V

HOMOTOPY GROUPS OF SOME MAPPING TELESCOPES

Donald M. Davis and Mark Mahowald[1]

§1. *INTRODUCTION*

If b is an integer and $t>b$ is an integer or ∞, there is a spectrum P_b^t which when $b\geq 0$ is the suspension spectrum of stunted real projective space RP^t/RP^{b-1}. We will prove in Section 2 that for all odd integers b and even or infinite integers t there are maps

$$\ell_{b,t}: P_b^t \longrightarrow P_{b-8}^{t-8}$$

of Adams filtration 4, nontrivial on the bottom cell.

DEFINITION 1.1. $\bar{P}_b^t$ *is the mapping telescope of the sequence*

$$P_b^t \xrightarrow{\ell_{b,t}} P_{b-8}^{t-8} \xrightarrow{\ell_{b-8,t-8}} P_{b-16}^{t-16} \xrightarrow{\ell_{b-16,t-16}} P_{b-24}^{t-24} \longrightarrow$$

The main purpose of this paper is to calculate the homotopy groups $\pi_*(\bar{P}_b^t)$. In [11] (unpublished) this calculation was used to prove that $\bar{P}_b^t$ is the K-theory localization of

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P_b^t, and some related results. A more elementary, and less computational, proof of these localization results was achieved by H. Miller and will appear in [12]. The present paper is culled from [11], for it is felt that the method of calculation of $\pi_*(\bar{P}_b^t)$ is of independent interest.

Let bo denote the spectrum for connective real K-theory localized at 2, $\Sigma^4 bsp$ the spectrum obtained from bo by killing the first 3 homotopy groups, and J the fiber of $\psi^3-1: bo \to \Sigma^4 bsp$ ([15],[17]). The following result is proved in Section 3 using bo-resolutions.

THEOREM 1.2. *The Hurewicz homomorphism* $\pi_*(\bar{P}_b^t) \to J_*(\bar{P}_b^t)$ *is an isomorphism.*

In Section 2, we will calculate $J_*(\bar{P}_b^t)$, and hence by 1.2 $\pi_*(\bar{P}_b^t)$. A novel feature of this calculation will be the use of Adams-type homotopy charts with negative (as well as positive) filtrations.

In Section 4 some K_*-localization results in addition to those of [12] are discussed. For example, it is proved that $\Sigma\bar{P}_1^{8n}$ is the K_*-localization of the mod 2^{4n} Moore spectrum, and that the K_*-localization of S^0 fits into a cofibration with $\bar{P}_1$ and the rational Moore spectrum.

In Section 5, we show that if $\wedge$ bu is applied to the K_*-localization map $S^{-1}/2^\infty \to \bar{P}_1$, then the cofibre is essentially a Brown-Comenetz dual ([16]) of $\Sigma^3 bu$. We generalize this argument and obtain the following universal coefficient theorem.

THEOREM 1.3. *If* G *is any divisible torsion abelian group, such as* $\mathbb{Z}/p^\infty$ *or* $\mathbb{Q}/\mathbb{Z}$, *and* X *is any spectrum, there is an exact sequence*

$$\to ku^n(X;G) \to KU^n(X;G) \to \mathrm{Hom}(ku_{n-2}(X),G) \to ku^{n+1}(X;G) \to .$$

We would like to express our gratitude to the University of Warwick and especially John Jones, who organized a seminar in Autumn, 1982, to study [19], out of which many of these ideas originated. The first author thanks the Institute for Advanced Study, where [11] was written, and Haynes Miller.

§2. CALCULATING $J_*(\bar{P}^t_b)$

The spectrum P^t_b can be defined as the Thom spectrum $T(b\xi_{t-b})$, where b is possibly negative and ξ_{t-b} is the Hopf bundle over RP^{t-b}. Alternatively it may be defined using James periodicity as in [3].

A map has (Adams) filtration $\geq s$ if it can be written as a composite of s maps, each trivial in $\mathbb{Z}_2$-cohomology. We use $\mathbb{Z}_2$ and $\mathbb{Z}/2$ interchangeably. A map $S^n \to X$ has filtration s if it is detected in

$\mathrm{Ext}^s_A(H^*X, \mathbb{Z}_2)$. Let $X^{\langle s\rangle}$ denote the spectrum obtained from X by killing Ext classes of filtration less than s. We often write P^∞_b as P_b. The following result is similar to one used by Lam in [13].

PROPOSITION 2.1. *For all odd integers* β *and even or infinite integers* t, *there are filtration-4 maps*

$$\ell_{b,t}: P^t_b \longrightarrow P^{t-8}_{b-8}$$

which induce isomorphisms in K_* *and* K^*.

Proof (*H. Miller*). Since it has filtration 4, 16ι lifts to a map $P^t_b \xrightarrow{f} (P^t_{b-8})^{\langle 4\rangle}$. By Adams' edge ([2]), $(P^t_{b-8})^{\langle 4\rangle}$ is $(b-1)$-connected and so the composite

$$P^{b-1}_{b-8} \xrightarrow{i} P^t_{b-8} \xrightarrow{f} (P^t_{b-8})^{\langle 4\rangle}$$

is trivial. Hence f factors as

$$P^t_{b-8} \xrightarrow{c} P^t_b \xrightarrow{\tilde{f}} (P^t_{b-8})^{\langle 4\rangle},$$

where c, as always, denotes the collapse map.

If $t = \infty$, we are done. Otherwise, let P^T_{B-8} be an S-dual of P^t_{b-8}. (Here $T = L - b + 7$ and $B = L - t + 7$, where L is highly 2-divisible.) Then the argument of the preceding paragraph gives a filtration-4 map $P^T_B \xrightarrow{\tilde{f}} P^T_{B-8}$, whose S-dual is a filtration-4 map $P^t_{b-8} \xrightarrow{D\tilde{f}} P^{t-8}_{b-8}$, which can be lifted to $P^t_{b-8} \to (P^{t-8}_{b-8})^{\langle 4\rangle}$. The connectivity argument of the previous paragraph can be applied to factor this map through

$$P_b^t \xrightarrow{\ell} (P_{b-8}^{t-8})^{\langle 4\rangle} .$$

The diagram

$$\begin{tikzcd} P_B^T \arrow[rr, "\tilde f"] & & P_{B-8}^T \\ & P_{B-8}^T \arrow[ul, "c"] \arrow[ur, "\cdot 16"'] & \end{tikzcd}$$

shows $\tilde{f}_*$ is injective in $K_{-1}(\ )$, and hence $\widetilde{(Df)}$ is surjective in $K^0(\ )$. Now the diagram

$$\begin{tikzcd} & P_{b-8}^t \arrow[dl, "c"'] \arrow[dr, "\widetilde{Df}"] & \\ P_b^t \arrow[rr, "\ell"] & & P_{b-8}^{t-8} \end{tikzcd}$$

shows ℓ^* is an isomorphism in $K^0(\ )$.

It is clear from 1.1 that there are equivalences

$$\bar{P}_b^t \longrightarrow \bar{P}_{b-8i}^{t-8i} .$$

That $\bar{P}_b^t$ is independent of the choice of maps $\ell_{b,t}$ satisfying 2.1 is not so clear, but it follows from the uniqueness of K_*-localization. See [11] or [12].

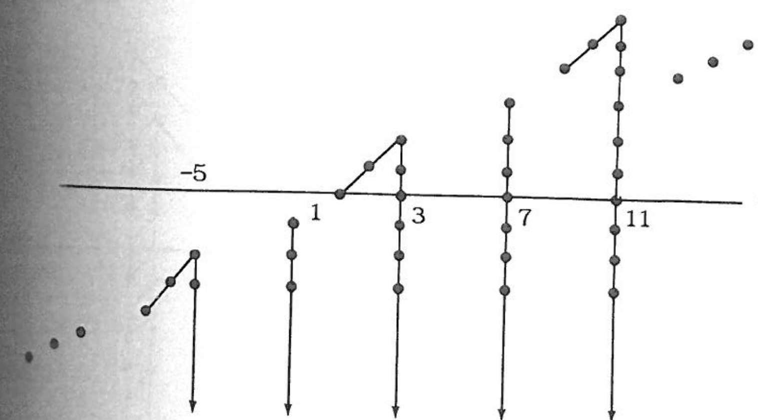
We use homotopy charts $E_r^{s,t}(X)$ with the usual (t-s,s)-coordinates (e.g., [15; pp. 93-95], [8; p.41], [10; p. 149]). For a sequence of filtration-4 maps such as those in 1.1

$$X_0 \to X_1 \to X_2 \to \dots ,$$

we form a chart for the mapping telescope $\bar{X}$ by

$$E_r^{s,t}(\bar{X}) = \lim_{\overrightarrow{i}} E_r^{s-4i,t-4i}(X_i) .$$

For example, all homomorphisms in the sequence for $\pi_*(\bar{P}_1 \wedge bo) = ko_*(\bar{P}_1)$ are injective, yielding the following chart for $E_2 = E_\infty$. (See [7], [10], or [15].)



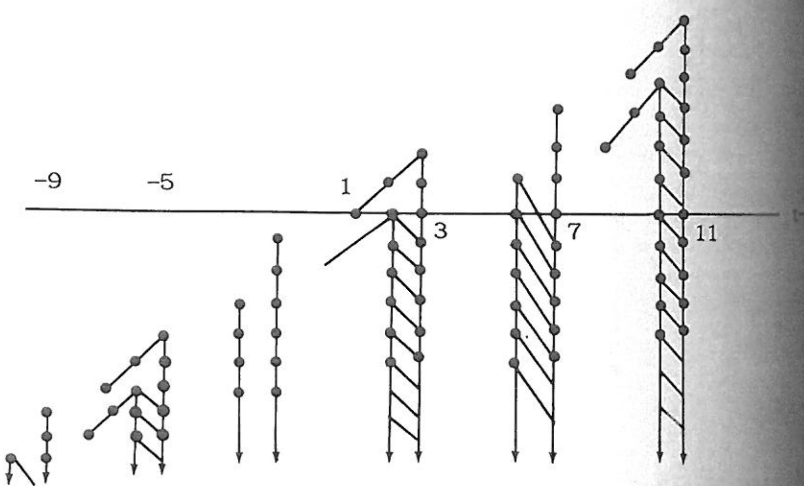
Thus $ko_i(\bar{P}_1) = \begin{cases} \mathbb{Z}/2^\infty & i \equiv 3(4) \\ \mathbb{Z}/2 & i \equiv 1,2(8) \\ 0 & \text{otherwise} \end{cases}$.

The negative filtrations are due to our reindexing; they seem essential to the utilization of charts for mapping telescopes.

Charts of $ko_*(\bar{P}_b)$ for other odd b are constructed similarly, as are $\pi_*(\bar{P}_b \wedge \Sigma^4 bsp)$. A chart for $\pi_*(\bar{P}_b \wedge J)$ is formed with

$$E_1^{s,t}(\bar{P}_b \wedge J) = E_2^{s,t}(\bar{P}_b \wedge bo) \oplus E_2^{s-1,t}(\bar{P}_b \wedge \Sigma^4 bsp) ,$$

and for $r \geq 1$, $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is nonzero on towers in dimensions $t-s$ satisfying $\nu_2(t-s+1) = r+1$, where $\nu_2(\ )$ denotes the exponent of 2. Such charts are not Ext charts, but by [15; 7.1] they do correspond to a direct limit of charts derived from resolutions of $P_{b-8i}^{t-8i} \wedge J$. The differentials were established in [15; 7.7]. This yields as part of the chart for $J_*(\bar{P}_1)$:



The homotopy groups are read off as follows.

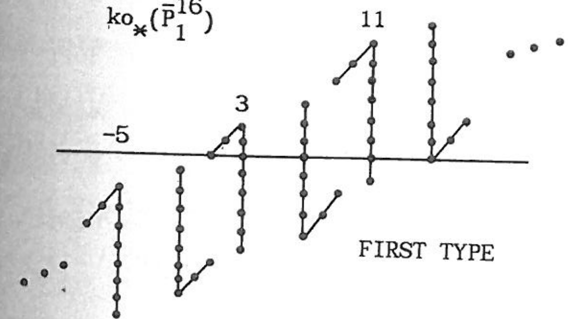
THEOREM 2.2.

$$J_j(\bar{P}_{4i\pm1}) \approx \begin{cases} \mathbb{Z}/2^{\nu(j+1)+1} & j \equiv 3(4), \quad j \neq -1 \\ \mathbb{Z}/2^{\infty} & j = -1 \text{ or } -2 \\ \mathbb{Z}/2 & j \equiv 4i \text{ or } 4i+2 \ (8) \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & j \equiv 4i+1 \ (8) \\ 0 & \text{otherwise} \end{cases}$$

Next we calculate $J_*(\bar{P}_b^t)$ when t is finite. The first step is to determine $ko_*(\bar{P}_b^t)$, which can be found from charts of $ko_*(P_{b-8i}^{t-8i})$ as in [7; §3]. Typical are the

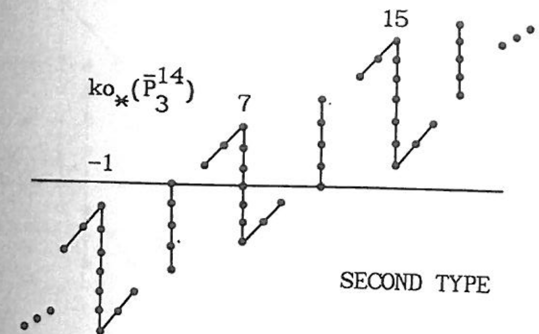
following charts:

$ko_*(\bar{P}_1^{16})$



FIRST TYPE

$ko_*(\bar{P}_3^{14})$

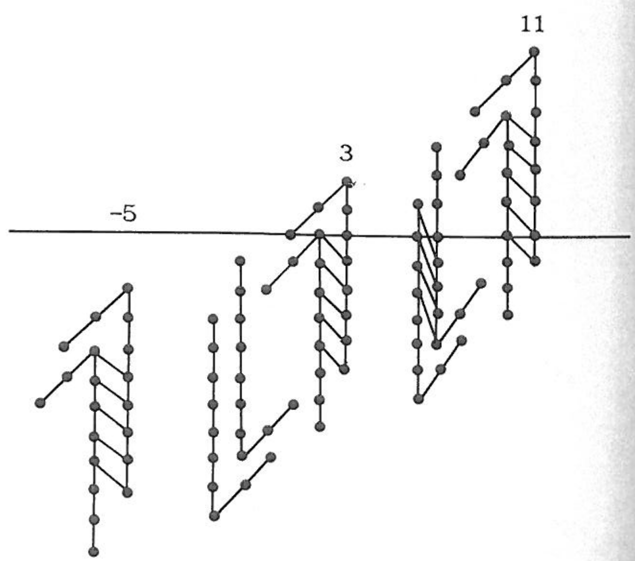


SECOND TYPE

All $ko_*(\bar{P}_b^t)$-charts are of one of these two types. Write $b = 4B \pm 1$ and $t = 4T - 1 \pm 1$. Bottoms [ ] are on towers in dimension $\equiv 4T-1$ (8), and tops [ ] are on towers in dimensions $\equiv 4B + 3$ (8). If $B \equiv T(2)$, a chart of the first type is obtained; $ko_i(\bar{P}_b^t) \approx \mathbb{Z}/2^{(t-b+1)/2}$ for all $i \equiv 3(4)$. If $B \not\equiv T(2)$, then the chart has the second type;

$$ko_i(\bar{P}_b^t) \approx \begin{cases} \mathbb{Z}/2^{(t-b+3)/2} & \text{if } i \equiv 4T-1(8) \\ \mathbb{Z}/2^{(t-b-1)/2} & \text{if } i \equiv 4T+3(8) \end{cases}$$

A chart for $J_*(\bar{P}_b^t)$ is obtained by summing two copies of $ko_*(\bar{P}_b^t)$, one unshifted and the other shifted one unit to the left and two units down, and inserting differentials by the same rule as was used in establishing 2.2. For example, a portion of the chart for $J_*(\bar{P}_1^{16})$ is as below.



Then one can easily write out results such as

PROPOSITION 2.3. *If* $\epsilon, \Delta \in \{0,1\}$, *then*

$$J_i(\bar{P}_{1-2\Delta}^{8n-2\epsilon}) \approx \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i \equiv 0,1(8) \\ \mathbb{Z}/2^{m(i)} & i \equiv 3,6(8) \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2^{m(i)} & i \equiv 2,7(8) \\ 0 & i \equiv 4,5(8) \end{cases}$$

where $m(i) = \min(4n-\epsilon+\Delta,\ 1+\max(\nu_2(i+1),\ \nu_2(i+2)))$.

§3. *PROOF OF THEOREM 1.2.*

The proof that $\pi_*(\bar{P}_b^t) \to J_*(\bar{P}_b^t)$ is injective uses bo-resolutions. That it is surjective is proved by constructing homotopy classes. We begin with the injectivity.

After possibly reindexing, it suffices to show that if $\alpha : S^n \to P_b^t$ becomes trivial in $P_b^t \wedge J$, then for some k the composite $S^n \xrightarrow{\alpha} P_b^t \xrightarrow{\ell_{b,t}} \cdots \xrightarrow{\ell_{b-8k+8,\,t-8k+8}} P_{b-8k}^{t-8k}$ is trivial. By duality it is equivalent to show that if $f : \Sigma^{n+1}P_{-t-1}^{-b-1} \to S^0$ becomes trivial in J then for some k the composite

$$S^0 \xleftarrow{f} \Sigma^{n+1}P_{-t-1}^{-b-1} \xleftarrow[\ell_1]{\ell_{-b+7,-t+7}} \cdots \xleftarrow{\ell_{-b+8k-1,-t+8k-1}} \Sigma^{n+1}P_{-t+8k-1}^{-b+8k-1} \tag{3.1}$$

is trivial.

We use bo-resolutions as introduced in [14]. There is a cofibre triangle $I = \Sigma^{-1}\overline{bo} \to S^0 \to bo \to \overline{bo}$. Let I^k be defined inductively by $I = I^1$, $I^k = I \wedge I^{k-1}$. Then I^k is $(3k-1)$-connected. The bo-resolution of S^0 is

$$\begin{array}{ccccccccc} S^0 & \xleftarrow{p_0} & I^1 & \xleftarrow{p_1} & I^2 & \xleftarrow{p_2} & I^3 & \xleftarrow{p_3} & \cdots \\ \downarrow & & \downarrow{\scriptstyle q_1} & & \downarrow & & \downarrow & & \\ bo & & I^1\wedge bo & & I^2\wedge bo & & I^3\wedge bo & & \end{array}$$

The main theorem of bo-resolutions, as interpreted in [8; 3.6.1] and corrected in [9], is

THEOREM 3.2. *Suppose there are no nontrivial differentials through dimension* N *in the Adams spectral sequence* $\mathrm{Ext}_A(H^*(D_N X \wedge bo), \mathbb{Z}_2) \Rightarrow \pi_*(D_N X \wedge bo)$, *where* $D_N X$ *is a stable* N-*dual of* X. *If* $s \geq 2$, *and* $X \xrightarrow{f_s} I^s$ *has Adams filtration* ≥ 2, *then there is a map* $X \xrightarrow{f_{s+1}} I^{s+1}$ *such that* $p_{s-1}p_s f_{s+1} \simeq p_{s-1}f_s$. *The same is true if* f_s *has Adams filtration* 1 *and* $\dim X < 5s$. *The same is also true if* $s = 1$ *and the first component of the horizontal composite*

$$\begin{array}{ccccc} & \Sigma^{-1}bo & & & \\ & & \searrow & & \\ X & \xrightarrow{f_1} & I & \longrightarrow I \wedge bo \longrightarrow & \Sigma^3 bsp \vee W \end{array}$$

factors through $\Sigma^{-1}bo$.

Now suppose f is as in (3.1). Its triviality in J implies that it lifts to a map f_1 satisfying the hypothesis in the last sentence of 3.2. Similarly to [8; 4.2], the first hypothesis of 3.2 is satisfied. Since each ℓ_i has Adams filtration 4, 3.2 implies that $f\ell_1 \cdots \ell_K$ lifts to a map $\Sigma^{n+1} P^{-b+8K-1}_{-t+8K-1} \to I^{2K+1}$. Choose $K = [\frac{n-b-3}{2}]$. Then further liftings satisfy the hypothesis $\dim X < 5s$ of 3.2. Hence $f\ell_1 \cdots \ell_K \ell_{K+1} \cdots \ell_k$ lifts to a map $\Sigma^{n+1}P^{-b+8k-1}_{-t+8k-1} \to I^{2K+4(k-K)+1}$. Choose $k = n-b-2$. Then

$$(\text{dimension of } \Sigma^{n+1}P^{-b+8k-1}_{-t+8k-1}) \leq (\text{connectivity of } I^{4k-2K+1}),$$

and hence $f\ell_1 \cdots \ell_k$ factors through a trivial map.

The proof that $\pi_i \bar{P}^t_b \to J_i \bar{P}^t_b$ is surjective follows, for the most part, as in [15; 7.14 and 7.18]. We begin with the case $t = \infty$.

The $\mathbb{Z}_2$'s in $i \equiv 0, 1, 2(4)$ $\left[\right.$  $\left.\right]$ are easily handled. Let $\epsilon = \pm 1$ and $d = 8, 9,$ or 10. The top arrow in the diagram below is surjective by [16; Tables 8.2, 8.4, 8.6, 8.8].

$$\begin{array}{ccc} \pi_{4k+d}P_{4k+\epsilon} & \longrightarrow & J_{4k+d}P_{4k+\epsilon} \\ \downarrow & & \downarrow\approx \\ \pi_{4k+d}\bar{P}_{4k+\epsilon\pm 8i} & \longrightarrow & J_{4k+d}\bar{P}_{4k+\epsilon\pm 8i} \end{array}$$

Therefore, so is the bottom one.

When $i \equiv 3(4)$, our work is directed toward proving

THEOREM 3.3. *Suppose* $\nu_2(N) = 4e + \epsilon \geq 2$ *with* $1 \leq \epsilon \leq 4$. *For*

$$d = \begin{cases} 7 \\ 9 \\ 11 \\ 13 \end{cases}, \quad \text{let } f(d) = \begin{cases} 3 \\ 4 \\ 6 \\ 7 \end{cases}.$$

Let $b = N - 8e - 8y - d$ *with* $y \geq 0$. *Then* $\pi_{N-1}(P_b) \to ko_{N-1}(P_b)$ *maps onto all elements of filtration* $\geq 4y + f(d) - \epsilon$.

The surjectivity of $\pi_i\bar{P}_b \to J_i\bar{P}_b$ follows from the observation that, in the notation of 3.3, $J_{N-1}(P_b) \to ko_{N-1}(P_b)$ is injective with image exactly that described in 3.3. Surjectivity of $\pi_i\bar{P}_b \to J_i\bar{P}_b$ when $i < b$ (and for perhaps an isolated $i > b$) requires surjectivity of $\pi_iP_{b-8k} \to J_iP_{b-8k}$ for appropriate $k > 0$, but this is also covered of course by 3.3 and the sentence preceding this one.

Proof of 3.3. By use of the filtration 4 maps $\ell_{b,\infty}$, it suffices to prove 3.3 when $y = 0$.[2] The argument is exactly that of [15; 7.14, 7.15], which we review.

There is a commutative diagram

$$\begin{tikzcd}
M_{4n+5} \arrow[r] \arrow[d] & B_{4n+2} \arrow[d] & & \\
P_{4n+3} \arrow[r, "g_3"] \arrow[d] & P_{4n+2} \arrow[d] \arrow[r, "g_2"] & P_{4n+1} \arrow[r, "f_1"] & P_{4n-1} \\
Q_{4n+3} \arrow[r, "f_3"'] & Q_{4n+2} \arrow[ur, "f_2"'] & &
\end{tikzcd} \tag{3.4}$$

such that

i) the vertical maps are cofibrations which define the spectra Q (called C in [15; 7.15]);

ii) $M_{4n+5} = S^{4n+5} \cup_2 e^{4n+6}$ and

$B_{4n+2} = S^{4n+2} \cup_\eta e^{4n+4} \cup_2 e^{4n+5}$;

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[2] Except for the case ϵ=4, d=7, where we need to use y=1, and this case is implied by the case ϵ=4, y=0, d=13.

iii) the maps f_i and g_i have Adams filtration 1;

iv) the maps f_i induce monomorphisms in $ko_*(\ )$.

Remark 3.5.

a) H^*Q_{4n+3} and H^*Q_{4n+2} as A_1-modules are $\Sigma^{4n+3}P_{3,\square,\infty}$ and $\Sigma^{4n+3}P_{2,\square,\infty}$ of [7;3.6].

b) The proof of the existence and properties of (3.4) is quite clearly presented in [15; p. 104] and so is omitted here.

PROPOSITION 3.6. *If* $\nu_2(N) = 4e+\epsilon \geq 2$, *there is a map nontrivial in* $H^*(\ ;\mathbb{Z}_2)$

$$S^{N-1} \to Q_{N-8e-4-\epsilon} \qquad \epsilon=1 \text{ or } 2$$
$$S^{N-1} \to P_{N-8e-1-2\epsilon} \qquad \epsilon=3 \text{ or } 4 .$$

We return to the proof of 3.6 after using it to deduce 3.3. We exemplify with the case $\epsilon = 1$: Following the map $S^{N-1} \to Q_{N-8e-5}$ by, respectively, f_2f_3, $f_1f_2f_3$, $g_2g_3f_1f_2f_3$, and $f_1g_2g_3f_1f_2f_3$ (where these refer to appropriate maps in (3.4)) yield maps of filtration 2, 3, 5, and 6 into $P_{N-8e-(7,9,11,13)}$, as stated in 3.3. ▩

Proof of 3.6. We begin with the familiar case $\epsilon = 3$ or 4. The S-dual of $P^{N-1}_{N-8e-1-2\epsilon}$ is $P^{-N+8e+2\epsilon}_{-N}$ which is the Thom complex $T(-N\xi_{8e+2\epsilon})$. Since $\widetilde{KO}(P^{8e+2\epsilon}) \approx \mathbb{Z}/2^{4e+\epsilon}$ for $\epsilon = 3$ or 4, the bundle $-N\xi_{8e+2\epsilon}$ is stably trivial and hence there is a splitting map $P_{-N}^{-N+8e+2\epsilon} \xrightarrow{s} S^{-N}$ for the bottom cell of its Thom complex. Our desired map is the composite

$$S^{N-1} \xrightarrow{D(s)} P_{N-8e-1-2\epsilon}^{N-1} \hookrightarrow P_{N-8e-1-2\epsilon} .$$

The case $\epsilon = 2$: We construct the map $P_{N-8e-6} \to Q_{N-8e-6}$ of (3.4) with some care. There is a map $P^{8e+5} \xrightarrow{f} \hat{B} = S^{8e+2} \cup_2 e^{8e+3} \cup_\eta e^{8e+5}$ of degree 2 on the top cell. This can be seen either by dualizing or by studying $\left[P_{8e+3}^{8e+5}, \hat{B}\right]$. The homomorphism $f^* : \widetilde{KO}(\hat{B}) \to \widetilde{KO}(P^{8e+5})$ is an injection $\mathbb{Z}/2 \to \mathbb{Z}/2^{4e+3}$. This can be seen by the Atiyah-Hirzebruch spectral sequence ([1]) or by dualizing and calculating $ko_*(Df)$ as in [7]. Fit f into a cofibration sequence $\hat{Q} \xrightarrow{i} P^{8e+5} \xrightarrow{f} \hat{B}$. Then there is also a cofibration $T(i^!(-N\xi)) \to P_{-N}^{-N+8e+5} \to \Sigma^{-N}\hat{B}$. Since $-N\xi$ is an element of order 2 in $KO(P^{8e+5})$, $i^!(-N\xi)$ is trivial, and hence there is a splitting map $T(i^!(-N\xi)) \to S^{-N}$. Dualize to get

$$\begin{array}{ccccc} Q_{N-8e-6}^{N-1} & \longleftarrow & P_{N-8e-6}^{N-1} & \xleftarrow{j} & B_{N-8e-6} . \\ \nwarrow {\scriptstyle s} & & & & \\ & S^{N-1} & & & \end{array}$$

Extend to infinite spaces by letting Q_{N-8e-6} be the cofibre of the composite $B_{N-8e-6} \xrightarrow{j} P_{N-8e-6}^{N-1} \hookrightarrow P_{N-8e-6}^{\infty}$.

The case $\epsilon = 1$ is entirely analogous, the key point being that there is a map $P^{8e+4} \xrightarrow{f} S^{8e+1} \cup_2 e^{8e+2}$ such that the image of $\widetilde{KO}(f)$ is elements of order ≤ 4, and $-N\xi_{8e+4}$ has order 4. ▨

The proof that $\pi_* \bar{P}_b^t \to J_* \bar{P}_b^t$ is surjective when $t = \infty$ will be completed by proving it is true when $i = -2$. This case is not included in [15]. It suffices to prove

LEMMA 3.7. *For any* $i > 0$ $\pi_{-2}(P_{-16i+1}) \xrightarrow{h} J_{-2}(P_{-16i+1})$ *maps onto all elements of filtration* $\geq 4i-1$.

For then, utilizing the filtration 4 maps ℓ, all elements of filtration $\geq -4i - 4k - 1$ in $J_{-2}(\bar{P}_{8k+1})$ are in $\mathrm{im}(h)$, and i can be chosen arbitrarily large.

Proof of 3.7. Let $N = 2^{4i}$. Let $\alpha \in \pi_{N-2}(P_{N-16i+1}^{N-2})$ denote the attaching map for the top cell of $P_{N-16i+1}^{N-1}$. Then $h(\alpha)$ has filtration $4i-1$ in $J_{N-2}(P_{N-16i+1}^{N-2})$ by [18; 5.4.1] and [15; 8.3]. In the exact sequence

$$\pi_{N-2}P_{N-16i+1}^{N-8i-2} \xrightarrow{j_*} \pi_{N-2}P_{N-16i+1}^{N-2} \xrightarrow{c_*} \pi_{N-2}P_{N-8i-1}^{N-2}$$

$c_*(\alpha) = 0$ by [18; 5.4.1], and hence there is $\hat{\alpha}$ such that $j_*\hat{\alpha} = \alpha$. Since $N\xi_{8i-3}$ is trivial, $P_{N-16i+1}^{N-8i-2} \simeq \Sigma^N P_{-16i+1}^{-8i-2}$, and hence $\hat{\alpha}$ corresponds to an element $\alpha' \in \pi_{-2}(P_{-16i+1}^{-8i-2})$ whose image in $J_{-2}(P_{-16i+1}^{-8i-2})$ also has filtration $4i-1$. The inclusion $P_{-16i+1}^{-8i-2} \to P_{-16i+1}$ completes the proof. ▨

Finally we show that surjectivity of $\pi_*\bar{P}^t_b \to J_*\bar{P}^t_b$ when t is finite follows from that when $t = \infty$ by using the following diagram as in [15; 7.18].

$$\begin{tikzcd}
\pi_{*+1}P_{T+1} \arrow[r] \arrow[d, "h_1"] & \pi_*P^T_B \arrow[r, "j_1"] \arrow[d, "h_2"] & \pi_*P_B \arrow[r, "c"] \arrow[d, "h_3"] & \pi_*P_{T+1} \arrow[d] \\
J_{*+1}P_{T+1} \arrow[r, "\partial"] & J_*P^T_B \arrow[r, "j_2"] & J_*P_B \arrow[r] & J_*P_{T+1}
\end{tikzcd} \tag{3.8}$$

Here $T = i-8i$, $B = b-8i$ for appropriate i.

Elements in $\mathrm{im}(\partial)$ are clearly in $\mathrm{im}(h_2)$ by surjectivity of h_1. If $j_2x \neq 0$ and $\ell^\infty s \neq 0$ in $J_*\bar{P}^T_B$, then there is an element y of π_*P_B with filtration equal to that of x such that $h_3(y) = j_2(x)$. It is easily verified that $c(y)$ lies above the Adams edge of π_*P_{T+1} and hence is 0. An easy diagram chase completes the proof. ▨

§4. K_*-*LOCALIZATION RESULTS*

The following result, proved by another method in [12], is an immediate consequence of 1.2 and [12; Lemma 1.1].

THEOREM 4.1. $\bar{P}^t_b$ *is* K_**-local; indeed,* $\bar{P}^t_b = (P^t_b)_K$. See [12] for terminology.

We can identify various $\bar{P}^t_b$ and closely related spectra as K_*-localizations of certain Moore spectra. Recall that if G is an abelian group, the Moore spectrum S^nG is characterized by the properties: $\pi_i(S^nG) = 0$ for $i < n$, $H_n(S^nG) \approx G$, and $H_i(S^nG) = 0$ if $i \neq n$. If E is a spectrum, EG is defined to be $E \wedge S^0G$. We abbreviate $E\mathbb{Z}/p^n$ to E/p^n, $E\mathbb{Z}_{(p)}$ to $E_{(p)}$, and S^0G to SG.

THEOREM 4.2. $\bar{P}_1 \approx \bar{P}_{-1} \approx (S^{-1}/2^\infty)_K$;
$\bar{P}^{8n}_1 \approx \bar{P}^{8n-2}_{-1} \approx (S^{-1}/2^{4n})_K$;
$\bar{P}^{8n-2}_1 \approx (S^{-1}/2^{4n-1})_K$;
$\bar{P}^{8n}_{-1} \approx (S^{-1}/2^{4n+1})_K$
$\bar{P}^{8n}_3 \cup_a C(\bar{P}^6_5) \approx (S^{-1}/2^{4n-2})_K$, *where* $a : \bar{P}^6_5 \to \bar{P}^{8n}_3$ *is an extension of the nonzero element of* $\pi_5(\bar{P}^{8n}_3)$.

Proof. Because a K_*-equivalence $X \longrightarrow Y$ implies $X_K \simeq Y_K$, it suffices by 4.1 to construct K_*-equivalences

(a) $S^{-1}/2^{4n+\epsilon} \to P^0_{1-8n-2\epsilon}$ $\quad(\epsilon = 0$ or $1)$

(b) $S^{-1}/2^{4n-1} \to P^{-2}_{1-8n}$

(c) $S^{-1}/2^{4n-2} \to P^0_{3-8n} \cup CP^{6-8n}_{5-8n}$

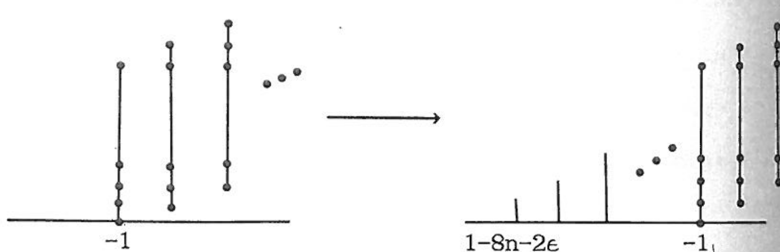
(d) $S^{-1}/2^\infty \to \bar{P}^\infty_{-1}$

Since P^{-1}_{-m} is the Thom complex of the stable normal bundle of RP^{m-1}, its top cell splits, giving a map $S^{-1} \xrightarrow{f} P^0_{-m}$ nontrivial in $H_*(\ ;\mathbb{Z})$. The spectrum

$P^0_{1-8n-2\epsilon}$ has order $2^{4n+\epsilon}$ by [20], and so f factors through a map $S^{-1}/2^{4n+\epsilon} \xrightarrow{\tilde{f}} P^0_{1-8n-2\epsilon}$. To see that $K_*(\tilde{f})$ is an isomorphism, first note that $K_0(S^{-1}/2^{4n+\epsilon}) = 0 = K_0(P^0_{1-8n-2\epsilon})$. There is a commutative diagram

$$\begin{tikzcd}
ku_{-1}(S^{-1}/2^{4n+\epsilon}) \arrow[r, "\tilde{f}_*"] \arrow[d, "\approx"] & ku_{-1}(P^0_{1-8n-2\epsilon}) \arrow[d, "\approx"] \\
K_{-1}(S^{-1}/2^{4n+\epsilon}) \arrow[r, "\tilde{f}_*"] & K_{-1}(P^0_{1-8n-2\epsilon})
\end{tikzcd}$$

The upper $\tilde{f}_*$ is an isomorphism by a chart calculation similar to those of Section 2.



Thus so is the lower $\tilde{f}_*$, proving (a).

The Adams edge theorem ([2]) shows that $2^{4n-2}\pi_{-1}(P^0_{3-8n} \cup CP^{6-8n}_{5-8n}) = 0$. Thus the composite $S^{-1} \longrightarrow P^0_{3-8n} \longrightarrow P^0_{3-8n} \cup CP^{6-8n}_{5-8n}$ factors through a map $\tilde{f}$ on $S^{-1}/2^{4n-2}$, and a calculation similar to that of the preceding paragraph shows that $K_*(\tilde{f})$ is an isomorphism, proving (c).

Since $2\pi_{-1}(P^0_{-1}) = 0$, $2[f]$ factors through P^{-2}_{1-8n}, giving the map required for (b).

Proof of (d): Since $\pi_{-1}(\bar{P}^{\infty}_{-1}) = \mathbb{Z}/2^{\infty}$, one can inductively construct nontrivial maps q_n so that the following diagram commutes

$$\begin{tikzcd}
\vdots \arrow[d] & \\
S^{-1}/2^n \arrow[d] \arrow[r, "q_n"] & \bar{P}^{\infty}_{-1} \\
S^{-1}/2^{n+1} \arrow[d] \arrow[ur, "q_{n+1}"'] & \\
\vdots &
\end{tikzcd}$$

Thus we obtain $S^{-1}/2^{\infty} \to \bar{P}^{\infty}_{-1}$. This induces an isomorphism in $K_*(\ \ )$ since $K_{-1}(\bar{P}^{\infty}_{-1}) \approx K_{-1}(P^{\infty}_{-1}) \approx \mathbb{Z}/2^{\infty}$ and each $K_{-1}(q_n)$ must be injective by induction. ▩

The case of $\bar{P}^0_{-1} = (S^{-1}/2)_K$ of 4.2 is [5; 4.2]. The $\bar{P}_1$ case is closely related to [19; 9.1].

Applying K_*-localization to the cofibration $S^{-1}Q \to S^{-1}/2^{\infty} \to S^0_{(2)}$, and using [5; 2.8],. we obtain

COROLLARY 4.3. *There is a cofibration* $S^{-1}Q \to \bar{P}_1 \to S_{K(2)}$.

Let KO denote the spectrum for nonconnective real K-theory, and $\mathcal{J}_{(2)}$ the fibre of the Adams operation

$\psi^3-1:KO_{(2)} \longrightarrow KO_{(2)}$. Comparing our description of $S_{K_{(2)}}$ with Bousfield's ([5] or [12]) yields

COROLLARY 4.4. *There is a cofibration* $S^{-2}Q \vee S^{-1}Q \to \bar{P}_1 \to \mathcal{J}_{(2)}$.

§5. *BROWN-COMENETZ DUALITY*

Let T denote the cofibre of the K_*-localization map $S/2^\infty \to \Sigma\bar{P}_1$. Then $K_*(T) = 0$, of course, but methods of Section 2 imply that

$$ku_i(T) = \begin{cases} \mathbb{Z}/2^\infty & i \text{ negative and even} \\ 0 & \text{otherwise} \end{cases}$$

This motivates us to try to prove that $T \wedge bu$ is a Brown-Comenetz dual ([16]) of $\Sigma^2 bu$. This will be proved (5.8) after modifying the definition slightly, and the proof leads us to the universal coefficient theorem 1.3.

Throughout this section, let G be any divisible torsion abelian group. Similarly to [6; 2.1, 1.2, 1.3, 1.6] we have the following two results. We need G to be divisible in order that $\mathrm{Hom}(E_*(\ ),G)$ be an exact functor.

THEOREM 5.1.

i) *There is a spectrum* c_GS *with* $\pi_0(c_GS) \approx G$ *such that for any spectrum* X, $\{X, c_GS\} \xrightarrow{\pi_0} \mathrm{Hom}(\pi_0X,G)$ *is an isomorphism.*

ii) *There is an additive antiexact functor* c_G *from the stable category to itself such that, for any spectrum* E, $c_G(E)$ *is the function spectrum* $F(E,c_GS)$.

COROLLARY 5.2. *There is a map* $E \wedge c_GE \to c_GS$ *which, for any spectrum* X, *induces an isomorphism* $(c_GE)^i(X) \approx \mathrm{Hom}(E_iX,G)$. *In particular,* $\pi_n(c_GE) \approx \mathrm{Hom}(\pi_{-n}E,G)$.

LEMMA 5.3. *If* K *is the spectrum for non-connective K-theory, then* $c_G(K) \approx KG$.

Proof. The multiplication $K \wedge K \to K$ gives a map $K \wedge KG \to KG$. By 5.1(i), the induced homomorphism in $\pi_0(\ )$ determines a map $K \wedge KG \to c_GS$, and hence a map $KG \xrightarrow{d} F(K,c_GS)$, which is easily verified to induce an isomorphism in $\pi_*(\ )$. ▩

LEMMA 5.4.

i) $SG \wedge S_K \wedge bu \approx KG$.

ii) $c_G(K(-\infty,0]) \approx buG$, where $K(-\infty,0]$ is defined by a map $K \to K(-\infty,0]$ which induces an isomorphism in $\pi_i(\ )$ for $i \le 0$ and $\pi_i(K(-\infty,0]) = 0$ for $i > 0$.

Proof.

i) There is a cofibration $SG \wedge S_K \wedge bu \to SG \wedge S_K \wedge K \to SG \wedge S_K \wedge K(-\infty,-2]$. The middle spectrum is KG. The third spectrum is contractible by Lemma 5.6.

ii) In the diagram

$$\begin{tikzcd} K \wedge buG \arrow[r, "q"] \arrow[d] & K(-\infty,0] \wedge buG \arrow[d, dashed] \\ K \wedge KG \arrow[r] & c_G S \end{tikzcd}$$

the dotted arrow exists because the fibre of q, $\Sigma^2 bu \wedge buG$, is 1-connected, so that $[\Sigma^2 bu \wedge buG, c_G S] = 0$ by 5.1(i). ▨

LEMMA 5.5. *If* HA *is the Eilenberg-MacLane spectrum of a torsion abelian group* A, *and* K *is the spectrum for non-connective K-theory, then* $K \wedge HA$ *is the trivial spectrum.*

Proof. It is equivalent to show that $H_*(K;A) = 0$. Since A is the direct limit of its finite subgroups, it suffices to show $H_*(K; \mathbb{Z}/p^n) = 0$. The Atiyah-Hirzebruch spectral sequence converging to $K_*(H\mathbb{Z}/p)$ has

$$E_2^{s,t} \approx \begin{cases} H_s(H\mathbb{Z}/p;\mathbb{Z}) \approx H_s(H\mathbb{Z};\mathbb{Z}/p) & t \text{ even} \\ 0 & t \text{ odd} \end{cases}$$

and first nonzero differential d_{2p-1} given by the action of the Milnor primitive $Q_1 = \tau_1'$. By [4; p. 194] $E_{2p}^{*,*} = 0$. Thus $H_*(K; \mathbb{Z}/p) \approx K_*(H\mathbb{Z}/p) = 0$, and by induction $H_*(K;\mathbb{Z}/p^n) = 0$. ▨

LEMMA 5.6 *If* X *is a spectrum such that* $\pi_i X = 0$ *for* $i > M$ *and* $\pi_i X$ *is a torsion abelian group for all* i, *then* $X \wedge S_K$ *is contractible.*

Proof. Let $X[-n,\infty)$ satisfy $\pi_i(X[-n,\infty)) \approx \begin{cases} \pi_i(X) & i \geq -n \\ 0 & i < -n \end{cases}$. Then $X[-n,\infty)$ is a finite Postnikov system with torsion homotopy, so that by 5.5 $K \wedge X[-n,\infty) = *$. Then $K \wedge X = \varinjlim_n K \wedge X[-n,\infty) = *$. Since X is K_*-acyclic, its K_*-localization is trivial. ▨

The universal coefficient sequence 1.3 is obtained by applying [X,] to the following result.

THEOREM 5.7. *There is a cofibration* $buG \to KG \to c_G(\Sigma^2 bu)$.

Proof. Apply $c_G(\ )$ to the cofibration $\Sigma^2 bu \to K \to K(-\infty,0]$ and use 5.4(ii) and 5.3. ▨

PROPOSITION 5.8. $c_{\mathbb{Z}/2^\infty}(\Sigma^2 bu) \approx T \wedge bu$, *where* T *is as defined at the beginning of this section.*

Proof. There is a cofibration $S/2^\infty \wedge bu \to S/2^\infty \wedge S_K \wedge bu \to T \wedge bu$. The middle spectrum is $K/2^\infty$ by 5.4(i), and the first is $bu/2^\infty$. This map $bu/2^\infty \to K/2^\infty$ induces an isomorphism in $\pi_i(\ \ )$ for $i \geq 0$ and hence agrees with the map of 5.7. Equating cofibres yields the results.

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Donald M. Davis
Lehigh University
Bethlehem, PA 18015

Mark Mahowald
Northwestern University
Evanston, IL 60201