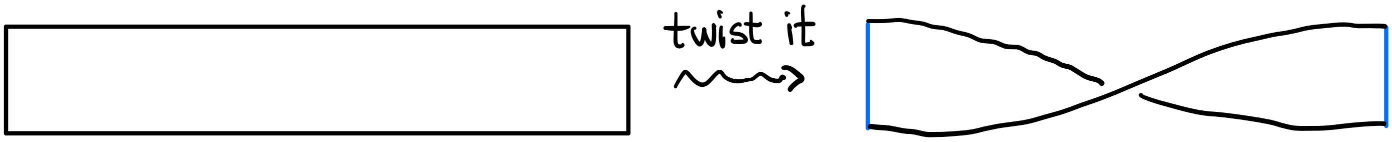


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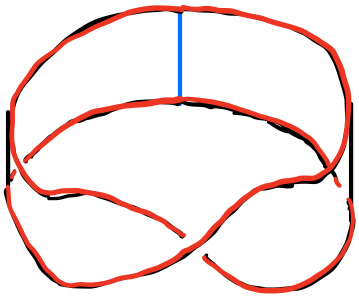

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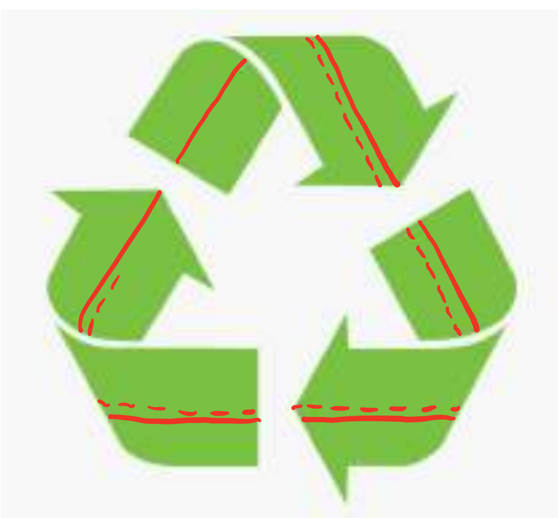

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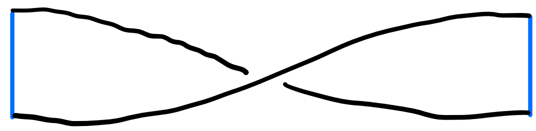

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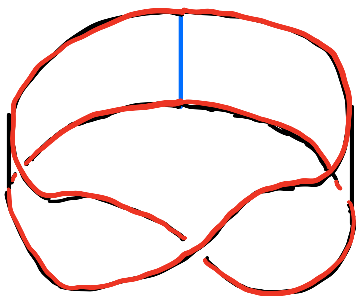

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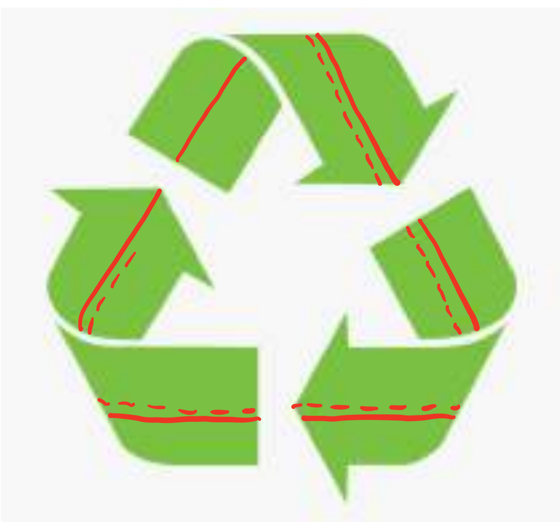

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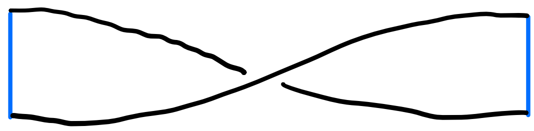

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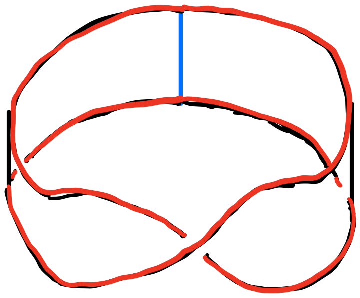

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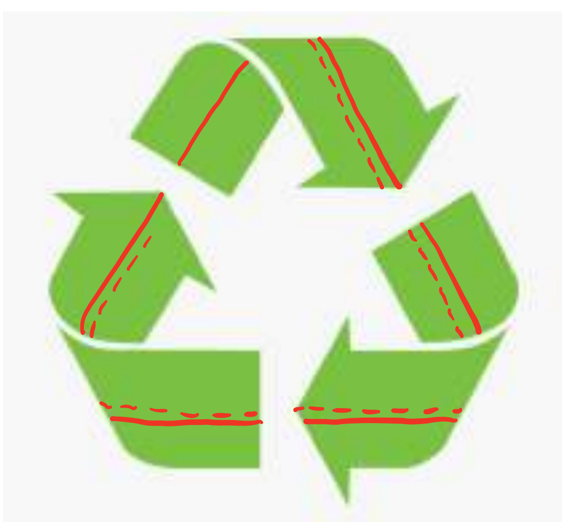

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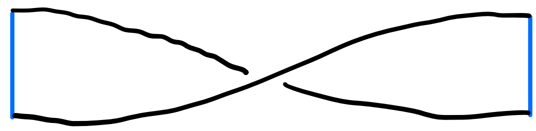

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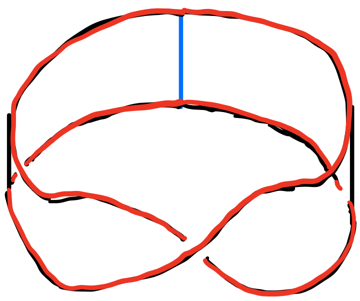

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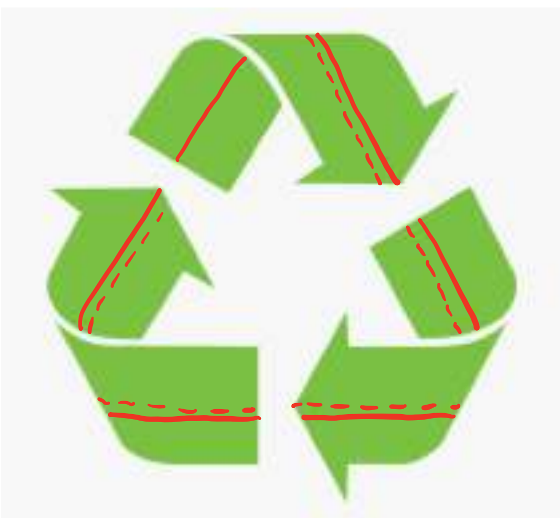

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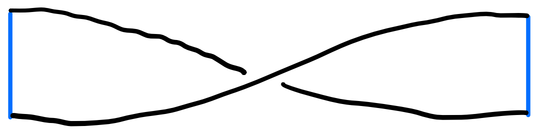

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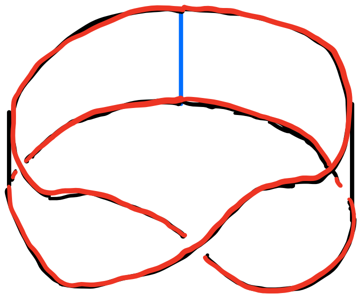

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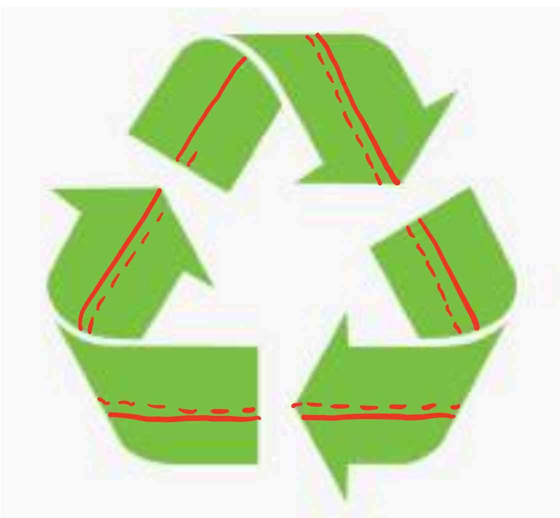

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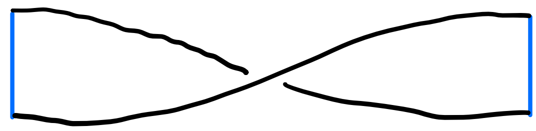

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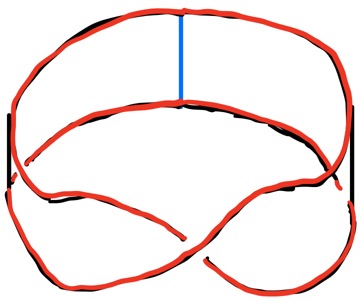

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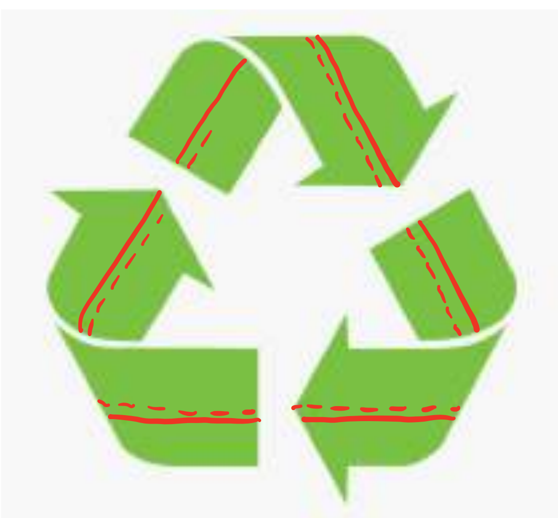

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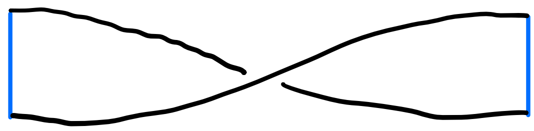

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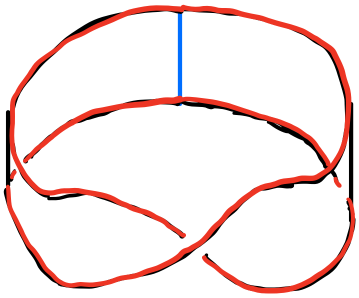

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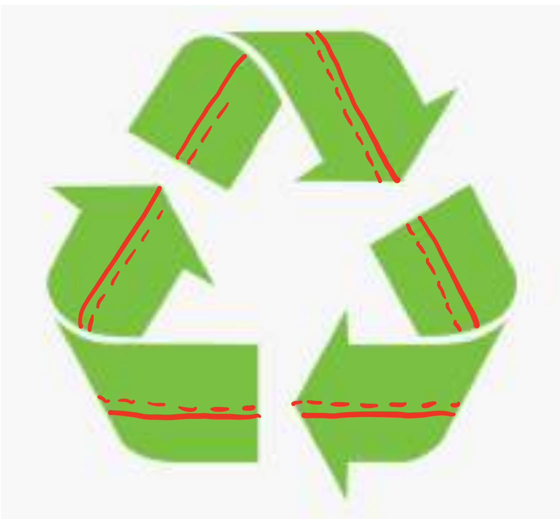

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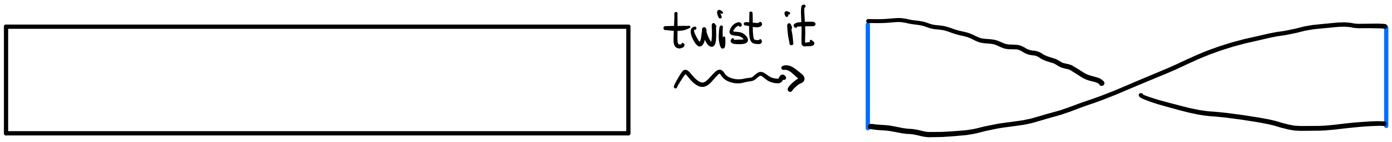


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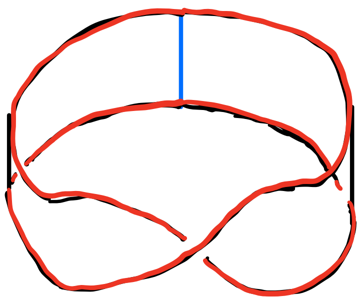

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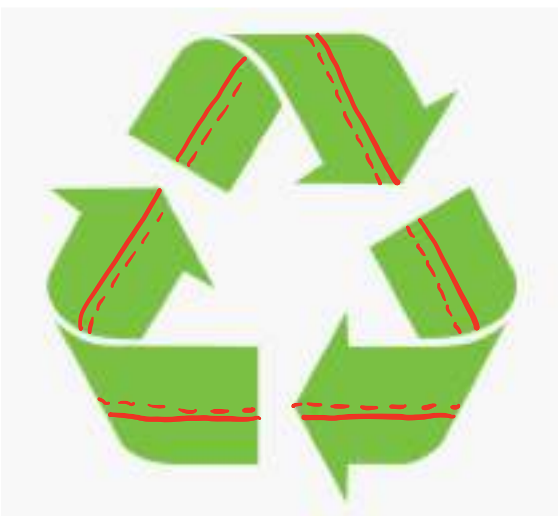

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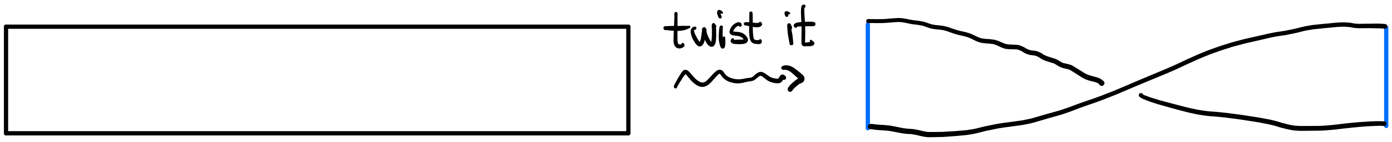


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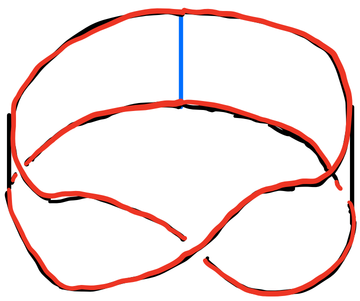

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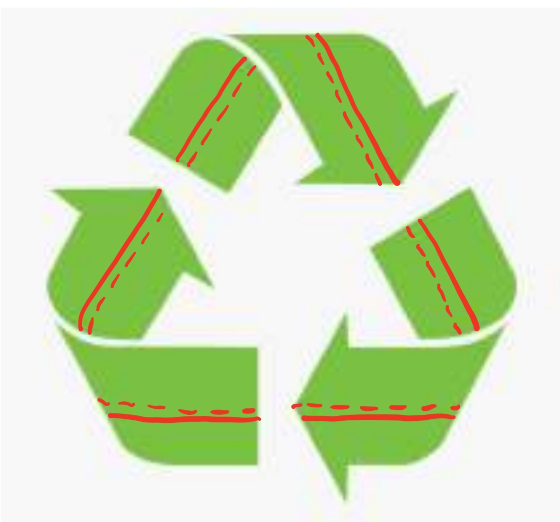

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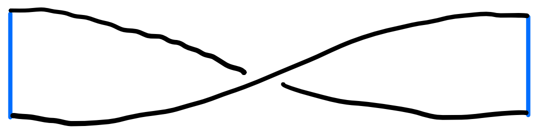

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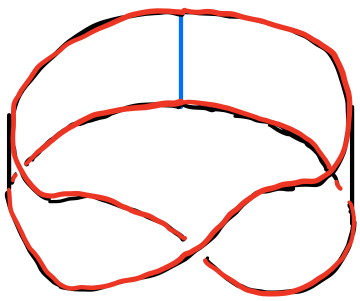

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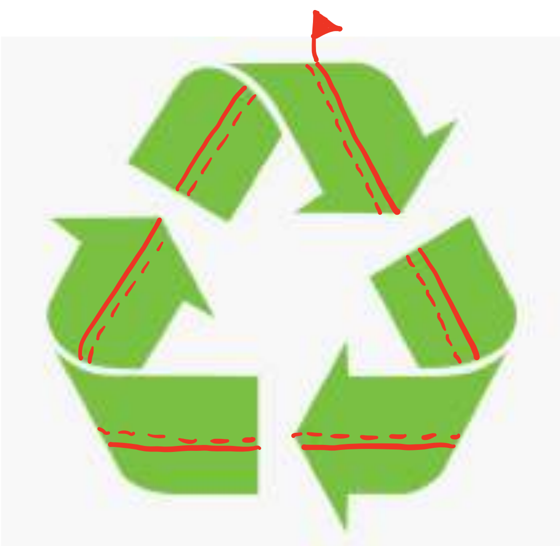

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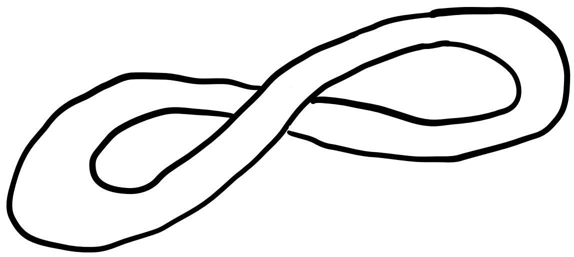

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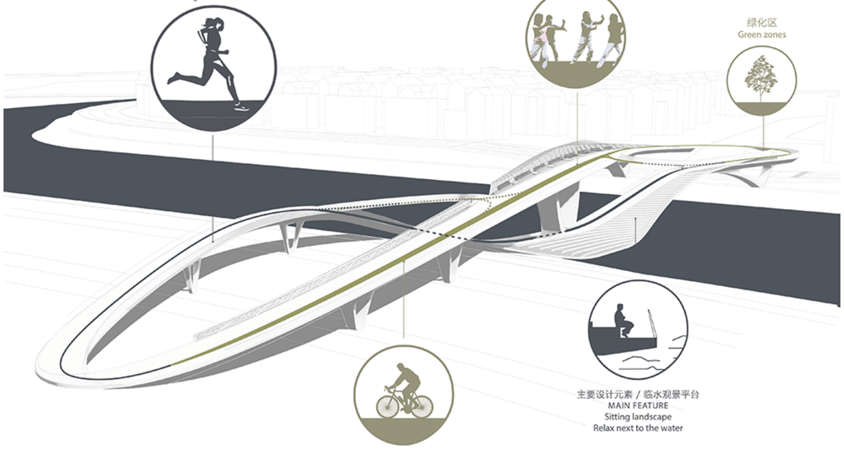


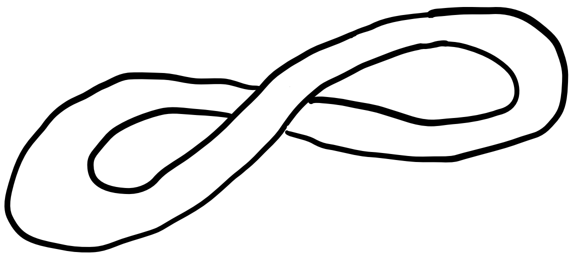

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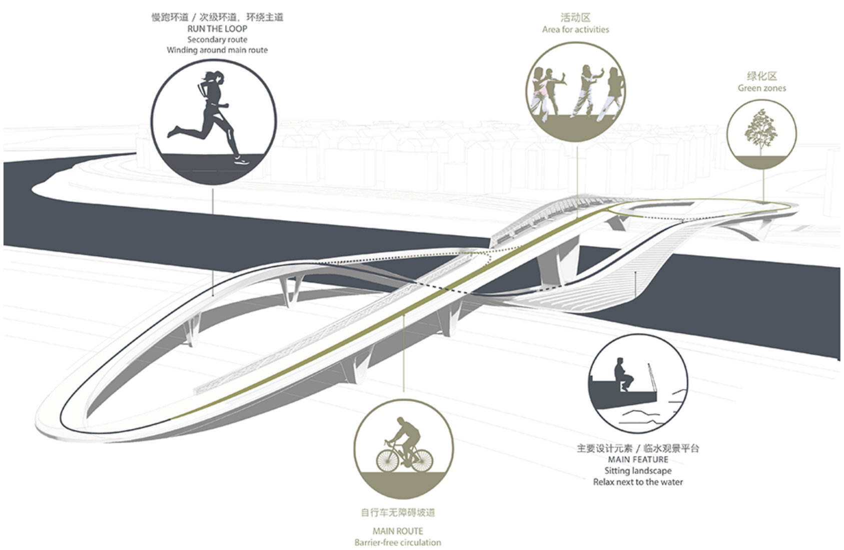


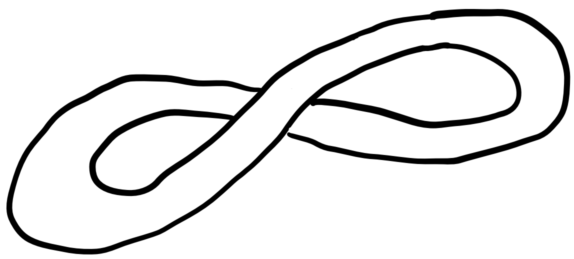

2022打卡点！

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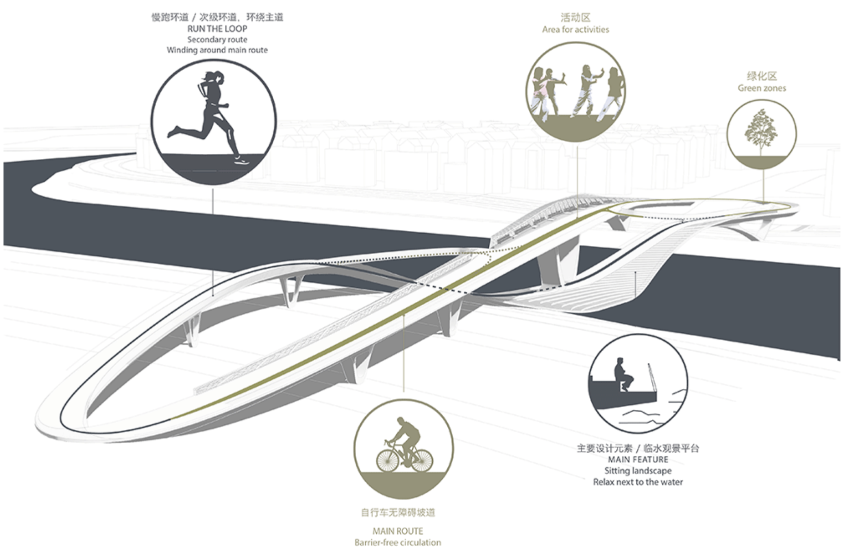


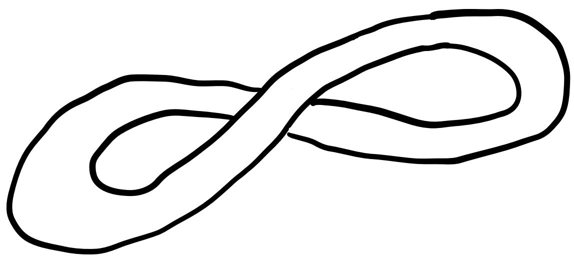

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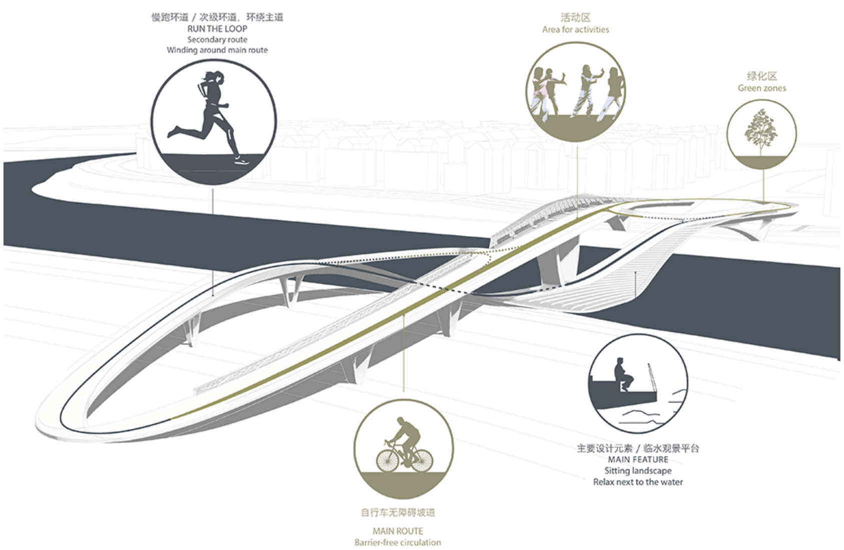


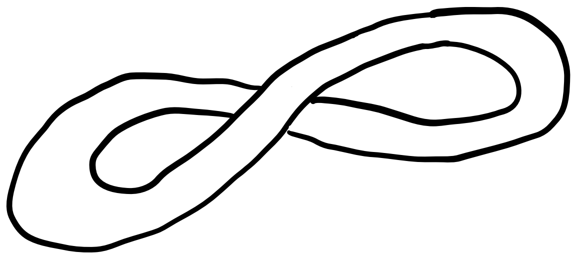

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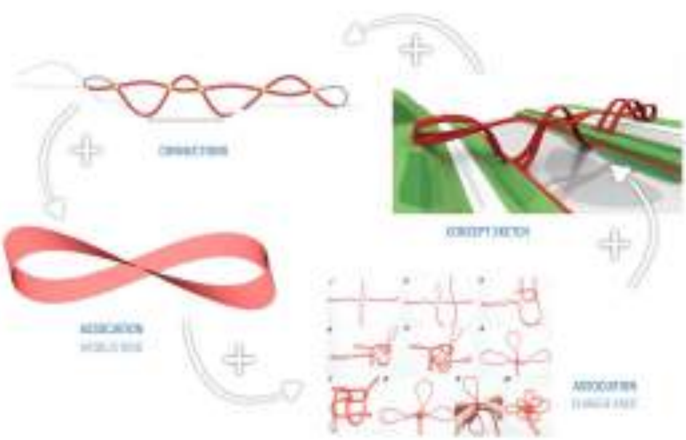

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Inspired~

MOBIUS

The design evolved from the Mobius strip which is a surface with only one side and only one boundary.  It has the mathematical property of being unorientable

The circular interior sits beneath the organic form.  Floor to ceiling glass doors circulate the open plan living space and lead you out to the pool area.  A circular kitchen is at the centre point of the Mobius house with a sky light that mirrors the diameter of the kitchen shape directly above.  A twisted staircase leads you up onto the roof terrace that follows the form of the Hempcrete internal walls of the structure

The large roof top creates another area of equal size to the interior space providing many options for its use as well as an area to view the surrounding nature.  A large eclipse shape swimming pool follows the form of the house, accessed from both sides of the building.  The twisting driveway to the property takes you down into the garage which is situated directly below the building with a second staircase that takes you back up to the main interior

# Architecture designs by Antony Gibbon

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TENDRIL GALLERY

# Klein bottle

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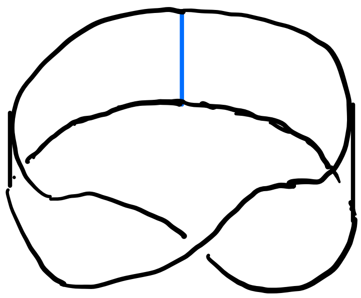

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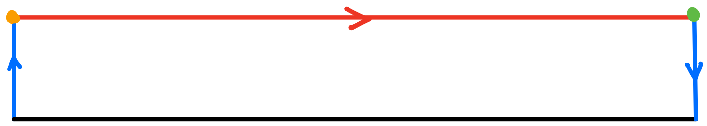

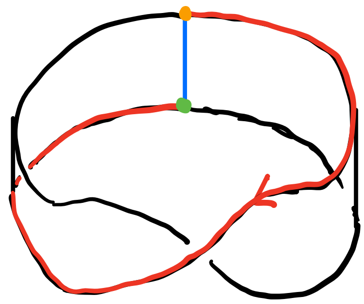

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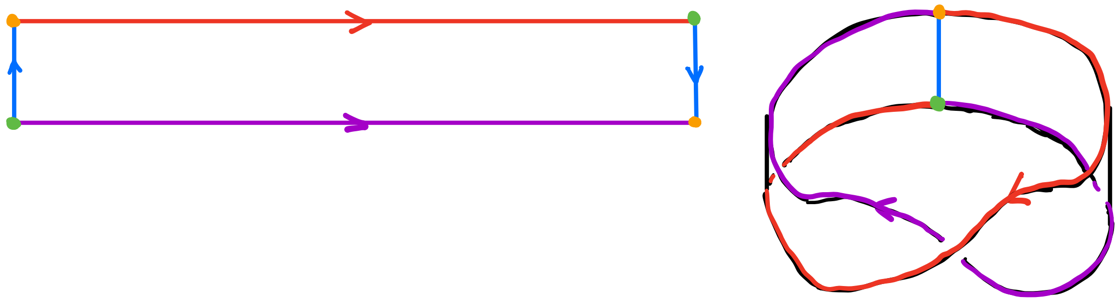

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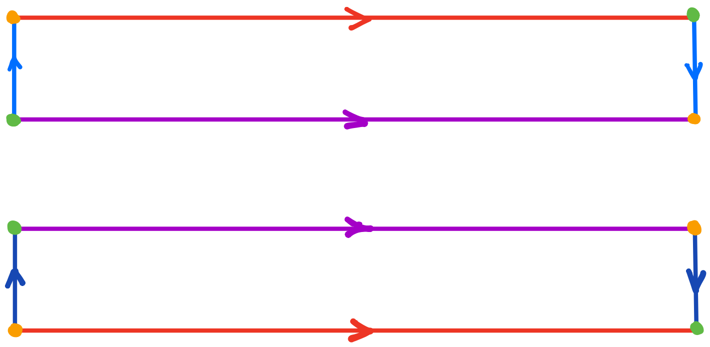

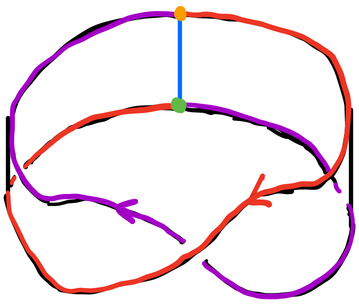

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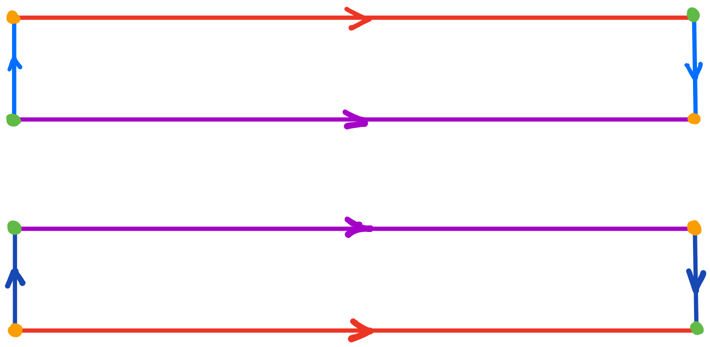

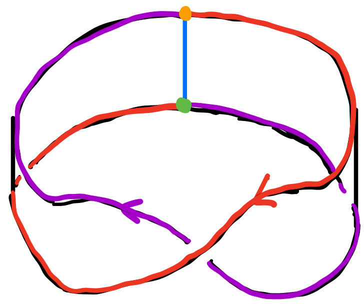

glue along } purple edges

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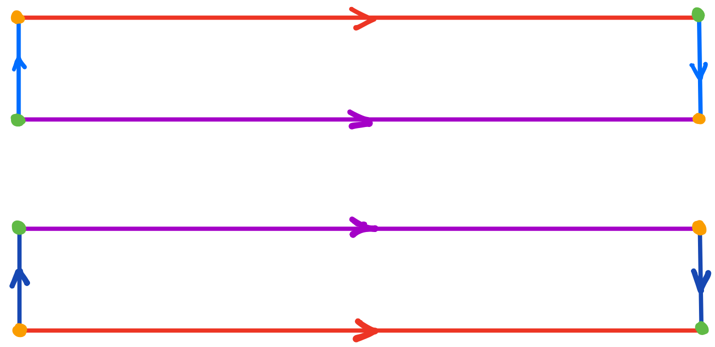

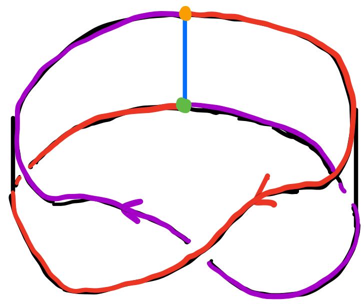

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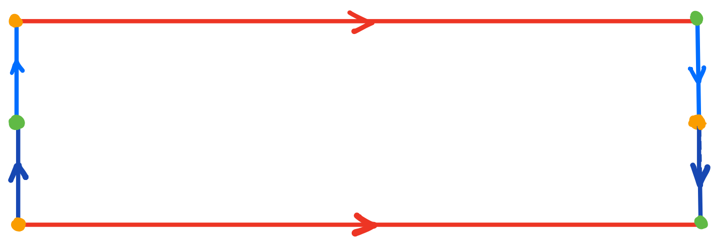

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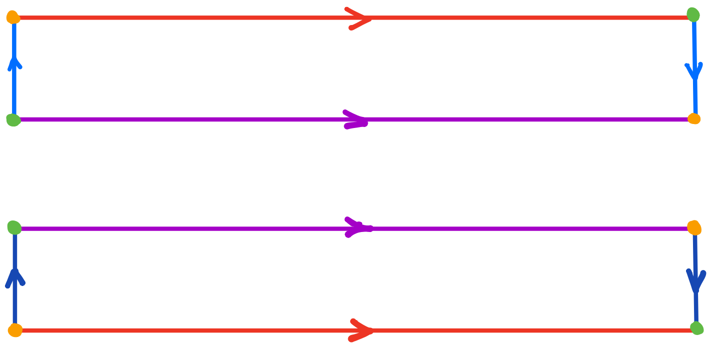

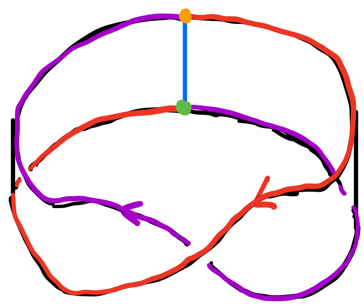

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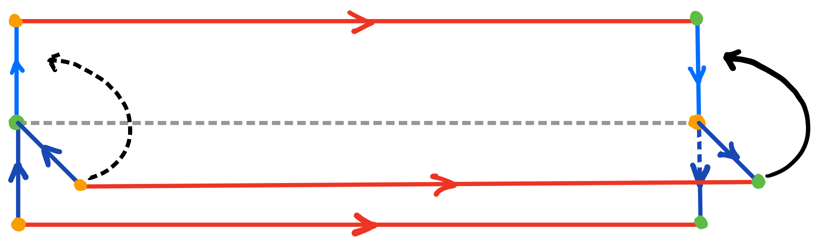

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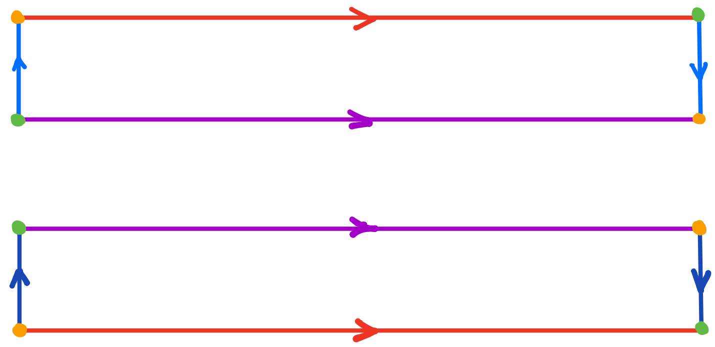

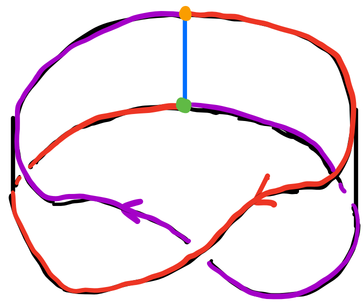

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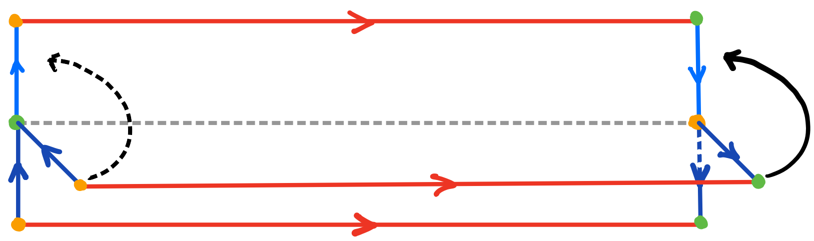

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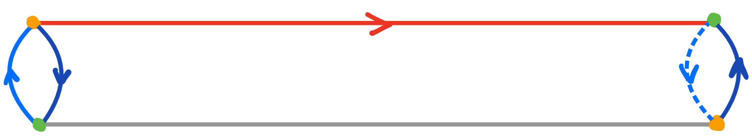

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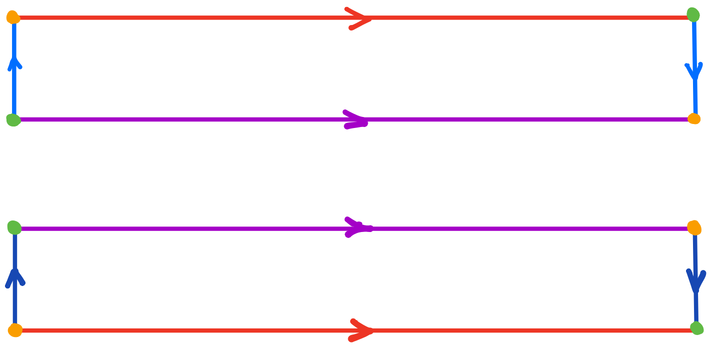
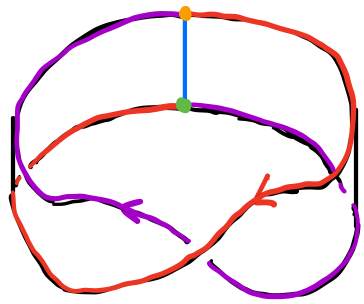
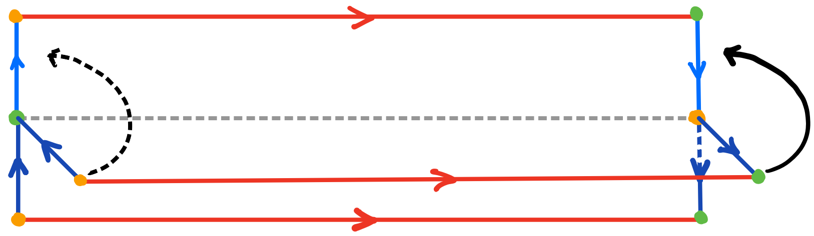




glue along $\Big\}\downarrow$ purple edges



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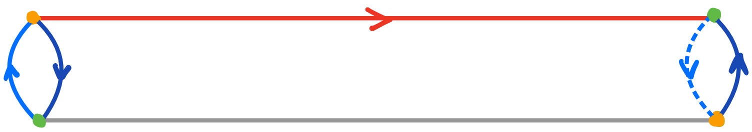

bend $\rightsquigarrow$

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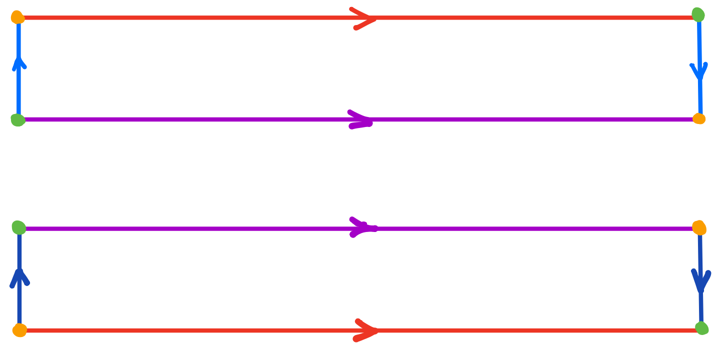

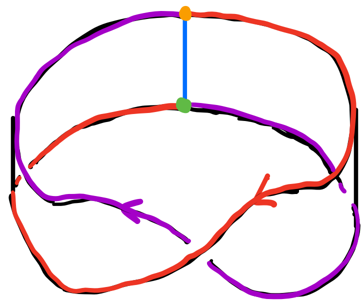

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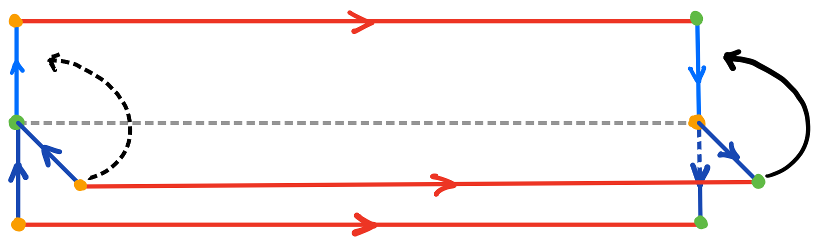

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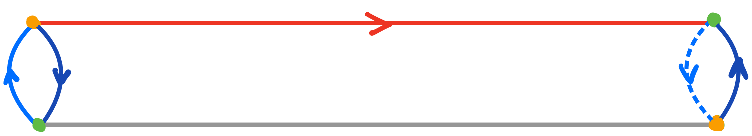

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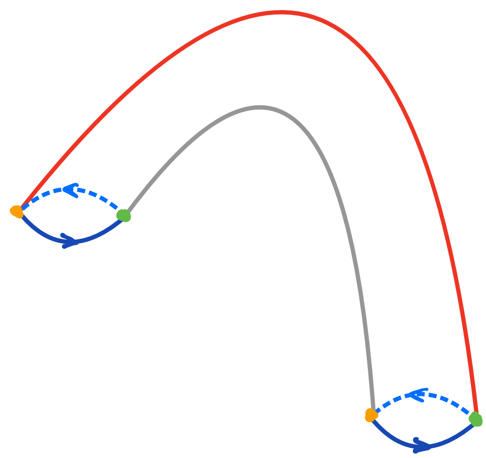

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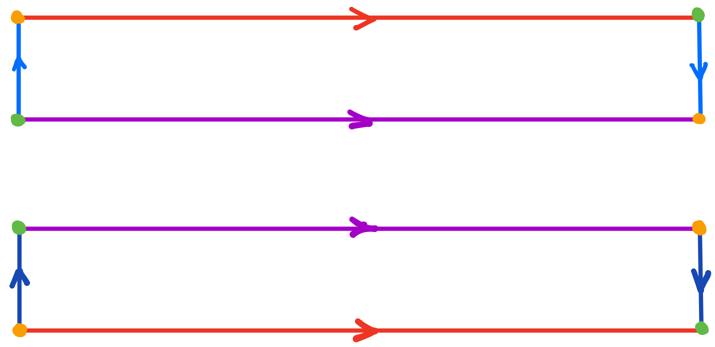

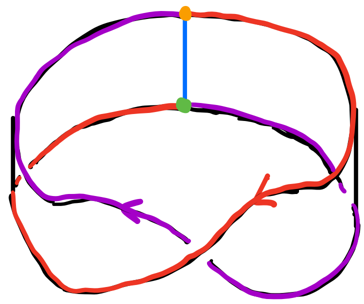

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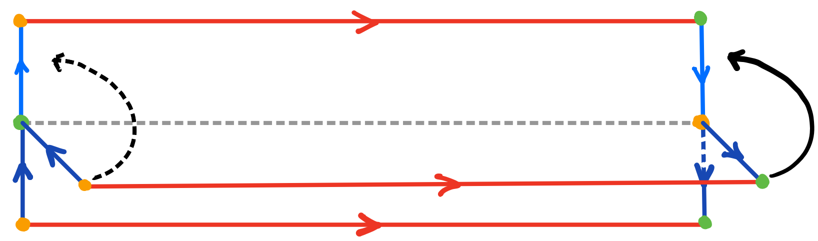

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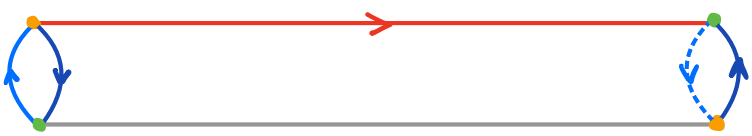

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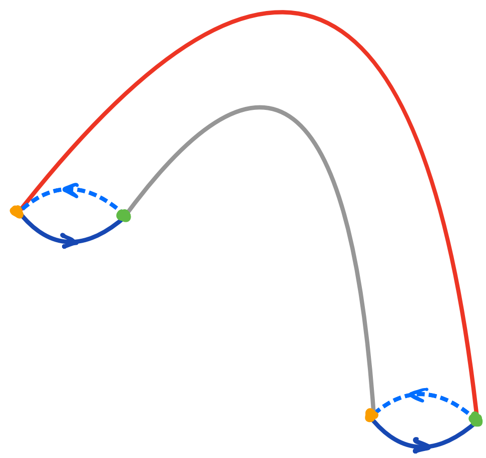

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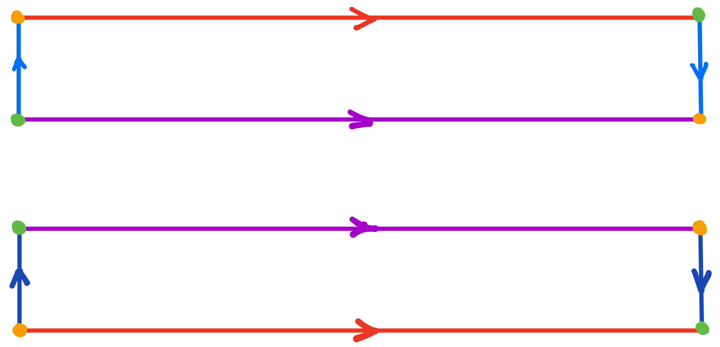

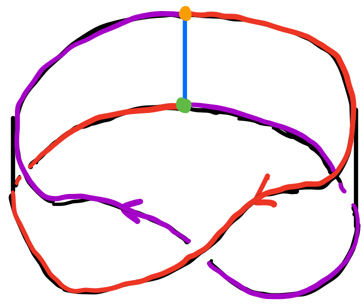

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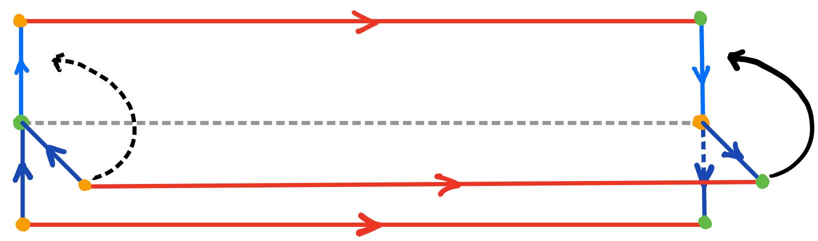

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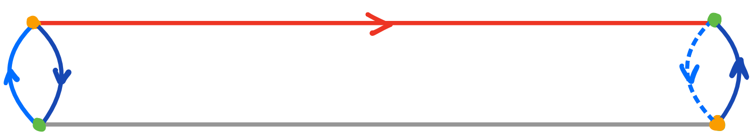

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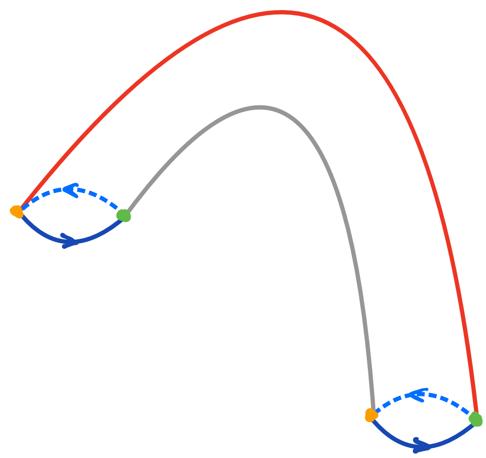

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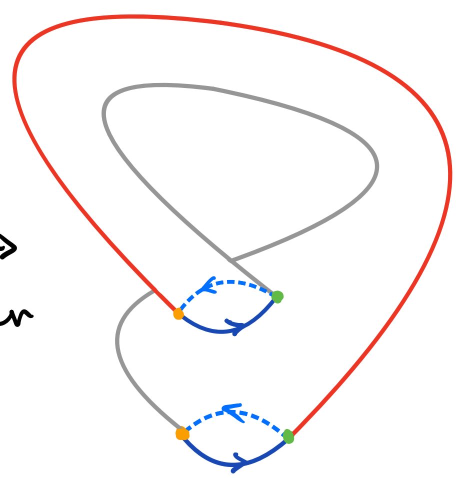

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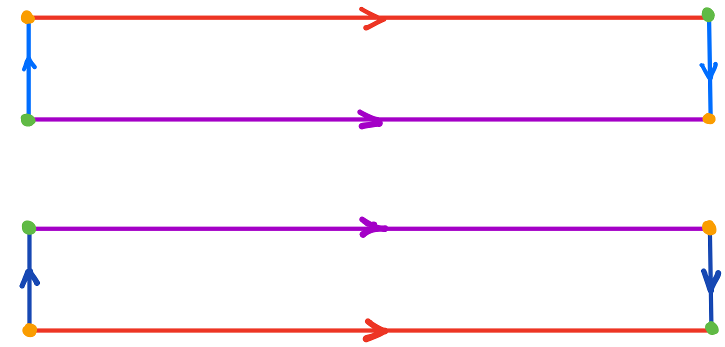

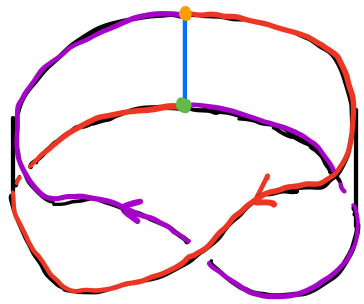

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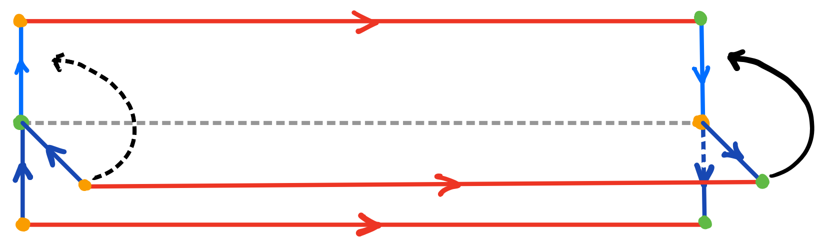

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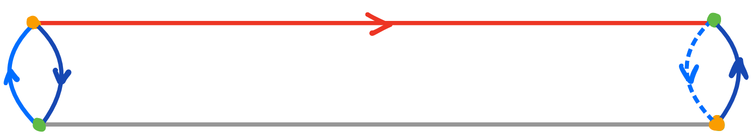

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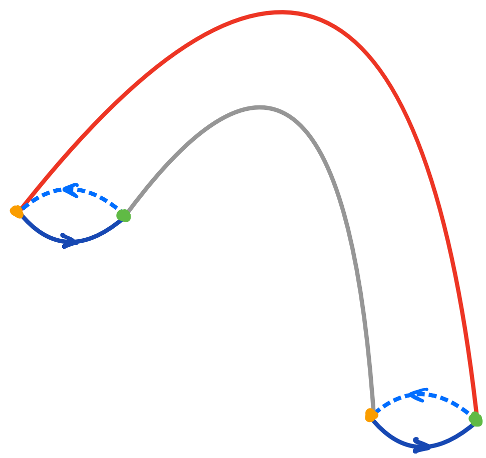

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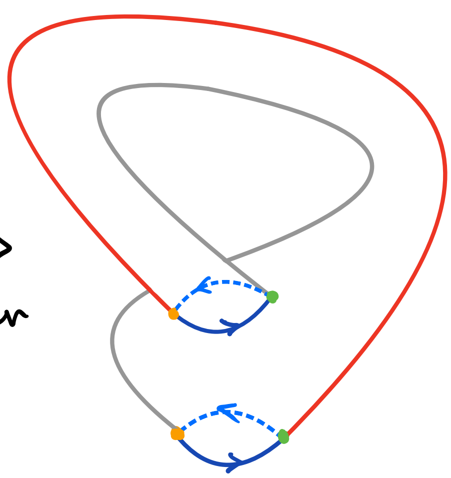

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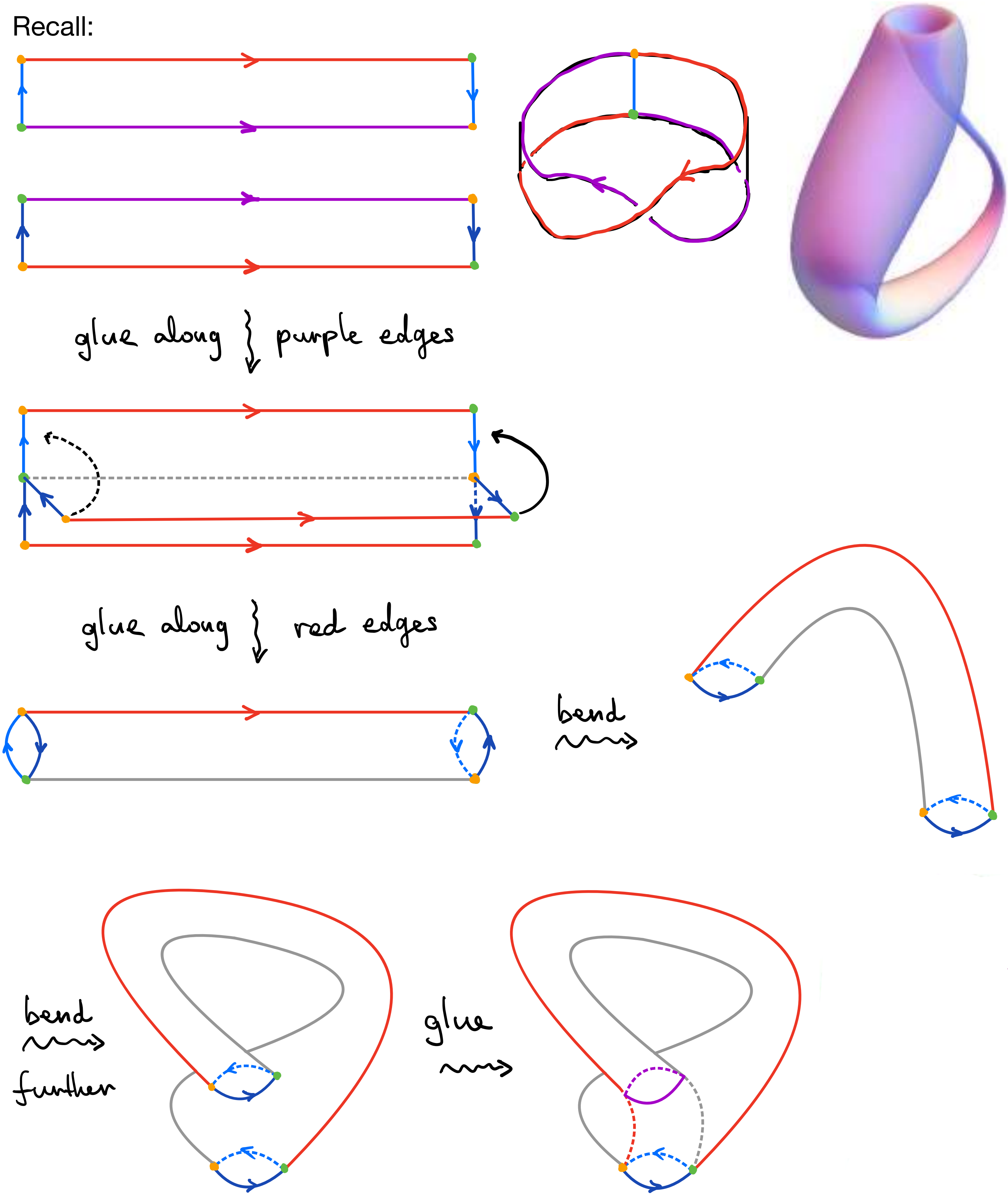

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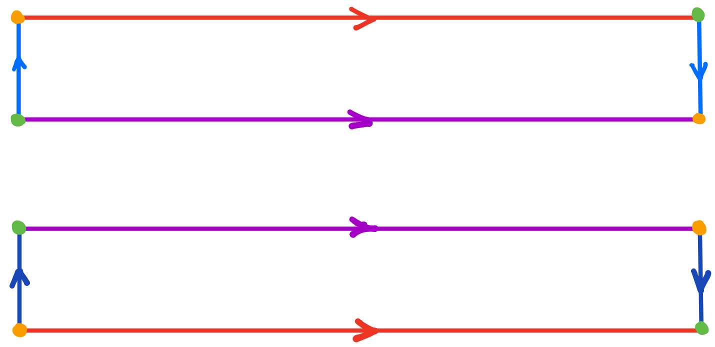

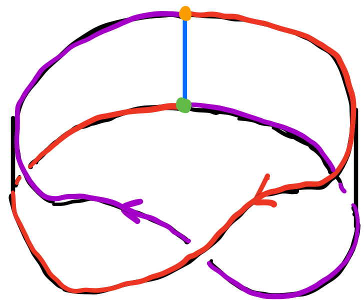

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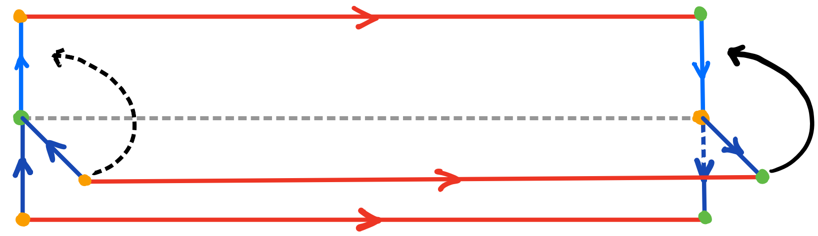

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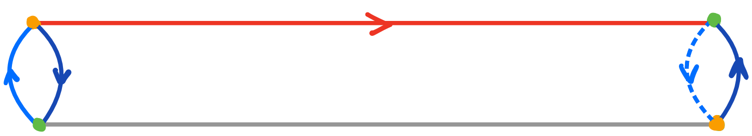

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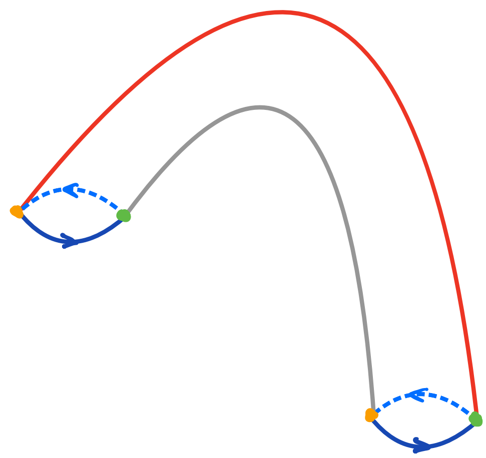

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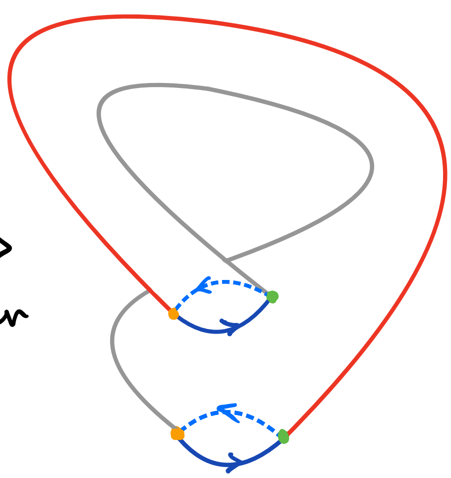

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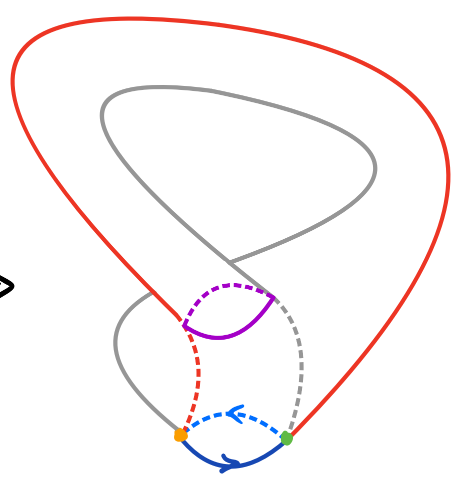

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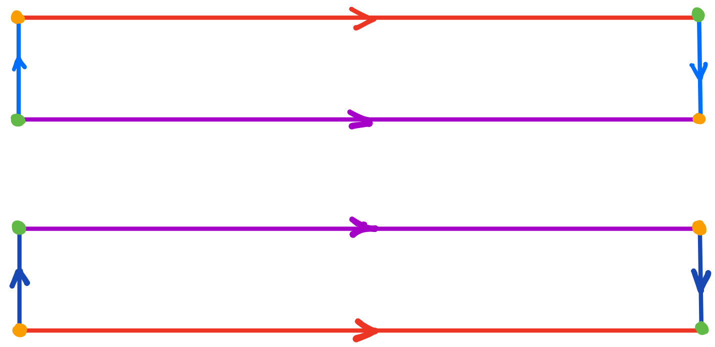

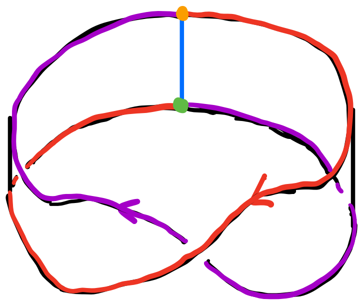

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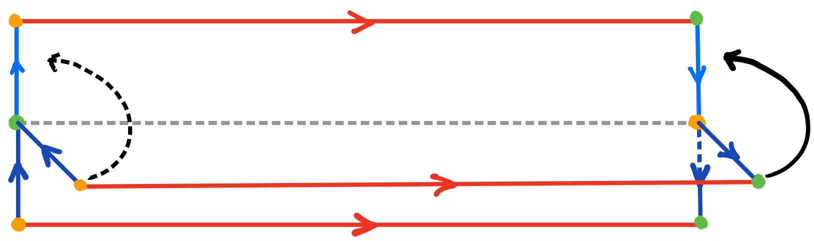

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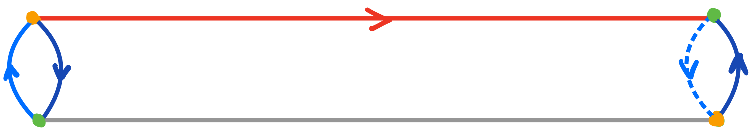
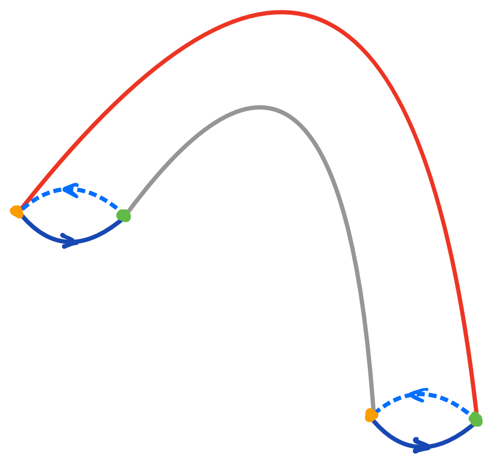


bend ⇝



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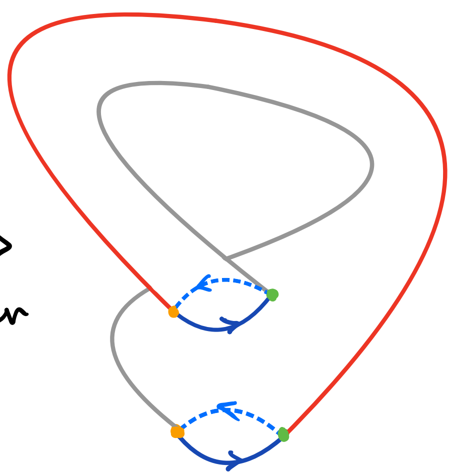

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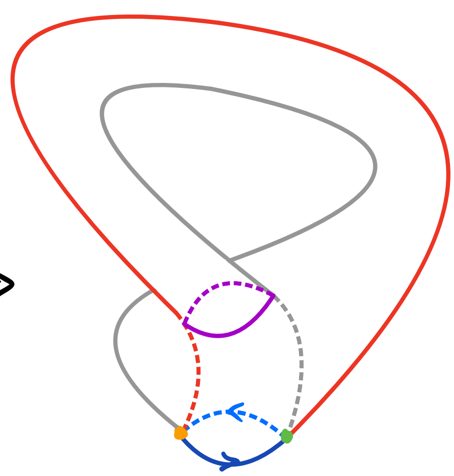

In 3D, has to intersect itself!

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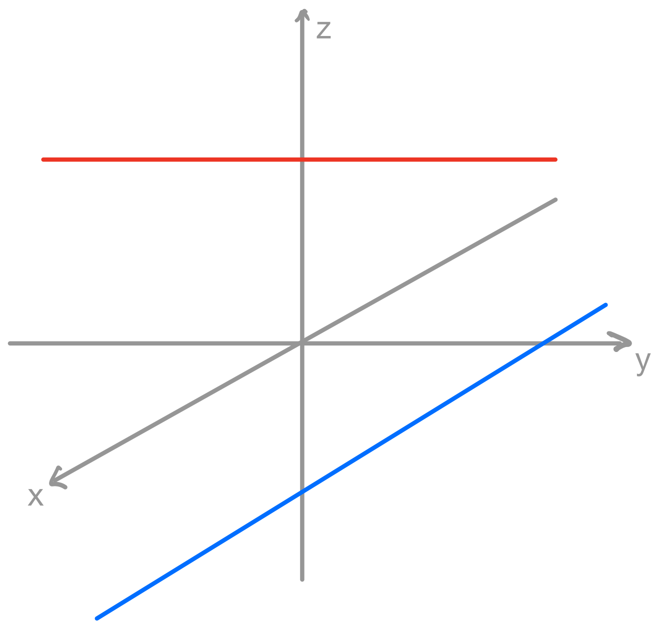

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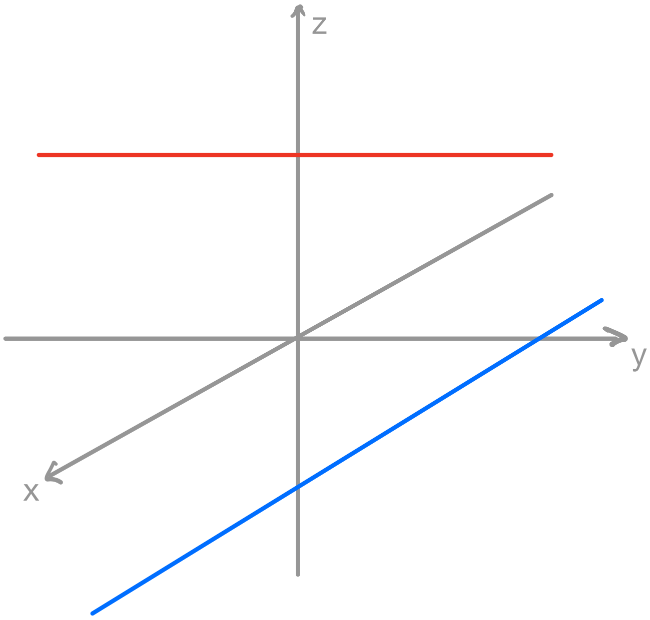


project
⇝
to 2D

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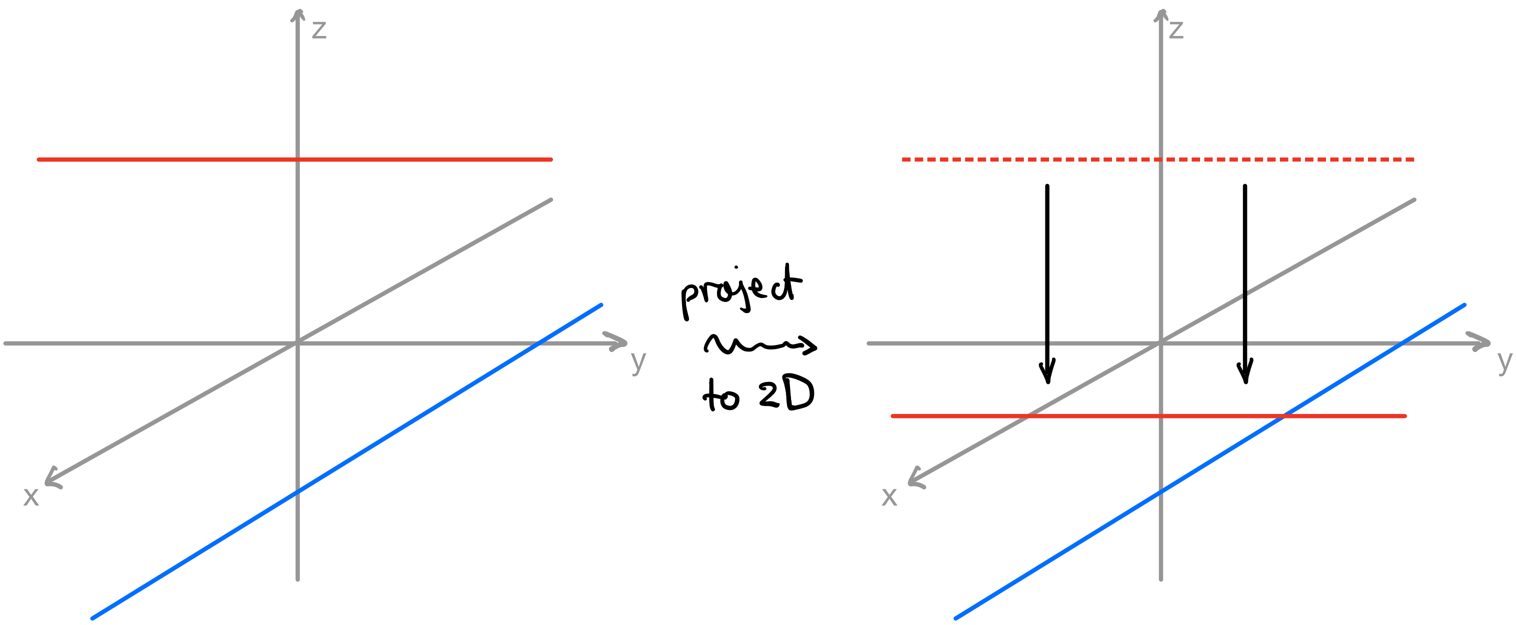

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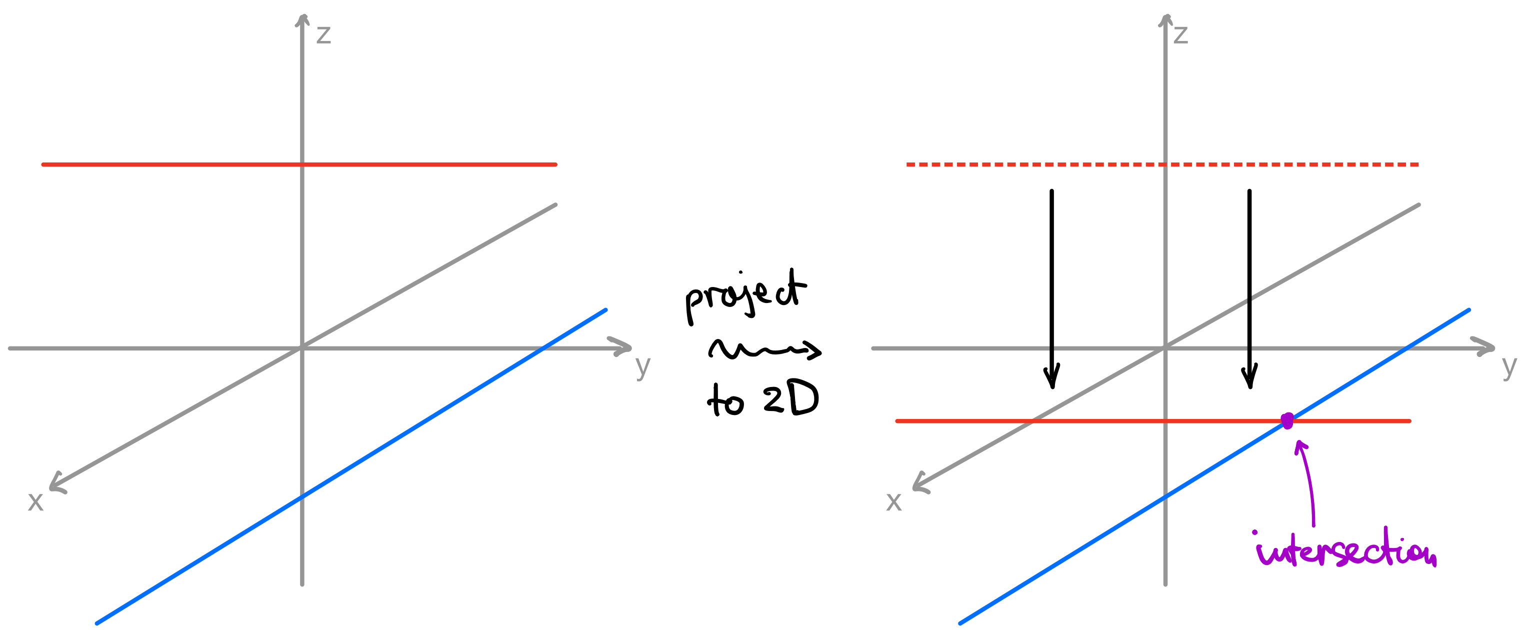

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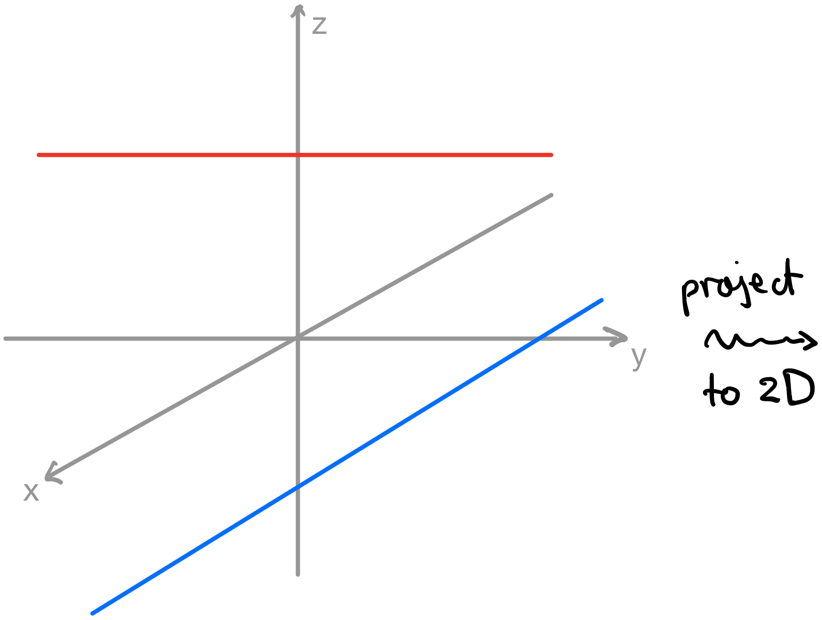


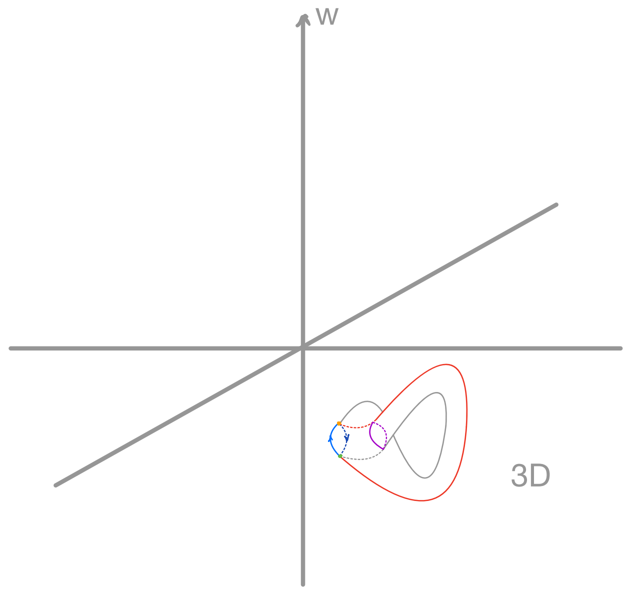

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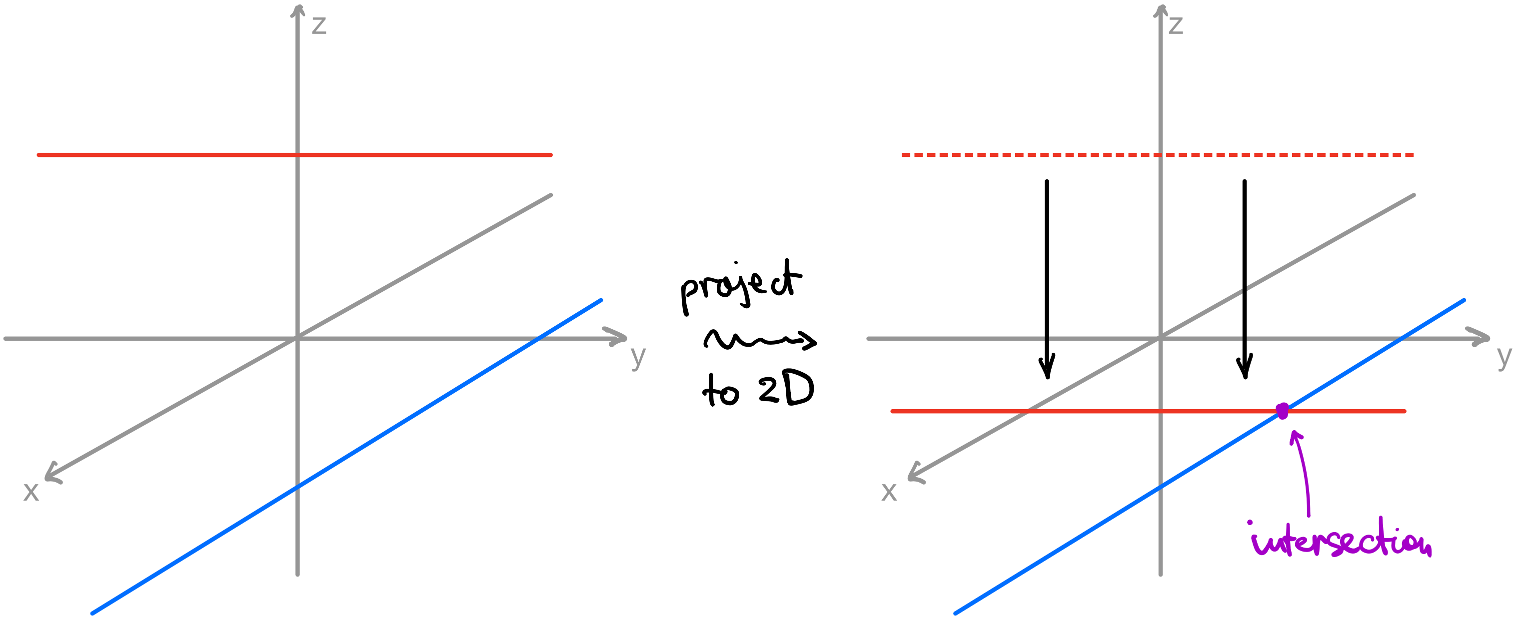


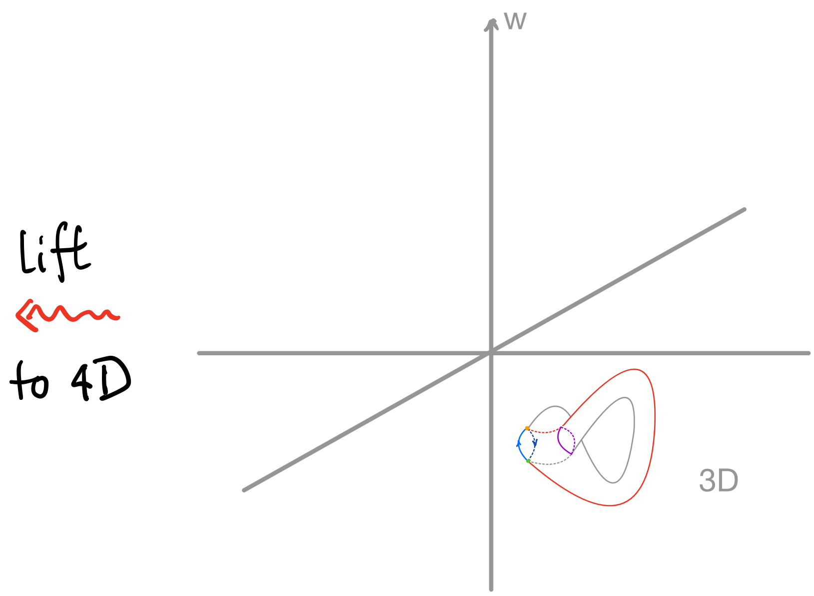

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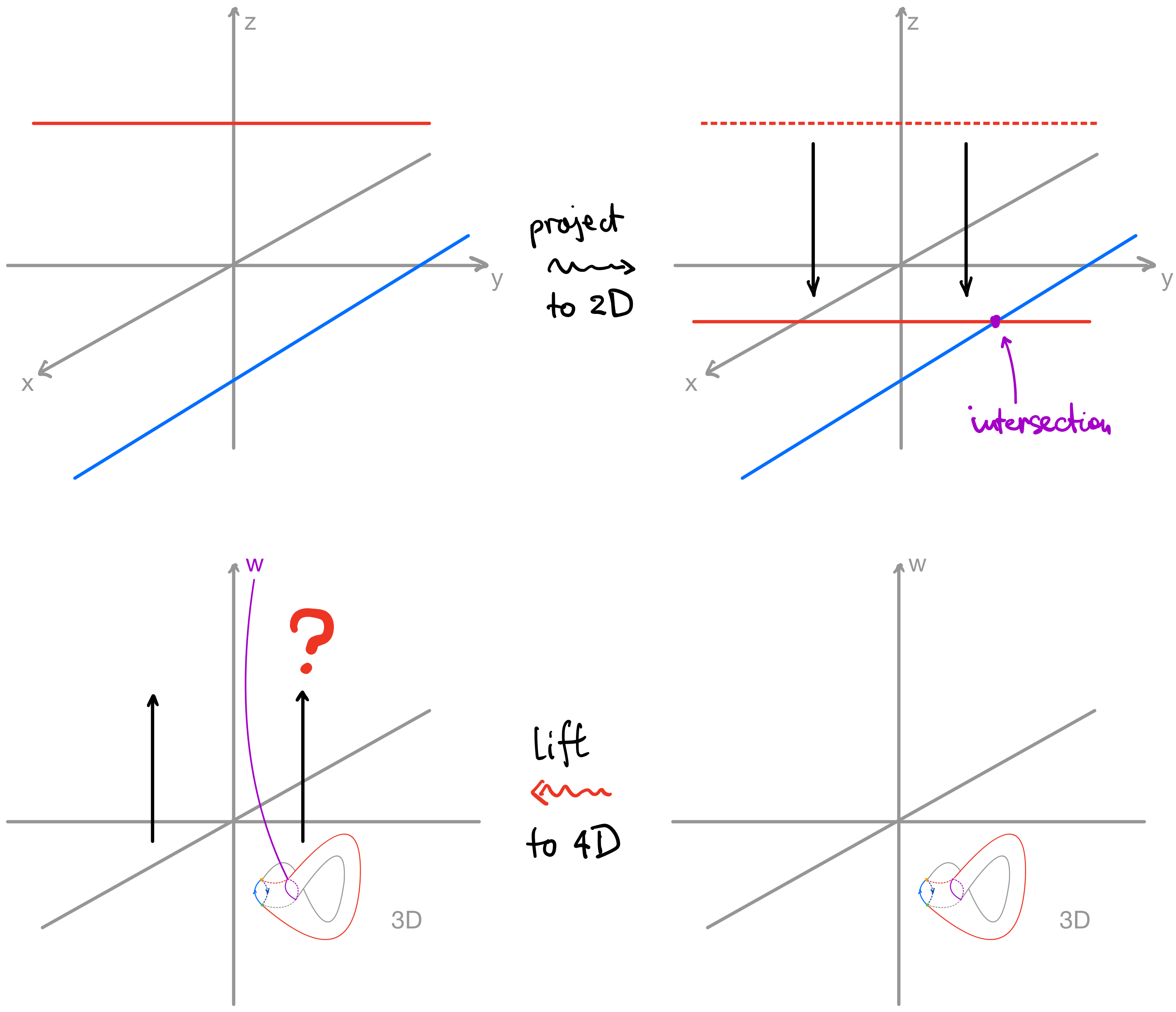

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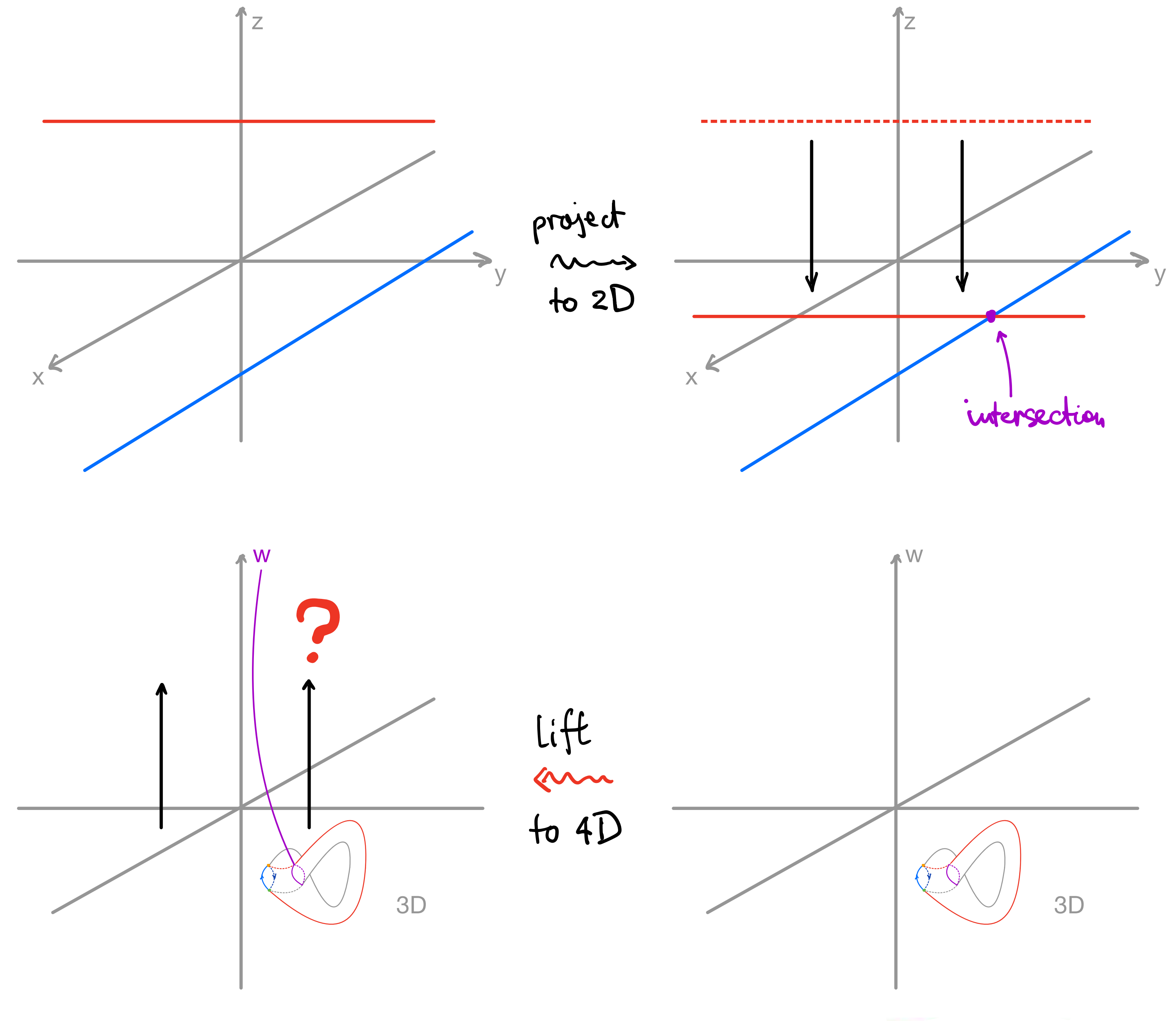


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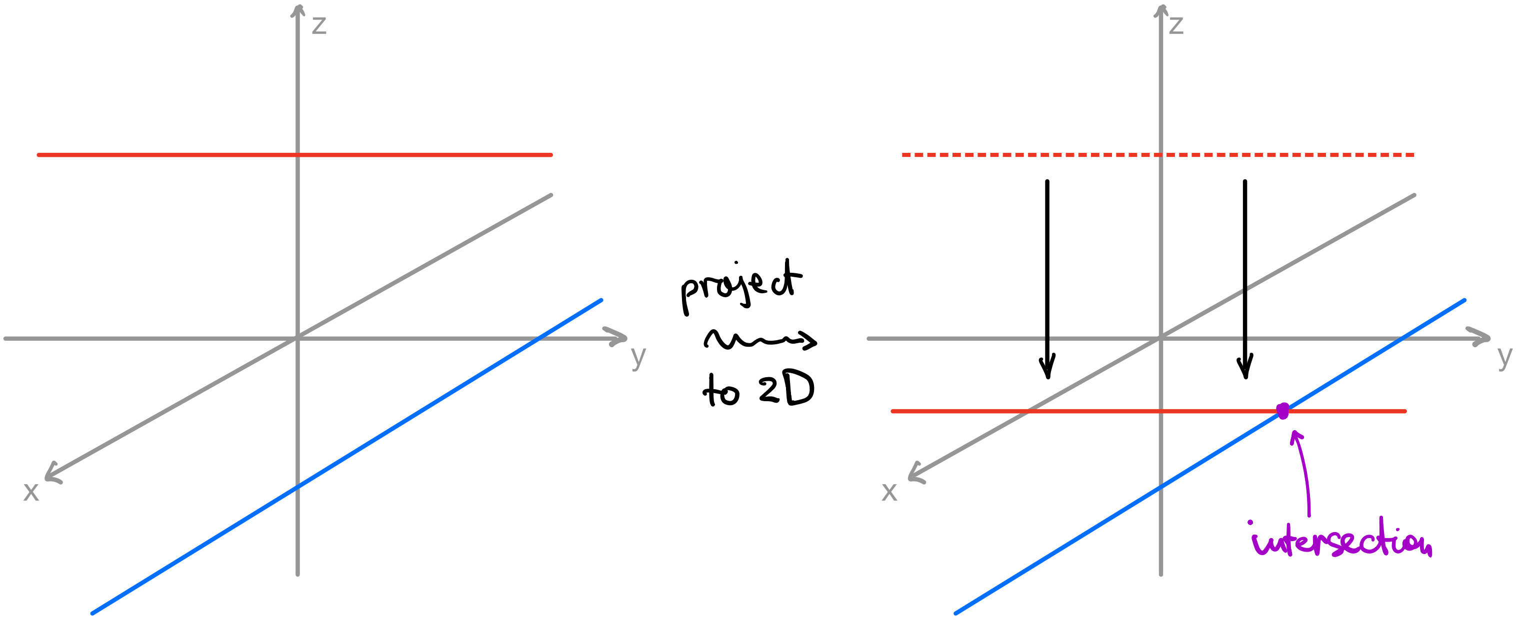


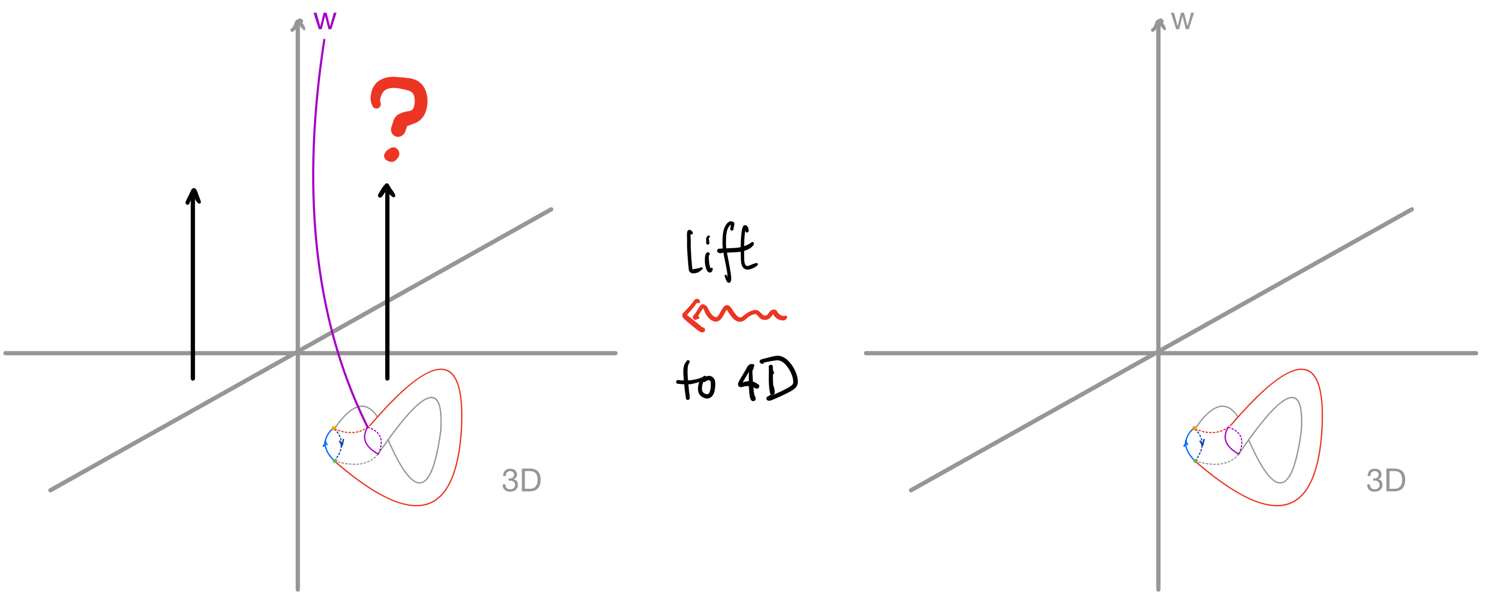


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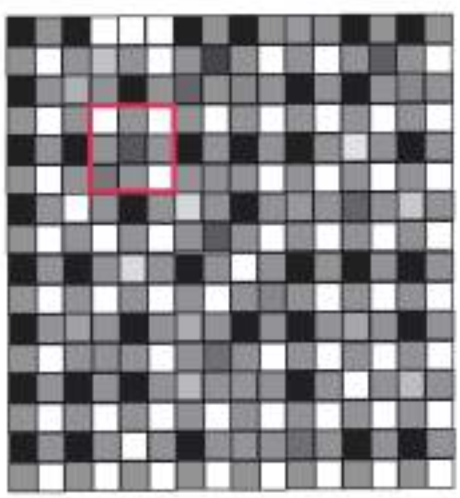

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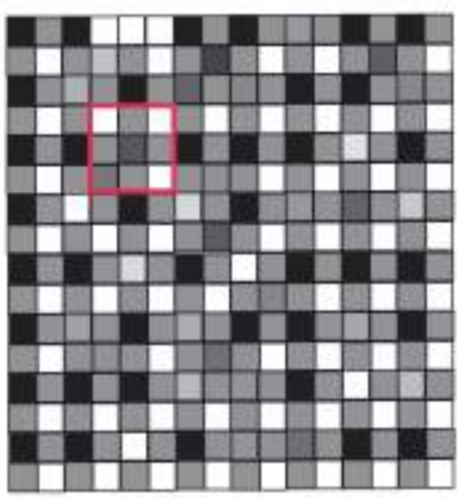

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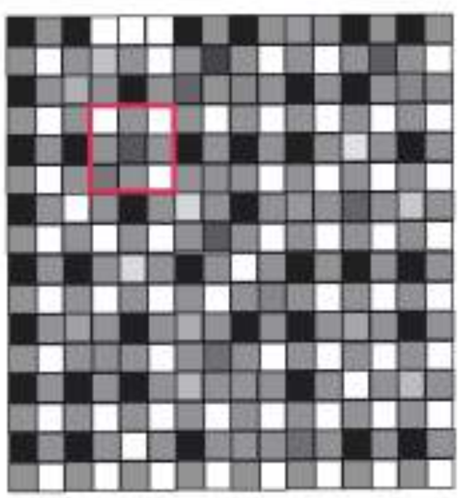

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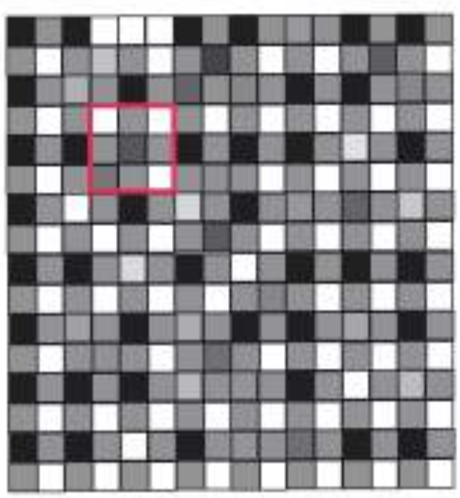

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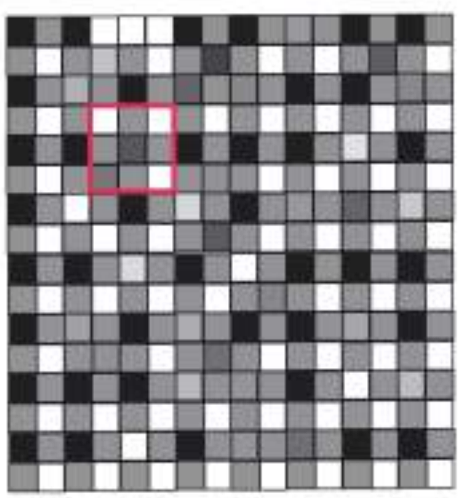

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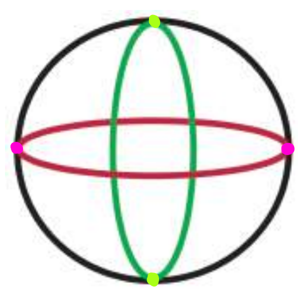

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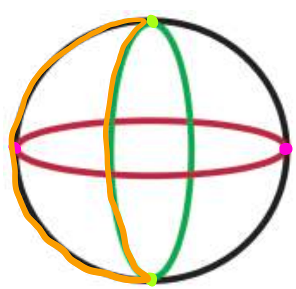

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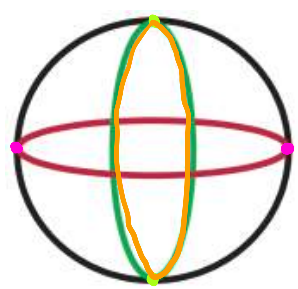

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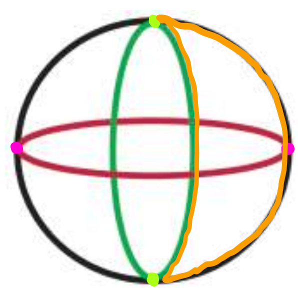

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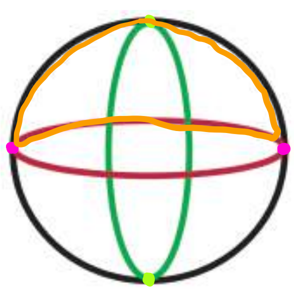

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- So, data now lie on a 7-dimensional sphere, $\cong S^7$

**Result**: Point cloud data $M$ lying on a sphere in $\mathbb{R}^8$

Carlsson–Ishkhanov–de Silva–Zomorodian wish to analyze it with "*persistent homology*" to understand it qualitatively

**First observation**: The points fill out $S^7$ in the sense that every point in $S^7$ is "close" to a point in $M$

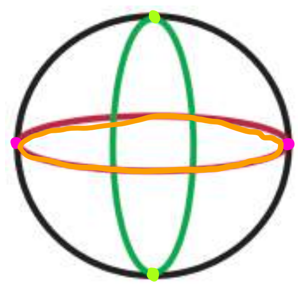
However, density of points varies a great deal from region to region

How to analyze? Set *thresholds* for $M$.

- Define $M[T] \subset M$ by $M[T] = \{x \mid x \text{ is in } T\text{-th percentile of densest points}\}$

By computing the persistent homology of these $M[T]$'s, they reveal

1. $5 \times 10^4$ points, $T = 25$:
   There are 5 independent 1-dimensional cycles on $M[T]$
   Red and green circles do not touch, each touches black circle



THREE CIRCLE MODEL

Lee–Mumford–Pedersen study only high contrast patches:

- Collect approximately $4.5 \times 10^6$ high contrast patches from a collection of images obtained by van Hateren and van der Schaaf

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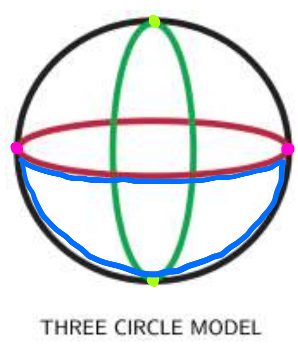

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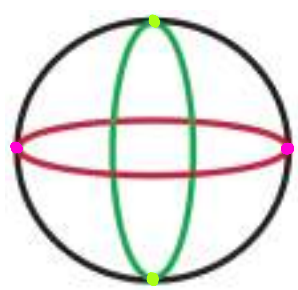

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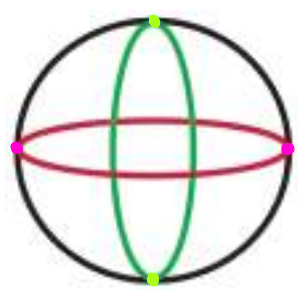
2. $4.5 \times 10^6$ points, $T = 10$:
   There are one 0D cycle (connected), two 1D cycles (loops), and one 2D cycle (surface)



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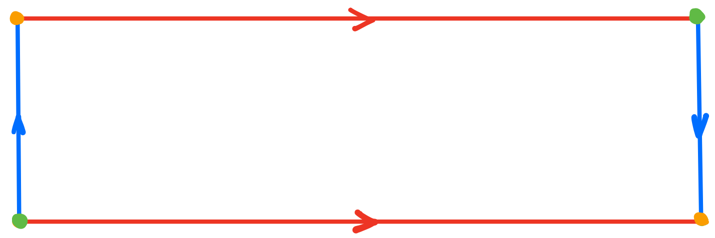

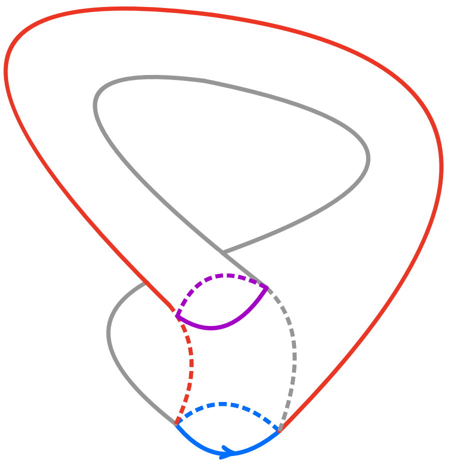

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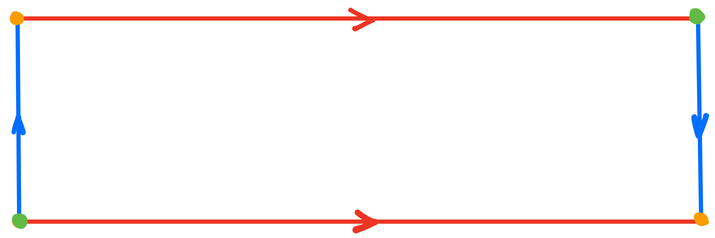

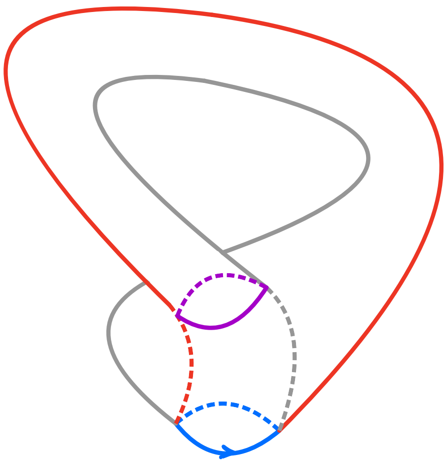

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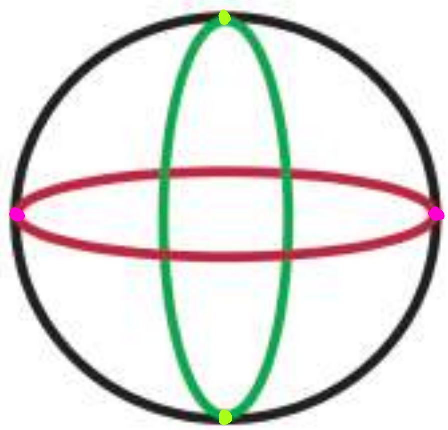

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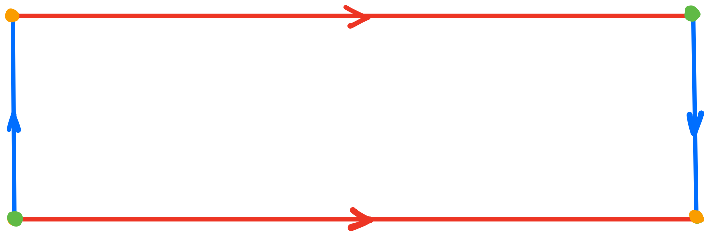

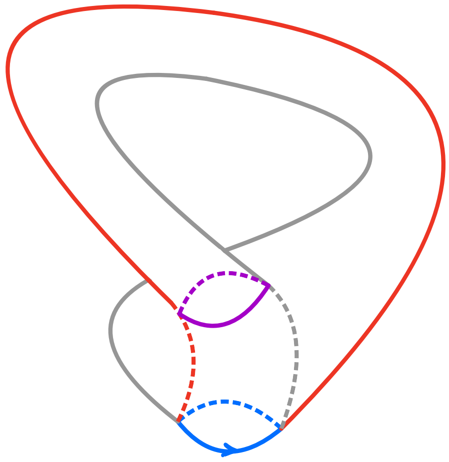

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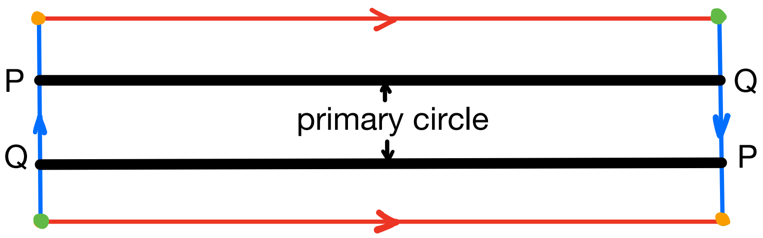


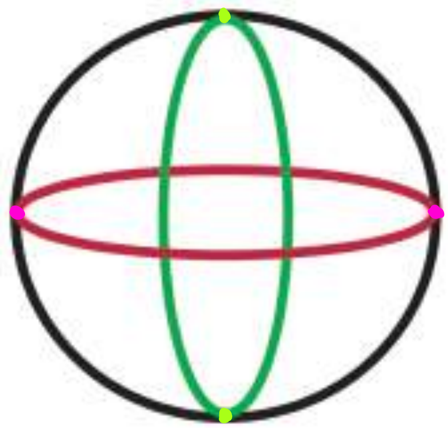

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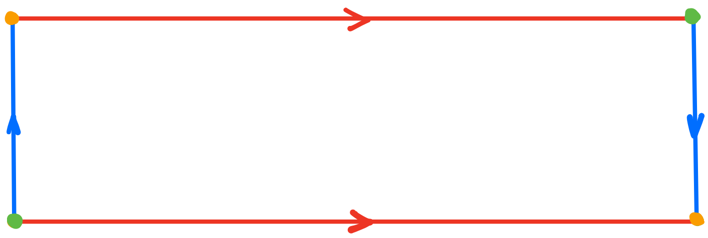

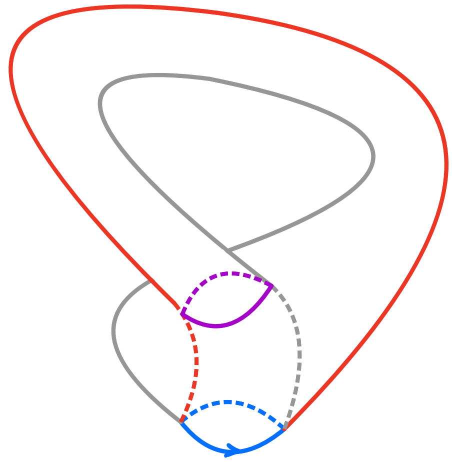

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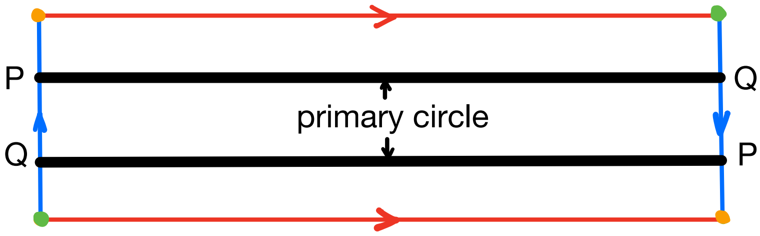


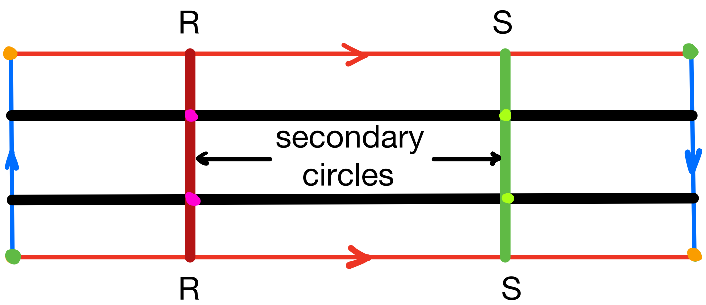


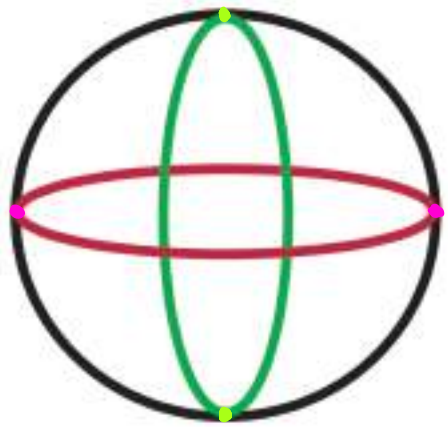

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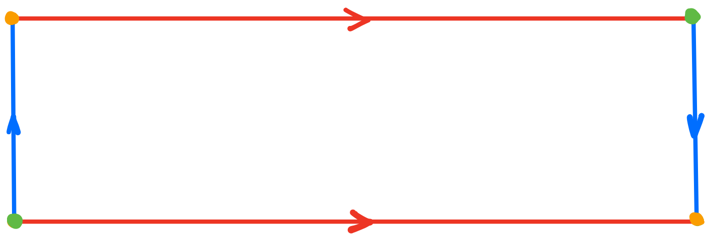

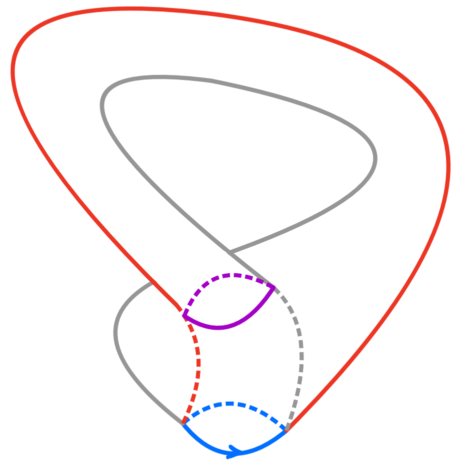

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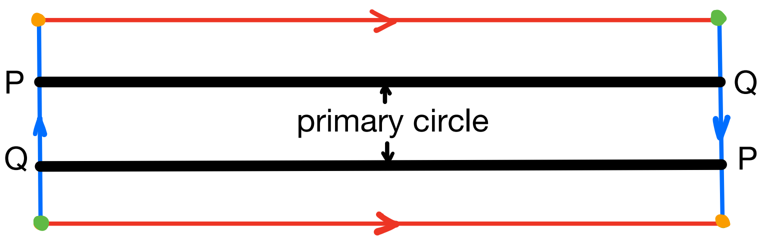


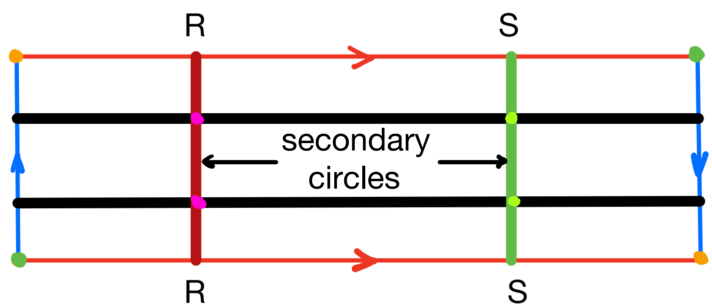


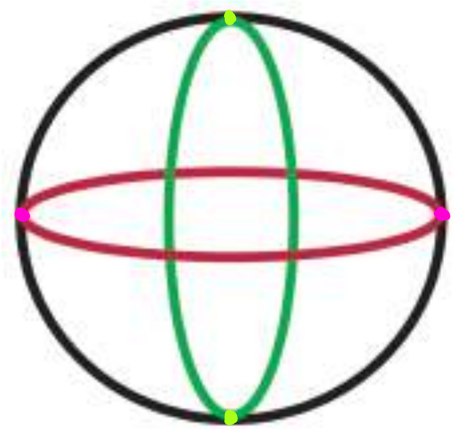

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- Klein bottle appears as where local data from digital camera images sit. It is as natural as what you see in everyday life, not just an idle game!

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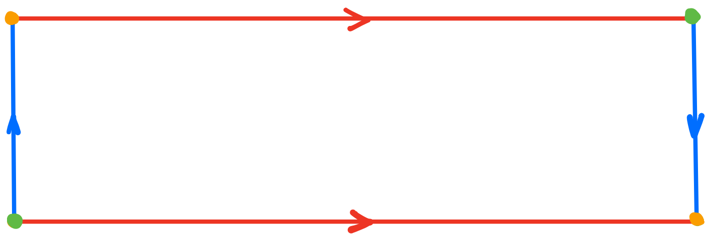

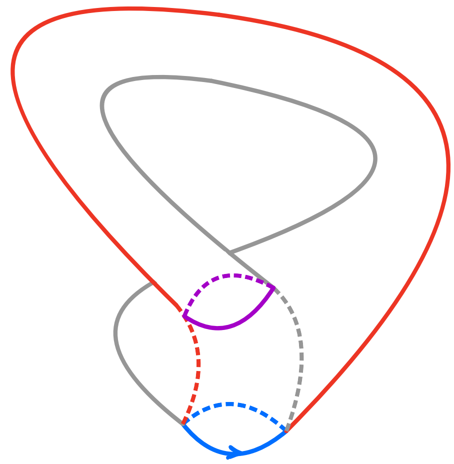

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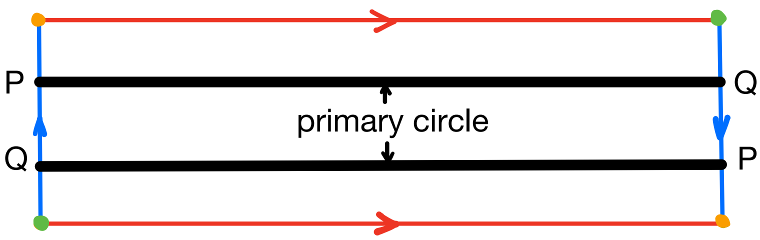


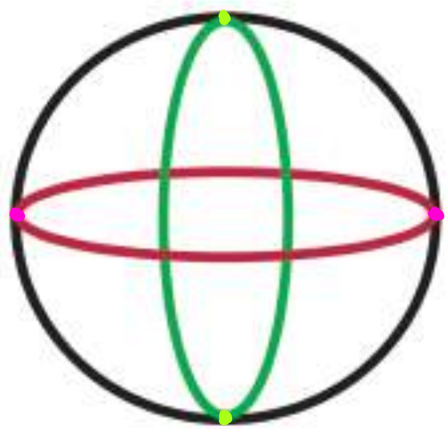

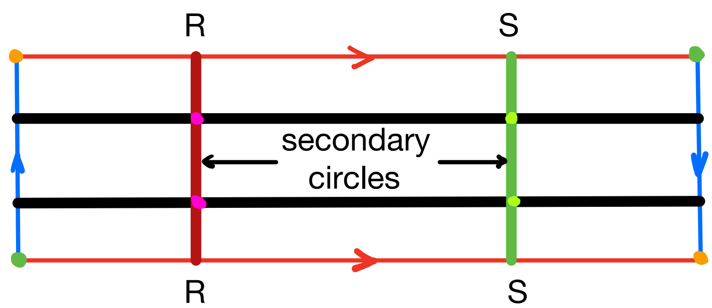


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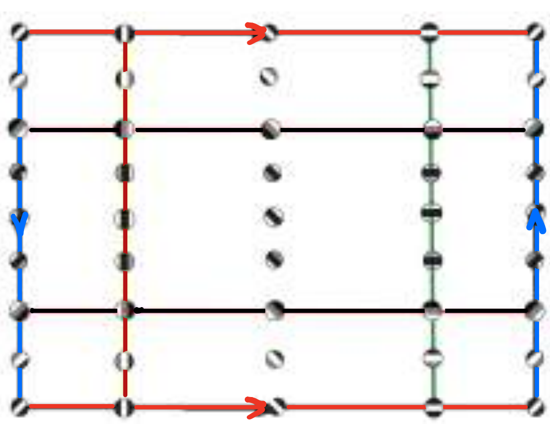

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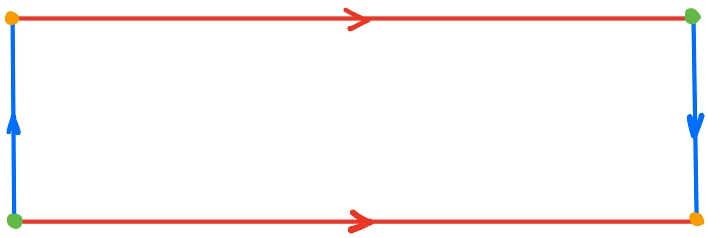

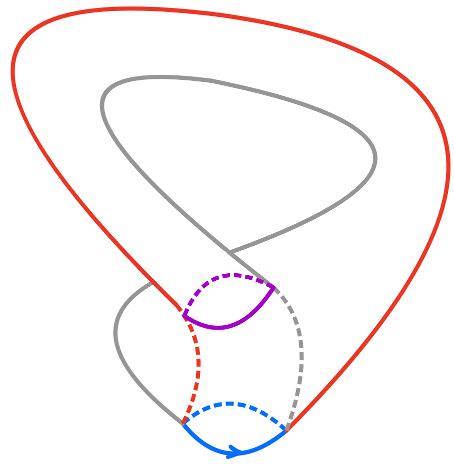

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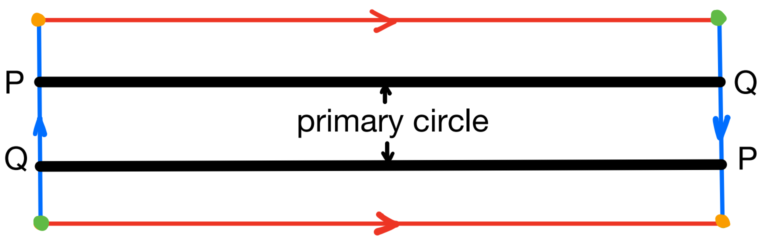


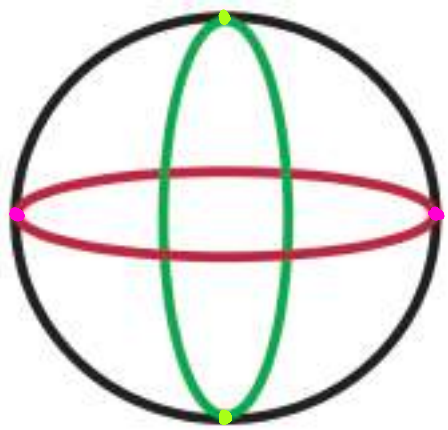

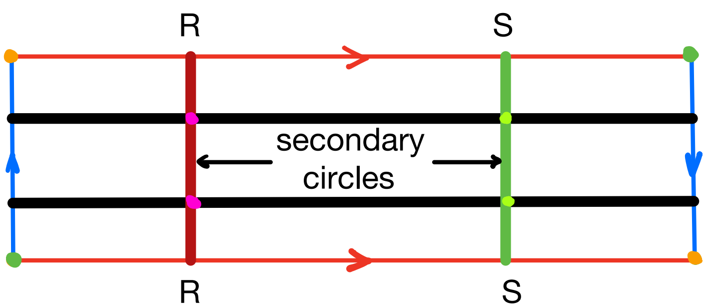


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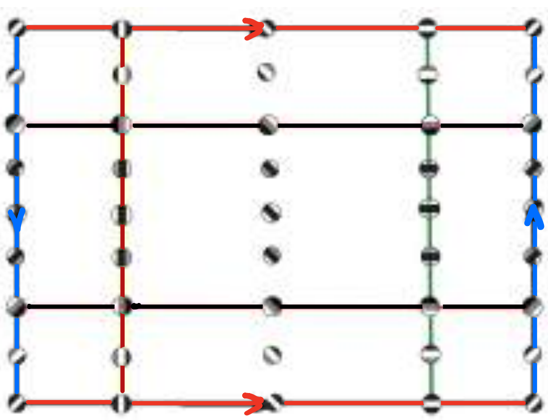

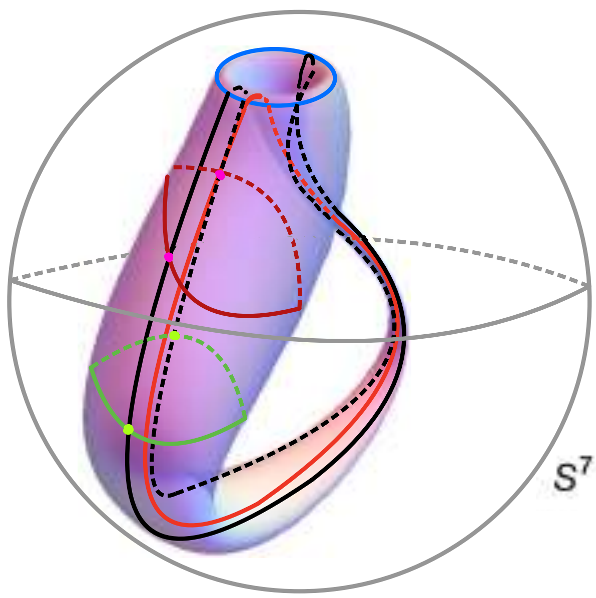

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Other weird surfaces contain **singularity**

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## Swallowtail

- 2D motion visualization in Mathematica
- 3D handicrafted model

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Holography featured prominently in the Beijing Winter Olympics opening ceremony

Holography is made possible through special optical devices and materials

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Topological Metasurface by Encircling an Exceptional Point

围绕奇异点的拓扑超表面

16:00 – 17:00
2021年10月14日（星期四）
工学院南楼326

Qinghua Song 宋清华
Assistant Professor
Tsinghua Shenzhen International Graduate School

报告摘要 ABSTRACT

This talk presents a plasmonic topological metasurface that introduces an additional degree of freedom to address optical phase engineering by exploiting the topological features of non-Hermitian matrices operating near their singular points. Choosing metasurface building blocks to encircle a singularity following an arbitrarily closed trajectory in parameter space, it is able to engineer a topologically protected full-2π-phase on a specific reflected polarization channel. The ease of implementation together with its compatibility with other phase-addressing mechanisms bring topological properties into the realm of industrial applications in optical frequencies and prove that metasurface technology represents a convenient test bench to study and validate topological photonic concepts.

个人简介 BIOGRAPHY

Qinghua Song received the B.Sc. and Ph.D. degrees from Xi'an Jiaotong University and Université Paris-Est in 2013 and 2017, respectively. Then he worked as a postdoc at Nanyang Technological University in Singapore in 2017 and CNRS-CRHEA, France in 2019. Since 2021 he has been with Tsinghua Shenzhen International Graduate School, Tsinghua University, China, where he is currently Assistant Professor. His research interests include optical metasurfaces, meta-holograms, non-Hermitian optics, topological photonics, tunable metasurface, antenna design, etc. He has published some papers as first author in Science, Science Advances, Nature Communications, etc.

A topological classification for intersection singularities of exceptional surfaces in pseudo-Hermitian systems

Hongwei Jia[1], Rao-Yang Zhang[1], Jing Hu, Yixin Xiao, Yifei Zhu[*], C. T. Chan[1]

Abstract: Exceptional degeneracy plays a pivotal role in the topology of non-Hermitian systems, and recently many efforts have been devoted to classifying exceptional points and exploring the intriguing physics. However, intersections of exceptional surfaces, which are commonly present in non-Hermitian systems with parity-time symmetry or chiral symmetry, were not classified. Here we classify generic pseudo-Hermitian systems, for which the momentum space is partitioned by exceptional surfaces, and those surfaces intersect stably in momentum space. The topology of such gapless structure can be viewed from its quotient space, which is "figure eight," by considering the equivalence relations of eigenstates in energy gaps and on exceptional surfaces. We reveal that the topology of such systems can be described by a free non-Abelian group composed of products of two generators. The topological invariants in the group are well associated with the spin rotation of eigenstates via adiabatic transformations. Our classification does not rely on specific bandgaps and is thus a global topological description. Importantly, the classification predicts a new phase of matter, and can systematically explain how the exceptional surfaces and their intersections evolve against deformations to the Hamiltonian. Our work opens a new pathway for designing systems with robust topological phases, and is potentially a guidance for applications to sensing and lasing devices utilizing exceptional surfaces and intersections.

Observation of $\boxed{\text{swallowtail catastrophe singularity}}$ in non-Hermitian bands and its topological origin

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Swallowtail and other singularities play a pivotal role in designing such.


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## Concluding prose:

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mathematicians check their equations
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They are inspired. Without dreams
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Topological Metasurface by Encircling an Exceptional Point

围绕奇异点的拓扑超表面

16:00 – 17:00
2021年10月14日（星期四）
工学院南楼326

Qinghua Song 宋清华
Assistant Professor
Tsinghua Shenzhen International Graduate School

ABSTRACT

This talk presents a plasmonic topological metasurface that introduces an additional degree of freedom to address optical phase engineering by exploiting the topological features of non-Hermitian matrices operating near their singular points. Choosing metasurface building blocks to encircle a singularity following an arbitrarily closed trajectory in parameter space, it is able to engineer a topologically protected full-2π-phase on a specific reflected polarization channel. The ease of implementation together with its compatibility with other phase-addressing mechanisms bring topological properties into the realm of industrial applications in optical frequencies and prove that metasurface technology represents a convenient test bench to study and validate topological photonic concepts.

个人简介 BIOGRAPHY

Qinghua Song received the B.Sc. and Ph.D. degrees from Xi'an Jiaotong University and Université Paris-Est in 2013 and 2017, respectively. Then he worked as a postdoc at Nanyang Technological University in Singapore in 2017 and CNRS-CRHEA, France in 2019. Since 2021 he has been with Tsinghua Shenzhen International Graduate School, Tsinghua University, China, where he is currently Assistant Professor. His research interests include optical metasurfaces, meta-holograms, non-Hermitian optics, topological photonics, tunable metasurface, antenna design, etc. He has published some papers as first author in Science, Science Advances, Nature Communications, etc.

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Thank you, and Happy π Day!

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- Natural image statistics example, including images, adapted from Gunnar Carlsson, *Topology of data*, available at http://www.stat.uchicago.edu/~lekheng/meetings/mmds/slides2008/carlsson.pdf
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- Beijing Winter Olympics picture: http://en.kremlin.ru/events/president/news/67715
- Qinghua Song lecture poster: Department of Electronic and Electrical Engineering, Southern University of Science and Technology
- Hongwei Jia, Ruo-Yang Zhang, Jing Hu, Yixin Xiao, Yifei Zhu, and C. T. Chan, *A topological classification for intersection singularities of exceptional surfaces in pseudo-Hermitian systems*, preprint, https://yifeizhu.github.io/2-band. pdf (supplementary materials, https://yifeizhu.github.io/2-band-supp.pdf), 2022
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