

MAT8021, Algebraic Topology

Assignment 11

Due in-class on Friday, May 7

1. Given a map $f: X \rightarrow Y$, $b \in H^p(Y)$, and $x \in H_n(X)$, show that

$$f_*(f^*(b) \frown x) = b \frown f_*(x)$$

(This formula goes by many names: the “projection formula,” or “Frobenius reciprocity.” The special case when $p = n$ gives $\langle f^*b, x \rangle = \langle b, f_*x \rangle$ for the *Kronecker pairing* $\langle -, - \rangle: H^p(X; R) \otimes H_p(X; R) \rightarrow R$ induced by the evaluation map.)

2. Let I be a directed set, L an abelian group, and $A: I \rightarrow \mathbf{Ab}$ an I -directed diagram of abelian groups, with bonding maps $f_{ij}: A_i \rightarrow A_j$ for $i \leq j$. Show that a map $A \rightarrow c_L$, the constant functor at L , given by compatible maps $f_i: A_i \rightarrow L$, is a direct limit if and only if

- (a) for any $b \in L$ there exists $i \in I$ and $a_i \in A_i$ such that $f_i a_i = b$, and
- (b) for any $a_i \in A_i$ such that $f_i a_i = 0 \in L$, there exists $j \geq i$ such that $f_{ij} a_i = 0 \in A_j$.

3. (a) Embed \mathbb{Z}/p^n into \mathbb{Z}/p^{n+1} by sending 1 to p , and write \mathbb{Z}_{p^∞} for the union. It is called the Prüfer group (at p). Show that $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[1/p]/\mathbb{Z}$ and that

$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}_{p^\infty}$$

where the sum runs over the prime numbers.

- (b) Compute $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A$ for A each of the following abelian groups: \mathbb{Z}/n , $\mathbb{Z}[1/q]$ (for q a prime), and \mathbb{Z}_{q^∞} (for q a prime).
 - (c) Compute $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}[1/p])$ and $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}_{p^\infty})$ for any abelian group M in terms of the self-map $p: M \rightarrow M$.
4. Show that if $f: X \rightarrow Y$ induces an isomorphism in homology with coefficients in the prime fields \mathbb{F}_p (for all primes p) and \mathbb{Q} , then it induces an isomorphism in homology with coefficients in \mathbb{Z} .