HANDBOOK OF ALGEBRAIC TOPOLOGY

Edited by
I.M. James

NORTH-HOLLAND

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edited by
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Foreword

Henri Poincaré may be regarded as the father of Topology. Of course many of the ideas which he developed originated earlier, with Bernhard Riemann above all. But in his monumental "Analysis Situs" Poincaré organized the subject for the first time. In the centenary year of its publication it seems appropriate to dedicate this Handbook to his memory.

In Poincaré's work the discussion is mainly conducted in geometric terms. It was not until much later that the value of a more algebraic approach became recognized. By the thirties the terms "Algebraic Topology" and "Geometric Topology" had come into use, although the two parts of the subject remained closely related, as they do to this day. This Handbook deals only with the algebraic side.

Since Algebraic Topology is still developing rapidly any attempt to cover the whole subject would soon be out-of-date. So instead of a comprehensive overview, which would be bound to occupy several volumes, it seemed better to put together a collection of articles, dealing with most of the areas in which research is active at the present time. Indeed many new results, and new ways of looking at known results, will be found in the pages of this volume. Some of the articles are more technical than others but that is in the nature of the subject. It did not seem necessary to cover all the topics which can be found in the standard textbooks and monographs.

So this Handbook is addressed to the reader who already has some knowledge of Algebraic Topology and wishes to know more about what is happening closer to the frontiers of research. Some overlap between different articles cannot be avoided if each is to be readable on its own but this has been kept to a minimum. In any case almost every article looks at the subject from a somewhat different viewpoint. Some areas of the subject are much better understood than others but it is in the latter, of course, that research activity tends to be most intense.

Algebraic topology is very much an international subject and this is reflected in the background of the various contributors. When I was first invited to become Editor of this volume in the North Holland series of Handbooks I thought it would be a daunting task but instead it has been a pleasure, thanks to the willing cooperation of those who have contributed to it.

I.M. James

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CHAPTER 1

Homotopy Types

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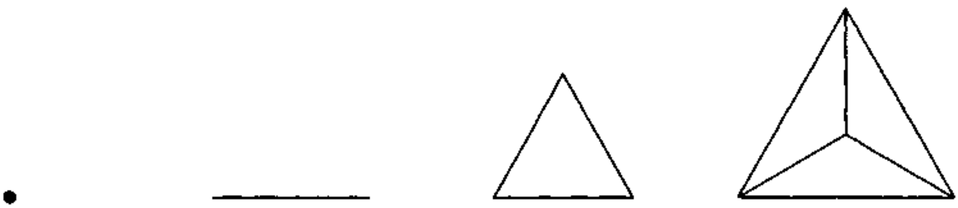
The theory of homotopy types is one of the most basic parts of topology and geometry. At the center of this theory stands the concept of algebraic invariants. In what follows we give a general introduction to this subject including recent results and explicit examples. There are three main topics:

Homotopy types with nontrivial fundamental group (Sections 2–5).
Homotopy types with trivial fundamental group (Sections 6–9, 12).
Stable homotopy types (Sections 10, 11).

Almost all definitions and notations below are explicitly described and statements of results are complete. Prerequisites are elementary topology, elementary algebra and some basic notions from category theory.

1. What are homotopy types

For each number $n = 0, 1, 2, \dots$ one has the *simplex* Δ^n which is the convex hull of the unit vectors $e_0, e_1, \dots, e_n$ in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$. Hence Δ^0 is a point, Δ^1 an interval, Δ^2 a triangle, Δ^3 a tetrahedron, and so on:



Δ^0 Δ^1 Δ^2 Δ^3

The dimension of Δ^n is n. A point $x \in \Delta^n$ is given by barycentric coordinates,

$$x = \sum_{i=0}^{n} t_i e_i \quad \text{with} \quad \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geqslant 0.$$

The name simplex describes an object which is supposed to be very simple; indeed, natural numbers and simplexes both have the same kind of innocence. Yet once the simplex was created, algebraic topology had to emerge:

For each subset $a \subset \{0, 1, \dots, n\}$ with $a = \{a_0 < \dots < a_r\}$ one has the r-dimensional *face* $\Delta_a \subset \Delta^n$ which is the convex hull of the set of vertices $e_{a_0}, \dots, e_{a_r}$. Hence the set of all subsets of the set $[n] = \{0, 1, \dots, n\}$ can be identified with the set of faces of the simplex Δ^n. There are "substructures" S of the simplex obtained by the union of several faces, that is,

$$S = \Delta_{a_1} \cup \Delta_{a_2} \cup \dots \cup \Delta_{a_k} \subset \Delta^n.$$

Finite polyhedra are topological spaces X homeomorphic to such substructures S of simplexes Δ^n, $n \geqslant 0$. A homeomorphism $S \approx X$ is called a *triangulation* of X. Hence a polyhedron X is just a topological space in which we do not see any simplexes. We can introduce simplexes via a triangulation, but this must be seen as an artifact similar to the choice of coordinates in a vector space or manifold (compare H. Weyl, *Philosophy of Mathematics and Natural Science*, 1949: "The introduction of numbers as coordinates ... is an act of violence ..."). Finite polyhedra form a large universe of objects. One is not interested in a particular individual object of the universe but in the classification of species. A system of such species and subspecies is obtained by the equivalence classes

homotopy types and homeomorphism types.

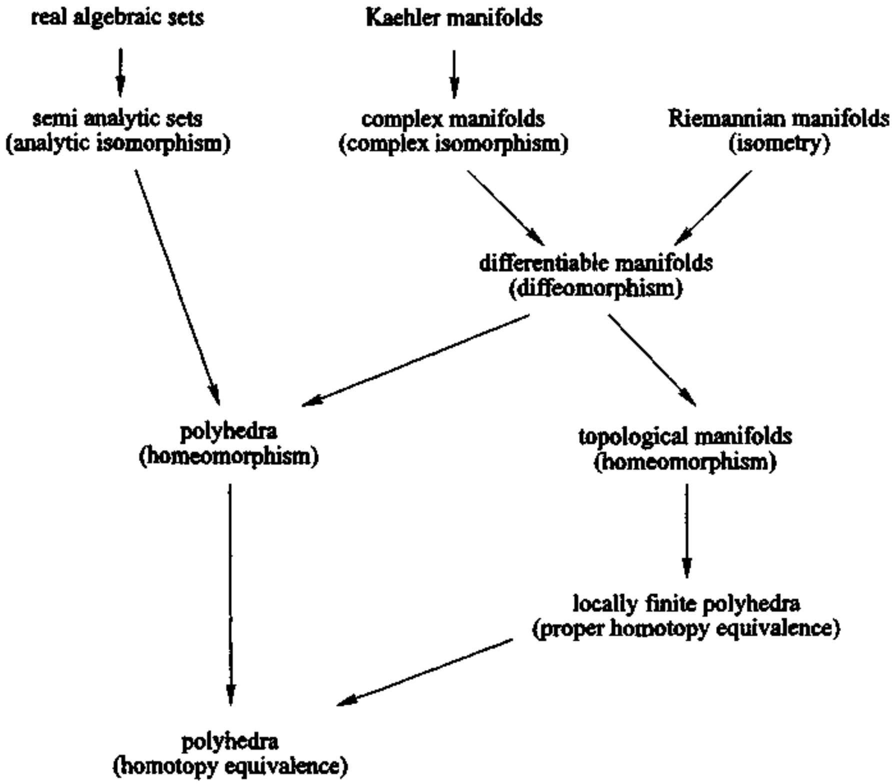
Recall that two spaces X, Y are *homeomorphic*, $X \approx Y$, if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composites $fg = 1_Y$ and $gf = 1_X$ are the identity maps. A class of homeomorphic spaces is called a *homeomorphism type*. The initial problem of algebraic topology – Seifert and Threlfall [82] called it the main problem – was the classification of homeomorphism types of finite polyhedra. Up to now such a classification was possible only in a very small number of special cases. One might compare this problem with the problem of classifying all *knots* and *links*. Indeed the initial datum for a finite polyhedron is just a set $\{a_1, \dots, a_k\}$ of subsets $a_i \subset [n]$ as above and the initial datum to describe a link, namely a finite sequence of neighboring pairs $(i, i+1)$ or $(i+1, i)$ in $[n]$ (specifying the crossings of $n+1$ strands) is of similar or even higher complexity. But we must emphasize that such a description of an object like a polyhedron or a link cannot be identified with the object itself: there are in general many different ways to describe the same object, and we care only about the equivalence classes of objects, not about the choice of description.

Homotopy types are equivalence classes of spaces which are considerably larger than homeomorphism types. To this end we use the notion of deformation or homotopy. The principal idea is to consider 'nearby' objects (that is, objects, which are 'deformed' or 'perturbed' continuously a little bit) as being similar. This idea of perturbation is a common one in mathematics and science; properties which remain valid under small perturbations are considered to be the stable and essential features of an object. The equivalence relation generated by 'slight continuous perturbations' has its precise definition by the notion of homotopy equivalence: Two spaces X and Y are *homotopy equivalent*, $X \simeq Y$, if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composites fg and gf are homotopic to the identity maps, $fg \simeq 1_Y$ and $gf \simeq 1_X$. (Two maps $f, g : X \to Y$ are *homotopic*, $f \simeq g$, if there is a family of maps $f_t : X \to Y$, $0 \leqslant t \leqslant 1$, with $f_0 = f$, $f_1 = g$ such that the map $(x, t) \mapsto f_t(x)$ is continuous as a function of two variables.) A class of homotopy equivalent spaces is called a *homotopy type*.

Using a category C in the sense of S. Eilenberg and Saunders Mac Lane [35] one has the general notion of isomorphism type. Two objects X, Y in C are called equivalent or *isomorphic* if there are morphisms $f : X \to Y$, $g : Y \to X$ in C such that $fg = 1_Y$ and $gf = 1_X$. An *isomorphism type* is a class of isomorphic objects in C. We may consider isomorphism types as being special entities: for example, the isomorphism types in the category of finite sets are the *numbers*. A homeomorphism type is then an isomorphism type in the category *Top* of topological spaces and continuous maps, whereas a homotopy type is an isomorphism type in the homotopy category $Top/\simeq$ in which the objects are topological spaces and the morphisms are not individual maps but homotopy classes of ordinary continuous maps.

The Euclidean spaces $\mathbb{R}^n$ and the simplexes Δ^n, $n \geqslant 1$, all represent different homeomorphism types but they are *contractible*, i.e. homotopy equivalent to a point. As a further example, the homeomorphism types of connected 1-dimensional polyhedra are the *graphs* which form a world of their own, but the homotopy types of such polyhedra correspond only to numbers since each graph is homotopy equivalent to the one point union of a certain number of circles S^1.

Homotopy types of polyhedra are archetypes underlying most geometric structures. This is demonstrated by the following table which describes a hierarchy of structures based on homotopy types of polyhedra. The arrows indicate the forgetful functors.



This hierarchy can be extended in many ways by further structures. Each kind of object in the table has its own notion of isomorphism; again as in the case of polyhedra not the individual object but its isomorphism type is of main interest. We only sample a few properties of these objects.

Some of the arrows in the table correspond to results in the literature. For example, every differentiable manifold is a polyhedron, see J.H.C. Whitehead [97] or Munkres [72]. Any (metrizable) topological manifold is proper homotopy equivalent to a locally finite polyhedron though a topological manifold needs not to be a polyhedron, see Kirby and Siebenmann [60]. Any semi-analytic set is a polyhedron, see Lojasiewicz [64]. There are also connections between the objects in the table in terms of realizability. For example, each differentiable manifold admits the structure of a Riemannian manifold, or each closed differentiable manifold has the structure of an irreducible real algebraic set (in fact, infinitely many birationally non isomorphic structures), see Bochnak and Kucharz [11].

The famous Poincaré conjecture states that the homotopy type of a 3-sphere contains only one homeomorphism type of a topological manifold. Clearly not every finite polyhedron is homotopy equivalent to a closed topological manifold. For this the polyhedron has to be, at least, a Poincaré complex; yet there are also many Poincaré complexes which are not homotopy equivalent to topological manifolds. By the result of M.H. Freedman [39] all simply connected 4-dimensional Poincaré complexes have the homotopy type of closed topological manifolds, they do not in general have the structure of a differentiable manifold by the work of Donaldson [31]. Homotopy types of Kähler manifolds are very much restricted by the fact that their (real) homotopy type is 'formal', see Deligne, Griffiths, Morgan and Sullivan [29].

Now one might argue that the set given by diffeomorphism types of closed differentiable manifolds is more suitable and restricted than the vast variety of homotopy types of finite polyhedra. This, however, turned out not to be true. Surgery theory showed that homotopy types of arbitrary simply connected finite polyhedra play an essential role for the understanding of differentiable manifolds. In particular, one has the following embedding of a set of homotopy types into the set of diffeomorphism types: Let X be a finite simply connected n-dimensional polyhedron, $n > 2$. Embed X into an Euclidean space $\mathbb{R}^{k+1}$, $k \geqslant 2n$, and let $N(X)$ be the boundary of a regular neighborhood of $X \subset \mathbb{R}^{k+1}$. This construction yields a well defined function $\{X\} \mapsto \{N(X)\}$ which carries homotopy types of simply connected n-dimensional finite polyhedra to diffeomorphism types of k-dimensional manifolds. Moreover for $k = 2n + 1$ this function is injective, see Kreck and Schafer [61]. Hence the set of simply connected diffeomorphism types is at least as complicated as the set of homotopy types of simply connected finite polyhedra.

In dimension $\geqslant 5$ the classification of simply connected diffeomorphism types (up to connected sum with homotopy spheres) is reduced via surgery to problems in homotopy theory which form the unsolved hard core of the question. This kind of reduction of geometric questions to problems in homotopy theory is an old and standard operating procedure. Further examples are the classification of fibre bundles and the determination of the ring of cobordism classes of manifolds.

All this underlines the fundamental importance of homotopy types of polyhedra. There is no good intuition what they actually are, but they appear to be entities as genuine and basic as numbers or knots. In my book [3] I suggested an axiomatic background for the theory of homotopy types; A. Grothendieck [45] commented:

> "Such suggestion was of course quite interesting for my present reflections, as I do have the hope indeed that there exists a 'universe' of schematic homotopy types..."

Moreover J.H.C. Whitehead [101] in his talk at the International Congress of Mathematicians 1950 in Harvard said with respect to homotopy types and the homotopy category of polyhedra:

> "The ultimate object of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry".

Today, 45 years later, this idea still remains a dream which has not yet come true. The full realization seems far beyond the reach of existing knowledge and techniques. Some progress in several directions will be described below.

2. How to build homotopy types

There are many different topological and combinatorial devices which can be used to construct the homotopy types of connected polyhedra, for example, simplicial complexes, simplicial sets, CW-complexes, topological spaces, simplicial groups, small categories, and partially ordered sets.

Up to now we have worked with finite polyhedra by viewing them as substructures of a simplex. One needs also polyhedra which are not finite since for example the universal covering space of a finite polyhedron, in general, is not finite, also the Euclidean spaces $\mathbb{R}^n$, $n \geqslant 1$, are nonfinite polyhedra. Infinite polyhedra are defined by 'simplicial complexes'. The following abstract notion of a simplicial complex is just a recipe for joining many simplexes together to obtain a space which is called the 'realization' of the simplicial complex.

2.1. DEFINITION. A *simplicial complex* X is a set of finite sets closed under formation of subsets. Equivalently X is a set of finite subsets of a set U such that U is the union of all sets in X and for $a \in X$, $b \subset a$ also $b \in X$. The set $U = X^0$ is called the set of *vertices* of X. The simplicial complex X is a partially ordered set by inclusion.

We obtain the realization of a simplicial complex X by associating with each element $a \in X$ a simplex Δ_a which is the convex hull of the set a in the real vector space with basis X^0. The vertices of Δ_a are elements of a. For $b \subset a$ the simplex $\Delta_b \subset \Delta_a$ is a face of Δ_a. The *realization* of X is the union of sets

$$|X| = \bigcup_{a \in X} \Delta_a \tag{2.2}$$

with the topology induced by the topology of the simplexes. That is, a subset in $|X|$ is open if and only if the intersection with all simplexes is open. If X is finite we can choose a bijection $X^0 \approx \{0, 1, \ldots, N\}$ such that $|X|$ coincides with the substructure $\bigcup\{\Delta_{j(a)},\ a \in X\}$ in the simplex Δ^N. The realization $|X|$ is compact if and only if X is finite.

2.3. DEFINITION. A *polyhedron* is a topological space homeomorphic to the realization of a simplicial complex.

Simplicial complexes have the disadvantage that for a subcomplex $Y \subset X$ the quotient space $|X| \setminus |Y|$ is not the realization of a simplicial complex. This is one of the reasons to introduce 'simplicial sets' which are considerably more flexible than simplicial complexes. Again a simplicial set X is a combinatorial affair, i.e. a family of sets and maps between them from which again may be deduced a topological space $|X|$. There is a more general notion of a 'simplicial object' which actually became one of the most influential notions of algebraic topology.

2.4. DEFINITION. The simplicial category Δ is the following subcategory of the category of sets. The objects are the finite sets $[n] = \{0, 1, \dots, n\}$, $n \geqslant 0$, and the morphisms $\alpha : [n] \to [m]$ are the order preserving functions, i.e. $x \leqslant y$ implies $\alpha(x) \leqslant \alpha(y)$. A *simplicial object* X in a category C is a contravariant functor from Δ to the category C; we also write

$$X : \Delta^{op} \to C$$

where Δ^{op} is the opposite category of Δ. Hence X is determined by objects $X[n]$, $n \geqslant 0$, in C and by morphisms $\alpha^* : X[m] \to X[n]$ one for each order preserving function $\alpha : [n] \to [m]$. Morphisms in the category sC of simplicial objects are the natural transformations.

Hence *simplicial sets*, *simplicial groups* and *simplicial spaces* are the simplicial objects in the category of sets, *Set*, groups, *Gr*, and topological spaces, *Top*, respectively. A simplicial set is also a simplicial space by using the discrete topology functor $Set \subset Top$. A simplicial space X is *good* if every surjective map α in Δ induces a 'cofibration' $\alpha^* : X[m] \to X[n]$. For example the inclusion $|B| \subset |A|$ given by a simplicial subcomplex B of a simplicial complex A is a cofibration. We define the realization of a good simplicial space X by the following quotient of the disjoint union of products $X[n] \times \Delta^n$ in *Top*,

$$|X| = \Big(\bigcup_{n \geqslant 0} X[n] \times \Delta^n \Big) \Big/ \sim . \tag{2.5}$$

Here the equivalence relation is generated by $(a, \alpha_* x) \sim (\alpha^* a, x)$ for $\alpha : [n] \to [m]$, $a \in X[m]$, $x \in \Delta^n$ where $\alpha_* : \Delta^n \to \Delta^m$ is the restriction of the linear map given on vertices by α. For different realizations of simplicial spaces compare the Appendix of Segal [81].

There are the following basic examples of simplicial sets. For any topological space X we obtain the simplicial set

$$SX : \Delta^{op} \to Set, \quad \begin{cases} (SX)[n] = \{a : \Delta^n \to X \in Top\}, \\ \alpha^*(a) = a \circ \alpha_*, \end{cases} \tag{2.6}$$

which is called the *singular set* of X. One has the canonical map

$$T : |SX| \to X, \quad T(a, x) = a(x),$$

which is a homotopy equivalence if X is a polyhedron. Moreover T is a weak homotopy equivalence for any space X (that is, T induces isomorphisms of homotopy groups with respect to all base points). Clearly the singular set SX is very large. This, however, has the advantage that SX is a 'Kan set'; for such Kan sets it is possible to describe homotopy theory purely combinatorially, see Curtis [28] and May [67].

In the next example we use the morphisms d_i, s_i which *generate the category* $\boldsymbol{\Delta}$ multiplicatively. The maps d_i are the unique injective maps $d_i : [n-1] \to [n] - \{i\} \subset [n]$, and the maps s_i are the unique surjective maps $s_i : [n] \to [n-1]$ with $s_i(i) = s_i(i+1) = i \in [n-1]$.

For any small category X we obtain the simplicial set

$$Nerve(X) : \boldsymbol{\Delta}^{op} \to Set \tag{2.7}$$

which is called the *nerve* of X. Here $Nerve(X)[n]$, $n \geqslant 1$, is the set of all sequences $(\lambda_1, \dots, \lambda_n)$ of n composable morphisms

$$X_0 \xleftarrow{\lambda_1} X_1 \longleftarrow \dots \xleftarrow{\lambda_n} X_n$$

in X. For $n = 0$ let $Nerve(X)[0]$ be the set of objects of X. The functor $Nerve(X)$ is defined on generating morphisms of $\boldsymbol{\Delta}$ by $s_0^*(A) = 1_A$ for $A \in Nerve(X)[0]$ and

$$s_i^*(\lambda_1, \dots, \lambda_n) = (\lambda_1, \dots, \lambda_{i-1}, 1, \lambda_i, \dots, \lambda_n)$$

where 1 is the appropriate identity. Moreover

$$d_i^*(\lambda) = \begin{cases} A, & i = 0, \\ B, & i = 1, \end{cases}$$

for $\lambda : A \to B \in Nerve(X)[1]$ and for $n \geqslant 2$

$$d_i^*(\lambda_1, \dots, \lambda_n) = \begin{cases} (\lambda_2, \dots, \lambda_n), & \text{for } i = 0, \\ (\lambda_1, \dots, \lambda_i\lambda_{i+1}, \dots, \lambda_n), & \text{for } i = 1, \dots, n-1, \\ (\lambda_1, \dots, \lambda_{n-1}), & \text{for } i = n. \end{cases}$$

There is a more formal way to define the simplicial set $Nerve(X)$ as follows. For this recall that any *partially ordered set* has the structure of a small category: objects are the elements of the set and there is a unique morphism $a \to b$ iff $a \geqslant b$. This way one obtains a functor $H : \boldsymbol{\Delta} \to Cat$ where Cat is the category of small categories and functors. The functor H carries the object $[n]$ to the category $H[n]$ given by the ordered set $[n]$. Using H we define the functor

$$Nerve(X) : \boldsymbol{\Delta}^{op} \to Set, \begin{cases} Nerve(X)[n] = \{a : H[n] \to X \in Cat\}, \\ \alpha^*(a) = a \circ \alpha_* \text{ with } \alpha_* = H(\alpha), \end{cases}$$

which coincides with the definition above; compare Gabriel and Zisman [41]. The realization $|Nerve\, X|$ is also called the classifying space of X.

Since products in the category *Top* of topological spaces do not behave well with respect to quotient maps we shall use in the next definition the full subcategory *Top*(cg) of spaces whose topology is compactly generated. The product $X \times Y$ in *Top*(cg) yields the structure of a monoidal category. The usefulness of compactly generated spaces was observed by Brown [17] and Steenrod [91].

If X is a small *topological category*, i.e. a category enriched over the monoidal category *Top*(cg), then $Nerve(X)$ is a simplicial space given by $N_n = Nerve(X)[n]$ above. For $n = 0$ the set N_0 is discrete and N_n, $n \geqslant 1$, is the union of products $X(X_1, X_0) \times \cdots \times X(X_n, X_{n-1})$ where $X(A, B) \in$ *Top*(cg) is the space of morphisms $A \to B$ in X. In particular, if $H \in$ *Top*(cg) is a *topological monoid*, i.e. a topological category with a single object, then the simplicial space $Nerve(H)$ is the *geometric bar construction* of H, see, e.g., Baues [2]. This is a good simplicial space if the inclusion of the neutral element $\{1\} \subset H$ is a closed cofibration (i.e. H is well pointed). For a well pointed topological group $G \in$ *Top*(cg) the realization

$$B(G) = |Nerve(G)| \tag{2.8}$$

is the *classifying space* of G which is the Eilenberg–Mac Lane space $K(G, 1)$ if G is discrete, see Milgram [70]. This classifying space is homeomorphic to the infinite projective space $\mathbb{R}P_\infty, \mathbb{C}P_\infty$ and $\mathbb{H}P_\infty$ in case the topological group G is $\mathbb{Z}/2$, S^1 and S^3 respectively.

A simplicial complex X is a partially ordered set and hence also a small category and we can form the simplicial set $Nerve(X)$. The realizations

$$|X| \approx |Nerve(X)| \tag{2.9}$$

are homeomorphic. In fact, $|Nerve(X)|$ can be identified with the barycentric subdivision of $|X|$.

Simplicial complexes and simplicial sets both are of combinatorial nature, but they tend to be very large objects even if one wants to describe simple spaces like products of spheres. J.H.C. Whitehead observed that for many purposes only the 'cell structure' of spaces is needed. In some sense 'cells' play a role in topology which is similar to the role of 'generators' in algebra. Let

$$\begin{aligned} &D^n = \{x \in \mathbb{R}^n, \|x\| \leqslant 1\}, \\ &\mathring{D}^n = \{x \in \mathbb{R}^n, \|x\| < 1\}, \quad \partial D^n = D^n - \mathring{D}^n = S^{n-1}, \end{aligned} \tag{2.10}$$

be the closed and open n-dimensional disk and the $(n-1)$-dimensional sphere. An (open) *n-cell* e, $n \geqslant 1$, in a space X is a homeomorphic image of the open disk $\mathring{D}^n$ in X, a 0-cell is a point in X. As a set a 'CW-complex' is the disjoint union of such cells. A CW-complex is not just a combinatorial affair since the 'attaching maps' in general may have very complicated topological descriptions.

2.11. Definition. A *CW-complex* X with skeleta $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X$ is a topological space constructed inductively as follows:

(a) X^0 is a discrete space whose elements are the 0-cells of X.

(b) X^n is obtained by attaching to X^{n-1} a disjoint union of n-disks D_i^n via continuous functions $\varphi_i : \partial(D_i^n) \to X^{n-1}$, i.e. take the disjoint union $X^{n-1} \smile \cup D_i^n$ and pass to the quotient space given by the identifications $x \sim \varphi_i(x)$, $x \in \partial D_i^n$. Each $\overset{\circ}{D}{}_i^n$ then projects homeomorphically to an n-cell e_i^n of X. The map φ_i is called the attaching map of e_i^n.

(c) X has the weak topology with respect to the filtration of skeleta.

The realization $|X|$ of a simplicial complex is a CW-complex with the n-cells given by elements $a \in X$ with $dim(\Delta_a) = n$. Also the realization $|X|$ of a simplicial set is a CW-complex with the n-cells given by 'non-degenerate' elements in $X[n]$. Here an element is *degenerate* if and only if it is in the image if one of the functions $s_i^* : X[n-1] \to X[n]$, $i \in [n-1]$. A CW-complex, however, need not be a polyhedron, see Metzler [68], but a CW-complex is always homotopy equivalent to a polyhedron. A *CW-space* is a topological space homotopy equivalent to a CW-complex. We now describe some of the many ways to create homotopy types of polyhedra.

2.12. THEOREM. *Homotopy types of polyhedra are the same as the homotopy types of the spaces in* (a) … (f) *respectively:*

(a) *realizations* $|X|$ *of simplicial complexes* X,

(b) *realizations* $|X|$ *of simplicial sets* X,

(c) *realizations* $|SX|$ *of singular sets of topological spaces* X,

(d) *classifying spaces* $|Nerve(X)|$ *of small categories* X,

(e) *classifying spaces* $|Nerve(X, \leqslant)|$ *of partially ordered sets* $(X, \leqslant)$,

(f) *CW-complexes.*

CW-complexes X, Y have a compactly generated topology and the product $X \times Y$ in *Top*(cg) is again a CW-complex (this does not hold for the product in *Top*). A *CW-monoid* is a CW-complex X which is also a monoid in *Top*(cg) such that the neutral element is a 0-cell and such that the multiplication is cellular. For example a simplicial group G yields the realization $|G|$ which is a CW-monoid. Here G, considered as a simplicial set, is a group object in *sSet* with a multiplication $G \times G \to G$ in *sSet* inducing the multiplication $|G| \times |G| = |G \times G| \to |G|$ in *Top*(cg).

A simplicial group F is called a *free simplicial group* if for each $n \geqslant 0$ the group $F[n]$ is a free group with a given basis and if all s_i^* carry basis elements to basis elements, compare Curtis [28].

2.13. THEOREM. *Homotopy types of connected polyhedra are the same as the homotopy types of the spaces* (a) *and* (b) *respectively:*

(a) *classifying spaces* $B(H) = |Nerve(H)|$ *of CW-monoids* H *for which the set* $\pi_0(H)$ *of path components is a group,*

(b) *classifying spaces* $B(|G|)$ *where* $|G|$ *is the realization of a free simplicial group.*

Hence free simplicial groups suffice to describe all homotopy types of connected polyhedra. This yields a very significant algebraic tool to construct such homotopy types. Computations in free simplicial groups, however, are still extremely complicated. It is shown in Baues [4] that the complexity of simplicial groups can be reduced considerably in case one studies homotopy types of connected 4-dimensional polyhedra. The connection between free simplicial groups and CW-complexes was described by Kan [58]:

2.14. THEOREM. *Let X be a CW-complex with trivial 0-skeleton $X^0 = *$. Then there is a free simplicial group G with $X \simeq B(|G|)$ such that the set of non-degenerate generators in $G[n]$ coincides with the set of $(n+1)$-cells in X, $n \geqslant 0$.*

This illuminates the role of cells as generators in topology. Unfortunately the free group $G[n]$ has also all the degenerate generators coming from cells in dimension $\leqslant n$. Therefore the free group $G[n]$ is very large already for CW-complexes with a few cells. We call G a *free simplicial group associated* to X if $X \simeq B(|G|)$ as in the theorem. There is, in fact, an algebraic homotopy theory of free simplicial groups which via the functors $G \mapsto B(|G|)$ is equivalent to the homotopy theory of connected polyhedra (compare Curtis [28] and Quillen [77]).

2.15. REMARK. Further methods of representing homotopy types were introduced by Smirnov [86] (compare also Smith [87]) and Kapranov and Voevodskii [59].

3. Whitehead's realization problem

The main problem and the hard core of algebraic topology is the 'classification' of homotopy types of polyhedra. Here the general idea of *classification* is to attach to each polyhedron 'invariants', which may be numbers, or objects endowed with algebraic structures (such as groups, rings, modules, etc.) in such a way that homotopy equivalent polyhedra have the same invariants (up to isomorphism in the case of algebraic structures). Such invariants are called *homotopy invariants*. The ideal would be to have an algebraic invariant which actually characterizes a homotopy type completely. The fascinating task of homotopy theory is thus the investigation of 'algebraic principles' hidden in homotopy types. We may be confident that such principles are of importance in mathematics far beyond the scope of topology as for example shown by the development of 'homological algebra' which now plays a role in ring theory, algebraic geometry, number theory and many other fields. A further very recent example is the use of 'operads' outside topology; compare, e.g., Getzler and Jones [42] and Ginzburg and Kapranov [43].

The main numerical invariants of a homotopy type are 'dimension' and 'degree of connectedness'.

3.1. DEFINITION. The *dimension* $\mathrm{Dim}(X) \leqslant \infty$ of a CW-complex is defined by $\mathrm{Dim}(X) \leqslant n$ if $X = X^n$ is the n-skeleton. The dimension $\mathrm{Dim}(X)$ of the homotopy type $\{X\}$ is defined by $\mathrm{Dim}(X) \leqslant \mathrm{Dim}(Y)$ for all CW-complexes Y homotopy equivalent to X.

3.2. DEFINITION. A space X is *(path) connected* or 0-connected if any two points in X can be joined by a path in X, this is the same as saying that any map $\partial D^1 \to X$ can be extended to a map $D^1 \to X$ where D^1 is the 1-dimensional disc. This notion has an obvious generalization: A space X is *k-connected* if for all $n \leqslant k+1$ any map

$\partial D^n \to X$ can be extended to a map $D^n \to X$ where D^n is the n-dimensional disc. The 1-connected spaces are also called *simply connected.*

The dimension is related to homology since all homology groups above the dimension are trivial, whereas the degree of connectedness is related to homotopy since below this degree all homotopy groups vanish. It took a long time in the development of algebraic topology to establish homology and homotopy groups as the main invariants of a homotopy type. For completeness we recall the definitions of these groups.

3.3. DEFINITION. Let *Top** be the category of topological spaces with basepoint $*$ and basepoint preserving maps. The set $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$ in *Top**. Choosing a basepoint in the sphere S^n we obtain the *homotopy groups*

$$\pi_n(X) = [S^n, X].$$

This is a set for $n = 0$ and a group for $n \geqslant 1$, abelian for $n \geqslant 2$. The group structure is induced by the map $\mu : S^n \to S^n \vee S^n$ obtained by identifying the equator of S^n to a point, that is for $\alpha, \beta \in \pi_n(X)$ we define $\alpha + \beta = (\alpha, \beta) \circ \mu$. The set $\pi_0(X)$ is the set of path components of X and $\pi_1(X)$ is called the *fundamental group* of X. An element $f \in [X, Y]$ induces $f_* : \pi_n X \to \pi_n Y$ by $f_*\alpha = f \circ \alpha$ so that π_n is a functor on the category $Top^*/\simeq$.

3.4. DEFINITION. For a simplicial set X let $C_n X$ be the free abelian group generated by the set $X[n]$ and let

$$\partial_n : C_n X \to C_{n-1} X$$

be the homomorphism defined on basis elements $x \in X[n]$ by

$$\partial_n(x) = \sum_{i=0}^{n} (-1)^i d_i^*(x).$$

Then one can check that $\partial_n \partial_{n+1} = 0$ so that the quotient group

$$H_n X = \text{kernel}\, \partial_n / \text{image}\, \partial_{n+1}$$

is defined. This is the n-th homology group of X. For a topological space X we define the *homology* $H_n X = H_n SX$ by use of the singular set. The homology H_n yields a functor from the homotopy category $Top/\simeq$ to the category of abelian groups.

The crucial importance of homotopy groups and homology groups relies on the following results due to J.H.C. Whitehead.

3.5. THEOREM. A) *A connected CW-space X is contractible if and only if for a basepoint in X all homotopy groups $\pi_n(X)$, $n \geqslant 1$, are trivial.*

B) *A simply connected CW-space X is contractible if and only if all homology groups $H_n(X)$, $n \geqslant 2$, are trivial.*

The theorem shows that homotopy groups and in the simply connected case also homology groups are able to detect the trivial homotopy type. In fact, homotopy groups and homology groups are able to decide whether two spaces have the same homotopy type:

3.6. WHITEHEAD THEOREM. *Let X and Y be connected CW-spaces and let $f : X \to Y$ be a map. Then f is a homotopy equivalence in Top/ $\simeq$ if and only if, for a basepoint in X, condition* A) *or equivalently* B) *holds.*

A) *The map f induces an isomorphism between homotopy groups, $f_* : \pi_n X \cong \pi_n Y$, $n \geqslant 1$.*

B) *The map f induces an isomorphism between fundamental groups, $f_* : \pi_1 X \cong \pi_1 Y$, and the induced map $\tilde{f} : \tilde{X} \to \tilde{Y}$ between universal coverings induces an isomorphism between homology groups, $\tilde{f}_* : H_n\tilde{X} \cong H_n\tilde{Y}$, $n \geqslant 2$.*

Hence homotopy groups constitute a system of algebraic invariants which, in a certain sense, are sufficiently powerful to characterize the homotopy type of a CW-space. This does not mean that $X \simeq Y$ just because there exist isomorphisms $\pi_n X \cong \pi_n Y$ for every $n = 1, 2, \dots$. The crux of the matter is not merely that $\pi_n X \cong \pi_n Y$, but that a certain family of isomorphisms, $\phi_n : \pi_n X \cong \pi_n Y$, has a *geometrical realization* $f : X \to Y$. That is to say, the latter map f induces all isomorphisms ϕ_n via the functor π_n, namely $\phi_n = \pi_n(f)$ for $n \geqslant 1$. Therefore the emphasis is shifted to the following problem; compare Whitehead [101].

3.7. REALIZATION PROBLEM OF WHITEHEAD. *Find necessary and sufficient conditions in order that a given set of isomorphisms or, more generally, homomorphisms, $\phi_n : \pi_n X \to \pi_n Y$, have a geometrical realization $X \to Y$.*

The Whitehead theorem shows that also the invariants $\pi_1 X$, $H_n\tilde{X}$ are sufficiently powerful to detect homotopy types. Therefore there is a realization problem for these invariants in a similar way. In particular, within the category of simply connected CW-spaces the functors π_n could be replaced by H_n. The realization problem of Whitehead above is highly unsolved, and is indeed one of the hardest problems of algebraic topology. We shall describe below solutions for some special cases; see 10.11. Using simplicial groups Kan gave a purely combinatorial description of Whitehead's realization problem. For this we need the following Moore chain complex of a simplicial group.

3.8. DEFINITION. A *chain complex* (C, ∂) of groups is a sequence of homomorphisms

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots, \quad n \in \mathbb{Z},$$

in the category of groups with image ∂_{n+1} a normal subgroup of kernel ∂_n. For each n, the homology $H_n(C, \partial)$ is defined to be the quotient group kernel (∂_n)/image (∂_{n+1}). For each simplicial group G one has the *Moore chain complex*, NG, with

$$N_n(G) = \bigcap_{i<n} \operatorname{kernel}(d_i^*),$$

$$\partial_n = d_n^* \quad \text{(restricted to } N_n G).$$

We define *homotopy groups* of G by $\pi_n G = H_n(NG)$.

A basic theorem of Kan [56] shows that homotopy groups of simplicial groups, in fact, correspond exactly to homotopy groups of connected CW spaces:

3.9. THEOREM. *Let G be a simplicial group. Then there is a natural isomorphism* $(n \geqslant 0)$

$$\pi_n(G) = \pi_n|G| = \pi_{n+1}B(|G|).$$

Hence if G_X is associated to the connected CW-space X, that is $X \cong B(|G_X|)$, we can compute $\pi_{n+1}(X) = \pi_n(G_X)$ by the Moore chain complex $N(G_X)$. For example, let $G_{S^{n+1}}$ be the free simplicial group with only one nondegenerate generator in degree n, then $G_{S^{n+1}}$ is associated to the sphere S^{n+1} and

$$\pi_{n+k}(G_{S^{n+1}}) = H_{n+k}N(G_{S^{n+1}}) = \pi_{n+k+1}S^{n+1}$$

gives us a purely combinatorial description of homotopy groups of spheres. This way Kan gave a new proof of Hopf's result $\pi_3 S^2 = \mathbb{Z}$. In general, however, free simplicial groups are so complicated that this formula was not suitable for computing homotopy groups of spheres. Theorem 3.9 leads to the following interpretation of Whitehead's realization problem.

3.10. THEOREM. *Let X, Y be connected CW-spaces and let G_X, G_Y be free simplicial groups associated to X and Y respectively. Then a set of homomorphisms $\phi_n : \pi_n X \to \pi_n Y$ is realizable by a map $X \to Y$ if and only if there is a map $f : G_X \to G_Y$ in sGr inducing for $n \geqslant 0$ the homomorphism*

$$\phi_{n+1} : \pi_{n+1}X = \pi_n G_X \xrightarrow{f_*} \pi_n G_Y = \pi_{n+1}Y.$$

We say that two simplicial groups G, G' are *weakly equivalent* if there is a map $f : G \to G'$ in *sGr* inducing isomorphisms $f_* : \pi_n G \cong \pi_n G'$. This yields actually an equivalence relation for free simplicial groups. As usual a 1–1 *correspondence* is a function which is injective and surjective. The next result is a consequence of 3.10 and 2.13.

3.11. COROLLARY. *There is a 1–1 correspondence between homotopy types of connected CW-spaces and weak equivalence classes of free simplicial groups. The correspondence is given by $X \mapsto G_X$ with the inverse $G \mapsto B(|G|)$.*

We point out that 'weak equivalence' generates an equivalence relation for all simplicial groups and that weak equivalence classes of all simplicial groups are the same as weak equivalence classes of free simplicial groups. In fact, for any simplicial group G' there is a free simplicial group G and a weak equivalence $G \to G'$ which is called a *free model* of G'.

3.12. DEFINITION. Let C be a category with a given class of morphisms called weak equivalences. Then the *localization* or *homotopy category* of C is the category $Ho(C)$ together with a functor $q : C \to Ho(C)$ having the following universal property: For every weak equivalence f the morphism $q(f)$ is an isomorphism; given any functor $F : C \to B$ with $F(f)$ an isomorphism for all weak equivalences f, there is a unique functor $\Theta : Ho(C) \to B$ such that $\Theta q = F$. Except for set theoretic difficulties the category $Ho(C)$ exists, see Gabriel and Zisman [41].

3.13. THEOREM. *Let spaces be the category of connected CW-spaces with basepoint and let spaces/ $\simeq$ be the corresponding homotopy category. Then there is an equivalence of categories*

$$Ho(sGr) \xrightarrow{\sim} \mathit{spaces}/\simeq$$

which carries a simplicial group G to the classifying space $B(|G|)$.

The results 3.9–3.13 are due to Kan, see Curtis [28] and Quillen [77].

4. Algebraic models of n-types

When studying a CW-complex or a polyhedron X it is natural to consider in succession the skeleta $X^1, X^2, \ldots$, where X^n consists of all the cells in X of at most n-dimensions. Now the homotopy type of X^n is not an invariant of the homotopy type of X. Therefore J.H.C. Whitehead introduced the n-type, this being a homotopy invariant of X, which depends only on X^{n+1}. There are two ways to present n-types. On the one hand they are certain equivalence classes of $(n+1)$-dimensional CW-complexes, on the other hand they are homotopy types of certain spaces.

4.1. DEFINITION. Let CW be the category of connected CW-complexes X with basepoint $* \in X^0$ and of basepoint preserving cellular maps. Let CW^{n+1} be the full subcategory of CW consisting of $(n+1)$-dimensional objects. For maps $F, G : X^{n+1} \to Y^{n+1}$ in CW^{n+1} let $F|X^n$, $G|X^n : X^n \to Y^{n+1}$ be the restrictions. Then we obtain an equivalence relation $\sim$ by setting $F \sim G$ iff there is a homotopy $F|X^n \simeq G|X^n$ in Top^*. Let $CW^{n+1}/\sim$ be the quotient category. Now an *n-type* in the sense of J.H.C. Whitehead is an isomorphism type in the category $CW^{n+1}/\sim$.

4.2. DEFINITION. Recall that *spaces* is the category of connected CW-spaces with basepoint and pointed maps. Let

$$n\text{-}\mathit{types} \subset \mathit{spaces}/\simeq$$

be the full subcategory consisting of spaces X with $\pi_i(X) = 0$ for $i > n$. Such spaces or their homotopy types are also called *n-types*.

The two definition of n-types are compatible since there is an equivalence of categories

$$P_n : CW^{n+1}/\sim \xrightarrow{\sim} n\text{-}\mathit{types}. \tag{4.3}$$

We define the functor P_n by use of the following *n-th Postnikov functor*

$$P_n : CW/\simeq \to n\text{-}\mathit{types}.$$

For X in *CW* we obtain P_nX by 'killing homotopy groups', that is, we choose a CW-complex P_nX with $(n+1)$-skeleton

$$(P_nX)^{n+1} = X^{n+1}$$

and with $\pi_i(P_nX) = 0$ for $i > n$. For a cellular map $F : X \to Y$ in *CW* we choose a map $PF^{n+1} : P_nX \to P_nY$ which extends the restriction $F^{n+1} : X^{n+1} \to Y^{n+1}$ of F. This is possible since $\pi_iP_nY = 0$ for $i > n$. The functor P_n in 4.4 and 4.3 carries X to P_nX and carries F to the homotopy class of P_nF. Different choices for P_nX yield canonically isomorphic functors P_n.

Isomorphism types in $\mathit{CW}^{n+1}/\sim$ were originally called '$(n+1)$-types', they are now called n-types since they correspond to homotopy types for which only $\pi_1, \ldots, \pi_n$ might be non trivial.

There is an important relationship between n-types and homotopy types of $(n+1)$-dimensional CW-spaces. Two $(n+1)$-dimensional connected CW-spaces X^{n+1}, Y^{n+1} have the same n-type iff one of the following conditions (A) and (B) is satisfied:

(A) There is a map $F : X^{n+1} \to Y^{n+1}$ which induces isomorphisms $\pi_i(F)$ for $i \leqslant n$.

(B) There is a homotopy equivalence $P_nX^{n+1} \simeq P_nY^{n+1}$.

4.4. THEOREM (J.H.C. Whitehead [103]). *Let X^{n+1}, Y^{n+1} be two finite $(n+1)$-dimensional CW-complexes which have the same n-type. Then there exist $a, b < \infty$ such that the one point unions*

$$X^{n+1} \vee \bigvee_a S^{n+1} \simeq Y^{n+1} \vee \bigvee_b S^{n+1}$$

are homotopy equivalent.

The theorem shows that each n-type Q determines a connected tree $\mathrm{HT}(Q, n+1)$ which we call the *tree of homotopy types* for $(Q, n+1)$. The vertices of this tree are the homotopy types $\{X^{n+1}\}$ of finite $(n+1)$-dimensional CW-complexes with $P_nX^{n+1} \simeq Q$. The vertex $\{X^{n+1}\}$ is connected by an edge to the vertex $\{Y^{n+1}\}$ if Y^{n+1} has the homotopy types of $X^{n+1} \vee S^{n+1}$. The *roots* of this tree are the homotopy types $\{Y^{n+1}\}$ which do not admit a homotopy equivalence $Y^{n+1} \simeq X^{n+1} \vee S^{n+1}$. Theorem (4.4) shows that the tree $\mathrm{HT}(Q, n+1)$ is connected. For a proof of theorem (4.4) see II. § 6 in Baues [4].

REMARK. There are various results on the tree $\mathrm{HT}(Q, n+1)$ in case $Q = K(\pi, 1)$ is an Eilenberg–Mac Lane space of degree 1. In this case the tree is determined by the group π. Results of Metzler [69], Sieradski [84] and Sieradski and Dyer [85] show that for $n \geqslant 1$ there exist trees $\mathrm{HT}(K(\pi, 1), n+1)$ with at least two roots.

As pointed out by Whitehead [99] one has to consider the hierarchy of categories and functors

$$1\text{-}types \xleftarrow{P} 2\text{-}types \xleftarrow{P} 3\text{-}types \xleftarrow{P} \cdots \tag{4.5}$$

where the functor P is given by the Postnikov functor above. Since 1-types are the same as Eilenberg–Mac Lane spaces $K(\pi, 1)$ we can identify a 1-type with an abstract group. In fact, the fundamental group π_1 gives us the equivalence of categories

$$\pi_1 : 1\text{-}types \xrightarrow{\sim} Gr. \tag{4.6}$$

From this point of view n-types are natural objects of higher complexity extending abstract groups. Following up on this idea Whitehead looked for a purely algebraic equivalent of an n-type, $n \geqslant 2$. An important requirement for such an algebraic system is 'realizability', in two senses. In the first sense this means that there is an n-type which is in the appropriate relation to a given one of these algebraic systems, just as there is a 1-type whose fundamental group is isomorphic to a given group. The second sense is the 'realizability' of homomorphisms between such algebraic systems by maps of the corresponding n-types.

Mac Lane and Whitehead [65] showed that a 'crossed module' is a purely algebraic equivalent of a 2-type:

4.7. Definition. An N-*group* or an *action* of a group N on a group M is a homomorphism f from N to the group of automorphisms of M. For $x \in M$, $\alpha \in N$ we denote the action by $x^\alpha = f(\beta)(x)$ where β is the inverse of α. Then a *pre-crossed module* $\partial : M \to N$ is a group homomorphism together with an action of N on M such that

$$\partial(x^\alpha) = \alpha^{-1}\partial(x)\alpha,$$

that is, ∂ is equivariant with respect to the action of N on N by inner automorphisms. A *Peiffer commutator* in M is the element

$$\langle x, y\rangle = x^{-1}y^{-1}x\left(y^{\partial x}\right) \quad \text{for } x, y \in M.$$

Now ∂ is a *crossed module* if all Peiffer commutators are trivial. A morphism between crossed modules (or pre crossed modules) is a commutative diagram in Gr

$$\begin{array}{ccc} M & \xrightarrow{g} & M' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ N & \xrightarrow{f} & N' \end{array}$$

where g is f-equivariant, that is $g(x^\alpha) = (gx)^{(f\alpha)}$. This is a *weak equivalence* if (f, g) induces isomorphisms $\pi_i(\partial) \cong \pi_i(\partial')$ for $i = 1, 2$ where $\pi_1(\partial) = \text{cokernel}(\partial)$ and $\pi_2(\partial) = \text{kernel}(\partial)$.

4.8. Theorem. *Let cross be the category of crossed modules and let $Ho(cross)$ be the localization with respect to weak equivalences. Then there is an equivalence of categories*

$$2\text{-}types \xrightarrow{\sim} Ho(cross).$$

For a proof of this result compare (III.8.2) in Baues [4]. Many further properties of crossed modules are described in this book, in particular, crossed modules lead to algebraic models which determine the homotopy types of connected 3-dimensional polyhedra.

Using Kan's result 3.13 a simplicial group G with $\pi_i(G) = 0$ for $i \geqslant 2$ is also an algebraic model of a 2-type. The crossed module ∂_G associated to G is obtained by the Moore chain complex $N(G)$ in (3.8). We have

$$\partial_G : N_1(G)/dN_2(G) \to N_0(G) \tag{4.9}$$

with $N_0(G) = G[0]$. Here $G[0]$ acts on $N_1(G)$ by $x^\alpha = s_0^*(\alpha)^{-1} \cdot x \cdot s_0^*(\alpha)$ so that $d : N_1(G) \to N_0(G)$ is a pre-crossed module. The normal subgroup $dN_2(G)$ of $N_1(G)$ contains all Peiffer commutators so that ∂_G induced by d is a well defined crossed module. Hence ∂_G reduces the complexity of the simplicial group G considerably, so that a crossed module describes the algebra behind a 2-type more precisely and simply than a simplicial group.

After Step Two in the hierarchy of n-types was achieved by Mac Lane and Whitehead in 1950 one had to consider Step Three. The solution for Step Three was obtained recently in Baues [4] where 'quadratic modules' are shown to be the appropriate algebraic models of 3-types.

4.10. Definition. A *quadratic module* $\sigma = (w, \delta, \partial)$ is a diagram of N-groups and N-equivariant homomorphisms

$$C \otimes C \xrightarrow{w} L \xrightarrow{\delta} M \xrightarrow{\partial} N$$

satisfying the equations

$$\begin{cases} \partial\delta = 0, \\ x^{-1}y^{-1}x(y^{\partial x}) = \delta w(\{x\} \otimes \{y\}), \\ a^{-1}b^{-1}ab = w(\{\delta a\} \otimes \{\delta b\}), \\ a^{\partial x} = a \cdot w(\{\delta a\} \otimes \{x\} + \{x\} \otimes \{\delta a\}), \end{cases}$$

for $a, b \in L$ and $x, y \in M$. Here C is the abelianization of the quotient group $M/P_2(\partial)$ where $P_2(\partial)$ is the subgroup of M generated by Peiffer commutators $\langle x, y\rangle$ in the pre-crossed module ∂. The element $\{x\} \in C$ is represented by $x \in M$ and the action of $\alpha \in N$ on the $\mathbb{Z}$-tensor product $C \otimes C$ is given by $(\{x\} \otimes \{y\})^\alpha = \{x^\alpha\} \otimes \{y^\alpha\}$. A morphism

$$\varphi : \sigma = (w, \delta, \partial) \to \sigma' = (w', \delta', \partial')$$

between quadratic modules with $\varphi = (l, m, n)$ is given by a commutative diagram in *Gr*

$$\begin{tikzcd} C \otimes C \arrow[r, "w"] \arrow[d, "\varphi_* \otimes \varphi_*"] & L \arrow[r, "\delta"] \arrow[d, "l"] & M \arrow[r, "\partial"] \arrow[d, "m"] & N \arrow[d, "n"] \\ C' \otimes C' \arrow[r, "w'"'] & L' \arrow[r, "\delta'"'] & M' \arrow[r, "\partial'"'] & N' \end{tikzcd}$$

where (m, n) is a map between pre-crossed modules which induces $\varphi_* : C \to C'$ and where l is n-equivariant. This is a *weak equivalence* if φ induces isomorphisms $\varphi_* : \pi_i(\sigma) \cong \pi_i(\sigma')$ for $i = 1, 2, 3$ where

$$\pi_1(\sigma) = \text{cokernel}\,\partial,$$
$$\pi_2(\sigma) = \text{kernel}\,\partial/\text{image}\,\delta,$$
$$\pi_3(\sigma) = \text{kernel}\,\delta.$$

4.11. THEOREM. *Let quad be the category of crossed modules and let $Ho(quad)$ be the localization with respect to weak equivalences. Then there is an equivalence of categories*

$$3\text{-types} \xrightarrow{\sim} Ho(quad).$$

Compare (IV. § 10) in Baues [4]. In this book many further properties and examples of quadratic modules are described, in particular quadratic modules lead to algebraic models which determine homotopy types of connected 4-dimensional polyhedra. One can deduce from a simplicial group G with $\pi_i(G) = 0$ for $i \geqslant 3$ the associated quadratic module σ_G as follows: We derive from the Moore chain complex $N(G)$ in 3.8 the quadratic module $\sigma_G = (w, \delta, \partial)$ with

$$C \otimes C \xrightarrow{w} N_2(G)/U \xrightarrow{\delta} N_1(G)/P_3(\partial) \xrightarrow{\partial} N_0(G). \tag{4.12}$$

Here the action of $N_0(G) = G[0]$ is obtained by s_0^* and $s_1^* s_0^*$ as in (4.9) and δ and ∂ are induced by the boundary maps in $N(G)$. Moreover $P_3(\partial)$ is the subgroup of $N_1(G)$ generated by triple Peiffer commutators $\langle x, \langle y, z\rangle\rangle$ and $\langle\langle x, y\rangle, z\rangle$ in the pre crossed module $\partial = d_1^*$, see (4.9). We define for $x, y \in N_1(G)$ the formal Peiffer bracket $\langle x, y\rangle \in N_2(G)$ by

$$\langle x, y\rangle = s_1^*\left(x^{-1}y^{-1}x\right)\left(s_0^*x\right)^{-1}\left(s_1^*y\right)\left(s_0^*x\right).$$

Then $d_2\langle x, y\rangle = \langle x, y\rangle$ holds. Now U is the subgroup of $N_2(G)$ generated by formal triple brackets $\langle x, \langle y, z\rangle\rangle$, $\langle\langle x, y\rangle, z\rangle$ and by elements $d_3(u)$ with $u \in N_3(G)$. Finally the function w is defined by $w(\{x\} \otimes \{y\}) = \{\langle x, y\rangle\}$ where $\langle x, y\rangle$ is the formal Peiffer bracket. See also (IV. B. 11) in Baues [4].

Again a quadratic module is a considerable simplification of a simplicial group G representing a 3-type. In fact, we restrict G to degrees $\leqslant 2$ and we are even allowed to divide out triple Peiffer commutators and formal triple Peiffer commutators in the Moore chain complex. We therefore say that a quadratic module has 'nilpotency degree two', a crossed module has 'nilpotency degree one'.

REMARK. Theorem 4.8 goes back to the work of J.H.C. Whitehead [100] and Mac Lane and Whitehead [65] though they do not formulate the result as an equivalence of categories. In the literature there are two ways to generalize crossed modules in order to obtain models of n-types, $n \geqslant 2$. On the one hand Loday [63] defines algebraic systems called 'cat^n-groups' (see also Porter [75] and Bullejos, Cegarra and Duskin [20]), on the other hand Conduché [26] considers 'crossed modules of length 2' representing 3-types which were generalized by Carrasco [21] and Carrasco and Cegarra [22] for n-types; this approach of Conduché and Carrasco describes additional structure for the Moore chain complex $N(G)$ which is sufficient to determine the simplicial group G. Moreover Brown and Gilbert [18] and Joyal and Tierney obtained further algebraic models of 3-types. But the quadratic modules above are the only models of 3-types which have nilpotency degree 2.

A 'nilpotent' algebraic model for 4-types is not known. For simply connected n-types, however, we can use the work of Curtis [27] for the construction of nilpotent models.

4.13. Definition. For a group G let $\Gamma_{m+1}G$ be the subgroup of all iterated commutators of length $m+1$. Then G has *nilpotency degree* m or equivalently is a nil(m)*-group* if $\Gamma_{m+1}G$ is trivial. Let *nil*(m) be the full subcategory in *Gr* consisting of nil(m)-groups. A free nil(m)-group, i.e. a free object in *nil*(m), is the same as the quotient $F/\Gamma_{m+1}F$ where F is a free group. Let *snil*(m) be the category of simplicial nil(m)-groups with *weak equivalences* defined as in *sGr*. A free simplicial nil(m)-group is defined in a similar way to a free simplicial group, see Section 2.

Let $\{a\}$ be the least integer $\geqslant a$.

4.14. Theorem. *For $2 \leqslant n \leqslant 1 + \{\log_2(m+1)\}$ let $T(n,m)$ be the full subcategory of snil(m) consisting of objects G with $\pi_i G = 0$ for $i = 0$ and $i \geqslant n$. Then there exists an equivalence of categories*

$$n\text{-}types_2 \xrightarrow{\sim} Ho\, T(n,m).$$

Here the left hand side denotes the full homotopy category of simply connected n-types.

For $m = 2$ and $n = 3$ the result is also a consequence of 4.11. This indicates that there might be a suitable generalization of both Theorems 4.11 and 4.14, available for n-types which are not simply connected.

Theorem 4.14, as it stands, is not contained in the work of Curtis. The equivalence in the theorem carries the n-type X to a free simplicial nil(m)-group $\overline{G}_X$ with $\pi_i\overline{G}_x = 0$ for $i \geqslant n$ and for which

$$\overline{G}_X^n = (G_X/\Gamma_{m+1}G_X)^n.$$

Here both sides denote the corresponding subobjects generated by basis elements in degree $\leqslant n$. Hence $\overline{G}_X$ is the 'n-type' of $G_X/\Gamma_{m+1}G_X$ in the category *snil*(m), compare the construction of the Postnikov section in 4.4. The result of Curtis [27] implies that there is a natural isomorphism $(i \geqslant 0)$

$$\pi_i(\overline{G}_X) = \pi_{i+1}(X)$$

for all simply connected n-types X. The inverse of the functor $X \mapsto \overline{G}_X$ carries the simplicial group G in $T(n,m)$ to the classifying space $B(|G|)$.

We have seen that the category 2-*types* has the algebraic model category *cross* in 4.8. This generalizes as follows.

4.15. DEFINITION. A *crossed complex* ρ is a sequence

$$\cdots \xrightarrow{d_4} \rho_3 \xrightarrow{d_3} \rho_2 \xrightarrow{d_2} \rho_1$$

of homomorphisms between ρ_1-groups where d_2 is a crossed module and ρ_n, $n \geqslant 3$, is abelian and a π_1-module via the action of ρ_1 where $\pi_1 = \text{cokernel}(d_2)$. Moreover $d_{n-1}d_n = 0$ for $n \geqslant 3$. A morphism $f : \rho \to \rho'$ is a sequence of homomorphisms $f_n : \rho_n \to \rho'_n$ which commute with d_n and are f_1-equivariant. Let $\pi_n(\rho) = \text{kernel}(d_n)/\text{image}(d_{n+1})$ be the homology of ρ. Then f is a weak equivalence if $\pi_n(f)$ is an isomorphism for all n. Let cross^n be the category of crossed chain complexes ρ with $\rho_i = 0$ for $i > n$ and $\pi_i(\rho) = 0$ for $1 < i < n$ so that $\mathit{cross}^2 = \mathit{cross}$.

The next result is a consequence of the work Huebschmann [54] and of Brown and Higgins [19]; see also 3.3.6 in Carrasco and Cegarra [22].

4.16. THEOREM. *Let $K_1^n \subset n$-types be the full homotopy category of n-types X with $\pi_i X = 0$ for $1 < i < n$, $n \geqslant 2$. Then there is an equivalence of categories*

$$K_1^n \xrightarrow{\sim} Ho(\mathit{cross}^n).$$

For $n = 2$ this is exactly the result in 4.8. The objects in cross^n which are by 4.16 models of special n-types have only nilpotency degree 1. In particular 3-types X with $\pi_2 X$ have a model in cross^3 so that in this case a quadratic module σ as in 4.10 is not needed to determine the homotopy type. We can associate with σ the crossed chain complex $\rho(\sigma)$,

$$L/w(C \otimes C) \xrightarrow{\delta} M/\delta w(C \otimes C) \xrightarrow{\partial} N, \tag{4.17}$$

obtained by dividing out the 'quadratic part'. If $\pi_2(\sigma) = 0$ then $\rho(\sigma)$ determines the homotopy type of σ. Therefore the quadratic structure w of σ is only relevant if $\pi_2 \neq 0$. In the next section we study the category K_1^n from a different point of view.

5. Cohomology of groups and cohomology of categories

We show that the classical cohomology of groups is related to special homotopy types. We also introduce the cohomology of categories with coefficients in a natural system, which generalizes the cohomology of groups and which turned out to have deep impact on homotopy classification. We shall need the cohomology of categories in particular for the comparison of Postnikov invariants and boundary invariants; see 8.11 below.

Let π be a group. A (right) π-*module* M, also denoted by the pair (π, M), is an abelian group M together with an action of π on M. As usual the homotopy groups $\pi_n(X)$, $n \geqslant 2$, are actually $\pi_1(X)$-modules. Let *Mod* and *mod* be the following categories. Objects in both are the modules (π, M) as above. Morphisms $(\pi, M) \to (\pi', M')$ are pairs

$$(a, f) = (a : \pi \to \pi',\ f : M \to M') \in \mathit{Mod},$$
$$(b, g) = (b : \pi' \to \pi,\ g : M \to M') \in \mathit{mod},$$

where a, b are maps between groups and f, g are maps between abelian groups such that $f(x^\alpha) = f(x)^{a(\alpha)}$ and $g(x^{b(\beta)}) = g(x)^\beta$ for $x \in M$, $\alpha \in \pi$, $\beta \in \pi'$. Using *homotopy groups* one has a functor $(n \geqslant 2)$

$$(\pi_1, \pi_n) : \mathit{Top}^* \to \mathit{Mod}. \tag{5.1}$$

The *cohomology of groups* is a functor (see K.S. Brown [16] and 5.12 below)

$$H^n : \mathit{mod} \to \mathit{Ab} \tag{5.2}$$

which carries (π, M) to $H^n(\pi, M)$. Let b^*M be the π'-module M given by $x^\beta = x^{b(\beta)}$. Then $(b, 1) : (\pi, M) \to (\pi', b^*M)$ is a morphism in *mod* which induces $b^* = H^n(b, 1)$,

$$b^* : H^n(\pi, M) \to H^n(\pi', b^*M).$$

On the other hand $(1, f) : (\pi, M) \to (\pi, a^*M')$ in *mod* induces $f_* = H^n(1, f)$,

$$f_* : H^n(\pi, M) \to H^n(\pi, a^*M').$$

We use the cohomology of groups for the definition of the following category, which is the 'Grothendieck construction' of the functor H^n in (5.2).

5.3. DEFINITION. The objects in the category $\mathit{Gro}(H^n)$ are triples (π, M, k) where (π, M) is a π-module and $k \in H^n(\pi, M)$. Morphisms $(\pi, M, k) \to (\pi', M', k')$ are maps $(a, f) : (\pi, M) \to (\pi', M')$ in *Mod* which satisfy the equation

$$a^*(k') = f_*(k) \in H^n(\pi, a^*M').$$

Composition is defined as in *Mod*; the forgetful functor $\mathit{Gro}(H^n) \to \mathit{Mod}$ is faithful.

The objects in $\mathit{Gro}(H^{n+1})$ are in fact algebraic models of special n-types.

5.4. THEOREM. *For the full homotopy category $K_1^n \subset n$-types of n-types X with $\pi_i X = 0$ for $1 < i < n$ there is a functor*

$$T^n : K_1^n \to \mathit{Gro}(H^{n+1})$$

with the following properties: The functor T^n is full and reflects isomorphisms and for each object (π, M, k) in $\mathit{Gro}(H^{n+1})$ there is X in K_1^n and an isomorphism $(\pi, M, k) \cong T^n(X)$ in $\mathit{Gro}(H^{n+1})$. The functor T^n is defined by $T^n(X) = (\pi_1(X), \pi_n(X), k(X))$ where $k(X)$ is the k-invariant.

In consequence of these properties of the functor T^n an object in $\mathit{Gro}(H^{n+1})$ may be described as an algebraic equivalent of a n-type in K_1^n; that is, T^n induces a 1–1 correspondence between homotopy types in K_1^n and isomorphism types in $Gro(H^{n+1})$. Theorem 5.4 is due to Mac Lane and Whitehead [65] for $n = 2$ and Eilenberg and Mac Lane [36] for $n \geqslant 3$. It is also a consequence of the 'Postnikov tower' of a space, see, e.g., Baues [1]. The theorem yields a special solution of Whitehead's realization problem 3.7:

5.5. COROLLARY. *Let X, Y be objects in K_1^n, then $\phi_* : \pi_*X \to \pi_*Y$ has a geometrical realization $X \to Y$ if and only if (ϕ_1, ϕ_n) is a morphism in Mod and the equation*

$$(\phi_1)^*k(Y) = (\phi_N)_*k(X)$$

holds where $k(X)$, $k(Y)$ are the k-invariants.

In view of Theorem 5.4 elements in the cohomology of groups can be considered as representatives of special n-types. We now recall the following notation which partially already was used in the theorem above.

5.6. NOTATION. Let $F : C \to K$ be a functor. We say that F is *full*, resp. *faithful* if the induced map on morphism sets $F : C(X, Y) \to K(FX, FY)$ is surjective, resp. injective for all objects X, Y in C. Moreover F *reflects isomorphisms* if f in C is an isomorphism if and only if $F(f)$ in K is an isomorphism. The functor F is *representative* if for each object Y in K there is an object X in C and an isomorphism $F(X) \cong Y$. We call X a 'realization' of Y. We say that F is a *detecting functor* if F reflects isomorphisms, is full and representative. A detecting functor which is faithful is the same as an equivalence of categories.

The properties of the functor T^n in 5.4 just say that T^n is a detecting functor. One readily checks that every detecting functor $F : C \to K$ induces a 1–1 correspondence between isomorphism types of objects in C and isomorphism types of objects in K. The functor T^n has actually a further nice property which is less well known, namely T^n is a 'linear extension' of categories. To this end we recall from Baues [3] the following concept of a linear extension which plays a crucial role in topology and algebra.

5.7. NOTATION. Let C be a category. The *category of factorizations* in C, denoted by FC, is given as follows: Objects are the morphisms $f, g, \ldots$ in C and morphisms $f \to g$ are pairs (α, β) for which

$$\begin{tikzcd}
A \arrow[r, "\alpha"] & A' \\
B \arrow[u, "f"] & B' \arrow[l, "\beta"] \arrow[u, "g"']
\end{tikzcd}$$

commutes in C. Hence $\alpha f \beta = g$ is a factorization of g. Composition is defined by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. We clearly have $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$. A *natural system* (of abelian groups) on C is a functor

$$D : FC \to Ab$$

from the category of factorizations to the category of abelian groups. The functor D carries the object f to $D_f = D(f)$ and carries the morphism $(\alpha, \beta) : f \to g$ to the induced homomorphism

$$D(\alpha, \beta) = \alpha_* \beta^* : D_f \to D_{\alpha f \beta} = D_g$$

where $D(\alpha, 1) = \alpha_*$, $D(1, \beta) = \beta^*$. We say that

$$D \xrightarrow{+} E \xrightarrow{p} C$$

is a *linear extension* of C by the natural system D if the following properties hold. The categories E and C have the same objects and p is a full functor which is the identity on objects. For each morphism $f : B \to A$ in C the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in E with $p^{-1}(f) \subset E(B, A)$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$. Moreover, the action satisfies the *linear distributivity law*:

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha.$$

Two linear extensions E, E' are equivalent if there is an isomorphism $\varepsilon : E \cong E'$ of categories with $p'\varepsilon = p$ and $\varepsilon(f_0 + \alpha) = \varepsilon(f_0) + \alpha$. The extension E is *split* if there is a functor $s : C \to E$ with $ps = 1$.

As an example we obtain the natural system

$$H^n : Mod \to Ab \tag{5.8}$$

which carries the object $(a, f) : (\pi, M) \to (\pi', M')$ to the abelian group

$$H^n_{(a,f)} = H^n(\pi, a^* M')$$

which is the cohomology of π with coefficients in $a^* M'$. Hence $H^n_{(a,f)}$ depends on a and not on f. Induced maps are given by $(a', f')_*(x) = (f')_*(x)$ and $(a'', f'')^*(x) = (a'')^*(x)$ for $x \in H^n_{(a,f)}$. The natural system H^n on Mod yields also a natural system H^n on $Gro(H^{n+1})$ via the forgetful functor in 5.3. Using the functor T^n in 5.4 we identify isomorphism types in K_1^n and in $Gro(H^{n+1})$ so that this way T^n is the identity on objects. The next result is a consequence of (VIII.2.5) in Baues [3].

5.9. THEOREM. *The category K_1^n is part of a linear extension of categories*

$$H^n \xrightarrow{+} K_1^n \xrightarrow{T^n} Gro(H^{n+1})$$

which is not split.

The result classifies maps in K_1^n completely in terms of the cohomology of groups. Since the functor T^n is not split the extension, however, is nontrivial. We now introduce the cohomology of categories which classifies linear extensions. In analogy to the category *mod* in (5.2) we obtain the category *nat* of natural systems: Objects are pairs (C, D) where D is a natural system of the small category C. Morphisms are pairs

$$(\phi^{op}, \tau) : (C, D) \to (C', D') \tag{5.10}$$

where $\phi : C' \to C$ is a functor and where $\tau : \phi^* D \to D'$ is a natural transformation. Here $\phi^* D : FC' \to Ab$ is given by $(\phi^* D)_f = D_{\phi f}$ and $\alpha_* = \phi(\alpha)_*$, $\beta^* = \phi(\beta)^*$. A natural transformation $T : D \to \tilde{D}$ yields as well the natural transformation $\phi^* t : \phi^* D \to \phi^* \tilde{D}$. Now morphisms in *nat* are composed by the formula

$$(\psi^{op}, \sigma)(\phi^{op}, \tau) = ((\phi\psi)^{op}, \sigma \circ \psi^* \tau).$$

The *cohomology of categories* (introduced in Baues and Wirsching [10] and Baues [3]) is the functor

$$H^n : nat \to Ab \tag{5.11}$$

defined in 5.13 below. One has the full inclusion of categories

$$mod \subset nat$$

which carries (π, M) to (C, D) where $C = \pi$ is the category given by the group π and where D is the natural system on C with $D_f = M$ for $f \in \pi$ and $\alpha^* =$ identity and $\beta_*(x) = x^\beta$ for $x \in M$, $\beta \in \pi$. Then the composition of functors

$$mod \subset nat \xrightarrow{H^n} Ab \tag{5.12}$$

coincides with the cohomology of groups in (5.2). In fact, we may consider the cohomology of categories as a canonical generalization of the cohomology of groups.

5.13. DEFINITION. Let X be a small category and let D be a natural system on X. The n-th cochain group F^n is the abelian group of all functions

$$f : Nerve(X)[n] \to \bigcup_{g \in \mathrm{Mor}(X)} D_g$$

with $f(\lambda_1, \dots, \lambda_n) \in D_{\lambda_1 \circ \dots \circ \lambda_n}$ and $f(A) \in D_{1_A}$ for $n = 0$. The right hand side denotes the disjoint union of all abelian groups D_g with g a morphism in X. Addition in F^n is given by adding pointwise in the abelian groups D_g. The coboundary $\delta = \delta^{n-1} : F^{n-1} \to F^n$ is defined by the formula

$$(\delta f)(\lambda) = \lambda_* f(A) - \lambda^* f(B) \quad \text{for } \lambda : A \to B,\ n = 1,$$

$$\begin{aligned}(\delta f)(\lambda_1, \dots, \lambda_n) = {} & (\lambda_1)_* f(\lambda_2, \dots, \lambda_n) \\ & + \sum_{i=1}^{n-1} (-1)^i f(\lambda_1, \dots, \lambda_i \lambda_{i+1}, \dots, \lambda_n) \\ & + (-1)^n (\lambda_n)^* f(\lambda_1, \dots, \lambda_{n-1}).\end{aligned}$$

One can check that $\delta\delta = 0$ so that the *cohomology*

$$H^n(X, D) = \text{kernel}\,\delta^n/\text{image}\,\delta^{n-1}$$

is defined. Induced maps $(\phi^{op}, \tau)_* = \phi^*\tau_*$ for the functor H_n in (5.11) are given by

$$(\phi^*\tau_* f)(\lambda'_1, \ldots, \lambda'_n) = \tau_f \circ f(\phi\lambda'_1, \ldots, \phi\lambda'_n).$$

This completes the definition of the functor H^n in (5.11).

It is proved in Baues and Wirsching [10] that an equivalence of categories ϕ induces an isomorphism ϕ^* for cohomology groups as above. Moreover a crucial property of this cohomology is the next result:

5.14. THEOREM. *Let $M(X, D)$ be the set of equivalence classes of linear extensions $D \to E \to X$ where X is a small category. Then there is a natural bijection*

$$\phi : M(X, D) = H^2(X, D)$$

which carries the split extension to the trivial element.

If $X = G$ is a group this is the classical result on the classification of extensions of G. We define the bijection ϕ as follows. Let $s : \text{Mor}(X) \to \text{Mor}(E)$ be a function with $ps(f) = f$. For $(\lambda_1, \lambda_2) \in Nerve(X)[2]$ there is a unique element $c(\lambda_1, \lambda_2) \in D_{\lambda_1\lambda_2}$ satisfying

$$s(\lambda_1\lambda_2) = s(\lambda_1)s(\lambda_2) + c(\lambda_1, \lambda_2).$$

This defines a cocycle $c \in F^2$ which represents the cohomology class $\phi\{E\} = \{c\}$. By 'change of universe' we can define the cohomology above also in case X is not small so that 5.14 remains true.

As an example we now consider the linear extension 5.9 which represents a nontrivial cohomology class $\phi\{K_1^n\}$; in fact, the functors H^n, H^{n+1} and this cohomology class determine the category K_1^n up to equivalence. Pirashvili [74] computed the following restrictions of the class $\phi\{K_1^n\}$.

5.15. THEOREM. *Let π be a finite group and let $Gro(H^{n+1})_\pi$ be the subcategory of $Gro(H^{n+1})$ consisting of objects (π, M, k) and morphisms $(1_\pi, f)$. Moreover let K_π^n be the corresponding subcategory of K_1^n. Then one has the linear extension*

$$H^n \xrightarrow{+} K_\pi^n \to Gro\big(H^{n+1}\big)_\pi$$

which is a restriction of the linear extension 5.9. This extension represents the generator

$$\phi\{K_\pi^n\} \in H^2\big(Gro\big(H^{n+1}\big)_\pi, H^n\big) = \mathbb{Z}/|\pi|$$

where the right hand side is a cyclic group of order $|\pi|$ = number of elements of π. Moreover the cohomology groups

$$0 = H^i\big(Gro(H^{n+1})_\pi, H^n\big), \quad i \neq 2,$$

are trivial otherwise.

These examples may suffice to show that cohomology of groups and cohomology of categories are both important ingredients of the homotopy classification problem. Further applications of the cohomology of categories above can for example be found in Jibladze and Pirashvili [55], Dwyer and Kan [32], Moerdijk and Svensson [71], Pavešic [73]. Basic properties are described in Baues and Wirsching [10], Baues [3], Baues [4] and Baues and Dreckmann [7].

6. Simply connected homotopy types and $H\pi$-duality

Any group can be obtained as the fundamental group of a polyhedron. This yields a multifaceted relationship between homotopy theory and group theory. There are natural restrictions to avoid the full complexity of homotopy theory. For example, one can restrict to homotopy types which are determined by the fundamental group; such homotopy types are the *acyclic spaces* for which the universal covering space is contractible. Many basic examples in geometry deal with acyclic spaces in which the complexities of higher homotopy theory do not arise. From the point of view of homotopy theory acyclic spaces are extremely special since they are just 1-types or Eilenberg–MacLane spaces $K(G,1)$, $G \in Gr$.

In contrast to acyclic spaces it is natural to consider simply connected spaces which avoid the complexities of group theory arising from the fundamental group. Indeed, for spaces with fundamental group π one has to use the theory of group rings $\mathbb{Z}[\pi]$ and $\mathbb{Z}[\pi]$-modules which are highly intricate algebraic objects. For simply connected spaces only the ring $\mathbb{Z}$ and abelian groups are needed. From now on we deal with simply connected homotopy types.

An important feature of the theory of simply connected homotopy types is an *$H\pi$-duality* between homology groups and homotopy groups. Though the definitions of these groups are completely different in nature it turned out that they have many properties which are "dual" to each other. This kind of duality is different from Eckmann–Hilton duality discussed in Hilton [51]. We shall describe various examples of $H\pi$-dual properties though a complete axiomatic characterization is not known. The starting point is again the theorem of Whitehead which yields $H\pi$-dual properties as follows: A simply connected CW-space X is contractible if and only if its homology groups, or equivalently homotopy groups vanish so that

$$H_*(X) = 0 \Longleftrightarrow \pi_*(X) = 0. \tag{6.1}$$

Here H_* denotes the reduced homology. A map $f : X \to Y$ between simply connected CW-spaces is a homotopy equivalence if and only if f induces an isomorphism for homology groups, or equivalently homotopy groups, hence

$$H_*(f) \text{ is iso} \iff \pi_*(f) \text{ is iso.} \tag{6.2}$$

Moreover for any abelian group A and $n \geqslant 2$ there are simply connected CW-spaces X, Y with

$$\begin{cases} H_n(X) \cong A \text{ and } H_iX = 0 & \text{for } i \neq n, \\ \pi_n(Y) \cong A \text{ and } \pi_iX = 0 & \text{for } i \neq n. \end{cases} \tag{6.3}$$

The homotopy types of X, Y are well defined by (A, n) and $X = M(A, n)$ is called a *Moore space* and $Y = K(A, n)$ is called an *Eilenberg–MacLane space*. The next result shows that these spaces are important building blocks for simply connected homotopy types. First we observe by (6.3) the following realizability result. Let A_i be a sequence of abelian groups, $i \in \mathbb{Z}$, with $A_i = 0$ for $i \leqslant 1$. Then there exist simply connected CW-spaces X, Y with

$$\begin{cases} H_i(X) = A_i & \text{for } i \geqslant 0, \\ \pi_i(Y) = A_i & \text{for } i \geqslant 0. \end{cases} \tag{6.4}$$

For this we take

$$X = \bigvee_{n \geqslant 2} M(A_n, n)$$

to be the one point union of Moore spaces and we take

$$Y = \prod_{n \geqslant 2} K(A_n, n)$$

to be the product of Eilenberg–MacLane space (with the CW-topology). All simply connected homotopy types can be obtained by 'twisting' these constructions, see 6.7 below.

In the category *Top** of pointed spaces one has the notions of *fibration* and *cofibration* which are Eckmann–Hilton dual to each other. Compare, e.g., Baues [3]. We consider pull backs and push outs in *Top** respectively,

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull}"] & X \arrow[d, "a"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[dr, phantom, "\text{push}"] & X'' \\ Y \arrow[r, "g"'] \arrow[u, "b"] & Y'' \arrow[u] \end{tikzcd}$$

where a is a fibration and b is a cofibration. If X is contractible we call $X' \to Y' \to Y$ a *fiber sequence* and $Y \to Y'' \to X''$ a *cofiber sequence*. If also Y', Y'' are contractible we write

$$X' = \Omega(Y) = \textit{loop space of } Y,$$
$$X'' = \Sigma(Y) = \textit{suspension of } Y.$$

We have the $H\pi$-dual properties

$$\begin{cases} \Sigma M(A,n) = M(A,n+1), \\ \Omega K(A,n) = K(A,n-1) \end{cases} \tag{6.5}$$

of Moore spaces and Eilenberg–MacLane spaces respectively. Moreover if f and g are null-homotopic we get

$$X' \simeq Y' \times \Omega(Y),$$
$$X'' \simeq Y'' \vee \Sigma(Y),$$

where the right hand side is a product and a one point union respectively. If f and g are not null homotopic we consider X' and X'' as 'twisted' via f and g. Then f is called a *classifying map* for X' and g is called a *coclassifying map* for X''.

6.6. DEFINITION. Let $A_* = (A_n,\ n \geqslant 2)$ be a sequence of abelian groups. A *homotopy decomposition* associated to A_* is a system of fiber sequences $(n \geqslant 3)$

$$Y_n \to Y_{n-1} \xrightarrow{k_n} K(A_n, n+1)$$

with $Y_2 = K(A_2, 2)$. This implies that Y_n is an n-type and therefore k_n induces the trivial homomorphism on homotopy groups. A *homology decomposition* associated to A_* is a system of cofiber sequences $(n \geqslant 3)$

$$X_n \leftarrow X_{n-1} \xleftarrow{k'_n} M(A_n, n-1)$$

with $X_2 = M(A_2, 2)$ where k'_n is required to induce the trivial homomorphism on homology groups.

Homology and homotopy decompositions are $H\pi$-dual constructions for which the following classical result holds (due to Postnikov [76] and Eckmann and Hilton [34], Brown and Copeland [15]). Let $\varinjlim$ and $\varprojlim$ be the direct and inverse limits in *Top*.

6.7. THEOREM. *Let X be a simply connected CW-space. Then there exists a homology decomposition associated to H_*X and a map*

$$\varinjlim X_n \to X$$

which induces an isomorphism of homology groups. Moreover there exist a homotopy decomposition associated to $\pi_* X$ and a map

$$X \longrightarrow \varprojlim Y_n$$

which induces isomorphisms of homotopy groups.

Hence each simply connected homotopy type X can be constructed in two ways, either by a homology decomposition or by a homotopy decomposition. The space $Y_n \simeq P_n X$ may also be obtained by the Postnikov functor in (4.3). Using the Postnikov decomposition Schön [83] showed that an 'effective' classification of homotopy types of simply connected compact polyhedra is possible. The Whitehead theorem 6.2 and Theorem 6.7 somehow show that a simply connected homotopy type is 'generated' by homology groups and in a dual way also by homotopy groups. For this compare also the minimal models in 12.9, 12.11 below. Theorem 6.7, however, does not tell us how to compare two homology decompositions or two homotopy decompositions respectively, that is, we do not know under which condition two such decompositions represent the same homotopy type. For this one has to solve Whitehead's realization problem.

6.8. REMARK. Dwyer, Kan and Smith [33] construct for a graded abelian group A_* (with $A_i = 0$ for $i \leqslant 1$) a space $B(A_*)$ which parameterizes all homotopy decompositions associated to A_*. More precisely the set of path components, $\pi_0 B(AS_*)$, coincides with the set of all homotopy types X for which there exists an isomorphism $A_* \cong \pi_*(X)$. The fundamental group of the path component $B_X \subset B(A_*)$, corresponding to X, is the same as the group of homotopy equivalences $\pi_0 E(X)$ of X. In fact, the path component B_X has the homotopy type of the classifying space $B(E(X))$ where $E(X)$ is the topological monoid of homotopy equivalences of X, i.e. $B_X \simeq B(E(X))$.

We now consider the functorial properties of Moore spaces and Eilenberg–MacLane spaces respectively. Let Ab be the category of abelian groups and for $n \geqslant 2$ let

$$\mathit{K}^n,\ \mathit{M}^n \subset \mathit{Top}/\simeq \tag{6.9}$$

be the full homotopy categories consisting of spaces $K(A, n)$ and $M(A, n)$ respectively with $A \in \mathit{Ab}$.

6.10. LEMMA. *The n-th homotopy group functor*

$$\pi_n : \mathit{K}^n \xrightarrow{\sim} \mathit{Ab}$$

is an equivalence of categories. The n-th homology group functor

$$H_n : \mathit{M}^n \longrightarrow \mathit{Ab}$$

is not an equivalence but a detecting functor, see 5.6.

6.11. REMARK. In fact there is a functor $\mathit{Ab} \to \mathit{Top}$ which carries an abelian group A to a space $K(A, n)$. For this we observe that the classifying space $B(H)$ of an abelian topological monoid H is again an abelian topological monoid in a canonical way. Hence we can iterate the classifying space construction and obtain the n-fold classifying space

$$K(A,n) = B \dots B(A).$$

Compare Segal [81]. On the other hand there is no functor $\mathit{Ab} \to \mathit{Top}/\simeq$ which carries A to $M(A,n)$ and which is compatible with the homology H_n. For this we observe that there is actually a linear extension of categories

$$E^n \to M^n \to \mathit{Ab}$$

which represents a nontrivial class in $H^2(\mathit{Ab}, E^n)$, see Baues [3]. The bifunctor E^n on Ab is given by

$$E^n(A,B) = Ext(A, \Gamma_1^n B)$$

where $\Gamma_1^n B = B \oplus \mathbb{Z}/2$ for $n \geqslant 2$ and $\Gamma_1^2(B) = \Gamma(B)$ is the quadratic construction of J.H.C. Whitehead [102].

The lemma and the remark describe a lack of $H\pi$-duality. We shall describe many further examples of $H\pi$-dual properties; yet this duality does not cover all important features of homotopy groups and homology groups respectively. In particular homology is often computable while there is still no simply connected (noncontractible) finite polyhedron known for which all homotopy groups are computed. The homotopy groups $\pi_* M(A,n)$ of a Moore space are $H\pi$-dual to the homology groups $H_* K(A,n)$. If A is finitely generated it is a fundamental unsolved problem to compute $\pi_* M(A,n)$. The computation of $H_* K(A,n)$, however, was achieved in the work of Eilenberg and MacLane [36] and Cartan [23]. For example, we have

$$H_{n+2} K(A,n) = \pi_{n+1} M(A,n) = \Gamma_1^n(A) \tag{6.12}$$

where we use Γ_1^n as in 6.11. Recall that $[X,Y]$ denotes the set of homotopy classes of pointed maps $X \to Y$. The homology $H_* K(A,n)$ is used for the computation of the groups

$$[K(A,n),\ K(B,m)]$$

whose elements are also called *cohomology operations.* In particular the first nontrivial classifying map in a homotopy decomposition is such an operation. Applications of cohomology operations are discussed by Steenrod [90]. On the other hand the groups

$$[M(A,n),\ M(B,m)]$$

are not at all understood; for $A = B = \mathbb{Z}$ these are the homotopy groups of spheres.

A further lack of $H\pi$-duality is the following result on decompositions in 6.7.

6.13. PROPOSITION. *The homotopy decomposition of X can be chosen in Top to be functorial in X. The homology decomposition of X cannot be chosen to be functorial, neither in Top nor in the homotopy category $\mathit{Top}/\simeq$.*

Using Eilenberg–MacLane spaces and Moore spaces we obtain the groups ($n \geqslant 2$)

$$\begin{aligned} H^n(X,A) &= [X, K(A,n)], \\ \pi_n(A,X) &= [M(A,n), X] \end{aligned} \tag{6.14}$$

which are called the *cohomology group* of X, resp. the *homotopy group* of X in dimension n *with coefficients* in the abelian group A. Hence the decompositions of X in 6.7 yield elements

$$\begin{aligned} k_nX &= k_n \in H^{n+1}(P_{n-1}X, \pi_nX), \\ k'_nX &= k'_n \in \pi_{n-1}(H_nX, X_{n-1}). \end{aligned} \tag{6.15}$$

Here k_nX is actually an *invariant* of the homotopy type of X in the sense that a map $f : X \to Y$ satisfies

$$(P_{n-1}f)^* k_nY = (\pi_n f)_* k_nX \tag{6.16}$$

in $H^{n+1}(P_{n-1}X, \pi_nY)$. Here we use the Postnikov functor P_{n-1} and the naturality of the Postnikov decomposition in 6.13. The element k_nX in (6.15) is called the n-th *k-invariant* or *Postnikov invariant* of X. The element k'_nX given by a homology decomposition of X is not an invariant of X since the homotopy type of X_n is not well defined by the homotopy type of X. We shall describe below new invariants of X which are $H\pi$-dual to Postnikov invariants and which we call boundary invariants of X. They are given by the 'invariant portion' of the elements k'_nX; see 8.10 below.

The groups in (6.14) are part of natural short exact sequences which are $H\pi$-dual to each other:

$$\begin{aligned} &Ext(H_{n-1}X, A) \overset{\Delta}{\rightarrowtail} H^n(X,A) \overset{\mu}{\twoheadrightarrow} Hom(H_nX, A), \\ &Ext(A, \pi_{n+1}X) \overset{\Delta}{\rightarrowtail} \pi_n(A,X) \overset{\mu}{\twoheadrightarrow} Hom(A, \pi_nX). \end{aligned} \tag{6.17}$$

Here the surjection μ carries $\varphi : X \to K(A,n)$, resp. $\psi : M(A,n) \to X$, to the induced map

$$\begin{aligned} &H_n\varphi : H_nXC \to H_nK(A,n) = A, \quad \text{resp.} \\ &\pi_n\psi : A = \pi_nM(A,n) \to \pi_nX. \end{aligned}$$

The exact sequence for $H^n(X,A)$ is always split (unnaturally) while the exact sequence for $\pi_n(A,X)$ needs not to be split. We point out that the cohomology $H^n(X,A)$ may also be defined by

$$H^n(X,A) = [C_*X, C_*M(A,n)]. \tag{6.18}$$

Here C_* is the singular chain complex and the right hand side denotes the set of homotopy classes of chain maps. Dually we define the *pseudo-homology*

$$H_n(A,X) = [C_*M(A,n), C_*X] \tag{6.19}$$

which yields a well defined bifunctor $Ab^{op} \times Top \to Ab$. This is the analogue of $\pi_n(A,X)$ in the category of chain complexes. As in (6.17) one has the natural short exact sequence

$$Ext(A, H_{n+1}X) \overset{\Delta}{\rightarrowtail} H_n(A,X) \overset{\mu}{\twoheadrightarrow} Hom(A, H_nX) \tag{6.20}$$

which is always split (unnaturally).

7. The Hurewicz homomorphism

Homology groups and homotopy groups are connected by the Hurewicz homomorphism

$$h = h_n X : \pi_n X \to H_n X. \tag{7.1}$$

This is the special case $A = \mathbb{Z}$ of the homomorphism

$$h^A = h_n(A, X) : \pi_n(A, X) \to H_n(A, X)$$

which carries $\psi : M(A, n) \to X$ to the induced chain map $C_*\psi$. These homomorphisms are compatible with the short exact $\Delta - \mu$-sequences in (6.17) and 6.7, and they are natural in X and hence invariants of the homotopy type of X. In fact, the next result shows that the Hurewicz homomorphism has a strong impact on homotopy types.

7.2. PROPOSITION. *Let X be a simply connected CW-space. Then* (A) *and* (B) *hold.*

(A) *The Hurewicz homomorphism $h_n X$ is split injective for all n if and only if X has the homotopy type of a product of Eilenberg–MacLane spaces.*

(B) *Moreover $h_n X$ is split surjective for all n if and only if X has the homotopy type of a one point union of Moore spaces.*

Properties (A) and (B) form a further nice example of $H\pi$-duality.

PROOF. (A) Let r_n be a retraction of $h_n X$ and let $f_n \in H^n(X, \pi_n X)$ be a map with $\mu(f_n) = r_n$, see (6.17). Then the collection $\{f_n\}$ defines a map

$$f : X \longrightarrow \prod_{n \geqslant 2} K(\pi_n X, n)$$

which is a homotopy equivalence by the Whitehead theorem.

(B) Let s_n be a splitting of $h_n X$ and let $g_n \in \pi_n(H_n X, X)$ be a map with $\mu(g_n) = s_n$. Then the collection $\{g_n\}$ defines a map

$$g : \bigvee_{n \geqslant 2} M(H_n X, n) \longrightarrow X$$

which is a homotopy equivalence by the Whitehead theorem. □

We now discuss topological analogues of the Hurewicz homomorphism. We consider for a simply connected CW-complex X the *infinite symmetric product* $SP_\infty = \lim SP_n X$ where

$$SP_n X = X^n / S_n \tag{7.3}$$

is the space of orbits of the action of the symmetric group S_n on the n-fold product $X^n = X \times \cdots \times X$ obtained by permuting coordinates. The map $SP_{n-1} X \to SP_n X$ is induced by the inclusion $X^{n-1} = X^{n-1} \times * \subset X^n$ where $*$ is the base point of X. The inclusion

$$X = SP_1 X \to SP_\infty X$$

induces the Hurewicz homomorphism

$$h_n X : \pi_n X \to \pi_n SP_\infty X = H_n X$$

where the right hand side is the Dold–Thom isomorphism [30]. Let ΓX be the homotopy fiber of $X \subset SP_\infty X$ so that

$$\Gamma X \to X \to SP_\infty X \tag{7.4}$$

is a fiber sequence. Using the simplicial group GX there is an alternative way to obtain this fiber sequence by the short exact sequence

$$\Gamma_2 X \rightarrowtail GX \twoheadrightarrow AX \tag{7.4$'$}$$

where AX is the abelianization and where $\Gamma_2 X$ is the commutator subgroup of GX. Then $\Gamma X \simeq B|\Gamma_2 X|$ is the classifying space of the realization of $\Gamma_2 X$ and the functor $B|\ \ |$ applied to (7.4′) yields (7.4) up to homotopy equivalence. For this compare Kan [57] who proved that $GX \to AX$ induces the Hurewicz homomorphism. Using the skeleta X^n of a CW-complex J.H.C. Whitehead introduced the Γ-groups of X given by

$$\Gamma_n X = \operatorname{image}\big(\pi_n X^{n-1} \to \pi_n X^n\big) \tag{7.5}$$

where the homomorphism is induced by the inclusion $X^{n-1} \subset X^n$. Moreover we introduce in Baues [6] the Γ-groups with coefficients $\Gamma_n(A, X)$ by the following push-pull diagram derived from the $\Delta - \mu$-sequence 6.7.

$$\begin{tikzcd}
Ext(A,\Gamma_{n+1}X) \arrow[d, tail, "\Delta"'] \arrow[dr, phantom, "\text{push}"] & Ext(A,\pi_{n+1}X^n) \arrow[l, "p_*"'] \arrow[d, tail] \arrow[dr, "\Delta"] & \\
\Gamma_n(A,X) \arrow[dr, "\mu"'] & P \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull}"] & \pi_n(A;X^n) \arrow[d, "\mu"] \\
 & Hom(A,\Gamma_n X) \arrow[r, "i_*"'] & Hom(A,\pi_n X^n)
\end{tikzcd}$$

Here $i : \Gamma_n X \subset \pi_n X^n$ is the inclusion and $\varphi : \pi_{n+1} X^n \twoheadrightarrow \Gamma_{n+1} X$ is the projection defined by (7.5).

7.6. PROPOSITION. *Let X be a simply connected CW-complex. Then there are natural isomorphisms*
(a) $\Gamma_n X = \pi_n \Gamma X$,
(b) $\Gamma_n(A, X) = \pi_n(A, \Gamma X)$,
(c) $H_n X = \pi_n SP_\infty X$,
(d) $H_n(A, X) = \pi_n(A, SP_\infty X)$.
The isomorphisms which we shall use as identifications are compatible with $\Delta - \mu$ exact sequences above.

Here (a) and (c) are due to Kan [57] and Dold and Thom [30], respectively. Hence the long exact sequence of homotopy groups for the fiber sequence (7.4) yields by identification, as in 7.6, the exact sequences

$$\begin{gathered} \ldots \longrightarrow H_{n+1} X \xrightarrow{b} \Gamma_n X \xrightarrow{i} \pi_n X \xrightarrow{h} H_n X \xrightarrow{b} \ldots \text{ and} \\ \ldots \longrightarrow H_{n+1}(A, X) \xrightarrow{b^A} \Gamma_n(A, X) \xrightarrow{i^A} \pi_n(A, X) \xrightarrow{h^A} H_n(A, X) \xrightarrow{b^A} \ldots \end{gathered} \tag{7.7}$$

in which all operators are compatible with the $\Delta - \mu$ exact sequences. We call these the *Γ-sequence* and the *Γ-sequence with coefficients in A* respectively. Hence kernels and cokernels of the Hurewicz homomorphisms can be determined by the operators i, b in these sequences. Here i and i^A are induced by $X^n \subset X$ and b is the *secondary boundary operator* of J.H.C. Whitehead. In Baues [6] (II.3.5) we give also an explicit description of the operator b^A. The Γ-sequence coincides with the classical certain exact sequence of J.H.C. Whitehead which is the special case, $A = \mathbb{Z}$, of the second exact sequence. Clearly both exact sequences are invariants of the homotopy type of X. In fact, J.H.C. Whitehead [102] used part of the Γ-sequence as a classifying invariant of a simply connected 4-dimensional homotopy type.

The definition of $\Gamma_n X$ in (7.5) shows that this group depends only on the $(n-1)$-type of X, in fact we have the natural isomorphism

$$\Gamma_k(X) = \Gamma_k(P_{n-1}X), \quad k \leqslant n, \tag{7.8}$$

induced by a map $p_{n-1} : X \to P_{n-1}X$ which extends the inclusion $X^n \subset P_{n-1}X$, see (4.3). Moreover the map p_{n-1} applied to the Γ-sequence of X yields natural isomorphisms

$$\begin{aligned} H_n P_{n-1} X &= \Gamma''_{n-1} P_{n-1} X = \Gamma''_{n-1}(X), \\ H_{n+1} P_{n-1} X &= \Gamma_n P_{n-1} X = \Gamma_n(X) \end{aligned} \tag{7.9}$$

where $\Gamma''_{n-1} X = \text{kernel}(\Gamma_{n-1} X \to \pi_{n-1} X)$. These groups are used in the following result on the '*realizability of Hurewicz homomorphisms*', proved in III.4.7 of Baues [6].

7.10. THEOREM. *Let Y be a simply connected $(n-1)$-type and let*

$$H_1 \longrightarrow \Gamma_n Y \longrightarrow \pi \longrightarrow H_0 \longrightarrow \Gamma''_{n-1} Y \longrightarrow 0 \tag{$*$}$$

be an exact sequence of abelian groups where H_1 is free abelian. Then there exists an $(n+1)$-dimensional complex X and a map $p : X \to Y$ inducing isomorphisms $\pi_k X \cong \pi_k Y$ for $k \leqslant n-1$ together with a commutative diagram

$$\begin{tikzcd}
H_{n+1}X \arrow[r] \arrow[d, "\cong"'] & \Gamma_n X \arrow[r] \arrow[d, "\cong"', "p_*"] & \pi_n X \arrow[r, "h"] \arrow[d, "\cong"'] & H_n X \arrow[r] \arrow[d, "\cong"'] & \Gamma''_{n-1} X \arrow[r] \arrow[d, "\cong"', "p_*"] & 0 \\
H_1 \arrow[r] & \Gamma_n Y \arrow[r] & \pi \arrow[r] & H_0 \arrow[r] & \Gamma''_{n-1} Y \arrow[r] & 0
\end{tikzcd}$$

in which all vertical arrows are isomorphisms. The top row is part of the Γ-sequence of X. The space Y together with the sequence $(*)$ in general does not determine the homotopy type of X.

The result shows exactly what kind of abstract homomorphisms $\pi \to H_0$ between abelian groups can be realized as a Hurewicz homomorphism $\pi_n \to H_n$ of a space with a given $(n-1)$-type. It also demonstrates to what extent homotopy groups and homology groups depend on each other.

7.11. Example. We may choose for Y in 7.10 an Eilenberg–MacLane space

$$Y = K(A,k) \quad \text{with } 2 \leqslant k \leqslant n-1.$$

Then the groups, see (7.9),

$$\begin{aligned} \Gamma_n Y &= H_{n+1}K(A,k), \\ \Gamma''_{n-1} Y &= H_n K(A,k) \end{aligned}$$

are known by the work of Eilenberg, MacLane and Cartan. Hence any exact sequence

$$H_1 \longrightarrow H_{n+1}K(A,k) \longrightarrow \pi \longrightarrow H_0 \longrightarrow H_n K(A,k) \longrightarrow 0$$

with H_1 free abelian can be realized as a Γ-sequence of an $(n+1)$-dimensional CW-complex X with $P_{n-1}X = K(A,k)$. For example, for $k=5$, $n=9$ we have

$$\begin{aligned} H_{10}K(A,5) &= \Lambda^2(A) \oplus A * \mathbb{Z}/6, \\ H_9 K(A,5) &= A \otimes \mathbb{Z}/6, \end{aligned}$$

where Λ^2 is the exterior square and $\mathbb{Z}/6$ is the cyclic group of order 6. Hence for any exact sequence

$$H_{10} \longrightarrow \Lambda^2(A) \oplus A * \mathbb{Z}/6 \longrightarrow \pi_9 \longrightarrow H_9 \longrightarrow A \otimes \mathbb{Z}/6 \longrightarrow 0$$

of abelian groups with H_{10} free abelian there exists a 10-dimensional CW-complex X with $\pi_5 X = A$, $\pi_i = 0$ for $i < 5$ and $5 < i < 9$, such that this sequence is part of the Γ-sequence of X.

8. Postnikov invariants and boundary invariants

Recall that the n-th Postnikov invariant of a simply connected space X is an element

$$k_n X \in H^{n+1}(P_{n-1}X, \pi_n X). \tag{8.1}$$

This element is highly related to the Γ-sequence of X. For this we observe that by (6.17) and (7.9) we obtain the natural short exact sequence

$$Ext(\Gamma''_{n-1}X, A) \overset{\Delta}{\rightarrowtail} H^{n+1}(P_{n-1}X, A) \overset{\mu}{\twoheadrightarrow} Hom(\Gamma_n X, A). \tag{8.2}$$

Each element $k \in H^{n+1}(P_{n-1}X, A)$ yields elements

$$k_* = \mu(k) \in Hom(\Gamma_n X, A),$$
$$k_\dagger = \Delta^{-1} q_*(k) \in Ext(\Gamma''_{n-1}X, cok\, k_*),$$

where $q : A \twoheadrightarrow cok(k_*)$ is the projection of the cokernel of k_*. A bifunctor in X and A is given by $(X, A) \mapsto H^{n+1}(P_{n-1}X, A)$.

8.3. THEOREM ON POSTNIKOV INVARIANTS. *To each 1-connected CW-space X there is canonically associated a sequence of elements $(k_3, k_4, \dots)$ with*

$$k_n = k_n X \in H^{n+1}(P_{n-1}X, \pi_n X)$$

such that the following properties are satisfied:

(a) Naturality: *For a map $F : X \to Y$ we have*

$$(\pi_n F)_*(k_n X) = F^*(k_n Y) \in H^{n+1}(P_{n-1}X, \pi_n Y).$$

(b) Compatibility with $i_n X$ in the Γ-sequence:

$$(k_n X)_* = i_n X \in Hom(\Gamma_n X, \pi_n X).$$

(c) Compatibility with the extension $H_n X$ in the Γ-sequence:

$$(k_n X)_\dagger = \{H_n X\} \in Ext(\Gamma''_{n-1}X, cok\, i_n X)$$

Here the extension element $\{H_n X\}$ is given by the exact Γ-sequence of X,

$$\Gamma_n X \xrightarrow{i_n X} \pi_n X \to H_n X \to \Gamma''_{n-1}X \to 0.$$

(d) Vanishing condition: *All Postnikov invariants are trivial if and only if X has the homotopy type of a product of Eilenberg–MacLane spaces.*

This result which partially seems to be unknown is proved in II.5.10 of Baues [6]. We now introduce new invariants which are $H\pi$-dual to the Postnikov invariants above. For this we first define the subgroup

$$\Gamma''_{n-1}(A, X) \subset \Gamma_{n-1}(A, X) \tag{8.4}$$

obtained by all elements $\alpha \in \Gamma_{n-1}(A, X)$ for which $\mu(\alpha)(A) \subset \Gamma''_{n-1}X$. Hence one has the short exact sequence

$$Ext(A, \Gamma_n X) \overset{\Delta}{\rightarrowtail} \Gamma''_{n-1}(A, X) \overset{\mu}{\twoheadrightarrow} Hom(A, \Gamma''_{n-1}X). \tag{8.5}$$

Here Γ''_{n-1} is actually a bifunctor in $A \in Ab$ and $X \in Top^* / \simeq$. To see this we observe that the map $p_{n-1} : X \to P_{n-1}X$ induces a binatural isomorphism

$$\Gamma''_{n-1}(A, X) = H_n(A, P_{n-1}X). \tag{8.6}$$

Here the right hand side is the pseudo homology and we use the Γ-sequence with coefficients in A and (7.9). Since $b_n X : H_n X \to \Gamma_{n-1}X$ yields a surjection $b_n X : H_n X \twoheadrightarrow \Gamma''_{n-1}X$ we see that the boundary operator b^A in the Γ-sequence with coefficients maps to $\Gamma''_{n-1}(A, X)$. Hence we obtain the following commutative diagram which is natural in $A \in Ab$ and simply connected spaces X.

$$\begin{array}{ccccc}
Ext(A, H_{n+1}X) & \xrightarrow{\Delta} & H_n(A, X) & \xrightarrow{\mu} & Hom(A, H_n X) \\
\downarrow{\scriptstyle (b_{n+1}X)_*} & & \downarrow{\scriptstyle b^A} & & \downarrow{\scriptstyle (b_n X)_*} \\
Ext(A, \Gamma_n X) & \xrightarrow{\Delta} & \Gamma''_{n-1}(A, X) & \longrightarrow & Hom(A, \Gamma''_{n-1}X)
\end{array}$$

8.7. DEFINITION. Consider this diagram for $A = H_n X$ and let $1_n \in H_n(H_n X, X)$ be an element with $\mu(1_n) = $ identity of $H_n X$. Then the coset of $b^A(1_n)$ modulo the image of $\Delta(b_{n+1}X)_*$ is the *boundary invariant* $\beta_n X$ of X, that is

$$\beta_n X = \{b^A(1_n)\} \in \frac{\Gamma''_{n-1}(H_n X, X)}{im(\Delta(b_{n+1}X)_*)}.$$

We have the short exact sequence

$$Ext(A, cok\, b_{n+1}X) \overset{\Delta}{\rightarrowtail} \frac{\Gamma''_{n-1}(A, X)}{im(\Delta(b_{n+1}X)_*)} \xrightarrow{\mu} Hom(A, \Gamma''_{n-1}X) \tag{8.8}$$

which is natural in $A \in Ab$ and simply connected spaces X. This sequence is the $H\pi$-dual of the sequence in (8.2) above. Each element

$$\beta \in \frac{\Gamma''_{n-1}(A, X)}{im(\Delta(b_{n+1}X)_*)}$$

yields elements

$$\beta_* = \mu(\beta) \in Hom(A, \Gamma''_{n-1}X),$$
$$\beta_\dagger = \Delta^{-1} j^*(\beta) \in Ext(ker\, \beta_*, cok\, b_{n+1}X)$$

where $j : ker\, \beta_* \subset A$ is the inclusion of the kernel of β_*. The next result is the $H\pi$-dual of the 'theorem on Postnikov invariants' in 8.3.

8.9. THEOREM ON BOUNDARY INVARIANTS. *To each* 1*-connected CW-space* X *there is canonically associated a sequence of elements* $(\beta_3, \beta_4, \dots)$ *with*

$$\beta_n = \beta_n X \in \frac{\Gamma''_{n-1}(H_n X, X)}{im\, \Delta(b_{n+1}X)_*}$$

such that the following properties are satisfied:

(a) Naturality: *For a map* $F : X \to Y$ *we have*

$$(H_n F)^*(\beta_n Y) = F_*(\beta_n X) \in \frac{\Gamma''_{n-1}(H_n X, Y)}{im\, \Delta(b_{n+1}X)_*}.$$

(b) Compatibility with $b_n X$ in the Γ-sequence:

$$(\beta_n X)_* = b_n X \in Hom(H_n X, \Gamma''_{n-1}X).$$

(c) Compatibility with the extension $\pi_n X$ in the Γ-sequence:

$$(\beta_n X)_\dagger = \{\pi_n X\} \in Ext(ker\, b_n X, cok\, b_{n+1}X).$$

Here the extension element $\{\pi_n X\}$ *is determined by the exact* Γ*-sequence of* X,

$$H_{n+1} \xrightarrow{b_{n+1}X} \Gamma_n X \to \pi_n X \to H_n X \xrightarrow{b_n X} \Gamma''_{n-1}X.$$

(d) Vanishing condition: *All boundary invariants are trivial if and only if* X *has the homotopy type of a one point union of Moore spaces.*

This result is proved in II.6.9 of Baues [6].

8.10. REMARK. The boundary invariants have the following connection with the coclassifying maps k'_n in a homology decomposition of X. For this let $X = \lim X_n$ be given by a homology decomposition. Then X is a CW-complex with skeleta X^n and there are inclusions

$$X^{n-1} \subset X_n \subset X^n.$$

Moreover the classifying map k'_n can be chosen such that the following diagram commutes, $H_n = H_n X$.

$$\begin{tikzcd}
 & X_{n-1} \\
M(H_n, n-1) \arrow[ur, "k'_n"] \arrow[r, "\beta"] & X^{n-1} \arrow[u] \\
M(H_n, n-1)^{n-2} \arrow[u] \arrow[r] & X^{n-2} \arrow[u]
\end{tikzcd}$$

Hence β represents an element in $\Gamma_{n-1}(H_n, X)$ (using the definition in (7.5)) and this element represents the boundary invariant $\beta_n X$. Therefore 8.9(c) yields an explicit formula for deriving $\pi_n X$ from k'_n.

8.11. REMARK. Let C be a homotopy category of simply connected spaces. Then we have the functors

$$\Gamma_n, \Gamma''_{n-1} : C \to Ab \tag{1}$$

which both appear in a dual fashion in the natural exact sequences (8.2) and (8.5). There is an obstruction $\mathcal{O}$ for the existence of a splitting of (8.2) which is natural in $X \in C$ and $A \in Ab$. This obstruction is an element in the cohomology of C,

$$\mathcal{O} \in H^1\big(C, Ext(\Gamma''_{n-1}, \Gamma_m)\big). \tag{2}$$

Here $Ext(\Gamma''_{n-1}, \Gamma_n)$ is the natural system which carries $f : X \to Y$ in C to the abelian group $Ext(\Gamma''_{n-1}X, \Gamma_n Y)$. The element $\mathcal{O}$ determines the extension (8.2) as a bifunctor in $X \in C$, $A \in Ab$ up to equivalence. On the other hand there is an obstruction $\mathcal{O}'$ for the existence of a splitting of (8.5) which is natural in $X \in C$ and $A \in Ab$. This obstruction also turns out to be an element in the cohomology (2),

$$\mathcal{O}' \in H^1\big(C, Ext(\Gamma''_{n-1}, \Gamma_n)\big). \tag{3}$$

Again $\mathcal{O}'$ determines the extension (8.5) as a functor in $X \in C$, $A \in Ab$ up to equivalence. Now the extension (8.2) and (8.5) are dual in the explicit sense that the elements (2) and (3) actually coincide; that is $\mathcal{O} = \mathcal{O}'$. This is proved in III. § 3 of Baues [6]. In the next section we use the extensions (8.2) and (8.5) in a crucial way to obtain models of homotopy types.

9. The classification theorems

We now show that k-invariants and boundary invariants both can be used to classify homotopy types. For this we choose a full subcategory

$$C \subset (n-1)\text{-}types \tag{9.1}$$

consisting of simply connected $(n-1)$-types. For example we can take for $1 < k < n$ the category $C = K^k \cong Ab$ consisting of Eilenberg–MacLane spaces $K(A,k)$ with $A \in Ab$. We consider the functor

$$P_n : spaces^{n+1}(C) \to n\text{-}types(C) \tag{9.2}$$

where the left hand side is the full homotopy category of $(n+1)$-dimensional CW-spaces U for which the $(n-1)$-Postnikov section $P_{n-1}U$ is in C, similarly the right hand side is the full homotopy category of n-types V for which $P_{n-1}V$ is in C. The functor P_n is the restriction of the Postnikov functor in (4.3). In the next definition we use the new word 'kype' which is an amalgamation of k-invariant and type.

9.3. DEFINITION. Let C be a category as in (9.1). A *C-kype*

$$\overline{X} = (X, \pi, k, H, b)$$

is a tuple consisting of an object X in C, abelian groups π, H and elements

$$k \in H^{n+1}(X,\pi), \quad b \in Hom(H, \Gamma_n X)$$

such that the sequence

$$H \xrightarrow{b} \Gamma_n X \xrightarrow{k_*} \pi$$

is exact, see (8.2). A *morphism* between C-kypes

$$(f,\varphi,\psi) : (X,\pi,k,H,b) \to (X',\pi',k',H',b')$$

is given by a map $f : X \to X'$ in C and homomorphisms $\varphi : \pi \to \pi'$, $\psi : H \to H'$ between abelian groups such that

$$f^*(k') = \varphi_*(k),$$
$$(\Gamma_n f)b = b'\psi.$$

The C-kype $\overline{X}$ is *free*, resp. *injective*, if H is free abelian, resp. b is an injective homomorphism. Let $\mathit{Kypes}(C)$, resp. $\mathit{kypes}(C)$ be the categories of free, resp. injective C-kypes with morphisms as above. We have the forgetful functor

$$\phi : \mathit{Kypes}(C) \to \mathit{kypes}(C)$$

which carries (X,π,k,H,b) to $(X,\pi,k.H',b')$ where H' is the image of b and where b' is the inclusion of this image. The functor ϕ is easily seen to be full and representative.

Recall that a 'detecting' functor is a functor which reflects isomorphisms and is full and representative.

9.4. CLASSIFICATION BY POSTNIKOV INVARIANTS. *There are detecting functors Λ, λ for which the following diagram of functors commutes up to natural isomorphism.*

$$\begin{tikzcd} \mathit{spaces}^{n+1}(C) \arrow[r, "\Lambda"] \arrow[d, "P_n"] & \mathit{Kypes}(C) \arrow[d, "\phi"] \\ n\text{-}\mathit{types}(C) \arrow[r, "\lambda"] & \mathit{kypes}(C) \end{tikzcd}$$

Here the functor Λ carries the space X to the free C-kype

$$\Lambda(X) = (P_{n-1}X, \pi_n X, k_n X, H_{n+1}X, b_{n+1}X) \tag{9.5}$$

given by the Postnikov invariant (8.1), see 8.3. We point out that only the detecting functor λ is a classical result of Postnikov, the existence of the detecting functor Λ seems to be a new property of k-invariants which did not appear in the literature. Theorem 9.4 is proved in III.4.4 of Baues [6].

Using boundary invariants we obtain the $H\pi$-dual of the classification theorem above. We are now going to use a new word 'bype' which is an amalgamation of boundary invariant and type.

9.6. DEFINITION. Let C be a category as in (9.1). A *C-bype*

$$\overline{X} = (X, H_0, H_1, b, \beta)$$

is a tuple consisting of an object X in C, abelian groups H_0, H_1 and elements

$$b \in Hom(H_1, \Gamma_n X),$$
$$\beta \in \frac{\Gamma''_{n-1}(H_0, X)}{im(\Delta b_*)}.$$

Here we use Δ in (8.5) and $b_* : Ext(H_0, H_1) \to Ext(H_0, \Gamma_n X)$. Moreover the induced homomorphism

$$\beta_* = \mu(\beta) : H_0 \twoheadrightarrow \Gamma''_{n-1} X$$

is surjective, see (8.5). A *morphism* between C-bypes

$$(f, \varphi_0, \varphi_1) : (X, H_0, H_1, b, \beta) \to (X', H'_0, H'_1, b', \beta')$$

is given by a morphism $f : X \to X'$ in C and by homomorphisms $\varphi_0 : H_0 \to H'_0$, $\varphi_1 : H_1 \to H'_1$ such that

$$(\Gamma_n f) b = b' \varphi_0,$$
$$f_*(\beta) = \varphi_0^*(\beta').$$

The C-bype $\overline{X}$ is *free*, resp. *injective*, if H_1 is a free abelian group, resp. b is an injective homomorphism. Let $Bypes(C)$, resp. $bypes(C)$, be the categories of free, resp. injective, C-bypes with morphisms as above. We have the forgetful functor

$$\phi : Bypes(C) \to bypes(C)$$

which carries (X, H_0, H_1, b, β) to $(H, H_0, H'_0, b', \beta)$ where H'_1 is the image of b and where b' is the inclusion of this image. The functor ϕ is full and representative.

9.7. CLASSIFICATION BY BOUNDARY INVARIANTS. *There are detecting functors Λ', λ' for which the following diagram of functors commutes up to natural isomorphism.*

$$\begin{array}{ccc} spaces^{n+1}(C) & \xrightarrow{\Lambda'} & Bypes(C) \\ \downarrow{\scriptstyle P_n} & & \downarrow{\scriptstyle \phi} \\ n\text{-}types(C) & \xrightarrow{\lambda'} & bypes(C) \end{array}$$

Here the functor Λ' carries the space U to the free C-bype

$$\Lambda'(U) = (P_{n-1}X, H_nX, H_{n+1}X, b_{n+1}X, \beta_nX) \tag{9.8}$$

given by the boundary invariant β_nX in 8.7, see 8.9. The classification theorem 9.7 is proved in III.4.4 of Baues [6]. It shows that boundary invariants can be used in the same way as Postnikov invariants for the classification of homotopy types. In the book Baues [6] we give many explicit examples of applications for the classification theorems above.

9.9. REMARK. J.H.C. Whitehead [102] obtained for the homotopy category of simply connected 4-dimensional CW-spaces two detecting functors. These coincide exactly with Λ and Λ' above if we take $n = 3$ and $C = K^2$. This is, in fact, a very simple case of the classification theorems above for which we use

$$\Gamma_3K(A,2) = \Gamma(A),$$
$$\Gamma_2''K(A,2) = 0.$$

We leave this as an exercise to the reader, see also 10.8 below. In Baues [6] we use 9.7 for the classification of simply connected 5-dimensional homotopy types.

10. Stable homotopy types

The *suspension* Σ is an endofunctor of the homotopy category $\mathit{Top}^*/\simeq$ given by the quotient space

$$\Sigma X = I \times X/(\{0\} \times X \cup I \times * \cup \{1\} \times X) \tag{10.1}$$

where $I = [0,1]$ is the unit interval, see also (6.5). The functor Σ carries a map $f : X \to Y$ to $\Sigma f : \Sigma X \to \Sigma Y$ with $(\Sigma f)(t,x) = (t, fx)$ for $t \in I$, $x \in X$. It is easy to see that Σ carries homotopic maps to homotopic maps. We say that two finite dimensional CW-complexes X, Y are *stably homotopy equivalent* if there is $k \geqslant 0$ and a homotopy equivalence $\Sigma^kX \simeq \Sigma^kY$ where Σ^k is the k-fold suspension. A *stable homotopy type* is a class of stably homotopy equivalent CW-complexes. Since the classification of homotopy types is so hard Spanier and Whitehead [88] supposed that stable homotopy types might give a first approximation of the homotopy classification problem which is easier to understand. For this the 'stable homotopy theory of spectra' was invented which, however, turned out to be still an extremely complicated world, see G.W. Whitehead [96].

The impact of the suspension operator Σ comes from a classical result of Freudenthal which we state in the following form.

10.2. FREUDENTHAL SUSPENSION THEOREM. *Let* spaces_n^k *be the full homotopy category in* $\mathit{Top}^*/\simeq$ *consisting of* $(n-1)$*-connected* $(n+k)$*-dimensional CW-complexes,* $n \geqslant 1$, $k \geqslant 0$. *Then the suspension yields a functor*

$$\Sigma : \mathit{spaces}_n^k \to \mathit{spaces}_{n+1}^k$$

which is an equivalence of (additive) categories for $k+1<n$ and which is a detecting functor for $k+1=n$. Moreover for $k=n$ this functor is representative.

Compare, for example, Gray [44].

For $n \geqslant 2$ the functor Σ in the theorem reflects isomorphisms. This follows from the Whitehead theorem (6.2) since the (reduced) homology of a suspension satisfies

$$H_n \Sigma X = H_{n-1} X \quad \text{for all } n. \tag{10.3}$$

As pointed out in Section 3 the main numerical invariants of a homotopy type are dimension and degree of connectedness. These invariants are of particular importance in the theory of manifolds. Therefore it is natural to consider for given n, k the properties of $(n-1)$-connected $(n+k)$-dimensional CW-complexes which J.H.C. Whitehead [98] called *A_n^k-polyhedra*. The A_n^k-polyhedra, $n \geqslant 1$, are the objects in the homotopy categories of the sequence

$$\mathit{spaces}_1^k \xrightarrow{\Sigma} \mathit{spaces}_2^k \to \cdots \mathit{spaces}_n^k \xrightarrow{\Sigma} \mathit{spaces}_{n+1}^k \to \tag{10.4}$$

which by Freudenthal's theorem above 'stabilizes' for $n \geqslant k+2$. Hence there are only $k+2$ different categories in this sequence. This also shows that the stable homotopy types of A_n^k-polyhedra ($n \geqslant 0$) can be identified with the homotopy types in the category spaces_n^k, $n \geqslant k+1$. We say that A_n^k-polyhedra are *stable* if $n \geqslant k+1$.

Each homotopy type of an A_n^k-polyhedron can be represented by a (reduced) CW-complex X with $X^{n-1} = *$ and $\mathit{dim}(X) = n+k$. Hence $X - \{*\}$ has only cells in dimension n, $n+1, \ldots, n+k$. For $k=0$ the CW-complex X is thus a one point union of n-spheres. This also shows that one has equivalences of categories

$$\begin{aligned} &\mathit{spaces}_1^0 = \text{category of free groups}, \\ &\mathit{spaces}_2^0 = \text{category of free abelian groups} \end{aligned} \tag{10.5}$$

where Σ coincides with the abelianization functor for groups. For $k>0$ the algebraic models of the categories in (10.4) get more complicated. J.H.C. Whitehead [102], [98], [100] studied the case $k=2$ and we study the case $k=3$ in Baues [4], [6], see 10.8 and 10.11 below. Moreover Unsöld [95] considers for $k=4$, $n \geqslant 3$ the subcategory of spaces_n^4 consisting of CW-complexes with finitely generated torsion free homology. It would be unreasonable to try and extend these calculations for large values of k. It will, however, increase our knowledge on the nature of homotopy types considerably if we are able to discuss in detail homotopy types of A_n^k-polyhedra for small k, say $k \leqslant 5$. This for example includes, for $n=2$, simply connected 7-dimensional homotopy types.

10.6. REMARK. M.J. Hopkins [53] discusses new global methods to study stable homotopy types. For this a fundamental filtration of the stable homotopy category $\mathcal{C}_0$ of 'p-local finite spectra':

$$\mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n \supset \mathcal{C}_{n+1} \supset \cdots$$

is considered where $\mathcal{C}_n$ contains all objects which are acyclic with respect to the Morawa K-theory $K(n-1)$. The classical dimension filtration of the stable homotopy category, coming from the sequence (10.4), is more related to problems like the classification of manifolds in a particular dimension. J.H.C. Whitehead [102] obtained the following algebraic models of stable A_n^2-polyhedra, $n \geqslant 3$.

10.7. DEFINITION. An A^2*-system*

$$S = (H_0, H_2, \pi_1, b_2, \eta)$$

is a tuple consisting of abelian groups H_0, H_2, π_1 and elements

$$b_2 \in Hom(H_2, H_0 \otimes \mathbb{Z}/2),$$
$$\eta \in Hom(H_0 \otimes \mathbb{Z}/2, \pi_1)$$

such that the sequence

$$H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_1$$

is exact. A morphism

$$(\varphi_0, \varphi_2, \varphi_{pi}) : (H_0, H_2, \pi_1, b_2, \eta) \to \left(H_0', H_2', \pi_1', b_2', \eta'\right)$$

is given by homomorphisms $\varphi_i : H_i \to H_i'$ for $i = 0, 2$ and $\varphi_\pi : \pi_1 \to \pi_1'$ such that the diagram

$$\begin{tikzcd}
H_2 \arrow[r, "b_2"] \arrow[d, "\varphi_2"'] & H_0 \otimes \mathbb{Z}/2 \arrow[r, "\eta"] \arrow[d, "\varphi_0 \otimes \mathbb{Z}/2"] & \pi_1 \arrow[d, "\varphi_\pi"] \\
H_2' \arrow[r, "b_2'"] & H_0' \otimes \mathbb{Z}/2 \arrow[r, "\eta'"] & \pi_1'
\end{tikzcd}$$

commutes. The A^2-system S is free, resp. injective, if H_2 is free abelian resp. b_2 is injective. Let A^2*-Systems*, resp. A^2*-systems* be the categories of free, resp. injective A^2-systems with morphisms as above. We have a forgetful functor

$$\phi : A^2\text{-}\mathit{Systems} \to A^2\text{-}\mathit{systems}$$

which carries $(H_0, H_2, \pi_1, b_2, \eta)$ to $(H_0, H_2', \pi_1, b_2', \eta)$ where H_2' is the image of b_2 and b_2' is the inclusion of this image.

Let types_n^k be the full homotopy category of $(n-1)$-connected $(n+k)$-types and let

$$P_n^k : \mathit{spaces}_n^k \to \mathit{types}_n^{k-1}$$

be the restriction of the Postnikov functor.

10.8. CLASSIFICATION OF J.H.C. WHITEHEAD. *For $n \geqslant 3$ there exist detecting functors Λ, λ for which the following diagram of functors commutes up to natural isomorphism.*

$$\begin{array}{ccc} \mathit{spaces}_n^2 & \xrightarrow{\Lambda} & A^2\text{-}\mathit{Systems} \\ {\scriptstyle P_n^2}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathit{types}_n^1 & \xrightarrow{\lambda} & A^2\text{-}\mathit{systems} \end{array}$$

This result is an easy application of 9.4, compare 9.9 and (6.12). The functor Λ carries a space X to part of the Γ-sequence of X,

$$H_{n+2}X \xrightarrow{b} \Gamma_{n+1}X \xrightarrow{\eta} \pi_{n+1}X$$

where $\Gamma_{n+1}X = H_nX \otimes \mathbb{Z}/2$. Here η can be identified with the Postnikov invariant $\eta = k_{n+1}X$.

Next we describe algebraic models of stable A_n^3-polyhedra, $n \geqslant 4$. For this let $\mathbb{Z}/2$ be the cyclic group of two elements and let

$$Hom(\otimes\mathbb{Z}/2, -) : Ab^{op} \times Ab \to Ab$$

be the functor which carries H, L to $Hom(H \otimes \mathbb{Z}/2, L)$. Moreover let

$$Gro\big(Hom(\otimes\mathbb{Z}/2, -)\big) \tag{10.9}$$

be the Grothendieck construction of this functor. Objects in the category (10.9) are triples $\eta = (H, L, \eta)$ with $\eta \in Hom(H \otimes \mathbb{Z}/2, L)$ and morphisms $(\psi_1, \psi_0) : \eta \to \eta'$ are homomorphisms $\psi_1 : L \to L'$, $\psi_0 : H \to H'$ with $\psi_1\eta = \eta'(\psi_0 \otimes \mathbb{Z}/2)$. We point out that there is an obvious equivalence of categories

$$Gro\big(Hom(\otimes\mathbb{Z}/2, -)\big) \xrightarrow{i} A^2\text{-}\mathit{systems}. \tag{1}$$

For each abelian group A we have the short exact sequence

$$A \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(A) \overset{\mu}{\twoheadrightarrow} A * \mathbb{Z}/2 \tag{2}$$

associated to the natural homomorphisms

$$\tau_A : A * \mathbb{Z}/2 = \{x \in A,\ 2x = 0\} \subset A \twoheadrightarrow A/2A = A \otimes \mathbb{Z}/2.$$

The abelian extension (2) is determined up to equivalence by $\Delta^{-1}(2\mu^{-1}(x)) = \tau_A(x)$ for $x \in A * \mathbb{Z}/2$. Let

$$G \subset G' \subset Gro\big(Hom(\otimes\mathbb{Z}/2, -)\big) \tag{3}$$

be the following subcategories. Objects in G are the triple $A = (A, G(A), \Delta)$ given by (2) and morphisms are pairs $(\varphi, \overline{\varphi})$ which are compatible with (2), that is $\mu\overline{\varphi} = (\varphi * \mathbb{Z}/2)\mu$. There is a full forgetful functor

$$G \to Ab \tag{4}$$

which carries $(A, G(A), \Delta)$ to A and there is an equivalence $G = M^n$, $n \geqslant 3$, where M^n is the homotopy category of Moore spaces in degree n; see (6.9). Moreover G' in (3) is the full subcategory consisting of objects $\eta = (H, L, \eta)$ for which there exists a factorization $\eta : H \otimes \mathbb{Z}/2 \overset{\Delta}{\rightarrowtail} G(H) \to L$. We shall need the group $G(\eta)$ defined by the push out diagram

$$\begin{tikzcd}
L \otimes \mathbb{Z}/2 \arrow[r, "\Delta"] & G(\eta) \arrow[r, "\mu"] & H * \mathbb{Z}/2 \\
H \otimes \mathbb{Z}/2 \arrow[u, "\eta\otimes 1"] \arrow[r] & G(H) \arrow[u] \arrow[r] & H * \mathbb{Z}/2 \arrow[u, equal]
\end{tikzcd} \tag{5}$$

Moreover we shall use a canonical bifunctor

$$\overline{G} : G^{op} \times G' \to Ab \tag{6}$$

which carries the pair of objects (A, η) to an abelian group $\overline{G}(A, \eta)$. Here we only define this group if A or H is finitely generated; for a complete definition of $\overline{G}$ see VIII.1.3 (B) in Baues [6]. Using (2) we have the dual extension

$$\begin{tikzcd}
Ext(A, \mathbb{Z}/2) \arrow[r, tail, "\Delta"] \arrow[d, equal] & Hom(G(A), \mathbb{Z}/4) \arrow[r, two heads, "\mu"] \arrow[d, equal] & Hom(A, \mathbb{Z}/2) \arrow[d, equal] \\
Hom(A * \mathbb{Z}/2, \mathbb{Z}/4) \arrow[r, tail] & Hom(G(A), \mathbb{Z}/4) \arrow[r, two heads] & Hom(A \otimes \mathbb{Z}/2, \mathbb{Z}/4)
\end{tikzcd} \tag{7}$$

which we use in the following push out diagram for the definition of $\overline{G}(A, \eta)$.

$$\begin{tikzcd}
Ext(A, L) \arrow[r, tail, "\Delta"] & \overline{G}(A, \eta) \arrow[r, two heads, "\mu"] & Hom(A, H \otimes \mathbb{Z}/2) \\
Ext(A, H \otimes \mathbb{Z}/2) \arrow[u, "\eta_*"] \arrow[r, phantom, "\text{push}"] & & \\
Ext(A, \mathbb{Z}/2) \otimes H \arrow[u, equal] \arrow[r] & Hom(G(A), \mathbb{Z}/4) \otimes H \arrow[uu] \arrow[r, two heads] & Hom(A, \mathbb{Z}/2) \otimes H \arrow[uu, equal]
\end{tikzcd} \tag{8}$$

The bottom row is obtained by applying the functor $- \otimes H$ to (7). The top row is short exact. Induced homomorphisms for the functor $\overline{G}$ are defined by

$$\begin{aligned} (\varphi, \bar{\varphi})^* &= Ext(\varphi, L) \oplus Hom(\bar{\varphi}, \mathbb{Z}/4) \otimes H, \\ (\psi_1, \psi)_* &= Ext(A, \psi_1) \oplus Hom(G(A), \mathbb{Z}/4) \otimes \psi_0. \end{aligned} \tag{9}$$

Using these constructions of $G(\eta)$ and $\overline{G}(A, \eta)$ we are now ready to define the following algebraic models of stable A_n^3-polyhedra.

10.10. DEFINITION. An A^3-*system*

$$S = (H_0, H_2, H_3, \pi_1, b_2, \eta, b_3, \beta) \tag{1}$$

is a tuple consisting of abelian groups H_0, H_2, H_3, π_1 and elements

$$\begin{aligned} &b_2 \in Hom(H_2, H_0 \otimes \mathbb{Z}/2),\\ &\eta \in Hom(H_0 \otimes \mathbb{Z}/2, \pi_1),\\ &b_3 \in Hom\big(H_3, G(\eta)\big),\\ &\beta \in \overline{G}(H_2, \eta_\sharp). \end{aligned} \tag{2}$$

Here $\eta_\sharp = q\Delta(\eta \otimes 1)$ is the composition

$$\eta_\sharp : H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta\otimes 1} \pi_1 \otimes \mathbb{Z}/2 \xrightarrow{\Delta} G(\eta) \twoheadrightarrow cok(b_3) \tag{3}$$

where q is the quotient map for the cokernel of b_3. These elements satisfy the following conditions (4) and (5). The sequence

$$H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_1 \tag{4}$$

is exact and β satisfies

$$\mu(\beta) = b_2 \tag{5}$$

where μ is the operator on $\overline{G}$ in (10.9) (8). A morphism

$$(\varphi_0, \varphi_2, \varphi_3, \varphi_\pi, \varphi_\Gamma) : S \to S' \tag{6}$$

between A^3-systems is a tuple of homomorphisms

$$\begin{cases} \varphi_i : H_i \to H_i' \ (i = 0, 2, 3), \\ \varphi_\pi : \pi_1 \to \pi_1', \\ \varphi_\Gamma : G(\eta) \to G(\eta'), \end{cases}$$

such that the following diagrams (7), (8), (9) commute and such that the equation (10) holds.

$$\begin{tikzcd} H_2 \arrow[r, "b_2"] \arrow[d, "\varphi_2"] & H_0 \otimes \mathbb{Z}/2 \arrow[r, "\eta"] \arrow[d, "\varphi_0\otimes 1"] & \pi_1 \arrow[d, "\varphi_\pi"] \\ H_2' \arrow[r, "b_2'"'] & H_0' \otimes \mathbb{Z}/2 \arrow[r, "\eta'"'] & \pi_1' \end{tikzcd} \tag{7}$$

$$\begin{tikzcd} \pi_1 \otimes \mathbb{Z}/2 \arrow[r, "\Delta"] \arrow[d, "\varphi_\pi\otimes 1"] & G(\eta) \arrow[r, "\mu"] \arrow[d, "\varphi_\Gamma"] & H_0 * \mathbb{Z}/2 \arrow[d, "\varphi_0 * 1"] \\ \pi_1' \otimes \mathbb{Z}/2 \arrow[r, "\Delta"'] & G(\eta') \arrow[r, "\mu"'] & H_0' * \mathbb{Z}/2 \end{tikzcd} \tag{8}$$

$$\begin{array}{ccc} H_3 & \xrightarrow{b_3} & G(\eta) \\ \downarrow{\scriptstyle\varphi_3} & & \downarrow{\scriptstyle\varphi_\Gamma} \\ H_3' & \xrightarrow[b_3']{} & G(\eta') \end{array} \tag{9}$$

Hence φ_Γ induces $\varphi_\Gamma : cok(b_3) \to cok(b_3')$ such that $(\varphi_0, \varphi_\Gamma) : q\Delta(\eta \otimes 1) \to q\Delta(\eta' \otimes 1)$ is a morphism in G' which induces $(\varphi_0, \varphi_\Gamma)_*$ as in (10.9) (9). We have

$$(\varphi_0, \varphi_\Gamma)_*(\beta) = (\varphi_2, \overline{\varphi}_2)^*(\beta') \tag{10}$$

in $\overline{G}(H_2, q\Delta(\eta' \otimes 1))$. In (10) we choose $\overline{\varphi}_2$ for φ_2. The right hand side of (10) does not depend on the choice of $\overline{\varphi}_2$.

An A^3-system S as above is *free* if H_3 is free abelian, and S is *injective* if $b_3 : H_3 \to G(\eta)$ is injective. Let A^3-*Systems*, resp. A^3-*systems*, be the full category of free, resp. injective, A^3-systems. We have the canonical forgetful functor

$$\phi : A^3\text{-}\mathit{Systems} \to A^3\text{-}\mathit{systems} \tag{11}$$

which replaces $b_3 : H_3 \to G(\eta)$ by the inclusion $b_3(H_3) \subset G(\eta)$ of the image of b_3. One readily checks that this forgetful functor ϕ is full and representative. We associate with an A^3-system S the exact Γ-sequence

$$H_3 \xrightarrow{b_3} G(\eta) \to \pi_2 \to H_2 \xrightarrow{b_2} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta} \pi_1 \to H_1 \to 0. \tag{12}$$

Here $H_1 = cok(\eta)$ is the cokernel of η and the extension

$$cok(b_3) \rightarrowtail \pi_2 \twoheadrightarrow ker(b_2)$$

is obtained by the element β, that is, the group π_2 is given by the extension element $\beta_\dagger \in Ext(ker(b_2), cok(b_3))$ defined by

$$\beta_\dagger = \Delta^{-1}(j, \bar{j})^*(\beta).$$

Here $j : ker(b_2) \subset H_2$ is the inclusion.

10.11. Classification theorem. *For $n \geqslant 4$ there exist detecting functors Λ', λ' for which the following diagram of functors commutes up to natural isomorphism*

$$\begin{array}{ccc} \mathit{spaces}_n^3 & \xrightarrow{\Lambda'} & A^3\text{-}\mathit{Systems} \\ {\scriptstyle P_n^3}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathit{types}_n^2 & \xrightarrow{\lambda'} & A^3\text{-}\mathit{systems} \end{array}$$

Moreover for $S = \Lambda'(X)$, $X \in spaces^3_n$, the Γ-sequence of S describes part of the Γ-sequence of X, that is $H_0 = H_n X$ and

$$\begin{tikzcd}
H_3 \arrow[r] \arrow[d, equal] & G(\eta) \arrow[r] \arrow[d, equal] & \pi_2 \arrow[r] \arrow[d, equal] & H_2 \arrow[r] \arrow[d, equal] & H_0 \otimes \mathbb{Z}/2 \arrow[r, "\eta"] \arrow[d, equal] & \pi_1 \arrow[r, two heads] \arrow[d, equal] & H_1 \arrow[d, equal] \\
H_{n+3}X \arrow[r] & \Gamma_{n+2}X \arrow[r] & \pi_{n+2}X \arrow[r] & H_{n+2}X \arrow[r] & \Gamma_{n+1}X \arrow[r] & \pi_{n+1}X \arrow[r, two heads] & H_{n+1}X
\end{tikzcd}$$

In addition $\overline{G}(A,\eta) = \Gamma_{n+1}(A, X)$.

In Baues [6] we prove similar theorems also for $n = 2, 3$. We point out that the functor λ' classifies all homotopy types Y for which at most the homotopy groups $\pi_n Y$, $\pi_{n+1}Y$, $\pi_{n+2}Y$ are non trivial, i.e. $Y \in types^2_n$. The functor Λ' carries X to the A^3-system

$$(H_n X,\ H_{n+2}X,\ H_{n+3}X,\ \pi_{n+1}X,\ b_{n+2}X,\ \eta = k_{n+1}X,\ b_{n+3}X,\ \beta_{n+2}X)$$

given by the Γ-sequence of X and the boundary invariant $\beta_{n+2}X$. In fact, the classification theorem 10.11 is an application of 8.9; see VIII.1.6 in Baues [6] and (4.10) in Baues and Hennes [8].

10.12. Example. Let $\mathbb{R}P_4$ be the real projective space of dimension 4. Then the iterated suspension $\Sigma^{n-1}\mathbb{R}P_4$ is an object in *spaces*^3_n which satisfies

$$\Lambda'(\Sigma^{n-1}\mathbb{R}P_4) = (\mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, 1, 0, \Delta(1)).$$

Therefore $G(\eta) = \mathbb{Z}/4$ and the extension

$$G(\eta) = \mathbb{Z}/4 \rightarrowtail \pi_2 \twoheadrightarrow H_2 = \mathbb{Z}/2$$

is nontrivial so that $\pi_2 = \mathbb{Z}/8$. This yields a new proof that $\pi_{n+2}\Sigma^{n-1}\mathbb{R}P_4 = \mathbb{Z}/8$; see for example G.W. Whitehead [96].

The classification theorem 10.11 shows exactly what homology homomorphisms are realizable by maps between stable A^3_n-polyhedra. Hence 10.11 yields a partial solution of Whitehead's realization problem described in 3.7.

10.13. Remark. One of the deepest problems of homotopy theory is the computation of *homotopy groups of spheres* $\pi_{n+k}(S^n)$. Ravenel [80] writes

"The study of the homotopy groups of spheres can be compared with astronomy. The groups themselves are like distant stars waiting to be discovered by the determined observer, who is constantly building better telescopes to see further into the distant sky. The telescopes are spectral sequences and other algebraic constructions of various sorts. Each time a better instrument is built new discoveries are made and our perspective changes. The more we find the more we see how complicated the problem really is."

For us elements of homotopy groups of spheres, $\alpha \in \pi_{n+k-1}(S^n)$, yield very special elementary A_n^k-polyhedra

$$X = S^n \cup_\alpha e^{n+k}$$

obtained by attaching via α an $(n+k)$-cell to the sphere S^n. Such A_n^k-polyhedra with $k \geqslant 2$ are determined by the homological condition

$$H_i(X) = 0 \quad \text{for } i \neq n, n+k,$$
$$H_i(X) = \mathbb{Z} \quad \text{for } i = n, n+k,$$

and the homotopy type of X essentially can be identified with the homotopy class α. Hence the 'telescopes' above are directed to only a very small but distinguished section of the universe of homotopy types. In view of 'Freyd's generating hypothesis' [40] one might speculate that the classification of finite stable homotopy types is of similar complexity as the computation of all stable homotopy groups of spheres.

11. Decomposition of stable homotopy types

Given a class of objects with certain properties one would like to furnish a complete list of isomorphism types of such objects. This is an ultimate objective of classification. In mathematics indeed many classification problems arise but complete solutions are extremely rare. We here describe a complete list of homotopy types of $(n-1)$-connected $(n+k)$-dimensional polyhedra which are finite and stable with $k \leqslant 3$. This also yields a list of all $(n-1)$-connected $(n+k)$-types with finitely generated homotopy groups and $k \leqslant 2$, $n \geqslant k+2$.

Let C be a category with an initial object $*$ and assume sums, denoted by $A \vee B$, exist in C. An object X in C is *decomposable* if there exists an isomorphism $X \cong A \vee B$ in C where A and B are not isomorphic to $*$. Hence an object X is *indecomposable* if $X \cong A \vee B$ implies $A \cong *$ or $B \cong *$. A *decomposition* of X is an isomorphism

$$X \cong A_1 \vee \cdots \vee A_n, \quad n < \infty, \tag{11.1}$$

in C where A_i is indecomposable for all $i \in \{1, \ldots, n\}$. The decomposition of X is *unique* (*up to permutation*) if $B_1 \vee \cdots \vee B_m \cong X \cong A_1 \vee \cdots \vee A_n$ implies that $m = n$ and that there is a permutation σ with $B_{\sigma_i} \cong A_i$ for all i. A morphism f in C is *indecomposable* if the object f is indecomposable in the category $\mathit{Pair}(C)$. The objects of $\mathit{Pair}(C)$ are the morphisms of C and the morphisms $f \to g$ in $\mathit{Pair}(C)$ are the pairs (α, β) of morphism in C with $g\alpha = \beta f$. The sum of f and g is the morphism $f \vee g = (i_1 f, i_2 g)$. In a similar way we define decompositions with respect to products in a category C. Below we consider decompositions of CW-spaces in the homotopy category $C = \mathit{Top}^*/\simeq$ where the operation $\vee$ is either the one point union or the product of spaces. The main (and perhaps hopeless) purpose of *representation theory* is the determination of indecomposable objects in the category of R-modules satisfying some finiteness restraint. By the classical Grushko–Neumann theorem each finitely generated group has a unique decomposition with respect to the sum (i.e. free product) of groups. In the next result we obtain a unique decomposition of certain spaces with respect to the sum (i.e. one point union) in $Top^* / \simeq$.

11.2. THEOREM. *Let $k \leqslant 3$ and $n \geqslant k+1$ and let X be an $(n-1)$-connected $(n+k)$-dimensional finite CW-complex. Then there exists a decomposition $X \simeq X_1 \vee \cdots \vee X_r$, $r < \infty$, where the one point union of indecomposable CW-complexes X_i on the right hand side is unique up to permutation.*

Hence homotopy types in the theorem admit a unique prime factorization with respect to the operation of 'one point union'. The prime factors are called *indecomposable A_n^k-polyhedra, $k \leqslant 3$*. For $k \geqslant 4$ a unique prime factorization as in the theorem does not exist. For this we describe the following example. Let α be the generator of the cyclic group $\pi_{n+3}S^n = \mathbb{Z}/24$ where $n \geqslant 5$. Then the spaces $X_{t\alpha} = S^n \cup_{t\alpha} e^{n+4}$ are indecomposable for $0 < t < 24$ but there is a homotopy equivalence

$$X_{2\alpha} \vee X_{3\alpha} \simeq S^n \vee S^{n+4} \vee X_{5\alpha}$$

which shows that in this case the decomposition is not unique. The homotopy equivalence is obtained in 4.25 of Cohen [25]. In the presence of only one prime such decompositions are unique; see 12.12. Below we give a complete list of all indecomposable stable A_n^k-polyhedra, $k \leqslant 3$, which are the prime factors in 11.2.

General remarks on stable indecomposable polyhedra can be found in Chapter 4 of Cohen [25].

11.3. THEOREM. *Let $k \leqslant 2$ and $n \geqslant k+2$ and let Y be an $(n-1)$-connected $(n+k)$-type with finitely generated homotopy groups. Then there exists a decomposition*

$$Y \simeq K_1 \times \cdots \times K_r, \quad r < \infty,$$

where the product of indecomposable CW-spaces K_i on the right hand side is unique up to permutation.

Thus homotopy types in this theorem admit a unique prime factorization with respect to the product operation. We call the prime factors *indecomposable $\mathfrak{a}_n^k$-types, $k \leqslant 2$*. For $k \geqslant 3$ a unique prime factorization as in the theorem does not exist. The next result shows that the prime factors in 11.2 correspond exactly to the prime factors in 11.3; this is a consequence of 4.4.

11.4. THEOREM. *Let $k \leqslant 3$ and $n \geqslant k+1$. Then the Postnikov functor P_{n+k-1} yields a bijection*

$$Ind(A_n^k) - \{S^{n+k}\} \approx Ind(\mathfrak{a}_n^{k-1})$$

where the left hand side is the set of all indecomposable A_n^k-homotopy types different from the sphere S^{n+k} and the right hand side is the set of all indecomposable $\mathfrak{a}_n^{k-1}$-homotopy types.

These results are proved in Chapter X of Baues [6].

The *elementary Moore spaces* are the spheres S^m and the Moore spaces $M(\mathbb{Z}/p^i, m)$ where p^i is a power of a prime p. The *elementary Eilenberg–MacLane spaces* are $K(\mathbb{Z}, m)$ and $K(\mathbb{Z}/p^i, m)$. The following result is easy to prove.

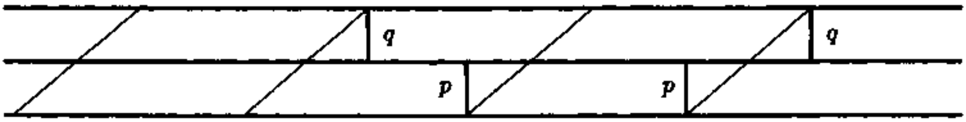
11.5. PROPOSITION. *For $k = 0$, $n \geqslant 1$ there is only one indecomposable A_n^0-polyhedron namely the sphere S^n. For $k = 1$, $n \geqslant 2$ the indecomposable A_n^1-polyhedra are exactly the elementary Moore spaces. For $k = 0$, $n \geqslant 2$ the indecomposable $\mathfrak{a}_n^0$-types are the elementary Eilenberg–MacLane spaces.*

The first nontrivial case is described in the next result due to J.H.C. Whitehead [102] and Chang [24]. For this we define the

11.6. ELEMENTARY CHANG COMPLEXES. Let η_n be the Hopf map in $\pi_{n+1}S^n$ and let p and q be powers of 2. The elementary Chang complex X in the list below is the mapping cone of the corresponding attaching map where i_1, i_2 denote the inclusion of S^n, S^{n+1} in $S^n \vee S^{n+1}$.

X	attaching map
$X(\eta) = S^n \cup e^{n+2}$	$\eta_n : S^{n+1} \to S^n$
$X(\eta q) = S^n \vee S^{n+1} \cup e^{n+2}$	$qi_1 + i_2\eta_n : S^{n+1} \to S^{n+1} \vee S^n$
$X({}_p\eta) = S^n \cup e^{n+1} \cup e^{n+2}$	$(\eta_n, p) : S^{n+1} \vee S^n \to S^n$
$X({}_p\eta q) = S^n \vee S^{n+1} \cup e^{n+1} \cup e^{n+2}$	$(qi_1 + i_2\eta_n, pi_2) : S^{n+1} \vee S^n \to S^{n+1} \vee S^n$

These complexes are also discussed in the books of Hilton [50], [51]. Our notation of the elementary Chang complexes above in terms of the "words" η, ηq, ${}_p\eta$, ${}_p\eta q$ is compatible with the notation on elementary A_n^3-complexes below. These words can also be visualized by the following graphs where vertical edges are associated with numbers p, q and where the edge, connecting level 0 and 2, is denoted by η.



η ηq ${}_p\eta$ ${}_p\eta q$

Hence the elementary Chang complexes correspond to all subgraphs (or subwords) of ${}_p\eta q$ which contain η. We shall describe the elementary A_n^3-polyhedra by subgraphs (or subwords) of more complicated graphs.

11.7. THEOREM. *Let $n \geqslant 3$. The elementary Moore spaces and the elementary Chang complexes furnish a complete list of all indecomposable A_n^2-polyhedra.*

11.8. ELEMENTARY CHANG TYPES. Let p, q be powers of 2 and let $\eta : \mathbb{Z} \to \mathbb{Z}/2 \to \mathbb{Z}/q$ and $\eta' : \mathbb{Z}/p \to \mathbb{Z}/2 \to \mathbb{Z}/q$ be the unique non trivial homomorphisms. The elementary

Chang types $K(\mathbb{Z}, \mathbb{Z}/q, n)$ and $K(\mathbb{Z}/p, \mathbb{Z}/q, n)$ are the $(n-1)$-connected $(n+1)$-types with k-invariant η and η' respectively.

Using 11.4 we get the following application of Chang's theorem.

11.9. COROLLARY. *Let $n \geqslant 3$. The elementary Eilenberg–MacLane spaces and the elementary Chang types furnish a complete list of indecomposable $\mathfrak{a}_n^1$-types. Moreover the bijection in* 11.4 *is given by the following list.*

X	$P_{n+1}X$
S^n	$K(\mathbb{Z}, \mathbb{Z}/2, n)$
S^{n+1}	$K(\mathbb{Z}, n+1)$
$M(\mathbb{Z}/p, n)$	$K(\mathbb{Z}/p, \mathbb{Z}/2, n)$
$M(\mathbb{Z}/q, n+1)$	$K(\mathbb{Z}/q, n+1)$
$X(\eta)$	$K(\mathbb{Z}, n)$
$X({}_p\eta)$	$K(\mathbb{Z}/p, n)$
$X(\eta q)$	$K(\mathbb{Z}, \mathbb{Z}/2q, n)$
$X({}_p\eta q)$	$K(\mathbb{Z}/p, \mathbb{Z}/2q, n)$

Moreover P_{n+1} carries an elementary Moore space of odd primes in A_n^2 to the corresponding elementary Eilenberg–MacLane space.

We say that a CW-space X is finite if there is a finite CW-complex homotopy equivalent to X. Let *spaces*$_n^k$ (finite) be the full homotopy category of finite $(n-1)$-connected $(n+k)$-dimensional CW-spaces X. Then *Spanier–Whitehead duality* [89] is an endofunctor D of this category with $n \geqslant k+2$ satisfying $DD =$ identity. We say that the space X is *self-dual* if there is a homotopy equivalence $DX \simeq X$.

EXAMPLE. Let $X = M - *$ be obtained by deleting a point in an $(n-1)$-connected closed differential manifold M of dimension $2n+k$ with $n \geqslant k+2$. Then X is self dual. Compare Baues [5] and Stöcker [92]. Hence self-dual CW-spaces play an important role in the classification of highly connected manifolds.

Spanier–Whitehead duality carries a one point union to a one point union, i.e. $D(X \vee Y) = D(X) \vee D(Y)$, and hence D carries indecomposable polyhedra to indecomposable polyhedra. In particular we have the following properties of elementary Chang complexes.

11.10. PROPOSITION. *The Spanier–Whitehead duality functor $D : A_n^2 \cong A_n^2$ satisfies $DX(\eta) = X(\eta)$, $DX(\eta q) = X({}_q\eta)$, $DX({}_p\eta) = X(\eta p)$, $DX({}_p\eta q) = X({}_q\eta p)$. Hence the Spanier–Whitehead duality turns the graphs in* 11.6 *around by* 180 *degrees. For example, $X({}_p\eta p)$, $X(\eta)$ and $X({}_p\eta) \vee X(\eta p)$ are self-dual. While clearly $X({}_p\eta)$ is not self-dual.*

For the description of the indecomposable objects in A_n^3, $n \geqslant 4$, we use certain words. Let L be a set, the elements of which are called 'letters'. A word with letters in L is an element in the free monoid generated by L. Such a word a is written $a = a_1 a_2 \dots a_n$

with $a_i \in L$, $n \geqslant 0$; for $n = 0$ this is the empty word ϕ. Let $b = b_1 \dots b_k$ be a word. We write $w = \dots b$ if there is a word a with $w = ab$, similarly we write $w = b\dots$ if there is a word c with $w = bc$ and we write $w = \dots b \dots$ if there exists words a and c with $w = abc$. A *subword* of an infinite sequence $\dots a_{-2}a_{-1}a_0a_1a_2\dots$ with $a_i \in L$, $i \in \mathbb{Z}$, is a finite connected subsequence $a_na_{n+1}\dots a_{n+k}$, $n \in \mathbb{Z}$. For the word $a = a_1 \dots a_n$ we define the word $-a = a_na_{n-1}\dots a_1$ by reversing the order in a.

11.11. DEFINITION. We define a collection of finite words $w = w_1w_2\dots w_k$. The letters w_i of w are symbols ξ, η, ε or natural numbers $t, s_i, r_i, i \in \mathbb{Z}$, which are powers of 2. We write the letters s_i as upper indices, the letters r_i as lower indices, and the letter t in the middle of the line since we have to distinguish between these numbers. For example, $\eta 4\xi^2\eta_8$ is such a word with $t = 4, r_1 = 8, s_1 = 2$. A basic sequence is defined by

$$\xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\dots \tag{1}$$

This is the infinite product $a(1)a(2)\dots$ of words $a(i) = \xi^{s_i}\eta_{r_i}$, $i \geqslant 1$. A *basic word* is any subword of (1). A central sequence is defined by

$$\dots{}^{s-2}\xi_{r_{-2}}\eta^{s-1}\xi_{r_{-1}}\eta t\xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\dots \tag{2}$$

A *central word* w is any subword of (2) containing the number t. Hence a central word w is of the form $w = atb$ where $-a$ and b are basic words. An ε-sequence is defined by

$$\dots{}^{s-2}\xi_{r_{-2}}\eta^{s-1}\xi_{r_{-1}}\varepsilon^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\dots \tag{3}$$

An *ε-word* w is any subword of (3) containing the letter ε; again we can write $w = a\varepsilon b$ where $-a$ and b are basic words.

A *general word* is a basic word, a central word or an ε-word.

A general word w is called *special* if w contains at least one of the letters ξ, η or ε and if the following conditions (i), D(i), (ii) and D(ii) are satisfied in case $w = a\varepsilon b$ is an ε-word. We associate with b the tuple

$$s(b) = (s_1^b, s_2^b, \dots) = \begin{cases} (s_1, \dots, s_m, \infty, 1, 1, \dots) & \text{if } b = \dots\xi, \\ (s_1, \dots, s_m, 1, 1, 1, \dots) & \text{otherwise,} \end{cases}$$

$$r(b) = (r_1^b, r_2^b, \dots) = \begin{cases} (r_1, \dots, r_\ell, \infty, 1, 1, \dots) & \text{if } b = \dots\eta, \\ (r_1, \dots, r_\ell, 1, 1, 1, \dots) & \text{otherwise,} \end{cases}$$

where $s_1 \dots s_m$ and $r_1 \dots r_\ell$ are the words of upper indices and lower indices respectively given by b. In the same way we get $s(-a) = (s_1^{-a}, s_2^{-a}, \dots)$ and $r(-a) = (r_1^{-a}, r_2^{-a}, \dots)$ with $s_i^{-a} \in \{s_{-i}, \infty, 1\}$ and $r_i^{-a} \in \{r_{-i}, \infty, 1\}$, $i \in \mathbb{N}$. The conditions in question on the ε-word $w = a\varepsilon b$ are:

$$b = \phi \Longrightarrow a \neq \xi_2, \tag{i}$$

$$a = \phi \Longrightarrow b \neq {}^2\eta. \tag{D(i)}$$

Moreover if $a \neq \phi$ and $b \neq \phi$ we have:

$$s_1 = 2 \Longrightarrow r_{-1} \geqslant 4 \tag{ii}$$

and

$$\begin{aligned}&\left(2r_1^b, -s_2^b, r_2^b, -s_3^b, r_3^b, \dots, -s_i^b, r_i^b, \dots\right)\\ &\quad < \left(r_1^{-a}, -s_1^{-a}, r_2^{-a}, -s_2^{-a}, r_3^{-a}, -s_3^{-a}, \dots, r_i^{-a}, -s_i^{-a}, \dots\right)\end{aligned}$$

$$r_{-1} = 2 \Longrightarrow s_1 \geqslant 4 \tag{D(ii)}$$

and

$$\begin{aligned}&\left(-s_1^b, r_1^b, -s_2^b, r_2^b, -s_3^b, r_3^b, \dots, -s_i^b, r_i^b, \dots\right)\\ &\quad < \left(-2\cdot s_1^{-a}, r_2^{-a}, -s_2^{-a}, r_3^{-a}, -s_3^{-a}, \dots, r_i^{-a}, -s_i^{-a}, \dots\right).\end{aligned}$$

The index i runs through $i = 2, 3, \dots$ as indicated. In (ii) and D(ii) we use the lexicographical ordering from the left, that is $(n_1, n_2, \dots) < (m_1, m_2, \dots)$ if and only if there is $t \geqslant 1$ with $n_j = m_j$ for $j < t$ and $n_t < m_t$.

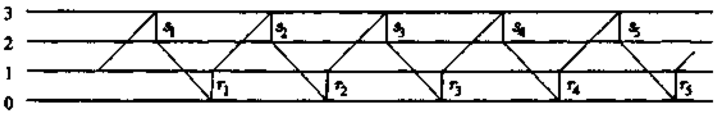
Finally we define a *cyclic word* by a pair (w, φ) where w is a basic word of the form $(p \geqslant 1)$

$$w = \xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\dots\xi^{s_p}\eta_{r_p} \tag{4}$$

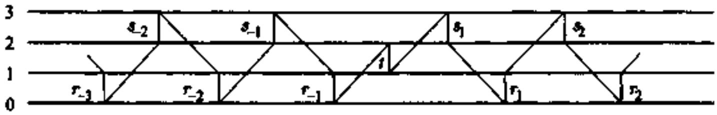
and where φ is an automorphism of a finite dimensional $\mathbb{Z}/2$-vector space $V = V(\varphi)$. Two cyclic words (w, φ) and (w', φ') are *equivalent* if w' is a cyclic permutation of w, that is

$$w' = \xi^{s_i}\eta_{r_i}\dots\xi^{s_p}\eta_{r_p}\xi^{s_1}\eta_{r_1}\dots\xi^{s_{i-1}}\eta_{r_{i-1}},$$

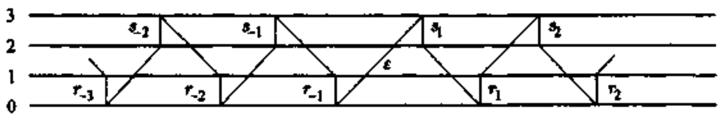
and if there is an isomorphism $\Psi : V(\varphi) \cong V(\varphi')$ with $\varphi = \Psi^{-1}\varphi'\Psi$. A cyclic word (w, φ) is a *special cyclic word* if φ is an indecomposable automorphism and if w is not of the form $w = w'w'\dots w'$ where the right hand side is a j-fold power of a word w' with $j > 1$.



basic sequence



central sequence



ε-sequence

The sequences (1), (2), (3) can be visualized by the infinite graphs sketched below. The letters s_i, resp. r_i, correspond to vertical edges connecting the levels 2 and 3, resp. the levels 0, 1. The letters η, resp. ξ, correspond to diagonal edges connecting the levels 0 and 2, resp. the levels 1 and 3. Moreover ε connects the levels 0 and 3 and t the levels 1 and 2. We identify a general word with the connected finite subgraph of the infinite graphs below. Therefore the *vertices of level* i of a general word are defined by the vertices of level i of the corresponding graph, $i \in \{0,1,2,3\}$. We also write $|x| = i$ if x is a vertex of level i.

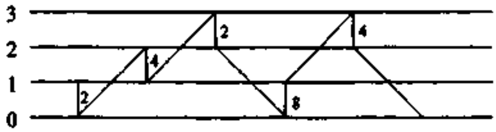
REMARK. There is a simple rule which creates exactly all graphs corresponding to general words. Draw in the plane $\mathbb{R}^2$ a connected finite graph of total height at most 3 that alternatingly consists of vertical edges of height one and diagonal edges of height 2 or 3. Moreover endow each vertical edge with a power of 2. An *equivalence relation* on such graphs is generated by reflection at a vertical line. One readily checks that the equivalence classes of such graphs are in $1-1$ correspondence to all general words.

11.12. DEFINITION. Let w be a basic word, a central word or an ε-word. We obtain the *dual word* $D(w)$ by reflection of the graph w at a horizontal line and by using the equivalence defined in the remark. Then $D(w)$ is again a basic word, a central word, or an ε-word, respectively. Clearly the reflection replaces each letter ξ in w by the letter η and vice versa, moreover it turns a lower index into an upper index and vice versa. We define the *dual cyclic word* $D(w,\varphi)$ as follows. For the cyclic word (w,φ) in 11.11 (4) let $D(w,\varphi) = (w', (\varphi^*)^{-1})$. Here we set

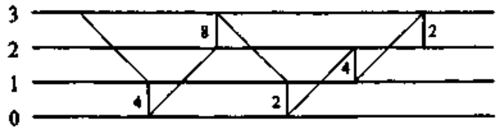
$$w' = \xi^{r_1}\eta_{s_2}\xi^{r_2}\dots\eta_{s_p}\xi^{r_p}\eta_{s_1}$$

and we set $\varphi^* = Hom(\varphi, \mathbb{Z}/2)$ with $V(\varphi^*) = Hom(V(\varphi), \mathbb{Z}/2)$. Up to cyclic permutation w' is just $D(w)$ defined above. We point out that the dual words $D(w)$ and $D(w,\varphi)$ are special if and only if w and (w,φ) are special.

As an example we have the special words $w = {}_2\eta 4\xi^2\eta_8\xi^4\eta$ and $D(w) = \xi_4\eta^8\xi_2\eta 4\xi^2$ which are dual to each other, they correspond to the graphs



$w = {}_2\eta 4\xi^2\eta_8\xi^4\eta$



$D(w) = \xi_4\eta^8\xi_2\eta 4\xi^2$

Hence the dual graph $D(w)$ is obtained by turning around the graph of w.

We are going to construct certain A_n^3-polyhedra, $n \geqslant 4$, associated to the words in 2.1. To this end we first define the homology of a word.

11.13. Definition. Let w be a general word and let $r_\alpha \dots r_\beta$ and $s_\mu \dots s_\nu$ be the words of lower indices and of upper indices respectively given by w. We define the *torsion groups* of w by

$$T_0(w) = \mathbb{Z}/r_\alpha \oplus \cdots \oplus \mathbb{Z}/r_\beta, \tag{1}$$

$$T_1(w) = \mathbb{Z}/t \quad \text{if } w \text{ is a central word}, \tag{2}$$

$$T_2(w) = \mathbb{Z}/s_\mu \oplus \cdots \oplus \mathbb{Z}/s_\nu, \tag{3}$$

and we set $T_i(w) = 0$ otherwise. We define the *integral homology* of w by

$$H_i(w) = \mathbb{Z}^{L_i(w)} \oplus T_i(w) \oplus \mathbb{Z}^{R_i(w)}. \tag{4}$$

Here $\beta_i(w) = L_i(w) + R_i(w)$ is the *Betti number* of w; this is the number of end points of the graph w which are vertices of level i and which are not vertices of vertical edges; we call such vertices x *spherical vertices* of w. If w has spherical vertices let $L(w)$, resp. $R(w)$, be the *left*, resp. *right*, spherical vertex of w. Now we set $L_i(w) = 1$ if $|L(w)| = i$ and $R_i(w) = 1$ if $|R(w)| = i$, moreover $L_i(w) = 0$ and $R_i(w) = 0$ otherwise.

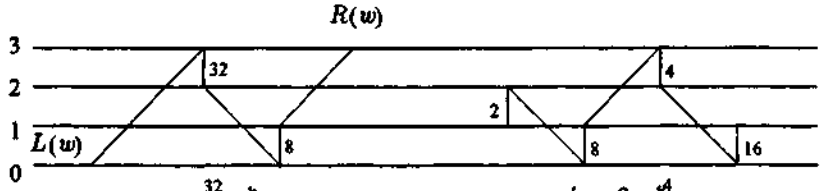
Using the equation (4) we have specified an *ordered basis* B_i of $H_i(w)$. We point out that

$$\beta_0(w) + \beta_1(w) + \beta_2(w) + \beta_3(w) \leqslant 2. \tag{5}$$

For a cyclic word (w, φ) we set

$$H_i(w,\varphi) = \bigoplus_v T_i(w) \tag{6}$$

where $v = dim\, V(\varphi)$ and where the right hand side is the v-fold direct sum of $T_i(w)$. As an example we consider the special words



$w = \varepsilon^{32}\eta_8\xi$ $\qquad$ $w' = 2\eta_8\xi^4\eta_{16}$

The homology of these words is:

	$w = \varepsilon^{32}\eta_8\xi$	$w' = 2\eta_8\xi^4\eta_{16}$
H_3	$\mathbb{Z}$	0
H_2	$\mathbb{Z}/32$	$\mathbb{Z}/4$
H_1	0	$\mathbb{Z}/2$
H_0	$\mathbb{Z}\oplus\mathbb{Z}/8$	$\mathbb{Z}/8\oplus\mathbb{Z}/16$

Here w has 2 spherical vertices while w' has no spherical vertex. We point out that the numbers 2^k attached to vertical edges correspond to cyclic groups $\mathbb{Z}/2^k$ in the homology. We describe many further examples below.

For the construction of polyhedra $X(w)$ associated to words w we use the following generators.

11.14. GENERATORS OF HOMOTOPY GROUPS. Let r, s be powers of 2. We have the Hopf maps

$$\eta = \eta_n : S^{n+1} \to S^n,\ \xi = \eta_{n+1} : S^{n+2} \to S^{n+1},\ \varepsilon = \eta_n^2 : S^{n+2} \to S^n.$$

We use the compositions

$$\eta = i\eta_n : S^{n+1} \to M(\mathbb{Z}/r, n), \quad \xi = \eta_{n+1}q : M(\mathbb{Z}/r, n+1) \to S^{n+1}$$

which are $(2n+1)$-dual. Moreover we have the $(2n+2)$-dual groups, $n \geqslant 4$

$$\left[S^{n+2}, M(\mathbb{Z}/r,n)\right] = \begin{cases} \mathbb{Z}/4\xi_2 & \text{for } r = 2,\\ \mathbb{Z}/2\xi_r + \mathbb{Z}/2\varepsilon_r & \text{for } r \geqslant 4,\end{cases}$$

$$\left[M(\mathbb{Z}/s, n+1), S^n\right] = \begin{cases} \mathbb{Z}/4\eta^2 & \text{for } r = 2,\\ \mathbb{Z}/2\eta^s + \mathbb{Z}/2\varepsilon^s & \text{for } s \geqslant 4,\end{cases}$$

where $\varepsilon_r = i\eta_n^2$ and $\varepsilon^s = \eta_n^2 q$ and $\xi_r = \chi_r^2\xi_2$ and $\eta^s = \eta^2\chi_2^s$. Next we use

$$\left[M(\mathbb{Z}/s, n+1), M(\mathbb{Z}/r, n)\right] = \begin{cases} \mathbb{Z}/2\xi_2^2 \oplus \mathbb{Z}/2\,\eta_2^2 & \text{for } s = r = 2,\\ \mathbb{Z}/4\xi_2^s \oplus \mathbb{Z}/2\,\eta_2^s & \text{for } s \geqslant 4,\ r = 2,\\ \mathbb{Z}/2\xi_r^2 \oplus \mathbb{Z}/4\,\eta_r^2 & \text{for } s = 2,\ r \geqslant 4,\\ \mathbb{Z}/2\xi_r^s \oplus \mathbb{Z}/2\eta_r^s \oplus \mathbb{Z}/2\,\varepsilon_r^s & \text{otherwise.}\end{cases}$$

Here we have $\xi_r^s = \chi_r^2\xi_2 q$, $\eta_r^s = i\eta^2\chi_2^s$ and $\varepsilon_r^s = i\eta_n^2 q$. We have the $(2n+2)$-dualities $D(\xi_r^s) = \eta_s^r$ and $D(\varepsilon_r^s) = \varepsilon_s^r$.

11.15. DEFINITION. Let $n \geqslant 4$ and let w be a general word. We define the A_n^3-polyhedron $X(w) = C_f$ by the mapping cone C_f of a map $f = f(w) : A \to B$ where

$$\begin{cases} A = M(H_3, n+2) \vee M(H_2, n+1) \vee S_c^{n+1},\\ B = M(H_0, n) \vee S_c^{n+1} \vee S_b^{n+1}.\end{cases} \tag{1}$$

Here $H_i = H_i(w)$ is the homology group above. We set $S_c^{n+1} = S^{n=1}$ if w is a central word and we set $S_c^{n+1} = *$ otherwise, moreover we set $S_b^{n+1} = S^{n+1}$ if w is a basic word of the form $w = \xi \dots$ and we set $S_b^{n+1} = *$ otherwise. The attaching map

$$\begin{aligned} f = f(w) &: M(H_3, n+2) \vee M(H_2, n+1) \vee S_c^{n+1} \\ &\to M(H_0, n) \vee S_c^{n+1} \vee S_b^{n+1} \end{aligned} \tag{2}$$

is constructed exactly via the pattern defined by the word w or the associated graph w. For this we subdivide the graph of w by a horizontal line between level 1 and 2; all edges crossing this line are summands in the attaching map $f(w)$. For example consider the graphs $\varepsilon^{32}\eta_8\xi$, $2\eta_8\xi^4\eta_{16}$ and ${}_2\eta 4\xi^2\eta_8\xi^4\eta$ above. Then we get

$$f(\varepsilon^{32}\eta_8\xi) = \begin{tikzcd} M(\mathbb{Z}/32,n+1) \arrow[d, "\varepsilon"] \arrow[drr, "\eta"] & \vee & S^{n+2} \arrow[d, "\xi"] \\ S^n & \vee & M(\mathbb{Z}/8,n) \end{tikzcd}$$

$$f(2\eta_8\xi^4\eta_{16}) = \begin{tikzcd} S^{n+1} \arrow[d, "2"] \arrow[drr, "i\eta_n"] & \vee & & M(\mathbb{Z}/4,n+1) \arrow[dl, "\xi"] \arrow[dr, "\eta"] & \\ S^{n+1} & \vee & M(\mathbb{Z}/8,n) & \vee & M(\mathbb{Z}/16,n) \end{tikzcd}$$

$$f({}_2\eta 4\xi^2\eta_8\xi^4\eta) = \begin{tikzcd} S^{n+1} \arrow[d, "i\eta_n"] \arrow[drr, "4"] & \vee & M(\mathbb{Z}/2,n+1) \arrow[d, "\xi"] \arrow[drr, "\eta"] & \vee & M(\mathbb{Z}/4,n+1) \arrow[d, "\xi"] \arrow[drr, "\eta"] & & \\ M(\mathbb{Z}/2,n) & \vee & S^{n+1} & \vee & M(\mathbb{Z}/8,n) & \vee & S^n \end{tikzcd}$$

Here ξ, η, ε are the corresponding generators in 11.14. For a cyclic word (w, φ) the construction of $X(w, \varphi)$ is slightly different; see Baues and Hennes [8]. Clearly the homology of $X(w)$ or $X(w, \varphi)$ is the homology in 11.13.

11.16. Theorem. *Let $n \geqslant 4$. The elementary Moore spaces, the complexes $X(w)$ where w is a special word, and the complexes $X(w, \varphi)$ where (w, φ) is a special cyclic word furnish a complete list of all indecomposable A_n^3-polyhedra. For two complexes X, X' in this list there is a homotopy equivalence $X \simeq X'$ if and only if there are equivalent special cyclic words*

$$(w, \varphi) \sim (w', \varphi')$$

with $X = X(w, \varphi)$ and $X' = X(w', \varphi')$. Moreover Spanier–Whitehead duality D satisfies

$$D\big(X(w)\big) = X(Dw),$$

$$D\big(X(w,\varphi)\big) = X\big(D(w,\varphi)\big),$$

where the right hand side is given by the dual words in 11.12.

The proof of this theorem relies on the classification by A^3-systems in 10.11. The result is then obtained by classifying the indecomposable A^3-systems with finitely generated homology; this being a purely algebraic question can be considered as a problem of representation theory. For a complete proof see Baues and Hennes [8].

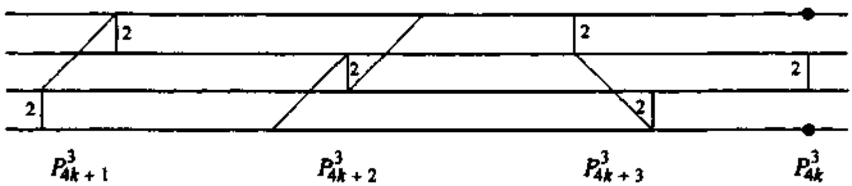
11.17. EXAMPLE. Let

$$P_n^3 = \mathbb{R}P_{n+3}/\mathbb{R}P_{n-1}$$

be the truncated real projective space. Then one has stable equivalences, $n \geqslant 1$,

$$P_n^3 \sim \begin{cases} X({}_2\xi^2) & \text{for } n \equiv 1(4), \\ X(\eta 2\xi) & \text{for } n \equiv 2(4), \\ X({}^2\eta_2) & \text{for } n \equiv 3(4), \\ S^n \vee S^{n+3} \vee M(\mathbb{Z}/2, n+1) & \text{for } n \equiv 0(4). \end{cases}$$

Hence the graphs of these stable spaces are ($k \geqslant 0$)



where P_{4k}^3 with $k \geqslant 1$ is a one point union of Moore spaces.

We now give an application of the classification theorem 11.16. We describe explicitly all indecomposable $(n-1)$-connected $(n+3)$-dimensional homotopy types X, $n \geqslant 4$, for which all homology groups H_iX are cyclic, $i \geqslant 0$.

Let $H_* = (H_0, H_1, H_2, H_3)$ be a tuple of finitely generated abelian groups with H_3 free abelian and let $N(H_*)$ be the number of all indecomposable homotopy types X as above with homology groups $H_{n+i}(X) \cong H_i$ for $i \in \{0,1,2,3\}$.

11.18. COROLLARY. *Let $n \geqslant 4$. The indecomposable $(n-1)$-connected $(n+3)$-dimensional homotopy types X, for which all homology groups $H_i(X)$ are cyclic, are exactly the elementary Moore spaces, the elementary Chang complexes, and the spaces $X(w)$ where w is one of the words in the following list.*

$H_* = (H_0$	H_1	H_2	$H_3)$	$N(H_*)$	w with $H_*X(w) \cong H_*$
$\mathbb{Z}/r$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	$\mathbb{Z}$	3	$\xi_r\eta t\xi^s,\ t\xi^s\eta_r\xi,\ {}^s\xi_r\eta t\xi$
$\mathbb{Z}/r$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	0	3	${}_r\eta t\xi^s,\ t\xi^s\eta_r,\ {}^s\xi_r\eta t$
$\mathbb{Z}/r$	$\mathbb{Z}/t$	0	$\mathbb{Z}$	2	${}_r\eta t\xi,\ \xi_r\eta t$
$\mathbb{Z}/r$	$\mathbb{Z}$	$\mathbb{Z}/s$	$\mathbb{Z}$	1	$\xi^s\eta_r\xi$
$\mathbb{Z}/r$	$\mathbb{Z}$	$\mathbb{Z}/s$	0	1	$\xi^s\eta_r$
$\mathbb{Z}/r$	0	$\mathbb{Z}/s$	$\mathbb{Z}$	$\begin{cases} 2, & r = s = 2 \\ 3, & rs \geqslant 8 \end{cases}$	${}^s\eta_r\xi,\ {}^s\xi_r\varepsilon$ and $\xi_r\varepsilon^s$ for $rs \geqslant 8$,
$\mathbb{Z}/r$	0	$\mathbb{Z}/s$	0	$\begin{cases} 3, & r = s = 2 \\ 4, & rs \geqslant 8 \end{cases}$	${}_r\xi^s,\ {}^s\eta_r,\ (\eta^s\xi_r, 1)$ and ${}_r\varepsilon^s$ for $rs \geqslant 8$,
$\mathbb{Z}/r$	0	$\mathbb{Z}$	$\mathbb{Z}$	1	$\eta_r\xi$
$\mathbb{Z}/r$	0	0	$\mathbb{Z}$	2	${}_r\xi,\ {}_r\varepsilon$
$\mathbb{Z}$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	0	2	$\eta t\xi^s,\ t\xi^s\eta$
$\mathbb{Z}$	$\mathbb{Z}/t$	0	$\mathbb{Z}$	1	$\eta t\xi$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/s$	0	1	$\xi^s\eta$
$\mathbb{Z}$	0	$\mathbb{Z}/s$	0	2	$\eta^r,\ \xi^r$
$\mathbb{Z}$	0	0	$\mathbb{Z}$	1	ε

The list describes all w ordered by the homology $H_* \cong H_*(X(w))$. The attaching map for $X(w)$ is obtained by 11.15. Let (r, t, s) be powers of 2.

All words in the list are special words, except the word $(\eta^s\xi_r, 1)$ which is a special cyclic word associated to the automorphism 1 of $\mathbb{Z}/2$.

Example. Let $n \geqslant 4$ and let $H_* = (H_0, H_1, H_2, H_3)$ be a tuple of cyclic groups with $H_3 \in \mathbb{Z}, 0$. Then it is easy to describe (by use of 11.18) all simply connected homotopy types X with $H_{n+i}(X) = H_i$ for $0 \leqslant i \leqslant 3$ and $i > n + 3$. In fact all such homotopy types are in a canonical way one point unions of the indecomposable homotopy types in the list above. For example, for $H_* = (\mathbb{Z}/6, \mathbb{Z}/2, \mathbb{Z}/2, 0)$ there exist exactly 9 such homotopy types X which are:

$$
\begin{aligned}
&M(\mathbb{Z}/6, n) \vee M(\mathbb{Z}/2, n+1) \vee M(\mathbb{Z}/2, n+2),\\
&M(\mathbb{Z}/6, n) \vee X(2\xi^2),\\
&M(\mathbb{Z}/3, n) \vee X({}_2\eta 2) \vee M(\mathbb{Z}/2, n+2),\\
&M(\mathbb{Z}/3, n) \vee X({}_2\xi^2) \vee M(\mathbb{Z}/2, n+1),\\
&M(\mathbb{Z}/3, n) \vee X({}^2\eta_2) \vee M(\mathbb{Z}/2, n+1),\\
&M(\mathbb{Z}/3, n) \vee X(\eta^2\xi_2, 1) \vee M(\mathbb{Z}/2, n+1),\\
&M(\mathbb{Z}/3, n) \vee X({}_2\eta 2\xi^2),\\
&M(\mathbb{Z}/3, n) \vee X(2\xi^2\eta_2),\\
&M(\mathbb{Z}/3, n) \vee X({}^2\xi_2\eta 2).
\end{aligned}
$$

Similarly we see that there are 24 homotopy types X for $H_* = (\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z})$; we leave this as an exercise.

Next we describe explicitly all indecomposable $(n-1)$-connected $(n+2)$-types X, $n \geqslant 4$, for which all homotopy groups are cyclic. For this we use the bijection 11.4 and the computation of $\pi_{n+2}X$, $\pi_{n+1}X$, $\pi_n X$ in 10.11. Let $\pi_* = (\pi_0, \pi_1, \pi_2)$ be a tuple of finitely generated abelian groups and let $N(\pi_8)$ be the number of all indecomposable homotopy types X with homotopy groups $\pi_{n+i}(X) \cong \pi_i$ for $i = 0, 1, 2$ and $\pi_j(X) = 0$ otherwise, $n \geqslant 4$.

11.19. Corollary. *Let $n \geqslant 4$. The indecomposable $(n-1)$-connected $(n+2)$-types X for which all homotopy groups $\pi_i(X)$ are cyclic are exactly the elementary Eilenberg–Mac Lane spaces, the elementary Chang types, and the spaces $P_{n+2}X(w)$ where w is one of the words in the following list.*

$\pi_* = (\pi_0$	π_1	π_2	$N(\pi_*)$	w with $\pi_* X(w) \cong \pi_*$
$\mathbb{Z}$	0	$\mathbb{Z}$	1	η
$\mathbb{Z}/r$	0	$\mathbb{Z}$	1	$\eta_r\xi$
$\mathbb{Z}$	0	$\mathbb{Z}/s$	1	${}^{2s}\eta$
$\mathbb{Z}/r$	0	$\mathbb{Z}/s$	3	$\begin{cases} {}^{2}\eta_r \text{ for } s = 2,\ {}^{2}\eta_r\xi^{s'} \text{ for } s = 2s' \geqslant 4 \\ {}^{2s}\eta_r\xi,\ (\eta^s\xi_r, 1) \end{cases}$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/s$	1	$\xi^s\eta$
$\mathbb{Z}/r$	$\mathbb{Z}$	$\mathbb{Z}/s$	1	$\xi^s\eta_r\xi$
$\mathbb{Z}$	$\mathbb{Z}/t$	$\mathbb{Z}/s$	1	$\begin{cases} P_{n+2}S^n,\ t = s = 2 \\ \eta t',\ t = 2t' \geqslant 4,\ s = 2 \\ \varepsilon^{s'},\ t = 2,\ s = 2s' \geqslant 4 \\ \eta t'\xi^{s'},\ t = 2t' \geqslant 4,\ s = 2s' \geqslant 4 \end{cases}$
$\mathbb{Z}/r$	$\mathbb{Z}/t$ $t \geqslant 4$	$\mathbb{Z}/s$ $s \geqslant 4$	2	$\begin{cases} \xi_r\eta t'\xi^{s'},\ {}^{s'}\xi_r\eta t'\xi \\ \text{with } t = 2t',\ s = 2s' \end{cases}$
$\mathbb{Z}/r$	$\mathbb{Z}/t$ $t \geqslant 4$	$\mathbb{Z}/2$	1	$\xi_r\eta t',\ t = 2t'$
$\mathbb{Z}/r$ $r \geqslant 4$	$\mathbb{Z}/2$	$\mathbb{Z}/s$ $s \geqslant 4$	2	$\begin{cases} {}^{s'}\xi_r\varepsilon \text{ and} \\ \xi_r\varepsilon^{s'},\ s = 2s' \end{cases}$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/s$ $s \geqslant 4$	2	$\begin{cases} P_{n+2}M(\mathbb{Z}/2, n) \text{ for } s = 4 \text{ and} \\ {}^{s'}\xi_2\varepsilon \text{ for } s = 2s' \geqslant 4, \text{ and} \\ \xi^{s''} \text{ for } s = 4s'' \geqslant 8 \end{cases}$
$\mathbb{Z}/r$ $r \geqslant 4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	2	$\begin{cases} {}_r\varepsilon \text{ and} \\ {}_r\xi \end{cases}$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	1	${}_2\varepsilon$

The list describes all w of the theorem ordered by the homotopy groups $\pi_* \cong \pi_* X(w)$. Let $r, t, s \geqslant 2$ be powers of 2 and for $t, s \geqslant 4$ let $2t' = t$ and $2s' = s$.

For all tuples of cyclic groups $\pi_* = (\pi_0, \pi_1, \pi_2)$, $\pi_0 \neq 0$, $\pi_2 \neq 0$ which are not in the list we have $N(\pi_*) = 0$. All words in the list are special words, except the word $(\eta^s \xi_r, 1)$ which is a special cyclic word associated to the identity automorphism 1 of $\mathbb{Z}/2$.

EXAMPLE. Let $n \geqslant 4$ and let $\pi_* = (\pi_0, \pi_1, \pi_2)$ be a tuple of cyclic groups. Then it is easy to describe all homotopy types X with $\pi_{n+i}(X) \cong \pi_i$ for $i = 0, 1, 2$ and $\pi_j X = 0$ for $j < n$ and $j > n+2$. In fact all such homotopy types are in a canonical way products of the indecomposable homotopy types in 11.19. For example, for $\pi_* = (\mathbb{Z}/6, \mathbb{Z}/2, \mathbb{Z}/2)$ there exist exactly 7 such homotopy types X which are

$$\begin{aligned}
&K(\mathbb{Z}/6, n) \times K(\mathbb{Z}/2, n+1) \times K(\mathbb{Z}/2, n+2),\\
&K(\mathbb{Z}/6, n) \times K(\mathbb{Z}/2, \mathbb{Z}/2, n+1),\\
&K(\mathbb{Z}/3, n) \times K(\mathbb{Z}/2, \mathbb{Z}/2, n) \times K(\mathbb{Z}/2, n+1),\\
&K(\mathbb{Z}/3, n) \times K(\mathbb{Z}/2, n+1) \times P_{n+2}X({}^2\eta_2),\\
&K(\mathbb{Z}/3, n) \times K(\mathbb{Z}/2, n+1) \times P_{n+2}X({}^4\eta_2\xi),\\
&K(\mathbb{Z}/3, n) \times K(\mathbb{Z}/2, n+1) \times P_{n+2}X(\eta^2\xi_2, 1),\\
&K(\mathbb{Z}/3, n) \times P_{n+2}X({}_2\varepsilon).
\end{aligned}$$

It is clear how to compute the homology H_n, H_{n+1} and H_{n+2} of these spaces and, in fact, we can easily describe the A^3-system of these spaces. We leave it to the reader to consider other cases, for example for $\pi_* = (\mathbb{Z}_4, \mathbb{Z}_{10}, \mathbb{Z})$ there exist exactly 3 homotopy types X with $\pi_* \cong \pi_* X$.

Finally we have the following applications of the classification theorem 11.16 which single out spaces which are highly desuspendable.

11.20. THEOREM. *The stable homotopy types of connected compact 4-dimensional polyhedra coincide with finite one point unions $X_1 \vee \cdots \vee X_r$ where the X_i are elementary Moore spaces in A_n^3 or the spaces $X(t\xi^s)$, $X(t\xi)$, $X(\xi^s)$, $X(\xi)$, and $X({}_r\xi^s)$. Here r, s, t are powers of 2 and $r \geqslant s$.*

For this compare V Appendix A in Baues [6]. The theorem shows that only a few spaces arise as prime factors in the stabilization of 4-dimensional polyhedra. This, for example, has the practical effect that the computation of generalized homology and cohomology groups of 4-dimensional polyhedra can be easily achieved by computing these groups only for the elementary spaces in 11.20.

11.21. THEOREM. *The stable homotopy types of simply connected compact 5-dimensional polyhedra coincide with finite one point unions $X_1 \vee \cdots \vee X_r$ where the X_i are elementary Moore spaces in A_n^3 or the elementary spaces $X(w)$, $X(w, \varphi)$. Here the special words satisfy the following conditions* (1), (2),

(1) $w \neq \eta^s \ldots$ *and* $w \neq \ldots {}^s\eta$,

(2) *for each subword of the form* ${}_r\eta^s$ *or* ${}^s\eta_r$ *of* w *(that is,* $w = \dots {}_r\eta^s \dots$ *or* $w = \dots {}^s\eta_r \dots$*) we have* $2r \leqslant s$.

See X.7.3 in Baues [6].

12. Localization

A generalized homology theory k_* (as, for example, defined in Gray [44]) can be used to define equivalence classes of spaces which are called 'k_*-local homotopy types'. We assume that k_* satisfies the *limit axiom*, namely that for all CW-complexes X the map $\varinjlim k_*(X_\alpha) \to k_*X$ is an isomorphism where the X_α run over all finite subcomplexes of X. We consider mainly the classical homology theory

$$k_*(X) = H_*(X, R) = H_*(SX \otimes_{\mathbb{Z}} R) \tag{12.1}$$

given by the homology of X with coefficients in a ring R; compare 3.4.

12.2. DEFINITION. Let *spaces* be the full subcategory of *Top* consisting of CW-spaces. A *CW-pair* (X, A) is a cofibration $A \rightarrowtail X$ in *Top* for which A and X are CW-spaces. For example, a CW-complex X together with a subcomplex A is a CW-pair. A map $f : X \to Y$ between CW-spaces is a *k_*-equivalence* if f induces an isomorphism

$$f_* : k_*(X) \cong k_*(Y).$$

A CW-space A is *k_*-local* if each CW-pair (X, A) for which $A \rightarrowtail$ is an k_*-equivalence admits a retraction $A \to X$. A map $g : Y \to A$ is called a *k_*-localization* if A is k_*-local and g is a k_*-equivalence.

Recall that we introduced the localized category $Ho(C)$ in 3.12. The next result is due to Bousfield [12].

12.3. THEOREM. *For all CW-spaces there exist k_*-localizations. Moreover there is an equivalence of categories*

$$Ho_{k_*}(\mathit{spaces}) \xrightarrow{\sim} \mathit{spaces}_{k_*}/\simeq$$

where the left hand side is the localization with respect to k_*-equivalences and the right hand side in the full homotopy category in Top/ $\simeq$ consisting of k_*-local CW-complexes. The equivalence carries a CW-space to its k_*-localization.

We refer the reader also to I.5.10 in Baues [3] where we consider k_*-equivalences as weak equivalences in a 'cofibration category'. The k_*-equivalences generate an equivalence relation for CW-spaces as follows. We say that CW-spaces X, Y are *k_*-equivalent* if there exist finitely many CW-spaces X_i, $i = 1, \dots, n$, together with k_*-equivalences α_i,

$$X = X_1 \xleftarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \xleftarrow{\alpha_3} \cdots X_n = Y,$$

where α_i and α_{i+1} have opposite directions. The theorem shows that the corresponding k_*-equivalence classes can be identified with the homotopy types of k_*-local CW-spaces which are called *k_*-local homotopy types*. The k_*-local homotopy type of a CW-space X singles out the k_*-specific properties of X. This turned out to be a very successful technique of homotopy theory.

12.4. THEOREM. *Let R be a subring of $\mathbb{Q}$ and let X be a simply connected CW-space. Then X is $H_*(-,R)$-local if and only if* (a) *or equivalently* (b) *is satisfied:*

(a) *The homotopy groups $\pi_n X$ are R-modules.*

(b) *The homology groups $H_n X$ are R-modules.*

Moreover an $H_*(-,R)$-localization $\ell : X \to X_R$ induces isomorphisms

$$\pi_n(X) \otimes_{\mathbb{Z}} R \cong \pi_n(X_R),$$
$$H_n(X) \otimes_{\mathbb{Z}} R \cong H_n(X_R)$$

which carries $\xi \otimes 1$ to $\ell_*(\xi)$.

A proof can be found, e.g., in Hilton, Mislin and Roitberg [52]. Spaces as in the theorem are also called *R-local*, these are the *rational* spaces if $R = \mathbb{Q}$. Moreover for a prime p these are the *p-local* spaces if $R = \mathbb{Z}_p$ is the subring of $\mathbb{Q}$ generated by $1/q$ where q runs over all primes different from p. The classification theorems in Section 9 are actually compatible with R-localization, $R \subset \mathbb{Q}$. For this we define for the category C in (9.1) the full subcategory

$$C_R \subset (n-1)\text{-types} \tag{12.5}$$

consisting of R-localizations X_R of objects X in C. Let

$$\ell_R : C \to C_R$$

be the localization functor. A C_R-kype $\overline{X}_R = (X_R, \pi, k, H, b)$ is *R-local* if π and H are R-modules, and $\overline{X}_R$ is *R-free* if H is a free R-module. Similarly a C_R-bype $\overline{Y} = (Y_R, H_0, H_1, b, \beta)$ in *R-local* if H_0, H_1 are R-modules, and $\overline{Y}_R$ is *R-free* if H_1 is a free R-module. Let

$$\mathit{spaces}_R^{n+1}(C_R)$$

be the full homotopy category of R-local CW-spaces X with $P_{n-1}X \in C_R$ and with $H_i(X_R) = 0$ for $i > n+1$ and $H_{n+1}(X_R)$ a free R-module.

12.6. CLASSIFICATION THEOREM. *There are detecting functors Λ_R, Λ'_R for which the following diagrams of functors commute up to natural isomorphism.*

$$\begin{array}{ccc} \mathit{spaces}^{n+1}(C) & \xrightarrow{\Lambda} & \mathit{Kypes}(C) \\ {\scriptstyle \ell_R}\downarrow & & \downarrow{\scriptstyle \ell_R} \\ \mathit{spaces}_R^{n+1}(C_R) & \xrightarrow{\Lambda_R} & \mathit{Kypes}_R(C_R) \end{array}$$

Here $\mathit{Kypes}_R(C_R)$ *is the category of free R-kypes and* ℓ_R *denotes the obvious localization functors.*

$$\begin{array}{ccc} \mathit{spaces}^{n+1}(C) & \xrightarrow{\Lambda'} & \mathit{Bypes}(C) \\ {\scriptstyle \ell_R}\big\downarrow & & \big\downarrow{\scriptstyle \ell_R} \\ \mathit{spaces}^{n+1}_R(C_R) & \xrightarrow{\Lambda'_R} & \mathit{Bypes}_R(C_R) \end{array}$$

Here $\mathit{Bypes}_R(C_R)$ *is the category of free R-bypes and* ℓ_R *denotes again the localization functors.*

For the definition of Λ_R, Λ'_R we use the Γ-sequence of X_R which coincides with (Γ-sequence of X) $\otimes R$. The theorem shows:

12.7. COROLLARY. *The Postnikov invariants of the localization* X_R *are obtained by R-localizing the Postnikov invariants of* X. *The boundary invariants of the localization* X_R *are obtained by R-localizing the boundary invariants of* X.

If $R = \mathbb{Q}$ is the ring of rational numbers the theory of Postnikov invariants and boundary invariants is completely understood. In fact Postnikov invariants correspond to the differential in the 'minimal model of Sullivan' and boundary invariants correspond to the differential in the 'Quillen minimal model' constructed in Baues and Lemaire [9]. Compare Quillen [79], Sullivan [94] and Chapter I in Baues [3].

12.8. DEFINITION. Let V be a graded $\mathbb{Q}$-vector space with $V_i = 0$ for $i \leqslant 0$. Let $T(V) = \bigoplus\{V^{\otimes n}, n \geqslant 0\}$ be the tensor algebra of V which is a Lie algebra by

$$[x, y] = xy - (-1)^{|x||y|}yx.$$

The *free Lie algebra* $L(V)$ is the Lie subalgebra of $(T(V), [\,,\,])$ generated by V. Let $[L(V), L(V)] \subset L(V)$ be the subset of all brackets $[x, y]$ with $x, y \in L(V)$ and let

$$d : L(V) \to \big[L(V), L(V)\big] \subset L(V)$$

be a $\mathbb{Q}$-linear map of degree -1 satisfying $dd = 0$ and $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$. Then $(L(V), d)$ is called a *Quillen minimal model* with differential d. A morphism between such models is a $\mathbb{Q}$-linear map of degree 0 compatible with brackets and differentials.

12.9. THEOREM. *Homotopy types of* 1*-connected rational spaces* X *are in* 1–1 *correspondence with isomorphism types of Quillen minimal models* $(L(V), d)$ *where* $V_i = H_{i+1}(X, \mathbb{Q})$ *and* $H_i(L(V), d) = \pi_{i+1}X$ *for* $i \geqslant 1$.

12.10. DEFINITION. Let V be a graded $\mathbb{Q}$-vector space such that V^i is finitely generated and $V^i = 0$ for $i \leqslant 1$. Let $\Lambda(V)$ be the *free graded-commutative algebra* generated by V, that is

$$\Lambda(V) = \text{Exterior algebra}\big(V^{\text{odd}}\big) \otimes \text{Symmetric algebra}\big(V^{\text{even}}\big).$$

Let $\widetilde{\Lambda}(V) \cdot \widetilde{\Lambda}(V)$ be the subset of products $x \cdot y$ with $x, y \in \Lambda(V), |x|, |y| \geqslant 1$ and let

$$d : \Lambda(V) \to \widetilde{\Lambda}(V) \cdot \widetilde{\Lambda}(V) \subset \Lambda(V)$$

be a $\mathbb{Q}$-linear map of degree $+1$ satisfying $dd = 0$ and $d(xy) = (dx)y + (-1)^{|x|}x(dy)$. Then $(\Lambda(V), d)$ is called a *Sullivan minimal model* with differential d. A morphism between such models is a $\mathbb{Q}$-linear map of degree 0 compatible with multiplications and differentials.

12.11. THEOREM. *Homotopy types of* 1*-connected rational spaces X for which $H_n X$ is a finitely generated $\mathbb{Q}$-vector space, $n \in \mathbb{Z}$, are in* 1–1 *correspondence with isomorphism types of Sullivan minimal models $(\Lambda(V), d)$ where $V_i = Hom(\pi_i(X), \mathbb{Q})$ and $H^i(\Lambda(V), d) = Hom(H_i(X), \mathbb{Q})$ for $i \geqslant 1$.*

These minimal models yield solutions of Whitehead's realization problem for rational spaces, see 3.7. They illustrate again that homology groups and homotopy groups respectively both 'generate' a homotopy type in a mutually $H\pi$-dual way. The Baues–Lemaire conjecture [9] (recently proved by Majewski [66]) describes the algebraic nature of this $H\pi$-duality. The minimal models allow a deep analysis of the rational properties of a simply connected space. For example, we refer the reader to the wonderful torsion gap result of Halperin [47] or to the alternative 'hyperbolic-elliptic' for rational spaces in Felix [37].

There are p-local analogues of A_n^k-polyhedra as follows. We say that a p-local CW-space X is a pA_n^k *polyhedron* if X is $(n-1)$-connected, $n \geqslant 2$, and the homology $H_i X$ is trivial for $i > n + k$ and is a free $\mathbb{Z}_p$-module for $i = n + k$. Moreover X is a *finite* pA_n^k-polyhedron if in addition all $H_i X$ are finitely generated $\mathbb{Z}_p$-modules. In the stable range we have by 3.6 (2) in Wilkerson [104] unique decompositions as follows.

12.12. THEOREM. *Let p be a prime and $n \geqslant k + 1 \geqslant 2$. Then each finite pA_n^k-polyhedron X admits a homotopy equivalence*

$$X \simeq X_1 \vee \cdots \vee X_r$$

where the one point union of p-local indecomposable CW-spaces on the right hand side is unique up to permutation.

12.13. REMARK. Generalizing the result of Chang 11.7 Henn [48] furnished a complete list of indecomposable pA_n^k-polyhedra for $k = 4p - 5$ and p odd. Such spaces are detected by primary cohomology operations while the A_n^3-polyhedra in (11.16) are not detected by primary cohomology operations. The classification of Henn uses implicitly the boundary invariants of X.

12.14. REMARK. For the ring $R = \mathbb{Z}/p$ where p is a prime the $H_*(-, \mathbb{Z}/p)$-localization X_p of a simply connected space X is the *p-completion* of Bousfield and Kan [14]. If in addition X has finite type then X_p is the p-profinite completion for which $\pi_n X_p$ is given by the p-profinite completion of $\pi_n X$; compare Sullivan [93] and Quillen [78]. Recently Goerss [46] considered simplicial coalgebras as models of $H_*(-,\mathbb{F})$-local spaces where $\mathbb{F}$ is an algebraically closed field; see also Kriz [62]. Moreover Bousfield [13] and Franke [38] consider algebraic models of k_*-local spaces with $k_* = K$-theory; they restrict attention, however, to the stable range.

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CHAPTER 2

Homotopy Theories and Model Categories

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Contents

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1. Introduction

This paper is an introduction to the theory of "model categories", which was developed by Quillen in [22] and [23]. By definition a model category is just an ordinary category with three specified classes of morphisms, called *fibrations*, *cofibrations* and *weak equivalences*, which satisfy a few simple axioms that are deliberately reminiscent of properties of topological spaces. Surprisingly enough, these axioms give a reasonably general context in which it is possible to set up the basic machinery of homotopy theory. The machinery can then be used immediately in a large number of different settings, as long as the axioms are checked in each case. Although many of these settings are geometric (spaces (§8), fibrewise spaces (3.11), G-spaces [11], spectra [3], diagrams of spaces [10] ...), some of them are not (chain complexes (§7), simplicial commutative rings [24], simplicial groups [23] ...). Certainly each setting has its own technical and computational peculiarities, but the advantage of an abstract approach is that they can all be studied with the same tools and described in the same language. What is the suspension of an augmented commutative algebra? One of incidental appeals of Quillen's theory (to a topologist!) is that it both makes a question like this respectable and gives it an interesting answer (11.3).

We have tried to minimize the prerequisites needed for understanding this paper; it should be enough to have some familiarity with CW-complexes, with chain complexes, and with the basic terminology associated with categories. Almost all of the material we present is due to Quillen [22], but we have replaced his treatment of suspension functors and loop functors by a general construction of homotopy pushouts and homotopy pullbacks in a model category. What we do along these lines can certainly be carried further. This paper is not in any sense a survey of everything that is known about model categories; in fact we cover only a fraction of the material in [22]. The last section has a discussion of some ways in which model categories have been used in topology and algebra.

Organization of the paper. Section 2 contains background material, principally a discussion of some categorical constructions (limits and colimits) which come up almost immediately in any attempt to build new objects of some abstract category out of old ones. Section 3 gives the definition of what it means for a category $\mathbf{C}$ to be a model category, establishes some terminology, and sketches a few examples. In §4 we study the notion of "homotopy" in $\mathbf{C}$ and in §5 carry out the construction of the homotopy category $\mathrm{Ho}(\mathbf{C})$. Section 6 gives $\mathrm{Ho}(\mathbf{C})$ a more conceptual significance by showing that it is equivalent to the "localization" of $\mathbf{C}$ with respect to the class of weak equivalences. For our purposes the "homotopy theory" associated to $\mathbf{C}$ is the homotopy category $\mathrm{Ho}(\mathbf{C})$ together with various related constructions (§10).

Sections 7 and 8 describe in detail two basic examples of model categories, namely the category $\mathbf{Top}$ of topological spaces and the category $\mathbf{Ch}_R$ of non-negative chain complexes of modules over a ring R. The homotopy theory of $\mathbf{Top}$ is of course familiar, and it turns out that the homotopy theory of $\mathbf{Ch}_R$ is what is usually called homological algebra. Comparing these two examples helps explain why Quillen called the study of model categories "homotopical algebra" and thought of it as a generalization of homological algebra. In §9 we give a criterion for a pair of functors between two model categories to induce equivalences between the associated homotopy categories; pinning down the meaning of "induce" here leads to the definition of derived functor. Section 10 constructs homotopy pushouts and homotopy pullbacks in an arbitrary model category in terms of derived functors. Finally, §11 contains some concluding remarks, sketches some applications of homotopical algebra, and mentions a way in which the theory has developed since Quillen.

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2. Categories

In this section we review some basic ideas and constructions from category theory; for more details see [17]. The reader might want to skip this section on first reading and return to it as needed.

2.1. *Categories.* We will take for granted the notions of *category*, *subcategory*, *functor* and *natural transformation* [17, I]. If $\mathbf{C}$ is a category and X and Y are objects of $\mathbf{C}$, we will assume that the morphisms $f : X \to Y$ in $\mathbf{C}$ form a *set* $\mathrm{Hom}_{\mathbf{C}}(X, Y)$, rather than a class, a collection, or something larger. These morphisms are also called *maps* or *arrows* in $\mathbf{C}$ from X to Y. Some categories that come up in this paper are:

(i) the category **Set** whose objects are sets and whose morphisms are functions from one set to another,
(ii) the category **Top** whose objects are topological spaces and whose morphisms are continuous maps,
(iii) the category Mod_R whose objects are left R-modules (where R is an associative ring with unit) and whose morphisms are R-module homomorphisms.

2.2. *Natural equivalences.* Suppose that $F, F' : \mathbf{C} \to \mathbf{D}$ are two functors, and that t is a natural transformation from F to F'. The transformation t is called a *natural equivalence* [17, p. 16] if the morphism $t_X : F(X) \to F'(X)$ is an isomorphism in $\mathbf{D}$ for each object X of $\mathbf{C}$. The functor F is said to be an *equivalence of categories* if there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that the composites FG and GF are related to the appropriate identity functors by natural equivalences [17, p. 90].

2.3. *Full and faithful.* A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be *full* (resp. *faithful*) if for each pair (X, Y) of objects of $\mathbf{C}$ the map

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}}\big(F(X), F(Y)\big)$$

given by F is an epimorphism (resp. a monomorphism) [17, p. 15]. A *full subcategory* $\mathbf{C}'$ of $\mathbf{C}$ is a subcategory with the property that the inclusion functor $i : \mathbf{C}' \to \mathbf{C}$ is full (the functor i is always faithful). A full subcategory of $\mathbf{C}$ is determined by the objects in $\mathbf{C}$ which it contains, and we will sometimes speak of the full subcategory of $\mathbf{C}$ *generated* by a certain collection of objects.

2.4. *Opposite category.* If $\mathbf{C}$ is a category then the *opposite category* $\mathbf{C}^{\mathrm{op}}$ is the category with the same objects as $\mathbf{C}$ but with one morphism $f^{\mathrm{op}} : Y \to X$ for each morphism $f : X \to Y$ in $\mathbf{C}$ [17, p. 33]. The morphisms of $\mathbf{C}^{\mathrm{op}}$ compose according to the formula $f^{\mathrm{op}}g^{\mathrm{op}} = (gf)^{\mathrm{op}}$. A functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ is the same thing as what is sometimes called a *contravariant functor* $\mathbf{C} \to \mathbf{D}$. For example, for any category $\mathbf{C}$ the assignment $(X, Y) \mapsto \mathrm{Hom}_{\mathbf{C}}(X, Y)$ gives a functor

$$\mathrm{Hom}_{\mathbf{C}}(-,-) : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}.$$

2.5. *Smallness and functor categories.* A category $\mathbf{D}$ is said to be *small* if the collection $\mathrm{Ob}(\mathbf{D})$ of objects of $\mathbf{D}$ forms a set, and *finite* if $\mathrm{Ob}(\mathbf{D})$ is a finite set and $\mathbf{D}$ has only a finite number of morphisms between any two objects. If $\mathbf{C}$ is a category and $\mathbf{D}$ is a small category, then there is a *functor category* $\mathbf{C}^{\mathbf{D}}$ in which the objects are functors $F : \mathbf{D} \to \mathbf{C}$ and the morphisms are natural transformations; this is also called the category of *diagrams in* $\mathbf{C}$ *with the shape of* $\mathbf{D}$. For example, if $\mathbf{D}$ is the category $\{a \to b\}$ with two objects and one nonidentity morphism, then the objects of $\mathbf{C}^{\mathbf{D}}$ are exactly the morphisms $f : X(a) \to X(b)$ of $\mathbf{C}$ and a map $t : f \to g$ in $\mathbf{C}^{\mathbf{D}}$ is a commutative diagram

$$\begin{array}{ccc} X(a) & \xrightarrow{t_a} & Y(a) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ X(b) & \xrightarrow{t_b} & Y(b) \end{array} .$$

In this case $\mathbf{C}^{\mathbf{D}}$ is called the *category of morphisms* of $\mathbf{C}$ and is denoted $\mathbf{Mor}(\mathbf{C})$.

2.6. *Retracts.* An object X of a category $\mathbf{C}$ is said to be a *retract* of an object Y if there exist morphisms $i : X \to Y$ and $r : Y \to X$ such that $ri = \mathrm{id}_X$. For example, in the category Mod_R an object X is a retract of Y if and only if there exists a module Z such that Y is isomorphic to $X \oplus Z$. If f and g are morphisms of $\mathbf{C}$, we will say that f is a retract of g if the object of $\mathbf{Mor}(\mathbf{C})$ represented by f is a retract of the object of $\mathbf{Mor}(\mathbf{C})$ represented by g (see the proof of the next lemma for a picture of what this means).

2.7. LEMMA. *If g is an isomorphism in* $\mathbf{C}$ *and f is a retract of g, then f is also an isomorphism.*

PROOF. Since f is a retract of g, there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array}$$

in which the composites ri and $r'i'$ are the appropriate identity maps. Since g is an isomorphism, there is a map $h : Y' \to Y$ such that $hg = \mathrm{id}_Y$ and $gh = \mathrm{id}_{Y'}$. It is easy to check that $k = rhi'$ is the inverse of f. □

2.8. *Adjoint functors.* Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be a pair of functors. An *adjunction* from F to G is a collection of isomorphisms

$$\alpha_{X,Y} : \mathrm{Hom}_{\mathbf{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathbf{C}}\big(X, G(Y)\big), \quad X \in \mathrm{Ob}(\mathbf{C}),\ Y \in \mathrm{Ob}(\mathbf{D}),$$

natural in X and Y, i.e. a collection which gives a natural equivalence (2.2) between the two indicated Hom-functors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$ (see 2.4). If such an adjunction exists we write

$$F : \mathbf{C} \Longleftrightarrow \mathbf{D} : G$$

and say that F and G are *adjoint functors* or that (F, G) is an *adjoint pair*, F being the *left adjoint* of G and G the *right adjoint* of F. Any two left adjoints of G (resp. right adjoints of F) are canonically naturally equivalent, so we speak of "the" left adjoint or right adjoint of a functor (if such a left or right adjoint exists) [17, p. 81]. If $f : F(X) \to Y$ (resp. $g : X \to G(Y)$), we denote its image under the bijection $\alpha_{X,Y}$ by $f^\sharp : X \to G(Y)$ (resp. $g^\flat : F(X) \to Y$).

2.9. EXAMPLE. Let $G : \mathrm{Mod}_R \to \mathbf{Set}$ be the forgetful functor which assigns to each R-module its underlying set. Then G has a left adjoint $F : \mathbf{Set} \to \mathrm{Mod}_R$ which assigns to each set X the free R-module generated by the elements of X. The functor G does not have a right adjoint.

2.10. EXAMPLE. Let $G : \mathbf{Top} \to \mathbf{Set}$ be the forgetful functor which assigns to each topological space X its underlying set. Then G has a left adjoint, which is the functor which assigns to each set Y the topological space given by Y with the discrete topology. The functor G also has a right adjoint, which assigns to each set Y the topological space given by Y with the indiscrete topology (cf. [17, p. 85]).

2.11. *Colimits*

We introduce the notion of the colimit of a functor. Let $\mathbf{C}$ be a category and $\mathbf{D}$ a small category. Typically, $\mathbf{C}$ is one of the categories in 2.1 and $\mathbf{D}$ is from the following list.

2.12. *Shapes of colimit diagrams*

(i) A category with a set $\mathcal{I}$ of objects and no nonidentity morphisms.
(ii) The category $\mathbf{D} = \{a \leftarrow b \to c\}$, with three objects and the two indicated nonidentity morphisms.

(iii) The category $\mathbf{Z}^+ = \{0 \to 1 \to 2 \to 3 \to \ldots\}$ with objects the non-negative integers and a single morphism $i \to j$ for $i \leqslant j$.

There is a diagonal or "constant diagram" functor

$$\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{D}},$$

which carries an object $X \in \mathbf{C}$ to the constant functor $\Delta(X) : \mathbf{D} \to \mathbf{C}$ (by definition, this "constant functor" sends each object of $\mathbf{D}$ to X and each morphism of $\mathbf{D}$ to id_X). The functor Δ assigns to each morphism $f : X \to X'$ of $\mathbf{C}$ the constant natural transformation $t(f) : \Delta_X \to \Delta_{X'}$ determined by the formula $t(f)_d = f$ for each object d of $\mathbf{D}$.

2.13. Definition. Let $\mathbf{D}$ be a small category and $F : \mathbf{D} \to \mathbf{C}$ a functor. A *colimit* for F is an object C of $\mathbf{C}$ together with a natural transformation $t : F \to \Delta(C)$ such that for every object X of $\mathbf{C}$ and every natural transformation $s : F \to \Delta(X)$, there exists a unique map $s' : C \to X$ in $\mathbf{C}$ such that $\Delta(s')t = s$ [17, p. 67].

Remark. The universal property of a colimit implies as usual that any two colimits for F are canonically isomorphic. If a colimit of F exists we will speak of "the" colimit of F and denote it $\mathrm{colim}(F)$. The colimit is sometimes called the direct limit, and denoted $\varinjlim F$, $\varinjlim^{\mathbf{D}} F$ or $\mathrm{colim}^{\mathbf{D}} F$. Roughly speaking, $\Delta(\mathrm{colim}(F))$ is the constant diagram which is most efficient at receiving a map from F, in the sense that any map from F to a constant diagram extends uniquely over the universal map $F \to \Delta(\mathrm{colim}(F))$.

2.14. Remark. A category $\mathbf{C}$ is said to *have all small* (resp. *finite*) *colimits* if $\mathrm{colim}(F)$ exists for any functor F from a small (resp. finite) category $\mathbf{D}$ to $\mathbf{C}$. The categories **Set**, **Top** and $\mathbf{Mod}_R$ have all small colimits. Suppose that $\mathbf{D}$ is a small category and $F : \mathbf{D} \to \mathbf{Set}$ is a functor. Let U be the disjoint union of the sets which appear as values of F, i.e., let U be the set of pairs (d, x) where $d \in \mathrm{Ob}(\mathbf{D})$ and $x \in F(d)$. Then $\mathrm{colim}(F)$ is the quotient of U with respect to the equivalence relation "$\sim$" generated by the formulas $(d, x) \sim (d', F(f)(x))$, where $f : d \to d'$ is a morphism of $\mathbf{D}$. If $F : \mathbf{D} \to \mathbf{Top}$ is a functor, then $\mathrm{colim}(F)$ is an analogous quotient space of the space U which is the disjoint union of the spaces appearing as values of F. If $F : \mathbf{D} \to \mathrm{Mod}_R$ is a functor, then $\mathrm{colim}(F)$ is an analogous quotient module of the module U which is the direct sum of the modules appearing as values of F.

Remark. If $\mathrm{colim}(F)$ exists for every object F of $\mathbf{C}^{\mathbf{D}}$, an argument from the universal property (2.13) shows that the various objects $\mathrm{colim}(F)$ of $\mathbf{C}$ fit together into a functor $\mathrm{colim}(-)$ which is left adjoint to Δ:

$$\mathrm{colim} : \mathbf{C}^{\mathbf{D}} \Longleftrightarrow \mathbf{C} : \Delta.$$

We will now give some examples of colimits [17, p. 64].

2.15. *Coproducts.* Let $\mathbf{D}$ be the category of 2.12(i), so that a functor $X : \mathbf{D} \to \mathbf{C}$ is just a collection $\{X_i\}_{i \in \mathcal{I}}$ of objects of $\mathbf{C}$. The colimit of X is called the *coproduct* of the collection and written $\coprod_i X_i$ or, if $\mathcal{I} = \{0, 1\}$, $X_0 \coprod X_1$. If $\mathbf{C}$ is **Set** or **Top** the coproduct is disjoint union; if $\mathbf{C}$ is $\mathbf{Mod}_R$, coproduct is direct sum. If $\mathcal{I} = \{0, 1\}$, then the definition of colimit (2.13) gives natural maps $\mathrm{in}_0 : X_0 \to X_0 \coprod X_1$ and $\mathrm{in}_1 : X_1 \to X_0 \coprod X_1$; given maps $f_i : X_i \to Y$ ($i = 0, 1$) there is a unique map $f : X_0 \coprod X_1 \to Y$ such that $f \cdot \mathrm{in}_i = f_i$ ($i = 0, 1$). The map f is ordinarily denoted $f_0 + f_1$.

2.16. *Pushouts.* If $\mathbf{D}$ is the category of 2.12(ii), then a functor $X : \mathbf{D} \to \mathbf{C}$ is a diagram $X(a) \leftarrow X(b) \to X(c)$ in $\mathbf{C}$. In this case the colimit of X is called the *pushout* P of the diagram $X(a) \leftarrow X(b) \to X(c)$. It is the result of appropriately gluing $X(a)$ to $X(c)$ along $X(b)$. The definition of colimit gives a natural commutative diagram

$$\begin{array}{ccc} X(b) & \xrightarrow{i} & X(c) \\ j \downarrow & & j' \downarrow \\ X(a) & \xrightarrow{i'} & P \end{array}$$

Any diagram isomorphic to a diagram of this form is called a *pushout diagram*; the map i' is called the *cobase change* of i (along j) and the map j' is called the *cobase change* of j (along i). Given maps $f_a : X(a) \to Y$ and $f_c : X(c) \to Y$ such that $f_a j = f_c i$, there is a unique map $f : P \to Y$ such that $fj' = f_c$ and $fi' = f_a$.

2.17. *Sequential colimits.* If $\mathbf{D}$ is the category of 2.12(iii), a functor $X : \mathbf{D} \to \mathbf{C}$ is a diagram of the following form

$$X(0) \to X(1) \to \cdots \to X(n) \to \cdots$$

in $\mathbf{C}$; this is called a *sequential direct system* in $\mathbf{C}$. The colimit of this direct system is called the sequential colimit of the objects $X(n)$, and denoted $\mathrm{colim}_n X(n)$. If $\mathbf{C}$ is one of the categories **Set**, **Top** or $\mathbf{Mod}_R$ and each one of the maps $X(n) \to X(n+1)$ is an inclusion, then $\mathrm{colim}_n X(n)$ can be interpreted as an increasing union of the $X(n)$; if $\mathbf{C} = \mathbf{Top}$ a subset of this union is open if and only if its intersection with each $X(n)$ is open.

2.18. *Limits*

We next introduce the notion of the limit of a functor [17, p. 68]. This is strictly dual to the notion of colimit, in the sense that a limit of a functor $F : \mathbf{D} \to \mathbf{C}$ is the same as a colimit of the "opposite functor" $F^{\mathrm{op}} : \mathbf{D}^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}}$. From a logical point of view this may be everything there is to say about limits, but it is worthwhile to make the construction more explicit and work out some examples.

Let $\mathbf{C}$ be a category and $\mathbf{D}$ a small category. Typically, $\mathbf{C}$ is as before (2.1) and $\mathbf{D}$ is one of the following.

2.19. *Shapes of limit diagrams*

(i) A category with a set $\mathcal{I}$ of objects and no nonidentity morphisms.
(ii) The category $\mathbf{D} = \{a \to b \leftarrow c\}$, with three objects and the two indicated nonidentity morphisms.

Let $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}$ be as before (2.11) the "constant diagram" functor.

2.20. DEFINITION. Let $\mathbf{D}$ be a small category and $F : \mathbf{D} \to \mathbf{C}$ a functor. A *limit* for F is an object L of $\mathbf{C}$ together with a natural transformation $t : \Delta(L) \to F$ such that for every object X of $\mathbf{C}$ and every natural transformation $s : \Delta(X) \to F$, there exists a unique map $s' : X \to L$ in $\mathbf{C}$ such that $t\Delta(s') = s$.

REMARK. The universal property of a limit implies as usual that any two limits for F are canonically isomorphic. If a limit of F exists we will speak of "the" limit of F and denote it $\lim(F)$. The limit is sometimes called the inverse limit, and denoted $\varprojlim F$, $\varprojlim_{\mathbf{D}} F$ or $\lim_{\mathbf{D}} F$. Roughly speaking, $\Delta(\lim(F))$ is the constant diagram which is most efficient at originating a map to F, in the sense that any map from a constant diagram to F lifts uniquely over the universal map $\Delta(\operatorname{colim}(F)) \to F$.

2.21. REMARK. A category $\mathbf{C}$ is said to *have all small* (resp. *finite*) *limits* if $\lim(F)$ exists for any functor F from a small (resp. finite) category $\mathbf{D}$ to $\mathbf{C}$. The categories **Set**, **Top** and Mod_R have all small limits. Suppose that $\mathbf{D}$ is a small category and $F : \mathbf{D} \to \mathbf{Set}$ is a functor. Let P be the product of the sets which appear as values of F, i.e., let U be the set of pairs (d, x) where $d \in \mathrm{Ob}(\mathbf{D})$ and $x \in F(d)$, $q : U \to \mathrm{Ob}(\mathbf{D})$ the map with $q(d, x) = d$, and P the set of all functions $s : \mathrm{Ob}(\mathbf{D}) \to U$ such that qs is the identity map of $\mathrm{Ob}(\mathbf{D})$. For $s \in P$ write $s(d) = (d, s_1(d))$, with $s_1(d) \in F(d)$. Then $\lim(F)$ is the subset of P consisting of functions s which satisfy the equation $s_1(d') = F(f)(s_1(d))$ for each morphism $f : d \to d'$ of $\mathbf{D}$. If $F : \mathbf{D} \to \mathbf{Top}$ is a functor, then $\lim(F)$ is the corresponding subspace of the space P which is the product of the spaces appearing as values of F. If $F : \mathbf{D} \to \mathrm{Mod}_R$ is a functor, then $\lim(F)$ is the corresponding submodule of the module U which is the direct product of the modules appearing as values of F.

REMARK. If $\lim(F)$ exists for every object F of $\mathbf{C}^{\mathbf{D}}$, an argument from the universal property (2.20) shows that various objects $\lim(F)$ of $\mathbf{C}$ fit together into a functor $\lim(-)$ which is right adjoint to Δ:

$$\Delta : \mathbf{C} \Longleftrightarrow \mathbf{C}^{\mathbf{D}} : \lim .$$

We will now give two examples of limits [17, p. 70].

2.22. *Products.* Let $\mathbf{D}$ be the category of 2.19(i), so that a functor $X : \mathbf{D} \to \mathbf{C}$ is just a collection $\{X_i\}_{i \in \mathcal{I}}$ of objects of $\mathbf{C}$. The limit of X is called the *product* of the collection and written $\prod_i X_i$ or, if $\mathcal{I} = \{0, 1\}$, $X_0 \times X_1$ (the notation "$X_0 \prod X_1$" is more logical but seems less common). If $\mathbf{C}$ is **Set** or **Top** the product is what is usually called direct product or cartesian product. If $\mathcal{I} = \{0, 1\}$ then the definition of limit (2.20) gives natural maps $\mathrm{pr}_0 : X_0 \times X_1 \to X_0$ and $\mathrm{pr}_1 : X_0 \times X_1 \to X_1$; given maps $f_i : Y \to X_i$ $(i = 0, 1)$ there is a unique map $f : Y \to X_0 \times X_1$ such that $\mathrm{pr}_i \cdot f = f_i$ $(i = 0, 1)$. The map f is ordinarily denoted (f_0, f_1).

2.23. *Pullbacks.* If $\mathbf{D}$ is the category of 2.19(ii), then a functor $X : \mathbf{D} \to \mathbf{C}$ is a diagram $X(a) \to X(b) \leftarrow X(c)$ in $\mathbf{C}$. In this case the limit of X is called the *pullback* P of the diagram $X(a) \to X(b) \leftarrow X(c)$. The definition of limit gives a natural commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{i'} & X(c) \\ j' \downarrow & & j \downarrow \\ X(a) & \xrightarrow{i} & X(b) \end{array}$$

Any diagram isomorphic to a diagram of this form is called a *pullback diagram*; the map i' is called the *base change* of i (along j) and the map j' is called the *base change* of j (along i). Given maps $f_a : Y \to X(a)$ and $f_c : Y \to X(c)$ such that $if_a = jf_c$, there is a unique map $f : Y \to P$ such that $i'f = f_c$ and $j'f = f_a$.

2.24. *Some remarks on limits and colimits*

An object $\varnothing$ of a category $\mathbf{C}$ is said to be an *initial object* if there is exactly one map from $\varnothing$ to any object X of $\mathbf{C}$. Dually, an object $*$ of $\mathbf{C}$ is said to be a *terminal object* if there is exactly one map $X \to *$ for any object X of $\mathbf{C}$. Clearly initial and terminal objects of $\mathbf{C}$ are unique up to canonical isomorphism. The proof of the following statement just involves unraveling the definitions.

2.25. PROPOSITION. *Let* $\mathbf{C}$ *be a category,* $\mathbf{D}$ *the empty category (i.e. the category with no objects), and* $F : \mathbf{D} \to \mathbf{C}$ *the unique functor. Then* $\operatorname{colim}(F)$, *if it exists, is an initial object of* $\mathbf{C}$ *and* $\lim(F)$, *if it exists, is a terminal object of* $\mathbf{C}$.

Suppose that $\mathbf{D}$ is a small category, that $X : \mathbf{D} \to \mathbf{C}$ is a functor, and that $F : \mathbf{C} \to \mathbf{C}'$ is a functor. If $\operatorname{colim}(X)$ and $\operatorname{colim}(FX)$ both exist, then it is easy to see that there is a natural map $\operatorname{colim}(FX) \to F(\operatorname{colim} X)$. Similarly, if $\lim(F)$ and $\lim(FX)$ both exist, then it is easy to see that there is a natural map $F(\lim X) \to \lim(FX)$. The functor F is said to *preserve colimits* if whenever $X : \mathbf{D} \to \mathbf{C}$ is a functor such that $\operatorname{colim}(X)$ exists, then $\operatorname{colim}(FX)$ exists and the natural map $\operatorname{colim}(FX) \to F(\operatorname{colim} X)$ is an isomorphism. The functor F is said to *preserve limits* if the corresponding dual condition holds [17, p. 112]. The following proposition is a formal consequence of the definition of an adjoint functor pair.

2.26. PROPOSITION [17, pp. 114–115]. *Suppose that*

$$F : \mathbf{C} \Longleftrightarrow \mathbf{C}' : G$$

is an adjoint functor pair. Then F *preserves colimits and* G *preserves limits.*

REMARK. Proposition 2.26 explains why the underlying set of a product (2.22) or pullback (2.23) in the category $\mathbf{Mod}_R$ or **Top** is the same as the product or pullback of the underlying sets: in each case the underlying set (or forgetful) functor is a right adjoint (2.9, 2.10) and so preserves limits, e.g., products and pullbacks. Conversely, 2.26 pins down why the forgetful functor G of 2.9 cannot possibly be a left adjoint or equivalently cannot possibly have a right adjoint: G does not preserve colimits, since, for instance, it does not take coproducts of R-modules (i.e. direct sums) to coproducts of sets (i.e. disjoint unions).

We will use the following proposition in §10.

2.27. LEMMA [17, p. 112]. *Suppose that* $\mathbf{C}$ *has all small limits and colimits and that* $\mathbf{D}$ *is a small category. Then the functor category* $\mathbf{C}^{\mathbf{D}}$ *also has small limits and colimits.*

REMARK. In the situation of 2.27 the colimits and limits in $\mathbf{C}^{\mathbf{D}}$ can be computed "pointwise" in the following sense. Suppose that $X : \mathbf{D}' \to \mathbf{C}^{\mathbf{D}}$ is a functor. Then for each object d of $\mathbf{D}$ there is an associated functor $X_d : \mathbf{D}' \to \mathbf{C}$ given by the formula $X_d(d') = (X(d'))(d)$. It is not hard to check that for each $d \in \mathrm{Ob}(\mathbf{D})$ there are natural isomorphisms $(\mathrm{colim} X)(d) \cong \mathrm{colim}(X_d)$ and $(\lim X)(d) \cong \lim(X_d)$.

3. Model categories

In this section we introduce the concept of a model category and give some examples. Since checking that a category has a model category structure is not usually trivial, we defer verifying the examples until later (§7 and §8).

3.1. DEFINITION. Given a commutative square diagram of the following form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle p}\downarrow \\ B & \xrightarrow{g} & Y \end{array} \tag{3.2}$$

a *lift* or *lifting* in the diagram is a map $h : B \to X$ such that the resulting diagram with five arrows commutes, i.e. such that $hi = f$ and $ph = g$.

3.3. DEFINITION. A *model category* is a category $\mathbf{C}$ with three distinguished classes of maps:

(i) *weak equivalences* ($\xrightarrow{\sim}$),
(ii) *fibrations* ($\twoheadrightarrow$), and
(iii) *cofibrations* ($\hookrightarrow$)

each of which is closed under composition and contains all identity maps. A map which is both a fibration (resp. cofibration) and a weak equivalence is called an *acyclic fibration* (resp. *acyclic cofibration*). We require the following axioms.

MC1 Finite limits and colimits exist in $\mathbf{C}$ (2.14, 2.21).

MC2 If f and g are maps in **C** such that gf is defined and if two of the three maps f, g, gf are weak equivalences, then so is the third.

MC3 If f is a retract of g (2.6) and g is a fibration, cofibration, or a weak equivalence, then so is f.

MC4 Given a commutative diagram of the form 3.2, a lift exists in the diagram in either of the following two situations: (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.

MC5 Any map f can be factored in two ways: (i) $f = pi$, where i is a cofibration and p is an acyclic fibration, and (ii) $f = pi$, where i is an acyclic cofibration and p is a fibration.

REMARK. The above axioms describe what in [22] is called a "closed" model category; since no other kind of model category comes up in this paper, we have decided to leave out the word "closed". In [22] Quillen uses the terms "trivial cofibration" and "trivial fibration" instead of "acyclic cofibration" and "acyclic fibration". This conflicts with the ordinary homotopy theoretic use of "trivial fibration" to mean a fibration in which the total space is equivalent to the product of the base and fibre; in geometric examples of model categories, the "acyclic fibrations" of 3.3 usually turn out to be fibrations with a trivial fibre, so that the total space is equivalent to the base. We have followed Quillen's later practice in using the word "acyclic". The axioms as stated are taken from [23].

3.4. REMARK. By **MC1** and 2.25, a model category **C** has both an initial object $\varnothing$ and a terminal object $*$. An object $A \in \mathbf{C}$ is said to be *cofibrant* if $\varnothing \to A$ is a cofibration and *fibrant* if $A \to *$ is a fibration. Later on, when we define the homotopy category $\mathrm{Ho}(\mathbf{C})$, we will see that $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, B)$ is in general a quotient of $\mathrm{Hom}_{\mathbf{C}}(A, B)$ only if A is cofibrant and B is fibrant; if A is not cofibrant or B is not fibrant, then there are not in general a sufficient number of maps $A \to B$ in **C** to represent every map in the homotopy category.

The factorizations of a map in a model category provided by **MC5** are not required to be functorial. In most examples (e.g., in cases in which the factorizations are constructed by the small object argument of 7.12) the factorizations can be chosen to be functorial.

We now give some examples of model categories.

3.5. EXAMPLE (see §8). The category **Top** of topological spaces can be given the structure of a model category by defining $f : X \to Y$ to be

(i) a *weak equivalence* if f is a weak homotopy equivalence (8.1)
(ii) a *cofibration* if f is a retract (2.6) of a map $X \to Y'$ in which Y' is obtained from X by attaching cells (8.8), and
(iii) a *fibration* if f is a Serre fibration (8.2).

With respect to this model category structure, the homotopy category $\mathrm{Ho}(\mathbf{Top})$ is equivalent to the usual homotopy category of CW-complexes (cf. 8.4).

The above model category structure seems to us to be the one which comes up most frequently in everyday algebraic topology. It puts an emphasis on CW-structures; every object is fibrant, and the cofibrant objects are exactly the spaces which are retracts of generalized CW-complexes (where a "generalized CW-complex" is a space built up from cells, without the requirement that the cells be attached in order by dimension.) In some topological situations, though, weak homotopy equivalences are not the correct maps to focus on. It is natural to ask whether there is another model category structure on **Top** with respect to which the "weak equivalences" are the ordinary homotopy equivalences. There is a beautiful paper of Strom [26] which gives a positive answer to this question. If B is a topological space, call a subspace inclusion $i : A \to B$ a *closed Hurewicz cofibration* if A is a closed subspace of B and i has the homotopy extension property, i.e. for every space Y a lift (3.1) exists in every commutative diagram

$$\begin{array}{ccc} B \times 0 \cup A \times [0,1] & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B \times [0,1] & \longrightarrow & * \end{array}$$

Call a map $p : X \to Y$ a *Hurewicz fibration* if p has the homotopy lifting property, i.e. for every space A a lift exists in every commutative diagram

$$\begin{array}{ccc} A \times 0 & \longrightarrow & X \\ \downarrow & & {\scriptstyle p}\downarrow \\ A \times [0,1] & \longrightarrow & Y \end{array}$$

3.6. EXAMPLE ([26]). The category **Top** of topological spaces can be given the structure of a model category by defining a map $f : X \to Y$ to be

(i) a *weak equivalence* if f is a homotopy equivalence,
(ii) a *cofibration* if f is a closed Hurewicz cofibration, and
(iii) a *fibration* if f is a Hurewicz fibration.

With respect to this model category structure, the homotopy category Ho(**Top**) is equivalent to the usual homotopy category of topological spaces.

REMARK. The model category structure of 3.6 is quite different from the one of 3.5. For example, let W be the "Warsaw circle"; this is the compact subspace of the plane given by the union of the interval $[-1, 1]$ on the y-axis, the graph of $y = \sin(1/x)$ for $0 < x \leqslant 1$, and an arc joining $(1, \sin(1))$ to $(0, -1)$. Then the map from W to a point is a weak equivalence with respect to the model category structure of 3.5 but not a weak equivalence with respect to the model category structure of 3.6.

It turns out that many purely algebraic categories also carry model category structures. Let R be a ring and $\mathbf{Ch}_R$ the category of non-negatively graded chain complexes over R.

3.7. EXAMPLE (see §7). The category $\mathbf{Ch}_R$ can be given the structure of a model category by defining a map $f : M \to N$ to be

(i) a *weak equivalence* if f induces isomorphisms on homology groups,

(ii) a *cofibration* if for each $k \geqslant 0$ the map $f_k : M_k \to N_k$ is a monomorphism with a projective R-module (§7.1) as its cokernel, and
(iii) a *fibration* if for each $k \geqslant 1$ the map $f_k : M_k \to N_k$ is an epimorphism.

The cofibrant objects in $\mathbf{Ch}_R$ are the chain complexes M such that each M_k is a projective R-module. The homotopy category $\mathrm{Ho}(\mathbf{Ch}_R)$ is equivalent to the category whose objects are these cofibrant chain complexes and whose morphisms are ordinary chain homotopy classes of maps (cf. proof of 7.3).

Given a model category, it is possible to construct many other model categories associated to it. We will do quite a bit of this in §10. Here are two elementary examples.

3.8. EXAMPLE. Let $\mathbf{C}$ be a model category. Then the opposite category $\mathbf{C}^{\mathrm{op}}$ (2.4) can be given the structure of a model category by defining a map $f^{\mathrm{op}} : Y \to X$ to be

(i) a *weak equivalence* if $f : X \to Y$ is a weak equivalence in $\mathbf{C}$,
(ii) a *cofibration* if $f : X \to Y$ is a fibration in $\mathbf{C}$,
(iii) a *fibration* if $f : X \to Y$ is a cofibration in $\mathbf{C}$.

3.9. *Duality.* Example 3.8 reflects the fact that the axioms for a model category are self-dual. Let P be a statement about model categories and P^* the dual statement obtained by reversing the arrows in P and switching "cofibration" with "fibration". If P is true for all model categories, then so is P^*.

REMARK. The duality construction in 3.9 corresponds via 3.5 or 3.6 to what is usually called "Eckmann–Hilton" duality in ordinary homotopy theory. Since there are interesting true statements P about the homotopy theory of topological spaces whose Eckmann–Hilton dual statements P^* are not true, it must be that there are interesting facts about ordinary homotopy theory which cannot be derived from the model category axioms. Of course this is something to be expected; the axioms are an attempt to codify what all homotopy theories might have in common, and just about any particular model category has additional properties that go beyond what the axioms give.

If $\mathbf{C}$ is a category and A is an object of $\mathbf{C}$, the under category [17, p. 46] (or comma category) $A{\downarrow}\mathbf{C}$ is the category in which an object is a map $f : A \to X$ in $\mathbf{C}$. A morphism in this category from $f : A \to X$ to $g : A \to Y$ is a map $h : X \to Y$ in $\mathbf{C}$ such that $hf = g$.

3.10. REMARK. Let $\mathbf{C}$ be a model category and A an object of $\mathbf{C}$. Then it is possible to give $A{\downarrow}\mathbf{C}$ the structure of a model category by defining $h : (A \to X) \to (A \to Y)$ in $A{\downarrow}\mathbf{C}$ to be

(i) a *weak equivalence* if $h : X \to Y$ is a weak equivalence in $\mathbf{C}$,
(ii) a *cofibration* if $h : X \to Y$ is a cofibration in $\mathbf{C}$, and
(iii) a *fibration* if $h : X \to Y$ is a fibration in $\mathbf{C}$.

REMARK. Let $\mathbf{Top}$ have the model category structure of 3.6 and as usual let $*$ be the terminal object of $\mathbf{Top}$, i.e. the space with one point. Then $*{\downarrow}\mathbf{Top}$ is the category of pointed spaces, and an object X of $*\downarrow\mathbf{Top}$ is cofibrant if and only if the basepoint of X is closed and nondegenerate [25, p. 380]. Thus (3.7) from the point of view of model categories, having a nondegenerate basepoint is for a space what being projective is for a chain complex!

3.11. REMARK. In the situation of 3.10, we leave it to the reader to define the *over category* $\mathbf{C}\downarrow A$ and describe a model category structure on it. If $\mathbf{C}$ is the category of spaces (3.5 and 3.6), the model category structure on $\mathbf{C}\downarrow A$ is related to fibrewise homotopy theory [15].

In the remainder of this section we make some preliminary observations about model categories.

3.12. *Lifting properties.* A map $i : A \to B$ is said to have the *left lifting property* (LLP) with respect to another map $p : X \to Y$ and p is said to have the *right lifting property* (RLP) with respect to i if a lift exists (3.1) in any diagram of the form 3.2.

3.13. PROPOSITION. *Let* $\mathbf{C}$ *be a model category.*

(i) *The cofibrations in* $\mathbf{C}$ *are the maps which have the LLP with respect to acyclic fibrations.*
(ii) *The acyclic cofibrations in* $\mathbf{C}$ *are the maps which have the LLP with respect to fibrations.*
(iii) *The fibrations in* $\mathbf{C}$ *are the maps which have the RLP with respect to acyclic cofibrations.*
(iv) *The acyclic fibrations in* $\mathbf{C}$ *are the maps which have the RLP with respect to cofibrations.*

PROOF. Axiom **MC4** implies that an (acyclic) cofibration or an (acyclic) fibration has the stated lifting property. In each case we need to prove the converse. Since the four proofs are very similar (and in fact statements (iii) and (iv) follow from (i) and (ii) by duality), we only give the first proof. Suppose that $f : K \to L$ has the LLP with respect to all acyclic fibrations. Factor f as a composite $K \hookrightarrow L' \xrightarrow{\sim} L$ as in **MC5**(i), so $i : K \to L'$ is a cofibration and $p : L' \to L$ is an acyclic fibration. By assumption there exists a lifting $g : L \to L'$ in the following diagram:

$$\begin{array}{ccc} K & \xrightarrow{i} & L' \\ {\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow\sim \\ L & \xrightarrow{\mathrm{id}} & L \end{array} .$$

This implies that f is a retract of i:

$$\begin{array}{ccccc} K & \xrightarrow{\mathrm{id}} & K & \xrightarrow{\mathrm{id}} & K \\ {\scriptstyle f}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle f}\downarrow \\ L & \xrightarrow{g} & L' & \xrightarrow{p} & L \end{array} .$$

By **MC3** we conclude that f is a cofibration. □

REMARK. Proposition 3.13 implies that the axioms for a model category are overdetermined in some sense. This has the following practical consequence. If we are trying to set up a model category structure on some given category and have chosen the fibrations and the weak equivalences, then the class of cofibrations is pinned down by property 3.13(i). Dually, if we have chosen the cofibrations and weak equivalences, the class of fibrations is pinned down by property 3.13(iii). Verifying the axioms then comes down in part to checking certain consistency conditions.

3.14. PROPOSITION. *Let* **C** *be a model category.*

(i) *The class of cofibrations in* **C** *is stable under cobase change* (2.16).
(ii) *The class of acyclic cofibrations is stable under cobase change.*
(iii) *The class of fibrations is stable under base change* (2.23).
(iv) *The class of acyclic fibrations is stable under base change.*

PROOF. The second two statements follow from the first two by duality (3.9), so we only prove the first and indicate the proof of the second. Assume that $i : K \hookrightarrow L$ is a cofibration, and pick a map $f : K \to K'$. Construct a pushout diagram (cf. **MC1**):

$$\begin{array}{ccc} K & \xrightarrow{f} & K' \\ i\downarrow & & j\downarrow \\ L & \xrightarrow{g} & L' \end{array} .$$

We have to prove that j is a cofibration. By (i) of the previous proposition it is enough to show that j has the LLP with respect to an acyclic fibration. Let $p : E \to B$ be an acyclic fibration and consider a lifting problem

$$\begin{array}{ccc} K' & \xrightarrow{a} & E \\ j\downarrow & & p\downarrow \\ L' & \xrightarrow{b} & B \end{array} \qquad (3.15)$$

Enlarge this to the following diagram

$$\begin{array}{ccccc} K & \xrightarrow{f} & K' & \xrightarrow{a} & E \\ i\downarrow & & & & p\downarrow \\ L & \xrightarrow{g} & L' & \xrightarrow{b} & B \end{array}$$

Since i is a cofibration, there is a lifting $h : L \to E$ in the above diagram. By the universal property of pushouts, the maps $h : L \to E$ and $a : K' \to E$ induce the desired lifting in 3.15. The proof of part (ii) is analogous, the only difference being that we need to invoke 3.13(ii) instead of 3.13(i). □

4. Homotopy relations on maps

In this section $\mathbf{C}$ is some fixed model category, and A and X are objects of $\mathbf{C}$. Our goal is to exploit the model category axioms to construct some reasonable homotopy relations on the set $\mathrm{Hom}_{\mathbf{C}}(A, X)$ of maps from A to X. We first study a notion of *left homotopy*, defined in terms of *cylinder objects*, and then a dual (3.9) notion of *right homotopy*, defined in terms of *path objects*. It turns out (4.21) that the two notions coincide in what will turn out to be the most important case, namely if A is cofibrant and X is fibrant.

4.1. *Cylinder objects and left homotopy*

4.2. DEFINITION. A *cylinder object* for A is an object $A \wedge I$ of $\mathbf{C}$ together with a diagram (**MC1**, 2.15):

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

which factors the folding map $\mathrm{id}_A + \mathrm{id}_A : A \coprod A \to A$ (2.15). A cylinder object $A \wedge I$ is called

(i) a *good cylinder object*, if $A \coprod A \to A \wedge I$ is a cofibration, and
(ii) a *very good cylinder object*, if in addition the map $A \wedge I \to A$ is a (necessarily acyclic) fibration.

If $A \wedge I$ is a cylinder object for A, we will denote the two structure maps $A \to A \wedge I$ by $i_0 = i \cdot \mathrm{in}_0$ and $i_1 = i \cdot \mathrm{in}_1$ (cf. 2.15).

4.3. REMARK. By **MC5**, at least one very good cylinder object for A exists. The notation $A \wedge I$ is meant to suggest the product of A with an interval (Quillen even uses the notation "$A \times I$" for a cylinder object). However, a cylinder object $A \wedge I$ is not necessarily the product of A with anything in $\mathbf{C}$; it is just an object of $\mathbf{C}$ with the above formal property. An object A of $\mathbf{C}$ might have many cylinder objects associated to it, denoted, say, $A \wedge I$, $A \wedge I', \ldots$, etc. We do not assume that there is some preferred natural cylinder object for A; in particular, we do not assume that a cylinder object can be chosen in a way that is functorial in A.

4.4. LEMMA. *If A is cofibrant and $A \wedge I$ is a good cylinder object for A, then the maps $i_0, i_1 : A \to A \wedge I$ are acyclic cofibrations.*

PROOF. It is enough to check this for i_0. Since the identity map $\mathrm{id}_A : A \to A$ factors as $A \xrightarrow{i_0} A \wedge I \xrightarrow{\sim} A$, it follows from **MC2** that i_0 is a weak equivalence. Since $A \coprod A$ is defined by the following pushout diagram (2.16)

$$\begin{array}{ccc} \varnothing & \longrightarrow & A \\ {\scriptstyle\text{cofibration}}\downarrow & & \downarrow{\scriptstyle \mathrm{in}_0} \\ A & \xrightarrow{\mathrm{in}_1} & A \coprod A \end{array}$$

it follows from 3.14 that the map in_0 is a cofibration. Since i_0 is thus the composite

$$A \xrightarrow{\mathrm{in}_0} A \coprod A \to A \wedge I,$$

of two cofibrations, it itself is a cofibration. □

DEFINITION. Two maps $f, g : A \to X$ in **C** are said to be *left homotopic* (written $f \overset{l}{\sim} g$) if there exists a cylinder object $A \wedge I$ for A such that the sum map $f + g : A \coprod A \to X$ (2.15) extends to a map $H : A \wedge I \to X$, i.e. such that there exists a map $H : A \wedge I \to X$ with $H(i_0 + i_1) = f + g$. Such a map H is said to be a *left homotopy* from f to g (via the cylinder object $A \wedge I$). The left homotopy is said to be *good* (resp. *very good*) if $A \wedge I$ is a good (resp. very good) cylinder object for A.

EXAMPLE. Let **C** be the category of topological spaces with the model category structure described in 3.5. Then one choice of cylinder object for a space A is the product $A \times [0, 1]$. The notion of left homotopy with respect to this cylinder object coincides with the usual notion of homotopy. Observe that if A is not a CW-complex, $A \times [0, 1]$ is not usually a good cylinder object for A.

4.5. REMARK. If $f \overset{l}{\sim} g$ via the homotopy H, then it follows from **MC2** that the map f is a weak equivalence if and only if g is. To see this, note that as in the proof of 4.4 the maps i_0 and i_1 are weak equivalences, so that if $f = Hi_0$ is a weak equivalence, so is H and hence so is $g = Hi_1$.

4.6. LEMMA. *If $f \overset{l}{\sim} g : A \to X$, then there exists a good left homotopy from f to g. If in addition X is fibrant, then there exists a very good left homotopy from f to g.*

PROOF. The first statement follows from applying **MC5**(i) to the map $A \coprod A \to A \wedge I$, where $A \wedge I$ is the cylinder object in some left homotopy from f to g. For the second, choose a good left homotopy $H : A \wedge I \to X$ from f to g. By **MC5**(i) and **MC2**, we may factor $A \wedge I \xrightarrow{\sim} A$ as

$$A \wedge I \overset{\sim}{\hookrightarrow} A \wedge I' \overset{\sim}{\twoheadrightarrow} A.$$

Applying **MC4** to the following diagram

$$\begin{array}{ccc} A \wedge I & \xrightarrow{H} & X \\ \downarrow & & \downarrow \\ A \wedge I' & \longrightarrow & * \end{array}$$

gives the desired very good homotopy $H' : A \wedge I' \to X$. □

4.7. LEMMA. *If A is cofibrant, then $\overset{l}{\sim}$ is an equivalence relation on* $\mathrm{Hom}_{\mathbf{C}}(A, X)$.

PROOF. Since we can take A itself as a cylinder object for A, we can take f itself as a left homotopy between f and f. Let $s : A\coprod A \to A\coprod A$ be the map which switches factors (technically, $s = \mathrm{in}_1 + \mathrm{in}_0$). The identity $(g+f) = (f+g)s$ shows that if $f \overset{l}{\sim} g$, then $g \overset{l}{\sim} f$. Suppose that $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$. Choose a good (4.6) left homotopy $H : A \wedge I \to X$ from f to g (i.e. $Hi_0 = f, Hi_1 = g$) and a good left homotopy $H' : A \wedge I' \to X$ from g to h (i.e. $H'i'_0 = g, H'i'_1 = h$). Let $A \wedge I''$ be the pushout of the following diagram:

$$A \wedge I \xleftarrow[\sim]{i_1} A \xrightarrow[\sim]{i'_0} A \wedge I'.$$

Since the maps $i_1 : A \to A \wedge I$ and $i'_0 : A \to A \wedge I'$ are acyclic cofibrations, it follows from 3.14 and the universal property of pushouts (2.16) that $A \wedge I''$ is a cylinder object for A. Another application of 2.16 to the maps H and H' gives the desired homotopy $H'' : A \wedge I'' \to X$ from f to h. □

Let $\pi^l(A, X)$ denote the set of equivalence classes of $\mathrm{Hom}_{\mathbf{C}}(A, X)$ under the equivalence relation generated by left homotopy.

4.8. REMARK. The word "generated" in the above definition of $\pi^l(A, X)$ is important. We will sometimes consider $\pi^l(A, X)$ even if A is not cofibrant; in this case left homotopy, taken on its own, is not necessarily an equivalence relation on $\mathrm{Hom}_{\mathbf{C}}(A, X)$.

4.9. LEMMA. *If A is cofibrant and $p : Y \to X$ is an acyclic fibration, then composition with p induces a bijection:*

$$p_* : \pi^l(A, Y) \to \pi^l(A, X), \quad [f] \mapsto [pf].$$

PROOF. The map p_* is well defined, since if $f, g : A \to Y$ are two maps and H is a left homotopy from f to g, then pH is a left homotopy from pf to pg. To show that p_* is onto, choose $[f] \in \pi^l(A, X)$. By **MC4**(i), a lift $g : A \to Y$ exists in the following diagram:

$$\begin{array}{ccc} \varnothing & \longrightarrow & Y \\ \downarrow & & p\downarrow\sim \\ A & \xrightarrow{f} & X \end{array}.$$

Clearly $p_*[g] = [pg] = [f]$. To prove that p_* is one to one, let $f, g : A \to Y$ and suppose that $pf \overset{l}{\sim} pg : A \to X$. Choose (4.6) a good left homotopy $H : A \wedge I \to X$ from pf to pg. By **MC4**(i), a lifting exists in the following diagram

$$\begin{array}{ccc} A\coprod A & \xrightarrow{f+g} & Y \\ \downarrow & & p\downarrow\sim \\ A \wedge I & \xrightarrow{H} & X \end{array}$$

and gives the desired left homotopy from f to g. □

4.10. LEMMA. *Suppose that X is fibrant, that f and g are left homotopic maps $A \to X$, and that $h : A' \to A$ is a map. Then $fh \overset{l}{\sim} gh$.*

PROOF. By 4.6, we can choose a very good left homotopy $H : A \wedge I \to X$ between f and g. Next choose a good cylinder object for A':

$$A' \coprod A' \xrightarrow{j} A' \wedge I \xrightarrow{\sim} A'.$$

By **MC4**, there is a lifting $k : A' \wedge I \to A \wedge I$ in the following diagram:

$$\begin{tikzcd} A' \coprod A' \arrow[r, "h \coprod h"] \arrow[d, "j"] & A \coprod A \arrow[r, "i"] & A \wedge I \arrow[d, "\sim"] \\ A' \wedge I \arrow[r, "\sim"] & A' \arrow[r, "h"] & A \end{tikzcd}$$

It is easy to check that Hk is the desired homotopy. □

4.11. LEMMA. *If X is fibrant, then the composition in $\mathbf{C}$ induces a map:*

$$\pi^l(A', A) \times \pi^l(A, X) \to \pi^l(A', X), \quad ([h], [f]) \mapsto [fh].$$

PROOF. Note that we are not assuming that A is cofibrant, so that two maps $A \to X$ which represent the same element of $\pi^l(A, X)$ are not necessarily directly related by a left homotopy (4.8). Nevertheless, it is enough to show that if $h \overset{l}{\sim} k : A' \to A$ and $f \overset{l}{\sim} g : A \to X$ then fh and gk represent the same element of $\pi^l(A', X)$. For this it is enough to check both that $fh \overset{l}{\sim} gh : A' \to X$ and that $gh \overset{l}{\sim} gk : A' \to X$. The first homotopy follows from the previous lemma. The second is obtained by composing the homotopy between h and k with g. □

4.12. *Path objects and right homotopies*

By duality (3.9), what we have proved so far in this section immediately gives corresponding results "on the other side".

DEFINITION. A *path object* for X is an object X^I of $\mathbf{C}$ together with a diagram (2.22)

$$X \xrightarrow{\sim} X^I \xrightarrow{p} X \times X$$

which factors the diagonal map $(\mathrm{id}_X, \mathrm{id}_X) : X \to X \times X$. A path object X^I is called

(i) a *good path object*, if $X^I \to X \times X$ is a fibration, and
(ii) a *very good path object*, if in addition the map $X \to X^I$ is a (necessarily acyclic) cofibration.

4.13. REMARK. By **MC5**, at least one very good path object exists for X. The notation X^I is meant to suggest a space of paths in X, i.e. a space of maps from an interval into X. However a path object X^I is not in general a function object of any kind; it is just some object of **C** with the above formal property. An object X of **C** might have many path objects associated to it, denoted X^I, $X^{I'}, \ldots$, etc.

We denote the two maps $X^I \to X$ by $p_0 = \mathrm{pr}_0 \cdot p$ and $p_1 = \mathrm{pr}_1 \cdot p$ (cf. 2.22).

4.14. LEMMA. *If X is fibrant and X^I is a good path object for X, then the maps $p_0, p_1 : X^I \to X$ are acyclic fibrations.*

DEFINITION. Two maps $f, g : A \to X$ are said to be right homotopic (written $f \overset{r}{\sim} g$) if there exists a path object X^I for X such that the product map $(f, g) : A \to X \times X$ lifts to a map $H : A \to X^I$. Such a map H is said to be a right homotopy from f to g (via the path object X^I). The right homotopy is said to be *good* (resp. *very good*) if X^I is a good (resp. very good) path object for X.

EXAMPLE. Let the category of topological spaces have the structure described in 3.5. Then one choice of path object for a space X is the mapping space $\mathrm{Map}([0,1], X)$.

4.15. LEMMA. *If $f \overset{r}{\sim} g : A \to X$ then there exists a good right homotopy from f to g. If in addition A is cofibrant, then there exists a very good right homotopy from f to g.*

4.16. LEMMA. *If X is fibrant, then $\overset{r}{\sim}$ is an equivalence relation on* $\mathrm{Hom}_{\mathbf{C}}(A, X)$.

Let $\pi^r(A, X)$ denote the set of equivalence classes of $\mathrm{Hom}_{\mathbf{C}}(A, X)$ under the equivalence relation generated by right homotopy.

4.17. LEMMA. *If X is fibrant and $i : A \to B$ is an acyclic cofibration, then composition with i induces a bijection:*

$$i^* : \pi^r(B, X) \to \pi^r(A, X).$$

4.18. LEMMA. *Suppose that A is cofibrant, that f and g are right homotopic maps from A to X, and that $h : X \to Y$ is a map. Then $hf \overset{r}{\sim} hg$.*

4.19. LEMMA. *If A is cofibrant then the composition in **C** induces a map $\pi^r(A, X) \times \pi^r(X, Y) \to \pi^r(A, Y)$.*

4.20. *Relationship between left and right homotopy*

The following lemma implies that if A is cofibrant and X is fibrant, then the left and right homotopy relations on $\mathrm{Hom}_{\mathbf{C}}(A, X)$ agree.

4.21. LEMMA. *Let $f, g : A \to X$ be maps.*

(i) *If A is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.*
(ii) *If X is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

4.22. *Homotopic maps.* If A is cofibrant and X is fibrant, we will denote the identical right homotopy and left homotopy equivalence relations on $\mathrm{Hom}_{\mathbf{C}}(A, X)$ by the symbol "$\sim$" and say that two maps related by this relation are *homotopic*. The set of equivalence classes with respect to this relation is denoted $\pi(A, X)$.

PROOF OF 4.21. Since the two statements are dual, we only have to prove the first one. By 4.6 there exists a good cylinder object

$$A \coprod A \xrightarrow{i_0+i_1} A \wedge I \xrightarrow{j} A$$

for A and a homotopy $H : A \wedge I \to X$ from f to g. By 4.4 the map i_0 is an acyclic cofibration. Choose a good path object (4.13)

$$X \xrightarrow{q} X^I \xrightarrow{(p_0,p_1)} X \times X$$

for X. By **MC4** it is possible to find a lift $K : A \wedge I \to X^I$ in the diagram

$$\begin{array}{ccc} A & \xrightarrow{qf} & X^I \\ {\scriptstyle i_0} \downarrow & & \downarrow {\scriptstyle (p_0,p_1)} \\ A \wedge I & \xrightarrow{(fj,H)} & X \times X \end{array}$$

The composite $Ki_1 : A \to X^I$ is the desired right homotopy from f to g. □

4.23. REMARK. Suppose that A is cofibrant, X is fibrant, $A \wedge I$ is some *fixed* good cylinder object for A and X^I is some *fixed* good path object for X. Let $f, g : A \to X$ be maps. The proof of 4.21 shows that $f \sim g$ if and only if $f \overset{r}{\sim} g$ via the fixed path object X^I. Dually, $f \sim g$ if and only if $f \overset{l}{\sim} g$ via the fixed cylinder object $A \wedge I$.

We will need the following observation later on.

4.24. LEMMA. *Suppose that $f : A \to X$ is a map in $\mathbf{C}$ between objects A and X which are both fibrant and cofibrant. Then f is a weak equivalence if and only if f has a homotopy inverse, i.e. if and only if there exists a map $g : X \to A$ such that the composites gf and fg are homotopic to the respective identity maps.*

PROOF. Suppose first that f is a weak equivalence. By **MC5** we can factor f as a composite

$$A \underset{\sim}{\overset{q}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} X \tag{4.25}$$

in which by **MC2** the map p is also a weak equivalence. Because $q : A \to C$ is a cofibration and A is fibrant, an application of **MC4** shows that there exists a left inverse for q, i.e. a morphism $r : C \to A$ such that $rq = \mathrm{id}_A$. By Lemma 4.17, q induces a bijection $q^* : \pi^r(C, C) \to \pi^r(A, C)$, $[g] \mapsto [gq]$. Since $q^*([qr]) = [qrq] = [q]$, we conclude that $qr \overset{r}{\sim} 1_C$ and hence that r is a two-sided right (equivalently left) homotopy inverse for q. A dual argument (3.9) gives a two-sided homotopy inverse of p, say s. The composite rs is a two sided homotopy inverse of $f = pq$.

Suppose next that f has a homotopy inverse. By **MC5** we can find a factorization $f = pq$ as in 4.25. Note that the object C is both fibrant and cofibrant. By **MC2**, in order to prove that f is a weak equivalence it is enough to show that p is a weak equivalence. Let $g : X \to A$ be a homotopy inverse for f, and $H : X \wedge I \to X$ a homotopy between fg and id_X. By **MC4** we can find a lift $H' : X \wedge I \to C$ in the diagram

$$\begin{array}{ccc} X & \xrightarrow{qg} & C \\ {\scriptstyle i_0}\downarrow & & {\scriptstyle p}\downarrow \\ X \wedge I & \xrightarrow{H} & X \end{array} .$$

Let $s = H'i_1$, so that $ps = \mathrm{id}_X$. The map q is a weak equivalence, so by the result just proved above q has a homotopy inverse, say r. Since $pq = f$, composing on the right with r gives $p \sim fr$ (4.11). Since in addition $s \sim qg$ by the homotopy H', it follows (4.11, 4.19) that

$$sp \sim qgp \sim qgfr \sim qr \sim \mathrm{id}_C.$$

By 4.5, then, sp is a weak equivalence. The commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{\mathrm{id}_C} & C & \xrightarrow{\mathrm{id}_C} & C \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle sp} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{s} & C & \xrightarrow{p} & X \end{array}$$

shows that p is a retract (2.6) of sp, and hence by **MC3** that the map p is a weak equivalence. □

5. The homotopy category of a model category

In this section we will use the machinery constructed in §4 to give a quick construction of the *homotopy category* $\mathrm{Ho}(\mathbf{C})$ associated to a model category $\mathbf{C}$.

We begin by looking at the following six categories associated to $\mathbf{C}$.

$\mathbf{C}_c$ – the full (2.3) subcategory of $\mathbf{C}$ generated by the cofibrant objects in $\mathbf{C}$.

$\mathbf{C}_f$ – the full subcategory of $\mathbf{C}$ generated by the fibrant objects in $\mathbf{C}$.

$\mathbf{C}_{cf}$ – the full subcategory of $\mathbf{C}$ generated by the objects of $\mathbf{C}$ which are both fibrant and cofibrant.

$\pi\mathbf{C}_c$ – the category consisting of the cofibrant objects in $\mathbf{C}$ and whose morphisms are right homotopy classes of maps (see 4.19).

$\pi\mathbf{C}_f$ – the category consisting of fibrant objects in $\mathbf{C}$ and whose morphisms are left homotopy classes of maps (see 4.11).

$\pi\mathbf{C}_{cf}$ – the category consisting of objects in $\mathbf{C}$ which are both fibrant and cofibrant, and whose morphisms are homotopy classes (4.22) of maps.

These categories will be used as tools in defining $\mathrm{Ho}(\mathbf{C})$ and constructing a canonical functor $\mathbf{C} \to \mathrm{Ho}(\mathbf{C})$. For each object X in $\mathbf{C}$ we can apply **MC5**(i) to the map $\varnothing \to X$ and obtain an acyclic fibration $p_X : QX \xrightarrow{\sim} X$ with QX cofibrant. We can also apply **MC5**(ii) to the map $X \to *$ and obtain an acyclic cofibration $i_X : X \xrightarrow{\sim} RX$ with RX fibrant. If X is itself cofibrant, let $QX = X$; if X is fibrant, let $RX = X$.

5.1. LEMMA. *Given a map $f : X \to Y$ in $\mathbf{C}$ there exists a map $\tilde{f} : QX \to QY$ such that the following diagram commutes:*

$$\begin{array}{ccc} QX & \xrightarrow{\tilde{f}} & QY \\ {\scriptstyle p_X}\downarrow\sim & & {\scriptstyle p_Y}\downarrow\sim \\ X & \xrightarrow{f} & Y \end{array} .$$

The map $\tilde{f}$ depends up to left homotopy or up to right homotopy only on f, and is a weak equivalence if and only if f is. If Y is fibrant, then $\tilde{f}$ depends up to left homotopy or up to right homotopy only on the left homotopy class of f.

PROOF. We obtain $\tilde{f}$ by applying **MC4** to the diagram:

$$\begin{array}{ccc} \varnothing & \longrightarrow & QY \\ \downarrow & & \sim\downarrow{\scriptstyle p_Y} \\ QX & \xrightarrow{f\cdot p_X} & Y \end{array} .$$

The statement about the uniqueness of $\tilde{f}$ up to left homotopy follows from 4.9. For the statement about right homotopy, observe that QX is cofibrant, and so by 4.21(i) two maps with domain QX which are left homotopic are also right homotopic. The weak equivalence condition follows from **MC2**, and the final assertion from 4.11. □

5.2. REMARK. The uniqueness statements in 5.1 imply that if $f = \mathrm{id}_X$ then $\tilde{f}$ is right homotopic to id_{QX}. Similarly, if $f : X \to Y$ and $g : Y \to Z$ and $h = gf$, then $\tilde{h}$ is right homotopic to $\tilde{g}\tilde{f}$. Hence we can define a functor $Q : \mathbf{C} \to \pi\mathbf{C}_c$ sending $X \to QX$ and $f : X \to Y$ to the right homotopy class $[\tilde{f}] \in \pi^r(QX, QY)$.

The dual (3.9) to 5.1 is the following statement.

5.3. LEMMA. *Given a map $f : X \to Y$ in $\mathbf{C}$ there exists a map $\bar{f} : RX \to RY$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow\sim & & {\scriptstyle i_Y}\downarrow\sim \\ RX & \xrightarrow{\bar{f}} & RY \end{array} .$$

The map $\bar{f}$ depends up to right homotopy or up to left homotopy only on f, and is a weak equivalence if and only if f is. If X is cofibrant, then $\bar{f}$ depends up to right homotopy or up to left homotopy only on the right homotopy class of f.

5.4. REMARK. The uniqueness statements in 5.3 imply that if $f = \mathrm{id}_X$ then $\bar{f}$ is left homotopic to id_{RX}. Moreover, if $f : X \to Y$ and $g : Y \to Z$ and $h = gf$, then $\bar{h}$ is left homotopic to $\bar{g}\bar{f}$, Hence we can define a functor $R : \mathbf{C} \to \pi\mathbf{C}_f$ sending $X \to RX$ and $f : X \to Y$ to the left homotopy class $[\bar{f}] \in \pi^l(RX, RY)$.

5.5. LEMMA. *The restriction of the functor $Q : \mathbf{C} \to \pi\mathbf{C}_c$ to $\mathbf{C}_f$ induces a functor $Q' : \pi\mathbf{C}_f \to \pi\mathbf{C}_{cf}$. The restriction of the functor $R : \mathbf{C} \to \pi\mathbf{C}_f$ to $\mathbf{C}_c$ induces a functor $R' : \pi\mathbf{C}_c \to \pi\mathbf{C}_{cf}$.*

PROOF. The two statements are dual to one another, and so we will prove only the second. Suppose that X and Y are cofibrant objects of $\mathbf{C}$ and that $f, g : X \to Y$ are maps which represent the same map in $\pi\mathbf{C}_c$; we must show that $Rf = Rg$. It is enough to do this in the special case $f \overset{r}{\sim} g$ in which f and g are directly related by a right homotopy; however in this case it is a consequence of 5.3. □

5.6. DEFINITION. *The homotopy category* Ho(**C**) of a model category **C** is the category with the same objects as **C** and with

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, Y) = \mathrm{Hom}_{\pi\mathbf{C}_{cf}}(R'QX, R'QY) = \pi(RQX, RQY).$$

5.7. REMARK. There is a functor $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ which is the identity on objects and sends a map $f : X \to Y$ to the map $R'Q(f) : R'Q(X) \to R'Q(Y)$. If each of the objects X and Y is both fibrant and cofibrant, then by construction the map $\gamma : \mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, Y)$ is surjective and induces a bijection $\pi(X, Y) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, Y)$.

It is natural to ask whether or not dualizing the definition of Ho(**C**) by replacing the composite functor $R'Q$ by $Q'R$ would give anything different. The answer is that it would not; rather than prove this directly, though, we will give a symmetrical construction of the homotopy category in the next section. There are some basic observations about Ho(**C**) that will come in handy later on.

5.8. PROPOSITION. *If f is a morphism of* **C**, *then $\gamma(f)$ is an isomorphism in* Ho(**C**) *if and only if f is a weak equivalence. The morphisms of* Ho(**C**) *are generated under composition by the images under γ of morphisms of* **C** *and the inverses of images under γ of weak equivalences in* **C**.

PROOF. If $f : X \to Y$ is a weak equivalence in **C**, then $R'Q(f)$ is represented by a map $f' : RQ(X) \to RQ(Y)$ which is also a weak equivalence (see 5.1 and 5.3); by 4.24, then, the map f' has an inverse up to left or right homotopy and represents an isomorphism in $\pi\mathbf{C}_{cf}$. This isomorphism is exactly $\gamma(f)$. On the other hand, if $\gamma(f)$ is an isomorphism then f' has an inverse up to homotopy and is therefore a weak equivalence by 4.24; it follows easily that f is a weak equivalence.

Observe by the above that for any object X of $\mathbf{C}$ the map $\gamma(i_{QX})\gamma(p_X)^{-1}$ in $\mathrm{Ho}(\mathbf{C})$ is an isomorphism from X to $RQ(X)$. Moreover, for two objects X and Y of $\mathbf{C}$, the functor γ induces an epimorphism (5.7)

$$\mathrm{Hom}_{\mathbf{C}}(RQ(X), RQ(Y)) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(RQ(X), RQ(Y)).$$

Consequently, any map $f : X \to Y$ in $\mathrm{Ho}(\mathbf{C})$ can be represented as a composite

$$f = \gamma(p_Y)\gamma(i_{QY})^{-1}\gamma(f')\gamma(i_{QX})\gamma(p_X)^{-1}$$

for some map $f' : RQ(X) \to RQ(Y)$ in $\mathbf{C}$. □

Proposition 5.8 has the following simple but useful consequence.

5.9. COROLLARY. *If F and G are two functors $\mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ and $t : F\gamma \to G\gamma$ is a natural transformation, then t also gives a natural transformation from F to G.*

PROOF. It is necessary to check that for each morphism h of $\mathrm{Ho}(\mathbf{C})$ an appropriate diagram $D(h)$ commutes. By assumption $D(h)$ commutes if $h = \gamma(f)$ or $h = \gamma(g)^{-1}$ for some morphism f in $\mathbf{C}$ or weak equivalence g in $\mathbf{C}$. It is easy to check that if $h = h_1h_2$, the $D(h)$ commutes if $D(h_1)$ commutes and $D(h_2)$ commutes. The lemma then follows from the fact (5.8) that any map of $\mathrm{Ho}(\mathbf{C})$ is a composite of maps of the form $\gamma(f)$ and $\gamma(g)^{-1}$. □

5.10. LEMMA. *Let $\mathbf{C}$ be a model category and $F : \mathbf{C} \to \mathbf{D}$ be a functor taking weak equivalences in $\mathbf{C}$ into isomorphisms in $\mathbf{D}$. If $f \overset{l}{\sim} g : A \to X$ or $f \overset{r}{\sim} g : A \to X$, then $F(f) = F(g)$ in $\mathbf{D}$.*

PROOF. We give a proof assuming $f \overset{l}{\sim} g$, the other case is dual. Choose (4.6) a good left homotopy $H : A \wedge I \to X$ from f to g, so that $A \wedge I$ is a good cylinder object for A:

$$A \coprod A \xhookrightarrow{i_0+i_1} A \wedge I \xrightarrow[\sim]{w} A\,.$$

Since $wi_0 = wi_1 (= \mathrm{id}_A)$, we have $F(w)F(i_0) = F(w)F(i_1)$. Since w is a weak equivalence, the map $F(w)$ is an isomorphism and it follows that $F(i_0) = F(i_1)$. Hence $F(f) = F(H)F(i_0)$ is the same as $F(g) = F(H)F(i_1)$. □

5.11. PROPOSITION. *Suppose that A is a cofibrant object of $\mathbf{C}$ and X is a fibrant object of $\mathbf{C}$. Then the map $\gamma : \mathrm{Hom}_{\mathbf{C}}(A, X) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, X)$ is surjective, and induces a bijection $\pi(A, X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, X)$.*

PROOF. By 5.10 and 5.8 the functor γ identifies homotopic maps and so induces a map $\pi(A, X) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, X)$. Consider the commutative diagram

$$\begin{array}{ccc}
\pi(RA, QX) & \longrightarrow & \pi(A, X) \\
\gamma \downarrow & & \gamma \downarrow \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(RA, QX) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, X)
\end{array}$$

in which the horizontal arrows are induced by the pair (i_A, p_X). By 5.8 the lower horizontal map is a bijection, while by 4.9 and 4.17 the upper horizontal map is a bijection. As indicated in 5.7, the left-hand vertical map is also a bijection. The desired result follows immediately. □

6. Localization of categories

In this section we will give a conceptual interpretation of the homotopy category of a model category. Surprisingly, this interpretation depends *only* on the class of weak equivalences. This suggests that in a model category the weak equivalences carry the fundamental homotopy theoretic information, while the cofibrations, fibrations, and the axioms they satisfy function mostly as tools for making various constructions (e.g., the constructions later on in §10). This also suggests that in putting a model category structure on a category, it is most important to focus on picking the class of weak equivalences; choosing fibrations and cofibrations is a secondary issue.

6.1. DEFINITION. Let $\mathbf{C}$ be a category, and $W \subseteq \mathbf{C}$ a class of morphisms. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be *a localization of* $\mathbf{C}$ *with respect to* W if

(i) $F(f)$ is an isomorphism for each $f \in W$, and
(ii) whenever $G : \mathbf{C} \to \mathbf{D}'$ is a functor carrying elements of W into isomorphisms, there exists a unique functor $G' : \mathbf{D} \to \mathbf{D}'$ such that $G'F = G$.

Condition 6.1(ii) guarantees that any two localizations of $\mathbf{C}$ with respect to W are canonically isomorphic. If such a localization exists, we denote it by $\mathbf{C} \to W^{-1}\mathbf{C}$.

EXAMPLE. Let $\mathbf{Ab}$ be the category of abelian groups, and W the class of morphisms $f : A \to B$ such that $\ker(f)$ and $\operatorname{coker}(f)$ are torsion groups. Let $\mathbf{D}$ be the category with the same objects, but with $\operatorname{Hom}_{\mathbf{D}}(A, B) = \operatorname{Hom}_{\mathbf{Ab}}(\mathbf{Q} \otimes A, \mathbf{Q} \otimes B)$. Define $F : \mathbf{Ab} \to \mathbf{D}$ to be the functor which sends an object A to itself and a map f to $\mathbf{Q} \otimes f$. It is an interesting exercise to verify directly that F is the localization of $\mathbf{Ab}$ with respect to W [12, p. 15].

6.2. THEOREM. *Let* $\mathbf{C}$ *be a model category and* $W \subseteq \mathbf{C}$ *the class of weak equivalences. Then the functor* $\gamma : \mathbf{C} \to \operatorname{Ho}(\mathbf{C})$ *is a localization of* $\mathbf{C}$ *with respect to* W.

More informally, Theorem 6.2 says that if $\mathbf{C}$ is a model category and $W \subseteq \mathbf{C}$ is the class of weak equivalences, then $W^{-1}\mathbf{C}$ exists and is isomorphic to $\operatorname{Ho}(\mathbf{C})$.

PROOF OF 6.2. We have to verify the two conditions in 6.1 for γ. Condition 6.1(i) is proved in 5.8. For 6.1(ii), suppose given a functor $G : \mathbf{C} \to \mathbf{D}$ carrying weak equivalences to isomorphisms. We must construct a functor $G' : \operatorname{Ho}(\mathbf{C}) \to \mathbf{D}$ such that $G'\gamma = G$, and show that G' is unique. Since the objects of $\operatorname{Ho}(\mathbf{C})$ are the same as the objects of $\mathbf{C}$, the effect of G' on objects is obvious. Pick a map $f : X \to Y$ in $\operatorname{Ho}(\mathbf{C})$, which is represented by a map $f' : RQ(X) \to RQ(Y)$, well defined up to homotopy (4.22).

Observe by 5.10 that $G(f')$ depends only on the homotopy class of f', and therefore only on f. Define $G'(f)$ by the formula

$$G'(f) = G(p_Y)G(i_{QY})^{-1}G(f')G(i_{QX})G(p_X)^{-1}.$$

It is easy to check that G' is a functor, that is, respects identity maps and compositions. If f is the image of a map $h : X \to Y$ of $\mathbf{C}$, then (5.1 and 5.3) after perhaps altering f' up to right homotopy we can find a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{i_{QX}} & RQ(X) \\ h\downarrow & & \tilde{h}\downarrow & & f'\downarrow \\ Y & \xleftarrow{p_Y} & QY & \xrightarrow{i_{QY}} & RQ(Y) \end{array} .$$

Applying G to this diagram shows that $G'(f) = G(h)$ and thus that G' extends G, that is, $G'\gamma = G$. The uniqueness of G' follows immediately from the second statement in 5.8 □

7. Chain complexes

Suppose that R is an associative ring with unit and let Mod_R denote the category of left R-modules. Recall that the category $\mathbf{Ch}_R$ of (non-negatively graded) chain complexes of R-modules is the category in which an object M is a collection $\{M_k\}_{k\geqslant 0}$ of R-modules together with boundary maps $\partial : M_k \to M_{k-1}$ $(k \geqslant 1)$ such that $\partial^2 = 0$. A morphism $f : M \to N$ is a collection of R-module homomorphisms $f_k : M_k \to N_k$ such that $f_{k-1}\partial = \partial f_k$. In this section we will construct a model category structure (7.2) on $\mathbf{Ch}_R$ and give some indication (7.3) of how the associated homotopy theory is related to homological algebra.

7.1. *Preliminaries.* For an object M of $\mathbf{Ch}_R$, define the k-dimensional cycle module $\text{Cy}_k(M)$ to be M_0 if $k = 0$ and $\ker(\partial : M_k \to M_{k-1})$ if $k > 0$. Define the k-dimensional boundary module $Bd_k(M)$ to be $\text{image}(\partial : M_{k+1} \to M_k)$. There are *homology functors* $\text{H}_k : \mathbf{Ch}_R \to \text{Mod}_R$ $(k \geqslant 0)$ given by $\text{H}_k M = \text{Cy}_k(M)/Bd_k(M)$ (we think of these homology groups as playing the role for chain complexes that homotopy groups do for a space). A chain complex M is *acyclic* if $\text{H}_k M = 0$ $(k \geqslant 0)$. Recall that an R-module P is said to be *projective* [6] if the following three equivalent conditions hold:

(i) P is a direct summand of a free R-module,
(ii) every epimorphism $f : A \to P$ of R-modules has a right inverse, or
(iii) for every epimorphism $A \to B$ of R-modules, the induced map

$$\text{Hom}_{\text{Mod}_R}(P, A) \to \text{Hom}_{\text{Mod}_R}(P, B)$$

is an epimorphism.

The first goal of this section is to prove the following result.

7.2. THEOREM. *Define a map $f : M \to N$ in $\mathbf{Ch}_R$ to be*

(i) *a weak equivalence if the map f induces isomorphisms $\mathrm{H}_k M \to \mathrm{H}_k N$ $(k \geqslant 0)$,*
(ii) *a cofibration if for each $k \geqslant 0$ the map $f_k : M_k \to N_k$ is a monomorphism with a projective R-module as its cokernel, and*
(iii) *a fibration if for each $k > 0$ the map $f_k : M_k \to N_k$ is an epimorphism.*

Then with these choices $\mathbf{Ch}_R$ is a model category.

After proving this we will make the following calculation. If A is an R-module, let $K(A,n)$ $(n \geqslant 0)$ denote the object M of $\mathbf{Ch}_R$ with $M_n = A$ and $M_k = 0$ for $k \neq n$ (these are the chain complex analogues of Eilenberg–MacLane spaces).

7.3. PROPOSITION. *For any two R-modules A and B and non-negative integers m, n there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Ch}_R)}\big(K(A,m), K(B,n)\big) \cong \mathrm{Ext}_R^{n-m}(A,B).$$

Here Ext_R^k is the usual Ext functor from homological algebra [6]. We take it to be zero if $k < 0$.

7.4. *Proof of* MC1–MC3

We should first note that the classes of weak equivalences, fibrations and cofibrations clearly contain all identity maps and are closed under composition. It is easy to see that limits and colimits in $\mathbf{Ch}_R$ can be computed degreewise, so that **MC1** follows from the fact that Mod_R has all small limits and colimits. Axiom **MC2** is clear. Axiom **MC3** follows from the fact that in Mod_R a retract of an isomorphism, monomorphism or epimorphism is another morphism of the same type (cf. 2.7). It is also necessary to observe that a retract (i.e. direct summand) of a projective R-module is projective.

7.5. *Proof of* MC4(i)

We need to show that a lift exists in every diagram of chain complexes:

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\downarrow & & \sim\downarrow{\scriptstyle p} \\ B & \xrightarrow{h} & Y \end{array}, \tag{7.6}$$

in which i is a cofibration and p is an acyclic fibration. By the definition of fibration, p_k is onto for $k > 0$. But since $(p_0)_* : H_0(X) \to H_0(Y)$ is an isomorphism, an application of the five lemma [17, p. 198] shows that p_0 is also onto. Hence there is a short exact sequence of chain complexes

$$0 \to K \to X \to Y \to 0$$

and it follows from the associated long exact homology sequence [6], [25, p. 181] that K is acyclic.

We will construct the required map $f_k : B_k \to X_k$ by induction on k. It is easy to construct a plausible map f_0, since, by 7.1 and the definition of cofibration, the module B_0 splits up to isomorphism as a direct sum $A_0 \oplus P_0$, where P_0 is a projective module; the map f_0 is chosen to be g_0 on the factor A_0 and any lifting $P_0 \to X_0$ of the given map $P_0 \to Y_0$ on the factor P_0. Assume that $k > 0$ and that for $j < k$ maps $f_j : B_j \to X_j$ with the following properties have been constructed:

(i) $\partial f_j = f_{j-1}\partial, \quad 1 \leqslant j < k,$
(ii) $p_j f_j = h_j, \quad 0 \leqslant j < k,$
(iii) $f_j i_j = g_j, \quad 0 \leqslant j < k.$

Proceeding as for $k = 0$ we can write $B_k \cong A_k \oplus P_k$ and construct a map $\tilde{f}_k : B_k \to X_k$ with properties (ii) and (iii) above. Let $\mathcal{E} : B_k \to X_{k-1}$ be the difference map $\partial \tilde{f}_k - f_{k-1}\partial$, so that the map $\mathcal{E}$ measures the failure of $\tilde{f}_k$ to satisfy (i). Then

(a) $\partial \cdot \mathcal{E} = 0$ because f_{k-1} satisfies (i),
(b) $p_{k-1} \cdot \mathcal{E} = 0$ because $p_k \tilde{f}_k = h_k$ commutes with ∂, and
(c) $\mathcal{E} \cdot i_k = 0$ because $\tilde{f}_k i_k = g_k$ commutes with ∂.

It follows that $\mathcal{E}$ induces a map

$$\mathcal{E}' : B_k/i_k(A_k) \cong P_k \to \mathrm{Cy}_{k-1}(K).$$

However, the chain complex K is acyclic and so the boundary map $K_k \to \mathrm{Cy}_{k-1}(K)$ is an epimorphism. Since P_k is a projective, $\mathcal{E}'$ can be lifted to a map $\mathcal{E}'' : P_k \to K_k$, which, after precomposition with the surjection $B_k \to P_k$ and postcomposition with the injection $K_k \to X_k$, gives a map $\mathcal{E}''' : B_k \to X_k$. It is straightforward to check that setting $f_k = \tilde{f}_k - \mathcal{E}'''$ gives a map $B_k \to X_k$ which satisfies all conditions (i)–(iii). This allows the induction to continue. □

7.7. *Proof of* MC4(ii)

This depends on a definition and a few lemmas. Suppose that A is an R-module. For $n \geqslant 1$ define the object $D_n(A)$ of $\mathbf{Ch}_R$ to be the chain complex with

$$D_n(A)_k = \begin{cases} 0, & k \neq n,\ n-1, \\ A, & k = n,\ n-1. \end{cases}$$

The boundary map $D_n(A)_n \to D_n(A)_{n-1}$ is the identity map of A. The letter "D" in this notation stands for "disk".

7.8. LEMMA. *Let A be an R-module and M an object of $\mathbf{Ch}_R$. Then the map*

$$\mathrm{Hom}_{\mathbf{Ch}_R}(D_n(A), M) \to \mathrm{Hom}_{\mathbf{Mod}_R}(A, M_n)$$

which sends f to f_n is an isomorphism.

This is obvious by inspection. In fact, the functor $D_n(-)$ is left adjoint to the functor from $\mathbf{Ch}_R$ to Mod_R which sends M to M_n.

7.9. REMARK. Lemma 7.8 immediately implies that if A is a projective R-module then $D_n(A)$ is what might be called a "projective chain complex", in the sense that if $p : M \to N$ is an epimorphism of chain complexes (or even an epimorphism in degrees $\geqslant 1$), then any map $D_n(A) \to N$ lifts over p to a map $D_n(A) \to M$. Similarly, any chain complex sum of the form $\bigoplus_i D_{n_i}(A_i)$ is a "projective chain complex" as long as each A_i is a projective R-module.

7.10. LEMMA. *Suppose that P is an acyclic object of $\mathbf{Ch}_R$ such that each P_k is a projective R-module. Then each module $\mathrm{Cy}_k P$ $(k \geqslant 0)$ is projective, and P is isomorphic as a chain complex to the sum $\bigoplus_{k\geqslant 1} D_k(\mathrm{Cy}_{k-1}P)$.*

PROOF. For $k \geqslant 1$ let $P^{(k)}$ be the chain subcomplex of P which agrees with P above degree $k-1$, contains $Bd_{k-1}P$ in degree $k-1$, and vanishes below degree $k-1$. The acyclicity condition gives isomorphisms $P^{(k)}/P^{(k+1)} \cong D_k(\mathrm{Cy}_{k-1}P)$. It is clear that $\mathrm{Cy}_0(P) = P_0$ is a projective R-module, and so by 7.9 there is an isomorphism $P = P^{(1)} \cong P^{(2)} \oplus D_1(\mathrm{Cy}_0 P)$. Since any direct factor of a projective R-module is projective, it follows that $P^{(2)}$ is a chain complex which satisfies the conditions of the lemma but vanishes in degree 0. Repeating the above argument in degree 1 gives an isomorphism $P^{(2)} \cong P^{(3)} \oplus D_2(\mathrm{Cy}_1 P)$. The proof is now completed by continuing along these lines. □

7.11. REMARK. Lemma 7.10 implies that if P is an acyclic object of $\mathbf{Ch}_R$ with the property that each P_k is a projective R-module, then P is a "projective chain complex" in the sense of 7.9.

Now we are ready to handle **MC4**(ii). We need to show that a lift exists in every diagram of the form 7.6 in which i is an acyclic cofibration and p is a fibration. By the definition of cofibration, the map i is a monomorphism of chain complexes and the cokernel P of i is a chain complex with the property that each P_k is a projective R-module. By the long exact homology sequence [6] associated to the short exact sequence

$$0 \to A \to B \to P \to 0$$

of chain complexes, P is acyclic. It follows from 7.11 that P is a "projective chain complex" in the sense of 7.9, so that B is isomorphic to the direct sum $A \oplus P$, and the desired lift can be obtained by using the map g on the factor A and, as far as the other factor is concerned, picking any lift $P \to X$ of the given map $P \to Y$. □

7.12. *The small object argument*

It is actually not hard to prove **MC5** in the present case by making very elementary constructions. We have decided, however, to give a more complicated proof that works in a variety of circumstances. This proof depends on an argument, called the "small object argument", that is due to Quillen and is very well adapted to producing factorizations with lifting properties. For the rest of this subsection we will assume that **C** is a category with all small colimits.

Given a functor $B : \mathbf{Z}^+ \to \mathbf{C}$ (2.12(iii)) and an object A of **C**, the natural maps $B(n) \to \operatorname{colim} B$ induce maps $\operatorname{Hom}_{\mathbf{C}}(A, B(n)) \to \operatorname{Hom}_{\mathbf{C}}(A, \operatorname{colim} B)$ which are compatible enough for various n to give a canonical map (2.17)

$$\operatorname{colim}_n \operatorname{Hom}_{\mathbf{C}}\big(A, B(n)\big) \to \operatorname{Hom}_{\mathbf{C}}\big(A, \operatorname{colim}_n B(n)\big). \tag{7.13}$$

7.14. Definition. An object A of **C** is said to be *sequentially small* if for every functor $B : \mathbf{Z}^+ \to \mathbf{C}$ the canonical map 7.13 is a bijection.

7.15. Remark. A set is sequentially small if and only if it is finite. An R-module is sequentially small if it has a finite presentation, i.e. it is isomorphic to the cokernel of a map between two finitely generated free R-modules. An object M of $\mathbf{Ch}_R$ is sequentially small if only a finite number of the modules M_k are non zero, and each M_k has a finite presentation.

Let $\mathcal{F} = \{f_i : A_i \to B_i\}_{i \in \mathcal{I}}$ be a *set* of maps in **C**. Suppose that $p : X \to Y$ is a map in **C**, and suppose that we desire to factor p as a composite $X \to X' \to Y$ in such a way that the map $X' \to Y$ has the RLP (3.12) with respect to all of the maps in $\mathcal{F}$. Of course we could choose $X' = Y$, but the secondary goal is to find a factorization in which X' is as close to X as reasonably possible. We proceed as follows. For each $i \in \mathcal{I}$ consider the set $S(i)$ which contains all pairs of maps (g, h) such that the following diagram commutes:

$$\begin{array}{ccc} A_i & \xrightarrow{g} & X \\ {\scriptstyle f_i}\downarrow & & {\scriptstyle p}\downarrow \\ B_i & \xrightarrow{h} & Y \end{array} . \tag{7.16}$$

We define the *Gluing Construction* $G^1(\mathcal{F}, p)$ to be the object of **C** given by the pushout diagram

$$\begin{array}{ccc} \coprod_{i \in \mathcal{I}} \coprod_{(g,h) \in S(i)} A_i & \xrightarrow{+_i +_{(g,h)} g} & X \\ \coprod f_i \downarrow & & i_1 \downarrow \\ \coprod_{i \in \mathcal{I}} \coprod_{(g,h) \in S(i)} B_i & \xrightarrow{+_i +_{(g,h)} h} & G^1(\mathcal{F}, p) \end{array} .$$

This is reminiscent of a "singular complex" construction; we are gluing a copy of B_i to X along A_i for *every* commutative diagram of the form 7.16. As indicated, there is a natural map $i_1 : X \to G^1(\mathcal{F}, p)$. By the universal property of colimits, the commutative diagrams 7.16 induce a map $p_1 : G^1(\mathcal{F}, p) \to Y$ such that $p_1 i_1 = p$. Now repeat the process: for $k > 1$ define objects $G^k(\mathcal{F}, p)$ and maps $p_k : G^k(\mathcal{F}, p) \to Y$ inductively by setting $G^k(\mathcal{F}, p) = G^1(\mathcal{F}, p_{k-1})$ and $p_k = (p_{k-1})_1$. What results is a commutative diagram

$$\begin{array}{ccccccccccc}
X & \xrightarrow{i_1} & G^1(\mathcal{F},p) & \xrightarrow{i_2} & G^2(\mathcal{F},p) & \xrightarrow{i_3} & \cdots & \xrightarrow{i_k} & G^k(\mathcal{F},p) & \longrightarrow & \cdots \\
{\scriptstyle p}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & & & {\scriptstyle p_k}\downarrow & & \\
Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & \cdots & \xrightarrow{=} & Y & \xrightarrow{=} & \cdots
\end{array}$$

Let $G^\infty(\mathcal{F}, p)$, the Infinite Gluing Construction, denote the colimit (2.17) of the upper row in the above diagram; there are natural maps $i_\infty : X \to G^\infty(\mathcal{F}, p)$ and $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ such that $p_\infty i_\infty = p$.

7.17. PROPOSITION. *In the above situation, suppose that for each $i \in \mathcal{I}$ the object A_i of* **C** *is sequentially small. Then the map $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ has the RLP* (3.12) *with respect to each of the maps in the family $\mathcal{F}$.*

PROOF. Consider a commutative diagram which gives one of the lifting problems in question:

$$\begin{array}{ccc}
A_i & \xrightarrow{g} & G^\infty(\mathcal{F},p) \\
{\scriptstyle f_i}\downarrow & & {\scriptstyle p_\infty}\downarrow \\
B_i & \xrightarrow{h} & Y
\end{array} .$$

Since A_i is sequentially small, there exists an integer k such that the map g is the composite of a map $g' : A_i \to G^k(\mathcal{F}, p)$ with the natural map $G^k(\mathcal{F}, p) \to G^\infty(\mathcal{F}, p)$. Therefore the above commutative diagram can be enlarged to another one

$$\begin{array}{ccccccc}
A_i & \xrightarrow{g'} & G^k(\mathcal{F},p) & \xrightarrow{i_{k+1}} & G^{k+1}(\mathcal{F},p) & \longrightarrow & G^\infty(\mathcal{F},p) \\
{\scriptstyle f_i}\downarrow & & {\scriptstyle p_k}\downarrow & & {\scriptstyle p_{k+1}}\downarrow & & {\scriptstyle p_\infty}\downarrow \\
B_i & \xrightarrow{h} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y
\end{array}$$

in which the composite all the way across the top row is g. However, the pair (g', h) contributes itself as an index in the construction of $G^{k+1}(\mathcal{F}, p)$ from $G^k(\mathcal{F}, p)$; what it indexes is in fact a gluing of B_i to $G^k(\mathcal{F}, p)$ along A_i. By construction, then, there exists a map $B_i \to G^{k+1}(\mathcal{F}, p)$ which makes the appropriate diagram commute. Composing with the map $G^{k+1}(\mathcal{F}, p) \to G^\infty(\mathcal{F}, p)$ gives a lifting in the original square. □

7.18. *Proof of* MC5

For $n \geqslant 1$, let D^n (the "n-disk") denote the chain complex $D_n(R)$ (7.7) and for $n \geqslant 0$ let S^n (the "n-sphere") denote the chain complex $K(R, n)$ (7.3). There is an evident inclusion $j_n : S^{n-1} \to D_n$ which is the identity on the copy of R in degree $(n-1)$. Let D^0 denote the chain complex $K(R, 0)$, let S^{-1} denote the zero chain complex, and let $j_0 : S^{-1} \to D^0$ be the unique map. Note that the chain complexes D^n and S^n are sequentially small (7.15).

The following proposition is an elementary exercise in diagram chasing.

7.19. PROPOSITION. *A map $f : X \to Y$ in $\mathbf{Ch}_R$ is*

(i) *a fibration if and only if it has the RLP with respect to the maps $0 \to D^n$ for all $n \geqslant 1$, and*
(ii) *an acyclic fibration if and only if it has the RLP with respect to the maps $j_n : S^{n-1} \to D^n$ for all $n \geqslant 0$.*

To verify **MC5**(i), let $f : X \to Y$ be the map to be factored, and let $\mathcal{F}$ be the set of maps $\{j_n\}_{n \geqslant 0}$. Consider the factorization of f provided by the small object argument (7.12):

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{F}, f) \xrightarrow{p_\infty} Y.$$

It is immediate from 7.17 and 7.19 that p_∞ is an acyclic fibration, so what we have to check is that i_∞ is a cofibration. This is essentially obvious; in each degree n, $G^{k+1}(\mathcal{F}, f)$ is by construction the direct sum of $G^k(\mathcal{F}, f)$ with a (possibly large) number of copies of R; passing to the colimit shows that $G^\infty(\mathcal{F}, f)_n$ is similarly the direct sum of X_n with copies of R.

The proof of **MC5**(ii) is very similar: let $f : X \to Y$ be the map to be factored, let $\mathcal{F}'$ be the set of maps $\{0 \to D_n\}_{n \geqslant 1}$ and consider the factorization of f provided by the small object argument:

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{F}', f) \xrightarrow{p_\infty} Y.$$

Again it is immediate from 7.17 and 7.19 that p_∞ is a fibration. We leave it to the reader to check that i_∞ in this case is an acyclic cofibration. □

PROOF OF 7.3. We will only treat the case in which $m = 0$ and $n > 0$; the general case is similar. Use **MC5**(i) to find a weak equivalence $P \to K(A, 0)$, where P is some cofibrant object of $\mathbf{Ch}_R$. There are bijections

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}\big(K(A,0), K(B,n)\big) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}\big(P, K(B,n)\big) \cong \pi\big(P, K(B,n)\big)$$

where the first comes from the fact (5.8) that the map $P \to K(A, 0)$ becomes an isomorphism in $\mathrm{Ho}(\mathbf{C})$, and the second is from 5.11. Let X denote the good path object for $K(B, n)$ given by

$$X_i = \begin{cases} B \oplus B, & i = n, \\ B, & i = n-1, \\ 0, & \text{otherwise,} \end{cases}$$

with boundary map $X_n \to X_{n-1}$ sending (b_0, b_1) to $b_1 - b_0$. The path object structure maps $q : K(B,n) \to X$ and $p_0, p_1 : X \to K(B,n)$ are determined in dimension n by the formulas $q(b) = (b,b)$ and $p_i(b_0, b_1) = b_i$. According to 4.23, two maps $f, g : P \to K(B,n)$ represent the same class in $\pi(P, K(B,n))$ if and only if they are related by right homotopy with respect to X, that is, if and only if there is a map $H : P \to X$ such that $p_0 H = f$ and $p_1 H = g$.

In the language of homological algebra, P is a projective resolution of A. A map $f : P \to K(B,n)$ amounts by inspection to a module map $f_n : P_n \to B$ such that $f_n \partial = 0$. Two maps $f, g : P \to K(B,n)$ are related by a right homotopy with respect to X if and only if there exists a map $h : P_{n-1} \to B$ such that $h\partial = f_n - g_n$. A comparison with the standard definition of $\mathrm{Ext}^*_R(A, -)$ in terms of a projective resolution of A [6] now shows that $\pi(P, K(B,n))$ is in natural bijective correspondence with $\mathrm{Ext}^n_R(A, B)$. □

8. Topological spaces

In this section we will construct the model category structure 3.5 on the category **Top** of topological spaces.

8.1. Definition. A map $f : X \to Y$ of spaces is called a *weak homotopy equivalence* [25, p. 404] if for each basepoint $x \in X$ the map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is a bijection of pointed sets for $n = 0$ and an isomorphism of groups for $n \geqslant 1$.

8.2. Definition. A map of spaces $p : X \to Y$ is said to be a *Serre fibration* [25, p. 375] if, for each CW-complex A, the map p has the RLP (3.12) with respect to the inclusion $A \times 0 \to A \times [0,1]$.

8.3. Proposition. *Call a map of topological spaces*

(i) *a* weak equivalence *if it is a weak homotopy equivalence,*
(ii) *a* fibration *if it is a Serre fibration, and*
(iii) *a* cofibration *if it has the LLP with respect to acyclic fibrations (i.e. with respect to each map which is both a Serre fibration and a weak homotopy equivalence).*

Then with these choices **Top** *is a model category.*

After proving this we will make the following calculation.

8.4. Proposition. *Suppose that A is a CW-complex and that X is an arbitrary space. Then the set* $\mathrm{Hom}_{\mathbf{Ho(Top)}}(A, X)$ *is in natural bijective correspondence with the set of (conventional) homotopy classes of maps from A to X.*

Remark. In the model category structure of 8.3, every space is weakly equivalent to a CW-complex.

We will need two facts from elementary homotopy theory (cf. 7.19). Let D^n $(n \geqslant 1)$ denote the topological n-disk and S^n $(n \geqslant 0)$ the topological n-sphere. Let D^0 be a single point and S^{-1} the empty space. There are standard (boundary) inclusions $j_n : S^{n-1} \to D^n$ $(n \geqslant 0)$.

8.5. LEMMA [14, Theorem 3.1, p. 63]. *Let $p : X \to Y$ be a map of spaces. Then p is a Serre fibration if and only if p has the RLP with respect to the inclusions $D^n \to D^n \times [0, 1]$, $n \geqslant 0$.*

8.6. LEMMA. *Let $p : X \to Y$ be a map of spaces. Then the following conditions are equivalent:*

(i) *p is both a Serre fibration and a weak homotopy equivalence,*
(ii) *p has the RLP with respect to every inclusion $A \to B$ such that (B, A) is a relative CW-pair, and*
(iii) *p has the RLP with respect to the maps $j_n : S^{n-1} \to D^n$ for $n \geqslant 0$.*

This is not hard to prove with the arguments from [25, p. 376]. We will also need a fact from elementary point-set topology.

8.7. LEMMA. *Suppose that*

$$X_0 \to X_1 \to X_2 \to \cdots \to X_n \to \cdots$$

is a sequential direct system of spaces such that for each $n \geqslant 0$ the space X_n is a subspace of X_{n+1} and the pair (X_{n+1}, X_n) is a relative CW-complex [25, p. 401]. *Let A be a finite CW-complex. Then the natural map* (7.13)

$$\mathrm{colim}_n \mathrm{Hom}_{\mathbf{Top}}(A, X_n) \to \mathrm{Hom}_{\mathbf{Top}}(A, \mathrm{colim}_n X_n)$$

is a bijection (of sets).

8.8. REMARK. In the situation of 8.7, we will refer to the natural map $X_0 \to \mathrm{colim}_n X_n$ as a *generalized relative CW inclusion* and say that $\mathrm{colim}_n X_n$ *is obtained from X_0 by attaching cells.* It follows easily 8.6 that any such generalized relative CW inclusion is a cofibration with respect to the model category structure described in 8.3. There is a partial converse to this.

8.9. PROPOSITION. *Every cofibration in* **Top** *is a retract of a generalized relative CW inclusion.*

8.10. *Proof of* **MC1–MC3**. It is easy to see directly that the classes of weak equivalences, fibrations and cofibrations contain all identity maps and are closed under composition. Axiom **MC1** follows from the fact that **Top** has all small limits and colimits (2.14, 2.21). Axiom **MC2** is obvious. For the case of weak equivalences, **MC3** follows from functoriality and 2.6. The other two cases of **MC3** are similar, so we will deal only with cofibrations. Suppose that f is a retract of a cofibration f'. We need to show that a lift exists in every diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow \\ B & \xrightarrow{b} & Y \end{array} \tag{8.11}$$

in which p is an acyclic fibration. Consider the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{i} & A' & \xrightarrow{r} & A & \xrightarrow{a} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & & & {\scriptstyle p}\downarrow \\ B & \xrightarrow{j} & B' & \xrightarrow{s} & B & \xrightarrow{b} & Y \end{array}$$

in which maps i, j, r and s are retraction constituents. Since f' is a cofibration, there is a lifting $h : B' \to X$ in this diagram. It is now easy to see that hj is the desired lifting in the diagram 8.11. □

The proofs of **MC4**(ii) and **MC5**(ii) depend upon a lemma.

8.12. LEMMA. *Every map $p : X \to Y$ in* **Top** *can be factored as a composite $p_\infty i_\infty$, where $i_\infty : X \to X'$ is a weak homotopy equivalence which has the LLP with respect to all Serre fibrations, and $p_\infty : X' \to Y$ is a Serre fibration.*

PROOF. Let $\mathcal{F}$ be the set of maps $\{D^n \times 0 \to D^n \times [0,1]\}_{n \geqslant 0}$. Consider the Gluing Construction $G^1(\mathcal{F}, p)$ (see 7.12). It is clear that $i_1 : X \to G^1(\mathcal{F}, p)$ is a relative CW inclusion and a deformation retraction; in fact, $G^1(\mathcal{F}, p)$ is obtained from X by taking (many) solid cylinders and attaching each one to X along one end. It follows from the definition of Serre fibration that the map i_1 has the LLP with respect to all Serre fibrations. Similarly, for each $k \geqslant 1$ the map $i_{k+1} : G^k(\mathcal{F}, p) \to G^{k+1}(\mathcal{F}, p)$ is a homotopy equivalence which has the LLP with respect to all Serre fibrations. Consider the factorization

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{F}, p) \xrightarrow{p_\infty} Y$$

provided by the Infinite Gluing Construction. It is immediate that i_∞ has the LLP with respect to all Serre fibrations: given a lifting problem, one can inductively find compatible solutions on the spaces $G^k(\mathcal{F}, p)$ and then use the universal property of colimit to obtain a solution on $G^\infty(\mathcal{F}, p) = \operatorname{colim}_k G^k(\mathcal{F}, p)$. The proof of Proposition 7.17 shows that p_∞ has the RLP with respect to the maps in $\mathcal{F}$ and so (8.5) is a Serre fibration; it is only necessary to observe that although the spaces D^n are not in general sequentially small, they are (8.7) small with respect to the particular sequential colimit that comes up here. Finally, by 8.7 any map of a sphere into $G^\infty(\mathcal{F}, p)$ or any homotopy involving such maps must actually lie in $G^k(\mathcal{F}, p)$ for some k; it follows that i_∞ is a weak homotopy equivalence because (by the remarks above) each of the maps $X \to G^k(\mathcal{F}, p)$ is a weak homotopy equivalence. □

PROOF OF **MC5**. Axiom **MC5**(ii) is an immediate consequence of 8.12. The proof of **MC5**(i) is similar to the proof of 8.12. Let p be the map to be factored, let $\mathcal{F}$ be the set

$$\mathcal{F} = \{j_n : S^{n-1} \to D^n\}_{n \geqslant 0}$$

and consider the factorization $p = p_\infty i_\infty$ of p provided by the Infinite Gluing Construction $G^\infty(\mathcal{F}, p)$. By 8.6 each map $i_{k+1} : G^k(\mathcal{F}, p) \to G^{k+1}(\mathcal{F}, p)$ has the LLP with respect to Serre fibrations which are weak homotopy equivalences; by induction and a colimit argument the map i_∞ has the same LLP and so by definition is a cofibration. By 8.7 and the proof of 7.17, the map p_∞ has the RLP with respect to all maps in the set $\mathcal{F}$, and so (8.6) is a Serre fibration and a weak equivalence. □

PROOF OF **MC4**. Axiom **MC4**(i) is immediate from the definition of cofibration. For **MC4**(ii) suppose that $f : A \to B$ is an acyclic cofibration; we have to show that f has the LLP with respect to fibrations. Use 8.12 to factor f as a composite pi, where p is a fibration and i is weak homotopy equivalence which has the LLP with respect to all fibrations. Since $f = pi$ is by assumption a weak homotopy equivalence, it is clear that p is also a weak homotopy equivalence. A lift $g : B \to A'$ exists in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & A' \\ {\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow\sim \\ B & \xrightarrow{\mathrm{id}} & B \end{array} \tag{8.13}$$

because f is a cofibration and p is an acyclic fibration. (Recall that by definition every cofibration has the LLP with respect to acyclic fibrations.) This lift g expresses the map f as a retract (2.6) of the map i. The argument in 8.10 above can now be used to show that the class of maps which have the LLP with respect to all Serre fibrations is closed under retracts; it follows that f has the LLP with respect to all Serre fibrations because i does. □

PROOF OF 8.4. Since A is cofibrant (8.6) and X is fibrant, the set $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top})}(A, X)$ is naturally isomorphic to $\pi(A, X)$ (see 5.11). It is also easy to see from 8.6 that the product $A \times [0, 1]$ is a good cylinder object for A. By 4.23, two maps $f, g : A \to X$ represent the same element of $\pi(A, X)$ if and only if they are left homotopic via the cylinder object $A \times [0, 1]$, in other words, if and only if they are homotopic in the conventional sense. □

PROOF OF 8.9. Let $f : A \to B$ be a cofibration in **Top**. The argument in the proof of **MC5(i)** above shows that f can be factored as a composite pi, where $i : A \to A'$ is a generalized relative CW inclusion and $p : A' \to B$ is an acyclic fibration. Since f is a cofibration, a lift $g : B \to A'$ exists in the resulting diagram 8.13, and this lift g expresses f as a retract of i. □

9. Derived functors

Let $\mathbf{C}$ be a model category and $F : \mathbf{C} \to \mathbf{D}$ a functor. In this section we define the *left and right derived functors* of F; if they exist, these are functors

$$LF, RF : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$$

which, up to natural transformation on one side or the other, are the best possible approximations to an "extension of F to $\mathrm{Ho}(\mathbf{C})$", that is, to a factorization of F through $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$. We give a criterion for the derived functors to exist, and study a condition under which a pair of adjoint functors (2.8) between two model categories induces, via a derived functor construction, adjoint functors between the associated homotopy categories. The homotopy pushout and homotopy pullback functors of §10 will be constructed by taking derived functors of genuine pushout or pullback functors.

9.1. DEFINITION. Suppose that $\mathbf{C}$ is a model category and that $F : \mathbf{C} \to \mathbf{D}$ is a functor. Consider pairs (G, s) consisting of a functor $G : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ and natural transformation $s : G\gamma \to F$. A *left derived functor* for F is a pair (LF, t) of this type which is universal from the left, in the sense that if (G, s) is any such pair, then there exists a unique natural transformation $s' : G \to LF$ such that the composite natural transformation

$$G\gamma \xrightarrow{s' \circ \gamma} (LF)\gamma \xrightarrow{t} F \tag{9.2}$$

is the natural transformation s.

REMARK. A *right derived functor* for F is a pair (RF, t), where $RF : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ is a functor and $t : F \to (RF)\gamma$ is a natural transformation with the analogous property of being "universal from the right".

REMARK. The universal property satisfied by a left derived functor implies as usual that any two left derived functors of F are canonically naturally equivalent. Sometimes we will refer to LF as the left derived functor of F and leave the natural transformation t understood. If F takes weak equivalences in $\mathbf{C}$ into isomorphisms in $\mathbf{D}$, then there is a functor $F' : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ with $F' = F\gamma$ (6.2), and it is not hard to see that in this case F' itself (with the identity natural transformation $t : F'\gamma \to F$) is a left derived functor of F. The next proposition shows that sometimes LF exists even though a functor F' as above does not.

9.3. PROPOSITION. *Let $\mathbf{C}$ be a model category and $F : \mathbf{C} \to \mathbf{D}$ a functor with the property that $F(f)$ is an isomorphism whenever f is a weak equivalence between cofibrant objects in $\mathbf{C}$. Then the left derived functor (LF, t) of F exists, and for each cofibrant object X of $\mathbf{C}$ the map*

$$t_X : LF(X) \to F(X)$$

is an isomorphism.

The proof depends on a lemma, which for future purposes we state in slightly greater generality than we actually need here.

9.4. LEMMA. *Let* $\mathbf{C}$ *be a model category and* $F : \mathbf{C}_c \to \mathbf{D}$ (§5) *a functor such that* $F(f)$ *is an isomorphism whenever* f *is an acyclic cofibration between objects of* $\mathbf{C}_c$. *Suppose that* $f, g : A \to B$ *are maps in* $\mathbf{C}_c$ *such that* f *is right homotopic to* g *in* $\mathbf{C}$. *Then* $F(f) = F(g)$.

PROOF. By 4.15 there exists a right homotopy $H : A \to B^I$ from f to g such that B^I is a very good path object for B. Since the path object structure map $w : B \to B^I$ is then an acyclic cofibration and B by assumption is cofibrant, it follows that B^I is cofibrant and hence that $F(w)$ is defined and is an isomorphism. The rest of the proof is identical to the dual of the proof of 5.10. First observe that there are equalities $F(p_0)F(w) = F(p_1)F(w) = F(\mathrm{id}_B)$ and then use the fact that $F(w)$ is an isomorphism to cancel $F(w)$ and obtain $F(p_0) = F(p_1)$. The equality $F(f) = F(g)$ then follows from applying F to the equalities $f = p_0H$ and $g = p_1H$. □

PROOF OF 9.3. By Lemma 9.4, F identifies right homotopic maps between cofibrant objects of $\mathbf{C}$ and so induces a functor $F' : \pi\mathbf{C}_c \to \mathbf{D}$. By assumption, if g is a morphism of $\pi\mathbf{C}_c$ which is represented by a weak equivalence in $\mathbf{C}$ then $F'(g)$ is an isomorphism. Recall from 5.2 that there is a functor $Q : \mathbf{C} \to \pi\mathbf{C}_c$ with the property (5.1) that if f is a weak equivalence in $\mathbf{C}$ then $g = Q(f)$ is a right homotopy class which is represented by a weak equivalence in $\mathbf{C}$. It follows that the composite functor $F'Q$ carries weak equivalences in $\mathbf{C}$ into isomorphisms in $\mathbf{D}$. By the universal property (6.2) of $\mathrm{Ho}(\mathbf{C})$, the composite $F'Q$ induces a functor $\mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$, which we denote LF. There is a natural transformation $t : (LF)\gamma \to F$ which assigns to each X in $\mathbf{C}$ the map $F(p_X) : LF(X) = F(QX) \to F(X)$. If X is cofibrant then $QX = X$ and the map t_X is the identity; in particular, t_X is an isomorphism.

We now have to show that the pair (LF, t) is universal from the left in the sense of 9.1. Let $G : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ be a functor and $s : G\gamma \to F$ a natural transformation. Consider a hypothetical natural transformation $s' : G \to LF$, and construct (for each object X of $\mathbf{C}$) the following commutative diagram which in the horizontal direction involves the composite of $s' \circ \gamma$ and t;

$$
\begin{array}{ccccc}
G(QX) & \xrightarrow{s'_{QX}} & LF(QX) & \xrightarrow{t_{QX}=\mathrm{id}} & F(QX) \\
{\scriptstyle G(\gamma(p_X))} \downarrow & & \downarrow {\scriptstyle LF(\gamma(p_X))=\mathrm{id}} & & \downarrow {\scriptstyle F(p_X)} \\
G(X) & \xrightarrow{s'_X} & LF(X) & \xrightarrow{t_X=F(p_X)} & F(X)
\end{array}
$$

If s' is to satisfy the condition of 9.1, then the composite across the top row of this diagram must be equal to s_{QX}, which gives the equality $s'_X = s_{QX}G(\gamma p_X)^{-1}$ and proves that there is at most one natural transformation s' which satisfies the required condition. However, it is obvious that setting $s'_X = s_{QX}G(\gamma p_X)^{-1}$ does give a natural transformation $G\gamma \to (LF)\gamma$, and therefore (5.9) it also gives a natural transformation $G \to LF$. □

9.5. DEFINITION. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between model categories. A *total left derived functor* $\mathbf{L}F$ for F is a functor

$$\mathbf{L}F : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$$

which is a left derived functor for the composite $\gamma_{\mathbf{D}} \cdot F : \mathbf{C} \to \mathrm{Ho}(\mathbf{D})$. Similarly, a *total right derived functor* $\mathbf{R}F$ for F is a functor $\mathbf{R}F : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ which is a right derived functor for the composite $\gamma_{\mathbf{D}} \cdot F$.

REMARK. As usual, total left or right derived functors are unique up to canonical natural equivalence.

9.6. EXAMPLE. Let R be an associative ring with unit, and $\mathbf{Ch}_R$ the chain complex model category constructed in §7. Suppose that M is a *right* R-module, so that $M \otimes -$ gives a functor $F : \mathbf{Ch}_R \to \mathbf{Ch}_\mathbf{Z}$. Proposition 9.3 can be used to show that the total derived functor $\mathbf{L}F$ exists (see 9.11). Let N be a left R-module and $K(N,0)$ (cf. 7.3) the corresponding chain complex. The final statement in 9.3 implies that $\mathbf{L}F(K(N,0))$ is isomorphic in $\mathrm{Ho}(\mathbf{Ch}_\mathbf{Z})$ to $F(P)$, where P is any cofibrant chain complex with a weak equivalence $P \xrightarrow{\sim} K(N,0)$. Such a cofibrant chain complex P is exactly a projective resolution of N in the sense of homological algebra, and so we obtain natural isomorphisms

$$\mathrm{H}_i\mathbf{L}F\big(K(N,0)\big) \cong \mathrm{Tor}_i^R(M,N), \quad i \geqslant 0,$$

where $\mathrm{Tor}_i^R(M,-)$ is the usual i'th left derived functor of $M \otimes_R -$. This gives one connection between the notion of total derived functor in 9.5 and the standard notion of derived functor from homological algebra.

9.7. THEOREM. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be model categories, and*

$$F : \mathbf{C} \Longleftrightarrow \mathbf{D} : G$$

a pair of adjoint functors (2.8). *Suppose that*

(i) F *preserves cofibrations and* G *preserves fibrations.*

Then the total derived functors

$$\mathbf{L}F : \mathrm{Ho}(\mathbf{C}) \Longleftrightarrow \mathrm{Ho}(\mathbf{D}) : \mathbf{R}G$$

exist and form an adjoint pair. If in addition we have

(ii) *for each cofibrant object* A *of* $\mathbf{C}$ *and fibrant object* X *of* $\mathbf{D}$, *a map* $f : A \to G(X)$ *is a weak equivalence in* $\mathbf{C}$ *if and only if its adjoint* $f^\flat : F(A) \to X$ *is a weak equivalence in* $\mathbf{D}$,

then $\mathbf{L}F$ *and* $\mathbf{R}G$ *are inverse equivalences of categories.*

REMARK. In this paper we will not use the last statement of 9.7, but this criterion for showing that two model categories have equivalent homotopy categories is used heavily by Quillen in [23]. There are various other structures associated to a model category besides its homotopy category; these include fibration and cofibration sequences [22], Toda brackets [22], various homotopy limits and colimits (§10), and various function complexes [9]. All such structures that we know of are preserved by adjoint functors that satisfy the two conditions above.

9.8. REMARK. Condition 9.7(i) is equivalent to *either* of the following two conditions:

(i′) G preserves fibrations and acyclic fibrations.
(i″) F preserves cofibrations and acyclic cofibrations.

Assume, for instance, that F preserves acyclic cofibrations. Let $f : A \to B$ be an acyclic cofibration in $\mathbf{C}$ and $g : X \to Y$ a fibration in $\mathbf{D}$. Suppose given the commutative diagram on the left together with its "adjoint" diagram (2.8) on the right:

$$\begin{array}{ccc} A & \xrightarrow{u} & G(X) \\ {\scriptstyle f}\downarrow & & {\scriptstyle G(g)}\downarrow \\ B & \xrightarrow{v} & G(Y) \end{array} \qquad\qquad \begin{array}{ccc} F(A) & \xrightarrow{u^\flat} & X \\ {\scriptstyle F(f)}\downarrow & & {\scriptstyle g}\downarrow \\ F(B) & \xrightarrow{v^\flat} & Y \end{array}$$

Since F preserves acyclic cofibrations, a lift $w : F(B) \to X$ exists in the right-hand diagram. Its adjoint $w^\sharp : B \to G(X)$ is then a lift in the left-hand diagram. It follows that $G(g)$ has the RLP with respect to all acyclic cofibrations in $\mathbf{C}$, and therefore by 3.13(iii) that $G(g)$ is a fibration. This gives 9.7(i). Running the argument in reverse and using 3.13(ii) shows the converse: if G preserves fibrations then F preserves acyclic cofibrations.

The proof of 9.7 depends on a lemma that is also useful in verifying the hypotheses of 9.3.

9.9. LEMMA (K. Brown). *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between model categories. If F carries acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

PROOF. Let $f : A \to B$ be a weak equivalence in $\mathbf{C}$ between cofibrant objects. By **MC5**(i) we can factor the coproduct (2.15) map $f + \mathrm{id}_B : A \coprod B \to B$ as a composite pq, where $q : A \coprod B \to C$ is a cofibration and $p : C \to B$ is an acyclic fibration. It follows from the fact that A and B are cofibrant (cf. 4.4) that the composite maps $q \cdot \mathrm{in}_0 : A \to C$ and $q \cdot \mathrm{in}_1 : B \to C$ are cofibrations. Since $pq \cdot \mathrm{in}_i$ is a weak equivalence for $i = 0, 1$ and p is a weak equivalence, it is clear from **MC2** that $q \cdot \mathrm{in}_i$ is a weak equivalence, $i = 0, 1$. By assumption, then $F(q \cdot \mathrm{in}_0)$, $F(q \cdot \mathrm{in}_1)$ and $F(pq \cdot \mathrm{in}_1) = F(\mathrm{id}_B)$ are weak equivalences in $\mathbf{D}$. It follows that the maps $F(p)$ and hence $F(pq \cdot \mathrm{in}_0) = F(f)$ are also weak equivalences. □

PROOF OF 9.7. In view of 9.8, 9.9 and the dual (3.9) of 9.9, Proposition 9.3 and its dual guarantee that the total derived functors $\mathbf{L}F$ and $\mathbf{R}G$ exist. Since F is a left adjoint it preserves colimits (2.26) and therefore (2.25) initial objects. Since G is a right adjoint it preserves limits and therefore terminal objects. It then follows as in 9.8 that F carries cofibrant objects in $\mathbf{C}$ into cofibrant objects in $\mathbf{D}$, and that G carries fibrant objects in $\mathbf{D}$ into fibrant objects in $\mathbf{C}$.

Suppose that A is a cofibrant object in $\mathbf{C}$ and that X is a fibrant object in $\mathbf{D}$. We will show that the adjunction isomorphism $\mathrm{Hom}_{\mathbf{C}}(A, G(X)) \cong \mathrm{Hom}_{\mathbf{D}}(F(A), X)$ respects the homotopy equivalence relation (4.21) and gives a bijection

$$\pi\big(A, G(X)\big) \cong \pi\big(F(A), X\big). \tag{9.10}$$

If $f, g : A \to G(X)$ represent the same class in $\pi(A, G(X))$, then f is left homotopic to g via a left homotopy $H : A \wedge I \to G(X)$ in which the cylinder object $A \wedge I$ is good (4.6) and hence cofibrant (4.4). It then follows from 9.8(i″) that $F(A \wedge I)$ is a cylinder object for $F(A)$ and hence that $H^\flat : F(A \wedge I) \to X$ is a left homotopy between $f^\flat$ and $g^\flat$. Thus $f^\flat \sim g^\flat$. A dual argument with right homotopies shows that if $f^\flat \sim g^\flat$ then $f \sim g$ and establishes the isomorphism 9.10.

Let Q be the construction of 5.2 for $\mathbf{C}$ and S the construction of 5.4 for $\mathbf{D}$. (We have temporarily changed the letter denoting this functor from "R" to "S" in order to avoid confusion with the notation for right derived functors). In view of the construction of $\mathbf{L}F$ and $\mathbf{R}G$ given by the proof of 9.3 and its dual, the isomorphism 9.10 gives for every object A of $\mathbf{C}$ and object X of $\mathbf{D}$ a bijection

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}\big(A, \mathbf{R}G(X)\big) \xrightarrow{(\gamma p_A)^*} \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}\big(QA, G(SX)\big) \\ &\cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{D})}\big(F(QA), SX\big) \xrightarrow{((\gamma i_X)^{-1})_*} \mathrm{Hom}_{\mathrm{Ho}(\mathbf{D})}\big(\mathbf{L}F(A), X\big) . \end{aligned}$$

It is clear that this bijection gives a natural equivalence of functors from $\mathbf{C}^{\mathrm{op}} \times \mathbf{D}$ to **Sets**, and the argument of 5.9 shows that it also gives a natural equivalence of functors $\mathrm{Ho}(\mathbf{C})^{\mathrm{op}} \times \mathrm{Ho}(\mathbf{D}) \to$ **Sets**. This provides the adjunction between $\mathbf{L}F$ and $\mathbf{R}G$.

Suppose that condition (ii) is satisfied. Let A be an cofibrant object of $\mathbf{C}$. The map $i^\sharp_{F(A)} : A \to G(SF(A))$ is then a weak equivalence in $\mathbf{C}$ because its adjoint $i_{F(A)} : F(A) \to SF(A)$ is a weak equivalence in $\mathbf{D}$. Let

$$\varepsilon_A = \mathrm{id}^\sharp_{\mathbf{L}F(A)} : A \to \mathbf{R}G(\mathbf{L}F(A))$$

denote the map in $\mathrm{Ho}(\mathbf{C})$ which is adjoint to the identity map of $\mathbf{L}F(A)$ in $\mathrm{Ho}(\mathbf{D})$. It follows from the above constructions that ε_A is an isomorphism. Since every object of $\mathrm{Ho}(\mathbf{C})$ is isomorphic to A for a cofibrant object A of $\mathbf{C}$, we conclude that ε_A is an isomorphism for any object A of $\mathrm{Ho}(\mathbf{C})$ and thus that the composite $(\mathbf{R}G)(\mathbf{L}F)$ is naturally equivalent to the identity functor of $\mathrm{Ho}(\mathbf{C})$. A dual argument shows that the composite $(\mathbf{L}F)(\mathbf{R}G)$ is naturally equivalent to the identity functor of $\mathrm{Ho}(\mathbf{D})$. This proves that $\mathbf{L}F$ and $\mathbf{R}G$ are inverse equivalences of categories. □

9.11. EXAMPLE. Let $F : \mathbf{Ch}_R \to \mathbf{Ch}_{\mathbf{Z}}$ be the functor of 9.6. In order to use 9.3 to show that the total derived functor $\mathbf{L}F$ exists, it is necessary to show that F carries weak equivalences between cofibrant objects to weak equivalences. By 9.9 it is enough to check this for acyclic cofibrations between cofibrant objects. Let $i : A \to B$ be a acyclic cofibration between cofibrant objects in $\mathbf{Ch}_R$. The quotient B/A is then an acyclic chain complex which satisfies the hypotheses of 7.10, so that by 7.11 there is an isomorphism $B \cong A \oplus (B/A)$ and (7.10) a further isomorphism between B/A and a direct sum of chain complexes of the form $D_k(P)$. Since F respects direct sums we conclude that $F(B)$ is isomorphic to the direct sum of $F(A)$ with a number of chain complexes of the form $F(D_k(P))$. By inspection $F(D_k(P))$ is acyclic, and so $F(i)$ is a weak equivalence.

10. Homotopy pushouts and homotopy pullbacks

The constructions in this section are motivated by the fact that pushouts and pullbacks are not usually well-behaved with respect to homotopy equivalences. For example, in the category **Top** of topological spaces, let D^n $(n \geqslant 1)$ denote the n-disk, $j_n : S^{n-1} \to D^n$ the inclusion of the boundary $(n-1)$-sphere, and $*$ the one-point space. There is a commutative diagram

$$\begin{array}{ccccc} D^n & \xleftarrow{j_n} & S^{n-1} & \xrightarrow{j_n} & D^n \\ \downarrow & & \mathrm{id}\downarrow & & \downarrow \\ * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array} \tag{10.1}$$

in which all three vertical arrows are homotopy equivalences. The pushout (2.16) or colimit of the top row is homeomorphic to S^n, the pushout of the bottom row is the space "$*$", and the map $S^n \to *$ induced by the diagram is not a homotopy equivalence.

Faced with diagram 10.1, a seasoned topologist would probably say that the pushout of the top row has the "correct" homotopy type and invoke the philosophy that to give a pushout homotopy significance the maps involved should be replaced if necessary by cofibrations. In this section we work in an arbitrary model category $\mathbf{C}$ and find a conceptual basis for this philosophy. The strategy is this. Let $\mathbf{D}$ be the category $\{a \leftarrow b \to c\}$ of 2.12 and $\mathbf{C}^{\mathbf{D}}$ the category of functors $\mathbf{D} \to \mathbf{C}$ (2.5). An object of $\mathbf{C}^{\mathbf{D}}$ is pushout data

$$X(a) \leftarrow X(b) \to X(c)$$

in $\mathbf{C}$ and a morphism $f : X \to Y$ is a commutative diagram

$$\begin{array}{ccccc} X(a) & \longleftarrow & X(b) & \longrightarrow & X(c) \\ f_a\downarrow & & f_b\downarrow & & f_c\downarrow \\ Y(a) & \longleftarrow & Y(b) & \longrightarrow & Y(c) \end{array} \tag{10.2}$$

The pushout or colimit construction gives a functor colim : $\mathbf{C}^{\mathbf{D}} \to \mathbf{C}$. We will construct a model category structure on $\mathbf{C}^{\mathbf{D}}$ with respect to which a weak equivalence is a map f whose three components (f_a, f_b, f_c) are weak equivalences in $\mathbf{C}$. As 10.1 illustrates, in this setting the functor $\mathrm{colim}(-)$ is not usually homotopy invariant (i.e. does not usually carry weak equivalences in $\mathbf{C}^{\mathbf{D}}$ to weak equivalences is $\mathbf{C}$) and so $\mathrm{colim}(-)$ does not directly induce a functor $\mathrm{Ho}(\mathbf{C}^{\mathbf{D}}) \to \mathrm{Ho}(\mathbf{C})$. However, it turns out that $\mathrm{colim}(-)$ does have a *total left derived functor* (9.5)

$$\mathbf{L}\mathrm{colim} : \mathrm{Ho}(\mathbf{C}^{\mathbf{D}}) \to \mathrm{Ho}(\mathbf{C})$$

which in a certain sense (9.1) is the best possible homotopy invariant approximation to $\mathrm{colim}(-)$. We will call $\mathbf{L}\mathrm{colim}$ the *homotopy pushout functor*; it is left adjoint to the functor

$$\mathrm{Ho}(\Delta) : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C}^{\mathbf{D}})$$

induced by the "constant diagram" (2.11) construction $\Delta : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}^{\mathbf{D}}$. By 9.3, computing $\mathbf{L}\mathrm{colim}(X)$ for a diagram X involves computing $\mathrm{colim}(X')$, where X' is a cofibrant object of $\mathbf{C}^{\mathbf{D}}$ which is weakly equivalent to X. It turns out that finding such a cofibrant X' involves replacing $X(b)$ by a cofibrant object and replacing the maps $X(b) \to X(a)$ and $X(b) \to X(c)$ by cofibrations, and so in the end what we do is more or less recover, in this abstract setting, the standard philosophy. In fact, it becomes clear (see 9.6) that this philosophy is no different from the philosophy in homological algebra that a cautious practitioner should usually replace a module by a projective resolution before, for instance, tensoring it with something.

Working dually gives a construction of the *homotopy pullback functor*. At the end of the section we make a few remarks about more general homotopy colimits or limits in $\mathbf{C}$.

10.3. Remark. In the above situation, there is a natural functor $\mathrm{Ho}(\mathbf{C}^{\mathbf{D}}) \to \mathrm{Ho}(\mathbf{C})^{\mathbf{D}}$, but this functor is usually *not* an equivalence of categories (and much of the subtlety of homotopy theory lies in this fact). Consequently, the homotopy pushout functor $\mathbf{L}\mathrm{colim}$ does *not* provide "pushouts in the homotopy category", that is, it is *not* a left adjoint to constant diagram functor

$$\Delta_{\mathrm{Ho}(\mathbf{C})} : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C})^{\mathbf{D}}.$$

10.4. *Homotopy pushouts*

Let $\mathbf{C}$ be a model category, $\mathbf{D}$ be the category $\{a \leftarrow b \to c\}$ above, and $\mathbf{C}^{\mathbf{D}}$ the category of functors $\mathbf{D} \to \mathbf{C}$. Given a map $f : X \to Y$ of $\mathbf{C}^{\mathbf{D}}$ as in 10.2, let $\partial_b(f)$ denote $X(b)$ and define objects $\partial_a(f)$ and $\partial_c(f)$ of $\mathbf{C}$ by the pushout diagrams

$$\begin{array}{ccc} X(b) & \longrightarrow & X(a) \\ {\scriptstyle f_b}\downarrow & & \downarrow \\ Y(b) & \longrightarrow & \partial_a(f) \end{array} \qquad \begin{array}{ccc} X(b) & \longrightarrow & X(c) \\ {\scriptstyle f_b}\downarrow & & \downarrow \\ Y(b) & \longrightarrow & \partial_c(f) \end{array} \quad . \tag{10.5}$$

The commutative diagram 10.2 induces maps $i_a(f) : \partial_a(f) \to Y(a)$, $i_b(f) : \partial_b(f) \to Y(b)$, and $i_c(f) : \partial_c(f) \to Y(c)$.

10.6. PROPOSITION. *Call a morphism $f : X \to Y$ in $\mathbf{C}^{\mathbf{D}}$*

(i) *a weak equivalence, if the morphisms f_a, f_b and f_c are weak equivalences in* $\mathbf{C}$,
(ii) *a fibration if the morphisms f_a, f_b and f_c are fibrations in* $\mathbf{C}$, *and*
(iii) *a cofibration if the maps $i_a(f)$, $i_b(f)$ and $i_c(f)$ are cofibrations in* $\mathbf{C}$.

Then these choices provide $\mathbf{C}^{\mathbf{D}}$ with the structure of a model category.

PROOF. Axiom **MC1** follows from 2.27. Axiom **MC2** and the parts of **MC3** dealing with weak equivalences and fibrations are direct consequences of the corresponding axioms in $\mathbf{C}$. It is not hard to check that if f is a retract of g, then the maps $i_a(f)$, $i_b(f)$ and $i_c(f)$ are respectively retracts of $i_a(g)$, $i_b(g)$ and $i_c(g)$, so that the part of **MC3** dealing with cofibrations is also a consequence of the corresponding axiom for $\mathbf{C}$. For **MC4**(i), consider a commutative diagram

$$\begin{array}{ccc}
(A(a) \leftarrow A(b) \rightarrow A(c)) & \longrightarrow & (X(a) \leftarrow X(b) \rightarrow X(c)) \\
f \downarrow & & p \downarrow \\
(B(a) \leftarrow B(b) \rightarrow B(c)) & \longrightarrow & (Y(a) \leftarrow Y(b) \rightarrow Y(c))
\end{array}$$

in which f is a cofibration and p is an acyclic fibration. This diagram consists of three slices:

$$\begin{array}{ccc}
A(a) & \longrightarrow & X(a) \\
f_a \downarrow & & p_a \downarrow \\
B(a) & \longrightarrow & Y(a),
\end{array}
\qquad
\begin{array}{ccc}
A(b) & \longrightarrow & X(b) \\
f_b \downarrow & & p_b \downarrow \\
B(b) & \longrightarrow & Y(b),
\end{array}
\qquad
\begin{array}{ccc}
A(c) & \longrightarrow & X(c) \\
f_c \downarrow & & p_c \downarrow \\
B(c) & \longrightarrow & Y(c)
\end{array}$$

Since f is a cofibration and p is an acyclic fibration, we can obtain the desired lifting in the middle slice by applying **MC4**(i) in $\mathbf{C}$; this lifting induces maps $u : \partial_a(f) \to X(a)$ and $v : \partial_c(f) \to X(c)$. Liftings in the other two slices can now be constructed by applying **MC4**(i) in $\mathbf{C}$ to the squares

$$\begin{array}{ccc}
\partial_a(f) & \xrightarrow{u} & X(a) \\
i_a(f) \downarrow & & p_a \downarrow \\
B(a) & \longrightarrow & Y(a)
\end{array}
\qquad
\begin{array}{ccc}
\partial_c(f) & \xrightarrow{v} & X(c) \\
i_c(f) \downarrow & & p_c \downarrow \\
B(c) & \longrightarrow & Y(c)
\end{array}$$

in which each left-hand arrow is a cofibration. The proof of the second part of **MC4**(ii) is analogous; in this case the fact that the maps $i_c(f)$ and $i_a(f)$ are acyclic cofibrations follows easily from the fact that the class of acyclic cofibrations in $\mathbf{C}$ is closed under cobase change (3.14).

To prove **MC5**(ii), suppose that we have a morphism $f : A \to B$. Use **MC5**(ii) in $\mathbf{C}$ to factor the map $f_b : A(b) \to B(b)$ as $A(b) \xrightarrow{\sim} Y \twoheadrightarrow B(b)$. Let X be the pushout of the diagram $A(a) \leftarrow A(b) \rightarrow Y$ and Z the pushout of $Y \leftarrow A(b) \rightarrow A(c)$. There is a commutative diagram

$$\begin{array}{ccccc}
A(a) & \longleftarrow & A(b) & \longrightarrow & A(c) \\
\sim\downarrow & & \sim\downarrow & & \sim\downarrow \\
X & \longleftarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
B(a) & \longleftarrow & B(b) & \longrightarrow & B(c)
\end{array}$$

in which the lower outside vertical arrows are constructed using the universal property of pushouts. Now use **MC5**(ii) in **C** again to factor the lower outside vertical arrows as $X \overset{\sim}{\hookrightarrow} X' \twoheadrightarrow B(a)$ and $Z \overset{\sim}{\hookrightarrow} Z' \twoheadrightarrow B(c)$. It is not hard to see that the object $X' \leftarrow Y \rightarrow Z'$ of $\mathbf{C}^{\mathbf{D}}$ provides the intermediate object for the desired factorization of f. The proof of **MC5**(i) is similar. □

10.7. PROPOSITION. *The adjoint functors*

$$\operatorname{colim} : \mathbf{C}^{\mathbf{D}} \Longleftrightarrow \mathbf{C} : \Delta$$

satisfy condition (i) *of Theorem* 9.7. *Hence the total derived functors* Lcolim *and* $\mathbf{R}\Delta$ *exist and form an adjoint pair*

$$\mathrm{Lcolim} : \mathrm{Ho}(\mathbf{C}^{\mathbf{D}}) \Longleftrightarrow \mathrm{Ho}(\mathbf{C}) : \mathbf{R}\Delta.$$

PROOF. This is clear from 9.8, since the functor Δ preserves both fibrations and acyclic fibrations. □

This completes the construction of the *homotopy pushout functor* Lcolim : $\mathrm{Ho}(\mathbf{C}^{\mathbf{D}}) \rightarrow \mathrm{Ho}(\mathbf{C})$. According to 9.3, $\mathrm{Lcolim}(X)$ is isomorphic to $\mathrm{colim}(X)$ if X is a cofibrant object of $\mathbf{C}^{\mathbf{D}}$; in general $\mathrm{Lcolim}(X)$ is isomorphic to $\mathrm{colim}(X')$ for any cofibrant object X' of $\mathbf{C}^{\mathbf{D}}$ weakly equivalent to X.

10.8. *Homotopy pullbacks*

The following results on homotopy pullbacks are dual (3.9) to the above ones on homotopy pushouts, so we state them without proof.

Let **C** be a model category, let **D** be the category $\{a \rightarrow b \leftarrow c\}$, and $\mathbf{C}^{\mathbf{D}}$ the category of functors $\mathbf{D} \rightarrow \mathbf{C}$. Given a map $f : X \rightarrow Y$ of $\mathbf{C}^{\mathbf{D}}$

$$\begin{array}{ccccc}
X(a) & \longrightarrow & X(b) & \longleftarrow & X(c) \\
f_a\downarrow & & f_b\downarrow & & f_c\downarrow \\
Y(a) & \longrightarrow & Y(b) & \longleftarrow & Y(c)
\end{array}, \qquad (10.9)$$

let $\delta_b(f)$ denote $X(b)$ and define objects $\delta_a(f)$ and $\delta_c(f)$ of $\mathbf{C}$ by the pullback diagrams

$$\begin{array}{ccccccc} \delta_a(f) & \longrightarrow & X(b) & \qquad & \delta_c(f) & \longrightarrow & X(b) \\ \downarrow & & f_b \downarrow & & \downarrow & & f_b \downarrow \\ Y(a) & \longrightarrow & Y(b) & & Y(c) & \longrightarrow & Y(b) \end{array} . \tag{10.10}$$

The commutative diagram 10.9 induces maps $p_a(f) : X(a) \to \delta_a(f)$, $p_b(f) : X(b) \to \delta_b(f)$, and $p_c(f) : X(c) \to \delta_c(f)$.

10.11. PROPOSITION. *Call a morphism $f : X \to Y$ in $\mathbf{C}^{\mathbf{D}}$*

(i) *a weak equivalence, if the morphisms f_a, f_b and f_c are weak equivalences in* $\mathbf{C}$,
(ii) *a cofibration if the morphisms f_a, f_b and f_c are cofibrations in* $\mathbf{C}$, *and*
(iii) *a fibration if the maps $p_a(f)$, $p_b(f)$ and $p_c(f)$ are fibrations in* $\mathbf{C}$.

Then these choices provide $\mathbf{C}^{\mathbf{D}}$ with the structure of a model category.

10.12. PROPOSITION. *The adjoint functors*

$$\Delta : \mathbf{C}^{\mathbf{D}} \Longleftrightarrow \mathbf{C} : \lim$$

satisfy condition (i) *of Theorem* 9.7. *Hence the total derived functors* $\mathbf{R}\mathrm{lim}$ *and* $\mathbf{L}\Delta$ *exist and form an adjoint pair*

$$\mathbf{L}\Delta : \mathrm{Ho}(\mathbf{C}^{\mathbf{D}}) \Longleftrightarrow \mathrm{Ho}(\mathbf{C}) : \mathbf{R}\mathrm{lim}.$$

This completes the construction of the *homotopy pullback functor* $\mathbf{R}\mathrm{lim} : \mathrm{Ho}(\mathbf{C}^{\mathbf{D}}) \to \mathrm{Ho}(\mathbf{C}^{\mathbf{D}})$. According to 9.3, $\mathbf{R}\mathrm{lim}(X)$ is isomorphic to $\lim(X)$ if X is a fibrant object of $\mathbf{C}^{\mathbf{D}}$; in general $\mathbf{R}\mathrm{lim}(X)$ is isomorphic to $\lim(X')$ for any fibrant object X' of $\mathbf{C}^{\mathbf{D}}$ weakly equivalent to X.

10.13. *Other homotopy limits and colimits*

Say that a category $\mathbf{D}$ is *very small* if it satisfies the following conditions

(i) $\mathbf{D}$ has a finite number of objects,
(ii) $\mathbf{D}$ has a finite number of morphisms, and
(iii) there exists an integer N such that if

$$A_0 \xrightarrow{f_1} A_1 \to \cdots \to A_n$$

is a string of composable morphisms of $\mathbf{D}$ with $n > N$, then some f_i is an identity morphism.

Propositions 10.6 and 10.11 can be generalized to give two distinct model category structures on the category $\mathbf{C}^{\mathbf{D}}$ whenever $\mathbf{D}$ is very small. These structures share the same weak equivalences (and therefore have isomorphic homotopy categories) but they differ in their fibrations and cofibrations. One of these structures is adapted to constructing **L**colim and the other to constructing **R**lim. We leave this as an interesting exercise for the reader. The generalization of 10.6(iii) is as follows. For each object d of $\mathbf{D}$, let ∂d denote the full subcategory of $\mathbf{D}\downarrow d$ (3.11) generated by all the objects *except* the identity map of d. There is a functor $j_d : \partial d \to \mathbf{D}$ which sends an object $d' \to d$ of ∂d to the object d' of $\mathbf{D}$. If X is an object of $\mathbf{C}^{\mathbf{D}}$, let $X|_{\partial d}$ denote the composite of X with j_d and let $\partial_d(X)$ denote the object of $\mathbf{C}$ given by $\operatorname{colim}(X|_{\partial d})$. There is a natural map $\partial_d(X) \to X(d)$. If $f : X \to Y$ is a map of $\mathbf{C}^{\mathbf{D}}$, define $\partial_d(f)$ by the pushout diagram

$$\begin{array}{ccc} \partial_d(X) & \longrightarrow & X(d) \\ \downarrow & & \downarrow \\ \partial_d(Y) & \longrightarrow & \partial_d(f) \end{array}$$

and observe that there is a natural map $i_d(f) : \partial_d(f) \to Y(d)$. Then the generalization of 10.6(iii) is the condition that the map $i_d(f)$ be a cofibration for every object d of $\mathbf{D}$.

Suppose that $\mathbf{D}$ is an arbitrary small category. It seems unlikely that $\mathbf{C}^{\mathbf{D}}$ has a natural model category structure for a general model category $\mathbf{C}$. However, $\mathbf{C}^{\mathbf{D}}$ does have a model category structure if $\mathbf{C}$ is the category of simplicial sets (11.1) [5, XI, §8]. The arguments of §8 can be used to construct a parallel model category structure on $\mathbf{Top}^{\mathbf{D}}$. In these special cases the homotopy limit and colimit functors have been studied by Bousfield and Kan [5]; they deal explicitly only with the case of simplicial sets, but the topological case is very similar.

11. Applications of model categories

In this section, which is less self-contained than the rest of the paper, we will give a sampling of the ways in which model categories have been used in topology and algebra. For an exposition of the theory of model categories from an alternate point of view see [16]; for a slightly different approach to axiomatic homotopy theory see, for example, [1].

11.1. *Simplicial sets.* Let $\boldsymbol{\Delta}$ be the category whose objects are the ordered sets $[\mathbf{n}] = \{0, 1, \ldots, n\}$ ($n \geqslant 0$) and whose morphisms are the order-preserving maps between these sets. (Here "order-preserving" means that $f(i) \leqslant f(j)$ whenever $i \leqslant j$). The category *s***Set** of *simplicial sets* is defined to be the category of functors $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathbf{Set}$; the morphisms, as usual (2.5), are natural transformations. Recall from 2.4 that a functor $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ is the same as a contravariant functor $\boldsymbol{\Delta} \to \mathbf{Set}$. For an equivalent but much more explicit description of what a simplicial set is see [18, p. 1]. If X is a simplicial set it is customary to denote the set $X([\mathbf{n}])$ by X_n and call it the *set of n-simplices* of X.

A simplicial set is a combinatorial object which is similar to an abstract simplicial complex with singularities. In an abstract simplicial complex [21, p. 15], [25, p. 108], for instance, an n-simplex has $(n+1)$ distinct vertices and is determined by these vertices; in a simplicial set X, an n-simplex $x \in X_n$ does have $n+1$ "vertices" in X_0 (obtained from x and the $(n+1)$ maps $[\mathbf{n}] \to [\mathbf{0}]$ in $\boldsymbol{\Delta}^{\mathrm{op}}$) but these vertices are not necessarily distinct and they in no way determine x. Let Δ_n denote the standard topological n-simplex, considered as the space of formal convex linear combinations of the points in the set $[\mathbf{n}]$. If Y is a topological space, it is possible to construct an associated simplicial set $\mathrm{Sing}(Y)$ by letting the set of n-simplices $\mathrm{Sing}(Y)_n$ be the set of all continuous maps $\Delta_n \to Y$; this is a set-theoretic precursor of the singular chain complex of Y. The functor $\mathrm{Sing} : \mathbf{Top} \to s\mathbf{Set}$ has a left adjoint, which sends a simplicial set X to a space $|X|$ called the *geometric realization* of X [18, Ch. III]; this construction is a generalization of the geometric realization construction for simplicial complexes. Call a map $f : X \to Y$ of simplicial sets

(i) a *weak equivalence* if $|f|$ is a weak homotopy equivalence (8.1) of topological spaces,
(ii) a *cofibration* if each map $f_n : X_n \to Y_n$ $(n \geqslant 0)$ is a monomorphism, and
(iii) a *fibration* if f has the RLP with respect to acyclic cofibrations (equivalently, f is a *Kan fibration* [18, §7]).

Quillen [22] proves that with these definitions the category $s\mathbf{Set}$ is a model category. He also shows that the adjoint functors

$$|?| : s\mathbf{Set} \Longleftrightarrow \mathbf{Top} : \mathrm{Sing}$$

satisfy both conditions of Theorem 9.7 and so induce an equivalence of categories $\mathrm{Ho}(s\mathbf{Set}) \to \mathrm{Ho}(\mathbf{Top})$ (this is of course with respect to the model category structure on $\mathbf{Top}$ from §8). This shows that the category of simplicial sets is a good category of algebraic or combinatorial "models" for the study of ordinary homotopy theory.

11.2. *Simplicial objects.* There is an obvious way to extend the notion of simplicial set: if $\mathbf{C}$ is a category, the category $s\mathbf{C}$ of *simplicial objects in* $\mathbf{C}$ is defined to be the category of functors $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathbf{C}$ (with natural transformations as the morphisms). The usual convention, if $\mathbf{C}$ is the category of groups, for instance, is to call an object of $s\mathbf{C}$ a "simplicial group". The category $\mathbf{C}$ is embedded in $s\mathbf{C}$ by the "constant diagram" functor (2.11) and in dealing with simplicial objects it is common to identify $\mathbf{C}$ with its image under this embedding. Suppose that $\mathbf{C}$ has an "underlying set" or forgetful functor $U : \mathbf{C} \to \mathbf{Set}$ (cf. 2.9). Call a map $f : X \to Y$ in $s\mathbf{C}$

(i) a *weak equivalence* if $U(f)$ is a weak equivalence in $s\mathbf{Set}$,
(ii) a *fibration* if $U(f)$ is a fibration in $s\mathbf{Set}$, and
(iii) a *cofibration* if f has the LLP with respect to acyclic fibrations.

In [22, Part II, §4] Quillen shows that in all common algebraic situations (e.g., if $\mathbf{C}$ is the category of groups, abelian groups, associative algebras, Lie algebras, commutative algebras, ...) these choices give $s\mathbf{C}$ the structure of a model category; he also characterizes the cofibrations [22, Part II, p. 4.11].

Consider now the example $\mathbf{C} = \mathrm{Mod}_R$. It turns out that there is a normalization functor $N : s\mathrm{Mod}_R \to \mathbf{Ch}_R$ [18, §22] which is an equivalence of categories and translates the model category structure on $s\mathrm{Mod}_R$ above into the model category structure on $\mathbf{Ch}_R$ from §7. Thus the homotopy theory of $s\mathrm{Mod}_R$ is ordinary homological algebra over R.

For a general category $\mathbf{C}$ there is no such normalization functor, and so it is natural to think of an object of $s\mathbf{C}$ as a substitute for a chain complex in $\mathbf{C}$, and consider the homotopy theory of $s\mathbf{C}$ as homological algebra, or better homotopical algebra, over $\mathbf{C}$. This leads to the conclusion (11.1) that homotopical algebra over the category of sets is ordinary homotopy theory!

11.3. *Simplicial commutative rings.* Let $\mathbf{C}$ be the category of commutative rings. In [24] Quillen uses the model category structure on $s\mathbf{C}$ which was described above in order to construct a cohomology theory for commutative rings (now called André–Quillen cohomology). This has been studied extensively by Miller [19] and Goerss [13] because of the fact that if X is a space the André–Quillen cohomology of $\mathrm{H}^*(X;\mathcal{F}_p)$ plays a role in various unstable Adams spectral sequences associated to X. In this way the homotopical algebra of the commutative ring $\mathrm{H}^*(X;\mathcal{F}_p)$ leads back to information about the homotopy theory of X itself; this is parallel to the way in which, if Y is a spectrum, the homological algebra of $\mathrm{H}^*(Y;\mathcal{F}_p)$ as a module over the Steenrod algebra leads to information about the homotopy theory of Y.

We can now answer a question from the introduction. Suppose that k is a field. Let $\mathbf{C}$ be the category of commutative augmented k-algebras and let R be an object of $\mathbf{C}$. Recall that $\mathbf{C}$ can be identified with a subcategory of $s\mathbf{C}$ by the constant diagram construction. Topological intuition suggests that the *suspension* ΣR of R should be the homotopy pushout (§10) of the diagram $* \leftarrow R \rightarrow *$, where $*$ is a terminal object in $s\mathbf{C}$. Since this terminal object is k itself, ΣR should be the homotopy pushout in $s\mathbf{C}$ of $k \leftarrow R \rightarrow k$. It is not hard to compute this; up to homotopy ΣR is given by the bar construction [19, Section 5] [13, p. 51] and the i'th homotopy group of the underlying simplicial set of ΣR is $\mathrm{Tor}_i^R(k,k)$.

11.4. *Rational homotopy theory.* A simplicial set X is said to be 2-reduced if X_i has only a single point for $i < 2$. Call a map $f : X \to Y$ between 2-reduced simplicial sets

(i) a *weak equivalence* if $\mathrm{H}_*(|f|;\mathbf{Q})$ is an isomorphism,
(ii) a *cofibration* if each map $f_k : X_k \to Y_k$ is a monomorphism, and
(iii) a *fibration* if f has the RLP with respect to acyclic cofibrations.

In [23], Quillen shows that these choices give a model category structure on the category $s\mathbf{Set}_2$ of 2-reduced simplicial sets. A differential graded Lie algebra X over $\mathbf{Q}$ is said to be 1-reduced if $X_0 = 0$. Call a map $f : X \to Y$ between 1-reduced differential graded Lie algebras over $\mathbf{Q}$

(i) a *weak equivalence* if $\mathrm{H}_*(f)$ is an isomorphism,
(ii) a *fibration* if $f_k : X_k \to Y_k$ is surjective for each $k > 1$, and
(iii) a *cofibration* if f has the LLP with respect to acyclic fibrations.

These choices give a model category structure on the category $\mathbf{DGL}_1$ of 1-reduced differential graded Lie algebras over $\mathbf{Q}$. By repeated applications of Theorem 9.7, Quillen shows [23] that the homotopy categories $\mathrm{Ho}(s\mathbf{Set}_2)$ and $\mathrm{Ho}(\mathbf{DGL}_1)$ are equivalent. It is not hard to relate the category $s\mathbf{Set}_2$ to the category $\mathbf{Top}_1$ of 1-connected topological spaces (there is a slight difficulty in that $\mathbf{Top}_1$ is not closed under colimits or limits and so cannot be given a model category structure). What results is a specific way in which objects of $\mathbf{DGL}_1$ can be used to model the rational homotopy types of 1-connected spaces. For a dual approach based on differential graded algebras see [4] and for an attempt to eliminate some denominators [7]. There is a large amount of literature in this area.

11.5. *Homology localization.* Let h_* be a homology theory on the category of spaces which is represented in the usual way by a spectrum. Call a map $f : X \to Y$ in $s\mathbf{Set}$

(i) a *weak h_*-equivalence* if $h_*(|f|)$ is an isomorphism,
(ii) an *h_*-cofibration* if f is a cofibration with respect to the conventional model category structure (11.1) on $s\mathbf{Set}$, and
(iii) an *h_*-fibration* if f has the RLP with respect to each map which is both a weak h_*-equivalence and an h_*-cofibration.

Bousfield shows [2, Appendix] that these choices give a model category structure on $s\mathbf{Set}$, called, say the h_*-structure. The hardest part of the proof is verifying **MC5**(ii). Bousfield does this by an interesting generalization of the small object argument (7.12). He first shows that there is a *single* map $i : A \to B$ which is both a weak h_*-equivalence and a h_*-cofibration, such that f is a h_*-fibration if and only if f has the RLP with respect to i. (Actually he finds a set $\{i_\alpha\}$ of such test maps, but there is nothing lost in replacing this set by the single map $\coprod_\alpha i_\alpha$.) Now the domain A of i is potentially quite large, and so A is not necessarily sequentially small. However, if η is the cardinality of the set $\coprod_n A_n$ of simplices of A, the functor $\mathrm{Hom}_{s\mathbf{Set}}(A, -)$ does commute with colimits indexed by transfinite ordinals of cofinality greater than η. Bousfield then proves **MC5**(ii) by using the general idea in the proof of 7.17 but applying the gluing construction $G(\{i\}, -)$ transfinitely; this involves applying the gluing construction itself at each successor ordinal, and taking a colimit of what has come before at each limit ordinal.

Let $\mathbf{Ho}$ denote the conventional homotopy category of simplicial sets (11.1). Say that a simplicial set X is h_*-local if any weak h_*-equivalence $f : A \to B$ induces a bijection $\mathrm{Hom}_{\mathbf{Ho}}(B, X) \to \mathrm{Hom}_{\mathbf{Ho}}(A, X)$. It is not hard to show that a simplicial set which is fibrant with respect to the h_*-structure above is also h_*-local. It follows that using **MC5**(ii) (for the h_*-structure) to factor a map $X \to *$ as a composite $X \xrightarrow{\sim} X' \twoheadrightarrow *$ gives an *h_*-localization* construction on $s\mathbf{Set}$, i.e. gives for any simplicial set X a weak h_*-equivalence $X \to X'$ from X to an h_*-local simplicial set X'. Since the factorization can be done explicitly with a (not so) small object argument, we obtain an h_*-localization functor on $s\mathbf{Set}$. It is easy to pass from this to an analogous h_*-localization functor on $\mathbf{Top}$. These functors extract from a simplicial set or space exactly the fraction of its homotopy type which is visible to the homology theory h_*.

11.6. *Feedback.* We conclude by describing a way to apply the theory of model categories to itself (see [8] and [9]). The intuition behind this application is the idea that almost any simple algebraic construction should have a (total) derived functor (§9), even, for instance, the localization construction (§6) which sends a pair $(\mathbf{C}, W)$ to the localized category $W^{-1}\mathbf{C}$. In fact it *is* possible to construct a total left derived functor of $(\mathbf{C}, W) \mapsto W^{-1}\mathbf{C}$, although this involves using Proposition 9.3 in a "meta" model category in which the objects themselves are categories enriched over simplicial sets [17, p. 181]! If **C** is a model category with weak equivalences W, let $L(\mathbf{C}, W)$ denote the result of applying this derived functor to the pair $(\mathbf{C}, W)$. The object $L(\mathbf{C}, W)$ is a category enriched over simplicial sets (or, with the help of the geometric realization functor, a category enriched over topological spaces) with the same collection of objects as **C**. For any pair of objects $X, Y \in \mathrm{Ob}(\mathbf{C})$ there is a natural bijection

$$\pi_0 \mathrm{Hom}_{L(\mathbf{C},W)}(X,Y) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X,Y)$$

which exhibits the set $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X,Y)$ as just the lowest order invariant of an entire simplicial set or space of maps from X to Y which is created by the localization process. The homotopy types of these "function spaces" $\mathrm{Hom}_{L(\mathbf{C},W)}(X,Y)$ can be computed by looking at appropriate simplicial resolutions of objects of **C** [9, §4]; these function spaces seem to capture most if not all of the higher order structure associated to **C** which was envisaged and partially investigated by Quillen [22, part I, p. 0.4], [22, part I, §2, §3].

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CHAPTER 3

Proper Homotopy Theory

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Contents

HANDBOOK OF ALGEBRAIC TOPOLOGY
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1. Introduction

Proper homotopy theory is both an old and a fairly new area of algebraic topology. Its origins go back to the classification of noncompact surfaces by Kerékjártó in 1923, but it is probably fair to say that it 'got off the ground' as a distinct area of algebraic topology as a result of the geometric work of Larry Siebenmann in 1965. There some of the distinctive features of the subject began to emerge: the study of invariants of ends, use of proper, not merely continuous, maps, etc. The problem Siebenmann tackled was the following:

If M is a smooth manifold with boundary, ∂M, then $M\backslash\partial M$ is an open manifold. Suppose instead that we are given an open manifold, N, is it possible to find a compact manifold M with $M\backslash\partial M \cong N$? If not, why not? Find some obstruction whose vanishing will be necessary and sufficient for such an M to exist.

In his thesis, [66], Siebenmann showed that necessary conditions include that the manifold have a finite number of ends, that the system of fundamental groups of connected open neighborhoods of each 'end' be 'essentially constant' and that there exist arbitrarily small open 'neighborhoods of ∞' homotopically dominated by finite complexes. When the manifold has dimension greater than five and has a single such end, there is an obstruction to the manifold having a boundary. It lies in $\tilde{K}_0(\pi_1(\infty))$, the projective class group of the fundamental group at ∞.

Similar ideas had been applied to this 'missing boundary problem' slightly earlier (cf. Brin and Thickstun [13] for a discussion) but the hypotheses used had not involved invariants of proper homotopy type. Concepts related to proper homotopy and proper homotopy invariants were promoted in Siebenmann's further work ([67], [68], [69]), by Farrell and Wagoner ([37] and [38]) and then by E.M. Brown [14]. The importance of proper homotopy equivalences became evident about the same time with results on non-compact manifolds where the hypotheses were proper homotopy theoretic (particularly 'at infinity'), but having homeomorphism type conclusions. Brown and Tucker [16] cite several papers which are examples of this; their paper is another.

Proper homotopy theory as such was relatively slow to catch on, but in the period after 1972, Chapman published work on infinite dimensional manifolds that was to give it a boost, [20], [21], [22]. He proved the topological invariance of Whitehead torsion in simple homotopy theory using methods involving the shape theory of Borsuk. He then proved a 'complement theorem' which can be interpreted as a sort of duality theory between shape and a weak form of proper homotopy. Borsuk's theory of shape is a 'homotopy theory' whose relation to ordinary homotopy theory is that of Čech homology to singular homology. It is essentially a Čech homotopy theory. The Chapman complement theorem gave a homotopy theoretic equivalence reminiscent of Alexander or Lefschetz duality. Any compact metric space, X, can be embedded in the Hilbert cube

$$Q = \prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n} \right].$$

The subset

$$s = \prod_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right)$$

of Q is called its pseudo interior. Suppose that X and Y are embedded as compact subsets of s, then they have the same shape (in the sense of Borsuk) if and only if $Q \setminus X$ and $Q \setminus Y$ have the same weak proper homotopy type and hence if and only if $Q \setminus X$ and $Q \setminus Y$ are *homeomorphic*.

The methods and perspectives of shape theory interacted with those of proper homotopy theory for the mutual benefit of both. This led, in 1976, to the publication by Edwards and Hastings of their lecture notes [28], which laid down a theoretical framework for studying proper homotopy theory, that is still actively used today. Their proof that the procategory, **proTop** supported a Quillen model category structure raised the prospect of applying many methods from 'classical' homotopy to the proper case via an embedding theorem. The repercussions of that work are continuing, as research tries to adapt methods from the modern theory of algebraic homotopy (cf. Baues [7]), and results on algebraic models for n-types, to attempt to push through proper analogues of J.H.C. Whitehead's original program for a 'combinatorial homotopy theory', and to understand the geometry behind the structures revealed by the procategorical approach.

As, later, we go more slowly through this material, what should be noted is the way that geometric structure interacts strongly with quite complex algebraic structure, even at a quite elementary level. The route we will take is not intended to be historical nor exhaustive and many valuable geometric aspects of the earlier development of the subject will be omitted, partially through lack of space, but mainly due to lack of sufficient expertise in those areas on the part of the author.

2. Initiation

(The following section is loosely based on parts of a series of lectures given by Larry Siebenmann at Logroño in November, 1991. Any errors are, of course, my own responsibility.)

The basic hypothesis will be that X is a connected locally compact Hausdorff space, which is also locally connected. Later on we will usually consider σ-compact spaces, X, so that there will be an increasing sequence, $\{K_n\}$, of compact subspaces with each K_n in the interior of K_{n+1} and such that

$$X = \bigcup_{n=0}^{\infty} K_n.$$

These spaces will often be locally finite simplicial complexes, in which case the K_n can be taken to be subcomplexes.

To illustrate the idea of the ends of a space, X, we note that naively $\mathbb{R}$ has two ends (i.e. two ways of going to 'infinity'), whilst $\mathbb{R}^2$ has really only one as $\mathbb{R}^2 = S^2 \setminus \{\infty\}$ via stereographic projection.

More exactly, consider the system of spaces

$$\varepsilon(X) = \{cl(X \setminus K)\colon\ K \text{ compact} \subset X\}$$

where cl denotes closure. By 'system', we mean 'inverse system', since if K, L are compact subsets of X, and $K \subset L$, then there is an induced inclusion

$$cl(X \setminus L) \to cl(X \setminus K),$$

and so the various spaces in the system are linked by maps.

Applying the connected component functor, π_0 to this system of spaces gives $\pi_0\varepsilon(X) = \{\pi_0 cl(X \setminus K)\colon\ K \text{ compact} \subset X\}$ and then taking the inverse limit of $\pi_0\varepsilon(X)$,

$$e(X) = \lim\ \pi_0\varepsilon(X),$$

we get the *set of ends* of X. In general, $e(X)$ should be given the inverse limit topology, if the information that set contains is to be useful. Before we look at this point in detail, note the following result:

LEMMA 2.1. *If X is as above, K is compact in X and U is an open set containing K, then only finitely many components of $X \setminus K$ are not contained in U.*

The proof is reasonably simple. The result implies that if

$$\pi'\varepsilon(X) = \{c'(X \setminus K)\colon\ K \text{ compact} \subset X\}$$

and $c'(X \setminus K)$ = set of unbounded components of $cl(X \setminus K)$ then $\lim \pi'\varepsilon(X) = \lim \pi_0\varepsilon(X)$. However each $c'(X \setminus K)$ is a finite set and hence is compact in the discrete topology. The inverse limit, $e(X)$, can thus be formed as a closed subset of the product, $\prod c'(X \setminus K)$, and so $e(X)$ is compact and totally disconnected in this inverse limit topology. With this topology, we will say $e(X)$ is the *space of (Freudenthal) ends* of X.

EXAMPLES

1) Let X_8 be the figure eight space, the one-point union of two circles, and let X be its universal cover. This is an 'infinite Hawthorn bush'. It has infinitely many ends, in fact

$$e(X) \cong 2^{\aleph_0}.$$

2) Let M be a compact manifold with boundary ∂M and let $X = M \setminus \partial M$, then $e(X) = \pi_0(\partial M)$.

To investigate whether or not this construction e gives a (useful) invariant, first note that $e(\mathbb{R}) = \{-\infty, \infty\}$, whilst e of any compact space X is empty, so e cannot be a functor on the category of spaces and *continuous* maps, as the contracting map $\mathbb{R} \to \{0\}$ is clearly continuous but there is no function from a two-point set to the empty set! The problem is that although continuous maps $f : X \to Y$ preserve compactness in as much as, if $C \subset X$ is compact, then so is $f(C)$, continuity is 'really' about inverse images and inverse images of compact sets need not be compact, as the above simple example shows.

If, however, we restrict to maps $f : X \to Y$, such that if K is compact in Y, then $f^{-1}(K)$ is compact in X, these are the *proper* maps, then not only is our trivial example excluded, but f induces a morphism of inverse systems

$$\varepsilon(f) : \varepsilon(X) \to \varepsilon(Y)$$

and hence a continuous map of the endspaces,

$$e(f) : e(X) \to e(Y).$$

As we have not yet made precise what is the exact meaning of inverse system, nor of a morphism of such things, we cannot be more precise on this just yet. Accepting that for the moment, we see that we have a functor e from some category $\mathbf{P}$ of spaces and proper maps, to the category, **Profin**, of 'profinite' spaces, i.e. compact totally disconnected spaces.

LEMMA 2.2. *For any space X, the natural inclusions of X into $X \times I$, $e_i(x) = (x, i)$, $i = 0, 1$, and the projection map from $X \times I$ to X are proper maps.*

We thus can make the obvious definition.

DEFINITION 2.3. If $f, g : X \to Y$ are proper maps, then a proper homotopy between them is a proper map

$$H : X \times I \to Y$$

such that $He_0 = f$, $He_1 = g$.

All the usual results of elementary homotopy theory go across to the proper case without difficulty once one notes that the composite of proper maps is proper, that homeomorphisms are proper, and one or two other similar observations. *Proper homotopy equivalences* are defined in the obvious way.

This is a convenient place to set up the 'Proper Category' and various associated categories, before returning to the functor, e.

DEFINITION 2.4 (cf. Edwards and Hastings [28, p. 214]). Let $\mathbf{P}$ be the category of locally compact Hausdorff spaces and proper maps and $\mathbf{Ho(P)}$ be the associated proper homotopy category. Restricting to σ-compact spaces will give corresponding categories $\mathbf{P}_\sigma$ and $\mathbf{Ho(P}_\sigma\mathbf{)}$.

Although a proper map $f : X \to Y$ induces a continuous map $e(f) : e(X) \to e(Y)$, it is clear that one does not need f to be defined on the whole of X for this to be so. We only need f to be defined on a 'neighborhood of ∞' and this leads to the definition of a germ at ∞ of a proper map.

Suppose X is a locally Hausdorff space and $A \subset X$. The inclusion $j : A \to X$ will be said to be *cofinal* if the closure of the complement of A, $cl(X \setminus A)$, is compact. (We also say A is cofinal in X.) In this case the inclusion j is clearly proper and induces an isomorphism between $\varepsilon(A)$ and $\varepsilon(X)$, since eventually these two inverse systems are the same.

Let Σ be the class of all cofinal inclusions in $\mathbf{P}$ and let $\mathbf{P}_\infty = \mathbf{P}(\Sigma^{-1})$, the quotient category obtained by formally inverting the cofinal inclusions. This will be called the *proper category at* ∞. As $(\mathbf{P}, \Sigma)$ admits a calculus of right fractions (in the sense of Gabriel and Zisman), any morphism from X to Y in $\mathbf{P}_\infty$ can be represented by a diagram

$$X \xleftarrow{j} A \xrightarrow{f} Y$$

with j a cofinal inclusion, i.e. f is defined on some 'neighborhood of the end of X'. A morphism in $\mathbf{P}_\infty$ is called a *germ at* ∞ *of a proper map*. Two diagrams

$$X \xleftarrow{j'} A' \xrightarrow{f'} Y \quad \text{and} \quad X \xleftarrow{j''} A'' \xrightarrow{f''} Y$$

represent the same germ if $f' \mid A = f'' \mid A$ for some cofinal subspace $A \supset A' \cup A''$ of X. Composition is defined in the obvious way. Passing to proper homotopy classes gives us $\mathbf{Ho}(\mathbf{P}_\infty)$, the corresponding homotopy category. There are also variants $\mathbf{P}_{\sigma,\infty}$ and $\mathbf{Ho}(\mathbf{P}_{\sigma,\infty})$, obtained by restricting to σ-compact spaces.

We can now state more neatly the functoriality of e by saying that it is a functor from $\mathbf{P}_\infty$ to **Profin**. In fact, if $f, g : X \to Y$ are proper homotopic, then since $e(X) \cong e(X \times I)$, we have $e(f) = e(g)$ up to isomorphism. This is true whether f, g are proper maps or merely germs of such, so we have e is a functor from $\mathbf{Ho}(\mathbf{P})$, or $\mathbf{Ho}(\mathbf{P}_\infty)$, to **Profin**.

Given X, one can attempt to sew in $e(X)$ as if it were a boundary to get a compact space $\widehat{X}$, but even if X is a manifold, $\widehat{X}$ may not be one as it may have singularities and the study of these singularities reduces to the study of the relationship between X and $e(X)$, i.e. to the study of the various $cl(X \setminus K)$ with K compact.

REMARK. A tantalizing question is raised by the fact that $e(X)$ is a profinite space. By Stone duality, this must be the maximal ideal space of a Boolean ring. Goldman in the late 1960's in his Yale thesis, looked at a ring, R, of 'almost continuous maps' from X to $\mathbb{Z}/2\mathbb{Z}$ and showed that $Max(R)$ and $e(X)$ were homeomorphic. This raises a lot of interesting questions, especially given the greater understanding of Stone duality and 'Stone spaces' that there is today.

As $e(X) = \lim \pi_0(\varepsilon(X))$ and so uses π_0, the next obvious 'invariant' to try should use π_1. For this, we need a base point in each $X \setminus K$. For simplicity, we will assume that

X has only one end (so X is 'connected at ∞'). We will also suppose X is σ-compact and will specify an increasing sequence

$$K_1 \subset K_2 \subset \cdots \quad \text{with} \bigcup K_i = X$$

and with each $K_i \subset \operatorname{Int} K_{i+1}$. We set $U_i = X \setminus K_i$, $U_0 = X$ to get $U_0 \supset U_1 \supset \cdots$.

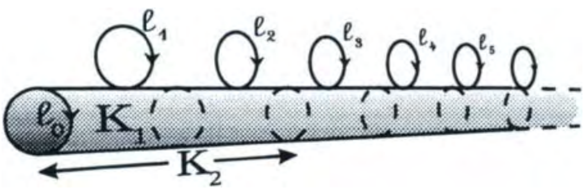
Pick a base point x_i in each U_i. Then we can define $\pi_1(U_i, x_i)$, but we still cannot define an inverse limit group since we do not yet have an inverse sequence of groups. To get that, join x_i to x_{i+1} in U_i by some arc, α_i. Then we get an induced homomorphism

$$\alpha_{i\#} : \pi_1(U_{i+1}, x_{i+1}) \to \pi_1(U_i, x_i)$$

and hence an inverse sequence, S, of groups, $(\pi_1(U_i, x_i), \alpha_{i\#})$. We note that the α_i combine to give a proper map $\alpha : [0, \infty) \to X$, a *base ray* rather than a base point.

This S is a well defined invariant of X up to (i) passage to a subsequence, and (ii) conjugacy, but one still has problems as regards to change of 'base ray' within the end, as the following example shows.

EXAMPLE. Let X be an infinite cylinder with an infinite string of circles attached via a proper ray $\alpha : [0, \infty) \to X$.



Let l_0 be a path once around the circle $S^1 \times \{0\}$ and let l_i be the i^{th} loop of the string of circles. Let α' be the base ray that goes along α_{i-1} then round l_i, then along α_i and so on – it goes along the top of the cylinder in the picture looping once around each l_i as it gets to it. The single Freudenthal end of X is determined by the points $x_k = \alpha(k)$, $k = 1, 2, \ldots$. By analogy with π_0 of a space, different choices of base ray 'should' make no difference to the answer – if $e(X)$ is truly the proper analogue of $\pi_0(X)$. However we can calculate the limits of the inverse systems, $S = (\pi_1(U_i, x_i), \alpha_{i\#})$, and the corresponding $S' = (\pi_1(U_i, x_i), \alpha'_{i\#})$. In both cases $\pi_1(U_i, x_i)$ is the free group $F(l_0, l_i, l_{i+1}, \ldots)$ on the set of loops l_0, plus all l_k, for $k \geqslant i$, but whilst $\alpha_{i\#}$ corresponds to the inclusion of

$$l_0, l_{i+1}, \ldots \to l_0, l_i, l_{i+1}, \ldots,$$

the other 'bonding' homomorphism conjugates by l_i so $\alpha'_{i,\#}(l_k) = l_i l_k l_i^{-1}$ for $k \geqslant i+1$, and for $k = 0$. Comparing the limits of S and S', we find:

– $\lim S \cong F(l_0)$ as only the l_0-loop survives to infinity;

– $\lim S' \cong$ the trivial group, since the only feasible nontrivial element would have been that corresponding to l_0, but that cannot be there as its projections to the various terms of the sequence would need to be conjugated an infinite number of times.

This means that $\lim S$ is not an invariant of the end. There are however cases where S does not depend on the choice of arcs making up the base ray. This occurs if the inverse system of groups, S, is *Mittag-Leffler*. The definition of this condition is well known but we will include it for completeness.

DEFINITION 2.5. An inverse sequence of groups $\underline{G} = (G_n, p_n^m)$ satisfies the Mittag-Leffler condition provided that for any n, there is an $n' > n$ such that for any $n'' > n$, we have $p_n^{n''}(G_{n''}) = p_n^{n'}(G_{n'})$.

In other words, in a Mittag-Leffler inverse system, the images of terms from far down the sequence do not get smaller. It is well known and easy to prove that if $\underline{G}$ is a Mittag-Leffler inverse sequence of groups then it is *essentially epimorphic*, i.e. it is isomorphic to an inverse system of groups which has epimorphic structure maps $p_n^m : G_m \to G_n$. There are similar 'internal' conditions equivalent to the system being 'essentially monomorphic' or 'essentially constant'. The essentially constant systems are isomorphic to constant systems, that is ones in which every object is the same and all the bonding morphisms are the identity on that one object. These systems are also called 'stable'. This is discussed fully in [26].

As a final comment on Freudenthal ends, we mention the Waldhausen boundary. If X and Y are locally compact Hausdorff spaces, one cannot form a space of proper maps from X to Y in any meaningful way, but one can form a simplicial set, $\underline{\mathbf{P}}(X,Y)$, with $\underline{\mathbf{P}}(X,Y)_n = \mathbf{P}(X \times \Delta^n, Y)$, which acts as if it were the singular complex of the mythical space of proper maps from X to Y. The Waldhausen boundary of X is the simplicial set $\underline{\mathbf{P}}([0,\infty), X)$. There is an epimorphism from $\pi_0(\underline{\mathbf{P}}([0,\infty), X))$ to $e(X)$. Thus in our example of the cylinder with the string of circles attached, $\pi_0(\underline{\mathbf{P}}([0,\infty), X))$ is uncountable and $\pi_1(\underline{\mathbf{P}}([0,\infty), X))$ maps onto $\lim S$. When X has a single end and $\pi_0(\varepsilon(X))$ is Mittag-Leffler, then $\pi_0(\underline{\mathbf{P}}([0,\infty), X))$ is a single point, i.e. all possible base rays are properly homotopic.

As the sequence of fundamental groups π_1, based at a sequence of points, behaves less well than one might expect, it is usual to specify a base ray $\alpha : [0,\infty) \to X$ that will link the sequence $\{\alpha(n) : n \in \mathbb{N}\}$. Proper homotopy classes of these correspond to components of the Waldhausen boundary and the limiting fundamental group construction is a proper invariant of the base-rayed space, (X, α). Clearly there are analogues of this construction for other classical base pointed invariants, so we can define the limit homotopy groups at infinity by $\lim \pi_n(\varepsilon(X), \varepsilon(\alpha))$.

Just as with Čech homology, one cannot expect limit groups to give exact sequences and the other construction we have seen, using the Waldhausen boundary, looks more promising. It gives a definition of some homotopy groups of a base-rayed space (X, α) by the simple method of taking $\pi_n(\underline{\mathbf{P}}([0,\infty), X))$. We will denote these groups by $\underline{\pi}_n(X, \alpha)$. Clearly there should be a variant based on germs and that will be denoted $\underline{\underline{\pi}}_n^{\infty}(X, \alpha)$ with $\underline{\mathbf{P}}_{\infty}([0,\infty), X)$ as the corresponding simplicial set. These constructions clearly raise some interesting homotopy theoretic questions, but as yet our theoretical framework is not properly in place and so these must wait until later.

3. Strings of spheres

Before we set-up a 'framework' for a theoretical analysis of the constructions we have glanced at above, we will look at a construction of Ed M. Brown, given in [14]. When we examined the set of components of the Waldhausen boundary, it was clear that each of its vertices, α, determined an end, i.e. determined an element of the Freudenthal end space, $e(X)$. However the end was really determined by much less information, namely the sequence $\{\alpha(n) : n \in \mathrm{N}\}$ of images of the natural numbers. A 1-simplex in $\underline{\mathbf{P}}([0,\infty), X)$ is a proper map from $[0,\infty) \times \Delta^1$ to X and so two vertices in the same component of the Waldhausen boundary determine the same end (as we have seen before). However the existence of such a 1-simplex is a much stronger condition than is necessary since if α, β are two such vertices, it is sufficient to have arcs $a(n) : \alpha(n) \to \beta(n)$ for each $n \in \mathrm{N}$ to ensure that $\{\alpha(n) : n \in \mathrm{N}\}$ and $\{\beta(n) : n \in \mathrm{N}\}$ determine the same Freudenthal end. This suggests that one might use an infinite ladder

$$\underline{I} = \big([0,\infty) \times \partial I\big) \cup (\mathrm{N} \times I)$$

then α and β determine the same end if there is a proper map $h : \underline{I} \to X$ such that $h \mid [0,\infty) \times \{0\} = \alpha$, whilst $h \mid [0,\infty) \times \{1\} = \beta$. More generally, one could form spaces

$$\underline{\Delta}^n = \big([0,\infty) \times \Delta_0^n\big) \cup \big(\mathrm{N} \times \Delta^n\big)$$

and form the corresponding simplicial set by considering $\mathbf{P}(\underline{\Delta}^n, X)$ as its set of n-simplices. Then the homotopy groups of this simplicial set would be invariants of X. In fact E.M. Brown showed in 1974, [14], that it is easier to define the resulting homotopy groups directly. The interaction between them and the homotopy groups of the Waldhausen boundary give a lot of insight into the phenomena involved in going 'out to infinity' in locally compact spaces.

Let S^n denote, as always, the n-sphere, which will be considered as being pointed at some t_0.

DEFINITION 3.1. Let $\underline{S}^n$ denote a half line together with a distinct n-sphere attached at each integer point, i.e. $\underline{S}^n = [0,\infty) \times \{t_0\} \cup (\mathrm{N} \times S^n)$.

The fixed base ray in a 'rayed' space will from now on usually be denoted $* : [0,\infty) \to X$. A proper germ of pairs $\alpha : (\underline{S}^n, [0,\infty)) \to (X, *)$ will mean a proper germ $\alpha : \underline{S}^n \to X$ so that the germ of $\alpha \mid [0,\infty)$ is $*$. If β is another such, we say that α and β are *germ homotopic rel* $*$ if α and β have representatives which are proper homotopic rel. $*$ in the obvious sense. The set of such germ homotopy classes is denoted $\underline{\pi}_n^\infty(X, *)$. It is clear that

$$\underline{\pi}_n^\infty(X, *) \cong \mathbf{Ho}(\mathbf{P}_\infty)\big((\underline{S}^n, [0,\infty)), (X, *)\big)_{pairs}.$$

With this description, it is clear that $\underline{\pi}_n^\infty$ is functorial on based rayed spaces and that it only depends on the choice of the class of $*$ within $e(X)$, not on the particular ray used.

Brown proved, [14], a Whitehead type theorem for a combination of these groups with the classical compactly based groups.

THEOREM 3.1 ([14]). *Let K, L be finite dimensional connected locally finite simplicial complexes, then a proper map $f : K \to L$ is a proper homotopy equivalence if and only if:*

(i) $e(f) : e(K) \to e(L)$ *is a homeomorphism;*
(ii) *for each n, $\pi_n(f) : \pi_n(K, *(0)) \to \pi_n(L, f * (0))$ is an isomorphism;*
(iii) *for each n, and each base ray $*$, in K, $\underline{\pi}_n^\infty(f) : \underline{\pi}_n^\infty(K, *) \to \underline{\pi}_n^\infty(L, f*)$ is an isomorphism.*

Brown points out that if one removes the condition of finite dimensionality, the result no longer holds, but claims that the situation can be retrieved by using the space $\underline{S}^\infty$, which is $[0, \infty)$ with for each $k \in \mathbb{N}$, a copy of S^k attached at that integer. This gives a group $\underline{\pi}_\infty(X, *)$ and the necessary amendment to 3 of Theorem 2.2 is claimed to be to allow $n = \infty$. There is however a subtle counterexample to this due to Edwards, Geoghegan, and Hastings. (A version of the construction of this counterexample is given in [28, pp. 195–202]. Several authors have slipped up at exactly this point so that when consulting the published literature it is advisable to take care when algebraic invariants of infinite dimensional complexes are mentioned.)

Brown's $\mathcal{P}$-functor
Suppose given $X, * : [0, \infty) \to X$. As before we can, theoretically, calculate the inverse sequence of groups, $\pi_n(\varepsilon(X)) \cong \{\pi_n(U_k, *(k)) : k \in \mathbb{N}\}$. Brown gave, again in [14], a method that constructs $\underline{\pi}_n^\infty(X, *)$ from $\pi_n(\varepsilon(X))$. The method has an easy derivation using categories of inverse sequences, but as we have yet to meet this in detail, we will use Brown's original description and return to the other later on.

Let $\underline{G} = \{G_n, p_n^m\}$ be an inverse sequence (tower) of groups with $G_0 = 1$ for simplicity. Consider all sequences $\{g_{k(n)}\}$ with $g_{k(n)} \in G_{k(n)}$ where $k(n)$ is a sequence of natural numbers such that $k(n) \to \infty$ as $n \to \infty$. Given two such sequences, $\{g_{k(n)}\}$, $\{g'_{l(n)}\}$, we say they are equivalent if there is a third sequence $m(n)$, $m(n) \to \infty$ as $n \to \infty$, $m(n) \leqslant \min(k(n), l(n))$ and $p_{m(n)}^{k(n)} g_{k(n)} = p_{m(n)}^{l(n)} g'_{l(n)}$ for all $n \in \mathbb{N}$. We let $\mathcal{P}(\underline{G})$ be the set of equivalence classes. It has a natural group structure and one obtains the following facts:

(i) Let $X = \bigcup K_n$, $U_n = X \setminus K_n$. Suppose $* : [0, \infty) \to X$ is chosen so that $*[n, \infty) \subset U_n$. Set $G_n = \pi_k(U_n, *(n))$ with $G_n \to G_{n-1}$ induced by the inclusion and change of base point along $*([n-1, n])$. Then

$$\underline{\pi}_n^\infty(X, *) \cong \mathcal{P}(\underline{G}).$$

(ii) For any tower of groups $\underline{G}$ as above, there is an action of the group $\underline{F} = \underline{\pi}_1(\underline{S}^1, [0, \infty))$ on $\mathcal{P}(\underline{G})$.
(iii) (Chipman [23].) Let $\underline{G}$, $\underline{H}$ be towers of finitely generated groups, then $\underline{G}$ is isomorphic to $\underline{H}$ if and only if there is an isomorphism from $\mathcal{P}(\underline{G})$ to $\mathcal{P}(\underline{H})$

commuting with the operators of $\underline{F}$. (Notice that initially no morphism of towers from $\underline{G}$ to $\underline{H}$ is given!)

Grossman's reduced power construction
Grossman [40] gave an alternative construction of $\mathcal{P}(\underline{G})$ and indeed of $\mathcal{P}(\underline{X})$ for any tower of sets, $\underline{X}$.

Given a set X, let $X^{\mathbb{N}}$ be the set of infinite sequences of elements of X. On $X^{\mathbb{N}}$ put the following equivalence relation:

$$\underline{a} \equiv \underline{b} \quad \text{if and only if} \quad \{i \in \mathbb{N} : a_i \neq b_i\} \text{ is finite}$$

(if X is a group, this determines a normal subgroup of $X^{\mathbb{N}}$, namely $N = \{\underline{a} \mid \underline{a} \equiv \underline{1}\}$, where $\underline{1}$ is the constant sequence with value the identity, 1, of the group X, hence $N = \{\underline{a} \mid a_i \neq 1 \text{ for only finitely many } i \in \mathbb{N}\}$. If X is abelian, $N = X^{(\mathbb{N})}$, the direct sum of a countable family of copies of X.) We write $I(X)$ for the set (group, abelian group, etc.) of equivalence classes.

Grossman proves that if $\underline{X} = \{X_s\}$ is a tower of sets (groups, etc) then

$$\mathcal{P}(\underline{X}) \cong \lim I(X_s).$$

Thus if $\underline{X}$ is a constant tower, $\mathcal{P}(\underline{X}) \cong I(X_1)$.

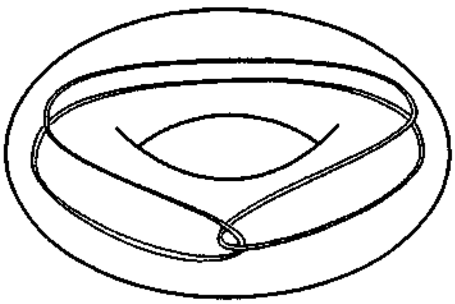
Brown had pointed out, in the original [14], that if M was a compact PL-manifold with boundary ∂M and $K = M \setminus \partial M$, then

$$\underline{\pi}_k^{\infty}(K,*) \cong \mathcal{P}\big(\pi_k(N, x_0)\big) \cong I\big(\pi_k(N,*)\big)$$

where N is the component of ∂M determined by $*$, and x_0 is a base point in N. It is worth noting that a similar reduced power construction is used by Farrell, Taylor, and Wagoner in their definition of their Δ-homotopy groups in [36].

Our last mention of these Brown–Grossman homotopy groups for the moment relates to Brown's later work [15]. As the spaces considered are noncompact 3-manifolds, only the case $n = 1$ is needed. Before stating the results which are quite surprising, it is worth placing these results in context.

J.H.C. Whitehead had given in [73], an example of an open contractible 3-manifold, which was not homeomorphic to $\mathbb{R}^3$. His construction was to take one solid torus inside another so that the first linked itself around the central hole of the second.



Now repeat with the second torus linking inside a third – and so on. Let W be the union of all the solid tori. Then W is contractible but is not homeomorphic to $\mathbb{R}^3$. McMillan [56] generalized Whitehead's construction by putting a knot in the torus before self linking it. By varying the knot at each stage, this gives an uncountable family of contractible open subsets of $\mathbb{R}^3$, no two of which are homeomorphic. In the paper referred to above, Brown proved:

Let M and N be contractible open 3-manifolds which are irreducible and eventually end-irreducible. Let $f : M \to N$ be a proper map which induces an isomorphism $\underline{\pi}_1(f) : \underline{\pi}_1(M) \to \underline{\pi}_1(N)$. Then f is properly homotopic to a homeomorphism.

This to some degree shows the power of these invariants, but fails to say how one can gain any useful algebraic information on the Whitehead–McMillan examples, even though they are explicitly given.

4. Categories of proobjects, towers, shape and strong shape

In this section we will briefly look at some technicalities needed for defining more rigorously, some of the ideas we have already met. We also will introduce shape and strong shape. Although the treatment will be, of necessity, very cursory, it will be useful to refer to this 'dual' theory for comparison.

Procategories

Although in practice in proper homotopy theory, one only needs to use towers of objects, the generalities of procategory theory indicate how proper homotopy theory relates to areas such as the étale homotopy theory of Artin and Mazur and the strong shape of nonmetric compact spaces.

Let $\mathbb{I}$ be a small category (so the class of objects in $\mathbb{I}$ is a set). We say that $\mathbb{I}$ is *filtering* if

(i) given any $i, i' \in \mathbb{I}$ there is an i'' and morphisms

$$\begin{tikzcd} & i \\ i'' \arrow[ur] \arrow[dr] & \\ & i' \end{tikzcd}$$

(ii) given any two maps

$$\begin{tikzcd} i' \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & i, \end{tikzcd}$$

there is a morphism $\gamma : i'' \to i'$ such that $\alpha\gamma = \beta\gamma : i'' \to i$.

A proobject in a category $\mathbf{C}$ is a functor $F : \mathbb{I} \to \mathbf{C}$ for some small filtering category, $\mathbb{I}$. It will sometimes be convenient to write $(\mathbb{I}, F)$ for the proobject $F : \mathbb{I} \to \mathbf{C}$. Later on we shall restrict to *towers*, that is proobjects in which $\mathbb{I} = \mathbb{N}$, the filtering category of natural numbers with a single morphism from n to m if and only if $n \geqslant m$.

If $(\mathbb{I}, F)$ and $(\mathbb{J}, G)$ are proobjects in $\mathbf{C}$, the set of morphisms of proobjects between them is defined to be

$$\mathbf{pro}(\mathbf{C})\big((\mathbb{I}, F), (\mathbb{J}, G)\big) = \lim_{\mathbb{J}} \operatorname*{colim}_{\mathbb{I}} \mathbf{C}\big(F(i), G(j)\big).$$

(For a discussion of where the definition (in this form) comes from, see [26, Section 2.3].)

This definition of morphism can be 'domesticated' as follows:
an element of $\mathbf{pro}(\mathbf{C})((\mathbb{I}, F), (\mathbb{J}, G))$ consists of a function $\theta : \mathbb{J} \to \mathbb{I}$ (not necessarily order preserving) and a $\mathbb{J}$-indexed family of morphisms in $\mathbf{C}$,

$$\big\{f_j : F\theta(j) \to G(j)\big\}_{j \in \mathbb{J}}$$

such that if

$$j' \xrightarrow{\alpha} j$$

is a morphism in $\mathbb{J}$, there is some i and morphisms in $\mathbb{I}$,

$$\begin{tikzcd} & \theta(j) \\ i \arrow[ur, "\beta"] \arrow[dr, "\beta'"'] & \\ & \theta(j') \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd} & F\theta(j) \arrow[r, "f_j"] & G(j) \\ F(i) \arrow[ur, "F(\beta)"] \arrow[dr, "F(\beta')"'] & & \\ & F\theta(j') \arrow[r, "f_{j'}"] & G(j') \arrow[uu, "G(\alpha)"'] \end{tikzcd}$$

is commutative. We write $\mathbf{proC}$ or $\mathbf{pro}(\mathbf{C})$ for the category of proobjects in $\mathbf{C}$ and morphisms between them.

Examples of morphisms

(i) Any natural transformation $\eta : F \to G$ of functors from $\mathbb{I}$ to $\mathbf{C}$ defines a promorphism from $(\mathbb{I}, F)$ to $(\mathbb{I}, G)$ given by:

$$\theta = identity, \; f_i : F(i) \to G(i) \text{ is } \eta(i).$$

We call such a promorphism a *level map*.

(ii) If $\mathbb{I}, \mathbf{J}$ are small filtering categories and $\phi : \mathbf{J} \to \mathbb{I}$ is a functor, we say that ϕ is *cofinal* if:

given any i in $\mathbb{I}$, there is a j in $\mathbb{J}$ and a map $\phi(j) \to i$ in $\mathbb{I}$.

The functor ϕ induces a morphism from $(\mathbb{I}, F)$ to $(\mathbf{J}, F\phi)$ for any proobject $(\mathbb{I}, F)$ indexed by $\mathbb{I}$. If ϕ is cofinal, this morphism is an isomorphism, which will be called a *cofinality isomorphism*.

(iii) The reindexing lemma (below, cf. Artin and Mazur [1, Appendix]) shows that these two types of morphism generate all the promorphisms.

REINDEXING LEMMA. *Given and $f : (\mathbb{I}, F) \to (\mathbb{J}, G)$ in* $\mathbf{pro}(\mathbf{C})$, *there is a filtering category $M(f)$ with cofinal functors $\phi_{\mathbb{I}} : M(f) \to \mathbb{I}, \phi_{\mathbb{J}} : M(f) \to \mathbb{J}$ and a natural transformation*

$$\eta : F\phi_{\mathbb{I}} \to G\phi_{\mathbb{J}}$$

such that the diagram

$$\begin{tikzcd} F \arrow[r, "f"] & G \\ F\phi_{\mathbb{I}} \arrow[u, "\cong"] \arrow[r, "\eta"] & G\phi_{\mathbb{J}} \arrow[u, "\cong"'] \end{tikzcd}$$

is commutative within $\mathbf{pro}(\mathbf{C})$, *where the vertical arrows are cofinality isomorphisms.*

This result thus says that any promap can be reindexed to give a level map and so one can think of $\mathbf{pro}(\mathbf{C})$ as being made up of copies of the functor categories, $\mathbf{C}^{\mathbb{I}}$, glued together by cofinality isomorphisms.

Towers

A tower in $\mathbf{C}$ is a proobject of the form $(\mathbb{N}, F)$, where $\mathbb{N}$ is the category of natural numbers with a single morphism $n \to m$ if and only if $n \geqslant m$. The full subcategory of $\mathbf{pro}(\mathbf{C})$ determined by the towers will be denoted $\mathbf{tow}(\mathbf{C})$.

Key example
Let X be a locally compact Hausdorff space, then, as before

$$\varepsilon(X) = \left\{cl(X \setminus A) \mid A \text{ compact in } X\right\}$$

is a proobject in **Top**. If $f : X \to Y$ is proper, then the 'recipe' for a promap requires that any corresponding $\varepsilon(f) : \varepsilon(X) \to \varepsilon(Y)$ consist of a function

$$\theta : \text{Compacta in } Y \to \text{Compacta in } X$$

and for each compacta B in Y, a continuous map

$$f_B : cl\big(X \setminus \theta(B)\big) \to cl(Y \setminus B).$$

The obvious way to do this is to take $\theta(B) = f^{-1}(B)$ and f_B to be f restricted to $cl(X \setminus f^{-1}(B))$. This also works if f is merely a germ of a proper map.

Conversely, if $(\theta, f_B) : \varepsilon(X) \to \varepsilon(Y)$ is any promap then, as the structural maps in $\varepsilon(X)$ and $\varepsilon(Y)$ are inclusions, all the f_B are restrictions of $f_\varnothing$ and θ must be given by $\theta(B) = f_\varnothing^{-1}(B)$ up to equivalence. We have almost proved (cf. Edwards and Hastings [28]) that $\varepsilon : \mathbf{P}_\infty \to \mathbf{pro(Top)}$ is a full embedding. If we restrict to $\mathbf{P}_{\sigma,\infty}$, we get an embedding $\mathbf{P}_{\sigma,\infty} \to \mathbf{tow(Top)}$, by a choice of ascending filtration in each space. A different choice gives an equivalent embedding. If we look at those $f : X \to Y$ which are *globally defined* then we need to specify that $f^{-1}(\varnothing) = \varnothing$, i.e. f is everywhere defined. The natural way to handle this is then to consider the category $(\mathbf{pro(Top)}, \mathbf{Top})$ with objects the promorphisms $(\mathbb{I}, F) \to (\mathbf{1}, X)$, with codomain a constant object, then to a locally compact Hausdorff space, X, one can assign the promap

$$\varepsilon(X) \to X$$

that has $\theta(1) = \varnothing \in$ Compacta in X and has $f_1 =$ identity. However not only does a proper map $f : X \to Y$ induce a morphism of promaps

$$\begin{tikzcd} \varepsilon(X) \arrow[r] \arrow[d, "\varepsilon(f)"] & X \arrow[d, "f"'] \\ \varepsilon(Y) \arrow[r] & Y \end{tikzcd},$$

but conversely any morphism in $(\mathbf{pro(Top)}, \mathbf{Top})((\varepsilon\mathbf{X}, \mathbf{X}), (\varepsilon\mathbf{Y}, \mathbf{Y}))$ comes from a globally defined proper map and so we get embeddings

$$\mathbf{P} \longrightarrow (\mathbf{pro(Top)}, \mathbf{Top}),$$

$$\mathbf{P}_\sigma \longrightarrow (\mathbf{tow(Top)}, \mathbf{Top}).$$

Any geometrically minded reader may quite reasonably protest that it would seem that the nice 'small' geometric objects of **P** have thus been put in a very abstract categorical setting and ask what is the advantage in such a process. The answer is simply that many constructions in homotopy theory require certain limits and colimits of spaces to exist. Now **P** and $\mathbf{P}_\sigma$ have very few useful limits and colimits, especially the latter, as colimits tend to 'glue' or 'crush' spaces and this destroys end information, but **pro(Top)** has all filtered limits by construction (see Cordier and Porter [26], for instance). In fact, it is a sort of free completion of **Top** with respect to such constructions. Because of this, it is often easier to mimic 'ordinary homotopy theoretic' constructions in **pro(Top)** or (**pro(Top)**,**Top**), than in $\mathbf{P}_\infty$ or **P**. The constructions may start with a space or spaces, but may lead to an object which 'formally' plays the role of a space, although the construction of a space with those properties may be difficult or even impossible. This would, for instance, be the case when considering mapping cylinder constructions – one can sometimes do it with care in **P** or $\mathbf{P}_\infty$, but it is very easy to do it in (**pro(Top)**,**Top**) or **pro(Top)**.

Conditions on a space X in $\mathbf{P}_\sigma$ can thus be imposed using conditions on the end $\varepsilon(X)$ within **pro(Top)**. Of particular use are the conditions that $\varepsilon(X)$ be 'movable' or 'stable'. These correspond to ensuring that the towers $\pi_i(\varepsilon(X))$ satisfy the Mittag-Leffler condition (and hence are essentially epimorphic) or that $\varepsilon(X)$ is isomorphic in **Ho(proTop)** to a constant proobject. (We refer the reader to [26] for a discussion of the algebraic and categorical aspects of these ideas. We will consider some geometric aspects later.)

Returning to **proC** in more generality, we next turn to the definition of homotopy theories on **proC** or (**proC**,**C**) given one on **C**. We will illustrate this with **C** = **Top**. The reindexing lemma interprets as saying that **proC** can be thought of as being made up of many copies of various $\mathbf{C}^{\mathbb{J}}$ for different filtering categories, $\mathbb{J}$, linked together by cofinality isomorphisms. Suppose in each $\mathbf{Top}^{\mathbb{J}}$, we invert the level weak equivalences, then if $\phi : \mathbb{I} \to \mathbb{J}$ is a functor, one has an obvious induced functor

$$\mathbf{Ho}(\mathbf{Top}^{\mathbb{J}}) \to \mathbf{Ho}(\mathbf{Top}^{\mathbb{I}}),$$

since reindexing along ϕ respects the level weak equivalences. We can thus try to glue these homotopy categories together via cofinality. We do this by inverting those promaps $f : \underline{A} \to \underline{B}$ which are isomorphic (via reindexing) to a level weak equivalence. This gives us the category **Ho(pro(Top))**. Edwards and Hastings [28] showed that if **C** has a reasonably nice Quillen model category structure, then so have **proC** and (**proC**,**C**). In particular, this is true for **C** = **Top**. Their remarkable deep result was that the embeddings mentioned earlier pass to the homotopy category to give

$$\begin{aligned}
&\mathbf{HoP} &&\longrightarrow \mathbf{Ho(pro(Top),Top)},\\
&\mathbf{HoP}_\sigma &&\longrightarrow \mathbf{Ho(tow(Top),Top)},\\
&\mathbf{HoP}_\infty &&\longrightarrow \mathbf{Ho(proTop)},\\
&\mathbf{HoP}_{\sigma,\infty} &&\longrightarrow \mathbf{Ho(towTop)}.
\end{aligned}$$

Thus, for instance, if $f : X \to Y$ is in **P** and f is a weak equivalence and $\varepsilon(f)$ is a weak equivalence at the end, then it is a proper homotopy equivalence. (Compare this with Brown's version of the Whitehead theorem given earlier.)

REMARK. The category **Ho(proTop)** is not the same as the category **proHoTop**. The objects of the second of these are of the form $(\mathbb{I}, F)$ with $F : \mathbb{I} \to$ **HoTop** and so are homotopy commutative diagrams. It is not possible, in general, to replace such a homotopy commutative diagram by a commutative diagram in **Top**. The difficulty comes from the choice of the various homotopies. If these can be chosen to be 'coherent' then the theory of homotopy coherent diagrams (developed by Vogt [71], Cordier [24], Cordier and Porter [25], and in some of the work of Dwyer and Kan, for example, [27]) shows that there is an actual commutative diagram with the same homotopy type. By Vogt's result [71] (see also [25]) the category $\mathbf{Ho}(\mathbf{Top}^{\mathbf{I}})$ interprets as a category of homotopy coherent diagrams. Hence **Ho(proTop)** and **Ho(proTop,Top)** have similar interpretations. This can be extremely useful for arguments in proper homotopy theory.

Homotopy limits. The limit lim : **proAb** $\to$ **Ab** is right adjoint to the inclusion functor Yon : **Ab** $\to$ **proAb**. The analogous limit in **Top** exists, but does not interact well with homotopy. To use a limiting construction in homotopy, we turn to the homotopy limit,

$$holim : \mathbf{Ho}(\mathbf{proTop}) \to \mathbf{HoTop},$$

right adjoint to $Ho(Yon) : \mathbf{HoTop} \to \mathbf{Ho}(\mathbf{proTop})$.

To the geometrically minded topologist, $holim$ can initially seem an abomination, but simple examples of homotopy limits and colimits are well known. In any case the construction of homotopy limits can also be domesticated for use in proper homotopy theory. Suppose X is a space in **P**, using the end functor, we get that $\varepsilon(X)$ in **Ho(proTop)** is an invariant of its proper homotopy type at infinity. Applying $holim$ gives us a space, representing some of that same information. What are the homotopy groups of that space?

First pick a base ray in X, $* : [0, \infty) \to X$. Applying ε gives $\varepsilon[0, \infty) \simeq \{x_0\}$ in **Ho(proTop)**, since each $[k, \infty) = cl([0, \infty) \setminus [0, k])$ is contractible. Thus the effect of having a base ray is the same as that of taking ε to have values in $\mathbf{Ho}(\mathbf{proTop_0})$, the analogous homotopy procategory constructed using pointed spaces as starting point and pointed maps as morphisms. (In this situation we will use the pointed version of the homotopy limit functor.) Now

$$\begin{aligned}\pi_n\big(holim\,\varepsilon(X)\big) &\cong \mathbf{Ho}(\mathbf{Top_0})\big(S^n, holim\,\varepsilon(X)\big)\\ &\cong \mathbf{Ho}(\mathbf{proTop_0})\big(c(S^n), \varepsilon(X)\big)\\ &\cong \mathbf{Ho}(\mathbf{proTop_0})\big(\varepsilon(S^n \times [0,\infty)), \varepsilon(X)\big)\\ &\cong \mathbf{Ho}(\mathbf{P}_{\infty,0})\big(S^n \times [0,\infty), X\big)\\ &\cong \underline{\pi}_n^{\infty}(X, *),\end{aligned}$$

i.e. the homotopy groups of the homotopy limit of the end of X are the same as those of the 'germ' version of the Waldhausen boundary, and correspond to proper homotopy classes of germs of proper maps from an semi-infinite cylinder, $S^n \times [0, \infty)$ to X.

Shape, strong shape and proper homotopy

We started this survey by asking questions relating to $M \setminus \partial M$ for M a manifold. Many examples of noncompact spaces arise by removing compact subsets from a compact space. The geometric significance of the subspace that is removed may be that it is perhaps a set of singular points in a generalized manifold, or a boundary, or an attractor for a dynamical system on the ambient space. In each case, it is not just the ambient space, X, that matters, nor the removed subspace, Y, but the combination of both. If one avoids problems relating to the embeddings by passing to infinite codimension, and one equally well avoids any interaction of the homotopy type of the ambient space by restricting to the case when it is contractible, then one is essentially looking at the shape theory of Y.

Any compact metric space, Y can be embedded in the Hilbert cube, Q, so $Q \setminus Y$ gives one an instance of the above idea. Borsuk's shape theory of Y works with polyhedral neighborhoods of Y within Q, thus attaching to Y a system of polyhedra. This enables one to find a functor from the category of compact spaces to the category **proHo**(**Top**). For the details of this approach the book [55] by Mardešić and Segal is a good source. Another approach uses the homotopy analogue of the Čech construction from homology theory.

Given a compact space X, and an open cover $\mathcal{U}$ of X, we can form a simplicial set $N(\mathcal{U})$ or $N(X;\mathcal{U})$ if more precision is required. This 'nerve' of $\mathcal{U}$ has as a typical n-simplex an ordered subset $\{U_0, \ldots, U_n\}$ of the cover $\mathcal{U}$ satisfying the condition that it has non-empty intersection. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a mapping $\phi : \mathcal{V} \to \mathcal{U}$ such that $\phi(V) \supseteq V$ for each V in $\mathcal{V}$. This ϕ enables one to define a simplicial map from $N(\mathcal{V})$ to $N(\mathcal{U})$; another choice of mapping ϕ' will give a different but homotopic map. Because of this, if we denote by $Cov(X)$ the ordered set of open covers of X, then N gives us a functor from $Cov(X)$ to **Ho**(**Simp**), the homotopy category of simplicial sets. Somewhat similarly, varying the space along a continuous map, the induced map depends, up to homotopy, on the choices made. Thus N gives a functor from **Comp**.**Top**. to **proHo**(**Simp**) and by geometric realization to **proHo**(**Top**). In fact analysis of the homotopies involved shows that they can be chosen to be coherent. This can also be proved using the dual 'Vietoris' construction, $V(X;\mathcal{U})$, in which $V(X;\mathcal{U})_n$ consists of $(n+1)$-tuples of points, $(x_0, \ldots, x_n)$, such that there is some $U \in \mathcal{U}$ with $x_i \in U$ for all i. This does behave well under refinement and under continuous mappings and one gets a functor V : **Comp**.**Top**. $\to$ **Ho**(**proSimp**) and thus to **Ho**(**proTop**.). Dowker proved that $| V(X;\mathcal{U}) |$ and $| N(X;\mathcal{U}) |$ are homotopically equivalent, so the two constructions do give the same prohomotopy type. The shape category has compact spaces as its objects and, if X and Y are compact spaces,

$$\mathbf{Sh}(X,Y) = \mathbf{proHo}(\mathbf{Top})\big(\mid N(X;-) \mid, \mid N(Y;-) \mid \big).$$

Edwards and Hastings [28] define the strong shape category by

$$\mathbf{StSh}(X,Y) = \mathbf{Ho}(\mathbf{proTop})\big(\mid V(X;-) \mid, \mid V(Y;-) \mid \big).$$

It is clear that there should be some connection between Strong Shape (for metric spaces) and proper homotopy theory. An analysis of Chapman's complement theorem, [20], made by Edwards and Hastings, showed that X and Y had the same strong shape if and only if, when embedded in s, the pseudo-interior of Q, $Q\backslash X$ and $Q\backslash Y$ had the same proper homotopy type. Better still, this strong form of the Chapman complement theorem linked the proper homotopy category with the strong shape category of compact metric spaces (see §6.5 of [28]).

REMARK. The links between strong shape and proper homotopy theory are exploited in the homology of groups. For this see Geoghegan [39]. We will look briefly at this connection later in this survey.

5. Applications of the Edwards–Hastings embedding

The proof that **proTop** had a Quillen model category structure together with the embeddings of $\mathbf{Ho}(\mathbf{P}_\infty)$ into $\mathbf{Ho}(\mathbf{proTop})$ and of $\mathbf{Ho}(\mathbf{P})$ into $\mathbf{Ho}(\mathbf{proTop},\mathbf{Top})$ opened up new possibilities. The first use of this theory that we will examine clarifies the connections between the two main types of proper homotopy groups.

As we have seen, given a base rayed space, $(X, * : [0,\infty) \to X)$, one can study the homotopy groups of the Waldhausen boundary, defined by

$$\underline{\underline{\pi}}_n(X,*) = \mathbf{Ho}(\mathbf{P}_\bullet)\big(S^n \times [0,\infty), X\big)$$

(where the $_0$ in $\mathbf{P}_\bullet$ again indicates 'pointed' maps). The study of these groups had been suggested by Waldhausen (personal communication) and independently by Čerin [19] who called them Steenrod homotopy groups. He proved that they correspond to the local homotopy groups of Hu [50] of the Freudenthal compactification of X based at the point at infinity determined by the ray $*$. (The reason for calling them Steenrod homotopy groups came from the use of the term Steenrod homotopy theory by Edwards and Hastings [28, Chapter 8]. They are related in the Shape context to Steenrod homology.)

These groups or rather their shape theoretic analogues had earlier (1973) been studied by Quigley [64]. He had shown that they were linked with other groups analogous to Brown's proper homotopy groups, and with Borsuk's shape groups, in a long exact sequence. This Quigley exact sequence was generalized to $\mathbf{Ho}(\mathbf{proTop})$ in [62] and thus gave, via the Edwards–Hastings embedding, a result in proper homotopy theory linking the two main forms of proper homotopy groups.

The Quigley exact sequence
(We will assume for simplicity that X is a locally finite simplicial complex.)

The description of $\underline{\underline{\pi}}_n(X,*)$ as π_n of a *holim* of a tower has an immediate consequence. The Bousfield–Kan theory of homotopy limits gives a spectral sequence with E_2-term of the form $\lim^p \pi_q(\varepsilon(X))$. As the indexing category of $\varepsilon(X)$ is countable, all the higher derived limits, $\lim^p$, vanish for $p > 1$ and the spectral sequence collapses to the Milnor-type short exact sequence:

$$0 \to \lim{}^1 \pi_{t+1}\big(\varepsilon(X)\big) \to \underline{\underline{\pi}}_t(X) \to \lim\, \pi_t\varepsilon(X) \to 0.$$

As it stands here, this short exact sequence seems to have little obvious geometric content, but it embeds in the Quigley exact sequence, which *is* geometric.

The Steenrod homotopy groups (i.e. the homotopy groups of the Waldhausen boundary) are defined by mapping a semi-infinite cylinder, $S^n \times [0,\infty)$, into X; the Brown–Grossman groups are defined by mapping an infinite string of n-spheres, $\underline{S}^n$, into X. There is an obvious proper inclusion $r_n : \underline{S}^n \to S^n \times [0,\infty)$. (Clearly this 'should' be a proper cofibration in some homotopy structure, but as yet in this article we have not introduced a candidate for such a notion, as the homotopy structure we have introduced lives in **proTop** not in **P**.) Passing to **proTop**, using ε, we introduce $\underline{\Sigma}^n$, the prospace with

$$\underline{\Sigma}^n(k) = \bigvee_{l \geqslant k} S^n,$$

the infinite wedge of n-spheres, labeled by the natural numbers $l \geqslant k$, where the bonding map

$$\underline{\Sigma}^n(k) \to \underline{\Sigma}^n(k-1)$$

is the obvious inclusion.

Clearly $\varepsilon(\underline{S}^n) \simeq \underline{\Sigma}^n$ in $\mathbf{proTop_0}$. (There is an obvious level map contracting $[k,\infty)$.) We have already noted that $\varepsilon(S^n \times [0,\infty)) \simeq c(S^n)$, the constant sequence with 'value' S^n, so the proper map

$$r_n : \underline{S}^n \to S^n \times [0,\infty)$$

induces the 'fold' map

$$\varepsilon(r_n) : \underline{\Sigma}^n \to S^n,$$

that is the identity on each part of the infinite wedge.

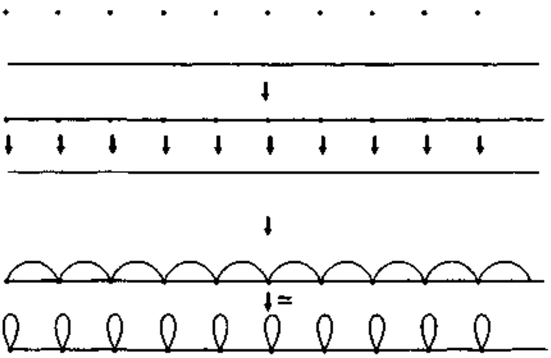
There are other geometrically defined proper maps around in this situation. There is a shift map

$$Shift : \underline{S}^n \to \underline{S}^n$$

that shifts the whole string one place further out towards infinity and there is a quotient map,

$$q_n : S^n \times [0,\infty) \to \underline{S}^{n+1}$$

given by identifying each vertical slice $S^n \times \{k\}$ in $S^n \times [0,\infty)$ to the point $\{k\}$ in $[0,\infty)$ and then sliding back the resulting $(n+1)$-sphere to have contact only at $\{k-1\}$ with $[0,\infty)$. This is easiest to see in dimension 0:



THEOREM 5.1 (Quigley exact sequence in the proper category). *If X is a base rayed σ-compact space, there is a long exact sequence*

$$\cdots \longrightarrow \underline{\underline{\pi}}_n(X) \xrightarrow{r_{n*}} \underline{\pi}_n(X) \xrightarrow{\mathrm{Id\text{-}Shift}} \underline{\pi}_n(X) \xrightarrow{q_{n*}} \underline{\underline{\pi}}_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow \underline{\pi}_1(X) \xrightarrow{\mathrm{Id\text{-}Shift}} \underline{\pi}_1(X) \xrightarrow{q_{1*}} \underline{\underline{\pi}}_0(X) \longrightarrow \underline{\pi}_0(X)$$

It would be feasible to prove this by direct analysis of the sequence, but this would be 'reinventing the wheel'. An efficient proof is to note that $f = \varepsilon(r_0) : \underline{\Sigma}^0 \to S^0$ in $\mathbf{Ho}(\mathbf{proTop}_0)$ leads to a cofibration sequence

$$\underline{\Sigma}^0 \xrightarrow{f} S^0 \longrightarrow C_f \longrightarrow \underline{\Sigma}^1 \longrightarrow S^1 \longrightarrow \Sigma C_f \longrightarrow \cdots$$

in the usual way. It remains only for one to identify $C_f \simeq \underline{\Sigma}^1$, etc., and to analyze the maps (cf. [62] and [63]).

If we split up this sequence into smaller bits, we first note that

$$Ker(Id - Shift) \cong \lim \pi_n \varepsilon(X),$$

whilst

$$Coker(Id - Shift) \cong \lim{}^1 \pi_n \varepsilon(X).$$

This gives one back the Milnor-type short exact sequence we saw earlier but now with a geometrical interpretation on the end terms.

Calculating sets of proper homotopy classes

Suppose that we have two σ-compact spaces, X and Y, together with chosen proper base rays, $*_X$ and $*_Y$ respectively. It is convenient to denote the image of X under the embedding $(\varepsilon, Id) : \mathbf{Ho}(\mathbf{P}_{\sigma,0}) \to \mathbf{Ho}(\mathbf{proTop_0}, \mathbf{Top_0})$ by $\rho(X) : \varepsilon(X) \to X$, thus

$$\mathbf{Ho}(\mathbf{P}_{\sigma,0})(X,Y) \cong \left[\begin{array}{ccc} \varepsilon(X) & & \varepsilon(Y) \\ {\scriptstyle\rho(X)}\big\downarrow & & \big\downarrow{\scriptstyle\rho(Y)} \\ X & & Y \end{array}\right]_0$$

where the zero-suffix indicates the pointed connected version.

Earlier we briefly mentioned the theory of homotopy coherence and that, by Vogt's theorem [71] and the Reindexing lemma, one can interpret elements in this set $[\rho(X), \rho(Y)]_0$ as diagrams

$$\begin{tikzcd} \varepsilon(X) \arrow[r, "f_1"] \arrow[d, "\rho(X)"'] & \varepsilon(Y) \arrow[d, "\rho(Y)"] \\ X \arrow[r, "f_0"'] \arrow[ur, Rightarrow, "F"] & Y \end{tikzcd}$$

where f_1 is a homotopy coherent map between the proobjects $\varepsilon(X)$ and $\varepsilon(Y)$, and F is a homotopy coherent homotopy between the composites $f_0\rho(X)$ and $\rho(Y)f_1$. Given the complex nature of homotopy coherence data, this will in general be impossible to handle, however we can immediately see some examples in which it can be simplified.

(a) Y *contractible*

Here we have

$$\left[\begin{array}{ccc} \varepsilon(X) & & \varepsilon(Y) \\ \big\downarrow & & \big\downarrow \\ X & & Y \end{array}\right]_0 \cong \left[\begin{array}{ccc} \varepsilon(X) & & \varepsilon(Y) \\ \big\downarrow & & \big\downarrow \\ X & & * \end{array}\right]_0,$$

hence our typical element looks like

$$\begin{tikzcd} \varepsilon(X) \arrow[r, "f_1"] \arrow[d] & \varepsilon(Y) \arrow[d] \\ X \arrow[r, "f_0"'] \arrow[ur, Rightarrow, "F"] & * \end{tikzcd}$$

and we get

$$\mathbf{Ho}(\mathbf{P}_{\sigma,0})(X,Y) \cong \big[\varepsilon(X), \varepsilon(Y)\big]_0$$

as we would expect. If either of the ends is stable (i.e. essentially constant and so isomorphic to a constant proobject), then further information can be obtained.

(i) if $\varepsilon(X)$ is stable and stabilizes to K say, then the set of pointed proper homotopy classes is isomorphic to $\mathbf{Ho(Top_0)}(K, holim\,\varepsilon(Y))$.

(ii) if $\varepsilon(Y)$ is stable, and stabilizes to L, then

$$[\varepsilon(X), \varepsilon(Y)]_0 \cong [\varepsilon(X), L]_0 \cong colim[X \setminus X_j, L]_0.$$

(b) *X has stable ends.*

The condition that X has stable ends also allows one to get information out even without assumptions on Y. Suppose $\varepsilon(X) \cong K$ in $\mathbf{Ho(proTop_*)}$. Then

$$\left[\begin{array}{cc} \varepsilon(X) & \varepsilon(Y) \\ \downarrow & \downarrow \\ X & Y \end{array}\right]_0 \cong \left[\begin{array}{cc} K & \varepsilon(Y) \\ \downarrow & \downarrow \\ X & Y \end{array}\right]_0 \cong \left[\begin{array}{cc} K & holim\,\varepsilon(Y) \\ \downarrow & \downarrow{\scriptstyle \bar{\rho}(Y)} \\ X & Y \end{array}\right]_0$$

where $\bar{\rho}(Y) : holim\,\varepsilon(Y) \to Y$ is well defined up to homotopy. Methods from the more classical case can then be used to try to calculate this since an element is a homotopy class of diagrams

$$\begin{tikzcd} K \arrow[r, "f_1"] \arrow[d] & holim\,\varepsilon(Y) \arrow[d, "\bar{\rho}(Y)"] \\ X \arrow[r, "f_0"'] \arrow[ur, Rightarrow, "F"] & Y \end{tikzcd} .$$

As an example, suppose X and Y are open orientable surfaces of genus $g(X)$ and $g(Y)$ respectively and that both have one end. Both spaces have stable ends with homotopy type a circle, and X and Y have homotopy types a bouquet of $2g(X)$ and $2g(Y)$ circles respectively. Hence we can replace the above calculation by that of finding classes of maps and homotopies (f_0, f_1, F) such that the relevant square is homotopy coherent. This is now amenable to an attack by simple obstruction theoretic techniques.

(c) *X contractible*

In this case, one is looking at diagrams

$$\begin{tikzcd} \varepsilon(X) \arrow[r, "f_1"] \arrow[d] & \varepsilon(Y) \arrow[d, "\rho(Y)"] \\ * \arrow[r] \arrow[ur, Rightarrow, "F"] & Y \end{tikzcd} .$$

If we form the homotopy fibre, $F(\rho(Y))$, of $\rho(Y)$ within $\mathbf{Ho}(\mathbf{proTop_0})$, the construction gives a map, $\widehat{F} : \varepsilon(X) \to F(\rho(Y))$ induced by F. This is well defined up to homotopy. Then $\mathbf{Ho}(\mathbf{P}_{\sigma,0})(X, Y) \cong [\varepsilon(X), F(\rho(Y))]_0$. Note that $F(\rho(Y))$ is still a proobject. Its homotopy type is of geometric importance as it contains the information on the interaction between the 'space at ∞', i.e. $\varepsilon(Y)$, and the homotopy type of Y.

Proper pointed maps from $\mathbb{R}^{n+1}$ to a σ-compact space

Let X be a σ-compact space with base ray $* : [0, \infty) \to X$ and use $x = *(0)$ as a base point for X. Consider $\mathbb{R}^{n+1}$ as being given the line $\{(t, 0, \ldots, 0) : t \in [0, \infty)\}$ as base ray. The argument above shows that

$$\mathbf{Ho}(\mathbf{P}_{\sigma,0})\big((\mathbb{R}^{n+1}, *), (X, *)\big) \cong \underline{\pi}_n F\rho(X).$$

Thus $\mathbf{Ho}(\mathbf{P}_{\sigma,0})((\mathbb{R}^{n+1}, *), (X, *))$ is a group and

$$\underline{\pi}_1 F\rho(X) \to \underline{\pi}_1(\varepsilon(X))$$

is a crossed module. The group $\underline{\pi}_1 F\rho(X)$ was studied by Brin and Thickstun, [12], and the general π_n-version by Hernández [41] who denoted it by $\underline{\tau}_n(X, *)$. It is a 'relative Steenrod group' in as much as it is "$\underline{\pi}_n(X, \varepsilon(X))$". This group fits into a long exact sequence

$$\cdots \to \pi_{n+1}(X, x) \to \underline{\tau}_n(X, *) \to \underline{\pi}_n(X, *) \to \pi_n(X, x) \to \cdots$$

linking the classical homotopy groups with the Steenrod–Čerin homotopy groups. How general can these groups be?

Given any $f : A \to B$ with A, B compact CW-complexes, there is a space with $\pi_0(A)$ stable ends such that $\rho(X)$ is isomorphic to f. In fact take the mapping cylinder, M_f, of f and glue on a copy of $A \times [0, \infty)$ to get X. Thus the 'generality' of $\underline{\tau}_n(X, *)$ is at least as great as that of the homotopy groups of homotopy fibres of maps between compact CW-complexes.

Using both the fibration sequence of $\rho(X)$ and the Quigley exact sequence / cofibration sequence, one can get a lattice of exact sequences. For lack of space, the diagrams will be left out here and the reader referred to [46] and [47]. Such diagrams have only limited use for calculation, but they do provide a very convenient framework for presenting the linkage between the homotopy group information 'at the end', with that calculated globally and the corresponding classical homotopy groups.

Whitehead theorems, stability and related problems

One of the advantages of the Edwards–Hastings embedding is that it concentrates attention on the abstract setting of procategories as a means to understanding the geometric problems of $\mathbf{P}$ or $\mathbf{P}_\sigma$. For instance, to prove that a proper map $f : X \to Y$ is a proper homotopy equivalence, one need only prove (i) that it is a homotopy equivalence and (ii) that $\varepsilon(f)$ is a homotopy equivalence in **proTop**. The category **proTop** and its subcategory **towTop** thus have come in for a lot of attention. (One good reason for proving a result in **Ho(towTop)** is that one gets at the same time a result on the proper homotopy of σ-compact spaces and on the strong shape of metric compacta.) In their lecture notes of 1976, Edwards and Hastings [28] give a good treatment of several of these abstract results together with the relevant applications to shape and proper homotopy. Of particular note is the discussion of versions of the Whitehead Theorem and of the Stability Problem. Both problems, in the context of proper homotopy theory, were implicit in Siebenmann's work. As mentioned earlier, various 'Whitehead theorems' for proper homotopy were published in the early 1970's. Within the context of the procategory, **proHo(CW)**, Artin and Mazur [1] had proved a general result and had pointed out the difficulty that if a CW-complex has an infinite Postnikov system, $X^\natural = \{X_n\}$, then within **proHo(CW)**, it is difficult to distinguish the constant system X and the inverse system $\{X_n\}$. The following version of their result includes extra information due to Edwards and Hastings.

THEOREM 5.2 ([1] and [28]). *Let* $\mathbf{CW_0}$ *be the category of pointed connected CW-complexes, then if a map* $f : \underline{X} \to \underline{Y}$ *in* $\mathbf{proHo}(\mathbf{CW_0})$ *induces isomorphisms,*

$$\pi_i(f) : \pi_i(\underline{X}) \to \pi_i(\underline{Y})$$

in **pro(Groups)** *for* $i \geqslant 1$, *then* f *induces an isomorphism* $f^\natural : X^\natural \to Y^\natural$ *of Postnikov systems in* $\mathbf{proHo}(\mathbf{CW_\bullet})$. *If, in addition,* $f : \underline{X} \to \underline{Y}$ *satisfies either of the following conditions:*

(a) *the dimensions of the* X_j *and* Y_k *are bounded above; or*

(b) *each* X_j *and each* Y_k *is finite dimensional and* f *is movable,*

then f *is an isomorphism in* $\mathbf{proHo}(\mathbf{CW_0})$.

Movability is a condition introduced by Borsuk in Shape Theory. It is a condition that guarantees that the progroups $\pi_i(\underline{X})$ and $\pi_i(\underline{Y})$ are Mittag-Leffler in a nice consistent way, see for instance [26].

This result in $\mathbf{proHo}(\mathbf{CW_0})$ cannot be immediately applied to $\mathbf{Ho}(\mathbf{towCW_0})$ and so does not directly give a result in proper homotopy theory.

THEOREM 5.3 ([28, p. 226]). *Let* $f : X \to Y$ *be a proper map of one-ended, connected, countable locally finite simplicial complexes, which is an ordinary homotopy equivalence and induces isomorphisms between* $\pi_i\varepsilon(X)$ *and* $\pi_i\varepsilon(Y)$ *in* **pro(Groups)** *for* $i \geqslant 1$, *then* f *is a proper homotopy equivalence if either of the following additional conditions is satisfied:*

(a) $\dim X < \infty$ *and* $\dim Y < \infty$;

(b) f *is movable.*

Again the reader is referred to [28] for more discussion.

The Proper Stability Problem is to decide when a space X has stable ends, i.e. $\varepsilon(X) \simeq c(K)$ for some space K. (Recall that $c(K)$ is the constant system with value K.) The problem again is really a problem that lives in **Ho(towTop)**. Various people worked on this problem including the author [58] and for a reasonably full bibliography, see any of the books [26], [28] and [55]. Various equivalent solutions have been found. For simplicity only one will be given here although a different context might make other forms more appropriate. The point to note is that, if $\varepsilon(X)$ is to stabilize to anything, it should stabilize to $holim\,\varepsilon(X)$, i.e. to the homotopy type of the Waldhausen boundary (provided the space is 'locally nice'). The attack is thus to look at the natural map from $holim\,\varepsilon(X)$ to $\varepsilon(X)$ and then to use the Whitehead Theorem to find when it is an isomorphism. To apply this Whitehead theorem, you need the 'homotopy dimension' of $holim\,\varepsilon(X)$ to be finite, which explains the domination condition (cf. Wall [72]).

THEOREM 5.4. *Let X be a one ended locally finite simplicial complex and let $holim\,\varepsilon(X) \to \varepsilon(X)$ be the canonical map. If $\pi_i\varepsilon(X)$ is stable / essentially constant for each $i \geqslant 1$ then $h^\natural$ is an isomorphism in* $\mathbf{Ho(towCW_0)}$ *and $\varepsilon(X)$ will be stable if either X is finite dimensional or $\varepsilon(X)$ is dominated in* $\mathbf{proHo(CW_0)}$ *by an object of* $\mathbf{Ho(CW_0)}$.

Proper n-types

Clearly a certain amount of effort has gone into analyzing proper analogues of useful classical results of algebraic topology, for instance the proper Hurewicz theorem exists in several forms, see for instance Extremiana, Hernández and Rivas [32] and Baues [9]. One approach to Proper Homotopy would be to attack the 'key' results of classical algebraic topology in turn, attempting to prove proper variants of each. The snag is that it is not clear if this is always a useful exercise, nor, in any case, which of the myriad results should be attacked. However the programme for algebraicizing homotopy theory put forward by J.H.C. Whitehead in 1950 gives a possible structured plan that helps guide such an approach. One of the key elements in that plan was the use of n-types. The proper homotopy analogue of n-types arose first outside of proper homotopy and completely independently of any abstract approach to classifying proper homotopy types.

Before we look at that, it is necessary to counsel caution when it comes to terminology. The original definition of n-types, for instance in Whitehead's paper [74], corresponds to $(n-1)$-types in today's usage. For instance the important paper, [54] by Mac Lane and Whitehead is on what are there called '3-types', but would now be called '2-types'. We give a definition in the modern terminology.

DEFINITION 5.1. Suppose X, Y are simplicial or CW-complexes, and $f, g : X \to Y$ two maps. We say that f is *n-homotopic* to g (written $f \simeq_n g$) if for every map ϕ of an arbitrary CW-complex, P, of dimension $\leqslant n$ into X, $f\phi$ is homotopic to $g\phi$. Two CW-complexes X and Y are said to be of the same n-type if there are (cellular) maps

$$\phi : X^{n+1} \to Y^{n+1},$$

$$\phi' : Y^{n+1} \to X^{n+1}$$

such that $\phi'\phi \simeq_n 1$, $\phi\phi' \simeq_n 1$.

Since in $\mathbf{CW}$, there is a cellular approximation theorem, the n-type is a homotopy invariant. (In [49], Hernández and the author have collected some of the main classical results on n-types in a consistent notation.)

The analogous definition for proper homotopy theory is almost obvious, but not quite, because the proper analogue of the cellular approximation theorem fails in the category of locally finite CW-complexes and proper maps. We will therefore restrict to the category $\mathbf{SC}_\sigma$ of σ-compact locally compact simplicial complexes and proper maps where no such difficulty arise. Another solution to this problem might involve strongly locally finite CW-complexes. (See Farrell, Taylor and Wagoner [36] for an earlier reference to proper n-equivalences and proper cellular approximation.)

We next look at how this proper notion of n-types arose in combinatorial group theory. Consider an infinite but finitely generated group G. Let X be a $K(G,1)$ with finite 1-skeleton, then the 1-skeleton $\tilde{X}^1$ of its universal cover is locally compact and is, in fact, the Cayley graph associated with a finite set of generators of G. The cardinality of the set of end of $\tilde{X}^1$ is well known to be 1, 2 or infinite. If G is infinite and finitely presented, then $\underline{\underline{\pi}}_0(\tilde{X}^2)$ depends only on G, not on the choice of X. Johnson [51] proves in this situation that the proper 1-type of $\tilde{X}$ is independent of the choice of X. More generally if X^n can be chosen to be finite (technically G is said to be of type $\mathcal{F}(n)$ in this case, see Geoghegan [39]) then the proper $(n-1)$-type of X is independent of the choice of X. If G is of type $\mathcal{F}$, i.e. X is finite, then the proper homotopy type of $\tilde{X}$ itself is independent of the choice of X. (For more on this area see the above mentioned paper by Geoghegan.)

The theory of n-types and their algebraic models was considered by Whitehead to be at the center of his 'algebraic homotopy' approach to algebraic topology. In the category of CW-complexes, one can take a space X and, by adding high dimensional cells, one can kill off the high dimensional homotopy groups and thus embed X in a space with the same n-type, but which is $n+1$-coconnected.

The proper analogue of this is not clear. Which version of the homotopy groups should one use? How does one attach cells to kill elements in, say, $\underline{\pi}_m(X,*)$? There are answers, but they are not obvious. Two approaches have been tried. One by Baues and his students will be briefly looked at slightly later. Here we embed the problem in the procategory and provide an approach using simplicial sets. Both the $\underline{\underline{\pi}}_m(X)$ and the $\underline{\pi}_m(X)$ groups can be calculated from the progroups $\pi_m(\varepsilon(X))$. Within the category of simplicial sets, the analogue of killing off higher order homotopy groups can be done functorially via the coskeleta functors. Combining these two observations one can fairly easily prove an embedding theorem of $\mathbf{Ho_n}((\mathbf{SC}_\sigma)_\infty)$ into $\mathbf{Ho(ProSS)}$ or $\mathbf{Ho_n(proSS)}$, where $\mathbf{Ho_n}$ in each case indicated the category obtained by formally inverting the n-homotopy equivalences [49]. This leads to various results such as a Whitehead theorem for proper n-types, and a proof that a proper n-equivalence $f: X \to Y$ induces isomorphisms on $\underline{\underline{\pi}}_i$ for $i < n$ and on $\underline{\underline{\pi}}_i^\infty$ for $i \leqslant n$. (If the ends of X and Y are movable then f induces an isomorphism on $\underline{\underline{\pi}}_n^\infty$ as well.) This opens the way to finding algebraic 'promodels' for proper n-types. The immediate problems in this area are to explore the geometric interpretation of these algebraic promodels, and to find efficient ways of extracting algebraic *models* from these *promodels*, e.g., by versions of the homotopy limit spectral sequence or Brown's $\mathcal{P}$-functor. The models for 2-types should most probably be crossed modules and here one can hope for progress, but with the many different models for 3-types (crossed squares, quadratic complexes, 2-crossed modules, etc.), the situation is less clear. For some classes of spaces, the problem is simpler as the algebraic n-type models can be chosen to be truncated crossed complexes (cf. [48] and [49]). The resulting theory of the proper analogues of the J_n-complexes of Whitehead is quite useful even though of limited generality.

Extensions, obstructions and classification of proper maps

Given the importance of the various forms of obstruction theory in the continuous setting, it is not surprising that analogues have been sought in the proper category. Both Hernández and his coworkers, and Baues and his team, have worked on aspects of this. Hernández has used the Edwards–Hastings embedding and procategories and so his theory most easily fits here.

In 1985, Hernández studied the obstruction to extensions [43]. For this he needed to develop a new cohomology theory with coefficients in a pro-abelian group for $\mathbf{P}_\infty$ and also (for work globally, i.e. in $\mathbf{P}$) in a morphism from a proabelian group to an abelian group, i.e. an object of (**proAb**,**Ab**). Let S_* denote the usual singular chain complex functor and $proS_*$ its extension to a functor from **proTop** to $\mathbf{ProC_{Ab}}$. This gives a composite:

$$\mathbf{P} \to (\mathbf{proTop}, \mathbf{Top}) \to (\mathbf{proC_{Ab}}, \mathbf{C_{Ab}})$$

then given an object $\pi' \to \pi$ in (**ProAb**,**Ab**), one can define a cochain complex:

$$S^*(X) = (\mathbf{proAb},\mathbf{Ab})\begin{pmatrix} proS_*\varepsilon X & \pi' \\ \downarrow & \downarrow \\ S_*X & \pi \end{pmatrix}.$$

The cohomology, $\mathcal{H}^m(X;\pi' \to \pi) = H^m(S^*(X))$ is the m^{th} cohomology with coefficients in $\pi' \to \pi$. (This cohomology is related to one defined by myself in 1977, [61]. That cohomology had been introduced in order to construct an obstruction theory for strong shape. There is a natural transformation from $\mathcal{H}^m(X;\pi' \to \pi)$ to $H^m(S\varepsilon X, \pi')$ where $S\varepsilon X$ is the singular pro-simplicial set associated to εX.) The development of the obstruction theory follows a traditional path, but with some surprising twists due to the fact that the category of towers of abelian groups has projective dimension 2, see later. This upsets attempts to extend classical results like the Universal Coefficient Theorem using a direct translation of the classical proofs, cf. [42]. The sort of result obtained by Hernández in [43] is the following:

THEOREM 5.5. *Let K be a second countable, locally compact cell complex and L a subcomplex. Writing K^n for the n-skeleton of K, suppose given a proper map $f : K^n \cup L \to Y$ where Y is a pathwise connected space with one Freudenthal end (so $\sharp(e(Y)) = 1$). Suppose the progroup $\pi_1\varepsilon(Y)$ acts trivially on $\pi_n\varepsilon(Y)$ and that Y is n-simple in the classical sense. Then there is an obstruction class*

$$\gamma^{n+1}(f) \in \mathcal{H}^{n+1}(K, L; \pi_n\varepsilon(Y) \to \pi_n Y)$$

such that f can be properly extended to a proper map defined on $K^{n+1} \cup L$ if and only if $\gamma^{n+1}(f) = 0$.

Hernández has applied this theory to calculate the set of proper homotopy classes of proper maps from a non-compact connected surface to $\mathbb{R}^2$, see [43].

The main problem of working with this obstruction theory is the lack of good machinery to calculate the cohomology groups in anything like a general situation. A 'top-down' approach of imposing conditions so as to improve the computability of the cohomology is not that successful, even though conditions such as the spaces having one stable end do allow progress to be made.

Classical obstruction theory is most easily used with CW-complexes and Hernández and his co-workers have suggested an approach which leads to a class of spaces that to some extent take the place of CW-complexes in the proper context and for which this proper obstruction theory simplifies. They call these spaces *proper* CW-complexes, but as this term is also used by Baues [9] for a wider class of spaces (see later) we will use the term PCW-complex instead.

One of the problems with noncompact locally finite CW-complexes is that infinitely many cells need to be used. A simple example of this is $[0, \infty)$ itself with its usual cell decomposition, yet $[0, \infty)$ is the same as $[0, 1)$ and that is almost a CW-complex with only two cells! The Steenrod–Čerin groups, $\underline{\underline{\pi}}_n(X)$ or $\underline{\underline{\pi}}_n^\infty(X)$ depend on $S^n \times [0, \infty)$ and this suggests, by analogy with the classical construction of CW-complexes, that one construct spaces using cones on $S^n \times [0, \infty)$ in some sense. The resulting class of spaces is explored by Extremiana, Hernández and Rivas in [29], [30], and [33]. The spaces that result will not exhaust all proper homotopy types, far from it, but they are combinatorially defined and therefore allow calculations to be made more effectively than for general spaces. The idea is thus to construct spaces by assembling both compact and noncompact cells. More precisely such a PCW-complex consists of a space, X together with a filtration, (X^n). The subspace of vertices, X^0, is discrete, and those cells in X^n, but not in X^{n-1} come in two lists, A_n and B_n. For each $\alpha \in A_n$, one attaches a copy of a closed n-cell, E^n, by some characteristic map in the usual way, but if $\beta \in B_n$, one attaches a copy of $E^{n-1} \times [0, 1)$, i.e. a noncompact cell, by a proper map. These PCW-complexes have very nice properties. For instance, if X is a finite PCW-complex, then its Freudenthal compactification (that is X with $e(X)$ attached) is a finite standard CW-complex. If M is a compact PL-manifold, then it is clear that $M \setminus \partial M$ has the structure of a PCW-complex. If K is a compact simplicial complex and L is any subcomplex, then $K \setminus L$ is a PCW-complex. (In both these examples, a standard decomposition of the space would require an infinite number of cells.) Note, a locally compact CW-complex is always locally finite, but a locally compact PCW-complex need not be.

For regular PCW-complexes, unlike for locally finite CW-complexes, there is a cellular approximation theorem. It is thus feasible to define homology groups for the former spaces using either singular or cellular chains. For both, there are variants using compact and/or noncompact generators, and thus one gets homology groups:

$$H_q(X) \quad \text{– based on compact oriented cells;}$$

$J_q(X)$ – based on all oriented cells;
$E^q(X)$ – based on all noncompact oriented cells.

There are, of course, singular versions of these, applicable to general spaces. There is a long exact sequence,

$$\cdots \to H_{n+1}(X) \to J_{n+1}(X) \to E_{n+1}(X) \to H_n(X) \to \cdots,$$

which corresponds in homology to the homotopy exact sequence (for $* : [0,\infty) \to X$, a base ray):

$$\cdots \to \pi_n(X, *(0)) \to \tau_n(X, *) \to \underline{\underline{\pi}}_n(X, *) \to \pi_{n-1}(X, *(0)) \to \cdots,$$

that we saw earlier (cf. [65]).

The advantage of finite regular PCW-complexes is that the corresponding groups are more easily computed and this can be important in obstruction theory. The restriction to such complexes also allows one to handle some finitely ended spaces. The resulting theory is developed in [30] and [33], and is applied to the classification problem for proper maps $f : X \to Y$, where X is a finite regular PCW-complex and Y is a topological space with a finite number of ends.

Clearly for the obstruction theory described by Hernández, the restriction to PCW-complexes simplifies the description greatly. Firstly, suppose that X is a space. Then to keep track of the ends of X when mapping it to a space Y with finitely many ends, it is convenient to label the ends of X by a labeling function $g : e(X) \to F$ where $F = \{e_1, \dots e_k\}$ is a finite set. Then one can form a chain complex with, in dimension n, the group $C_n(X, g^{-1}(e_i))$ of singular cubical chains on X either mapping in E^n or mapping the end of $E^{n-1} \times [0,1)$ to $g^{-1}(e_i)$. This defines a homology group denoted $J_n(X, g^{-1}(e_i))$. There are similarly defined groups $E_n(X, g^{-1}(e_i))$ and they are linked via a long exact sequence as before. Together with $S_*(X)$, the complex of singular cubical chains in$X$, which is contained in all the $C_*(X, g^{-1}(e_i))$, this gives a diagram

$$\begin{tikzcd}
 & C_n(X, g^{-1}(e_1)) \\
 & C_n(X, g^{-1}(e_2)) \\
S_n(X) \arrow[uur] \arrow[ur] \arrow[ddr] & \\
 & \vdots \\
 & C_n(X, g^{-1}(e_k)),
\end{tikzcd}$$

in which all the objects are abelian groups. Diagrams of this kind form the abelian category, $\mathbf{Ab}^{\pi_k}$, where π_k is a cone on a set of k objects.

$$\pi_k = \begin{tikzcd} & 1 \\ & 2 \\ 0 \arrow[uur] \arrow[ur] \arrow[ddr] & \\ & \vdots \\ & k \end{tikzcd}$$

Diagram categories of this kind are quite well understood, cf. Mitchell [57] and correspond to categories of modules over a particular ring of $(k+1) \times (k+1)$ matrices over $\mathbb{Z}$.

The basic idea of the resulting obstruction theory should now begin to be clear. If (X, A) is a PCW-complex pair, Y is a similarly labeled pointed space, and $g : e(X) \to F \subset e(Y)$ is a labeling, then the obstructions to extending a proper map $f : A \to Y$ live in cohomology groups of (X, A, g) with coefficients in the diagram, $\phi_n(Y)$ over π_k with $\phi_n(Y)(0) = \pi_n(Y, y_0)$, $\phi_n(Y)(j) = \tau_n(Y, \alpha_j)$ where $\alpha_j : [0, \infty) \to Y$ satisfies $\alpha_j(0) = y_0$ and α_j represents the j^{th} element of F. The simplicity of the category π_k means that it is possible to classify proper maps with a certain ease in those situations where the theory applies and in particular for $k = 1$ and 2.

REMARKS. (i) It is worth noting that picking a basepoint in Y and a finite set of ends, effectively determines a tree within Y. This idea is one of the basic building blocks of Baues' work; see later.

(ii) The use of matrix rings to handle proper homotopy is not new, as it already occurs in Farrell and Wagoner's version [37] and [38] of Siebenmann's simple proper homotopy theory [69]. We take up this theme in the next section.

In the last two sections we will briefly look at some very recent developments in proper homotopy theory. At the time of writing, not all the results have been published and some revision is likely before they are. Because of this, the descriptions will be kept fairly brief and general. There are two approaches, and although these are related, the exact translation between them is not as yet completely clear.

6. Monoids of infinite matrices, M-simplicial sets and a proper singular complex

Although the basic level of the work of Hernández and Beattie relates to *monoids* of infinite matrices, the more easily approachable aspect of this work is in the additive case, where the monoids become rings. This is also the oldest aspect as it already appears in the work of Farrell and Wagoner [37] and [38] on simple proper homotopy theory and its link with algebraic K-theory.

Given a σ-compact space X, it is natural to consider not only its tower of homotopy groups (assuming a base ray has been given), but also the homology groups of $\varepsilon(X)$, which will give an object in **towAb**. This category of towers of abelian groups is an abelian category and so it is natural to look for an embedding of this category into a category of modules. (Such an embedding is known to exist by the Freyd–Lubkin–Mitchell embedding theorem.) If **A** is an abelian category, the usual method of embedding it in a category of the form **Mod**-R is to find what is called a faithful projective generator, P, then to set $R = End(P)$, the endomorphism ring of P. The ring structure on $End(P) = \mathbf{A}(P,P)$ comes, in part, from the additive structure of the 'hom-set' $\mathbf{A}(P,P)$ with the multiplication coming from the composition, i.e. the natural *monoid* structure on this set. Thus one looks for a faithful projective generator in **towAb**. The obvious candidate is the object $C(\mathbb{Z})$ defined by

$$C(\mathbb{Z})_i = \mathbb{Z}[e_i, e_{i+1}, \ldots],$$

i.e. the free abelian group on infinitely many generators, $\{e_j : j \geqslant i\}$. The bonding morphisms, $p_i^{i+1} : C(\mathbb{Z})_{i+1} \to C(\mathbb{Z})_i$ are induced by inclusions of the corresponding sets of generators. It is worthwhile noting the similarity of this object to the 'string of spheres' used as a basis for the Brown–Grossman definition of proper homotopy groups. There is a close connection. If one prefers to work with (**towAb**,**Ab**), so as to get a 'global' version, one merely notes down $C(\mathbb{Z})_0$ as well.

Now let $\ell\mathbb{Z}$ denote the ring of locally finite matrices over $\mathbb{Z}$, that is, infinite integer matrices such that each row and each column has only finitely many nonzero entries. The rows and columns are thought of as being indexed by $\mathbb{N}$. This ring of infinite matrices has a two sided ideal, $m\mathbb{Z}$, made up of those matrices with all but finitely many entries zero. Let $\mu\mathbb{Z}$ be the quotient ring $\ell\mathbb{Z}/m\mathbb{Z}$. Both Beattie and Hernández noted that

LEMMA 6.1. (i) $\mathbf{towAb}(C(\mathbb{Z}), C(\mathbb{Z})) \cong \mu\mathbb{Z}$,
(ii) $(\mathbf{towAb}, \mathbf{Ab})(C(\mathbb{Z}), C(\mathbb{Z})) \cong \ell\mathbb{Z}$.

The general abstract theory of embedding an abelian category into a module category next points out that for any tower of abelian groups, A, the abelian group $\mathbf{towAb}(C(\mathbb{Z}), A)$ has a natural $\mu\mathbb{Z}$-module structure coming from composition. The functor $\mathbf{towAb}(C(\mathbb{Z}), -)$ then gives an embedding of **towAb** into **Mod**-$\mu\mathbb{Z}$. A similar result holds of course between (**towAb**,**Ab**) and **Mod**-$\ell\mathbb{Z}$ (see [10] for a detailed elementary treatment of this and much more).

These embeddings are not immediately useful. Much more useful is the fact that the finitely presented objects in **towAb** and (**towAb**, **Ab**) form a category *equivalent* to that of the finitely presented modules over $\mu\mathbb{Z}$ and $\ell\mathbb{Z}$ respectively. The sense of 'finitely presented' is well known for modules: M is a finitely presented right R-modules if there is an exact sequence

$$R^m \xrightarrow{\kappa} R^n \to M \to 0$$

with m, n finite. For towers one merely replaces R by the projective generator, $P = C(\mathbb{Z})$. In both cases here, one easily sees that $R \oplus R \cong R$ and $P \oplus P \cong P$, so finitely presented objects have presentations with m and n both equal to 1. Certain homological properties relevant to the proper homotopy of locally finite simplicial complexes can be explained as being due to the homological algebra of $\ell\mathbb{Z}$ or $\mu\mathbb{Z}$. For a finitely presented *abelian group*, the short exact sequence above can always be chosen to have κ a monomorphism. This is conveniently summarized by saying that $\mathbb{Z}$ has projective dimension one. This is important geometrically, since it means that Moore spaces are unique up to homotopy. Ayala, Domínguez et al. (see [2] and [3], but beware errors in this latter paper; and Beattie [11]) noted that **towAb** and (**towAb**,**Ab**), and thus $\mu\mathbb{Z}$ and $\ell\mathbb{Z}$, have projective dimension 2. This is related to the fact that $\lim^1 \underline{M}$ may be nonzero.

Let $n \geqslant 4$. A finite dimensional locally finite CW-complex is called a proper Moore space (in dimension n) if its 1-skeleton is $[0, \infty)$ (which will be considered as a base ray) and its homology towers all vanish except possibly in dimension n (cf. Beattie [10] and [11] and Ayala et al. [2]). Beattie [10] gives a category of algebraic models for such proper Moore spaces. The classical models were constructed using presentations of the abelian group that was the n^{th}-homology group, and a realization functor using wedges of n-spheres modeling the maps of the presentation gave the equivalence between the algebraic models and the spaces. If $\underline{M}$ is a tower, i.e. is in (**towAb**,**Ab**), and is isomorphic to the n^{th} homology tower of some finite dimensional locally finite CW-complex, then it is finitely presented in the sense mentioned earlier. As $\ell\mathbb{Z}$ has projective dimension 2, it has a resolution,

$$\underline{P}_2 \to \underline{P}_1 \to \underline{P}_0 \to \underline{M}$$

where each $\underline{P}_i$ is isomorphic to $C(\mathbb{Z})$. Such a resolution allows one to construct a Moore space $M(\underline{M}, n)$, however there may be many different proper homotopy types of $M(\underline{M}, n)$ for a given $\underline{M}$. Beattie shows how to adapt the resolution, augmenting it with further data, so that the algebraic category of such gadgets successfully mirrors all the Moore spaces for this N. (Ayala et al. give another version of this result in [2]. Beattie has given a brief summary of his approach in [11] with a full version in his thesis [10].)

Returning to more general considerations, it is important to note that

$$\mathbf{towAb}\big(C(\mathbb{Z}), A\big) \cong \mathcal{P}(A),$$

where $\mathcal{P}$ is Brown's $\mathcal{P}$ functor for abelian groups mentioned earlier. Some of the proof of the equivalence between $(\mathbf{towAb})_{fp}$ and $\mathbf{Mod}_{fp} - \mu\mathbb{Z}$ is thus related to Grossman's results on isomorphisms between towers of (abelian) groups. The fact that both the $\mathcal{P}$-functor and Grossman's results relate to nonadditive situations as well, suggests that a nonadditive version of the above equivalence results should hold, and recent results of Hernández show this to be the case [44] and [45].

Let **C** be any one of the categories:

Sets of sets,

$\mathbf{Sets}_*$ of pointed sets,

Grps of groups,

or

Ab of abelian groups.

Then **C** has a small projective generator, G. These are respectively a singleton, $*$; a 'doubleton', S^0; $\mathbb{Z}$ as a free group on one element; and $\mathbb{Z}$ as a free abelian group on one element. Each of these categories has infinite coproducts and so, as previously in **Ab**, we can form $C(G)$ in **towC** by

$$C(G)_i = \coprod_{j \geqslant i} G.$$

The theory now runs parallel to the additive case. One forms **towC**$(C(G), C(G)) = M$. If $\mathbf{C} = \mathbf{Sets}$, M is a monoid; if $\mathbf{C} = \mathbf{Sets}_*$, it is a 0-monoid; if $\mathbf{C} = \mathbf{Grps}$, it is a near ring, and, of course, for $\mathbf{C} = \mathbf{Ab}$, $M = \mu\mathbb{Z}$, as before. Brown's $\mathcal{P}$-functor is then **towC**$(C(G), -)$, and this has a natural action of M on it. This functor $\mathcal{P}$ is faithful and its restriction to finitely generated towers is also full. (A tower X is *finitely generated* if there is an effective epimorphism $Q \to X$ with Q a finite coproduct of copies of $C(G)$.) In [44], Hernández analyzes the categorical properties of $\mathcal{P}$, in particular constructing left adjoints to $\mathcal{P}$ in all the four cases. In the sequel, [45], Hernández uses the above to relate **towSimp** to a category of simplicial M-sets. This gives rise to proper singular and realization functors.

The proper singular functor is defined by firstly considering the monoid $M = \mathbf{P}(\mathrm{N}, \mathrm{N})$, where N is given the discrete topology. This also gives a left M-set structure to $\mathrm{N} \times \Delta^q$. If X is a space, then its proper singular complex is defined by

$$S_{\mathbf{P}}(X)_q = \mathbf{P}\big(\mathrm{N} \times \Delta^q, X\big),$$

with the obvious face and degeneracy maps. The simplicial set $S_{\mathbf{P}}(X)$ has a natural M-action, so

$$S_{\mathbf{P}} : \mathbf{P} \to \mathbf{Simp}(\mathbf{Sets}_{\mathrm{M}}).$$

Using an equivariant singular functor from $\mathbf{Top}_{\mathrm{M}}$ to $\mathbf{Simp}(\mathbf{Sets}_{\mathrm{M}})$, Hernández shows that $\mathbf{P}(\mathrm{N}, X)$, denoted $X^{\mathrm{N}}_{\mathbf{P}}$, can be interpreted as a subspace of $\widehat{X}^{\widehat{\mathrm{N}}}_{\mathrm{c}}$, where $\widehat{X}$ is the Alexandroff one-point compactification of X, similarly for N and the c denotes that the compact-open topology is given to the set of continuous functions from $\widehat{\mathrm{N}}$ to $\widehat{X}$. The functor $S_{\mathbf{P}}$ is then a composite of the functor $(\ )^{\mathrm{N}}$ and an equivariant singular functor on $\mathbf{Top}_{\mathrm{M}}$.

Suppose one defines a 'new' q-th proper homotopy group of a space X based at a proper map, $\sigma : \mathrm{N} \to X$, to be

$${}^{\mathbf{P}}\pi_q(X, \sigma) = \pi_q \overline{S}_{\mathbf{P}} X$$

where $\overline{S}_{\mathbf{P}}$ is the underlying simplicial set of the simplicial M-set, $S_{\mathbf{P}}$. Then $\underline{\pi}_q(X, \alpha)$ for $\alpha : [0, \infty) \to X$, is isomorphic to $\mathbf{P}\pi_q(X, \alpha \mid \mathrm{N})$. If α and β are two rays, which coincide on N then they determine the same end, and $\underline{\pi}_q(X,\alpha) \cong \underline{\pi}_q(X,\beta)$ as groups. Of course, the action of $\underline{\pi}_q(\underline{S}^q,*)$ on the two groups may differ, leading to the sort of phenomenon that was noted earlier with the infinite cylinder with $\underline{S}^1$ attached. In particular the shift map may not be the same.

Using these proper homotopy groups, Hernández proves a proper Hurewicz theorem relatively simply. This proper singular functor, $\overline{S}_{\mathbf{P}}$, therefore looks to have great potential, but only the beginnings of its development are available at the time of writing. This whole M-set approach of Hernández also suggests links with equivariant homotopy theory. This may introduce a new set of tools for handling proper homotopy problems.

7. Proper algebraic homotopy theory

The other approach to proper homotopy theory currently being developed is based on the Algebraic Homotopy Theory of Baues [7] and [8]. Baues himself has collected up material from earlier sources, together with a wealth of new material, in a draft manuscript [9]. The main points of this approach include:

- the use of the language and results of the theory of cofibration categories;
- the use of strongly locally finite CW-complexes;
- the important role played by trees;
- the use of the theory of algebraic theories as a means of managing the large quantity of algebraic structure in this new setting.

The main source for this theory is Baues' manuscript [9] which is, at the time of writing, about 160 pages long. There is not room here to give an adequate treatment of this, but we will briefly look at the four aspects mentioned above. Certain of these are also handled in the thesis of J. Zobel [75] and in papers by Ayala, Domínguez and Quintero [4] and [5].

We have already mentioned Quillen's axiomatization of homotopy theory in connection with the results of Edwards and Hastings [28]. Quillen's axiom system involved three classes of morphisms the weak equivalences, the fibrations and the cofibrations. K. Brown [17] introduced a weakened form of axiom system involving weak equivalences and fibrations only. This theory, or rather the dual theory involving weak equivalences and cofibrations, formed the theoretical underpinning for the development of the obstruction theory in $\mathbf{proKan}_0$ mentioned earlier; see [59], [60], and [61]. This dual theory was adapted by Baues as the foundation of his algebraic homotopy theory [7] which attempts to complete Whitehead's programme. Baues' definition of a cofibration category is as follows:

A cofibration category is a category $\mathbf{C}$ with an additional structure ($\mathbf{C}$, *cof*, *w.e.*), subject to axioms C1–C4 below. The notation *cof* stands for a class of morphisms called *cofibrations*, and *w.e* for a class of morphisms called *weak equivalences*.

(C1) *Composition Axiom*: The isomorphisms in $\mathbf{C}$ are both weak equivalences and cofibrations. For two maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

if any two of f, g and gf are weak equivalences, then so is the third. The composite of cofibrations is a cofibration.

(C2) *Pushout Axiom*: For a cofibration

$$i : B \rightarrowtail A$$

and map $f : B \to Y$, there is a pushout in $\mathbf{C}$,

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "\overline{i}"] \\ A \arrow[r, "\overline{f}"] & A \cup_B Y \end{tikzcd}$$

and $\overline{i}$ is a cofibration. Moreover

(a) if f is a weak equivalence, so is $\overline{f}$,

(b) if i is a weak equivalence, so is $\overline{i}$.

(C3) *Factorization Axiom*: For a map $f : B \to Y$ in $\mathbf{C}$, there is a factorization $f = gi$ with i a cofibration and g a weak equivalence.

(C4) *Axiom on Fibrant Models*: For each object X in $\mathbf{C}$, there is a trivial cofibration (i.e. a cofibration that is also a weak equivalence) $X \to RX$, where RX is *fibrant* in $\mathbf{C}$. More exactly, RX is such that each trivial cofibration $i : RX \to Q$ in $\mathbf{C}$ admits a retraction $r : Q \to RX$, $ri = Id_{RX}$.

In [7], Baues gives a large number of examples of cofibration categories arising naturally within algebraic topology. He also discusses, in detail, the comparison of this structure with that of Quillen. The classical topological example of a cylinder based homotopy theory generates an 'obvious' cofibration category structure with *cof* being the class of closed cofibrations and *w.e.* the class of homotopy equivalences. This is related to the Quillen structure given by Strøm [70]. Baues introduces a notion of a category with a natural cylinder, based on ideas of Kan [53] which were extensively developed by Kamps in a series of articles, (see the bibliography of the forthcoming book [52]).

An I-category is a category $\mathbf{C}$ with the structure $(\mathbf{C}, cof, I, \emptyset)$, where *cof* is a class of morphisms in $\mathbf{C}$, called cofibrations, I is a functor $\mathbf{C} \to \mathbf{C}$ together with natural transformations i_0, i_1 and p, and $\emptyset$ is the initial object in $\mathbf{C}$. This data is to satisfy 5 axioms (to be found on pp. 18 and 19 of [7]). Such I-category structures induce cofibration categories structures.

An I-category structure for the category of spaces and *perfect* maps, based on the obvious cylinder functor has been given by Ayala, Domíguez and Quintero [5]. In fact as they work with *perfect* rather than *proper* maps, their results only apply to proper maps when the spaces concerned are locally compact Hausdorff. (A map is *perfect* if inverse images of points are compact.) This is probably not a serious restriction in applications, but must be kept in mind. Cabeza, Elvira and Hernández [18] have given a different cofibration category structure for the category $\mathbf{P}$ of spaces and proper maps.

As a consequence of this cofibration category structure, for a fixed space, A, the category, $(\mathbf{P}^A)_c$, also has such a structure. This category has as objects the proper cofibrations, $i : A \to X$ and has the obvious morphisms, namely if $i' : A \to X'$, $f : i \to i'$ is to be a proper map $f : X \to X'$ such that $fi = i'$. The most interesting case is when $A = T$, an infinite tree. Baues [9] studies in detail the properties of such trees following ideas already introduced briefly by Farrell and Wagoner. If X is a space then a proper map $T \to X$ determines a closed subset of $e(X)$. The category $(\mathbf{P}^T)_c$ is a proper analogue of the category of well pointed spaces. From this perspective, the base rayed spaces considered earlier in this survey are a very special case.

The analogues of the Brown–Grossman groups $\underline{\pi}_k(X,*)$ can be defined using the *spherical objects*, S^n_α, under T. Here $\alpha : E \to T^0$ is a finite-to-one function onto the nodes of T and $\{S^n_e : e \in E\}$ is a collection of n-spheres, $S^n_e = S^n$. S^n_α is obtained by gluing each S^n_e to the node $\alpha(e)$ of T. (Thus $\underline{S}^n$, the string of n-spheres used earlier, is a special case of this.) This gives proper homotopy groups $\Pi^\alpha_n(X)$ if X is in $(\mathbf{P}^T)_c$.

The n-dimensional spherical objects under T and proper homotopy classes of maps (under T) between them form a category with finite sums. It forms a theory of cogroups, a particular type of a many sorted algebraic theory [6]. Baues exploits the general ideas of algebraic theories, developing descriptions of the categories of models of such 'theories of cogroups' as models of proper homotopy types. Abelianization and various other natural operations are generalized to this context and Zobel [75] also considers analogues of crossed module structures that occur naturally in this context. These theories encode many important geometrically defined operations and their study is clearly one of the most important areas for future research in this subject.

As this gives a wide range of tree-based algebra and, in particular, a large number of subtly interrelated homotopy groups, based on spherical objects, it is natural that the 'CW-complexes' in this context are built up with proper cones on spherical objects. The resulting T-CW-complexes provide models for all proper homotopy types (under T) of connected strongly locally finite CW-complexes. These T-CW-complexes allow for the generalization of many results from 'classical' CW-complex theory, for instance results using the proper Blakers–Massey theorem, and proper T-analogues of exact sequences due to Whitehead. Invariants such as the higher homotopy groups, that, for classical theory, are abelian groups, here take on the structure of abelian group-valued models of the algebraic theory. They thus correspond to modules over ringoids, i.e. rings with many objects.

The overall structure of both this theory and that of Hernández sketched out earlier, therefore, corresponds to a change of perspective. In the procategory theory approach, homology towers of abelian groups are thought of as sequences of invariants; here they are modules over a 'ringoid'. This more global perspective would seem to have important consequences for proper homotopy theory. It may 'feed back' into other areas of algebraic topology as well. As yet it has led to no surprising results or calculations, and mostly has produced new elegant versions of old theorems, but it is clearly a conceptual clarification and simplification of the foundations of the subject. Much work is still being done in this area, both by Hernández and his team and by Baues, together with others in Germany, Spain and the UK.

Further reading and acknowledgements
Much of this survey resulted from discussions with Baues, Beattie, Hernández, Siebenmann and Zobel at the Workshop on Proper Homotopy Theory, Colegio Universitaria de La Rioja (November 1991). I have extensively used the survey written for the workshop by Extremiana, Hernádez and Rivas [34] but have deliberately chosen to emphasize other aspects. That source contains an extensive and valuable bibliography and is to be recommended. The authors were also the organizers of the Workshop and edited the proceedings [35] which also contains articles by Beattie and myself on this area.

I would like to thank them and also Hans Baues, Joe Zobel and Larry Siebenmann for their help. I should also thank the organizations who provided financial assistance for that meeting.

Finally I should point out once again that this article does not claim to be inclusive and the work of several other authors could have justifiably been described. In particular Mihalik's work on applications to combinatorial and homological group theory, Čerin's ideas on conditions related to movability of ends, and the whole area of simple proper homotopy theory have been omitted. To some extent, these areas are described in [34], and the relevant papers are listed in the bibliography of that article.

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CHAPTER 4

Introduction to Fibrewise Homotopy Theory

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HANDBOOK OF ALGEBRAIC TOPOLOGY
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As a subject in its own right fibrewise homotopy theory is quite a recent development. It dates back to around 1970 when several people independently realized its value, in particular Becker [1], [2], Dold [12], [13], McClendon [23], Smith [29] and myself [15]. Since then research activity in the theory has been fairly continuous so that a comprehensive account would be far too long for the present volume. What follows, therefore, is simply an introduction.

Ideally any exposition of fibrewise homotopy theory should be based on a fibrewise version of topology, as in [18], but for our limited purposes ordinary topology will be sufficient. Elsewhere in this volume Dwyer and Spalinski, developing earlier ideas of Quillen, have shown that it is possible to work in a general framework which encompasses equivariant homotopy theory, fibrewise homotopy theory and much else. At our level, however, it seems better to give a self-contained account rather than appeal to this general theory for what are, for the most part, rather elementary results.

1. Fibrewise spaces

Let us work over a base space B. A *fibrewise space* over B consists of a space X together with a map $p : X \to B$, called the *projection*. Usually X alone is sufficient notation. We regard any subspace of X as a fibrewise space over B by restricting the projection. When p is a fibration we describe X as *fibrant*, and this class of fibrewise spaces has special properties.

We regard B as a fibrewise space over itself using the identity as projection. We regard the topological product $T \times B$, for any space T, as a fibrewise space over B using the second projection.

Let X be a fibrewise space over B. For each point b of B the fibre over b is the subset $X_b = p^{-1}b$ of X; fibres may be empty since we do not require p to be surjective. Also for each subspace B' of B we regard $X_{B'} = p^{-1}B'$ as a fibrewise space over B' with projection p' determined by p.

Fibrewise spaces over B are the objects of a category with the following notion of morphism. Let X and Y be fibrewise spaces over B with projections p and q, respectively. A *fibrewise map* of X to Y is a map $\phi : X \to Y$ in the ordinary sense such that $q \circ \phi = p$, in other words such that $\phi X_b \subset Y_b$ for each point b of B. The fibrewise map ϕ is said to be *fibrewise constant* if $\phi = t \circ p$ for some section $t : B \to Y$. Equivalences in the category of fibrewise spaces are called *fibrewise topological equivalences* or *fibrewise homeomorphisms*.

If $\phi : X \to Y$ is a fibrewise map over B then the restriction $\phi_{B'} : X_{B'} \to Y_{B'}$ is a fibrewise map over B' for each subspace B' of B. Thus a functor is defined from the category of fibrewise spaces over B to the category of fibrewise spaces over B'.

Given an indexed family $\{X_j\}$ of fibrewise spaces over B the *fibrewise product* $\prod_B X_j$ is defined, as a fibrewise space over B, and comes equipped with a family of fibrewise projections

$$\pi_j : \prod_B X_j \to X_j.$$

The fibres of the fibrewise product are just the products of the corresponding fibres of the factors. The fibrewise product is characterized by the following cartesian property: for each fibrewise space X over B the fibrewise maps

$$\phi : X \to \prod_B X_j$$

correspond precisely to the families of fibrewise maps $\{\phi_j\}$, where

$$\phi_j = \pi_j \circ \phi : X \to X_j.$$

For example if $X_j = X$ for each index j the *diagonal*

$$\Delta : X \to \prod_B X$$

is defined so that $\pi_j \circ \Delta = id_X$ for each j.

If $\{X_j\}$ is as before the *fibrewise coproduct* $\coprod_B X_j$ is also defined, as a fibrewise space over B, and comes equipped with a family of fibrewise insertions

$$\sigma_j : X_j \to \coprod_B X_j.$$

The fibres of the fibrewise coproduct are just the coproducts of the corresponding fibres of the summands. The fibrewise coproduct is characterized by the following cocartesian property: for each fibrewise space X over B the fibrewise maps

$$\psi : \coprod_B X_j \to X$$

correspond precisely to the family of fibrewise maps $\{\psi_j\}$, where

$$\psi_j = \psi \circ \sigma_j : X_j \to X.$$

For example if $X_j = X$ for each index j the *codiagonal*

$$\nabla : \coprod_B X \to X$$

is defined so that $\nabla \circ \sigma_j = id_X$ for each j.

The notations $X \times_B Y$ and $X +_B Y$ are used for the fibrewise product and fibrewise coproduct in the case of a family $\{X, Y\}$ of two fibrewise spaces, and similarly for finite families generally. When $X = Y$ the switching maps

$$X \times_B X \to X \times_B X, \quad X +_B X \to X +_B X$$

are defined with components (π_2, π_1) and (σ_2, σ_1), respectively.

Given a map $\lambda : B' \to B$, for any space B', we can regard B' as a fibrewise space over B. For each fibrewise space X over B we denote by $\lambda^* X$ the fibrewise product $X \times_B B'$, regarded as a fibrewise space over B' using the second projection, and similarly for fibrewise maps. Thus λ^* constitutes a functor from the category of fibrewise spaces over B to the category of fibrewise spaces over B'. When B' is a subspace of B and λ the inclusion this is equivalent to the restriction functor mentioned earlier.

A fibrewise space X over B is said to be *trivial* if X is fibrewise homeomorphic to $T \times B$ for some space T, and then a fibrewise homeomorphism $\phi : X \to T \times B$ is called a *trivialization* of X. A fibrewise space X over B is said to be *locally trivial* if there exists an open covering of B such that X_V is trivial over V for each member V of the covering. A locally trivial fibrewise space is the simplest form of fibre bundle, or bundle of spaces.

As Dold [12] has shown the theory of fibre bundles is improved if it is confined to the class of numerable bundles, i.e. bundles which are trivial over each member of some numerable covering of the base. Derwent [7] and tom Dieck [9] have pointed out that such a covering may be taken to be countable, thus facilitating inductive arguments.

A more sophisticated form of the notion of fibre bundle involves a topological group G, the structural group. A *principal G-bundle* over the base space B is a locally trivial fibrewise space P over B on which G acts freely. Moreover the action is fibre-preserving, so that each of the fibres is homeomorphic to G. Such a principal G-bundle P over B determines a functor $P_\#$ from the category of G-spaces to the category of fibre bundles over B. Specifically $P_\#$ transforms each G-space A into the associated bundle $P \times_G A$ with fibre A, and similarly with G-maps. We refer to $P_\#$ as the *associated bundle functor*.

The theory of fibre bundles is dealt with in the standard textbooks, such as Steenrod [31], where a large variety of examples are discussed. Some of these will be appearing later in this article.

From our point of view it is only natural to proceed a stage further and develop a fibrewise version of the theory of fibre bundles as in [22]. Thus let X and T be fibrewise spaces over B. By a *fibrewise fibre bundle* over X, with fibrewise fibre T, we mean a fibrewise space E together with a fibrewise map $p : E \to X$ which is *fibrewise locally trivial*, in the sense that there exists a covering of X such that E_V is fibrewise homeomorphic to $V \times_B T$ over B, for each member V of the covering. This is the simplest form of the definition, but of course there is a more sophisticated form, involving a fibrewise structural group. Details will be found in [22].

Various solutions have been given, in the literature, to the problem of constructing a right adjoint to the fibrewise product. Specifically, the problem is to find an appropriate topology for the fibrewise mapping-space

$$\mathrm{map}_B(X, Z) = \coprod_{b \in B} \mathrm{map}(X_b, Z_b),$$

where X and Z are fibrewise spaces over B. Although this can be done in general as in [17], the case when $X = T \times B$ admits simpler treatment. In fact maps of $T \times \{b\}$ into Z_b can be regarded as maps of T into Z, in the obvious way. and so $\mathrm{map}_B(T \times B, Z)$ can be topologized as a subspace of $\mathrm{map}(T, Z)$, with the compact-open topology. It is easy to check that for any fibrewise space Y over B a fibrewise map

$$T \times Y = (T \times B) \times_B Y \to Z$$

determines a fibrewise map

$$Y \to \mathrm{map}_B(T \times B, Z),$$

through the standard formula, and the converse holds when T is compact Hausdorff.

2. Fibrewise homotopy

Fibrewise homotopy is an equivalence relation between fibrewise maps. Specifically consider fibrewise maps $\theta, \phi : X \to Y$, where X and Y are fibrewise spaces over B. A *fibrewise homotopy* of θ into ϕ is a homotopy in the ordinary sense which is a fibrewise map at each stage. If there exists a fibrewise homotopy of θ into ϕ we say that θ is *fibrewise homotopic* to ϕ. In this way an equivalence relation is defined on the set of fibrewise maps from X to Y, and the set of equivalence classes is denoted by $\pi_B(X, Y)$. Formally π_B constitutes a binary functor from the category of fibrewise spaces to the category of sets, contravariant in the first entry and covariant in the second.

Recall from §1 that for each principal G-bundle P over B, where G is a topological group, the associated bundle functor $P_\#$ is defined from the category of G-spaces to the category of fibrewise spaces over B. This not only transforms G-maps into fibrewise maps but also transforms G-homotopies into fibrewise homotopies.

The operation of composition for fibrewise maps induces a function

$$\pi_B(Y, Z) \times \pi_B(X, Y) \to \pi_B(X, Z),$$

for any fibrewise spaces X, Y and Z over B. Postcomposition with a fibrewise map $\psi : Y \to Z$ induces a function

$$\psi_* : \pi_B(X, Y) \to \pi_B(X, Z),$$

while precomposition with a fibrewise map $\phi : X \to Y$ induces a function

$$\phi^* : \pi_B(Y, Z) \to \pi_B(X, Z).$$

The fibrewise map $\phi : X \to Y$ is a *fibrewise homotopy equivalence* if there exists a fibrewise map $\psi : Y \to X$ such that $\psi \circ \phi$ is fibrewise homotopic to id_X and $\phi \circ \psi$ is fibrewise homotopic to id_Y. Thus an equivalence relation between fibrewise spaces is defined; the equivalence classes are the *fibrewise homotopy types*.

A fibrewise homotopy into a fibrewise constant is a *fibrewise nulhomotopy*. A fibrewise space is said to be *fibrewise contractible* if it has the same fibrewise homotopy type as the base space, in other words if the identity is fibrewise nulhomotopic. A subspace of a fibrewise space is said to be *fibrewise categorical* if the inclusion is fibrewise nulhomotopic.

Let $\phi : X \to Y$ and $\psi : Y \to X$ be fibrewise maps such that $\psi \circ \phi$ is fibrewise homotopic to id_X. Then ψ is a left inverse of ϕ, up to fibrewise homotopy, and ϕ is a right inverse of ψ, up to fibrewise homotopy. If ϕ admits both a left inverse ψ and a right inverse ψ', up to fibrewise homotopy, then ψ is fibrewise homotopic to ψ' and so ϕ is a fibrewise homotopy equivalence.

Examples can easily be given of fibrewise maps which are homotopic as ordinary maps but are not fibrewise homotopic. Thus take $X = (I \times \{0,1\}) \cup (\{0\} \times I)$ and $B = I$, with the first projection. Although X is contractible, as an ordinary space, it is not fibrewise contractible since the fibres over points of $(0,1]$ are not contractible.

It can be shown, however, that for a large class of fibrewise spaces X there exists an integer m such that for each fibrewise map $\phi : X \to X$ which is nulhomotopic on each fibre the m-fold composition $\phi \circ \ldots \circ \phi$ is fibrewise nulhomotopic. Details are given in [10] and [24]. Another result which might be mentioned here concerns the group $G(X)$ of fibrewise homotopy classes of fibrewise homotopy equivalences of X with itself. Under similar conditions it is shown, in [17] and [24], that $G_1(X)$ is nilpotent of class less than m, where $G_1(X)$ denotes the normal subgroup of $G(X)$ consisting of fibrewise homotopy equivalences which are homotopic to the identity on each fibre.

3. Fibrewise pointed spaces

A *fibrewise pointed space* over B consists of a space X together with maps

$$B \xrightarrow{s} X \xrightarrow{p} B$$

such that $p \circ s = id_B$. In other words X is a fibrewise space over B with section s. Note that the projection is necessarily a quotient map and the section is necessarily an embedding. To simplify the exposition in what follows let us assume, once and for all, that the embedding is closed, as is always the case when X is a Hausdorff space. We regard any subspace of X containing the section as a fibrewise pointed space in the obvious way; no other subspaces will be admitted.

A fibrewise pointed space reduces to a pointed space or space with basepoint when the base space is just a point. It is not customary to require that subspaces must contain the basepoint but from our point of view this is an essential condition. It is also not customary to require the basepoint to be closed.

Let X be a fibrewise pointed space over B, as above. For each subspace B' of B we regard $X_{B'}$ as a fibrewise pointed space over B' with section $s_{B'}$. In particular we regard the fibre over a point b of B as a pointed space with basepoint $s(b)$.

Fibrewise pointed spaces over B are the objects of a category with the following notion of morphism. Let X and Y be fibrewise pointed spaces over B with sections s and t respectively. A *fibrewise pointed map* of X to Y is a fibrewise map $\phi : X \to Y$ which is section-preserving in the sense that $\phi \circ s = t$, in other words such that $\phi_b : X_b \to Y_b$ is a pointed map for each point b of B. Equivalences in the category of fibrewise pointed spaces are called *fibrewise pointed topological equivalences* or *fibrewise pointed homeomorphisms*.

If $\phi : X \to Y$ is a fibrewise pointed map over B then the restriction $\phi_{B'} : X_{B'} \to Y_{B'}$ is a fibrewise pointed map over B' for each subspace B' of B. Thus a functor is defined from the category of fibrewise pointed spaces over B to the category of fibrewise pointed spaces over B'.

For each fibrewise pointed space X over B the pull-back $\lambda^* X$ is regarded as a fibrewise pointed space over B', in the obvious way, for each space B' and map $\lambda : B' \to B$, and similarly for fibrewise pointed maps. Thus λ^* constitutes a functor from the category of fibrewise pointed spaces over B to the category of fibrewise pointed spaces over B'. When B' is a subspace of B and λ the inclusion this is equivalent to the restriction functor of the previous paragraph.

Let X be a fibrewise space over B and let A be a closed subspace of X. We can define the fibrewise quotient space $X/_B A$ of $X +_B B$ by identifying points of A with their images under the projection. We refer to $X/_B A$ as the *fibrewise collapse* of A in X. If A is closed in X the fibrewise collapse $X/_B A$ becomes a fibrewise pointed space with section given by $B = A/_B A \to X/_B A$.

Let X and Y be fibrewise pointed spaces over B with sections s and t, respectively. We regard the fibrewise product $X \times_B Y$ as a fibrewise pointed space with section given by $b \mapsto (s(b), t(b))$. The subspace

$$X \times_B B \cup B \times_B Y \subset X \times_B Y$$

is denoted by $X \vee_B Y$ and called the *fibrewise pointed coproduct* (or *fibrewise wedge product*) of X and Y, while the fibrewise collapse

$$X \wedge_B Y = (X \times_B Y)/_B(X \vee_B Y)$$

is called the *fibrewise smash product*. Of course these constructions are functorial in nature. The fibrewise smash product distributes over the fibrewise wedge product. Unlike the fibrewise wedge product the fibrewise smash product is not in general associative. However associativity holds for a reasonably large class of fibrewise pointed spaces (see [17]).

In particular we may consider the fibrewise smash product with X of a fibrewise pointed space of the form $T \times B$, where T is pointed space. When $T = I$, the unit interval, with $\{0\}$ as basepoint, this is called the (reduced) *fibrewise cone* on X, denoted by $\Gamma_B^B(X)$. When $T = I/\dot{I}$, the circle, this is called the (reduced) *fibrewise suspension* of X, denoted by $\Sigma_B^B(X)$. Of course these constructions are functorial.

A fibrewise pointed space X over B is said to be *trivial* if X is fibrewise pointed homeomorphic to $T \times B$ for some pointed space T, and then a fibrewise pointed homeomorphism $\phi : X \to T \times B$ is called a *trivialization* of X. A fibrewise pointed space X over B is said to be *locally trivial* if there exists an open covering of B such that X_V is trivial over V for each member V of the covering. A locally trivial fibrewise pointed space is the simplest form of *sectioned fibre bundle* or *bundle of pointed spaces*.

A more sophisticated form involves a structural group G. A principal G-bundle P over B determines a functor $P_\#$ from the category of pointed G-spaces to the category of sectioned fibre bundles over B. Specifically $P_\#$ transforms each pointed G-space A into the associated bundle $P \times_G A$ with fibre A and similarly with pointed G-maps.

If X is a sectioned fibre bundle over B then λ^*X is a sectioned fibre bundle over B' for each space B' and map $\lambda : B' \to B$. The triviality covering in the case of B' is just the pull-back of the triviality covering in the case of B.

We may refer to a sectioned fibre bundle, as above, as a bundle of pointed spaces. The question naturally arises as to whether a bundle of (nonpointed) spaces which admits a section is then a bundle of pointed spaces. When the fibre is a manifold an affirmative answer can be given, as shown in [6]:

PROPOSITION 3.1. *Let B be a space and let X be a fibre bundle over B with a manifold as fibre. If X admits a section then X (with this section) is locally trivial as a fibrewise pointed space.*

In other words a sectionable bundle of spaces with fibre a manifold is equivalent, as a fibrewise pointed space, to a bundle of pointed spaces. Moreover the complement of the section is a fibre bundle with fibre the punctured manifold.

The fibrewise mapping-space, as in §1, serves as an adjoint to the fibrewise product, for the category of fibrewise spaces. The fibrewise pointed mapping-space, to be defined here, serves as an adjoint to the fibrewise smash product, for the category of fibrewise pointed spaces. Specifically, let X and Z be fibrewise pointed spaces over B. Disregarding the sections, for a moment, the fibrewise mapping-space

$$\mathrm{map}_B(X, Z) = \coprod_{b\in B} \mathrm{map}(X_b, Z_b)$$

is defined, as a fibrewise space. The subspace

$$\mathrm{map}_B^B(X, Z) = \coprod_{b\in B} \mathrm{map}_*(X_b, Z_b),$$

where map_* denotes the pointed maps, has a section given by the constant map in each fibre. In this way the fibrewise pointed mapping-space is defined.

As explained in §1 the case $X = T \times B$, for any T, admits of simple treatment. Taking T to be pointed it is easily seen that for any fibrewise pointed space Y over B a fibrewise pointed map

$$(T \times B) \wedge_B Y \to Z$$

determines a fibrewise pointed map

$$Y \to \mathrm{map}_B^B(T \times B, Z),$$

through the standard formula, and the converse holds when T is compact Hausdorff.

Two special cases should be noted. When T is the unit interval I, with basepoint $\{0\}$, the fibrewise pointed mapping-space $\mathrm{map}_B^B(I \times B, Z)$ is called the *fibrewise path-space* of Z and denoted by $\Lambda_B(Z)$. When T is the circle $I/\dot{I}$ the fibrewise pointed mapping-space is called the *fibrewise loop-space* of Z and denoted by $\Omega_B(Z)$. Thus Λ_B is adjoint to the fibrewise cone Γ_B^B and Ω_B to the fibrewise suspension Σ_B^B.

4. Fibrewise pointed homotopy

Fibrewise pointed homotopy is an equivalence relation between fibrewise pointed maps. Specifically let $\theta, \phi : X \to Y$ be fibrewise pointed maps, where X and Y are fibrewise pointed spaces over B. A *fibrewise pointed homotopy* of θ into ϕ is a homotopy $f_t : X \to Y$ of θ into ϕ which is fibrewise pointed for all $t \in I$.

If there exists a fibrewise pointed homotopy of θ into ϕ we say that θ is *fibrewise pointed homotopic* to ϕ. In this way an equivalence relation is defined on the set of fibrewise pointed maps of X into Y, and the pointed set of equivalence classes is denoted by $\pi_B^B(X,Y)$. Formally π_B^B constitutes a binary functor from the category of fibrewise pointed spaces to the category of pointed sets, contravariant in the first entry and covariant in the second.

The operation of composition for fibrewise pointed maps induces a function

$$\pi_B^B(Y,Z) \times \pi_B^B(X,Y) \to \pi_B^B(X,Z)$$

for fibrewise pointed spaces X, Y, Z over B. Postcomposition with a fibrewise pointed map $\psi : Y \to Z$ induces a function

$$\psi_* : \pi_B^B(X,Y) \to \pi_B^B(X,Z),$$

while precomposition with a fibrewise pointed map $\phi : X \to Y$ induces a function

$$\phi^* : \pi_B^B(Y,Z) \to \pi_B^B(X,Z).$$

The fibrewise pointed map $\phi : X \to Y$ is called a *fibrewise pointed homotopy equivalence* if there exists a fibrewise pointed map $\psi : Y \to X$ such that $\psi \circ \phi$ is fibrewise pointed homotopic to id_X and $\phi \circ \psi$ is fibrewise pointed homotopic to id_Y. Thus an equivalence relation between fibrewise pointed spaces is defined; the equivalence classes are called *fibrewise pointed homotopy types*.

A fibrewise pointed homotopy into the fibrewise constant is called a *fibrewise pointed nulhomotopy*. A fibrewise pointed space is said to be *fibrewise pointed contractible* if it has the same fibrewise pointed homotopy type as the base space, in other words if the identity is fibrewise pointed nulhomotopic. For example, the fibrewise cone $\Gamma_B^B(Y)$ and fibrewise path-space $\Lambda_B(Y)$ on a fibrewise pointed space Y are fibrewise pointed contractible. A subspace of a fibrewise pointed space is said to be *fibrewise pointed categorical* if the inclusion is fibrewise pointed nulhomotopic.

Let $\phi : X \to Y$ and $\psi : Y \to X$ be fibrewise pointed maps such that $\psi \circ \phi$ is fibrewise pointed homotopic to id_X. Then ψ is a left inverse to ϕ, up to fibrewise pointed homotopy, and ϕ is a right inverse to ψ, in the same sense. If ϕ admits both a left inverse ψ and a right inverse ψ', up to fibrewise pointed homotopy, then ψ and ψ' are fibrewise pointed homotopic and so ϕ is a fibrewise pointed homotopy equivalence.

Of course the associated bundle functor $P_\#$ discussed earlier transforms pointed G-homotopy classes of pointed G-maps into fibrewise pointed homotopy classes of fibrewise pointed maps, for each principal G-bundle P over B.

Given a fibrewise pointed space X over B a fibrewise pointed map $m : X \times_B X \to X$ is called a *fibrewise multiplication*. If m is fibrewise pointed homotopic to $m \circ t$, where $t : X \times_B X \to X \times_B X$ switches factors, we say that m is *fibrewise homotopy-commutative*. If

$$m \circ (m \times id), \quad m \circ (id \times m) : X \times_B X \times_B X \to X$$

are fibrewise pointed homotopic we say that m is *fibrewise homotopy-associative*. If

$$m \circ (id \times c) \circ \Delta, \quad m \circ (c \times id) \circ \Delta : X \to X$$

are fibrewise pointed homotopic to id_X we say that m is a *fibrewise Hopf structure* on X and that X, with this structure, is a *fibrewise Hopf space*.

Of course the associated bundle functor $P_\#$ transforms Hopf structures in the equivariant sense into Hopf structures in the fibrewise sense.

Cook and Crabb [5] have studied the problem of the existence of fibrewise Hopf structures in the case of sectioned q-sphere-bundles over a given base. This is only possible when $q = 1, 3$ or 7, since otherwise the fibre S^q does not admit Hopf structure. Cook and Crabb show that the fibrewise suspension of an orthogonal $(q-1)$-sphere bundle always admits fibrewise Hopf structure when $q = 1$, does so when $q = 3$ provided the bundle is orientable, and does so when $q = 7$ provided the structural group of the bundle can be reduced to the exceptional Lie group G_2. These observations depend on the properties of the classical Hopf structure on S^q in these dimensions. In the case of S^1 this is given by complex multiplication, which is $O(1)$-equivariant, in the case of S^3 by quaternionic multiplication, which is SO(3)-equivariant, and in the case of S^7 by Cayley multiplication, which is G_2-equivariant.

A *fibrewise homotopy right inverse* for a fibrewise multiplication m on X is a fibrewise pointed map $u : X \to X$ such that the composition

$$X \xrightarrow{\Delta} X \times_B X \xrightarrow{id \times u} X \times_B X \xrightarrow{m} X$$

is fibrewise pointed nulhomotopic. *Fibrewise homotopy left inverses* are defined similarly. When m is fibrewise homotopy-associative a fibrewise homotopy right inverse is also a fibrewise homotopy left inverse, and the term *fibrewise homotopy inverse* may be used.

A fibrewise homotopy-associative fibrewise Hopf space for which the fibrewise multiplication admits a fibrewise homotopy inverse is called a *fibrewise group-like space*. For example the topological product $T \times B$ is fibrewise group-like for each group-like space T. For another example the fibrewise loop-space $\Omega_B(Y)$ on a fibrewise pointed space Y is fibrewise group-like.

A fibrewise multiplication on the fibrewise pointed space Y over B determines a multiplication on the pointed set $\pi_B^B(X, Y)$ for all fibrewise pointed spaces X. If the former is fibrewise homotopy-commutative than the latter is commutative, and similarly with the other conditions. Thus $\pi_B^B(X, Y)$ is a group if Y is fibrewise group-like.

In this area of fibrewise homotopy theory the formal duality of Eckmann–Hilton operates satisfactorily. Thus given a fibrewise pointed space X over B a fibrewise pointed map $m : X \to X \vee_B X$ is called a *fibrewise comultiplication*. If m is fibrewise pointed homotopic to $t \circ m$, where $t : X \vee_B X \to X \vee_B X$ switches factors, we say that m is *fibrewise homotopy-commutative*. If $(m \vee id) \circ m$, $(id \vee m) \circ m : X \to X \vee_B X \vee_B X$ are fibrewise pointed homotopic we say that m is *fibrewise homotopy-associative*. If

$$\nabla \circ (id \vee c) \circ m, \quad \nabla \circ (c \vee id) \circ m : X \to X$$

are fibrewise pointed homotopic to id_X we say that m is a *fibrewise coHopf structure* on X and that X, with this structure, is a *fibrewise coHopf space*.

Of course the associated bundle functor transforms coHopf structures in the equivariant sense into coHopf structures in the fibrewise sense. The problem of the existence of fibrewise coHopf structures has been studied by Sunderland [32] and myself [19], particularly in the case of sectioned sphere-bundles over spheres.

A fibrewise homotopy right inverse for a fibrewise comultiplication m is a fibrewise pointed map $u : X \to X$ such that the composition

$$X \xrightarrow{m} X \vee_B X \xrightarrow{id \vee u} X \vee_B X \xrightarrow{\nabla} X$$

is fibrewise pointed nulhomotopic. Fibrewise homotopy left inverses are defined similarly. When m is fibrewise homotopy-associative a fibrewise homotopy right inverse is always a fibrewise homotopy left inverse, and the term fibrewise homotopy inverse may be used.

A fibrewise homotopy-associative fibrewise coHopf space for which the fibrewise comultiplication admits a fibrewise homotopy inverse is called a *fibrewise cogroup-like space*. For example the topological product $T \times B$ is fibrewise cogroup-like for each cogroup-like space T.

A fibrewise comultiplication on the fibrewise pointed space X over B determines a multiplication on the pointed set $\pi^B_B(X, Y)$ for all fibrewise pointed spaces Y. If the former is fibrewise homotopy-commutative then the latter is commutative, and similarly with the other conditions we have mentioned. Thus $\pi^B_B(X, Y)$ is a group if X is fibrewise cogroup-like.

If X is a fibrewise coHopf space and Y is a fibrewise Hopf space then the multiplication on $\pi^B_B(X, Y)$ determined by the fibrewise comultiplication on X coincides with the multiplication determined by the fibrewise multiplication on Y. Furthermore the multiplication is both commutative and associative.

By the distributive law for the fibrewise smash product a fibrewise comultiplication on X determines a fibrewise comultiplication on $X \wedge_B Y$ for all fibrewise pointed spaces Y. If the former is fibrewise homotopy-commutative then so is the latter, and similarly with the other conditions. Thus $X \wedge_B Y$ is fibrewise cogroup-like if X is fibrewise cogroup-like.

For example, take $X = S^1 \times B$, which is fibrewise cogroup-like since S^1 is cogroup-like. We see that the fibrewise suspension $\Sigma^B_B(Y)$ is fibrewise cogroup-like for all fibrewise pointed spaces Y.

5. Fibrewise cofibrations

Let $u : A \to X$ be a fibrewise pointed map, where A and X are fibrewise pointed spaces over B. Suppose that for each fibrewise pointed space E, fibrewise pointed map $f : X \to E$, and fibrewise pointed homotopy $g_t : A \to E$ of $f \circ u$, there exists a fibrewise homotopy $h_t : X \to E$ of f such that $h_t \circ u = g_t$. Then we say that u is a *fibrewise cofibration*. For example the identity $X \to X$ is a fibrewise cofibration, also the section $s : B \to X$. A fibrewise cofibration is not necessarily a cofibration in the ordinary sense.

It is not difficult to show that a fibrewise cofibration is necessarily an embedding, so that the case when A is a subspace of X and u the inclusion is typical. Moreover we have

PROPOSITION 5.1. *Let X be a fibrewise pointed space over B and let A be a closed subspace of X. The inclusion $A \to X$ is a fibrewise cofibration if and only if $(\{0\} \times X) \cup (I \times A)$ is a fibrewise retract of $I \times X$.*

This characterization enables us to see that the associated bundle functor $P_\#$ transforms cofibrations in the equivariant sense into cofibrations in the fibrewise sense. Specifically, let P be a principal G-bundle over B. Let X be a pointed G-space and let A be a closed invariant subspace of X. Suppose that the inclusion $A \to X$ is a G-fibration. Then the inclusion $P_\# A \to P_\# X$ is a fibrewise cofibration.

Unsurprisingly there is a fibrewise version of the well-known Puppe sequence. This concerns sequences

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots$$

of fibrewise pointed spaces and fibrewise pointed maps, over the given base space B. We describe such a sequence as *exact*, in this context, if the induced sequence

$$\pi_B^B(X_1, E) \xleftarrow{f_1^*} \pi_B^B(X_2, E) \xleftarrow{f_2^*} \pi_B^B(X_3, E) \leftarrow \cdots$$

is exact for all fibrewise pointed spaces E.

Given a fibrewise pointed map $\phi : X \to Y$, where X and Y are fibrewise pointed spaces over B, the fibrewise mapping-cone $\Gamma_B^B(\phi)$ of ϕ is defined to be the push-out of the cotriad

$$\Gamma_B^B(X) \supset X \xrightarrow{\phi} Y.$$

Now $\Gamma_B^B(\phi)$ comes equipped with a fibrewise embedding

$$\phi' : Y \to \Gamma_B^B(\phi),$$

and we easily see that the sequence

$$\pi_B^B(X, E) \xleftarrow{\phi^*} \pi_B^B(Y, E) \xleftarrow{\phi'^*} \pi_B^B(\Gamma_B^B(\phi), E)$$

is exact, for all fibrewise pointed spaces E. Obviously the procedure can be iterated so as to obtain exact sequences of unlimited length, but that in itself is not of great interest.

To understand the situation better consider the case of a fibrewise cofibration $u : A \to X$, where A and X are fibrewise pointed spaces over B. Then the natural projection

$$\rho : \Gamma_B^B(u) \to \Gamma_B^B(u)/_B \Gamma_B^B(A) = X/_B A$$

is a fibrewise pointed homotopy equivalence. Moreover if u' is derived from u in the way that ϕ' above is derived from ϕ then $\rho \circ u' : X \to X/_B A$ is just the fibrewise collapse.

Returning to the general case, where $\phi : X \to Y$, we observe that the embedding $\phi' : Y \to \Gamma_B^B(\phi)$ is a fibrewise cofibration. In fact the embedding $X \to \Gamma_B^B(X)$ is a fibrewise cofibration, from first principles, and so the conclusion follows from the observation that the push-out of a fibrewise cofibration is again a fibrewise cofibration.

By combining these last two results we see that the fibrewise mapping-cone $\Gamma_B^B(\phi')$ is equivalent to the fibrewise suspension

$$\Sigma_B^B(X) = \Gamma_B^B(\phi)/_B Y = \Gamma_B^B(\phi')/_B \Gamma_B^B(\phi),$$

up to fibrewise pointed homotopy equivalence. In the process, moreover, $(\phi')'$ is transformed into the fibrewise pointed map

$$\phi'' : \Sigma_B^B(\phi) \to \Sigma_B^B(X).$$

Repeating the procedure we find that $\Gamma_B(\phi')'$ is equivalent to the fibrewise suspension $\Sigma_B^B(Y)$, in the same sense. In the process, moreover, $((\phi')')'$ is transformed into the fibrewise suspension

$$\Sigma_B^B(\phi) : \Sigma_B^B(X) \to \Sigma_B^B(Y)$$

of ϕ, precomposed with the fibrewise reflection in which $(t, x) \mapsto (1 - t, x)$. This last does not affect the exactness property and so we arrive at an exact sequence of the form

$$X \xrightarrow{\phi} Y \to \Gamma_B^B(\phi) \to \Sigma_B^B(X) \xrightarrow{\Sigma_B^B(\phi)} \Sigma_B^B(Y) \to \cdots$$

When the given fibrewise pointed map ϕ is varied by a fibrewise pointed homotopy the exact sequence varies similarly. In particular, if ϕ is fibrewise pointed nulhomotopic the sequence has the same fibrewise pointed homotopy type (in an obvious sense) as in the case of the fibrewise constant, where the sequence reduces to the form

$$X \to Y \to Y \vee_B \Sigma_B^B(X) \to \Sigma_B^B(X) \to \cdots$$

6. Fibrewise fibrations

Let $p : E \to X$ be a fibrewise pointed map, where E and X are fibrewise pointed spaces over B. Suppose that for each fibrewise pointed space A, each fibrewise pointed map $f : A \to E$ and each fibrewise pointed homotopy $g_t : A \to X$ such that $g_0 = p \circ f$ there exists a fibrewise pointed homotopy $h_t : A \to E$ of f such that $g_t = p \circ h_t$. Then we say that p is a *fibrewise fibration*. For example the second projection $T \times_B X \to X$ is a fibrewise fibration for each fibrewise pointed space T. In particular take $X = B$; we see that for every fibrewise pointed space T over B the projection $T \to B$ is a fibrewise fibration.

Consider the fibrewise path-space

$$\Lambda_B(X) = \mathrm{map}_B^B(I \times B, X).$$

By evaluating at $t \in I$ we obtain projections

$$\rho_t : \Lambda_B(X) \to X.$$

It is a formal exercise in the use of adjoints to show that ρ_t is a fibrewise fibration for $t = 0, 1$. To fix ideas let us prefer ρ_0, in this situation. Then for any fibrewise pointed space E and fibrewise pointed map $p : E \to X$ the corresponding fibrewise pointed map

$$p^* \Lambda_B(X) \to E$$

is a fibrewise fibration. Now by the cartesian property of pull-backs we have a fibrewise pointed map

$$k : \Lambda_B(E) \to p^* \Lambda_B(X),$$

with components $\Lambda_B(p)$ and ρ_0. The following characterization of fibrewise fibrations is fundamental.

PROPOSITION 6.1. *Let $p : E \to X$ be a fibrewise pointed map, where E and X are fibrewise pointed spaces over B. Then p is a fibrewise fibration if and only if the fibrewise pointed map $k : \Lambda_B(E) \to p^* \Lambda_B(X)$ admits a right inverse.*

We can use this to show that the associated bundle functor transform fibrations in the equivariant sense into fibrations in the fibrewise sense. Again this is just a routine exercise in the use of adjoints.

At a formal level the Eckmann–Hilton duality between fibrewise cofibrations and fibrewise fibrations is a useful guide. Thus we find a fibrewise version of the Nomura exact sequence which is dual to the fibrewise version of the Puppe exact sequence described in §5. More significant, however, are the results which do not dualize, such as

PROPOSITION 6.2. *Let $p : E \to X$ be a fibrewise fibration, where E and X are fibrewise pointed spaces over B. Let $\theta, \phi : X' \to X$ be fibrewise pointed maps, where X' is* *fibrewise pointed over B. If θ and ϕ are fibrewise pointed homotopic then $\theta^* E$ and $\phi^* E$ have the same fibrewise pointed homotopy type over X'.*

COROLLARY 6.3. *Let $p : E \to X$ be a fibrewise fibration, where E and X are fibrewise pointed spaces over B. Let $\theta : X' \to X$ be a fibrewise pointed nulhomotopic map, where X' is fibrewise pointed over B. Then $\theta^* E$ is equivalent to $X' \times_B T$, in the sense of fibrewise pointed homotopy type over X', where T is the fibrewise fibre of X.*

7. Special types of fibrewise space

In the Quillen model a special role is played by cofibrant objects. In the case of a fibrewise pointed space X over B cofibrant means that the section $s : B \to X$ is a cofibration, in the ordinary sense. We use the terms *well-sectioned* or *fibrewise well-pointed*, in preference to cofibrant. For example B is always fibrewise well-pointed, as a fibrewise space over itself. For another example, suppose that the inclusion $u : A \to X$ is a fibrewise cofibration, where X is a fibrewise space over B and A is a subspace of X. Then the fibrewise collapse $X/_B A$ is fibrewise well-pointed.

The associated bundle functor, as before, transforms well-pointed spaces in the equivariant sense into well-pointed spaces in the fibrewise sense.

Fibrewise well-pointed spaces have a number of useful properties. For example, let $\phi : X \to Y$ be a fibrewise map, where X and Y are fibrewise pointed spaces over B with sections s and t, respectively. Suppose that $\phi \circ s$ is fibrewise homotopic to t. Also suppose that X is fibrewise well-pointed. Then ϕ is fibrewise homotopic to a fibrewise pointed map.

This result is just a straightforward application of the homotopy extension property. With rather more effort such an argument can be used to prove

PROPOSITION 7.1. *Let X be a fibrewise well-pointed space over B. Let $\theta : X \to X$ be a fibrewise pointed map which is fibrewise homotopic to the identity. Then there exists a fibrewise pointed map $\theta' : X \to X$ such that $\theta' \circ \theta$ is fibrewise pointed homotopic to the identity.*

This leads to the important

THEOREM 7.2. *Let $\phi : X \to Y$ be a fibrewise pointed map, where X and Y are fibrewise well-pointed spaces over B. If ϕ is a fibrewise homotopy equivalence then ϕ is a fibrewise pointed homotopy equivalence.*

This result, which is due to Dold [12], is proved as follows. Let $\psi : Y \to X$ be a fibrewise map which is an inverse of ϕ, up to fibrewise homotopy. Since $\psi \circ t = \psi \circ \phi \circ s$, which is fibrewise homotopic to s, we can deform ψ into a fibrewise pointed map $\psi' : Y \to X$ by a fibrewise homotopy. Since $\psi' \circ \phi$ is fibrewise homotopic to the identity there exists, by (7.1), a fibrewise pointed map $\psi'' : X \to Y$ such that $\psi'' \circ \psi' \circ \phi$ is fibrewise pointed homotopic to the identity. Thus ϕ admits a left inverse $\phi' = \psi'' \circ \psi'$ up to fibrewise pointed homotopy.

Now ϕ' is a fibrewise homotopy equivalence, since ϕ is a fibrewise homotopy equivalence, and so the same argument, applied to ϕ' instead of ϕ, shows that ϕ' admits a left inverse ϕ'', up to fibrewise pointed homotopy. Thus ϕ' admits both a right inverse ϕ and a left inverse ϕ'', up to fibrewise pointed homotopy. Hence ϕ' is a fibrewise pointed homotopy equivalence and so ϕ itself is a fibrewise pointed homotopy equivalence.

The class of fibrewise well-pointed spaces has many good properties. For example, the fibrewise coproduct of fibrewise well-pointed spaces is fibrewise well-pointed. Less obviously, the fibrewise product and fibrewise smash product of fibrewise well-pointed spaces are fibrewise well-pointed.

However, the class of fibrewise well-pointed spaces is too restrictive for some purposes and there is another, wider, class which also has some good properties, as follows. Consider a fibrewise pointed space X over B with section $s : B \to X$. We denote by $\check{X}_B$ the fibrewise mapping cylinder of s, regarded as a fibrewise pointed space with section the insertion σ_1. The inclusion $\sigma : X \to \check{X}_B$ is a fibrewise map, in fact a fibrewise homotopy equivalence, but not a fibrewise pointed map. The natural projection $\rho : \check{X}_B \to X$, which fibrewise collapses $I \times B$, is a fibrewise pointed map as well as a fibrewise homotopy equivalence. When ρ is a fibrewise pointed homotopy equivalence we describe X as *fibrewise nondegenerate*.

For example, fibrewise well-pointed spaces are fibrewise nondegenerate, by (7.2). It can be shown (see §22 of [18]) that the fibrewise product and fibrewise smash product of fibrewise nondegenerate spaces are fibrewise nondegenerate.

When a fibrewise space X over B admits a section, and so can be regarded as a fibrewise pointed space, the fibrewise pointed homotopy type will generally depend on the choice of section. However if $s_0, s_1 : B \to X$ are vertically homotopic sections then the fibrewise mapping cylinders of s_0 and s_1 have the same fibrewise pointed homotopy type. Hence if both the fibrewise pointed spaces obtained from X by using these sections are fibrewise nondegenerate then we can conclude that they have the same fibrewise pointed homotopy type. When X admits more than one vertical homotopy class of section, however, examples can be given to show that the fibrewise pointed homotopy type depends on the choice of section; in fact X may be fibrewise Hopf or fibrewise coHopf with one choice of section but not with another.

8. Theorems of tom Dieck and Dold

There is a well-known theorem of Dold [12] which forms a bridge between fibrewise homotopy theory and ordinary homotopy theory. As it stands this is already of great significance in fibrewise homotopy theory but from our point of view a fibrewise pointed version of Dold's theorem is still more significant, namely

PROPOSITION 8.1. *Let $\phi : X \to Y$ be a fibrewise pointed map, where X and Y are fibrewise pointed spaces over B. Suppose that B admits a numerable covering such that the restriction $\phi_V : X_V \to Y_V$ is a fibrewise pointed homotopy equivalence over V for each member V of the covering. Then ϕ is a fibrewise pointed homotopy equivalence over B.*

In the original version of this result there are no sections to be considered. Eggar [14] made the modifications to Dold's argument which are necessary for the version given here. To be quite accurate it is not Eggar's main result which is relevant but rather the second remark after the statement where it is assumed that the base space is paracompact. In fact this assumption is unnecessary since numerable coverings can always be shrunk in the way required for this purpose.

When the fibrewise spaces in Dold's theorem are fibrant, as is often the case in the applications, the following consequence of the main result is convenient.

COROLLARY 8.2. *Let $\phi : X \to Y$ be a fibrewise pointed map, where X and Y are fibrant fibrewise pointed spaces over B. Suppose that B admits a numerable categorical covering. Also suppose that the restriction of ϕ to the fibre is a homotopy pointed equivalence. Then ϕ is a fibrewise pointed homotopy equivalence.*

For let V be a categorical subspace of B. Since X is fibrant the restriction X_V of X has the same fibrewise pointed homotopy type over V as $V \times X_b$ where b is the basepoint of B, similarly Y_V has the same fibrewise pointed homotopy type as $V \times Y_b$. By hypothesis the restriction $\phi_b : X_b \to Y_b$ of ϕ to the fibres is a homotopy equivalence and so it follows that the restriction $\phi_V : X_V \to Y_V$ is a fibrewise pointed homotopy equivalence over V. Since this is true for each member of the numerable categorical covering the hypothesis of (8.1) is satisfied and the conclusion of (8.2) is obtained. We shall now give a few applications of this key result.

It is not always easy to decide what is the right fibrewise version of a condition in ordinary homotopy theory. Take the condition of path-connectedness, for example. One fibrewise version is obviously vertical connectedness, where all sections are required to be vertically homotopic. But there is another condition, significantly easier to fulfill, which also reduces to path-connectedness when B is a point. Giving preference to the latter we describe a fibrewise pointed space X over B as *polarized* if every section of X which does not meet the standard section is vertically homotopic to the standard section.

PROPOSITION 8.3. *Let X be a fibrewise well-pointed space X over B which admits a numerable fibrewise categorical covering. Suppose that X is polarized in the above sense. Then any fibrewise Hopf structure on X admits a right inverse and a left inverse, up to fibrewise pointed homotopy.*

For let $m : X \times_B X \to X$ be a fibrewise Hopf structure. By using the fibrewise homotopy extension property we may suppose, with no real loss of generality, that the section $s : B \to X$ is a strict neutral section for m, in the sense that $m \circ (c \times id) \circ \Delta = id$, where $c = s \circ p$ is the fibrewise constant. We regard $X \times_B X$ as a fibrewise pointed space over X using the first projection π_1 and the section $(c \times id) \circ \Delta$. Then $X \times_B X$ is fibrant over X since X is fibrant over B. Also X is numerably fibrewise categorical. Hence the fibrewise shearing map

$$k : X \times_B X \to X \times_B X,$$

where $\pi_1 \circ k = \pi_1$ and $\pi_2 \circ k = m$, is a fibrewise homotopy equivalence, by (8.1). Also $X \times_B X$ is fibrewise well-pointed over X, since X is fibrewise well-pointed over B, and so k is a fibrewise pointed homotopy equivalence, by (8.2). Hence the composition

$$X \xrightarrow{u} X \times_B X \xrightarrow{l} X \times_B X \xrightarrow{\pi_2} X$$

provides a right inverse for the fibrewise Hopf structure, up to fibrewise pointed homotopy, where u is given by $(id \times c) \circ \Delta$ and l is the right inverse of k, up to fibrewise pointed homotopy. Similarly m admits a left inverse, in the same sense. When m is fibrewise homotopy-associative the left and right inverses are equivalent, up to fibrewise pointed homotopy.

Similar ideas are used in the proof of

PROPOSITION 8.4. *Let X be a fibrant fibrewise well-pointed space over B. Suppose that B admits a numerable categorical covering. If a Hopf structure on the fibre of X can be extended to a fibrewise multiplication on X then it can be extended to a fibrewise Hopf structure.*

For let $\theta : X \times_B X \to X$ be a fibrewise multiplication extending a given Hopf structure on X_b, where $b \in B$ is the basepoint. Then the restriction to X_b of the fibrewise pointed map $\theta \circ \sigma_j : X \to X$ $(j = 1, 2)$ is pointed homotopic to the identity. Hence $\theta \circ \sigma_j$ is a fibrewise homotopy equivalence by Dold's theorem (8.2) and so a fibrewise pointed homotopy equivalence by (7.2). So let $\alpha_j : X \to X$ be an inverse of $\theta \circ \sigma_j$, up to fibrewise pointed homotopy. Then

$$\theta \circ (\alpha_1 \times \alpha_2) : X \times_B X \to X$$

is a fibrewise Hopf structure on X which extends the given structure on X_b. This proves (8.4).

Dold's theorem may be compared with a series of results of tom Dieck [10], such as the following.

PROPOSITION 8.5. *Let $\phi : X \to Y$ be a fibrewise map, where X and Y are fibrewise spaces over B. Let $\{X_j\}$ and $\{Y_j\}$ be J-indexed numerable coverings of X and Y, respectively, such that $\phi X_j \subset Y_j$ for each index j. Suppose that the fibrewise map $\phi_\sigma : X_\sigma \to Y_\sigma$ determined by ϕ is a fibrewise homotopy equivalence for each finite subset σ of J. Then ϕ is a fibrewise homotopy equivalence.*

Here we use the convention that X_σ, for $\sigma \subset J$, denotes the intersection $\cap X_j$, where j runs through σ, and similarly with Y_σ and ϕ_σ.

9. The fibrewise Freudenthal theorem

In ordinary homotopy theory the class of CW-spaces is adequate for most purposes. Stasheff [30] has shown that a fibrant fibrewise space over a CW-space is also a CW-space if the fibre is a CW-space. For example, a sphere-bundle over a CW-space is a

CW-space. In fibrewise homotopy theory certain results can be proved for this class of fibrewise spaces using the methods of ordinary homotopy theory. To extend them to fibrewise spaces more generally a fibrewise version of the notion of CW-space is required, and this is developed in the monograph mentioned above. For present purposes, however, the ordinary notion will suffice.

PROPOSITION 9.1. *Let B be a CW-space. Let (K, L) be a fibrant fibrewise pointed pair over B with CW fibres. Let (X, Y) be a fibrant pair over B such that*

$$H^n\big(K, L; \pi_n(X, Y)\big) = 0$$

for all n. Then every fibrewise pointed map

$$f : (K, L) \to (X, Y)$$

is fibrewise homotopic, relative to L, to a fibrewise pointed map of K into Y.

The proof is a straightforward exercise in ordinary obstruction theory. We use (9.1) to prove

THEOREM 9.2. *Let B be a CW-space. Let K be a fibrant fibrewise pointed space over B with fibre a CW-space. Let $u : E \to F$ be a k-connected fibrewise pointed map, where E and F are fibrant fibrewise pointed spaces over B. Then the function*

$$u_* : \pi^B_B(K, E) \to \pi^B_B(K, F)$$

is injective when $\dim K < k$, *surjective when* $\dim K \leqslant k$.

Dimension, here, means cohomological dimension. To deduce (9.2) from (9.1) first observe that without real loss of generality we may suppose, after taking the fibrewise mapping cylinder, that $E \subset F$ and u is the inclusion. Surjectivity, in (9.2), follows at once from (9.1) applied to the pair $(K, \varnothing)$, while injectivity follows from (9.1) applied to the pair $(I \times K, \{0\} \times K)$. Of course a relative version of this result can be proved in the same way.

We use (9.2) to prove a fibrewise version of the Freudenthal suspension theorem, as follows.

PROPOSITION 9.3. *Let B be a CW-space. Let K be a fibrant fibrewise pointed space over B with CW fibres. Let E be a sectioned fibre bundle over B with $(m-1)$-connected fibre. Then the fibrewise suspension*

$$\pi^B_B(K, E) \to \pi^B_B\big(\Sigma^B_B(K), \Sigma^B_B(E)\big)$$

is injective for $\dim K < 2m$, *surjective for* $\dim K \leqslant 2m$.

First observe that since E is a fibre-bundle over B so is $\Sigma^B_B(E)$ (in fact this is true for fibre spaces but is not so obvious) and so the fibrewise loop-space $\Omega_B \Sigma^B_B(E)$ is fibrant.

Since E is $(m-1)$-connected the classical Freudenthal suspension theorem tells us that the adjoint

$$u : E \to \Omega_B \Sigma_B^B(E)$$

of the identity is $(2m-1)$-connected, and so (9.3) follows at once from (9.2).

COROLLARY 9.4. *Let K be a sectioned k-sphere bundle and L a sectioned l-sphere-bundle over the CW-complex B. Then for each sectioned sphere-bundle N the fibrewise smash product*

$$N_\# : \pi_B^B(K, L) \to \pi_B^B(N \wedge_B K, N \wedge_B L)$$

is injective when $2l - k \geqslant \dim B + 2$, *surjective when* $2l - k \geqslant \dim B + 1$.

First recall that if ξ is a Euclidean bundle with fibrewise compactification N there exists a Euclidean bundle ξ' with fibrewise compactification N' such that the Whitney sum $\xi' \oplus \xi$ is equivalent to the trivial q-plane bundle $\mathbb{R}^q \times B$, for q sufficiently large, and so the fibrewise smash product $N' \wedge_B N$ is equivalent to $S^{q-1} \times B$, as a fibrewise pointed space. Therefore the function

$$(N' \wedge_B N)_\# : \pi_B^B(K, L) \to \pi_B^B(N' \wedge_B N \wedge_B K,\ N' \wedge_B N \wedge_B L)$$

is equivalent to the $(q-1)$-fold fibrewise suspension, and so is injective when $2l - k \geqslant \dim B + 2$, surjective when $2l - k \geqslant \dim B + 1$. By associativity of the fibrewise smash product we have $(N' \wedge_B N)_\# = N'_\# \circ N_\#$, where

$$N_\# : \pi_B^B(K, L) \to \pi_B^B(N \wedge_B K,\ N \wedge_B L),$$

$$N'_\# : \pi_B^B(N \wedge_B K,\ N \wedge_B L) \to \pi_B^B(N' \wedge_B N \wedge_B K,\ N' \wedge_B N \wedge_B L).$$

Hence $N_\#$ is injective when $2l - k \geqslant \dim B + 2$, surjective when $2l - k \geqslant \dim B + 1$.

By repeating this argument with N' instead of N and $(N \wedge_B K,\ N \wedge_B L)$ instead of (K, L) we find that $N'_\#$ is injective as well as surjective when $2l - k \geqslant \dim B + 1$. But $N'_\# \circ N_\#$ is surjective when $2l - k \geqslant \dim B + 1$, as we have seen, and so $N_\#$ is surjective when $2l - k \geqslant \dim B + 1$. This completes the proof of (9.4).

Given K and L, as in (9.4), we can always choose N so that the fibrewise smash product $N \wedge_B L$ is trivial (see [8]) and hence express $\pi_B^B(K, L)$, when $2l - k \geqslant \dim B + 2$, in terms of cohomotopy sets in the ordinary sense.

The fibrewise Freudenthal theory leads to a fibrewise version of stable homotopy theory and to a notion of *fibrewise spectra*, as developed by Clapp [3] and Clapp and Puppe [4] who prefer the term *parameterized spectra*.

10. Fibrewise homology

Homology with local coefficients is a concept which dates from the forties if not earlier. However the more general concept of fibrewise homology does not seem to have originated before the late sixties, when it was developed by several researchers independently, notably Becker [2], Dold [13], Hodgkin (unpublished), Johannson (unpublished) and Smith [29]. Later Clapp [3] and Clapp and Puppe [4] developed the theory further.

The basic ideas of fibrewise homology and cohomology will be presented here; many variants are possible.

Given a category C we denote by C' the comma category of C, so that the objects of C' are the morphisms of C and so on. Let $\mathcal{A}$ be an abelian category, for example the category of abelian groups. We consider sequences $h : \{h_n : n \in \mathbb{N}\}$ of functors $C' \to \mathcal{A}$ which are equipped with sequences $\partial = \{\partial_n : n \in \mathbb{N}\}$ of natural transformations having the following exactness property: whenever $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ are morphisms of C the sequence

$$\cdots \to h_{n+1}(\psi) \xrightarrow{\partial_{n+1}} h_n(\phi) \xrightarrow{\psi_*} h_n(\psi \circ \phi) \xrightarrow{\phi_*} h_n(\psi) \xrightarrow{\partial_n} h_{n-1}(\phi) \to \cdots$$

is exact. Here, as usual, we write ϕ_* and ψ_* for the transforms of the commutative squares

$$\begin{array}{ccc} X & \xrightarrow{\psi\circ\phi} & Z \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle id} \\ Y & \xrightarrow[\psi]{} & Z \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow[\psi\circ\phi]{} & Z \end{array}$$

In particular take C to be the category of fibrewise spaces over B. By a *fibrewise homology functor* over B I mean a functor, as above, defined on the comma category C', which is invariant with respect to fibrewise homotopy.

Given a fibrewise homology functor h it is convenient, in the case of a fibrewise pair (X, A), to write $h(X, A)$ in place of $h(u)$, where $u : A \subset X$. For $A = \varnothing$ we write $h(X, \varnothing) = h(X)$. Thus if $\phi : X \to Y$ is a fibrewise map, where X and Y are fibrewise spaces, we have an exact sequence of the form

$$\cdots \to h_{n+1}(\phi) \to h_n(X) \xrightarrow{\phi_*} h_n(Y) \to h_n(\phi) \to h_{n-1}(X) \to \cdots$$

Examples of fibrewise homology functors can easily be constructed. One may simply take $h(X)$, for the fibrewise space X, to be the singular homology of X, as an ordinary space, and similarly for fibrewise maps. More generally one may choose a fibrewise space T and then take $h(X)$ to be the ordinary homology of the fibrewise product $X \times_B T$.

The *excision condition*, for fibrewise homology functors, is defined in the same way as for ordinary homology functors, so that fibrewise relative homeomorphisms, satisfying appropriate conditions, induce isomorphisms of the relative fibrewise homology. Mayer–Vietoris exact sequences arise in the usual way. Note that if h is constructed out of singular homology as above then h satisfies the excision condition.

If h is a fibrewise homology functor over B then for each family $\{X_j\}$ of fibrewise spaces we have a homomorphism

$$\prod h(X_j) \to h(\coprod_B X_j)$$

induced by the standard insertions of the coproduct. If the homomorphism is an isomorphism for finite families we describe h as *additive*. In fact the excision condition implies additivity. If the homomorphism is an isomorphism for all families we describe h as *strongly additive*. Note that if h is constructed as before by taking the singular homology of the fibrewise product with a given fibrewise space then h is strongly additive.

There is one more condition we need to discuss. Following Becker [2] and Dold [13] we call this the *cylinder condition*. Consider the cylinder $I \times X$ on a given space X, with the standard maps

$$i_t : X \to I \times X \quad (t \in I)$$

given by $i_t(x) = (t, x)$. If X is a fibrewise space with projection $p : X \to B$ then $I \times X$ is regarded as a fibrewise space with projection $p \circ \pi$, where $\pi : I \times X \to X$ is given by $\pi(t, x) = x$. Suppose, however, that $I \times X$ is a fibrewise space with some projection $r : I \times X \to B$. Then X may be regarded as a fibrewise space with projection $r_t = r \circ i_t$ for any $t \in I$. To avoid any possibility that this second situation might be confused with the first we refer to $I \times X$ in the second case as a cylinder over B.

CONDITION 10.1. *The fibrewise homology functor h over B satisfies the cylinder condition if for each space X and cylinder $I \times X$ over B the inclusion $i_0 : X \to I \times X$ has trivial fibrewise homology $h(i_0)$.*

Here, of course, we are regarding X as a fibrewise space with projection r_0, where $r : I \times X \to B$ is the projection of the cylinder.

For example, if h is derived from singular homology theory by taking fibrewise products $\times_B T$ with a given fibrewise space T, as above, then h satisfies the condition when T is fibrant, although not in general. Essentially this follows from the homotopy property of induced fibre spaces.

When the cylinder condition is satisfied it follows by exactness that the induced homomorphisms

$$i_{t*} : h(X) \to h(I \times X) \quad (t = 0, 1)$$

are isomorphisms. Hence an automorphism $\alpha = (i_{1*})^{-1} \circ (i_{0*})$ of $h(X)$ is defined. Of course α depends on the projection r used to represent $I \times X$ as a fibrewise space. In particular if X, at the start, is given as a fibrewise space and $r = \pi$ then α is trivial. We see, therefore, that the cylinder condition implies fibrewise homotopy invariance. For this reason Clapp [3] and others use the term *strong fibrewise homotopy invariance* for the cylinder condition.

PROPOSITION 10.2. *Let h be a fibrewise homology functor satisfying the cylinder condition. Let $\phi : X \to X'$ be a fibrewise map, where X and X' are fibrewise spaces over B. Suppose that ϕ is a homotopy equivalence, in the ordinary sense, of X with X'. Then*

$$\phi_* : h(X) \approx h(X').$$

For let $\phi' : X' \to X$ be a homotopy inverse of ϕ, so that $f \circ i_0 = id$, $f \circ i_1 = \phi \circ \phi'$ for some homotopy $f : I \times X' \to X'$. Regard $I \times X'$ as a cylinder over B through $p' \circ f$. Then f_* is defined, since f is fibrewise, and is an isomorphism, since i_{0*} and $f_* \circ i_{0*}$ are isomorphisms.

Now consider the map $i_1 : X'' \to I \times X'$, where $X'' = X'$ as a space but is regarded as a fibrewise space with projection $p \circ \phi'$ rather than p'. Then i_1 is fibrewise since $p' \circ f \circ i_1 = p' \circ \phi \circ \phi' = p \circ \phi$, and so i_{1*} is defined and is an isomorphism, where $\phi' : X'' \to X$. Thus $\phi_* \circ \phi'_*$ is an isomorphism, hence ϕ'_* is a monomorphism and ϕ_* is an epimorphism. Applying the same argument to ϕ', instead of ϕ, which is also a homotopy equivalence, we obtain that ϕ'_* is an epimorphism, and so ϕ_* is an isomorphism, as asserted.

Among the contractible fibrewise spaces over B a special role is played by those in which the total space reduces to a point. When B is path-connected there is just one fibrewise homotopy type of these "points over B"; in general there is one type for each path-component.

Given a fibrewise homology functor h over B satisfying the cylinder condition, also the excision condition and strong additivity, we may follow procedures closely similar to those used in the case of ordinary homology. For example if K is a CW-complex over B we may analyze $h(K)$ through the relative groups $h(K^q, K^{q-1})$ which can be computed as follows.

Consider the pair (D^q, S^{q-1}) consisting of the q-ball D^q and the boundary $(q-1)$-sphere S^{q-1}. Regard (D^q, S^{q-1}) as a fibrewise pair by choosing a projection $D^q \to B$. The argument used in the case of ordinary homology shows that $h(D^q, S^{q-1})$ is equivalent to $h(p)$, where $p \in S^{q-1}$.

Now for each q-cell e_j of K we have a characteristic map

$$f_j : (D^q, S^{q-1}) \to (K^q, K^{q-1});$$

let (D^q_j, S^{q-1}_j) denote (D^q, S^{q-1}) regarded as a fibrewise pair through f_j. By excision the fibrewise relative homeomorphism

$$f : \left(\coprod D^q_j, \coprod S^{q-1}_j\right) \to (K^q, K^{q-1}),$$

given by $f \mid (D^q_j, S^{q-1}_j) = f_j$, induces an isomorphism in relative fibrewise homology.

Therefore $h(K^q, K^{q-1})$ is isomorphic to the direct sum

$$\bigoplus_j h(p_j),$$

where p_j is a point of S_j^{q-1}.

To illustrate these remarks we prove

PROPOSITION 10.3. *Let h and k be strongly additive fibrewise homology functors over B satisfying the excision and cylinder conditions. Let $\Phi : h \to k$ be a natural transformation which is an equivalence for all points p over B. Then Φ is an equivalence for all CW-complexes over B.*

This result is due to Dold [13], who calls it the *comparison theorem*, and uses it to give an elegant proof of the Thom isomorphism theorem for vector bundles. Briefly Dold's argument is as follows.

To establish the result in general it is sufficient to establish that Φ gives an equivalence in the absolute case, since then the conclusion can be reached by a five lemma argument. So let K be a CW-complex over B. We prove by induction that

$$\Phi(K^q) : h(K^q) \approx k(K^q),$$

where K^q is the q-section of K. The inductive step from $q-1$ to q amounts to showing that

$$\Phi(K^q, K^{q-1}) : h(X^q, K^{q-1}) \approx k(K^q, K^{q-1}),$$

and this follows from the analysis given above. This proves the assertion when K is finite dimensional. For the general case we use the telescope technique and it is here that strong additivity is required. It should be noted that if B is path-connected then the condition in the statement of (10.3) only needs to be verified for one point.

Turning now to the dual concept, the term *fibrewise cohomology functor* is defined in the obvious way, with appropriate changes of notation, also the dual forms of the three conditions we have been discussing. Examples can be constructed by taking the singular cohomology of the fibrewise product with a given fibrewise space. The excision condition and strong additivity are always satisfied. The cylinder condition is satisfied when the given fibrewise space is fibrant.

Suppose, in particular, that the range category $\mathcal{A}$ of our fibrewise cohomology functor h is the category of modules over some (commutative) ring Λ. Then we may require h to be multiplicative in the sense that for each pair of fibrewise spaces X, Y there is a natural exterior product

$$h(X) \otimes_\Lambda h(Y) \to h(X \times_B Y)$$

satisfying the standard conditions. Of course the exterior product leads to a natural ring structure on $h(X)$ in the usual way. Specifically the product $\alpha \cup \beta \in h^{p+q}(X)$ of elements $\alpha \in h^p(X)$, $\beta \in h^q(X)$ is defined to be the image of $\alpha \otimes \beta$ under the homomorphism

$$h^{p+q}(X \times_B X) \to h^{p+q}(X)$$

induced by the diagonal.

Examples of fibrewise cohomology theories with this multiplicative structure are obtained by taking the singular cohomology of the fibrewise product with a given fibrewise space.

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CHAPTER 5

Coherent Homotopy over a Fixed Space

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0. Introduction

The homotopy theory of categories of diagrams was studied initially by Edwards and Hastings [10]. The homotopy category $Ho(Top^I)$ of the diagram category Top^I for a small category I is obtained in their system by inverting those diagram maps which are at each level a homotopy equivalence. A disadvantage of this definition is that through the formality of the morphisms the geometric meaning associated with composition and inverse is obscured. For a 'concrete' description of the morphisms of $Ho(Top^I)$ one may apply a theorem of Vogt to the effect that $Ho(Top^I)$ is equivalent to a category of homotopy coherent diagrams and homotopy classes of homotopy coherent maps between them. Vogt proved this result in 1973 (cf. [26]). Cordier [5] simplified Vogt's description of homotopy coherent diagrams and with Porter [6] provided a new and simpler proof of Vogt's main result. Meanwhile the first author [11], [12] had already observed that the coherent approach in the special case $I = 1$, the ordered set $\{0 < 1\}$, yielded a simple and natural alternative to the *pair homotopy* theory of Eckmann and Hilton [9]. It seemed that it would be advantageous to embark on a systematic study of (homotopy theoretic aspects of) the simplest diagram categories, partly with the view to the light their properties might throw on more complex categories, but also with a view to studying their interrelations in the hope that feedback of results would eventually yield new techniques for study of $Ho(Top)$ itself.

The category Top_B of spaces over a fixed space B can be regarded as a (nonfull) subcategory of the category of pairs Top^1. Some interest is attached to the associated homotopy category $Ho(Top_B)$, which is obtained by inverting the maps over B that are homotopy equivalences in their own right. Although it seems not to be possible to deduce the fact from Vogt's theorem (at least in its present formulation), it turns out [16] that $Ho(Top_B)$ is also equivalent to a category $\mathcal{H}_B$ whose objects are spaces over B and whose morphisms are coherent homotopy classes of homotopy equivalent triangles with sink vertex B. In this article we discuss principally these examples but try to exhibit features that they share generally with other coherent homotopy categories.

We lay a special emphasis on the coherent homotopy theory over B and show how the basic ingredients of elementary homotopy theory, as for example Dold's theorem on fibre homotopy equivalences, the homotopy theorem for fibrations and model categorical properties, arise in this theory in a very natural and transparent way.

For convenience of the reader we work in the category Top of topological spaces. However the methods and results can be transferred to pointed topological spaces, or indeed to abstract homotopy theory in any category C which is equipped with a suitable cubical enrichment, see, e.g., [19], [20], [15].

1. The coherent homotopy categories

Let B be a fixed space. The classical homotopy category over B (fibre homotopy category), Top_Bh, is a quotient category of the category Top_B of spaces over B. Thus the objects of Top_Bh are the spaces over B, i.e. the maps $f : X \to B$ with codomain B and the morphisms of $Top_B h$ are represented by maps over B, $h : f \to g$, i.e. commutative triangles

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & E \arrow[dl, "g"] \\ & B & \end{tikzcd} \quad . \tag{1.1}$$

Two maps over B, $h, h' : f \to g$, represent the same element of $Top_B h$, if they are homotopic over B (fibrewise homotopic) that means if there is a homotopy over B, $h_t : h \simeq_B h'$, i.e. a homotopy $h_t : h \simeq h'$ such that $gh_t = f$.

The coherent analogue, $\mathcal{H}_B$, the *track category over* B is defined as follows. The objects of $\mathcal{H}_B$ are the spaces over B. (Thus $\mathcal{H}_B$ and $Top_B h$ have the same objects.) If

$$f : X \to B \quad \text{and} \quad g : E \to B$$

are spaces over B, then the set $\mathcal{H}_B(f, g)$ of morphisms $f \to g$ (also denoted $\pi(f, g/B)$) is obtained from the set of squares of the form

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & E \arrow[d, "g"] \\ B \arrow[r, equal] \arrow[ur, phantom, "\{h_t\}"] & B \end{tikzcd} \quad , \tag{1.2}$$

where $\{h_t\}$ is the track (relative homotopy class) of a homotopy $h_t : f \simeq gh$, by factoring out by the equivalence relation

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & E \arrow[d, "g"] \\ B \arrow[r, equal] \arrow[ur, phantom, "\{h_t\}"] & B \end{tikzcd} \quad \sim \quad \begin{tikzcd} X \arrow[r, "h'"] \arrow[d, equal] & E \arrow[d, equal] \\ X \arrow[r, "h"] \arrow[d, "f"'] \arrow[ur, phantom, "\{h'_t\}"] & E \arrow[d, "g"] \\ B \arrow[r, equal] \arrow[ur, phantom, "\{h_t\}"] & B \end{tikzcd} \quad ,$$

where $h'_t : h \simeq h'$ is a homotopy and the diagram on the right is the composite in the obvious sense of the two squares. To simplify the notation we henceforth omit the curly brackets from representations of squares such as (1.2). We use the abbreviated notation $\{h, h_t\}$ for the element represented by the square (1.2).

If A is a fixed space, then we have a classical homotopy category under A, $Top^A h$, as well as a coherent analogue, the track category under A, $\mathcal{H}^A$. The morphisms of $Top^A h$ are represented by commutative triangles of the form

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "h"'] & & Y \end{tikzcd} \quad ,$$

whereas the morphisms of $\mathcal{H}^A$ are represented by homotopy commutative squares of the form

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"'] \arrow[dr, phantom, "\{h_t\}"] & A \arrow[d, "j"] \\ X \arrow[r, "h"'] & Y \end{tikzcd},$$

where $\{h_t\}$ is the track of a homotopy $h_t : hi \simeq j$. Recall that two maps under A, $h, h' : i \to j$, represent the same element of $Top^A h$ if they are homotopic under A, that is if there is a homotopy $h_t : h \simeq h'$ such that $h_t i = j$.

The objects of $\mathcal{H}^1$, the category of *homotopy pairs* (our coherent version of $Ho(Top^1)$ mentioned in the introduction), are the maps of *Top*. Analogously, if

$$f : X \to Y \quad \text{and} \quad f' : X' \to Y'$$

are two maps, a morphism $f \to f'$ is an equivalence class of squares of form

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\{h_t\}"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "k"'] & Y' \end{tikzcd}, \tag{1.3}$$

where $h_t : kf \simeq f'h$, under the relation

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\{h_t\}"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "k"'] & Y' \end{tikzcd} \quad \sim \quad \begin{tikzcd} X \arrow[r, "h'"] \arrow[d, equal] \arrow[dr, phantom, "\{h'_t\}"] & X' \arrow[d, equal] \\ X \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\{h_t\}"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "k"] \arrow[d, equal] \arrow[dr, phantom, "\{k'_t\}"] & Y' \arrow[d, equal] \\ Y \arrow[r, "k'"'] & Y' \end{tikzcd}.$$

We use the notation $\{k, h, h_t\}$ for the element represented by the square (1.3).

REMARK. In the definition of the categories $\mathcal{H}_B$ and $\mathcal{H}^A$ the category Top of topological spaces can be replaced by an arbitrary groupoid enriched category, i.e. any 2-category with invertible 2-cells.

In particular we can replace the category Top by the category Top^A and fix an object b of Top^A, i.e. a map $b : A \to B$. The resulting track homotopy category over b can be described as a partially coherent homotopy category $\mathcal{H}_b$, under A and over B. An object of $\mathcal{H}_b$ is a factorization (σ, ρ) of b, i.e. a diagram

$$A \xrightarrow{\sigma} X \xrightarrow{\rho} B,$$

where σ, ρ are maps of spaces such that $\rho\sigma = b$. An arrow from (σ, ρ) to (τ, η) in $\mathcal{H}_b$ is an equivalence class $\{u, \rho_t\}$ of a diagram of the form

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\sigma"'] & A \arrow[d, "\tau"] \\
X \arrow[r, "u"] \arrow[d, "\rho"'] \arrow[r, phantom, shift right=6ex, "\{\rho_t\}^A \nearrow"] & Y \arrow[d, "\eta"] \\
B \arrow[r, equal] & B
\end{tikzcd}
\quad , \tag{1.4}
$$

where the upper square commutes (i.e. u is a map under A from σ to τ) and $\{\rho_t\}^A$ is the *track under* A (relative homotopy class under A) of a homotopy under A $\rho_t : \rho \simeq^A \eta u$. If we write $(u, \{\rho_t\}^A)$ for the diagram (1.4) then the equivalence relation is given by the formula

$$\left(u, \{\rho_t\}^A\right) \sim \left(u', \{\rho_t + \eta u_t\}^A\right)$$

whenever $u_t : u \simeq^A u'$ is a homotopy under A. Here $+$ refers to the usual track addition of homotopies.

Note that in the case $b = 1_B$, our category $\mathcal{H}_b$ provides a (partially) coherent version of the category of ex-spaces (see also [1]).

2. Vogt's lemma and the characterization of isomorphisms

In order to characterize the isomorphisms in our categories the following result due to Vogt [25] is invaluable.

2.1. LEMMA. *Let $f : X \to Y$, $g : Y \to X$ be homotopy inverses and let $h_t : gf \simeq 1_X$ be a homotopy. Then there exists a homotopy $k_t : fg \simeq 1_Y$ such that $\{fh_t\} = \{k_t f\}$ and $\{h_t g\} = \{gk_t\}$.*

PROOF. By hypothesis there is a homotopy $\psi_t : fg \simeq 1_Y$. The trick is to choose $k_t = \psi_{1-t} fg + fh_t g + \psi_t$. Then the lemma is a consequence of 2-categorical properties of the track composition. For the rectangle

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] \arrow[r, phantom, shift right=5ex, "h_{1-t} \nearrow"] & Y \arrow[rr, equal] \arrow[d, "g"] \arrow[dr, "g"] & & Y \arrow[ddd, equal] \arrow[dl, phantom, "\psi_t \nearrow"] \\
X \arrow[r, equal] \arrow[dd, equal] \arrow[dr, "f"'] & X \arrow[r, equal] \arrow[d, "f"] \arrow[r, phantom, shift right=5ex, "h_t \nearrow"] & X \arrow[d, equal] & \\
& Y \arrow[r, "g"] \arrow[d, equal] \arrow[r, phantom, shift right=5ex, "\psi_{1-t} \nearrow"] & X \arrow[dr, "f"] & \\
X \arrow[r, "f"] & Y \arrow[rr, equal] & & Y
\end{tikzcd}
$$

certainly represents the track $\{fh_{1-t} + k_t f\}$. However, examining it one sees that the tracks can be cancelled so that $\{fh_t\} = \{k_t f\}$. Next, by making various internal cancellations we see that the following two squares represent the same composite track.

$$
\begin{tikzcd}
Y \arrow[rr, equal] \arrow[d, "g"'] \arrow[drr, phantom, "\psi_t \nearrow"] & & Y \arrow[r, "g"] \arrow[d, equal] & X \arrow[dd, "f"] \\
X \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "h_t \nearrow"] & X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "h_{1-t} \nearrow"] & Y \arrow[d, "g"] & \\
Y \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\psi_{1-t} \nearrow"] & X \arrow[r, equal] \arrow[d, "f"] \arrow[dr, phantom, "h_t \nearrow"] & X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "h_{1-t} \nearrow"] & Y \arrow[d, "g"] \\
Y \arrow[r, equal] & Y \arrow[r, "g"] & X \arrow[r, equal] & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, "g"'] \arrow[dr, phantom, "\psi_t \nearrow"] & Y \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\psi_{1-t} \nearrow"] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] \arrow[d, "f"'] & Y \arrow[r, equal] \arrow[d, "g"'] \arrow[dr, phantom, "\psi_t \nearrow"] & Y \arrow[d, equal] \\
Y \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\psi_{1-t} \nearrow"] & X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "g"] \\
Y \arrow[r, equal] & Y \arrow[r, "g"] & X
\end{tikzcd}
$$

However, by cancelling the central square of the left diagram with the square immediately below it, we find that the left diagram represents the track $\{gk_t + h_{1-t}g\}$. Since the right hand square cancels to the trivial track, we have proved that

$$\{gk_t\} = \{h_t g\},$$

as required. □

We now turn to the characterization of isomorphisms in our coherent homotopy categories.

2.2. Proposition. *Let $\{h, h_t\} \in \mathcal{H}_B(f, g)$ be represented by the homotopy commutative square* (1.2). *Then $\{h, h_t\}$ is an isomorphism of $\mathcal{H}_B$ if and only if h is a homotopy equivalence.*

Proof. If $\{h', h'_t\}$ is inverse to $\{h, h_t\}$ in $\mathcal{H}_B$ then certainly h' is a homotopy inverse of h. Conversely, suppose that h is a homotopy equivalence. Then by Vogt's lemma we may choose a homotopy inverse h' of h and homotopies

$$k_t : h'h \simeq 1_X, \qquad k'_t : hh' \simeq 1_E$$

such that $\{hk_t\} = \{k'_t h\}$. Then the diagram

$$
\begin{tikzcd}
E \arrow[r, "h'"] \arrow[d, equal] \arrow[dr, phantom, "k'_{1-t} \nearrow"] & X \arrow[r, equal] \arrow[d, "h"] \arrow[dr, phantom, "h_{1-t} \nearrow"] & X \arrow[dd, "f"] \\
E \arrow[r, equal] \arrow[d, "g"'] & E \arrow[d, "g"'] & \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
$$

represents an element of $\mathcal{H}_B(g, f)$ which is inverse to $\{h, h_t\}$. This can be seen by composing diagrams, observing that in the square

$$\begin{tikzcd}
X \arrow[rrr, equal] \arrow[d, equal] \arrow[dr, phantom, "k_t"] & & & X \arrow[d, equal] \\
X \arrow[r, "h"] \arrow[ddd] \arrow[dr, phantom, "h_t"] & E \arrow[r, "h'"] \arrow[d, equal] \arrow[dr, phantom, "k'_{1-t}"] & X \arrow[r, equal] \arrow[d, "h"] \arrow[dr, phantom, "h_{1-t}"] & X \arrow[ddd, "f"] \\
& E \arrow[r, equal] \arrow[dd, "g"] & E \arrow[dd, "g"] & \\
& & & \\
B \arrow[r, equal] & B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}$$

we may first cancel hk_t and $k'_{1-t}h$, and then h_t and h_{1-t}. □

2.3. COROLLARY. (a) *If $f, f' : X \to B$ are spaces over B such that $f \simeq f'$ then f and f' are isomorphic in $\mathcal{H}_B$.* (b) *Every object in $\mathcal{H}_B$ is isomorphic to a fibration.*

PROOF. (a) Choose a homotopy $h_t : f \simeq f'$ and consider diagram (1.2) with h, E and g replaced by 1_X, X and f'. (b) In the mapping track factorization of $f : X \to B$

$$X \xrightarrow{j} E \xrightarrow{p} B$$

(see, e.g., [7, (5.27)]) j is a homotopy equivalence and p is a fibration. Now consider diagram (1.2) with h and g replaced by j and p, h_t being the constant homotopy. □

REMARK. Proposition 2.2 allows us to relate the track homotopy category over B, $\mathcal{H}_B$, and the homotopy category $Ho(Top_B)$ mentioned in the introduction. By definition $Ho(Top_B)$ is the category of fractions obtained from the category Top_B of spaces over B by formally inverting those maps over B, $h : f \to g$ (see diagram (1.1)) such that $h : X \to E$ is an ordinary homotopy equivalence.

Let $\gamma : Top_B \to \mathcal{H}_B$ be the functor which is the identity on objects and which maps a commutative diagram (1.1) into the class of the corresponding diagram (1.2) with the track of the constant homotopy $f \simeq f$. Then by Proposition 2.2 and the universal property of a category of fractions we obtain an induced functor

$$\bar{\gamma} : Ho(Top_B) \to \mathcal{H}_B$$

which is the identity on objects. It has been proved [16] that $\bar{\gamma}$ is an isomorphism of categories.

In the next proposition we characterize the isomorphisms of $\mathcal{H}^1$.

2.4. PROPOSITION. *Let $\{k, h, h_t\} \in \mathcal{H}^1(f, f')$ be represented by the homotopy commutative square* (1.3). *Then $\{k, h, h_t\}$ is an isomorphism of $\mathcal{H}^1$ if and only if k and h are homotopy equivalences.*

PROOF. If $\{k', h', h'_t\}$ is inverse to $\{k, h, h_t\}$ in $\mathcal{H}^1$ then certainly k' and h' are homotopy inverses of k and h respectively. Conversely, suppose that k and h are homotopy equivalences. Then by Vogt's lemma we may choose homotopy inverses k' and h' of k and h respectively and homotopies

$$\phi_t : kk' \simeq 1_{Y'}, \quad \phi' : k'k \simeq 1_Y, \quad \psi_t : hh' \simeq 1_{X'}, \quad \psi_t : h'h \simeq 1_X$$

such that $\{\phi_t k\} = \{k\phi'_t\}$ and $\{\psi_t h\} = \{h\psi'_t\}$. Then an inverse for $\{k, h, h_t\}$ is given by the square

$$\begin{tikzcd}
X' \arrow[r, "h'"] \arrow[d, equal] \arrow[dr, phantom, "\psi_{1-t}"] & X \arrow[r, equal] \arrow[d, "h"] \arrow[dr, phantom, "h_{1-t}"] & X \arrow[r, equal] \arrow[d, "f"] & X \arrow[d, "f"] \\
X' \arrow[r, equal] \arrow[d, "f'"'] & X' \arrow[d, "f'"'] & Y \arrow[r, equal] \arrow[d, "k"'] \arrow[dr, phantom, "\phi'_t"] & Y \arrow[d, equal] \\
Y' \arrow[r, equal] & Y' \arrow[r, equal] & Y' \arrow[r, "k'"'] & Y
\end{tikzcd},$$

as can be seen by composing diagrams and cancelling tracks appropriately in an argument similar to that given in the proof of Proposition 2.2. □

2.5. COROLLARY. (a) *If $f, f' : X \to Y$ are maps such that $f \simeq f'$, then f and f' are isomorphic as objects of $\mathcal{H}^1$.* (b) *Every object f in $\mathcal{H}^1$ is isomorphic to a fibration.* (c) *Every object f in $\mathcal{H}^1$ is isomorphic to a cofibration.*

The proof is similar to that given for Corollary 2.3, except that for part (c) we have to use the mapping cylinder factorization of f into a cofibration followed by a homotopy equivalence (see, e.g., [7, (1.27)]).

Finally the isomorphisms in $\mathcal{H}_b$ are characterized as follows.

2.6. PROPOSITION. *An arrow $\{u, \rho_t\}$ of $\mathcal{H}_b$ represented by diagram* (1.4) *is an isomorphism in $\mathcal{H}_b$ if and only if u, viewed as a map under A, $u : \sigma \to \tau$, is a homotopy equivalence under A (i.e. u represents an isomorphism of $Top^A h$).*

REMARK. For a general investigation of homotopy equivalences in 2-categories we refer to Marcum [21].

3. Coherent homotopy over B and fibrewise homotopy

In this section we describe the relation between our coherent homotopy categories and the corresponding 'rigid' homotopy categories based on strictly commutative diagrams.

Let B be a topological space. The functor $\gamma : Top_B \to \mathcal{H}_B$ described in Section 2 induces a functor

$$\Theta : Top_B h \to \mathcal{H}_B.$$

If we denote the set of morphisms in $Top_B h$ from f to g by $[f,g]_B$ and write $[h]_B$ for the element of $[f,g]_B$ represented by (1.1), then Θ is given by the formula

$$\Theta[h]_B = \{h, f\},$$

where f denotes also the (track of) the constant homotopy $f \simeq f$.

3.1. PROPOSITION. *If $g : E \to B$ is a fibration, then, for any space over B, f, the map $\Theta : [f,g]_B \to \mathcal{H}_B(f,g)$ is a bijection.*

For the proof we need the following fundamental lemma.

3.2. LEMMA. *In the diagram*

$$\begin{tikzcd} X \arrow[rr, shift left, "h"] \arrow[rr, shift right, "h'"'] \arrow[dr, "f"'] & & E \arrow[dl, "g"] \\ & B & \end{tikzcd},$$

let $h, h' : f \to g$ be maps over B. Suppose g is a fibration. Then if $h \simeq h'$ via a homotopy $h_t : h \simeq h'$ such that $\{gh_t\} = \{f\}$ it follows that $h \simeq_B h'$.

PROOF. Let $H_{t,s} : X \times I \times I \to B$ be a homotopy of homotopies such that

$$H_{t,0} = gh_t, \quad H_{t,1} = H_{0,s} = H_{1,s} = f.$$

Since g is a fibration we can lift $H_{t,s}$ to a homotopy

$$K_{t,s} : X \times I \times I \to E$$

such that $K_{t,0} = h_t$. Then we have homotopies over B, $K_{0,s} : h \simeq_B K_{0,1}$, $K_{t,1} : K_{0,1} \simeq_B K_{1,1}$, $K_{1,1-s} : K_{1,1} \simeq_B h'$. It follows that $h \simeq_B h'$. □

PROOF OF 3.1. Let $\{h, h_t\} \in \mathcal{H}_B(f,g)$ be represented by the square (1.2). Since g is a fibration there exists a homotopy $h'_t : h'_0 \simeq h$ such that $gh'_t = h_t$. Then $[h'_0]_B \in [f,g]_B$ and we have

$$\Theta[h'_0]_B = \{h'_0, f\} = \{h, h_t\}$$

by definition of the relation $\sim$ in Section 1. This proves that Θ is surjective. To see that Θ is injective let $[h]_B, [h']_B \in [f,g]_B$ and suppose $\Theta[h]_B = \Theta[h']_B$. Then by definition of $\sim$ there exists a homotopy $h_t : h \simeq h'$ such that $\{gh_t\} = \{f\}$. Since g is a fibration, we can apply Lemma 3.2 and obtain $[h]_B = [h']_B$. □

REMARK. Both Lemma 3.2 and Proposition 3.1 hold under the weaker assumption that g is an h-fibration (see [7, (6.4)]).

As a corollary to (2.2) and (3.1) we obtain a proof of a theorem of Dold ([8, 6.1]) separating arguments involving homotopy equivalences from arguments using fibrations in a transparent way.

3.3. COROLLARY. *Let*

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

be a commutative diagram of maps and suppose that p and p' are fibrations. Then if h is a homotopy equivalence, h is a fibre homotopy equivalence (i.e. $[h]_B$ is an isomorphism of Top_Bh).

PROOF. We have to prove that the set maps

$$[h]_{B*} : [p,p]_B \to [p,p']_B, \quad [h]_{B*} : [p',p]_B \to [p',p']_B$$

induced by composition of maps over B are bijective. In the case of the first we may argue that in the commutative diagram of sets

$$\begin{tikzcd} {[p,p]_B} \arrow[r, "{[h]_{B*}}"] \arrow[d, "\Theta"'] & {[p,p']_B} \arrow[d, "\Theta"] \\ \mathcal{H}_B(p,p) \arrow[r, "{\{h,p\}_*}"'] & \mathcal{H}_B(p,p') \end{tikzcd},$$

where the bottom map is induced by composition in $\mathcal{H}_B$, the other three arrows are bijections. The proof of the second bijection is similar. □

Finally, by Corollary 2.3(b) and Proposition 3.1 we obtain:

3.4. COROLLARY. *Let $\mathcal{F}_Bh$ denote the full subcategory of Top_Bh whose objects are all spaces over B which are fibrations. Then the functor $\Theta : Top_Bh \to \mathcal{H}_B$ restricts to an equivalence of categories $\mathcal{F}_Bh \to \mathcal{H}_B$.*

The rigid analogue of the category $\mathcal{H}^1$ is the classical *pair-homotopy category Top^1h*. The assignment which sends a commutative square of maps to the corresponding homotopy commutative square (endowed with the track of the constant homotopy) induces a functor

$$\Theta : Top^1h \to \mathcal{H}^1$$

which is the identity on objects. We list some properties of Θ obtained by arguments similar to those given above.

3.5. PROPOSITION. *$\Theta : Top^1 h(f, f') \to \mathcal{H}^1(f, f')$ is a bijection if either f' is a fibration or f is a cofibration.*

3.6. COROLLARY. *Let $Top^1_{\mathcal{F}} h$ (resp. $Top^1_{\mathcal{C}} h$) denote the full subcategory of $Top^1 h$ whose objects are all fibrations (resp. cofibrations). Then the functor Θ restricts to an equivalence of categories $Top^1_{\mathcal{F}} h \to \mathcal{H}^1$ (resp. $Top^1_{\mathcal{C}} h \to \mathcal{H}^1$).*

If $b : A \to B$ is a map and

$$A \xrightarrow{\sigma} X \xrightarrow{\rho} B, \quad A \xrightarrow{\tau} X \xrightarrow{\eta} B,$$

are objects of $\mathcal{H}_b$ (so that $\rho\sigma = \eta\tau = b$), then we have a canonical map

$$\Theta : \left[(\sigma, \rho), (\tau, \eta)\right]^A_B \to \mathcal{H}_b\big((\sigma, \rho), (\tau, \eta)\big), \tag{3.7}$$

where $[(\sigma, \rho), (\tau, \eta)]^A_B$ denotes the classical homotopy set under A and over B (see [7, (0.26)]). As before Θ is induced by an assignment which views a commutative diagram of the form

$$\begin{tikzcd} & A \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ X \arrow[rr, "u"] \arrow[dr, "\rho"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd} \tag{3.8}$$

as a homotopy commutative diagram. The following result corresponds to 3.1 and 3.5.

3.9. PROPOSITION. *If $(\sigma \times 1_{[0,1]}, \eta)$, hence also (σ, η) has the CHEP (covering homotopy extension property), then the map Θ in (3.7) is a bijection.*

Recall that a pair of maps (σ, η), $\sigma : A \to X, \eta : Y \to B$, is said to have the CHEP if for any map $f : X \to Y$ and any pair of homotopies (ϕ_t, ψ_t) such that $\phi_t : f\sigma \simeq \phi_1$, $\psi_t : \eta f \simeq \psi_1$ and $\eta\phi_t = \psi_t\sigma$ there exists a homotopy $\Phi_t : f \simeq \Phi_1$ such that $\Phi_t\sigma = \phi_t$ and $\eta\Phi_t = \psi_t$.

Note that by a result of Strøm ([24, Theorem 4]) a pair (σ, η) has the CHEP if σ is a closed cofibration and η is a fibration. Further examples are listed in [3, 2.2].

Combining Proposition 2.6, the Eckmann–Hilton dual of Corollary 3.3 and Proposition 3.9, the method of proof of Corollary 3.3 leads to the following comparison (bridging) theorem.

3.10. COROLLARY. *Let (3.8) be a commutative diagram in Top such that σ, τ are closed cofibrations and ρ, η are fibrations. Then if u is a homotopy equivalence, u is a homotopy equivalence under A and over B.*

4. Model categorical aspects

In this section we show how certain model categorical properties (cf. Quillen [23], K.S. Brown [4], Baues [2]) of topological spaces arise in our theory in a natural and transparent way.

4.1. PROPOSITION. *Let*

$$\begin{tikzcd} E' \arrow[r, "\beta"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "\alpha"'] & B \end{tikzcd} \tag{4.2}$$

be a pullback with p a fibration. Then (a) *if p is a homotopy equivalence, so is p';* (b) *if α is a homotopy equivalence, so is β.*

Baues [2, Chapter 1], shows that 4.1(a) and 4.1(b) are equivalent in the presence of his axioms (F1) and (F3). We shall see that 4.1(a) is a consequence of Dold's theorem 3.3 whereas 4.1(b) follows from Vogt's lemma 2.1. (An alternative proof of 4.1(b) based on homotopy pullback theory will be given in Section 5.)

Let $\alpha : B' \to B$ be a map. Then by choosing a pullback $p' = \alpha^*(p) : E' \to B'$ for each object $p : E \to B$ of Top_B we obtain a functor

$$\alpha^* : Top_B \to Top_{B'}.$$

Note that α^* transforms a homotopy over B into a homotopy over B'. Hence there is an induced functor $\alpha^* : Top_Bh \to Top_{B'}h$. Consider the diagram

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "p"'] & & B \arrow[dl, "1_B"] \\ & B & \end{tikzcd}$$

If p is a homotopy equivalence and a fibration, then by Corollary 3.3, $p : p \to 1_B$ is a fibre homotopy equivalence, i.e. $[p]_B$ is an isomorphism of Top_Bh. It follows that in diagram (4.2) p' is a fibre homotopy equivalence, in particular a homotopy equivalence.

PROOF OF 4.1(b). Let $\alpha' : B \to B'$ be a homotopy inverse of α. By Vogt's lemma there are homotopies

$$\phi_t : 1_B \simeq \alpha\alpha', \quad \phi'_t : 1_{B'} \simeq \alpha'\alpha$$

such that $\{\alpha\phi'_t\} = \{\phi_t\alpha\}$. Since p and hence p' are fibrations, there are homotopies

$$\psi_t : E \times I \to E, \quad \psi'_t : E' \times I \to E'$$

such that $\psi_0 = 1_E$, $\psi_0' = 1_{E'}$, $p\psi_t = \phi_t p$, $p'\psi_t' = \phi_t' p'$. Since $p\psi_1 = \phi_1 p = \alpha\alpha' p$, by the pullback property, we have an induced map

$$\beta : E \to E'$$

such that $\beta\beta' = \psi_1$ and $p'\beta' = \alpha' p$. We claim that β' is a homotopy inverse of β. Since $\beta\beta' = \psi_1 \simeq \psi_0 = 1_E$ it remains to show that $\beta'\beta \simeq 1_{E'}$. We compare $\beta\beta'\beta$ and $\beta\psi_1'$. We observe that

$$p\beta\beta'\beta = \alpha p'\beta'\beta = \alpha\alpha' p\beta = \alpha\alpha'\alpha p' = \alpha\phi_1' p' = \alpha p'\psi_1' = p\beta\psi_1'.$$

Hence $\beta\beta'\beta$ and $\beta\psi_1'$ can be viewed as maps over B. Choose a homotopy

$$\gamma_t \in \{\psi_{1-t}\beta + \beta\psi_t'\}, \qquad \gamma_t : \beta\beta'\beta \simeq \psi_1'.$$

Then we have

$$\begin{aligned}\{p\gamma_t\} &= \{p\psi_{1-t}\beta + p\beta\psi_t'\} = \{\phi_{1-t}p\beta + \alpha p'\psi_t'\} = \{\phi_{1-t}\alpha p' + \alpha\phi_t' p'\}\\ &= \{\alpha\alpha'\alpha p'\}.\end{aligned}$$

Since p is a fibration, it follows by Lemma 3.2 that $\beta\beta'\beta \simeq_B \beta\psi_1'$. We choose a homotopy over B, $\rho_t : \beta\beta'\beta \simeq_B \beta\psi_1'$. By the pullback property we obtain a homotopy over B', $\rho_t' : \beta'\beta \simeq_{B'} \psi_1'$, hence $\beta\beta' \simeq \psi_1' \simeq \psi_0' = 1_{E'}$, as required. □

5. Homotopy pullback

In this section we explain how homotopy pullbacks can be interpreted as products in the category $\mathcal{H}_B$. As a corollary we obtain the homotopy theorem for fibrations (see [7, (7.22)]). The notion of a homotopy pullback has been defined in Mather [22]. We slightly reformulate the definition in terms of homotopy commutative squares.

5.1. Definition. A homotopy commutative square

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\{g_t\}"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & B \end{tikzcd} , \qquad (5.2)$$

where $g_t : g\beta \simeq f\alpha$ is called a homotopy pullback if the following holds:

(HPB 1) If

$$\begin{tikzcd} Z \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\{h_t\}"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & B \end{tikzcd} \qquad (5.3)$$

is another homotopy commutative square where $h_t : gv \simeq fu$, then there exists a map $\phi : Z \to P$ and a decomposition of (5.3)

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, equal] & Z \arrow[r, "u"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\{k_t\}"] & X \arrow[d, equal] \\ Z \arrow[r, "\phi"] \arrow[d, "v"'] \arrow[dr, phantom, "\{j_t\}"] & P \arrow[r, "\alpha"] \arrow[d, "\beta"] \arrow[dr, phantom, "\{g_t\}"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, "g"'] & B \end{tikzcd} \tag{5.4}$$

(i.e. the track represented by (5.4) is equal to $\{h_t\}$).

(HPB 2) If we have another decomposition (5.4′) of (5.3) with a map $\phi' : Z \to P$ and homotopies $k'_t : \alpha\phi' \simeq u$, $j'_t : v \simeq \beta\phi'$, then there is a homotopy

$$\phi_t : \phi \simeq \phi'$$

such that $\{\alpha\phi_{1-t} + k_t\} = \{k'_t\}$ and $\{j_t + \beta\phi_t\} = \{j'_t\}$.

Any homotopy commutative square (5.2) gives rise to a diagram

$$\begin{tikzcd} Y \arrow[d, "g"'] & P \arrow[l, "\beta"'] \arrow[r, "\alpha"] \arrow[d, "h"'] \arrow[dr, phantom, "\{g_t\}"] & X \arrow[d, "f"] \\ B \arrow[r, equal] & B \arrow[r, equal] & B, \end{tikzcd}$$

where h is the composite $g\beta$, representing a diagram in $\mathcal{H}_B$:

$$g \xleftarrow{\pi_g} h \xrightarrow{\pi_f} f \ , \tag{5.5}$$

where $\pi_g = \{\beta, h\}$, $\pi_f = \{\alpha, g_t\}$. Then the relevant observation is

5.6. Proposition. *If* (5.2) *is a homotopy pullback, then* (5.5) *is a product diagram of f and g in the category $\mathcal{H}_B$.*

The proof of Proposition 5.6 is straightforward but lengthy (see [13]).

Now let $p : E \to B$ be a fibration and let $\alpha, \beta : A \to B$ be maps which are homotopic, $\alpha \simeq \beta$. Suppose that

$$\begin{tikzcd} D \arrow[r, "\alpha'"] \arrow[d, "\alpha^*(p)"'] & E \arrow[d, "p"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\beta'"] \arrow[d, "\beta^*(p)"'] & E \arrow[d, "p"] \\ A \arrow[r, "\beta"'] & B \end{tikzcd} \tag{5.7}$$

are pullbacks. We want to compare the induced fibrations $\alpha^*(p)$ and $\beta^*(p)$. By Corollary 2.3(a), α and β are isomorphic objects of $\mathcal{H}_B$. (If $h_t : \alpha \simeq \beta$ is a homotopy, then $\{1_A, h_t\} : \alpha \to \beta$ is an isomorphism.) Since p is a fibration, the diagrams (5.7) are homotopy pullbacks ([22, Lemma 19]). Thus Proposition 5.6 and the uniqueness property of a product in a category together with Proposition 2.2 allows one to deduce the existence of a homotopy equivalence $u : D \to F$ such that the diagram

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[d, "\alpha^*(p)"'] & F \arrow[d, "\beta^*(p)"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

is homotopy commutative. It follows by Proposition 2.2 that $\alpha^*(p)$ and $\beta^*(p)$ are isomorphic in $\mathcal{H}_A$. Hence by Proposition 3.1, they are isomorphic in $Top_A h$, proving the homotopy theorem for fibrations.

Proposition 5.6 can also be applied to give an alternative proof of Proposition 4.1(b). Since p is a fibration the diagram (4.2) is a homotopy pullback. So then

$$p\beta = \alpha \times p \approx 1_B \times p \approx p$$

in $\mathcal{H}_B$. But this implies that β is a homotopy equivalence.

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CHAPTER 6

Modern Foundations for Stable Homotopy Theory

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Contents

HANDBOOK OF ALGEBRAIC TOPOLOGY
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0. Introduction

It is a truism that algebraic topology is a very young subject. In some of its most fundamental branches, the foundations have not yet reached a state of shared consensus. Our theme will be stable homotopy theory and an emerging consensus on what its foundations should be. The consensus is different than would have been the case as recently as a decade ago. We shall illustrate the force of the change of paradigm with new constructions of some of the most basic objects in modern algebraic topology, namely the various spectra and cohomology theories that can be derived from complex cobordism. The two following articles will give introductions to completions in stable homotopy theory and to equivariant stable homotopy theory. The three papers have a common theme: the relationship between commutative algebra and stable homotopy theory, both relations of analogy and relations of application.

Stable homotopy theory began around 1937 with the Freudenthal suspension theorem. In simplest terms, it states that, if q is small relative to n, then $\pi_{n+q}(S^n)$ is independent of n. Stable phenomena had of course appeared earlier, at least implicitly: reduced homology and cohomology are examples of functors that are invariant under suspension without limitation on dimension. Stable homotopy theory emerged as a distinct branch of algebraic topology with Adams' introduction of his eponymous spectral sequence and his spectacular conceptual use of the notion of stable phenomena in his solution to the Hopf invariant one problem. Its centrality was reinforced by two related developments that occurred at very nearly the same time, in the late 1950's. One was the introduction of generalized homology and cohomology theories and especially K-theory, by Atiyah and Hirzebruch. The other was the work of Thom which showed how to reduce the problem of classifying manifolds up to cobordism to a problem, more importantly, a solvable problem, in stable homotopy theory.

The reduction of geometric phenomena to solvable problems in stable homotopy theory has remained an important mathematical theme, the most recent major success being Stolz's use of Spin cobordism to study the classification of manifolds with positive scalar curvature. In an entirely different direction, the early 1970's saw Quillen's introduction of higher algebraic K-theory and the recognition by Segal and others that it could be viewed as a construction in stable homotopy theory. With algebraic K-theory as an intermediary, there has been a growing volume of work that relates algebraic geometry to stable homotopy theory. With Waldhausen's introduction of the algebraic K-theory of spaces in the late 1970's, stable homotopy became a bridge between algebraic K-theory and the study of diffeomorphisms of manifolds. Within algebraic topology, the study of stable homotopy theory has been and remains the focus of much of the best work in the subject. The study of nilpotence and periodic phenomena by Hopkins, Mahowald, Ravenel, and many others has been especially successful.

We shall focus on the study of structured ring, module, and algebra spectra. This study plays a significant role in all of the directions of work that we have just mentioned and would have been technically impossible within the foundational consensus that existed a decade ago.

Stable homotopy theory demands a category in which to work. One could set up the ordinary Adams spectral sequence ad hoc, as Adams did, but it would be ugly at best to set up the Adams spectral sequence based on a generalized homology theory that way. One wants objects – called spectra – that play the role of spaces in unstable homotopy theory, and one wants a category in which all of the usual constructions on spaces are present and, up to homotopy, the suspension functor is an equivalence. At this point, we introduce a sharp distinction: there is a category of point-set level objects, and there is an associated derived category. There has been consensus on what the latter should be, up to equivalence of categories, since the fundamental work of Boardman in the 1960's. The change in paradigm concerns the point-set level category that underlies the stable homotopy category. There is a growing recognition that one needs a good point-set level category in order to study stable topological algebra seriously.

There is an analogy with algebra that is fundamental to an understanding of this area of mathematics. Suppose given a (discrete) commutative ring R. It has an associated category $\mathscr{M}_R$ of ($\mathbb{Z}$-graded) chain complexes, there is a notion of homotopy between maps of chain complexes, and there is a resulting homotopy category $h\mathscr{M}_R$. However, this is not the category that algebraists are interested in. For example, if R-modules M and N are regarded as chain complexes concentrated in degree zero, then, in the derived category, the homology of their tensor product should be their torsion product $\mathrm{Tor}^R_*(M,N)$. Formally, the fundamental invariants of chain complexes are their homology groups, and one constructs a category that reflects this. A map of chain complexes is said to be a quasi-isomorphism if it induces an isomorphism of homology groups. The derived category $\mathscr{D}_R$ is obtained by adjoining formal inverses to the quasi-isomorphisms. The best way to make this rigorous is to introduce a notion of cell R-module such that every quasi-isomorphism between cell R-modules is a chain homotopy equivalence (Whitehead theorem) and every chain complex is quasi-isomorphic to a cell R-module. Then $\mathscr{D}_R$ is equivalent to the ordinary homotopy category of cell R-modules. See [15], [21]. This is a topologist's way of thinking about the appropriate generalization to chain complexes of projective resolutions of modules.

We think of the sphere spectrum S as the analog of R. We think of spectra as analogs of chain complexes, or rather as a first approximation to the definitive analogs, which will be S-modules. We let $\mathscr{S}$ denote the category of spectra. There is a notion of homotopy of maps between spectra, and there is a resulting homotopy category $h\mathscr{S}$. The fundamental invariants of spectra are their homotopy groups, and a map of spectra is a weak equivalence if it induces an isomorphism of homotopy groups. The stable homotopy category, which we denote by $\bar{h}\mathscr{S}$, is obtained by formally inverting the weak equivalences. This is made rigorous by introducing CW spectra. A weak equivalence between CW spectra is a homotopy equivalence and every spectrum is weakly equivalent to a CW spectrum. Then $\bar{h}\mathscr{S}$ is equivalent to the ordinary homotopy category of CW spectra.

Now the category $\mathscr{M}_R$ has an associative and commutative tensor product. If we regard R as a chain complex concentrated in degree zero, then R is a unit for the tensor product. A differential R-algebra A is a chain complex with a unit $R \longrightarrow A$ and product $A \otimes_R A \longrightarrow A$ such that the evident associativity and unity diagrams commute. It is commutative if the evident commutativity diagram also commutes. These are, obviously enough, point-set level structures. Algebraists would have trouble taking seriously the idea of an algebra defined in $\mathscr{D}_R$, with unit and product only defined in that category.

The category $\mathscr{S}$ has a smash product but, in contrast with the tensor product, it is not associative, commutative, or unital. The induced smash product on the stable homotopy category $\bar{h}\mathscr{S}$ is associative and commutative, and it has S as unit. Topologists routinely study ring spectra, which are objects E of $\bar{h}\mathscr{S}$ with a unit $\eta : S \longrightarrow E$ and product $\phi : E \wedge E \longrightarrow E$ such that the evident unit diagrams commute; that is, $\phi \circ (\eta \wedge \mathrm{id}) = \mathrm{id} = \phi \circ (\mathrm{id} \wedge \eta)$ in $\bar{h}\mathscr{S}$. Similarly, E is associative or commutative if the appropriate diagrams commute in $\bar{h}\mathscr{S}$. Given that the point-set level smash product is not associative or commutative, it would seem at first sight that these up to homotopy notions are the only ones possible.

It is a recent discovery that there is a category $\mathscr{M}_S$ of S-modules that has an associative, commutative, and unital smash product $\wedge_S$ [11]. Its objects are spectra with additional structure, and we say that a map of S-modules is a weak equivalence if it is a weak equivalence as a map of spectra. The derived category $\mathscr{D}_S$ is obtained from $\mathscr{M}_S$ by formally inverting the weak equivalences, and $\mathscr{D}_S$ is equivalent to the stable homotopy category $\bar{h}\mathscr{S}$. Again, this is made rigorous by a theory of CW S-modules that is just like the theory of CW spectra.

In the category $\mathscr{M}_S$, we have a point-set level notion of an S-algebra R that is defined in terms of maps $\eta : S \longrightarrow R$ and $\phi : R \wedge_S R \longrightarrow R$ in $\mathscr{M}_S$ such that the standard unit and associativity diagrams commute on the point-set level; we say that R is commutative if the standard commutativity diagram also commutes. There were earlier notions with a similar flavor, namely the A_∞ and E_∞ ring spectra introduced in [19], [20]. Here "A_∞" stands historically for "associative up to an infinite sequence of higher homotopies"; similarly, "E_∞" stands for "homotopy everything", meaning that the product is associative and commutative up to all higher coherence homotopies. With the definitions just given, the higher homotopies are hidden in the definition of the associative and commutative smash product in $\mathscr{M}_S$, but these definitions are essentially equivalent to the earlier ones, in which the higher homotopies were exhibited in terms of an "operad action". It is tempting to simply call these objects associative and commutative ring spectra, but that would be a mistake. These terms have long established meanings, as associative and commutative rings in the stable homotopy category, and the more precise point-set level notions do not make the older notions obsolete: there are plenty of examples of associative or commutative ring spectra that do not admit structures of A_∞ or E_∞ ring spectra. It is part of the new paradigm that one must always be aware of when one is working in the derived category and when one is working on the point-set level.

Now fix an S-algebra R. An R-module M is an S-module together with a map $\mu : R \wedge_S M \longrightarrow M$ such that the evident unit and transitivity diagrams commute. Let $\mathscr{M}_R$ be the category of R-modules. Again we have a homotopy category $h\mathscr{M}_R$ and a derived category $\mathscr{D}_R$ that is obtained from it by inverting the weak equivalences, by which we mean the maps of R-modules that are weak equivalences of underlying spectra. The construction of $\mathscr{D}_R$ is made rigorous by a theory of cell R-modules, the one slight catch being that, unless R is connective, in the sense that its homotopy groups are zero in negative degrees, we cannot insist that cells be attached only to cells of lower dimension, so that our cell R-modules cannot be restricted to be CW R-modules. These categories enjoy all of the good properties that we have described in the special case

$R = S$. There is an associative, commutative, and unital smash product over R. We can therefore go on to define R-algebras and commutative R-algebras A in terms of point-set level associative, unital, and commutative multiplications $A \wedge_R A \longrightarrow A$. We can also define derived category level associative and commutative R-ring spectra A, exactly like the classical associative and commutative ring spectra in the stable homotopy category.

It is the derived category $\mathscr{D}_R$ that we wish to focus on in describing the current state of the art in stable homotopy theory. We can mimic classical commutative algebra in this category. In particular, for an ideal I and multiplicatively closed subset Y in the coefficient ring $R_* = \pi_*(R)$, we will show how to construct quotients M/IM and localizations $M[Y^{-1}]$. When applied with R taken to be the representing spectrum MU for complex cobordism, these constructions specialize to give simple constructions of various spectra that are central to modern stable homotopy theory, such as the Morava K-theory spectra. Moreover, we shall see that these spectra are MU-ring spectra.

This account is largely a summary of parts of the more complete and technical paper [11], to which the reader is referred for further background, detailed proofs, and many more applications.

1. Spectra and the stable homotopy category

We here give a bare bones summary of the construction of the stable homotopy category, referring to [16] and [11] for details and to [22] for a more leisurely exposition. We aim to give just enough of the basic definitional framework that the reader can feel comfortable with the ideas.

By Brown's representability theorem [6], if E^* is a reduced cohomology theory on based spaces, then there are CW complexes E_n such that, for CW complexes X, $E^n(X)$ is naturally isomorphic to the set $[X, E_n]$ of homotopy classes of based maps $X \longrightarrow E_n$. The suspension isomorphism $E^n(X) \cong E^{n+1}(\Sigma X)$ gives rise to a homotopy equivalence $\tilde{\sigma}_n : E_n \longrightarrow \Omega E_{n+1}$. The object $E = \{E_n, \tilde{\sigma}_n\}$ is called an Ω-spectrum. A map $f : E \longrightarrow E'$ of Ω-spectra is a sequence of maps $f_n : E_n \longrightarrow E'_n$ that are compatible up to homotopy with the equivalences $\tilde{\sigma}_n$ and $\tilde{\sigma}'_n$. The category of Ω-spectra is equivalent to the category of cohomology theories on based spaces and can be thought of as an intuitive first approximation to the stable homotopy category. However, this category does not have a usable theory of cofibration sequences and is not suitable for either point-set level or homotopical work. For that, one needs more precise objects and morphisms that are defined without use of homotopies but that still represent cohomology theories and their maps. More subtly, one needs a coordinate-free setting in order to define smash products sensibly. The nth space E_n relates to the n-sphere and thus to $\mathbb{R}^n$. Restricting to spaces E_n is very much like restricting to the standard basis of $\mathbb{R}^\infty$ when doing linear algebra.

A coordinate-free spectrum is indexed on the set of finite dimensional subspaces V of a "universe" U, namely a real inner product space isomorphic to the sum $\mathbb{R}^\infty$ of countably many copies of $\mathbb{R}$. In detail, writing $W - V$ for the orthogonal complement of V in W, a spectrum E assigns a based space EV to each finite dimensional subspace

V of U, with (adjoint) structure maps

$$\tilde{\sigma}_{V,W} : EV \xrightarrow{\cong} \Omega^{W-V} EW$$

when $V \subset W$, where $\Omega^W X$ is the function space $F(S^W, X)$ of based maps $S^W \longrightarrow X$ and S^W is the one-point compactification of W. The structure maps are required to satisfy an evident transitivity relation when $V \subset W \subset Z$, and they are required to be *homeomorphisms*. A map of spectra $f : E \rightarrow E'$ is a collection of maps of based spaces $f_V : EV \rightarrow E'V$ for which each of the following diagrams commutes:

$$\begin{tikzcd} EV \arrow[r, "f_V"] \arrow[d, "\tilde{\sigma}_{V,W}"'] & E'V \arrow[d, "\tilde{\sigma}'_{V,W}"] \\ \Omega^{W-V} EW \arrow[r, "\Omega^{W-V} f_W"] & \Omega^{W-V} E'W \end{tikzcd} .$$

We obtain the category $\mathscr{S} = \mathscr{S}U$ of spectra indexed on U. We obtain an equivalent category if we restrict to any cofinal family of indexing spaces. If we drop the requirement that the maps $\tilde{\sigma}_{V,W}$ be homeomorphisms, we obtain the notion of a prespectrum and the category $\mathscr{P} = \mathscr{P}U$ of prespectra indexed on U. The forgetful functor $\ell : \mathscr{S} \longrightarrow \mathscr{P}$ has a left adjoint L. When the structure maps $\tilde{\sigma}$ are inclusions, $(LE)(V)$ is just the union of the spaces $\Omega^{W-V} EW$ for $V \subset W$. We write $\sigma : \Sigma^{W-V} EV \longrightarrow EW$ for the adjoints of the maps $\tilde{\sigma}$, where $\Sigma^V X = X \wedge S^V$.

EXAMPLES 1.1. Let X be a based space. The suspension prespectrum $\Pi^\infty X$ is the prespectrum whose Vth space is $\Sigma^V X$; the structure maps σ are the evident identifications $\Sigma^{W-V} \Sigma^V X \cong \Sigma^W X$. The suspension spectrum of X is $\Sigma^\infty X = L\Pi^\infty X$. Let $QX = \cup \Omega^V \Sigma^V X$, where the union is taken over the inclusions obtained from the adjoints of the cited identifications. Then $(\Sigma^\infty X)(V) = Q(\Sigma^V X)$. The functor Σ^∞ from based spaces to spectra is left adjoint to the functor that assigns the zeroth space $E_0 = E(\{0\})$ to a spectrum E. More generally, for a fixed subspace $Z \subset U$, define $\Pi^\infty_Z X$ to be the analogous prespectrum whose Vth space is $\Sigma^{V-Z} X$ if $Z \subset V$ and a point otherwise and define $\Sigma^\infty_Z X = L\Pi^\infty_Z X$. Then Σ^∞_Z is left adjoint to the functor that sends a spectrum to its Zth space EZ; these functors are generally called "shift desuspensions".

Functors on prespectra that do not preserve spectra are extended to spectra by applying the functor L. For example, for a based space X and a prespectrum E, we have the prespectrum $E \wedge X$ specified by $(E \wedge X)(V) = EV \wedge X$. When E is a spectrum, the structure maps for this prespectrum level smash product are not homeomorphisms, and we understand the smash product $E \wedge X$ to be the spectrum $L(\ell E \wedge X)$. Function spectra are easier. We set $F(X, E)(V) = F(X, EV)$ and find that this functor on prespectra preserves spectra. If we topologize the set $\mathscr{S}(E, E')$ as a subspace of the product over V of the function spaces $F(EV, E'V)$ and let $\mathscr{T}$ be the category of based spaces with sets of maps topologized as function spaces, then there result homeomorphisms

$$\mathscr{S}(E \wedge X, E') \cong \mathscr{T}(X, \mathscr{S}(E, E')) \cong \mathscr{S}(E, F(X, E')).$$

Recall that a category is said to be cocomplete if it has all colimits and complete if it has all limits.

PROPOSITION 1.2. *The category $\mathscr{S}$ is complete and cocomplete.*

PROOF. Limits and colimits are defined on prespectra spacewise. Limits preserve spectra, and colimits of spectra are obtained by use of the left adjoint L. □

We write Y_+ for the union of a space Y and a disjoint basepoint. A homotopy in the category of spectra is a map $E \wedge I_+ \longrightarrow E'$. We have cofibration and fibration sequences that are defined exactly as on the space level (e.g., [29]) and enjoy the same homotopical properties. Let $[E, E']$ denote the set of homotopy classes of maps $E \longrightarrow E'$; we shall later understand that, when using this notation, E must be of the homotopy type of a CW spectrum. For based spaces X and Y with X compact, we have

$$[\Sigma^\infty X, \Sigma^\infty Y] \cong \operatorname*{colim}_n [\Sigma^n X, \Sigma^n Y].$$

Fix a copy of $\mathbb{R}^\infty$ in U. In the equivariant generalization of the present theory, it is essential not to insist that $\mathbb{R}^\infty$ be all of U, but the reader may take $U = \mathbb{R}^\infty$ here. We write $\Sigma_n^\infty = \Sigma_{\mathbb{R}^n}^\infty$. For $n \geqslant 0$, the sphere spectrum S^n is $\Sigma^\infty S^n$. For $n > 0$, the sphere spectrum S^{-n} is $\Sigma_n^\infty S^0$. We write S for the zero sphere spectrum. The nth homotopy group of a spectrum E is the set $[S^n, E]$ of homotopy classes of maps $S^n \longrightarrow E$, and this fixes the notion of a weak equivalence of spectra. The adjunctions of Examples 1.1 make it clear that a map f of spectra is a weak equivalence if and only if each of its component maps f_Z is a weak equivalence of spaces. The stable homotopy category $\bar{h}\mathscr{S}$ is constructed from the homotopy category of spectra by adjoining formal inverses to the weak equivalences, a process that is made rigorous by CW approximation.

The theory of CW spectra is developed by taking sphere spectra as the domains of attaching maps of cells $CS^n = S^n \wedge I$ [16, I§5]. The one major difference from the space level theory of CW complexes is that we have to construct CW spectra as unions $E = \cup E_n$, where E_0 is the trivial spectrum and where we are allowed to attach cells of arbitrary dimension when constructing E_{n+1} from E_n. There results a notion of a cell spectrum. We define a CW spectrum to be a cell spectrum whose cells are attached only to cells of lower dimension. Thus CW spectra have two filtrations, the sequential filtration $\{E_n\}$ that gives the order in which cells are attached, and the skeletal filtration $\{E^q\}$, where E^q is the union of the cells of dimension at most q. We say that a map between CW spectra is cellular if it preserves both filtrations. In fact, by redefining the sequential filtration appropriately, we can always arrange that the sequential filtration is preserved. We have three basic results, whose proofs are very little different from their space level counterparts.

THEOREM 1.3 (Whitehead). *If E is a CW spectrum and $f : F \longrightarrow F'$ is a weak equivalence of spectra, then $f_* : [E, F] \longrightarrow [E, F']$ is an isomorphism. Therefore a weak equivalence between CW spectra is a homotopy equivalence.*

THEOREM 1.4 (Cellular approximation). *Let A be a subcomplex of a CW spectrum E, let F be a CW spectrum, and let $f : E \longrightarrow F$ be a map whose restriction to A is cellular.*

Then f is homotopic relative to A to a cellular map. Therefore any map $E \longrightarrow F$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.

THEOREM 1.5 (Approximation by CW spectra). *For a spectrum E, there is a CW spectrum ΓE and a weak equivalence $\gamma : \Gamma E \longrightarrow E$. On the homotopy category $h\mathscr{S}$, Γ is a functor such that γ is natural.*

It follows that the stable category $\bar{h}\mathscr{S}$ is equivalent to the homotopy category of CW spectra. Homotopy-preserving functors on spectra that do not preserve weak equivalences are transported to the stable category by first replacing their variables by weakly equivalent CW spectra.

Observe that there has been no mention of space level CW complexes in our development so far. The total lack of hypotheses on the spaces and structural maps of our prespectra allows considerable point-set level pathology, even if, as usual in modern algebraic topology, we restrict attention to compactly generated weak Hausdorff spaces. Recall that a space X is weak Hausdorff if the diagonal subspace is closed in the compactly generated product $X \times X$. More restrictively, a space X is said to be LEC (locally equiconnected) if the inclusion of the diagonal subspace is a cofibration. We record the following list of special kinds of prespectra both to prepare for our discussion of smash products and to compare our definitions with those adopted in the original treatments of the stable homotopy category.

DEFINITION 1.6. A prespectrum D is said to be

(i) Σ-cofibrant if each $\sigma : \Sigma^{W-V} DV \to DW$ is a based cofibration (that is, satisfies the based homotopy extension property).
(ii) CW if it is Σ-cofibrant and each DV is LEC and has the homotopy type of a CW complex.
(iii) strictly CW if each DV is a based CW complex and the structure maps σ are the inclusions of subcomplexes.

A spectrum E is said to be Σ-cofibrant if it is isomorphic to LD for some Σ-cofibrant prespectrum D; E is said to be tame if it is of the homotopy type of a Σ-cofibrant spectrum.

If E is a spectrum, then the maps $\tilde{\sigma}$ are homeomorphisms. Therefore the underlying prespectrum ℓE is not Σ-cofibrant unless it is trivial. However, it is a very weak condition on a spectrum that it be tame. We shall see that this weak condition is enough to avoid serious point-set topological problems. If D is a Σ-cofibrant prespectrum, then the maps $\tilde{\sigma}$ are inclusions and therefore $LD(V)$ is just the union of the spaces $\Omega^{W-V} DW$. We have the following relations between CW prespectra and CW spectra. Remember that CW spectra are defined in terms of spectrum level attaching maps.

THEOREM 1.7. *If D is a CW prespectrum, then LD has the homotopy type of a CW spectrum. If E is a CW spectrum, then each space EV has the homotopy type of a CW complex and E is homotopy equivalent to LD for some CW prespectrum D. Thus a* *spectrum has the homotopy type of a CW spectrum if and only if it has the homotopy type of LD for some CW prespectrum D.*

In particular, spectra of the homotopy types of CW spectra are tame.

Implicitly or explicitly, early constructions of the stable homotopy category restricted attention to the spectra arising from strict CW prespectra. This is far too restrictive for serious point-set level work, and it is also too restrictive to admit a sensible equivariant analogue. Note that such a category cannot possibly be complete or have well-behaved point-set level function spectra.

One reason for focusing on Σ-cofibrant spectra is that they are built up out of their component spaces in a simple fashion.

PROPOSITION 1.8. *If $E = LD$, where D is a Σ-cofibrant prespectrum, then*

$$E \cong \operatorname*{colim}_V \Sigma_V^\infty DV,$$

where the colimit is computed as the prespectrum level colimit of the maps

$$\Sigma_W^\infty \sigma : \Sigma_V^\infty DV \cong \Sigma_W^\infty \Sigma^{W-V} DV \longrightarrow \Sigma_W^\infty DW.$$

That is, the prespectrum level colimit is a spectrum that is isomorphic to E. The maps of the colimit system are shift desuspensions of based cofibrations.

Another reason is that general spectra can be replaced functorially by weakly equivalent Σ-cofibrant spectra.

PROPOSITION 1.9. *There is a functor $K : \mathscr{P}U \longrightarrow \mathscr{P}U$, called the cylinder functor, such that KD is Σ-cofibrant for any prespectrum D, and there is a natural spacewise weak equivalence of prespectra $KD \longrightarrow D$. On spectra E, define $KE = LK\ell E$. Then there is a natural weak equivalence of spectra $KE \longrightarrow E$.*

In practice, if one is given a prespectrum D, perhaps indexed only on integers, and one wishes to construct a spectrum from it that retains homotopical information, one forms $E = LKD$. Then

$$\pi_n(E) = \operatorname*{colim}_q \pi_{n+q} D_q.$$

If D is an Ω-spectrum that represents a given cohomology theory on spaces, then $E = LKD$ is a genuine spectrum that represents the same theory.

2. Smash products and twisted half-smash products

The construction of the smash product of spectra proceeds by internalization of an "external smash product". The latter is an associative and commutative pairing

$$\mathscr{S}U \times \mathscr{S}U' \to \mathscr{S}(U \oplus U')$$

for any pair of universes U and U'. It is constructed by starting with the prespectrum level definition

$$(E \wedge E')(V \oplus V') = EV \wedge E'V'.$$

The structure maps fail to be homeomorphisms when E and E' are spectra, and we apply the spectrification functor L to obtain the desired spectrum level smash product.

In order to obtain smash products internal to a single universe U, we exploit the "twisted half-smash product". The input data for this functor consist of two universes U and U', an unbased space A with a given map $\alpha : A \to \mathscr{I}(U, U')$, and a spectrum E indexed on U. The output is the spectrum $A \ltimes E$, which is indexed on U'. It must be remembered that the construction depends on α and not just on A, although different choices of α lead to equivalent functors on the level of stable categories. When A is a point, α is a choice of a linear isometry $f : U \longrightarrow U'$ and we write f_* for the twisted half-smash product. For a prespectrum D,

$$(f_*D)(V') = D(V) \wedge S^{V' - f(V)}, \quad \text{where } V = f^{-1}(V' \cap \operatorname{im} f).$$

For a spectrum E, f_*E is obtained by application of L to the prespectrum level construction. The functor f_* is left adjoint to the more elementary functor f^* specified by $(f^*E')(V) = E'(f(V))$. For general A and α, the intuition is that $A \ltimes E$ is obtained by suitably topologizing the union of the $\alpha(a)_*(E)$. Another intuition is that the twisted half-smash product is a generalization to spectra of the "untwisted" functor $A_+ \wedge X$ on based spaces X. This intuition is made precise by the following "untwisting formula" relating twisted half-smash products and shift desuspensions.

PROPOSITION 2.1. *For a map $A \longrightarrow \mathscr{I}(U, U')$ and an isomorphism $V \cong V'$, where $V \subset U$ and $V' \subset U'$, there is an isomorphism of spectra*

$$A \ltimes \Sigma^\infty_V X \cong A_+ \wedge \Sigma^\infty_{V'} X$$

that is natural in spaces A over $\mathscr{I}(U, U')$ and based spaces X.

The twisted-half smash product functor enjoys essentially the same formal properties as the space level functor $A_+ \wedge X$. The functor $A \ltimes E$ is homotopy-preserving in E, and it therefore preserves homotopy equivalences in the variable E. However, it only preserves homotopies over $\mathscr{I}(U, U')$ in A. Nevertheless, it very often preserves homotopy equivalences in the variable A. The following central technical result is an easy consequence of Propositions 1.8 and 2.1.

THEOREM 2.2. *Let $E \in \mathscr{S}U$ be tame and let A be a space over $\mathscr{I}(U, U')$. If $\phi : A' \longrightarrow A$ is a homotopy equivalence, then $\phi \ltimes \mathrm{id} : A' \ltimes E \longrightarrow A \ltimes E$ is a homotopy equivalence.*

Since $A \ltimes E$ is a CW spectrum if A is a CW complex and E is a CW spectrum, this has the following consequence.

COROLLARY 2.3. *Let $E \in \mathscr{S}U$ be a spectrum that has the homotopy type of a CW spectrum and let A be a space over $\mathscr{I}(U, U')$ that has the homotopy type of a CW complex. Then $A \ltimes E$ has the homotopy type of a CW spectrum.*

Now, as before, restrict attention to a particular universe U and write $\mathscr{S} = \mathscr{S}U$; again, the reader may think of U as $\mathbb{R}^\infty$. We are especially interested in twisted half-smash products defined in terms of the following spaces of linear isometries.

NOTATIONS 2.4. Let U^j be the direct sum of j copies of U and let $\mathscr{L}(j) = \mathscr{I}(U^j, U)$. The space $\mathscr{L}(0)$ is the point i, where $i : \{0\} \to U$, and $\mathscr{L}(1)$ contains the identity map $1 = \mathrm{id}_U : U \to U$. The left action of Σ_j on U^j by permutations induces a free right action of Σ_j on the contractible space $\mathscr{L}(j)$. Define maps

$$\gamma : \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \longrightarrow \mathscr{L}(j_1 + \cdots + j_k)$$

by

$$\gamma(g; f_1, \ldots, f_k) = g \circ (f_1 \oplus \cdots \oplus f_k).$$

The spaces $\mathscr{L}(j)$ form an operad [18, p. 1] with structural maps γ, called the linear isometries operad. Points $f \in \mathscr{L}(j)$ give functors f_* that send spectra indexed on U^j to spectra indexed on U. Applied to a j-fold external smash product $E_1 \wedge \cdots \wedge E_j$, there results an internal smash product $f_*(E_1 \wedge \cdots \wedge E_j)$. All of these smash products become equivalent in the stable category $\bar{h}\mathscr{S}$, but none of them are associative or commutative on the point set level. More precisely, the following result holds.

THEOREM 2.5. *Let $\mathscr{S}_t \subset \mathscr{S}$ be the full subcategory of tame spectra and let $h\mathscr{S}_t$ be its homotopy category. On $\mathscr{S}_t$, the internal smash products $f_*(E \wedge E')$ determined by varying $f \in \mathscr{L}(2)$ are canonically homotopy equivalent, and $h\mathscr{S}_t$ is symmetric monoidal under the internal smash product. For based spaces X and tame spectra E, there is a natural homotopy equivalence $E \wedge X \simeq f_*(E \wedge \Sigma^\infty X)$.*

This implies formally that we have arrived at a stable situation. As for spaces, the suspension functor Σ is given by $\Sigma E = E \wedge S^1$ and is left adjoint to the loop functor Ω given by $\Omega E = F(S^1, E)$. The cofibre Cf of a map $f : E \longrightarrow E'$ of spectra is the pushout $E' \cup_f CE$.

THEOREM 2.6. *The suspension functor $\Sigma : h\mathscr{S}_t \longrightarrow h\mathscr{S}_t$ is an equivalence of categories. A cofibre sequence $E \xrightarrow{f} E' \longrightarrow Cf$ in $\mathscr{S}_t$ gives rise to a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_q(E) \longrightarrow \pi_q(E') \longrightarrow \pi_q(Cf) \longrightarrow \pi_{q-1}(E) \longrightarrow \cdots.$$

PROOF. For based spaces X, $\Sigma^\infty X$ is naturally isomorphic to $(\Sigma_1^\infty X) \wedge S^1$ because the structural homeomorphisms $\tilde{\sigma} : E_0 \longrightarrow \Omega E_1$ on spectra give an isomorphism between their right adjoints. Thus, for tame spectra E, the previous theorem gives a natural homotopy equivalence

$$E = E \wedge S^0 \simeq f_*(E \wedge \Sigma^\infty S^0) \cong f_*(E \wedge \Sigma_1^\infty S^0) \wedge S^1.$$

Therefore Σ is an equivalence of categories with inverse obtained by smashing with the (-1)-sphere spectrum $S^{-1} = \Sigma_1^\infty S^0$. It follows categorically that $\Omega E \simeq f_*(E \wedge S^{-1})$ and that the unit and counit

$$\eta : E \longrightarrow \Omega\Sigma E \quad \text{and} \quad \varepsilon : \Sigma\Omega E \longrightarrow E$$

of the adjunction are homotopy equivalences. The last statement is a standard consequence of the fact that maps can now be desuspended. □

Note that only actual homotopy equivalences, not weak ones, are relevant to the last two results. For this reason among others, $h\mathscr{S}_t$ is a technically convenient halfway house between the homotopy category of spectra and the stable homotopy category.

3. The category of $\mathbb{L}$-spectra

We need a category of spectra with a canonical smash product. The category of $\mathbb{L}$-spectra that we introduce here will be shown in the next section to have an associative and commutative smash product $\wedge_{\mathscr{L}}$. This product is not quite unital, but there is a natural unit weak equivalence $\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$. The S-modules will be the $\mathbb{L}$-spectra such that λ is an isomorphism.

For $f \in \mathscr{L}(j)$ and $E_i \in \mathscr{S}_t$, Theorem 2.2 implies that the inclusion $\{f\} \subset \mathscr{L}(j)$ induces a homotopy equivalence

$$f_*(E_1 \wedge \cdots \wedge E_j) \longrightarrow \mathscr{L}(j) \ltimes (E_1 \wedge \cdots \wedge E_j).$$

The proof of Theorem 2.5 above is entirely based on the use of such equivalences. It therefore seems natural to think of

$$\mathscr{L}(j) \ltimes (E_1 \wedge \cdots \wedge E_j)$$

as a canonical j-fold smash product. It is still not associative, but it seems closer to being so. However, to take this idea seriously, we must take note of the difference between E and its "1-fold smash product" $\mathscr{L}(1) \ltimes E$. The space $\mathscr{L}(1)$ is a monoid under composition, and the formal properties of twisted half-smash products imply a natural isomorphism

$$\mathscr{L}(1) \ltimes \big(\mathscr{L}(1) \ltimes E\big) \cong \big(\mathscr{L}(1) \times \mathscr{L}(1)\big) \ltimes E,$$

where, on the right, $\mathscr{L}(1) \times \mathscr{L}(1)$ is regarded as a space over $\mathscr{L}(1)$ via the composition product. This product induces a map $\mu : (\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes E \longrightarrow \mathscr{L}(1) \ltimes E$, and the inclusion $\{1\} \longrightarrow \mathscr{L}(1)$ induces a map $\eta : E \longrightarrow \mathscr{L}(1) \ltimes E$. Thus it makes sense to consider spectra E with an action $\xi : \mathscr{L}(1) \ltimes E \longrightarrow E$ of the monoid $\mathscr{L}(1)$. It is required that the following diagrams commute:

$$\begin{tikzcd}
(\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes E \arrow[r, "\mu"] \arrow[d, "\mathscr{L}(1) \ltimes \xi"'] & \mathscr{L}(1) \ltimes E \arrow[d, "\xi"] \\
\mathscr{L}(1) \ltimes E \arrow[r, "\xi"'] & E
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
E \arrow[r, "\eta"] \arrow[dr, "="'] & \mathscr{L}(1) \ltimes E \arrow[d, "\xi"] \\
 & E
\end{tikzcd}$$

DEFINITION 3.1. An $\mathbb{L}$-spectrum is a spectrum E together with an action ξ of $\mathscr{L}(1)$. A map $f : E \to E'$ of $\mathbb{L}$-spectra is a map of spectra such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{L}(1) \ltimes E \arrow[r, "\mathscr{L}(1) \ltimes f"] \arrow[d, "\xi_E"'] & \mathscr{L}(1) \ltimes E' \arrow[d, "\xi_{E'}"] \\
E \arrow[r, "f"'] & E'
\end{tikzcd}$$

We let $\mathscr{S}[\mathbb{L}]$ denote the category of $\mathbb{L}$-spectra.

A number of basic properties of the category of spectra are directly inherited by the category of $\mathbb{L}$-spectra.

THEOREM 3.2. *The category of $\mathbb{L}$-spectra is complete and cocomplete, with both limits and colimits created in the underlying category $\mathscr{S}$. If X is a based space and M is an $\mathbb{L}$-spectrum, then $M \wedge X$ and $F(X, M)$ are $\mathbb{L}$-spectra, and the spectrum level fibre and cofibre of a map of $\mathbb{L}$-spectra are $\mathbb{L}$-spectra.*

A homotopy in the category of $\mathbb{L}$-spectra is a map $M \wedge I_+ \longrightarrow M'$. A map of $\mathbb{L}$-spectra is a weak equivalence if it is a weak equivalence as a map of spectra. The stable homotopy category $\bar{h}\mathscr{S}[\mathbb{L}]$ is constructed from the homotopy category of $\mathbb{L}$-spectra by adjoining formal inverses to the weak equivalences. There is a theory of CW $\mathbb{L}$-spectra that is exactly like the theory of CW spectra, and, again, the construction of $\bar{h}\mathscr{S}[\mathbb{L}]$ is made rigorous by CW approximation. We have a free functor $\mathbb{L}$ from spectra to $\mathbb{L}$-spectra specified by $\mathbb{L}E = \mathscr{L}(1) \ltimes E$. The "sphere $\mathbb{L}$-spectra" that we take as the domains of attaching maps when defining CW $\mathbb{L}$-spectra are the free $\mathbb{L}$-spectra $\mathbb{L}S^n$. Using the freeness adjunction

$$\mathscr{S}[\mathbb{L}](\mathbb{L}E, M) \cong \mathscr{S}(E, M),$$

it is easy to prove Whitehead, cellular approximation, and approximation by CW $\mathbb{L}$-spectra theorems exactly like those stated for spectra in Section 1, and $\bar{h}\mathscr{S}[\mathbb{L}]$ is equivalent to the homotopy category of CW $\mathbb{L}$-spectra. There is one catch: although S and all other suspension spectra are $\mathbb{L}$-spectra in a natural way, using the untwisting isomorphism of Proposition 2.1 and the projection $\mathscr{L}(1) \longrightarrow \{*\}$, S does not have the homotopy type of a CW $\mathbb{L}$-spectrum. However, it is not hard to see that the categories $\bar{h}\mathscr{S}$ and $\bar{h}\mathscr{S}[\mathbb{L}]$ are equivalent.

THEOREM 3.3. *The following conclusions hold.*

(i) *The free functor* $\mathbb{L} : \mathscr{S} \longrightarrow \mathscr{S}[\mathbb{L}]$ *carries CW spectra to CW* $\mathbb{L}$*-spectra.*
(ii) *The forgetful functor* $\mathscr{S}[\mathbb{L}] \longrightarrow \mathscr{S}$ *carries* $\mathbb{L}$*-spectra of the homotopy types of CW* $\mathbb{L}$*-spectra to spectra of the homotopy types of CW spectra.*
(iii) *Every CW* $\mathbb{L}$*-spectrum* M *is homotopy equivalent as an* $\mathbb{L}$*-spectrum to* $\mathbb{L}E$ *for some CW spectrum* E.
(iv) *If* $E \in \mathscr{S}_t$, *for example if* E *is a CW spectrum, then* $\eta : E \longrightarrow \mathbb{L}E$ *is a homotopy equivalence of spectra.*
(v) *If* M *has the homotopy type of a CW* $\mathbb{L}$*-spectrum, then* $\xi : \mathbb{L}M \longrightarrow M$ *is a homotopy equivalence of* $\mathbb{L}$*-spectra.*

Therefore the free and forgetful functors establish an adjoint equivalence between the stable homotopy categories $\bar{h}\mathscr{S}$ *and* $\bar{h}\mathscr{S}[\mathbb{L}]$.

4. The smash product of $\mathbb{L}$-spectra and function $\mathbb{L}$-spectra

One of the most surprising developments of recent years is the discovery of an associative and commutative smash product $\wedge_{\mathscr{L}}$ in the category of $\mathbb{L}$-spectra. We proceed to define it. To begin with, observe that the monoid $\mathscr{L}(1) \times \mathscr{L}(1)$ acts from the right on $\mathscr{L}(2)$ and acts from the left on $\mathscr{L}(i) \times \mathscr{L}(j)$, via instances of the structural maps γ of the linear isometries operad. Another instance of γ gives rise to a map

$$\bar{\gamma} : \mathscr{L}(2) \times_{\mathscr{L}(1)\times\mathscr{L}(1)} \mathscr{L}(i) \times \mathscr{L}(j) \longrightarrow \mathscr{L}(i+j). \tag{4.1}$$

The space on the left is the balanced product (formally a coequalizer) of the two specified actions by $\mathscr{L}(1) \times \mathscr{L}(1)$. The essential, elementary, point is that this map is a homeomorphism if $i \geqslant 1$ and $j \geqslant 1$. To see this, choose linear isometric isomorphisms $s : U \longrightarrow U^i$ and $t : U \longrightarrow U^j$. Composition on the right with $s \oplus t$ gives vertical homeomorphisms in the commutative diagram

$$\begin{tikzcd}
\mathscr{L}(2) \times_{\mathscr{L}(1)\times\mathscr{L}(1)} \mathscr{L}(i) \times \mathscr{L}(j) \arrow[r, "\bar{\gamma}"] \arrow[d] & \mathscr{L}(i+j) \arrow[d] \\
\mathscr{L}(2) \times_{\mathscr{L}(1)\times\mathscr{L}(1)} \mathscr{L}(1) \times \mathscr{L}(1) \arrow[r, "\bar{\gamma}"] & \mathscr{L}(2)
\end{tikzcd},$$

and the lower map $\bar{\gamma}$ is clearly a homeomorphism. Note also that $\mathscr{L}(1)$ acts from the left on $\mathscr{L}(2)$ and that this action commutes with the right action of $\mathscr{L}(1) \times \mathscr{L}(1)$.

Regard $\mathscr{L}(1) \times \mathscr{L}(1)$ as a space over $\mathscr{I}(U^2, U^2)$ via the direct sum of isometries map. If M and N are $\mathbb{L}$-spectra, then $\mathscr{L}(1) \times \mathscr{L}(1)$ acts from the left on the external smash product $M \wedge N$ via the map

$$\xi : \big(\mathscr{L}(1) \times \mathscr{L}(1)\big) \ltimes (M \wedge N) \cong \big(\mathscr{L}(1) \ltimes M\big) \wedge \big(\mathscr{L}(1) \ltimes N\big) \xrightarrow{\xi \wedge \xi} M \wedge N.$$

The operadic smash product of M and N is simply the balanced product (again, formally a coequalizer)

$$M \wedge_{\mathscr{L}} N = \mathscr{L}(2) \ltimes_{\mathscr{L}(1) \times \mathscr{L}(1)} (M \wedge N). \tag{4.2}$$

The left action of $\mathscr{L}(1)$ on $\mathscr{L}(2)$ induces a left action of $\mathscr{L}(1)$ on $M \wedge_{\mathscr{L}} N$ that gives it a structure of L-spectrum. Use of the transposition $\sigma \in \Sigma_2$ and the commutativity of the external smash product easily gives a commutativity isomorphism

$$\tau : M \wedge_{\mathscr{L}} N \longrightarrow N \wedge_{\mathscr{L}} M.$$

More substantially, there is a natural associativity isomorphism

$$(M \wedge_{\mathscr{L}} N) \wedge_{\mathscr{L}} P \cong M \wedge_S (N \wedge_{\mathscr{L}} P).$$

In fact, using the case $i = 2$ and $j = 1$ of the homeomorphism $\bar{\gamma}$, we obtain isomorphisms

$$\begin{aligned}(M \wedge_{\mathscr{L}} N) \wedge_{\mathscr{L}} P &\cong \mathscr{L}(2) \ltimes_{\mathscr{L}(1)^2} \\ &\qquad \big(\mathscr{L}(2) \ltimes_{\mathscr{L}(1)^2} (M \wedge N)\big) \wedge \big(\mathscr{L}(1) \ltimes_{\mathscr{L}(1)} P\big) \\ &\cong \big(\mathscr{L}(2) \times_{\mathscr{L}(1)^2} \mathscr{L}(2) \times \mathscr{L}(1)\big) \ltimes_{\mathscr{L}(1)^3} (M \wedge N \wedge P) \\ &\cong \mathscr{L}(3) \ltimes_{\mathscr{L}(1)^3} M \wedge N \wedge P.\end{aligned}$$

The symmetric argument shows that this is also isomorphic to $M \wedge_{\mathscr{L}} (N \wedge_{\mathscr{L}} P)$. In view of the generality of the homeomorphisms (4.1), the argument iterates to give

$$M_1 \wedge_{\mathscr{L}} \cdots \wedge_{\mathscr{L}} M_j \cong \mathscr{L}(j) \ltimes_{\mathscr{L}(1)^j} (M_1 \wedge \cdots \wedge M_j), \tag{4.3}$$

where the iterated smash product on the left is associated in any fashion.

On passage to the derived category $\bar{h}\mathscr{S}[\mathbb{L}]$, the smash product of L-spectra just constructed can be used interchangeably with the internal smash product on the stable category $\bar{h}\mathscr{S}$. To see this, one defines the latter by use of a linear isometric isomorphism $f : U^2 \longrightarrow U$ (not just an isometry). With this choice, it is not hard to check the following result.

PROPOSITION 4.4. *For spectra E and F, there are isomorphisms of $\mathbb{L}$-spectra*

$$\mathbb{L}E \wedge_{\mathscr{L}} \mathbb{L}F \cong \mathscr{L}(2) \ltimes E \wedge F \cong \mathbb{L}f_*(E \wedge F).$$

For CW $\mathbb{L}$-spectra M and N, $M \wedge_{\mathscr{L}} N$ is a CW $\mathbb{L}$-spectrum with one $(p+q)$-cell for each p-cell of M and q-cell of N.

However, we need a deeper result, one that depends on the fine structure of the linear isometries operad, to complete the comparison of smash products. By arguments like those in the proof of Theorem 2.6, its first statement implies its second statement.

PROPOSITION 4.5. *For* $\mathbb{L}$*-spectra* N*, there is a natural weak equivalence of* $\mathbb{L}$*-spectra* $\omega : \mathbb{L}S \wedge_{\mathscr{L}} N \longrightarrow N$*, and* $\Sigma : \pi_n(N) \longrightarrow \pi_{n+1}(\Sigma N)$ *is an isomorphism for all integers* n*. Therefore the unit* $\eta : N \longrightarrow \Omega\Sigma N$ *and counit* $\varepsilon : \Sigma\Omega N \longrightarrow N$ *of the* (Σ, Ω)*-adjunction are weak equivalences and any cofibre sequence*

$$N \xrightarrow{f} N' \longrightarrow Cf$$

of $\mathbb{L}$*-spectra gives rise to a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_q(N) \longrightarrow \pi_q(N') \longrightarrow \pi_q(Cf) \longrightarrow \pi_{q-1}(N) \longrightarrow \cdots.$$

It is a pleasant technical feature of the theory that this result holds whether or not the given $\mathscr{L}$-spectra are tame. In particular, we have the following consequence, which is the $\mathbb{L}$-spectrum analog of the algebraic statement that, when computing torsion products, one need only resolve one of the tensor factors by a projective resolution.

PROPOSITION 4.6. *If* M *is a CW* $\mathbb{L}$*-spectrum and* $\phi : N \longrightarrow N'$ *is a weak equivalence of* $\mathbb{L}$*-spectra, then* $\mathrm{id} \wedge_{\mathscr{L}} \phi : M \wedge_{\mathscr{L}} N \longrightarrow M \wedge_{\mathscr{L}} N'$ *is a weak equivalence of* $\mathbb{L}$*-spectra.*

The previous results lead easily to the promised comparison between the internal smash product of spectra and the operadic smash product of $\mathbb{L}$-spectra.

THEOREM 4.7. *For* $\mathbb{L}$*-spectra* M *and* N*, there is a natural map of spectra*

$$\alpha : f_*(M \wedge N) \longrightarrow M \wedge_{\mathscr{L}} N,$$

and α *is a weak equivalence when* M *is a CW* $\mathbb{L}$*-spectrum and* N *is a tame spectrum. For any* $\mathbb{L}$*-spectrum* N*, the functors* $(?) \wedge_{\mathscr{L}} N$ *and* $f_*(? \wedge N)$ *from* $\bar{h}\mathscr{S}[\mathbb{L}]$ *to* $\bar{h}\mathscr{S}$ *are naturally isomorphic.*

Thus, under the forgetful functor, the operadic smash product in $\bar{h}\mathscr{S}[\mathbb{L}]$ agrees with the internal smash product in $\bar{h}\mathscr{S}$.

There is a function $\mathbb{L}$-spectrum functor to go with the operadic smash product. The twisted half-smash product functor $A \ltimes E$ has a right adjoint twisted function spectrum functor $F[A, E')$ and the external smash product has a right adjoint function spectrum functor. Using these functors and appropriate equalizer diagrams, dual to the coequalizer diagrams that were implicit in the definition of $\wedge_{\mathscr{L}}$, we obtain the following result.

THEOREM 4.8. *There is a function* $\mathbb{L}$*-spectrum functor* $F_{\mathscr{L}}(M, N)$ *such that*

$$\mathscr{S}[\mathbb{L}](M \wedge_{\mathscr{L}} N, P) \cong \mathscr{S}[\mathbb{L}](M, F_{\mathscr{L}}(N, P))$$

for $\mathbb{L}$*-spectra* M*,* N*, and* P*.*

Given the adjunction, we can deduce the homotopical behavior of $F_{\mathscr{L}}$ from that of $\wedge_{\mathscr{L}}$. There is an internal function spectrum functor F that is induced from the external spectrum functor by use of our chosen linear isometric isomorphism $f : U^2 \longrightarrow U$. Our function $\mathscr{L}$-spectrum functor gives a canonical substitute.

PROPOSITION 4.9. *If M is a CW $\mathbb{L}$-spectrum and $\phi : N \to N'$ is a weak equivalence of $\mathbb{L}$-spectra, then*

$$F_{\mathscr{L}}(\mathrm{id}, \phi) : F_{\mathscr{L}}(M, N) \longrightarrow F_{\mathscr{L}}(M, N')$$

is a weak equivalence of $\mathbb{L}$-spectra.

THEOREM 4.10. *For $\mathbb{L}$-spectra M and N, there is a natural map of spectra*

$$\tilde{\alpha} : F_{\mathscr{L}}(M, N) \longrightarrow F(M, f^*N),$$

and $\tilde{\alpha}$ is a weak equivalence when M is a CW $\mathbb{L}$-spectrum. The forgetful functor $\bar{h}\mathscr{S}[\mathbb{L}] \longrightarrow \bar{h}\mathscr{S}$ carries the function $\mathbb{L}$-spectrum functor $F_{\mathscr{L}}$ to the internal function spectrum functor F.

We must still address the question of units.

PROPOSITION 4.11. *For $\mathbb{L}$-spectra N, there is a natural unit map of $\mathbb{L}$-spectra $\lambda : S \wedge_{\mathscr{L}} N \longrightarrow N$. It is a weak equivalence for any N, and it is a homotopy equivalence of $\mathbb{L}$-spectra if N is a CW $\mathbb{L}$-spectrum.*

PROOF. Consider the map $\bar{\gamma}$ of (4.1). It is a nontrivial property of the linear isometries operad that $\bar{\gamma}$, although not a homeomorphism, is a homotopy equivalence when $i = 0$ and $j > 0$. When N is the free S-module $\mathbb{L}E = \mathscr{L}(1) \ltimes E$ generated by a spectrum E, λ is given by the map

$$\begin{aligned} S \wedge_S \mathbb{L}E &= \mathscr{L}(2) \ltimes_{\mathscr{L}(1)\times\mathscr{L}(1)} \big(\mathscr{L}(0) \ltimes S^0\big) \wedge \big(\mathscr{L}(1) \ltimes E\big) \\ &\cong \big(\mathscr{L}(2) \times_{\mathscr{L}(1)\times\mathscr{L}(1)} \mathscr{L}(0) \times \mathscr{L}(1)\big) \ltimes \big(S^0 \wedge E\big) \\ &\xrightarrow{\bar{\gamma}\ltimes\mathrm{id}} \mathscr{L}(1) \ltimes E = \mathbb{L}E. \end{aligned}$$

Since $\bar{\gamma}$ is a homotopy equivalence, Theorem 2.2 implies that λ is a homotopy equivalence when $E \in \mathscr{S}_t$. For general N, the map just constructed for $\mathbb{L}N$ induces the required map for N by a comparison of coequalizer diagrams. Although the arguments are not transparent, the rest can be deduced from this. □

There is one important case when λ is an isomorphism. It turns out that the map $\bar{\gamma}$ of (4.1) is a homeomorphism when $i = j = 0$; that is, nonobviously since $\mathscr{L}(1)$ is a monoid but not a group, the domain $\mathscr{L}(2)/\mathscr{L}(1) \times \mathscr{L}(1)$ of (4.1) is then a point. This implies that $S \wedge_{\mathscr{L}} S \cong S$. More generally, it implies that the smash product over $\mathscr{L}$ precisely generalizes the smash product of based spaces, in the sense that

$$\Sigma^\infty X \wedge_{\mathscr{L}} \Sigma^\infty Y \cong \Sigma^\infty(X \wedge Y).$$

5. The category of S-modules

Here, finally, is the promised definition of S-modules.

DEFINITION 5.1. Define an S-module to be an $\mathbb{L}$-spectrum M which is unital in the sense that $\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$ is an isomorphism. Let $\mathscr{M}_S$ denote the full subcategory of $\mathscr{S}[\mathbb{L}]$ whose objects are the S-modules. For S-modules M and N, define

$$M \wedge_S N = M \wedge_{\mathscr{L}} N \quad \text{and} \quad F_S(M,N) = S \wedge_{\mathscr{L}} F_{\mathscr{L}}(M,N).$$

The justification for the name "S-module" is given by the commutative diagrams

$$\begin{tikzcd} S \wedge_S S \wedge_S M \arrow[r, "\lambda\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\lambda"'] & S \wedge_S M \arrow[d, "\lambda"] \\ S \wedge_S M \arrow[r, "\lambda"'] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "\lambda^{-1}"] \arrow[dr, "\mathrm{id}"'] & S \wedge_S M \arrow[d, "\lambda"] \\ & M \end{tikzcd}$$

We consistently retain the notation $M \wedge_{\mathscr{L}} N$ when the given $\mathbb{L}$-spectra M and N are not restricted to be S-modules. We have the following examples of S-modules.

PROPOSITION 5.2. *For any based space X, $\Sigma^\infty X$ is an S-module, and*

$$\Sigma^\infty X \wedge_S \Sigma^\infty Y \cong \Sigma^\infty(X \wedge Y).$$

For any S-module M and any $\mathbb{L}$-spectrum N, $M \wedge_{\mathscr{L}} N$ is an S-module. In particular, $S \wedge_{\mathscr{L}} N$ is an S-module for any $\mathbb{L}$-spectrum N.

We have the following categorical relationship between $\mathscr{S}[\mathbb{L}]$ and $\mathscr{M}_S$.

LEMMA 5.3. *The functor*

$$S \wedge_{\mathscr{L}} (?) : \mathscr{S}[\mathbb{L}] \longrightarrow \mathscr{M}_S$$

is left adjoint to the functor $F_{\mathscr{L}}(S, ?) : \mathscr{M}_S \longrightarrow \mathscr{S}[\mathbb{L}]$ and right adjoint to the inclusion $\ell : \mathscr{M}_S \longrightarrow \mathscr{S}[\mathbb{L}]$.

This implies that to lift right adjoint functors from $\mathscr{S}[\mathbb{L}]$ to $\mathscr{M}_S$, we must first forget down to $\mathscr{S}[\mathbb{L}]$, next apply the given functor, and then apply the functor $S \wedge_{\mathscr{L}} (?)$. For example, limits in $\mathscr{M}_S$ are created in this fashion.

PROPOSITION 5.4. *The category of S-modules is complete and cocomplete. Its colimits are created in $\mathscr{S}[\mathbb{L}]$. Its limits are created by applying the functor $S \wedge_S (?)$ to limits in $\mathscr{S}[\mathbb{L}]$. If X is a based space and M is an S-module, then $M \wedge X$ is an S-module, and the spectrum level cofibre of a map of S-modules is an S-module. For a based space X and S-modules M and N,*

$$\mathscr{M}_S(M \wedge X, N) \cong \mathscr{M}_S(M, S \wedge_{\mathscr{L}} F(X, N)).$$

Moreover,

$$M \wedge X \cong M \wedge_S \Sigma^\infty X \quad and \quad S \wedge_{\mathscr{L}} F(X, M) \cong F_S(\Sigma^\infty X, M).$$

Lemma 5.3 also explains our definition of function S-modules. Its second adjunction and the adjunction of Theorem 4.8 compose to give the adjunction displayed in the following theorem.

THEOREM 5.5. *The category $\mathscr{M}_S$ is symmetric monoidal under $\wedge_S$, and*

$$\mathscr{M}_S(M \wedge_S N, P) \cong \mathscr{M}_S(M, F_S(N, P))$$

for S-modules M, N, and P.

A homotopy in the category of S-modules is a map $M \wedge I_+ \longrightarrow N$. A map of S-modules is a weak equivalence if it is a weak equivalence as a map of spectra. The derived category $\mathscr{D}_S$ of S-modules is constructed from the homotopy category $h\mathscr{M}_S$ by adjoining formal inverses to the weak equivalences; again, the process is made rigorous by CW approximation. We define sphere S-modules

$$S_S^n \equiv S \wedge_{\mathscr{L}} \mathbb{L}S^n \tag{5.6}$$

and use them as the domains of attaching maps when defining cell and CW S-modules. From here, the theory of cell and CW S-modules is exactly like the theory of cell and CW spectra and is obtained by specialization of the theory of cell R-modules to be discussed shortly. A weak equivalence of cell S-modules is a homotopy equivalence, any S-module is weakly equivalent to a CW S-module, and $\mathscr{D}_S$ is equivalent to the homotopy category of CW S-modules. Again, the S-module S does not have the homotopy type of a CW S-module. When working homotopically, we replace it with $S_S \equiv S_S^0$.

The following comparison between CW S-modules and CW $\mathbb{L}$-spectra establishes an equivalence between $\mathscr{D}_S$ and $\bar{h}\mathscr{S}[\mathbb{L}]$ and thus between $\mathscr{D}_S$ and $\bar{h}\mathscr{S}$. It is largely a recapitulation of results already discussed.

THEOREM 5.7. *The following conclusions hold.*

(i) *The functor $S \wedge_{\mathscr{L}} (?) : \mathscr{S}[\mathbb{L}] \longrightarrow \mathscr{M}_S$ carries CW $\mathbb{L}$-spectra to CW S-modules.*
(ii) *The forgetful functor $\mathscr{M}_S \longrightarrow \mathscr{S}[\mathbb{L}]$ carries S-modules of the homotopy types of CW S-modules to $\mathbb{L}$-spectra of the homotopy types of CW $\mathbb{L}$-spectra.*
(iii) *Every CW S-module M is homotopy equivalent as an S-module to $S \wedge_S N$ for some CW $\mathbb{L}$-spectrum N.*
(iv) *The unit $\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$ is a weak equivalence for all $\mathbb{L}$-spectra M and is a homotopy equivalence of $\mathbb{L}$-spectra if M has the homotopy type of a CW $\mathbb{L}$-spectrum.*

The functors $S \wedge_{\mathscr{L}} (?)$ and the forgetful functor establish an adjoint equivalence between the stable homotopy category $\bar{h}\mathscr{S}[\mathbb{L}]$ and the derived category $\mathscr{D}_S$. This equivalence of categories preserves smash products and function spectra.

When doing classical homotopy theory, we can work interchangeably in any of the categories $\bar{h}\mathscr{S}$, $\bar{h}\mathscr{S}[\mathbb{L}]$, or $\mathscr{D}_S$. These three categories are equivalent, and the equivalences preserve all structure in sight. When working on the point set level, we have reached a nearly ideal situation with our construction of $\mathscr{M}_S$, and the rest of the article will describe how to exploit this.

6. S-algebras and their categories of modules

Intuitively, S-algebras are as near to associative rings with unit as one can get in stable homotopy theory, and commutative S-algebras are as near as one can get to commutative rings.

DEFINITION 6.1. An S-algebra is an S-module R together with maps of S-modules $\eta : S \longrightarrow R$ and $\phi : R \wedge_S R \to R$ such that the following diagrams of S-modules commute:

$$\begin{tikzcd} S \wedge_S R \arrow[r, "\eta\wedge_S \mathrm{id}"] \arrow[dr, "\lambda"'] & R \wedge_S R \arrow[d, "\phi"] & R \wedge_S S \arrow[l, "\mathrm{id}\wedge_S \eta"'] \arrow[dl, "\lambda\tau"] \\ & R & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R\wedge_S R \wedge_S R \arrow[r, "\mathrm{id}\wedge_S\phi"] \arrow[d, "\phi\wedge_S \mathrm{id}"'] & R\wedge_S R \arrow[d, "\phi"] \\ R\wedge_S R \arrow[r, "\phi"] & R \end{tikzcd} ;$$

R is commutative if the following diagram also commutes:

$$\begin{tikzcd} R\wedge_S R \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & R\wedge_S R \arrow[dl, "\phi"] \\ & R. & \end{tikzcd}$$

We shall not review the older definitions of A_∞ and E_∞ ring spectra. It turns out that they are equivalent to the structures that are given by the definition above, with the single exception that the unit map λ of an A_∞ or E_∞ ring spectrum need not be an isomorphism. In other words, the natural ground category for A_∞ and E_∞ ring spectra is the category of $\mathbb{L}$-spectra rather than the category of S-modules. We state this formally.

THEOREM 6.2. *An S-algebra or commutative S-algebra is an A_∞ or E_∞ ring spectrum which is also an S-module. If A is an A_∞ ring spectrum, then $S \wedge_{\mathscr{L}} A$ is a weakly equivalent S-algebra. If A is an E_∞ ring spectrum, then $S \wedge_{\mathscr{L}} A$ is a weakly equivalent commutative S-algebra.*

This means that we can use the older theory to construct examples. For example, the classical Thom spectra occur in nature as E_∞ ring spectra, and [20] gives a machine for manufacturing A_∞ and E_∞ ring spectra from space level data. It shows that the Eilenberg–MacLane spectrum Hk of a ring k is an A_∞ ring spectrum and is an E_∞ ring spectrum if k is commutative and that the algebraic K-theory spectrum Kk of a commutative ring k is an E_∞ ring spectrum. Similarly, the spectra ko and ku that represent real and complex connective K-theory are E_∞ ring spectra.

Since it is very convenient to have strict units, we shall always work with S-algebras.

DEFINITION 6.3. Let R be an S-algebra. A (left) R-module M is an S-module together with a map $\mu : R \wedge_S M \to M$ of S-modules such that the following diagrams commute:

$$\begin{tikzcd} S \wedge_S M \arrow[r, "\eta\wedge_S \mathrm{id}"] \arrow[dr, "\lambda"'] & R \wedge_S M \arrow[d, "\mu"] \\ & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \wedge_S R \wedge_S M \arrow[r, "\mathrm{id}\wedge_S \mu"] \arrow[d, "\phi\wedge_S \mathrm{id}"'] & R \wedge_S M \arrow[d, "\mu"] \\ R \wedge_S M \arrow[r, "\mu"] & M \end{tikzcd}$$

A map $f : M \longrightarrow M'$ of R-modules is a map of S-modules such that the following diagram commutes:

$$\begin{tikzcd} R \wedge_S M \arrow[r, "\mathrm{id}\wedge f"] \arrow[d, "\mu"'] & R \wedge_S M' \arrow[d, "\mu'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

We let $\mathscr{M}_R$ denote the category of R-modules.

If R is commutative, then an R-module is the same thing as a left module over R regarded as an S-algebra, exactly as in algebra. From here, we can mimic vast areas of algebra, one particularly striking direction being the development of topological Hochschild homology. However, we shall concentrate on the generalized analog of stable homotopy theory that we obtain by studying the homotopy theory of R-modules for a fixed commutative S-algebra R. Everything that makes sense is also true in the noncommutative case.

THEOREM 6.4. *The category of R-modules is complete and cocomplete, with both limits and colimits created in the underlying category $\mathscr{M}_S$. Let X be a based space, K be an S-module, and M and N be R-modules. Then the following conclusions hold, where the displayed isomorphisms are obtained by restriction of the corresponding isomorphisms for S-modules.*

(i) *$M \wedge X$ is an R-module and the spectrum level cofibre of a map of R-modules is an R-module.*

(ii) *$S \wedge_{\mathscr{L}} F(X, N)$ is an R-module and*

$$\mathscr{M}_R(M \wedge X, N) \cong \mathscr{M}_R\big(M, S \wedge_{\mathscr{L}} F(X, N)\big).$$

(iii) *$M \wedge_S K$ and $F_S(K, N)$ are R-modules and*

$$\mathscr{M}_R(M \wedge_S K, N) \cong \mathscr{M}_R\big(M, F_S(K, N)\big).$$

(iv) *$F_S(M, K)$ is an R-module.*

(v) *As R-modules,*

$$M \wedge X \cong M \wedge_S \Sigma^\infty X \quad \textit{and} \quad S \wedge_{\mathscr{L}} F(X, N) \cong F_S\big(\Sigma^\infty X, N\big).$$

A homotopy in the category of R-modules is a map $M \wedge I_+ \longrightarrow M'$. A map of R-modules is a weak equivalence if it is a weak equivalence as a map of spectra. The derived category $\mathscr{D}_R$ is constructed from the homotopy category $h\mathscr{M}_R$ by adjoining formal inverses to the weak equivalences; again, the process is made rigorous by the approximation of general R-modules by cell R-modules.

Cell theory is based on the free R-module functor $\mathbb{F}_R$ from spectra to R-modules that is specified by $\mathbb{F}_R X = R \wedge_S \mathbb{F}_S X$, where $\mathbb{F}_S X = S \wedge_{\mathscr{L}} \mathbb{L}X$. The term "free" is a slight misnomer, in view of the following result.

PROPOSITION 6.5. *The functor $\mathbb{F}_R : \mathscr{S} \longrightarrow \mathscr{M}_R$ is left adjoint to the functor that sends an R-module M to the spectrum $F_{\mathscr{L}}(S, M)$, and there is a natural map of R-modules $\xi : \mathbb{F}_R M \longrightarrow M$ whose adjoint $M \longrightarrow F_{\mathscr{L}}(S, M)$ is a weak equivalence of spectra. Therefore*

$$\pi_n(M) \cong h\mathscr{M}_R(\mathbb{F}_R S^n, M).$$

In the stable homotopy category $\bar{h}\mathscr{S}$, $\mathbb{F}_R X$ is naturally isomorphic to the internal smash product $R \wedge X$ when X is tame.

Thus $\mathbb{F}_R$ is left adjoint to a functor that is weakly equivalent to the obvious forgetful functor. This is the price to be paid for insisting on strict units, and it introduces no serious complications in the theory. Homotopically, the functor $\mathbb{F}_R$ behaves as one would expect. Generalizing (5.6), we define sphere R-modules by

$$S_R^n = \mathbb{F}_R S^n, \tag{6.6}$$

and we use them as the domains of attaching maps when developing the cell theory of R-modules. For cells, we note that the cone functor $CE = E \wedge I$ commutes with $\mathbb{F}_R$, so that $CS_R^n \cong \mathbb{F}_R CS^n$. Thus, via the adjunction, maps out of sphere R-modules and their cones are induced by maps on the spectrum level. Using this, we can simply parrot the theory of cell spectra in the context of R-modules, reducing proofs to the spectrum level via adjunction. We easily obtain the Whitehead theorem for cell R-modules, and the approximation theorem to the effect that any R-module is weakly equivalent to a cell R-module. The category $\mathscr{D}_R$ is equivalent to the homotopy category of cell R-modules. If R is connective, but not otherwise, we obtain the cellular approximation theorem when we restrict attention to CW R-modules, namely cell R-modules such that cells are only attached to cells of lower dimension.

The category $\mathscr{D}_R$ has all homotopy limits and colimits; the former are created as the corresponding constructions on the underlying diagrams of spectra; the latter require application of the functor $S \wedge_{\mathscr{L}}$ (?). Thus we have enough information to quote the categorical form of Brown's representability theorem given in [6]. Adams' analog [3] for functors defined only on finite CW spectra also applies in our context, with the same proof.

THEOREM 6.7 (Brown). *A contravariant functor $k : \mathscr{D}_R \rightarrow$ Sets is representable in the form $k(M) \cong \mathscr{D}_R(M, N)$ for some R-module N if and only if k converts wedges to products and converts homotopy pushouts to weak pullbacks.*

THEOREM 6.8 (Adams). *A contravariant group-valued functor k on the homotopy category of finite cell R-modules is representable in the form $k(M) \cong \mathscr{D}_R(M, N)$ for some R-module N if and only if k converts finite wedges to direct products and converts homotopy pushouts to weak pullbacks of underlying sets.*

In fact, Brown's theorem is the kind of formal result that can be derived in any (closed) model category in the sense of Quillen (see [8] for a good exposition), and we have the following result. Serre fibrations of spectra are maps that satisfy the covering homotopy property with respect to the set of cone spectra

$$\{\Sigma_q^\infty CS^n \mid q \geqslant 0 \text{ and } n \geqslant 0\}.$$

Relative cell R-modules $M \longrightarrow N$ are constructed exactly like cell R-modules, except that one starts the inductive construction of $N = \cup N_n$ with $N_0 = M$.

We write q-cofibrations and q-fibrations here to avoid confusion with cofibrations (HEP) and fibrations (CHP); the ambiguous use of the same term for both the classical and the model theoretic concepts is one of the banes of the literature.

THEOREM 6.9. *The category of R-modules is a model category. Its weak equivalences are the maps of R-modules that are weak equivalences of spectra. Its q-cofibrations are the retracts of relative cell R-modules. Its q-fibrations are the maps $M \longrightarrow N$ such that $F_{\mathscr{L}}(S, M) \longrightarrow F_{\mathscr{L}}(S, N)$ is a Serre fibration of spectra.*

7. The smash product of R-modules and function R-modules

Continuing to work with our fixed commutative S-algebra R, we mimic the definition of tensor products of modules over algebras.

DEFINITION 7.1. For R-modules M and N, define $M \wedge_R N$ to be the coequalizer displayed in the following diagram of S-modules:

$$M \wedge_S R \wedge_S N \underset{\mathrm{id} \wedge_S \nu}{\overset{\mu \wedge_S \mathrm{id}}{\rightrightarrows}} M \wedge_S N \longrightarrow M \wedge_R N$$

where μ and ν are the given actions of R on M and N. Then $M \wedge_R N$ has a canonical R-module structure induced from the R-module structure of M or, equivalently, N.

Of course, S is a commutative S-algebra and our new $M \wedge_S N$ coincides with our old $M \wedge_S N$. The functor $\wedge_R$ preserves colimits in each of its variables, and smash products with spaces commute with $\wedge_R$, in the sense that

$$X \wedge (M \wedge_R N) \cong (X \wedge M) \wedge_R N.$$

Therefore the functor $\wedge_R$ commutes with cofibre sequences in each of its variables. We have analogous relations with smash products over S and an adjunction that can be thought of as completing Proposition 5.4.

PROPOSITION 7.2. *For an S-module K,*

$$K \wedge_S (M \wedge_R N) \cong (K \wedge_S M) \wedge_R N$$

and

$$\mathscr{M}_S(M \wedge_R N, K) \cong \mathscr{M}_R\big(N, F_S(M, K)\big).$$

The associativity, commutativity, and unity of the smash product over S is inherited by the smash product over R.

THEOREM 7.3. *Under the smash product over R, the category of R-modules is symmetric monoidal with unit R.*

We can deduce not only formal but also homotopical properties of $\wedge_R$ from corresponding properties of $\wedge_S$. As in Section 4, we use an isomorphism of universes $f : U \oplus U \to U$ to define the internal smash product $f_*(E \wedge F)$.

PROPOSITION 7.4. *Let X and Y be spectra and let N be an R-module. There is a natural isomorphism of R-modules*

$$\mathbf{F}_R X \wedge_R N \cong \mathbf{F}_S X \wedge_S N.$$

There is also a natural isomorphism of R-modules

$$\mathbf{F}_R X \wedge_R \mathbf{F}_R Y \cong \mathbf{F}_R f_*(X \wedge Y).$$

If M and N are cell R-modules, then $M \wedge_R N$ is a cell R-module with one $(p+q)$-cell for each p-cell of M and q-cell of N.

THEOREM 7.5. *If M is a cell R-module and $\phi : N \longrightarrow N'$ is a weak equivalence of R-modules, then*

$$\mathrm{id} \wedge_R \phi : M \wedge_R N \longrightarrow M \wedge_R N'$$

is a weak equivalence of R-modules.

We construct $\wedge_R$ as a functor $\mathscr{D}_R \times \mathscr{D}_R \to \mathscr{D}_R$ by approximating one of the variables by a cell R-module.

We have a function spectrum functor F_R to go with our smash product. It is defined as the equalizer of a certain pair of maps $F_S(M, N) \longrightarrow F_S(R \wedge_S M, N)$. The details are dictated by the expected adjunction. Again, $F_R(M, N)$ inherits a structure of R-module from M or, equivalently, N.

PROPOSITION 7.6. *For R-modules N and P and an S-module K,*

$$\mathscr{M}_R(K \wedge_S N, P) \cong \mathscr{M}_S\big(K, F_R(N, P)\big).$$

If M is an R-module, then

$$\mathscr{M}_R(M \wedge_R N, P) \cong \mathscr{M}_R\big(M, F_R(N, P)\big).$$

Therefore

$$F_R(M \wedge_R N, P) \cong F_R\big(M, F_R(N, P)\big).$$

Formal arguments from the adjunction, as in algebra, give a natural associative and unital composition pairing

$$\pi : F_R(N, P) \wedge_R F_R(M, N) \longrightarrow F_R(M, P). \tag{7.7}$$

Parenthetically, we note that this gives rise to a host of examples of S-algebras; in fact, R itself need not be commutative in the following result.

PROPOSITION 7.8. *For an R-module M, $F_R(M, M)$ is an S-algebra; For R-modules M and N, $F_R(M, N)$ is an $(F_R(N, N), F_R(M, M))$-bimodule.*

PROPOSITION 7.9. *Let X be a spectrum and M be an R-module. There is a natural isomorphism of R-modules*

$$F_R(\mathbb{F}_R X, M) \cong F_S(\mathbb{F}_S X, M).$$

The functor $F_R(M, N)$ converts colimits and cofibre sequences in M to limits and fibre sequences. It preserves limits and fibre sequences in N. Using the previous result to deal with sphere R-modules, we obtain the analog of Theorem 7.5.

PROPOSITION 7.10. *If M is a cell R-module and $\phi : N \longrightarrow N'$ is a weak equivalence of R-modules, then*

$$F_R(\mathrm{id}, \phi) : F_R(M, N) \longrightarrow F_R\big(M, N'\big)$$

is a weak equivalence of R-modules.

In the derived category $\mathscr{D}_R$, $F_R(M, N)$ means $F_R(\Gamma M, N)$, where ΓM is a cell approximation of M.

Summarizing, we obtain the following derived category level conclusion.

THEOREM 7.11. *The derived category $\mathscr{D}_R$ is symmetric monoidal under the product derived from $\wedge_R$, and*

$$\mathscr{D}_R(M \wedge_R N, P) \cong \mathscr{D}_R\big(M, F_R(N, P)\big).$$

There is a formal theory of duality (explained in [16, Chapter III]) that now applies to $\mathscr{D}_R$. We define the dual of an R-module M to be $D_R M = F_R(M, R)$. We have an evaluation map $\varepsilon : D_R M \wedge_R M \longrightarrow R$ and a map $\eta : R \to F_R(M, M)$, namely the adjoint of $\lambda : R \wedge_R M \longrightarrow M$. There is also a natural map

$$\nu : F_R(L, M) \wedge_R N \longrightarrow F_R(L, M \wedge_R N). \tag{7.12}$$

By composition with $F_R(\mathrm{id}, \lambda)$, ν specializes to a map

$$\nu : D_R M \wedge_R M \longrightarrow F_R(M, M). \tag{7.13}$$

We say that M is "strongly dualizable" if it has a coevaluation map $\bar\eta : R \longrightarrow M \wedge_R D_R M$ such that the following diagram commutes in $\mathscr{D}_R$:

$$\begin{tikzcd} R \arrow[r, "\bar\eta"] \arrow[d, "\eta"'] & M \wedge_R D_R M \arrow[d, "\tau"] \\ F_R(M, M) & D_R M \wedge_R M \arrow[l, "\nu"] \end{tikzcd} \tag{7.14}$$

The definition has many purely formal implications. The map ν of (7.12) is an isomorphism in $\mathscr{D}_R$ if either L or N is strongly dualizable. The map ν of (7.13) is an isomorphism if and only if M is strongly dualizable, and the coevaluation map $\bar\eta$ is then the composite $\tau\nu^{-1}\eta$ in (7.14). The natural map

$$\rho : M \longrightarrow D_R D_R M$$

is an isomorphism if M is strongly dualizable. The natural map

$$\wedge : F_R(M, N) \wedge_R F_R(M', N') \longrightarrow F_R(M \wedge_R M', N \wedge_R N')$$

is an isomorphism if M and M' are strongly dualizable or if M is strongly dualizable and $N = R$.

Say that a cell R-module N is a wedge summand up to homotopy of a cell R-module M if there is a homotopy equivalence of R-modules between M and $N \vee N'$ for some cell R-module N'. We say that N is semi-finite if it is a wedge summand up to homotopy of a finite cell R-module. In contrast with the usual stable homotopy category, a semi-finite R-module need not have the homotopy type of a finite cell R-module.

THEOREM 7.15. *A cell R-module is strongly dualizable if and only if it is semi-finite.*

The analogy with finitely generated projective modules in algebra should be clear.

8. Tor and Ext in topology and algebra

Still restricting for definiteness to a commutative S-algebra R and its modules, we define Tor and Ext groups as the homotopy groups of derived smash product and function modules.

DEFINITION 8.1. For R-modules M and N, define

$$\mathrm{Tor}_n^R(M,N) = \pi_n(M \wedge_R N)$$

and

$$\mathrm{Ext}_R^n(M,N) = \pi_{-n}\big(F_R(M,N)\big).$$

Note that $\mathrm{Tor}_*^R(M,N)$ and $\mathrm{Ext}_R^*(M,N)$ are $\pi_*(R)$-modules.

We emphasize that the smash product and function spectra are understood to be taken in the derived category $\mathscr{D}_R$. At this point in our exposition, we act as traditional topologists, taking it for granted that all spectra and modules are to be approximated as cell modules, without change of notation, whenever necessary. Various properties reminiscent of those of the classical Tor and Ext functors follow directly from the definition and the results of the previous sections. The intuition is that the definition gives an analogue of the differential Tor and Ext functors (alias hyperhomology and cohomology functors) in the context of differential graded modules over differential graded algebras. In particular, the grading should not be thought of as the resolution grading of the classical torsion product, but rather as a total grading that sums a resolution degree and an internal degree; this idea will be made precise by the grading of the spectral sequences that we shall describe for the calculation of these functors.

PROPOSITION 8.2. $\mathrm{Tor}_*^R(M,N)$ *satisfies the following properties.*

(i) *If R, M, and N are connective, then* $\mathrm{Tor}_n^R(M,N) = 0$ *for* $n < 0$.

(ii) *A cofibre sequence $N' \to N \to N''$ gives rise to a long exact sequence*

$$\cdots \to \mathrm{Tor}_n^R(M,N') \to \mathrm{Tor}_n^R(M,N) \to \mathrm{Tor}_n^R(M,N'') \\ \to \mathrm{Tor}_{n-1}^R(M,N') \to \cdots.$$

(iii) $\mathrm{Tor}_*^R(M,R) \cong \pi_*(M)$ *and, for a spectrum X,*

$$\mathrm{Tor}_*^R(M,\mathbb{F}X) \cong \pi_*(M \wedge X).$$

(iv) *The functor* $\mathrm{Tor}_*^R(M,?)$ *carries wedges to direct sums.*

The commutativity and associativity relations for the smash product imply various further properties. We content ourselves with the following examples:

$$\mathrm{Tor}_*^R(M,N) \cong \mathrm{Tor}_*^R(N,M)$$

and

$$\mathrm{Tor}_*^R(M \wedge_R N, P) \cong \mathrm{Tor}_*^R(M, N \wedge_R P).$$

Say that a spectrum N is coconnective if $\pi_n(N) = 0$ for $n > 0$.

PROPOSITION 8.3. $\mathrm{Ext}^*_R(M,N)$ *satisfies the following properties.*

(i) *If R and M are connective and N is coconnective, then* $\mathrm{Ext}^n_R(M,N) = 0$ *for* $n < 0$.

(ii) *Fibre sequences $N' \to N \to N''$ and cofibre sequences $M' \to M \to M''$ give rise to long exact sequences*

$$\cdots \to \mathrm{Ext}^n_R(M,N') \to \mathrm{Ext}^n_R(M,N) \to \mathrm{Ext}^n_R(M,N'') \\ \to \mathrm{Ext}^{n+1}_R(M,N') \to \cdots$$

and

$$\cdots \to \mathrm{Ext}^n_R(M'',N) \to \mathrm{Ext}^n_R(M,N) \to \mathrm{Ext}^n_R(M',N) \\ \to \mathrm{Ext}^{n+1}_R(M'',N) \to \cdots.$$

(iii) $\mathrm{Ext}^*_R(R,N) \cong \pi_{-*}(N)$ *and, for a spectrum X,*

$$\mathrm{Ext}^*_R(\mathbf{F}X,N) \cong \pi_{-*}\big(F(X,N)\big).$$

(iv) *The functor* $\mathrm{Ext}^*_R(?,N)$ *carries wedges to products.*

Passing to homotopy groups from the pairings (7.7), we obtain the following further property. As usual, for a spectrum E, abbreviate

$$E_n = \pi_n(E) = E^{-n}.$$

PROPOSITION 8.4. *There is a natural, associative, and unital system of pairings of R^*-modules*

$$\pi^* : \mathrm{Ext}^*_R(M,N) \otimes_{R^*} \mathrm{Ext}^*_R(L,M) \longrightarrow \mathrm{Ext}^*_R(L,N).$$

The formal duality theory of the previous section implies the following result, together with various other such isomorphisms.

PROPOSITION 8.5. *For a finite cell R-module M and any R-module N,*

$$\mathrm{Tor}^R_n(D_R M, N) \cong \mathrm{Ext}^{-n}_R(M,N).$$

Thinking of the derived category $\mathscr{D}_R$ as a stable homotopy category, we may change notations and reinterpret the functors Tor and Ext as prescribing homology and cohomology theories in this category.

DEFINITION 8.6. Let M and E be R-modules. Define

$$E^R_n(M) = \pi_n(E \wedge_R M) \quad \text{and} \quad E^n_R(M) = \pi_{-n}\big(F_R(M,E)\big).$$

The properties of Tor and Ext translate directly to statements about homology and cohomology. All of the standard homotopical machinery is available to us, and the previous result now takes the form of Spanier–Whitehead duality.

COROLLARY 8.7. *For a finite cell R-module M and any R-module E,*

$$E_n^R(D_R M) \cong E_R^{-n}(M).$$

Since the equivalence between the classical stable homotopy category and the derived category of S-modules preserves smash products and function spectra, we obtain all of the usual homology and cohomology theories by taking $R = S$.

We also obtain the classical algebraic Tor and Ext groups as special cases, by specialization to Eilenberg–MacLane spectra. Thus let R be a discrete commutative ring for a moment. Recall that HR denotes a spectrum whose zeroth homotopy group is R and whose remaining homotopy groups are zero. It follows from multiplicative infinite loop space theory [20] that the Eilenberg–MacLane spectrum $HR = K(R, 0)$ is an E_∞ ring spectrum. Analogously, if M is an R-module, then HM can be constructed as an HR-module. We shall see a quick and easy construction shortly. Granting this, we have the following result.

THEOREM 8.8. *For a discrete commutative ring R and R-modules M and N,*

$$\mathrm{Tor}_*^R(M, N) \cong \mathrm{Tor}_*^{HR}(HM, HN)$$

and

$$\mathrm{Ext}_R^*(M, N) \cong \mathrm{Ext}_{HR}^*(HM, HN).$$

Under the second isomorphism, the topologically defined pairing

$$\mathrm{Ext}_{HR}^*(HM, HN) \otimes_R \mathrm{Ext}_{HR}^*(HL, HM) \longrightarrow \mathrm{Ext}_{HR}^*(HL, HN)$$

coincides with the algebraic Yoneda product.

The proof is clear enough: we just check the axioms for Tor and Ext.

We can elaborate this result to an equivalence of derived categories. Recall from [28] or [15, Chapter III] that the derived category $\mathscr{D}_R$ is obtained from the homotopy category of chain complexes over R by formally inverting the quasi-isomorphisms, exactly as we obtained the category $\mathscr{D}_{HR}$ from the homotopy category of HR-modules by inverting the weak equivalences. The algebraic theory of cell and CW chain complexes over R in the latter source makes the analogy precise and gives a treatment of tensor products and Hom functors in $\mathscr{D}_R$ that exactly parallels our treatment of $\wedge_{HR}$ and F_{HR}. The proof of the equivalence is quite easy. The category $\mathscr{D}_{HR}$ is equivalent to the homotopy category of CW HR-modules and cellular maps. It is a simple matter to see that CW HR-modules have associated chain complexes. This gives a functor $\mathscr{D}_{HR} \longrightarrow \mathscr{D}_R$. An inverse functor Φ is obtained by applying Brown's representability theorem. Indeed, for a given chain complex X, the functor k on $\mathscr{D}_{HR}$ specified by $k(M) = \mathscr{D}_R(C_*(M), X)$ satisfies the hypotheses of that result, and we let $\Phi(X)$ represent this functor. Specialization to R-modules regarded as chain complexes concentrated in degree zero gives the promised construction of Eilenberg–MacLane HR-modules from R-modules.

THEOREM 8.9. *The cellular chain functor C_* on HR-modules induces an equivalence of categories $\mathscr{D}_{HR} \longrightarrow \mathscr{D}_R$. The functor C_* satisfies $H_*(C_*(M)) \cong \pi_*(M)$ and carries the functors $\wedge_{HR}$ and F_{HR} to the functors $\otimes_R$ and* Hom_R. *The inverse equivalence Φ satisfies $\pi_*(\Phi(X)) \cong H_*(X)$ and carries the functors $\otimes_R$ and* Hom_R *to the functors $\wedge_{HR}$ and F_{HR}.*

PROOF. By construction, we have an adjunction

$$\mathscr{D}_R\big(C_*(M), X\big) \cong \mathscr{D}_{HR}\big(M, \Phi(X)\big),$$

and one checks that its unit and counit are isomorphisms. The statements relating $\wedge_{HR}$ and F_{HR} to $\otimes_R$ and Hom_R are all consequences of the fact that if M and N are CW HR-modules, then $M \wedge_{HR} N$ is a CW HR-module such that

$$C_*(M \wedge_{HR} N) \cong C_*(M) \otimes_R C_*(N).$$

$\square$

9. Universal coefficient and Künneth spectral sequences

Returning to our general commutative S-algebra R, we find spectral sequences for the calculation of our Tor and Ext groups that are analogous to the Eilenberg–Moore (or hyperhomology) spectral sequences in differential homological algebra. Compare [9], [13], [15]. They may be viewed as giving universal coefficient and Künneth spectral sequences for homology and cohomology theories on R-modules, and they specialize to give such spectral sequences for homology and cohomology theories on spectra.

THEOREM 9.1. *For R-modules M and N, there are natural spectral sequences of differential R_*-modules*

$$E^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(M_*, N_*) \Longrightarrow \mathrm{Tor}^R_{p+q}(M, N)$$

and

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{R^*}(M^*, N^*) \Longrightarrow \mathrm{Ext}^{p+q}_R(M, N).$$

Moreover, the pairing $F_R(M,N) \wedge_R F_R(L,M) \to F_R(L,N)$ induces a pairing of spectral sequences that coincides with the algebraic Yoneda pairing

$$\mathrm{Ext}^{*,*}_{R^*}(M^*, N^*) \otimes_{R^*} \mathrm{Ext}^{*,*}_{R^*}(L^*, M^*) \longrightarrow \mathrm{Ext}^{*,*}_{R^*}(L^*, N^*)$$

on the E_2-level and that converges to the induced pairing of Ext groups.

The Tor spectral sequence is of standard homological type, with

$$d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

It lies in the right half-plane, and it converges strongly. The Ext spectral sequence is of standard cohomological type, with

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

It lies in the right half plane. In the language of Boardman [5] (see also [12, App. B]), it is conditionally convergent. It therefore converges strongly if, for each fixed (p,q), there are only finitely many r such that d_r is nonzero on $E_r^{p,q}$.

Setting $M = \mathbb{F}_R X$ in the two spectral sequences of Theorem 8.1, we obtain a universal coefficient spectral sequence. We have written the stars to indicate the way the grading is usually thought of in cohomology.

THEOREM 9.2 (Universal coefficient). *For an R-module N and any spectrum X, there are spectral sequences of the form*

$$\mathrm{Tor}^{R_*}_{*,*}(R_*(X), N_*) \Longrightarrow N_*(X)$$

and

$$\mathrm{Ext}^{*,*}_{R^*}(R_{-*}(X), N^*) \Longrightarrow N^*(X).$$

Of course, replacing R and N by Eilenberg–MacLane spectra HR and HN for a ring R and R-module N, we obtain the classical universal coefficient theorems. Replacing N by $\mathbf{F}_R Y$ and by $F_R(\mathbb{F}_R Y, R)$ in the two universal coefficient spectral sequences, we arrive at Künneth spectral sequences.

THEOREM 9.3 (Künneth). *For any spectra X and Y, there are spectral sequences of the form*

$$\mathrm{Tor}^{R_*}_{*,*}(R_*(X), R_*(Y)) \Longrightarrow R_*(X \wedge Y)$$

and

$$\mathrm{Ext}^{*,*}_{R^*}(R_{-*}(X), R^*(Y)) \Longrightarrow R^*(X \wedge Y).$$

Adams [1] first observed that one can derive Künneth spectral sequences from universal coefficient spectral sequences, and he observed that, by duality, the four spectral sequences of Theorems 9.2 and 9.3 imply two more universal coefficient and two more Künneth spectral sequences. He derived spectral sequences of this sort under the hypothesis that his given ring spectrum E is the colimit of finite subspectra E_α such that $H^*(E_\alpha; E^*)$ is E^*-projective and the Atiyah–Hirzebruch spectral sequence converging from $H^*(E_\alpha; E^*)$ to $E^*(E_\alpha)$ satisfies $E_2 = E_\infty$. Of course, this is an ad hoc calculational hypothesis that requires case-by-case verification. It covers some cases that are not covered by the results above, and conversely. The cited paper of Adams, and his later book [2], are prime sources for the first flowering of stable homotopy theory. While some of their foundational parts may be obsolete, their applications and calculational parts certainly are not.

The following generalized Künneth theorem admits a number of variants; see [11].

THEOREM 9.4. *Let E and R be commutative S-algebras and M and N be R-modules. Then there is a spectral sequence of differential $E_*(R)$-modules of the form*

$$\mathrm{Tor}^{E_*(R)}_{p,q}(E_*(M), E_*(N)) \Longrightarrow E_{p+q}(M \wedge_R N).$$

10. Algebraic constructions in the derived category of R-modules

If we replace the pair (S, R) by a pair (R, A) in Definition 6.1, we arrive at the notion of an algebra A over a commutative S-algebra R. For example, the S-algebras $F_R(M, M)$ of Proposition 7.8 are actually R-algebras. Again, if A is an algebra over a discrete commutative ring R, then HA is an HR-algebra. Proceeding in this line, we can, for instance, construct R-modules whose homotopy groups realize the Hochschild homology of A with coefficients in (A, A)-bimodules.

However, we now proceed in a more homotopical direction, thinking of the derived category of R-modules as an analog of the stable homotopy category. From this point of view, we have the notion of an R-ring spectrum, which is just like the classical notion of a ring spectrum in the stable homotopy category.

DEFINITION 10.1. An R-ring spectrum A is an R-module A with unit $\eta : R \longrightarrow A$ and product $\phi : A \wedge_R A \longrightarrow A$ in $\mathscr{D}_R$ such that the following left and right unit diagram commutes in $\mathscr{D}_R$.

$$\begin{tikzcd}
R \wedge_R A \arrow[r, "\eta \wedge \mathrm{id}"] \arrow[dr, "\lambda"'] & A \wedge_R A \arrow[d, "\phi"] & A \wedge_R R \arrow[l, "\mathrm{id} \wedge \eta"'] \arrow[dl, "\lambda\tau"] \\
& A &
\end{tikzcd}$$

A is associative or commutative if the appropriate diagram commutes in $\mathscr{D}_R$.

LEMMA 10.2. *If A and B are R-ring spectrum, then so is $A \wedge_R B$. If A and B are associative or commutative, then so is $A \wedge_R B$.*

By neglect of structure, an R-ring spectrum A is a ring spectrum in the sense of classical stable homotopy theory; its unit is the composite of the unit of R and the unit of A and its product is the composite of the product of A and the canonical map

$$A \wedge A \simeq A \wedge_S A \longrightarrow A \wedge_R A.$$

Similarly, for an R-algebra A, we have the evident homotopical notion of an A-module spectrum. These structures play a role in the study of $\mathscr{D}_R$ analogous to the role played by ring spectra and their module spectra in classical stable homotopy theory. When $R = S$,

S-ring spectra and their module spectra are equivalent to classical ring spectra and their module spectra.

We show in this section how to construct quotients M/IM and localizations $M[Y^{-1}]$ of modules over a commutative S-algebra R and indicate in the next section when these constructions inherit a structure of R-ring spectrum from an R-ring spectrum structure on M. When specialized to MU, these results give highly structured versions of spectra that in the past were constructed by means of the Baas–Sullivan theory of manifolds with singularities or the Landweber exact functor theorem. At least at odd primes, the results give an entirely satisfactory, and very simple, treatment of MU-ring structures on the resulting MU-modules.

We are interested in homotopy groups, and we make use of the isomorphisms

$$M_n = h\mathscr{S}(S^n, M) \cong h\mathscr{M}_S(S^n_S, M) \cong h\mathscr{M}_R(S^n_R, M) \tag{10.3}$$

to represent elements as maps of R-modules. For $x \in R_n$, the composite map of R-modules

$$S^n_R \wedge_R M \xrightarrow{x \wedge \mathrm{id}} R \wedge_R M \xrightarrow{\lambda} M \tag{10.4}$$

is a module theoretic version of the map $x\cdot : \Sigma^n M \longrightarrow M$, and we agree to write $\Sigma^n M$ for $S^n_R \wedge_R M$ in this section. By Proposition 7.4, $S^n_R \wedge_R M$ is isomorphic as an R-module to $S^n_S \wedge_S M$ and, by Theorem 4.7, $S^n_S \wedge_S M$ is weakly equivalent as a spectrum to $S^n \wedge M$. Therefore, the R-module $\Sigma^n M$ is a model for the spectrum level suspension of M.

DEFINITION 10.5. Define M/xM to be the cofibre of the map (10.4) and let $\rho : M \longrightarrow M/xM$ be the canonical map. Inductively, for a finite sequence $\{x_1, \ldots, x_n\}$ of elements of R_*, define

$$M/(x_1, \ldots, x_n)M = N/x_n N, \quad \text{where} \quad N = M/(x_1, \ldots, x_{n-1})M.$$

For a (countably) infinite sequence $X = \{x_i\}$, define M/XM to be the telescope of the $M/(x_1, \ldots, x_n)M$, where the telescope is taken with respect to the successive canonical maps ρ.

Clearly we have a long exact sequence

$$\cdots \longrightarrow \pi_{q-n}(M) \xrightarrow{x\cdot} \pi_q(M) \xrightarrow{\rho_*} \pi_q(M/xM) \longrightarrow \pi_{q-n-1}(M) \longrightarrow \cdots. \tag{10.6}$$

If x is not a zero divisor for $\pi_*(M)$, then ρ_* induces an isomorphism of R_*-modules

$$\pi_*(M)/x \cdot \pi_*(M) \cong \pi_*(M/xM). \tag{10.7}$$

If $\{x_1, \ldots, x_n\}$ is a regular sequence for $\pi_*(M)$, in the sense that x_i is not a zero divisor for $\pi_*(M)/(x_1, \ldots, x_{i-1})\pi_*(M)$ for $1 \leqslant i \leqslant n$, then

$$\pi_*(M)/(x_1, \ldots, x_n)\pi_*(M) \cong \pi_*\big(M/(x_1, \ldots, x_n)M\big), \tag{10.8}$$

and similarly for a possibly infinite regular sequence $X = \{x_i\}$. The following result implies that M/XM is independent of the ordering of the elements of the set X. We write R/X instead of R/XR.

LEMMA 10.9. *For a set X of elements of R_*, there is a natural weak equivalence*

$$(R/X) \wedge_R M \longrightarrow M/XM.$$

In particular, for a finite set $X = \{x_1, \ldots, x_n\}$,

$$R/(x_1, \ldots, x_n) \simeq (R/x_1) \wedge_R \cdots \wedge_R (R/x_n).$$

If I denotes the ideal generated by X, then it is reasonable to define

$$M/IM = M/XM. \tag{10.10}$$

However, this notation must be used with caution since, if we fail to restrict attention to regular sequences X, the homotopy type of M/XM will depend on the set X and not just on the ideal it generates. For example, quite different modules are obtained if we repeat a generator x_i of I in our construction.

We next construct localizations of R-modules at countable multiplicatively closed subsets Y of R_*. Let $\{y_i\}$ be any cofinal sequence of Y, with $y_i \in R_{n_i}$, so that every $y \in Y$ divides some y_i. We may represent y_i by an R-map $S^0_R \longrightarrow S^{-n_i}_R$, which we also denote by y_i. Let $q_0 = 0$ and, inductively, $q_i = q_{i-1} + n_i$. The R-map

$$S^0_R \wedge_R M \xrightarrow{y_i \wedge \mathrm{id}} S^{-n_i}_R \wedge_R M$$

represents $y_i \cdot$. Smashing over R with $S^{-q_{i-1}}_R$, we obtain a sequence of R-maps

$$S^{-q_{i-1}}_R \wedge_R M \longrightarrow S^{-q_i}_R \wedge_R M. \tag{10.11}$$

DEFINITION 10.12. Define the localization of M at Y, denoted $M[Y^{-1}]$, to be the telescope of the sequence of maps (10.11). Since $M \cong S^0_R \wedge_R M$ in $\mathscr{D}_R$, we may regard the inclusion of the initial stage $S^0_R \wedge_R M$ of the telescope as a natural map $\lambda : M \longrightarrow M[Y^{-1}]$.

Since homotopy groups commute with localization, we see immediately that λ induces an isomorphism of R_*-modules

$$\pi_*\big(M[Y^{-1}]\big) \cong \pi_*(M)[Y^{-1}]. \tag{10.13}$$

As in Lemma 10.9, the localization of M is the smash product of M with the localization of R.

LEMMA 10.14. *For a multiplicatively closed set Y of elements of R_*, there is a natural equivalence*

$$R[Y^{-1}] \wedge_R M \longrightarrow M[Y^{-1}].$$

Moreover, $R[Y^{-1}]$ is independent of the ordering of the elements of Y. For sets X and Y, $R[(X \cup Y)^{-1}]$ is equivalent to the composite localization $R[X^{-1}][Y^{-1}]$.

11. R-ring structures on localizations and on quotients by ideals

The behavior of localizations with respect to R-ring structures is immediate.

PROPOSITION 11.1. *Let Y be a multiplicatively closed set of elements of R_*. If A is an R-ring spectrum, then $A[Y^{-1}]$ is an R-ring spectrum such that $\lambda : A \longrightarrow A[Y^{-1}]$ is a map of R-ring spectra. If A is associative or commutative, then so is $A[Y^{-1}]$.*

PROOF. By Lemmas 10.2 and 10.14, it suffices to observe that $R[Y^{-1}]$ is an associative and commutative R-ring spectrum with unit λ and product the equivalence

$$R[Y^{-1}] \wedge_R R[Y^{-1}] \simeq R[Y^{-1}][Y^{-1}] \simeq R[Y^{-1}].$$

□

This doesn't work for quotients since $(R/X)/X$ is not equivalent to R/X. However, we can analyze the problem by analyzing the deviation, and, by Lemma 10.9, we may as well work one element at a time. We have a necessary condition for R/x to be an R-ring spectrum that is familiar from classical stable homotopy theory.

LEMMA 11.2. *Let A be an R-ring spectrum. If A/xA admits an R-ring spectrum structure such that $\rho : A \longrightarrow A/xA$ is a map of R-ring spectra, then $x : A/xA \longrightarrow A/xA$ is null homotopic as a map of R-modules.*

Thus, for example, the Moore spectrum $S/2$ is not an S-ring spectrum since the map $2 : S/2 \longrightarrow S/2$ is not null homotopic. To give a criterion for when R/x does have an R-ring spectrum structure, we first note an easy formal lemma.

LEMMA 11.3. *Let $\rho : R \longrightarrow M$ be any map of R-modules. Then*

$$(\rho \wedge \mathrm{id}) \circ \rho \simeq (\mathrm{id} \wedge \rho) \circ \rho : R \longrightarrow M \wedge_R M.$$

THEOREM 11.4. *Let $x \in R_m$ and assume that $\pi_{m+1}(R/x) = 0$ and $\pi_{2m+1}(R/x) = 0$. Then R/x admits a structure of R-ring spectrum with unit $\rho : R \longrightarrow R/x$. Therefore A/XA admits a structure of R-ring spectrum such that $\rho : A \longrightarrow A/XA$ is a map of R-ring spectra for every R-ring spectrum A and every sequence X of elements of R_* such that $\pi_{m+1}(R/x) = 0$ and $\pi_{2m+1}(R/x) = 0$ if $x \in X$ has degree m.*

PROOF. Consider the following diagram in the derived category $\mathscr{D}_R$:

$$
\begin{tikzcd}
 & & & \Sigma^{2m+1}R \arrow[d, "x"] & \\
 & & & \Sigma^{m+1}R \arrow[r, "x"] \arrow[d, "\rho"] \arrow[dl, dashed, "\nu"'] & \Sigma R \arrow[d, "\rho"] \\
\Sigma^m(R/x) \arrow[r, "x"] & R/x \arrow[r, shift left, "\rho\wedge \mathrm{id}"] & (R/x)\wedge_R(R/x) \arrow[l, shift left, "\phi"] \arrow[r, shift left, "\pi"] & \Sigma^{m+1}(R/x) \arrow[l, shift left, "\sigma"] \arrow[r, "x"] \arrow[d, "\pi'"] & \Sigma(R/x) \\
 & & & \Sigma^{2m+2}R. &
\end{tikzcd}
\tag{11.5}
$$

The map x is that specified by (10.4). The bottom row is the cofibre sequence that results from the equivalence

$$(R/x)\wedge_R(R/x)\simeq(R/x)/x$$

of Lemma 10.9, and the column is also a cofibre sequence. The composite $x\circ\rho$ is null homotopic since $\rho\circ x$ is null homotopic and the square commutes. Therefore there is a map ν such that $\pi\circ\nu=\rho$, and ν is unique since $\pi_{m+1}(R/x)=0$. Since $\pi\circ\nu\circ x=\rho\circ x=0$, $\nu\circ x$ factors through a map $\Sigma^{2m+1}R\longrightarrow R/x$. Since $\pi_{2m+1}(R/x)=0$, such maps are null homotopic. Thus $\nu\circ x$ is null homotopic. Therefore there is a map σ such that $\sigma\circ\rho=\nu$. Now $\pi\circ\sigma\circ\rho=\pi\circ\nu=\rho$, hence $(\pi\circ\sigma-\mathrm{id})\rho=0$. Therefore $\pi\circ\sigma-\mathrm{id}$ factors through a map $\Sigma^{2m+2}R\longrightarrow\Sigma^{m+1}(R/x)$. Again, such maps are null homotopic. Therefore $\pi\circ\sigma=\mathrm{id}$. Thus the bottom cofibre sequence splits (proving in passing that $x:\Sigma^m(R/x)\longrightarrow R/x$ is null homotopic, as it must be). A choice ϕ of a splitting gives a product on R/x. The unit condition $\phi\circ(\rho\wedge\mathrm{id})=\mathrm{id}$ is automatic. To see that $\phi\circ(\mathrm{id}\wedge\rho)=\mathrm{id}$, we observe that, by the lemma,

$$\phi\circ(\mathrm{id}\wedge\rho)-\mathrm{id})\circ\rho=\phi\circ(\mathrm{id}\wedge\rho-\rho\wedge\mathrm{id})\circ\rho=0.$$

Therefore $\phi\circ(\mathrm{id}\wedge\rho)-\mathrm{id}$ factors through a map $\Sigma^{m+1}R\longrightarrow R/x$. Again, such maps are null homotopic, hence $\phi\circ(\mathrm{id}\wedge\rho)=\mathrm{id}$. This completes the proof that R/x is an R-ring spectrum with unit ρ. The rest follows from Lemmas 10.9 and 10.2. $\square$

The product on R/x can be described a little more concretely. The wedge sum

$$(\rho\wedge\mathrm{id})\vee\sigma:(R/x)\vee\Sigma^{m+1}(R/x)\longrightarrow(R/x)\wedge_R(R/x)\tag{11.6}$$

is an equivalence. The product ϕ restricts to the identity on the first wedge summand and to the trivial map on the second wedge summand. Thus the product is determined by the choice of σ, and two choices of σ differ by a composite

$$\Sigma^{m+1}(R/x) \xrightarrow{\pi'} \Sigma^{2m+2}R \longrightarrow (R/x) \wedge_R (R/x). \tag{11.7}$$

By the splitting (10.6) and the assumption that $\pi_{m+1}(R/x) = 0$, we can view the second map as an element of $\pi_{2m+2}(R/x)$. If x is not a zero divisor, then $\pi'_* = 0$ on homotopy groups and any two products have the same effect on homotopy groups.

For an R-ring spectrum A and an element x as in the theorem, we give $A/xA \simeq (R/x) \wedge_R A$ the product induced by one of our constructed products on R/x and the given product on A. We refer to any such product as a "canonical" product on A/xA. Observe that, by first using the product on A, the product on A/xA can be factored through

$$\phi \wedge_R \mathrm{id} : (R/x) \wedge_R (R/x) \wedge_R A \longrightarrow (R/x) \wedge_R A.$$

This allows us to smash any diagram giving information about the product on R/x with A and so obtain information about the product on A/xA. Obviously any diagram so constructed is a diagram of right A-modules via the product action of A on itself. This smashing with A can kill obstructions. Clearly, a map of A-modules $\Sigma^q A \longrightarrow M$ is determined by its restriction $S^q \longrightarrow M$ along the unit of A regarded as a map of spectra (or S-modules), which is just an element of $\pi_q(M)$. This leads to the following result.

THEOREM 11.8. *Let $x \in R_m$ and assume that $\pi_{m+1}(R/x) = 0$ and $\pi_{2m+1}(R/x) = 0$. Let A be an R-ring spectrum and assume that $\pi_{2m+2}(A/xA) = 0$. Then there is a unique canonical product on A/xA. If A is commutative, then A/xA is commutative. If A is associative and $\pi_{3m+3}(A/xA) = 0$, then A/xA is associative.*

PROOF. The second arrow of (11.7) becomes zero after smashing with A since it is then given by an element of $\pi_{2m+2}(A/xA) = 0$. This proves the uniqueness statement. The commutativity statement follows since if ϕ is a canonical product on A/xA, then so is $\phi\tau$. The associativity statement requires consideration of the restriction of the iterated product to the wedge summands of $A/xA \wedge_R A/xA \wedge_R A/xA$. The details are similar to, but simpler than, those in the proof of Theorem 11.4. □

Iterating and observing that passage to telescopes can kill obstructions, we arrive at the following fundamental conclusion.

THEOREM 11.9. *Assume that $R_i = 0$ if i is odd. Let X be a sequence of nonzero divisors in R_* such that $\pi_*(R/X)$ is concentrated in degrees congruent to zero* mod 4. *Then R/X has a unique canonical structure of R-ring spectrum, and it is commutative and associative.*

12. The specialization to MU-modules and MU-ring spectra

The classical Thom spectra arise in nature as E_∞ ring spectra. In fact, it was inspection of their prespectrum level definition in terms of Grassmannians that first led to the theory of E_∞ ring spectra [19]. Applying the functor $S \wedge_{\mathscr{L}} (?)$, we obtain models for Thom spectra which are commutative S-algebras. Of course, the homotopy groups of MU are concentrated in even degrees, and every nonzero element is a non zero divisor. Thus the results above have the following immediate corollary.

THEOREM 12.1. *Let X be a regular sequence in MU_*, let I be the ideal generated by X, and let Y be any sequence in MU_*. Then there is an MU-ring spectrum $(MU/X)[Y^{-1}]$ and a natural map of MU-ring spectra (the unit map)*

$$\eta : MU \longrightarrow (MU/X)[Y^{-1}]$$

such that

$$\eta_* : MU_* \longrightarrow \pi_*((MU/X)[Y^{-1}])$$

realizes the natural homomorphism of MU_*-algebras

$$MU_* \longrightarrow (MU_*/I)[Y^{-1}].$$

If MU_*/I is concentrated in degrees congruent to zero mod 4, *then there is a unique canonical product on $(MU/X)[Y^{-1}]$, and this product is commutative and associative.*

In comparison with earlier constructions of this sort based on the Baas–Sullivan theory of manifolds with singularities or on Landweber's exact functor theorem (where it applies), we have obtained a simpler proof of a substantially stronger result. We emphasize that an MU-ring spectrum is a much richer structure than just a ring spectrum and that commutativity and associativity in the MU-ring spectrum sense are much more stringent conditions than mere commutativity and associativity of the underlying ring spectrum.

We illustrate by explaining how BP appears in this context. Fix a prime p and write $(?)_p$ for localization at p. Let BP be the Brown–Peterson spectrum at p. We are thinking of Quillen's idempotent construction [24], and we have the splitting maps $i : BP \longrightarrow MU_p$ and $e : MU_p \longrightarrow BP$. These are maps of commutative and associative ring spectra such that $e \circ i = \mathrm{id}$. Let I be the kernel of the composite

$$MU_* \longrightarrow MU_{p*} \longrightarrow BP_*.$$

Then I is generated by a regular sequence X, and our MU/X is a canonical integral version of BP. For the moment, let $BP' = (MU/X)_p$. Let $\xi : BP \longrightarrow BP'$ be the composite

$$BP \xrightarrow{i} MU_p \xrightarrow{\eta_p} BP'.$$

It is immediate that ξ is an equivalence. In effect, since we have arranged that η_p has the same effect on homotopy groups as e, ξ induces the identity map of $(MU_*/I)_p$ on homotopy groups. By the splitting of MU_p and the fact that self-maps of MU_p are determined by their effect on homotopy groups [2, II.9.3], maps $MU_p \longrightarrow BP$ are determined by their effect on homotopy groups. This implies that $\xi \circ e = \eta_p : MU_p \longrightarrow BP'$. The product on BP is the composite

$$BP \wedge BP \xrightarrow{i \wedge i} MU_p \wedge MU_p \xrightarrow{\phi} MU_p \xrightarrow{e} BP.$$

Since η_p is a map of MU-ring spectra and thus of ring spectra, a trivial diagram chase now shows that the equivalence $\xi : BP \longrightarrow BP'$ is a map of ring spectra.

We conclude that our BP' is a model for BP that is an MU-ring spectrum, commutative and associative if $p > 2$. The situation for $p = 2$ is interesting. We conclude from the equivalence that BP' is commutative and associative as a ring spectrum, although we do not know that it is commutative or associative as an MU-ring spectrum.

Recall that $\pi_*(BP) = \mathbb{Z}_{(p)}[v_i | deg(v_i) = 2(p^i - 1)]$, where the generators v_i come from $\pi_*(MU)$ (provided that we use the Hazewinkel generators). We list a few of the spectra derived from BP, with their coefficient rings. Let $\mathbb{F}_p$ denote the field with p elements.

$BP\langle n \rangle$	$\mathbb{Z}_{(p)}[v_1, \ldots, v_n]$	$E(n)$	$\mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$
$P(n)$	$\mathbb{F}_p[v_n, v_{n+1}, \ldots]$	$B(n)$	$\mathbb{F}_p[v_n^{-1}, v_n, v_{n+1}, \ldots]$
$k(n)$	$\mathbb{F}_p[v_n]$	$K(n)$	$\mathbb{F}_p[v_n, v_n^{-1}]$

By the method just illustrated, we can construct canonical integral versions of the $BP\langle n \rangle$ and $E(n)$. All of these spectra fit into the context of Theorem 11.1. If $p > 2$, they all have unique canonical commutative and associative MU-ring spectrum structures. Further study is needed when $p = 2$. In any case, this theory makes it unnecessary to appeal to Baas–Sullivan theory or to Landweber's exact functor theorem for the construction and analysis of spectra such as these.

With more sophisticated techniques, the second author [14] has proven that BP can be constructed as an commutative S-algebra, and in fact admits uncountably many distinct such structures. There is much other ongoing work on the construction and application of new commutative S-algebras, by Hopkins, Miller, McClure, and others, and we have recently proven that the periodic K-theory spectra KO and KU can be constructed as commutative S-algebras. The enriched multiplicative structures on rings and modules that we have discussed are rapidly becoming a standard tool in the study of periodicity phenomena in stable homotopy theory.

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CHAPTER 7

Completions in Algebra and Topology

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0. Introduction

Localization and completion are among the fundamental first tools in commutative algebra. They play a correspondingly fundamental role in algebraic topology. Localizations and completions of spaces and spectra have been central tools since the 1970's. Some basic references are [3], [17], [24]. These constructions start from ideals in the ring of integers and are very simple algebraically since $\mathbb{Z}$ is a principal ideal domain. Localizations and completions that start from ideals in the representation ring or the Burnside ring of a compact Lie group play a correspondingly central role in equivariant topology. These rings are still relatively simple algebraically since, when G is finite, they are Noetherian and of Krull dimension one. A common general framework starts from ideals in the coefficient ring of a generalized cohomology theory. We shall explain some old and new algebra that arises in this context, and we will show how this algebra can be mimicked topologically. The topological constructions require the foundations described in the previous article, which deals with the algebraically familiar theory of localization at multiplicatively closed subsets. We here explain the deeper and less familiar theory of completion, together with an ideal theoretic variant of localization. There is a still more general theory of localization of spaces and spectra at spectra, due to Bousfield [1], [2], and we shall see how our theory of localizations and completions with respect to ideals in coefficient rings fits into this context.

Consider an ideal I in a commutative ring A and the completions $M_I^\wedge = \lim M/I^kM$ of R-modules M. The algebraic fact that completion is not exact in general forces topologists to work with the derived functors of completion, and we shall explain how topological completions of spectra mimic an algebraic description of these derived functors in terms of "local homology groups". These constructs are designed for the study of cohomology theories, and we will describe dual constructs that are designed for the study of homology theories and involve Grothendieck's local cohomology groups. There are concomitant notions of "Čech homology and cohomology groups", which fit into algebraic fibre sequences that we shall mimic by interesting fibre sequences of spectra. These lead to a theory of localizations of spectra away from ideals. When specialized to MU-module spectra, these new localizations shed considerable conceptual light on the chromatic filtration that is at the heart of the study of periodic phenomena in stable homotopy theory.

1. Algebraic definitions: Local and Čech cohomology and homology

Suppose to begin with that A is a commutative Noetherian ring and that $I = (\alpha_1, \ldots, \alpha_n)$ is an ideal in A. There are a number of cases of topological interest where we must deal with non-Noetherian rings and infinitely generated ideals, but in these cases we attempt to follow the Noetherian pattern.

We shall be concerned especially with two naturally occurring functors on A-modules: the I-power torsion functor and the I-adic completion functor.

The I-power torsion functor Γ_I is defined by

$$M \longmapsto \Gamma_I(M) = \{x \in M \mid I^k x = 0 \text{ for some positive integer } k\}.$$

It is easy to see that the functor Γ_I is left exact.

We say that M is an I-power torsion module if $M = \Gamma_I M$. This admits a useful reinterpretation. Recall that the support of M is the set of prime ideals $\wp$ of A such that the localization $M_\wp$ is nonzero. We say that M is supported over I if every prime in the support of M contains I. This is equivalent to the condition that $M[1/\alpha] = 0$ for each $\alpha \in I$. It follows that M is an I-power torsion module if and only if the support of M lies over I.

The I-adic completion functor is defined by

$$M \longmapsto M_I^\wedge = \lim_k M/I^k M,$$

and M is said to be I-adically complete if the natural map $M \longrightarrow M_I^\wedge$ is an isomorphism. The Artin–Rees lemma states that I-adic completion is exact on finitely generated modules, but it is neither right nor left exact in general.

Since the functors that arise in topology are exact functors on triangulated categories, it is essential to understand the algebraic functors at the level of the derived category, which is to say that we must understand their derived functors. The connection with topology comes through one particular way of calculating the derived functors $R^*\Gamma_I$ of Γ_I and L_*^I of I-adic completion. Moreover, this particular method of calculation provides a connection between the two sets of derived functors and makes available various inductive proofs.

In this section, working in an arbitrary commutative ring A, we use our given finite set $\{\alpha_1, \ldots, \alpha_n\}$ of generators of I to define various homology groups. We shall explain why a different set of generators gives rise to isomorphic homology groups, but we postpone the conceptual interpretations of our definitions until the next section.

For a single element α, we may form the flat stable Koszul cochain complex

$$K^\bullet(\alpha) = (A \longrightarrow A[1/\alpha]),$$

where the nonzero modules are in cohomological degrees 0 and 1. The word stable is included since this complex is the colimit over s of the unstable Koszul complexes

$$K_s^\bullet(\alpha) = (\alpha^s : A \longrightarrow A).$$

When defining local cohomology, it is usual to use the complex $K^\bullet(\alpha)$ of flat modules.

However, we shall need a complex of projective A modules in order to define certain dual local homology modules. Accordingly, we take a projective approximation $PK^\bullet(\alpha)$ to $K^\bullet(\alpha)$. A good way of thinking about this is that, instead of taking the colimit of the $K_s^\bullet(\alpha)$, we take their telescope [13, p. 447]. This places the algebra in the form relevant to the topology. However, we shall use the model for $PK^\bullet(\alpha)$ displayed as the upper row in the quasi-isomorphism

$$\begin{tikzcd} A \oplus A[x] \arrow[r, "(1,\alpha x-1)"] \arrow[d, "(1,0)"'] & A[x] \arrow[d, "g"] \\ A \arrow[r] & A[1/\alpha] \end{tikzcd},$$

where $g(x^i) = 1/\alpha^i$, because, like $K^\bullet(\alpha)$, this choice of $PK^\bullet(\alpha)$ is nonzero only in cohomological degrees 0 and 1.

The Koszul cochain complex for a sequence $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ is obtained by tensoring together the complexes for the elements, so that

$$K^\bullet(\boldsymbol{\alpha}) = K^\bullet(\alpha_1) \otimes \cdots \otimes K^\bullet(\alpha_n),$$

and similarly for the projective complex $PK^\bullet(\boldsymbol{\alpha})$.

LEMMA 1.1. *If β is in the ideal $I = (\alpha_1, \alpha_2, \dots, \alpha_n)$, then $K^\bullet(\boldsymbol{\alpha})[1/\beta]$ is exact.*

PROOF. Since homology commutes with colimits, it suffices to show that some power of β acts as zero on the homology of $K_s^\bullet(\boldsymbol{\alpha}) = K_s^\bullet(\alpha_1) \otimes \cdots \otimes K_s^\bullet(\alpha_n)$. However, $(\alpha_i)^s$ annihilates $H^*(K_s^\bullet(\alpha_i))$, and it follows easily that $(\alpha_i)^{2s}$ annihilates $H^*(K_s^\bullet(\boldsymbol{\alpha}))$. Writing β as a linear combination of the n elements α_i, we see that β^{2sn} is a linear combination of elements each of which is divisible by some $(\alpha_i)^{2s}$, and the conclusion follows. □

Note that, by construction, we have an augmentation map

$$\varepsilon : K^\bullet(\boldsymbol{\alpha}) \longrightarrow A.$$

COROLLARY 1.2. *Up to quasi-isomorphism, the complex $K^\bullet(\boldsymbol{\alpha})$ depends only on the ideal I.*

PROOF. The lemma implies that the augmentation $K^\bullet(\boldsymbol{\alpha}, \beta) \longrightarrow K^\bullet(\boldsymbol{\alpha})$ is a quasi-isomorphism if $\beta \in I$. It follows that we have quasi-isomorphisms

$$K^\bullet(\boldsymbol{\alpha}) \longleftarrow K^\bullet(\boldsymbol{\alpha}) \otimes K^\bullet(\boldsymbol{\alpha}') \longrightarrow K^\bullet(\boldsymbol{\alpha}')$$

if $\boldsymbol{\alpha}'$ is a second set of generators for I. □

We therefore write $K^\bullet(I)$ for $K^\bullet(\boldsymbol{\alpha})$. Observe that $K^\bullet(\boldsymbol{\alpha})$ is unchanged if we replace the elements α_i by powers $(\alpha_i)^k$. Thus $K^\bullet(I)$ depends only on the radical of the ideal I. Since $PK^\bullet(\boldsymbol{\alpha})$ is a projective approximation to $K^\bullet(\boldsymbol{\alpha})$, it too depends only on the radical of I. We also write $K_s^\bullet(I) = K_s^\bullet(\alpha_1) \otimes \cdots \otimes K_s^\bullet(\alpha_n)$, but this is an abuse of notation since its homology groups do depend on the choice of generators.

The local cohomology and homology of an A-module M are then defined by

$$H_I^*(A;M) = H^*\big(PK^\bullet(I)\otimes M\big)$$

and

$$H_*^I(A;M) = H_*\big(\operatorname{Hom}\big(PK^\bullet(I), M\big)\big).$$

We usually omit the ring A from the notation. In particular, we write $H_I^*(A) = H_I^*(A;A)$. Note that we could equally well use the flat stable Koszul complex in the definition of local cohomology, as is more common. It follows from Lemma 1.1 that $H_I^*(M)$ is supported over I and is thus an I-power torsion module.

We observe that local cohomology and homology are invariant under change of base ring. While the proof is easy enough to leave as an exercise, the conclusion is of considerable calculational value.

LEMMA 1.3. *If $A \longrightarrow A'$ is a ring homomorphism, I' is the ideal $I\cdot A'$ and M' is an A'-module regarded by pullback as an A-module, then*

$$H_I^*(A;M') \cong H_{I'}^*(A';M') \quad \textit{and} \quad H_*^I(A;M') \cong H_*^{I'}(A';M').$$

We next define the Čech cohomology and homology of the A-module M. We will motivate the name at the end of the next section. Observe that $\varepsilon : K^\bullet(\alpha) \longrightarrow A$ is an isomorphism in degree zero and define the flat Čech complex $\check{C}^\bullet(I)$ to be the complex $\Sigma(\ker\varepsilon)$. Thus, if $i \geqslant 0$, then $\check{C}^i(I) = K^{i+1}(I)$. For example, if $I = (\alpha,\beta)$, then

$$\check{C}^\bullet(I) = \big(A[1/\alpha]\oplus A[1/\beta] \longrightarrow A\big[1/(\alpha\beta)\big]\big).$$

The differential $K^0(I) \longrightarrow K^1(I)$ specifies a chain map $A \longrightarrow \check{C}^\bullet(I)$ whose fibre is exactly $K^\bullet(I)$; see [11, pp. 439, 440]. Thus we have a fibre sequence

$$\boxed{K^\bullet(I) \longrightarrow A \longrightarrow \check{C}^\bullet(I)}\,.$$

We define the projective version $P\check{C}^\bullet(I)$ similarly, using the kernel of the composite of ε and the quasi-isomorphism $PK^\bullet(I) \longrightarrow K^\bullet(I)$; note that $P\check{C}^\bullet(I)$ is nonzero in cohomological degree -1.

The Čech cohomology and homology of an A-module M are then defined by

$$\check{C}H_I^*(A;M) = H^*\big(P\check{C}^\bullet(I)\otimes M\big)$$

and

$$\check{C}H_*^I(A;M) = H_*\big(\operatorname{Hom}\big(P\check{C}^\bullet(I), M\big)\big).$$

The Čech cohomology can also be defined by use of the flat Čech complex and is zero in negative degrees, but the Čech homology is usually nonzero in degree -1.

The fibre sequence $PK^\bullet(I) \longrightarrow A \longrightarrow P\check{C}^\bullet(I)$ gives rise to long exact sequences relating local and Čech homology and cohomology. These reduce to

$$0 \longrightarrow H_I^0(M) \longrightarrow M \longrightarrow \check{C}H_I^0(M) \longrightarrow H_I^1(M) \longrightarrow 0$$

and

$$0 \longrightarrow H_1^I(M) \longrightarrow \check{C}H_0^I(M) \longrightarrow M \longrightarrow H_0^I(M) \longrightarrow \check{C}H_{-1}^I(M) \longrightarrow 0,$$

together with isomorphisms

$$H_I^i(M) \cong \check{C}H_I^{i-1}(M) \quad \text{and} \quad H_i^I(M) \cong \check{C}H_{i-1}^I(M) \quad \text{for } i \geqslant 2.$$

Using the Čech theory, we may splice together local homology and local cohomology to define "local Tate cohomology" $\hat{H}_I^*(A;M)$, which has attractive formal properties; we refer the interested reader to [9].

2. Connections with derived functors; calculational tools

We gave our definitions in terms of specific chain complexes, but we gave our motivation in terms of derived functors. The meaning of the definitions appears in the following two theorems.

THEOREM 2.1 (Grothendieck [15]). *If A is Noetherian, then the local cohomology groups calculate the right derived functors of the left exact functor $M \longmapsto \Gamma_I(M)$. In symbols,*

$$H_I^n(A;M) = (R^n\Gamma_I)(M).$$

Since $\Gamma_I(M)$ is clearly isomorphic to $\operatorname{colim}_r(\operatorname{Hom}(A/I^r, M)$, these right derived functors can be expressed in more familiar terms:

$$(R^n\Gamma_I)(M) \cong \operatorname*{colim}_r \operatorname{Ext}_A^n(A/I^r, M).$$

THEOREM 2.2 (Greenlees and May [13]). *If A is Noetherian, then the local homology groups calculate the left derived functors of the (not usually right exact) I-adic completion functor $M \longmapsto M_I^\wedge$. In symbols,*

$$H_n^I(A;M) = \big(L_n(\cdot)_I^\wedge\big)(M).$$

The conclusion of Theorem 2.2 is proved in [13] under much weaker hypotheses. There is a notion of "pro-regularity" of a sequence α for a module M [13, (1.8)], and [13, (1.9)] states that local homology calculates the left derived functors of completion provided that A has bounded α_i torsion for all i and α is pro-regular for A. Moreover, if this is the case and if α is also pro-regular for M, then $H_0^I(A;M) = M_I^\wedge$ and

$H_i^I(A;M) = 0$ for $i > 0$. We shall refer to a module for which the local homology is its completion concentrated in degree zero as *tame*. By the Artin–Rees lemma, any finitely generated module over a Noetherian ring is tame.

The conclusion of Theorem 2.1 is also true under similar weakened hypotheses [10].

An elementary proof of Theorem 2.1 can be obtained by induction on the number of generators of I. This uses the spectral sequence

$$H_I^*\big(H_J^*(M)\big) \Longrightarrow H_{I+J}^*(M)$$

that is obtained from the isomorphism $PK^\bullet(I+J) \cong PK^\bullet(I) \otimes PK^\bullet(J)$. This means that it is only necessary to prove the result when I is principal and to verify that if Q is injective then $\Gamma_I Q$ is also injective. The proof of Theorem 2.2 can also be obtained like this, although it is more complicated because the completion of a projective module will usually not be projective.

One is used to the idea that I-adic completion is often exact, so that L_0^I is the most significant of the left derived functors. However, it is the top nonvanishing right derived functor of Γ_I that is the most significant. Some idea of the shape of these derived functors can be obtained from the following result. Observe that the complex $PK^\bullet(\alpha)$ is nonzero only in cohomological degrees between 0 and n. This shows immediately that local homology and cohomology are zero above dimension n. A result of Grothendieck usually gives a much better bound. Recall that the Krull dimension of a ring is the length of its longest strictly ascending sequence of prime ideals and that the I-depth of a module M is the length of the longest regular M-sequence in I.

THEOREM 2.3 (Grothendieck [14]). *If A is Noetherian of Krull dimension d, then*

$$H_I^n(M) = 0 \quad \textit{and} \quad H_n^I(M) = 0 \quad \textit{if } n > d.$$

Let $\mathrm{depth}_I(M) = m$. *With no assumptions on A and M,*

$$H_I^i(M) = 0 \quad \textit{if } i < m.$$

If A is Noetherian, M is finitely generated, and $IM \neq M$, then

$$H_I^m(M) \neq 0.$$

PROOF. The vanishing theorem for local cohomology above degree d follows from the fact that we can re-express the right derived functors of Γ_I in terms of algebraic geometry and apply a vanishing theorem that results from geometric considerations. Indeed, if $X = Spec(A)$ is the affine scheme defined by A, Y is the closed subscheme determined by I with underlying space $V(I) = \{\wp | \wp \supset I\} \subset X$, and $\tilde{M}$ is the sheaf over X associated to M, then $\Gamma_I(M)$ can be identified with the space $\Gamma_Y(\tilde{M})$ of sections of $\tilde{M}$ with support in Y. For sheaves $\mathscr{F}$ of Abelian groups over X, the cohomology groups $H_Y^*(X;\mathscr{F})$ are defined to be the right derived functors $(R^*\Gamma_Y)(\mathscr{F})$, and we conclude that

$$H_I^*(A;M) \cong H_Y^*(X;\tilde{M}).$$

The desired vanishing of local cohomology groups is now a consequence of a general result that can be proven by using flabby sheaves to calculate sheaf cohomology: for any sheaf $\mathscr{F}$ over any Noetherian space of dimension d, $H^n(X;\mathscr{F}) = 0$ for $n > d$ [14, 3.6.5] (or see [16, III.2.7]). The vanishing result for local homology follows from that for local cohomology by use of the universal coefficient theorem that we shall discuss shortly.

The vanishing of local cohomology below degree m is elementary, but we give the proof since we shall later make a striking application of this fact. We proceed by induction on m. The statement is vacuous if $m = 0$. Choose a regular sequence $\{\beta_1, \ldots, \beta_m\}$ in I. Consider the long exact sequence of local cohomology groups induced by the short exact sequence

$$0 \longrightarrow M \xrightarrow{\beta_1} M \longrightarrow M/\beta_1 M \longrightarrow 0.$$

Since $\{\beta_2, \ldots, \beta_m\}$ is a regular sequence for $M/\beta_1 M$, the induction hypothesis gives that $H_I^i(M/\beta_1 M) = 0$ for $i < m - 1$. Therefore multiplication by β_1 is a monomorphism on $H_I^i(M)$ for $i < m$. Since $H_I^i(M)[1/\beta_1] = 0$, by Lemma 1.1, this implies that $H_I^i(M) = 0$. The fact that $H_I^m(M) \neq 0$ under the stated hypotheses follows from standard alternative characterizations of depth and local cohomology in terms of Ext [20, §16]. □

It follows directly from the chain level definitions that there is a third quadrant universal coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^s\big(H_I^{-t}(A), M\big) \Longrightarrow H^I_{-t-s}(A; M), \tag{2.4}$$

with differentials $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$. This generalizes Grothendieck's local duality spectral sequence [15]; see [13] for details.

We record a consequence of the spectral sequence that is implied by the vanishing result of Theorem 2.3. Recall that the nicest local rings are the regular local rings, whose maximal ideals are generated by a regular sequence; Cohen–Macaulay local rings, which have depth equal to their dimension, are more common. The following result applies in particular to such local rings.

COROLLARY 2.5. *If A is Noetherian and* $\mathrm{depth}_I(A) = \dim(A) = d$, *then*

$$L_s^I M = \mathrm{Ext}^{d-s}\left(H_I^d(A), M\right).$$

For example, if $A = \mathbb{Z}$ and $I = (p)$, then $H^*_{(p)}(\mathbb{Z}) = H^1_{(p)}(\mathbb{Z}) = \mathbb{Z}/p^\infty$. Therefore the corollary states that

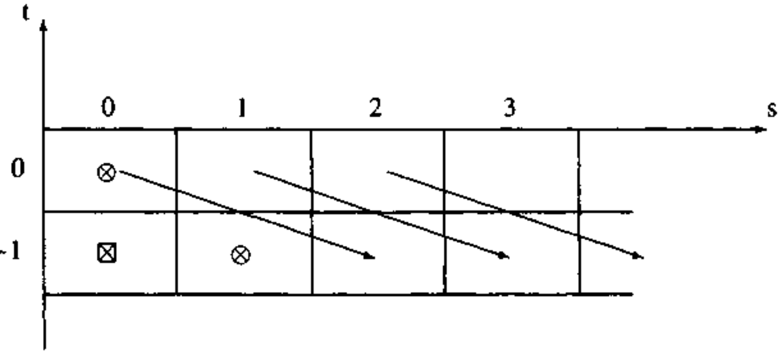
$$L_0^{(p)} M = \mathrm{Ext}(\mathbb{Z}/p^\infty, M) \quad \text{and} \quad L_1^{(p)} M = \mathrm{Hom}(\mathbb{Z}/p^\infty, M),$$

as was observed in Bousfield and Kan [3, VI.2.1].

There is a precisely similar universal coefficient theorem for calculating Čech homology from Čech cohomology. Together with Theorem 2.3, this implies vanishing theorems for the Čech theories.

COROLLARY 2.6. *If A is Noetherian of Krull dimension $d \geqslant 1$, then $\check{C}H^i_I(M)$ is only nonzero if $0 \leqslant i \leqslant d-1$ and $\check{C}H^I_i(M)$ is only nonzero if $-1 \leqslant i \leqslant d-1$. If $d = 0$ the Čech cohomology may be nonzero in degree 0 and the Čech homology may be nonzero in degrees 0 and -1.*

When R is of dimension one, the spectral sequence (2.4) can be pictured as follows:



Here the two boxes marked $\otimes$ contribute to H^I_0, and that marked $\boxtimes$ is H^I_1. Since there is no local homology in negative degrees, the first of the d_2 differentials must be an epimorphism and the remaining d_2 differentials must be isomorphisms. Thus we find an exact sequence

$$0 \to \operatorname{Ext}^1\big(H^1_I(A), M\big) \to H^I_0(M) \to \operatorname{Hom}\big(H^0_I(A), M\big)$$
$$\to \operatorname{Ext}^2\big(H^1_I(A), M\big) \to 0$$

and an isomorphism

$$H^I_1(M) \cong \operatorname{Hom}\big(H^1_I(A), M\big).$$

Another illuminating algebraic fact is that local homology and cohomology are invariant under the completion $M \longrightarrow M^\wedge_I$ of a tame module M. This can be used in conjunction with completion of A and I in view of Lemma 1.3. However, all that is relevant to the proof is the vanishing of the higher local homology groups, not the identification of the zeroth group.

PROPOSITION 2.7. *If $H^I_q(M) = 0$ for $q > 0$, then the natural map $M \longrightarrow H^I_0(M)$ induces isomorphisms on application of $H^*_I(\cdot)$ and $H^I_*(\cdot)$.*

PROOF. The natural map $\varepsilon^* : M \longrightarrow \operatorname{Hom}(PK^\bullet(I), M)$ induces a quasi-isomorphism

$$\begin{aligned}\operatorname{Hom}\big(PK^\bullet(I), M\big) &\longrightarrow \operatorname{Hom}\big(PK^\bullet(I), \operatorname{Hom}\big(PK^\bullet(I), M\big)\big)\\ &\cong \operatorname{Hom}\big(PK^\bullet(I) \otimes PK^\bullet(I), M\big)\end{aligned}$$

since the projection $PK^\bullet(I) \otimes PK^\bullet(I) \longrightarrow PK^\bullet(I)$ is a quasi-isomorphism of projective complexes by Corollary 1.2. We obtain a collapsing spectral sequence converging from $E^2_{p,q} = H^I_p(H^I_q(M))$ to the homology of the complex in the middle, and the invariance statement in local homology follows.

For local cohomology we claim that ε also induces a quasi-isomorphism

$$K^\bullet(I) \otimes M \longrightarrow K^\bullet(I) \otimes \mathrm{Hom}\left(PK^\bullet(I), M\right).$$

The right side is a double complex, and there will result a collapsing spectral sequence that converges from $E_2^{p,q} = H^p_I((H^I_{-q}(M))$ to its homology. This will give the invariance statement in local cohomology. The fibre of the displayed map is $K^\bullet(I) \otimes \mathrm{Hom}(P\check{C}^\bullet(I), M)$, and we must show that this complex is exact. However $K^\bullet(I)$ is a direct limit of the finite self-dual unstable Koszul complexes $K^\bullet_s(I)$ so it is enough to see that $\mathrm{Hom}(P\check{C}^\bullet(I) \otimes K^\bullet_s(I), M)$ is exact. Since the complex $P\check{C}^\bullet(I) \otimes K^\bullet_s(I)$ is projective, it suffices to show that it is exact. However, it is quasi-isomorphic to $\check{C}^\bullet(I) \otimes K^\bullet_s(I)$, which has a finite filtration with subquotients $A[1/\beta] \otimes K^\bullet_s(I)$ with $\beta \in I$. We saw in the proof of Lemma 1.1 that some power of β annihilates the homology of $K^\bullet_s(I)$. Therefore the homology of $A[1/\beta] \otimes K^\bullet_s(I)$ is zero and the conclusion follows. □

We must still explain why we called $\check{C}^\bullet(I)$ a Čech complex. In fact, this complex arises by using the Čech construction to calculate cohomology from a suitable open cover. More precisely, let Y be the closed subscheme of $X = Spec(A)$ determined by I, as in the proof of Theorem 2.3. The space $V(I) = \{\wp | \wp \supset I\}$ decomposes as $V(I) = V(\alpha_1) \cap \cdots \cap V(\alpha_n)$, and there results an open cover of the open subscheme $X - Y$ as the union of the complements $X - Y_i$ of the closed subschemes Y_i determined by the principal ideals (α_i). However, $X - Y_i$ is isomorphic to the affine scheme $Spec(A[1/\alpha_i])$. Since affine schemes have no higher cohomology,

$$H^*\left(Spec\left(A[1/\alpha_i]\right)\right) = H^0\left(Spec\left(A[1/\alpha_i]\right)\right) = A[1/\alpha_i].$$

Thus the E_1 term of the Mayer–Vietoris spectral sequence for this cover collapses to the chain complex $\check{C}^\bullet(I)$, and $H^*(X - Y; M) \cong \check{C}H^*_I(M)$.

3. Topological analogs of the algebraic definitions

We suppose given a commutative S-algebra R, where S is the sphere spectrum. (As explained in [7], this is essentially the same thing as an E_∞ ring spectrum, but adapted to a more algebraically precise topological setting.) We imitate the algebraic definitions of Section 1 in the category of R-modules to construct a variety of useful spectra. Here we understand R-modules in the point-set level sense discussed in the preceding article [7]. The discussion in this section and the next is exactly like that first given for the equivariant sphere spectrum in [11], before the appropriate general context of modules was available.

For $\beta \in \pi_* R$, we define the Koszul spectrum $K(\beta)$ by the fibre sequence

$$K(\beta) \longrightarrow R \longrightarrow R[1/\beta].$$

Here

$$R[1/\beta] = \operatorname{hocolim}\left(R \xrightarrow{\beta} R \xrightarrow{\beta} \cdots\right)$$

is a module spectrum and the inclusion of R is a module map, hence $K(\beta)$ is an R-module. Analogous to the filtration at the chain level, we obtain a filtration of the R-module $K(\beta)$ by viewing it as

$$\Sigma^{-1}\left(R[1/\beta] \cup CR\right).$$

Next we define the Koszul spectrum for the sequence $\beta_1, \ldots, \beta_n$ by

$$K(\beta_1, \ldots, \beta_n) = K(\beta_1) \wedge_R \cdots \wedge_R K(\beta_n).$$

The topological analogue of Lemma 1.1 states that if $\gamma \in J$ then

$$K(\beta_1, \ldots, \beta_n)[1/\gamma] \simeq *;$$

this follows from Lemma 1.1 and the spectral sequence (3.2) below (or from Lemma 3.6). We may now use precisely the same proof as in the algebraic case to conclude that the homotopy type of $K(\beta_1, \ldots, \beta_n)$ depends only on the radical of the ideal $J = (\beta_1, \ldots, \beta_n)$. We therefore write $K(J)$ for $K(\beta_1, \ldots, \beta_n)$.

We should remark that we are now working over the graded ring $R_* = \pi_*(R)$. All of the algebra in the previous two sections applies without change in the graded setting, but all of the functors defined there are now bigraded, with an internal degree coming from the grading of the given ring and its modules. As usual, we write $M_q = M^{-q}$.

With motivation from Theorems 2.1 and 2.2, we define the homotopy J-power torsion (or local cohomology) and homotopy J-completion (or local homology) modules associated to an R-module M by

$$\Gamma_J(M) = K(J) \wedge_R M \quad \text{and} \quad M_J^{\wedge} = F_R\left(K(J), M\right). \tag{3.1}$$

In particular, $\Gamma_J(R) = K(J)$.

Because the construction follows the algebra so precisely, it is easy to give methods of calculation for the homotopy groups of these R-modules. We use the product of the filtrations of the $K(\beta_i)$ given above and obtain spectral sequences

$$E^2_{s,t} = H_J^{-s,-t}(R_*; M_*) \Rightarrow \pi_{s+t}(\Gamma_J M) \tag{3.2}$$

with differentials $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ and

$$E_2^{s,t} = H^J_{-s,-t}(R^*; M^*) \Rightarrow \pi_{-(s+t)}(M_J^{\wedge}) \tag{3.3}$$

with differentials $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$.

Similarly, we define the Čech spectrum by the cofibre sequence

$$\boxed{K(J) \longrightarrow R \longrightarrow \check{C}(J)}. \tag{3.4}$$

With motivation deferred until Section 5, we define the homotopical localization (or Čech cohomology) and Čech homology modules associated to an R-module M by

$$M[J^{-1}] = \check{C}(J) \wedge_R M \quad \text{and} \quad \Delta^J(M) = F_R(\check{C}(J), M). \tag{3.5}$$

In particular, $R[J^{-1}] = \check{C}(J)$. Once again, we have spectral sequences for calculating their homotopy groups from the analogous algebraic constructions.

We can now give topological analogues of some basic pieces of algebra that we used in Section 1. Recall that the algebraic Koszul complex $K^\bullet(J)$ is a direct limit of unstable complexes $K_s^\bullet(J)$ that are finite complexes of free R_*-modules with homology annihilated by a power of J. We remind the reader that, in contrast with $K^\bullet(J)$, the homology of the modules $K_s^\bullet(J)$ depends on the choice of generators we use. We say that an R-module M is a J-power torsion module if its R_*-module M_* of homotopy groups is a J-power torsion module; equivalently, M_* must have support over J.

LEMMA 3.6. *The R-module $K(J)$ is a homotopy direct limit of finite R-modules $K_s(J)$, each of which has homotopy groups annihilated by some power of J. Therefore $K(J)$ is a J-power torsion module.*

PROOF. It is enough to establish the result in the principal ideal case and then take smash products over R. Let

$$K_s(\beta) = \Sigma^{-1} R/\beta^s$$

denote the fibre of $\beta^s : R \longrightarrow R$, and observe that its homotopy groups are annihilated by β^{2s}. Now observe that

$$\big(R \longrightarrow R[1/\beta]\big) = \operatorname*{hocolim}_s \big(R \xrightarrow{\beta^s} R\big),$$

and so their fibres are also equivalent:

$$K(\beta) \simeq \operatorname*{hocolim}_s K_s(\beta).$$

□

The following lemma is an analogue of the fact that $\check{C}^\bullet(J)$ is a chain complex which is a finite sum of modules $R[1/\beta]$ for $\beta \in J$.

LEMMA 3.7. *The R-module $\check{C}(J)$ has a finite filtration by R-submodules with subquotients that are suspensions of modules of the form $R[1/\beta]$ with $\beta \in J$.*

These lemmas are useful in combination.

COROLLARY 3.8. *If M is a J-power torsion module then $M \wedge_R \check{C}(J) \simeq *$; in particular $K(J) \wedge_R \check{C}(J) \simeq *$.*

PROOF. Since $M[1/\beta] \simeq *$ for $\beta \in J$, Lemma 3.7 gives the conclusion for M. □

We remark that the corollary leads via [9, B.2] to the construction of a topological J-local Tate cohomology module $t_J(M)$ that has formal properties like those of its algebraic counterpart studied in [9].

4. Completion at ideals and Bousfield localization

As observed in the proof of Lemma 3.6, we have $K(\beta) = \mathrm{hocolim}_s \Sigma^{-1}R/\beta^s$ and therefore

$$M^{\wedge}_{(\beta)} = F_R\big(\operatorname*{hocolim}_s \Sigma^{-1}R/\beta^s, M\big) = \operatorname*{holim}_s M/\beta^s.$$

If $J = (\beta, \gamma)$, then

$$M^{\wedge}_J = F_R\big(K(\beta) \wedge_R K(\gamma), M\big) = F_R\big(K(\beta), F_R(K(\gamma), M)\big) = \big(M^{\wedge}_{(\gamma)}\big)^{\wedge}_{(\beta)},$$

and so on inductively. This should help justify the notation $M^{\wedge}_J = F_R(K(J), M)$.

When $R = S$ is the sphere spectrum and $p \in \mathbb{Z} \cong \pi_0(S)$, $K(p)$ is a Moore spectrum for $\mathbb{Z}/p^\infty$ in degree -1 and we recover the usual definition

$$X^{\wedge}_p = F(S^{-1}/p^\infty, X)$$

of p-completions of spectra as a special case. The standard short exact sequence for the calculation of the homotopy groups of $X^{\wedge}_p$ in terms of 'Ext completion' and 'Hom completion' follows directly from Corollary 2.5.

Since p-completion has long been understood to be an example of a Bousfield localization, our next task is to show that completion at J is a Bousfield localization in general. The arguments are the same as in [11, §2], which dealt with the (equivariant) case $R = S$.

We must first review definitions. They are usually phrased homologically, but we shall give the spectrum level equivalents so that the translation to other contexts is immediate. Fix a spectrum E. A spectrum A is *E-acyclic* if $A \wedge E \simeq *$; a map $f : X \longrightarrow Y$ is an *E-equivalence* if its cofibre is E-acyclic. A spectrum X is *E-local* if $E \wedge T \simeq *$ implies $F(T, X) \simeq *$. A map $Y \longrightarrow L_E Y$ is a *Bousfield E-localization* of Y if it is an E-equivalence and $L_E Y$ is E-local. This means that $Y \longrightarrow L_E Y$ is terminal among E-equivalences with domain Y, and the Bousfield localization is therefore unique if it exists. Bousfield has proved that $L_E Y$ exists for all E and Y, but we shall construct the localizations that we need directly.

We shall need two variations of the definitions. First, we work in the category of R-modules, so that $\wedge$ and $F(\cdot,\cdot)$ are replaced by $\wedge_R$ and $F_R(\cdot,\cdot)$. It is proven in [8] that Bousfield localizations always exist in this setting. Second, we allow E to be replaced by a class $\mathscr{E}$ of R-modules, so that our conditions for fixed E are replaced by conditions for each $E \in \mathscr{E}$. When the class $\mathscr{E}$ is a set, it is equivalent to work with the single module given by the wedge of all $E \in \mathscr{E}$. Bousfield localizations at classes need not always exist, but the language will be helpful in explaining the conceptual meaning of our examples. The following observation relates the spectrum level and module level notions of local spectra.

LEMMA 4.1. *Let $\mathscr{E}$ be a class of R-modules. If an R-module N is $\mathscr{E}$-local as an R-module, then it is $\mathscr{E}$-local as a spectrum.*

PROOF. Let $\mathbb{F}$ be the free functor from spectra to R-modules. If $E \wedge T \simeq *$ for all E, then $E \wedge_R \mathbb{F}T \simeq *$ for all E and therefore $F(T,N) \simeq F_R(\mathbb{F}T,N) \simeq *$. □

The class that will concern us most is the class J-**Tors** of finite J-power torsion R-modules M. Thus M must be a finite cell R-module, and its R_*-module M_* of homotopy groups must be a J-power torsion module.

THEOREM 4.2. *For any finitely generated ideal J of R_* the map $M \longrightarrow M_J^\wedge$ is Bousfield localization in the category of R-modules in each of the following equivalent senses:*

(i) *with respect to the R-module $\Gamma_J(R) = K(J)$.*
(ii) *with respect to the class J-**Tors** of finite J-power torsion R-modules.*
(iii) *with respect to the R-module $K_s(J)$ for any $s \geqslant 1$.*

Furthermore, the homotopy groups of the completion are related to local homology groups by a spectral sequence

$$E^2_{s,t} = H^J_{s,t}(M_*) \Longrightarrow \pi_{s+t}(M_J^\wedge).$$

If R_* is Noetherian, the E^2 term consists of the left derived functors of J-adic completion: $H^J_s(M_*) = L^J_s(M_*)$.

PROOF. The statements about calculations are repeated from (3.3) and Theorem 2.2. We prove (i). Since

$$F_R\big(T, M_J^\wedge\big) \simeq F_R\big(T \wedge_R K(J), M\big),$$

it is immediate that $M_J^\wedge$ is $K(J)$-local. We must prove that the map $M \longrightarrow M_J^\wedge$ is a $K(J)$-equivalence. The fibre of this map is $F(\check{C}(J),M)$, so we must show that

$$F\big(\check{C}(J),M\big) \wedge_R K(J) \simeq *.$$

By Lemma 3.6, $K(J)$ is a homotopy direct limit of terms $K_s(J)$. Each $K_s(J)$ is in J-**Tors**, and we see by their definition in terms of cofibre sequences and smash products that their duals $D_RK_s(J)$ are also in J-**Tors**, where $D_R(M) = F_R(M,R)$. Since $K_s(J)$ is a finite cell R-module,

$$F_R\big(\check{C}(J),M\big)\wedge_R K_s(J) = F_R\big(\check{C}(J)\wedge_R D_RK_s(J),M\big),$$

and $\check{C}(J)\wedge_R D_RK_s(J)\simeq *$ by Corollary 3.8. Parts (ii) and (iii) are similar but simpler. For (iii), observe that we have a cofibre sequence $R/\beta^s \longrightarrow R/\beta^{2s} \longrightarrow R/\beta^s$, so that all of the $K_{js}(J)$ may be constructed from $K_s(J)$ using a finite number of cofibre sequences. □

5. Localization away from ideals and Bousfield localization

Bousfield localizations include both completions at ideals and localizations at multiplicatively closed sets, but one may view these Bousfield localizations as falling into the types typified by completion at p and localization away from p. Thinking in terms of $Spec(R_*)$, this is best viewed as the distinction between localization at a closed set and localization at the complementary open subset. We dealt with the closed sets in the previous section, and we deal with the open sets in this one. Observe that, when $J = (\beta)$, $M[J^{-1}]$ is just $R[\beta^{-1}]\wedge_R M = M[\beta^{-1}]$. However, the nonvanishing of higher Čech cohomology groups gives the construction for general finitely generated ideals a quite different algebraic flavor, and $M[J^{-1}]$ is generally not a localization of M at a multiplicatively closed subset of R_*. To characterize this construction as a Bousfield localization, we consider the class J-**Inv** of R-modules M for which there is an element $\beta\in J$ such that $\beta: M \longrightarrow M$ is an equivalence.

THEOREM 5.1. *For any finitely generated ideal $J = (\beta_1,\dots,\beta_n)$ of R_*, the map $M \longrightarrow M[J^{-1}]$ is Bousfield localization in the category of R-modules in each of the following equivalent senses:*

(i) *with respect to the R-module $R[J^{-1}] = \check{C}(J)$.*
(ii) *with respect to the class J-**Inv**.*
(iii) *with respect to the set $\{R[1/\beta_1],\dots,R[1/\beta_n]\}$.*

Furthermore, the homotopy groups of the localization are related to Čech cohomology groups by a spectral sequence

$$E^2_{s,t} = \check{C}H_J^{-s,-t}(M_*) \Longrightarrow \pi_{s+t}\big(M[J^{-1}]\big).$$

If R_* is Noetherian, the E^2 term can be viewed as the cohomology of $Spec(R_*)\setminus V(J)$ with coefficients in the sheaf associated to M_*.

PROOF. The spectral sequence is immediate from the construction of $M[J^{-1}]$, and the last paragraph of Section 2 gives the final statement.

To see that $M[J^{-1}]$ is local, suppose that $T \wedge_R \check{C}(J) \simeq *$. We must show that $F_R(T, M[J^{-1}]) \simeq *$. By the cofibre sequence defining $\check{C}(J)$ and the hypothesis, it suffices to show that $F_R(K(J) \wedge_R T, M[J^{-1}]) \simeq *$. By Lemma 3.6,

$$F_R\big(K(J) \wedge_R T, M\big[J^{-1}\big]\big) \simeq \operatorname{holim}_s F_R\big(K_s(J) \wedge_R T, \check{C}(J) \wedge_R M\big).$$

Observing that

$$F_R\big(K_s(J) \wedge_R T, \check{C}(J) \wedge_R M\big) \simeq F_R\big(T, D_R K_s(J) \wedge_R \check{C}(J) \wedge_R M\big),$$

we see that the conclusion follows from Corollary 3.8. The map $M \longrightarrow M[J^{-1}]$ is a $\check{C}(J)$-equivalence since its fibre is $\Gamma_J(M) = K(J) \wedge_R M$ and $K(J) \wedge_R \check{C}(J) \simeq *$ by Corollary 3.8. Parts (ii) and (iii) are proved similarly. □

Translating the usual terminology, we say that a localization L on R-modules is *smashing* if $L(N) \simeq N \wedge_R L(R)$ for all R-modules N. The following fact is obvious.

LEMMA 5.2. *Localization away from J is smashing.*

It is also clear that completion at J will not usually be smashing.

We complete the general theory with an easy, but tantalizing, result that will specialize to give part of the proof of the Chromatic Convergence Theorem of Hopkins and Ravenel [23]. It well illustrates how the algebraic information in Section 2 can have nonobvious topological implications. Observe that if $J' = J + (\beta)$, we have an augmentation map $\varepsilon : K(J') \simeq K(J) \wedge_R K(\beta) \longrightarrow K(J)$ over R. Applying $F_R(\cdot, M)$, we obtain an induced map

$$M^{\wedge}_J \longrightarrow M^{\wedge}_{J'}.$$

A comparison of cofibre sequences in the derived category of R-modules gives a dotted arrow ζ such that the following diagram commutes:

$$\begin{tikzcd}
\Gamma_{J'}(M) \arrow[r] \arrow[d, "\varepsilon"'] & M \arrow[r] \arrow[d] & M[J'^{-1}] \arrow[r] \arrow[d, dashed, "\zeta"] & \Sigma\Gamma_{J'}(M) \arrow[d] \\
\Gamma_J(M) \arrow[r] & M \arrow[r] & M[J^{-1}] \arrow[r] & \Sigma\Gamma_J(M)
\end{tikzcd}$$

Here the cofibre of ε is $\Gamma_J(M)[\beta^{-1}]$ and the cofibre of ζ is $\Sigma\Gamma_J(M)[\beta^{-1}]$. If an ideal $\mathscr{J}$ is generated by a countable sequence $\{\beta_i\}$ and J_n is the ideal generated by the first n generators, we may define

$$M^{\wedge}_{\mathscr{J}} = \operatorname{hocolim}_n M^{\wedge}_{J_n} \quad \text{and} \quad M\big[\mathscr{J}^{-1}\big] = \operatorname{holim}_n M\big[J_n^{-1}\big].$$

We say that $\mathscr{J}$ is of infinite depth if $\operatorname{depth}_{J_n}(R_*) \longrightarrow \infty$; this holds, for example, if $\{\beta_i\}$ is a regular sequence.

PROPOSITION 5.3. *If M is a finite cell R-module and $\mathcal{J}$ is of infinite depth, then $M \simeq M[\mathcal{J}^{-1}]$.*

PROOF. It suffices to prove that $\mathrm{holim}_n \Gamma_{J_n}(M) \simeq *$, and, since M is finite, it is enough to prove this when $M = R$. We show that the system of homotopy groups $\pi_*(K(J_n))$ is pro-zero. This just means that, for any n, there exists $q > n$ such that the map $K(J_q) \longrightarrow K(J_n)$ induces zero on homotopy groups, and it implies that both $\lim_n \pi_*(K(J_n)) = 0$ and $\lim^1_n \pi_*(K(J_n)) = 0$. By the $\lim^1$ exact sequence for the computation of the homotopy groups of a homotopy inverse limit, this will give the conclusion. Since J_n is finitely generated, there is a d such that $H^i_{J_n}(R_*) = 0$ for $i \geqslant d$. By hypothesis, we may choose q such that $\mathrm{depth}_{J_q}(R_*) > d$. Then, by Theorem 2.3, $H^i_{J_q}(R_*) = 0$ for $i \leqslant d$. Now the spectral sequence (3.2) for $\pi_*(K(J_n))$ is based on the filtration

$$\cdots \subseteq F_{-s} \subseteq F_{-s+1} \subseteq \cdots F_1 \subseteq F_0 = \pi_*\bigl(K(J_n)\bigr)$$

in which F_{-s} is the group of elements arising from $H^i_{J_n}(R_*)$ for $i > s$. The map $K(J_q) \longrightarrow K(J_n)$ is filtration preserving, hence the filtration corresponding to $s = d$ is mapped to 0. By the choice of q, this filtration is all of $\pi_*(K(J_q))$. $\square$

6. The specialization to ideals in MU_*

We specialize to the commutative S-algebra $R = MU$ in this section, taking [7, §11] as our starting point. Recall that $MU_* = \mathbb{Z}[x_i \mid i \geqslant 1]$, where $deg\, x_i = 2i$, and that MU_* contains elements v_i of degree $2(p^i - 1)$ that map to the Hazewinkel generators of $BP_* = \mathbb{Z}_{(p)}[v_i \mid i \geqslant 1]$. We let I_n denote the ideal $(v_0, v_1, \ldots, v_{n-1})$ in $\pi_*(MU)$, where $v_0 = p$; we prefer to work in MU rather than BP because of its canonical S-algebra structure. As explained in [7, §11], BP is an MU-ring spectrum whose unit $MU \longrightarrow BP$ factors through the canonical retraction $MU_{(p)} \longrightarrow BP$. We also have MU-ring spectra $E(n)$ such that $E(0)_* = \mathbb{Q}$ and

$$E(n)_* = \mathbb{Z}_{(p)}\bigl[v_1, \ldots, v_n, v_n^{-1}\bigr]$$

if $n > 0$. The Bousfield localization functor $L_n = L_{E(n)}$ on spectra plays a fundamental role in the "chromatic" scheme for the inductive study of stable homotopy theory, and we have the following result.

THEOREM 6.1. *When restricted to MU-modules M, the functor L_n coincides with localization away from I_{n+1}:*

$$L_n M \simeq M\bigl[I_{n+1}^{-1}\bigr].$$

PROOF. By [23, 7.3.2], localization at $E(n)$ is the same as localization at $BP[(v_n)^{-1}]$ or at the wedge of the $K(i)$ for $0 \leqslant i \leqslant n$. This clearly implies that localization at $E(n)$ is the same as localization at the wedge of the $BP[(v_i)^{-1}]$ for $0 \leqslant i \leqslant n$, and this is the same as localization at the wedge of the $MU[(v_i)^{-1}]$ for $0 \leqslant i \leqslant n$. By Lemma 4.1, we conclude that $M[I_{n+1}^{-1}]$ is $E(n)$-local. To see that the localization $M \longrightarrow M[I_{n+1}^{-1}]$ is an $MU[(v_i)^{-1}]$-equivalence for $0 \leqslant i \leqslant n$, note that its fibre is $\Gamma_{I_{n+1}}(M)$ and $\Gamma_{I_{n+1}}(M)[w^{-1}] \simeq *$ for any $w \in I_{n+1}$. Consider $MU_*(MU) = (MU \wedge MU)_*$ as a left MU_*-module, as usual, and recall from [23, B.5.15] that the right unit $MU_* \longrightarrow (MU \wedge MU)_*$ satisfies

$$\eta_R(v_i) \equiv v_i \bmod I_i \cdot MU_*(MU), \quad \text{hence} \quad \eta_R(v_i) \in I_{i+1} \cdot MU_*(MU).$$

We have

$$\Gamma_{I_{n+1}}(M) \wedge MU \simeq \Gamma_{I_{n+1}}(M) \wedge_{MU} (MU \wedge MU)$$

and can deduce inductively that $\Gamma_{I_{n+1}}(M) \wedge MU[w^{-1}] \simeq *$ for any $w \in I_{n+1}$ since $\Gamma_{I_{n+1}}(M)[w^{-1}] \simeq *$ for any such w. □

When $M = BP$, this result is essentially a restatement in our context of Ravenel's theorem [23, 8.1.1] (see also [21, §§5, 6] and [22]) on the geometric realization of the chromatic resolution for the calculation of stable homotopy theory. To explain the connection between our constructions and his, we offer the following dictionary:

$$N_nBP \simeq \Sigma^n \Gamma_{I_n} BP,$$

$$M_nBP \simeq \Sigma^n \Gamma_{I_n} BP\big[(v_n)^{-1}\big],$$

$$L_nBP \simeq BP\big[I_{n+1}^{-1}\big].$$

In fact, for any spectrum X, Ravenel defines M_nX and N_nX inductively by

$$N_0X = X, \quad M_nX = L_nN_nX,$$

and the cofibre sequences

$$N_nX \longrightarrow M_nX \longrightarrow N_{n+1}X. \tag{6.2}$$

He also defines C_nX to be the fibre of the localization $X \longrightarrow L_nX$ (where, to start inductions, $L_{-1}X = *$ and $C_{-1}X = X$). Elementary formal arguments given in [21, 5.10] show that the definition of Bousfield localization, the cofibrations in the definitions just given, and the fact that $L_mL_n = L_m$ for $m \leqslant n$ [21, 2.1] imply that

$$N_nX = \Sigma^n C_{n-1}X$$

and there is a cofibre sequence

$$\Sigma^{-n} M_nX \longrightarrow L_nX \longrightarrow L_{n-1}X. \tag{6.3}$$

The claimed identifications follow inductively from our description of L_nBP and the fact (implied by Lemma 1.1) that, for any MU-module M,

$$\Gamma_{I_n}(M)\big[I_{n+1}^{-1}\big] \simeq \Gamma_{I_n}(M)\big[(v_n)^{-1}\big].$$

In fact, the evident cofibrations of MU-modules

$$\Sigma^n \Gamma_{I_n} BP \longrightarrow \Sigma^n \Gamma_{I_n} BP\big[{v_n}^{-1}\big] \longrightarrow \Sigma^{n+1} \Gamma_{I_{n+1}}(BP)$$

and

$$\Gamma_{I_n} BP\big[(v_n)^{-1}\big] \longrightarrow BP\big[(I_{n+1})^{-1}\big] \longrightarrow BP\big[(I_n)^{-1}\big]$$

realize the case $X = BP$ of the cofibrations displayed in (6.2) and (6.3). Moreover, it is immediate from our module theoretic constructions that the homotopy groups are given inductively by

$$(N_0BP)_* = BP_*, \quad (M_nBP)_* = (N_nBP)_*\big[(v_n)^{-1}\big],$$

and the short exact sequences

$$0 \longrightarrow (N_nBP)_* \longrightarrow (M_nBP)_* \longrightarrow (N_{n+1}BP)_* \longrightarrow 0. \tag{6.4}$$

Ravenel's original arguments were substantially more difficult because, not having the new category of MU-modules to work in, he had to work directly in the classical stable homotopy category.

Although BP is not a finite cell MU-module, the retraction from $MU_{(p)}$ makes it clear that the proof of Proposition 5.3 applies to give the following conclusion.

PROPOSITION 6.5. *Let $\mathscr{I}$ be generated by $\{v_i | i \geqslant 0\}$. Then*

$$BP \simeq BP\big[\mathscr{I}^{-1}\big] \simeq \operatorname{holim} L_nBP.$$

The chromatic filtration theorem of Hopkins and Ravenel [23, 7.5.7] asserts that a finite p-local spectrum X is equivalent to $\operatorname{holim} L_nX$; the previous result plays a key role in the proof (in the guise of [23, 8.6.5]).

We close with a result about completions. We have the completion $M \longrightarrow M^\wedge_{I_n}$ on the category of MU-modules M. There is another construction of a completion at I_n which extends to all p-local spectra, and the two constructions agree when both are defined. We recall the other construction. For a sequence $\mathbf{i} = (i_0, i_1, \ldots, i_{n-1})$, we may attempt to construct generalized Toda–Smith spectra

$$M_{\mathbf{i}} = M\big(p^{i_0}, v_1^{i_1}, \ldots v_{n-1}^{i_{n-1}}\big)$$

inductively, starting with S, continuing with the cofibre sequence

$$M(p^{i_0}) \longrightarrow S \xrightarrow{p^{i_0}} S,$$

and, given $L = M_{(i_0,i_1,\ldots,i_{n-2})}$, concluding with the cofibre sequence

$$M_{\mathbf{i}} \longrightarrow L \xrightarrow{v_{n-1}^{i_{n-1}}} L.$$

Here $M_{\mathbf{i}}$ is a finite complex of type n and hence admits a v_n-self map by the Nilpotence Theorem [5], [17], [19], and $v_n^{i_n}$ is shorthand for such a map. These spectra do not exist for all sequences $\mathbf{i}$, but they do exist for a cofinal set of sequences, and Devinatz has shown [4] that there is a cofinal collection all of which are ring spectra. These spectra are not determined by the sequence, but it follows from the Nilpotence Theorem that they are asymptotically unique in the sense that $\mathrm{hocolim}_{\mathbf{i}} M_{\mathbf{i}}$ is independent of all choices. Hence we may define a completion for all p-local spectra X by

$$X_{I_n}^{\wedge} = F(\operatorname*{hocolim}_{\mathbf{i}} M_{\mathbf{i}}, X).$$

We shall denote the spectrum $\mathrm{hocolim}_{\mathbf{i}} M_{\mathbf{i}}$ by $\Gamma_{I_n}(S)$, although its construction is considerably more sophisticated than that of our local cohomology spectra.

PROPOSITION 6.6. *Localize all spectra at p. Then there is an equivalence of MU-modules*

$$MU \wedge \Gamma_{I_n}(S) \simeq \Gamma_{I_n}(MU).$$

Therefore, for any MU-module M, there is an equivalence of MU-modules between the two completions $M_{I_n}^{\wedge}$.

PROOF (Sketch). It is proven in [8] that localization at p, and indeed any other Bousfield localization, preserves commutative S-algebras. The second statement follows from the first since

$$F_{MU}\big(MU \wedge \Gamma_{I_n}(S), M\big) \simeq F\big(\Gamma_{I_n}(S), M\big)$$

as MU-modules. It suffices to construct compatible equivalences

$$MU \wedge M_{\mathbf{i}} \simeq MU/p^{i_0} \wedge_{MU} MU/v_1^{i_1} \wedge_{MU} \cdots \wedge_{MU} MU/v_{n-1}^{i_{n-1}}.$$

By [7, 9.9], the right side is equivalent to $MU/I_{\mathbf{i}}$, where $I_{\mathbf{i}} = (p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}) \subset I_n$. A v_n-self map $v : X \longrightarrow X$ on a type n finite complex X can be characterized as a map such that, for some i, $BP_*(v^i) : BP_*(X) \longrightarrow BP_*(X)$ is multiplication by v_n^j for some j. Since $MU_*(X) = MU_* \otimes_{BP_*} MU_*(X)$, we can use MU instead of BP. Using MU, we conclude that the two maps of spectra $\mathrm{id} \wedge v^i$ and $v_n^j \wedge \mathrm{id}$ from $MU \wedge X$ to itself induce the same map on homotopy groups. The cofibre of the first is $MU \wedge Cv^i$ and the cofibre of the second is $MU/(v_n^j) \wedge X$. In the case of our generalized Moore spectra, a nilpotence technology argument based on results in [19] shows that some powers of these two maps are homotopic, hence the cofibres of these powers are equivalent. The conclusion follows by induction. □

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CHAPTER 8

Equivariant Stable Homotopy Theory

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0. Introduction

The study of symmetries on spaces has always been a major part of algebraic and geometric topology, but the systematic homotopical study of group actions is relatively recent. The last decade has seen a great deal of activity in this area. After giving a brief sketch of the basic concepts of space level equivariant homotopy theory, we shall give an introduction to the basic ideas and constructions of spectrum level equivariant homotopy theory. We then illustrate ideas by explaining the fundamental localization and completion theorems that relate equivariant to nonequivariant homology and cohomology.

The first such result was the Atiyah–Segal completion theorem which, in its simplest terms, states that the completion of the complex representation ring $R(G)$ at its augmentation ideal I is isomorphic to the K-theory of the classifying space BG: $R(G)^{\wedge}_{I} \cong K(BG)$. A more recent homological analogue of this result describes the K-homology of BG. As we shall see, this can best be viewed as a localization theorem. These are both consequences of equivariant Bott periodicity, although full understanding depends on the localization away from I and the completion at I of the spectrum K_G that represents equivariant K-theory. We shall explain a still more recent result which states that a similar analysis works to give the same kind of localization and completion theorems for the spectrum MU_G that represents a stabilized version of equivariant complex cobordism and for all module spectra over MU_G. We shall also say a little about equivariant cohomotopy, a theory for which the cohomological completion theorem is true, by Carlsson's proof of the Segal conjecture, but the homological localization theorem is false.

1. Equivariant homotopy

We shall not give a systematic exposition of equivariant homotopy theory. There are several good books on the subject, such as [12] and [17], and a much more thorough expository account will be given in [53]. Some other expository articles are [49], [1]. We aim merely to introduce ideas, fix notations, and establish enough background in space level equivariant homotopy theory to make sense of the spectrum level counterpart that we will focus on later.

The group

We shall restrict our attention to compact Lie groups G, although the basic unstable homotopy theory works equally well for general topological groups. To retain the homeomorphism between orbits and homogeneous spaces we shall always restrict attention to *closed* subgroups.

The class of compact Lie groups has two big advantages: the subgroup structure is reasonably simple ('nearby subgroups are conjugate'), and there are enough representations (any sufficiently nice G-space embeds in one). We shall sometimes restrict to finite groups to avoid technicalities, but most of what we say applies in technically modified form to general compact Lie groups. The reader unused to equivariant topology may find it helpful to concentrate on the case when G is a group of order 2. Even this simple case well illustrates most of the basic ideas.

G-spaces and G-maps

All of our spaces are to be compactly generated and weak Hausdorff.

A G-space is a topological space X with a continuous left action by G; a based G-space is a G-space together with a basepoint fixed by G. These will be our basic objects. We frequently want to convert unbased G-spaces Y into based ones, and we do so by taking the topological sum of Y and a G-fixed basepoint; we denote the result by Y_+.

We give the product $X \times Y$ of G-spaces the diagonal action, and similarly for the smash product $X \wedge Y$ of based G-spaces. We use the notation $\mathrm{map}(X, Y)$ for the G-space of continuous maps from X to Y; G acts via $(\gamma f)(x) = \gamma f(\gamma^{-1}x)$; we let $F(X, Y)$ denote the subspace of based maps. The usual adjunctions apply.

A map of based G-spaces is a continuous basepoint preserving function which commutes with the action of G. A homotopy of based G-maps $f_0 \simeq f_1$ is a G-map $X \wedge I_+ \longrightarrow Y$ whose composites with the inclusions of $X \wedge \{0\}_+$ and $X \wedge \{1\}_+$ are f_0 and f_1. We use the notation $[X, Y]_G$ to denote the set of homotopy classes of based G-maps $X \longrightarrow Y$.

Cells, spheres, and G-CW complexes

We shall be much concerned with cells and spheres. There are two important sorts of these, arising from homogeneous spaces and from representations, and the interplay between the two is fundamental to the subject.

Given any closed subgroup H of G we may form the homogeneous space G/H and its based counterpart, G/H_+. These are treated as 0-dimensional cells, and they play a role in equivariant theory analogous to the role of a point in nonequivariant theory. We form the n-dimensional cells from these homogeneous spaces. In the unbased context, the cell-sphere pair is

$$\left(G/H \times D^n, G/H \times S^{n-1}\right),$$

and in the based context

$$\left(G/H_+ \wedge D^n, G/H_+ \wedge S^{n-1}\right).$$

We shall always use different notation for different actions, so that when we write D^n and S^n we understand that G acts trivially.

Starting from these cell-sphere pairs, we form G-CW complexes exactly as nonequivariant CW-complexes are formed from the cell-sphere pairs (D^n, S^{n-1}). The usual theorems transcribe directly to the equivariant setting, and we shall say more about them below. Smooth compact G-manifolds are triangulable as finite G-CW complexes, but topological G-manifolds need not be.

We also have balls and spheres formed from orthogonal representations V of G. We shall be concerned especially with the one-point compactification S^V of V, with ∞ as the basepoint; note in particular that the usual convention that n denotes the trivial n-dimensional real representation gives S^n the usual meaning. We may also form the unit disc

$$D(V) = \{v \in V \mid \|v\| \leqslant 1\},$$

and the unit sphere

$$S(V) = \{v \in V \mid \|v\| = 1\};$$

we think of them as unbased G-spaces. There is a homeomorphism $S^V \cong D(V)/S(V)$. The resulting cofibre sequence

$$S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V$$

can be very useful in inductive arguments since there is an equivariant homotopy equivalence $D(V)_+ \simeq S^0$.

Fixed points and quotients

There are a number of ways to increase or decrease the size of the ambient group. If $f : G_1 \longrightarrow G_2$ is a group homomorphism we may regard a G_2-space Y as a G_1-space f^*Y by pullback along f, and we usually omit f^* when the context makes it clear. The most common cases of this are when G_1 is a subgroup of G_2 and when G_2 is a quotient of G_1; in particular every space may be regarded as a G-fixed G-space.

The most important construction on G-spaces is passage to fixed points:

$$X^H = \{x \in X \mid hx = x \text{ for all } h \in H\}.$$

For example, $F(X,Y)^G$ is the space of based G-maps $X \longrightarrow Y$. It is easy to check that the fixed point spaces for the conjugates of H are all homeomorphic; indeed, multiplication by g induces a homeomorphism $g : X^{g^{-1}Hg} \longrightarrow X^H$. In particular X^H is invariant under the action of the normalizer $N_G(H)$, and hence it has a natural action of the Weyl group $W_G(H) = N_G(H)/H$. Passage to H-fixed point spaces is a functor from G-spaces to $W_G(H)$-spaces.

Dually, we have the quotient space X/H of X by H. This is actually a standard abuse of notation, since $H\backslash X$ would be more consistent logically; for example, we are using G/H to denote the quotient of G by its *right* action by H. Again, multiplication by g gives a homeomorphism $X/g^{-1}Hg \longrightarrow X/H$. Thus X/H also has a natural action of the Weyl group, and passage to the quotient by H gives a functor from G-spaces to $W_G(H)$-spaces.

If N is a normal subgroup of G, then it is easy to verify that passage to N-fixed points is right adjoint to pullback along $G \longrightarrow G/N$ and that passage to the quotient by N is left adjoint to this pullback.

LEMMA 1.1. *For G-spaces X and G/N-spaces Y, there are natural homeomorphisms*

$$G\text{-map}(Y, X) \cong G/N\text{-map}(Y, X^N)$$

and

$$G/N\text{-map}(X/N, Y) \cong G\text{-map}(X, Y),$$

and similarly in the based context.

The particular case

$$G\text{-map}(G/H, X) \cong X^H$$

helps explains the importance of the fixed point functor.

Isotropy groups and universal spaces

An unbased G-space is said to be *G-free* if $X^H = \varnothing$ whenever $H \neq 1$. A based G-space is *G-free* if $X^H = *$ whenever $H \neq 1$. More generally, for $x \in X$ the *isotropy group* at x is the stabilizer G_x; given any collection $\mathscr{F}$ of subgroups of G, we say that X is an $\mathscr{F}$-space if $G_x \in \mathscr{F}$ for every non-basepoint $x \in X$. Thus a G-space is free if and only if it is a $\{1\}$-space. It is usual to think of a G-space as built up from the G-fixed subspace X^G by adding points with successively smaller and smaller isotropy groups. This gives a stratification in which the pure strata consist of points with isotropy group in a single conjugacy class.

A collection $\mathscr{F}$ of subgroups of G closed under passage to conjugates and subgroups is called a *family* of subgroups. For each family, there is an unbased $\mathscr{F}$-space $E\mathscr{F}$, required to be of the homotopy type of a G-CW complex, which is universal in the sense that there is a unique homotopy class of G-maps $X \longrightarrow E\mathscr{F}$ for any $\mathscr{F}$-space X of the homotopy type of a G-CW complex. It is characterized by the fact that the fixed point set $(E\mathscr{F})^H$ is contractible for $H \in \mathscr{F}$ and empty for $H \notin \mathscr{F}$. For example, if $\mathscr{F}$ consists of only the trivial group, then $E\{1\}$ is the universal free G-space EG, and if $\mathscr{F}$ is the family of all subgroups, then $EAll = *$. Another case of particular interest is the family $\mathscr{P}$ of all proper subgroups. If G is finite, then

$$E\mathscr{P} = \bigcup_{k \geqslant 0} S(kV),$$

where V is the reduced regular representation of G, and in general $E\mathscr{P} = \operatorname{colim}_V S(V)$ where V runs over all finite dimensional representations V of G such that $V^G = \{0\}$; to be precise, we restrict V to lie in some complete G-universe (as defined in the next section). Such universal spaces exist for any family and may be constructed either by killing homotopy groups or by using a suitable bar construction [20]. In the based case we consider $E\mathscr{F}_+$, and a very basic tool is the isotropy separation cofibering

$$\boxed{E\mathscr{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathscr{F}},$$

where the first map is obtained from $E\mathscr{F} \longrightarrow *$ by adding a disjoint basepoint. Note that the mapping cone $\tilde{E}\mathscr{F}$ may alternatively be described as the join $S^0 * E\mathscr{F}$; it is

$\mathscr{F}$-contractible in the sense that it is H-contractible for every $H \in \mathscr{F}$. We think of this cofibering as separating a space X into the $\mathscr{F}$-space $E\mathscr{F}_+ \wedge X$ and the $\mathscr{F}$-contractible space $\tilde{E}\mathscr{F} \wedge X$.

Induced and coinduced spaces

We can use the fact that G is both a left and a right G-space to define induced and coinduced G-space functors. If H is a subgroup of G and Y is an H-space, we define the induced G-space $G \times_H Y$ to be the quotient of $G \times Y$ by the equivalence relation $(gh, y) \sim (g, hy)$ for $g \in G$, $y \in Y$, and $h \in H$; the G-action is defined by $\gamma[g, y] = [\gamma g, y]$.

Similarly the coinduced G-space $\mathrm{map}_H(G, Y)$ is the subspace of $\mathrm{map}(G, Y)$ consisting of those maps $f : G \longrightarrow Y$ such that $f(hg) = hf(g)$ for $h \in H$ and $g \in G$; the G-action is defined by $(\gamma f)(g) = f(g\gamma)$. When these constructions are applied to a G-space, the actions may be untwisted, and it is well worth writing down the particular homeomorphisms.

LEMMA 1.2. *If X is a G-space then there are homeomorphisms*

$$G \times_H X \cong G/H \times X \quad \textit{and} \quad \mathrm{map}_H(G, X) \cong \mathrm{map}(G/H, X),$$

natural for G-maps of X.

PROOF. In the first case, the maps are $[g, x] \longmapsto (gH, gx)$ and $[g, g^{-1}x] \longleftarrow (gH, x)$. In the second case, $f \longmapsto a(f)$, where $a(f)(gH) = gf(g^{-1})$, and $b(f') \longleftarrow f'$, where $b(f')(g) = gf'(g^{-1}H)$. We encourage the reader to make the necessary verifications. □

The induced space functor is left adjoint to the forgetful functor and the coinduced space functor is right adjoint to it.

PROPOSITION 1.3. *For G-spaces X and H-spaces Y, there are natural homeomorphisms*

$$G\text{-}\mathrm{map}(G \times_H Y, X) = H\text{-}\mathrm{map}(Y, X)$$

and

$$H\text{-}\mathrm{map}(X, Y) = G\text{-}\mathrm{map}\big(X, \mathrm{map}_H(G, Y)\big).$$

PROOF. The unit and counit for the first adjunction are the H-map $\eta : Y \longrightarrow G \times_H Y$ given by $y \longmapsto [e, y]$ and the G-map $\varepsilon : G \times_H X \longrightarrow X$ given by $[g, x] \longmapsto gx$. For the second, they are the G-map $\eta : X \longrightarrow \mathrm{map}_H(G, X)$ given by $\eta(x)(g) = gx$ and the H-map $\varepsilon : \mathrm{map}_H(G, Y) \longrightarrow Y$ given by $\varepsilon(f) = f(e)$. We encourage the reader to make the necessary verifications. □

Analogous constructions and homeomorphisms apply in the based case. If Y is a based H-space, it is usual to write $G_+ \wedge_H Y$ or $G \ltimes_H Y$ for the induced based G-space, and $F_H(G_+, Y)$ or $F_H[G, Y)$ for the coinduced based G-space.

Homotopy groups, weak equivalences, and the G-Whitehead theorem
One combination of the above adjunctions is particularly important. To define H-equivariant homotopy groups, we might wish to define them G-equivariantly as $[G/H_+ \wedge S^n, \cdot]_G$, or we might wish to define them H-equivariantly as $[S^n, \cdot]_H$; fortunately these agree, and we define

$$\pi_n^H(X) = \left[G/H_+ \wedge S^n, X\right]_G \cong \left[S^n, X\right]_H \cong \left[S^n, X^H\right].$$

Using the second isomorphism, we may apply finiteness results from nonequivariant homotopy theory. For example, if X and Y are finite G-CW complexes and double suspensions, then $[X, Y]_G$ is a finitely generated abelian group.

A G-map $f : X \longrightarrow Y$ is a weak G-equivalence if $f^H : X^H \longrightarrow Y^H$ is a weak equivalence for all closed subgroups H. As in the nonequivariant case one proves that any G-CW pair has the homotopy extension and lifting property and deduces that a weak equivalence induces a bijection of $[T, \cdot]_G$ for every G-CW complex T. The G-Whitehead theorem follows: a weak G-equivalence of G-CW complexes is a G-homotopy equivalence. Similarly, the cellular approximation theorem holds: any map between G-CW complexes is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic. Also, by the usual construction, any G-space is weakly equivalent to a G-CW complex.

The generalization to families $\mathscr{F}$ is often useful. We say that a G-map f is a weak $\mathscr{F}$-equivalence if f^H is a weak equivalence for $H \in \mathscr{F}$; the principal example of an $\mathscr{F}$-equivalence is the map $E\mathscr{F}_+ \wedge X \longrightarrow X$. A based $\mathscr{F}$-CW complex is a G-CW complex whose cells are all of the form $G/H_+ \wedge S^n$ for $H \in \mathscr{F}$; note that an $\mathscr{F}$-CW complex is an $\mathscr{F}$-space. The usual proofs show that a weak $\mathscr{F}$-equivalence induces a bijection of $[T, \cdot]_G$ for every $\mathscr{F}$-CW complex T and that any G-space is $\mathscr{F}$-equivalent to an $\mathscr{F}$-CW complex.

To state a quantitative version of the G-Whitehead theorem, we consider functions n on the set of subgroups of G with values in the set $\{-1, 0, 1, 2, 3, \ldots, \infty\}$ that are constant on conjugacy classes. For example if X is a G-space, we can view dimension and connectivity as giving such functions by defining $dim(X)(H) = dim(X^H)$ and $conn(X)(H)$ to be the connectivity of X^H. The value -1 allows the possibility of empty or of nonconnected fixed point spaces. Now the standard proof gives the following result.

THEOREM 1.4. *If T is a G-CW complex and $f : X \longrightarrow Y$ is n-connected, then the induced map*

$$f_* : [T, X]_G \longrightarrow [T, Y]_G$$

is surjective if $dim(T^H) \leqslant n(H)$ for all $H \subseteq G$, and bijective if $dim(T^H) \leqslant n(H) - 1$.

The G-Freudenthal suspension theorem
In the stable world, we shall want to desuspend by spheres of representations. Accordingly, for any orthogonal representation V, we define the Vth suspension functor by

$\Sigma^V X = X \wedge S^V$. This gives a map

$$\Sigma^V : [X, Y]_G \longrightarrow [\Sigma^V X, \Sigma^V Y]_G.$$

We shall be content to give the version of the Freudenthal Theorem, due to Hauschild [36], that gives conditions under which this map is an isomorphism. However, we note in passing that the presence of S^V gives the codomain a richer algebraic structure than the domain, and it is natural to seek a theorem stating that Σ^V may be identified with an algebraic enrichment of the domain even when it is not an isomorphism. L.G. Lewis [38] has proved versions of the Freudenthal Theorem along these lines when X is a sphere.

Just as nonequivariantly, we approach the Freudenthal Theorem by studying the adjoint map $\eta : Y \longrightarrow \Omega^V \Sigma^V Y$.

THEOREM 1.5. *The map $\eta : Y \longrightarrow \Omega^V \Sigma^V Y$ is an n-equivalence if n satisfies the following two conditions:*

(1) *$n(H) \leqslant 2conn(Y^H) + 1$ for all subgroups H with $V^H \neq 0$, and*

(2) *$n(H) \leqslant conn(Y^K)$ for all pairs of subgroups $K \subseteq H$ with $V^K \neq V^H$.*

Therefore the suspension map

$$\Sigma^V : [X, Y]_G \longrightarrow [\Sigma^V X, \Sigma^V Y]_G$$

is surjective if $dim(X^H) \leqslant n(H)$ for all H, and bijective if $dim(X^H) \leqslant n(H) - 1$.

This is proven by reduction to the nonequivariant case and obstruction theory. When G is finite and X is finite dimensional, it follows that if we suspend by a sufficiently large representation, then all subsequent suspensions will be isomorphisms.

COROLLARY 1.6. *If G is finite and X is finite dimensional, there is a representation $V_0 = V_0(X)$ such that, for any representation V,*

$$\Sigma^V : [\Sigma^{V_0} X, \Sigma^{V_0} Y]_G \xrightarrow{\cong} [\Sigma^{V_0 \oplus V} X, \Sigma^{V_0 \oplus V} Y]_G$$

is an isomorphism.

If X and Y are finite G-CW complexes, this stable value $[\Sigma^{V_0} X, \Sigma^{V_0} Y]_G$ is a finitely generated abelian group. If G is a compact Lie group and X has infinite isotropy groups, there is usually no representation V_0 for which all suspensions Σ^V are isomorphisms, and the colimit of the $[\Sigma^V X, \Sigma^V Y]_G$ is usually not finitely generated.

The direct limit $\mathrm{colim}_V [S^V, S^V]_G$ is a ring under composition, and it turns out to be isomorphic to the Burnside ring $A(G)$. When G is finite, $A(G)$ is defined to be the Grothendieck ring associated to the semiring of finite G-sets, and it is the free Abelian group with one generator $[G/H]$ for each conjugacy class of subgroups of G. When G is a general compact Lie group, $A(G)$ is more complicated to define, but it turns out to be a free Abelian group, usually of infinite rank, with one basis element $[G/H]$ for each conjugacy class of subgroups H such that $W_G H$ is finite.

Eilenberg–MacLane G-spaces and Postnikov towers

The homotopy groups $\pi_n^H(X)$ of a G-space X are related as H varies, and we must take all of them into account to develop obstruction theory. Let $\mathscr{O}$ denote the orbit category of G-spaces G/H and G-maps between them, and let $h\mathscr{O}$ be its homotopy category. By our first description of homotopy groups, we see that the definition $\underline{\pi}_n(X)(G/H) = \pi_n^H(X)$ gives a set-valued contravariant functor on $h\mathscr{O}$; it is group-valued if $n = 1$ and Abelian group-valued if $n \geqslant 2$. An Eilenberg–MacLane G-space $K(\underline{\pi}, n)$ associated to such a contravariant functor $\underline{\pi}$ on $h\mathscr{O}$ is a G-space such that $\underline{\pi}_n(K(\underline{\pi}, n)) = \underline{\pi}$ and all other homotopy groups of $K(\underline{\pi}, n)$ are zero. Either by killing homotopy groups or by use of a bar construction [20], one sees that Eilenberg–MacLane G-spaces exist for all $\underline{\pi}$ and n.

Recall that a space X is simple if it is path connected and if $\pi_1(X)$ is Abelian and acts trivially on $\pi_n(X)$ for $n \geqslant 2$. More generally, X is nilpotent if it is path connected and if $\pi_1(X)$ is nilpotent and acts nilpotently on $\pi_n(X)$ for $n \geqslant 2$. A G-space X is said to be simple or nilpotent if each X^H is simple or nilpotent. Exactly as in the nonequivariant situation, simple G-spaces are weakly equivalent to inverse limits of simple Postnikov towers and nilpotent G-spaces are weakly equivalent to inverse limits of nilpotent Postnikov towers.

Ordinary cohomology theory; localization and completion

We define a "coefficient system" M to be a contravariant Abelian group-valued functor on $h\mathscr{O}$. There are associated cohomology theories on pairs of G-spaces, denoted $H_G^*(X, A; M)$. They satisfy and are characterized by the equivariant versions of the usual axioms: homotopy, excision, exactness, wedge, weak equivalence, and dimension; the last states that

$$H_G^*(G/H; M) \cong M(G/H),$$

functorially on $h\mathscr{O}$. This is a manifestation of the philosophy that orbits play the role of points. There are also homology theories, denoted $H_*^G(X, A; N)$, but these must be defined using covariant functors $N : h\mathscr{O} \longrightarrow \mathscr{A}b$.

By the weak equivalence axiom, it suffices to define these theories on G-CW pairs. The cohomology of such a pair (X, A) is the reduced cohomology of X/A, so it suffices to deal with G-CW complexes X. These have cellular chain coefficient systems that are specified by

$$\underline{C}_n(X)(G/H) = H_n\big((X^n)^H, (X^{n-1})^H; \mathbb{Z}\big);$$

the differential d_n is the connecting homomorphism of the triple

$$\big((X^n)^H, (X^{n-1})^H, (X^{n-2})^H\big).$$

The homology and cohomology groups of X are then calculated from chain and cochain complexes of Abelian groups given by

$$\underline{C}_*(X) \otimes_{h\mathscr{O}} N \quad \text{and} \quad \mathrm{Hom}_{h\mathscr{O}}\big(\underline{C}_*(X), M\big).$$

Here $\mathrm{Hom}_{h\mathcal{O}}(\underline{C}_n(X), M)$ is the group of natural transformations $\underline{C}_n(X) \longrightarrow M$, and the tensor product over $h\mathcal{O}$ is described categorically as a coend of functors.

Alternatively, for based G-CW complexes X, one has the equivalent description of reduced cohomology as

$$\tilde{H}^n(X; M) = [X, K(M, n)]_G.$$

From here, it is an exercise to transcribe classical obstruction theory to the equivariant context. This was first done by Bredon [11], who introduced these cohomology theories.

One can localize or complete nilpotent G-spaces at a set of primes. One first works out the construction on $K(\underline{\pi}, n)$'s, and then proceeds by induction up the Postnikov tower. See [55], [57]. When G is finite, one can algebraicize equivariant rational homotopy theory, by analogy with the nonequivariant theory. See [63]. Bredon cohomology is the basic tool in these papers.

While the theory we have described looks just like nonequivariant theory, we emphasize that it behaves very differently calculationally. For example, a central calculational theorem in nonequivariant homotopy theory states that the rationalization of a connected Hopf space splits, up to homotopy, as a product of Eilenberg–MacLane spaces. The equivariant analogue is false [64].

2. The equivariant stable homotopy category

The entire foundational framework described in [22] works equally well in the presence of a compact Lie group G acting on all objects in sight. We here run through the equivariant version of [22], with emphasis on the new equivariant phenomena that appear. From both the theoretical and calculational standpoint, the main new feature is that the equivariant analogs of spheres are the spheres associated to representations of G, so that there is a rich interplay between the homotopy theory and representation theory of G. The original sources for most of this material are the rather encyclopedic [42] and the nonequivariantly written [22]; a more leisurely and readable exposition will appear in [53].

By a G-universe U, we understand a countably infinite dimensional real inner product space with an action of G through linear isometries. We require that U be the sum of countably many copies of each of a set of representations of G and that it contain a trivial representation and thus a copy of $\mathbb{R}^\infty$. We say that U is complete if it contains a copy of every irreducible representation of G. At the opposite extreme, we say that U is G-fixed if $U^G = U$. When G is finite, the sum of countably many copies of the regular representation $\mathbb{R}G$ gives a canonical complete universe. We refer to a finite dimensional sub G-inner product space of U as an indexing space.

A G-spectrum indexed on U consists of a based G-space EV for each indexing space V in U together with a transitive system of based G-homeomorphisms

$$\tilde{\sigma} : EV \xrightarrow{\cong} \Omega^{W-V} EW$$

for $V \subset W$. Here $\Omega^V X = F(S^V, X)$ and $W - V$ is the orthogonal complement of V in W. A map of G-spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : EV \to E'V$ which commute with the respective structure maps.

We obtain the category $G\mathscr{S} = G\mathscr{S}U$ of G-spectra indexed on U. Dropping the requirement that the maps $\tilde{\sigma}_{V,W}$ be homeomorphisms, we obtain the notion of a G-prespectrum and the category $G\mathscr{P} = G\mathscr{P}U$ of G-prespectra indexed on U. The forgetful functor $\ell : G\mathscr{S} \longrightarrow G\mathscr{P}$ has a left adjoint L. When the structure maps $\tilde{\sigma}$ are inclusions, $(LE)(V)$ is just the union of the G-spaces $\Omega^{W-V}EW$ for $V \subset W$. We write $\sigma : \Sigma^{W-V}EV \longrightarrow EW$ for the adjoint structure maps.

EXAMPLES 2.1. Let X be a based G-space. The suspension G-prespectrum $\Pi^\infty X$ has Vth space $\Sigma^V X$, and the suspension G-spectrum of X is $\Sigma^\infty X = L\Pi^\infty X$. Let $QX = \cup \Omega^V \Sigma^V X$, where the union is taken over the indexing spaces $V \subset U$; a more accurate notation would be $Q_U X$. Then $(\Sigma^\infty X)(V) = Q(\Sigma^V X)$. The functor Σ^∞ is left adjoint to the zeroth space functor. More generally, for an indexing space $Z \subset U$, let $\Pi_Z^\infty X$ have Vth space $\Sigma^{V-Z}X$ if $Z \subset V$ and a point otherwise and define $\Sigma_Z^\infty X = L\Pi_Z^\infty X$. The "shift desuspension" functor Σ_Z^∞ is left adjoint to the Zth space functor from G-spectra to G-spaces.

For a G-space X and G-spectrum E, we define G-spectra $E \wedge X$ and $F(X, E)$ exactly as in the nonequivariant situation. There result homeomorphisms

$$G\mathscr{S}(E \wedge X, E') \cong G\mathscr{T}(X, \mathscr{S}(E, E')) \cong G\mathscr{S}(E, F(X, E')),$$

where $G\mathscr{T}$ is the category of based G-spaces.

PROPOSITION 2.2. *The category $G\mathscr{S}$ is complete and cocomplete.*

A homotopy between maps $E \longrightarrow F$ of G-spectra is a map $E \wedge I_+ \longrightarrow F$. Let $[E, F]_G$ denote the set of homotopy classes of maps $E \longrightarrow F$. For example, if X and Y are based G-spaces and X is compact, then

$$[\Sigma^\infty X, \Sigma^\infty Y]_G \cong \operatorname*{colim}_V \, [\Sigma^V X, \Sigma^V Y]_G.$$

Fix a copy of $\mathbb{R}^\infty$ in U and write $\Sigma_n^\infty = \Sigma_{\mathbb{R}^n}^\infty$. For $n \geqslant 0$, the sphere G-spectrum S^n is $\Sigma^\infty S^n$. For $n > 0$, the sphere G-spectrum S^{-n} is $\Sigma_n^\infty S^0$. We shall often write S_G rather than S^0 for the zero sphere G-spectrum. Remembering that orbits are the analogues of points, we think of the G-spectra $G/H_+ \wedge S^n$ as generalized spheres. Define the homotopy groups of a G-spectrum E by

$$\pi_n^H(E) = [G/H_+ \wedge S^n, E]_G.$$

A map $f : E \longrightarrow F$ of G-spectra is said to be a weak equivalence if $f_* : \pi_*^H(E) \longrightarrow \pi_*^H(F)$ is an isomorphism for all H. Here serious equivariant considerations enter for the first time.

THEOREM 2.3. *A map $f : E \longrightarrow F$ of G-spectra is a weak equivalence if and only if $f_V : EV \longrightarrow FV$ is a weak equivalence of G-spaces for all indexing spaces $V \subset U$.*

This is obvious when the universe U is trivial, but it is far from obvious in general. To see that a weak equivalence of G-spectra is a spacewise weak equivalence, one sets up an inductive scheme and uses the fact that spheres S^V are triangulable as G-CW complexes [42, I.7.12].

The equivariant stable homotopy category $\bar{h}G\mathscr{S}$ is constructed from the homotopy category $hG\mathscr{S}$ of G-spectra by adjoining formal inverses to the weak equivalences, a process that is made rigorous by G-CW approximation. The theory of G-CW spectra is developed by taking the sphere G-spectra as the domains of attaching maps of cells $G/H_+ \wedge CS^n$, where $CE = E \wedge I$ [42, I§5]. This works just as well equivariantly as nonequivariantly, and we arrive at the following theorems.

THEOREM 2.4 (Whitehead). *If E is a G-CW spectrum and $f : F \longrightarrow F'$ is a weak equivalence of G-spectra, then $f_* : [E, F]_G \longrightarrow [E, F']_G$ is an isomorphism. Therefore a weak equivalence between G-CW spectra is a homotopy equivalence.*

THEOREM 2.5 (Cellular approximation). *Let A be a subcomplex of a G-CW spectrum E, let F be a G-CW spectrum, and let $f : E \longrightarrow F$ be a map whose restriction to A is cellular. Then f is homotopic relative to A to a cellular map. Therefore any map $E \longrightarrow F$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.*

THEOREM 2.6 (Approximation by G-CW spectra). *For a G-spectrum E, there is a G-CW spectrum ΓE and a weak equivalence $\gamma : \Gamma E \longrightarrow E$. On the homotopy category $hG\mathscr{S}$, Γ is a functor such that γ is natural.*

Thus the stable category $\bar{h}G\mathscr{S}$ is equivalent to the homotopy category of G-CW spectra. As in the nonequivariant context, we have special kinds of G-prespectra that lead to a category of G-spectra on which the smash product has good homotopical properties. Of course, we define cofibrations of G-spaces via the homotopy extension property in the category of G-spaces. For example, X is G-LEC if its diagonal map is a G-cofibration.

DEFINITION 2.7. A G-prespectrum D is said to be

(i) Σ-cofibrant if each $\sigma : \Sigma^{W-V}DV \to DW$ is a based G-cofibration.
(ii) G-CW if it is Σ-cofibrant and each DV is G-LEC and has the homotopy type of a G-CW complex.

A G-spectrum E is said to be Σ-cofibrant if it is isomorphic to LD for some Σ-cofibrant G-prespectrum D; E is said to be tame if it is of the homotopy type of a Σ-cofibrant G-spectrum.

There is no sensible counterpart to the nonequivariant notion of a strict CW prespectrum for general compact Lie groups, and any such notion is clumsy at best even for finite groups. The next few results are restated from [22]. Their proofs are the same equivariantly as nonequivariantly.

THEOREM 2.8. *If D is a G-CW prespectrum, then LD has the homotopy type of a G-CW spectrum. If E is a G-CW spectrum, then each space EV has the homotopy type of a G-CW complex and E is homotopy equivalent to LD for some G-CW prespectrum D. Thus a G-spectrum has the homotopy type of a G-CW spectrum if and only if it has the homotopy type of LD for some G-CW prespectrum D.*

In particular, G-spectra of the homotopy types of G-CW spectra are tame.

PROPOSITION 2.9. *If $E = LD$, where D is a Σ-cofibrant G-prespectrum, then*

$$E \cong \operatorname*{colim}_V \Sigma_V^\infty DV,$$

where the colimit is computed as the prespectrum level colimit of the maps

$$\Sigma_W^\infty \sigma : \Sigma_V^\infty DV \cong \Sigma_W^\infty \Sigma^{W-V} DV \longrightarrow \Sigma_W^\infty DW.$$

That is, the prespectrum level colimit is a G-spectrum that is isomorphic to E. The maps of the colimit system are shift desuspensions of based G-cofibrations.

PROPOSITION 2.10. *There is a functor $K : G\mathscr{P}U \longrightarrow G\mathscr{P}U$ such that KD is Σ-cofibrant for any G-prespectrum D, and there is a natural spacewise weak equivalence of G-prespectra $KD \longrightarrow D$. On G-spectra E, define $KE = LK\ell E$. Then there is a natural weak equivalence of G-spectra $KE \longrightarrow E$.*

For G-universes U and U', there is an associative and commutative smash product

$$G\mathscr{S}U \times G\mathscr{S}U' \to G\mathscr{S}(U \oplus U').$$

It is obtained by applying the spectrification functor L to the prespectrum level definition

$$(E \wedge E')(V \oplus V') = EV \wedge E'V'.$$

We internalize by use of twisted half-smash products. For G-universes U and U', we have a G-space $\mathscr{I}(U,U')$ of linear isometries $U \longrightarrow U'$, with G acting by conjugation. For a G-map $\alpha : A \to \mathscr{I}(U,U')$, the twisted half-smash product assigns a G-spectrum $A \ltimes E$ indexed on U' to a G-spectrum E indexed on U. While the following result is proven the same way equivariantly as nonequivariantly, it has different content: for a given $V \subset U$, there may well be no $V' \subset U'$ that is isomorphic to V.

PROPOSITION 2.11. *For a G-map $A \longrightarrow \mathscr{I}(U,U')$ and an isomorphism $V \cong V'$, where $V \subset U$ and $V' \subset U'$, there is an isomorphism of G-spectra*

$$A \ltimes \Sigma_V^\infty X \cong A_+ \wedge \Sigma_{V'}^\infty X$$

that is natural in G-spaces A over $\mathscr{I}(U,U')$ and based G-spaces X.

Propositions 2.9 and 2.11 easily imply the following fundamental technical result.

THEOREM 2.12. *Let $E \in G\mathscr{S}U$ be tame and let A be a G-space over $\mathscr{I}(U, U')$, where the universe U' contains a copy of every indexing space $V \subset U$. If $\phi : A' \longrightarrow A$ is a homotopy equivalence, then $\phi \ltimes \mathrm{id} : A' \ltimes E \longrightarrow A \ltimes E$ is a homotopy equivalence.*

If A is a G-CW complex and E is a G-CW spectrum, then $A \ltimes E$ is a G-CW spectrum when G is finite and has the homotopy type of a G-CW spectrum in general, hence this has the following consequence.

COROLLARY 2.13. *Let $E \in G\mathscr{S}U$ have the homotopy type of a G-CW spectrum and let A be a G-space over $\mathscr{I}(U, U')$ that has the homotopy type of a G-CW complex. Then $A \ltimes E$ has the homotopy type of a G-CW spectrum.*

We define the equivariant linear isometries operad $\mathscr{L}$ by letting $\mathscr{L}(j)$ be the G-space $\mathscr{I}(U^j, U)$, exactly as in [22, 2.4]. A G-linear isometry $f : U^j \to U$ defines a G-map $\{*\} \longrightarrow \mathscr{L}(j)$ and thus a functor f_* that sends G-spectra indexed on U^j to G-spectra indexed on U. Applied to a j-fold external smash product $E_1 \wedge \cdots \wedge E_j$, there results an internal smash product $f_*(E_1 \wedge \cdots \wedge E_j)$.

THEOREM 2.14. *Let $G\mathscr{S}_t \subset G\mathscr{S}$ be the full subcategory of tame G-spectra and let $hG\mathscr{S}_t$ be its homotopy category. On $G\mathscr{S}_t$, the internal smash products $f_*(E \wedge E')$ determined by varying $f : U^2 \to U$ are canonically homotopy equivalent, and $hG\mathscr{S}_t$ is symmetric monoidal under the internal smash product. For based G-spaces X and tame G-spectra E, there is a natural homotopy equivalence $E \wedge X \simeq f_*(E \wedge \Sigma^\infty X)$.*

We can define $\Sigma^V E = E \wedge S^V$ for any representation V. This functor is left adjoint to the loop functor Ω^V given by $\Omega^V E = F(S^V, E)$. For $V \subset U$, *and only for such* V, we also have the shift desuspension functor Σ^∞_V and therefore a $(-V)$-sphere $S^{-V} = \Sigma^\infty_V S^0$. Now the proof of [22, 2.6] applies to show that we have arrived at a stable situation *relative to* U.

THEOREM 2.15. *For $V \subset U$, the suspension functor $\Sigma^V : hG\mathscr{S}_t \longrightarrow hG\mathscr{S}_t$ is an equivalence of categories with inverse given by smashing with S^{-V}. A cofiber sequence $E \xrightarrow{f} E' \longrightarrow Cf$ in $G\mathscr{S}_t$ gives rise to long exact sequences of homotopy groups*

$$\cdots \longrightarrow \pi_q^H(E) \longrightarrow \pi_q^H(E') \longrightarrow \pi_q^H(Cf) \longrightarrow \pi_{q-1}^H(E) \longrightarrow \cdots.$$

From here, the theory of $\mathbb{L}$-spectra, S-modules, S-algebras, and modules over S-algebras that was summarized in [22, §§3–7] applies verbatim equivariantly, with one striking exception: duality theory only works when one restricts to cell R-modules that are built up out of sphere R-modules $G/H_+ \wedge S^n_R$ such that G/H embeds as a sub G-space of U. We shall focus on commutative S_G-algebras later, but we must first explain the exception just noted, along with various other matters where considerations of equivariance are central to the theory.

3. Homology and cohomology theories and fixed point spectra

In the previous section, the G-universe U was arbitrary, and we saw that the formal development of the stable category $\bar{h}G\mathscr{S}U$ worked quite generally. However, there is very different content to the theory depending on the choice of universe. We focus attention on a complete G-universe U and its fixed point universe U^G. We call G-spectra indexed on U^G "naive G-spectra" since these are just spectra with G-action in the most naive sense. Examples include nonequivariant spectra regarded as G-spectra with trivial action. Genuine G-spectra are those indexed on U, and we refer to them simply as G-spectra. Their structure encodes the interrelationship between homotopy theory and representation theory that is essential for duality theory and most other aspects of equivariant stable homotopy theory.

$RO(G)$-graded homology and cohomology

Some of this interrelationship is encoded in the notion of an $RO(G)$-graded cohomology theory, which will play a significant role in our discussion of completion theorems. To be precise about this, one must remember that virtual representations are formal differences of *isomorphism classes* of orthogonal G-modules; we refer the interested reader to [53] for details and just give the idea here. For a virtual representation $\nu = W - V$, we can form the sphere G-spectrum $S^\nu = \Sigma^W S^{-V}$. We then define the homology and cohomology groups represented by a G-spectrum E by

$$E_\nu^G(X) = [S^\nu, E \wedge X]_G \tag{3.1}$$

and

$$E^\nu_G(X) = [S^{-\nu} \wedge X, E]_G = [S^{-\nu}, F(X, E)]_G. \tag{3.2}$$

If we think just about the $\mathbb{Z}$-graded part of a cohomology theory on G-spaces, then $RO(G)$-gradability amounts to the same thing as naturality with respect to stable G-maps.

Underlying nonequivariant spectra

To relate such theories to nonequivariant theories, let $i : U^G \longrightarrow U$ be the inclusion. We have the forgetful functor $i^* : G\mathscr{S}U \longrightarrow G\mathscr{S}U^G$ specified by $i^*E(V) = E(i(V))$ for $V \subset U^G$; that is, we forget about the indexing spaces with nontrivial G-action. The "underlying nonequivariant spectrum" of E is i^*E with its action by G ignored. Recall that i^* has a left adjoint $i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$ that builds in nontrivial representations. Using an obvious notation to distinguish suspension spectrum functors, we have $i_*\Sigma^\infty_{U^G} X \cong \Sigma^\infty_U X$. These change of universe functors play a critical role in relating equivariant and nonequivariant phenomena. Since, with G-actions ignored, the universes are isomorphic, the following result is intuitively obvious.

LEMMA 3.3. *For $D \in G\mathscr{S}U^G$, the unit G-map $\eta : D \longrightarrow i^*i_*D$ of the (i_*, i^*) adjunction is a nonequivariant equivalence. For $E \in G\mathscr{S}U$, the counit G-map $\varepsilon : i_*i^*E \longrightarrow E$ is a nonequivariant equivalence.*

Fixed point spectra and homology and cohomology
We define the fixed point spectrum D^G of a naive G-spectrum D by passing to fixed points spacewise, $D^G(V) = (DV)^G$. This functor is right adjoint to the forgetful functor from naive G-spectra to spectra (compare Lemma 1.1):

$$G\mathscr{S}U^G(C, D) \cong \mathscr{S}U^G(C, D^G) \quad \text{for } C \in \mathscr{S}U^G \text{and } D \in G\mathscr{S}U^G. \tag{3.4}$$

It is essential that G act trivially on the universe to obtain well-defined structural homeomorphisms on D^G. For $E \in G\mathscr{S}U$, we define $E^G = (i^*E)^G$. Composing the (i_*, i^*)-adjunction with (3.4), we obtain

$$G\mathscr{S}U(i_*C, E) \cong \mathscr{S}U^G(C, E^G) \quad \text{for } C \in \mathscr{S}U^G \text{ and } E \in G\mathscr{S}U. \tag{3.5}$$

The sphere G-spectra $G/H_+ \wedge S^n$ in $G\mathscr{S}U$ are obtained by applying i_* to the corresponding sphere G-spectra in $G\mathscr{S}U^G$. When we restrict (3.1) and (3.2) to integer gradings and take $H = G$, we see that (3.5) implies

$$E_n^G(X) \cong \pi_n((E \wedge X)^G) \tag{3.6}$$

and

$$E_G^n(X) \cong \pi_{-n}(F(X, E)^G). \tag{3.7}$$

Exactly as in (3.7), naive G-spectra D represent $\mathbb{Z}$-graded cohomology theories on naive G-spectra, or on G-spaces. In sharp contrast, we cannot represent interesting homology theories on G-spaces X in the form $\pi_*((D \wedge X)^G)$ for a naive G-spectrum D: smash products of naive G-spectra commute with fixed points, hence such theories vanish on X/X^G. For genuine G-spectra, there is a well-behaved natural map

$$E^G \wedge (E')^G \longrightarrow (E \wedge E')^G, \tag{3.8}$$

but, even when E' is replaced by a G-space, it is not an equivalence. Similarly, there is a natural map

$$\Sigma^\infty(X^G) \longrightarrow (\Sigma^\infty X)^G, \tag{3.9}$$

which, by Theorem 3.10 below, is the inclusion of a wedge summand but not an equivalence. Again, the fixed point spectra of free G-spectra are nontrivial. We shall shortly define a different G-fixed point functor that commutes with smash products and the suspension spectrum functor and which is trivial on free G-spectra.

Fixed point spectra of suspension G-spectra
Because the suspension functor from G-spaces to genuine G-spectra builds in homotopical information from representations, the fixed point spectra of suspension G-spectra are richer structures than one might guess. The following important result of tom Dieck [18] (see also [42, V§11]), gives a precise description.

THEOREM 3.10. *For based G-CW complexes X, there is a natural equivalence*

$$(\Sigma^\infty X)^G \simeq \bigvee_{(H)} \Sigma^\infty(EWH_+ \wedge_{WH} \Sigma^{\mathrm{Ad}(WH)} X^H),$$

where $WH = NH/H$ and $\mathrm{Ad}(WH)$ is its adjoint representation; the sum runs over all conjugacy classes of subgroups H.

Quotient spectra and free G-spectra

Quotient spectra D/G of naive G-spectra are constructed by first passing to orbits spacewise on the prespectrum level and then applying the functor L from prespectra to spectra. This orbit spectrum functor is left adjoint to the forgetful functor to spectra:

$$\mathscr{S}U^G(D/G,C) \cong G\mathscr{S}U^G(D,C) \quad \text{for } C \in \mathscr{S}U^G \text{ and } D \in G\mathscr{S}U^G. \tag{3.11}$$

Commuting left adjoints, we see that $(\Sigma^\infty X)/G \cong \Sigma^\infty(X/G)$. There is no useful quotient functor on genuine G-spectra in general, but there is a suitable substitute for free G-spectra.

Recall that a based G-space is said to be free if it is free away from its G-fixed basepoint. A G-spectrum, either naive or genuine, is said to be free if it is equivalent to a G-CW spectrum built up out of free cells $G_+ \wedge CS^n$. The functors

$$\Sigma^\infty : \mathscr{T} \longrightarrow G\mathscr{S}U^G \quad \text{and} \quad i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$$

carry free G-spaces to free naive G-spectra and free naive G-spectra to free G-spectra. In all three categories, X is homotopy equivalent to a free object if and only if the canonical G-map $EG_+ \wedge X \longrightarrow X$ is an equivalence. A free G-spectrum E is equivalent to i_*D for a free naive G-spectrum D, unique up to equivalence; the orbit spectrum D/G is the appropriate substitute for E/G. A useful mnemonic slogan is that "free G-spectra live in the G-fixed universe". For free naive G-spectra D, it is clear that $D^G = *$. However, this is false for free genuine G-spectra. For example, Theorem 3.10 specializes to give that $(\Sigma^\infty X)^G \simeq (\Sigma^{\mathrm{Ad}(G)} X)/G$ if X is a free G-space. Thus the fixed point functor on free G-spectra has the character of a quotient.

More generally, for a family $\mathscr{F}$, we say that a G-spectrum E is $\mathscr{F}$-free, or is an $\mathscr{F}$-spectrum, if E is equivalent to a G-CW spectrum all of whose cells are of orbit type in $\mathscr{F}$. Thus free G-spectra are $\{1\}$-free. We say that a map $f : D \longrightarrow E$ is an $\mathscr{F}$-equivalence if $f^H : D^H \longrightarrow E^H$ is an equivalence for all $H \in \mathscr{F}$ or, equivalently by the Whitehead theorem, if f is an H-equivalence for all $H \in \mathscr{F}$.

Split G-spectra

It is fundamental to the passage back and forth between equivariant and nonequivariant phenomena to calculate the equivariant cohomology of free G-spectra in terms of the nonequivariant cohomology of orbit spectra. To explain this, we require the subtle and important notion of a "split G-spectrum".

DEFINITION 3.12. A naive G-spectrum D is said to be split if there is a nonequivariant map of spectra $\zeta : D \longrightarrow D^G$ whose composite with the inclusion of D^G in D is homotopic to the identity. A genuine G-spectrum E is said to be split if i^*E is split.

The K-theory G-spectra K_G and KO_G are split. Intuitively, the splitting is obtained by giving nonequivariant bundles trivial G-action. Similarly, equivariant Thom spectra are split. The naive Eilenberg–MacLane G-spectrum HM that represents Bredon cohomology with coefficients in M is split if and only if the restriction map $M(G/G) \longrightarrow M(G/1)$ is a split epimorphism; this implies that G acts trivially on $M(G/1)$, which is usually not the case. The suspension G-spectrum $\Sigma^\infty X$ of a G-space X is split if and only if X is stably a retract up to homotopy of X^G, which again is usually not the case. In particular, however, the sphere G-spectrum $S = \Sigma^\infty S^0$ is split. The following consequence of Lemma 3.3 gives more examples.

LEMMA 3.13. *If $D \in G\mathscr{S}U^G$ is split, then $i_*D \in G\mathscr{S}U$ is also split. In particular, i_*D is split if D is a nonequivariant spectrum regarded as a naive G-spectrum with trivial action.*

The notion of a split G-spectrum is defined in nonequivariant terms, but it admits the following equivariant interpretation.

LEMMA 3.14. *If E is a G-spectrum with underlying nonequivariant spectrum D, then E is split if and only if there is a map of G-spectra $i_*D \longrightarrow E$ that is a nonequivariant equivalence.*

THEOREM 3.15. *If E is a split G-spectrum and X is a free naive G-spectrum, then there are natural isomorphisms*

$$E_n^G(i_*X) \cong E_n\big((\Sigma^{\mathrm{Ad}(G)}X)/G\big) \quad \text{and} \quad E_G^n(i_*X) \cong E^n(X/G),$$

where $\mathrm{Ad}(G)$ is the adjoint representation of G and E_* and E^* denote the theories represented by the underlying nonequivariant spectrum of E.

The cohomology isomorphism holds by inductive reduction to the case $X = G_+$. The homology isomorphism is deeper, and we shall say a bit more about it later.

Geometric fixed point spectra
There is a "geometric" fixed-point functor

$$\Phi^G : G\mathscr{S}U \longrightarrow \mathscr{S}U^G$$

that enjoys the properties

$$\Sigma^\infty(X^G) \simeq \Phi^G(\Sigma^\infty X) \tag{3.16}$$

and

$$\Phi^G(E) \wedge \Phi^G(E') \simeq \Phi^G(E \wedge E'). \tag{3.17}$$

It is trivial on free G-spectra and, more generally, on $\mathscr{P}$-spectra, where $\mathscr{P}$ is the family of proper subgroups of G. Recall that, for a family $\mathscr{F}$, $\tilde{E}\mathscr{F}$ is the cofibre of the natural map $EG_+ \longrightarrow S^0$. We define

$$\Phi^G(E) = (E \wedge \tilde{E}\mathscr{P})^G, \tag{3.18}$$

where $\mathscr{P}$ is the family of proper subgroups of G. Here $E \wedge \tilde{E}\mathscr{P}$ is H-trivial for all $H \in \mathscr{P}$. The isomorphism (3.16) is clear from Theorem 3.10.

We call Φ^G the "geometric" fixed point functor because its properties are like those of the space level G-fixed point functor and because it corresponds to the direct prespectrum level construction that one is likely to think of first. Restricting to finite groups G for simplicity and indexing G-prespectra on multiples of the regular representation, we can define a prespectrum level fixed point functor Φ^G by $(\Phi^G D)(\mathbb{R}^n) = (D(n\mathbb{R}G))^G$. If D is tame, then $(\Phi^G)(LD)$ is equivalent to $L\Phi^G D$. Therefore, if we start with a G-spectrum E, then $\Phi^G(E)$ is equivalent to $L\Phi^G(K\ell E)$, where K is the cylinder functor. This alternative description leads to the proof of (3.17). It also leads to a proof that

$$[E, F \wedge \tilde{E}\mathscr{P}]_G \cong \left[\Phi^G(E), \Phi^G(F)\right] \quad \text{for } G\text{-spectra } E \text{ and } F. \tag{3.19}$$

Euler classes and a calculational example

As an illuminating example of the use of $RO(G)$-grading to allow descriptions invisible to the $\mathbb{Z}$-graded part of a theory, we record how to compute $E^G_*(X \wedge \tilde{E}\mathscr{P})$ in terms of $E^G_*(X)$ for a ring G-spectrum E and any G-spectrum X. When $X = S$, it specializes to a calculation of

$$E^G_*(\tilde{E}\mathscr{P}) = \pi_*(\Phi^G E).$$

The example may look esoteric, but it is at the heart of the completion theorems that we will discuss later. We use the Euler classes of representations, which appear ubiquitously in equivariant theory. For a representation V, we define the Euler class $\chi_V \in E^G_{-V} = E^V_G(S^0)$ to be the image of $1 \in E^0_G(S^0) \cong E^V_G(S^V)$ under $e(V)^*$, where $e(V) : S^0 \longrightarrow S^V$ sends the basepoint to the point at ∞ and the non-basepoint to 0.

PROPOSITION 3.20. *Let E be a ring G-spectrum and X be any G-spectrum. Then $E^G_*(X \wedge \tilde{E}\mathscr{P})$ is isomorphic to the localization of the E^G_*-module $E^G_*(X)$ obtained by inverting the Euler classes of all representations V such that $V^G = \{0\}$.*

PROOF. A check of fixed points, using the cofibrations $S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V$, shows that we obtain a model for $\tilde{E}\mathscr{P}$ by taking the colimit Y of the spaces S^V as V ranges over the indexing spaces $V \subset U$ such that $V^G = \{0\}$. The point is that if H is a proper subgroup of G, then $V^H \neq \{0\}$ for all sufficiently large V, so that $Y^H \simeq *$. Therefore

$$E^G_\nu(X \wedge \tilde{E}\mathscr{P}) \cong \operatorname*{colim}_V E^G_{-\nu}(X \wedge S^V) \cong \operatorname*{colim}_V E^G_{\nu-V}(X).$$

Since the colimit is taken over iterated products with χ_V, it coincides algebraically with the cited localization. □

4. Change of groups and duality theory

So far, we have discussed the relationship between G-spectra and 1-spectra, where 1 is the trivial group. We must consider other subgroups and quotient groups of G.

Induced and coinduced G-spectra

First, consider a subgroup H. Since any representation of NH is a summand in a restriction of a representation of G and since a WH-representation is just an H-fixed NH-representation, the H-fixed point space U^H of our given complete G-universe U is a complete WH-universe. We define

$$E^H = (i^*E)^H, \quad i : U^H \subset U. \tag{4.1}$$

This gives a functor $G\mathscr{S}U \longrightarrow (WH)\mathscr{S}U^H$. For $D \in (NH)\mathscr{S}U^H$, the orbit spectrum D/H is also a WH-spectrum.

Exactly as on the space level, we have induced and coinduced G-spectra generated by an H-spectrum $D \in H\mathscr{S}U$. These are denoted by

$$G \ltimes_H D \quad \text{and} \quad F_H[G, D).$$

The "twisted" notation $\ltimes$ is used because there is a little twist in the definitions to take account of the action of G on indexing spaces. As on the space level, these functors are left and right adjoint to the forgetful functor $G\mathscr{S}U \longrightarrow H\mathscr{S}U$: for $D \in H\mathscr{S}U$ and $E \in G\mathscr{S}U$, we have

$$G\mathscr{S}U(G \ltimes_H D, E) \cong H\mathscr{S}U(D, E) \tag{4.2}$$

and

$$H\mathscr{S}U(E, D) \cong G\mathscr{S}U\big(E, F_H[G, D)\big). \tag{4.3}$$

Again, as on the space level, for a G-spectrum E, we have

$$G \ltimes_H E \cong (G/H)_+ \wedge E \tag{4.4}$$

and

$$F_H[G, E) \cong F(G/H_+, E). \tag{4.5}$$

We can now deduce as on the space level that

$$\pi_n^H(E) \equiv \big[G/H_+ \wedge S^n, E\big]_G \cong \big[S^n, E\big]_H \cong \pi_n\big(E^H\big). \tag{4.6}$$

We also have a geometric H-fixed point functor Φ^H. It is obtained by regarding G-spectra as NH-spectra and setting

$$\Phi^H(E) = \left(E \wedge \tilde{E}\mathscr{F}[H]\right)^H,$$

where $\mathscr{F}[H]$ is the family of subgroups of NH that do not contain H. Again, $\Phi^H E$ is an NH-spectrum indexed on U^H. While the Whitehead theorem appeared originally as a statement about homotopy groups and thus about the genuine fixed point functors, it implies a version in terms of the Φ-fixed point functors.

THEOREM 4.7. *Let $f : E \longrightarrow F$ be a map between G-CW spectra. Then the following statements are equivalent.*

(i) *f is a G-homotopy equivalence.*
(ii) *Each f^H is a nonequivariant homotopy equivalence.*
(iii) *Each $\Phi^H f$ is a nonequivariant homotopy equivalence.*

Subgroups and the Wirthmüller isomorphism
In cohomology, the isomorphism (4.2) gives

$$E_G^*(G \ltimes_H D) \cong E_H^*(D). \tag{4.8}$$

We shall not be precise, but we can interpret this in terms of $RO(G)$ and $RO(H)$ graded cohomology theories. The isomorphism (4.3) does not have such a convenient interpretation as it stands. However, there is an important change of groups result, called the Wirthmüller isomorphism, which in its most conceptual form is given by a calculation of the functor $F_H[G, D)$. It leads to the following homological complement of (4.8). Let $L(H)$ be the tangent H-representation at the identity coset of G/H. Then

$$E_*^G(G \ltimes_H D) \cong E_*^H\left(\Sigma^{L(H)} D\right). \tag{4.9}$$

THEOREM 4.10 (Generalized Wirthmüller isomorphism). *For H-spectra D, there is a natural equivalence of G-spectra*

$$F_H\left[G, \Sigma^{L(H)} D\right) \xrightarrow{\simeq} G \ltimes_H D.$$

Therefore, for G-spectra E,

$$\left[E, \Sigma^{L(H)} D\right]_H \cong [E, G \ltimes_H D]_G.$$

The last isomorphism complements the isomorphism from (4.2):

$$[G \ltimes_H D, E]_G \cong [D, E]_H. \tag{4.11}$$

We deduce (4.8) by replacing E in (4.9) by a sphere, replacing D by $E \wedge D$, and using the generalization $G \ltimes_H (D \wedge E) \cong (G \ltimes_H D) \wedge E$ of (4.4).

Quotient groups and the Adams isomorphism

Now let N be a normal subgroup of G with quotient group J. In practice, one is often thinking of a quotient map $NH \longrightarrow WH$ rather than $G \longrightarrow J$. There is an analogue of the Wirthmüller isomorphism, called the Adams isomorphism, that compares orbit and fixed-point spectra. It involves the change of universe functors associated to the inclusion $i : U^N \longrightarrow U$ and requires restriction to N-free G-spectra. We emphasize that U^N is not a complete G-universe. We have generalizations of the adjunctions (3.4) and (3.11): for $D \in J\mathscr{S}U^N$ and $E \in G\mathscr{S}U^N$,

$$G\mathscr{S}U^N(D,E) \cong J\mathscr{S}U^N(D,E^N) \tag{4.12}$$

and

$$J\mathscr{S}U^N(E/N,D) \cong G\mathscr{S}U^N(E,D). \tag{4.13}$$

Here we suppress notation for the pullback functor $J\mathscr{S}U^N \longrightarrow G\mathscr{S}U^N$. An N-free G-spectrum E indexed on U is equivalent to i_*D for an N-free G-spectrum D indexed on U^N, and D is unique up to equivalence. Thus our slogan that "free G-spectra live in the G-fixed universe" generalizes to the slogan that "N-free G-spectra live in the N-fixed universe". This gives force to the following version of (4.12). It compares maps of J-spectra indexed on U^N with maps of G-spectra indexed on U.

THEOREM 4.14. *Let $J = G/N$. For N-free G-spectra E indexed on U^N and J-spectra D indexed on U^N,*

$$[E/N,D]_J \cong [i_*E,i_*D]_G.$$

The conjugation action of G on N gives rise to an action of G on the tangent space of N at e; we call this representation $\mathrm{Ad}(N)$, or $\mathrm{Ad}(N;G)$. The following result complements the previous one, but is considerably deeper. When $N = G$, it is the heart of the proof of the homology isomorphism of Theorem 3.15.

THEOREM 4.15 (Generalized Adams isomorphism). *Let $J = G/N$. For N-free G-spectra $E \in G\mathscr{S}U^N$, there is a natural equivalence of J-spectra*

$$E/N \xrightarrow{\simeq} \left(\Sigma^{-\mathrm{Ad}(N)} i_*E\right)^N.$$

Therefore, for $D \in J\mathscr{S}U^N$,

$$[D,E/N]_J \cong \left[i_*D,\Sigma^{-\mathrm{Ad}(N)} i_*E\right]_G.$$

The last two results admit homological and cohomological interpretations, like those of Theorem 3.15, that are based on a generalization of the notion of a split G-spectrum. We shall not go into that here; see [42, Chapter II].

Spanier–Whitehead and Atiyah duality

Recall that the dual of a G-space or G-spectrum X is $DX = F(X,S)$. This is defined for any universe, but we observe the striking fact that if we work over U^G, then the sphere S has trivial G-action and $F(X,S) = F(X/G,S)$; in particular, the dual of every orbit G/H_+ is S. We must therefore work in the complete universe U to give useful content to the formal theory of duality, and the first thing we must do is to identify the duals of orbits. In fact, this identification is the real content of the Wirthmüller isomorphism, which implies that

$$D(G/H_+) \simeq G \ltimes_H S^{-L(H)}. \tag{4.16}$$

In particular, orbits are self-dual if G is finite.

It follows that finite G-CW spectra are strongly dualizable, and the Spanier–Whitehead duality theorem is a formal consequence.

THEOREM 4.17 (Spanier–Whitehead duality). *If X is a wedge summand of a finite G-CW spectrum and E is any G-spectrum, then*

$$\nu : DX \wedge E \xrightarrow{\cong} F(X,E)$$

is an isomorphism in $\bar{h}G\mathscr{S}U$. Therefore, for any virtual representation ν,

$$E^G_\nu(DX) \cong E^{-\nu}_G(X).$$

By developing a space level analysis of how to identify dual G-spectra, one can generalize the identification of duals of orbits to an identification of the duals of smooth G-manifolds. Working on the space level, one has a notion of V-duality between spaces X and Y. It involves evaluation and coevaluation maps $Y \wedge X \longrightarrow S^V$ and $S^V \longrightarrow X \wedge Y$ and implies that $\Sigma^{-V}\Sigma^\infty Y$ is dual to $\Sigma^\infty X$.

THEOREM 4.18 (Atiyah duality). *If M is a smooth closed G-manifold embedded in a representation V with normal bundle ν, then M_+ is V-dual to the Thom complex $T\nu$. If M is a smooth compact G-manifold with boundary ∂M, $V = V' \oplus \mathbb{R}$, and $(M, \partial M)$ is properly embedded in $(V' \times [0,\infty), V' \times \{0\})$ with normal bundles ν' of ∂M in V' and ν of M in V, then $M/\partial M$ is V-dual to $T\nu$, M_+ is V-dual to $T\nu/T\nu'$, and the cofibration sequence*

$$T\nu' \longrightarrow T\nu \longrightarrow T\nu/T\nu' \longrightarrow \Sigma T\nu'$$

is V-dual to the cofibration sequence

$$\Sigma(\partial M)_+ \longleftarrow M/\partial M \longleftarrow M_+ \longleftarrow (\partial M)_+.$$

We display the coevaluation map $\eta : S^V \longrightarrow M_+ \wedge T\nu$ explicitly in the closed case. By the equivariant tubular neighborhood theorem, we may extend the embedding of M in

V to an embedding of the normal bundle ν and apply the Pontryagin–Thom construction to obtain a map $t : S^V \longrightarrow T\nu$. The diagonal map of the total space of ν induces the Thom diagonal $\Delta : T\nu \longrightarrow M_+ \wedge T\nu$, and η is just the composite $\Delta \circ t$.

Specializing to $M = G/H$, we have

$$\tau = G \times_H L(H) \quad \text{and} \quad T\tau = G_+ \wedge_H S^{L(H)}.$$

If G/H is embedded in V with normal bundle ν and W is the orthogonal complement to $L(H)$ in the fiber over the identity coset, then $\nu = G \times_H W$ and therefore $\Sigma_V^\infty T\nu \simeq G \ltimes_H S^{-L(H)}$. Observe that we have a composite map

$$S^V \xrightarrow{t} T\nu \longrightarrow T(\nu \oplus \tau) \cong G/H_+ \wedge S^V. \tag{4.19}$$

This is called the "transfer map" associated to the projection $G/H \longrightarrow *$.

We can deduce equivariant versions of the Poincaré and Lefschetz duality theorems by combining Spanier–Whitehead duality, Atiyah duality, and the Thom isomorphism. However, the results are more subtle and less algebraically tractable than their nonequivariant analogs because G-manifolds are not homogeneous: they look locally like $G \times_H W$ for a subgroup H and H-representation W, which means that there is generally no natural "dimension" in which the orientation class or fundamental class of a manifold should lie. We refer the reader to [42, Chapter III] for discussion.

5. Mackey functors, $K(M, n)$'s, and $RO(G)$-graded cohomology

We have considered the ordinary cohomology $H_G^*(X; M)$ of a G-space X with coefficients in a coefficient system M. We can construct an additive category $\mathbb{Z}[h\mathscr{O}]$ from the homotopy category $h\mathscr{O}$ of orbits by applying the free Abelian group functor. The resulting category is isomorphic to the full subcategory of naive orbit spectra $\Sigma^\infty G/H_+$ in the stable homotopy category $\bar{h}G\mathscr{S}U^G$ of naive G-spectra. Clearly, a coefficient system is the same thing as an additive contravariant functor $\mathbb{Z}[h\mathscr{O}] \longrightarrow \mathscr{A}b$. Just as nonequivariantly, we can construct naive Eilenberg–MacLane G-spectra $HM = K(M, 0)$ associated to coefficient systems M and so extend our cohomology theories on G-spaces to cohomology theories on naive G-spectra.

It is natural to ask when these cohomology theories can be extended to $RO(G)$-graded cohomology theories on genuine G-spectra. The answer is suggested by the previous paragraph. Define $h\mathscr{O}\mathscr{S}$ to be the full subcategory of orbit spectra $\Sigma^\infty G/H_+$ in the stable homotopy category $\bar{h}G\mathscr{S}U$ of genuine G-spectra. Define a Mackey functor to be an additive contravariant functor $M : \bar{h}\mathscr{O}\mathscr{S} \longrightarrow \mathscr{A}b$; we abbreviate $M(G/H) = M(\Sigma^\infty G/H_+)$. This is the appropriate definition for general compact Lie groups, but we shall describe an equivalent algebraic definition when G is finite. It turns out that the cohomology theory $H_G^*(\cdot, M)$ can be extended to an $RO(G)$-graded theory if and only if the coefficient system M extends to a Mackey functor [40].

The idea can be made clear by use of the transfer map (4.18). If $H_G^*(\cdot; M)$ is $RO(G)$-gradable, then, for based G-spaces X, the transfer map induces homomorphisms

$$\begin{array}{c} \tilde{H}_G^n(G/H_+ \wedge X; M) \cong \tilde{H}_G^{n+V}(\Sigma^V(G/H_+ \wedge X); M) \\ \downarrow \\ \tilde{H}_G^n(X; M) \cong \tilde{H}_G^{n+V}(\Sigma^V X; M) \end{array} \tag{5.1}$$

Taking $n = 0$ and $X = S^0$, we obtain a transfer homomorphism

$$M(G/H) \longrightarrow M(G/G).$$

An elaboration of this argument shows that the coefficient system M must extend to a Mackey functor.

Algebraic description of Mackey functors

For finite groups G, calculational analysis of the category $h\mathscr{O}\mathscr{S}$ leads to an algebraic translation of our topological definition. Let $\mathscr{F}$ denote the category of finite G-sets and G-maps and let $h\mathscr{F}\mathscr{S}$ be the full subcategory of the stable category whose objects are the $\Sigma^\infty X_+$ for finite G-sets X. Then $h\mathscr{O}\mathscr{S}$ embeds as a full subcategory of $h\mathscr{F}\mathscr{S}$, and every object of $h\mathscr{F}\mathscr{S}$ is a finite wedge of objects of $h\mathscr{O}\mathscr{S}$. Since an additive functor necessarily preserves any finite direct sums in its domain, it is clear that an additive contravariant functor $h\mathscr{O}\mathscr{S} \longrightarrow \mathscr{A}b$ determines and is determined by an additive contravariant functor $h\mathscr{F}\mathscr{S} \longrightarrow \mathscr{A}b$. In turn, an additive contravariant functor $h\mathscr{F}\mathscr{S} \longrightarrow \mathscr{A}b$ determines and is determined by a Mackey functor in the algebraic sense defined by Dress [19]. Precisely, such a Mackey functor M consists of a contravariant functor M^* and a covariant functor M_* from finite G-sets to Abelian groups. These functors have the same object function, denoted M, and M converts disjoint unions to direct sums. Write $M^*\alpha = \alpha^*$ and $M_*\alpha = \alpha_*$. For pullback diagrams of finite G-sets

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "\gamma"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd},$$

it is required that $\alpha^* \circ \beta_* = \delta_* \circ \gamma^*$. For an additive contravariant functor $M : h\mathscr{F}\mathscr{S} \longrightarrow \mathscr{A}b$, the maps induced by the projections $G/H \longrightarrow G/K$ for $H \subset K$ and the corresponding transfer maps specify the contravariant and covariant parts of the corresponding algebraic Mackey functor, and conversely. The algebraic notion has applications to many areas of mathematics in which finite group actions are studied.

In the compact Lie case it is hard to prove that an algebraically defined coefficient system extends to a Mackey functor, but there is one important example.

PROPOSITION 5.2. *Let G be any compact Lie group. There is a unique Mackey functor $\underline{\mathbb{Z}} : h\mathscr{O}\mathscr{S} \longrightarrow \mathscr{A}b$ such that the underlying coefficient system of $\underline{\mathbb{Z}}$ is constant at $\mathbb{Z}$ and* *the homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ induced by the transfer map $\Sigma^\infty G/K_+ \longrightarrow \Sigma^\infty G/H_+$ associated to an inclusion $H \subset K$ is multiplication by the Euler characteristic $\chi(K/H)$.*

Construction of $RO(G)$-graded cohomology theories and $K(M,0)$'s

Returning to our original problem of constructing an $RO(G)$-graded ordinary cohomology theory and thinking on the spectrum level, we see that we want to construct a genuine Eilenberg–MacLane G-spectrum $HM = K(M,0)$. It is clear that the coefficient system $M = \underline{\pi}_0(HM)$ must be a Mackey functor since, by our homotopical definition of Mackey functors, the homotopy group system $\underline{\pi}_n(E)$ must be a Mackey functor for any G-spectrum E. The following result was first proven in [40].

THEOREM 5.3. *For a Mackey functor M, there is an Eilenberg–MacLane G-spectrum $HM = K(M,0)$, unique up to isomorphism in $\bar{h}G\mathscr{S}$. For Mackey functors M and M', $[HM, HM']_G$ is the group of maps of Mackey functors $M \longrightarrow M'$.*

We prove this by constructing a $\mathbb{Z}$-graded cohomology theory on G-spectra. By Brown's representability theorem, its degree zero part can be represented. The representing G-spectrum is our HM, and, since it is a genuine G-spectrum, it must of course represent an $RO(G)$-graded theory. The details that we use to construct the desired cohomology theories are virtually identical to those that we used to construct ordinary theories in the first place.

We start with G-CW spectra X. They have skeletal filtrations, and we define a Mackey-functor valued cellular chain complex by setting

$$\underline{C}_n(X) = \underline{\pi}_n(X^n/X^{n-1}). \tag{5.4}$$

Of course, X^n/X^{n-1} is a wedge of n-sphere G-spectra $G/H_+ \wedge S^n$, and the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}) specifies the required differential. For a Mackey functor M, we define

$$C^n_G(X;M) = \mathrm{Hom}_{\mathscr{M}}\left(\underline{C}_n(X), M\right) \quad \text{with } \delta = \mathrm{Hom}_{\mathscr{M}}(d, \mathrm{Id}). \tag{5.5}$$

Then $C^*_G(X;M)$ is a cochain complex of Abelian groups. We denote its cohomology by $H^*_G(X;M)$. The evident cellular versions of the homotopy, excision, exactness, and wedge axioms admit exactly the same derivations as on the space level, and we use G-CW approximation to extend from G-CW spectra to general G-spectra: we have a $\mathbb{Z}$-graded cohomology theory on $G\mathscr{S}U$. It satisfies the dimension axiom

$$H^*_G(\Sigma^\infty G/H_+;M) = H^0_G(\Sigma^\infty G/H_+;M) = M(G/H), \tag{5.6}$$

and these isomorphisms give an isomorphism of Mackey functors. The zeroth term is represented by a G-spectrum HM, and we read off its homotopy groups from (5.6):

$$\underline{\pi}_0(HM) = M \quad \text{and} \quad \underline{\pi}_n(HM) = 0 \quad \text{if } n \neq 0.$$

The uniqueness of HM is evident, and the calculation of $[HM, HM']_G$ follows easily from the functoriality in M of the theories $H^*_G(X;M)$.

We should observe that spectrum level obstruction theory works exactly as on the space level, modulo connectivity assumptions to ensure that one has a dimension in which to start inductions.

For G-spaces X, we have now given two meanings to the notation $H^*_G(X;M)$: we can regard our Mackey functor as a coefficient system and take the ordinary cohomology of X as in §1, or we can take our newly constructed cohomology. We know by the axiomatic characterization of ordinary cohomology that these must in fact be isomorphic, but it is instructive to check this directly. At least after a single suspension, we can approximate any G-space by a weakly equivalent based G-CW complex, with based attaching maps. The functor Σ^∞ takes based G-CW complexes to G-CW spectra, and we find that the space level and spectrum level chain complexes are isomorphic. Alternatively, we can check on the represented level:

$$[\Sigma^\infty X, \Sigma^n HM]_G \cong [X, \Omega^\infty \Sigma^n HM]_G \cong [X, K(M,n)]_G.$$

The Conner conjecture

Lest this all seem too abstract, let us us retrieve a direct and important space level consequence of this machinery, namely the Conner conjecture.

THEOREM 5.7 (The Conner conjecture). *Let X be a finite dimensional G-space with finitely many orbit types, where G is any compact Lie group, and let A be any Abelian group. If $\tilde{H}^*(X;A) = 0$, then $\tilde{H}^*(X/G;A) = 0$.*

This was first proven by Oliver [60], using Čech cohomology and wholly different techniques. It was known early on that the conjecture would hold if one could construct a suitable transfer map. It is now easy to do so [40].

THEOREM 5.8. *Let X be a G-space and $\pi : X/H \longrightarrow X/G$ be the projection, where $H \subseteq G$. For any $n \geqslant 0$ and any Abelian group A, there is a natural transfer homomorphism*

$$\tau : H^n(X/H;A) \longrightarrow H^n(X/G;A)$$

such that $\tau \circ \pi^*$ is multiplication by the Euler characteristic $\chi(G/H)$.

PROOF. Tensoring the Mackey functor $\underline{\mathbb{Z}}$ of Proposition 5.2 with A, we obtain a Mackey functor $\underline{A}$ whose underlying coefficient system is constant at A. The map $\underline{A}(G/H) \longrightarrow \underline{A}(G/G)$ associated to the stable transfer map $G/G_+ \longrightarrow G/H_+$ is multiplication by $\chi(G/H)$. By the axiomatization, the ordinary G-cohomology of a G-space X with coefficients in a constant coefficient system is isomorphic to the ordinary nonequivariant cohomology of its orbit space X/G:

$$H^n_G(X;\underline{A}) \cong H^n(X/G;A)$$

and

$$H^n_G(G/H \times X;\underline{A}) \cong H^n_H(X;\underline{A}|H) \cong H^n(X/H;A).$$

Taking $M = \underline{A}$, (5.1) displays the required transfer map. □

How does the Conner conjecture follow? Conner [15] proved it when G is a finite extension of a torus, the methods being induction and use of Smith theory: one proves that both X^G and X/G are A-acyclic. For example, the result for a torus reduces immediately to the result for a circle. Here the "finitely many orbit types" hypothesis implies that $X^G = X^C$ for C cyclic of large enough order, so that we are in the realm where classical Smith theory can be applied. Assuming that the result holds when G is a finite extension of a torus, let N be the normalizer of a maximal torus in G. Then N is a finite extension of a torus and $\chi(G/N) = 1$. The composite

$$\tau \circ \pi^* : \tilde{H}^n(X/G; A) \longrightarrow \tilde{H}^n(X/N; A) \longrightarrow \tilde{H}^n(X/G; A)$$

is the identity, and that's all there is to it.

The rational equivariant stable category

Exactly as for simple spaces and for spectra, we can use our Eilenberg–MacLane G-spectra to show that any G-spectrum can be approximated as the homotopy inverse limit of a Postnikov tower constructed out of $K(M, n)$'s and k-invariants, where $K(M, n) = \Sigma^n HM$. For finite groups, the k-invariants vanish rationally.

THEOREM 5.9. *Let G be finite. Then, for rational G-spectra E, there is a natural equivalence $E \xrightarrow{\simeq} \prod K(\underline{\pi}_n(E), n)$.*

Counterexamples of Triantafillou [64] show that, unless G is cyclic of prime power order, the conclusion is false for naive G-spectra. A counterexample of Haeberly [34] shows that the conclusion is also false for genuine G-spectra when G is the circle group, the rationalization of KU_G furnishing a counterexample.

The proof of Theorem 5.9 depends on two facts, one algebraic and one topological. Assume that G is finite.

PROPOSITION 5.10. *All objects are projective and injective in the Abelian category of rational Mackey functors.*

The analogue for coefficient systems is false, and so is the analogue for general compact Lie groups. One of us has recently studied what does happen for compact Lie groups [27]. The following result is easy for finite groups and false for compact Lie groups, as we see from Theorem 3.10.

PROPOSITION 5.11. *For $H \subseteq G$ and $n \neq 0$, $\underline{\pi}_n(G/H_+) \otimes \mathbb{Q} = 0$.*

Let $\mathscr{M} = \mathscr{M}[G]$ denote the Abelian category of Mackey functors over G. For G-spectra E and F, there is an evident natural map

$$\theta : [E, F]_G \longrightarrow \prod \mathrm{Hom}_{\mathscr{M}}\left(\underline{\pi}_n(E), \underline{\pi}_n(F)\right).$$

Let F be rational. By the previous result and the Yoneda lemma, θ is an isomorphism when $E = \Sigma^\infty G/H_+$ for any H. Clearly, we can extend θ to a graded map

$$\theta : F^q_G(E) = [E,F]^q_G = [\Sigma^{-q}E, F]_G \longrightarrow \prod \operatorname{Hom}_{\mathscr{M}}\big(\underline{\pi}_n(\Sigma^{-q}E), \underline{\pi}_n(F)\big).$$

It is still an isomorphism when E is an orbit. We obtain the same groups if we replace E and the Mackey functors $\underline{\pi}_n(\Sigma^{-q}E)$ by their rationalizations. Since the Mackey functors $\underline{\pi}_n(F)$ are injective, the right hand side is a cohomology theory on G-spectra E. Clearly θ is a map of cohomology theories and this already implies the following result. With $F = \prod K(\underline{\pi}_n(E), n)$, Theorem 5.9 is a direct consequence.

THEOREM 5.12. *Let G be finite. If F is rational, then θ is a natural isomorphism.*

This classifies rational G-spectra and one can go on to classify maps between them and so obtain a complete algebraization of the rational equivariant stable category. We refer the reader to [30, App. A].

6. Philosophy of localization and completion theorems

We shall work with reduced homology and cohomology theories in the rest of this article.

It is natural to want to know about the homology and cohomology of classifying spaces, as invariants of groups, as homes of characteristic classes, and as groups of bordism classes of G-manifolds.

One reason that it is difficult to calculate $k^*(BG_+)$ or $k_*(BG_+)$ is that BG_+ is an infinite complex. The conventional approach to calculation is based on the skeletal filtration of BG_+, which gives rise to Atiyah–Hirzebruch spectral sequences. One problem with this approach is that ordinary cohomology is not the most natural way to look at BG, and much of its good behaviour when viewed by other cohomology theories is invisible to ordinary cohomology.

An attractive alternative is to consider equivariant forms of k-theory. We shall say that $k^*_G(\cdot)$ is an equivariant form of $k^*(\cdot)$ if it is represented by a split G-spectrum k_G whose underlying spectrum k represents $k^*(\cdot)$. This means in particular that there is a map $k^* \longrightarrow k^*_G$ and also that for any free G-spectrum X there is a natural isomorphism $k^*_G(X) = k^*(X/G)$.

Typically, there will be many equivariant versions of $k^*(\cdot)$, and some will serve our purposes better than others. Perhaps the most obvious version is i_*k, but that is usually not the most useful version. We suppose that one particular version has been chosen in the following discussion. For example, the nicest equivariant form of topological K-theory is the Atiyah–Segal equivariant K-theory defined using equivariant bundles [62].

The point of thinking equivariantly is that

$$k^*_G(EG_+) = k^*(BG_+) \quad \text{and} \quad k^G_*(EG_+) = k_*\big(EG_+ \wedge_G S^{\mathrm{Ad}(G)}\big),$$

so that we have moved the problem into the equivariant world: we have to understand the homology and cohomology of free G-spectra, and we may hope to do so in general, allowing effective use of finite G-CW complexes to obtain information about our infinite G-CW complex EG. To carry out this idea, we introduce a parameter G-space X. By introducing equivariance, we have made available the comparison map

$$\pi^* : k_G^*(X) \longrightarrow k_G^*(EG_+ \wedge X) = k^*(EG_+ \wedge_G X),$$

induced by the projection $\pi : EG_+ \longrightarrow S^0$. It is appropriate to think of X as finite, so that the domain is easily calculated, whilst the codomain is the cohomology of an infinite complex. The motivating case $X = S^0$ gives the map

$$\pi^* : k_G^* \longrightarrow k_G^*(EG_+) \cong k^*(BG_+).$$

It is only slightly over-optimistic to hope that this is an isomorphism, as we now explain.

To obtain some algebraic control, we assume that $k^*(\cdot)$ and $k_G^*(\cdot)$ are ring theories, and that the splitting map is a ring map. More generally, we assume given module theories $m^*(\cdot)$ and $m_G^*(\cdot)$ over $k^*(\cdot)$ and $k_G^*(\cdot)$, with suitable splitting maps. Then all groups $m_G^*(X)$ are modules over the coefficient ring k_G^*. It turns out that the ideal theoretic geometry of the k_G^*-module $m_G^*(X)$ is the controlling structure. We discussed the algebra that we have in mind in the previous article [31].

Consider the augmentation ideal

$$J = \ker\left(res_1^G : k_G^* = k_G^*(S^0) \longrightarrow k_*^G(G_+) \cong k^*\right),$$

which by definition acts as zero on $k_G^*(G_+)$ and therefore on $m_G^*(G_+)$. Since any free G-spectrum is constructed from cells $S^n \wedge G_+$ it follows that a power of J acts as zero on $m_G^*(X)$ whenever X is finite and free. We emphasize that we are thinking about *$\mathbb{Z}$-graded, but $RO(G)$-gradable*, equivariant cohomology theories. If we allowed $RO(G)$-grading in our definition of J, the discussion would still make sense, but the results would often be trivial to prove and useless in practice.

Now observe that EG_+ is a direct limit of finite free complexes and consider its cohomology. If there are no $\lim^1$ problems, $m_G^*(EG_+)$ is an inverse limit of J-nilpotent modules, and therefore the nicest answer we could hope to have is that π^* is completion, so that

$$m_G^*(EG_+ \wedge X) = \left(m_G^*(X)\right)_J^\wedge.$$

However the algebra has already warned us against this: the topology guarantees that the left hand side is an exact functor of X, whereas the right hand side is only known to be exact when k_G^* is Noetherian and $m_G^*(X)$ is finitely generated. The solution is to replace J-completion by the associated functor on the derived category: this will be exact in a suitable sense and its homology groups will be calculated by left derived functors of completion. We gave the relevant descriptions of derived functors in [31].

NICEST POSSIBLE ANSWER 6.1. *For any G-spectrum X, $m_G^*(EG_+ \wedge X)$ is the 'homotopical J-completion' of the k_G^*-module $m_G^*(X)$ and hence there is a spectral sequence*

$$E_2^{*,*} = H_*^J\left(m_G^*(X)\right) \Longrightarrow m_G^*(EG_+ \wedge X).$$

If this nicest possible answer is the correct answer we say that the completion theorem *holds for* $m_G^*(\cdot)$.

Now consider the situation in homology. In any case, $m_*^G(EG_+)$ is a direct limit of J-nilpotent modules. The nicest functor of this form is the J-power torsion functor, but we saw in the previous article that this is rarely exact, and so even in the best cases we need to take derived functors into account.

NICEST POSSIBLE ANSWER 6.2. *For any G-spectrum X, $m_*^G(EG_+ \wedge X)$ is the 'homotopical J-power torsion' of the m_*^G-module $m_G^*(X)$ and hence there is a spectral sequence*

$$E^2_{*,*} = H_J^*(m_*^G(X)) \Longrightarrow m_*^G(EG_+ \wedge X).$$

If this nicest possible answer is the correct answer we say that the localization theorem *holds for* $m_*^G(\cdot)$.

One of us used to call this a 'local cohomology theorem' [24]. We shall explain in the next section why we now understand it to be a 'localization theorem'. We shall also recall what we mean by 'homotopy J-completion' and 'homotopy J-power torsion' and describe how one can hope to prove that theories $m_G^*(\cdot)$ and $m_*^G(\cdot)$ enjoy such good behaviour. However, the statements about spectral sequences are perfectly clear as they stand; the initial terms of the spectral sequences are local homology and local cohomology groups, respectively, as defined in [31, §1].

The entire discussion just given applies equally well to the calculation of $m_G^*(E\mathscr{F}_+)$ and $m_*^G(E\mathscr{F}_+)$ for an arbitrary family $\mathscr{F}$, provided that the ideal J is replaced by

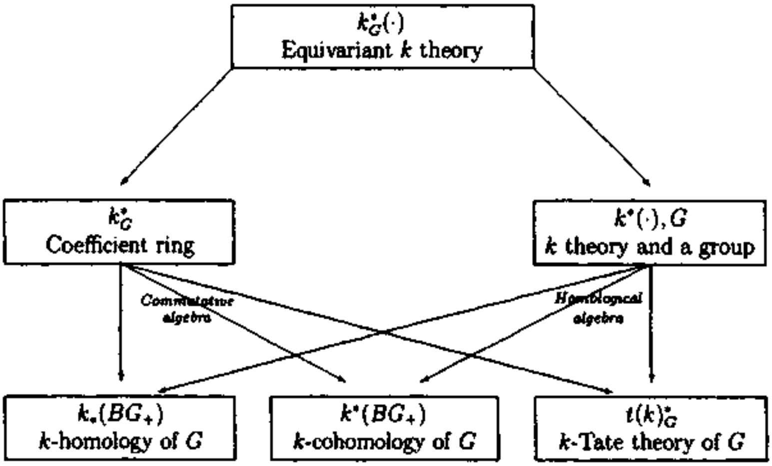
$$J\mathscr{F} = \bigcap_{H\in\mathscr{F}} \ker(k_G^* \longrightarrow k_H^*).$$

This case cannot usually be reduced to a nonequivariant statement, but it often has its own applications. For example, it leads to calculations of the cohomology and homology of equivariant classifying spaces and thus to determinations of equivariant characteristic classes.

We consider the alternative methods of calculation available to us in the following schematic diagram, restricting attention to our given ring theory $k_G^*(\cdot)$.

In this picture, the conventional (Atiyah–Hirzebruch) homological algebra route takes as input the nonequivariant k-theory together with the group structure on G; it results in a calculation of infinite homological dimension and with infinitely many extension problems. Where it applies, the more favorable route through commutative algebra takes as input the equivariant augmented coefficient ring $k_G^* \longrightarrow k^*$; the calculation usually has finite homological dimension, and in favorable cases the spectral sequences collapse and there are no extension problems.

There is an undefined term here, namely the Tate theory $t(k)_G^*$ [30]. It fits into a long exact sequence whose other two terms are $k_*(BG_+)$ and $k^*(BG_+)$. Returning to the context of module theories and remembering that every theory is a module theory



over stable cohomotopy, we have the following remarkable relationship between our two Nicest Possible Answers.

THEOREM 6.3. *Let G be finite and let J be the augmentation ideal of the Burnside ring $A(G)$. Regard a G-spectrum m_G as a module over the sphere G-spectrum S_G and recall that $A(G) \cong \pi_0^G(S_G)$. The localization theorem for the calculation of $m_*(BG_+)$ is true if and only if the completion theorem for the calculation of $m^*(BG_+)$ is true and $t(m)_G^*$ is rational.*

The Tate theory is relatively easy to compute. It is a direct consequence of Theorem 3.10 that the Tate theory $t(S)_G^*$ is not rational, so that one cannot hope to prove the localization theorem in stable homotopy, although the completion theorem is true in stable cohomotopy. We shall say no more about the Tate theory here, referring the interested reader to [30].

7. How to prove localization and completion theorems

We now outline a strategy for proving that the Nicest Possible Answer applies in both homology and cohomology [24]. One limitation of the method is obvious: it cannot apply to theories like stable homotopy.

The calculational restriction that we will shortly place on our homology theory and that will rule out stable homotopy is that the theory should have Thom isomorphisms for complex representations V:

$$R_*^G(S^V \wedge X) \cong R_*^G(S^{|V|} \wedge X) \tag{7.1}$$

as R_*^G-modules, where $|V|$ denotes the real dimension of V. The point is that localization theorems are often automatic, by arguments like the proof of Proposition 3.20, if one grades over the representation ring. Thom isomorphisms allow us to reinterpret that result in terms of integer grading.

There are two further assumptions. The first is fundamental to the general strategy: we assume that we are working in the category of modules over a commutative S_G-algebra R_G with underlying nonequivariant commutative S-algebra R. (Remember that commutative S_G-algebras are essentially the same things as E_∞ ring G-spectra.) We have switched notation from k to R to emphasize this assumption. Without it, we cannot make the constructions we need except under very favorable circumstances.

The second is made solely to simplify the exposition: we assume that the ring R_G^* is Noetherian. If this is not the case, the outline of the argument is the same but its implementation is considerably more complicated since one must use topological arguments to show that the relevant ideals can be replaced by finitely generated ones; at present, these arguments only apply to the trivial family $\mathscr{F} = \{1\}$.

The idea of the proofs is to model the algebra in topology; the model is so chosen that formal arguments imply that constructions on isotropy types are directly related to constructions on ideals in commutative rings. The necessary topological constructions are described in [31, §3].

We restrict attention to the augmentation ideal

$$J = \ker\left(res_1^G : R_*^G \to R_*\right)$$

and consider the canonical map

$$\kappa' : EG_+ \wedge K(J) \to S^0 \wedge K(J)$$

of R_G-modules. The module $K(J) = \Gamma_J(R_G)$ encodes homotopical J-power torsion. By our Noetherian assumption, we may take $J = (\beta_1, \ldots, \beta_n)$. Then $K(J)$ is the smash product over R_G of the fibers $K(\beta_i)$ of the localizations $R_G \longrightarrow (R_G)[1/\beta_i]$. Since the β_i are trivial as nonequivariant maps, we have the following observation.

LEMMA 7.2. *The natural map $K(J) \to R_G$ is a nonequivariant equivalence.*

Thus $EG_+ \wedge K(J) \simeq EG_+ \wedge R_G$ and κ' induces a map of R_G-modules

$$\boxed{\kappa : EG_+ \wedge R_G \longrightarrow K(J)}. \tag{7.3}$$

The homotopy groups of $R_G \wedge EG_+$ are $R_*^G(EG_+)$. More generally, we consider an R_G-module M_G with underlying nonequivariant R-module M, and we have

$$(EG_+ \wedge R_G) \wedge_{R_G} M_G \simeq EG_+ \wedge M_G$$

and

$$F_{R_G}(EG_+ \wedge R_G, M_G) \simeq F(EG_+, M_G).$$

Recall the definitions

$$\Gamma_J(M_G) = K(J) \wedge_{R_G} M_G \quad \text{and} \quad (M_G)^{\wedge}_J = F_{R_G}(K(J), M_G).$$

The homotopy groups of these modules may be calculated by the spectral sequences [31, (3.2) and (3.3)]. Clearly the map κ induces maps

$$EG_+ \wedge M_G \longrightarrow \Gamma_J(M_G) \quad \text{and} \quad (M_G)^{\wedge}_J \longrightarrow F(EG_+, M_G),$$

and these maps are equivalences if κ is an equivalence. Therefore, if we can prove that κ is a homotopy equivalence, we can deduce the spectral sequences of the Nicest Possible Answers for both $M^G_*(EG_+)$ and $M^*_G(EG_+)$ for all R_G-modules M_G. Given a G-spectrum X, we can replace M_G by $X \wedge M_G$ and $F(X, M_G)$ and so arrive at the the Nicest Possible Answers as stated in 6.1 and 6.2.

We pause to describe the role of localization away from J. We have the cofibre sequence

$$K(J) \longrightarrow R_G \longrightarrow \check{C}(J).$$

Smashing over R_G with M_G, recalling that $M_G[J^{-1}] = \check{C}(J) \wedge_{R_G} M_G$, and using a standard comparison of cofibre sequences argument in the category of R_G-modules, we obtain a map of cofibre sequences

$$\begin{tikzcd}
EG_+ \wedge M_G \arrow[r] \arrow[d] & M_G \arrow[r] \arrow[d, "\mathrm{id}"] & \tilde{E}G \wedge M_G \arrow[d] \\
\Gamma_J(M_G) \arrow[r] & M_G \arrow[r] & M_G[J^{-1}]
\end{tikzcd}$$

Clearly the left arrow is an equivalence if and only if the right arrow is an equivalence. This should be interpreted as stating that the 'topological' localization of M_G away from its free part is equivalent to the 'algebraic' localization of M_G away from J. This is why we call our Nicest Possible Answer in homology a localization theorem. The parallel with the completion theorem, which states that the 'algebraic' completion $M^{\wedge}_J$ is equivalent to the 'topological' completion $F(EG_+, M_G)$ of M_G at its free part, is now apparent.

The strategy for proving that the map κ of (7.3) is an equivalence is an inductive scheme. To set it up, we need to know that if we restrict κ to a subgroup H, we obtain an analogous map of H-spectra. We have

$$K(\beta, \ldots, \beta_n)\,|_H = K(\beta_1|_H, \ldots, \beta_n|_H);$$

the latter is defined with respect to $R_H = R_G\,|_H$. That is, if we write J_G instead of J, as we shall often do to clarify inductive arguments,

$$\Gamma_{J_G}(R_G)\,|_H \simeq \Gamma_{\mathrm{res}(J_G)}(R_H),$$

where res : $R_*^G \longrightarrow R_*^H$ is restriction. It is rarely the case (even for cohomotopy, when one is looking at Burnside rings) that $\mathrm{res}(J_G) = J_H$, but these ideals do have the same radical.

THEOREM 7.4. *Assume that G is finite and each $(R_H)_*$ is Noetherian. Then*

$$\sqrt{\mathrm{res}(J_G)} = \sqrt{J_H}$$

for all subgroups $H \subseteq G$.

We therefore have the equivalence of H-spectra

$$\Gamma_{J_G}(R_G)|_H \simeq \Gamma_{J_H} R_H.$$

SKETCH PROOF OF THEOREM 7.4. For theories such as cohomotopy and K-theory, where we understand all primes of R_*^G, this can be verified algebraically.

In general, if G acts freely on a product of spheres, one may check that J_G is the radical of the ideal generated by all Euler classes and deduce the result. This covers the case when G is a p-group, and general finite groups can then be dealt with by transfer.

The argument just sketched requires considerable elaboration, and it can be the main technical obstruction to the implementation of our strategy when we work more generally with compact Lie groups and non-Noetherian coefficient rings.

THEOREM 7.5 (Localization and completion theorem). *Assume that G is finite and each $(R_H)_*$ is Noetherian. If all of the theories $R_*^H(\cdot)$ admit Thom isomorphisms (7.1), then the map of R_G-module G-spectra*

$$\kappa : EG_+ \wedge R_G \xrightarrow{\simeq} K(J)$$

is an equivalence. Therefore, for any R_G-module M_G and any G-spectrum X, there are spectral sequences

$$E^2_{*,*} = H^*_J(R_*^G; M_*^G(X)) \Rightarrow M_*^G(EG_+ \wedge X)$$

and

$$E_2^{*,*} = H_*^J(R_G^*; M_G^*(X)) \Rightarrow M_G^*(EG_+ \wedge X).$$

PROOF. Write J_G instead of J, and observe from the original construction of κ' that the cofibre of κ is $\tilde{E}G \wedge K(J_G)$. We must prove that this is contractible.

We proceed by induction on the order of the group. By Theorem 7.4, we have

$$(\tilde{E}G \wedge K(J_G))\,|_H \simeq \tilde{E}H \wedge K(J_H),$$

and so our inductive assumption implies that

$$G/H_+ \wedge \tilde{E}G \wedge K(J_G) \simeq *$$

for all proper subgroups $H \subset G$.

We now use the idea in Proposition 3.20 and its proof. We take $\tilde{E}\mathscr{P} = \mathrm{colim}_V S^V$, where the colimit is taken over indexing G-spaces $V \subset U$ such that $V^G = \{0\}$. Since G is finite, we may restrict attention to copies of the reduced regular representation of G. Since $(\tilde{E}\mathscr{P})^G = S^0$, $\tilde{E}\mathscr{P}/S^0$ is triangulable as a G-CW complex whose cells are of the form $G/H_+ \wedge S^n$ with H proper. Therefore

$$\tilde{E}\mathscr{P}/S^0 \wedge \tilde{E}G \wedge K(J_G) \simeq *$$

by the inductive assumption, hence

$$\tilde{E}G \wedge K(J_G) \simeq \tilde{E}\mathscr{P} \wedge \tilde{E}G \wedge K(J_G).$$

Since $\tilde{E}\mathscr{P} \wedge S^0 \longrightarrow \tilde{E}\mathscr{P} \wedge \tilde{E}G$ is an equivalence, we have established the following useful reduction.

LEMMA 7.6 (Carlsson's reduction). *It suffices to show that* $\tilde{E}\mathscr{P} \wedge K(J_G) \simeq *$. □

Now recall that we have Euler classes $\chi_V \in R^G_{-V}(S^0)$ obtained by applying $e(V)^*$, $e(V) : S^0 \longrightarrow S^V$, to the unit $1 \in E^0(S^0) \cong E^V(S^V)$. At this point, our Thom isomorphisms (7.1) come into play, allowing us to move these Euler classes into integer gradings. Thus let $\chi(V) \in R^G_{-|V|}$ be the image of χ_V under the Thom isomorphism. When $V \neq \{0\}$, $e(V)$ is nonequivariantly null homotopic and therefore $\chi(V)$ is in J_G. Via the Thom isomorphism, Proposition 3.20 implies that, for any G-spectrum X, $\pi^G_*(\tilde{E}\mathscr{P} \wedge X)$ is the localization of $\pi^G_*(X)$ obtained by inverting the Euler classes $\chi(V)$. Here we may restrict everything to lie in integer gradings. With $X = K(J_G)$, the localization is zero since the $\chi(V)$ are in J_G [31, 1.1]. From the spectral sequence [31, (3.2)], we see that

$$\pi^G_*(\tilde{E}\mathscr{P} \wedge K(J_G)) = 0.$$

Since $\tilde{E}\mathscr{P}$ is H-equivariantly contractible for all proper subgroups H, this shows that $\tilde{E}\mathscr{P} \wedge K(J_G) \simeq *$, as required. □

8. Examples of localization and completion theorems

The discussion in the previous section was very general. In this section we consider a number of important special cases in a little more detail. In each case, we give some history, state precise theorems, discuss their import, and comment on wrinkles in their proofs. We refer the reader to [53] for precise descriptions of the representing G-spectra and more extended discussions of these results and their proofs.

8.1. *K-theory*

Historically this was the beginning of the whole subject. Atiyah [5] first proved the completion theorem for finite groups, by the conventional homological algebra route. Full use of equivariance appeared in the 1969 paper of Atiyah and Segal [8], which gave the completion theorem for compact Lie groups in essentially the following form. Let I be the augmentation ideal of the representation ring $R(G)$.

THEOREM 8.1 (Atiyah–Segal). *If G is a compact Lie group and X is a finite G-CW complex, then*

$$K_G^*(X)_I^\wedge \cong K_G^*(EG_+ \wedge X).$$

Their proof, like any other, depends fundamentally on the equivariant Bott periodicity theorem, which provides Thom isomorphisms via isomorphisms

$$K_G(\Sigma^V X) \cong K_G(X)$$

for complex representations V. The coefficient ring is $K_G^* = K_G^0[\beta, \beta^{-1}]$, and $K_G^0 = R(G)$. Since nonequivariant K-theory is also periodic, the augmentation ideal is $J = I[\beta, \beta^{-1}]$, and the completion theorem is therefore stated using I. The ring $R(G)$ is Noetherian [61], and Theorem 7.4 holds for all compact Lie groups G.

Atiyah and Segal used an inductive scheme in which they first proved the result for a torus, then used holomorphic induction to deduce it for a unitary group, and finally deduced the general case from the case of unitary groups. A geodesic route from Bott periodicity to the conclusion, basically a cohomological precursor of the homological argument sketched in the previous section, is given in [2]. That paper also gives the generalization of the result to arbitrary families of subgroups in G. A remarkable application of that generalization has been given by McClure [57]: restriction to finite subgroups detects equivariant K-theory.

THEOREM 8.2 (McClure). *For a compact Lie group G and a finite G-CW complex X, restriction to finite subgroups F specifies a monomorphism*

$$K_G^*(X) \longrightarrow \prod K_F^*(X).$$

It is not known that K_G is a commutative S_G-algebra in general, although recent work shows that this does hold when G is finite [23]. Therefore the techniques of the previous section do not apply in general. The arguments in [8] and [2] prove the isomorphism of Theorem 8.1 directly in cohomology. The trick that recovers enough exactness to make this work is to study pro-group valued cohomology theories.

A pro-group is just an inverse system of (Abelian) groups. There is an Abelian category of pro-groups, and the inverse limit functor is exact in that category. For a cohomology theory k_G^* on G-CW complexes, one obtains a pro-group valued theory $\mathbf{k}_G^*$ by letting $\mathbf{k}_G^*(X)$ be the system $\{k_G^*(X_\alpha)\}$, where X_α runs through the finite subcomplexes of X.

Working with pro-groups has an important bonus: for a finite G-CW complex X, the system $\{K_G^*(X)/I^n\}$ clearly satisfies the Mittag-Leffler condition. One proves that this system is pro-isomorphic to the system $k_G^*(EG_+ \wedge X)$, and one is entitled to conclude that

$$K_G^*(EG_+ \wedge X) \cong \lim_n K_G^*(EG_+^n \wedge X).$$

That is, the relevant $\lim^1$ term vanishes.

Various people have deduced calculations of the K-homology of classifying spaces for finite groups using suitable universal coefficient theorems, but the use of local cohomology and the proof via the localization theorem were first given in [24].

THEOREM 8.3. *If G is finite, then the localization and completion theorems hold for equivariant K-theory. Therefore, for any G-spectrum X, there are short exact sequences*

$$0 \longrightarrow H_I^1(K_*^G(\Sigma X)) \longrightarrow K_*^G(EG_+ \wedge X) \longrightarrow H_I^0(K_*^G(X)) \longrightarrow 0$$

and

$$0 \longrightarrow L_1^I K_G^*(\Sigma X) \longrightarrow K_G^*(EG_+ \wedge X) \longrightarrow L_0^I K_G^*(X) \longrightarrow 0.$$

In [24], the strategy of the previous section was applied to K_G regarded as an S_G-module: we have the permutation representation homomorphism $A(G) \longrightarrow R(G)$, and the completion of an $R(G)$-module at the augmentation ideal of $R(G)$ is isomorphic to its completion at the augmentation ideal of $A(G)$ [28, 4.5]. Using the new result that K_G is a commutative S_G-algebra when G is finite, the strategy can now be applied directly: Theorem 8.3 is an application of Theorem 7.5. The collapse of the relevant spectral sequences to short exact sequences results from the fact that $A(G)$ and $R(G)$ have Krull dimension 1 when G is finite.

There is an alternative strategy. In view of Theorems 6.3 and 8.1, one can prove Theorem 8.2 by proving directly that the Tate theory $t(K)_G^*$ is rational. This approach is carried out in [30]. It has the bonus that the topology carries out the commutative algebra of calculating the local cohomology groups, leading to the following succinct conclusion. Let $\mathbb{C}G$ be the regular representation of G; the ideal it generates in $R(G)$ is a free abelian group of rank 1, and the composite $I \longrightarrow R(G) \longrightarrow R(G)/(\mathbb{C}G)$ is an isomorphism.

THEOREM 8.4. *Let G be finite. Then $K_0(BG) \cong \mathbb{Z}$, with generator the image of $\mathbb{C}G$, and*

$$K_1(BG) \cong \big(R(G)/(\mathbb{C}G)\big)_I^\wedge \otimes (\mathbb{Q}/\mathbb{Z}).$$

When G is a p-group, I-adic and p-adic completion agree on $I \cong R(G)/(\mathbb{C}G)$, and explicit calculations in both K-homology and K-cohomology are easily obtained.

For general compact Lie groups, these strategies all fail: we do not know that K_G is a commutative S_G-algebra, and the alternative based on use of S_G fails since $A(G)$ has

Krull dimension 1 and is non-Noetherian in general, whereas $R(G)$ is Noetherian but has Krull dimension $r + 1$, where r is the rank of G [61]. The localization theorem is not known to hold in general.

8.2. *Bordism*

The case of bordism is the greatest success of the method outlined in Section 7. The correct equivariant form of bordism to use is tom Dieck's homotopical equivariant bordism [16]. A completion theorem for the calculation of $MU^*(BG)$ for Abelian compact Lie groups was proven by Löffler [46], [47] soon after the Atiyah–Segal completion theorem appeared, but there was no further progress until quite recently.

It is easy to describe the representing G-spectrum MU_G. Consider the usual model for the prespectrum with associated spectrum MU. The spaces comprising it are the Thom complexes of the Grassmannian models for universal vector bundles. Now carry out the construction using indexing spaces in a complete G-universe. The Vth space is defined using $|V|$ dimensional subspaces of the appropriate Grassmannian and therefore, up to G-homeomorphism, depends only on the dimension of V. This fact leads to the Thom isomorphisms required by our general strategy. Moreover, the explicit construction leads to a quick proof that the Thom G-spectrum MU_G is in fact a commutative S_G-algebra. Our general strategy applies [32].

THEOREM 8.5 (Greenlees–May). *Let G be finite. Then the localization and completion theorems hold for any module M_G over MU_G. Thus there are equivalences*

$$M_G \wedge EG_+ \simeq \Gamma_J(M_G) \quad \textit{and} \quad F(EG_+, M_G) \simeq (M_G)^\wedge_J$$

and, for any G-spectrum X, there are spectral sequences

$$E^2_{*,*} = H^*_J(MU^G_*; M^G_*(X)) \Rightarrow M^G_*(EG_+ \wedge X)$$

and

$$E_2^{*,*} = H_*^J(MU^*_G; M^*_G(X)) \Rightarrow M^*_G(EG_+ \wedge X).$$

We have several comments on this theorem, beginning with comments on its proof. An immediate difficulty is that MU^*_G is certainly not Noetherian. Furthermore, we have no good reason to think that the augmentation ideal $J \subset MU^*_G$ is finitely generated unless G is abelian. We modify our strategy accordingly, proving the theorem for any sufficiently large finitely generated subideal of J. By definition, the stated constructions based on J mean the relevant constructions based on such a sufficiently large subideal. When G is a p-group, the arguments of the previous section apply to ideals generated by a finite number of Euler classes. Rather elaborate multiplicative transfer and double coset formula arguments allow us to deduce the result for general finite groups using ideals that are generated by the transfers of the Euler classes from all p-Sylow subgroups and finitely many more elements. We expect that the result for an arbitrary compact Lie group can be proved by similar methods, but we do not yet see how to use these methods to give the result for arbitrary families.

Next we comment on the meaning of the theorem. Its most striking feature is its generality. The methods explained in [22, §11] apply to give equivariant forms of all of the important modules over MU, such as ku, K, BP, $BP(n)$, $E(n)$, $P(n)$, $B(n)$, $k(n)$ and $K(n)$. The equivariant and nonequivariant constructions are so closely related that we can deduce MU_G-ring spectrum structures on the equivariant spectra from the MU-ring spectra structures on the nonequivariant spectra. There are a variety of nonequivariant calculations of the homology and cohomology of classifying spaces with coefficients in one or another of these spectra in the literature, and our theorem gives a common framework for all such calculations.

We should comment on the specific case of connective K-theory. Here it is known that the completion theorem is false for connective equivariant K-theory: $ku^*(BG_+)$ is not a completion of ku_G^* at its augmentation ideal. However the theorem is consistent, since the equivariant form of ku constructed by the methods of [22, §11] is not the connective cover of equivariant K-theory. Indeed connective equivariant K-theory does not have Thom isomorphisms and is therefore not a module over MU_G.

We should also note that the coefficient ring MU_G^* is only known in the abelian case, and even then only in a rather awkward algebraic form. On the other hand, $M^*(BG_+)$ is known in a good many other cases, and in reasonably attractive form. Thus the theorem does not at present give a useful way of calculating $M^*(BG_+)$. However, there are several ways that it might be used for calculational purposes. For example, in favorable cases, such as $M = MU$ for Abelian groups G, one can work backwards to deduce that M_G^* is tame, in the sense that its local homology is its completion concentrated in degree zero. The local cohomology of M_G^* is then the same as that of its completion [31, 2.7], hence one can hope to calculate its local cohomology as well and to use this information to study $M_*(BG_+)$. The point is that, nonequivariantly, the calculation of homology is often substantially more difficult than the calculation of cohomology. Again, if M is an MU-ring spectrum, then one can use invariance under change of base [31, 1.3] to calculate the local cohomology and local homology over M_G^*; it sometimes turns out that M_G^* is a ring of small Krull dimension, and this gives vanishing theorems that make calculation more feasible.

These comments are speculative: the theorem is too recent to have been assimilated calculationally. Certainly it renews interest in the connection through cobordism between algebraic and geometric topology.

8.3. *Cohomotopy*

Soon after the Atiyah–Segal theorem was proved, Segal conjectured that the analogous result would hold for stable cohomotopy, at least in degree 0. In simplest terms, the idea is that the Burnside ring $A(G)$ plays a role in equivariant cohomotopy analogous to the role that $R(G)$ plays in equivariant K-theory and should therefore play an analogous role in the calculation of the nonequivariant cohomotopy groups of classifying spaces.

We restrict attention to finite groups G. Then the elements of positive degree in the homotopy ring π_*^G are nilpotent, so that it is natural to take its degree zero part $\pi_0^G \cong A(G)$ as our base ring; A(G) is Noetherian, and we let I denote its augmentation ideal $\ker(A(G) \longrightarrow \mathbb{Z})$. Theorem 7.4 applies.

Segal's original conjecture was simply that $A(G)_I^\wedge \cong \pi^0(BG_+)$. However, it quickly became apparent that, to prove the conjecture, it would be essential to extend it to a statement concerning the entire graded module $\pi^*(BG_+)$. In view of Theorem 3.10, we have enough information to formulate the conjecture in entirely nonequivariant terms [41], but it was the equivariant formulation that led to a proof.

In accordance with our philosophy we make a spectrum level statement and take the algebraic statement as a corollary, although the proofs proceed the opposite way.

THEOREM 8.6 (Carlsson). *For any finite group G and any G-spectrum X there is an equivalence of G-spectra*

$$(DX)_I^\wedge \xrightarrow{\cong} D(EG_+ \wedge X).$$

If X is finite, then

$$\pi_G^*(X)_I^\wedge \cong \pi_G^*(EG_+ \wedge X);$$

in general, there is a short exact sequence

$$0 \longrightarrow L_1^I \pi_G^*(\Sigma X) \longrightarrow \pi_G^*(EG_+ \wedge X) \longrightarrow L_0^I \pi_G^*(X) \longrightarrow 0.$$

We have already remarked that the localization theorem for stable homotopy fails and that cohomotopy does not have Thom isomorphisms. Therefore the strategy of proof must be quite different from that presented in Section 7. We first note that the generality of our statement is misleading: it was observed in [28, 4.1] that the statement for general X is a direct consequence of the statement for $X = S_G$. One reason for working on the G-spectrum level is to allow such deductions.

Taking $X = S_G$, it suffices to prove that the map $\varepsilon : S_G \longrightarrow D(EG_+)$ induced by the projection $EG_+ \longrightarrow S^0$ induces an isomorphism on homotopy groups. Proceeding by induction on the order of G and using Theorem 7.4, we may assume that the homotopy groups π_*^H for proper subgroups H are mapped isomorphically, so that we need only consider the groups π_*^G. As with the Atiyah–Segal theorem, we think cohomologically and control exactness by working with pro-groups. We find that it suffices to show that ε induces an isomorphism of pro-groups

$$\{\pi_G^*(S^0)/I^n\} \xrightarrow{\cong} \pi_G^*(EG_+).$$

At this point, a useful piece of algebra comes into play. In the context of Mackey functors, there is a general framework for proving induction theorems, due to Dress [19]. An induction theorem for I-adically complete Mackey functors was proven in [54], and it directly reduces the problem at hand to the study of p-groups and p-adic completion.

A more sophisticated reduction process, developed in [3], shows that the generalization of the Segal conjecture to arbitrary families of subgroups of G also reduces to this same special case.

This reduces the problem to what Carlsson actually proved [13]. Fix a p-group G, assume the theorem for all proper subgroups of G, and write $\pi_G^*(X)$ and $[X,Y]_G^*$ for the pro-group valued, p-adically completed, versions of these groups, where p-adic completion is understood in the pro-group sense. We replace G-spaces by their suspension G-spectra without change of notation. What Carlsson proved is that

$$\pi_G^*(S^0) \xrightarrow{\cong} \pi_G^*(EG_+)$$

is a pro-isomorphism.

A first reduction (see Lemma 7.6) shows that it suffices to prove that $\pi_G^*(\tilde{E}\mathscr{P}) = [\tilde{E}\mathscr{P}, S^0]_G^*$ is pro-zero. The cofibre sequence $EG_+ \longrightarrow S^0 \longrightarrow \tilde{E}G$ gives rise to a long exact sequence

$$\begin{aligned} \cdots \longrightarrow [\tilde{E}\mathscr{P}, EG_+]_G^q \longrightarrow [\tilde{E}\mathscr{P}, S^0]_G^q \longrightarrow [\tilde{E}\mathscr{P}, \tilde{E}G]_G^q \\ \xrightarrow{\delta} [\tilde{E}\mathscr{P}, EG_+]_G^{q+1} \longrightarrow \cdots. \end{aligned}$$

The $\tilde{E}G$ terms carry the singular part of the problem; the EG_+ terms carry the free part. It turns out that if G is *not* elementary Abelian, then both $[\tilde{E}\mathscr{P}, EG_+]_G^*$ and $[\tilde{E}\mathscr{P}, \tilde{E}G]_G^*$ are pro-zero. This is not true when G is elementary abelian, but then the connecting homomorphism δ is a pro-isomorphism.

The calculation of the groups $[\tilde{E}\mathscr{P}, \tilde{E}G]_G^*$ involves a functorial filtered approximation with easily understood subquotients of the singular subspace SX of a G-space X. Here SX consists of the elements of X with nontrivial isotropy groups; it is relevant since, on the space level,

$$[X, \tilde{E}G \wedge Y]_G \cong [SX, Y]_G.$$

A modification of Carlsson's original approximation given in [14] shows that SX depends only on the fixed point spaces X^E for elementary Abelian subgroups E of G, and this analysis reduces the vanishing of the $[\tilde{E}\mathscr{P}, \tilde{E}G]_G^*$ when G is not elementary Abelian to direct application of the induction hypothesis.

Recall the description of $\tilde{E}\mathscr{P}$ as the union $\cup S^{nV}$, where V is the reduced regular representation of G. One can describe $[S^{nV}, EG_+]_G^*$ as the homotopy groups of a nonequivariant Thom spectrum BG^{-nV} (see [52]) and so translate the calculation of the free part to a nonequivariant problem that can be attacked by use of an inverse limit of Adams spectral sequences. The vanishing of $[\tilde{E}\mathscr{P}, EG_+]_G^*$ when G is not elementary abelian is an Euler class argument: a theorem of Quillen implies that $\chi(V) \in H^*(BG; \mathbb{F}_p)$ is nilpotent if G is not elementary Abelian, and this implies that the E_2 term of the relevant inverse limit of Adams spectral sequences is zero.

When G is elementary Abelian, it turns out that all of the work in the calculation of $[\tilde{E}\mathscr{P}, EG_+]_G^*$ lies in the calculation of the E_2 term of the relevant inverse limit of Adams spectral sequences. When G is $\mathbb{Z}_2$ or $\mathbb{Z}_p$, the calculation is due to Lin [44], [45] and Gunawardena [33], respectively, and they were the first to prove the Segal conjecture in these cases. For general elementary Abelian p-groups, the calculation is due to Adams, Gunawardena and Miller [4]. While these authors were the first to prove the elementary Abelian case of the Segal conjecture, they didn't publish their argument, which started from the nonequivariant formulation of the conjecture. A simpler proof within Carlsson's context was given in [14], which showed that the connecting homomorphism δ is an isomorphism by comparing it to the corresponding connecting homomorphism for a theory, Borel cohomology, for which the completion theorem holds tautologously.

The Segal conjecture has been given a number of substantial generalizations, such as those of [40], [3], [56]. The situation for general compact Lie groups is still only partially understood; Lee and Minami have given a good survey [43]. One direction of application has been the calculation of stable maps between classifying spaces. The Segal conjecture has the following implication [40], [51], which reduces the calculation to pure algebra.

Let G and Π be finite groups and let $A(G,\Pi)$ be the Grothendieck group of Π-free finite $(G \times \Pi)$-sets. Observe that $A(G,\Pi)$ is an $A(G)$-module.

THEOREM 8.7. *There is a canonical isomorphism*

$$A(G,\Pi)^\wedge_I \cong \left[\Sigma^\infty BG_+, \Sigma^\infty B\Pi_+\right].$$

Many authors have studied the relevant algebra [59], [48], [35], [10], [65], which is now well understood. One can obtain an analog with Π allowed to be compact Lie [56], and even with G and Π both allowed to be compact Lie [58].

8.4. *The cohomology of groups*

We have emphasized the use of ideas and methods from commutative algebra in equivariant stable homotopy theory. We close with a remark on equivariant cohomology which shows that ideas and methods from equivariant stable homotopy theory can have interesting things to say about algebra.

The best known equivariant cohomology theory is simply the ordinary cohomology of the Borel construction:

$$H_G^*(X) = H^*(EG_+ \wedge_G X; k),$$

where we take k to be a field. The coefficient ring is the cohomology ring $H_G^*(S^0) = H^*(G)$ of the group G, and the augmentation ideal J consists of the elements of positive degree. Of course, this theory can be defined algebraically in terms of chain complexes. As far as completion theorems are concerned, this case has been ignored since the cohomology ring is obviously complete for the J-adic topology and the completion theorem is thus a tautology.

However, once one has formulated the localization theorem, it is easy to give a proof along the lines sketched above, using either topology or algebra. We give an algebraic statement proven in [26].

THEOREM 8.8. *For any finite group G and any bounded below chain complex M of kG-modules there is a spectral sequence with cohomologically graded differentials*

$$E_2^{p,q} = H_J^{p,q}\big(H^*(G;M)\big) \Longrightarrow H_{-(p+q)}(G;M).$$

It would be perverse to attempt to use the theorem to calculate $H_*(G;M)$, but if we consider the case when the coefficient ring is Cohen-Macaulay, so that the only nonvanishing local cohomology group occurs for $d = \dim H^*(G)$, we see that the theorem for $M = k$ states that

$$H_n(G) = H_J^{d,-n-d}\big(H^*(G)\big).$$

In particular, using that $H_*(G)$ is the k-dual of $H^*(G)$, this duality theorem implies that the ring $H^*(G)$ is also Gorenstein, which is a theorem originally proven by Benson and Carlson [9].

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CHAPTER 9

The Stable Homotopy Theory of Finite Complexes

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Contents

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1. Introduction

The object of this chapter is to describe some recent progress in one of the oldest and most difficult problems of algebraic topology, that of computing the stable homotopy groups of a finite complex. While these groups are elementary to define, there is still no nontrivial example of a finite complex for which they are completely known! The situation with unstable or ordinary homotopy groups is essentially the same; with the exception of some spaces (such as S^1 and surfaces other than S^2 and RP^2) which are known to be $K(\pi, 1)$'s, there is no example for which the problem has been completely solved.

There are numerous computational techniques (most notably the Adams spectral sequence) for getting partial information about this problem. For a given finite dimensional space X and a given integer k, one can often find $\pi_k(X)$ if k is not too large and one is willing to work hard enough. It is not our purpose here to discuss these methods here.

Our focus instead will be on the overall structure of these groups. This is a new and promising field of study, although one cannot hope to pursue it without a working knowledge of the computational methods that we are suppressing. Thus this chapter should be regarded as an illustration of what these methods can lead to without an explanation of the methods themselves. Our rationale for this approach is that the time allotted is enough to cover either the methods or the conclusions (but not both), and the latter are of interest to a wider audience.

Twenty years ago nobody suspected that stable homotopy groups had any general structure. Published results in the subject did not admit to any systematic interpretation. The first hints of such were Adams' work [2] on the J-homomorphism in the 1960's and the work of Toda [80] and L. Smith [70] on periodic families in the 1970's. That such families are the rule rather than the exception was suggested in an algebraic context by [48].

In 1977 the author made several conjectures concerning the stable homotopy theory of finite complexes [58]. By 1986 all but one of these had been proved, mainly by the remarkable work of Devinatz, Hopkins and J. Smith [22], [27] and [29]. *This work is the main subject of this chapter.*

The one conjecture not proved by Hopkins et al. was the telescope conjecture (6.5.5). It was *disproved* by the author in 1990, and we will discuss it briefly in §6.

In §2 we will give the elementary definitions in homotopy theory needed to state the main results, the nilpotence theorem (2.4.2) and the periodicity theorem (2.5.4). The latter implies the existence of a global structure in the homotopy groups of many spaces called the chromatic filtration. This is the subject of §3, which begins with a review of some classical results about homotopy groups.

The nilpotence theorem says that the complex bordism functor reveals a great deal about the homotopy category. This functor and the algebraic category ($\mathbf{C}\Gamma$, defined in 4.3.2) in which it takes its values are the subject of §4. This discussion is of necessity quite algebraic with the theory of formal group laws playing a major role.

In $\mathbf{C}\Gamma$ it is easy to enumerate all the thick subcategories (defined in 4.4.1). The thick subcategory theorem (4.4.3) says that there is a similar enumeration in the homotopy category itself. This result is extremely useful; it means that certain statements about a large class of spaces can be proved by verifying them only for very carefully chosen examples.

The thick subcategory theorem is derived from the nilpotence theorem in §5. In §5.4 we prove that the set of spaces satisfying the periodicity theorem forms a thick subcategory; this requires some computations in certain noncommutative rings. This thickness statement reduces the proof of the theorem to the construction of a few examples; this requires some modular representation theory due to Jeff Smith. Details can be found in [61, Chapter 6].

In §6 we introduce the concepts of Bousfield localization (6.1.1 and 6.1.3) and Bousfield equivalence (6.2.1). These are useful both for understanding the structure of the homotopy category and for proving the nilpotence theorem. The proof of the nilpotence theorem itself is outlined in §7, a more complete account can be found in [22] and in [61, Chapter 9].

This manuscript is an abbreviated version of the author's book [61], which the interested reader should consult for more background and detailed proofs.

2. The main theorems

The aim of this section is to state the nilpotence and periodicity theorems (2.4.2 and 2.5.4) with as little technical fussing as possible. Readers familiar with homotopy theory can skip the first three subsections, which contain some very elementary definitions.

2.1. *Homotopy*

A basic problem in homotopy theory is to classify continuous maps up to homotopy. Two continuous maps from a topological space X to Y are homotopic if one can be continuously deformed into the other. A more precise definition is the following.

DEFINITION 2.1.1. Two continuous maps f_0 and f_1 from X to Y are *homotopic* if there is a continuous map (called a homotopy)

$$X \times [0,1] \xrightarrow{h} Y$$

such that for $t = 0$ or 1, the restriction of h to $X \times \{t\}$ is f_t. If f_1 is a constant map, i.e. one that sends all of X to a single point in Y, then we say that f_0 is *null homotopic* and that h is a null homotopy. A map which is not homotopic to a constant map is *essential*. The set of homotopy classes of maps from X to Y is denoted by $[X, Y]$.

For technical reasons it is often convenient to consider maps which send a specified point $x_0 \in X$ (called the *base point*) to a given point $y_0 \in Y$, and to require that homotopies between such maps send all of $\{x_0\} \times [0,1]$ to y_0. Such maps and homotopies are said to be *base point preserving*. The set of equivalence classes of such maps (under base point preserving homotopies) is denoted by $[(X, x_0), (Y, y_0)]$.

Under mild hypotheses (needed to exclude pathological cases), if X and Y are both path-connected and Y is simply connected, the sets $[X, Y]$ and $[(X, x_0), (Y, y_0)]$ are naturally isomorphic.

In many cases, e.g., when X and Y are compact manifolds or algebraic varieties over the real or complex numbers, this set is countable. In certain cases, such as when Y is a topological group, it has a natural group structure. This is also the case when X is a suspension (2.3.1 and 3.1.2).

In topology two spaces are considered identical if there is a homeomorphism (a continuous map which is one-to-one and onto and which has a continuous inverse) between them. A homotopy theorist is less discriminating than a point set topologist; two spaces are identical in his eyes if they satisfy a much weaker equivalence relation defined as follows.

DEFINITION 2.1.2. Two spaces X and Y are *homotopy equivalent* if there are continuous maps $f: X \to Y$ and $g: Y \to X$ such that gf and fg are homotopic to the identity maps on X and Y. The maps f and g are *homotopy equivalences*. A space that is homotopy equivalent to a single point is *contractible*. Spaces which are homotopy equivalent have the same *homotopy type*.

For example, every real vector space is contractible and a solid torus is homotopy equivalent to a circle.

2.2. *Functors*

In algebraic topology one devises ways to associate various algebraic structures (groups, rings, modules, etc.) with topological spaces and homomorphisms of the appropriate sort with continuous maps.

DEFINITION 2.2.1. A *covariant functor* F from the category of topological spaces $\mathcal{T}$ to some algebraic category $\mathcal{A}$ (such as that of groups, rings, modules, etc.) is a function which assigns to each space X an object $F(X)$ in $\mathcal{A}$ and to each continuous map $f: X \to Y$ a homomorphism $F(f): F(X) \to F(Y)$ in such a way that $F(fg) = F(f)F(g)$ and F sends identity maps to identity homomorphisms. A *contravariant functor* G is similar function which reverses the direction of arrows, i.e. $G(f)$ is a homomorphism from $G(Y)$ to $G(X)$ instead of the other way around. In either case a functor is *homotopy invariant* if it takes isomorphic values on homotopy equivalent spaces and sends homotopic maps to the same homomorphism.

Familiar examples of such functors include ordinary homology, which is covariant and cohomology, which is contravariant. Both of these take values in the category of graded abelian groups. Definitions of them can be found in any textbook on algebraic topology. We will describe some less familiar functors which have proved to be extremely useful below.

These functors are typically used to prove that some geometric construction does *not* exist. For example one can show that the 2-sphere S^2 and the torus T^2 (doughnut-shaped surface) are not homeomorphic by computing their homology groups and observing that they are not the same.

Each of these functors has that property that if the continuous map f is null homotopic then the homomorphism $F(f)$ is trivial, but *the converse is rarely true*. Some of the best theorems in the subject concern special situations where it is. One such result is the nilpotence theorem (2.4.2), which is the main subject of this chapter.

Other results of this type in the past decade concern cases where at least one of the spaces is the classifying space of a finite or compact Lie group. A comprehensive book on this topic has yet to be written. A good starting point in the literature is the J.F. Adams issue of *Topology* (Vol. 31, No. 1, January 1992), specifically [16], [23], [32], [46], and [10].

The dream of every homotopy theorist is a solution to the following.

PROBLEM 2.2.2. *Find a functor F from the category of topological spaces to some algebraic category which is reasonably easy to compute and which has the property that $F(f) = 0$ if and only if f is null homotopic.*

We know that this is impossible for several reasons. First, the category of topological spaces is too large. One must limit oneself to a restricted class of spaces in order to exclude many pathological examples which would otherwise make the problem hopeless. Experience has shown that a reasonable class is that of *CW-complexes*. A definition is given in [61, A.1.1]. This class includes all the spaces that one is ever likely to want to study in a geometric way, e.g., all manifolds and algebraic varieties (with or without singularities) over the real or complex numbers. It does not include spaces such as the rational numbers, the p-adic integers or the Cantor set. An old result of Milnor [50] (see [61, A.1.4]) asserts that the space of maps from one compact CW-complex to another is homotopy equivalent to a CW-complex. Thus we can include, for example, the space of closed curves on a manifold.

The category of CW-complexes (and spaces homotopy equivalent to them) is a convenient place to do homotopy theory, but in order to have any chance of solving 2.2.2 we must restrict ourselves further by requiring that our complexes be *finite*, which essentially means compact up to homotopy equivalence.

It is convenient to weaken the problem somewhat further. We need another elementary definition from homotopy theory.

2.3. *Suspension*

DEFINITION 2.3.1. The *suspension* of X, ΣX is the space obtained from $X \times [0,1]$ by identifying all of $X \times \{0\}$ to a single point and all of $X \times \{1\}$ to another point. Given a continuous map $f: X \to Y$, we define

$$X \times [0,1] \xrightarrow{\tilde{f}} Y \times [0,1]$$

by $\tilde{f}(x,t) = (f(x),t)$. This $\tilde{f}$ is compatible with the identifications above and gives a map

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y.$$

This construction can be iterated and the i-th iterate is denoted by Σ^i. If $\Sigma^i f$ is null homotopic for some i we say that f is *stably null homotopic*; otherwise it is *stably essential*.

One can use the suspension to convert $[X,Y]$ to a graded object $[X,Y]_*$, where $[X,Y]_i = [\Sigma^i X, Y]$. (We will see below in 3.1.2 that this set has a natural group structure for $i > 0$.) It is also useful to consider the group of *stable homotopy classes of maps*,

$$[X,Y]_i^S = \varinjlim \left[\Sigma^{i+j} X, \Sigma^j Y\right].$$

If X has a base point x_0, we will understand ΣX to be the *reduced suspension*, which is obtained from the suspension defined above by collapsing all of $\{x_0\} \times [0,1]$ to (along with $X \times \{1\}$ and $X \times \{0\}$) a single point, which is the base point of ΣX. (Under mild hypotheses on X, the reduced and unreduced suspensions are homotopy equivalent, so we will not distinguish them notationally.)

Thus ΣX can be thought of as the double cone on X. If S^n (the n-sphere) denotes the space of unit vectors in $\mathbf{R}^{n+1}$, then it is an easy exercise to show that ΣS^n is homeomorphic to S^{n+1}.

Most of the functors we will consider are *homology theories* or, if they are contravariant, *cohomology theories*; the definition can be found in [61, A.3.3]. Ordinary homology and cohomology are examples of such, while homotopy groups (to be defined below in 3.1.1) are not. Classical K-theory is an example of a cohomology theory. Now we will point out the properties of such functors that are critical to this discussion.

A homology theory E_* is a functor from the category of topological spaces and homotopy classes of maps to the category of graded abelian groups. This means that for each space X and each integer i, we have an abelian group $E_i(X)$. $E_*(X)$ denotes the collection of these groups for all i. A continuous map $f : X \to Y$ induces a homomorphism

$$E_i(X) \xrightarrow{E_i(f)} E_i(Y)$$

which depends only on the homotopy class of f.

In particular one has a canonical homomorphism

$$E_*(X) \xrightarrow{\varepsilon} E_*(\text{pt.}),$$

called the *augmentation map*, induced by the constant map on X. Its kernel, denoted by $\overline{E}_*(X)$, is called the *reduced homology* of X, while $E_*(X)$ is sometimes called the *unreduced homology* of X.

Note that the augmentation is the projection onto a direct summand because one always has maps

$$\text{pt.} \longrightarrow X \longrightarrow \text{pt.}$$

whose composite is the identity. $E_*(\text{pt.})$ is nontrivial as long as E_* is not identically zero. A reduced homology theory vanishes on every contractible space.

One of the defining axioms of a homology theory (see [61, A.3.3]) implies that there is a natural isomorphism

$$\overline{E}_i(X) \xrightarrow{\sigma} \overline{E}_{i+1}(\Sigma X). \tag{2.3.2}$$

A *multiplicative homology theory* is one equipped with a ring structure on $E_*(\text{pt.})$ (which is called the *coefficient ring* and usually denoted simply by E_*), over which $E_*(X)$ has a functorial module structure.

PROBLEM 2.3.3. *Find a reduced homology theory $\overline{E}_*$ on the category of finite CW-complexes which is reasonably easy to compute and which has the property that $\overline{E}_*(f) = 0$ if and only if $\Sigma^i f$ is null homotopic for some i.*

In this case there is a long standing conjecture of Freyd [25, §9], known as the generating hypothesis, which says that stable homotopy (to be defined in 3.2.3) is such a homology theory. A partial solution to the problem, that is very much in the spirit of this book, is given by Devinatz in [21].

(The generating hypothesis was arrived in the following way. The stable homotopy category **FH** of finite complexes is additive, that is the set of morphisms between any two objects has a natural abelian group structure. Freyd gives a construction for embedding any additive category into an abelian category, i.e., one with kernels and cokernels. It is known that any abelian category is equivalent to a category of modules over some ring. This raises the question of identifying the ring thus associated with **FH**. It is natural to guess that it is π_*^S, the stable homotopy groups of spheres. This statement is equivalent to the generating hypothesis.)

Even if the generating hypothesis were known to be true, it would not be a satisfactory solution to 2.3.3 because stable homotopy groups are anything but easy to compute.

2.4. *Self-maps and the nilpotence theorem*

Now suppose that the map we want to study has the form

$$\Sigma^d X \xrightarrow{f} X$$

for some $d \geqslant 0$. Then we can iterate it up to suspension by considering the composites

$$\cdots \Sigma^{3d} X \xrightarrow{\Sigma^{2d} f} \Sigma^{2d} X \xrightarrow{\Sigma^{d} f} \Sigma^{d} X \xrightarrow{f} X.$$

For brevity we denote these composite maps to X by f, f^2, f^3, etc.

DEFINITION 2.4.1. A map $f: \Sigma^d X \to X$ is a *self-map* of X. It is *nilpotent* if some suspension of f^t for some $t > 0$ is null homotopic. Otherwise we say that f is *periodic*.

If we apply a reduced homology theory $\overline{E}_*$ to a self-map f, by 2.3.2 we get an endomorphism of $\overline{E}_*(X)$ that raises the grading by d.

Now we can state the nilpotence theorem of Devinatz, Hopkins and Smith [22].

THEOREM 2.4.2 (Nilpotence theorem, self-map form). *There is a homology theory MU_* such that a self-map f of a finite CW-complex X is stably nilpotent if and only if some iterate of $\overline{MU}_*(f)$ is trivial.*

Actually this is the weakest of the three forms of the nilpotence theorem; the other two (5.1.4 and 7.0.1) are equivalent and imply this one.

The functor MU_*, known as complex bordism theory, takes values in the category of graded modules over a certain graded ring L, which is isomorphic to $MU_*(\mathrm{pt.})$. These modules come equipped with an action by a certain infinite group Γ, which also acts on L. The ring L and the group Γ are closely related to the theory of formal group laws. $MU_*(X)$ was originally defined in terms of maps from certain manifolds to X, but this definition sheds little light on its algebraic structure. It is the algebra rather than the geometry which is central to our discussion. We will discuss this in more detail in §4 and more background can be found in [59, Chapter 4]. In practice it is not difficult to compute, although there are still plenty of interesting spaces for which it is still unknown.

2.5. *Morava K-theories and the periodicity theorem*

We can also say something about periodic self-maps.

Before doing so we must discuss localization at a prime p. In algebra one does this by tensoring everything in sight by $\mathbf{Z}_{(p)}$, the integers localized at the prime p; it is the subring of the rationals consisting of fractions with denominator prime to p. If A is a finite abelian group, then $A \otimes \mathbf{Z}_{(p)}$ is the p-component of A. $\mathbf{Z}_{(p)}$ is flat as a module over the integers $\mathbf{Z}$; this means that tensoring with it preserves exact sequences.

There is an analogous procedure in homotopy theory. The definitive reference is [14]; a less formal account can be found in [4]. For each CW-complex X there is a unique $X_{(p)}$ with the property that for any homology theory E_*, $\overline{E}_*(X_{(p)}) \cong \overline{E}_*(X) \otimes \mathbf{Z}_{(p)}$. We call $X_{(p)}$ the *p-localization* of X. If X is finite we say $X_{(p)}$ is a *p-local finite CW-complex*.

PROPOSITION 2.5.1. *Suppose X is a simply connected CW-complex such that $\overline{H}_*(X)$ consists entirely of torsion.*

(i) *If this torsion is prime to p then $X_{(p)}$ is contractible.*
(ii) *If it is all p-torsion then X is p-local, i.e. $X_{(p)}$ is equivalent to X. (In this case we say that X is a p-torsion complex.)*
(iii) *In general X is homotopy equivalent to the one-point union of its p-localizations for all the primes p in this torsion.*

If X is as above, then its p-localization will be nontrivial only for finitely many primes p. The cartesian product of any two of them will be the same as the one-point union. The smash product (defined below in 5.1.2)

$$X_{(p)} \wedge X_{(q)}$$

is contractible for distinct primes p and q.

The most interesting periodic self-maps occur when X is a finite p-torsion complex. In these cases it is convenient to replace MU_* by the Morava K-theories. These were invented by Jack Morava, but he never published an account of them. Most of the following result is proved in [34]; a proof of (v) can be found in [58].

PROPOSITION 2.5.2. *For each prime p there is a sequence of homology theories $K(n)_*$ for $n \geqslant 0$ with the following properties. (We follow the standard practice of omitting p from the notation.)*

(i) *$K(0)_*(X) = H_*(X; \mathbf{Q})$ and $\overline{K(0)}_*(X) = 0$ when $\overline{H}_*(X)$ is all torsion.*
(ii) *$K(1)_*(X)$ is one of $p-1$ isomorphic summands of mod p complex K-theory.*
(iii) *$K(0)_*(\mathrm{pt.}) = \mathbf{Q}$ and for $n > 0$, $K(n)_*(\mathrm{pt.}) = \mathbf{Z}/(p)[v_n, v_n^{-1}]$ where the dimension of v_n is $2p^n - 2$. This ring is a graded field in the sense that every graded module over it is free. $K(n)_*(X)$ is a module over $K(n)_*(\mathrm{pt.})$.*
(iv) *There is a Künneth isomorphism*

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*(\mathrm{pt.})} K(n)_*(Y).$$

(v) *Let X be a p-local finite CW-complex. If $\overline{K(n)}_*(X)$ vanishes, then so does $\overline{K(n-1)}_*(X)$.*
(vi) *If X is as above then*

$$\overline{K(n)}_*(X) = K(n)_*(\mathrm{pt.}) \otimes \overline{H}_*(X; \mathbf{Z}/(p))$$

for n sufficiently large. In particular it is nontrivial if X is simply connected and not contractible.

DEFINITION 2.5.3. A p-local finite complex X has *type n* if n is the smallest integer such that $\overline{K(n)}_*(X)$ is nontrivial. If X is contractible it has type ∞.

Because of the Künneth isomorphism, $K(n)_*(X)$ is easier to compute than $MU_*(X)$. Again there are still many interesting spaces for which this has not been done. See [64] and [28]. A corollary of the nilpotence theorem (2.4.2) says that the Morava K-theories, along with ordinary homology with coefficients in a field, are essentially the only homology theories with Künneth isomorphisms.

The Morava K-theories for $n > 0$ have another property which we will say more about below. Suppose we ignore the grading on $K(n)_*(X)$ and consider the tensor product

$$K(n)_*(X) \otimes_{K(n)_*(\mathrm{pt.})} \mathbf{F}_{p^n}$$

where $\mathbf{F}_{p^n}$ denotes the field with p^n elements, which is regarded as a module over $K(n)_*(\mathrm{pt.})$ by sending v_n to 1. Then this $\mathbf{F}_{p^n}$-vector space is acted upon by a certain p-adic Lie group S_n (not to be confused with the n-sphere S^n) which is contained in a certain p-adic division algebra.

The Morava K-theories are especially useful for detecting periodic self-maps. This is the subject of the second major result of this book, the periodicity theorem of Hopkins and Smith [29]. The proof is outlined in [27] and in §5.4.

THEOREM 2.5.4 (Periodicity theorem). *Let X and Y be p-local finite CW-complexes of type n* (2.5.3) *for n finite.*

(i) *There is a self-map $f\colon \Sigma^{d+i}X \to \Sigma^i X$ for some $i \geqslant 0$ such that $K(n)_*(f)$ is an isomorphism and $K(m)_*(f)$ is trivial for $m > n$.* (We will refer to such a map as a *v_n-map;* see p. 53. *When $n = 0$ then $d = 0$, and when $n > 0$ then d is a multiple of $2p^n - 2$.*

(ii) *Suppose $h\colon X \to Y$ is a continuous map. Assume that both have already been suspended enough times to be the target of a v_n-map. Let $g\colon \Sigma^e Y \to Y$ be a self-map as in* (i). *Then there are positive integers i and j with $di = ej$ such that the following diagram commutes up to homotopy.*

$$\begin{tikzcd} \Sigma^{di}X \arrow[r, "\Sigma^{di}h"] \arrow[d, "f^i"'] & \Sigma^{di}Y \arrow[d, "g^j"] \\ X \arrow[r, "h"] & Y \end{tikzcd}.$$

(*The integers i and j can be chosen independently of the map h.*)

Some comments are in order about the definition of a v_n-map given in the theorem. First, X is not logically required to have type n, but that is the only case of interest. If X has type $> n$, then the trivial map satisfies the definition, and if X has type $< n$, it is not difficult to show that no map satisfies it (4.3.11). Second, it does not matter if we require $K(m)_*(f)$ to be trivial or merely nilpotent for $m > n$. If it is nilpotent for each $m > n$, then some iterate of it will be trivial for *all* $m > n$. For $d > 0$ this follows because some iterate of $H_*(f)$ must be trivial for dimensional reasons, and

$$K(m)_*(f) = K(m)_* \otimes H_*(f) \quad \text{for } m \gg 0.$$

The case $d = 0$ occurs only when $n = 0$, for which the theorem is trivially true since the degree p map satisfies the definition.

The map h in (ii) could be the identity map, which shows that f is *asymptotically unique* in the following sense. Suppose g is another such periodic self-map. Then there are positive integers i and j such that f^i is homotopic to g^j. If X is a suspension of Y and f is a suspension of g, this shows that f is asymptotically central in that any map h commutes with some iterate of f.

3. Homotopy groups and the chromatic filtration

In this section we will describe the homotopy groups of spheres, which make up one of the messiest but most fundamental objects in algebraic topology. First we must define them.

3.1. *The definition of homotopy groups*

The following definition is originally due to Čech [18]. Homotopy groups were first studied systematically by Witold Hurewicz in [30] and [31].

DEFINITION 3.1.1. The n-th *homotopy group of* X, $\pi_n(X)$ is the set of homotopy classes of maps from the n-sphere S^n (the space of unit vectors in $\mathbf{R}^{n+1}$) to X which send a fixed point in S^n (called the base point) to a fixed point in X. (If X is not path-connected, then we must specify in which component its base point x_0 is chosen to lie. In this case the group is denoted by $\pi_n(X, x_0)$.) $\pi_1(X)$ is the *fundamental group* of X.

We define a group structure on $\pi_n(X)$ as follows. Consider the pinch map

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n$$

obtained by collapsing the equator in the source to a single point. Here $X \vee Y$ denotes the one-point union of X and Y, i.e. the union obtained by identifying the base point in X with the one in Y. We assume that the base point in the source S^n has been chosen to lie on the equator, so that the map above is base point preserving.

Now let $\alpha, \beta \in \pi_n(X)$ be represented by maps $f, g: S^n \to X$. Define $\alpha \cdot \beta \in \pi_n(X)$ to be the class of the composite

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{f \vee g} X.$$

The inverse α^{-1} is obtained by composing f with a base point preserving reflection map on S^n.

It is easy to verify that this group structure is well defined and that it is abelian for $n > 1$.

It is easy to construct a finite CW-complex of dimension $\leqslant 2$ whose π_1 is any given finitely presented group. This means that certain classification problems in homotopy theory contain problems in group theory that are known to be unsolvable.

REMARK 3.1.2. In a similar way one can define a group structure on the set of base point preserving maps from ΣX to Y for any space X (not just $X = S^{n-1}$ as above) and show that it is abelian whenever X is a suspension, i.e. whenever the source of the maps is a double suspension.

These groups are easy to define but, unless one is very lucky, quite difficult to compute. Of particular interest are the homotopy groups of the spheres themselves. These have been the subject of a great deal of effort by many algebraic topologists who have developed an arsenal of techniques for calculating them. Many references and details can be found in [59]. We will not discuss any of these methods here, but we will describe a general approach to the problem suggested by the nilpotence and periodicity theorems known as the chromatic filtration.

3.2. *Classical theorems*

First we need to recall some classical theorems on the subject.

THEOREM 3.2.1 (Hurewicz theorem, 1935). *The groups $\pi_n(S^m)$ are trivial for $n < m$, and $\pi_n(S^n) \cong \mathbf{Z}$; this group is generated by the homotopy class of the identity map.*

The next result is due to Hans Freudenthal [24].

THEOREM 3.2.2 (Freudenthal suspension theorem, 1937). *The suspension homomorphism* (see 2.3.1)

$$\sigma: \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphism for $k < n - 1$. The same is true if we replace S^n by any $(n-1)$-connected space X, i.e. any space X with $\pi_i(X) \cong 0$ for $i < n$.

This means that $\pi_{n+k}(\Sigma^n X)$ depends only on k if $n > k + 1$.

DEFINITION 3.2.3. The k-th *stable homotopy group of* X, $\pi_k^S(X)$, is

$$\pi_{n+k}(\Sigma^n X) \quad \text{for } n > k + 1.$$

In particular $\pi_k^S(S^0) = \pi_{n+k}(S^n)$, for n large, is called the *stable k-stem* and will be abbreviated by π_k^S.

The stable homotopy groups of spheres are easier to compute than the unstable ones. They are finite for $k > 0$. The p-component of π_k^S is known for $p = 2$ for $k < 60$ and for p odd for $k < 2p^3(p-1)$. Tables for $p = 2, 3$ and 5 can be found in [59]. Empirically we find that $\log_p |(\pi_k^S)_{(p)}|$ grows linearly with k.

The next result is due to Serre [66] and gives a complete description of $\pi_*(S^n)$ mod torsion.

THEOREM 3.2.4 (Serre finiteness theorem, 1953). *The homotopy groups of spheres are finite abelian except in the following cases:*

$$\pi_n(S^n) \cong \mathbf{Z} \qquad \text{and}$$
$$\pi_{4m-1}(S^{2m}) \cong \mathbf{Z} \oplus F_m$$

where F_m is finite abelian.

Before stating the next result we need to observe that π_*^S is a graded ring. If $\alpha \in \pi_k^S$ and $\beta \in \pi_\ell^S$ are represented by maps $f: S^{n+k} \to S^n$ and $g: S^{n+\ell} \to S^n$, then $\alpha\beta \in \pi_{k+\ell}^S$ is represented by the composite

$$S^{n+k+\ell} \xrightarrow{\Sigma^k g} S^{n+k} \xrightarrow{f} S^n.$$

This product is commutative up to the usual sign in algebraic topology, i.e. $\beta\alpha = (-1)^{k+\ell}\alpha\beta$.

The following was proved in [54].

THEOREM 3.2.5 (Nishida's theorem, 1973). *Each element in π_k^S for $k > 0$ is nilpotent, i.e. some power of it is zero.*

This is the special case of the nilpotence theorem for $X = S^n$. It also shows that π_*^S as a ring is very bad; it has no prime ideals other than (p). It would not be a good idea to try to describe it in terms of generators and relations. We will outline another approach to it at the end of this section.

The following result was proved in [17].

THEOREM 3.2.6 (Cohen–Moore–Neisendorfer theorem, 1979). *For p odd and $k > 0$, the exponent of $\pi_{2n+1+k}(S^{2n+1})_{(p)}$ is p^n, i.e. there are no elements of order p^{n+1}.*

3.3. *Cofibres*

By the early 1970's several examples of periodic maps had been discovered and used to construct infinite families of elements in the stable homotopy groups of spheres. Before we can describe them we need another elementary definition from homotopy theory.

DEFINITION 3.3.1. Let $f: X \to Y$ be a continuous map. Its *mapping cone*, or *cofibre*, C_f, is the space obtained from the disjoint union of $X \times [0,1]$ and Y by identifying all of $X \times \{0\}$ to a single point and $(x, 1) \in X \times [0,1]$ with $f(x) \in Y$.

If X and Y have base points x_0 and y_0 respectively with $f(x_0) = y_0$, then we define C_f to be as above but with all of $\{x_0\} \times [0,1]$ collapsed to a single point, which is defined to be the base point of C_f. (This C_f is homotopy equivalent to the one defined above.)

In either case, Y is a subspace of C_f, and the evident inclusion map will be denoted by i.

The following result is an elementary exercise.

PROPOSITION 3.3.2. *Let $i: Y \to C_f$ be the map given by* 3.3.1. *Then C_i is homotopy equivalent to ΣX.*

DEFINITION 3.3.3. A *cofibre sequence* is a sequence of spaces and maps of the form

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{j} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \cdots$$

in which each space to the right of Y is the mapping cone of the map preceding the map to it, and each map to the right of f is the canonical inclusion of a map's target into its mapping cone, as in 3.3.1.

If one has a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "i"] & C_f \\ X' \arrow[u, "g_1"] \arrow[r, "f'"] & Y' \arrow[u, "g_2"] \arrow[r, "i'"] & C_{f'} \arrow[u, dotted] \end{tikzcd}$$

then the missing map always exists, although it is not unique up to homotopy. Special care must be taken if f' is a suspension of f. Then the diagram extends to

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "i"] & C_f \arrow[r, "j"] & \Sigma X \\ \Sigma^d X \arrow[u, "g_1"] \arrow[r, "\Sigma^d f"] & \Sigma^d Y \arrow[u, "g_2"] \arrow[r, "\Sigma^d i"] & \Sigma^d C_f \arrow[u, "g_3"] \arrow[r, "(-1)^d \Sigma^d j"] & \Sigma^{d+1} X \arrow[u, "\Sigma g_1"] \end{tikzcd}$$

and the sign in the suspension of j is unavoidable.

Now suppose $g: Y \to Z$ is continuous and that gf is null homotopic. Then g can be extended to a map $\bar{g}: C_f \to Z$, i.e. there exists a $\bar{g}$ whose restriction to Y (which can be thought of as a subspace of C_f) is g. More explicitly, suppose $h: X \times [0,1] \to Z$ is a null homotopy of gf, i.e. a map whose restriction to $X \times \{0\}$ is constant and whose restriction to $X \times \{1\}$ is gf. Combining h and g we have a map to Z from the union of $X \times [0,1]$ with Y which is compatible with the identifications of 3.3.1. Hence we can use h and g to define $\bar{g}$.

Note that $\bar{g}$ depends on the homotopy h; a different h can lead to a different (up to homotopy) $\bar{g}$. The precise nature of this ambiguity is clarified by the following result, which describes three of the fundamental long exact sequences in homotopy theory.

PROPOSITION 3.3.4. *Let X and Y be path connected CW-complexes.*

(i) *For any space Z the cofibre sequence of* 3.3.3 *induces a long exact sequence*

$$[X,Z] \xleftarrow{f^*} [Y,Z] \xleftarrow{i^*} [C_f, Z] \xleftarrow{j^*} [\Sigma X, Z] \xleftarrow{\Sigma f^*} [\Sigma Y, Z] \longleftarrow \cdots$$

Note that each set to the right of $[C_f, Z]$ is a group, so exactness is defined in the usual way, but the first three sets need not have group structures. However each of them has a distinguished element, namely the homotopy class of the constant map. Exactness in this case means that the image of one map is the preimage of the constant element under the next map.

(ii) *Let E_* be a homology theory. Then there is a long exact sequence*

$$\cdots \xrightarrow{j_*} \overline{E}_m(X) \xrightarrow{f_*} \overline{E}_m(Y) \xrightarrow{i_*} \overline{E}_m(C_f) \xrightarrow{j_*} \overline{E}_{m-1}(X) \xrightarrow{f_*} \cdots$$

(iii) *Suppose X and Y are each $(k-1)$-connected (i.e. their homotopy groups vanish below dimension k) and let W be a finite CW-complex (see* [61, A.1.1]) *which is a double suspension with top cell in dimension less than $2k-1$. Then there is a long exact sequence of abelian groups*

$$[W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{i_*} [W,C_f] \xrightarrow{j_*} [W,\Sigma X] \xrightarrow{\Sigma f_*} [W,\Sigma Y] \longrightarrow \cdots.$$

This sequence will terminate at the point where the connectivity of the target exceeds the dimension of W.

COROLLARY 3.3.5. *Suppose X as in the periodicity Theorem* 2.5.4 *has type n. Then the cofibre of the map given by* 2.5.4 *has type $n+1$.*

PROOF. Assume that X has been suspended enough times to be the target of a v_n-map f and let W be its cofibre. We will study the long exact sequence

$$\begin{aligned}\cdots \xrightarrow{j_*} \overline{K(m)}_t(\Sigma^d X) &\xrightarrow{f_*} \overline{K(m)}_t(X) \xrightarrow{i_*} \overline{K(m)}_t(W)\\ &\xrightarrow{j_*} \overline{K(m)}_{t-1}(\Sigma^d X) \xrightarrow{f_*} \cdots\end{aligned}$$

for various m.

For $m<n$, $\overline{K(m)}_*(X)=0$, so $\overline{K(m)}_*(W)=0$. For $m=n$, f_* is an isomorphism, so again $\overline{K(m)}_*(W)=0$. For $m>n$, $f_*=0$ and $\overline{K(m)}_*(X)\neq 0$ by 2.5.2(v). It follows that

$$\overline{K(m)}_*(W) \cong \overline{K(m)}_*(X) \oplus \overline{K(m)}_*(\Sigma^{d+1}X),$$

so W has type $n+1$. □

3.4. *Motivating examples*

The following examples of periodic maps led us to conjecture the nilpotence and periodicity theorems.

EXAMPLE 3.4.1 (The earliest known periodic maps). (i) Regard S^1 as the unit circle in the complex numbers **C**. The *degree p* map on S^1 is the one which sends z to z^p. This map is periodic in the sense of 2.4.1, as is each of its suspensions. In this case n (as in the periodicity theorem) is zero.

(ii) Let $V(0)_k$ (known as the *mod p Moore space*) be the cofibre of the degree p map on S^k. Adams [2] and Toda [77] showed that for sufficiently large k there is a periodic map

$$\Sigma^q V(0)_k \xrightarrow{\alpha} V(0)_k$$

where q is 8 when $p=2$ and $2p-2$ for p odd. In this case the n of 2.5.4 is one. The induced map in $K(1)_*(V(0)_k)$ is multiplication by v_1 when p is odd and by v_1^4 for $p=2$.

For $p = 2$ there is no self-map inducing multiplication by a smaller power of v_1. One could replace the mod 2 Moore space by the mod 16 Moore space and still have a map α as above.

(iii) For $p \geqslant 5$, let $V(1)_k$ denote the cofibre of the map in (ii). Larry Smith [70] and H. Toda [80] showed that for sufficiently large k there is a periodic map

$$\Sigma^{2p^2-2}V(1)_k \xrightarrow{\beta} V(1)_k$$

which induces multiplication by v_2 in $K(2)$-theory.

(iv) For $p \geqslant 7$, let $V(2)_k$ denote the cofibre of the map in (iii). Smith and Toda showed that for sufficiently large k there is a periodic map

$$\Sigma^{2p^3-2}V(2)_k \xrightarrow{\gamma} V(2)_k$$

which induces multiplication by v_3 in $K(3)$-theory. We denote its cofibre by $V(3)_k$.

These results were not originally stated in terms of Morava K-theory, but in terms of complex K-theory in the case of (ii) and complex bordism in the case of (iii) and (iv). Attempts to find a self-map on $V(3)$ inducing multiplication by v_4 have been unsuccessful. The Periodicity Theorem guarantees that there is a map inducing multiplication by some power of v_4, but gives no upper bound on the exponent. References to some other explicit examples of periodic maps can be found in [59, Chapter 5].

Each of the maps in 3.4.1 led to an infinite family (which we also call *periodic*) of elements in the stable homotopy groups of spheres as follows.

EXAMPLE 3.4.2 (Periodic families from periodic maps).

(i) We can iterate the degree p map of 3.4.1(i) and get multiples of the identity map on S^k by powers of p, all of which are essential.

(ii) With the map α of 3.4.1(ii) we can form the following composite:

$$S^{k+qt} \xrightarrow{i_1} \Sigma^{qt}V(0)_k \xrightarrow{\alpha^t} V(0)_k \xrightarrow{j_1} S^{k+1}$$

where $i_1: S^k \to V(0)_k$ and $j_1: V(0)_k \to S^{k+1}$ are maps in the cofibre sequence associated with the degree p map. (We are using the same notation for a map and each of its suspensions.) This composite was shown by Adams [2] to be essential for all $t > 0$. The resulting element in π^S_{qt-1} is denoted by α_t for p odd and by α_{4t} for $p = 2$.

(iii) Let $i_2: V(0)_k \to V(1)_k$ and $j_2: V(1)_k \to \Sigma^{q+1}V(0)_k$ denote the maps in the cofibre sequence associated with α. Using the map β of 3.4.1(iii) for $p \geqslant 5$ we have the composite

$$S^{k+2(p^2-1)t} \xrightarrow{i_2 i_1} \Sigma^{k+2(p^2-1)t}V(1)_k \xrightarrow{\beta^t} V(1)_k \xrightarrow{j_1 j_2} S^{k+2p}$$

which is denoted by $\beta_t \in \pi_{2(p^2-1)t-2p}$. Smith [70] showed it is essential for all $t > 0$.

(iv) For $p \geqslant 7$ there is a similarly defined composite

$$S^{k+2(p^3-1)t} \longrightarrow \Sigma^{k+2(p^3-1)t} V(2)_k \xrightarrow{\gamma^t} V(2)_k \longrightarrow S^{k+(p+2)q+3}$$

which is denoted by γ_t. It was shown to be nontrivial for all $t > 0$ in [48].

In general a periodic map on a finite CW-complex leads to a periodic family of elements in π_*^S, although the procedure is not always as simple as in the above examples. Each of them has the following features. We have a CW-complex (defined in [61, A.1.1]) X of type n with bottom cell in dimension k and top cell in some higher dimension, say $k + e$. Thus we have an inclusion map $i_0 : S^k \to X$ and a pinch map $j_0 : X \to S^{k+e}$. Furthermore the composite

$$S^{k+td} \xrightarrow{i_0} \Sigma^{td} X \xrightarrow{f^t} X \xrightarrow{j_0} S^{k+e} \tag{3.4.3}$$

is essential for each $t > 0$, giving us a nontrivial element in π_{td-e}^S. This fact does *not* follow from the nontriviality of f^t; in each case a separate argument (very difficult in the case of the γ_t) is required.

If the composite (3.4.3) is null, we can still get a nontrivial element in $\pi_{td-\varepsilon}^S$ (for some ε between e and $-e$) as follows. At this point we need to be in the stable range, i.e. we need $k > td + e$, so we can use 3.3.4(iii). This can be accomplished by suspending everything in sight enough times.

For $k \leqslant r \leqslant s \leqslant k + e$, X_r^s will denote the cofibre of the inclusion map $X^{r-1} \to X^s$. In particular, $X_k^{k+e} = X$ and X_s^s is a wedge of s-spheres, one for each s-cell in X. We will use the letter i to denote any inclusion map $X_r^s \to X_r^{s'}$ with $s' > s$, and the letter j to denote any pinch map $X_r^s \to X_{r'}^s$ with $s \geqslant r' > r$.

Now let $f_e = f^t$ and consider the diagram

$$\begin{tikzcd}
\Sigma^{td} X \arrow[r, "f_e"] \arrow[dr, dotted, "f_{e-1}"] \arrow[ddr, dotted, "f_{e-2}"'] & X_k^{k+e} = X \arrow[r, "j"] & X_{k+e}^{k+e} = S^{k+e} \\
& X_k^{k+e-1} \arrow[u, "i"] \arrow[r, "j"] & X_{k+e-1}^{k+e-1} \\
& X_k^{k+e-2} \arrow[u, "i"] \arrow[r, "j"] & X_{k+e-2}^{k+e-2} \\
& {} \arrow[u, "i"] &
\end{tikzcd} \tag{3.4.4}$$

If the composite jf_e is null, then by 3.3.4(iii) there is a map f_{e-1} with $if_{e-1} = f_e$. Similarly if jf_{e-1} is null then there is a map f_{e-2} with $if_{e-2} = f_{e-1}$. We proceed in this way until the composite

$$\Sigma^{td}X \xrightarrow{f_{e_1}} X_k^{k+e_1} \xrightarrow{j} X_{k+e_1}^{k+e_1}$$

is essential. This must be the case for some e_1 between 0 and e, because if all of those composites were null, then 3.3.4(iii) would imply that f^t is null.

Now let $g_0 = jf_{e_1}$ and consider the diagram

$$\begin{tikzcd}
\Sigma^{td}X_k^k \arrow[r, "i"] & \Sigma^{td}X \arrow[r, "g_0"] \arrow[d, "j"'] & X_{k+e_1}^{k+e_1} \\
\Sigma^{td}X_{k+1}^{k+1} \arrow[r, "i"] & \Sigma^{td}X_{k+1}^{k+e} \arrow[d, "j"'] \arrow[ur, dotted, "g_1"] & \\
\Sigma^{td}X_{k+2}^{k+2} \arrow[r, "i"] & \Sigma^{td}X_{k+2}^{k+e} \arrow[d, "j"'] \arrow[uur, dotted, "g_2"'] & \\
 & {} &
\end{tikzcd} \tag{3.4.5}$$

This time we use 3.3.4(i) instead of 3.3.4(iii). It says that if $g_0 i$ is null then there is a map g_1 with $g_1 j = g_0$. Similarly if $g_1 i$ is null there is a map g_2 with $g_1 = g_2 j$. The composites $g_m i$ for $0 \leqslant m \leqslant e$ cannot all be null because g_0 is essential. Let e_2 be the integer between 0 and e such that $g_{e_2} i$ is essential.

Summing up, we have a diagram

$$\begin{tikzcd}
 & \Sigma^{td}X \arrow[r, "f_{e_1}"] \arrow[d, "j"] & X_k^{k+e_1} \arrow[r, "i"] \arrow[d, "j"] & X \\
\Sigma^{td}X_{k+e_2}^{k+e_2} \arrow[r, "i"] & \Sigma^{td}X_{k+e_2}^{k+e} \arrow[r, "g_{e_2}"] & X_{k+e_1}^{k+e_1} &
\end{tikzcd}$$

where $if_{e_1} = f^t$. The source and target of $g_{e_2}i$ are both wedges of spheres, so this is the promised stable homotopy element. Its dimension is $td + e_2 - e_1$ with $0 \leqslant e_1, e_2 \leqslant e$.

The simplest possible outcome of this procedure is the case $e_1 = e$ and $e_2 = 0$; this occurs in each of the examples in 3.4.2. In any other outcome, the construction is riddled with indeterminacy, because the maps f_{e_1} and g_{e_2} are not unique.

In any case the outcome may vary with the exponent t. In every example that we have been able to analyze, the behavior is as follows. With a finite number of exceptions (i.e. for t sufficiently large), the outcome depends only on the congruence class of t modulo some power of the prime p.

3.5. *The chromatic filtration*

These examples led us to ask if every element in the stable homotopy groups of spheres is part of such a family. In [48] we explored an algebraic analog of this question. The Adams–Novikov spectral sequence ([61, A.6.3]) is a device for computing π_*^S and its E_2-term was shown there to have such an organization using a device called the chromatic spectral sequence(see [61, B.8]), which is also described in [59, Chapter 5]. In [58] we explored the question of making this algebraic structure more geometric. It was clear that the periodicity theorem would be essential to this program, and that the former would be false if there were a counter example to the nilpotence theorem. Now that the nilpotence and periodicity theorems have been proved, we can proceed directly to the geometric construction that we were looking for in [58] without dwelling on the details of the chromatic spectral sequence.

Suppose Y is a p-local complex and $y \in \pi_k(Y)$ is represented by a map $g\colon S^k \to Y$. If all suspensions of y have infinite order, then it has a nontrivial image in $\pi_k^S(Y) \otimes \mathbf{Q}$. In the case where Y is a sphere, this group can be read off from 3.2.4. In general this group is easy to compute since it is known to be isomorphic to $\overline{H}_*(Y;\mathbf{Q})$.

On the other hand, if some suspension of y has order p^i, then it factors through the cofibre of the map of degree p^i on the corresponding suspension of S^k, which we denote here by $W(1)$. For the sake of simplicity we will ignore suspensions in the rest of this discussion. The map from $W(1)$ to Y will be denoted by g_1.

The complex $W(1)$ has type 1 and therefore a periodic self-map

$$f_1\colon \Sigma^{d_1} W(1) \to W(1)$$

which induces a $K(1)_*$-equivalence. Now we can ask whether g_1 becomes null homotopic when composed with some iterate of f_1 or not. If all such composites are stably essential then g_1 has a nontrivial image in the direct limit obtained by taking homotopy classes of maps from the inverse system

$$W(1) \xleftarrow{f_1} \Sigma^{d_1} W(1) \xleftarrow{f_1} \Sigma^{2d_1} W(1) \longleftarrow \cdots$$

which gives a direct system of groups

$$[W(1),Y]_*^S \xrightarrow{f_1^*} [\Sigma^{d_1}W(1),Y]_*^S \xrightarrow{f_1^*} [\Sigma^{2d_1}W(1),Y]_*^S \xrightarrow{f_1^*} \cdots, \qquad (3.5.1)$$

which we denote by $v_1^{-1}[W(1),Y]_*^S$. Note that the second part of the periodicity theorem implies that this limit is independent of the choice of f_1.

This group was determined in the case when Y is a sphere for the prime 2 by Mahowald in [43] and for odd primes by Miller in [47]. More precise calculations not requiring any suspensions of the spaces in question in the case when Y is an odd-dimensional sphere were done for $p = 2$ by Mahowald in [44] and for p odd by Thompson in [76]. In general it appears to be an accessible problem. For more details, see [7], [8], [9], [19] and [20].

There is a definition of $v_1^{-1}\pi_*(Y)$ which is independent of the exponent i. We have an inverse system of cofibre sequences

$$\begin{tikzcd}
S^k \arrow[d, "p^i"'] & S^k \arrow[l, "p"'] \arrow[d, "p^{i+1}"] & \cdots \arrow[l] \\
S^k \arrow[d] & S^k \arrow[l, "\simeq"'] \arrow[d] & \cdots \arrow[l] \\
C_{p^i} & C_{p^{i+1}} \arrow[l, "\rho"'] & \cdots \arrow[l]
\end{tikzcd}$$

which induces a direct system of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & {[C_{p^i},Y]} \arrow[r] \arrow[d, "\rho^*"] & \pi_k(Y) \arrow[r, "p^i"] \arrow[d, "\simeq"] & \pi_k(Y) \arrow[r] \arrow[d, "p"] & \cdots \\
\cdots \arrow[r] & {[C_{p^{i+1}},Y]} \arrow[r] \arrow[d, "\rho^*"] & \pi_k(Y) \arrow[r, "p^{i+1}"] \arrow[d, "\simeq"] & \pi_k(Y) \arrow[r] \arrow[d, "p"] & \cdots \\
& {} & {} & {} &
\end{tikzcd}$$

The limit of these is a long exact sequence of the form

$$\cdots \longrightarrow \pi_k(Y) \longrightarrow p^{-1}\pi_k(Y) \longrightarrow \pi_k(Y)/(p^\infty) \longrightarrow \cdots$$

where

$$\pi_k(Y)/(p^\infty) = \varinjlim \left[\Sigma^{-i} C_{p^i}, Y\right].$$

Note that since Y is p-local,

$$p^{-1}\pi_k(Y) = \pi_k(Y) \otimes \mathbf{Q}.$$

We can define $v_1^{-1}\pi_*(Y)/(p^\infty)$ by using some more detailed information about v_1-maps on the Moore spaces C_{p^i}. For sufficiently large k (independent of i), there are v_1-maps

$$\Sigma^{2p^{i-1}(p-1)} C_{p^i} \xrightarrow{f_{1,i}} C_{p^i}$$

such that the following diagram commutes.

$$\begin{tikzcd}
\Sigma^{2p^i(p-1)} C_{p^{i+1}} \arrow[r, "f_{1,i+1}"] \arrow[d, "\rho"'] & C_{p^{i+1}} \arrow[d, "\rho"] \\
\Sigma^{2p^i(p-1)} C_{p^i} \arrow[r, "f_{1,i}^p"] & C_{p^i}
\end{tikzcd}$$

This means we can define the groups $v_1^{-1}[C_{p^i}, Y]_*$ compatibly for various i, and their direct limit is $v_1^{-1}\pi_*(Y)/(p^\infty)$. More details of this construction can be found in [20].

Returning to (3.5.1), suppose that some power of f_1 annihilates g_1, i.e. some composite of the form $g_1 f_1^{j_1}$ is stably null homotopic. In this case, let $W(2)$ be the cofibre of $f_1^{j_1}$ and let g_2 be an extension of g_1 to $W(2)$.

Then $W(2)$ has type 2 and therefore it admits a periodic self-map

$$\Sigma^{d_2} W(2) \xrightarrow{f_2} W(2)$$

which is detected by $K(2)_*$. This leads us to consider the group

$$v_2^{-1}[W(2), Y]_*^S.$$

This group is not yet known for any Y. There is some machinery (see 6.5) available for computing what was thought to be a close algebraic approximation. The relation of this approximation to the actual group in question was the subject of the telescope conjecture (6.5.5), which has recently been disproved by the author. (It is known to be true in the v_1 case.) The algebraic computation in the case where Y is a sphere, $W(1)$ is a mod p Moore space and $p \geqslant 5$ has been done by Shimomura and Tamura in [67] and [68].

Summing up, we have a diagram

$$\begin{tikzcd}
Y & S^k \arrow[l, "g"'] \arrow[d] & S^k \arrow[l, "p^i"'] \\
& W(1) = C_{p^i} \arrow[ul, dotted, "g_1"'] \arrow[d] & \Sigma^{d_1}W(1) \arrow[l, "f_1"'] \\
& W(2) = C_{f_1^{i_1}} \arrow[uul, dotted, "g_2"] \arrow[d] & \Sigma^{d_2}W(2) \arrow[l, "f_2"'] \\
& {} &
\end{tikzcd}$$

One could continue this process indefinitely. At the n-th stage one has an extension g_n of the original map g to a complex $W(n)$ of type n which has a periodic self-map f_n. Then one asks if g_n is annihilated stably by some iterate of f_n. If the answer is no, then the process stops and g_n has a nontrivial image in the group $v_n^{-1}[W(n), Y]_*^S$. On the other hand, if g_n is annihilated by a power of f_n then we can move on to the $(n+1)$-th stage.

In view of this we make the following definitions.

DEFINITION 3.5.2. If an element $y \in \pi_*^S(Y)$ extends to a complex $W(n)$ of type n as above, then y is *v_{n-1}-torsion*. If in addition y does *not* extend to a complex of type $n+1$, it is *v_n-periodic*. The *chromatic filtration* of $\pi_*^S(Y)$ is the decreasing family of subgroups consisting of the v_n-torsion elements for various $n \geqslant 0$.

We use the word 'chromatic' here for the following reason. The n^{th} subquotients in the chromatic filtration consists of v_n-periodic elements. As illustrated in 3.4.2, these elements fall into periodic families. The chromatic filtration is thus like a spectrum in the astronomical sense in that it resolves the stable homotopy groups of a finite complex into periodic families of various periods. Comparing these to the colors of the rainbow led us to the word 'chromatic.'

The construction outlined above differs slightly from that used in 3.4.2. Suppose for example that we apply chromatic analysis to the map

$$\alpha_1 : S^{m+2p-3} \to S^m$$

for an odd prime p. This element in π_{q-1}^S has order p so the map extends to the mod p Moore space $V(0)_{m+q-1}$, which has the Adams self-map α of 3.4.1(ii). We find that all iterates of α when composed with α_1 are essential, so α_1 is v_1-periodic. The composite

$$S^{m+qi+q-1} \longrightarrow V(0)_{m+qi+q-1} \xrightarrow{\alpha^i} V(0)_{m+q-1} \xrightarrow{\tilde{\alpha}_1} S^m$$

is the map α_{i+1} of 3.4.1(ii). In the chromatic analysis no use is made of the map $j: V(0)_m \to S^{m+1}$.

More generally, suppose $g: S^k \to Y$ is v_n-periodic and that it extends to $g_n: W(n) \to Y$. There is no guarantee that the composite

$$S^K \xrightarrow{e} \Sigma^{d_n i} W(n) \xrightarrow{f_n^i} W(n) \xrightarrow{g_n} Y$$

(where e is the inclusion of the bottom cell in $W(n)$) is essential, even though $g_n f_n^i$ is essential by assumption. (This is the case in each of the examples of 3.4.2.) If this composite is null homotopic then $g_n f_n^i$ extends to the cofibre of e. Again, this extension may or may not be essential on the bottom cell of C_e. However, $g_n f_n^i$ must be nontrivial on one of the 2^n cells of $W(n)$ since it is an essential map. (To see this, one can make a construction similar to that shown in (3.4.4) and (3.4.5).) Thus for each i we get some nontrivial element in $\pi_*^S(Y)$.

DEFINITION 3.5.3. Given a v_n-periodic element $y \in \pi_*^S(Y)$, the elements described above for various $i > 0$ constitute the *v_n-periodic family associated with y.*

One can ask if the chromatic analysis of a given element terminates after a finite number of steps. For a reformulation of this question, see the chromatic convergence theorem, 6.5.7.

4. MU-theory and formal group laws

In this section we will discuss the homology theory MU_* used in the nilpotence theorem. $MU_*(X)$ is defined in terms of maps of manifolds into X as will be explained presently. Unfortunately the geometry in this definition does not appear to be relevant to the applications we have in mind. We will be more concerned with some algebraic properties of the functor which are intimately related to the theory of formal group laws.

4.1. *Complex bordism*

DEFINITION 4.1.1. Let M_1 and M_2 be smooth closed n-dimensional manifolds, and let $f_i: M_i \to X$ be continuous maps for $i = 1, 2$. These maps are *bordant* if there is a map $f: W \to X$, where W is a compact smooth manifold whose boundary is the disjoint union of M_1 and M_2, such that the restriction of f to M_i is f_i. f is a *bordism* between f_1 and f_2.

Bordism is an equivalence relation and the set of bordism classes forms a group under disjoint union, called the n-th *bordism group of* X.

A manifold is *stably complex* if it admits a complex linear structure in its stable normal bundle, i.e. the normal bundle obtained by embedding in a large dimensional Euclidean space. (The term stably *almost* complex is often used in the literature.) A complex analytic manifold (e.g., a nonsingular complex algebraic variety) is stably complex, but the notion of stably complex is far weaker than that of complex analytic.

DEFINITION 4.1.2. $MU_n(X)$, the n-th *complex bordism group of* X, is the bordism group obtained by requiring that all manifolds in sight be stably complex.

The fact that these groups are accessible is due to some remarkable work of Thom in the 1950's [75]. More details can be found in [61, B.2]. A general reference for cobordism theory is Stong's book [74].

The groups $MU_*(X)$ satisfy all but one of the axioms used by Eilenberg and Steenrod to characterize ordinary homology; see [61, A.3]. They fail to satisfy the dimension axiom, which describes the homology of a point. If X is a single point, then the map from the manifold to X is unique, and $MU_*(\mathrm{pt.})$ is the group of bordism classes of stably complex manifolds, which we will denote simply by MU_*. It is a graded ring under Cartesian product and its structure was determined independently by Milnor [51] and Novikov ([55] and [56]).

THEOREM 4.1.3. *The complex bordism ring, MU_* is isomorphic to*

$$\mathbf{Z}[x_1, x_2, \ldots]$$

where $\dim x_i = 2i$.

It is possible to describe the generators x_i as complex manifolds, but this is more trouble than it is worth. The complex projective spaces CP^i serve as polynomial generators of $\mathbf{Q} \otimes MU_*$.

Note that $MU_*(X)$ is an MU_*-module as follows. Given $x \in MU_*(X)$ represented by $f: M \to X$ and $\lambda \in MU_*$ represented by a manifold N, λx is represented by the composite map

$$M \times N \longrightarrow M \xrightarrow{f} X.$$

4.2. *Formal group laws*

DEFINITION 4.2.1. A *formal group law* over a commutative ring with unit R is a power series F(x, y) over R that satisfies the following three conditions.

(i) $F(x, 0) = F(0, x) = x$ (identity),
(ii) $F(x, y) = F(y, x)$ (commutativity) and
(iii) $F(F(x, y), z) = F(x, F(y, z))$ (associativity).

(The existence of an inverse is automatic. It is the power series $i(x)$ determined by the equation $F(x, i(x)) = 0$.)

EXAMPLE 4.2.2.

(i) $F(x, y) = x + y$. This is called the *additive* formal group law.

(ii) $F(x, y) = x + y + xy = (1 + x)(1 + y) - 1$. This is called the *multiplicative* formal group law.

(iii)
$$F(x, y) = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1 - \varepsilon x^2 y^2}$$
where
$$R(x) = 1 - 2\delta x^2 + \varepsilon x^4.$$

This is the formal group law associated with the elliptic curve
$$y^2 = R(x),$$
a Jacobi quartic, so we call it the *elliptic* formal group law. It is defined over $\mathbf{Z}[1/2][\delta, \varepsilon]$. This curve is nonsingular mod p (for p odd) if the discriminant $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$ is invertible. This example figures prominently in elliptic cohomology theory; see [38] for more details.

The theory of formal group laws from the power series point of view is treated comprehensively in [26]. A short account containing all that is relevant for the current discussion can be found in [59, Appendix 2].

The following result is due to Lazard [39].

THEOREM 4.2.3 (Lazard's theorem). (i) *There is a universal formal group law defined over a ring L of the form*
$$G(x, y) = \sum_{i,j} a_{i,j} x^i y^j \quad \text{with } a_{i,j} \in L$$
such that for any formal group law F over R there is a unique ring homomorphism θ from L to R such that
$$F(x, y) = \sum_{i,j} \theta(a_{i,j}) x^i y^j.$$

(ii) *L is a polynomial algebra $\mathbf{Z}[x_1, x_2, \ldots]$. If we put a grading on L such that $a_{i,j}$ has degree $2(1 - i - j)$ then x_i has degree $-2i$.*

The grading above is chosen so that if x and y have degree 2, then $G(x, y)$ is a homogeneous expression of degree 2. Note that L is isomorphic to MU_* except that the grading is reversed. There is an important connection between the two.

Associated with the homology theory MU_* there is a cohomology theory MU^*. This is a contravariant functor bearing the same relation to MU_* that ordinary cohomology bears to ordinary homology. When X is an m-dimensional manifold, $MU^*(X)$ has a geometric description; an element in $MU^k(X)$ is represented by a map to X from an $(m-k)$-dimensional manifold with certain properties. The conventions in force in algebraic topology require that $MU^*(\text{pt.})$ (which we will denote by MU^*) be the same as $MU_*(\text{pt.})$ but with the grading reversed. Thus MU^* is isomorphic to the Lazard ring L.

This isomorphism is natural in the following sense. $MU^*(X)$, like $H^*(X)$, comes equipped with cup products, making it a graded algebra over MU^*. Of particular interest is the case when X is the infinite-dimensional complex projective space $\mathbf{C}P^\infty$. We have

$$MU^*(\mathbf{C}P^\infty) \cong MU^*[[x]]$$

where $\dim x = 2$, and

$$MU^*\big(\mathbf{C}P^\infty \times \mathbf{C}P^\infty\big) \cong MU^*[[x \otimes 1, 1 \otimes x]].$$

The space $\mathbf{C}P^\infty$ is an abelian topological group, so there is a map

$$\mathbf{C}P^\infty \times \mathbf{C}P^\infty \xrightarrow{f} \mathbf{C}P^\infty$$

with certain properties. ($\mathbf{C}P^\infty$ is also the classifying space for complex line bundles and the map in question corresponds to the tensor product.) Since MU^* is contravariant we get a map

$$MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) \xleftarrow{f^*} MU^*(\mathbf{C}P^\infty)$$

which is determined by its behavior on the generator $x \in MU^2(\mathbf{C}P^\infty)$. The power series

$$f^*(x) = F(x \otimes 1, 1 \otimes x)$$

can easily be shown to be a formal group law. Hence by Lazard's theorem 4.2.3 it corresponds to a ring homomorphism $\theta: L \to MU^*$. The following was proved by Quillen [57] in 1969.

THEOREM 4.2.4 (Quillen's theorem). *The homomorphism*

$$\theta: L \to MU^*$$

above is an isomorphism. In other words, the formal group law associated with complex cobordism is the universal one.

Given this isomorphism (and ignoring the reversal of the grading), we can regard $MU_*(X)$ as an L-module.

4.3. *The category* $\mathbf{C}\Gamma$

Now we define a group Γ which acts in an interesting way on L.

DEFINITION 4.3.1. Let Γ be the group of power series over $\mathbf{Z}$ having the form

$$\gamma = x + b_1x^2 + b_2x^3 + \cdots$$

where the group operation is functional composition. Γ acts on the Lazard ring L of 1.5 as follows. Let $G(x, y)$ be the universal formal group law as above and let $\gamma \in \Gamma$. Then $\gamma^{-1}(G(\gamma(x), \gamma(y)))$ is another formal group law over L, and therefore is induced by a homomorphism from L to itself. Since γ is invertible, this homomorphism is an automorphism, giving the desired action of Γ on L.

For reasons too difficult to explain here, Γ also acts naturally on $MU_*(X)$ compatibly with the action on $MU_*(\text{pt.})$ defined above. (For more information about this, see [61, B.3 and B.4].) That is, given $x \in MU_*(X)$, $\gamma \in \Gamma$ and $\lambda \in L$, we have

$$\gamma(\lambda x) = \gamma(\lambda)\gamma(x)$$

and the action of Γ commutes with homomorphisms induced by continuous maps.

For algebraic topologists we can offer some explanation for this action of Γ. It is analogous to the action of the Steenrod algebra in ordinary cohomology. More precisely, it is analogous to the action of the group of multiplicative cohomology operations, such as (in the mod 2 case) the total Steenrod square, $\sum_{i\geqslant 0} \mathrm{Sq}^i$. Such an operation is determined by its effect on the generator of $H^1(\mathbf{R}P^\infty; \mathbf{Z}/(2))$. Thus the group of multiplicative mod 2 cohomology operations embeds in $\Gamma_{Z/(2)}$, the group of power series over $\mathbf{Z}/(2)$ analogous to Γ over the integers.

DEFINITION 4.3.2. Let $\mathbf{C}\Gamma$ denote the category of finitely presented graded L-modules equipped with an action of Γ compatible with its action on L as above, and let **FH** denote the category of finite CW-complexes and homotopy classes of maps between them.

Thus we can regard $\overline{MU}_*$ as a functor from **FH** to $\mathbf{C}\Gamma$. The latter category is much more accessible. We will see that it has some structural features which reflect those of **FH** very well. The nilpotence, periodicity and chromatic convergence theorems are examples of this.

In order to study $\mathbf{C}\Gamma$ further we need some more facts about formal group laws. Here are some power series associated with them.

DEFINITION 4.3.3. For each integer n the *n-series* $[n](x)$ is given by

$$\begin{aligned}
[1]\,(x) &= x,\\
[n]\,(x) &= F\big(x, [n-1]\,(x)\big) \quad \text{for } n > 1 \quad \text{and}\\
[-n]\,(x) &= i\big([n]\,(x)\big).
\end{aligned}$$

These satisfy

$$\begin{aligned}[n](x) &\equiv nx \bmod (x^2),\\ [m+n](x) &= F\big([m](x),[n](x)\big) \quad \text{and}\\ [mn](x) &= [m]\big([n](x)\big).\end{aligned}$$

For the additive formal group law 4.2.2(i), we have $[n](x) = nx$, and for the multiplicative formal group law, $[n](x) = (1+x)^n - 1$.

Of particular interest is the p-series. In characteristic p it always has leading term ax^q where $q = p^h$ for some integer h. This leads to the following.

DEFINITION 4.3.4. Let $F(x,y)$ be a formal group law over a ring in which the prime p is not a unit. If the mod p reduction of $[p](x)$ has the form

$$[p](x) = ax^{p^h} + \text{higher terms}$$

with a invertible, then we say that F has *height h at p*. If $[p](x) \equiv 0 \bmod p$ then the height is infinity.

For the additive formal group law we have $[p](x) = 0$ so the height is ∞. The multiplicative formal group law has height 1 since $[p](x) = x^p$. The mod p reduction (for p odd) of the elliptic formal group law of 4.2.2(iii) has height one or two depending on the values of δ and ε. For example, if $\delta = 0$ and $\varepsilon = 1$ then the height is one for $p \equiv 1 \bmod 4$ and two for $p \equiv 3 \bmod 4$. (See [59, pages 373–374].)

The following classification theorem is due to Lazard [40].

THEOREM 4.3.5 (Classification of formal group laws). *Two formal group laws over the algebraic closure of $\mathbf{F}_p$ are isomorphic if and only if they have the same height.*

Let $v_n \in L$ denote the coefficient of x^{p^n} in the p-series for the universal formal group law; the prime p is omitted from the notation. This v_n is closely related to the v_n in the Morava K-theories (2.5.2); the precise relation is explained in [61, B.7]. It can be shown that v_n is an indecomposable element in L, i.e. it could serve as a polynomial generator in dimension $2p^n - 2$. Let $I_{p,n} \subset L$ denote the prime ideal $(p, v_1, \ldots v_{n-1})$.

The following result is due to Morava [53] and Landweber [36].

THEOREM 4.3.6 (Invariant prime ideal theorem). *The only prime ideals in L which are invariant under the action of Γ are the $I_{p,n}$ defined above, where p is a prime integer and n is a non-negative integer, possibly ∞. ($I_{p,\infty}$ is by definition the ideal $(p, v_1, v_2, \ldots)$ and $I_{p,0}$ is the zero ideal.)*

Moreover in $L/I_{p,n}$ for $n > 0$ the subgroup fixed by Γ is $\mathbf{Z}/(p)[v_n]$. In L itself the invariant subgroup is $\mathbf{Z}$.

This shows that the action of Γ on L is very rigid. L has a bewildering collection of prime ideals, but the only ones we ever have to consider are the ones listed in the theorem. This places severe restriction on the structure of modules in $\mathbf{C}\Gamma$.

Recall that a finitely generated module M over a Noetherian ring R has a finite filtration

$$0 = F_0M \subset F_1M \subset F_2M \subset \cdots F_kM = M$$

in which each subquotient $F_iM/F_{i-1}M$ is isomorphic to R/I_i for some prime ideal $I_i \subset R$. Now L is not Noetherian, but it is a direct limit of Noetherian rings, so finitely presented modules over it admit similar filtrations. For a module in $\mathbf{C}\Gamma$, the filtration can be chosen so that the submodules, and therefore the prime ideals, are all invariant under Γ. The following result is due to Landweber [37].

THEOREM 4.3.7 (Landweber filtration theorem). *Every module M in $\mathbf{C}\Gamma$ admits a finite filtration by submodules in $\mathbf{C}\Gamma$ as above in which each subquotient is isomorphic to a suspension (recall that the modules are graded) of $L/I_{p,n}$ for some prime p and some finite n.*

These results suggest that, once we have localized at a prime p, the only polynomial generators of MU_* which really matter are the $v_n = x_{p^n-1}$. In fact the other generators act freely on any module in $\mathbf{C}\Gamma$ and hence provide no information. We might as well tensor them away and replace the theory of L-modules with Γ-action by a corresponding theory of modules over the ring

$$V_p = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]. \tag{4.3.8}$$

This has been done and the ring V_p is commonly known as BP_*, the coefficient ring for Brown–Peterson theory. There are good reasons for doing this from the topological standpoint, from the formal group law theoretic standpoint, and for the purpose of making explicit calculations useful in homotopy theory. Indeed all of the current literature on this subject is written in terms of BP-theory rather than MU-theory.

However it is *not* necessary to use this language in order to describe the subject conceptually as we are doing here. There is one technical problem with BP-theory which makes it awkward to discuss in general terms. There is no BP-theoretic analogue of the group Γ. It has to be replaced instead by a certain groupoid, and certain Hopf algebras associated with MU-theory have to be replaced by Hopf algebroids (see [61, B.3]).

The following are easy consequences of the Landweber filtration theorem.

COROLLARY 4.3.9. *Suppose M is a p-local module in $\mathbf{C}\Gamma$ and $x \in M$.*

(i) *If x is annihilated by some power of v_n, then it is annihilated by some power of v_{n-1}, so if $v_n^{-1}M = 0$, i.e. if each element in M is annihilated by some power of v_n, then $v_{n-1}^{-1}M = 0$.*
(ii) *If x is nonzero, then there is an n so that $v_n^k x \neq 0$ for all k, so if M is nontrivial, then so is $v_n^{-1}M$ for all sufficiently large n.*
(iii) *If $v_{n-1}^{-1}M = 0$, then there is a positive integer k such that multiplication by v_n^k in M commutes with the action of Γ.*

(iv) *Conversely, if $v_{n-1}^{-1}M$ is nontrivial, then there is no positive integer k such that multiplication by v_n^k in M commutes with the action of Γ on x.*

The first two statements should be compared to the last two statements in 2.5.2. In fact the functor $v_n^{-1}\overline{MU}_*(X)_{(p)}$ is a homology theory (see [61, B.6.2]) which vanishes on a finite p-local CW-complex X if and only if $\overline{K(n)}_*(X)$ does. One could replace $K(n)_*$ by $v_n^{-1}MU_{(p)*}$ in the statement of the periodicity theorem. The third statement is an algebraic analogue of the periodicity theorem.

We can mimic the definition of type n finite spectra (2.5.3) and v_n-maps (2.5.4) in $\mathbf{C}\Gamma$.

DEFINITION 4.3.10. A p-local module M in $\mathbf{C}\Gamma$ has *type n* if n is the smallest integer with $v_n^{-1}M$ nontrivial. A homomorphism $f : \Sigma^d M \to M$ in $\mathbf{C}\Gamma$ is a *v_n-map* if it induces an isomorphism in $v_n^{-1}M$ and the trivial homomorphism in $v_m^{-1}M$ for $m \neq n$.

Another consequence of the Landweber filtration theorem is the following.

COROLLARY 4.3.11. *If M in $\mathbf{C}\Gamma$ is a p-local module with $v_{n-1}^{-1}M$ nontrivial, then M does not admit a v_n-map.*

SKETCH OF PROOF OF 4.3.9. (i) The statement about x is proved by Johnson and Yosimura in [35]. The statement about M can be proved independently as follows. The condition implies that each subquotient in the Landweber filtration is a suspension of $L/I_{p,m}$ for some $m > n$. It follows that each element is annihilated by some power of v_{n-1} as claimed.

(ii) We can choose n so that each Landweber subquotient of M is a suspension of $L/I_{p,m}$ for some $m \leqslant n$. Then no element of M is annihilated by any power of v_n.

(iii) If $v_{n-1}^{-1}M = 0$, then each Landweber subquotient is a suspension of $L/I_{p,m}$ for $m \geqslant n$. It follows that if the length of the filtration is j, then M is annihilated by $I_{p,n}^j$. For any $\gamma \in \Gamma$ we have

$$\gamma(v_n) = v_n + e \quad \text{with } e \in I_{p,n}.$$

It follows easily that

$$\gamma\left(v_n^{p^{j-1}}\right) = (v_n + e)^{p^{j-1}} = v_n^{p^{j-1}} + e' \quad \text{with } e' \in I_{p,n}^j.$$

This means that multiplication by $v_n^{p^{j-1}}$ is Γ-equivariant in $L/I_{p,n}^j$ and hence in M.

(iv) Suppose such an integer k exists. Then multiplication by v_n^k is Γ-equivariant on each Landweber subquotient. However by 4.3.6 this is not the case on $L/I_{p,m}$ for $m < n$. It follows that $v_{n-1}^{-1}M = 0$, which is a contradiction. □

PROOF OF 4.3.11. Suppose M has type m for $m < n$. This means that each Landweber subquotient of M is a suspension of $L/I_{p,k}$ for some $k \geqslant m$. Hence we see that $v_m^{-1}M$, $v_n^{-1}M$ and hence $v_m^{-1}v_n^{-1}M$ are all nontrivial. On the other hand, if f is a v_n-map, then $v_m^{-1}v_n^{-1}f$ must be both trivial and an isomorphism, which is a contradiction. □

4.4. *Thick subcategories*

Now we need to consider certain full subcategories of $\mathbf{C}\Gamma$ and $\mathbf{FH}$.

DEFINITION 4.4.1. A full subcategory $\mathbf{C}$ of $\mathbf{C}\Gamma$ is *thick* if it satisfies the following axiom:
If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $\mathbf{C}\Gamma$, then M is in $\mathbf{C}$ if and only if M' and M'' are. (In other words $\mathbf{C}$ is closed under subobjects, quotient objects, and extensions.)

A full subcategory $\mathbf{F}$ of $\mathbf{FH}$ is thick if it satisfies the following two axioms:

(i) If

$$X \xrightarrow{f} Y \longrightarrow C_f$$

is a cofibre sequence in which two of the three spaces are in $\mathbf{F}$, then so is the third.

(ii) If $X \vee Y$ is in $\mathbf{F}$ then so are X and Y.

Thick subcategories were called generic subcategories by Hopkins in [27].

Using the Landweber filtration theorem, one can classify the thick subcategories of $\mathbf{C}\Gamma_{(p)}$.

THEOREM 4.4.2. *Let $\mathbf{C}$ be a thick subcategory of $\mathbf{C}\Gamma_{(p)}$ (the category of all p-local modules in $\mathbf{C}\Gamma$). Then $\mathbf{C}$ is either all of $\mathbf{C}\Gamma_{(p)}$, the trivial subcategory (in which the only object is the trivial module), or consists of all p-local modules M in $\mathbf{C}\Gamma$ with $v_{n-1}^{-1}M = 0$. We denote the latter category by $\mathbf{C}_{p,n}$.*

We will give the proof of this result below.

There is an analogous result about thick subcategories of $\mathbf{FH}_{(p)}$, which is a very useful consequence of the nilpotence theorem.

THEOREM 4.4.3 (Thick subcategory theorem). *Let $\mathbf{F}$ be a thick subcategory of $\mathbf{FH}_{(p)}$, the category of p-local finite CW-complexes. Then $\mathbf{F}$ is either all of $\mathbf{FH}_{(p)}$, the trivial subcategory (in which the only object is a point) or consists of all p-local finite CW-complexes X with $v_{n-1}^{-1}\overline{MU}_*(X) = 0$. We denote the latter category by $\mathbf{F}_{p,n}$.*

The condition $v_{n-1}^{-1}\overline{MU}_*(X) = 0$ is equivalent to $\overline{K(n-1)}_*(X) = 0$, in view of 2.5.2(v) and 6.3.2(d) below.

Thus we have two nested sequences of thick subcategories,

$$\mathbf{FH}_{(p)} = \mathbf{F}_{p,0} \supset \mathbf{F}_{p,1} \supset \mathbf{F}_{p,2} \cdots \{\text{pt.}\} \tag{4.4.4}$$

and

$$\mathbf{C}\Gamma_{(p)} = \mathbf{C}_{p,0} \supset \mathbf{C}_{p,1} \supset \mathbf{C}_{p,2} \cdots \{0\}. \tag{4.4.5}$$

The functor $MU_*(\cdot)$ sends one to the other. Until 1983 it was not even known that the $\mathbf{F}_{p,n}$ were nontrivial for all but a few small values of n. Mitchell [52] first showed that all of the inclusions of the $\mathbf{F}_{p,n}$ are proper. Now it is a corollary of the periodicity theorem.

In §4.5 we will describe another algebraic paradigm analogous to 4.4.3 discovered in the early 70's by Jack Morava. It points to some interesting connections with number theory and was the original inspiration behind this circle of ideas.

In §5 we will derive the thick subcategory theorem from another form of the nilpotence theorem. This is easy since it uses nothing more than elementary tools from homotopy theory.

In §5.4 we will sketch the proof of the periodicity theorem. It is not difficult to show that the collection of complexes admitting periodic self maps for given p and n forms a thick subcategory. Given the thick subcategory theorem, it suffices to find just one nontrivial example of a complex of type n with a periodic self-map. This involves some hard homotopy theory. There are two major ingredients in the construction. One is the Adams spectral sequence, a computational tool that one would expect to see used in such a situation. The other is a novel application of the modular representation theory of the symmetric group described in as yet unpublished work of Jeff Smith.

Proof of Theorem 4.4.2. Note that $\mathbf{C}_{p,0} = \mathbf{C}\Gamma_{(p)}$ by convention and we have a decreasing filtration

$$\mathbf{C}\Gamma_{(p)} = \mathbf{C}_{p,0} \supset \mathbf{C}_{p,1} \supset \cdots \mathbf{C}_{p,n} \supset \cdots$$

with

$$\bigcap_{n\geqslant 0} \mathbf{C}_{p,n} = \{0\}$$

by Corollary 4.3.9(ii).

Now suppose $\mathbf{C} \subset \mathbf{C}\Gamma_{(p)}$ is thick. If $\mathbf{C} \neq \{0\}$, choose the largest n so that $\mathbf{C}_{p,n} \supset \mathbf{C}$. Then $\mathbf{C} \not\subset \mathbf{C}_{p,n+1}$, and we want to show that $\mathbf{C} = \mathbf{C}_{p,n}$, so we need to verify that $\mathbf{C} \supset \mathbf{C}_{p,n}$.

Let M be a comodule in $\mathbf{C}$ but not in $\mathbf{C}_{p,n+1}$. Thus $v_m^{-1}M = 0$ for $m < n$ but $v_n^{-1}M \neq 0$. Choosing a Landweber filtration of M in $\mathbf{C}\Gamma$,

$$0 = F_0M \subset F_1M \subset \cdots \subset F_kM = M,$$

all F_sM are in $\mathbf{C}$, hence so are all the subquotients

$$F_sM/F_{s-1}M = \Sigma^{d_s}MU_*/I_{p,m_s}.$$

Since $v_n^{-1}M \neq 0$, we must have

$$v_n^{-1}(MU_*/I_{p,m_s}) \neq 0$$

for some s, so some m_s is no more than n. This m_s must be n, since a smaller value would contradict the assumption that $\mathbf{C} \subset \mathbf{C}_{p,n}$. Hence we conclude that

$$MU_*/I_{p,n} \in \mathbf{C}. \tag{4.4.6}$$

Now let N be in $\mathbf{C}_{p,n}$; we want to show that it is also in $\mathbf{C}$. Then $v_{n-1}^{-1}M = 0$, so each subquotient of a Landweber filtration of N is a suspension of $MU_*/I_{p,m}$ for some $m \geqslant n$. Since $MU_*/I_{p,n} \in \mathbf{C}$ by (4.4.6), it follows that $MU_*/I_{p,n} \in \mathbf{C}$ for all $m \geqslant n$. Hence the Landweber subquotients of N are all in $\mathbf{C}$, so N itself is in $\mathbf{C}$. □

4.5. *Morava's picture of the action of Γ on L*

The action of the group Γ on the Lazard ring L (4.3.1) is central to this theory and the picture we will describe here sheds considerable light on it. Let $H_{\mathbf{Z}}L$ denote the set of ring homomorphisms $L \to \mathbf{Z}$. By 4.2.3 this is the set of formal group laws over the integers. Since L is a polynomial ring, a homomorphism $\theta \in H_{\mathbf{Z}}L$ is determined by its values on the polynomial generators $x_i \in L$. Hence $H_{\mathbf{Z}}L$ can be regarded as an infinite dimensional affine space over $\mathbf{Z}$. The action of Γ on L induces one on $H_{\mathbf{Z}}L$. The following facts about it are straightforward.

PROPOSITION 4.5.1. *Let $H_{\mathbf{Z}}L$ and the action of the group Γ on it be as above. Then*

(i) *Points in $H_{\mathbf{Z}}L$ correspond to formal group laws over $\mathbf{Z}$.*
(ii) *Two points are in the same Γ-orbit if and only if the two corresponding formal group laws are isomorphic over $\mathbf{Z}$.*
(iii) *The subgroup of Γ fixing point $\theta \in H_{\mathbf{Z}}L$ is the strict automorphism group of the corresponding formal group law.*
(iv) *The strict automorphism groups of isomorphic formal group laws are conjugate in Γ.*

We have not yet said what a strict automorphism of a formal group law F is.
An automorphism is a power series $f(x)$ satisfying

$$f\big(F(x,y)\big) = F\big(f(x), f(y)\big)$$

and $f(x)$ is *strict* if it has the form

$$f(x) = x + \text{ higher terms.}$$

The classification of formal group laws over the integers is quite complicated, but we have a nice classification theorem (4.3.5) over k, the algebraic closure of $\mathbf{F}_p$. Hence we want to replace $\mathbf{Z}$ by k in the discussion above. Let H_kL denote the set of ring homomorphisms $L \to k$; it can be regarded as an infinite dimensional vector space over k. Let Γ_k denote the corresponding group of power series. Then it follows that there is one Γ_k-orbit for each height n. Since $\theta(v_i) \in k$ is the coefficient of x_{p^i} in the power series $[p](x)$, the following is a consequence of the relevant definitions.

PROPOSITION 4.5.2. *The formal group law over k corresponding to $\theta \in H_k L$ has height n if and only if $\theta(v_i) = 0$ for $i < n$ and $\theta(v_n) \neq 0$. Moreover, each $v_n \in L$ is indecomposable, i.e. it is a unit (in $\mathbf{Z}_{(p)}$) multiple of*

$$x_{p^n-1} + \textit{decomposables.}$$

Let $Y_n \subset H_k L$ denote the height n orbit. It is the subset defined by the equations $v_i = 0$ for $i < n$ and $v_n \neq 0$ for finite n, and for $n = \infty$ it is defined by $v_i = 0$ for all $n < \infty$. Let

$$X_n = \bigcup_{n \leqslant i} Y_i$$

so we have a nested sequence of subsets

$$H_k L = X_1 \supset X_2 \supset X_3 \cdots X_\infty \tag{4.5.3}$$

which is analogous to (4.4.4) and (4.4.5).

4.6. *Morava stabilizer groups*

Now we want to describe the strict automorphism group S_n (called the n-th *Morava stabilizer group*) of a height n formal group law over k. It is contained in the multiplicative group over a certain division algebra D_n over the p-adic numbers $\mathbf{Q}_p$. To describe it we need to define several other algebraic objects.

Recall that $\mathbf{F}_{p^n}$, the field with p^n elements, is obtained from $\mathbf{F}_p$ by adjoining a primitive $(p^n - 1)$-st root of unity $\bar{\zeta}$, which is the root of some irreducible polynomial of degree n. The Galois group of this extension is cyclic of order n generated by the Frobenius automorphism which sends an element x to x^p.

There is a corresponding degree n extension $W(\mathbf{F}_{p^n})$ of the p-adic integers $\mathbf{Z}_p$, obtained by adjoining a primitive $(p^n - 1)$-st root of unity ζ (whose mod p reduction is $\bar{\zeta}$), which is also the root of some irreducible polynomial of degree n. The Frobenius automorphism has a lifting σ fixing $\mathbf{Z}_p$ with $\sigma(\zeta) = \zeta^p$ and

$$\sigma(x) \equiv x^p \bmod p$$

for any $x \in W(\mathbf{F}_{p^n})$.

We denote the fraction field of $W(\mathbf{F}_{p^n})$ by K_n; it is the unique unramified extension of $\mathbf{Q}_p$ of degree n. Let $K_n\langle S\rangle$ denote the ring obtained by adjoining a *noncommuting* power series variable S subject to the rule

$$Sx = \sigma(x)S$$

for $x \in K_n$. Thus S commutes with everything in $\mathbf{Q}_p$ and S^n commutes with all of K_n. The division algebra D_n is defined by

$$D_n = K_n\langle S\rangle/(S^n - p). \tag{4.6.1}$$

It is an algebra over $\mathbf{Q}_p$ of rank n^2 with center $\mathbf{Q}_p$. It is known to contain each degree n field extension of $\mathbf{Q}_p$ as a subfield. (This statement is 6.2.12 of [59], where appropriate references are given.)

It also contains a maximal order

$$E_n = W(\mathbf{F}_{p^n})\langle S\rangle/(S^n - p). \tag{4.6.2}$$

E_n is a complete local ring with maximal ideal (S) and residue field $\mathbf{F}_{p^n}$. Each element in $a \in E_n$ can be written uniquely as

$$a = \sum_{0\leqslant i\leqslant n-1} a_i S^i \tag{4.6.3}$$

with $a_i \in W(\mathbf{F}_{p^n})$, and also as

$$a = \sum_{i\geqslant 0} e_i S^i \tag{4.6.4}$$

where each $e_i \in W(\mathbf{F}_{p^n})$ satisfies the equation

$$e_i^{p^n} - e_i = 0,$$

i.e. e_i is either zero or a root of unity. The groups of units $E_n^\times \subset E_n$ is the set of elements with $e_0 \neq 0$, or equivalently with a_0 a unit in $W(\mathbf{F}_{p^n})$.

PROPOSITION 4.6.5. *The full automorphism group of a formal group law over k of height n is isomorphic to $E_n^\times$ above, and the strict automorphism group S_n is isomorphic to the subgroup of $E_n^\times$ with $e_0 = 1$.*

If we regard each coefficient e_i as a continuous $\mathbf{F}_{p^n}$-valued function on S_n, then it can be shown that the ring of all such functions is

$$S(n) = \mathbf{F}_{p^n}[e_i\colon i \geqslant 1]/(e_i^{p^n} - e_i). \tag{4.6.6}$$

This is a Hopf algebra over $\mathbf{F}_{p^n}$ with coproduct induced by the group structure of S_n. It should be compared to the Hopf algebra $\Sigma(n)$ of ([61, B.7.5]). This is a factor of $K(n)_*(K(n))$, the Morava K-theory analog of the dual Steenrod algebra. Its multiplicative structure is given by

$$\Sigma(n) = K(n)_*[t_1, t_2, \ldots]/(t_i^{p^n} - v_n^{p^i-1} t_i).$$

Hence we have

$$S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbf{F}_{p^n}$$

under the isomorphism sending t_i to e_i and v_n to 1.

Now we will describe the action of S_n on a particular height n formal group law F_n. To define F_n, let F be the formal group law over $\mathbf{Z}_{(p)}$ with logarithm

$$\log_F(x) = \sum_{i \geqslant 0} \frac{x^{p^{in}}}{p^i}. \tag{4.6.7}$$

F_n is obtained by reducing F mod p and tensoring with $\mathbf{F}_{p^n}$.

Now an automorphism e of F_n is a power series $e(x)$ over $\mathbf{F}_{p^n}$ satisfying

$$e\big(F_n(x, y)\big) = F_n\big(e(x), e(y)\big).$$

For

$$e = \sum_{i \geqslant 0} e_i S^i \in S_n$$

(with $e_0 = 1$) we define $e(x)$ by

$$e(x) = \sum_{i \geqslant 0}{}^{F_n} e_i x^{p^i}. \tag{4.6.8}$$

More details can be found in [59, Appendix 2].

5. The thick subcategory and periodicity theorems

In this section we will derive the thick subcategory theorem 4.4.3 from a variant of the nilpotence theorem (5.1.4 below) with the use of some standard tools from homotopy theory, which we must introduce before we can give the proof. The proof itself is identical to the one given by Hopkins in [27].

Then in §5.4 we will explain the relevance of the thick subcategory theorem to the periodicity theorem.

5.1. *Spectra*

First we have to introduce the category of spectra. These objects are similar to spaces and were invented to avoid qualifying statements (such as Definition 2.4.1) with phrases such as 'up to some suspension' and 'stably'. Since the category was introduced around 1960 [41], it has taken on a life of its own, as will be seen later in this chapter. We will say as little about it here as we can get away with; for more details see [61, A.2]). The use of the word 'spectrum' in homotopy theory has no connection with its use in analysis (the spectrum of a differential operator) or in algebraic geometry (the spectrum of a commutative ring). It also has no direct connection with the term 'spectral sequence'.

Most of the theorems in this paper that are stated in terms of spaces are really theorems about spectra that we have done our best to disguise. However we cannot keep up this act any longer.

DEFINITION 5.1.1. A *spectrum* X is a collection of spaces $\{X_n\}$ (defined for all large values of n) and maps $\Sigma X_n \to X_{n+1}$. The *suspension spectrum* of a space X is defined by $X_n = \Sigma^n X$ with each map being the identity. The *sphere spectrum* S^0 is the suspension spectrum of the space S^0, i.e. the n-th space is S^n. The i-th suspension $\Sigma^i X$ of X is defined by

$$\left(\Sigma^i X\right)_n = X_{n+i}$$

for any integer i. Thus any spectrum can be suspended or desuspended any number of times.

The homotopy groups of X are defined by

$$\pi_k(X) = \varinjlim \pi_{n+k}(X_n)$$

and the generalized homology $E_*(X)$ is defined by

$$E_k(X) = \varinjlim \overline{E}_{n+k}(X_n);$$

note that the homology groups on the right are reduced while those on the left are not. In the category of spectra there is no need to distinguish between reduced and unreduced homology.

In particular, $\pi_k(S^0)$ is the stable k-stem π_k^S of 3.2.3.

The generalized cohomology of a spectrum can be similarly defined.

A spectrum X is *connective* if its homotopy groups are bounded below, i.e. if $\pi_{-k}(X) = 0$ for $k \gg 0$. It has *finite type* if $\pi_k(X)$ is finitely generated for each k. It is *finite* if some suspension of it is equivalent to the suspension spectrum of a finite CW-complex.

The homotopy groups of spectra are much more manageable than those of spaces. For example, one has

$$\pi_k\left(\Sigma^i E\right) = \pi_{k-i}(E)$$

for all k and i, and a cofibre sequence (3.3.3) of spectra leads to a long exact sequence of homotopy groups as well as the usual long exact sequence of homology groups (3.3.4).

It is surprisingly difficult to give a correct definition of a map $E \to F$ of spectra. One's first guess, namely a collection of maps $E_n \to F_n$ for $n \gg 0$ making the obvious diagrams commute, turns out to be too restrictive. While such data does give a map of spectra, there are some maps one would dearly like to have that do *not* come from any such data. However this naive definition is adequate in the case where E and F are suspension spectra of finite CW-complexes, which is all we will need for this section. A correct definition is given in [61, A.2.5].

Next we need to discuss smash products. For spaces the definition is as follows.

DEFINITION 5.1.2. Let X and Y be spaces equipped with base points x_0 and y_0. The *smash product* $X \wedge Y$ is the quotient of $X \times Y$ obtained by collapsing $X \times \{y_0\} \cup \{x_0\} \times Y$ to a single point. The k-fold iterated smash product of X with itself is denoted by $X^{(k)}$. For $f: X \to Y$, $f^{(k)}$ denote the evident map from $X^{(k)}$ to $Y^{(k)}$. The map f is *smash nilpotent* if $f^{(k)}$ is null homotopic for some k.

The k-fold suspension $\Sigma^k X$ is the same as $S^k \wedge X$. For CW-complexes X and Y there is an equivalence

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$

Defining the smash product of two spectra is not as easy as one would like. If E is a suspension spectrum, then there is an obvious definition of the smash product $E \wedge F$, namely

$$(E \wedge F)_n = E_0 \wedge F_n.$$

A somewhat more flexible but still unsatisfactory definition is the following.

DEFINITION 5.1.3. For spectra E and F, the *naive smash product* is defined by

$$\begin{aligned}(E \wedge F)_{2n} &= E_n \wedge F_n,\\ (E \wedge F)_{2n+1} &= \Sigma E_n \wedge F_n\end{aligned}$$

where the map

$$\Sigma E_n \wedge \Sigma F_n = \Sigma(E \wedge F)_{2n+1} \to (E \wedge F)_{2n+2} = E_{n+1} \wedge F_{n+1}$$

is the smash product of the maps $\Sigma E_n \to E_{n+1}$ and $\Sigma F_n \to F_{n+1}$.

However the correct definition of the smash product of two spectra is very difficult; we refer the interested reader to the lengthy discussion in Adams [3, III.4]. In this section at least, the only smash products we need are with finite spectra, which are always suspension spectra, so the naive definition is adequate.

The nilpotence theorem can be stated in terms of smash products as follows.

THEOREM 5.1.4 (Nilpotence theorem, smash product form). *Let*

$$F \xrightarrow{f} X$$

be a map of spectra where F is finite. Then f is smash nilpotent if $MU \wedge f$ (i.e. the evident map $MU \wedge F \to MU \wedge X$) is null homotopic.

Both this and 2.4.2 will be derived from a third form of the nilpotence theorem in §7. A more useful form of it for our purposes is the following, which we will prove at the end of §5.2, using some methods from §6.

COROLLARY 5.1.5. *Let W, X and Y be p-local finite spectra with $f: X \to Y$. Then $W \wedge f^{(k)}$ is null homotopic for $k \gg 0$ if $K(n)_*(W \wedge f) = 0$ for all $n \geqslant 0$.*

It is from this result that we will derive the thick subcategory theorem.

5.2. *Spanier–Whitehead duality*

Next we need to discuss Spanier–Whitehead duality, which is treated in more detail in [3, III.5].

THEOREM 5.2.1. *For a finite spectrum X there is a unique finite spectrum DX (the* Spanier–Whitehead dual *of X) with the following properties.*

(i) *For any spectrum Y, the graded group $[X, Y]_*$ is isomorphic to $\pi_*(DX \wedge Y)$, and this isomorphism is natural in both X and Y. In particular, $DS^0 = S^0$. We say that the maps $S^n \to DX \wedge Y$ and $\Sigma^n X \to Y$ that correspond under this isomorphism are* adjoint *to each other. In particular when $Y = X$, the identity map on X is adjoint to a map $e: S^0 \to DX \wedge X$.*

(ii) *This isomorphism is reflected in Morava K-theory, namely (since $K(n)_*(X)$ is free over $K(n)_*$)*

$$\mathrm{Hom}_{K(n)_*}\big(K(n)_*(X), K(n)_*(Y)\big) \cong K(n)_*(DX \wedge Y).$$

In particular for $Y = X$, $K(n)_*(e) \neq 0$ when $K(n)_*(X) \neq 0$. Similar statements hold for ordinary mod p homology. For $X = S^0$, this isomorphism is the identity.

(iii) *$DDX \simeq X$ and $[X, Y]_* \cong [DY, DX]_*$.*

(iv) *For a homology theory E_*, there is a natural isomorphism between $E_k(X)$ and $E^{-k}(DX)$.*

(v) *Spanier–Whitehead duality commutes with smash products, i.e. for finite spectra X and Y, $D(X \wedge Y) = DX \wedge DY$.*

(vi) *The functor $X \mapsto DX$ is contravariant.*

The Spanier–Whitehead dual DX of a finite complex X is analogous to the linear dual $V^* = \mathrm{Hom}(V, k)$ of a finite dimensional vector space V over a field k. 5.2.1(i) is analogous to the isomorphism

$$\mathrm{Hom}(V, W) \cong V^* \otimes W$$

for any vector space W. 5.2.1(iii) is analogous to the statement that $(V^*)^* = V$ and 5.2.1(v) is analogous to the isomorphism

$$(V \otimes W)^* \cong V^* \otimes W^*.$$

The geometric idea behind Spanier–Whitehead duality is as follows. A finite spectrum X is the suspension spectrum of a finite CW-complex, which we also denote by X. The latter can always be embedded in some Euclidean space $\mathbf{R}^N$ and hence in S^N. Then DX is a suitable suspension of the suspension spectrum of the complement $S^N - X$. 5.2.1(iv) is a generalization of the classical Alexander duality theorem, which says that $H_k(X)$ is isomorphic to $H^{N-1-k}(S^N - X)$. A simple example of this is the case where $X = S^k$ and it it linearly embedded in S^N. Then its complement is homotopy equivalent to S^{N-1-k}. The Alexander duality theorem says that the complement has the same cohomology as S^{N-1-k} even when the embedding of S^k in S^N is not linear, e.g., when $k = 1$, $n = 3$ and $S^1 \subset S^3$ is knotted.

Before we can proceed with the proof of the thick subcategory theorem we need an elementary lemma about Spanier–Whitehead duality. For a finite spectrum X, let $f: W \to S^0$ be the map such that

$$W \xrightarrow{f} S^0 \xrightarrow{e} DX \wedge X$$

is a cofibre sequence. In the category of spectra, such maps always exist. W in this case is finite, and $C_f = DX \wedge X$.

LEMMA 5.2.2. *With notation as above, there is a cofibre sequence*

$$C_{f^{(k)}} \longrightarrow C_{f^{(k-1)}} \longrightarrow \Sigma W^{(k-1)} \wedge C_f$$

for each $k > 1$.

PROOF. A standard lemma in homotopy theory says that given maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

there is a diagram

$$\begin{tikzcd}
C_f \arrow[r] & \text{pt.} \arrow[r] & \Sigma C_f \\
Y \arrow[u] \arrow[r, "g"] & Z \arrow[u] \arrow[r] & C_g \arrow[u] \\
X \arrow[u, "f"] \arrow[r, "gf"] & Z \arrow[u, "\simeq"'] \arrow[r] & C_{gf} \arrow[u]
\end{tikzcd}$$

in which each row and column is a cofibre sequence. Setting $X = W^{(k)}$, $Y = W^{(k-1)}$, $Z = S^0$ and $g = f^{(k-1)}$, this diagram becomes

$$\begin{tikzcd}
W^{(k-1)} \wedge C_f \arrow[r] & \mathrm{pt.} \arrow[r] & \Sigma W^{(k-1)} \wedge C_f \\
W^{(k-1)} \arrow[r, "f^{(k-1)}"] \arrow[u] & S^0 \arrow[r] \arrow[u] & C_{f^{(k-1)}} \arrow[u] \\
W^{(k)} \arrow[r, "f^{(k)}"] \arrow[u, "W^{(k-1)}\wedge f"] & S^0 \arrow[r] \arrow[u, "\simeq"'] & C_{f^{(k)}} \arrow[u]
\end{tikzcd}$$

and the right hand column is the desired cofibre sequence. □

PROOF OF COROLLARY 5.1.5. Let $R = DW \wedge W$ and let $e : S^0 \to R$ be the adjoint of the identity map. R is a ring spectrum ([61, A.2.8]) whose unit is e and whose multiplication is the composite

$$R \wedge R = DW \wedge W \wedge DW \wedge W \xrightarrow{DW \wedge De \wedge W} DW \wedge S^0 \wedge W = R.$$

The map $f : X \to Y$ is adjoint to $\hat{f} : S^0 \to DX \wedge Y$, and $W \wedge f$ is adjoint to the composite

$$S^0 \xrightarrow{\hat{f}} DX \wedge Y \xrightarrow{e \wedge DX \wedge Y} R \wedge DX \wedge Y = F,$$

which we denote by g. The map $W \wedge f^{(i)}$ is adjoint to the composite

$$S^0 \xrightarrow{g^{(i)}} F^{(i)} = R^{(i)} \wedge DX^{(i)} \wedge Y^{(i)} \longrightarrow R \wedge DX^{(i)} \wedge Y^{(i)},$$

the latter map being induced by the multiplication in R.

By 5.1.4 it suffices to show that $MU \wedge g^{(i)}$ is null for large i. Let $T_i = R \wedge DX^{(i)} \wedge Y^{(i)}$ and let T be the direct limit of

$$S^0 \xrightarrow{g} T_1 \xrightarrow{T_1 \wedge \hat{f}} T_2 \xrightarrow{T_2 \wedge \hat{f}} T_3 \longrightarrow \cdots.$$

The desired conclusion will follow from showing that $MU \wedge T$ is contractible, and our hypothesis implies that $K(n) \wedge T$ is contractible for each n.

Now we need to use the methods of §6. Since we are in a p-local situation, it suffices to show that $BP \wedge T$ is contractible. Using 6.3.2 and the fact that $K(n) \wedge T$ is contractible, it suffices to show that $P(m) \wedge T$ is contractible for large m.

Now for large enough m,

$$K(m)_*(W \wedge f) = K(m)_* \otimes H_*(W \wedge f)$$

and

$$P(m)_*(W \wedge f) = P(m)_* \otimes H_*(W \wedge f).$$

Our hypothesis implies that both of these homomorphisms are trivial, so $P(m) \wedge T$ is contractible as required. □

5.3. *The proof of the thick subcategory theorem*

Let $\mathbf{C} \subset \mathbf{FH}_{(p)}$ be a thick subcategory. Choose n to be the smallest integer such that $\mathbf{C}$ contains a p-local finite spectrum X with $K(n)_*(X) \neq 0$. Equivalently (by 6.3.2(d) and 2.5.2(v)), n is the smallest integer such that $\mathbf{C}$ contains an X with $v_n^{-1}BP_*(X) \neq 0$. We want to show that $\mathbf{C} = \mathbf{F}_{p,n}$. It is clear from the choice of n that $\mathbf{C} \subset \mathbf{F}_{p,n}$, so it suffices to show that $\mathbf{C} \supset \mathbf{F}_{p,n}$.

Let Y be a p-local finite CW-spectrum in $\mathbf{F}_{p,n}$. From the fact that $\mathbf{C}$ is thick, it follows that $X \wedge F$ is in $\mathbf{C}$ for any finite F, so $X \wedge DX \wedge Y$ (or $C_f \wedge Y$ in the notation of 5.2.2) is in $\mathbf{C}$. Thus 5.2.2 implies that $C_{f^{(k)}} \wedge Y$ is in $\mathbf{C}$ for all $k > 0$.

It follows from 5.2.1(ii) that $K(i)_*(f) = 0$ when $K(i)_*(X) \neq 0$, i.e., for $i \geqslant n$. Since $K(i)_*(Y) = 0$ for $i < n$, it follows that $K(i)_*(Y \wedge f) = 0$ for all i. Therefore by 5.1.5, $Y \wedge f^{(k)}$ is null homotopic for some $k > 0$.

Now the cofibre of a null homotopic map is equivalent to the wedge of its target and the suspension of its source, so we have

$$Y \wedge C_{f^{(k)}} \simeq Y \vee \left(\Sigma Y \wedge W^{(k)}\right).$$

Since $\mathbf{C}$ is thick and contains $Y \wedge C_{f^{(k)}}$, it follows that Y is in $\mathbf{C}$, so $\mathbf{C}$ contains $\mathbf{F}_{p,n}$ as desired. □

5.4. *The periodicity theorem*

In this section we will outline part of the proof of the periodicity theorem (Theorem 2.5.4). Recall that a v_n-map $f : \Sigma^d X \to X$ on a p-local finite complex X is a map such that $K(n)_*(f)$ is an isomorphism and $K(m)_*(f) = 0$ for $m \neq n$. The case $n = 0$ is uninteresting; Theorem 2.5.4 is trivial because the degree p map, which is defined for any spectrum (finite or infinite), is a v_0-map. Hence *we assume throughout this section that* $n > 0$.

Let $\mathbf{V}_n$ denote the collection of p-local finite spectra admitting such maps. If $K(n)_*(X) = 0$, then the trivial map is a v_n-map, so we have

$$\mathbf{V}_n \supset \mathbf{F}_{p,n+1}.$$

On the other hand, we know for algebraic reasons (4.3.11) that X cannot admit a v_n-map if $K(n-1)_*(X) \neq 0$, so

$$\mathbf{F}_{p,n} \supset \mathbf{V}_n.$$

The periodicity theorem says that $\mathbf{V}_n = \mathbf{F}_{p,n}$. The proof falls into two steps. The first is to show that $\mathbf{V}_n$ is thick; this is Theorem 5.4.5. Thus by the thick subcategory theorem, this category is either $\mathbf{F}_{p,n}$, as asserted in the periodicity theorem, or $\mathbf{F}_{p,n+1}$.

The second and harder step in the proof is to construct an example of a spectrum of type n with a v_n-map. We will outline this at the end of this section; for details we refer the reader to [61, Chapter 6].

Now we will prove that $\mathbf{V}_n$ is thick. We begin by observing that a self-map $f: \Sigma^d X \to X$ is adjoint to $\hat{f}: S^d \to DX \wedge X$. We will abbreviate $DX \wedge X$ by R. Now R is a ring spectrum; see [61, A.2.8] for a definition. The unit is the map $e: S^0 \to DX \wedge X$ adjoint to the identity map on X (5.2.1). Since $DDX = X$ and Spanier–Whitehead duality commutes with smash products, e is dual to

$$X \wedge DX \xrightarrow{De} S^0.$$

The multiplication on R is the composite

$$DX \wedge X \wedge DX \wedge X \xrightarrow{DX \wedge De \wedge X} DX \wedge S^0 \wedge X = DX \wedge X.$$

Now we will state four lemmas, the second and fourth of which are used directly in the proof of 5.4.5. They will be proved below, and each one depends on the previous one.

LEMMA 5.4.1. *For a v_n-map f as above, there is an $i > 0$ such the map induced on $K(n)_*(X)$ by f^i is multiplication by some power of v_n.*

LEMMA 5.4.2. *For a v_n-map f as above, there is an $i > 0$ such that $\hat{f}^i$ is in the center of $\pi_*(R)$.*

LEMMA 5.4.3 (Uniqueness of v_n-maps). *If X has two v_n-maps f and g then there are integers i and j such that $f^i = g^j$.*

LEMMA 5.4.4 (Extended uniqueness). *If X and Y have v_n-maps f and g, then there are integers i and j such that the following diagram commutes for any map $h: X \to Y$.*

$$\begin{tikzcd}
\Sigma^{?} X \arrow[r, "h"] \arrow[d, "f^i"'] & \Sigma^{?} Y \arrow[d, "g^j"] \\
X \arrow[r, "h"] & Y
\end{tikzcd}$$

Note that 5.4.3 is the special case of this where h is the identity map on X. However, we will derive 5.4.4 from 5.4.3.

THEOREM 5.4.5. *The category $\mathbf{V}_n \subset \mathbf{FH}_{(p)}$ of finite p-local CW-spectra admitting v_n-maps is thick.*

PROOF. Suppose $X \vee Y$ is in $\mathbf{V}_n$ and

$$\Sigma^d(X \vee Y) \xrightarrow{f} X \vee Y$$

is a v_n-map. By 5.4.2 we can assume that f commutes with the idempotent

$$X \vee Y \longrightarrow X \longrightarrow X \vee Y$$

and it follows that the composite

$$\Sigma^d X \longrightarrow \Sigma^d(X \vee Y) \xrightarrow{f} X \vee Y \longrightarrow X$$

is a v_n-map, so X is in $\mathbf{V}_n$.

Now suppose $h: X \to Y$ where X and Y have v_n-maps f and g. By 5.4.4 we can assume that $hf \simeq gh$, so there is a map

$$\Sigma^d C_h \xrightarrow{\ell} C_h$$

making the following diagram commute.

$$\begin{tikzcd}
X \arrow[r, "h"] & Y \arrow[r] & C_h \\
\Sigma^d X \arrow[r, "h"] \arrow[u, "f"] & \Sigma^d Y \arrow[r] \arrow[u, "g"] & \Sigma^d C_h \arrow[u, "\ell"']
\end{tikzcd}$$

The 5-lemma implies that $K(n)_*(\ell)$ is an isomorphism.

We also need to show that $K(m)_*(\ell) = 0$ for $m \neq n$. This is *not* implied by the facts that $K(m)_*(f) = 0$ and $K(m)_*(g) = 0$. However, an easy diagram chase shows that they do imply that $K(m)_*(\ell^2) = 0$, so ℓ^2 is the desired v_n-map on C_h. It follows that C_h is in $\mathbf{V}_n$, so $\mathbf{V}_n$ is thick. □

Now we will give the proofs of the four lemmas stated earlier.

PROOF OF LEMMA 5.4.1. The ring $K(n)_*(R)$ is a finite-dimensional $K(n)_*$-algebra, so the ungraded quotient $K(n)_*(R)/(v_n - 1)$ is a finite ring with a finite group of units. It follows that the group of units in $K(n)_*(R)$ itself is an extension of the group of units of $K(n)_*$ by this finite group. Therefore some power of the unit $\hat{f}_*$ is in $K(n)_*$, and the result follows. □

PROOF OF LEMMA 5.4.2. Let A be a noncommutative ring, such as $\pi_*(R)$. Given $a \in A$ we define a map

$$\mathrm{ad}(a): A \longrightarrow A$$

by

$$\mathrm{ad}(a)(b) = ab - ba.$$

Thus a is in the center of A if $\mathrm{ad}(a) = 0$.

There is a formula relating $\mathrm{ad}(a^i)$ to $\mathrm{ad}^j(a)$, the j^{th} iterate of $\mathrm{ad}(a)$, which we will prove below, namely

$$\mathrm{ad}(a^i)(x) = \sum_{j=1}^{i} \binom{i}{j} \mathrm{ad}^j(a)(x)a^{i-j}. \tag{5.4.6}$$

(This is proved in [61, 6.1].) Now suppose $\mathrm{ad}(a)$ is nilpotent and $p^k a = 0$ for some k. We set $i = p^N$ for some large N. Then the terms on the right for large j are zero because $\mathrm{ad}(a)$ is nilpotent, and the terms for small j vanish because the binomial coefficient is divisible by p^k. Hence $\mathrm{ad}(a^i) = 0$ so a^i is in the center of A.

To apply this to the situation at hand, define

$$\Sigma^d R \xrightarrow{\mathrm{ad}(\hat{f})} R$$

to be the composite

$$S^d \wedge R \xrightarrow{\hat{f} \wedge R} R \wedge R \xrightarrow{1-T} R \wedge R \xrightarrow{m} R$$

where T is the map that interchanges the two factors. Then for $x \in \pi_*(R)$, $\pi_*(\mathrm{ad}(\hat{f}))(x) = \mathrm{ad}(\hat{f})(x)$. By 5.4.1 (after replacing $\hat{f}$ by a suitable iterate if necessary), we can assume that $K(n)_*(f)$ is multiplication by a power of v_n, so $K(n)_*(\hat{f})$ is in the center of $K(n)_*(R)$ and $K(n)_*(\mathrm{ad}(\hat{f})) = 0$. Hence 5.1.5 tells us that $\mathrm{ad}(\hat{f})$ is nilpotent and the argument above applies to give the desired result. □

PROOF OF LEMMA 5.4.3. Replacing f and g by suitable powers if necessary, we may assume that they commute with each other and that $K(m)_*(f) = K(m)_*(g)$ for all m. Hence $K(m)_*(f - g) = 0$ so $f - g$ is nilpotent. Hence there is an $i > 0$ with

$$(f - g)^{p^i} = 0.$$

Since f and g commute, we can expand this with the binomial theorem and get

$$f^{p^i} \equiv g^{p^i} \bmod p$$

from which it follows that

$$f^{p^{i+k}} \equiv g^{p^{i+k}} \bmod p^{k+1}$$

for any $k > 0$, so for sufficiently large k the two maps are homotopic. □

PROOF OF LEMMA 5.4.4. Let $W = DX \wedge Y$, so h is adjoint to an element $\hat{h} \in \pi_*(W)$. W has two v_n-maps, namely $DX \wedge g$ and $Df \wedge Y$, so by 5.4.3,

$$DX \wedge g^j \simeq Df^i \wedge Y$$

for suitable i and j.

Observe that W is a module spectrum over $DX \wedge X$, and the product

$$\hat{f}^i\hat{h} = (Df^i \wedge Y)\hat{h}$$

is the adjoint of hf^i. Moreover $g^j h$ is adjoint to $(DX \wedge g^j)\hat{h}$. Since these two maps are homotopic, the diagram of 5.4.4 commutes. □

Now we will outline the construction, due to Jeff Smith [71], of a type n finite complex which admits a v_n-map.

Let $X^{(k)}$ denote the k-fold smash product of a finite p-local spectrum X. The symmetric groups Σ_k acts by permutation of coordinates. Since we are in the stable category we can take $\mathbf{Z}_{(p)}$-linear combinations of maps, so we get an action of the p-local groups ring $S = \mathbf{Z}_{(p)}[\Sigma_k]$ on $X^{(k)}$. Now suppose $e \in S$ is idempotent ($e^2 = e$). Then $1 - e$ is also idempotent and for any S-module M (such as $\pi_*(X^{(k)})$) we get a splitting

$$M \cong eM \oplus (1 - e)M,$$

in which one of the summands may be trivial.

There is a standard construction in homotopy theory which gives a similar splitting of spectra

$$X^{(k)} \cong eX^{(k)} \oplus (1 - e)X^{(k)}.$$

Thus each idempotent element $e \in S$ leads to a splitting of $X^{(k)}$ for any X. We will use this to construct a a finite spectrum Y of type n that can be shown to admit a v_n-map, starting with a well known X.

Now suppose that V is a finite dimensional vector space over $\mathbf{Z}/(p)$, such as $H_*(X; \mathbf{Z}/(p))$. Then $W = V^{\otimes k}$ is an S-module, so we have a splitting

$$W \cong eW \oplus (1 - e)W.$$

and the rank of eW is determined by that of W. There are enough idempotents e to give the following, which is due to Jeff Smith.

THEOREM 5.4.7. *For each positive integer r there is an idempotent*

$$e_r \in \mathbf{Z}_{(p)}[\Sigma_k]$$

(where k depends on r) such that the rank of eW above is nonzero if and only if the rank of V is at least r.

Smith has generalized this result to graded vector spaces with permutations subject to the usual sign conventions.

For any spectrum Y, $H^*(Y;\mathbf{Z}/(p))$ is a module over the Steenrod algebra A; the best reference for this is the classic [73]. Using the Adams spectral sequence, it can be shown that of Y is finite and its cohomology is free over a certain subalgebra of A, then it has type n and admits a v_n-map.

To obtain such a Y, one starts with a finite X satisfying much milder conditions. An appropriate skeleton of the classifying space $B\mathbf{Z}/(p)$ will do. Then one a applies a Smith idempotent to an appropriate smash power of X and the resulting summand $Y = eX^{(k)}$ has the required properties.

This completes our outline of the proof of the periodicity theorem.

6. Bousfield localization and equivalence

In this section we will discuss localization with respect to a generalized homology theory. We attach Bousfield's name to it because the main theorem in the subject is due to him. He did invent the equivalence relation associated with it. It provides us with a very convenient language for discussing some of the concepts of this subject. A general reference for this material is [58].

6.1. *Basic definitions and examples*

DEFINITION 6.1.1. Let E_* be a generalized homology theory ([61, A.3]). A space Y is *E_*-local* if whenever a map $f: X_1 \to X_2$ is such that $E_*(f)$ is an isomorphism, the map

$$[X_1, Y] \xleftarrow{\quad f^* \quad} [X_2, Y]$$

is also an isomorphism. (For spectra, this is equivalent to the following condition: Y is E_*-local if $[X,Y]_* = 0$ whenever $E_*(X) = 0$.)

An *E_*-localization* of a space or spectrum X is a map η from X to an E_*-local space or spectrum X_E (which we will usually denote by $L_E X$) such that $E_*(\eta)$ is an isomorphism.

It is easy to show that if such a localization exists, it is unique up to homotopy equivalence. The following properties are immediate consequences of the definition.

PROPOSITION 6.1.2. *For any homology theory E_*:*

(i) *Any inverse limit* ([61, A.5]) *of E_*-local spectra is E_*-local.*

(ii) *If*

$$W \longrightarrow X \longrightarrow Y \longrightarrow \Sigma W$$

is a cofibre sequence and any two of W, X and Y are E_*-local, then so is the third.

(iii) *If $X \vee Y$ is E_*-local, then so are X and Y.*

On the other hand, a homotopy direct limit of local spectra need not be local.

The main theorem in this subject, that localizations always exist, was proved by Bousfield for spaces in [11] and for spectra in [13].

THEOREM 6.1.3 (Bousfield localization theorem). *For any homology theory E_* and any space or spectrum X, the localization $L_E X$ of* 6.1.1 *exists and is functorial in X.*

The idea of the proof is the following. It is easy to see that if $L_E X$ exists, then for any map $f: X \to X'$ with $E_*(f)$ an isomorphism (such a map is called an *E_*-equivalence*), the map $\eta: X \to L_E X$ extends uniquely to X'. In other words the map η is terminal among E_*-equivalences out of X. This suggests constructing $L_E X$ as the direct limit of all such X'; this idea is due to Adams. Unfortunately it does not work because there are too many such X'; they form a class rather than a set. Bousfield found a way around these set theoretic difficulties.

If E_* is represented by a connective spectrum E (i.e. all of its homotopy groups below a given dimension are trivial), and if X is connective spectrum or a simply connected space, then the localization is relatively straightforward; it is the same as localization or completion with respect to some set of primes. The homotopy and generalized homology groups of $L_E X$ are arithmetically determined by those of X.

If either E or X fails to be connective, then $L_E X$ is far more mysterious and deserving of further study. We offer two important examples.

EXAMPLE 6.1.4. (i) X is the sphere spectrum S^0 and E_* is the homology theory associated with classical complex K-theory. $L_K S^0$ was described in [58, Section 8] and it is *not* connective. In particular $\pi_{-2}(L_K S^0) \cong \mathbf{Q}/\mathbf{Z}$.

(ii) Let E_* be ordinary homology H_*. Let X be a finite spectrum (such as one of the examples of 3.4.1) satisfying $K(n)_*(X) \neq 0$ with a v_n-map f (2.5.4) and let $\hat{X}$ be the telescope obtained by iterating f, i.e., $\hat{X}$ is the direct limit of the system

$$X \xrightarrow{f} \Sigma^{-d} X \xrightarrow{f} \Sigma^{-2d} X \cdots.$$

Then $L_H \hat{X}$ is contractible since $H_*(f) = 0$ and therefore $H_*(\hat{X}) = 0$. On the other hand $\hat{X}$ is *not* contractible since

$$K(n)_*(\hat{X}) \cong K(n)_*(X) \neq 0.$$

LEMMA 6.1.5. *If E is a ring spectrum* ([61, A.2.8]) *then $E \wedge X$ is E_*-local for any spectrum X.*

PROOF. We need to show that for any spectrum W with $E_*(W) = 0$,

$$[W, E \wedge X] = 0.$$

Given any map $f : W \to E \wedge X$, we have a diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "\eta\wedge W"'] & E \wedge X \arrow[d, "\eta\wedge E\wedge X"'] \arrow[dr, "E\wedge X"] & \\ E \wedge W \arrow[r, "E\wedge f"'] & E \wedge E \wedge X \arrow[r, "m\wedge X"'] & E \wedge X \end{tikzcd}$$

Since $E \wedge W$ is contractible, f is null. □

DEFINITION 6.1.6. For a ring spectrum E, the class of *E-nilpotent* spectra is the smallest class satisfying the following conditions.

(i) E is E-nilpotent.
(ii) If N is E-nilpotent then so is $N \wedge X$ for any X.
(iii) The cofibre of any map between E-nilpotent spectra is E-nilpotent.
(iv) Any retract of an E-nilpotent spectrum is E-nilpotent.

A spectrum is *E-prenilpotent* if it is E_*-equivalent to an E-nilpotent one.

The definition of an E-nilpotent spectrum generalizes the notion of a finite Postnikov system; we replace Eilenberg–MacLane spectra by retracts of smash products $E \wedge X$. The following ([13, 3.8]) is an easy consequence of 6.1.5.

PROPOSITION 6.1.7. *Every E-nilpotent spectrum is E_*-local.*

6.2. *Bousfield equivalence*

Recall the smash product $X \wedge Y$ was defined in 5.1.2 and the wedge $X \vee Y$ was defined in 3.1.1.

DEFINITION 6.2.1. Two spectra E and F are *Bousfield equivalent* if for each spectrum X, $E \wedge X$ is contractible if and only if $F \wedge X$ is contractible. The Bousfield equivalence class of E is denoted by $\langle E \rangle$.

$\langle E \rangle \geqslant \langle F \rangle$ if for each spectrum X, the contractibility of $E \wedge X$ implies that of $F \wedge X$. We say $\langle E \rangle > \langle F \rangle$ if $\langle E \rangle \geqslant \langle F \rangle$ but $\langle E \rangle \neq \langle F \rangle$.

$\langle E \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle$ and $\langle E \rangle \vee \langle F \rangle = \langle E \vee F \rangle$. (We leave it to the reader to verify that these classes are well defined.)

A class $\langle E\rangle$ has a *complement* $\langle E\rangle^c$ if $\langle E\rangle \wedge \langle E\rangle^c = \langle \text{pt.}\rangle$ and $\langle E\rangle \vee \langle E\rangle^c = \langle S^0\rangle$, where S^0 is the sphere spectrum.

The operations $\wedge$ and $\vee$ satisfy the obvious distributive laws, namely

$$\big(\langle X\rangle \vee \langle Y\rangle\big) \wedge \langle Z\rangle = \big(\langle X\rangle \wedge \langle Z\rangle\big) \vee \big(\langle Y\rangle \wedge \langle Z\rangle\big)$$

and

$$\big(\langle X\rangle \wedge \langle Y\rangle\big) \vee \langle Z\rangle = \big(\langle X\rangle \vee \langle Z\rangle\big) \wedge \big(\langle Y\rangle \vee \langle Z\rangle\big).$$

The following result is an immediate consequence of the definitions.

PROPOSITION 6.2.2. *The localization functors L_E and L_F are the same if and only if $\langle E\rangle = \langle F\rangle$. If $\langle E\rangle \leqslant \langle F\rangle$ then $L_E L_F = L_E$ and there is a natural transformation $L_F \to L_E$.*

Notice that for any spectrum E,

$$\begin{aligned}
&\langle S^0\rangle \geqslant \langle E\rangle \geqslant \langle \text{pt.}\rangle,\\
&\langle S^0\rangle \wedge \langle E\rangle = \langle E\rangle,\\
&\langle S^0\rangle \vee \langle E\rangle = \langle S^0\rangle,\\
&\langle \text{pt.}\rangle \vee \langle E\rangle = \langle E\rangle, \quad \text{and}\\
&\langle \text{pt.}\rangle \wedge \langle E\rangle = \langle \text{pt.}\rangle,
\end{aligned}$$

i.e. $\langle S^0\rangle$ is the biggest class and $\langle \text{pt.}\rangle$ is the smallest.

Not all classes have complements, and there are even classes $\langle E\rangle$ which do not satisfy

$$\langle E\rangle \wedge \langle E\rangle = \langle E\rangle. \tag{6.2.3}$$

The following definition is due to Bousfield [12].

DEFINITION 6.2.4. **A** is the collection of all Bousfield classes. **DL** (for distributive lattice) is the collection of classes satisfying (6.2.3). **BA** (for Boolean algebra) is the collection of classes with complements.

Thus we have

$$\mathbf{BA} \subset \mathbf{DL} \subset \mathbf{A}$$

and both inclusions are proper. (Counterexamples illustrating this can be found in [58].) If E is connective then $\langle E\rangle \in \mathbf{DL}$, and if E is a (possibly infinite) wedge of finite complexes, then $\langle E\rangle \in \mathbf{BA}$. A partial description of **BA** is given below in 6.2.9.

Let $S^0\mathbf{Q}$ denote the rational sphere spectrum, $S^0_{(p)}$ the p-local sphere spectrum, and $S^0/(p)$ the mod p Moore spectrum. Then we have

PROPOSITION 6.2.5.

$$\begin{aligned} \langle S^0_{(p)}\rangle &= \langle S^0\mathbf{Q}\rangle \vee \langle S^0/(p)\rangle, \\ \langle S^0\mathbf{Q}\rangle \wedge \langle S^0/(p)\rangle &= \langle \mathrm{pt.}\rangle, \\ \langle S^0/(q)\rangle \wedge \langle S^0/(p)\rangle &= \langle \mathrm{pt.}\rangle \quad \textit{for } p \neq q, \textit{ and} \\ \langle S^0\rangle &= \langle S^0\mathbf{Q}\rangle \vee \bigvee_p \langle S^0/(p)\rangle. \end{aligned}$$

In particular each of these classes is in **BA**.

The following result is proved in [58].

PROPOSITION 6.2.6. (i) *If*

$$W \longrightarrow X \xrightarrow{f} Y \longrightarrow \Sigma W$$

is a cofibre sequence (3.3.3), *then*

$$\langle W\rangle \leqslant \langle X\rangle \vee \langle Y\rangle.$$

(ii) *If f is smash nilpotent* (5.1.2) *then*

$$\langle W\rangle = \langle X\rangle \vee \langle Y\rangle.$$

(iii) *For a self-map $f\colon \Sigma^d X \to X$, let C_f denote its cofibre and let*

$$\hat{X} = \varinjlim_i \Sigma^{-id} X$$

be the telescope obtained by iterating f. Then

$$\begin{aligned} \langle X\rangle &= \langle \hat{X}\rangle \vee \langle C_f\rangle \quad \textit{and} \\ \langle \hat{X}\rangle \wedge \langle C_f\rangle &= \langle \mathrm{pt.}\rangle. \end{aligned}$$

Two pleasant consequences of the thick subcategory theorem (4.4.3) are the following, which were the class invariance and Boolean algebra conjectures of [58].

THEOREM 6.2.7 (Class invariance theorem). *Let X and Y be p-local finite CW-complexes of types m and n respectively* (2.5.3). *Then $\langle X\rangle = \langle Y\rangle$ if and only if $m = n$, and $\langle X\rangle < \langle Y\rangle$ if and only if $m > n$.*

PROOF. Let $\mathbf{C}_X$ and $\mathbf{C}_Y$ be the smallest thick subcategories of $\mathbf{FH}_{(p)}$ containing X and Y respectively. In other words, $\mathbf{C}_X$ contains all finite complexes which can be built up from X by cofibrations and retracts. Hence each X' in $\mathbf{C}_X$ satisfies

$$\langle X'\rangle \leqslant \langle X\rangle.$$

Since $K(m-1)_*(X) = 0$, all complexes in $\mathbf{C}_X$ are $K(m-1)_*$-acyclic, so $\mathbf{C}_X$ is contained in $\mathbf{F}_m$. On the other hand, $\mathbf{C}_X$ is not contained in $\mathbf{F}_{m+1}$ since $K(m)_*(X) \neq 0$. Hence $\mathbf{C}_X$ must be $\mathbf{F}_m$ by the thick subcategory theorem. Similarly, $\mathbf{C}_Y = \mathbf{F}_n$.

It follows that if $m = n$ then $\mathbf{C}_X = \mathbf{C}_Y$ so $\langle X\rangle = \langle Y\rangle$ as claimed. The inequalities follow similarly. □

For a p-local finite CW-complex X_n of type n (2.5.3), the periodicity theorem (2.5.4) says there is a v_n-map $f: \Sigma^d X_n \to X_n$. We define the telescope $\hat{X}_n$ to be the direct limit of the system

$$X_n \xrightarrow{f} \Sigma^{-d} X_n \xrightarrow{f} \Sigma^{-2d} X_n \xrightarrow{f} \cdots \tag{6.2.8}$$

Since any two choices of f agree up to iteration (5.4.3), this telescope is independent of the choice of f. Moreover, 6.2.7 implies that its Bousfield classes $\langle X_n\rangle$ and $\langle \hat{X}_n\rangle$ are independent of the choice of X_n, for a fixed n and p.

THEOREM 6.2.9 (Boolean algebra theorem). *Let* $\mathbf{FBA} \subset \mathbf{BA}$ *be the Boolean subalgebra generated by finite spectra and their complements, and let* $\mathbf{FBA}_{(p)} \subset \mathbf{FBA}$ *denote the subalgebra of p-local finite spectra and their complements in* $\langle S^0_{(p)}\rangle$. *Then* $\mathbf{FBA}_{(p)}$ *is the free (under complements, finite unions and finite intersections) Boolean algebra generated by the classes of the telescopes* $\langle \hat{X}_n\rangle$ *defined above for* $n \geqslant 0$. *In particular, the classes represented by finite spectra are*

$$\langle X_n\rangle = \bigwedge_{0\leqslant i<n} \langle \hat{X}_i\rangle^c.$$

In other words $\mathbf{FBA}_{(p)}$ *is isomorphic to the Boolean algebra of finite and cofinite sets of natural numbers, with* $\langle \hat{X}_n\rangle$ *corresponding to the set* $\{n\}$.

Note that this is very similar to the Boolean algebra conjecture of [58, 10.8], in which $\langle \hat{X}_n\rangle$ was replaced by $\langle K(n)\rangle$. The recently disproved telescope conjecture (6.5.5) says that these two classes are the same, and 6.2.9 is phrased so that it is independent of 6.5.5.

PROOF OF 6.2.9. 6.2.6(iii) gives

$$\langle X_n\rangle = \langle \hat{X}_n\rangle \vee \langle X_{n+1}\rangle \quad \text{and}$$
$$\langle \hat{X}_n\rangle \wedge \langle X_{n+1}\rangle = \langle \text{pt.}\rangle.$$

This implies that

$$\langle \hat{X}_n\rangle = \langle \hat{X}_n\rangle \wedge \langle X_{n+1}\rangle^c,$$

so $\mathbf{FBA}_{(p)}$ contains the indicated Boolean algebra.

On the other hand, 6.2.6(iii) also implies that

$$\langle S^0_{(p)}\rangle = \langle X_0\rangle = \langle X_n\rangle \vee \bigvee_{0\leqslant i<n} \langle \hat{X}_i\rangle,$$

from which the identification of $\langle X_n \rangle$ follows. Hence the indicated Boolean algebra contains $\mathbf{FBA}_{(p)}$. □

6.3. *The structure of* $\langle MU \rangle$

The spectrum MU is described in [61, B.2]. It is known that its p-localization $MU_{(p)}$ splits into a wedge of suspensions of a 'smaller' spectrum BP, which is described in [61, B.5]. It follows that $\langle MU_{(p)} \rangle = \langle BP \rangle$ and 6.2.5 implies that

$$\langle MU \rangle = \bigvee_p \langle MU_{(p)} \rangle = \bigvee_p \langle BP \rangle$$

where the wedge on the right is over the BP's associated with the various primes p.

The class $\langle BP \rangle$ can be broken up further in terms of various spectra related to BP. A detailed account of this can be found in Section 2 of [58]. The relevant spectra for our purposes are all module spectra (see [61, A.2.8]) over BP, which means that they are characterized by the structure of their homotopy groups as modules over BP_*. First we have $P(n)$ with

$$\pi_*\big(P(n)\big) = BP_*/I_n.$$

In particular $P(0)$ is BP by definition. Würgler [81] has shown that each $P(n)$ is a ring spectrum. Using the construction of [61, A.2.10], we can form the telescopes $v_n^{-1}BP$ and $v_n^{-1}P(n)$, which is denoted in the literature by $B(n)$. Closely related to these are $E(n)$ and $K(n)$ (Morava K-theory) with

$$\begin{aligned} &E(n)_* = \mathbf{Z}_{(p)}\big[v_1, v_2, \dots, v_n, v_n^{-1}\big] \quad \text{and} \\ &K(n)_* = \mathbf{Z}/(p)\big[v_n, v_n^{-1}\big]. \end{aligned} \tag{6.3.1}$$

Finally we have $H/(p)$ the mod p Eilenberg–MacLane spectrum representing ordinary mod p homology.

The following result was proved in [58].

THEOREM 6.3.2. *With notation as above:*

(a) $\langle B(n) \rangle = \langle K(n) \rangle$.
(b) $\langle v_n^{-1}BP \rangle = \langle E(n) \rangle$.
(c) $\langle P(n) \rangle = \langle K(n) \rangle \vee \langle P(n+1) \rangle$.
(d) $\langle E(n) \rangle = \bigvee_{i=0}^n \langle K(i) \rangle$.
(e) $\langle K(m) \rangle \wedge \langle K(n) \rangle = \langle \text{pt.} \rangle$ *for* $m \neq n$ *and* $\langle H/(p) \rangle \wedge \langle K(n) \rangle = \langle \text{pt.} \rangle$.
(f) *For* $E = K(n)$ *or* $E = H/(p)$ *and for any* X, $\langle X \rangle \wedge \langle E \rangle$ *is either* $\langle E \rangle$ *or* $\langle \text{pt.} \rangle$.

6.4. *Some classes bigger than* $\langle MU\rangle$

For some time after conjecturing the nilpotence theorem, we tried to prove it by showing that $\langle MU\rangle = \langle S^0\rangle$. Eventually we disproved the latter by producing a nontrivial spectrum X with $MU_*(X) = 0$. The main tool in this construction is Brown–Comenetz duality, which was introduced in [15]. Their main result is the following.

THEOREM 6.4.1 (Brown–Comenetz duality theorem). *Let Y be a spectrum with finite homotopy groups. Then there is a spectrum cY (the Brown–Comenetz dual of Y) such that for any spectrum X,*

$$[X, cY]_{-i} = \mathrm{Hom}(\pi_i(X \wedge Y), \mathbf{R}/\mathbf{Z}).$$

In particular, $\pi_{-i}(cY) = \mathrm{Hom}(\pi_i(Y), \mathbf{R}/\mathbf{Z})$ and $cH/(p) = H/(p)$. Moreover c is a contravariant functor on spectra with finite homotopy groups which preserves cofibre sequences and satisfies $ccY = Y$.

From this it easily follows that if $[X, cY] = 0$, then $\pi_*(X \wedge ccY) = \pi_*(X \wedge Y) = 0$. Replacing Y by cY we see that if $[X, Y] = 0$ then $\pi_*(X \wedge cY) = 0$. Now if Y is a finite complex with trivial rational homology and $X = MU$, one can show by Adams spectral sequence methods that $[X, Y] = 0$, so we conclude that

PROPOSITION 6.4.2. *If Y is a finite complex with trivial rational homology then $MU_*(cY) = 0$.*

More details can be found in [58].

The existence of a nontrivial spectrum cY with $MU_*(cY) = 0$ means that $\langle MU\rangle < \langle S^0\rangle$.

Actually the situation is more drastic, as the following result (also proved in [58]) indicates.

THEOREM 6.4.3. *There are spectra $X(n)$ for $1 \leqslant n \leqslant \infty$ with $X(1) = S^0$ and $X(\infty) = MU$ such that*

$$\langle X(n)\rangle \geqslant \langle X(n+1)\rangle$$

for each n, with

$$\langle X(p^k - 1)_{(p)}\rangle > \langle X(p^k)_{(p)}\rangle$$

for each prime p and each $k \geqslant 0$.

The spectra $X(n)$ also figure in the proof of the nilpotence theorem, so we will describe them now. They are constructed in terms of vector bundles and Thom spectra. Some of the relevant background is given in [61, B.1]. Let SU denote the infinite special unitary group, i.e. the union of all the $SU(n)$'s. The Bott periodicity theorem gives us a homotopy equivalence

$$\Omega SU \longrightarrow BU$$

where BU is the classifying space of the infinite unitary group. Composing this with the loops on the inclusion of $SU(n)$ into SU, we get a map

$$\Omega SU(n) \longrightarrow BU.$$

The associated Thom spectrum is $X(n)$. A routine calculation gives

$$H_*(X(n)) = \mathbf{Z}[b_1, \dots, b_{n-1}]$$

where $|b_i| = 2i$ and these generators map to generators of the same name in $H_*(MU)$ as described in [61, B.1.15].

6.5. $E(n)$-*localization and the chromatic filtration*

Bousfield's theorem gives us a lot of interesting localization functors. Experience has shown that the case $E = E(n)$ (6.3.1), or equivalently (by 6.3.2(b)) $v_n^{-1}BP$, is particularly useful.

DEFINITION 6.5.1. L_nX is $L_{E(n)}X$ and C_nX denotes the fibre of the map $X \to L_nX$.

The following result enables us to compute $BP_*(L_nX)$ in terms of $BP_*(X)$.

THEOREM 6.5.2 (Localization theorem). *For any spectrum* Y,

$$BP \wedge L_nY = Y \wedge L_nBP.$$

In particular, if $v_{n-1}^{-1}BP_*(Y) = 0$, *then*

$$BP \wedge L_nY = Y \wedge v_n^{-1}BP,$$

i.e. $BP_*(L_nY) = v_n^{-1}BP_*(Y)$.

The proof of this theorem and a description of L_nBP will is given in [61, Chapter 8]. Using 6.2.2 and 6.3.2(d) we get a natural transformation $L_n \to L_{n-1}$.

DEFINITION 6.5.3. The *chromatic tower* for a p-local spectrum X is the inverse system

$$L_0X \longleftarrow L_1X \longleftarrow L_2X \longleftarrow \cdots X.$$

The *chromatic filtration* of $\pi_*(X)$ is given by the subgroups

$$\ker(\pi_*(X) \to \pi_*(L_nX)).$$

This definition of the chromatic filtration is *not* obviously the same as the one given in 3.5.2, which was in terms of periodic maps of finite complexes. The two definitions are equivalent if the telescope conjecture 6.5.5 is true. We will refer to these as the geometric (3.5.2) and algebraic (6.5.3) definitions of the chromatic filtration.

The geometric definition is the more natural of the two. The advantage of the algebraic one is that there are methods of computing $\pi_*(L_nX)$. In particular, suppose X is a p-local finite CW-complex of type n (2.5.3) with v_n-map f. Let $\hat{X}$ be the telescope as in (6.2.8). Then $K(n)_*(f)$ is an isomorphism. The same is true of $K(i)_*(f)$ for $i < n$ since $K(i)_*(X) = 0$. Hence $E(n)_*(f)$ is an equivalence by 6.3.2(d). This means that the map $X \to L_nX$ factors uniquely through the telescope $\hat{X}$, i.e. we have a map

$$\hat{X} \xrightarrow{\lambda} L_nX. \tag{6.5.4}$$

Moreover

$$BP_*(L_nX) = v_n^{-1}BP_*(X)$$

and λ is a BP_*-equivalence.

CONJECTURE 6.5.5 (Telescope conjecture). *Let X be a p-local finite CW-complex of type n. Then the map λ of* (6.5.4) *is an equivalence.*

For $n = 0$ this statement is a triviality. The map f can be taken to be the degree p map and it is clear that $\hat{X} = L_0$ for any p-local spectrum X.

For general n it is clear that the collection of p-local type n finite complexes satisfying 6.5.5 is thick, so by the thick subcategory theorem it suffices to prove or disprove it for a single such complex. For $n = 1$, it was proved for the mod p Moore spectrum by Mahowald [44] for $p = 2$ and by Haynes Miller [47] for $p > 2$. The author has recently disproved it for the type 2 complex $V(1)$ for $p \geqslant 5$; see [62] and [63]. In light of this, there is no reason to think it is true for $n > 2$.

Now the v_n-torsion subgroup of $\pi_*(X)$ as defined geometrically in 3.5.2 is the kernel of the map to $\pi_*(\hat{X})$, while the corresponding subgroup defined algebraically by 6.5.3 is the kernel of the map to $\pi_*(L_nX)$. These two subgroups would be the same if the telescope conjecture were true.

What can we say when the telescope conjecture is false? The existence of the map λ of (6.5.4) means that the algebraically defined subgroup contains the geometrically defined one. However we do not know that $\pi_*(\lambda)$ is either one-to-one or onto.

The localization L_nX is much better understood than the telescope $\hat{X}$. It was shown in [60] that in general $\pi_*(L_nY)$ can be computed with the Adams–Novikov spectral sequence. This is particularly pleasant in the case of a type n finite complex X. In that case there is some nice algebraic machinery for computing the E_2-term of the Adams–Novikov spectral sequence. *Indeed, it was this computability that motivated this whole program in the first place.*

We will illustrate first with the simplest possible example. Suppose our finite complex X is such that

$$BP_*(X) \cong BP_*/I_n.$$

Then $BP_*(L_nX) = v_n^{-1}BP_*/I_n$ and the E_2-term is

$$\mathrm{Ext}_{BP_*(BP)}\big(BP_*, v_n^{-1}BP_*/I_n\big).$$

This is known to be essentially the mod p continuous cohomology of the n-th Morava stabilizer group S_n, described in 4.6. This isomorphism is the subject of [59, Chapter 6] and a more precise statement (which would entail a distracting technical digression here) can be found there and in the change-of-rings isomorphism of [61, B.8.8].

More generally, if X is a p-local finite complex of type n, then the Landweber filtration theorem 4.3.7 tells us that $BP_*(X)$ has a finite filtration in which each subquotient is a suspension of BP_*/I_{n+i} for $i \geqslant 0$. When we pass to $v_n^{-1}BP_*(X)$, we lose the subquotients with $i > 0$ and the remaining ones get converted to suspensions of $v_n^{-1}BP_*/I_n$.

Hence the Landweber filtration leads to a spectral sequence for computing the Adams–Novikov spectral sequence E_2-term,

$$\mathrm{Ext}_{BP_*(BP)}\big(BP_*, BP_*(L_nX)\big),$$

in terms of $H^*(S_n)$. It is possible to formulate its E_2-term as the cohomology of S_n with suitable twisted coefficients.

Finally, we remark that the nature of the functor L_n is partially clarified by the following.

THEOREM 6.5.6 (Smash product theorem). *For any spectrum X,*

$$L_nX \cong X \wedge L_nS^0.$$

This will be proved in [61, Chapter 8]. We should point out here that in general L_EX is *not* equivalent to $X \wedge L_ES^0$. Here is a simple example. Let $E = H$, the integer Eilenberg–MacLane spectrum. Then it is easy to show that $L_HS^0 = S^0$. On the other hand we have seen examples (6.1.4(ii)) of nontrivial Y for which L_H is contractible, so

$$L_HX \ncong X \wedge L_HS^0$$

in general.

The smash product theorem is a special property of the functors L_n. They may be the only localization functors with this property. The spectrum L_1S^0 is well understood; its homotopy groups are given in [58]. Its connective cover is essentially (precisely at odd primes) the spectrum J. $\pi_*(L_nS^0)$ is not known for any larger value of n. The computations of Shimomura–Tamura ([67] and [68]) determine $\pi_*(L_2V(0))$ for $p \geqslant 5$, where $V(0)$ denotes the mod p Moore spectrum.

A consequence of the smash product theorem is the following, which is also proved in [61, Chapter 8].

THEOREM 6.5.7 (Chromatic convergence theorem). *For a p-local finite CW-complex X, the chromatic tower of* 6.5.3 *converges in the sense that*

$$X \simeq \varprojlim L_n X.$$

7. The proof of the nilpotence theorem

In this section we will outline the proof of the nilpotence theorem; a more detailed account is given in [61, Chapter 9]. We have previously stated it in two different guises, in terms of self-maps (2.4.2) and in terms of smash products (5.1.4). For our purposes here it is convenient to give a third statement, namely

THEOREM 7.0.1 (Nilpotence theorem, ring spectrum form). *Let R be a connective ring spectrum of finite type and let*

$$\pi_*(R) \xrightarrow{h} MU_*(R)$$

be the Hurewicz map ([61, A.3.4]). *Then $\alpha \in \pi_*(R)$ is nilpotent if $h(\alpha) = 0$.*

In [22] it is shown that the two previous statements are consequences of the one above. To show that 7.0.1 implies 2.4.2, let X be a finite complex and let $R = DX \wedge X$. Recall that a self-map $f\colon \Sigma^d X \to X$ is adjoint to a map $\hat{f}\colon S^d \to R$. We claim that $h(\hat{f})$ is nilpotent if $MU_*(f)$ is.

To see this, observe that if $MU_*(f) = 0$, then $MU \wedge f^{-1}X$ is contractible, where $f^{-1}X$ denotes the homotopy direct limit of

$$X \xrightarrow{f} \Sigma^{-d} X \xrightarrow{f} \Sigma^{-2d} X \xrightarrow{f} \cdots.$$

Since X is finite, this means that for large enough m, the composite

$$\Sigma^{md} X \xrightarrow{f^m} X \longrightarrow MU \wedge X$$

is null. Then $h(\widehat{f^m}) = h(\hat{f})^m = 0$, so $h(\hat{f})$ is nilpotent.

Theorem 2.4.2 is a special case of the following statement, which is derived from 7.0.1 in [22]. Suppose we have a sequence of maps of CW-spectra

$$\cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \longrightarrow \cdots$$

with $MU_*(f_n) = 0$ for each n, and suppose there are constants $m \leqslant 0$ and b such that each X_n is $(mn+b)$-connected. Then the homotopy direct limit $\varinjlim X_n$ is contractible.

PROOF THAT 7.0.1 IMPLIES 5.1.4. (In the following argument, MU could be replaced by any ring spectrum for which 7.0.1 holds.) In the former we are given a map

$$F \xrightarrow{f} X$$

with F finite. It is adjoint to a map

$$S^0 \xrightarrow{\hat{f}} X \wedge DF$$

where DF is the Spanier–Whitehead dual (5.2.1) of F. Now f is smash nilpotent if and only if $\hat{f}$ is, and $MU \wedge f$ is null if and only if $MU \wedge \hat{f}$ is.

This means that it suffices to prove 5.1.4 for the case $F = S^0$. The hypothesis that $MU \wedge f$ is null is equivalent (since MU is a ring spectrum) to the assumption that the composite

$$S^0 \xrightarrow{f} X \xrightarrow{\quad \eta \wedge X \quad} MU \wedge X$$

is null. Since X is a homotopy direct limit of finite subspectra X_α ([61, A.5.8]), both the map f and the null homotopy for the composite above factor through some finite X_α, i.e. we have

$$S^0 \xrightarrow{f} X_\alpha \xrightarrow{\quad \eta \wedge X_\alpha \quad} MU \wedge X_\alpha$$

and the composite is null.

Now let $Y = \Sigma^n X_\alpha$, where n is chosen so that Y is 0-connected. Let

$$R = \bigvee_{j \geqslant 0} Y^{(j)};$$

this is a connective ring spectrum of finite type with multiplication given by concatenation. Theorem 7.0.1 tells us that the element in $\pi_*(R)$ corresponding to f is nilpotent. This means that f itself is smash nilpotent, thereby proving Theorem 5.1.4. □

7.1. *The spectra $X(n)$*

Recall the spectrum $X(n)$ of 6.4.3, the Thom spectrum associated with $\Omega SU(n)$. It is a ring spectrum so we have a Hurewicz map

$$\pi_*(R) \xrightarrow{\quad h(n) \quad} X(n)_*(R).$$

In particular $X(1) = S^0$ so $h(1)$ is the identity map. The map $X(n) \to MU$ is a homotopy equivalence through dimension $2n - 1$. It follows that if $h(\alpha) = 0$, then $h(n)(\alpha) = 0$ for large n. Hence, the nilpotence theorem will follow from

THEOREM 7.1.1. *With notation as above, if $h(n+1)(\alpha) = 0$ then $h(n)(\alpha)$ is nilpotent.*

In order to prove this we need to study the spectra $X(n)$ more closely. Consider the diagram

$$\begin{tikzcd} \Omega SU(n) \arrow[r] & \Omega SU(n+1) \arrow[r] & \Omega S^{2n+1} \\ \Omega SU(n) \arrow[u] \arrow[r] & B_k \arrow[u] \arrow[r] & J_k S^{2n} \arrow[u] \end{tikzcd} \tag{7.1.2}$$

in which each row is a fibration. The top row is obtained by looping the fibration

$$SU(n) \longrightarrow SU(n+1) \xrightarrow{e} S^{2n+1}$$

where e is the evaluation map which sends a matrix $m \in SU(n+1)$ to mu where $u \in \mathbf{C}^{n+1}$ is fixed unit vector.

The loop space ΩS^{2n+1} was analyzed by James [33] and shown to be homotopy equivalent to a CW-complex with one cell in every dimension divisible by $2n$. $J_k S^{2n}$ denotes the k^{th} space in the James construction on S^{2n}, which is the same thing as the $2nk$-skeleton of ΩS^{2n+1}. It can also be described as a certain quotient of the Cartesian product $(S^{2n})^k$. The space B_k is the pullback, i.e. the $\Omega SU(n)$-bundle over $J_k S^{2n}$ induced by the inclusion map into ΩS^{2n+1}.

PROPOSITION 7.1.3. $H_*(\Omega SU(n)) = \mathbf{Z}[b_1, b_2, \ldots, b_{n-1}]$ *with* $|b_i| = 2i$, *and*

$$H_*(B_k) \subset H_*\bigl(\Omega SU(n+1)\bigr)$$

is the free module over it generated by b_n^i for $0 \leqslant i \leqslant k$.

Now the composite map

$$B_k \longrightarrow \Omega SU(n+1) \longrightarrow BU \tag{7.1.4}$$

gives a stable bundle over B_k and we denote the Thom spectrum by F_k. Thus we have $F_0 = X(n)$ and $F_\infty = X(n+1)$. We will be especially interested in F_{p^j-1}, which we will denote by G_j. These spectra interpolate between $X(n)$ and $X(n+1)$.

The following three lemmas clearly imply 7.1.1 and hence the nilpotence theorem. We will prove the first two.

LEMMA 7.1.5 (First lemma). *Let $\alpha^{-1}R$ be the telescope associated with $\alpha \in \pi_*(R)$. If $\alpha^{-1}R \wedge X(n)$ is contractible then $h(n)_*(\alpha)$ is nilpotent.*

LEMMA 7.1.6 (Second lemma). *If $h(n+1)(\alpha) = 0$ then $G_j \wedge \alpha^{-1}R$ is contractible for large j.*

The following is the hardest of the three and is the heart of the nilpotence theorem. We refer the reader to [61, Chapter 9] for the proof.

LEMMA 7.1.7 (Third lemma). *For each $j > 0$, $\langle G_j \rangle = \langle X(n) \rangle$. In particular $\langle G_j \rangle = \langle G_{j+1} \rangle$.*

PROOF OF THEOREM 7.1.1. We will now prove 7.1.1 assuming the three lemmas above. If $h(n+1)(\alpha) = 0$, then the telescope $\alpha^{-1}R \wedge G_j$ is contractible by 7.1.6. By 7.1.7 this means that $\alpha^{-1}R \wedge X(n)$ is also contractible. By 7.1.5, this means that $h(n)(\alpha)$ is nilpotent as claimed. □

7.2. *The proofs of the first two lemmas*

First we will prove 7.1.5. The map $\alpha: S^d \to R$ induces a self-map

$$\Sigma^d R \xrightarrow{\alpha} R.$$

The spectrum $\alpha^{-1}R \wedge X(n)$ is by definition the homotopy direct limit of

$$R \wedge X(n) \xrightarrow{\alpha\wedge X(n)} \Sigma^{-d}R \wedge X(n) \xrightarrow{\alpha\wedge X(n)} \cdots$$

It follows that each element of $X(n)_*(R)$, including $h(n)(\alpha)$, is annihilated after a finite number of steps, so $h(n)(\alpha)$ is nilpotent.

We will now outline the proof of 7.1.6. It requires the use of the Adams spectral sequence for a generalized homology theory. It is briefly introduced in [61, A.6], and a more thorough account is given in [59]. Fortunately all we require of it here is certain formal properties; we will not have to make any detailed computations.

We need to look at the Adams spectral sequence for $\pi_*(Y)$ based on $X(n+1)$-theory, for $Y = R \wedge G_j$, G_j and R. They have the following properties:

(i) The E_2-term, $E_2^{s,t}(Y)$ can be identified with a certain Ext group related to $X(n+1)$-theory, namely

$$\mathrm{Ext}^{s,t}_{X(n+1)_*(X(n+1))}\big(X(n+1)_*, X(n+1)_*(Y)\big).$$

This follows from the fact (proven in [22]) that $X(n+1)$ is a flat ring spectrum ([61, A.2.9]).

(ii) $E_2^{s,t}(Y)$ vanishes unless s is non-negative and $t-s$ exceeds the connectivity of Y

(iii) α corresponds to an element $x \in E_2^{s,s+d}(R)$ for some $s > 0$. This follows from the fact ([61, A.6.5]) that $h(n+1)(\alpha) = 0$. The group of permanent cycle in $E_2^{0,*}(Y)$ is precisely the Hurewicz image of $\pi_*(Y)$ in $X(n+1)_*(Y)$.

In addition we have the following property.

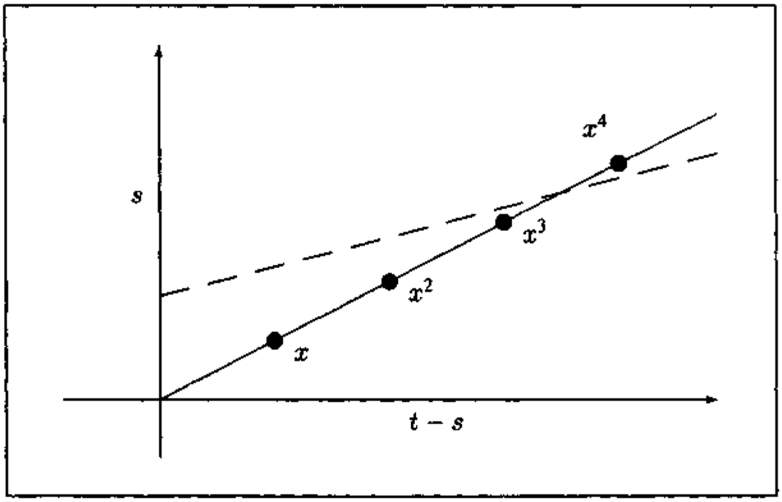
LEMMA 7.2.1. *$E_2^{s,t}(G_j)$ and $E_2^{s,t}(R \wedge G_j)$ vanish for all (s,t) above a certain line of slope*

$$\frac{1}{2p^j n - 1}.$$

(*This is called a* vanishing line.)

We will prove this at the end of this section.

The situation is illustrated in the following picture, which is intended to illustrate $E_2^{s,t}(R \wedge G_j)$. As usual the horizontal and vertical coordinates are $t-s$ and s respectively. The powers of x all lie on a line through the origin with slope s/d. The broken line represents the vanishing line for E_2. $E_2^{s,t} = 0$ for all points (s,t) above it. For large enough j, the vanishing line has slope less than s/d and the two lines intersect as shown. It follows that x and hence $\alpha \wedge G_j$ are nilpotent, thereby proving 7.1.6.



PROOF OF LEMMA 7.2.1. We will construct a noncanonical $X(n+1)$-based Adams resolution for G_j, i.e. a diagram of the form

$$\begin{tikzcd} G_j = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \cdots \arrow[l] \\ K_0 & K_1 & K_2 & \end{tikzcd} \tag{7.2.2}$$

as in [61, A.6.1], such that the spectrum K_s is $(2sp^j n - s)$-connected. This will give the desired vanishing line for $E_2(G_j)$. We can get a similar resolution for $R \wedge G_j$ by smashing (7.2.2) with R, thereby proving the vanishing line for $E_2(R \wedge G_j)$.

Recall that G_j is the p-local Thom spectrum of the bundle over B_{p^j-1}, which is the pullback of the fibre square

$$\begin{tikzcd}
B_{p^j-1} \arrow[r, "i_0"] \arrow[d] & \Omega SU(n+1) \arrow[r, "f"] \arrow[d] & \Omega S^{2p^j n+1} \arrow[d] \\
J_{p^j-1}S^{2n} \arrow[r, "i"] & \Omega S^{2n+1} \arrow[r, "H"] & \Omega S^{2p^j n+1}
\end{tikzcd} \tag{7.2.3}$$

The space $J_{p^j-1}S^{2n}$ is known (after localizing at p) to be fibre of the Hopf map H as shown. It follows that the same can be said of B_{p^j-1}.

The map f_0 of (7.2.2) is the Thomification of the map i_0 of (7.2.3). We will obtain the other maps f_s of (7.2.2) in a similar way. Let

$$\begin{aligned}
Y_0 &= B_{p^j-1},\\
L_0 &= \Omega SU(n+1),\\
Y_1 &= C_{i_0}.
\end{aligned}$$

For $s \geqslant 0$ we will construct cofibre sequences

$$Y_s \longrightarrow L_s \longrightarrow Y_{s+1} \tag{7.2.4}$$

which will Thomify to

$$\Sigma^s X_s \xrightarrow{f_s} \Sigma^s K_s \longrightarrow \Sigma^{s+1} X_{s+1} \tag{7.2.5}$$

where K_s is a wedge of suspensions of $X(n+1)$ with the desired connectivity.

Our definitions of Y_s and L_s are rather long winded. For simplicity let

$$\begin{aligned}
X &= B_{p^j-1},\\
E &= \Omega SU(n+1),\\
B &= \Omega S^{2p^j n+1}
\end{aligned}$$

and for $s \geqslant 0$ let

$$G_s = E \times \overbrace{B \times \cdots \times B}^{s \text{ factors}}.$$

Define maps $i_t : G_s \to G_{s+1}$ for $0 \leqslant t \leqslant s+1$ by

$$i_t(e, b_1, b_2, \cdots, b_s) = \begin{cases} (e, b_1, b_2, \cdots, b_s, *) & \text{if } t = 0, \\ (e, b_1, b_2, \cdots, b_t, b_t, b_{t+1}, \cdots b_s) & \text{if } 1 \leqslant t \leqslant s, \\ (e, f(e), b_1, b_2, \cdots, b_s) & \text{if } t = s+1. \end{cases}$$

(The astute reader will recognize this as the cosimplicial construction associated with the Eilenberg-Moore spectral sequence, due to Larry Smith [69] and Rector [65].)

Then for $s \geqslant 1$ we define

$$Y_s = G_{s-1}/\mathrm{im}\ i_0 \cup \mathrm{im}\ i_1 \cup \cdots \cup \mathrm{im}\ i_{s-1},$$
$$L_s = G_s/\mathrm{im}\ i_0 \cup \mathrm{im}\ i_1 \cup \cdots \cup \mathrm{im}\ i_{s-1}.$$

Then for $s \geqslant 0$, i_s induces a map $Y_s \to L_s$ giving the cofibre sequences of (7.2.4). For $s > 0$ there are reduced homology isomorphisms

$$\overline{H}_*(Y_s) = H_*(X) \otimes \overline{H}_*\big(B^{(s)}\big),$$
$$\overline{H}_*(L_s) = H_*(E) \otimes \overline{H}_*\big(B^{(s)}\big).$$

This shows L_s has the desired connectivity.

Projection onto the first coordinate gives compatible maps of the G_s to E, and hence a stable vector bundle over each of them. This means that we can Thomify the entire construction. We get the cofibre sequences (7.2.5) defining the desired Adams resolution by Thomifying (7.2.4). □

One can also prove this result by more algebraic methods by finding a vanishing line for the corresponding Ext groups; this is the approach taken in [22]. The slope one obtains is

$$\frac{1}{p^{j+1}n - 1}$$

which is roughly $2/p$ times the slope obtained above. In particular there is an element

$$b_{n,j} \in \mathrm{Ext}^{2,2p^{j+1}n}$$

which is closely related to a self-map of G_j to be defined below (7.3.3). (Proving the third lemma amounts to showing that this map is nilpotent.) All that we need to know about the slope here is that it can be made arbitrarily small by increasing n.

7.3. *The proof of the third lemma*

In this subsection we will outline the proof of the third lemma, 7.1.7. We need to show that $\langle G_j \rangle = \langle G_{j+1} \rangle$. Recall that $G_j = F_{p^j-1}$, and $H_*(F_k)$ is the free module over $H_*(X(n))$ generated by b_n^i for $0 \leqslant i \leqslant k$. One has inclusion maps

$$X(n) = F_0 \hookrightarrow F_1 \hookrightarrow F_2 \hookrightarrow \cdots$$

with cofibre sequences

$$F_{k-1} \longrightarrow F_k \longrightarrow \Sigma^{2kn} X(n).$$

From this it follows immediately that

$$\langle F_k \rangle \leqslant \langle X(n) \rangle$$

for all $k \geqslant 0$.

It can also be shown that (after localizing at p) there is a cofibre sequence

$$F_{kp^j-1} \longrightarrow F_{(k+1)p^j-1} \longrightarrow \Sigma^{2nkp^j} G_j.$$

In particular we have

$$G_j = F_{p^j-1} \hookrightarrow F_{2p^j-1} \hookrightarrow \cdots F_{(p-1)p^j-1} \hookrightarrow F_{p^{j+1}-1} = G_{j+1}$$

where the cofibre of each map is a suspension of G_j. This shows that

$$\langle G_j \rangle \geqslant \langle G_{j+1} \rangle. \tag{7.3.1}$$

It is also straightforward to show that there is a cofibre sequence

$$G_j \longrightarrow G_{j+1} \longrightarrow \Sigma^{2np^j} F_{(p-1)p^j-1}$$

which induces a short exact sequence in homology. Thus we can form the composite map

$$G_{j+1} \longrightarrow \Sigma^{2np^j} F_{(p-1)p^j-1} \longrightarrow \Sigma^{2np^j} G_{j+1}$$

in which the first map is surjective in homology while the second is monomorphic. We denote this map by $r_{n,j}$.

Then there are cofibre sequences

$$G_{j+1} \xrightarrow{\quad r_{n,j} \quad} \Sigma^{2np^j} G_{j+1} \longrightarrow K_{n,j} \tag{7.3.2}$$

and

$$\Sigma^{2np^{j+1}-2} G_j \xrightarrow{\quad b_{n,j} \quad} G_j \longrightarrow K_{n,j}. \tag{7.3.3}$$

The first of these shows that

$$\langle G_{j+1} \rangle \geqslant \langle K_{n,j} \rangle. \tag{7.3.4}$$

Using 6.2.6(iii), we see that *if the telescope $b_{n,j}^{-1} G_j$ is contractible* then we will have

$$\begin{aligned} \langle K_{n,j} \rangle &= \langle G_j \rangle && \text{so} \\ \langle G_{j+1} \rangle &= \langle G_j \rangle && \text{by (7.3.4) and (7.3.1).} \end{aligned}$$

Thus we have reduced the nilpotence theorem to the following.

LEMMA 7.3.5. *Let*

$$\Sigma^{2np^{j+1}-2}G_j \xrightarrow{b_{n,j}} G_j$$

be the map of (7.3.3). *It has a contractible telescope for each n and j.*

This is equivalent to the statement that for each finite skeleton of G_j, there is an iterate of $b_{n,j}$ whose restriction to the skeleton is null.

PROOF. We need to look again at (7.1.2) for $k = p^j - 1$. The map

$$J_{p^j-1}S^{2n} \longrightarrow \Omega S^{2n+1}$$

is known (after localizing at p) to be the inclusion of the fibre of a map

$$\Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np^j+1}.$$

Thus the diagram (7.1.2) can be enlarged to

$$\begin{tikzcd}
 & \Omega S^{2np^j+1} \arrow[r] & \Omega S^{2np^j+1} \\
\Omega SU(n) \arrow[r] & \Omega SU(n+1) \arrow[r] \arrow[u] & \Omega S^{2n+1} \arrow[u, "H"'] \\
\Omega SU(n) \arrow[r] \arrow[u] & B_{p^j-1} \arrow[r] \arrow[u] & J_{p^j-1}S^{2n} \arrow[u] \\
 & \Omega^2 S^{2np^j+1} \arrow[r] \arrow[u] & \Omega^2 S^{2np^j+1} \arrow[u]
\end{tikzcd}$$

in which each row and column is a fibre sequence.

Of particular interest is the map

$$\Omega^2 S^{2np^j+1} \longrightarrow B_{p^j-1}.$$

We can think of the double loop space $\Omega^2 S^{2np^j+1}$ as a topological group acting on the space B_{p^j-1}, so there is an action map

$$\Omega^2 S^{2np^j+1} \times B_{p^j-1} \longrightarrow B_{p^j-1}. \tag{7.3.6}$$

Recall that G_j is the Thom spectrum of a certain stable vector bundle over B_{p^j-1}. This means that (7.3.6) leads to a stable map

$$\Sigma^\infty \Omega^2 S_+^{2np^j+1} \wedge G_j \xrightarrow{\mu} G_j. \tag{7.3.7}$$

Here we are skipping over some technical details which can be found in [22, §3].

The space $\Omega^2 S^{2np^j+1}$ was shown by Snaith [72] to have a stable splitting. After localizing at p, this splitting has the form

$$\Sigma^\infty \Omega^2 S_+^{2np^j+1} \simeq \left(S^0 \vee S^{2np^j-1}\right) \wedge \bigvee_{i\geqslant 0} \Sigma^{i|b_{n,j}|} D_i$$

where each D_i is a certain finite complex (independent of n and j) with bottom cell in dimension 0. Moreover there are maps

$$S^0 = D_0 \xrightarrow{\ell} D_1 \xrightarrow{\ell} D_2 \xrightarrow{\ell} \cdots$$

of degree 1 on the bottom cell, and the limit, $\lim_{\rightarrow} D_i$, is known to be the mod p Eilenberg–MacLane spectrum $H/(p)$.

In [22, Proposition 3.19] it is shown that our map $b_{n,j}$ is the composite

$$\Sigma^{|b_{n,j}|} G_j \longrightarrow \Sigma^{|b_{n,j}|} D_1 \wedge G_j \longrightarrow \Sigma^\infty \Omega^2 S_+^{2np^j+1} \wedge G_j \xrightarrow{\mu} G_j$$

and $b_{n,j}^m$ is the composite

$$\Sigma^{m|b_{n,j}|} G_j \longrightarrow \Sigma^{m|b_{n,j}|} D_m \wedge G_j \longrightarrow \Sigma^\infty \Omega^2 S_+^{2np^j+1} \wedge G_j \xrightarrow{\mu} G_j.$$

Thus we get a diagram

$$\begin{tikzcd}
G_j \arrow[r, "\ell\wedge G_j"] \arrow[d] & D_1 \wedge G_j \arrow[r, "\ell\wedge G_j"] \arrow[d, "\mu"'] & D_2 \wedge G_j \arrow[r] \arrow[d, "\mu"'] & \cdots \\
G_j \arrow[r, "b_{n,j}"] & \Sigma^{-|b_{n,j}|} G_j \arrow[r, "b_{n,j}"] & \Sigma^{-2|b_{n,j}|} G_j \arrow[r] & \cdots
\end{tikzcd} \tag{7.3.8}$$

This means that the map

$$G_j \longrightarrow b_{n,j}^{-1} G_j$$

factors through $G_j \wedge H/(p)$.

Now consider the diagram

$$\begin{tikzcd}
G_j \arrow[r] \arrow[d, "b_{n,j}"'] & G_j \wedge H/(p) \arrow[r] \arrow[d, "b_{n,j} \wedge H/(p)"'] & b_{n,j}^{-1} G_j \arrow[d] \\
\Sigma^{-|b_{n,j}|} G_j \arrow[r] \arrow[d, "b_{n,j}"'] & \Sigma^{-|b_{n,j}|} G_j \wedge H/(p) \arrow[r] \arrow[d, "b_{n,j} \wedge H/(p)"'] & b_{n,j}^{-1} G_j \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

The middle vertical map is null because $b_{n,j}$ induces the trivial map in homology. Passing to the limit, we get

$$b_{n,j}^{-1} G_j \longrightarrow \text{pt.} \longrightarrow b_{n,j}^{-1} G_j$$

with the composite being the identity map on the telescope $b_{n,j}^{-1} G_j$. This shows that the telescope is contractible as desired. □

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CHAPTER 10

The EHP Sequence and Periodic Homotopy

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Contents

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1. Introduction

The problem we want to discuss is that of calculating the homotopy groups of spheres. We will give a somewhat historical approach to this problem. We will not give detailed calculations but will discuss the methods which yield such calculations. The emphasis will be on general theorems rather than particular detailed calculations.

The difference between stable and unstable calculations is a consequence of the Freudenthal suspension theorem, one of the early general results.

THEOREM 1.1. *If $k < n - 1$, then $\pi_{k+n}(S^n)$ is independent of n.*

These groups, $\pi_{k+n}(S^n)$, for $n - 1 > k$, are usually called the stable stems. Work of Adams produced methods to study these groups as a separate problem. This work was one of the important ingredients in the development of the stable category and spectra. Our emphasis will be on the connection between the unstable groups and the stable groups. The Adams spectral sequence will be discussed in connection with Theorem 1.9 and in other parts of this book.

Historically, the next major step was the introduction of the EHP sequence first introduced by James at the prime 2 and by Toda at odd primes.

THEOREM 1.2 (James). *There is a 2 primary fibration*

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}.$$

To state Toda's result we need a modified definition of an even sphere. Let

$$\hat{S}^{2n} = J_{p-1}S^{2n} = \left(\Omega S^{2n+1}\right)^{(2n(p-1))}.$$

This is a CW-complex with $p - 1$ cells, one in each dimension divisible by $2n$ up to $2n(p - 1)$.

THEOREM 1.3 (Toda). *There are p primary fibrations*

$$S^{2n-1} \to \Omega \hat{S}^{2n} \to \Omega S^{2np-1},$$

$$\hat{S}^{2n} \to \Omega S^{2n+1} \to \Omega S^{2np+1}.$$

The proofs of these results are easy calculations with the Serre spectral sequence. They represent an early success of the Serre spectral sequence technique.

We can piece together these exact sequences to get the EHP spectral sequence which is the spectral sequence of the filtration

$$\Omega^1 S^1 \to \Omega^2 \hat{S}^2 \to \cdots \ \Omega^n S^n \to \cdots \tag{1.1}$$

Toda and James' theorems allow us to identify the E_1 term of this spectral sequence.

THEOREM 1.4. *For $p = 2$ there is a spectral sequence converging to $\pi_*(QS^0)$ with*

$$E_1^{k,n} = \pi_{n+k}(S^{2n-1}). \tag{1.2}$$

For odd primes we have

$$E_1^{k,2m+1} = \pi_{2m+1+k}(S^{2pm+1})$$

and

$$E_1^{k,2m} = \pi_{2m+k}(S^{2pm-1}).$$

In both cases the indexing is such that

$$d_r : E_r^{k,n} \to E_r^{k-1,n-r}$$

and $E_\infty^{k,*}$ is the associated graded group for $\pi_k(QS^0)$ filtered by sphere of origin, i.e. by the images of $\pi_k(\Omega^n S^n)$ (with S^{2m} replaced by $\hat{S}^{2m}$ when p is odd).

When

$$\alpha \in \pi_k(QS^0)$$

corresponds to an element

$$\beta \in E_\infty^{k,n},$$

then α desuspends to the n-sphere and we will call β the Hopf invariant of α. We use the equation

$$\mathrm{HI}(\alpha) = \beta$$

to represent this connection; β can also be regarded as a coset in E_1.

There are several general results about this spectral sequence.

THEOREM 1.5. *At all primes $E_2^{s,t}$ is a $\mathbf{F}_p$ vector space.*

For p odd this follows from early work of Toda. For $p = 2$ it follows from work of James although not so directly. The details are in [5].

Theorem 1.5 has the following obvious corollary.

THEOREM 1.6. *Let $S^n\langle n\rangle$ be the n-connected cover of S^n. Then*

$$p^{2n}\pi_*(S^{2n+1}\langle 2n+1\rangle) = 0.$$

Cohen, Moore and Neisendorfer have improved this result to the following.

THEOREM 1.7. *For p odd, $p^n\pi_*(S^{2n+1}\langle 2n+1\rangle) = 0$.*

Gray [22] has shown that this is the best possible by constructing elements of the appropriate order.

In Section 4 we will discuss the 2-primary exponent question in some detail.

In [4] Adams shows that it makes sense to localize $B\Sigma_p$, the classifying space of the symmetric group, at a prime p. If $p = 2$ then $B\Sigma_2$ is already 2-local and is just the real projective space P. If p is odd, then we will denote this space by B. As a CW-complex, B has one cell in each dimension congruent to $0, -1 \bmod q$, where $q = 2(p-1)$. We will use B_b^t to denote the subquotient of B with top cell in dimension t, and bottom cell in dimension b. If $t = \infty$, then we will omit the superscript. If $b = q - 1$, then we will omit the subscript.

The following is due to Toda. We will discuss the proof in Section 2 where we give some additional applications of these results to embedding homotopy spheres.

THEOREM 1.8. i) *At $p = 2$, the partial filtrations $(\Omega^n S^{n+1}, \Omega^{n-k} S^{n+1-k})$ admit a map*

$$\left(\Omega^n S^{n+1}, \Omega^{n-k} S^{n+1-k}\right) \to \left(Q(\Sigma P^n_{n+1-k}), *\right)$$

which is a homotopy equivalence through dimension $3(n+1-k)-3$.

ii) *At p odd, the partial filtrations $(\Omega^{2n-1}\hat{S}^{2n}, \Omega^{2n-2k} S^{2n-2k+1})$ admit a map*

$$\left(\Omega^{2n-1}\hat{S}^{2n}, \Omega^{2n-2k} S^{2n-2k+1}\right) \to \left(Q\left(\Sigma B^{qn-1}_{q(n-k+1)-1}\right), *\right)$$

which is a homotopy equivalence through dimension $2(n-k+1)(p^2-1)-2$.

The Adams spectral sequence has turned out to be a powerful tool in studying $\pi_*(S^0)$, the stable groups. The main result is the following. For a fixed prime p, let A denote the mod p Steenrod algebra.

THEOREM 1.9. *There is a spectral sequence with*

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(F_p, F_p)$$

and such that

$$E_\infty^{s,t} = E_0\left(\pi_{t-s}(S^0) \otimes_Z Z_p\right)$$

where Z_p is the p-adic integers.

Novikov extended these ideas to other spectra and in particular, MU. This work has been extremely influential in subsequent work. This work is discussed in some detail in Ravenel [46]. We will concentrate on other approaches.

There have been several approaches to the problem of an unstable version of the Adams spectral sequence. There was the work of Massey and Peterson in [42] and [43], the approach based on the restricted lower central series taken in [9], and the most general approach carried out in [10].

The main unstable result for spheres is the following:

THEOREM 1.10. *For each n there is a spectral sequence with*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\mathcal{U}A}(F_p, F_p)$$

and such that

$$\{E_r^{s,t}\} \Rightarrow \pi_{t-s}(\Omega^n S^n) \otimes_Z Z_p.$$

The difference between this and 1.9 is that $H^*(S^n; F_p) = F_p$ is to be regarded as an object in the category of *unstable* A-modules and the Ext group is taken in this category. Refer to [42], [43], and [10] for a study of unstable A-modules. The construction given in [9] provides an explicit E_1-term for the sphere in terms of a certain graded F_p algebra called the Lambda algebra. This will be described in more detail in section 2. In particular this description of an E_1-term immediately leads to the following result which extends the unstable Adams spectral sequence for spheres to the EHP sequence. See [15].

THEOREM 1.11. *There is a map of unstable Adams spectral sequences*

$$E_r^{s,t}(S^n) \to E_r^{s,t}(S^{n+1}).$$

At E_∞ this map is compatible with the suspension homomorphism and at E_2 it fits into the following long exact sequence of E_2-terms:

$$\cdots \to E_2^{s,t}(S^n) \to E_2^{s,t}(S^{n+1}) \to E_2^{s-1,t-n-1}(S^{2n+1}) \to E_2^{s+1,t}(S^n) \to \cdots$$

In Section 4 we will discuss the applications of this result and the connection with Theorem 4.4.

Another important direction has been localizations. These ideas will require the introduction of several additional notions. We begin with the following two contrasting theorems.

THEOREM 1.12 (Nishida). *If $\alpha \in \pi_j(S^n)$ and $j \neq n$ then some smash product of α, $S^{jk} \xrightarrow{\alpha^k} S^{nk}$ is inessential.*

THEOREM 1.13 (Adams). *Let $S^4 \xrightarrow{\eta} S^3$ be the suspension of the Hopf map. Let*

$$\tilde{\eta} : S^4 \cup_2 e^5 = \Sigma^3 P^2 \to S^3$$

be an extension of η. Then all smash products $(\Sigma^3 P^2)^k \xrightarrow{\tilde{\eta}^k} S^{3k}$ are essential.

The first theorem suggests that, as a ring, $\pi_*(S^0)$ will be very bad. The second theorem says that $[M^*(2), S^0]$ has a chance of being nicer. This suggests the following definition. We let $M^j(p)$ denote the mod p Moore space with top cell in dimension j.

DEFINITION 1.14. Let $\pi_j(X; Z/p)$ be the homotopy classes of maps from $M^j(p)$ into X. Let $v : M^{j+q}(p) \to M^j(p)$ be a map such that all iterates of v are essential. Let $v^{-1}\pi_*(X; Z/p) = \pi_*(X; Z/p) \otimes_{Z[v]} Z[v, v^{-1}]$.

Adams constructed such a map in [1]. It is denoted by A or v_1 (v_1^4 if $p = 2$). All iterates of v_1 are essential due to the fact that this map induces an isomorphism in K-theory. Clearly $v_1^{-1}\pi_*(\ \ ; Z/p)$ is a homotopy theory in the sense that it is exact under fibrations and satisfies the other axioms for a homotopy theory. When we apply this theory to the EHP sequence we get complete answers. We will discuss this in Sections 4 and 5.

2. The Λ algebra

In [9], the authors construct the Λ algebra. This has been a very powerful tool in understanding EHP phenomena. We will give its definition and some properties. In addition, we conclude with a new result illustrating how it can be used to get EHP information.

The Λ algebra, for $p = 2$ is a bigraded F_2 algebra generated by elements, λ_i with $i = 0, 1, \ldots$ with bigrading given by

$$|\lambda_i| = (1, i + 1)$$

and which satisfy the relations

$$\lambda_i\lambda_j = \sum_{k\geqslant 0} \binom{j - 2i - 2 - k}{k} \lambda_{j-i-k-1}\lambda_{2i+k+1}$$

for $j > 2i$. A monomial is admissible if $2i_k \geqslant i_{k+1}$ and the relations imply that Λ has a F_2 basis consisting of admissible monomials.

In Λ there is a boundary operator given by

$$d\lambda_n = \sum_{i=1}^{[n/2]} \binom{n - i}{i} \lambda_{n-i}\lambda_{i-1}.$$

THEOREM 2.1. *The Λ algebra satisfies*

$$H^{*,*} = \mathrm{Ext}_A^{*,*}(F_2, F_2).$$

This is the E_2 term of the Adams spectral sequence.

Let $\Lambda(n) \subset \Lambda$ be the subspace generated by admissible monomials $\lambda_{i_1}, \ldots, \lambda_{i_s}$ with $i_1 \leqslant n$. Then we have:

THEOREM 2.2. *The subspace $\Lambda(n)$ is closed under the differential and*

$$H^{*,*}(\Lambda(n), d) = E_2^{*,*}(S^{n+1}).$$

In addition there is a short exact sequence of chain complexes

$$0 \to \Lambda(n-1) \to \Lambda(n) \to \lambda_n \Lambda(2n) \to 0$$

which gives rise to an EHP long exact sequence

$$\cdots \to E_2^{s,t}(S^n) \to E_2^{s,t}(S^{n+1}) \to E_2^{s-1,t-n-1}(S^{2n+1}) \\ \to E_2^{s+1,t}(S^n) \to \cdots.$$

For p odd the Lambda algebra is a generated over F_p by elements λ_i, $i \geqslant 1$, and μ_i, $i \geqslant 0$, of bidegrees $(1, qi-1)$ and $(1, qi)$, respectively. Here q stands for the ubiquitous integer $2(p-1)$. There are Adem relations analogous to those for the prime 2 (only a little more complicated to state – see [9]), and the result is that, as an F_p vector space, there is a basis consisting of admissible monomials. A monomial is considered admissible if, whenever $\lambda_i\lambda_j$ or $\lambda_i\mu_j$ occurs, then $j < pi$, and whenever $\mu_i\lambda_j$ or $\mu_i\mu_j$ occurs, then $j \leqslant pi$. There is a differential analogous to the one for $p = 2$, and again we have subcomplexes $\Lambda(n)$ defined as follows. $\Lambda(2m)$ is the subcomplex with basis consisting of admissible monomials beginning with λ_i, $i \leqslant m$, or μ_i, $i < m$. $\Lambda(2m+1)$ is the subcomplex with basis consisting of admissible monomials beginning with λ_i, $i \leqslant m$, or μ_i, $i \leqslant m$.

The theorem is as follows. Note the difference in indexing from the prime 2 case.

THEOREM 2.3. *The subspace $\Lambda(n)$ is closed under the differential and*

$$H^{*,*}(\Lambda(n), d) = E_2^{*,*}(S^n)$$

(actually $\widetilde{S}^{2m}$ if $n = 2m$). There are short exact sequences of chain complexes

$$0 \to \Lambda(2n-1) \to \Lambda(2n) \to \lambda_n \Lambda(2pn-1) \to 0,$$

$$0 \to \Lambda(2n) \to \Lambda(2n+1) \to \mu_n \Lambda(2pn+1) \to 0$$

which give rise to EHP long exact sequences

$$\cdots \to E_2^{s,t}(S^{2n-1}) \to E_2^{s,t}(\widetilde{S}^{2n}) \to E_2^{s-1,t-n-1}(S^{2np-1}) \\ \to E_2^{s+1,t}(S^{2n-1}) \to \cdots$$

$$\cdots \to E_2^{s,t}(\widetilde{S}^{2n}) \to E_2^{s,t}(S^{2n+1}) \to E_2^{s-1,t-n-1}(S^{2np+1}) \\ \to E_2^{s+1,t}(\widetilde{S}^{2n}) \to \cdots$$

The original paper [9] and Tangora's Memoir [48] are good places to follow up on the Λ algebra.

We close this section with a new result. For simplicity, we let $p = 2$. This is concerned with Toda's theorem [52]. M. Barratt constructed a map from $QS^n \to Q\Sigma^n P_n$ which factors through (QS^n, S^n) and is a homotopy equivalence in the Toda range. It is clear that this map can not be filtered so as to give a map of Adams spectral sequences. We can prove the following result. This is one of main results of [31] except that the range there is only $t - s < 2n - 2$.

THEOREM 2.4. *There is a mapping for* $t - s < 3n - 3$

$$\mathrm{Ext}_A^{s,t}(F_2, F_2) \to \mathrm{Ext}_A^{s-1,t-1}\big(H^*(P_n), F_2\big)$$

which projects to Toda's map.

PROOF. We begin with

$$\Lambda(n-1) \to \Lambda \to \Lambda/\Lambda(n-1).$$

This gives a long exact sequence

$$\cdots \to E_2^{s,t}\big(S^n\big) \to \mathrm{Ext}_A^{s,t}\big(F_2, F_2\big) \to H^{s,t}\big(\Lambda/\Lambda(n-1), d\big)$$
$$\to E_2^{s+1,t}\big(S^n\big) \to \cdots.$$

We write

$$\Lambda/\Lambda(n-1) = \bigoplus_{i\geqslant n} \lambda_i \Lambda(2i).$$

We also have a Λ algebra complex for the stable homotopy of P_n,

$$\Lambda(P_n) = \bigoplus_{i\geqslant n} e_i \Lambda.$$

The differential in $\Lambda(P_n)$ is given by

$$de_j\lambda_I = e_j d\lambda_I + \sum_{j-n\geqslant i\geqslant 1} \binom{j-i}{i} e_{j-i}\lambda_{i-1}\lambda_I$$

and the differential in $\Lambda/\Lambda(n-1)$ is given by

$$d\lambda_j\lambda_I = \lambda_j d\lambda_I + \sum_{j-n\geqslant i\geqslant 1} \binom{j-i}{i} \lambda_{j-i}\lambda_{i-1}\lambda_I.$$

The difference between these two differentials is that when expressions like $e_{j-i}\lambda_{i-1}\lambda_I$ are made admissible, the e_{j-i} does not change while in expressions like $\lambda_{j-i}\lambda_{i-1}\lambda_I$ the leading λ_{j-i} might change. To complete the proof we need only look at those dimensions where this change can not happen. This gives the theorem. □

3. The calculation of some Whitehead products

In this section we will show how the main theorem of the last section can be used to calculate some Whitehead products. We begin with the following commutative diagram

$$
\begin{array}{ccccc}
S^n & \longrightarrow & \Omega S^{n+1} & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow \\
S^n & \longrightarrow & QS^n & \longrightarrow & (QS^n, S^n)
\end{array}
$$

This shows that the boundary homomorphism in the long exact sequence in homotopy induced by the top row, usually denoted by P, factors through the boundary homomorphism of the bottom row. Thus P is the composite

$$\pi_j(\Omega S^{2n+1}) \to \pi_j(QS^n, S^n) \to \pi_{j-1}S^n.$$

The map P has a connection with the Whitehead product which we now define. Let $\alpha \in \pi_{p+1}(X)$ and $\beta \in \pi_{q+1}(X)$ be two homotopy classes. Choose representatives $f : E^{p+1}, \partial E^{p+1} \to X, *$ and $g : E^{q+1}, \partial E^{q+1} \to X, *$. Let

$$S = E^{p+1} \times \partial E^{q+1} \cup \partial E^{p+1} \times E^{q+1}.$$

Let $h : S \to X$ be defined by $h(x,y) = f(x)$ if $x \in E^{p+1}$ and $y \in \partial E^{q+1}$ and $h(x,y) = g(y)$ otherwise. The homotopy class of h represents a class

$$[\alpha, \beta] \in \pi_{p+q+1}(X).$$

This class is called the Whitehead product. For some properties see [53].

Let $\iota \in \pi_n(S^n)$ be a generator and let $\beta \in \pi_j(S^n)$ be some other class. We have the following result.

PROPOSITION 3.1. *If β is a suspension class then the Whitehead product $[\iota, \beta]$ is equal to the composite $S^{j+n-1} \to \Omega^2 S^{2n+1} \to S^n$ where the first map is the $n-1$ fold suspension of β.*

Putting together what we have so far gives us the following.

PROPOSITION 3.2. *Suppose $j < 2n-1$ and $\alpha \in \pi_j(S^n)$. If the composite $S^j \to S^n \to P_n$, where the first map is α and the second is the inclusion map, is inessential, then $[\iota, \alpha] = 0$.*

It would be nice if this would be necessary and sufficient but it is not. A complete set of necessary and sufficient conditions are not known. We will give some which allow one to settle most of the cases involving the first few stems.

PROPOSITION 3.3. *If there is a $k < n-1$ such that the composite $S^j \xrightarrow{\alpha} S^n \to P_n$ is not in the image of the pinch map $P_{n-k} \to P_n$ considered as a stable map, then $[\iota, \alpha] \neq 0$.*

We can apply these two results to the generator of the stable three stem, ν. We are only looking at the 2-primary part of the calculation too. The tables in [31] show that the map $S^{n+3} \to P_n$ representing ν is nonzero except if $n \equiv 7 \pmod 8$. Thus in S^{8i-1}, $[\iota, \nu] = 0$. A further look shows that in the remaining cases, with the exception of $n \equiv 5 \pmod 8$, the map $S^{n+3} \to P_n$ representing ν is not in the image of $P_{n-4} \to P_n$ and so in these cases $[\iota, \nu] \neq 0$. This leaves the troublesome case of $n \equiv 5 \pmod 8$. We have the following theorem which illustrates how to use Theorem 2.4 in answering questions like this.

THEOREM 3.4. *Suppose $n \equiv 5 \pmod 8$. Then $[\iota, \nu] \in \pi_{2n+2} S^n$ is not 0 unless $n = 2^i - 3$ and $h_i h_1$ is a permanent cycle in the Adams spectral sequence. In this case it is* 0.

PROOF. Suppose $n \equiv 5 \pmod 8$. Then the map $S^{n+3} \to P_n$ representing ν is essential and so we have to consider the boundary homomorphism $\partial : \pi_{n+3}(QS^n, S^n) \to \pi_{n+2} S^n$. It is enough to show that the map $S^{n+3} \to P_n$ representing ν either is or is not in the image of $\pi_{n+3}(QS^n) \to \pi_{n+3}(QS^n, S^n)$. To settle this we use 2.4. The map $S^{n+3} \to P_n$ representing ν has Adams filtration 1 and so if it is in the image of $\pi_{n+3}(QS^n) \to \pi_{n+3}(QS^n, S^n)$ there must be a class in $\mathrm{Ext}_A^{2,n+5}(F_2, F_2)$ which could map to this class in $\mathrm{Ext}_A^{1,n+4}(H^*(P_n), F_2)$. By Adams' calculation, this happens only if $n = 2^i - 3$. It is easy to verify that in this case $h_i h_1$ maps to the class in $Ext_A^{1,n+4}(H^*(P_n), F_2)$. It remains to show that $h_i h_1$ is a permanent cycle in the Adams spectral sequence. That argument is the content of the paper [35]. □

The work of Hsiang, Levine and Szczarba [27] gives some additional applications of these ideas. Consider the following diagram.

$$
\begin{tikzcd}
V_n \arrow[r, "i_1"] \arrow[d, "j_1"'] & BO(n) \arrow[r, "i_2"] \arrow[d, "j_2"'] & BO \arrow[d, "j_3"'] \\
FV_n \arrow[r, "k_1"] & BF(n) \arrow[r, "k_2"] & BF
\end{tikzcd}
$$

The top row is the usual fibration for vector bundles. The bottom row is the corresponding fibration for spherical fibrations. We use the facts that ΩBF is homotopically equivalent to the 1 component of QS^0 and ΩBF_n is homotopically equivalent to the 1 component of $\Omega^n S^n$. Then the work of the preceding section shows the following.

PROPOSITION 3.5. *In the above commutative diagram the map $V_n \to FV_n$ is a homotopy equivalence through dimension $2n - 2$.*

We can characterize the various spaces in this diagram by the following geometric properties:

- The space V_n classifies stably trivial n-plane bundles with a given trialization.
- The space $BO(n)$ classifies n-plane bundles.
- The space BO classifies stable vector bundles.
- The space FV_n classifies stably trivial $(n-1)$-spherical fibrations with a given trivialization.

– The space BF_n classifies $(n-1)$-spherical fibrations.
– The space BF classifies stable spherical fibrations.

Consider a homotopy class $S^j \to S^n \to V_n$, which we will call α. If $j < 2n-1$ then j_{1*} is an isomorphism. By a result of Barratt and Mahowald [6], i_{1*} is a monomorphism (for $j > 15$). So if $j_{2*}i_{1*}(\alpha) = 0$, then α represents a stably trivial n-plane bundle over S^j which is fiber homotopically trivial. One of the main results of [27] is that such bundles are normal bundles to embedded exotic spheres. In particular, since j_{1*} is an isomorphism in this range, we see that there is a classes $\bar{\alpha} \in \pi_{j+1}(BF) \simeq \pi_j(QS^0)$. Kervaire and Milnor [28] have shown that each homotopy class in $\pi_j(QS^0)$ corresponds to an exotic sphere or to a manifold with Kervaire invariant one. Let $\Sigma_{\bar{\alpha}}$ be the exotic sphere associated with $\bar{\alpha}$. The main result of [27] in this setting is the following.

THEOREM 3.6. *The exotic sphere $\Sigma_{\bar{\alpha}}$ embeds in R^{n+j} with normal bundle $i_{1*}(\alpha)$. If α is essential, then this normal bundle is essential.*

From the EHP sequence point of view we translate this result to the following. Let $\beta \in \pi_j(QS^0)$ desuspend to $\pi_{j+n+1}(S^{n+1})$ where it has a non trivial Hopf invariant $\mathrm{HI}(\beta) \in \pi_{j+n+1}(S^{2n+1})$. This gives the following diagram:

$$\begin{tikzcd}
\Omega^n S^n \arrow[r] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d] & \Omega^{n+1} S^{2n+1} \arrow[r, "\partial"] \arrow[d, "i_1"] & BF_n \arrow[d] \\
\Omega^n S^n \arrow[r] & QS^0 \arrow[r] & FV_n \arrow[r, "\partial"] & BF_n \\
O(n) \arrow[r] \arrow[u] & O \arrow[r] \arrow[u] & V_n \arrow[r, "\partial_1"] \arrow[u, "i_2"] & BO(n) \arrow[u] \\
O(n) \arrow[r] \arrow[u] & O(n+1) \arrow[r] \arrow[u] & S^n \arrow[r, "\partial"] \arrow[u] & BO(n) \arrow[u]
\end{tikzcd}$$

First note that $i_{1*}(\mathrm{HI}(\beta) \neq 0$. If we assume that $j < 2n-1$, then i_{2*} is an isomorphism and thus $\bar{\beta}$ defines a class $(i_1 i_2^{-1})_*(\mathrm{HI}(\beta)) \in \pi_j(V_n)$. Finally, ∂_1 is a monomorphism and so β with the above properties defines an n-plane bundle $\gamma = (\partial_1 i_1 i_2^{-1})(\mathrm{HI}(\beta))$. The Hsiang, Levine and Szczarba theorem asserts that if the exotic sphere is embedded in R^{j+n} then the normal bundle to that embedding is γ as constructed above. At the time of their work, there were only a few examples of this. The results of [35] give a large number of interesting examples. In particular, each class $\eta_j \in \pi_{2^j}(QS^0)$ corresponds to a class which desuspends to S^{2^j-2} where it has $\mathrm{HI}(\eta_j) = \nu \in \pi_{2^{j+1}-2}(S^{2^{j+1}-5})$. Thus the exotic sphere associated to η_j, Σ_{η_j} is of dimension 2^j. It embeds in $R^{2^{j+1}-3}$ with a nontrivial normal bundle classified by the composite $S^{2^j} \xrightarrow{\nu} S^{2^j-3} \xrightarrow{\tau} BO(2^j-3)$ where τ is the tangent bundle.

There is an interesting problem connected with these results. The bundle γ, associated with β, does not have geometric dimension 3 by a result of Massey [41]. We want to find the minimal k such that the bundle over S^j factors through $\pi_j(BO(k)$. It is clear that $3 < k < [j/2]$.

4. The v_1 EHP sequence

In this section we wish to look at the EHP sequence in v_1-periodic homotopy theory which is defined in Section 1. (See 1.14.) At odd primes the result is quite simple. We have the result of Thompson [50]. First we recall that for any space there is a James–Hopf map $j_p : \Omega^k \Sigma^k X \to QD_{k,p}X$ where $D_{k,p}X$ is the equivariant half smash product of X with a certain configuration space. See [13], [14] for a precise construction of j_p. In the special case where X is S^0 and $k = 2n+1$, we have that $D_{2n+1,p}S^0 = B^{qn}$ after localizing at p.

THEOREM 4.1. *There is a James–Hopf map*

$$j_p : \Omega^{2n+1} S^{2n+1} \longrightarrow Q(B^{nq})$$

which induces an isomorphism in v_1-periodic homotopy.

We wish to combine this with the following stable result [16].

THEOREM 4.2. *There is a stable map $M(p^n) \longrightarrow B^{nq}$ which induces an isomorphism in v_1-periodic homotopy.*

Using these two isomorphisms we have an isomorphism of the v_1-periodic homotopy of $\Omega^{2n+1} S^{2n+1}$ and the stable Moore space $M^0(p^n)$. Call this isomorphism ϕ_n.

THEOREM 4.3. *The isomorphism ϕ_n induces an isomorphism in v_1-periodic homotopy in the EHP sequence and the Bockstein sequence as in the following diagram:*

$$\begin{array}{ccccc}
\pi_*(W(n)) & \longrightarrow & \pi_*(S^{2n-1}) & \longrightarrow & \pi_*(\Omega^2 S^{2n+1}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_*(M^0(p)) & \longrightarrow & \pi_*(M^0(p^{n-1})) & \longrightarrow & \pi_*(M^0(p^n))
\end{array}$$

This observation summarizes the calculations of Thompson [50]. It gives a conceptual picture of the the unstable p-primary homotopy in spheres which is v_1-periodic. It gives some intuition about how homotopy exponents occur and why the image of J is of maximum order. It is useful to think of the sphere, S^{2n+1}, as such a Moore space $M^0(p^n)$ with the zero cell being considered the stable cell. It is the one which stabilizes to the sphere as $n \to \infty$. We think of the other cell as the unstable cell. It is the one on which the Bockstein spectral sequence is operating.

Next we want to look at the 2-primary version. The following theorems are analogous to the two preceding ones.

THEOREM 4.4 (Mahowald [34]). *There is a James–Hopf map*

$$j_2 : \Omega^{2n+1} S^{2n+1} \to QP^{2n}$$

which induces a v_1-periodic homotopy equivalence.

THEOREM 4.5 ([18]). *There are maps $M^0(2^{4n}) \to P^{8n}$ and $M^0(2^{4n-1}) \to P^{8n-2}$ which are v_1-periodic homotopy equivalences.*

The analogue of the odd primary case is the following eight fold suspension version of the EHP sequence.

$$\begin{array}{ccccc}
\pi_*(S^{8n-7}) & \longrightarrow & \pi_*(\Omega^8 S^{8n+1}) & \longrightarrow & \pi_*(\Omega^8 S^{8n+1}, S^{8n-7}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_*(M^0(2^{4n-4})) & \longrightarrow & \pi_*(M^0(2^{4n})) & \longrightarrow & M^1(16)
\end{array}$$

This shows that the unstable sphere S^{8n+1} is v_1-equivalent to the stable Moore space, $M^0(2^{4n})$. We think of the cell in dimension zero as the stable cell since in the above diagram it stabilizes to the stable sphere. The cell in dimension -1 is the unstable cell and the EHP sequence acts on that cell. The order of the identity map of $M^0(2^{4n})$ is 2^{4n}. The order of the maximum homotopy class in the v_1-periodic homotopy of $M^0(2^{4n})$ is also 2^{4n}.

It is quite interesting to see just how to fill in the four intermediate parts. As above, we want to think of S^{8n-1} as $M^0(2^{4n-1})$ with the zero cell being the stable cell. The cell in dimension -1 is the unstable cell. To go from S^{8n-1} to S^{8n+1} we again have a mod 2 Bockstein sequence in v_1-periodic homotopy

$$M^0(2^{4n-1}) \to M^0(2^{4n}) \to M^{-1}(2).$$

To see just what is happening in the other cases we need to introduce a new notion, called Brown–Comenetz duality [11].

The Brown–Comenetz dual IF of a finite spectrum F represents the functor

$$Y \mapsto \mathrm{Hom}(\pi_0 Y \wedge F, Q/Z).$$

If I denotes the Brown–Comenetz dual of S^0, then there is a weak equivalence $IF \approx \mathrm{Map}[F, I]$.

For connected spectra like S^0, the Brown–Comenetz dual is a very strange spectrum. Some properties are discussed in [45]. If F is periodic, like $v_1^{-1}P^{8n}$, then IF is not so strange. In particular we have the following.

THEOREM 4.6. *The Brown–Comenetz dual of $v_1^{-1}P^{8n}$ is $v_1^{-1}\Sigma^4 P_5^{8n+4}$.*

There are four things a $Z/2$ Moore space can do to such a complex. In going from S^{8n-7} to S^{8n-5} we change the unstable cell to the Brown–Comenetz dual cell. This increases the order of the identity map of the corresponding "Moore space" by 4. It does not increase the v_1-periodic homotopy order at all.

To go from S^{8n-5} to S^{8n-3} we do an ordinary Bockstein on the unstable (Brown–Comenetz dual) cell. This increases the order of the identity map by 2 and the v_1-periodic homotopy by 2.

To go from S^{8n-3} to S^{8n-1} we change the unstable Brown–Comenetz dual cell back to an ordinary unstable cell. This does not change the order of the identity map but increases the order of the v_1-periodic homotopy by 4.

Finally, to go from S^{8n-1} to S^{8n+1} we do an ordinary Bockstein on the unstable cell. This increases the order of the self map by 2 and increases the order of the v_1-periodic homotopy by 2.

After having done four steps we have increased both the order of the identity map and the maximum order of elements in the image of J by 16. This leads to the following conjecture which is attributed to Barratt and Mahowald in several places in the literature and is sometimes not correctly stated.

CONJECTURE 4.7. *The order of the self map of $\Omega^{2n+1}S^{2n+1}(2n+1)$ is 2^{n+a_n} where a_n is defined modulo 4 and is $0,1,1,0$ for $n \equiv 0,1,2,3 \pmod 4$.*

It is easy to see that the order of the self map of $\Omega^{2n+1}S^{2n+1}(2n+1)$ is at least the conjectured value.

One would expect that the maximum order of the homotopy would agree with this but we know that the maximum order of elements in the image of J on the S^{2n+1} sphere is given by the formula 2^{n+b_n} where b_n is defined modulo 4 and takes on the values $0,-1,-1,0$ for $n \equiv 0,1,2,3 \pmod 4$.

The fact that these two orders are different makes the exponent question very hard at the prime 2.

5. The proof of Theorems 4.1 and 4.4

In this section we will give a short sketch of the original proof of Theorems 4.1 and 4.4, making use of the unstable Adams spectral sequence and the Lambda algebra described in the previous sections. These theorems assert that the unstable v_1-periodic homotopy groups of spheres are isomorphic to the *stable* v_1-periodic homotopy groups of certain finite complexes, thus we will also review the theory of stable v_1-periodicity, which will involve a brief account of Bousfield localization.

Theorems 4.1 and 4.4 follow easily by induction on n from the following theorem which is concerned with $W(n)$, the fiber of the double suspension map $S^{2n-1} \to \Omega^2 S^{2n+1}$. Using Hopf invariants, a construction is given in [12] and [34] of a so-called secondary suspension map $W(n) \xrightarrow{c} \Omega^{2p}W(n+1)$. This map has degree one on the bottom cell which is in dimension $2np-3$. It is also shown that if this map is iterated, the mapping telescope of

$$W(n) \xrightarrow{c} \Omega^{2p}W(n+1) \xrightarrow{\Omega^{2p}c} \Omega^{4p}W(n+2) \to \cdots$$

is QM^{2np-2}. The map $W(n) \to QM^{2np-2}$ is compatible with the James–Hopf maps of 4.1 and 4.4 and we have

THEOREM 5.1. *The map $W(n) \to QM^{2np-2}$ induces an isomorphism in $v_1^{-1}\pi_*(\ ;Z/p)$.*

Note that this theorem describes the unstable (periodic) homotopy groups of a space in terms of the stable (periodic) homotopy groups of a certain spectrum. This is a key idea in the theory of periodic homotopy groups and has its origins in [31].

The key algebraic result which led to Theorem 5.1 is the following Lambda algebra calculation. This was done in [32] for $p = 2$. The odd primary analogue was done in [23]. Let $\Lambda(W(n))$ denote the quotient chain complex $\Lambda(2n)/\Lambda(2n-2)$ for $p = 2$, and $\Lambda(2n+1)/\Lambda(2n-1)$ for p odd. In either case, $\Lambda(W(n))$ is a chain complex which is the Lambda algebra analogue of the space $W(n)$.

THEOREM 5.2. *There is a map*

$$H^{s,t}(\Lambda(W(n))) \to \mathrm{Ext}^{s,t}_{A_*}(F_p, H_*(M))$$

which is an isomorphism above a line of slope $1/5$ *for* $p = 2$, *and* $1/2(p^2-1)$ *for* p *odd, in the* $(t-s, s)$ *plane.*

Since v_1 can be thought of as having bidegree $(2,1)$ for $p = 2$, or $(q, 1)$ for p odd, in the $(t-s, s)$ plane, the homomorphism of 5.2 is an isomorphism after inverting v_1. Thus 5.2 is an algebraic analogue of Theorem 5.1.

In order to deduce 5.1 from 5.2 it is necessary to analyze unstable Adams resolutions and their behaviour with respect to fibrations. We have the following definition from [34].

DEFINITION 5.3. A resolution of a space X is a tower of fibrations under X

$$\begin{tikzcd} X \arrow[d, "f_0"'] & X \arrow[l, "="'] \arrow[d, "f_1"'] & X \arrow[l, "="'] \arrow[d, "f_2"'] & \cdots \arrow[l, "="'] \\ X_0 & X_1 \arrow[l, "p_0"'] & X_2 \arrow[l, "p_1"'] & \cdots \arrow[l, "p_2"'] \end{tikzcd}$$

such that the fibers of the maps p_s are generalized Z/p Eilenberg–Mac Lane spaces. The resolution is called proper if $\ker p_s^* = \ker f_s^*$ in cohomology. If, in addition, the maps $f_s : X \to X_s$ are surjective in cohomology, we say the resolution is an Adams resolution.

Applying the functor π_* to a resolution yields a spectral sequence in the usual manner. Note that if we loop a resolution, the resulting tower of fibrations will again be a resolution, but looping need not preserve the properties of being proper or Adams. It is easy to see that if we are given a map $f : X \to Y$, and resolutions of X and Y such that the resolution of the source is a proper Adams resolution, then there is a map of resolutions covering f. Given resolutions of the source and target and given a map of resolutions covering the map f, a construction is given in [34] and [37] of a resolution of the homotopy fiber of f which yields a long exact sequence of E_2-terms with one of the maps corresponding to f.

We may apply this to the double suspension map $S^{2n-1} \to \Omega^2 S^{2n+1}$, where we take the canonical Adams resolution for the source and double loops on the canonical Adams resolution of S^{2n+1} for the target. By the above remarks we get a resolution of the fiber $W(n)$ which has $H^{*,*}(\Lambda(W(n)))$ as the E_2-term.

The next step is to construct a map of resolutions covering the secondary suspension map $W(n) \xrightarrow{c} \Omega^{2p} W(n+1)$. We take the resolution described above for the source, and the $2p$-fold loops on this resolution for the target. The resolution for $W(n)$ can be shown to be proper, but it is not an Adams resolutions. In order to construct a map of resolutions covering the map c, it is necessary to directly analyze the obstructions to constructing such a map. This analysis is carried out in [34], [39] and [50]. The key idea is to show that at each stage in the resolutions, the highest dimensional homotopy group in the resolution of the target is within the range for which the map f_s is surjective in cohomology in the resolution for the source. The fact that one can start with an Adams resolution, loop it down, and then obtain precise upper bounds on the range through which the maps f_s are surjective in cohomology, is the topic of [24]. This delicate analysis is carried out for the sphere S^m at $p = 2$ in [34], and at all primes for any nice space in [24].

The final step is to show that the map of resolutions so constructed induces the isomorphism of 5.2. It is shown that the map of resolutions covering c is essentially unique on E_2-terms. Again this follows from the analysis of [34] and [24] and some upper bounds established in [39] and [50].

In light of the fact that the preceding results describe unstable periodic homotopy in terms of stable periodic homotopy, we will now describe the theory which is used to compute the stable v_1-periodic homotopy groups mod p, of a spectrum. Thus in the remainder of this section we will be working in the stable homotopy category.

The key result of this stable theory is the computation, due to Mahowald for $p = 2$ [33] and Miller [44] for p odd, of the mod p v_1-periodic homotopy groups of the sphere spectrum. For p odd this is the same as the homotopy groups of the mapping telescope of the diagram

$$M \xrightarrow{A} \Sigma^{-q} M \xrightarrow{A} \Sigma^{-2q} M \xrightarrow{A} \cdots$$

where A denotes the Adams self map of a mod p Moore spectrum M (which corresponds to v_1). For $p = 2$ we have the mapping telescope of

$$M \xrightarrow{A} \Sigma^{-8} M \xrightarrow{A} \Sigma^{-16} M \xrightarrow{A} \cdots$$

where A corresponds to v_1^4. We denote this telescope by $v_1^{-1}M$. Mahowald's method of computing $\pi_* v_1^{-1}M$ consists of using the Adams spectral sequence based on *bo*, the connective K-theory spectrum. Miller's method consists of using the classical Adams spectral sequence, in conjunction with the algebraic Novikov spectral sequence. The answer is stated in the following theorem. For convenience, we will describe a modified version of the Adams map for $p = 2$ which results in a unified description of the mapping telescope for all primes. Let $Y = M \wedge (S^0 \cup_\eta e^2)$. Then Y admits an Adams self map $Y \xrightarrow{A} \Sigma^{-2} Y$ which corresponds to v_1.

THEOREM 5.4 (Mahowald, Miller). i) *For $p = 2$, $\pi_* v_1^{-1} Y = Z/2[v_1, v_1^{-1}] \otimes E(a)$ where E denotes an exterior algebra and the dimension of a is* 1.

ii) *For p odd, $\pi_* v_1^{-1} M = Z/p[v_1, v_1^{-1}] \otimes E(a)$ and the dimension of a is $q - 1$.*

We will now indicate how to use this computation of the v_1-periodic homotopy groups of the sphere spectrum, mod p, to obtain the v_1-periodic homotopy groups of an arbitrary spectrum X. For this we invoke Bousfield localization. In [2], Adams described a program for localizing the homotopy category with respect to any spectrum E; in other words, formally inverting the morphisms which are E_* isomorphisms. In [8] Bousfield carried this out, and discussed the example of $E = K$, complex K-theory, in some detail.

DEFINITION 5.5. Let E be a spectrum. A spectrum X is called E-local if, for any spectrum Y for which $E_*Y = 0$, we have $[Y, X] = 0$. For any X, the E-localization of X is an E_*-equivalence $X \to L_E X$, where $L_E X$ is E-local.

L_E can be thought of as an idempotent functor from the stable homotopy category to itself, equipped with a natural transformation from the identity to it. $L_E X$ is unique up to homotopy and represents that part of X which is detected by the homology theory E_*. The main theorem of [8] is that, for any E, the functor L_E exists.

Setting $E = K$, complex periodic K-theory, Bousfield shows in [8] that the map $M \to v_1^{-1}M$ is in fact localization with respect to K. There is a map $v_1^{-1}M \to L_K M$ by the universal properties of the localization. Bousfield calculates the homotopy groups of $L_K M$, and compares this with Mahowald and Miller's calculation of $\pi_* v_1^{-1}M$ and observes a homotopy equivalence. He also proves the convenient fact that for any spectrum X, $L_K X$ is just given by $L_K S^0 \wedge X$, i.e. localization with respect to K-theory is just smashing with the K local sphere spectrum. (This fact was proved independently in [45].) It follows that for any spectrum X, $L_K(M \wedge X) = v_1^{-1}M \wedge X$.

The following theorem is an immediate corollary of the above remarks. Ravenel's telescope conjecture, now known to be false for $n > 1$ ([47]), is the generalization of this statement to v_n, for all n.

THEOREM 5.6. *A map of spectra $f : X \to Y$ induces an isomorphism in mod p, stable v_1-periodic homotopy groups if and only if it induces an isomorphism in mod p, complex K-theory.*

Having described the link between v_1-periodicity and K-theory localization, we describe the link between these and the J-homomorphism. Recall that the J-homomorphism is a homomorphism from the (unstable) homotopy groups of the space SO to the stable homotopy groups of the sphere spectrum. The image of this homomorphism is one of the main topics of [1]. This image can be describe in purely stable terms by constructing a spectrum J which is the fiber of a certain primary operation in connective K-theory. That is, there is a fibration sequence

$$J \longrightarrow ku \xrightarrow{\theta} \Sigma^2 ku$$

where ku is connective complex K-theory and θ is a map constructed from certain readily described Adams operations.

There is a unit map $S^0 \to J$ which is an isomorphism in K-theory and maps the image of the J-homomorphism onto $\pi_* J$. The most complete way to describe the connection between J and the above discussion of localization is by means of the following result from [19], which is an extension of the above described results of Bousfield, Mahowald, Miller and Ravenel.

PROPOSITION 5.7. *A spectrum X is K-local if and only if the J homology Hurewicz homomorphism $\pi_* X \to J_* X$ is an isomorphism.*

Combining all of the above results yields the following.

THEOREM 5.8. *For any spectrum X,*

$$v_1^{-1}\pi_*(X; Z/p) = \pi_*\left(v_1^{-1}M \wedge X\right) = \pi_*\left(v_1^{-1}M \wedge X \wedge J\right) = v_1^{-1}J_*(X; Z/p).$$

Thus the v_1-periodic homotopy groups of a spectrum can be calculated as the J-homology groups with v_1 inverted, and these in turn can be calculated readily from the K-theory, by means of the fibration sequence $J \to ku \to \Sigma^2 ku$. Doing this for the skeleta of $B\Sigma_p$, localized at p, is a good exercise and yields the unstable v_1-periodic homotopy groups of spheres by Theorems 4.4 and 4.1.

6. Unstable periodization

A natural question arising from the preceding discussion is this: to what extent is Theorem 5.6 true unstably? An example which provides evidence of an unstable analogue of Theorem 5.6 was proved in [49], namely that the Adams map itself $M^{n+q} \to M^n$, as an unstable map of spaces, induces an isomorphism in v_1-periodic homotopy groups. (Note that this is not immediate from the definitions.) Further results along these lines were proved in [51] and [38]. By far, the most definitive answer to this question is given by Bousfield in [7]. In order to approach this, Bousfield studies the notion of the periodization of a space. In order to describe this we start with some preliminary definitions.

We are working in Ho_*, the homotopy category of pointed CW-complexes. Let $\mathrm{map}_*(X, Y)$ denote the space of pointed maps from X to Y.

DEFINITION 6.1. For a given map $f : A \to B$ we say that a space X is f-local if the induced map $\mathrm{map}_*(B, X) \to \mathrm{map}_*(A, X)$ is a homotopy equivalence. A map $X \to Y$ is called an f-equivalence if it induces a homotopy equivalence $\mathrm{map}_*(Y, W) \to \mathrm{map}_*(X, W)$ for all f-local spaces W. An f-localization of a space X is an f-equivalence into an f-local space.

THEOREM 6.2. *Given $f : A \to B$ and X, there exists an f-localization of X.*

This is proved in [7], and also in [21]. We denote the f-localization of X by $X \to L_f X$. We can think of L_f as a co-augmented, idempotent functor on Ho_*. Note that $L_f X$ is unique up to homotopy.

Bousfield localization with respect to a generalized homology theory E_*, described in the previous section, is a special case of this construction. Just take the map f to be the wedge of all E_* equivalences.

Another important special case occurs when the map f is the constant map $W \to *$ for some space W. In this case the f-localization of a space X is called the W-periodization of X, or the W-nullification of X, and is denoted $P_W X$. An f-equivalence is called a W-periodic equivalence. The functor P_W has been studied in depth in [7] and [20]. We are interested in the following special case:

DEFINITION 6.3. Let V_1 denote the cofiber of the Adams self map of a Z/p-Moore space. Letting $W = V_1$, the resulting localization functor associates to any space X a space which will be denoted by $P_{v_1}X$ and called the v_1-periodization of X.

The space $P_{v_1}X$ is a space whose mod p homotopy groups are all v_1-periodic, as is readily verified from the definitions. Roughly, $P_{v_1}X$ is constructed by taking the mapping cones of all maps of V_1 into X, and then iterating this ad infinitum. Since V_1 is K-theory acyclic, it is immediate from the construction that $X \to P_{v_1}X$ induces an isomorphism in K-theory. It is not so obvious that $X \to P_{v_1}X$ should induce an isomorphism in $v_1^{-1}\pi_*(\ ;Z/p)$, and we will sketch a proof of this fact.

For simplicity assume that p is odd. The following result is from [51]. This result is strengthened and generalized in [7]. Later in this section we will discuss Bousfield's result.

THEOREM 6.4 (2.1 of [51]). *Let X and Y be 3-connected spaces. Suppose $f : X \to Y$ is a map such that $\Omega^k f : \Omega^k X \to \Omega^k Y$ induces an isomorphism in K_* for $k = 0, 1, 2, 3$. Then f induces an isomorphism in $v_1^{-1}\pi_*(\ \ ;Z/p)$.*

PROOF. The Adams map induces a map

$$\mathrm{map}_*(M^k, X) \xrightarrow{v_1^*} \mathrm{map}_*(M^{k+q}, X).$$

By adjointness, $\pi_j\mathrm{map}_*(M^k, X) = [M^{k+j}, X] = \pi_{j+k}(X; Z/p)$ and v_1^* induces the action of v_1 on $\pi_*(X; Z/p)$, since X is 3-connected.

Let $V(X)$ denote the mapping telescope of the following diagram:

$$\mathrm{map}_*(M^3, X) \xrightarrow{v_1^*} \mathrm{map}_*(M^{3+q}, X) \xrightarrow{v_1^*} \mathrm{map}_*(M^{3+2q}, X) \xrightarrow{v_1^*} \cdots$$

Note that $\mathrm{map}_*(M^{3+kq}, X) = \Omega^{kq}\mathrm{map}_*(M^3, X)$, and that $V(X)$ is an infinite loop space which satisfies $\pi_*V(X) = v_1^{-1}\pi_*(X; Z/p)$ for $* > 0$. We need to show that

$$V(X) \xrightarrow{V(f)} V(Y)$$

is an equivalence. We have the following lemmas, where $\tilde{f}$ is the evident induced map.

LEMMA 6.5. *Under the hypothesis of Theorem* 6.4, *the map*

$$\tilde{f} : \mathrm{map}_*(M^3, X) \to \mathrm{map}_*(M^3, Y)$$

induces an isomorphism in $K_*(\ )$.

PROOF. The pointed mapping space functor converts a cofiber sequence in the first variable to a fiber sequence. Since $\mathrm{map}_*(S^k, X) = \Omega^k X$, there is a mapping of principal fiber sequences:

$$\begin{tikzcd}
\Omega^3 X \arrow[r] \arrow[d, "\Omega^3 f"'] & \Omega^3 X \arrow[r] \arrow[d, "\Omega^3 f"'] & \mathrm{map}_*(M^3, X) \arrow[d, "\bar{f}"] \\
\Omega^3 Y \arrow[r] & \Omega^3 Y \arrow[r] & \mathrm{map}_*(M^3, Y)
\end{tikzcd}$$

Since the maps on the fiber and total space induce isomorphisms in $K_*(\ )$ by hypothesis, the induced map of K-theory bar spectral sequences is an isomorphism and so the right hand vertical map is a $K_*(\ )$ isomorphism. □

LEMMA 6.6. *The space $V(X)$ is K-local, hence the map $i : \mathrm{map}_*(M^3, X) \to V(X)$ extends to a map $\bar{i} : [\mathrm{map}_*(M^3, X)]_K \to V(X)$.*

PROOF. Since $V(X)$ is a periodic infinite loop space, write $V(X) = \Omega^\infty T(X)$, where $T(X)$ is a periodic spectrum. Using Theorem 5.6, a criterion is given in [8] for deciding when a spectrum is K-local:

A spectrum Z is K-local if and only if the mod p homotopy groups of Z are periodic under the action of the Adams map. Since this is true for the space $V(X)$ by construction, it is true for the spectrum $T(X)$, at least in positive dimensions. But $T(X)$ is a periodic spectrum, so it is true for $\pi_* T(X)$ in all dimensions, hence $T(X)$ is a K-local spectrum. Since Ω^∞ of a K-local spectrum is a K-local space, $V(X)$ is K-local. □

After these preliminaries, we can complete the proof of Theorem 6.4. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{map}_*(M^3, X) \arrow[r, "\eta"] \arrow[d, "\bar{f}"'] & {[\mathrm{map}_*(M^3, X)]_K} \arrow[r, "\bar{i}"] \arrow[d, "\bar{f}_K", "\cong"'] & V(X) \arrow[d, "V(f)"] \\
\mathrm{map}_*(M^3, Y) \arrow[r, "\eta"'] & {[\mathrm{map}_*(M^3, Y)]_K} \arrow[r, "\bar{i}"'] & V(Y)
\end{tikzcd}$$

By Lemma 6.6, $\bar{i}$ exists and the horizontal composites are the inclusions into the mapping telescopes. By Lemma 6.5, the middle vertical map $\bar{f}_K$ is an equivalence. The right hand square commutes by the commutativity of the outer rectangle, the fact that $V(Y)$ is K-local, and the fact that

$$\mathrm{map}_*(M^3, X) \to [\mathrm{map}_*(M^3, X)]_K$$

is a K equivalence.

We will construct an inverse to the homomorphism of mod p homotopy groups

$$V(f)_* : \pi_*(V(X); Z/p) \to \pi_*(V(Y); Z/p).$$

Since $\pi_* V(X)$ and $\pi_* V(Y)$ are Z/p vector spaces, this implies that $V(f)$ is a homotopy equivalence. To construct $V(f)_*^{-1}$ let $\alpha : M^k \to V(Y)$ be an element in

$\pi_*(V(Y); Z/p)$. There exists j such that α lifts $M^k \xrightarrow{\alpha} \mathrm{map}_*(M^{3+qpj}, Y)$. By the results of [25] (see also [26]), the following diagram is homotopy commutative:

$$\begin{array}{ccc} M^{k+pqj} \wedge M^{3+pqj} & \xrightarrow{1\wedge v_1^{pj}} & M^{k+pqj} \wedge M^3 \\ \uparrow = & & \uparrow \cong \\ M^{k+pqj} \wedge M^{3+pqj} & \xrightarrow{v_1^{pj}\wedge 1} & M^k \wedge M^{3+pqj} \end{array}$$

Thus $v_1^{pj}\alpha$ lifts to a map $\tilde{\alpha} : M^{k+qjp} \to \mathrm{map}_*(M^3, Y)$.

Define $V(f)_*^{-1}(\alpha)$ to be $v_1^{-pj}(\tilde{i}_* \circ \tilde{f}_{K_*}^{-1} \circ \eta_*(\tilde{\alpha}))$. It is now straightforward to check that $V(f)_*^{-1}$ is well defined, and that $V(f)_*$ and $V(f)_*^{-1}$ are inverse isomorphisms of one another. This completes the proof of 6.4. □

Next we need to say something about the periodization of a loop space. It is very important to observe that the functor P_W, while we have been viewing it as a functor on the pointed homotopy category Ho_*, can be constructed as a functor on the category of *spaces*, which passes to a functor on the homotopy category.

The following is Proposition 3.1 of [7] and also Theorem B of [20].

PROPOSITION 6.7. *There is a natural homotopy equivalence*

$$\lambda : P_W(\Omega X) \to \Omega(P_{\Sigma W} X).$$

PROOF. (Brief sketch.) The proof uses G. Segal's idea of a 'special simplicial space', which is a way of recognizing loop spaces. See [3]. Let $X_\bullet$ be a simplicial space. For each n there are maps $i_k : X_n \to X_1$, $1 \leqslant k \leqslant n$, corresponding to the various ways of embedding Σ^1 as an edge of Σ^n. The product of these is a map $X_n \to X_1 \times X_1 \times \cdots \times X_1$. If this map is homotopy equivalence for each n, $X_0 \cong *$, and $\pi_0 X_1$ is a group, then we say that $X_\bullet$ is a special simplicial space. If $X_\bullet$ is special, then X_1 is a loop space, and the geometric realization of $X_\bullet$ satisfies $\Omega\|X_\bullet\| \cong X_1$. Furthermore, for any space X, the loop space ΩX is X_1 for some special simplicial space $X_\bullet$.

Now consider $P_W \Omega X$. Let $G_\bullet$ be a special simplicial space such that $G_1 = \Omega\|G_\bullet\| = \Omega X$. Since P_W is a functor on the category of spaces which preserves products up to homotopy, $P_W G_\bullet$ is a special simplicial space, hence $P_W \Omega X = P_W G_1$ is the loop space on the classifying space $\|P_W G_\bullet\|$. It is now formal to identify this classifying space as $P_{\Sigma W} X$ and the periodization map $\Omega X \to P_W \Omega X$ as loops on the periodization map $X \to P_{\Sigma W} X$. □

Now we can show that the v_1-periodization map induces an isomorphism in $v_1^{-1}\pi_*(\ ; Z/p)$. This exposition arose in work of the second author and D. Blanc. Assume that we are periodizing with respect to a cofiber V_1 of an Adams map which is at least a three fold suspension. Then for a 3-connected space X consider Ω^3 applied to the periodization: $\Omega^3 X \to \Omega^3 P_{v_1} X$. By iterating Proposition 6.7 this is just the periodization $P_{\Sigma^{-3} v_1}$ applied to the space $\Omega^3 X$. Since the periodization map induces an isomorphism in K-theory, we can apply Proposition 6.4 to conclude that $X \to P_{v_1} X$ is a $v_1^{-1}\pi_*(\ ; Z/p)$ isomorphism.

Putting this all together we have proved the following (weak) version of Bousfield's theorem:

THEOREM 6.8 (Bousfield). *Let* Ho_3 *denote the homotopy category of* 3*-connected spaces. Let* $\tilde{\Omega}$ *denote the functor from* Ho_3 *to itself given by taking the three connected cover of the loop space functor. Given a map* $f : X \to Y$ *in* Ho_3 *the following are equivalent:*

i) f *is a* P_{v_1}*-equivalence.*
ii) f *is a rational equivalence and induces an isomorphism in mod* p v_1*-periodic homotopy groups.*
iii) $\widetilde{\Omega^k} f$ *induces an isomorphism in* K*-theory for all* k.
iv) $\widetilde{\Omega^3} f$ *induces an isomorphism in* K*-theory.*

PROOF. What remains is to show that ii) implies i) and that ii) implies iii). If $X \to Y$ induces an isomorphism in v_1-periodic homotopy groups then, by the above, so does $P_{v_1}X \to P_{v_1}Y$. But these latter spaces are v_1-periodic which means that their periodic homotopy groups are their homotopy homotopy groups, hence $P_{v_1}X \to P_{v_1}Y$ is an equivalence. Finally, by the above, we have that an isomorphism in v_1-periodic homotopy groups is a K-theory isomorphism. Looping an isomorphism of homotopy groups is again an isomorphism of homotopy groups, so ii) implies iii). □

By going into a much deeper analysis of the effect of periodization on fibrations Bousfield's actual results in [7] are sharper than the above in the following significant ways. First, the restriction to p odd is unnecessary. Second, Bousfield considers periodization with respect to any cofiber of an Adams map, not just one that is a three-fold suspension. Thirdly, he generalizes to all $n \geqslant 1$. Finally, in the case where $n = 1$, the 3 in part iv) is lowered to 2. We will say more about this in the next section.

Theorem 6.8 has a number of striking applications. For instance, since maps inducing isomorphisms in v_1-periodic homotopy groups also induce isomorphisms in K-theory, it follows that the James–Hopf maps of 4.1 and 4.4 induce isomorphisms in K-theory, something that had been conjectured by Miller and Ravenel. Extending the calculations in [40], Lisa Langsetmo uses 6.8 to calculate the K-theory of $\Omega^k S^n$ for all $k < n$ [29].

Bousfield's theorem suggests another approach to computing unstable periodic homotopy groups, along somewhat different lines than those described in Section 5. For example, in [30], a proof is given of Theorem 4.1 which is based on Theorem 6.8. Let F^{2n+1} denote the fiber of the James–Hopf map $QS^{2n+1} \to Q\Sigma^{2n+1}B_{q(n+1)-1}$. There is an evident map $\iota : S^{2n+1} \to F^{2n+1}$. It can be shown that the v_1-periodic homotopy groups of the target are those of the target of the map in 4.1, thus by 6.8 it suffices to show that $\Omega^2\iota$ is a K-theory isomorphism. Details can be found in [30] or in [17] of this volume.

7. Higher periodicity

In this section we will briefly discuss some generalizations of the double suspension sequence and some calculations of unstable v_2-periodic homotopy groups made in [39]. We will also describe the generalization to higher periodicity of Theorem 6.8.

For concreteness, assume $p = 2$. In order to define v_2-periodic homotopy groups analogously to the definition of (mod 2) v_1-periodic homotopy groups, we need a self map of a finite complex analogous to the Adams map. The following result describes the particular finite complex used in [39]. The large number of cells is a technical necessity due to the method of proof. $K(n)$ denotes the Morava K-theory spectrum. Recall that $K(1)$ is just ordinary K-theory, mod p. A_1 denotes a spectrum whose mod 2 cohomology is isomorphic, as an A module, to $A(1)$, the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2.

THEOREM 7.1. *There exists a finite complex M with the following properties:*

i) *$K(1)_*M = 0$, $K(2)_*M \neq 0$ and M admits a v_2 self map, i.e. there is a map $\Sigma^{6k}M \to M$ which induces an isomorphism in $K(2)_*M$ and is nilpotent in $K(n)_*M$ for $n \neq 2$.*
ii) *Let $v_2^{-1}M$ denote the mapping telescope of the self map of* i). *Then $v_2^{-1}M \wedge A_1$ is a wedge of suspensions of eight copies of $v_2^{-1}M$. In particular it follows that $v_2^{-1}\pi_*(A_1; M)$ is a direct sum of eight copies of $\pi_*(v_2^{-1}M)$.*
iii) *$v_2^{-1}M$ is a flat ring spectrum.*

In Section 4 we described the double suspension spectral sequence with v_1 inverted. That is, QS^{2n-1} is filtered by double suspension, and $v_1^{-1}\pi_*(\ ; Z/p)$ is applied to this filtration to obtain an exact couple, hence a spectral sequence. This spectral sequence was then explicitly described. To extend this picture, use the secondary suspension map to filter the stable Moore space:

$$W(1) \subset \Omega^4 W(2) \subset \Omega^8 W(3) \subset \cdots \subset Q(M).$$

Apply $v_2^{-1}\pi_*(\ ; M)$ to this filtration to obtain the *secondary suspension spectral sequence.*

The homotopy of $W(1)$ (the 3-sphere) is slightly anomalous in this picture. To simplify things, consider the filtration of the pair $(QM, W(1))$:

$$\big(\Omega^4 W(2),\ W(1)\big) \subset \big(\Omega^8 W(3),\ W(1)\big) \subset \cdots \subset \big(Q(M),\ W(1)\big). \tag{7.1}$$

The complex A_1, whose cohomology is $A(1)$, can be constructed from P as follows. The degree 2 map $P_3^8 \to P_3^8$ extends to a map $P_5^8 \to P_3^8$. This map in turn lifts to a map $P_5^8 \to P_3^6$ and A_1 is the cofiber $P_3^6 \cup CP_5^8$. This gives a filtration of spectra:

$$P_3^6 \cup CP_5^8 \subset P_3^{10} \cup CP_5^{12} \subset \cdots \subset P_3 \cup CP_5 \tag{7.2}$$

and the subquotients are copies of A_1. The following is one of the main theorems of [39].

THEOREM 7.2. i) *The spectral sequence obtained by applying the homotopy theory $v_2^{-1}\pi_*(\ ; M)$ to 7.1 is isomorphic to the stable spectral sequence obtained by applying $v_2^{-1}\pi_*^S(\ ; M)$ to 7.2.*

ii) *$v_2^{-1}\pi_*^S(P_3 \cup CP_5; M)$ is isomorphic to a direct sum of four copies of $\pi_*(v_2^{-1}M)$ which pull back to $v_2^{-1}\pi_*^S(P_3^6 \cup CP_5^8; M)$.*

iii) $v_2^{-1}\pi_*^S(P_3^{4n-2} \cup CP_5^{4n}; M)$ *consists of eight copies of* $\pi_*(v_2^{-1}M)$, *four of which are described above, and four of which are in the kernel of the inclusion to* $v_2^{-1}\pi_*^S(P_3^{4n+2} \cup CP_5^{4n+4}; M)$. *The latter four map nontrivially to* $v_2^{-1}\pi_*^S(P_{4n-5}^{4n-2} \cup CP_{4n-3}^{4n}; M)$.

iv) *The spectral sequence collapses at* E_2.

The proof of this theorem involves essentially the same techniques as the proof of Theorems 4.1 and 4.4. With some extensions and adaptions. It would be interesting to know if this were some sort of Bockstein spectral sequence, analogous to the double suspension spectral sequence with v_1 inverted.

Finally, we will briefly describe one of the main results of [7], which is the generalization of Theorem 6.8, to v_n-periodicity for all n. Localize at any prime p. First notice that in the proof of Theorem 6.4 we used Theorem 5.6. Ravenel has recently shown that the generalization of 5.6 for $n > 1$ is false [47]. Thus we need to replace $K(n)$ by a spectrum which is designed to make 5.6 true. We can construct such a spectrum as follows. For each i such that $0 \leqslant i \leqslant n$, let M_i be a type i, finite complex. That is $K(j)_*M_i = 0$ for $j < i$ and $K(i)_*M_i \neq 0$. Let f_i be a v_i self map of M_i. Such self maps always exist by [26]. Let T_i denote the mapping telescope of f_i. Finally, let

$$\tilde{E}(n) = \bigvee_{i=0}^{n} T_i.$$

Now it can be shown (see, for example, [36]), that a map of spectra induces an isomorphism in $\tilde{E}(n)_*(\ )$ if and only if it induces an isomorphism in stable, mod M_i, v_i-periodic homotopy groups, for all i from 0 to n. This fact is the natural generalization of Theorem 5.6 for $n > 1$. In [7], Bousfield defines an integer $c(n)$ which is the dimension of the bottom cell of ΣW_n, where W_n is essentially a type $n+1$ space with minimal connectivity. The actual value of $c(n)$ is not known. It is known that $c(1) = 3$, and generally $c(n) \geqslant n + 2$. Let $P_{v_n} = P_{\Sigma W_n}$. There is the following:

THEOREM 7.3 (Bousfield). *Let* $\mathrm{Ho}_{c(n)}$ *denote the homotopy category of* $c(n)$*-connected spaces. Let* $\tilde{\Omega}$ *denote the functor from* $\mathrm{Ho}_{c(n)}$ *to itself given by taking the* $c(n)$*-connected cover of the loop space functor. Given a map* $f : X \to Y$ *in* $\mathrm{Ho}_{c(n)}$ *the following are equivalent:*

i) f *is a* P_{v_n}*-equivalence.*

ii) *For* $0 \leqslant i \leqslant n$, f *induces an isomorphism in* v_i*-periodic homotopy groups.*

iii) $\widetilde{\Omega^k} f$ *induces an isomorphism in* $\tilde{E}_*(\ )$ *for all* k.

iv) $\widetilde{\Omega^3} f$ *induces an isomorphism in* $\tilde{E}_*(\ )$.

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CHAPTER 11

Introduction to Nonconnective $Im(J)$-Theory

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0. Introduction

Classical K-theory has many applications to geometry and homotopy theory, e.g., the e-invariant of Adams and Toda, the J-homomorphism, the $J(X)$-groups or v_1-periodicity in stable homotopy. These applications are all related in some way to the kernel or cokernel of $\Psi^k - k^n$ where Ψ^k is the usual unstable Adams operation in K-theory. Rationally, that is as subspace of $K^*(X) \otimes \mathbf{Q}$, the eigenspace $ker(\Psi^k - k^n)$ is a cohomology theory (on compact spaces), namely ordinary rational (Čech-) cohomology. This is no longer true over the integers or p-locally since $ker(\Psi^k - k^n)$ is not exact. To get something close to $ker(\Psi^k - k^n)$ and $coker(\Psi^k - k^n)$ one may proceed as follows. Taking $k^{-n}\Psi^k$ on the $2n$-th term of the spectrum $K_{(p)}$ of p-local complex periodic K-theory induces the stable Adams operation ψ^k. Represent $\psi^k - 1$ as a stable self map on $K_{(p)}$ and define the spectrum Ad as its fibre. Thus Ad fits into the (co-) fibre sequence of spectra

$$\longrightarrow Ad \xrightarrow{D} K_{(p)} \xrightarrow{\psi^k - 1} K_{(p)} \xrightarrow{\Delta} \Sigma Ad \longrightarrow .$$

Here p is a fixed prime and k is chosen as an integer which for p odd reduces to a generator of $(\mathbf{Z}/p^2)^*$ whereas for $p = 2$ one can take $k = 3$. This choice gives the smallest eigenspaces $ker(\Psi^k - k^n) = ker(\psi^k - 1)$ on p-torsion classes in $K^{2n}(X)_{(p)}$. Because of its close relation to the image of the classical J-homomorphism the generalized homology theory defined by Ad is called $Im(J)$-theory.

This "secondary" cohomology theory Ad^* may now be used to formulate and derive the applications of K-theory mentioned above in a much more systematic and conceptional way. For example, the e-invariant is nothing but the Ad-theory Hurewicz or degree map, elements of small skeletal filtration in $Ad_n(X)$ are always stably spherical, and the result of Mahowald ($p = 2$) and Miller ($p \neq 2$) on the v_1-localization of stable homotopy becomes simply

$$v_1^{-1}\pi_*^S(X; \mathbf{F}_p) \cong Ad_*(X; \mathbf{F}_p).$$

Ad-theory is also closely related to the $K_{(p)}$-theory localization $L_K S^0$ of S^0: for $p \neq 2$ Ad and $L_K S^0$ differ only in one homotopy group. This implies that for spectra X with $H_*(X; \mathbf{Z}_{(p)})$ consisting of torsion, $L_K X$ is the same as $Ad \wedge X$.

The purpose of this article is to give an introduction to $Im(J)$-theory, to supply elementary proofs for some basic properties of Ad-theory and to study some examples. The main applications of $Im(J)$-theory are contained in [11] to which we refer simply as part II. In more detail, the contents are as follows:

We begin with the definition of Ad as a cofibre spectrum, describe its coefficient groups and introduce the Chern–Dold character for Ad-theory. In Section 2 we construct splittings of $K_{(p)} \wedge Ad$ and $Ad \wedge Ad$. The result is

$$Ad \wedge Ad \simeq Ad \vee \Sigma^{-1} Ad\mathbf{Q}.$$

We give a short but nonelementary proof using the relation of Ad to $L_K S^0$ and a somewhat longer but direct proof using well-known results on $K_*(K)$. A third approach for computing $Ad_*(K_{(p)})$ using $Ad_*(P_\infty\mathbf{C})$ and Snaith's theorem $P_\infty\mathbf{C}[\omega^{-1}] \simeq K$ is carried out in Section 6. Because of its close connection to $Ad_*(K_{(p)})$ we give a short discussion of the Ad-groups of the spectrum of connective K-theory bu and the Brown–Peterson spectrum BP at this place. Section 3 contains the universal coefficient formula for Ad-theory

$$\begin{aligned}0 &\longrightarrow Ext_{\mathbf{Z}_{(p)}}\big(Ad_{n-2}(X), \mathbf{Z}_{(p)}\big) \longrightarrow Ad^n(X) \\ &\longrightarrow Hom_{\mathbf{Z}_{(p)}}\big(Ad_{n-1}(X), \mathbf{Z}_{(p)}\big) \longrightarrow 0.\end{aligned}$$

This gives an easy computation of $Ad^*(Ad)$, the cohomology operations of Ad-theory. Section 5 contains the proof that Ad is a commutative ring spectrum. We show that the map $\mu_{Ad} : Ad \wedge Ad \to Ad$ appearing in the splitting of $Ad \wedge Ad$ is associative, commutative and has a two-sided unit. As long as we restrict to finite spectra the proof for this is very easy, but in the general case there are some technical difficulties caused by phantom maps.

The next three sections collect some examples where $Im(J)$-groups are known: As a basic and instructive example $Ad_*(P_\infty\mathbf{C})$ is treated in detail in Section 6. Because of their appealing number-theoretical interpretation we discuss in Section 7 the even dimensional Ad-groups of the classifying space BT^m of an m-torus. The close connection between representation theory and K-theory leads to a complete description of the groups $Ad_*(BG)$ for a finite group G. This is reviewed in Section 8.

There is a section dealing with $Im(J)$-theory of torsion-free spaces or spectra which belongs thematically to this article but is put at the end of part II since it is from there that it draws its examples.

As long as we work with complex $Im(J)$-theory there is no significant difference between $p = 2$ and p odd. At $p = 2$ the real version of $Im(J)$-theory is closer to geometry but we shall leave the necessary changes for deriving the corresponding results at $p = 2$ to the interested reader.

Detailed proofs are given for the universal coefficient formula, the splitting of $Ad \wedge Ad$ and the multiplicative properties of Ad since there is no published account of this. For the better documented results, such as the computation of $Ad_*(BG)$ for a finite group G or $Ad_{2n}(BT^m)$, proofs are only indicated or omitted entirely. The proofs for the main properties of $Im(J)$-theory such as the splitting of $Ad \wedge Ad$, the universal coefficient formula or the product structure may be skipped at first reading.

We shall always work at a fixed prime p and mostly suppress the symbol for p-localization from the notation, i.e. $Hom(A, B)$ will mean $Hom_{\mathbf{Z}_{(p)}}(A, B)$. All (co-)homology theories are taken as reduced. With the exception of Section 1 we shall assume that p is odd.

1. Definition of $Im(J)$-theory

As in the Introduction we shall define Ad-theory as the cofibre theory of $\psi^k - 1$. There is a bundle-theoretic approach to Ad-theory due to Seymour [25] aiming to produce also a product structure, but there are serious problems with this, see [26].

Let p be a fixed prime, choose $k \in \mathbf{N}$ generating $(\mathbf{Z}/p^2)^*$ (resp. $k = 3$ for $p = 2$) and let $\psi^k : K \longrightarrow K$ be the stable Adams operation in p-local complex K-theory. Then the spectrum Ad for p-local nonconnected $Im(J)$-theory is defined by the cofibre sequence

$$\longrightarrow Ad \xrightarrow{D} K \xrightarrow{\psi^k - 1} K \xrightarrow{\Delta} \Sigma Ad \longrightarrow . \tag{1.1}$$

For the definition of Ad the simplest notion of a spectrum will be sufficient; the spectrum maps $\epsilon_n : \Sigma Ad_n \to Ad_{n+1}$ may be defined as fill-in maps. But for any serious investigation of multiplicative structures, it is much more convenient to work in a suitable stable category of CW-spectra such as that described in [28], [2] or [20]. Therefore we shall work from now on in the category of CW-spectra ([28]) and assume that Ad is given as a CW-spectrum with the additional property that it is an Ω-spectrum.

From the known action of ψ^k on $K_{2n}(S^0)$ and the long exact sequence induced by (1.1) we easily deduce the coefficient groups $Ad_m(S^0)$: On $K_{2n}(S^0)$ the stable Adams operation ψ^k is multiplication by k^n. Denote by $\nu_p(n)$ the power of p in the prime factorization of n. By elementary number theory we have

$$\nu_p(k^n - 1) = \begin{cases} 1 + \nu_p(n) & \text{if } n \equiv 0 \bmod p-1 \\ 0 & \text{if } n \not\equiv 0 \bmod p-1 \end{cases} \quad \text{for } p \neq 2$$

and

$$\nu_p(k^n - 1) = \begin{cases} 2 + \nu_p(n) & \text{if } n \equiv 0 \bmod p \\ 1 & \text{if } n \not\equiv 0 \bmod p \end{cases} \quad \text{for } p = 2.$$

Define $\delta : Ad \longrightarrow \Sigma Ad$ to be $\delta = \Delta \circ D$, let $i \in Ad_0(S^0) = \mathbf{Z}_{(p)}$ and $i_{-1} \in Ad_{-1}(S^0) = \mathbf{Z}_{(p)}$ be generators satisfying $D(i) = 1 \in K_0(S^0)$ and $i_{-1} = \delta_*(i)$. Then for p odd with $q := 2p - 2$

$$\begin{aligned} Ad_0(S^0) &\cong \mathbf{Z}_{(p)} \text{ generated by } i, \\ Ad_{-1}(S^0) &\cong \mathbf{Z}_{(p)} \text{ generated by } i_{-1}, \\ Ad_{qt-1}(S^0) &\cong \mathbf{Z}/p^{1+\nu_p(t)} \ t \in \mathbf{Z} - \{0\}, \\ Ad_i(S^0) &= 0 \text{ otherwise.} \end{aligned}$$

This shows that $Im(J)$-theory is not a periodic theory like K-theory. However with mod p^a coefficients, periodicity shows up again. This will be discussed in part II. For example, $Im(J)$-theory with coefficients in $\mathbf{Z}/p$ is periodic with period q ($p \neq 2$) and the coefficients have the simple structure

$$Ad_*(S^0; \mathbf{Z}/p) = \mathbf{F}_p[\alpha] \otimes E(\delta)$$

with $\alpha \in Ad_q(S^0; \mathbf{Z}/p)$ and $\delta \in Ad_{q-1}(S^0; \mathbf{Z}/p)$.

The dependence of Ad on the choice of k is discussed in Section 2.

For an abelian group G denote the associated Moore spectrum by $M(G)$ and the Eilenberg–MacLane spectrum by HG. Ad-theory with coefficients in G will be denoted by $AdG = Ad \wedge M(G)$. From (1.1) and

$$H_n(K; \mathbf{Z}_{(p)}) = \begin{cases} \mathbf{Q}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

(e.g., see [28]) we easily deduce

$$H^n(Ad; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & n = 0, -1, \\ 0, & \text{otherwise}. \end{cases}$$

The canonical generators of $H^0(Ad; \mathbf{Q})$ and $H^{-1}(Ad; \mathbf{Q})$ induce the Chern–Dold character for Ad-theory. We shall denote its components by

$$\kappa_0 = ch_0^{Ad}\colon\ Ad \longrightarrow H\mathbf{Q},$$
$$\kappa_{-1} = ch_{-1}^{Ad}\colon\ Ad \longrightarrow \Sigma^{-1}H\mathbf{Q}.$$

The Chern–Dold character induces an isomorphism

$$Ad_n(X; \mathbf{Q}) \cong H_n(X; \mathbf{Q}) \oplus H_{n+1}(X; \mathbf{Q}).$$

Denote by $ch_{2i} \in H^{2i}(K; \mathbf{Q})$ the $2i$-th component of the classical Chern character. By definition of κ and ch the following diagrams will commute

$$\begin{tikzcd} Ad_n(X) \arrow[rr, "D"] \arrow[dr, "\kappa_0"'] & & K_n(X) \arrow[dl, "ch_0"] \\ & H_n(X;\mathbf{Q}) & \end{tikzcd} \qquad \begin{tikzcd} K_n(X) \arrow[rr, "\Delta"] \arrow[dr, "ch_0"'] & & Ad_{n-1}(X) \arrow[dl, "\kappa_{-1}"] \\ & H_n(X;\mathbf{Q}) & \end{tikzcd} \tag{1.2}$$

and therefore $\kappa_{-1} \circ \delta = \kappa_0, \quad \kappa_0 \circ \delta = 0$.

REMARK. The usual integrality theorem for the Chern character on torsion-free spaces implies that κ_0 is integral on $Ad_n(X)$ for X with p-torsion-free homology. Examples (e.g., see below) show that this is quite different for κ_{-1}.

We close this section with some comments on the definition of *nonconnective* $Im(J)$-*theory*.

A variant of the definition for Ad is sometimes useful: By well-known work of Adams ([3]) p-local K-theory splits into $p-1$ pieces

$$K_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} G$$

with $G_*(S^0) = \mathbf{Z}_{(p)}[v_1, v_1^{-1}]$, $|v_1| = q$. Since the Adams operation ψ^k commutes with the splitting maps, $\psi^k - 1$ restricts to a self map of G. Because $\psi^k - 1 : K_i(S^0) \to K_i(S^0)$ is an isomorphism for $i \not\equiv 0 \bmod q$ it follows easily that we may equally well define Ad by replacing K by G in (1.1):

$$\longrightarrow Ad \xrightarrow{D} G \xrightarrow{\psi^k - 1} G \xrightarrow{\Delta} \Sigma Ad \longrightarrow .$$

To $Im(J)$-theory Ad as defined above there is associated a connective theory A. For a CW-complex Y with n-skeleton $Y^{(n)}$ it may be defined by

$$A_n(Y) := im\big(Ad_n\big(Y^{(n)}\big) \to Ad_n\big(Y^{(n+1)}\big)\big). \tag{1.3}$$

This generalized homology theory gives the important connection to algebraic K-theory: If one chooses the number k in the definition of Ad to be a prime power, then A is nothing but the p-localization of Quillen's algebraic K-theory of the finite field $\mathbf{F}_k$ $(p \neq 2)$. Let $d : A_n(Y) \to Ad_n(Y)$ denote the canonical map. In general d is neither injective nor surjective. Also $A_*(Y)$ is usually much harder to calculate than $Ad_*(Y)$. In this article we shall concentrate on nonconnective $Im(J)$-theory.

The relation of Ad to the $K_{(p)}$-localization $L_K S^0$ of S^0 is discussed in detail in [8]. Here we only note the basic cofibre sequence $(p \neq 2)$.

THEOREM 1.4 ([8]).

$$\longrightarrow L_K S^0 \longrightarrow Ad \xrightarrow{k_{-1}} \Sigma^{-1} H\mathbf{Q} \longrightarrow$$

is a cofibre sequence of spectra.

So far we have seen three possibilities for the construction of $Im(J)$-theory: as the cofibre spectrum of $\psi^k - 1$, the bundle-theoretic approach and as a cofibre of a map $\Sigma^{-2} H\mathbf{Q} \to L_K S^0$. But one can also construct $Ad\mathbf{Q}/\mathbf{Z}$ without reference to K-theory at all as the direct limit of maps

$$F : B^\infty_{nq-1} \longrightarrow B^\infty_{(n-1)q-1}$$

see part II, Section 4. This is similar to Snaith's theorem for K-theory or the result that Ad-theory with $\mathbf{Z}/p^a$-coefficients is the v_1-localization of mod p^a stable homotopy (part II, Section 3). If one considers connective $Im(J)$-theory A as being the basic theory one may construct Ad-theory by gluing A to its Anderson dual ∇A by a map

$$\Sigma^{-2} \nabla A \longrightarrow A$$

with cofibre Ad (for the Anderson dual see, e.g., [30]).

2. The splitting of $Ad \wedge Ad$

In this section we construct the splitting of $Ad \wedge Ad$. We begin with a splitting of $Ad \wedge K$. Let $K\mathbf{Q}$ denote rational K-theory, $K\mathbf{Q} = K \wedge M(\mathbf{Q}) = K \wedge H\mathbf{Q}$ and $\mu : K \wedge K \to K$ the product map for K.

THEOREM 2.1. *The maps* $Ad \wedge K \xrightarrow{D\wedge 1} K \wedge K \xrightarrow{\mu} K$ *and* $Ad \wedge K \xrightarrow{\kappa_{-1}\wedge 1} \Sigma^{-1}H\mathbf{Q} \wedge K \simeq \Sigma^{-1}K\mathbf{Q}$ *induce a splitting*

$$Ad \wedge K \simeq K \vee \Sigma^{-1}K\mathbf{Q}.$$

The essential step in the proof is to determine the groups $Ad_*(K)$:

$$Ad_{2n}(K) = \mathbf{Z}_{(p)}, \quad Ad_{2n-1}(K) = \mathbf{Q}. \tag{2.2}$$

This may be done by computing the exact sequence

$$Ad_{2n}(K) \xrightarrow{D} K_{2n}(K) \xrightarrow{\psi^k - 1} K_{2n}(K) \xrightarrow{\Delta} Ad_{2n-1}(K) \tag{2.3}$$

or using the relation between Ad and $L_K S^0$, the K-theory localization of S^0.

We shall give both arguments, the one using $L_K S^0$ is short but nonelementary (and works only for odd primes) whereas the other is elementary but somewhat longer. A third approach using Snaith's theorem is carried out in Section 6.

FIRST PROOF OF 2.1. If X is a K-local spectrum, then $X \wedge L_K S^0 \simeq X$ [8] and since K itself is K-local we have $L_K S^0 \wedge K \simeq K$. The cofibre sequence (1.4) smashed with K gives

$$\begin{tikzcd}
L_K S^0 \wedge K \arrow[r] \arrow[d, "\simeq"] & Ad \wedge K \arrow[r] \arrow[dl, "\bar{\mu}"] & \Sigma^{-1}H\mathbf{Q} \wedge K \\
K & &
\end{tikzcd}$$

and the map $\bar{\mu} : Ad \wedge K \xrightarrow{D\wedge 1} K \wedge K \xrightarrow{\mu} K$ provides a splitting. □

SECOND PROOF OF 2.1. Denote by $u \in K_2(S^0)$ the Bott element and define $v := h_K(u) \in K_2(K)$ where $h_K : \pi_n(K) \longrightarrow K_n(K)$ is the K-theory Hurewicz map. Then $K_*(K; \mathbf{Q}) = \mathbf{Q}[u, u^{-1}, v, v^{-1}]$ and by [4] we have

a) $K_*(K) \xrightarrow{r} K_*(K; \mathbf{Q})$ is injective and the image consists of all Laurent-polynomials $f(u, v)$ satisfying

$$f(a \cdot t, b \cdot t) \in \mathbf{Z}_{(p)}\left[t, t^{-1}\right] \quad \text{for all } a, b \in \mathbf{Z}^*_{(p)}, \tag{2.4}$$

b) the binomial polynomial $\binom{w}{n}u^n$, $w = v/u$, is in $K_{2n}(K)$.

Since $K_1(K) = 0$ the sequence (2.3) shows $Ad_{2n}(K) \cong ker(\psi^k - 1)$. From $\psi^k(u) = k \cdot u$, $\psi^k(v) = v$ we have $\psi^k f(u,v) = f(k \cdot u, v)$. If

$$f(u,v) = \sum_{i+j=n} a_{i,j}u^i v^j \in K_{2n}(K)$$

is in $ker(\psi^k - 1)$ it follows $f(u,v) = a_{0,n}v^n$ and, since $f(0,1) = a_{0,n} \in \mathbf{Z}_{(p)}$ by (2.4), we see that $ker(\psi^k - 1) = \mathbf{Z}_{(p)}$ and $\overline{\mu}_* : Ad_{2n}(K) \to K_{2n}(S^0)$ induces an isomorphism. To compute $Ad_{2n-1}(K)$ observe first that

$$\kappa_{-1} : Ad_{2n-1}(K) \longrightarrow H_{2n}(K; \mathbf{Q}) = \mathbf{Q}$$

is onto. This is easily seen using (1.2) and evaluating ch_0 on $\binom{w}{m}u^n \in K_{2n}(K)$. We have

$$\binom{v/u}{m}u^n = \sum_{i=1}^{m} \frac{s(m,i)}{m!} v^i u^{n-i}$$

and therefore

$$ch_0\left(\binom{v/u}{m}u^n\right) = \frac{s(m,n)}{m!}.$$

Here $s(m,i)$ is a Stirling number of the first kind (e.g., see [10, §5.5]). Since for n fixed and $m > n$ the denominator of $s(m,n)/m!$ becomes arbitrarily large, κ_{-1} must be onto.

The last step is to prove injectivity of $\kappa_{-1} : Ad_{2n-1}(K) \longrightarrow H_{2n}(K; \mathbf{Q})$. Assume $\kappa_{-1} \circ \Delta(f(u,v)) = ch_0 f(u,v) = 0$ with

$$f(u,v) = \sum_{i+j=n} a_{i,j}u^i \cdot v^j, \quad a_{i,j} \in \mathbf{Q}.$$

Since $ch_0 f(u,v) = a_{0,n}$ we have $a_{0,n} = 0$. We may also assume that f is a polynomial in u, v since multiplication by u or v is bijective. Define

$$\check{f}(u,v) := \sum_{\substack{i+j=n \\ i \neq 0}} \frac{a_{i,j}u^i v^j - a_{i,j}v^n}{k^i - 1} \in K_{2n}(K; \mathbf{Q}).$$

Then $(\psi^k - 1)(\check{f}(u,v)) = f(u,v)$. We show $\check{f}(u,v) \in K_{2n}(K)$ using (2.4): For $a, b \in \mathbf{Z}^*_{(p)}$ write $a = k^s + c_a \cdot p^m$, $b = k^r + c_b \cdot p^m$, $r < s$, with m large enough to ensure $p^m a_{i,j}/(k^i - 1) \in \mathbf{Z}_{(p)}$ for all i, j. This is possible since k generates the p-adic units. Then $\check{f}(at, bt)$ and $\check{f}(k^s t, k^r t)$ differ by a polynomial in u, v with coefficients in $\mathbf{Z}_{(p)}$

and it suffices to show $\breve{f}(k^s t, k^r t) \in \mathbf{Z}_{(p)}[t]$. But

$$\begin{aligned}\breve{f}(k^s t, k^r t) &= \sum a_{i,j} \frac{(k^{s\cdot i} - k^{r\cdot i})k^{n-ri}}{k^i - 1} t^n = \sum a_{i,j} \frac{k^{(s-r)i} - 1}{k^i - 1} k^n \cdot t^n \\ &= \sum_{d=0}^{s-r-1} \sum_{i,j} a_{i,j} k^{i\cdot d} k^n \cdot t^n = \sum_{d=0}^{s-r-1} f(k^d t, t) k^n\end{aligned}$$

and this is in $\mathbf{Z}_{(p)}[t]$ since $f(u,v) \in K_{2n}(K)$. Therefore κ_{-1} is injective too and induces an isomorphism $Ad_{2n-1}(K) \cong \mathbf{Q}$. Since $\kappa_{-1|Ad_{2n}(K)} = 0$ and $\bar{\mu}_{|Ad_{2n-1}(K)} = 0$ the map

$$\bar{\mu} \vee \kappa_{-1} : Ad \wedge K \longrightarrow K \vee \Sigma^{-1} K\mathbf{Q}$$

induces an isomorphism on coefficients, hence is an equivalence. □

We now turn to the splitting of $Ad \wedge Ad$:

LEMMA 2.5. *The following diagram commutes.*

$$\begin{tikzcd} Ad \wedge K \arrow[r, "1\wedge(\psi^k - 1)"] \arrow[d, "\bar{\mu}"] & Ad \wedge K \arrow[d, "\bar{\mu}"] \\ K \arrow[r, "\psi^k - 1"] & K \end{tikzcd}$$

PROOF. The multiplicative properties of the Adams operations imply that

$$\begin{tikzcd} K \wedge K \arrow[r, "\psi^k\wedge\psi^k - 1"] \arrow[d, "\mu"] & K \wedge K \arrow[d, "\mu"] \\ K \arrow[r, "\psi^k - 1"] & K \end{tikzcd}$$

commutes. But $\psi^k \wedge \psi^k - 1 = (\psi^k - 1) \wedge (\psi^k - 1) + (\psi^k - 1) \wedge 1 + 1 \wedge (\psi^k - 1)$ and $(\psi^k - 1) \circ D = 0$ give that

$$\begin{tikzcd} Ad \wedge K \arrow[r, "1\wedge(\psi^k - 1)"] \arrow[d, "D\wedge 1"] & Ad \wedge K \arrow[d, "D\wedge 1"] \\ K \wedge K \arrow[r, "\psi^k\wedge\psi^k - 1"] & K \wedge K \end{tikzcd}$$

commutes. Since $\bar{\mu} = \mu \circ (D \wedge 1)$, this proves (2.5). □

Smashing the defining cofibre sequence (1.1) for Ad with Ad gives the diagram

$$\begin{tikzcd}
Ad\wedge Ad \arrow[r, "1\wedge D"] \arrow[d, "L_1"] & Ad\wedge K \arrow[r, "1\wedge(\psi^k-1)"] \arrow[d, "\bar{\mu}"] & Ad\wedge K \arrow[r, "1\wedge\Delta"] \arrow[d, "\bar{\mu}"] & \Sigma Ad\wedge Ad \arrow[d, "\Sigma L_1"] \\
Ad \arrow[r, "D"] & K \arrow[r, "\psi^k-1"] & K \arrow[r] & \Sigma Ad
\end{tikzcd} \tag{2.6}$$

and since the middle square commutes, there is a fill-in map L_1.

REMARK. Later on we shall see that this fill-in map $\mu_{Ad} := L_1$ is unique and defines a commutative, associative multiplication μ_{Ad} for Ad-theory.

COROLLARY 2.7. *The maps*

$$\mu_{Ad} : Ad \wedge Ad \longrightarrow Ad$$

and

$$\kappa_{-1} \wedge 1 : Ad \wedge Ad \longrightarrow \Sigma^{-1}H\mathbf{Q} \wedge Ad = \Sigma^{-1}Ad\mathbf{Q}$$

define a splitting

$$Ad \wedge Ad \simeq Ad \vee \Sigma^{-1}Ad\mathbf{Q}.$$

PROOF. The maps μ_{Ad} and $\kappa_{-1} \wedge 1$ fit into the commutative diagram

$$\begin{tikzcd}
Ad\wedge Ad \arrow[r] \arrow[d, "\mu_{Ad}\vee\kappa_{-1}\wedge 1"] & Ad\wedge K \arrow[r] \arrow[d, "\bar{\mu}\vee\kappa_{-1}\wedge 1"] & Ad\wedge K \arrow[r] \arrow[d] & \Sigma Ad\wedge Ad \arrow[d] \\
Ad\vee\Sigma^{-1}Ad\mathbf{Q} \arrow[r] & K\vee\Sigma^{-1}K\mathbf{Q} \arrow[r] & K\vee\Sigma^{-1}K\mathbf{Q} \arrow[r] & \Sigma Ad\vee Ad\mathbf{Q}
\end{tikzcd}$$

and the 5-lemma implies the result. □

Let A denote (-1)-connected $Im(J)$-theory ($p \neq 2$) and bu (-1)-connected p-local K-theory. Then similar arguments give $A \wedge K \simeq K$ and $A \wedge Ad \simeq Ad$, and may be used to determine $Ad \wedge bu$.

We only give the result for $Ad \wedge bu$, which follows also directly from the computation of $L_K bu$ in [22] by using (1.4).

PROPOSITION 2.8 ([22]). a) $Ad_{2n}(bu) = \mathbf{Z}_{(p)},\ n \geqslant 0$,
b) $Ad_{2n-1}(bu) = \mathbf{Q},\ n \geqslant 0$,
c) $Ad_{2n}(bu) = 0,\ n < 0$,
d) $Ad_{2n-1}(bu) = \mathbf{Q}/\mathbf{Z}_{(p)},\ n \leqslant -1$.

PROOF. The cases a) and b) are handled as in (2.1). For d) and $n \leqslant -1$ note that

$$\frac{v^{p^c(p-1)+n}}{p^c u^{p^c(p-1)}}$$

is in $ker(\psi^k - 1) = Ad_{2n}(bu; \mathbf{Q}/\mathbf{Z}_{(p)}) \subset K_{2n}(bu; \mathbf{Q}/\mathbf{Z}_{(p)})$, where $v = h_K(u) \in K_2(bu)$ is as in the proof of (2.1). It is easy to see that this gives all elements in $Ad_{2n}(bu; \mathbf{Q}/\mathbf{Z}_{(p)})$ and since $Ad_{2n}(bu; \mathbf{Q}/\mathbf{Z}_{(p)}) \cong Ad_{2n-1}(bu)$ the result follows. □

REMARK. The case of $Ad_*(BP)$ where BP is the Brown–Peterson spectrum at the prime p is similar: Likewise in [22] the K-theory localization $L_K BP$ of BP is determined, giving $Ad_*(BP)$ by (1.4). One may also use Lemma 4 in [17], where $Ad_*(BP; \mathbf{Q}/\mathbf{Z})$ is shown to be isomorphic to $v_1^{-1}BP_*(S^0; \mathbf{Q}/\mathbf{Z})$ giving $Ad_*(BP)$ by the Bockstein sequence.

We close this section by giving as an application of the splitting result a proof that $Im(J)$-theory does not depend on the particular value of k appearing in the defining cofibre sequence (1.1). For connective $Im(J)$-theory this was first proved in [27].

PROPOSITION 2.9. *Let p be an odd prime, k_1, k_2 integers generating $(\mathbf{Z}/p^2)^*$ and denote the fibre spectrum of $\psi^{k_i} - 1$ by $Ad(k_i)$. Then there exists a canonical equivalence $\varepsilon : Ad(k_1) \longrightarrow Ad(k_2)$ such that*

$$\begin{tikzcd} Ad(k_1) \arrow[r, "D"] \arrow[d, "\varepsilon"] & K \arrow[d, equal] \\ Ad(k_2) \arrow[r, "D"] & K \end{tikzcd}$$

commutes.

PROOF (sketch). Using the periodicity properties of Adams operations (e.g., see [1]) and that k_1 generates $(\mathbf{Z}/p^2)^*$ one easily shows that for arbitrary k_2 the composition

$$Ad(k_1) \xrightarrow{D} K \xrightarrow{\psi^{k_2}-1} K$$

is zero. Then the same proof as for (2.5) shows that

$$\begin{tikzcd} Ad(k_1) \wedge K \arrow[r, "\bar{\mu}"] \arrow[d, "1\wedge(\psi^{k_2}-1)"] & K \arrow[d, "\psi^{k_2}-1"] \\ Ad(k_1) \wedge K \arrow[r, "\bar{\mu}"] & K \end{tikzcd} \tag{2.10}$$

commutes. As in (2.6) we obtain a fill-in map L making $Ad(k_2)$ into an $Ad(k_1)$-module spectrum. Define now $\varepsilon : Ad(k_1) \longrightarrow Ad(k_2)$ by composing L with $Ad(k_1) \wedge S^0 \to Ad(k_1) \wedge Ad(k_2)$. If also k_2 generates $(\mathbf{Z}/p^2)^*$, then ε induces an isomorphism on coefficient groups, hence is an equivalence. This may be seen as follows. On π_0 and π_{-1} we get this from (2.10). On mod-p^i-homotopy groups ε induces an isomorphism by an argument using the fact that $\bar{\mu} \wedge id_{M(\mathbf{Z}/p^i)}$ is an equivalence (2.1). □

3. The universal coefficient formula for Ad-theory

In this section we derive the universal coefficient formula for Ad-theory:

THEOREM 3.1. *For every CW-spectrum X the following sequence is exact.*

$$0 \longrightarrow Ext_{\mathbf{Z}_{(p)}}\big(Ad_{n-2}(X), \mathbf{Z}_{(p)}\big) \longrightarrow Ad^n(X)$$

$$\xrightarrow{k_{Ad}} Hom_{\mathbf{Z}_{(p)}}\big(Ad_{n-1}(X), \mathbf{Z}_{(p)}\big) \longrightarrow 0.$$

There are corresponding universal coefficient formulas for other coefficient groups and also for $Ad_*(X)$ when X is a finite spectrum.

Let G be one of the abelian groups $\mathbf{Q}$, $\mathbf{Q}/\mathbf{Z}_{(p)}$ or $\mathbf{Z}_{(p)}$ and consider the Kronecker product

$$\langle\ ,\ \rangle_{Ad} : Ad^n(X;G) \otimes Ad_{n-1}(X) \longrightarrow Ad_{-1}\big(S^0;G\big) = G \tag{3.2}$$

which is induced by the pairing $\mu_{Ad} = L_1$ from (2.6) in the usual way. Then k_{Ad} : $Ad^n(X) \to Hom(Ad_{n-1}(X), \mathbf{Z}_{(p)})$ is the homomorphism adjoint to (3.2).

We start with $G = \mathbf{Q}/\mathbf{Z}_{(p)}$ or $\mathbf{Q}$ and let $M = M(G)$. The pairing μ_{Ad} was defined as a fill-in map L_1 in (2.6). Let L_2 be a fill-in map induced by the middle commutative square in

$$\begin{tikzcd} Ad\wedge Ad \arrow[r] \arrow[d, "L_2"] & K\wedge Ad \arrow[r, "(\psi^k-1)\wedge 1"] \arrow[d, "\mu(1\wedge D)"] & K\wedge Ad \arrow[r] \arrow[d, "\mu(1\wedge D)"] & \Sigma Ad\wedge Ad \arrow[d] \\ Ad \arrow[r] & K \arrow[r, "\psi^k-1"] & K \arrow[r] & \Sigma Ad \end{tikzcd} \tag{3.3}$$

Then

LEMMA 3.4. $L_1 \wedge 1_M \simeq L_2 \wedge 1_M : Ad \wedge Ad \wedge M \longrightarrow Ad \wedge M.$

LEMMA 3.5. $(L_2\wedge 1_M)\circ(\tau\wedge 1_M) \simeq L_1\wedge 1_M$ *for the switch map* $\tau : Ad\wedge Ad \longrightarrow Ad\wedge Ad.$

LEMMA 3.6. $L_1 \wedge 1_M$ *is associative, i.e.* $L_1(L_1 \wedge 1) \wedge 1_M \simeq L_1(1 \wedge L_1) \wedge 1_M$.

PROOF. We first consider the case $G = \mathbf{Q}/\mathbf{Z}_{(p)}$ and use $Ad \wedge Ad \wedge M \simeq Ad \wedge M$. The maps $L_1 \wedge 1_M$ and $L_2 \wedge 1_M$ are elements in $Ad^0(Ad \wedge Ad \wedge M; G)$. The universal coefficient formula for K-theory gives $K^{-1}(Ad \wedge Ad \wedge M; G) = 0$ which implies that

$$D : Ad^0(Ad \wedge Ad \wedge M; G) \longrightarrow K^0(Ad \wedge Ad \wedge M; G)$$

is injective. But $D \circ L_1 \simeq \mu \circ D \wedge D = D \circ L_2$, hence

$$(D \wedge 1_M) \circ (L_1 \wedge 1_M) \simeq (D \wedge 1_M) \circ (L_2 \wedge 1_M).$$

This proves (3.4) for $G = \mathbf{Q}/\mathbf{Z}_{(p)}$. If $G = \mathbf{Q}$ we use the facts that $L_1 - L_2 = \Delta(c)$ with $c \in K^{-1}(Ad \wedge Ad)$ and

$$K^{-1}(Ad \wedge Ad) \to K^{-1}(Ad \wedge Ad; \mathbf{Q})$$

is zero since

$$K^{-1}(Ad \wedge Ad) \cong Ext\big(K_0(Ad \wedge Ad), \mathbf{Z}_{(p)}\big)$$

and $Ext(-, \mathbf{Q}) = 0$. This gives

$$L_1 \simeq L_2 : Ad \wedge Ad \longrightarrow Ad \wedge M(\mathbf{Q})$$

and since $M(\mathbf{Q})$ is an associative ring spectrum we get $L_1 \wedge 1_M \simeq L_2 \wedge 1_M$. This proves 3.4 for $G = \mathbf{Q}$. The proofs for 3.5 and 3.6 are entirely analogous. □

PROPOSITION 3.7. *The Kronecker product* (3.2) *with $G = \mathbf{Q}$ or $G = \mathbf{Q}/\mathbf{Z}_{(p)}$ satisfies*

$$\big\langle a, \Delta(y)\big\rangle_{Ad} = \Delta\big\langle D(a), y\big\rangle_K,$$
$$\big\langle \Delta(a), b\big\rangle_{Ad} = \Delta\big\langle y, D(b)\big\rangle_K,$$

where $\langle\ ,\ \rangle_K$ is the K-theory Kronecker product analogous to (3.2).

PROOF. This follows from (3.3), (2.6), 3.4 and the definition of $\langle\ ,\ \rangle$. □

The main step in the proof of 3.1 is

PROPOSITION 3.8. *For every CW-spectrum X and $G = \mathbf{Q}$ or $G = \mathbf{Q}/\mathbf{Z}_{(p)}$ the homomorphism $k_{Ad} : Ad^n(X; G) \to Hom(Ad_{n-1}(X), G)$ induced by the Kronecker pairing* (3.2) *is an isomorphism.*

PROOF. We deduce this from the well-known statement that

$$k_K : K^n(X; G) \longrightarrow Hom(K_n(X), G)$$

is an isomorphism for X and G as above.

Let $c\psi^k$ be the operation which is adjoint to ψ^k with respect to $\langle\ ,\ \rangle_K$, i.e. $\langle c\psi^k x, y\rangle_K = \langle x, \psi^k y\rangle_K$. Then $c\psi^k = \psi^{1/k} = (\psi^k)^{-1}$ which follows from $\psi^k\langle a, b\rangle_K = \langle \psi^k a, \psi^k b\rangle_K$ and $\psi^k = 1$ on $K_0(S^0)$. By definition of $c\psi^k$ the following diagram commutes:

$$\begin{tikzcd}
K^n(X;G) \arrow[rr, "c\psi^k - 1"] \arrow[d, "k_K"] \arrow[drr, phantom, "(3)"] & & K^n(X;G) \arrow[d, "k_K"] \\
Hom(K_n(X), G) \arrow[rr, "Hom(\psi^k - 1, 1)"] & & Hom(K_n(X), G)
\end{tikzcd} \tag{3.9}$$

Denote by $\overline{Ad}$ the cofibre spectrum of $c\psi^k - 1$. The following commutative diagram

$$\begin{tikzcd}
\overline{Ad}^n(X;G) \arrow[r, "\overline{D}"] & K^n(X;G) \arrow[r, "c\psi^k-1"] & K^n(X;G) \arrow[r, "\overline{\Delta}"] & \overline{Ad}^{n+1}(X;G) \\
Ad^n(X;G) \arrow[r, "D"] \arrow[u, "F"] \arrow[ur, phantom, "(1)"] & K^n(X;G) \arrow[r, "\psi^k-1"] \arrow[u, "-\psi^k"', "\cong"] & K^n(X;G) \arrow[r, "\Delta"] \arrow[u, equal] \arrow[ur, phantom, "(2)"] & Ad^{n+1}(X;G) \arrow[u, "F"']
\end{tikzcd}$$

where F is induced by a fill-in map, shows that $\overline{Ad}$ and Ad are equivalent. We extend diagram (3.9) as follows:

$$\begin{tikzcd}
Hom(Ad_{n-1}(X),G) \arrow[d, "Hom(\Delta,1)"] & Ad^n(X;G) \arrow[l, "k_{Ad}", "(5)"'] \arrow[r, "F", "(1)"'] \arrow[d, "D"] & \overline{Ad}^n(X;G) \arrow[dd, "\overline{D}"] \\
Hom(K_n(X),G) \arrow[d, "-Hom(c\psi^k,1)", "\cong"'] & K^n(X;G) \arrow[l, "k_K", "(4)"'] \arrow[dr, "\cong", "-\psi^k"'] & \\
Hom(K_n(X),G) \arrow[d] & & K^n(X;G) \arrow[ll, "k_K", "(3)"'] \arrow[d, "c\psi^k-1"] \\
Hom(K_n(X),G) \arrow[dd] & & K^n(X;G) \arrow[ll, "k_K"] \arrow[dd, "\overline{\Delta}"] \\
 & K^n(X;G) \arrow[ul, "k_K", "(6)"'] \arrow[ur, "id"', "(2)"] \arrow[d, "\Delta"] & \\
Hom(Ad_n(X),G) & Ad^{n+1}(X;G) \arrow[l, "k_{Ad}"'] \arrow[r, "F"] & \overline{Ad}^{n+1}(X;G)
\end{tikzcd}$$

This diagram commutes: (1), (2), (3) have already appeared, (4) comes from $\langle\psi^k x, y\rangle_K = \langle x, c\psi^k y\rangle_K$ and (5), (6) follow from 3.7.

For $G = \mathbf{Q}$ or $G = \mathbf{Q}/\mathbf{Z}_{(p)}$ the left vertical sequence is exact since G is $\mathbf{Z}_{(p)}$-injective. Since k_K is an equivalence the result follows from the 5-lemma.

We may now complete the proof of 3.1 in the usual way:

Consider the Bockstein sequence associated to $\mathbf{Z}_{(p)} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}_{(p)}$:

$$\begin{tikzcd}
Ad^n(X;\mathbf{Q}) \arrow[r, "r"] \arrow[d, "\cong"', "k_1"] & Ad^n(X;\mathbf{Q}/\mathbf{Z}_{(p)}) \arrow[r, "\beta"] \arrow[d, "\cong"', "k_2"] & Ad^{n+1}(X) \arrow[r, "i"] & {} \\
Hom(Ad_{n-1}(X),\mathbf{Q}) \arrow[r, "Hom(1,r)"] & Hom(Ad_{n-1}(X),\mathbf{Q}/\mathbf{Z}_{(p)}) \arrow[r] & Ext^1(Ad_{n-1}(X),\mathbf{Z}_{(p)}) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\beta"] & Ad^{n+1}(X) \arrow[r, "i"] \arrow[d, "k_{Ad}"] & Ad^{n+1}(X;\mathbf{Q}) \arrow[r] \arrow[d, "\cong"', "k_1"] & Ad^{n+1}(X;\mathbf{Q}/\mathbf{Z}_{(p)}) \arrow[d, "\cong"', "k_2"] \\
0 \arrow[r] & Hom(Ad_n(X),\mathbf{Z}_{(p)}) \arrow[r] & Hom(Ad_n(X),\mathbf{Q}) \arrow[r] & Hom(Ad_n(X),\mathbf{Q}/\mathbf{Z}_{(p)})
\end{tikzcd}$$

Since k_1, k_2 are isomorphisms a diagram chase shows k_{Ad} to be onto and $ker(k_{Ad}) = ker(i) \cong im(\beta) = coker(r) \cong coker(Hom(1,r)) = Ext(Ad_{n-1}(X), \mathbf{Z}_{(p)})$. $\square$

4. *Ad*-theory cohomology operations

The universal coefficient formula 3.1 together with the splitting of $Ad \wedge Ad$ easily gives $Ad^* Ad$:

Recall that $\delta = \Delta \circ D : Ad \to K \to \Sigma Ad$ and denote the group $Ext_{\mathbf{Z}_{(p)}}(\mathbf{Q}, \mathbf{Z}_{(p)}) = \mathbf{Z}_p^\wedge/\mathbf{Z}_{(p)}$ by P. Note that P is torsion-free and divisible, hence a $\mathbf{Q}$-vector space. Then $Ad^0(Ad) \cong P \oplus \mathbf{Z}_{(p)} \cdot id$, $Ad^1(Ad) \cong Ad^0(Ad) \cdot \delta$ and $Ad^{1+qt}(Ad) = \mathbf{Z}/p^{1+\nu_p(t)}$ where $q = 2p-2$, $t \neq 0$ and $\nu_p(t)$ denotes the power of p in t. All other groups are zero.

Note that the subgroups $P \subset Ad^0(Ad)$ and $P \cdot \delta \subset Ad^1(Ad)$ consist of phantom operations, i.e. every $F \in P$ induces the zero map $F : Ad^*(X) \to Ad^*(X)$ if X is a finite complex (if X is finite, every $x \in Ad^*(X)$ factorizes through a finite subspectrum $Ad^{(r)}$ of Ad, but since $Ad^0(Ad^{(r)})$ is finitely generated over $\mathbf{Z}_{(p)}$, the restriction map $Ad^0(Ad) \to Ad^0(Ad^{(r)})$ must be zero on P). Hence up to phantom operations all operations in Ad-theory are given by multiplication with elements in the coefficient ring: e.g., δ is induced by multiplication with $i_{-1} \in Ad_{-1}(S^0)$ and the generator in $Ad^{1+tq}(Ad)$ is given by $Ad \wedge S^{-1-tq} \xrightarrow{1 \wedge \alpha} Ad \wedge Ad \xrightarrow{\mu} Ad$ where $\alpha \in Ad_{-1-tq}(S^0)$ is a generator.

This is exactly the result one expects from the close relation of Ad to $L_K S^0$.

5. The product structure on *Ad*

In this section we investigate the properties of μ_{Ad}. The proof that μ_{Ad} is commutative, associative with unit is straightforward but somewhat long. The reason is that $D : Ad^0(Ad \wedge Ad) \to K^0(Ad \wedge Ad)$ is not injective but has a nontrivial kernel consisting of phantom maps. If we ignore phantom maps, i.e. work only with finite complexes, then associativity and commutativity of μ_{Ad} becomes more or less trivial.

To compute $K^*(X)$ for X a product of copies of Ad and K we use the universal coefficient formula for K-theory and the splittings of $K \wedge Ad$, $Ad \wedge Ad$.

PROPOSITION 5.1. *μ_{Ad} is commutative and associative up to phantom maps.*

PROOF. $\mu_{Ad} - \mu_{Ad} \circ \tau$ is in

$$\ker\left(D : Ad^0(Ad \wedge Ad) \to K^0(Ad \wedge Ad)\right)$$

hence comes from $K^{-1}(Ad \wedge Ad)$. But this group is isomorphic to $P := Ext(\mathbf{Q}, \mathbf{Z}_{(p)})$ and consists of phantom maps (e.g., see Section 4). The same argument applies to associativity up to phantom maps. □

REMARK. In the same way one may show that the product induced on p-complete $Im(J)$-theory $Ad\mathbf{Z}_p^\wedge$ is commutative and associative.

PROPOSITION 5.2. *$i \in Ad_0(S^0)$ is a two-sided unit for μ_{Ad}.*

PROOF. The three maps

$$id_{Ad},\ \mu_{Ad} \circ (i \wedge id),\ \mu_{Ad}(id \wedge i) \in Ad^0(Ad)$$

are all mapped to D under $D : Ad^0(Ad) \longrightarrow K^0(Ad)$, hence they are equal since $K^{-1}(Ad) = 0$ implies that D is injective. □

Next we show that μ_{Ad} is uniquely determined by being a fill-in map in (2.6).

PROPOSITION 5.3. *There is only one fill-in map in* (2.6).

PROOF. The difference of two fill-in maps may be written as $\Delta(F)$ for a phantom map $F \in K^{-1}(Ad \wedge Ad) \cong P$. Denote by g the map

$$g : Ad \wedge Ad \xrightarrow{\kappa_{-1} \wedge \kappa_{-1}} \Sigma^{-1} H\mathbf{Q} \wedge \Sigma^{-1} H\mathbf{Q} \xrightarrow{\mu} \Sigma^{-2} H\mathbf{Q}.$$

Then $g_* : K_{-2}(Ad \wedge Ad) \to K_{-2}(\Sigma^{-2} H\mathbf{Q})$ is onto and the universal coefficient formula implies

$$g^* : K^1(H\mathbf{Q}) \to K^{-1}(Ad \wedge Ad)$$

is an isomorphism. That $\Delta(F)$ is the difference of two fill-in maps implies also

$$(1 \wedge \Delta)^* \Delta(F) = 0.$$

We show that $(1 \wedge \Delta)^*(\Delta F) = 0$ implies $F = 0$. This follows from the commutative diagram

$$\begin{tikzcd}
 & K^{-1}(Ad\wedge Ad) \arrow[r, "(1\wedge\Delta)^*"] \arrow[dl] & K^{-1}(Ad\wedge\Sigma^{-1}K) \arrow[r, "\Delta"] & Ad^0(Ad\wedge\Sigma^{-1}K) \\
Ad^0(Ad\wedge Ad) & K^1(H\mathbf{Q}) \arrow[u, "\cong"', "g^*"] \arrow[r, "ch_0\otimes\mathbf{Q}^*"] \arrow[d, "\Delta"] \arrow[ur, phantom, "(a)"] & K^1(H\mathbf{Q}\wedge K) \arrow[u, "\kappa_{-1}\wedge 1^*"] \arrow[r, "\Delta"] \arrow[d, "\Delta"] & Ad^0(\Sigma^{-2}H\mathbf{Q}\wedge K) \arrow[u, "\kappa_{-1}\wedge 1^*"] \arrow[d, equal] \\
 & Ad^2(H\mathbf{Q}) \arrow[ul, "g^*"] \arrow[r, "ch_0\otimes\mathbf{Q}^*"] & Ad^2(H\mathbf{Q}\wedge K) \arrow[r, equal] & Ad^2(H\mathbf{Q}\wedge K)
\end{tikzcd}$$

Here $ch_0 \otimes \mathbf{Q} : K\mathbf{Q} \simeq K \wedge H\mathbf{Q} \to H\mathbf{Q}$ is the rational Chern character and square (a) commutes because of (1.2). That $ch_0 \otimes \mathbf{Q}$ and $\kappa_{-1} \wedge 1$ induce monomorphisms follows from the fact that both maps are splitting maps. It is also easy to see that $\Delta : K^1(H\mathbf{Q}) \longrightarrow Ad^2(H\mathbf{Q})$ is injective. □

To prove commutativity and associativity we construct an equivalence

$$S : \Sigma^{-1}Ad\mathbf{Q} \vee Ad \longrightarrow Ad \wedge Ad$$

which is well behaved with respect to the map $\tau : Ad \wedge Ad \to Ad \wedge Ad$.

Step 1: The group $[Ad, Ad \wedge Ad]$

From the commutative diagram (with $\mu = \mu_{Ad}$, $M = M(\mathbf{Q}/\mathbf{Z}_{(p)})$, r = reduction mod $\mathbf{Q}$, β = Bockstein map) and the splitting of $Ad \wedge Ad$

$$\begin{tikzcd}
\mathbf{Z}_{(p)}=[S^0,Ad] & {[S^0,Ad\wedge Ad]} \arrow[l, "\mu_*"'] & \\
\mathbf{Z}_{(p)}\oplus P=[Ad,Ad] \arrow[u, "i^*"] & {[Ad,Ad\wedge Ad]=\mathbf{Z}_{(p)}\oplus P\oplus\mathbf{Q}} \arrow[l, "\mu_*"'] \arrow[u, "i^*"] \arrow[r, "i^*_{-1}"] & {[S^{-1},Ad\wedge Ad]=\mathbf{Z}_{(p)}\oplus\mathbf{Q}} \\
P=[Ad\mathbf{Q},Ad] \arrow[u, "r^*"] & {[Ad\mathbf{Q},Ad\wedge Ad]=P\oplus\mathbf{Q}} \arrow[l] \arrow[u, "r^*"] \arrow[r, "(\kappa_{-1})^*", "\cong"'] & {[\Sigma^{-1}M\mathbf{Q},Ad\wedge Ad]=P\oplus\mathbf{Q}} \\
{[Ad\mathbf{Q},\Sigma^{-1}Ad\wedge M]} \arrow[u, "\beta_*"] & {[Ad\mathbf{Q},\Sigma^{-1}Ad\wedge Ad\wedge M]} \arrow[l] \arrow[u, "\beta_*"] &
\end{tikzcd}$$

we derive a direct sum decomposition of $[Ad, Ad\wedge Ad]$ as $\mathbf{Z}_{(p)}\cdot(i\wedge 1)\oplus P\oplus\mathbf{Q}(i\wedge 1 - 1\wedge i)$ satisfying: $P = Ext(\mathbf{Q}, \mathbf{Z}_{(p)}) = im(r^*\beta_*)$ and $\ker(\mu_*) = \mathbf{Q}(i \wedge 1 - 1 \wedge i)$ with τ acting on $\ker(\mu_*)$ as multiplication by -1. Since $(\mu \wedge 1_M) \circ (\tau \wedge 1_M) \simeq \mu \wedge 1_M$ (3.5) we see that τ acts on P as the identity. We choose a generator k of $\ker(\mu_*)$ in $[Ad\mathbf{Q}, Ad \wedge Ad]$ with $r^*(k) = (i \wedge 1 - 1 \wedge i)$ and $\tau(k) = -k$ (r^* is injective and $i \wedge 1 - 1 \wedge i \neq 0$). Denote $i_{-1}^*(k)$ by $\overline{k}$ in $[\Sigma^{-1}M\mathbf{Q}, Ad \wedge Ad]$. Since i_{-1}^* is an isomorphism we must have $\overline{k} \circ \kappa_{-1} = k$.

Step 2: The equivalence S
Define $R := k \circ \delta + 1 \wedge \delta \circ \Sigma^{-1}(k) : \Sigma^{-1}Ad\mathbf{Q} \longrightarrow Ad \wedge Ad$ and let $S := R \vee (i \wedge 1) : \Sigma^{-1}Ad\mathbf{Q} \vee Ad \longrightarrow Ad \wedge Ad$.

LEMMA 5.4. *S is inverse to $\mu \vee \kappa_{-1} \wedge 1 : Ad \wedge Ad \longrightarrow Ad \vee \Sigma^{-1}Ad\mathbf{Q}$.*

PROOF. $\mu \circ S = id_{Ad} \vee 0$ follows from $\mu \circ (i \wedge 1) = id_{Ad}$ and $\mu \circ R = 0$ since $\mu(1 \wedge \delta) = \delta \circ \mu$ and $\mu \circ k = 0$. Then $\kappa_{-1} \wedge 1 \circ S = 0 \vee id$ follows from $\kappa_{-1} \circ i = 0$ and $(\kappa_{-1} \wedge 1) \circ R = id$. The last equation is seen by restricting to $i : \Sigma^{-1}M\mathbf{Q} \hookrightarrow \Sigma^{-1}Ad\mathbf{Q}$ and $i_{-1} : \Sigma^{-2}M\mathbf{Q} \hookrightarrow \Sigma^{-1}Ad\mathbf{Q}$. □

Step 3: The group $[\Sigma^{-1}Ad, Ad \wedge Ad]$
By using the splitting of $Ad \wedge Ad$ one easily derives the exactness of the sequence

$$0 \longrightarrow [Ad, Ad \wedge Ad] \xrightarrow{\delta^*} [\Sigma^{-1}Ad, Ad \wedge Ad] \xrightarrow{i_{-1}^*} [S^{-2}, Ad \wedge Ad] \longrightarrow 0$$

with $\pi_{-2}(Ad \wedge Ad) = \mathbf{Q}(i_{-1} \wedge i_{-1})$.

1. $\mu(1 \wedge i_{-1}) = \mu(i_{-1} \wedge 1)$
Since $i_{-1} \wedge 1 - 1 \wedge i_{-1}$ is in $\ker(i_{-1}^*)$ we may write

$$i_{-1} \wedge 1 - 1 \wedge i_{-1} = a(i \wedge 1) \circ \delta + F \circ \delta + c(i \wedge 1 - 1 \wedge i) \circ \delta$$

with $F \in P$. Restricting to $[S^{-1}, Ad \wedge Ad]$ and applying μ shows $a = 0$. The action of τ implies $2 \cdot F \cdot \delta = 0$. But P is torsion-free, hence $F\delta = 0$, and $i_{-1} \wedge 1 - 1 \wedge i_{-1} \in \ker(\mu_*)$ since $\mu(i \wedge 1 - 1 \wedge i) = 0$ by (5.2)

2. $\mu(\delta \wedge i) = \mu(i \wedge \delta)$
We have $\delta \wedge i - i \wedge \delta = \delta^*(1 \wedge i - i \wedge 1)$ and the claim follows again from (5.2).

4. $\mu(i \wedge \delta - 1 \wedge i_{-1}) = 0$
This follows easily from the defining diagram for μ.

5. Now $\ker(\mu_*)$ in $[\Sigma^{-1}Ad, Ad \wedge Ad]$ is the 2-dimensional $\mathbf{Q}$-vector space with basis $i_{-1} \wedge 1 - 1 \wedge i_{-1}$ and $i \wedge \delta - 1 \wedge i_{-1}$. This is seen by applying i^* and i_{-1}^*. In particular, $\ker(\mu_*)$ is invariant under τ.

LEMMA 5.5. $(1 \wedge \delta) \circ k = (\delta \wedge 1) \circ k$.

PROOF. Using the universal coefficient formula we see that the composition $r^* \circ (\kappa_{-1})^* : [\Sigma^{-2}M\mathbf{Q}, Ad \wedge Ad] \longrightarrow [\Sigma^{-1}Ad, Ad \wedge Ad]$ is injective. Therefore $\ker(\mu_*) \subset [\Sigma^{-2}M\mathbf{Q}, Ad \wedge Ad]$ is invariant under τ and τ acts by multiplication with -1. Since $(1 \wedge \delta) \circ k \in \ker(\mu_*)$ we see that $-\tau \circ (1 \wedge \delta) \circ \overline{k} = -\delta \wedge 1 \circ \tau \circ \overline{k} = \delta \wedge 1 \circ \overline{k}$ and this is $(1 \wedge \delta) \circ \overline{k}$. From $k = \overline{k} \circ \kappa_{-1}$ the lemma follows. □

Proposition 5.6. $\tau \circ R = -R$.

Proof. By definition $R := k \circ \delta + (1 \wedge \delta) \circ \Sigma^{-1}(k)$, therefore

$$\tau \circ R = -k \circ \delta + \delta \wedge 1 \circ \Sigma^{-1}(\tau k) = -\big(k \circ \delta + (1 \wedge \delta) \circ \Sigma^{-1}(k)\big)$$

by 5.5. $\square$

Corollary 5.7. $\mu_{Ad} \circ \tau = \mu_{Ad}$.

Proof. Under the equivalence S the projection $pr : \Sigma^{-1}Ad\mathbf{Q} \vee Ad \longrightarrow Ad$ corresponds to μ_{Ad}: $pr = \mu_{Ad} \circ S$. Now

$$\tau \circ S = -R \vee (1 \wedge i) = S - (2R \vee (i \wedge 1 - 1 \wedge i))$$

by 5.6 and therefore

$$\mu_{Ad} \circ \tau \circ S = \mu_{Ad} \circ S - \mu_{Ad}(2R \vee (i \wedge 1 - 1 \wedge i)) = \mu_{Ad} \circ S - 0,$$

hence $\mu_{Ad} \circ \tau = \mu_{Ad}$. $\square$

Corollary 5.8. *μ_{Ad} is associative.*

Proof. Define maps f_1, f_2 by the following diagram ($\mu = \mu_{Ad}$):

$$\begin{array}{ccc}
Ad \wedge (Ad \wedge Ad) \xleftarrow{1 \wedge S} Ad \wedge (Ad \vee \Sigma^{-1}Ad\mathbf{Q}) \simeq & Ad \wedge Ad \vee Ad \wedge \Sigma^{-1}Ad\mathbf{Q} \\
\| & f_1 \downarrow \; f_2 \swarrow\!\!\!\!\searrow \quad \downarrow \\
(Ad \wedge Ad) \wedge Ad \xleftarrow{S \wedge 1} (Ad \vee \Sigma^{-1}Ad\mathbf{Q}) \wedge Ad \simeq & Ad \wedge Ad \vee \Sigma^{-1}Ad\mathbf{Q} \wedge Ad \\
\downarrow \mu \wedge 1 & \downarrow pr \\
Ad \wedge Ad \qquad\qquad = & Ad \wedge Ad
\end{array}$$

Associativity of μ follows from $\mu \circ f_1 = \mu$ and $\mu \circ f_2 = 0$.

a) $f_1 = \mathrm{id}$

$1 \wedge S$ restricted to $Ad \wedge Ad$ is $Ad \wedge S^0 \wedge Ad \xrightarrow{1 \wedge i \wedge 1} Ad \wedge Ad \wedge Ad$. By definition f_1 is the restriction of $1 \wedge S$ to $Ad \wedge Ad$ composed with $pr \circ (S \wedge 1)^{-1} = \mu \wedge 1$ and this is $(\mu \wedge 1) \circ (1 \wedge i \wedge 1) = \mathrm{id}$.

b) $\mu \circ f_2 = 0$

$1 \wedge S$ restricted to $Ad \wedge \Sigma^{-1}Ad\mathbf{Q}$ is $1 \wedge R$. We have to compute $\mu \circ (\mu \wedge 1) \circ (1 \wedge R)$, which is determined by its restrictions to $S^0 \wedge \Sigma^{-1}Ad\mathbf{Q}$ and $S^{-1} \wedge \Sigma^{-1}Ad\mathbf{Q}$. For the first restriction we have

$$\mu \circ (\mu \wedge 1) \circ (i \wedge R) = \mu \circ (\mu \wedge 1) \circ (i \wedge 1 \wedge 1) \circ (1_{S^0} \wedge R) = \mu \circ R = 0 .$$

The second restriction is reduced to this by

$$\mu \circ (\mu \wedge 1) \circ (i_{-1} \wedge R) = \mu \circ (\mu \wedge 1) \circ (1 \wedge R) \circ (\delta \wedge 1)(i \wedge 1)$$

$$= \mu \circ (\mu \wedge 1) \circ \delta \wedge 1 \circ 1 \wedge R \circ i \wedge 1$$
$$= \delta \circ (\mu \circ \mu \wedge 1 \circ 1 \wedge R \circ i \wedge 1) = 0.$$

We have used $\mu \circ (\delta \wedge 1) = \mu \circ (1 \wedge \delta) = \delta \circ \mu$ which follows from 5.7 and 2.6. □

REMARK. If E is a commutative ring spectrum then its (-1)-connected cover is again a commutative ring spectrum. Hence (-1)-connected $Im(J)$-theory A is a commutative ring spectrum.

This follows also from the facts that p-localized Quillen K-theory of a finite field, $K\mathbf{F}_k$, k chosen as a prime power, is a model for A ($p \neq 2$) and $K\mathbf{F}_k$ is a commutative ring spectrum. It also follows from the facts that $L_K S^0$ is a commutative ring spectrum and A is the (-1)-connected cover of $L_K S^0$. A direct proof that A is a ring spectrum based on the Adams spectral sequence (but without convergence considerations) may be found in [29].

Denote by l the Adams summand in p-local (-1)-connected K-theory, $l_*(S^0) = \mathbf{Z}_{(p)}[v_1]$, with $v_1 = u^{p-1}$, and let Q be the operation with $v_1 \circ Q = \psi^k - 1$. Then A may also be defined by the cofibre sequence

$$\longrightarrow A \xrightarrow{D} l \xrightarrow{Q} \Sigma^q l \xrightarrow{\Delta} \Sigma A \longrightarrow$$

and the multiplication $\mu_A : A \wedge A \longrightarrow A$ fits into the commutative diagram

$$\begin{tikzcd}
\arrow[r] & A\wedge A \arrow[r, "D\wedge 1"] \arrow[d, "\mu_A"] & l\wedge A \arrow[r, "Q\wedge 1"] \arrow[d, "\bar\mu"] & \Sigma^q l\wedge A \arrow[r, "\Delta\wedge 1"] \arrow[d, "\bar\mu"'] & \Sigma A\wedge A \arrow[r] \arrow[d] & {} \\
\arrow[r] & A \arrow[r, "D"] & l \arrow[r, "Q"] & \Sigma^q l \arrow[r, "\Delta"] & \Sigma A \arrow[r] & {}
\end{tikzcd} \tag{5.9}$$

This is also proved in [29]. The proof that the middle square in (5.9) commutes is similar to (2.5) using in addition the cofibre sequence

$$\Sigma^q l \xrightarrow{v_1} l \longrightarrow H\mathbf{Z}_{(p)}.$$

Then we may define μ_A as a fill-in map in (5.9). To compare μ_A with the product μ'_A induced from Ad (and thus proving associativity and commutativity for μ_A) we use

LEMMA (see [29]). $[A \wedge A, A] \xrightarrow{D_*} [A \wedge A, l]$ *is injective.*

This may also be proved using the splitting of $l \wedge A$ [16] for computing $[A \wedge A, \Sigma^{q-1} l] = 0$ (and thus avoiding all convergence problems). Since μ'_A satisfies $D\mu'_A = D\mu_A$ we must have $\mu'_A = \mu_A$.

6. $Ad_*(P_\infty\mathbf{C})$

In this section we treat as an example the $Im(J)$-groups of the complex projective space $Ad_*(P_\infty\mathbf{C})$. This computation is contained implicitly in [14] and also follows from the determination of $L_K(P_\infty\mathbf{C})$ in [22]. Recently Hesselholt and Madsen [13] have given a computation of $Ad^*(P_\infty\mathbf{C})$. One possible approach for computing $Ad_*(P_\infty\mathbf{C})$ is to approximate $P_\infty\mathbf{C}$ by $B\mathbf{Z}/p^r$ and use the computation of $Ad_*(B\mathbf{Z}/p^r)$ [18], see §8. This is the method used in [13] or [14]. We give a direct calculation.

Let $x = H - 1 \in K^0(P_\infty\mathbf{C})$, H the Hopf line bundle, and denote by $b_n \in K_{2n}(P_\infty\mathbf{C})$ the element dual to x^n. If $\sigma \in \pi_2^s(P_\infty\mathbf{C}) = \mathbf{Z}$ is a generator, then $h_K(\sigma) = b_1$ up to sign. Denote $h_{Ad}(\sigma) \in Ad_2(P_\infty\mathbf{C})$ by $\tilde{b}_1$, then

Lemma 6.1. *$Ad_{2n}(P_\infty\mathbf{C}) \cong \mathbf{Z}_{(p)} \cdot \tilde{b}_1^n$ for $n > 0$.*

Proof. The n-th power of $\tilde{b}_1$ in the Pontryagin ring structure generates a subgroup $\mathbf{Z}_{(p)} \cdot \tilde{b}_1^n$ of $Ad_{2n}(P_\infty\mathbf{C})$ of maximal rank since $Ad_{2n}(P_\infty\mathbf{C};\mathbf{Q}) = \mathbf{Q}$. If $\tilde{b}_1^n = c \cdot z$ in $Ad_{2n}(P_\infty\mathbf{C})$ then $\langle x, D(z)\rangle_K \in \mathbf{Z}_{(p)}$ implies

$$\frac{1}{c}\langle x, D(\tilde{b}_1^n)\rangle_K = \frac{1}{c}\langle ch_{2n}(x), n![P_n\mathbf{C}]\rangle_{H_*(\ ,\mathbf{Q})} = \frac{1}{c} \in \mathbf{Z}_{(p)},$$

hence the result. □

Let $\langle\ ,\ \rangle_K : K^0(X) \otimes K_0(X;\mathbf{Q}/\mathbf{Z}_{(p)}) \to \mathbf{Q}/\mathbf{Z}_{(p)}$ be the Kronecker product, then

Lemma 6.2. *Any element $z \in ker(\psi^k - 1) \subset K_{2n}(P_\infty\mathbf{C};\mathbf{Q}/\mathbf{Z}_{(p)})$ is completely determined by the Kronecker products*

$$\gamma_i(z) := \langle H^{p^i}, z\rangle_K, \quad i = 0, 1, 2, \ldots .$$

Note first that on $K_0(X;\mathbf{Q})$ the eigenspace of ψ^j with respect to the eigenvalue j^n is independent of j as long as $j \neq \pm 1$. On $K_0(X;\mathbf{Q}/\mathbf{Z}_{(p)})$ we only have $ker(\psi^k - k^n) \subset ker(\psi^j - j^n)$ as long as $j \not\equiv 0(p)$ and k generates $(\mathbf{Z}/p^2)^*$. This follows easily from the periodicity properties of Adams operations discussed in [1].

If we identify $K_{2n}(X)$ with $K_0(X)$ then under this isomorphism $ker(\psi^k - 1)$ is the same as $ker(\psi^k - k^{-n}) = ker(k^n\psi^k - 1)$.

Proof of 6.2. Let $z \in K_0(P_m\mathbf{C};\mathbf{Q}/\mathbf{Z}_{(p)})$ be given. Since $K_0(P_m\mathbf{C}_+;\mathbf{Q}/\mathbf{Z}_{(p)})$ is isomorphic to $Hom(K^0(P_m\mathbf{C}_+);\mathbf{Q}/\mathbf{Z}_{(p)})$, z is determined by the Kronecker products with $1, x, x^2, \ldots, x^m$. But instead of $1, x, x^2, \ldots, x^m$ we may use the basis $1, H, H^2, \ldots, H^m$. Write $j = p^i \cdot s$, $s \not\equiv 0(p)$ and assume $z \in K_0(P_\infty\mathbf{C};\mathbf{Q}/\mathbf{Z}_{(p)})$ is in $ker(\psi^k - k^{-n})$ (i.e. z is in $ker(\psi^k - 1)$ if z is viewed as element in $K_{2n}(P_\infty\mathbf{C};\mathbf{Q}/\mathbf{Z}_{(p)})$). Then

$$\begin{aligned}\langle H^j, z\rangle_K &= \langle H^{p^i s}, z\rangle_K = \langle \psi^s H^{p^i}, z\rangle_K = \langle H^{p^i}, c\psi^s(z)\rangle_K \\ &= s^n\langle H^{p^i}, z\rangle_K = s^n\gamma_i(z).\end{aligned}$$

Here $c\psi^s$ is adjoint to ψ^s as in Section 3 and by the remark above $c\psi^s = \psi^{1/s}$ acts by multiplication with s^n on $ker(\psi^k - k^{-n})$. □

Denote by $\omega : S^2 \wedge P_\infty\mathbf{C} \to P_\infty\mathbf{C}$ the (stable) map inducing multiplication by σ. Then by Snaith's theorem (for an easy proof see [5]) the mapping telescope of ω, $P_\infty\mathbf{C}[\omega^{-1}]$, is complex K-theory

$$P_\infty\mathbf{C}[\omega^{-1}] \simeq K. \tag{6.3}$$

We now first reprove the splitting of $Ad \wedge K$ (2.1) using 6.2.

Since $\omega_*(\tilde{b}_1^i) = \tilde{b}_1^{i+1}$, 6.1 implies

$$Ad_{2n}(P_\infty\mathbf{C})\big[\omega_*^{-1}\big] = Ad_{2n}\big(P_\infty\mathbf{C}[\omega^{-1}]\big) = Ad_{2n}(K) = \mathbf{Z}_{(p)}.$$

Also the map

$$Ad_{2n}(P_\infty\mathbf{C}) \xrightarrow{l} Ad_{2n}(K) \xrightarrow{D} K_{2n}(K) \xrightarrow{\mu_*} K_{2n}(S^0)$$

is an isomorphism since $Dl(\tilde{b}_1^n) = h_K(u^n)$.

Next, $\kappa_{-1} : Ad_{2n-1}(P_\infty\mathbf{C}) \longrightarrow H_{2n}(P_\infty\mathbf{C};\mathbf{Q})$ is onto. This is seen by evaluating κ_{-1} on $\Delta(b_m u^{n-m})$:

From $b_1 \cdot b_n = nb_n \cdot u + (n+1)b_{n+1}$ in $K_{2n+2}(P_\infty\mathbf{C})$ it follows that

$$b_n = \binom{b_1/u}{n} u^n \tag{6.4}$$

and therefore

$$b_m = \sum_{k=1}^{m} \frac{s(m,k)}{m!}\, b_1^k u^{m-k}$$

with $s(m,k)$ as in Section 2. Then

$$\kappa_{-1}\Delta(b_m u^{n-m}) = ch_0(b_m u^{n-m}) = \frac{s(m,n)n!}{m!}$$

and the claim follows as in Section 2.

The Bockstein sequence associated to $\mathbf{Z}_{(p)} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}_{(p)}$ induces an exact sequence

$$\begin{aligned} Ad_{2n}(P_\infty\mathbf{C};\mathbf{Q}) \xrightarrow{r} Ad_{2n}\big(P_\infty\mathbf{C};\mathbf{Q}/\mathbf{Z}_{(p)}\big) &\xrightarrow{\beta} Ad_{2n-1}(P_\infty\mathbf{C}) \\ &\xrightarrow{\kappa_{-1}} \mathbf{Q} \to 0. \end{aligned} \tag{6.5}$$

We show, that if $z = \beta(y)$ in $Ad_{2n-1}(P_\infty\mathbf{C})$, then $\omega_*^i(z) = 0$ for i large enough. For this, it is enough to show $\omega_*^i(y) \in im(r)$ for i large. Since

$$D : Ad_{2n}(P_\infty\mathbf{C};\mathbf{Q}/\mathbf{Z}_{(p)}) \to K_{2n}(P_\infty\mathbf{C};\mathbf{Q}/\mathbf{Z}_{(p)})$$

is injective we may use the γ_i-sequences of 6.2 to check this.

LEMMA 6.6. $\gamma_i(\omega_*(y)) = p^i\gamma_i(y)$ *for* $y \in K_0(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)})$.

PROOF. $\omega^*(H^{p^i}) = p^i \cdot H^{p^i}$ implies this statement. □

Given $y \in Ad_{2n}(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)})$, then for t large enough, only $\gamma_0(\omega_*^t(y)) = \gamma_0(y)$ can be nonzero. On the other hand $\gamma_i(\tilde{b}_1^n/p^s) = p^{i\cdot n}/p^s$, hence for n large enough (compared with s) $\gamma_i(\tilde{b}_1^n/p^s) = 0$ for $i > 0$, too. This implies $\omega_*^t(y) \in im(r_*) = \mathbf{Q}/\mathbf{Z}_{(p)}\cdot\tilde{b}_1^n$ finishing the proof of 2.1.

It is now easy to complete the computation of $Ad_*(P_\infty\mathbf{C})$:

We would like to construct elements $z_{r,b} \in K_0(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)})$ in $ker(\psi^k - k^{-n})$ with γ_i-sequence

$$\gamma_i(z_{r,b}) = \delta_{i,r}/p^b \quad (\delta_{i,r} \text{ Kronecker symbol}).$$

If $\underline{H}^i$ is "dual" to H^i then

$$z_{r,b} = \frac{1}{p^b}\sum_{i\not\equiv 0(p)} i^n \underline{H}^{ip^r}, \quad b > 0,\ r \geqslant 0, \tag{6.7}$$

would be the right element. To make this precise identify $K^0(P_\infty\mathbf{C}_+)$ with $\mathbf{Z}_{(p)}[[x]]$ and define C by the exact sequence

$$0 \to \mathbf{Z}_{(p)}[x] \xrightarrow{i} \mathbf{Z}_{(p)}[[x]] \longrightarrow C \to 0.$$

Since $Hom(-, \mathbf{Q}/\mathbf{Z}_{(p)})$ is exact we get an exact sequence

$$\begin{aligned} 0 \leftarrow Hom(\mathbf{Z}_{(p)}[x], \mathbf{Q}/\mathbf{Z}_{(p)}) &\xleftarrow{i^*} Hom(\mathbf{Z}_{(p)}[[x]], \mathbf{Q}/\mathbf{Z}_{(p)}) \\ &\longleftarrow Hom(C, \mathbf{Q}/\mathbf{Z}_{(p)}) \leftarrow 0. \end{aligned}$$

Let

$$\begin{aligned} Hom_c(\mathbf{Z}_{(p)}[[x]], \mathbf{Q}/\mathbf{Z}_{(p)}) :&= \varinjlim Hom(K^0(P_m\mathbf{C}_+), \mathbf{Q}/\mathbf{Z}_{(p)}) \\ &\cong K_0(P_\infty\mathbf{C}_+; \mathbf{Q}/\mathbf{Z}_{(p)}). \end{aligned}$$

Then the canonical inclusion

$$Hom_c(\mathbf{Z}_{(p)}[[x]], \mathbf{Q}/\mathbf{Z}_{(p)}) \to Hom(\mathbf{Z}_{(p)}[[x]], \mathbf{Q}/\mathbf{Z}_{(p)})$$

composed with i^* gives an injection

$$s : Hom_c(\mathbf{Z}_{(p)}[[x]], \mathbf{Q}/\mathbf{Z}_{(p)}) \longrightarrow Hom(\mathbf{Z}_{(p)}[x], \mathbf{Q}/\mathbf{Z}_{(p)}). \tag{6.8}$$

Note that $\underline{H}^j/p^b$ defines a well defined element in $Hom_{\mathbf{Z}_{(p)}}(\mathbf{Z}_{(p)}[x], \mathbf{Q}/\mathbf{Z}_{(p)})$ namely the map

$$x^s \longmapsto \langle x^s, \underline{H}^j/p^b\rangle = (-1)^{s+j}\binom{s}{j}p^{-b}.$$

Also

$$\sum_{i\not\equiv 0(p)} i^n \underline{H}^{ip^r}/p^b \in Hom(\mathbf{Z}_{(p)}[x], \mathbf{Q}/\mathbf{Z}_{(p)})$$

is well defined since

$$\left\langle x^s, \sum_{i\not\equiv 0(p)} i^n \underline{H}^{ip^r}/p^b \right\rangle = \sum_{i\not\equiv 0(p)} (-1)^{s+ip^r} \binom{s}{ip^r} i^n/p^b$$

involves only finitely many values of i.

Define numbers

$$P_r(n,s) := \sum_{i\not\equiv 0(p)} (-1)^{s+ip^r} \binom{s}{ip^r} i^n \tag{6.9}$$

for $r \in \mathbf{N}$, $n \in \mathbf{Z}$ and denote the power of p in $m \in \mathbf{N}$ by $\nu_p(m)$. We need

LEMMA 6.10. $\lim_{s\to\infty} \nu_p(P_r(n,s)) = \infty$.

PROOF. For $i > 0, t > 0, s > 0$ and $r \geqslant 0$ let

$$a_r(i,t,s) := \sum_{j\geqslant 0} (-1)^{s+ip^r+jp^{r+t}} \binom{s}{ip^r + jp^{r+t}}.$$

Then it is easy to see – and carried out, for example, in (2.11) [18] – that

$$\lim_{s\to\infty} \nu_p\big(a_r(i,t,s)\big) = \infty.$$

But

$$P_r(n,s)/p^b \equiv \sum_{\substack{i=1 \\ i\not\equiv 0(p)}}^{p^r} a_r(i,b,s) i^n/p^b \bmod \mathbf{Z}$$

hence $P_r(n,s)/p^b \in \mathbf{Z}_{(p)}$ for s large enough and the lemma is proved. □

REMARK. More precise information on the numbers $P_r(n,s)$ is contained in [15].

Now

$$\left\langle x^s, \sum_{i\not\equiv 0(p)} i^n \underline{H}^{ip^r}/p^b \right\rangle = P_r(n,s)/p^b,$$

therefore only finitely many x^s evaluate nontrivially on $z_{r,b}$. Hence $z_{r,b}$ belongs to

$$Hom_c\big(\mathbf{Z}_{(p)}[[x]], \mathbf{Q}/\mathbf{Z}_{(p)}\big) = K_0\big(P_\infty\mathbf{C}_+; \mathbf{Q}/\mathbf{Z}_{(p)}\big).$$

With respect to the usual basis we have

$$z_{r,b} = \sum_s P_r(n,s)\cdot b_s/p^b. \tag{6.11}$$

LEMMA 6.12. *$(c\psi^k - k^n)(z_{r,b}) = 0$ and $\gamma_i(z_{r,b}) = \delta_{i,r}/p^b$.*

PROOF. Since the map s of (6.8) is injective it is enough to show

$$\big\langle H^j, (c\psi^k - k^n)(z_{r,b})\big\rangle_K = 0$$

for all $j > 0$. In $Hom(\mathbf{Z}_{(p)}[x], \mathbf{Q}/\mathbf{Z}_{(p)})$ we may use

$$\sum_{i\not\equiv 0(p)} i^n \underline{H}^{ip^r}/p^b$$

as a description of $z_{r,b}$. We have

$$\begin{aligned}\big\langle H^j, (c\psi^k - k^n)(z_{r,b})\big\rangle &= \big\langle \psi^k H^j - k^n H^j, z_{r,b}\big\rangle \\ &= \Big\langle H^{kj} - k^n H^j, \sum_{i\not\equiv 0(p)} i^n \underline{H}^{ip^r}/p^b\Big\rangle.\end{aligned}$$

If $j \neq sp^r$, $s \not\equiv 0(p)$, then this is trivially zero. If $j = sp^r$, $s \not\equiv 0(p)$, then

$$\Big\langle H^{ksp^r} - k^n H^{sp^r}, \sum_{i\not\equiv 0(p)} i^n \underline{H}^{ip^r}/p^b\Big\rangle = \big((ks)^n - k^n\cdot s^n\big)/p^b = 0.$$

For $\gamma_j(z_{r,b})$ we get

$$\gamma_j(z_{r,b}) = \Big\langle H^{p^j}, \sum_{i\not\equiv 0(p)} i^n \underline{H}^{ip^r}/p^b\Big\rangle = \delta_{r,j}/p^b. \qquad \square$$

Consider now $z_{r,b}$ as an element in $K_{2n}(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)})$, then 6.12 means $(\psi^k - 1)(z_{r,b}) = 0$ (see proof of 3.8). Since $D : Ad_{2n}(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)}) \to K_{2n}(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)})$ is injective, $z_{r,b}$ defines a well-defined element in $Ad_{2n}(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)})$, which we shall also denote by $z_{r,b}$.

PROPOSITION 6.13.

$$Ad_{2n}\big(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)}\big) = \bigoplus_{i=0}^{\infty} \mathbf{Q}/\mathbf{Z}_{(p)}$$

and the r-th summand is generated by the elements $z_{r,b}$, $b \in \mathbf{N}$.

PROOF. Let $m_p : P_\infty\mathbf{C} \to P_\infty\mathbf{C}$ be the p-th power map in the H-space structure of $P_\infty\mathbf{C}$. Then $m_p^* H^{p^i} = H^{p^{i+1}}$ implies:

$$m_{p*}(z_{r,b}) = z_{r-1,b} \quad \text{for } r \geqslant 1 \qquad \text{and} \qquad m_{p*}(z_{0,b}) = 0.$$

Hence there are no relations between the elements $z_{r,b}$. Any $z \in Ad_{2n}(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)})$ is determined by its γ_i-sequence $\gamma_i(z) = \langle H^{p^i}, D(z)\rangle_K \in \mathbf{Q}/\mathbf{Z}_{(p)}$ and only finitely many $\gamma_i(z)$ can be nonzero. Write $\gamma_i(z) = a_i/p^{b_i}$, $a_i \in \mathbf{Z}_{(p)}$, then $\sum a_i \cdot z_{i,b_i}$ and z must be equal since they have the same γ_i-sequence. □

COROLLARY 6.14.

$$Ad \wedge P_\infty\mathbf{C} \wedge M(\mathbf{Q}/\mathbf{Z}_{(p)}) \simeq \bigvee_{i=0}^{\infty} K\mathbf{Q}/\mathbf{Z}_{(p)}.$$

PROOF. The slant product with H^{p^i} (after applying $D : Ad \to K$) defines a map $Ad \wedge P_\infty\mathbf{C} \wedge M(\mathbf{Q}/\mathbf{Z}_{(p)}) \to K \wedge M(\mathbf{Q}/\mathbf{Z}_{(p)})$ which induces the invariant γ_i of 6.2 in homotopy. In order to get a map to

$$\bigvee_{i=0}^{\infty} K\mathbf{Q}/\mathbf{Z}_{(p)}$$

we have to make a change of basis and use the slant products with x^{p^i} where $x = H - 1$. Since $x^{p^i}_{|P_n\mathbf{C}} \simeq 0$ for i larger than n the maps $x^{p^i} : P_n\mathbf{C} \to K$ may be added up to give maps

$$P_n\mathbf{C} \to \bigvee_{i=0}^{\infty} K$$

with inverse limit

$$P_\infty\mathbf{C} \to \bigvee_{i=0}^{\infty} K.$$

The slant product with this map gives

$$Ad \wedge P_\infty\mathbf{C} \wedge M\big(\mathbf{Q}/\mathbf{Z}_{(p)}\big) \longrightarrow \bigvee_{i=0}^{\infty} K\mathbf{Q}/\mathbf{Z}_{(p)}$$

which induces an isomorphism in homotopy groups by 6.13. □

The exact sequence (6.5) implies:

COROLLARY 6.15.

$$Ad_{2n}(P_\infty\mathbf{C}) = \mathbf{Z}_{(p)}, \quad n > 0,$$

$$Ad_{2n}(P_\infty\mathbf{C}) = 0, \quad n \leqslant 0,$$

$$Ad_{2n-1}(P_\infty\mathbf{C}) = \mathbf{Q} \oplus \bigoplus_{r=1}^{\infty} \mathbf{Q}/\mathbf{Z}_{(p)}, \quad n > 0,$$

$$Ad_{2n-1}(P_\infty\mathbf{C}) = \bigoplus_{r=0}^{\infty} \mathbf{Q}/\mathbf{Z}_{(p)}, \quad n \leqslant 0.$$

The Pontryagin ring structure on $Ad_*(P_\infty\mathbf{C})$ now follows trivially.

Only products with powers of $\bar{b}_1$ can be nontrivial. The result of $\bar{b}_1^k \cdot z$ is easily derived from the fact that $\bar{b}_1^k \cdot z = \omega_*^k(z)$ and the knowledge of ω_* on $Ad_*(P_\infty\mathbf{C})$ 6.6 and $H_*(P_\infty\mathbf{C};\mathbf{Q})$.

The Ad-cohomology groups $Ad^*(P_\infty\mathbf{C})$ follow easily from 3.1.

The spectra $Ad \wedge P_\infty\mathbf{C}$ and $Ad \wedge bu$ are related as follows. Let $\varepsilon : P_\infty\mathbf{C} \to \Sigma^2 bu$ denote the canonical map. An easy consequence of 6.15 and 2.8 is that $\varepsilon_* : Ad_*(P_\infty\mathbf{C}) \to Ad_*(\Sigma^2 bu)$ is onto. A result of Ravenel [22, (9.2)] identifies the fibre of $1 \wedge \varepsilon : Ad \wedge P_\infty\mathbf{C} \to Ad \wedge \Sigma^2 bu$.

THEOREM 6.16 ([22]). *There exists a map*

$$g : \bigvee_{i=1}^{\infty} \Sigma^{-1} K\mathbf{Q}/\mathbf{Z}_{(p)} \to Ad \wedge P_\infty\mathbf{C}$$

such that

$$\bigvee_{i=1}^{\infty} \Sigma^{-1} K\mathbf{Q}/\mathbf{Z}_{(p)} \xrightarrow{g} Ad \wedge P_\infty\mathbf{C} \xrightarrow{1\wedge\varepsilon} Ad \wedge \Sigma^2 bu \tag{6.17}$$

is a cofibre sequence.

In [22] this is stated in terms of K-theory localization, but $L_K P_\infty\mathbf{C} \longrightarrow L_K \Sigma^2 bu$ and $Ad \wedge P_\infty\mathbf{C} \longrightarrow Ad \wedge \Sigma^2 bu$ have the same fibre, since ε is a rational equivalence. Later on we shall see that (6.17) does not split.

PROOF OF 6.16. We may deduce (6.17) from 6.14 as follows: Let $\Sigma^{-1}F$ be the fibre of the map $1 \wedge \varepsilon : Ad \wedge P_\infty\mathbf{C} \to Ad \wedge \Sigma^2 bu$. Then, since ε is a rational equivalence, F is also the fibre of

$$1 \wedge \varepsilon \wedge 1 : Ad \wedge P_\infty\mathbf{C} \wedge \mathbf{M}(\mathbf{Q}/\mathbf{Z}_{(p)}) \to Ad \wedge \Sigma^2 bu \wedge M(\mathbf{QZ}_{(p)}).$$

Mapping Ad and $\Sigma^2 bu$ into K and then using multiplication induces an equivalence f_2

$$Ad \wedge \Sigma^2 bu \wedge M(\mathbf{Q}/\mathbf{Z}_{(p)}) \simeq K\mathbf{Q}/\mathbf{Z}_{(p)}.$$

The composition of this equivalence with $1 \wedge \varepsilon \wedge 1$ is nothing but the slant product with $x = H - 1$. This shows that we have a commutative diagram

$$\begin{tikzcd}
Ad \wedge P_\infty \mathbf{C} \wedge M(\mathbf{Q}/\mathbf{Z}_{(p)}) \arrow[r, "1\wedge\varepsilon\wedge 1"] \arrow[d, "\simeq"', "f_1"] & Ad \wedge \Sigma^2 bu \wedge M(\mathbf{Q}/\mathbf{Z}_{(p)}) \arrow[d, "\simeq"', "f_2"] \\
\bigvee_{i=0}^{\infty} K\mathbf{Q}/\mathbf{Z}_{(p)} \arrow[r, "pr_1"] & K\mathbf{Q}/\mathbf{Z}_{(p)}
\end{tikzcd}$$

where f_1 is the equivalence of 6.14 and pr_1 the canonical map onto the first wedge summand. But clearly the fibre of pr_1 is $\bigvee_{i=1}^{\infty} K\mathbf{Q}/\mathbf{Z}_{(p)}$. □

7. $Ad_*(BT^m)$

Let BT^m denote the classifying space of an m-torus. In this section we discuss the groups $Ad_{2n}(BT^m)$, which admit a nice combinatorial and number-theoretic interpretation.

Call – as in [24] – a polynomial $f(x_1, x_2, \ldots, x_m) \in \mathbf{Q}[x_1, x_2, \ldots, x_m]$ *numerical* if $f(k_1, k_2, \ldots, k_m) \in \mathbf{Z}$ for all $k_i \in \mathbf{Z}$ and define

$$N^{(m)} := \{f \in \mathbf{Q}[x_1, x_2, \ldots, x_m] \mid f \text{ is numerical}\}.$$

It is well known that the usual binomial polynomials $\binom{x}{n} \in \mathbf{Q}[x]$ constitute a $\mathbf{Z}$-basis for $N^{(1)}$, the ring of rational polynomials which take integer values on integers. Similarly the products

$$\binom{x_1}{n_1}\binom{x_2}{n_2}\cdots\binom{x_m}{n_m}$$

give a $\mathbf{Z}$-basis for $N^{(m)}$. Then the basic observation (6.4)

$$b_n = \binom{b_1}{n} \quad \text{in } K_0(P_\infty \mathbf{C})$$

implies $K_0(P_\infty \mathbf{C}_+) \cong N^{(1)}$ and similarly $K_0(BT^m_+) \cong N^{(m)}$. This is an isomorphism of rings, where $K_0(BT^m_+)$ has the usual Pontryagin ring structure. The proofs are easy and references may be found, for example, in [7].

The p-local version is

$$K_0(BT^m_+)_{(p)} \cong N^{(m,p)} := \{f \in \mathbf{Q}[x_1, x_2, \ldots, x_m] \mid f(k_1, k_2, \ldots, k_m) \in \mathbf{Z}_{(p)} \text{ if all } k_i \in \mathbf{Z}_{(p)}\}.$$

In the following we shall work purely p-locally, call elements in $N^{(m,p)}$ p-numerical (or even numerical) and suppress the symbol for localization from the notation.

Let x_i have degree 1.

PROPOSITION 7.1. *$Ad_{2n}(BT_+^m) \cong \{f \in N^{(m,p)} \mid f$ is homogeneous of degree $n\}$ with Pontryagin multiplication on*

$$\bigoplus_{n\geqslant 0} Ad_{2n}(BT_+^m)$$

corresponding to polynomial multiplication.

PROOF. The defining sequence (1.1) for $X = BT^m$ shows

$$Ad_{2n}(BT^m) \cong ker(\psi^k - 1) \subset K_{2n}(BT^m).$$

But $ker(\psi^k - 1)$ on $K_{2n}(BT^m)$ is nothing but the eigenspace of ψ^k on $K_0(BT^m)$ with respect to the eigenvalue k^{-n}. On $K_0(P_\infty\mathbf{C})_{(p)}$ we have $\psi^k(b_1) = k^{-1}b_1$ hence $\psi^k(x_i) = k^{-1}x_i$. Therefore only homogeneous polynomials of degree n are in $ker(\psi^k - k^{-n}) \subset K_0(BT^m)$. □

The group $Ad_{2n}(BT_+^m)$ is a free $\mathbf{Z}_{(p)}$-module of rank $\binom{n+m-1}{m-1}$ and we may ask the following question.

PROBLEM. *Construct a $\mathbf{Z}_{(p)}$-basis of the free $\mathbf{Z}_{(p)}$-module*

$$\big\{f \in \mathbf{Q}[x_1, x_2, \dots, x_m] \mid f(k_1, \dots, k_m) \in \mathbf{Z}_{(p)} \text{ for all } k_i \in \mathbf{Z}_{(p)} \text{ and } f \text{ is homogeneous of degree } n\big\}.$$

The case $m = 1$ is trivial and has already appeared in 6.1. The case $m = 2$ was solved independently by L. Schwartz in [24] and the present author (unpublished, 1980, see [15]) and will be described below. Although this problem is formulated completely in elementary terms, a solution for $m > 2$ seems not to be known. Computations for three variables at the prime 3 for $n \leqslant 27$ do not suggest any reasonable answer.

To describe the solution for $m = 2$ we identify $Ad_{2n}(BT_+^2)$ with the group of homogeneous polynomials in x and y which are p-numerical. For $f \in Ad_{2n}(BT_+^2)$ and $n > 1$ define as in [14]

$$Q(f) := \big(f^p - f \cdot h_n(x,y)\big)/p$$

where $h_n(x,y) := x^{(p-1)n} + y^{(p-1)n} - x^{(p-1)}y^{(p-1)(n-1)}$. The polynomial h_n has the property that $h_n(s,t) \equiv 1 \bmod p$ for $s, t \in \mathbf{Z}$ not both 0 mod p. Any other polynomial with this property will also do. To prove that $Q(f)$ is p-numerical, let $s, t \in \mathbf{Z}_{(p)}$ be given. If both s and t are divisible by p we are done since f is homogeneous. If $s \not\equiv 0$

or $t \not\equiv 0 \bmod p$ then $h_n(s,t) \equiv 1 \bmod p$ and $f(s,t)^p \equiv f(s,t) \bmod p$ proves that $f^p - f \cdot h_n$ is divisible by p.

REMARK. Operators $Q : Ad_{2n}(BT^m) \to Ad_{2np}(BT^m)$ for $m > 2$ may be defined similarly.

Define now a series of p-numerical polynomials

$$P_i(x,y) \in Ad_{2n}(BT^2), \quad n = (p+1)p^{i-1},$$

inductively by

$$P_1(x,y) = (x^p y - xy^p)/p,$$
$$P_i(x,y) = Q(P_{i-1}(x,y)),$$

and for

$$m = \sum_{i=1}^{r} m_i p^{i-1}$$

with $0 \leqslant m_i < p$ let

$$P(m) := P_1^{m_1} \cdot P_2^{m_2} \cdots P_r^{m_r} \in Ad_{2n(p+1)}(BT^2).$$

Then

THEOREM 7.2 ([24], [15]). a) *$x, y, P_1, P_2, \dots P_i, \dots$ generate $\bigoplus_{n\geqslant 0} Ad_{2n}(BT^2)$ as a ring, or*

b) *$\{P(m)x^a y^b \mid m \geqslant 0,\ a \leqslant p-1 \text{ if } b > 0\}$ is a $\mathbf{Z}_{(p)}$-basis of $\bigoplus_{n\geqslant 0} Ad_{2n}(BT^2)$, or*

c) *Write $n = m(p+1) + l$, $0 \leqslant l \leqslant p$. We have $(l+1)$ polynomials*

$$P(m)y^l,\ P(m)xy^{l-1}, \dots, P(m)x^l$$

and for every s with $0 \leqslant s < m$ $(p+1)$ polynomials

$$P(s)y^t,\ P(s)xy^{t-1}, \dots, P(s)x^{p-1}y^{t-p+1},\ P(s)x^t$$

with $t = n - s(p+1)$. These are exactly $n+1 = (p+1)m + l + 1$ polynomials, which are a $\mathbf{Z}_{(p)}$-basis of $Ad_{2n}(BT^2)$ for $n > 0$.

For a proof, that given by Schwartz in [24] or [7] is recommended, it is much shorter than that in [15]. Applications of the polynomials $P_i(x,y)$ or 7.2 may be found in [7], [9] or [14].

The computation of $Ad_{2n}(BT^2)$ may now be used to show that the cofibre sequence (6.17)

$$\bigvee_{i\geqslant 1} \Sigma^{-1} K\mathbf{Q}/\mathbf{Z}_{(p)} \longrightarrow Ad \wedge P_\infty \mathbf{C} \xrightarrow{1\wedge\epsilon} Ad \wedge \Sigma^2 bu$$

does not split. For example, let $p = 3$ and $n = 10$ then

$$Ad_{2n}(P_\infty\mathbf{C} \wedge P_\infty\mathbf{C}) \xrightarrow{1\wedge\varepsilon_*} Ad_{2n-2}(P_\infty\mathbf{C} \wedge bu)$$

is not onto and $coker\,\varepsilon_*$ is a nontrivial finite group. Since

$$P_\infty\mathbf{C} \wedge bu \simeq \bigvee_{i\geqslant 1} \Sigma^{2i}bu$$

the groups $Ad_{2n-2}(P_\infty\mathbf{C} \wedge bu)$ are known by (2.8). In odd dimensions, the cofibre sequence (6.17) reduces to

$$\begin{aligned} 0 \to coker\,\varepsilon_* \to \bigoplus_{i\geqslant 1} K_{2n}\big(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}_{(p)}\big) \to Ad_{2n-1}(P_\infty\mathbf{C} \wedge P_\infty\mathbf{C}) \\ \to Ad_{2n-3}(P_\infty\mathbf{C} \wedge bu) \to 0 \end{aligned}$$

giving a description of $Ad_{2n-1}(BT^2)$.

The description of the elements of $Ad_{2n}(BT^m)$ by numerical polynomials has been extended by L. Schwartz to the case of a compact connected Lie group G, see [21].

8. $Ad_*(BG)$ for a finite group G

The close connection between the representation ring $R(G)$ and the K-theory of the classifying space of a finite group G extends to $Im(J)$-theory. The $Im(J)$-homology groups $Ad_*(BG)$ of BG are much simpler to write down than for example their ordinary homology groups. We only give a short review, details and proofs may be found in [18].

We shall concentrate on the Ad-homology groups, the groups $Ad^*(BG)$ are then given by the universal coefficient formula 3.1, e.g., for $n \neq 0$ $\quad Ad^{1+2n}(BG) \cong Ad_{2n-1}(BG)$. There is also a direct and different approach to $Ad^n(BG)$ for $n \leqslant 0$ due to Rector [23]: Since the p-localization of Quillen's K-theory $K\mathbf{F}_k$ of the finite field $\mathbf{F}_k$, where k is now chosen as a prime power, is isomorphic to connective $Im(J)$-theory A we have isomorphisms

$$K\mathbf{F}_k^i(BG)_{(p)} \cong A^i(BG) \cong Ad^i(BG)$$

for $i \leqslant 0$. Let $R_{\mathbf{F}_k}(G)$ denote the Grothendieck group of finitely generated $\mathbf{F}_k[G]$-modules, I the augmentation ideal of $R_{\mathbf{F}_k}(G)$ and $R_{\mathbf{F}_k}(G)^\wedge$ the I-adic completion of $R_{\mathbf{F}_k}(G)$. Then Rector proves an analogue of Atiyah's theorem (8.1):

$$K\mathbf{F}_k^0(BG) \cong R_{\mathbf{F}_k}(G)^\wedge$$

and $K\mathbf{F}_k^{2i}(BG) = 0$ for $i < 0$.

Let $R(G)$ be the complex representation ring of the finite group G, EG a contractible CW-complex with free cellular G action and $BG = EG/G$. Define a map $\alpha : R(G) \to$

$K^0(BG_+^m)$ by $\alpha(V - W) = EG^m \times_G V - EG^m \times_G W$ for complex representations V, W of G. For spaces like BG we have

$$K^*(BG) \cong \varprojlim K^*(BG^m)$$

and α extends to a map

$$\alpha : R(G) \longrightarrow K^0(BG_+).$$

The close relation between $R(G)$ and $K^*(BG)$ is then given by Atiyah's theorem [6]

$$K^0(BG_+) \cong R(G)^\wedge, \quad K^1(BG_+) = 0. \tag{8.1}$$

Here

$$R(G)^\wedge = \varprojlim R(G)/I(G)^m$$

is the completion of $R(G)$ with respect to the augmentation ideal $I(G) = \ker(\dim : R(G) \to \mathbf{Z})$ and the isomorphism in (8.1) is induced by α. Note that for G a p-group, $I(G)$-adic completion is simply p-adic completion, i.e. $K^0(BG) \cong I(G)_p^\wedge$; for general G the completion process is more complicated.

There is a corresponding result in K-homology, which may be deduced from (8.1) by duality as follows (for a direct approach see [12]):

The usual Kronecker pairing

$$\langle\ ,\ \rangle_R : R(G) \otimes R(G) \longrightarrow \mathbf{Z}$$

defined by $\langle \lambda, \mu \rangle_R := \dim Hom^G(\lambda, \mu)$ is nonsingular and induces an isomorphism

$$L_R : R(G) \otimes \mathbf{Q}/\mathbf{Z} \longrightarrow Hom(R(G), \mathbf{Q}/\mathbf{Z}).$$

From the K-theory Kronecker product

$$\langle , \rangle_K : K^i(X) \otimes K_i(X; \mathbf{Q}/\mathbf{Z}) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

we have a map

$$L : K_i(X; \mathbf{Q}/\mathbf{Z}) \longrightarrow Hom(K^i(X), \mathbf{Q}/\mathbf{Z}).$$

The map Ψ_G dual to α is defined by the composition

$$\Psi_G : K_0(BG_+; \mathbf{Q}/\mathbf{Z}) \xrightarrow{L} Hom(K^0(BG), \mathbf{Q}/\mathbf{Z}) \xrightarrow{\alpha^*} Hom(R(G), \mathbf{Q}/\mathbf{Z})$$
$$\xleftarrow{L_R} R(G) \otimes \mathbf{Q}/\mathbf{Z}.$$

PROPOSITION 8.2. *Ψ_G is a natural monomorphism and $K_1(BG;\mathbf{Q}/\mathbf{Z})$ vanishes. For a p-group G the monomorphism Ψ_G is also surjective after localization at p.*

The Bockstein sequence then gives easily $K_*(BG)$, in particular $K_0(BG) = 0$. As a consequence we have the exact sequence

$$\begin{aligned} 0 \to Ad_{2n-1}(BG) &\xrightarrow{D} K_{2n-1}(BG)_{(p)} \xrightarrow{\psi^k - 1} K_{2n-1}(BG)_{(p)} \\ &\xrightarrow{\Delta} Ad_{2n-2}(BG) \to 0. \end{aligned}$$

Since $R(G)$ is a λ-ring, Adams operations Ψ^i on $R(G)$ are defined in the usual way. For $i \not\equiv 0 \bmod p$ define $\psi^i_{2n} := i^n \Psi^i$ on $R(G) \otimes \mathbf{Q}/\mathbf{Z}_{(p)}$. Then $\Psi_G \circ \psi^i = \psi^i_{2n} \circ \Psi_G$ on $K_{2n}(BG_+;\mathbf{Q}/\mathbf{Z})_{(p)}$ provided $i \not\equiv 0 \bmod p$ and $(|G|, i) = 1$ and we may compute the groups $Ad_m(BG_+;\mathbf{Q}/\mathbf{Z})$ by means of $R(G) \otimes \mathbf{Q}/\mathbf{Z}_{(p)}$ and ψ^k_{2n}. From now on we shall assume that k is chosen to be prime to the group order of G. An immediate consequence is

PROPOSITION 8.3. *For $n \neq 0$ we have: $Ad_{2n-2}(BG) = 0$ and $Ad_{2n-1}(BG)$ is finite with $n|G| \cdot Ad_{2n-1}(BG) = 0$.*

REMARK. Note the wrong sign in the definition of ψ^i_{2n} in [18, p. 419, 427], which fortunately does not cause any serious problems.

EXAMPLE 8.4. Let p be an odd prime and $G = \mathbf{Z}/p^a$. Then $R(G) = \mathbf{Z}[\lambda]/(\lambda^{p^a} - 1)$ with λ the canonical 1-dimensional complex representation. For $i = 0, \dots, a-1$ let $s_i = (p-1)p^{a-i-1}$. Define

$$x^i_n := \sum_{s=0}^{s_i - 1} k^{ns} \cdot \lambda^{p^i k^s} / p^{a-i+\nu_p(n)}.$$

It is easy to see that this gives all elements in $ker(\psi^i_{2n} - 1)$ and therefore for $n \neq 0$

$$Ad_{2n-1}(B\mathbf{Z}/p^a) = \bigoplus_{i=0}^{a-1} \left(\mathbf{Z}/p^{a-i+\nu_p(n)}\right) \cdot x^i_n$$

and

$$Ad_{-1}(B\mathbf{Z}/p^a) \cong Ad_{-2}(B\mathbf{Z}/p^a) \cong \bigoplus_{i=0}^{a-1} \mathbf{Z}/p^\infty.$$

Note that the set of representations $\{\lambda^{p^i}, \lambda^{p^i \cdot k}, \lambda^{p^i \cdot k^2}, \dots\}$ involved in the definition of x^i_n is closed under the action of ψ^k_0. This is a general fact. The key observation is that if $(k, |G|) = 1$ then ψ^k_0 maps irreducible representations into irreducible representations, i.e. ψ^k_0 acts as a permutation on the set $Irr(G)$ of irreducible representations of G. The orbits of this action of ψ_0^k on $Irr(G)$ may now be used to describe the elements of $Ad_*(BG)$ as follows:

Let first G be a p-group. Denote by

$$V_i = \left\{\rho_1^{(i)}, \rho_2^{(i)}, \ldots, \rho_{s_i}^{(i)}\right\}, \quad i = 0, \ldots, w,$$

the orbits of the action of ψ_0^k on $Irr(G)$. By renumbering we may assume $\psi_0^k(\rho_m^{(i)}) = \rho_{m+1}^{(i)}$ for $m < s_i$ and $\psi_0^k(\rho_{s_i}^{(i)}) = \rho_1^{(i)}$. We also assume $\rho_1^{(0)} = 1$, the trivial representation and define the numbers s_i and w in this way. For an orbit $V = \{\rho_1, \rho_2, \ldots, \rho_s\}$ of the action of ψ_0^k on $Irr(G)$ define

$$x_n(V) := \frac{1}{k^{ns}-1}\sum_{j=0}^{s-1}\psi_{2n}^{k^j}(\rho_1) = \frac{1}{k^{ns}-1}\sum_{j=1}^{s}k^{n(j-1)}\rho_j$$

as an element in $R(G) \otimes \mathbf{Q}/\mathbf{Z}_{(p)} \cong K_{2n}(BG_+; \mathbf{Q}/\mathbf{Z}_{(p)})$. Then $x_n(V) \in ker(\psi_{2n}^k - 1)$ and since we may write

$$K_{2n}\big(BG_+; \mathbf{Q}/\mathbf{Z}_{(p)}\big) = \bigoplus_{\lambda \in Irr(G)} \mathbf{Z}/p^\infty \cdot \lambda$$

one gets:

Proposition 8.5. *Let G be a p-group. Then for $n \neq 0$*

$$Ad_{2n-1}(BG) \cong \bigoplus_{i=1}^{w}\big(\mathbf{Z}_{(p)}/(k^{n\cdot s_i} - 1)\big) \cdot x_n(V_i).$$

For a p-group G and $i > 0$ the numbers s_i are divisible by $p-1$, so that $\nu_p(k^{n\cdot s_i} - 1) = \nu_p(n \cdot s_i) + 1$.

Let $C(G)$ be the set of conjugacy classes of G and let ψ_0^k act on $C(G)$ as the k-th power map. It turns out that the two ψ_0^k-sets $C(G)$ and $Irr(G)$ are equivariantly isomorphic. This implies that the numbers s_i and w which describe the abelian group structure of $Ad_*(BG)$ may also be computed from the action of the k-th power map on $C(G)$. Let

$$W_i = \left\{c_1^{(i)}, c_2^{(i)}, \ldots, c_{h_i}^{(i)}\right\}, \quad i = 0, \ldots, r,$$

be the orbits of the k-th power map on $C(G)$ with the convention that $c_1^{(0)} = \{1\}$. Then up to renumbering $h_i = s_i$ and $r = w$.

This setting generalizes to an arbitrary finite group as follows:

Let G be a finite group with p-Sylow subgroup G_p and define

$$P := im\big(res : R(G) \longrightarrow R(G_p)\big)_{(p)}$$

where *res* is the restriction homomorphism induced by the inclusion $G_p \subset G$. Then

$$K_0(BG_+; \mathbf{Q}/\mathbf{Z}_{(p)}) \cong P \otimes \mathbf{Q}/\mathbf{Z}_{(p)}.$$

Now ψ_0^k acts on P and if P is a permutation representation with respect to this action, then the same argument as for a p-group can be made to work. That P is indeed a permutation representation with respect to the group action induced by ψ_0^k is proved by Kühlshammer in [19]. The proof in [19] is an existence proof, so except in special cases it is still not obvious how to construct a permutation basis of P. Nevertheless it is enough to deduce the group structure of $Ad_*(BG)$ (how to construct elements in $Ad_*(BG)$ is described in [18]): Namely if $C_p(G)$ denotes the set of conjugacy classes of p-elements of G (i.e. elements whose order is a power of p), and S a permutation basis of P then $C_p(G)$ and S have the same orbit structure with respect to the ψ_0^k-action and we may use the explicit permutation basis of $C_p(G)$ to describe $Ad_*(BG)$.

THEOREM 8.6. *Let G be a finite group and $C_p(G)$ the set of conjugacy classes of p-elements of G. Denote by*

$$B_i := \left\{c_1^{(i)}, c_2^{(i)}, \ldots, c_{h_i}^{(i)}\right\}, \quad i = 0, \ldots, r,$$

the orbits of the k-th power map on $C_p(G)$ with the convention that $c_1^{(0)} = \{1\}$. Then for $n \neq 0$

$$Ad_{2n-1}(BG) \cong \bigoplus_{i=1}^{r} \left(\mathbf{Z}_{(p)}/\left(k^{n \cdot h_i} - 1\right)\right)$$

and

$$Ad_{-1}(BG) \cong Ad_{-2}(BG) \cong \bigoplus_{i=1}^{r} \mathbf{Z}/p^\infty.$$

EXAMPLE 8.7. Let $G = D_{10}$ be the dihedral group of order 20 and $p = 5$. Then there are two conjugacy classes of elements of order 5 in D_{10}, which are permuted by ψ_0^k. Hence we have one orbit of length 2 (besides the orbit of $\{1\}$), so for $n \neq 0$

$$Ad_{2n-1}(BD_{10}) = \mathbf{Z}_{(5)}/\left(k^{2n} - 1\right) = \begin{cases} \mathbf{Z}/5^{1+\nu_5(n)}, & n \equiv 0\ (2), \\ 0, & n \equiv 1\ (2). \end{cases}$$

For applications of the computation of $Ad_*(BG)$ see [18]; we mention here only the main result of [18] giving the stably spherical classes in $K_1(BG)$ up to low dimensional exceptions, namely for every finite group G and odd prime p there exists a constant $n_0(G,p) \in \mathbf{N}$ such that for $n \geqslant n_0(G,p)$ the image of the Hurewicz map $h_K : \pi_{2n-1}^S(BG)_{(p)} \to K_{2n-1}(BG)_{(p)}$ is the subgroup $ker(\psi^k - 1) \cong Ad_{2n-1}(BG)$.

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CHAPTER 12

Applications of Nonconnective $Im(J)$-theory

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0. Introduction

This survey article is the continuation of "Introduction to nonconnective $Im(J)$-theory" [24] which we shall refer to simply as part I. As explained there the applications of $Im(J)$-theory arise as applications of K-theory. The applications we present here are, except perhaps for desuspension of $Im(J)$, all well known but usually not formulated in terms of $Im(J)$-theory. Only a small part of this material has appeared in text books up to now. The basic references are the "On the groups $J(X)$" papers of Adams and work of Mahowald [26].

We begin with applications to stable homotopy and recall in §1 some of the most important different definitions of the classical e-invariant of Adams and Toda. The e-invariant is closely related to the Adams–Novikov spectral sequence based on K-theory and we show how the E_2-term of this spectral sequence may be expressed and computed in terms of $Im(J)$-theory. The next section discusses the classical J-homomorphism and its extension to $Im(J)$-theory provided by a solution of the Adams conjecture. It is this extended J-map

$$j_A : Ad^0(Z) \longrightarrow \pi_S^0(Z)_{(p)}$$

which leads to the most interesting applications and makes the v_1-periodic part of stable homotopy very special: there is not only a detective device, the e-invariant or Ad-theory Hurewicz map h_A, but also a constructive device, namely j_A. The computation of the composition of e-invariant and J-homomorphism then has the simple reformulation that $h_A \circ j_A$ is a bijection.

As the first application of the existence of the j_A-map (and thus of the Adams conjecture) we discuss the Mahowald–Miller theorem on the v_1-localization of stable homotopy as presented in [13]. The original proofs used quite different techniques and seem to be more complicated but may have the advantage of being better suited for a generalization to the v_i situation for $i > 1$. The $im(j_A)$-technique used here connects the v_1-torsion order with the desuspendability properties of a stable map f and leads to an estimate of the number of iterates needed for $v_1^t \cdot f$ to vanish.

Our main application of the j_A-map is the construction of desuspensions of $im(J)$-classes. The original result at $p = 2$ is due to M. Mahowald [26] and used a quite different method. The odd primary case was first proved in [12] and is still unpublished. The approach presented here is a shortened version of [12].

The next section treats the relation of $Im(J)$-theory to the classical J-groups $J(X)$. We work out the orientability condition for Ad-theory and derive as a consequence at odd primes the isomorphism

$$J(Z)_{(p)} \cong im(\Delta) \subset Ad^1(Z).$$

A discussion of the examples $J(\Sigma^{2m} P_n \mathbf{C})_{(p)}$ follows. In the last section we consider the special case of torsion-free spaces or spectra. This section belongs thematically to part I but is placed here since it draws its examples mostly from part II.

We assume that the reader knows the basic properties of $Im(J)$-theory Ad and we shall use the notation, definitions and results introduced in part I without comment.

1. The e-invariant

The e-invariant was introduced by Adams [3] and Toda [30] in the early 60's and has turned out to be a very useful tool in stable homotopy. There are several different definitions of the e-invariant and we begin by recalling some of the most well known. We shall work stably although there are also unstable versions and applications. We shall also work in the cohomology setting; one then gets the homology versions by S-duality. In addition there are more refined versions using the connective theories. Those capture v_1-torsion phenomena, but we shall not go into this. The same applies to the real versions. Since we have defined $Im(J)$-theory only p-locally it seems to be more natural in this section to work p-locally throughout (without indicating this in the notation). Most definitions make sense over the integers.

Suppose X is a finite spectrum and $f : X \to S^0$ is a stable map, i.e. an element of $\pi_S^0(X)$. The case of a map $g : Z \to S^n$ may be reduced to the former case by using $X = \Sigma^{-n}Z$. We assume that f induces the trivial map in complex periodic K-theory. This is the same as assuming that f is in the kernel of the Hurewicz map

$$h_K : \pi_S^0(X) \longrightarrow K^0(X).$$

Consider the cofibre sequence of f

$$X \xrightarrow{f} S^0 \xrightarrow{i} C_f \xrightarrow{j} \Sigma X \xrightarrow{\Sigma f} S^1.$$

Since $f^* = 0$ this sequence induces the short exact sequence

$$0 \longrightarrow K^{-1}(X) \xrightarrow{j^*} K^0(C_f) \xrightarrow{i^*} K^0(S^0) \longrightarrow 0 \tag{1.1}$$

in K-theory. The e-invariant describes this extension.

DEFINITION 1. Let $\hat{1} \in K^0(C_f)$ be a preimage for $1 \in K^0(S^0)$. As in part I we write ψ^k for the stable Adams operation. Then $(\psi^k - 1)(\hat{1}) = j^*(x)$ for some $x \in K^{-1}(X)$ since $\psi^k(1) = 1$. The residue class of x in $K^{-1}(X)/(\psi^k - 1)K^{-1}(X)$ is well defined and independent of the choice of $\hat{1}$. Then define

$$e(f) := \text{ class of } x \text{ in } K^{-1}(X)/(\psi^k - 1)K^{-1}(X).$$

EXAMPLE 1.2. The cofibre of the classical Hopf map $\eta : S^3 \to S^2$ is the complex projective plane $C_\eta = P_2\mathbf{C}$ and (1.1) becomes (at $p = 2$):

$$0 \longrightarrow K^{-1}(S^1) \xrightarrow{j^*} K^0(\Sigma^{-2}P_2\mathbf{C}) \xrightarrow{i^*} K^0(S^0) \longrightarrow 0.$$

With $K^0(P_2\mathbf{C}) \cong Z[x]/x^3$ and u the Bott element we have $\hat{1} = u^{-1}x$ and

$$\begin{aligned}(\psi_0^k - 1)(u^{-1}x) &= u^{-1}(\psi_2^k - 1)(x) = u^{-1}(k^{-1}\Psi^k - 1)(x)\\ &= u^{-1}(k^{-1}((x+1)^k - 1) - x),\end{aligned}$$

which gives for $p = 2$ and $k = 3$

$$(\psi^k - 1)\hat{1} = j^* u^{-1} x^2.$$

Then $e(\eta) \neq 0$ in $K^0(S^2)/(\psi^k - 1)K^0(S^2) = \mathbf{Z}/2$.

DEFINITION 2. The extension (1.1) defines an element in a certain group of extensions which is denoted by $Ext^1(K^0(S^0), K^0(\Sigma X))$ in [3]. As mentioned in [29] (see also [8]), this group can be identified with $Ext^1_{K_*K}(K_*(X), K_*(S^0))$ which belongs to the E_2-term of the Adams–Novikov spectral sequence based on periodic complex K-theory. Then the e-invariant may be defined as the canonical map

$$ker(h_K) = F^1\pi_S^0(X) \longrightarrow E_\infty^1 \subset E_2^1 = Ext^1_{K_*K}(K_*(X), K_*(S^0)).$$

This is discussed in detail (for X a sphere) in [29] and [3]. The equivalence of Definitions 1 and 2 is proved at the end of this section.

DEFINITION 3. Let ch be the classical Chern character and ch_0 its component in $H^0(X;\mathbf{Q})$. Using the following commutative diagram with $t = ch - ch_0$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^0(\Sigma X) & \longrightarrow & K^0(C_f) & \longrightarrow & K^0(S^0) & \longrightarrow & 0\\ & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \\ 0 & \longrightarrow & H^*(\Sigma X;\mathbf{Q}) & \longrightarrow & H^*(C_f;\mathbf{Q}) & \longrightarrow & H^*(S^0;\mathbf{Q}) & \longrightarrow & 0\end{array}$$

we obtain a class $j^{*-1}(ch - ch_0)(\hat{1})$ in $H^*(\Sigma X;\mathbf{Q})/(ch - ch_0)K^0(\Sigma X)$. This is the functional Chern character definition of the e-invariant.

To see how it is related to the first definition let us define a natural transformation $T : K^0(Z) \to H^*(Z;\mathbf{Q})/H^0(Z;\mathbf{Q})$ by $T_{2n}(z) := (k^n - 1)^{-1} ch_{2n}(z)$ in $H^{2n}(Z;\mathbf{Q})$ for $n \neq 0$. Let $e(f)$ be the e-invariant as given in definition 1. Then $T(e(f))$ is just $j^{*-1}(ch - ch_0)(\hat{1})$. Hence the invariant of Definition 1 determines the invariant of Definition 3, and if X is torsion-free the converse is true.

DEFINITION 4. The following commutative diagram

$$\begin{array}{ccccccccc} & \longrightarrow & \pi_S^{-1}(X;\mathbf{Q}) & \longrightarrow & \pi_S^{-1}(X;\mathbf{Q}/\mathbf{Z}) & \longrightarrow & \pi_S^0(X) & \longrightarrow & \pi_S^0(X;\mathbf{Q})\\ & & \downarrow{\scriptstyle h_K} & & \downarrow{\scriptstyle h_K} & & \downarrow{\scriptstyle h_K} & & \\ K^{-1}(X) & \longrightarrow & K^{-1}(X;\mathbf{Q}) & \longrightarrow & K^{-1}(X;\mathbf{Q}/\mathbf{Z}) & \longrightarrow & K^0(X) & \longrightarrow & \end{array}$$

defines a map from $ker(h_K) \subset \pi_S^0(X)$ into $K^{-1}(X;\mathbf{Q})/(K^{-1}(X) + H^{-1}(X;\mathbf{Q}))$ which may be called the functional Hurewicz map. The equivalence of Definitions 3 and 4 will also be proved later.

REMARK 1.3. The functional Hurewicz map defined by using BP or MU, the spectrum of complex bordism, is determined by the invariant of Definition 4 provided X is torsion-free. This is an application of the Hattori–Stong theorem. Now the MU-theory functional Hurewicz map determines the E-theory functional Hurewicz map for any complex-oriented cohomology theory E. This shows that the e-invariant defined with a complex-oriented cohomology theory E is determined by the K-theory e-invariant for torsion-free X.

Since bordism has a more natural geometric interpretation than cobordism we turn to homology for the last definition of the e-invariant.

DEFINITION 5. The cofibre sequence of spectra $S^0 \to MU \to MU/S^0$ induces the well known exact sequence of bordism groups [11]:

$$\to \Omega_n^{fr}(X) \to \Omega_n^U(X) \to \Omega_n^{U,fr}(X) \xrightarrow{\partial} \Omega_{n-1}^{fr}(X) \to .$$

Here $\Omega_n^{fr}(X) \cong \pi_n^S(X_+)$ is framed bordism, $\Omega_n^U(X)$ complex bordism and $\Omega_n^{U,fr}(X)$ bordism of stably almost complex manifolds with framed boundary. To an element y in $\Omega_n^{U,fr}(X)$ represented by $[(W,\partial W), f, (\Phi,\alpha)]$ where $f : W \to X$ is a map, Φ an almost complex structure for the stable normal bundle ν of W and α a framing of $\nu_{|\partial W}$, associate the homology class

$$T'(y) := f_*\big([W,\partial W] \cap \big(Todd(\nu/\alpha) - 1\big)\big) \in \bigoplus_{i=0}^{n-1} H_i(X;\mathbf{Q}).$$

Here $Todd(\nu/\alpha)$ is the Todd polynomial of the vector bundle ν/α on $W/\partial W$ defined by the clutching construction with the stable trivialization α on ∂W and $\cap$ is the relative cap-product $H_n(W,\partial W) \otimes H^*(W,\partial W;\mathbf{Q}) \to H_*(W;\mathbf{Q})$. The map T' restricted to $\Omega_n^U(X)$ gives $T([M,f,\Phi]) = f_*([M] \cap Todd(\nu_M)) \in H_*(X;\mathbf{Q})$. For $x \in \Omega_{n-1}^{fr}(X)$, the bordism-theoretic definition of the e-invariant is then given by

$$T'(\partial^{-1}x) \quad \text{in } H_*(X;\mathbf{Q})/\big(T\big(\Omega_n^U(X)\big) + H_n(X;\mathbf{Q})\big).$$

The case $X = S^n$ is discussed in detail in [11]. The advantage of this definition is that it is sometimes easier to find a bounding manifold W than to work with the cofibre of f.

If X is torsion-free, then Definition 5 is equivalent to Definition 3. In general, the e-invariant of Definition 5 is determined by that of Definition 3, but may have a smaller range of definition. (It is only defined if $f_* = 0$ in $\Omega_*^U(\,)$ and this is in general stronger than the condition $f_* = 0$ in $K_*(\,)$.) The proof is left to the reader.

The secondary cohomology theory Ad may now be used to turn the secondary invariant e into a primary one:

PROPOSITION 1.4. *The following diagram commutes:*

$$\begin{tikzcd} ker(h_K) \arrow[d, "e"] & \subset & \pi^0_S(X) \arrow[d, "h_{Ad}"] \\ K^{-1}(X)/(\psi^k-1)K^{-1}(X) \arrow[rr, hook, "\Delta"] & & Ad^0(X) \end{tikzcd}$$

Since $(\psi^k - 1)K^{-1}(X) = ker(\Delta)$, the Ad-theory Hurewicz map h_{Ad} describes the e-invariant completely. Since also $D \circ h_{Ad} = h_K$, the Ad-theory Hurewicz map h_{Ad} combines the two invariants h_K and e into one invariant.

PROOF OF 1.4. Consider the following staircase built up by exact sequences induced by the cofibre sequence of f and exact sequences defining Ad-theory:

$$\begin{tikzcd} K^0(S^0) \arrow[d] & K^0(C_f) \arrow[l, "i^*"'] \arrow[d, "\psi^k-1"] & \arrow[l] & \\ K^0(S^0) & K^0(C_f) \arrow[l] & K^{-1}(X) \arrow[l, "j^*"'] \arrow[d, "\Delta"] & 0 \arrow[l] \\ & \arrow[r, phantom, ""] & Ad^0(X) \arrow[ll] & Ad^0(S^0) \arrow[l, "f^*"'] \arrow[d, "D", "\cong"'] \\ & & \arrow[r, phantom, ""] & K^0(S^0) \arrow[l] \end{tikzcd}$$

In such a situation, as we recall below, one has

$$e(f) = j^{*-1}\big(\psi^k - 1\big)\, i^{*-1}(1) = \Delta^{-1} f^* D^{-1}(1). \tag{1.5}$$

This equation holds in $K^{-1}(X)$ modulo the indeterminacy $(\psi^k - 1)K^{-1}(X)$. If we apply Δ to (1.5) we obtain

$$\Delta\big(e(f)\big) = f^* D^{-1}(1) = f^*(1) = h_{Ad}(f),$$

so proving 1.4.

The equation (1.5) is a special case of the following more general result.

Given three stable maps

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

satisfying $g \circ f \simeq 0$ and $h \circ g \simeq 0$ we can define the (stable) Toda bracket

$$\langle h, g, f\rangle : \Sigma A \longrightarrow D.$$

There are three equivalent ways of constructing elements in $\langle h, g, f\rangle$, depending on which map among $\{f, g, h\}$ is chosen to define the cofibre sequence which is used. We indicate these three possibilities in the diagrams below using the following notation. If $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is a cofibre sequence in the stable category and $h : Y \to W$ is a map with $h \circ \alpha \simeq 0$, then $\bar{h} : Z \to W$ denotes an extension of h, whereas $\underline{g} : V \to X$ denotes a coextension of a map $g : V \to Y$ satisfying $\beta \circ g \simeq 0$.

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r] \arrow[d, "g"'] & C_f \arrow[r] \arrow[dl, "\bar{g}"] & \Sigma A \arrow[ddll, "\overline{h\bar{g}}"] \\
& C \arrow[d, "h"'] & & \\
& D & &
\end{tikzcd} \tag{1.6}$$

$\overline{h \circ \bar{g}} \in \langle h, g, f\rangle$

$$\begin{tikzcd}
& & & \Sigma A \arrow[dl, "\underline{\Sigma f}"'] \arrow[d, "\Sigma f"] \arrow[dr, "0"] & \\
B \arrow[r, "g"] \arrow[dr, "0"'] & C \arrow[r] \arrow[d, "h"] & C_g \arrow[r] \arrow[dl, "\bar{h}"] & \Sigma B \arrow[r, "\Sigma g"] & \Sigma C \\
& D & & &
\end{tikzcd} \tag{1.7}$$

$\bar{h} \circ \underline{\Sigma f} \in \langle h, g, f\rangle$

$$\begin{tikzcd}
& & & \Sigma A \arrow[d, "\Sigma f"] \arrow[ddll, "\underline{\underline{\Sigma g \Sigma f}}"'] & \\
& & & \Sigma B \arrow[dl, "\underline{\Sigma g}"'] \arrow[d, "\Sigma g"] & \\
C \arrow[r, "h"] & D \arrow[r] & C_h \arrow[r] & \Sigma C \arrow[r, "\Sigma h"] & \Sigma D
\end{tikzcd} \tag{1.8}$$

$\underline{\underline{\Sigma g \Sigma f}} \in \langle h, g, f\rangle$

The theorem on Toda brackets then says that the indeterminacy in all three constructions is the same and that all three define the same element modulo this indeterminacy (at least up to sign).

Returning to the proof of (1.5), appropriately chosen data in definition (1.6) give $e(f) = j^{*-1}(\psi^k - 1)\, i^{*-1}(1)$, whereas definition (1.8) produces $\Delta^{-1} f^* D^{-1}(1)$. □

We now compare the Definitions 3 and 4. Let $e_4(f)$ be the e-invariant of f according to Definition 4. Replacing the coefficient sequence $\mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ in that definition by $\mathbf{Z} \xrightarrow{p^r} \mathbf{Z} \to \mathbf{Z}/p^r$ with $p^r \cdot f \simeq 0$, we obtain a class $e_4'(f)$ in $K^{-1}(X)/p^r K^{-1}(X) + \pi_S^{-1}(X)$. It is the stable Toda bracket belonging to

$$X \xrightarrow{p^r} X \xrightarrow{f} S^0 \longrightarrow K$$

with the cofibre of p^r used in the construction. We have $e_4(f) = e_4'(f)/p^r$ in $K^{-1}(X; \mathbf{Q})$. By the theorem on Toda brackets $e_4'(f)$ can also be constructed using the cofibre of f. Apply the transformation t to this construction, incorporating the factor p^{-r}. Then an easy diagram chase shows that we obtain the functional Chern character of Definition 3. This is the essential step in establishing the equivalence of the two definitions.

We conclude this section with a short discussion of the K_*-Adams–Novikov spectral sequence which is related to the e-invariant by Definition 2. The Adams–Novikov spectral sequence defined by p-local periodic K-theory has E_2-terms

$$Ext^{s,t}_{K_*K}\big(K_*(X), K_*(Y)\big)$$

which vanish for $s > 2$ and converges to $[L_K X, L_K Y]_*$, if X is finite. Since localization with respect to K-theory is usually a rather drastic process, the groups $[L_K X, L_K Y]_*$ are in most cases a long way from $[X, Y]_*$, and one does not expect this spectral sequence to have many direct applications to stable homotopy. But it is an important tool in the investigation of the homotopy types of K-local spectra, see [8]. The close relation between $Im(J)$-theory and this spectral sequence comes from the fact that the set of primitives in $K_n(X)$ is given by $D(Ad_n(X))$. Assume $p \neq 2$.

PROPOSITION 1.9. *For a spectrum X we have*

$$\begin{aligned} Hom^n_{K_*K}\big(K_*(S^0), K_*(X)\big) &= Ext^{0,n}_{K_*K}\big(K_*(S^0), K_*(X)\big) \\ &\cong DAd_n(X) = ker\big(\psi^k - 1\big). \end{aligned}$$

PROOF. The group $Hom^n_{K_*K}(K_*(S^0), K_*(X))$ may be identified with the set of coaction primitives

$$Pr_n K_*(X) = \big\{x \in K_n(X) \,|\, \psi(x) = 1 \otimes x\big\}.$$

Here $\psi : K_*(X) \to K_*(K) \otimes K_*(X)$ is the usual coaction map, e.g., see [29]. With $\eta_L, \eta_R : K \to K \wedge K$ the two standard maps, a simple reformulation is

$$Pr_n K_*(X) = ker\big(\eta_L - \eta_R : K_n(X) \to K_n(K \wedge X)\big).$$

Since stable K-theory operations are Kronecker dual to the elements of $K_*(K)$ and $\psi^k - 1$ is in $K^0(K)$ and different from id we must have $(\psi^k - 1)x = 0$ for $x \in Pr_n K_*(X)$. Hence $Pr_n K_*(X) \subset ker(\psi^k - 1) = D(Ad_n(X))$.

To prove the converse, assume that X is a finite spectrum and consider the commutative diagram

$$\begin{tikzcd} Ad_n(X) \arrow[r, "\eta_L - \eta_R"] \arrow[d, "D"] & Ad_n(Ad \wedge X) \arrow[d, "D \wedge D"] \\ K_n(X) \arrow[r, "\eta_L - \eta_R"] & K_n(K \wedge X) \end{tikzcd}$$

Using the splitting of $Ad \wedge Ad$ and properties of $\mu_{Ad} : Ad \wedge Ad \to Ad$ the map $\eta_L - \eta_R : Ad \to Ad \wedge Ad \simeq Ad \vee \Sigma^{-1} Ad\mathbf{Q}$ is easily seen to have vanishing first component and a nontrivial second component in $\Sigma^{-1} Ad\mathbf{Q}$. But $D \wedge D$ restricted to $\Sigma^{-1} Ad\mathbf{Q}$ must vanish for X finite. The general case follows by taking direct limits.

Alternatively one may construct a splitting of $G \wedge G$ into a wedge of copies of G

$$S : G \wedge G \longrightarrow \bigvee_{m=0}^{\infty} G$$

such that the resulting operations $pr_m \circ S \circ \eta_R : G \to G$ are given by

$$\left(\psi^k - \dot{k}^{-[\frac{m}{2}]}\right) \circ \left(\psi^k - \dot{k}^{1-[\frac{m}{2}]}\right) \circ \cdots \circ \left(\psi^k - 1\right) \circ \left(\psi^k - \dot{k}\right) \\ \circ \cdots \circ \left(\psi^k - \dot{k}^{m-1-[\frac{m}{2}]}\right)$$

for $m > 0$ (where $\dot{k} = k^{p-1}$). Formulated slightly differently this means that every stable operation of degree 0 in G-theory which vanishes on $1 \in G^0(S^0)$ has $\psi^k - 1$ as factor. From the identity

$$\eta_L - \eta_R = \bigvee_{m \geqslant 1} (pr_m \circ S \circ \eta_R)$$

we immediately see that x is primitive if and only if $(\psi^k - 1)x$ is zero. $\square$

Let $\overline{K}$ be the cofibre of the unit map $S^0 \to K$. Then the short exact sequence

$$0 \to K_*(X) \underset{\longrightarrow}{\overset{\mu}{\longleftarrow}} K_*(K \wedge X) \to K_*\left(\overline{K} \wedge X\right) \to 0$$

induces the long exact Ext-sequence in the following commutative diagram.

$$\begin{tikzcd}
Pr_{n+1}K_*(K\wedge X) \arrow[r] & Pr_{n+1}K_*(\overline{K}\wedge X) \arrow[r, "\partial_1"] & Ext^{1,n+1}_{K_*K}(K_*(S^0),K_*(X)) \arrow[r] & 0 \\
Ad_{n+1}(K\wedge X) \arrow[r] \arrow[u, "D"] & Ad_{n+1}(\overline{K}\wedge X) \arrow[r, "\partial_2"] \arrow[u, "D"] & Ad_n(X) \arrow[r, "\psi"] & Ad_n(K\wedge X) \\
 & K_{n+2}(\overline{K}\wedge X) \arrow[r, "\partial_3"] \arrow[u, "\Delta"] & K_{n+1}(X) \arrow[u, "\Delta"] &
\end{tikzcd} \tag{1.10}$$

Define

$$\varphi : Ext^{1,n+1}_{K_*K}\big(K_*(S^0), K_*(X)\big) \longrightarrow Ad_n(X) \quad \text{by } \varphi := \partial_2 \circ D^{-1} \circ \partial^{-1}.$$

Since the maps D are onto by 1.9 and $\partial_3 = 0$ this map is well defined and clearly injective. Its image is the same as $ker(\psi)$ which is isomorphic to

$$ker\big(D + \kappa_{-1} : Ad_n(X) \longrightarrow K_n(X) \oplus H_{n+1}(X;\mathbf{Q})\big),$$

as can be seen by using again the splittings of $Ad \wedge Ad$, $Ad \wedge K$ and $Ad\mathbf{Q} \simeq H\mathbf{Q} \vee \Sigma^{-1}H\mathbf{Q}$. We have proved:

PROPOSITION 1.11.

$$\begin{aligned} &Ext^{1,n+1}_{K_*K}\big(K_*(S^0), K_*(X)\big) \cong ker\big(D + \kappa_{-1} : Ad_n(X) \\ &\quad \to K_n(X) \oplus H_{n+1}(X;\mathbf{Q})\big) \subset Ad_n(X). \end{aligned}$$

A similar but easier argument gives

PROPOSITION 1.12. *$Ext^{2,n+2}_{K_*K}(K_*(S^0), K_*(X))$ is isomorphic to $ker(D + \kappa_{-1})$ on $Ad_{n+1}(\overline{K} \wedge X)$.*

With these propositions the reader may try to compute the Ext-groups for $X = S^0$ or $X = BG$, the classifying space of a finite group G.

From diagram (1.10) it is also clear that the e-invariant of Definition 2 composed with the map φ is just the Ad-theory Hurewicz map h_{Ad}. This shows the equivalence of Definitions 1 and 2.

2. The J-homomorphism

$Im(J)$ is the best understood part of stable homotopy. It is the J-homomorphism which makes the v_1-periodic part of stable homotopy special. The J-map was first defined by G.W. Whitehead in 1942 in its unstable form:

$$J : \pi_i\big(SO(n)\big) \longrightarrow \pi_{n+i}(S^n).$$

We consider here its stable (complex) version

$$J : K^{-1}(Z) \longrightarrow \pi_S^0(Z_+). \tag{2.1}$$

Here and in the rest of this section Z is a connected finite CW-complex (not a spectrum), and p-localization is indicated when needed. Define $J(z)$ as follows.

Represent z by a map $f : Z \to U(n)$ and let $\hat{f} : Z \times \mathbf{C}^n \to \mathbf{C}^n$ be its adjoint $\hat{f}(x, v) = f(x)(v)$. Then $\hat{f}$ induces a map on Thom spaces

$$T(\hat{f}) : Z^{\mathbf{C}^n} = \frac{Z \times D^{2n}}{Z \times S^{2n-1}} \longrightarrow \frac{D^{2n}}{S^{2n-1}} = S^{2n}.$$

We identify $Z^{\mathbf{C}^n}$ with $\Sigma^{2n}(Z_+)$. The stable map represented by $T(\hat{f})$ is independent of all choices and defines $J(z)$ in $\pi_S^0(Z_+)$. Note that $J(z)$ restricted to a point in Z is always a map of degree 1, hence $J(z)$ is in the group of units $1 + \pi_S^0(Z)$, and we define

$$\widetilde{J}(z) := J(z) - 1 \in \pi_S^0(Z).$$

There are variants of this definition: for example, one may apply the Hopf construction to $\hat{f} : Z \times S^{2n-1} \to S^{2n-1}$ to get a map $H(\hat{f}) : Z * S^{2n-1} \simeq \Sigma^{2n} Z \to \Sigma S^{2n-1} \simeq S^{2n}$. After stabilization this map is equivalent to $\widetilde{J}(z)$; for a detailed discussion see, e.g., [18].

For $X = S^n$ one has the well-known geometric description of J by reframing spheres. More precisely, the n-sphere with its standard framing θ coming from $S^n \subset \mathbf{R}^{n+1}$ represents the trivial element in framed bordism, i.e. with $[M, \xi]$ denoting the framed bordism class of a closed manifold M with framing ξ

$$0 = [S^n, \theta] \in \Omega_n^{fr}(*) \cong \pi_S^0(S^n).$$

Given $\alpha \in [S^n, U(m)]$ we may twist the framing θ by α (compose the stable trivialization $\theta : \nu S^n \to S^n \times \mathbf{C}^m$ with $\hat{\alpha}$). With the new framing θ_α we have

$$\widetilde{J}(\alpha) = [S^n, \theta_\alpha].$$

It is well known and follows easily from the definition above that J is exponential, i.e. satisfies

$$J(x + y) = J(x) \cdot J(y) \text{ or } \widetilde{J}(x + y) = \widetilde{J}(x) + \widetilde{J}(y) + \widetilde{J}(x) \cdot \widetilde{J}(y). \tag{2.2}$$

In particular, if $Z = \Sigma Y$ is a suspension then $\widetilde{J}$ is a homomorphism of additive groups. By (2.2) we have an obvious extension of J to a map $J : K^{-1}(Z)_{(p)} \to 1 + \pi_S^0(Z)_{(p)}$ on p-localizations.

The other important property of $\widetilde{J}(z)$ is that it figures in the following cofibre sequence of spectra

$$Z \xrightarrow{\widetilde{J}(z)} S^0 \xrightarrow{i} (\Sigma Z)^{\hat{E}_z} \xrightarrow{g} \Sigma Z \xrightarrow{\Sigma \widetilde{J}(z)} S^1. \tag{2.3}$$

Here E_z is the vector bundle on ΣZ defined by using $z : Z \to U(m)$ as a clutching function and $\widetilde{E}_z = E_z - m$, i.e. $\widetilde{E}_z$ represents z in $K^0(\Sigma Z)$ and $(\Sigma Z)^{\widetilde{E}_z}$ is the Thom spectrum of z (e.g., see [29, 19.27]). The map i is the standard inclusion of the Thom spectrum of $\bar{E}_z$ restricted to a point. This cofibre sequence allows us to compute the composition $h_{Ad} \circ \widetilde{J}$. Recall the Bott characteristic class $\rho^k(\xi)$, which is defined by the equation

$$\rho^k(\xi) \cup U_K(\xi) = \psi^k U_K(\xi)$$

where $U_K(\xi)$ is the standard Thom class of the complex vector bundle ξ.

PROPOSITION 2.4. *We have $h_{Ad} \circ \widetilde{J} = \Delta \circ (\rho^k - 1)$, that is the diagram*

$$\begin{tikzcd} K^{-1}(Z)_{(p)} \arrow[r, "\tilde{J}"] \arrow[d, "\rho^k - 1"] & \pi_S^0(Z)_{(p)} \arrow[d, "h_{Ad}"] \\ K^{-1}(Z)_{(p)} \arrow[r, "\Delta"] & Ad^0(Z) \end{tikzcd}$$

commutes.

Because $\Delta \circ e = h_{Ad}$ this result describes also the values of the e-invariant on $Im(J)$ (see [3]).

PROOF. Since E_z is K^*-orientable $i : S^0 \to (\Sigma Z)^{\bar{E}_z}$ induces a surjection in K-theory and (2.3) induces the short exact sequence

$$0 \to K^{-1}(Z)_{(p)} \xrightarrow{g^*} K^0\Big((\Sigma Z)^{\bar{E}_z}\Big)_{(p)} \xrightarrow{i^*} K^0(S^0)_{(p)} \to 0.$$

This implies $h_K(\widetilde{J}(z)) = 0$, so that the e-invariant of $\widetilde{J}(z)$ is defined. The result will follow from $e(\widetilde{J}(z)) = \rho^k(\widetilde{E}_z) - 1$ by 1.4. Using the relative version of the Thom isomorphism it easily follows that $g^*(x) = x \cup U_K(\widetilde{E}_z)$. Since we may choose $\hat{1} = U_K(\widetilde{E}_z)$ we have by definition of the e-invariant

$$e(\widetilde{J}(z)) = g^{*-1}(\psi^k - 1)(\hat{1}) = \rho^k(\widetilde{E}_z) - 1,$$

finishing the proof. $\square$

COROLLARY 2.5. *The Hurewicz map $h_{Ad} : \pi_n^S(S^0)_{(p)} \to Ad_n(S^0)$ is onto for $n \geqslant 0$ and $p \neq 2$.*

PROOF. It is enough to show that $h_{Ad} : \pi_S^0(S^{qt-1})_{(p)} \to Ad^0(S^{qt-1})$ is onto for $t > 0$. By 2.4 this will follow from the fact that $\rho^k - 1 : K^0(S^{qt})_{(p)} \to K^0(S^{qt})_{(p)}$ is an isomorphism. Now $ch \circ \rho^k(u^{2i}) = \pm(k^{2i} - 1)B_i/2i \cdot w^{4i}$ for $u^{2i} \in K^0(S^{4i})$ and $w^{4i} \in$

$H^{4i}(S^{4i};\mathbf{Z})$ generators and B_i the i-th Bernoulli number (see [2]). Since $\nu_p(B_i/2i) = 1 + \nu_p(i)$ if $i \equiv 0 \bmod (p-1)/2$ the map $\rho^k - 1$ is multiplication by a number which is nonzero mod p. □

Our next aim is to describe the factorization of the J-map through Ad-theory. For this we need the description of the J-map as a forgetful map from vector bundles to spherical fibrations. More details are to be found, for example, in [27, Chapter 3].

As before let Z be a finite connected complex and $SG(n+1)$ the space of degree 1 homotopy equivalences of the n-sphere. There are canonical inclusions of $SG(m)$ in $SG(m+1)$ and we put $SG = \varinjlim SG(m)$. Then $SG(m)$ has a classifying space $BSG(m)$ and with $BSG = \varinjlim BSG(m)$ we have that the homotopy set $[Z, BSG]$ is in one to one correspondence with the set of stable fibre homotopy classes of oriented spherical fibrations over Z. Let $SF(n)$ be the homotopy fibre of the evaluation map $E : SG(n+1) \to S^n$ and set $SF = \varinjlim SF(m)$. Since $SF(n)$ is homotopy equivalent to the 1-component of the n-fold loop space $\Omega^n S^n$ of the n-sphere we have bijections of based homotopy sets

$$[Z, SF(n)]_0 \cong [Z, \Omega_1^n S^n]_0 \cong [\Sigma^n Z, S^n]_0, \tag{2.6}$$

$$[Z, SF]_0 \cong [Z, \Omega_1^\infty S^\infty]_0 \cong [Z, \Omega_0^\infty S^\infty]_0 \cong \pi_S^0(Z).$$

The fibre of the evaluation map $E : SO(n+1) \to S^n$ for the special orthogonal group is $SO(n)$ and the natural inclusion of $SO(n+1)$ into $SG(n+1)$ induces a map $SO(n) \to SF(n)$ on fibres. Taking direct limits we get a map

$$J : SO \longrightarrow SF.$$

If we compose J with $U \to SO$ and use the bijections (2.6) we obtain a map

$$K^{-1}(Z) = [Z, U] \cong [Z, U]_0 \longrightarrow [Z, SF]_0 \cong \pi_S^0(Z)$$

which can be identified with the J-map of (2.1). The maps $U \to SF$ and $BU \to BSF$ will also be denoted by J. Since the connectivity of the inclusion $SF(n) \to SG(n+1)$ grows with n to infinity, we have $SF \simeq SG$ and $BSF \simeq BSG$. Now SG classifies stable spherical fibrations over ΣZ, and BSG those over Z and we may view J as a forgetful map from vector bundles to spherical fibrations. As such it is an H-map (the bijections in (2.6) change the H-space structure), so there is a map on p-localizations. Denote the homotopy fibre of $J : BU_{(p)} \to BSF_{(p)}$ by $SF_{(p)}/U_{(p)}$. The Adams conjecture in its p-local form is the statement that $J \circ (\psi^k - 1)$ is null homotopic (e.g., see [4, Chapter 5]). This was established in the early 70's and so we may set up the following commutative diagram of fibrations of classifying spaces (localized at p):

$$\begin{tikzcd}
U \arrow[r, "\Delta"] \arrow[d, equal] & ImJ \arrow[r, "D"] \arrow[d, "j_A"] & BU \arrow[rr, "\psi^k - 1"] \arrow[d, "\alpha"] & & BU \arrow[d, equal] & \\
U \arrow[r, "J"] & SF \arrow[r] & SF/U \arrow[rr] & & BU \arrow[r, "J"] & BSF
\end{tikzcd} \tag{2.7}$$

Here ImJ is the homotopy fibre of $\psi^k - 1$ which by our choice for the spectrum of $Im(J)$-theory has the property

$$[Z, ImJ]_0 \cong Ad^0(Z).$$

The existence of a lift α of $\psi^k - 1$ comes from the solution of the Adams conjecture. (There are explicit constructions for α; in the complex case α may be chosen as an H-map.) Let $j_A : ImJ \to SF$ be the induced map on fibres. For the following fix a choice for α and j_A. If we compose the maps J and j_A with the identifications $SF \simeq \Omega_1^\infty S^\infty \simeq \Omega_0^\infty S^\infty$ from (2.6) we obtain the required factorization of the J-map through $Im(J)$-theory: the following diagram commutes

$$\begin{tikzcd}
K^{-1}(Z)_{(p)} \arrow[rr, "\Delta"] \arrow[dr, "\widetilde{J}"'] & & Ad^0(Z) \arrow[dl, "j_A"] \\
& \pi_S^0(Z)_{(p)} &
\end{tikzcd} \tag{2.8}$$

Note that if Z is a suspension, then j_A is a homomorphism of additive groups. Observe also that $im(j_A)$ can be larger than $im(\widetilde{J})$; an example is discussed below.

Next consider the composition $h_{Ad} \circ j_A$.

THEOREM 2.9 ([31]). *Let Z be a connected CW-complex. Then for an odd prime p the composition*

$$Ad^0(Z) \xrightarrow{j_A} \pi_S^0(Z)_{(p)} \xrightarrow{h_A} Ad^0(Z)$$

is bijective.

PROOF. From 2.5 we know that $h_{Ad} \circ j_A$ is onto for $Z = S^n$, $n \geqslant 1$, hence bijective since $Ad^0(S^n)$ is finite. Therefore the corresponding map of classifying spaces is a weak homotopy equivalence proving the result. □

Since j_A is not a natural transformation of cohomology theories, there is no j_A-map in homology. For if there were such a J-map (defined on connective $Im(J)$-theory $j_A : A_n(X) \to \pi_n^S(X)_{(p)}$ with $h_A \circ j_A$ bijective) then this would imply $\pi_*^S(S^0)_{(p)} = A_*(S^0)$. (Use the cofibre sequence $S^0 \to BP \to \overline{BP} \to S^1$, the convergence of the Adams–Novikov spectral sequence for S^0 and that for X torsion-free h_A detects all elements of

Adams–Novikov filtration 1, see §7.) But by using S-duality and desuspension results one may, for example, deduce the following weak homology form of the existence of j_A:

PROPOSITION 2.10. *Let X be a 0-connected p-local spectrum with finite $2m$-skeleton $X^{(2m)}$. If $x \in Ad_{2n-1}(X)$ is in $im(Ad_{2n-1}(X^{(2m)}) \to Ad_{2n-1}(X))$ and $pm \leqslant n(p-1)$ then $x \in im(h_A)$.*

For the easy proof and applications see [23].

3. v_1-periodicity

In this section we give a short review of the Mahowald–Miller theorem on the v_1-localization of stable homotopy following [13]. Recall that Adams [3] constructed stable self maps

$$B_a : \Sigma^{s(a)} M_a \longrightarrow M_a$$

of the mod p^a-Moore spectrum $M_a = S^0 \cup_{p^a} e^1$ inducing isomorphisms in K-theory. Here $s(a)$ is defined as $s(a) = 2(p-1)p^{a-1}$ for $p \neq 2$. On $G_*(M_a) = \mathbf{Z}/p^a[v_1, v_1^{-1}]$ the map B_a induces multiplication by $v_1^{p^{a-1}}$, hence the name v_1-periodicity. The Moore spectrum is used to introduce mod p^a-coefficients into a cohomology theory E by using the spectrum $E \wedge M_a$. Then B_a operates on $E^*(X; \mathbf{Z}/p^a)$ and $E^*(X; \mathbf{Z}/p^a)[B_a^{-1}]$ is defined. Observe that simply by the 5-lemma B_a induces an isomorphism on $Ad^*(X; \mathbf{Z}/p^a)$, hence $Ad^*(X; \mathbf{Z}/p^a)[B_a^{-1}] \cong Ad^*(X; \mathbf{Z}/p^a)$. The Mahowald–Miller theorem as interpreted by Bousfield [9] computes $E \wedge M_a[B_a^{-1}]$:

THEOREM 3.1 ([26], [28] for $a = 1$). *The Ad-theory Hurewicz map induces an isomorphism $\pi_S^*(X; \mathbf{Z}/p^a)[B_a^{-1}] \cong Ad^*(X; \mathbf{Z}/p^a)$.*

We shall here restrict to the case of an odd prime; the 2-primary case is only technically more involved.

Adams constructed his maps B_a as extensions and coextensions of a generator of the image of the J-homomorphism in $\pi^S_{qp^{a-1}-1}(S^0)_{(p)}$. One extension or coextension can be saved by using the fact that except for $p = 2$ and $a = 1$ the Moore spectrum has a product structure. Here we shall construct B_a as an element in $im(j_A)$, where j_A is the factorization of the J-map through Ad-theory provided by a solution of the Adams conjecture (2.7). It is an example of a case where $im(j_A)$ is larger than $im(J)$, since B_a is in $im(j_A)$ but not in $im(J)$. This construction has the advantage that one gets control over compositions of B_a and a bound on the order of the v_1-torsion in $\pi_S^*(X; \mathbf{Z}/p^a)$. Thus, if $x \in \pi_S^*(X; \mathbf{Z}/p^a)$ is an element in $ker(h_{Ad})$, one can estimate the number of iterates needed for $B_a^t(x)$ to vanish.

The map B_a is an element in $\{\Sigma^{s(a)} M_a, M_a\} = \pi_S^0(\Sigma^{s(a)} M_a; \mathbf{Z}/p^a)$; therefore it is convenient to introduce a j_A-map with mod p^a-coefficients. The Moore spectrum is self dual up to a suspension: $D(M_a) \simeq \Sigma^{-1} M_a$, and we choose a fixed S-duality map for it. This induces an isomorphism $D_a : E^i(X;\mathbf{Z}/p^a) \cong E^{i+1}(X \wedge M_a)$ for every cohomology theory E. Define for $i \geqslant 2$ and Z a finite CW-complex

$$\begin{aligned} j_A^a : Ad^{-i}(Z;\mathbf{Z}/p^a) &\overset{D_a}{\cong} Ad^{2-i}(Z \wedge \Sigma M_a) \xrightarrow{j_A} \pi_S^{2-i}(Z \wedge \Sigma M_a) \\ &\overset{D_a}{\cong} \pi_S^{-i}(Z;\mathbf{Z}/p^a). \end{aligned}$$

Note that ΣM_a, and hence $Z \wedge \Sigma M_a$, is a space, so that j_A is defined. The properties of j_A^a are as follows. The map $j_A^a : Ad^0(W;\mathbf{Z}/p^a) \to \pi_S^0(W;\mathbf{Z}/p^a)$ is defined if W is a double suspension and a map $f : W \to Y$ between double suspensions commutes with j_A^a if $f \wedge 1_{M_a} : W \wedge \Sigma M_a \longrightarrow Y \wedge \Sigma M_a$ desuspends twice.

For the construction of B_a consider the following diagram ($t \geqslant 1$).

$$\begin{tikzcd}
Ad^{-ts(a)}(M_a;\mathbf{Z}/p^a) \arrow[d, "j_A^a"] & \\
\pi_S^{-ts(a)}(M_a;\mathbf{Z}/p^a) \arrow[d, "h_{Ad}"] \arrow[dr, "h_K"] & \\
Ad^{-ts(a)}(M_a;\mathbf{Z}/p^a) \arrow[r, "D"] & G^{-ts(a)}(M_a;\mathbf{Z}/p^a)
\end{tikzcd}$$

Then $G^{-ts(a)}(M_a;\mathbf{Z}/p^a) = \{\Sigma^{ts(a)}M_a, M_a \wedge G\} \cong \mathbf{Z}/p^a$ is generated by $\hat{v} := id_{M_a} \wedge v_1^{tp^{a-1}}$ and D is an isomorphism. Since $h_{Ad} \circ j_A^a$ is an isomorphism too, 2.9, there exists a unique element ${}^tb_a \in Ad^{-ts(a)}(M_a;\mathbf{Z}/p^a)$ mapping to $\hat{v}$ under $D \circ h_{Ad} \circ j_A^a$. For $t \geqslant 1$ define internal operators ${}^tB_a := j_A^a({}^tb_a)$, $B_a = {}^1B_a$ and external operators by t-fold composition $B_a^t := B_a \circ \cdots \circ B_a$. By construction it is clear that ${}^tB_a$ and B_a operate on $G^*(\ ;\mathbf{Z}/p^a)$ by multiplication with powers of v_1, hence induce isomorphisms in $G^*(\ ;\mathbf{Z}/p^a)$ and $Ad^*(\ ;\mathbf{Z}/p^a)$.

Suppose now that $x \in \pi_S^j(X;\mathbf{Z}/p^a)$ is represented by the stable map $f : X \to \Sigma^j M_a$. Then ${}^tB_a(x)$ is simply the composition

$$\Sigma^{ts(a)-j}X \xrightarrow{\Sigma^{ts(a)-j}(f)} \Sigma^{ts(a)}M_a \xrightarrow{{}^tB_a} M_a.$$

Consider ${}^tB_a \in \pi_S^0(\Sigma^{ts(a)}M_a;\mathbf{Z}/p^a)$ as an element in $im(j_A^a)$ with f operating on it. If now $\Sigma^{ts(a)-j}(f) : \Sigma^{ts(a)-j}X \longrightarrow \Sigma^{ts(a)}M_a$ commutes with the mod p^a J-map j_A^a, which is the case if the $(ts(a) - 1 - j)$-fold suspension of $f \wedge 1_{M_a}$ is induced by a map between spaces, then the composition ${}^tB_a \circ \Sigma^{ts(a)-j}(f)$ must be in $im(j_A^a)$ too:

$${}^tB_a(x) = f^*({}^tB_a) = f^* \circ j_A^a({}^tb_a) = j_A^a \circ f^*({}^tb_a). \tag{3.2}$$

Hence the effect of ${}^tB_a$ on x can be controlled in $Im(J)$-theory.

To obtain unstable maps representing a given stable map we use the following desuspension result (e.g., see [13]):

LEMMA 3.3. *Let p be an odd prime, Z a finite CW-complex of dimension n, and $m \geqslant 1$. Then the stabilization map*

$$[\Sigma Z, \Sigma^{2m-1}M_a] \longrightarrow \{Z, \Sigma^{2m-2}M_a\}$$

is onto provided $n \leqslant 2mp - 4$.

Using this, we easily get

THEOREM 3.4. *If x is in $\pi_S^j(Z;\mathbf{Z}/p^a)$ and t satisfies $t \cdot s(a) - j - 4 \geqslant 0$ and $(p-1) \cdot s(a) \cdot t \geqslant dimZ - j + 3$ then ${}^tB_a(x) \in im(j_A^a)$.*

A first consequence of 3.4 is:

PROPOSITION 3.5. *${}^tB_a = B_a^t$ for $t \geqslant 1$.*

PROOF. Apply Theorem 3.4 to the case of $x = B_a$ and use $B_a^*({}^tb_a) = {}^{t+1}b_a$. The necessary desuspensions are provided by 3.3. □

We can then deduce

COROLLARY 3.6. *If x is in $ker(h_{Ad} : \pi_S^j(Z;\mathbf{Z}/p^a) \to Ad^j(Z;\mathbf{Z}/p^a))$ and t satisfies $t \cdot s(a) - j - 4 \geqslant 0$ and $(p-1) \cdot s(a) \cdot t \geqslant dimZ - j + 3$ then ${}^tB_a(x) = B_a^t(x) = 0$.*

PROOF. If x is in $ker(h_{Ad})$, then $f^* : Ad^*(M_a;\mathbf{Z}/p^a) \longrightarrow Ad^*(Z;\mathbf{Z}/p^a)$ is zero, since Ad is a ring spectrum. Therefore $f^*({}^tb_a) = 0$ and by (3.2) we must have ${}^tB_a(x) = 0$.□

Now Theorem 3.1 is proved as follows:

By 3.6 it follows that B_a^t annihilates the kernel of h_{Ad}, and the existence of the j_A^a-map shows that

$$h_{Ad} : \pi_S^j(Z;\mathbf{Z}/p^a) \longrightarrow Ad^j(Z;\mathbf{Z}/p^a)$$

is onto for $j \leqslant -3$. □

For an application of 3.1 see [14].

We close this section with some comments on spaces or spectra X with vanishing Ad-theory. Beside most Eilenberg–MacLane complexes $K(\pi;n)$, $n > 2$, the Adams maps B_i produce the most important examples of such spectra. Let $V(1)$ be the cofibre of the Adams map $B_1 : \Sigma^q M_1 \to M_1$ then $V(1)$ is a CW-spectrum with four stable cells and vanishing $Im(J)$-theory:

$$Ad \wedge V(1) \simeq *.$$

Other examples are given by the cofibres of the map F introduced in §4. Let $H(\pi)$ denote the Eilenberg–MacLane spectrum of a countable torsion abelian group π then

$$Ad \wedge H(\pi) \simeq *$$

and the vanishing of $\left[H(\pi), Ad\right]_*$ or $[Ad, H(\pi)]_*$ follows directly from

$$Ad \wedge H(\mathbf{F}_p) \simeq Ad \wedge H(\mathbf{Z}) \wedge M_1 \simeq *$$

by induction. But the triviality of $Ad \wedge H(\mathbf{F}_p)$ is a direct consequence of the existence of B_1. For the map B_1 induces an isomorphism on $\pi_*^S(Ad \wedge H(\mathbf{F}_p))$ but the zero map in mod p homology (since it is a stable cohomology operation of negative degree for mod p cohomology).

4. Desuspension of the image of J

In this section we discuss applications of $Im(J)$-theory to unstable homotopy. To be exact, we consider the problem of desuspending elements in the image of the J-homomorphism $j_A : Ad^0(Z) \to \pi_S^0(Z)_{(p)}$ for an odd prime p. Maximal desuspensions for the elements in $Im(J) \subset \pi_S^0(S^n)$ at $p = 2$ were first constructed by M. Mahowald (see [26] and references there). For p odd and $x \in Im(J)$ B. Gray [19] constructed maximal desuspensions of stable homotopy classes having the same e-invariant as x. Here we shall follow the method introduced in [12] which is slightly different but much simpler (provided one accepts the Adams conjecture) than the original methods at $p = 2$ which used bo-resolutions and unstable computations with the lambda algebra. Nevertheless the principal idea, namely the use of periodicity operators on stunted projective spaces (lens spaces for p odd), originates in work of M. Mahowald. The main new ingredient in [12] is, as in §3, the use of the extension of the J-map given by the Adams conjecture. Instead of the more geometrically defined periodicity operators used in [12] we choose the by now more familiar periodicity operators F introduced in [19].

Let $B\Sigma_p$ be the classifying space of the symmetric group Σ_p. By [5] there is a simply connected CW-complex B which is (stably) p-equivalent to $B\Sigma_p$ and has one cell in each dimension congruent to 0 or -1 mod q. Note also that ΣB is contained in $\Sigma B\mathbf{Z}/p$ as a retract. We shall use B as a p-local substitute for $B\Sigma_p$. Define $B_{bq-1}^{aq} := B^{aq}/B^{bq-2}$ for $a \geqslant b > 0$ where B^m is the m-skeleton of B. The S-dual DB_{bq-1}^{aq} of B_{bq-1}^{aq} is a spectrum with bottom cell in dimension $-aq$ and top cell in dimension $1 - bq$. It may be identified with ΣB_{-aq-1}^{-bq} where $B_{nq-1}^{(n+k)q}$ for $n < 0$ is defined by $B_{nq-1}^{(n+k)q} = \Sigma^{(n-r)q} B_{rq-1}^{(r+k)q}$ with $r \equiv n \bmod p^k$ and r positive (see, for example, [7, Chapter V, §2]), but this identification is not used here.

The reduced transfer or Kahn–Priddy map $tr : B\Sigma_p \to S^0$ (e.g., see [5]) induces a stable map

$$\lambda : B^{rq} \to S^0 \tag{4.1}$$

and it is shown in [19], proof of Theorem 9, that λ is induced by an honest map after $2r + 1$ suspensions

$$\lambda : \Sigma^{2r+1} B^{rq} \to S^{2r+1}.$$

Next we recall from [19] the construction of a map $F : \Sigma^2 B \to \Sigma^2 B$ as a compression of the multiplication by p map on $\Sigma^2 B$. Since the identity of $\Sigma^{mq} M_1$ is of order p, the composition

$$\Sigma B^{mq} \xrightarrow{p} \Sigma B^{mq} \xrightarrow{j} \Sigma^{mq} M_1$$

is null homotopic. A null homotopy H defines a coextension

$$\begin{aligned} F_m : \Sigma^2 B^{mq} = C_+ \Sigma B^{mq} \cup C_- \Sigma B^{mq} &\xrightarrow{H \cup Cp} \Sigma^{mq} M_1 \cup_j C\Sigma B^{mq} \\ &\simeq \Sigma^2 B^{(m-1)q} \end{aligned}$$

such that the lower triangle in

$$\begin{tikzcd} \Sigma^2 B^{(m+1)q} \arrow[rr, "F"] & & \Sigma^2 B^{mq} \\ \Sigma^2 B^{mq} \arrow[u, "i"] \arrow[urr, "p"] \arrow[rr, "F"] & & \Sigma^2 B^{(m-1)q} \arrow[u, "i"'] \end{tikzcd}$$

commutes up to homotopy. The upper triangle commutes, since it does so if we compose with the inclusion into $\Sigma^2 B^{(m+1)q}$ and we can deform the homotopy into $\Sigma^2 B^{mq}$. Thus the maps F_m fit together to define F.

From this we obtain maps

$$F : \Sigma^2 B^{(a+1)q}_{(b+1)q-1} \longrightarrow \Sigma^2 B^{aq}_{bq-1} \tag{4.2}$$

which are easily seen to induce isomorphisms in K-theory. If $a = b$ then B^{aq}_{bq-1} is a suspension of the mod p Moore spectrum and F is exactly the Adams map B_1 of §3. Hence F acts as a v_1-periodicity map between the Moore space pieces of $B^{(a+1)q}_{(b+1)q-1}$ and B^{aq}_{bq-1}.

REMARK. Let V be the $p-1$ dimensional complex vector bundle on $B\Sigma_p$ corresponding to the orthogonal complement of a copy of $\mathbf{C}$ with trivial action in the permutation representation of Σ_p on $\mathbf{C}^p$. Then one may form the Thom spectrum B^{sV} for $s \in \mathbf{Z}$. By considering $B^{-sV}/*^{-sV}, s < 0$, instead of B one can define (stable) versions of the map F with $a = \infty$ and b negative

$$F : B^{\infty}_{bq-1} \longrightarrow B^{\infty}_{(b-1)q-1}.$$

It is then shown in [15], [16] that the mapping telescope of

$$B^{\infty}_{q-1} \xrightarrow{F} B^{\infty}_{-1} \xrightarrow{F} B^{\infty}_{-q-1} \xrightarrow{F} B^{\infty}_{-2q-1} \xrightarrow{F} \cdots$$

$\overline{B} := \varinjlim(B^{\infty}_{-nq-1}, F)$ is K-local, from which it easily follows that $\overline{B}$ represents $Im(J)$-theory with $\mathbf{Q}/\mathbf{Z}$-coefficients: $\overline{B} = \Sigma^{-1} Ad\mathbf{Q}/\mathbf{Z}$. To desuspend stable maps and spectra we shall use the following p-local desuspension results:

THEOREM 4.3 (see [6]). *Let X and Z be finite connected complexes such that X is $d-1$ connected, $d \geqslant 1$, and* $\dim Z \leqslant pd - 1$. *Then the stabilization map*

$$[\Sigma Z, \Sigma X_{(p)}]_0 \longrightarrow \{Z, X_{(p)}\}$$

is surjective.

THEOREM 4.4 (see [20]). *Suppose $\underline{X}$ is a $2n-2$ connected p-local spectrum and* $\dim \underline{X} < 2np - 1$. *Then there is a space X such that $\Sigma^\infty X \simeq \underline{X}$.*

One may also work with the classical global versions of these desuspension theorems obtaining slightly weaker estimates in the desuspension theorem for $im(j_A)$-classes below. We still need one piece of notation. For spectra X and Y denote the group of spectrum maps $\{X, Y \wedge Ad\}_*$ also by $Ad^*\{X;Y\}$. For Z finite with S-dual $D(Z)$ we have $Ad^*\{X;Z\} \cong Ad^*(X \wedge D(Z))$. The transfer map λ of (4.1) induces a homomorphism

$$\lambda_* : Ad^*\{X; B^{rq}\} \longrightarrow Ad^*\{X; S^0\} = Ad^*(X).$$

THEOREM 4.5 ([12]). *Let Z be a finite complex of dimension m, $X := \Sigma^n Z$, and suppose $x \in Ad^0(X)$ is in the image of $\lambda_* : Ad^0\{X; B^{sq}\} \to Ad^0(X)$. Then x desuspends to $[\Sigma^{2s+1}X, S^{2s+1}]_{(p)}$ provided*

$$n(p-1) - m \geqslant 2s\big(p^2 - (p-1)\big) + 3p. \tag{4.6}$$

PROOF. The main step is to establish commutativity of the basic diagram:

$$\begin{tikzcd}
\left[\Sigma^{2s+1}X, \Sigma^{2s+1}B^{(s+k)q}_{(k+1)q-1}\right] \arrow[r, "F^k_*"', "1"] \arrow[d, "\Sigma^\infty"', "3"] & \left[\Sigma^{2s+1}X, \Sigma^{2s+1}B^{sq}_{q-1}\right] \arrow[r, "\lambda_*", "2"'] \arrow[d, "\Sigma^\infty"'] & \left[\Sigma^{2s+1}X, S^{2s+1}\right] \arrow[d, "\Sigma^\infty"'] \\
\left\{X, B^{(s+k)q}_{(k+1)q-1}\right\} \arrow[r, "F^k_*"] \arrow[d, equal] & \left\{X, B^{sq}_{q-1}\right\} \arrow[r, "\lambda_*"] \arrow[d, equal] & \left\{X, S^0\right\} \arrow[d, equal] \\
\left\{X \wedge DB^{(s+k)q}_{(k+1)q-1}, S^0\right\} \arrow[r, "(DF^*)^k"] & \left\{X \wedge DB^{sq}_{q-1}, S^0\right\} \arrow[r, "D\lambda^*"] & \left\{X, S^0\right\} \\
\left\{X \wedge DB^{(s+k)q}_{(k+1)q-1}, Ad\right\} \arrow[r, "(DF^*)^k", "5"'] \arrow[u, "j_A", "4"'] \arrow[d, equal] & \left\{X \wedge DB^{sq}_{q-1}, Ad\right\} \arrow[r, "D\lambda^*", "6"'] \arrow[u, "j_A", "4"'] \arrow[d, equal] & \{X, Ad\} \arrow[u, "j_A"] \arrow[d, equal] \\
\left\{X, B^{(s+k)q}_{(k+1)q-1} \wedge Ad\right\} \arrow[r, "F^k_*"] & \left\{X, B^{sq}_{q-1} \wedge Ad\right\} \arrow[r, "\lambda_*"] & \{X, Ad\}
\end{tikzcd} \tag{4.7}$$

As already noted, λ is induced by a map $\lambda : \Sigma^{2s+1}B^{sq} \to S^{2s+1}$ so that the upper right hand square $\langle 2\rangle$ commutes. Since $F : \Sigma^2 B \to \Sigma^2 B$ is induced by a map the upper left hand square $\langle 1\rangle$ commutes as well. The squares of the second and fourth row commute by S-duality. The j_A-maps in the lower squares exist only if the spectra $X \wedge DB^{(s+a)q}_{(a+1)q-1}$ for $a = 0, \dots, k$ are equivalent to spaces and the corresponding squares commute provided DF^k and $D\lambda$ are induced by maps between these spaces. This will be guaranteed by numerical conditions on k, n and m, which we shall work out next using 4.3 and 4.4.

First of all, the stability map $\langle 3\rangle$ will be onto provided

$$kpq \geqslant n + m - qs - qp + p + 1. \tag{4.8}$$

To desuspend

$$DF : \Sigma^n DB^{(s+i-1)q}_{iq-1} \longrightarrow \Sigma^n DB^{(s+i)q}_{(i+1)q-1} \quad \text{for } i = 1, \dots, k$$

we impose the condition

$$kq(p-1) \leqslant (p-1)n - sqp - (p-1) - 2. \tag{4.9}$$

This implies also that $\Sigma^{n-1}DB^{(s+i-1)q}_{iq-1}$, $i = 1, \dots, k+1$, is realizable by a space and, hence, that all the j_A-maps in $\langle 4\rangle$ are defined. Since (4.9) is satisfied only if $qk \leqslant n - 2sp - 2$ we need a value for k with

$$n + m - qs - qp + p + 1 \leqslant kpq \leqslant pn - 2p^2 s - 2p.$$

We can find such a k if $(pn - 2p^2s - 2p) - (n + m - qs - qp + p + 1) \geqslant pq - 1$. This gives

$$n(p-1) - m \geqslant 2s\big(p^2 - (p-1)\big) + 3p. \tag{4.10}$$

To desuspend $D\lambda : S^n \longrightarrow \Sigma^n DB^{sq}_{q-1}$ and to realize $\Sigma^{n-1}DB^{sq}_{q-1}$ by a space we need $(p-1)n \geqslant pqs + p$ which is already implied by (4.10). Hence under condition (4.10) diagram (4.7) will commute and the stabilization map in $\langle 3\rangle$ will be onto. Note that F being a K-theory equivalence implies that F^k is an Ad-theory equivalence as well. This together implies that every $x \in im\,\lambda_* : Ad^0\{X; B^{sq}_{q-1}\} \to Ad^0(X)$ desuspends to $[\Sigma^{2s+1}X, S^{2s+1}]_{(p)}$. □

We next investigate the transfer condition $x \in im(\lambda_* : Ad^0\{X; B^{sq}_{q-1}\} \to Ad^0(X))$ for desuspendability. Define the spectrum R^{mq+1} by the upper cofibre sequence in the diagram

$$\begin{tikzcd}
B^{mq} \arrow[r, "\lambda"] & S^0 \arrow[r, "i"] & R^{mq+1} \arrow[r, "j"] \arrow[d, "p^m"] \arrow[dl, dotted, "\phi"'] & \Sigma B^{mq} \\
B^{mq} \arrow[r, "\lambda"] & S^0 \arrow[r] & R^{mq+1} \arrow[r, "j"] & \Sigma B^{mq}
\end{tikzcd} \tag{4.11}$$

Then $x \in im\,\lambda_*$ if and only if $i_*(x) = 0$ in $Ad^0\{X; R^{mq+1}\}$. We leave it as an exercise to compute $K_*(R^{mq+1})$:

$$K_i(R^{mq+1}) = \begin{cases} \mathbf{Z}_{(p)}, & i = 0, \\ 0, & i = 1, \end{cases}$$

and to check that i_* is of degree p^m. If p^m denotes the stable map inducing multiplication by p^m, then $j \circ p^m \simeq 0$ in (4.11) since the identity of B^{mq} is stably of order p^m. Hence there is a coextension $\phi : R^{mq+1} \to S^0$ in (4.11). Simple degree considerations imply that $\phi_* : K_*(R^{mq+1}) \to K_*(S^0)$ is an isomorphism and that $\phi \circ i \simeq p^m$. Hence the same is true in $Im(J)$-theory and we have the commutative diagram

$$\begin{tikzcd} Ad^0\{X; B^{mq}_{q-1}\} \arrow[r, "\lambda"] & Ad^0(X) \arrow[r, "i"] \arrow[dr, "p^m"'] & Ad^0\{X; R^{mq+1}\} \arrow[d, "\phi", "\cong"'] \\ & & Ad^0\{X; S^0\} \end{tikzcd}$$

COROLLARY 4.12. $im(\lambda_* : Ad^0\{X; B^{mq}_{q-1}\} \to Ad^0(X)) = \{x \in Ad^0(X) \mid p^m x = 0\}$.

Up to some low dimensional exceptions this and 4.5 give the maximal desuspension of elements in $Im(J) \subset \pi^S_*(S^o)_{(p)}$:

COROLLARY 4.13. *Assume $\nu_p(t) \geqslant s - 1 \geqslant 0$ and let $x \in Ad^0(S^{qt-1})$ be an element of order p^s. Then $j_A(x) \in \pi^S_{qt-1}(S^0)_{(p)}$ desuspends to $[S^{2s+1+qt-1}, S^{2s+1}]_{(p)}$.*

PROOF. Choose $X = S^0$. With the exceptions

$$\begin{aligned} &s = 1, t = 1, && \text{all } p, \\ &s = 1, t = 2 && \text{for } p = 3, \\ &s = 2, t = 3 && \text{for } p = 3, \end{aligned}$$

which have to be proved by other means, Theorem 4.5 gives all the required desuspensions. □

Another application of 4.5 is a desuspension of the Adams periodicity operators B_a of §3 by choosing as X a suitable suspension of $M_a \wedge D(M_a)$ (again with some low dimensional exceptions).

REMARKS. The case $p = 2$ is similar but more complicated, see [12]. Necessary conditions for desuspendability are given by the Hopf invariant; for a discussion of this topic in conjunction with 4.5 see also [12]. It turns out that in the range of (4.6) the transfer condition $x \in im\,\lambda_* : Ad^0\{X; B^{sq}_{q-1}\} \to Ad^0(X)$ is also a necessary condition.

5. J-theory

The material of this section is all contained, implicitly or explicitly, in the $J(X)$-papers of Adams. We assume $p \neq 2$ and begin by studying Ad^*-orientability for complex vector bundles. The condition for a complex vector bundle ξ to be orientable for Ad-theory turns out to be strong, namely

THEOREM 5.1. *Let ξ be an n-dimensional complex vector bundle on Z and $[\xi]$ its class in $K^0(Z)_{(p)}$. Then ξ has an Ad-theory Thom class if and only if $\Delta([\xi]) = 0$ in $Ad^1(Z)$.*

REMARK. The cofibre sequence (2.3) already shows quite clearly the interplay between the class of ξ represented in Ad-theory (here given by $h_{Ad}(\widetilde{J}(\xi))$, which is $\Delta(\xi)$ up to the bijection $h_{Ad} \circ j_A$) and Ad^*-orientability: i^* is onto (which is equivalent to orientability) if and only if $h_{Ad}(\widetilde{J}(\xi)) = 0$.

Preparing for the proof we derive the basic ρ^k-diagram for the Adams summand $G^0(Z)$ of p-local K-theory. Let Z be a finite connected CW-complex, choose a multiplicative Thom class $U_G(\xi)$ for complex vector bundles in G-theory and define the Bott characteristic class $\rho^k_G(\xi) \in G^0(Z_+)$ by the equation

$$\psi^k U_G(\xi) = \rho^k_G(\xi) \cup U_G(\xi).$$

Then ρ^k_G extends from $Vect^n(Z)$ to $K^0(Z)$ since k is invertible in $G^0(Z_+)$. The canonical extension of ρ^k_G to $K^0(Z)_{(p)}$ can be restricted to $G^0(Z)$ to define an exponential class

$$\rho^k_G : G^0(Z) \longrightarrow 1 + G^0(Z).$$

LEMMA 5.2. *Let Z be a finite connected CW-complex. Then ρ^k_G is bijective and the following diagrams are commutative:*

$$\begin{tikzcd} G^0(Z) \arrow[r, "\psi^k - 1"] \arrow[d, "\rho^k_G"] & G^0(Z) \arrow[d, "\rho^k_G"] \\ 1+G^0(Z)^{\otimes} \arrow[r, "\psi^k/1"] & 1+G^0(Z)^{\otimes} \end{tikzcd} \qquad \begin{tikzcd} K^0(Z)_{(p)} \arrow[r, "\psi^k - 1"] \arrow[d, "\rho^k_G"] & K^0(Z)_{(p)} \arrow[d, "\rho^k_G"] \\ 1+G^0(Z)^{\otimes} \arrow[r, "\psi^k/1"] & 1+G^0(Z)^{\otimes} \end{tikzcd} \qquad (5.3)$$

PROOF. The $\otimes$-sign indicates that we consider $1 + G^0(Z)$ multiplicatively, that is as a subgroup of the units in $G^0(Z_+)$. By the splitting principle and the fact that ρ^k_G is exponential the proof for $\psi^k \rho^k_G = \rho^k_G \psi^k$ is reduced to the case of a line bundle L. But on $P_\infty\mathbf{C}$ the Adams operation ψ^k may be induced by the map $m_k : P_\infty\mathbf{C} \to P_\infty\mathbf{C}$ which represents multiplication by k in the H-space structure. The result follows by naturality of ρ^k_G.

To see that ρ^k_G is bijective on $G^0(Z)$, it is enough to prove this for $Z = S^{qt}, t > 0$, because then ρ^k_G induces a weak homotopy equivalence between classifying spaces. Instead of doing the somewhat long calculation for ρ^k_G on $G^0(S^{qt})$ directly we shall, recalling Remark 1.3, reduce to the well known case of ρ^k on $K^0(S^{qt})$, which was used already in 2.5:

$$\begin{tikzcd}
G^{-1}(S^{qt-1}) \arrow[d, "\rho_G^k - 1"] & \subset & K^{-1}(S^{qt-1})_{(p)} \arrow[r, "\widetilde{J}"] & \pi_S^0(S^{qt-1})_{(p)} \arrow[d, "h_{Ad}"] \arrow[dl, "e_G"'] \\
G^{-1}(S^{qt-1}) \arrow[rr] & & \dfrac{G^{-1}(S^{qt-1})}{(\psi^k - 1)G^{-1}(S^{qt-1})} \arrow[r, "\Delta"] & Ad^0(S^{qt-1})
\end{tikzcd}$$

Here e_G denotes the e-invariant defined using G-theory. Commutativity is proved in the same way as (2.4) is established. This shows that $\rho_G^k - 1$ is surjective on $G^0(S^{qt}) = \mathbf{Z}_{(p)}$, hence bijective. □

PROOF OF THEOREM 5.1.

"$\Rightarrow$" Let $U_A(\xi) \in Ad^{2n}(Z^\xi)$ be a Thom class for ξ. Then $D(U_A(\xi))$ is a Thom class in $G^{2n}(Z^\xi)$ since $D : Ad^{2n}(S^{2n}) \to G^{2n}(S^{2n})$ is an isomorphism. Now two Thom classes for the same bundle differ by a unit, that is we have

$$e \cup D(U_A(\xi)) = U_G(\xi) \tag{5.4}$$

where $U_G(\xi)$ is the Thom class used above and $e \in 1 + G^0(Z)$ is a unit. Apply ψ^k to this equation to obtain with $\tilde{\xi} = \xi - n$

$$\rho_G^k(\tilde{\xi}) = \frac{\psi^k e}{e}$$

in $1 + G^0(Z)$. Write now $\tilde{\xi} = \xi_G + \xi_R$ with $\xi_G \in G^0(Z)$ and ξ_R in the complementary summand $G^0(Z)^\perp$. Since $\psi^k - 1$ is bijective on $G^0(Z)^\perp$ there is an element z_R with $(\psi^k - 1)(z_R) = \xi_R$. From $\rho_G^k(\xi_G + \xi_R) = \rho_G^k(\xi_G) \cdot \rho_G^k(\xi_R) = \psi^k(e)/e$ it follows by 5.2 that

$$\rho_G^k(\xi_G) = (\psi^k/1)\big(e \cdot \rho_G^k(-z_R)\big) = (\psi^k/1)\rho_G^k(w) = \rho_G^k\big((\psi^k - 1)w\big)$$

for some $w \in G^0(Z)$. But then $\xi_G = (\psi^k - 1)w$, and $\Delta(\xi) = 0$ follows.

"$\Leftarrow$" If $\Delta(\tilde{\xi}) = 0$ we can write $\tilde{\xi} = (\psi^k - 1)z$ for some $z \in K^0(Z)$. Apply ρ^k to this equation

$$\rho^k(\tilde{\xi}) = \rho^k(\psi^k z - z) = \rho^k(\psi^k z)/\rho^k(z) = \psi^k \rho^k(z)/\rho^k(z)$$

and define $U'_A(\tilde{\xi}) := \rho^k(-z) \cdot U_K(\tilde{\xi}) \in K^0(Z^\xi)$. Then

$$\psi^k U'_A(\tilde{\xi}) = \psi^k \rho^k(-z) \cdot \rho^k(\tilde{\xi}) \cdot U_K(\tilde{\xi}) = \rho^k(-z) \cdot U_K(\tilde{\xi}) = U'_A(\tilde{\xi}).$$

Hence $U'_A(\tilde{\xi}) \in ker(\psi^k - 1)$ and there exists a class $U_A(\tilde{\xi})$ with $D(U_A(\tilde{\xi})) = U'_A(\tilde{\xi})$. Then $U_A(\tilde{\xi})$ is a Thom class for $\tilde{\xi}$ in Ad-theory. □

REMARK. Let ξ be Ad-theory orientable with Thom class $U_A(\xi)$. Then a reformulation of the Adams conjecture is that this implies that $U_A(\xi)$ is stably spherical.

It is now easy to clarify the relation between the groups $J(X)$ of stable fibre-homotopy equivalence classes of sphere bundles of vector bundles and $Im(J)$-theory. Here we use the convention that $J(Z)$ means the reduced J-group of Z, i.e. $J(*) = 0$.

THEOREM 5.5. *For an odd prime p and Z a finite connected CW-complex there is a natural isomorphism*

$$J(Z)_{(p)} \cong im(\Delta) \subset Ad^1(Z).$$

PROOF. Define $J' : im(\Delta) \to J(Z)_{(p)}$ by $J'(z) = J(\Delta^{-1}(z))$. This map is well defined as a consequence of the Adams conjecture. It is trivially surjective. Assume now $J'(z) = 0$ and choose a complex vector bundle ξ such that $z = \Delta(\tilde{\xi})$, $\tilde{\xi} = \xi - \dim \xi$. Then $J(\tilde{\xi}) = 0$ means that ξ is stably fibre homotopy trivial at p. From this we easily get a Thom class for ξ in p-local stable cohomotopy. But then ξ is Ad^*-orientable and by 5.1 this implies $\Delta(\tilde{\xi}) = 0$, so proving injectivity. □

COROLLARY 5.6. *For a complex vector bundle ξ the following statements are equivalent.*
1) *ξ is stably fibre homotopy trivial at p.*
2) *ξ is orientable for $\pi^*_{S(p)}$.*
3) *ξ is orientable for Ad^*.*

REMARKS. 1. The proof of Theorem 5.1 may be translated into the commutative diagram

$$\begin{tikzcd}
G^0(Z) \arrow[r, "\psi^k - 1"] \arrow[d, "\rho^k_G"] & G^0(Z) \arrow[r] \arrow[d, "\rho^k_G"] & coker(\psi^k - 1) \arrow[r] \arrow[d, "\hat{\rho}^k_G", "\cong"'] & 0 \\
1 + G^0(Z)^{\otimes} \arrow[r, "\psi^k/1"] & 1 + G^0(Z)^{\otimes} \arrow[r] & coker(\psi^k/1) \arrow[r] & 0
\end{tikzcd}$$

Now the groups $coker(\psi^k - 1)$ and $coker(\psi^k/1)$ are precisely the groups $J''(Z)_{(p)}$ and $J'(Z)_{(p)}$ from the $J(X)$-papers of Adams which served as upper and lower bounds for $J(Z)_{(p)}$, and 5.5 is a translation of the squeezing technique used there to determine $J(Z)_{(p)}$.

2. Observe that for the conclusion of Theorem 5.5 one requires the Adams conjecture only for the particular space Z and not in full generality. In the case where $K^0(Z)$ is generated by line bundles the proof of the Adams conjecture is short and easy and already contained in [1].

3. Spherical fibrations at an odd prime p split into $Im(J)$ and $coker(J)$ parts. It may be shown that $coker(J)$-spherical fibrations are orientable for Ad-theory, and this shows the difference between Ad-theory and stable cohomotopy orientability. For Z a suspension this immediately follows from the sequence (2.3).

4. The statement $J(Z) \otimes \mathbf{Q} = 0$ for a connected finite complex Z now follows trivially from $Ad^1(Z) \otimes \mathbf{Q} \cong H^0(Z; \mathbf{Q}) \oplus H^{-1}(Z; \mathbf{Q})$.

6. Examples of J-groups

In this section we discuss as an example the J-groups of suspensions of complex projective spaces $\Sigma^{2m}P_n\mathbf{C}$ at an odd prime p. Observe first that since $K^1(P_n\mathbf{C}) = 0$ Theorem 5.5 implies that

$$J\big(\Sigma^{2m}P_n\mathbf{C}\big)_{(p)} \cong Ad^{1-2m}(P_n\mathbf{C}). \tag{6.1}$$

We begin by recalling what is known about $J(P_n\mathbf{C})_{(p)}$. The group order of $Ad^1(P_n\mathbf{C})$ follows directly from the defining sequence of Ad-theory (see part I) by computing the determinant of $\psi^k - 1$, which is

$$\nu_p\big(|Ad^1(P_n\mathbf{C})|\big) = \sum_{i=1}^{t}\big(1+\nu_p(i)\big) = \nu_p\big((tp)!\big)$$

with $t = [\frac{n}{p-1}]$ and $[x]$ denoting the greatest integer not exceeding x. The same argument shows that

$$Ad^1(P_n\mathbf{C}) \cong Ad^1\big(P_{t(p-1)}\mathbf{C}\big)$$

where $n = t\cdot(p-1)+s$ with $0 \leqslant s \leqslant p-2$.

The number of cyclic summands in the abelian group $Ad^1(P_n\mathbf{C})$ was computed in [21] and is equal to

$$\left[\frac{\log(n+1)}{\log p}\right]$$

(that is r, where $p^r \leqslant n+1 < p^{r+1}$). The argument is as follows. Assume $n = t(p-1)$. By the universal coefficient formula (§3, part I) $Ad^1(P_n\mathbf{C})$ and $Ad_{-1}(P_n\mathbf{C})$ have the same number of summands, which is the dimension of $Ad_0(P_n\mathbf{C}; F_p)$. But $\dim Ad_0(P_n\mathbf{C}; F_p) = \dim Ad_{2n}(P_n\mathbf{C}; F_p)$ by v_1-periodicity (§3). Then the dimension of $Ad_{2n}(P_n\mathbf{C}; F_p)$ is found by working out the skeleton filtration of the elements $z_{r,1}$ defined in §6 of part I. For this one only has to determine the mod p values of the numbers $P_r(n,s)$ introduced there.

Denote by $H = H_n$ the Hopf line bundle on $P_n\mathbf{C}$. Since $\Delta((\psi^i - 1)x) = 0$ for $i \not\equiv 0$ mod p, the classes $\Delta(H^{p^i} - 1)$, $i = 0, \dots, [\log_p n]$, form a generating set of $Ad^1(P_n\mathbf{C})$. The order of $J(H-1) = \Delta(H-1)$ is the well known Atiyah–Todd number

$$\nu_p\big(|J(H_n - 1)|\big) = \max\big\{r + \nu_p(r) \mid 0 \leqslant r \leqslant [n/(p-1)]\big\}.$$

The simplest derivation of this seems to be the argument given by Lam in [25] where the order of $J(H^{p^i} - 1)$ is also determined as

$$\nu_p\big(|J(H_n^{p^i} - 1)|\big) = \max\big\{r + \nu_p(r) \mid 0 \leqslant r \leqslant [n/p^i(p-1)]\big\} =: c(n,i).$$

The generating set $\{J(H_n^{p^i} - 1) \mid i \geqslant 0\}$ is in general not a basis for $Ad^1(P_n\mathbf{C})$: there can be nontrivial relations between these elements. Let C_n be the direct sum of cyclic groups of order $p^{c(n,i)}$ and generators h_i. Then the obvious map $\varphi_n : C_n \to Ad^1(P_n\mathbf{C})$ with $\varphi_n(h_i) = \Delta(H_n^{p^i} - 1)$ is always onto, but may have a nontrivial kernel. Comparing the orders of both groups gives the values of n where φ_n is an isomorphism. For example if $n = p^a - 1$, or more generally if $n = t \cdot (p-1)$ and

$$t = \sum_{i=0}^{b} \alpha_i p^i$$

with $0 \leqslant \alpha_i \leqslant p-1$ and all $\alpha_i \neq 0$, the map φ_n is injective. As a consequence the relations among the generators $\Delta(H_n^{p^i} - 1)$ in $Ad^1(P_n\mathbf{C})$ are always given by linear combinations of $\Delta((H-1)^b)$ with $b \in \{n+1, \ldots, c\}$, where c is the next value larger than n for which φ_c is injective. Besides the cases where φ_n is bijective there are other values of n where one can determine the group structure of $Ad^1(P_n\mathbf{C})$, but in general the problem of determining $Ad^1(P_n\mathbf{C})$ becomes combinatorially more and more involved as n grows and no general formula is known.

We now turn to $J(\Sigma^{2m}P_n\mathbf{C})_{(p)} \cong Ad^{1-2m}(P_n\mathbf{C})$ for $m \geqslant 1$. The group order is determined by the same method as for $m = 0$ and is given by

$$\begin{aligned}\nu_p\big(\big|Ad^{1-2m}(P_n\mathbf{C}_+)\big|\big) &= \sum_{s=0}^{n} \nu_p\big|Ad^{1-2m-2s}(S^0)\big| \\ &= \sum_{r\geqslant 0}\left[\frac{n+m}{p^r(p-1)}\right] - \sum_{r\geqslant 0}\left[\frac{m-1}{p^r(p-1)}\right].\end{aligned}$$

Also the number of cyclic summands in $Ad^{1-2m}(P_n\mathbf{C})$ may be computed by the method above. But the order functions for $J(u^m(H^{p^r} - 1)) \in J(\Sigma^{2m}P_n\mathbf{C})_{(p)}$, where u denotes Bott periodicity, follow a slightly different pattern.

PROPOSITION 6.2. *The order of $u^m(H_n - 1)$ in $J(\Sigma^{2m}P_n\mathbf{C})_{(p)}$ is given by*

$$\nu_p\big|\Delta u^m(H_n - 1)\big| = \left[\frac{n+m}{p-1}\right] - \nu_p\big((m-1)!\big) - f_m(n)$$

where $n \mapsto f_m(n)$ is a periodic function.

In the case $m = 1$, which seems to be known, we have $f_m(n) = 0$, so that

$$\big|J\big(u(H_n - 1)\big)_{(p)}\big| = p^{\left[\frac{n+1}{p-1}\right]}.$$

For the proof of (6.2) we reformulate the orientability condition for $Im(J)$-theory in terms of rational characteristic classes as in [2]. Choose a complex orientation e for p-local K-theory. Associated to e we have a formal group F with logarithm f and a multiplicative Thom class $U_F(\xi)$ for complex vector bundles. Denote the Euler class belonging to $U_F(\xi)$ by $e_F(\xi)$ and let $U_H(\xi)$ be the usual cohomology Thom class of ξ. Define the characteristic class $Todd_F(\xi)$ by the equation

$$chU_F(\xi) = Todd_F(\xi) \cup U_H(\xi)$$

in $H^*(Z_+;\mathbf{Q})$ and set

$$Bh_F(\xi) := ch^{-1}Todd_F(\xi) \in K^0(Z_+;\mathbf{Q}). \tag{6.3}$$

Then Bh_F is an exponential characteristic class and the standard proof applies to give

LEMMA 6.4. *$Bh_F(L) = e_F(L)/f(e_F(L))$ for a line bundle L.*

PROPOSITION 6.5. *Suppose Z is a connected finite complex with $H^*(Z;\mathbf{Z}_{(p)})$ torsion-free and let ξ be a complex vector bundle on Z. Then ξ is orientable for $Im(J)$-theory at the odd prime p if and only if $Bh_F(\xi)$ is integral.*

PROOF. If $U_A(\xi)$ is an $Im(J)$-theory Thom class for ξ we must have $chD(U_A(\xi)) = U_H(\xi)$, hence the equation $U_F(\xi) = Bh_F(\xi) \cup DU_A(\xi)$ shows that $Bh_F(\xi)$ is integral. Conversely, if $Bh_F(\xi)$ is integral, then $U_F(\xi) \cup Bh_F(\xi)^{-1}$ is in $ker(\psi^k - 1)$ and defines a Thom class in $Im(J)$-theory. □

Using the formal group F a straightforward calculation computes

$$Bh_F\big(u(L)\big) = Bh_F\big((H_1 - 1)\otimes L\big) = Bh_F(H_1 \otimes L)/Bh_F(1 \otimes L)$$

and shows:

PROPOSITION 6.6. *Let L be a complex line bundle on Z then*

$$\begin{aligned} &Bh_F\big(u(L-1)\big) \\ &\quad = 1 + u\left(\frac{1}{e_F(L)\cdot f'(e_F(L))} - \frac{1}{f(e_F(L))}\right) \quad \textit{in } K^0\big(\Sigma^2 Z_+;\mathbf{Q}\big). \end{aligned} \tag{6.7}$$

Note that both summands involve negative powers of $e_F(L)$, but the first summand $e_F(L)^{-1}\cdot f'(e_F(L))^{-1}$ is an integral Laurent polynomial.

EXAMPLE. The usual multiplicative formal group gives with $x = H - 1$

$$Bh(uH) = 1 + u\left(1 + \frac{1}{x} - \frac{1}{\log(x+1)}\right) \quad \text{in } K^0\big(\Sigma^2 P_n\mathbf{C}_+;\mathbf{Q}\big). \tag{6.8}$$

Better suited for p-local computations is a p-typical formal group. Let G denote the formal group with logarithm

$$\log_G(x) = \sum_{i=0}^{\infty} x^{p^i}/p^i.$$

Then $U_G(\xi)$ and $e_G(\xi)$ are already contained in the Adams summand $G^*(\ )$ of p-local K-theory and we have

$$\begin{aligned}&Bh_G\big(u(H-1)\big)\\&\quad=1+u\left(\frac{1}{e_G(H)\cdot(1+e_G(H)^{p-1}+e_G(H)^{p^2-1}+\cdots)}-\frac{1}{\log_G(e_G(H))}\right).\end{aligned}$$

To extend this to $m>1$ we shall use the stable map

$$\omega: S^2\wedge P_n\mathbf{C}_+\longrightarrow P_{n+1}\mathbf{C}$$

defined by taking the relevant components of $\Sigma(\mu)$ on

$$\Sigma\big(S^2\times P_n\mathbf{C}\big)\simeq S^3\wedge P_n\mathbf{C}\vee S^3\vee\Sigma P_n\mathbf{C}$$

for μ the restriction of the product map on $P_\infty\mathbf{C}$. Then $\omega^*(H_{n+1}-1)=u(H_n)$ in K-theory. In homology ω_* is multiplication with the generator $b_1\in H_2(P_n\mathbf{C})$ in the Pontryagin ring structure. Dually we find that ω^* is differentiation with respect to the generator $g\in H^2(P_n\mathbf{C})$. Therefore ω^* acts as differentiation with respect to $g=f(e_F(H))$ in $K^*(P_n\mathbf{C};Q)$. By induction we get

COROLLARY 6.9.

$$Bh_F\big(u^m(H_n-1)\big)=1+u^m\left[r(e_F)+(-1)^m\frac{(m-1)!}{f(e_F)^m}\right]$$

in $K^0(\Sigma^{2m}P_n\mathbf{C}_+;\mathbf{Q})$, *where* $r(e_F)$ *is a p-integral Laurent polynomial in* e_F.

Now the order of Δu^mH_n is given by the smallest number c such that $Bh_F(cu^mH_n)=Bh_F(u^mH_n)^c$ is integral. But on a suspension all reduced products vanish, hence c is the smallest number c such that $c\cdot Bh_F(u^mH_n)$ is integral. Therefore we have

COROLLARY 6.10. *The order of* Δu^mH_n *is given by the maximal denominator of the coefficients in the polynomial part of* $(m-1)!/f(e_G(H_n))^m$.

Proposition 6.2 follows from this corollary by an investigation of the combinatorial properties of the denominators of the coefficients in the series $f(e_G)^{-m}$.

For the p-typical formal group G define h_p by

$$\frac{\log_G(x)}{x}=h_p\big(x^{p-1}/p\big).$$

Then h_p is a power series with integral coefficients and the denominator of the coefficient $x^{t(p-1)}$ in $h_p(x^{p-1}/p)$ is thus bounded by p^t. Modulo p we have $h_p(y)\equiv 1+y$. This gives the mod p^{p-1} values of the coefficients $a_r^{(m)}$ in

$$h_p\big(x^{p-1}/p\big)^{-m}=\sum_{r=0}^{\infty}a_r^{(m)}x^{r(p-1)}/p^r \tag{6.11}$$

in terms of binomial coefficients. For $m = 1$ we get

$$a_r^{(1)} \equiv \binom{-1}{r} \not\equiv 0 \bmod p$$

and hence

$$\nu_p\left|J(uH_n)\right| = \left[\frac{n+1}{p-1}\right].$$

Let now $m \geqslant 1$. Then the exponent of p in the denominator of $Bh_G(u^m H_n)$ is bounded by

$$\left[\frac{n+m}{p-1}\right] - \nu_p\big((m-1)!\big).$$

Define f_m by

$$f_m(n) := \left[\frac{n+m}{p-1}\right] - \nu_p\big((m-1)!\big) - \nu_p\left|J\big(u^m H_n\big)\right|.$$

Then $f_m(t(p-1)-m) = 0$ if and only if $a_t^{(m)} \not\equiv 0 \bmod p$. Suppose $f_m(t(p-1)-m) \geqslant s$. Since $f_m((t+1)(p-1)-m) \leqslant f_m(t(p-1)-m)+1$ all the coefficients $a_t^{(m)}, a_{t-1}^{(m)}, \ldots, a_{t-s+1}^{(m)}$ must be zero mod p. But there are at most $m-1$ consecutive coefficients in $(1+y)^{-m}$ which are all congruent to 0 mod p. Therefore f_m is bounded by $m-1$ (at least for $n \geqslant m(p-1)$). To investigate f_m the coefficients $a_r^{(m)}$ may therefore be reduced mod p^m. But then the usual periodicity properties of binomial coefficients mod p^m imply that $a_r^{(m)}$ mod p^m is periodic; hence the same must be true for f_m. This finishes the proof of 6.2.

Similar arguments show that the exponent of the order function for $J(u^m H_n^{p^r})$ may be written as the difference of the monotonic function

$$\left[\frac{n+mp^r}{(p-1)p^r}\right] - \nu_p\big((m-1)!\big)$$

and a function periodic in n. The mod p^{p-1} information for the coefficients in (6.11) is sufficient to determine f_m explicitly for m small.

Although the order functions for $J(u^m H_n^{p^r})$ are more regular for $m \geqslant 1$ than for $m = 0$, the remarks concerning the group structure of $J(\Sigma^{2m} P_n \mathbf{C})_{(p)}$ apply also for $m \geqslant 1$. An exception is perhaps the case $m = 1$. If n is of the form $n = p^j(p-1)-1$, then we have

$$J\big(\Sigma^2 P_n \mathbf{C}\big)_{(p)} \cong \bigoplus \mathbf{Z}/p^{a_i} \quad \text{with } a_i = \left[\frac{n+1}{p^i(p-1)}\right].$$

This is presumably true for all n.

Other examples where J-groups are known include the lens spaces $L^n(p^a)$, see, for example, [17], suspensions of stunted lens spaces and some spherical space forms.

7. $Im(J)$-theory for torsion-free spaces

$Im(J)$-theory for torsion-free spaces is easier to handle than for general spaces. In this section we discuss some aspects of the theory for torsion-free spaces. It is, in general, much easier to compute in $ker(\psi^k - 1) \subset K_*(X)$ than to work with $coker(\psi^k - 1)$, that is with classes in $im(\Delta)$. We first show how to switch from $im(\Delta)$ to $ker(\psi^k - 1)$ in the torsion-free situation by introducing $\mathbf{Q}/\mathbf{Z}$ coefficients. Then we show that $Im(J)$-theory detects Adams–Novikov filtration one elements in $\pi_n^S(X)_{(p)}$ if X has torsion-free homology. The last topic is the Atiyah–Hirzebruch spectral sequence. Since this spectral sequence collapses for K-theory on a torsion-free space there is not much room for nontrivial differentials in the corresponding spectral sequence for Ad-theory. We give a formula for the remaining nontrivial differentials in terms of the Chern character.

PROPOSITION 7.1. *Suppose X is a finite spectrum and that $x \in Ad^{n+1}(X)$ is in $ker\, D \cap ker\, q$. Then $D\beta^{-1}(x) \in K^n(X; \mathbf{Q}/\mathbf{Z})$ is given by $r \circ (\psi^k - 1)^{-1} \circ q \circ \Delta^{-1}(x)$ mod $r \circ D(Ad^n(X))$, where the maps are from the following commutative diagram built up by Bockstein sequences and defining sequences for $Im(J)$-theory.*

$$
\begin{tikzcd}
K^n(X;\mathbf{Q}/\mathbf{Z}) & & & \\
K^n(X;\mathbf{Q}) \arrow[u, "r"] \arrow[r, "\psi^k - 1"] & K^n(X;\mathbf{Q}) \arrow[r, "\Delta"] & Ad^{n+1}(X;\mathbf{Q}) & \\
 & K^n(X) \arrow[u, "q"] \arrow[r, "\Delta"] & Ad^{n+1}(X) \arrow[u, "q"] & \\
 & & Ad^n(X;\mathbf{Q}/\mathbf{Z}) \arrow[u, "\beta"] \arrow[r, "D"] & K^n(X;\mathbf{Q}/\mathbf{Z})
\end{tikzcd}
$$

PROOF. This follows from the relation

$$
\begin{aligned}
&\left(\psi^k - 1\right)^{-1} \circ q \circ \Delta^{-1} \\
&\quad \equiv r^{-1} \circ D \circ \beta^{-1} \bmod qK^n(X) + DAd^n(X; \mathbf{Q})
\end{aligned} \tag{7.2}
$$

by applying r to both sides. The equivalence (7.2) may be proved as follows. Replacing the sequence of coefficients in the diagram above by $\mathbf{Z} \xrightarrow{p^a} \mathbf{Z} \to \mathbf{Z}/p^a$ one has the corresponding relation

$$
\left(\psi^k - 1\right)^{-1} \circ p^a \circ \Delta^{-1} \equiv r^{-1} \circ D \circ \beta^{-1} \bmod p^a K^n(X) + DAd^n(X), \tag{7.3}
$$

which is proved by applying the theorem on Toda brackets (see §1) to the sequence of maps

$$X \xrightarrow{p^a} X \xrightarrow{x} \Sigma^{n+1}Ad \xrightarrow{D} \Sigma^{n+1}Ad.$$

Then (7.2) follows by taking direct limits. □

EXAMPLE 1. From part I, §6, we know that elements in $Ad_{2n-1}(P_\infty\mathbf{C})$ can be described by their γ_i-sequences. Recall that

$$\gamma_i : Ad_{2n}(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

is defined by the Kronecker product

$$\gamma_i(z) = \left\langle H^{p^i}, D(z)\right\rangle_K.$$

If x in $Ad_{2n-1}(P_\infty\mathbf{C})$ is given as $\Delta(b_m u^n)$, where $1, b_1, b_2, \ldots, b_n$ is the basis of $K_0(P_n\mathbf{C})$ dual to the basis $1, x, x^2, \ldots, x^n$ of $K^0(P_n\mathbf{C})$, then 7.1 applied to the S-dual of $P_n\mathbf{C}$, together with 6.4 in part I for $m < n$, gives

$$\gamma_i\left(\beta^{-1}\Delta b_m u^n\right) = \gamma_i\left(r \circ \left(\psi^k - 1\right)^{-1} \circ q\left(b_m u^n\right)\right) = \frac{1}{m!}\sum_{j=1}^{m} \frac{s(m,j)\cdot p^{ij}}{k^{n-j}-1}$$

modulo $\gamma_i(rDAd_{2n}(P_\infty\mathbf{C};\mathbf{Q}))$. Here $s(m,j)$ is as in part I, §2, $Ad_{2n}(P_\infty\mathbf{C};\mathbf{Q}) \cong \mathbf{Q}\cdot\tilde{b}_1^n$ and $\gamma_i(\tilde{b}_1^n/p^a) = p^{-a+ni}$.

REMARK. This leads to a description and computation of $A_{2n-1}(P_\infty\mathbf{C})$, the connective $Im(J)$-theory groups of $P_\infty\mathbf{C}$, see [22].

EXAMPLE 2. If E is an m-dimensional complex vector bundle on ΣZ where Z is a finite connected complex with torsion-free homology, then $h_{Ad} \circ \widetilde{J}(E) \in Ad^0(Z)$ is defined and by (2.4) equal to $\Delta(\rho^k(E-n)-1)$. To write $h_{Ad} \circ \widetilde{J}(E)$ as $\beta(y)$ consider the rational characteristic class

$$Bh_F(E-n) \in K^0(\Sigma Z_+;\mathbf{Q})$$

introduced in (6.3). From the equality

$$\left(\psi^k - 1\right)Bh_F(E-n) = \rho_F^k(E-n) - 1$$

on a suspension and 7.1 it follows that

$$D \circ \beta^{-1} \circ h_{Ad} \circ \widetilde{J}(E) = r \circ Bh_F(E-n) - 1$$

in $K^{-1}(Z;\mathbf{Q}/\mathbf{Z})$. The indeterminacy is zero.

A special case of this is the S^1-transfer map $tr \in \pi_S^{-1}(P_\infty\mathbf{C}_+)$. The stable map tr may be identified with $\widetilde{J}(uH)$ (e.g., see [21]) and hence (6.8), or more generally (6.7), gives the e-invariant or the Hurewicz image $h_{Ad}(tr)$ of tr:

$$D \circ \beta^{-1} h_{Ad}(tr) = \frac{1}{x} - \frac{1}{\log(x+1)} =: z \in K^{-2}(P_\infty\mathbf{C}; \mathbf{Q}/\mathbf{Z}). \tag{7.4}$$

We can use this to compute

$$tr : Ad_n(X \wedge P_\infty\mathbf{C}_+) \longrightarrow Ad_{n+1}(X)$$

for torsion-free X: it is determined by the commutative diagram

$$\begin{tikzcd}
Ad_n(X \wedge P_\infty\mathbf{C}_+) \arrow[r, "tr"] \arrow[dd, "D"] & Ad_{n+1}(X) & \\
& & Ad_{n+2}(X; \mathbf{Q}/\mathbf{Z}) \arrow[ul, "\beta"'] \arrow[dl, "D"] \\
K_n(X \wedge P_\infty\mathbf{C}_+)_{(p)} \arrow[r, "\backslash z"] & K_{n+2}(\mathbf{X}; \mathbf{Q}/\mathbf{Z})_{(p)} &
\end{tikzcd}$$

with z defined in (7.4).

EXAMPLE 3. Again let X be a connective torsion-free CW-spectrum and $f : S^n \to X$ a stable map with cofibre C_f inducing the trivial map in K-theory. Suppose we know the action of the Steenrod powers $\mathcal{P}^i$ on $H_*(X; \mathbf{F}_p)$ and want to describe the extension

$$0 \to H_*(X; \mathbf{F}_p) \xrightarrow{j_*} H_*(C_f; \mathbf{F}_p) \xrightarrow{i_*} H_*(S^{n+1}; \mathbf{F}_p) \to 0 \tag{7.5}$$

as a module over the mod p Steenrod Algebra $\mathcal{A}_p$ ($p \neq 2$). This information can be obtained from the e-invariant $h_{Ad}(f)$ as follows. It is enough to determine $\chi\mathcal{P}^i(\hat{1})$, where χ is the canonical anti-automorphism of $\mathcal{A}_p$ and $\hat{1} \in H_{n+1}(C_f; \mathbf{F}_p)$ maps to the generator in $H_{n+1}(S^{n+1}; \mathbf{F}_p)$ under i_*.

For the corresponding exact sequence in K-theory

$$0 \to K_*(X) \xrightarrow{j_*} K_*(C_f) \xrightarrow{i_*} K_*(S^{n+1}) \to 0$$

we know by (1.5)

$$(\psi^k - 1)\hat{1} = j_*(z) \quad \text{with } \Delta(z) = h_{Ad}(f). \tag{7.6}$$

We now use the well-known integrality theorem for the Chern character, namely the fact that $p^i ch_{qi}(z)$ is p-integral with its mod p reduction equal to $\chi\mathcal{P}^i(ch_0(z))$ in

$H_{n+1-qi}(X;\mathbf{F}_p)$. Note that $ch_0(z)$ is integral since X is torsion-free. Applying $p^i ch_{qi}$ to (7.6) gives

$$\chi\mathcal{P}^i(\hat{1}) = j_* \circ red_p\left(\frac{p^i ch_{qi}(z)}{k^{(p-1)i}-1}\right).$$

In §1 we studied the relation between $Im(J)$-theory and the K-theory Adams spectral sequence. Better suited for applications to stable homotopy than this spectral sequence is the BP or MU-Adams–Novikov spectral sequence. We denote the associated Adams filtration by F^n_{BP}. For torsion-free spectra we show that $Im(J)$-theory may be used to describe elements in $Ext^{1,*}_{BP_*BP}(BP_*, BP_*(X))$ (see also Remark 1.3).

The Todd map $T: BP \to G$ induces a map between spectral sequences and thus a map

$$\begin{aligned} T_1 : Ext^{1,*}_{BP_*BP}(BP_*, BP_*(X)) &\to Ext^{1,*}_{K_*K}(K_*, K_*(X))_{(p)} \\ &= Ext^{1,*}_{G_*G}(G_*, G_*(X)). \end{aligned}$$

From 1.11 and diagram (1.10) we have a monomorphism

$$Ext^{1,n+1}_{K_*K}(K_*, K_*(X))_{(p)} \subset Ad_n(X).$$

Let $\bar{e}$ denote the composition of these two maps. The equivalence of Definitions 1 and 2 in §1 together with 1.4 imply that the composition of the natural map

$$F^1_{BP}\pi^S_n(X)_{(p)} \longrightarrow Ext^{1,*}_{BP_*BP}(BP_*, BP_*(X))$$

with $\bar{e}$ is the Ad-theory Hurewicz map h_{Ad}.

PROPOSITION 7.7 (see Lemma 3.7 in [21]). *Let X be a connective spectrum with torsion-free homology. Then*

$$\bar{e} : Ext^{1,n+1}_{BP_*BP}(BP_*, BP_*(X)) \longrightarrow Ad_n(X)$$

is injective and $h_{Ad}: F^1_{BP}\pi^S_n(X)_{(p)} \to Ad_n(X)$ has kernel F^2_{BP}.

PROOF. Denote by $Pr_n E_*(X)$ the coaction primitives for the homology theory E. Then we have a commutative diagram

$$\begin{tikzcd} Pr_{n+1}BP_*(X;\mathbf{Q}) \arrow[r] \arrow[d, "\cong"] & Pr_{n+1}BP_*(X;\mathbf{Q}/\mathbf{Z}) \arrow[r] \arrow[d, "T_0"] & Ext^{1,n+1}_{BP_*BP}(BP_*, BP_*(X)) \arrow[r] \arrow[d, "T_1"] & 0 \\ Pr_{n+1}K_*(X;\mathbf{Q}) \arrow[r] & Pr_{n+1}K_*(X;\mathbf{Q}/\mathbf{Z})_{(p)} \arrow[r] & Ext^{1,n+1}_{K_*K}(K_*, K_*(X))_{(p)} \arrow[r] & 0 \end{tikzcd}$$

and it is enough to show that T_0 is injective. An easy application of the Hattori–Stong theorem is that for torsion-free spectra X the K-theory Hurewicz map

$$h_G : BP_*(X;\mathbf{Q}/\mathbf{Z}) \longrightarrow G_*(BP \wedge X;\mathbf{Q}/\mathbf{Z})$$

is injective. (Consider the collapsing Atiyah–Hirzebruch spectral sequences for both groups.) In the following diagram, with $\bar{\mu} = \mu \circ 1 \wedge T, pr : BP \to \overline{BP}$ the cofibre map of $S^0 \to BP$ and $Y = X \wedge M(\mathbf{Q}/\mathbf{Z})$, the squares commute.

$$\begin{tikzcd}
Pr_n BP_*(Y) \arrow[r, "\subset"] \arrow[dd, "T_0"] & BP_n(Y) \arrow[r, "\eta_R - \eta_L"] \arrow[d, "h_G"] \arrow[ddl, "T"'] & BP_n(BP \wedge Y) \arrow[d] \\
 & G_n(BP \wedge Y) \arrow[r, "1\wedge\eta_R - 1\wedge\eta_L"] \arrow[d, "id"] & G_n(BP \wedge BP \wedge Y) \arrow[d, "\mu\wedge pr\wedge 1_*"] \\
G_n(Y) \arrow[r, "i_*"] & G_n(BP \wedge Y) \arrow[r, "pr_*"] & G_n(\overline{BP} \wedge Y)
\end{tikzcd}$$

This shows that h_G maps $Pr_n BP_*(Y)$ injectively into $i_* G_n(Y)$ and defines T_0. Since $\bar{\mu}$ is a splitting for i_*, we see that this map is the restriction of T. But clearly $T(Pr_n BP_*(Y)) \subset Pr_n G_*(Y)$. □

REMARK. A complete description of $Ext^{1,n+1}_{BP_*BP}(BP_*, BP_*(X))$ for a torsion-free spectrum X by $Im(J)$-theory is possible with connective $Im(J)$-theory A. The map $\bar{e}$ factorizes through $A_n(X)$ and its image is the kernel of $i_* : A_n(X) \to A_n(BP \wedge X)$ with i the unit map $i : S^0 \to BP$.

REMARK. We note without proof another property of torsion-free spectra. The canonical map

$$d : A_n(X) \to Ad_n(X)$$

from connective $Im(J)$-theory A to nonconnective $Im(J)$-theory is always injective.

The last topic of this section is the Atiyah–Hirzebruch spectral sequence for $Ad_*(X)$ with X a connective torsion-free spectrum. Homology and K-theory are localized at p without indication in the notation. We shall use the exact couple

$$\begin{tikzcd}
Ad_*(X, X^s) \arrow[dr, "\partial"'] & & Ad_*(X, X^{s-1}) \arrow[ll, "i_*"'] \\
 & Ad_*(X^s, X^{s-1}) \arrow[ur, "j_*"'] &
\end{tikzcd}$$

to set up the Atiyah–Hirzebruch spectral sequence for $Ad_*(X)$. By comparing the Atiyah–Hirzebruch spectral sequences for $Ad_*(X)$ and $K_*(X)$ one finds that only differentials of the type

$$d_r : E^r_{s,0} \longrightarrow E^r_{s-r,r-1}$$

can be nontrivial. As a consequence there is only one value of t for which $E^t_{l,m}$ can be the target of a nontrivial differential, namely $t = m + 1 = r + 1$. Hence d_r is defined on a subgroup of $H_s(X;\mathbf{Z})$ with target $H_{s-r}(X;Ad_{r-1}(S^0)) = E^r_{s-r,r-1}$ and $E^{r+1}_{s-r,r-1} = E^{\infty}_{s-r,r-1}$. Since $r \geqslant 2$ the Bockstein map $\beta : Ad_r(S^0;\mathbf{Q}/\mathbf{Z}) \to Ad_{r-1}(S^0)$ is an isomorphism and we may change the target of d_r as follows. For r even, identify $K_r(S^0;\mathbf{Q}/\mathbf{Z})$ with $\mathbf{Q}/\mathbf{Z}_{(p)}$ and define

$$\bar{d}_r : E^r_{s,0} \longrightarrow H_{s-r}(X;\mathbf{Q}/\mathbf{Z})$$

as $\bar{d}_r := 1 \otimes D \circ (1 \otimes \beta)^{-1} \circ d_r$. Since $H_*(X;\mathbf{Z})$ is torsion-free and $1 \otimes D$ is injective, $\bar{d}_r$ determines d_r completely. The canonical map from the cycles in the cellular complex to homology

$$\begin{aligned} K_s(X^{s-r}, X^{s-r-1};\mathbf{Q}/\mathbf{Z}) &\cong H_{s-r}(X^{s-r}, X^{s-r-1};\mathbf{Q}/\mathbf{Z}) \\ &\supset \ker d_1 \xrightarrow{\varphi} H_{s-r}(X;\mathbf{Q}/\mathbf{Z}) \end{aligned}$$

will be denoted by φ. Note that for X with only even cells the differential d_1 of the cellular complex vanishes so that $K_s(X^{s-r}, X^{s-r-1};\mathbf{Q}/\mathbf{Z})$ is canonically isomorphic to $H_{s-r}(X;\mathbf{Q}/\mathbf{Z})$. The proof of the following result simplifies if one uses the fact that every X with torsion-free homology is homotopy equivalent to a complex with $d_1 = 0$.

THEOREM 7.8. *Let X be a connective CW-spectrum with torsion-free homology and $x \in H_s(X;Ad_0(S^0))$ a class such that $d_r(x)$ is defined with r even. Take $ch^{-1}(x)$ in $K_s(X, X^{s-r-1};\mathbf{Q})$ and reduce mod $\mathbf{Z}_{(p)}$. The resulting class $red\, ch^{-1}(x)$ is in the image of*

$$j_* : K_s(X^{s-r}, X^{s-r-1};\mathbf{Q}/\mathbf{Z}) \to K_s(X, X^{s-r-1};\mathbf{Q}/\mathbf{Z})$$

and any class y with $j_*(y) = red\, ch^{-1}(x)$ represents $\bar{d}_r(x)$ under the map φ. In diagram form:

$$\begin{tikzcd} H_s(X;\mathbf{Z}_{(p)}) \arrow[rr, "\bar{d}_r"] \arrow[d, "ch^{-1}"] & & H_{s-r}(X;\mathbf{Q}/\mathbf{Z}) \\ K_s(X,X^{s-r-1};\mathbf{Q}) \arrow[r, "red"] & K_s(X,X^{s-r-1};\mathbf{Q}/\mathbf{Z}) & K_s(X^{s-r},X^{s-r-1};\mathbf{Q}/\mathbf{Z}) \arrow[l, "j_*"'] \arrow[u, "\varphi"'] \end{tikzcd}$$

We give only the outline of a proof.

By definition d_r is given by the diagram

$$\begin{tikzcd} H_s(X;Ad_0) \arrow[rrr, "d_r"] & & & H_{s-r}(X;Ad_{r-1}) \\ Ad_s(X^s,X^{s-1}) \arrow[u, "\varphi"] \arrow[dr, "j_*"'] & & & Ad_{s-1}(X^{s-r},X^{s-r-1}) \arrow[u, "\varphi"'] \\ & Ad_s(X,X^{s-1}) & Ad_s(X,X^{s-r}) \arrow[l, "i^{r-1}_*"'] \arrow[ur, "\partial"] & \end{tikzcd}$$

with φ defined only on $ker\, d_1$. The first step consists in showing that ∂ is given by the diagram:

$$
\begin{tikzcd}
Ad_s(X, X^{s-r}) \arrow[r, "\partial"] \arrow[d, "q"] & Ad_s(X^{s-r}, X^{s-r-1}) \\
Ad_s(X, X^{s-r}; \mathbf{Q}) & Ad_s(X^{s-r}, X^{s-r-1}; \mathbf{Q}/\mathbf{Z}) \arrow[u, "\beta", "\cong"'] \arrow[d, "j_*^{r-1}"] \\
Ad_s(X, X^{s-r-1}; \mathbf{Q}) \arrow[r, "red"] \arrow[u, "i_*", "\cong"'] & Ad_s(X, X^{s-r-1}; \mathbf{Q}/\mathbf{Z})
\end{tikzcd}
\tag{7.9}
$$

The proof is similar to that in 1.9 for comparing different definitions of stable Toda brackets; the only problem is to handle the indeterminacy.

For the next step we map a part of this diagram into K-theory

$$
\begin{tikzcd}
Ad_s(X, X^{s-r-1}; \mathbf{Q}) \arrow[d, "D"] & & Ad_s(X^{s-r}, X^{s-r-1}; \mathbf{Q}/\mathbf{Z}) \arrow[d, "D"] \\
K_s(X, X^{s-r-1}; \mathbf{Q}) \arrow[r, "red"] \arrow[d, "ch"] & K_s(X, X^{s-r-1}; \mathbf{Q}/\mathbf{Z}) & K_s(X^{s-r}, X^{s-r-1}; \mathbf{Q}/\mathbf{Z}) \arrow[l, "j_*^{r-1}"'] \arrow[d, "\varphi"] \\
H_s(X; \mathbf{Q}) & & H_{s-r}(X; \mathbf{Q}/\mathbf{Z})
\end{tikzcd}
$$

showing that $\bar{d}_r(x)$ is given by $\varphi \circ (j_*^{r-1})^{-1} \circ red \circ D \circ i_*^{-1} \circ q(x_1)$ where x_1 is a class in $Ad_s(X, X^{s-r})$ with $i_*^{r-1}(x_1) = j_* \varphi^{-1}(x)$ in $Ad_s(X^s, X^{s-1})$. In the last step naturality is used to show that $ch^{-1}(x)$ represents $D \circ i_*^{-1} \circ q(x_1)$. □

Theorem 7.8 applies easily to spectra such as $P_\infty \mathbf{C}$, BT^n, BP for which the Chern character is known explicitly. Observe that ch^{-1} is determined by $ch : K^*(X) \to H^*(X; \mathbf{Q})$ via Kronecker duality if one chooses bases in a compatible way.

EXAMPLES.

1. $X = P_\infty \mathbf{C}$.

With $K_0(P_\infty \mathbf{C}_+) = \mathbf{Z}_{(p)}\langle 1, b_1, b_2, \ldots\rangle$ and $H_*(P_\infty \mathbf{C}_+) = \mathbf{Z}_{(p)}\langle 1, [P_1\mathbf{C}], [P_2\mathbf{C}], \ldots\rangle$ we find $ch^{-1}([P_i\mathbf{C}]) = \sum a_{ij} b_j$ where

$$
\begin{aligned}
a_{ij} &= \left\langle x^j, \sum a_{ij} b_j \right\rangle_K = ch\left\langle x^j, \sum a_{ij} b_j \right\rangle_K = \langle ch(x^j), [P_i\mathbf{C}]\rangle_H \\
&= \langle (e^g - 1)^j, [P_i\mathbf{C}]\rangle_H = \text{coefficient of } g^i \text{ in } (e^g - 1)^j.
\end{aligned}
$$

Since

$$\left(e^g - 1\right)^n = \sum_{j=n}^{\infty} S(j,n) n! g^j / j!$$

with $S(j,n)$ a Stirling number of the second kind (e.g., see [10]) it follows

$$ch^{-1}\left([P_i\mathbf{C}]\right) = \sum_{j=1}^{i} \frac{S(i,j)j!}{i!} b_j$$

and

$$d_{2r}\left(m \cdot [P_s\mathbf{C}]\right) = m \cdot \frac{S(s,s-r)(s-r)!}{s!}[P_{s-r}\mathbf{C}] \in H_{2s-2r}(P_\infty\mathbf{C};\mathbf{Q}/\mathbf{Z}).$$

2. $X = BP$.

This case illustrates another method for computing ch^{-1}. We have

$$H_*(BP) = \mathbf{Z}_{(p)}[m_1, m_2, \ldots], \quad BP_*BP = BP_*[t_1, t_2, \ldots],$$

$$BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots], \quad K_*(BP) = K_*[t_1, t_2, \ldots],$$

with $t_i \in K_*(BP)$ the image of $t_i \in BP_*BP$ under the Todd map $T : BP \to K$. From the well known formulas

$$p \cdot m_n = v_n + \sum_{i=1}^{n-1} m_i \cdot v_{n-i}^{p^i}, \quad \eta_R(m_n) = \sum_{i=0}^{n} m_i \cdot t_{n-i}^{p^i},$$

$$T(v_i) = \begin{cases} v_i = u^{p-1} & \text{if } i = 1, \\ 0 & \text{if } i > 1, \end{cases}$$

we get

$$T(m_n) = p^{-n} \cdot v_1^{\frac{p^n-1}{p-1}}$$

so that the composition

$$H_*(BP;\mathbf{Q}) \cong BP_* \otimes \mathbf{Q} \xrightarrow{\eta_R} BP_*(BP;\mathbf{Q}) \xrightarrow{T} K_*(BP;\mathbf{Q})$$

maps m_n to

$$\sum_{i=0} t_{n-i}^{p^i} / p^i.$$

But if we identify $H_*(BP;\mathbf{Q})$ with $BP_* \otimes \mathbf{Q}$ the composition $T \circ \eta_R$ is nothing but ch^{-1}, hence

$$ch^{-1}(m_n) = \sum_{i=0}^{n} \frac{t_{n-i}^{p^i}}{p^i}.$$

REMARK. This also gives all differentials in the Atiyah–Hirzebruch spectral sequence for $A_*(BP)$, the connective $Im(J)$-theory of BP, and is the first step in an approach to describing $\pi_*^S(S^0)$ by computing the Atiyah–Hirzebruch spectral sequence for $\pi_*^S(BP)$ where the E^∞-terms are known but the E^2-terms involve the unknown groups $\pi_*^S(S^0)$.

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CHAPTER 13

Stable Homotopy and Iterated Loop Spaces

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1. Introduction

Homology theory has been a very effective tool in the study of homotopy invariants for topological spaces. An important reason for this is the fact that it is often easy to compute homology groups. For instance, if one is given a finite simplicial complex, computing its homology becomes a straightforward problem in the linear algebra of finitely generated free modules over the integers. More generally, homology groups admit long exact Mayer–Vietoris sequences, which describe the homology, $H_*(X)$, of a space X which is a union of open subsets U and V in terms of $H_*(U)$, $H_*(V)$, and $H_*(U \cap V)$. In addition, under quite general circumstances when $A \subseteq X$ is a closed subspace, there is a long exact sequence

$$\cdots \longrightarrow \tilde{H}_{i+1}(X/A) \longrightarrow \tilde{H}_i(A) \longrightarrow \tilde{H}_i(X) \longrightarrow \tilde{H}_i(X/A) \longrightarrow \tilde{H}_{i-1}(A) \longrightarrow \cdots$$

where X/A denotes the result of identifying A to a point. Iterated applications of these long exact sequences are quite effective in computing the homology of many spaces.

Homotopy groups are much more difficult to compute. For instance, there are no finite CW-complexes except for the classifying spaces of certain infinite groups, for example bouquet of circles or compact closed surfaces, whose homotopy groups are known completely. The difficulty in carrying out this calculation can be traced in part to the nonexistence of an excision theorem for homotopy groups, and the consequent nonexistence of long exact Mayer–Vietoris sequences and long exact sequences of cofibrations.

It turns out to be possible, using a theorem of Freudenthal [17], to modify the homotopy groups a bit via a process of stabilization, so as to allow excision. The stabilization procedure goes as follows. For any space X, we have a homomorphism $\sigma : \pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$, where ΣX denotes the suspension of X. σ applied to an element in $\pi_i(X)$ is obtained by suspending a representing map, and identifying ΣS^i with S^{i+1}. One can repeat this process and obtain a directed system

$$\cdots \longrightarrow \pi_{i+k}(\Sigma^k X) \longrightarrow \pi_{i+k+1}(\Sigma^{k+1} X) \longrightarrow \cdots$$

whose direct limit is defined to be $\pi_i^s(X)$, the i-th stable homotopy group of X. Freudenthal's theorem is that this limit is actually attained at a finite stage, in fact at $k = i$. A consequence of Freudenthal's theorem is that given a cofibration sequence $A \longrightarrow X \longrightarrow X/A$, one obtains a long exact sequence

$$\cdots \longrightarrow \pi_{i+1}^s(X/A) \longrightarrow \pi_i^s(A) \longrightarrow \pi_i^s(X) \longrightarrow \pi_i^s(X/A) \longrightarrow \pi_{i-1}^s(A) \longrightarrow \cdots$$

of stable homotopy groups, just as one would in the case of homology. For this reason, π_*^s is referred to as a generalized homology theory, since it now satisfies all the Eilenberg–Steenrod axioms for a homology theory with the exception of the dimension axiom, which identifies the value of the theory on a point. The generalized homology theory property is quite useful. It permits the construction of the Adams spectral sequence [3] and its variants, which are effective computational methods for stable homotopy groups. For instance, they have allowed the calculation of stable homotopy groups in a far larger range of dimensions than is currently possible for unstable groups.

The stabilization procedure described above for homotopy groups can also be carried out on the level of spaces, rather than groups. For any based space X, let ΩX denote the loop space of X, i.e. the set of based maps from the circle to X, equipped with the compact open topology (see [27]). Then suspension gives rise to maps $\sigma : \Omega^k \Sigma^k X \longrightarrow \Omega^{k+1} \Sigma^{k+1} X$, and hence homomorphisms

$$\pi_i(\sigma) : \pi_i(\Omega^k \Sigma^k X) \longrightarrow \pi_i(\Omega^{k+1} \Sigma^{k+1} X).$$

Via the standard adjoint identification $\pi_i(\Omega^k X) \cong \pi_{i+k}(X)$, we obtain a homomorphism $\pi_{i+k}(\Sigma^k X) \longrightarrow \pi_{i+k+1}(\Sigma^{k+1} X)$, which is easily seen to be equal to the map in the directed system defining $\pi_i^s(X)$. Freudenthal's theorem can now be interpreted as a statement about the connectivity of the inclusion $\Omega^k \Sigma^k X \longrightarrow \Omega^{k+1} \Sigma^{k+1} X$.

It has turned out to be possible to obtain very detailed information about the spaces $\Omega^k \Sigma^k X$. In fact one can give an explicit description of $H_*(\Omega^k \Sigma^k X)$ as a functor of $H_*(X)$, and produce explicit combinatorial constructions which are homotopy equivalent to the spaces $\Omega^k \Sigma^k X$. This line of work began with the James construction [19] for the case $k = 1$, and was extended to the case of all k by Milgram [24]. An alternate version, based on Boardman's "little cubes", was worked out by J.P. May [22]. Barratt and Eccles [6] developed a simplicial version for the limiting case $k = \infty$, and J. Smith [30] gave a simplicial version valid for all k.

The case $k = \infty$, i.e. $\lim_k \Omega^k \Sigma^k X$, is usually denoted $Q(X)$. It is called an "infinite loop space" since it is a k-fold loop space for all $k \geqslant 0$. Of course infinite loop spaces need not arise only in this way. What one needs are spaces Z_k, $k = 0, 1, 2, \ldots$, and identifications $Z_k \simeq \Omega Z_{k+1}$. The collection of spaces $\{Z_k\}_{k \geqslant 0}$ forms a *spectrum*. It turns out that a spectrum determines a generalized homology theory in the above sense. The spectrum $\{Q(S^k)\}_{k \geqslant 0}$ determines stable homotopy theory. Other spectra determine well known generalized homology theories such as K-theory, the various bordism theories, and of course ordinary singular homology theory.

The theory of iterated loop spaces described above can be used to give a structure on a space which assures that the space is the zeroth space in some spectrum. The relevant structure turns out to be a homotopy theoretic version of an abelian group structure. In particular, topological abelian groups are always infinite loop spaces. This result is J.P. May's "recognition principle" for the case $k = \infty$. It in turn allows the construction of spectra and hence generalized homology theories [11] out of category theoretic data, specifically from categories with a coherently commutative and associative sum operation. The category of finite sets gives stable homotopy theory under this construction.

In this chapter we discuss these ideas. The second section outlines the general homotopy theoretic information we will need. The third section gives a proof of Freudenthal's theorem and the generalized homology theory property of stable homotopy. Section 4 studies Spanier–Whitehead duality, which can be though of as a space level version of Lefschetz duality. Section 5 contains the James construction as well as results of Adams and Hilton [1] and Adams [2] concerning the structure of loop spaces of general spaces (not necessarily suspensions). In Section 6 we give a detailed discussion of double loop spaces. This serves to motivate and clarify the work in the following chapter, and the case $k = 2$ contains all the essential difficulties that occur for arbitrary k. Section 7 contains an extended discussion of all the models mentioned above for $\Omega^k \Sigma^k X$. Finally, in Section 8 we sketch May's recognition principle as well as Segal's Γ-space version, and describe the necessary category theoretic data for constructing spectra.

2. Prerequisites

We summarize some basic material from homotopy theory which we will be using. We assume the reader has the standard knowledge of homology theory, as well as of the definitions and elementary properties of homotopy groups.

2.1. *Basic homotopy theory*

Recall that the Hurewicz homomorphism $h_n : \pi_n(X,*) \longrightarrow H_n(X)$ is given by $h_n([f]) = H_n(f)(i_n)$ where i_n is the standard generator for $H_n(S^n)$. Throughout this paper $[x]$ will denote the equivalence class of x in various contexts. It should not create confusion.

DEFINITION 2.1.1. A space X is said to be *n-connected* if $\pi_i(X) = 0$ for $i \leqslant n$. A map $f : X \longrightarrow Y$ is said to be n-connected if $\pi_i(f)$ is an isomorphism for $i \leqslant n$ and $\pi_{n+1}(f)$ is surjective. A pair (X, Y) is said to be n-connected if the inclusion $Y \longrightarrow X$ is.

THEOREM 2.1.1 (Hurewicz, Absolute case). *If $\pi_n(X,*) = 0$ for $0 < n < N$, and X is connected, then $H_n(X) = 0$ for $0 < n < N$, and h_N is an isomorphism if $N \geqslant 2$. If $N = 1, h_N$ is just abelianization. Note that this also implies that if X is simply connected and $H_n(X) = 0$ for $0 < n < N$, then $\pi_n(X,*) = 0$ for $0 < n < N$.*

We shall also need the relative form of this theorem. First, recall the notion of the homotopy group (or set if $n = 1$) of a pair (A, B).

DEFINITION 2.1.2. Let (A, B) be a pair of spaces, i.e. B is a subspace of A. Then by $\pi_n(A, B)$, we mean the set of homotopy classes of maps of the standard n-cube which carry the boundary into B (and the bottom face to the basepoint). This is a set if $n = 1$, a (perhaps non-abelian) group if $n = 2$, and an abelian group if $n \geqslant 3$. We have a relative Hurewicz homomorphism $h_n(A, B) : \pi_n(A, B) \longrightarrow H_n(A, B)$defined in the obvious way.

We can now formulate the relative version of the Hurewicz theorem.

THEOREM 2.1.2 (Hurewicz, Relative form). *Suppose A and B are connected, $N \geqslant 2$ and $\pi_n(A, B) = 0$ for $0 < n < N$. Then, if $N \geqslant 3$, $H_n(A, B) = 0$ for $0 < n < N$ and $h_N : \pi_N(A, B) \longrightarrow H_N(A, B)$ is an isomorphism, and if $N = 2$, $h_2 : \pi_2(A, B) \longrightarrow H_2(A, B)$ is abelianization.*

COROLLARY 2.1.1 (Whitehead). *Let X and Y be CW complexes, and let $f : X \longrightarrow Y$ be a continuous map*[1]. *If X and Y are simply connected, and $H_n(f)$ is an isomorphism*

[1] Actually, all our maps are continuous so from here on we will simply call them maps.

for $0 < n < N$, with $N \geqslant 3$, then $\pi_n(f)$ is an isomorphism for $0 < n < N-1$. Conversely, if $\pi_n(f)$ is an isomorphism for $0 < n < N$, then $H_n(f)$ is an isomorphism for $0 < n < N-1$.

We also record the following standard result about CW complexes. Recall that a continuous map $f : X \longrightarrow Y$ is said to be a weak equivalence if $\pi_n(f)$ is an isomorphism for all n.

THEOREM 2.1.3. *Let X and Y be CW complexes, and suppose $f : X \longrightarrow Y$ is a weak equivalence. Then f is a homotopy equivalence. Also, suppose X and Y are simply connected, and suppose $H_n(f)$ is an isomorphism for all n. Then f is a homotopy equivalence.*

2.2. *Hurewicz fibrations*

We recall some parts of the theory of Hurewicz fibrations.

DEFINITION 2.2.1. A map $p : E \longrightarrow B$ is a Hurewicz fibration if for every pair of spaces (X, Y), and every commutative diagram

$$\begin{tikzcd} X \times \{0,1\} \cup Y \times I \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] & B \end{tikzcd}$$

there is a map $\tilde{H} : X \times I \longrightarrow E$ making both triangles commute. If $F = p^{-1}(b)$, for a point $b \in B$, and B is path connected, we obtain a long exact sequence on homotopy groups,

$$\cdots \longrightarrow \pi_i(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \longrightarrow \cdots.$$

Equivalently, $\pi_i(E, F) \longrightarrow \pi_i(B, b)$ is an isomorphism.

For us, a fibration will mean a Hurewicz fibration. In the case of a path-connected base space B, it follows directly from this definition that if b_0 and b_1 are points of B, then $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.

DEFINITION 2.2.2. Let X be a space. By a space over X we mean a space E together with a reference map $E \xrightarrow{r} X$. If $E_1 \xrightarrow{r_1} X$ and $E_2 \xrightarrow{r_2} X$ are spaces over X, then a map over X from $(E_1, r_1)$to (E_2, r_2) is a map $f : E_1 \longrightarrow E_2$ so that $r_1 = r_2 \circ f$.

For any space (E, r) over X, we have the space $(E \times I, r \circ p_E)$, over X, where $p_E : E \times I \longrightarrow E$ is the projection. With this construction, *homotopies over X* are defined in the evident way, as are homotopy equivalences.

- Note that a map f over X from (E_1, r_1) to (E_2, r_2) gives rise to a map $Cyl(f)$ from the mapping cylinder $Cyl(r_1)$ on r_1 to the mapping cylinder $Cyl(r_2)$ on r_2, and an

induced map $C(f)$ from the mapping cone $C(r_1)$ on r_1 to the mapping cone $C(r_2)$ on r_2.
- If f is a homotopy equivalence over X, then $C(f)$ is a homotopy equivalence.

DEFINITION 2.2.3. Let X and Y be spaces, and suppose (E, r) is a space over Y. Then if $f : X \longrightarrow Y$ is a continuous map, the pullback $f^*(E, r)$ is the space (f^*E, f^*r) over X, defined by letting f^*E be the subspace of $X \times E$ given by

$$f^*E = \{(x, e) \mid f(x) = r(e)\},$$

and letting f^*r be the composite $f^*E \longrightarrow X \times E \xrightarrow{\pi_X} X$. If r is a fibration, then so is f^*r.

The pullback operation has an important homotopy invariance property when r is a fibration.

PROPOSITION 2.2.1. *Suppose (E, r) is a space over Y, with r a fibration. Let $f, g : X \longrightarrow Y$ be homotopic continuous maps. Then $f^*(E, r)$ and $g^*(E, r)$ are homotopy equivalent spaces over X.*

PROOF. Let H be a homotopy from f to g, and consider the space $H^*(E, r)$ over $X \times I$. $i_0^* H^*(E, r) \cong f^*(E, r)$ and $i_1^* H^*(E, r) \cong g^*(E, r)$ as spaces over X. The homotopy lifting property for fibrations applied to the canonical homotopy from i_0 to i_1 gives a map α from $i_0^* H^*(E, r)$ to $i_1^* H^*(E, r)$ of spaces over X, and similarly we obtain a map $\beta : i_1^* H^*(E, r) \longrightarrow i_0^* H^*(E, r)$, also over X. We must show that $\alpha\beta$ and $\beta\alpha$ are homotopic to the identity over X. Consider $\beta\alpha$. From the way in which α and β were constructed, it is clear that there is a map $h : i_0^* H^*(E, r) \times I \longrightarrow H^*(E, r)$, so that the composite $H^*r \circ h$ is equal to $g \circ (i_0^* H^*r \times id)$, where $g : X \times I \longrightarrow X \times I$ is given by $g(x, t) = (x, 2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$, and $g(x, t) = (x, 2 - 2t)$ for $\frac{1}{2} \leqslant t \leqslant 1$, and so that $h \mid i_0^* H^*(E, r) \times 0$ is the inclusion, and $H \mid i_0^* H^*(E, r) \times 1$ is $\beta\alpha$ composed with the inclusion. In view of the fact that there is an evident homotopy from g to the constant homotopy $\hat{g^*}, \hat{g^*}(x, t) = (x, 0)$, we may use the homotopy lifting property again to obtain a map $\hat{h}$ from $i_0^* H^*(E, r) \times I \longrightarrow H^*(E, r)$, so that $\hat{h} \mid i_0^* H^*(E, r) \times 0 \cup i_0^* H^*(E, r) \times 1 = h \mid i_0^* H^*(E, r) \times 0 \cup i_0^* H^*(E, r) \times 1$ and so that $H^*r \circ \hat{h} = \hat{g} \circ (i_0^* H^*r \times id)$. $\hat{h}$ is now the required homotopy over X from the identity on $i_0^* H^*(E, r)$ to $\beta\alpha$. The procedure works similarly for $\alpha\beta$. □

COROLLARY 2.2.1. *Let X be a space, and let (E, r) be a space over X, with r a fibration. Suppose X is contractible. Then for any $x \in X$, (E, r) is homotopy equivalent over X to the space $(X \times r^{-1}(x), \pi_X)$ over X.*

PROOF. This is an easy application of the preceding result. □

PROPOSITION 2.2.2. *Let X be a CW-complex, and let A be a subcomplex. Let Y be a space. Let $F(X, Y)$ denote the space of maps from X to Y, with the compact open topology (see [27]), then the restriction map $F(X, Y) \longrightarrow F(A, Y)$ is a Hurewicz fibration. Moreover, the inverse image of the constant map from A to Y is identified with $F(X/A, Y)$.*

PROOF. This fact follows directly from the fact that inclusions of subcomplexes of CW-complexes have the *homotopy extension property*, which is a dual condition to the homotopy lifting property characterizing Hurewicz fibrations. It states that if we are given a map $f : X \longrightarrow Y$ and a homotopy $H : A \times I \longrightarrow Y$ so that $H \mid A \times 0 = f \mid A$, then there exists an extension $\hat{H} : X \times I \longrightarrow Y$ so that $\hat{H} \mid X \times 0 = f$ and $\hat{H} \mid A \times I = H$. That this property holds in the case of the inclusion of a subcomplex of a CW-complex is proved in [21]. □

REMARK. Generally, a map having the homotopy extension property is referred to as a cofibration.

2.3. *Serre fibrations*

A Serre fibration has the same definition as an Hurewicz fibration except the spaces X and Y are restricted to being finite polyhedral complexes. These are particularly useful when we are dealing with mapping spaces $X^Y = \{f : Y \to X \mid f \text{ continuous }\}$ which are assumed, as in 2.2.2, to have the compact-open topology.

Given any continuous map $f : Y \to X$ we have the associated Serre fibration $E^{\bar{X}}_{Y,\bar{X}} \xrightarrow{\pi} X$ where $\bar{X}$ is the mapping cone of f, $E^C_{A,B}$ is the space of paths in C that start in A, end in B and $\pi : E^C_{A,B} \to B$ is projection onto the endpoint. The fiber of π over the point x is the subspace $E^{\bar{X}}_{Y,x}$ and we have the commutative diagram

$$\begin{array}{ccccc} f^{-1}(x) & \hookrightarrow & Y & \xrightarrow{f} & X \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle =} \\ E^{\bar{X}}_{Y,x} & \hookrightarrow & E^{\bar{X}}_{Y,\bar{X}} & \xrightarrow{\pi} & X \end{array} \tag{2.1}$$

where i includes $y \in Y$ as the constant path at y.

2.4. *Quasifiberings*

DEFINITION 2.4.1. A continuous map $f : Y \to X$ is a quasifibration if and only if, for all $x \in X$, the map i above restricted to $f^{-1}(x)$ is a weak homotopy equivalence.

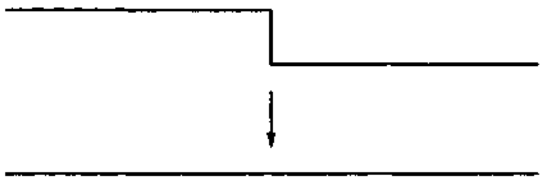
Using the 5-lemma this is equivalent to the condition

LEMMA 2.4.1. *$f : Y \to X$ is a quasifibration if and only if, for all $x \in X$ and $y \in f^{-1}(x)$, the induced map of homotopy groups*

$$f_* : \pi_*(Y, f^{-1}(x), y) \longrightarrow \pi_*(X, x)$$

is an isomorphism.

Basically, it turns out that the difference between quasifibrations and Hurewicz fibrations is that with an Hurewicz fibration one can lift homotopies "on the nose", however, in a quasifibration, the weak equivalence condition limits the homotopies to finite cell complexes and homotopies can be lifted, but only "up to a homotopy". A good example to keep in mind is the map



(2.2)

which is a quasifibration but not an Hurewicz fibration.

One has notions of the equivalence of two quasifibrations, principal quasifibrations, and the equivalence of principal quasifibrations similar to those for bundles. However, the construction of "associated quasifibrations" is more difficult.

DEFINITION 2.4.2. (i) Two quasifibrations, $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B'$ are said to be equivalent if there are weak homotopy equivalences $f : E \longrightarrow E'$, $\bar{f} : B \longrightarrow B'$ so that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\bar{f}} & B' \end{array} .$$

(ii) A quasifibration $p : E \longrightarrow B$ is a (left)-principal M-quasifibration if M is an associative, unitary H-space and there is a map $\mu : M \times E \longrightarrow E$ so that

a) $\mu(mm', e) = \mu(m, \mu(m', e))$ for all $m, m' \in M$, $e \in E$, (associative action).

b) $\mu(1, e) = e$ all $e \in E$ where $1 \in M$ is the unit, (unitary action).

c) $p(\mu(m, e)) = p(e)$ for all $e \in E$, $m \in M$, (fiber preserving).

d) $\mu(-, e) : M \longrightarrow p^{-1}p(e)$ is a weak homotopy equivalence for each $e \in E$.

(iii) Two principal M-quasifibrations $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ are called structurally equivalent if they are equivalent via f, $\bar{f}$ where f preserves the M-structure.

The best references for the structure of quasifibrations are [16], [15], [32] and we summarize the results from [16, §2] that we will need in the sequel now.

The main tool for constructing lifts up to homotopy is the following result.

LEMMA 2.4.2. *Let $p : F \longrightarrow U$ be continuous, $V \subset U$ and $G = p^{-1}(V)$. Let K be an*

r-cell ($r \geqslant 0$) and assume that for all $x \in U$, $y \in p^{-1}(x)$ we have

$$\begin{cases} p_r : \pi_r(F,G,y) \longrightarrow \pi_r(U,V,x) & \textit{is a monomorphism,} \\ p_{r+1} : \pi_{r+1}(F,G,y) \longrightarrow \pi_{r+1}(U,V,x) & \textit{is an epimorphism.} \end{cases}$$

Then p has the following homotopy lifting property: suppose that we are given three maps

(i) $\bar{H} : (K \times I, K \times 1) \to (U,V)$,
(ii) $h : (K \times 0 \cup \partial K \times I, \partial K \times 1) \to (F,G)$,
(iii) $d : ((K \times 0 \cup \partial K \times I) \times I, (\partial K \times 1) \times I) \to U,V)$

with $d(z,t,0) = \bar{H}(z,t)$, $d(z,t,1) = p \circ h(z,t)$ for all $z \in K$, $t \in K$. Then there is a map

$$H : (K \times I, K \times 1) \longrightarrow (F,G)$$

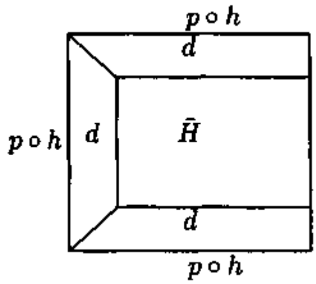
with $H|K \times 0 \cup \partial K \times 1$ equal to h, and a homotopy

$$D : (K \times I \times I, K \times 1 \times I) \longrightarrow (U,V)$$

filling in d in the sense that

$$\begin{aligned} D|(K \times 0 \cup \partial K \times 1) \times I &= d, \\ D(z,t,0) &= \bar{H}(z,t), \\ D(z,t,1) &= p \circ H(z,t). \end{aligned}$$

PROOF. h defines an element $a \in \pi_r(F,G)$ with $f(a) = 0 \in \pi_r(U,V)$ using $\bar{H}$ and d to construct the trivializing homotopy.



(2.3)

But since we assume that f_* on $\pi_r(F,G,y)$ is a monomorphism, it follows that $a = 0$, and there is a trivializing homotopy

$$H' : (K \times I, K \times 1) \longrightarrow (F,G)$$

with $H|(K \times 0) = h$. Adding the image of H' to the map in fig. 2.3, we have a map $H'' : (K \times I \times 0 \cup \partial(K \times I) \times I) \to U$ with $H''|\partial(K \times I) \times 1$ contained in V. H'' in turn defines an element $\gamma \in \pi_{r+1}(U, V, x)$ which may not be zero. However, we are free to modify the homotopy H' by any element $\beta \in \pi_{r+1}(F, G, y)$, and this will change γ to $\gamma + p_*(\beta)$. Consequently, since p_{r+1} is onto, we can assume γ represents 0 and the existence of the desired homotopy follows. □

We now give some geometric conditions which will guarantee that a map f is a quasifibration.

DEFINITION 2.4.3. Let $f : X \longrightarrow Y$ be a continuous map, and $U \subset Y$ be any subset. We say that U is distinguished for f if $f : f^{-1}(U) \longrightarrow U$ is a quasifibration.

LEMMA 2.4.3. *Suppose that $f : X \longrightarrow Y$ is a continuous map. Suppose that $Y' \subset Y$ is distinguished for f with $X' = f^{-1}(Y')$. Suppose that there are deformations*

$$\begin{aligned} D : I \times (X, X') &\longrightarrow (X, X'), \\ d : I \times (Y, Y') &\longrightarrow (Y, Y') \end{aligned}$$

so that $D_0 = id$, $d_0 = id$, $im(D_1) \subset X'$, $im(d_1) \subset Y'$, $f \circ D_1 = d_1 \circ f$, and finally, for every $x \in X$, $D_{1*} : \pi_*(f^{-1}(x)) \longrightarrow \pi_*(f^{-1}(d_1(x)))$ is an isomorphism. Then Y is distinguished for f, i.e. f is a quasifibration.

PROOF. d_1 and D_1 are deformations so d_{1*} and D_{1*} induce homotopy equivalences. Now, from the induced maps of pairs $(X, f^{-1}(y) \longrightarrow (X', f^{-1}(d_1(y)))$ and the five-lemma we have that $\pi_*(X, f^{-1}(y)) \cong \pi_*(X', f^{-1}(d_1(y)))$. But since Y' is distinguished for f we know $\pi_*(X', f^{-1}(y')) \cong \pi_*(Y', y')$, and d_1 shows that these groups are isomorphic to $\pi_*(Y, y)$. □

Perhaps the most important method of showing that f is a quasifibration is the following result.

THEOREM 2.4.1. *Let $f : X \longrightarrow Y$ be a continuous map, and suppose that there is a family $\mathcal{Y}$ of distinguished open sets for f, $U_i \subset Y$ with the following two properties:*

- *The sets $U_i \in \mathcal{Y}$ cover Y.*
- *For every pair $U_i, U_j \in \mathcal{Y}$ and $y \in U_i \cap U_j$ there is a $U_y \in \mathcal{Y}$ with $y \in U_y \subset U_i \cap U_j$.*

Then Y is distinguished for f.

(The idea of the proof is to modify the standard proof of (polyhedral) homotopy lifting for f if $\mathcal{Y}$ was a family of open sets for which $f^{-1}(U_i) = Y_i \times U_i$, i.e. the map has a local product structure. One covers the homotopy on the base by distinguished neighborhoods, and then refines the polyhedral decomposition so that each polygon has the form $P_i \times [a, b]$ and is contained in one of the distinguished neighborhoods. One then constructs the extension over skeleta, one cell at a time. The only difference here is that the lifting is not exact but involves a second homotopy. The homotopy extension lemma above provides the necessary tool.)

2.5. *Associated quasifibrations*

For ordinary (local product) fibrations one can associate a (left)-principal fibration to any fibration $F \longrightarrow E \longrightarrow B$, which we can write $H \longrightarrow \mathcal{E} \xrightarrow{p} B$ with fiber a subgroup $H \subset Aut(F)$, the group of homeomorphisms of F. Then given any Y with H-action $Y \times H \longrightarrow Y$, there is an associated fibration

$$Y \longrightarrow Y \times_H \mathcal{E} \longrightarrow B.$$

However, for quasifibrations this construction may not always result in a quasifibration. For one thing, since M is not a group in general, the operation $\times_M$ is not directly an equivalence relation. For another, even taking the associated equivalence relation, the local structure may be sufficiently bad that the map of the quotient to B is not a quasifibration.

The problem was studied by Stasheff in [32] and he introduced a classifying space construction there which made sense of the notion of associated quasifibrations. Basically, given a left M-space E, and a right M-space X, he constructs a space $E(X, M, E)$ with the following properties:

- $E(X, M, E)$ is natural in all three variables. For example, if $h : X \longrightarrow X'$ is a map of right M-spaces then there is an induced map

  $$E(h, 1, 1) : E(X, M, E) \longrightarrow E(X', M, E)$$

  and similarly for the other variables which satisfy the expected naturality properties. Also, if the maps are weak homotopy equivalences, then the resulting maps of $E(X, M, E)$ are also.
- $E(X, M, M) \simeq X$, $E(M, M, E) \simeq E$.
- If $E \longrightarrow B$ is a principal quasifibering then $E(M, M, E) \longrightarrow E(*, M, E)$ is a principal quasifibering which is structurally equivalent to $E \longrightarrow B$.
- If $E \longrightarrow B$ is a principal quasifibering, then $E(X, M, E) \longrightarrow E(*, M, E)$ is a quasifibering with fiber X.

An important example to keep in mind is the loop-path Serre fibration

$$\Omega X \longrightarrow E^X_{*,X} \xrightarrow{p_1} X.$$

These spaces are constructed as a limit over n of spaces constructed from the products $\sigma^n \times X \times M^n \times E$ by introducing the equivalence relation

$$(t, x, m_2, \ldots, m_{n+1}, e) \sim (t, x', m'_2, \ldots, m'_{n+1}, e')$$

where $m_i m_{i+1} = m'_i m'_{i+1}$ if $t_i = 0$, the t_i are barycentric coordinates for the simplex σ^n and in this relation $x = m_1$, $e = m_{n+2}$. One must be a bit careful with the topologies here. In particular Stasheff, following [15], gives the quotients a topology just strong enough for certain maps to be continuous. However, one can use the compactly generated topology in the quotient, and this will work as well. (For a complete study of the properties of this topology see [33].)

The construction has the property that it is graded and $E_n - E_{n-1}$ is a product $Y \times \mathrm{Int}(\sigma^n) \times N^n \times X$ where $N = M - *$, and that there is a neighborhood U_n of E_{n-1} in E_n together with a deformation retraction, D, of U_n onto E_{n-1} so that for any point

$$(x, t, n_2, \ldots, n_{n+1}, y) \in U \cap (E_n - E_{n-1})$$

$D_1(x, t, n_2, \ldots, n_{n+1}, y)$ lies in a product neighborhood $E_j - E_{j-1}$ for a unique j and there has the form $(xm_1, t', n'_2, \ldots, n'_{j+1}, m_2 y)$ with m_1, m_2 independent of x, y. In particular each fiber $X \times Y$ is mapped by a translation of the form $(x, y) \mapsto (xm_1, ym_2)$ and if we assume that the actions $M \times Y \to Y$, $X \times M \to X$ give rise to weak homotopy equivalences $y \mapsto my$, $x \mapsto xm$ for all $m \in M$ then the results of Dold and Thom above show that the construction gives a quasifibration.

There is one more property of these spaces which will be useful to us. If X also has a left N-action, then the space $E(X, M, E)$ becomes a left N-space from the action on passing to quotients. (The compactly generated topology again seems better here than Stasheff's original topology.)

3. The Freudenthal suspension theorem

The computation of homotopy groups is a notoriously difficult problem. Even for spheres, our knowledge is quite spotty compared with what might have been expected over forty years ago, when work on them began in earnest. An important simplification was made by Freudenthal, who proved his famous suspension theorem, which asserts that for $k < n$ the suspension homomorphism $\sigma : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism. On can therefore compute the value of infinitely many homotopy groups of spheres by computing one stable group, i.e. one group of the form $\pi_{n+k}(S^n)$, $k < n$.

Let Σ denote the reduced suspension functor. For any based space (X, a), we may define a suspension homomorphism

$$\sigma : \pi_i(X, x) \to \pi_{i+1}(\Sigma X, x)$$

and consequently, a directed system of groups $\{\pi_{i+l}(\Sigma^l X, x)\}_{l \geqslant 0}$ by the requirement that $\sigma[f] = [\Sigma f]$. $\varinjlim_l \pi_{i+l}(\Sigma^l X, x)$ is now an abelian group valued functor of spaces, which we denote by $\pi_i^s(X, x)$. It will follow from Freudenthal's result that this system eventually stabilizes, i.e. that for sufficiently large l, the suspension homomorphism $\pi_{i+l}(\Sigma^l X, x) \to \pi_{i+l+1}(\Sigma^{l+1} X, x)$ is an isomorphism. It also turns out that the graded group valued functor $\pi_*^s(-)$ is a generalized homology theory in X. This means that many of the methods used to compute integral homology so successfully also apply to stable homotopy theory; the only obstacle is that one cannot compute its value on a point.

In this section we will outline proofs of these fundamental results. We will assume that the reader is familiar with the standard theory of Hurewicz fibrations, presented in Section 2.2.

LEMMA 3.1. *Let $p : E \longrightarrow B$ be a Hurewicz fibration, where B is a path connected CW complex with preferred base point b. Suppose further that B is obtained from a subcomplex B_0 by attaching a single n-cell along a based map $f : S^{n-1} \longrightarrow B$, so $B = B_0 \cup e^n$. Finally, suppose $F = p^{-1}(b)$ is k-connected. Then the map of pairs $(E, p^{-1}(B_0)) \longrightarrow (B, B_0)$ induces isomorphisms on H_j for $j \leqslant n + k$.*

PROOF. Let $f^n \subseteq e^n$ denote the closed disc of radius $\frac{1}{2}$ centered at the origin. It is clear that B_0 is a deformation retract of $B - \mathring{f}^n$. It is therefore a direct consequence of the homotopy lifting property that $p^{-1}(B_0)$ is also a deformation retract of $p^{-1}(B - \mathring{f}^n)$. Consequently, the inclusions $(B, B_0) \longrightarrow (B, B - \mathring{f}^n)$ and $(E, p^{-1}(B_0)) \longrightarrow (E, p^{-1}(B - \mathring{f}^n))$ induce isomorphisms on relative homology. It therefore suffices to show that the homomorphism $H_j(E, p^{-1}(B - \mathring{f}^n)) \longrightarrow H_j(B, B - \mathring{f}^n)$ is an isomorphism when $0 \leqslant j \leqslant n + k$. Let ∂f^n denote the boundary of f^n. It is a direct consequence of the excision theorem for homology that the inclusions $(f^n, \partial f^n) \longrightarrow (B, B - \mathring{f}^n)$ and $(p^{-1}(f^n), p^{-1}(\partial f^n)) \longrightarrow (E, p^{-1}(B - \mathring{f}^n))$ induce isomorphisms on relative homology, H_i, for all i. It consequently suffices to show that the homomorphism

$$H_j(p^{-1}(f^n), p^{-1}(\partial f^n)) \longrightarrow H_j(f^n, \partial f^n)$$

is an isomorphism for $0 \leqslant j \leqslant n + k$.

Let $v \in f^n$ denote the center of the ball. Note that since B is path connected, it follows from the fact that F is k-connected that $p^{-1}(v)$ is. Since f^n is a contractible space, we have a homotopy equivalence over X from $p^{-1}(f^n)$, with the restriction of p as reference map, to $f_n \times p^{-1}(v)$, with projection on the first factor as reference map. It now follows that it suffices to show that the projection homomorphism

$$H_j(f_n \times p^{-1}(v), \partial f_n \times p^{-1}(v)) \longrightarrow H_j(f_n, \partial f_n)$$

is an isomorphism for $0 \leqslant j \leqslant n + k$. But this follows from the Künneth formula and the Hurewicz theorem. □

COROLLARY 3.1. *Suppose, as before, that we have a Hurewicz fibration $E \xrightarrow{p} B$, where B is a CW complex equipped with a preferred base point $b \in B$. Suppose that F is k-connected and B is n-connected. Then the natural map of pairs $(E, F) \longrightarrow (B, b)$ induces isomorphisms on H_j for $0 \leqslant j \leqslant n + k + 1$.*

PROOF. It is standard homotopy theory that there is a based homotopy equivalence $(B, b) \xrightarrow{\phi} (B', b')$, where B' is a CW complex with a unique 0-cell b', and which has no l-cells for $0 \leqslant l \leqslant n$. By pulling back E along a homotopy inverse to ϕ, we obtain from Proposition 2.2.1 an equivalent fibration E' over B'.

We are therefore free to suppose that B has b as unique 0-cell, and that B has no l-cells for $0 \leqslant l \leqslant n$. Let $B^{(t)}$ denote the t-skeleton of B. We will show inductively that the homomorphisms $H_j(p^{-1}(B^{(l)}), F)) \longrightarrow H_j(B^{(l)}, b)$ are isomorphisms for $0 \leqslant j \leqslant n+k+1$, and all l. For $l = 0$, this is trivial since $B^{(0)} = 0$ and therefore $p^{-1}(B^{(0)}) = F$, so both target and source of the homomorphisms in question are trivial groups. Now suppose the result is known for l, and we attempt to show that $H_j(p^{-1}(B^{(l+1)}), F) \longrightarrow H_j(B^{(l+1)}, b)$ is an isomorphism for $0 \leqslant j \leqslant n+k+1$. Consider the following commutative diagram

$$\begin{array}{ccccc}
H_{j+1}(p^{-1}(B^{(l+1)}), p^{-1}B^{(l)}) & \longrightarrow & H_j(p^{-1}(B^{(l)}), F) & \longrightarrow & H_j(p^{-1}(B^{l+1}), F) \\
\Big\downarrow{\scriptstyle \psi_1} & & \Big\downarrow{\scriptstyle \psi_2} & & \Big\downarrow{\scriptstyle \psi_3} \\
H_{j+1}(B^{(l+1)}, B^{(l)}) & \longrightarrow & H_j(B^{(l)}, b) & \longrightarrow & H_j(B^{l+1}, b)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H_j(p^{-1}(B^{(l+1)}), p^{-1}(B^{(l)})) & \longrightarrow & H_{j-1}(p^{-1}(B^{(l)}), F) \\
& \Big\downarrow{\scriptstyle \psi_4} & & \Big\downarrow{\scriptstyle \psi_5} \\
\longrightarrow & H_j(B^{(l+1)}, B^{(l)}) & \longrightarrow & H_{j-1}(B^{(l)}, b)
\end{array}$$

It is just an induced map of homology long exact sequences induces by p. Suppose $l < n$. Then, since $B^{(l)} = B^{(l+1)} = b$, it follows directly from this sequence that $H_j(p^{-1}(B^{(l+1)}), F) = H_j(B^{(l+1)}, b) = 0$, which gives the result in this case. If $l = n$, then ψ_2 and ψ_5 are both isomorphisms since their domains and images are trivial groups. On the other hand, ψ_1 and ψ_5 are isomorphisms by Lemma 3.1. The five lemma now shows that ψ_3 is an isomorphism. Finally, if $l > n$, then ψ_2 and ψ_5 are isomorphisms by the inductive hypothesis, and ψ_1 and ψ_4 are again isomorphisms by 3.1. This gives the result. □

We now wish to use these results to give proofs of Freudenthal's theorem and of the generalized homology theory property of π_*^s. Let $i : (Y, y_0) \longrightarrow (X, x_0)$ be a based cofibration, let $Cyl(i)$ and $C(i)$ denote the reduced mapping cylinder and reduced mapping cone construction on i, respectively. Thus, $Cyl(i) = Y \times [0,1] \cup X/\simeq$, where $\simeq$ is the equivalence relation generated by $(y, 0) \simeq i(y)$, and $(y_0, t) \simeq x_0$ for all t, and $C(i) = Cyl(i)/\mathrm{Image}(Y)$. Let E denote the space of maps $\phi : [0,1] \longrightarrow C(i)$ such that $\phi(0) \in X$, with the compact open topology. We have a projection map $p : E \longrightarrow C(i)$, given by $p(\phi) = \phi(1)$; it is a Hurewicz fibration. Let F denote the fibre over x_0 of p; thus, F is the space of maps $\phi : [0,1] \longrightarrow C(i)$ such that $\phi(1) = x_0$ and $\phi(0) \in X$. We now define a map $\lambda : Y \longrightarrow F$ by $\lambda(y) = \psi_Y$, where $\psi_Y(t) = [y, 1-t]$. Let $j : F \longrightarrow E$ be the inclusion; note that the composite $j \circ \lambda$ is homotopic, rel y_0, to the map $\mu : Y \longrightarrow E$ which sends y to the constant path with values $i(y)$. The homotopy is given by $H(s, y) = [y, 1-st]$. Of course, μ extends to a map $\bar{\mu} : X \longrightarrow E$, which sends x to the constant path with value x. We therefore have a map $Y \times [0,1] \longrightarrow X \longrightarrow E$, which is H on $Y \times [0,1]$ and is $\bar{\mu}$ on X, and which respects the equivalence relation defining $Cyl(i)$. Since the map restricts to λ on the image of $Y \times 0$, we have a map of pairs $(Cyl(i), Y) \longrightarrow (E, F)$. Further, the composite $(Cyl(i), Y) \longrightarrow (E, F) \longrightarrow (C(i), x_0)$ is just the identification map $Cyl(i) \longrightarrow C(i)$, which shrinks Y to a point.

THEOREM 3.1. *Let X, Y, and i be as above. Suppose that Y is k-connected and $C(i)$ is l-connected, with $k > 0$, $l > 1$. Then the map $\lambda : Y \longrightarrow F$ induces isomorphisms on π_j for $0 \leqslant j \leqslant k + l$.*

PROOF. We know by Lemma 3.1 and Corollary 3.1 that the homomorphism $H_j(E, F) \longrightarrow H_j(C(i), x_0)$ is an isomorphism for $0 \leqslant j \leqslant k + l + 1$. By the above description of the composite

$$\big(Cyl(i), Y\big) \longrightarrow (E, F) \longrightarrow \big(C(i), x_0\big),$$

and the excision theorem, we conclude that

$$H_j\big(Cyl(i), Y\big) \longrightarrow H_j(E, F) \longrightarrow H_j\big(C(i), x_0\big)$$

is an isomorphism for all j, and hence that $H_j(Cyl(i), Y) \longrightarrow H_j(E, F)$ is an isomorphism for $0 \leqslant j \leqslant k + l + 1$. Now consider the commutative diagram below

$$\begin{array}{ccccc}
H_{j+1}(Cyl(i)) & \longrightarrow & H_{j+1}(Cyl(i), Y) & \longrightarrow & H_j(Y) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
H_{j+1}(E) & \longrightarrow & H_{j+1}(E, F) & \longrightarrow & H_j(F)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H_j(Cyl(i)) & \longrightarrow & H_j(Cyl(i), Y) \\
 & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\
\longrightarrow & H_j(E) & \longrightarrow & H_j(E, F)
\end{array}$$

It is easy to check that the map $Cyl(i) \longrightarrow E$ is a homotopy equivalence, so α and δ are isomorphisms for $j \leqslant k + l$. β and ε are also isomorphisms, from the above discussion. The five lemma now shows that γ is an isomorphism. It follows easily from the long exact homotopy sequence of the fibration $F \longrightarrow E \longrightarrow C(i)$ that F is simply connected. Therefore, the relative Hurewicz theorem asserts that $\pi_j(Y) \longrightarrow \pi_j(F)$ is an isomorphism for $0 \leqslant j \leqslant k + l$. □

Let X be any connected CW complex. Define a based map $J : X \longrightarrow \Omega\Sigma X$, where ΣX denotes the reduced suspension of X by $x \mapsto [x]$ where $[x](t) = [x, t] \in \Sigma X$.

THEOREM 3.2 (Freudenthal). *If X is k connected then the homomorphism*

$$\pi_i(J) : \pi_i(X) \longrightarrow \pi_i(\Omega\Sigma X)$$

is an isomorphism for $0 \leqslant i < 2k + 1$.

PROOF. Apply Theorem 3.1 to the inclusion $X \hookrightarrow CX$; λ in this case is J. □

COROLLARY 3.2. *Let $\sigma : \pi_i(X,*) \longrightarrow \pi_{i+1}(\Sigma X,*)$ be the suspension homomorphism. Suppose X is k-connected and $i < 2k+1$. Then σ is an isomorphism.*

PROOF. Standard adjointness identifies $\pi_{i+1}(\Sigma X,*)$ with $\pi_i(\Omega\Sigma X,*)$; it is not hard to see that after this identification, σ corresponds to $\pi_i(J)$. □

We now prove the cofibration property.

THEOREM 3.3. *Let $i : Y \longrightarrow X$ be a cofibration and let $C(i)$ denote its reduced mapping cone. Then there is a long exact sequence*

$$\cdots \longrightarrow \pi^s_{i+1}\big(C(i)\big) \longrightarrow \pi^s_i(Y) \longrightarrow \pi^s_i(X) \longrightarrow \pi^s_i\big(C(i)\big) \longrightarrow \pi^s_{i-1}(Y) \longrightarrow \cdots$$

PROOF. Consider the map $\Sigma^k i : \Sigma^k Y \longrightarrow \Sigma^k X$. From the definitions, it is clearly seen that $\Sigma^k C(i)$ is naturally homeomorphic to $C(\Sigma^k i)$. Let $E(\Sigma^k i)$ denote the space of maps $\phi : [0,1] \longrightarrow C(\Sigma^k i)$ with $\phi(0) \in \Sigma^k X$; as before, the map $p : E(\Sigma^k i) \longrightarrow C(\Sigma^k i)$ is a fibration and we let $F(\Sigma^k i)$ denote the inverse image of the basepoint. There is an evident map $\Sigma F(\Sigma^k i) \longrightarrow F(\Sigma^{k+1} i)$.

We therefore obtain a directed system of groups $\{\pi_{i+k}(F(\Sigma^k i))\}$. It now follows from the long exact sequences of the fibrations $F(\Sigma^k i) \longrightarrow E(\Sigma^k i) \longrightarrow C(\Sigma^k i)$ that we have a long exact sequence

$$\cdots \longrightarrow \varinjlim_k \pi_{i+k+1}\big(C(\Sigma^k i)\big) \longrightarrow G_i \longrightarrow \pi^s_i(X) \longrightarrow \varinjlim_k \pi_{i+k}\big(C(\Sigma^k i)\big) \longrightarrow \cdots$$

From the identification $\Sigma^k C(i) \simeq C(\Sigma^k i)$, we see that $\lim_k \pi_{i+k}(C(\Sigma^k i)) \cong \pi^s_i(C(i))$. On the other hand, there are maps $\Sigma^k Y \longrightarrow F(\Sigma^k i)$ which give a homomorphism of directed systems of abelian groups

$$\big\{\pi_{i+k}(\Sigma^k Y)\big\}_{k\geqslant 0} \longrightarrow \big\{\pi_{i+k}\big(F(\Sigma^k i)\big)\big\}_{k\geqslant 0}$$

and hence a homomorphism

$$\pi^s_i(Y) \longrightarrow \varinjlim_k \pi_{i+k}\big(F(\Sigma^k i)\big) = G_i.$$

Theorem 3.1 now shows that for sufficiently large k, $\pi_{i+k}(\Sigma^k Y) \longrightarrow \pi_{i+k}(F(\Sigma^k i))$ is an isomorphism, hence so is the homomorphism $\pi^s_i(Y) \longrightarrow G_i$. This gives the required result. □

We obtain a corollary concerning the homology of iterated loop spaces.

COROLLARY 3.3. *Consider the iterated loop space $\Omega^k S^N$ where $k < N$. We have the map $S^{N-k} \xrightarrow{\lambda} \Omega^k S^N$, adjoint to the standard identification $\Sigma^k S^{N-k} \xrightarrow{=} S^N$. Then λ induces isomorphisms on H_j for $j < 2(N-k-1)$.*

PROOF. Consider $\pi_j(\lambda) : \pi_j(S^{N-k}) \longrightarrow \pi_j(\Omega^k S^N) \cong \pi_{j+k}(S^N)$. $\pi_j(\lambda)$ is identified with the k-fold suspension homomorphism, which is an isomorphism if $j < 2(N-k) - 1$ by Corollary 3.2. Thus, by the Whitehead theorem $H_j(\lambda)$ is an isomorphism if $j < 2(N-k) - 2$, which is the required result. □

4. Spanier–Whitehead duality

4.1. *The definition and main properties*

Let X be a based finite complex. One may consider the function space of based maps $X \longrightarrow S^N$, $F(X, S^N)$, as usual in the compact open topology. This space does not have the homotopy type of a finite complex. However, for N sufficiently large, there is a finite complex Y and a map $Y \longrightarrow F(X, S^N)$ which induces isomorphisms on homotopy groups in dimensions less than $2N - 2k$. One could also state the result as follows. We have natural suspension maps $\Sigma F(X, S^N) \longrightarrow F(X, S^{N+1})$, and hence a directed system of abelian groups $\{\pi_{i+k}(F(X, S^{N+k}))\}_{k \geqslant 0}$. We also have maps $\Sigma^k Y \longrightarrow F(X, S^{N+k})$, and these maps are compatible with respect to suspensions. This gives a homomorphism of abelian groups

$$\varinjlim_k \{\pi_{i+k}(\Sigma^k Y)\}_{k \geqslant 0} \longrightarrow \varinjlim_k \pi_{i+k}(F(X, S^{N+k})). \tag{4.1}$$

The statement will be that this homomorphism is in fact an isomorphism. This theorem and the general development is due to Spanier and Whitehead; see [31].

To study this situation, we first consider any two based CW complexes X and Y. Let S_*X and S_*Y denote the complexes of singular chains on X and Y respectively. We have the evaluation map $e : X \wedge F(X,Y) \longrightarrow Y$. Therefore we have a chain map $S_*e : S_*(X \wedge F(X,Y)) \longrightarrow S_*Y$. Let $\sigma : S_*(X) \otimes S_*(F(X,Y) \longrightarrow S_*(X \times F(X,Y))$ be any chain inverse to the Alexander–Whitney homomorphism, e.g., the shuffle homomorphism. $S_*e \circ \sigma$ is now a homomorphism $S_*(X) \otimes S_*(F(X,Y) \longrightarrow S_*(Y)$ and we may take its adjoint

$$S_*(F(X,Y)) \xrightarrow{\alpha(X,Y)} Hom(S_*(X), S_*(Y)).$$

Now let $Y = S^N$, and fix a generating cocycle c for $H^N(S^N) \cong \mathbf{Z}$. c now gives a chain map which we also call c from $C_*(S^N)$ to the chain complex D_* with $D_i = 0$ when $i \neq N$, and $D_N \cong \mathbf{Z}$, and c induces an isomorphism on H_N. $c \circ \alpha(X, S^N)$ is now a homomorphism from $S_*(F(X, S^N)$ to $Hom(S_*(X), D_*)$, and $H_i(Hom(S_*(X), D_*)) \cong H^{N-i}(X)$, as contravariant functors in X.

THEOREM 4.1.1. *Let X be a finite complex of dimension i. Then $c \circ \alpha(X, S^N)$ induces an isomorphism on H_j for $0 < j < 2N - 2i - 2$.*

PROOF. We first study the situation where X is an i-sphere. In this case,

$$H_*(F(X,S^N)) \cong H_*(\Omega^i S^N) \cong H_*(S^{N-i})$$

for $* < 2(N-i-1)$. In the range in question then, we are only required to verify that $c \circ \alpha(X, S^N)$ induces an isomorphism $H_{N-i}(F(X,S^N)) \cong \mathbf{Z}$. But from the definitions, this is equivalent to the assertion that

$$H_i(S^i) \otimes H_{N-i}(F(S^i,S^N)) \longrightarrow H_N(S^i \times F(S^i,S^N)) \xrightarrow{H_N(e)} H_N(S^N)$$

is a perfect pairing. Note further that if the composite

$$\beta : S^i \wedge S^{N-i} \longrightarrow S^i \wedge F(S^i,S^N) \xrightarrow{e} S^N$$

is the standard identification, then the homomorphism

$$H_i(S^i) \otimes H_{N-i}(S^{N-i}) \longrightarrow H_N(S^i \wedge S^{N-i}) \xrightarrow{\beta} H_N(S^N)$$

yields a perfect pairing. This gives the result for spheres in view of Corollary 3.3.

To deal with a general complex, we work by induction on the dimension i. The case $i = 0$ is trivial. Suppose the result is known for complexes of dimension $< i$, and consider an i-dimensional complex X. Let $X^{(i-1)}$ denote the $(i-1)$-skeleton. Then we have a fibration

$$\begin{array}{ccc} \prod_{\alpha\in A} \Omega^i S^N & \longrightarrow & F(X,S^N) \\ & & \downarrow \\ & & F(X^{(i-1)},S^N) \end{array}$$

where A is an indexing set for the collection of i-cells in X, and the vertical arrow is restriction to the $i-1$ skeleton. $F(X^{(i-1)},S^N)$ is $(N-i)$-connected and $\Omega^i S^N$ is $N-i-1$ connected, so, by Corollary 3.1, we have exact sequences

$$\begin{aligned} H_{j+1}(F(X^{(i-1)},S^N)) \to H_j\Big(\prod_{\alpha\in A} \Omega^i S^N\Big) \to H_j(F(X,S^N)) \\ \to H_j(F(X^{(i-1)},S^N)) \to H_j\Big(\prod_{\alpha\in A} \Omega^i S^N\Big) \end{aligned}$$

for $j < 2(N-i)-1$. These exact sequences map to the corresponding long exact sequences

$$\begin{aligned} H^{N-j-1}(X^{i-1}) \to H^{N-j}\Big(\bigvee_{\alpha\in A} S^i\Big) \to H^{N-j}(X) \\ \to H^{N-j}(X^{i-1}) \to H^{N-j-1}\Big(\bigvee_{\alpha\in A} S^i\Big) \end{aligned}$$

associated to the pair (X, X^{i-1}). The five lemma and the inductive hypothesis now give the result. □

Now, suppose we have two based finite complexes X and Y, with a map $X \wedge Y \xrightarrow{D} S^N$. Consider the composite

$$C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \wedge Y) \longrightarrow C_*(S^N)$$

where the left hand arrow is the same chain inverse to the Alexander–Whitney map which we chose earlier. We therefore obtain an adjoint chain map

$$C_*(Y) \xrightarrow{\Delta} Hom(C_*(X), C_*(S^N)).$$

We say the map D is an *S-duality map* if Δ is a chain equivalence, i.e. induces an isomorphism on homology, and we refer to Y as an S-dual to X.

PROPOSITION 4.1.1. *Suppose $D : X \wedge Y \to S^N$ is an S-duality map. Consider the adjoint map $adj(D) : Y \to F(X, S^N)$. Then, if X is i-dimensional $adj(D)$ induces an isomorphism on H_j for $j < 2N - 2i - 2$, and hence on π_j for $j < 2N - 2i - 3$.*

PROOF. We have the following commutative diagram of chain complexes

$$\begin{array}{ccccc} C_*(X) \otimes C_*(Y) & \longrightarrow & C_*(X \wedge Y) & \longrightarrow & C_*(S^N) \\ \downarrow{\scriptstyle l_1} & & \downarrow{\scriptstyle l_2} & & \downarrow{\scriptstyle =} \\ C_*(X) \otimes C_*(F(X, S^N)) & \longrightarrow & C_*(X \wedge F(X, S^N)) & \longrightarrow & C_*(S^N) \end{array}$$

where l_1 is the chain map $C_*(id) \otimes C_*(adj(D))$ and l_2 is $C_*(id \wedge adj(D))$. Therefore, we have another commutative diagram

$$\begin{array}{ccc} C_*(Y) & \longrightarrow & Hom(C_*(S), C_*(S^N)) \\ \downarrow & & \downarrow{\scriptstyle =} \\ C_*(F(X, S^N)) & \longrightarrow & Hom(C_*(X), C_*(S^N)) \end{array}$$

where the upper horizontal arrow induces isomorphisms on H_j for all j, and the lower horizontal arrow induces isomorphisms on H_j for $j < 2(N - i) - 2$. The result is now immediate. □

4.2. *Existence and construction of S-duals*

We must address the question of whether or not there exists an S-dual for a given finite complex X and some N. We first examine what happens when we attach one cell.

PROPOSITION 4.2.1. *Suppose we have an S-duality $X \wedge Y \longrightarrow S^N$, and a map $f : S^d \longrightarrow X$. Let $X' = X \cup_f e^{d+1}$. Suppose further that $dim(Y) < 2(N-d) - 1$. Then there is a finite based complex Y', of dimension $\leqslant max(dim(Y)+1, N-d+1)$ and an S-duality $X' \wedge Y' \longrightarrow S^N$.*

PROOF. We first consider the sequence of maps

$$Y \longrightarrow F(X, S^N) \xrightarrow{F(f,S^N)} \Omega^d S^N \cong F(S^d, S^N) \longleftarrow S^{N-d}.$$

Here the left arrow is the adjoint to the original S-duality, and the right one is the adjoint to the identification $S^d \wedge S^{N-d} \xrightarrow{\sim} S^N$. Since $dim(Y) < 2(N-d) - 1$, there is a map ϕ, $Y \xrightarrow{\phi} S^{N-d}$ which makes the diagram commute up to homotopy. Equivalently, we have a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[r] \arrow[d] & F(X,S^N) \arrow[d, "F(f,S^N)"] \\
S^{N-d} \simeq & Cyl(\phi) \arrow[r] & F(S^d,S^N)
\end{tikzcd}$$

where $Cyl(\phi)$ is the mapping cylinder of ϕ and the left vertical map is the inclusion on one end of the cylinder. Now consider the diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & F(X,S^N) \arrow[r] \arrow[d, "F(f,S^N)"] & \Omega F(X,S^{N+1}) \arrow[d, "\Omega F(f,S^{N+1})"] \\
Cyl(\phi) \arrow[r] & F(S^d,S^N) \arrow[r] & \Omega F(S^d,S^{N+1}) \arrow[d] \\
 & & F(X \cup_f e^{d+1}, S^{N+1})
\end{tikzcd}$$

where the right hand vertical sequence is the fibration sequence obtained via Proposition 2.2.2 by applying $F(-, S^{N+1})$ to the inclusion

$$X \cup_f e^{d+1} \longrightarrow X \cup_f e^{d+1} \cup CX \simeq \Sigma S^d \simeq S^{d+1}.$$

Since the composite

$$\Omega F(X, S^{N+1}) \longrightarrow \Omega F(S^d, S^{N+1}) \longrightarrow F(X \cup_f e^{d+1}, S^{N+1})$$

is null homotopic, the map $Y \longrightarrow F(X \cup_f e^{d+1}, S^{N+1})$ is null homotopic, and therefore the composite $Cyl(\phi) \longrightarrow \Omega F(S^d, S^{N+1}) \longrightarrow F(X \cup_f e^{d+1}, S^{N+1})$ extends over $Cyl(\phi) \cup CY$. We therefore have a commutative diagram

$$\begin{array}{ccccc}
Y & \longrightarrow & F(\Sigma X, S^{N+1}) & & (= \Omega F(X, S^{N+1})) \\
\downarrow & & \downarrow & & \\
Cyl(\phi) & \longrightarrow & F(S^{d+1}, S^{N+1}) & & (= \Omega F(S^d, S^{N+1})) \\
\downarrow & & \downarrow & & \\
Cyl(\phi) \cup CY & \xrightarrow{\alpha} & F(X \cup_f e^{d+1}, S^{N+1}) & &
\end{array}$$

Let $Y' = Cyl(\phi) \cup cY$, with a map $X' \wedge Y' \xrightarrow{D'} S^{N+1}$ given as the adjoint of α. We claim D' is an S-duality map. To see this, it is only required to show that the associated maps

$$H_i(Y') \longrightarrow H^{N+1-k}(X')$$

are isomorphisms. But this follows from the 5-lemma and the following diagram of long exact sequences:

$$\begin{array}{ccccccccc}
\to & H_k(Y) & \to & H_k(Cyl(\phi)) & \to & H_k(Y') & & & \\
& \downarrow & & \downarrow & & \downarrow & & & \\
\to & H^{N+1-k}(X) & \to & H^{N+1-k}(S^d) & \to & H^{N+1-k}(X') & & & \\
& & & & \to & H_{k-1}(Y) & \to & H_{k-1}(Cyl(\phi)) & \to \\
& & & & & \downarrow & & \downarrow & \\
& & & & \to & H^{N+2-k}(X) & \to & H^{N+2-k}(S^d) & \to
\end{array}$$

□

REMARK. S-duals are also unique in the following sense. Suppose we have a finite complex X, and S-duality maps $D : X \wedge Y \longrightarrow S^N$ and $D' : X \wedge Y' \longrightarrow S^{N'}$. Suppose $N' \geqslant N$. Then, for sufficiently large l there is a homotopy equivalence $\Sigma^{N'-N+l} Y \xrightarrow{\theta} \Sigma^l Y'$. Furthermore, it is characterized by the requirement that

$$\begin{tikzcd}
X \wedge \Sigma^{N'-N+l} Y \arrow[rr, "\Sigma^{N'-N+l}"] \arrow[dr, "id\wedge\theta"'] & & S^{N'+l} \\
& X \wedge \Sigma^l Y' \arrow[ur, "\Sigma^l D'"'] &
\end{tikzcd}$$

commutes up to homotopy.

It is also possible to describe the S-dual in a very concrete fashion. Let X be a finite CW complex. It is well known that it is possible to embed X in Euclidean space, $\mathbf{R}^N$, and from now on we view X as a subspace of $\mathbf{R}^N$. Let Y denote the complement

$\mathbf{R}^N - X$. For any pair of distinct points $v, w \in \mathbf{R}^N$, let $l(v,w) : \mathbf{R} \longrightarrow \mathbf{R}^N$ be given by $l(v,w)(t) = (1-t)v + tw$. Notice that since v and w are distinct, if we view S^N as the one point compactification of $\mathbf{R}^N$, then $l(v,w)$ defines a loop in S^N. l may therefore be viewed as a map from $E \subseteq \mathbf{R}^N \times \mathbf{R}^N$, $E = \{(v,w) \mid v \neq w\}$, to ΩS^N. Let $i : X \longrightarrow \mathbf{R}^N$ and $j : Y \longrightarrow \mathbf{R}^N$ be inclusions, then $X \times Y \xrightarrow{i \times j} \mathbf{R}^N \times \mathbf{R}^N$ factors through E, and we call the composite $X \times Y \longrightarrow E \xrightarrow{l} \Omega S^N$ the (preliminary) duality map, $\hat{D}$. Since X is compact, X is contained in some ball, B, in $\mathbf{R}^N$. Choose a basepoint y for Y outside that ball. Observe that $\hat{D}|X \times y$ extends over $B \times y$, since $y \notin B$. Since B is contractible, we obtain an extension $\tilde{D}$ from $X \times Y \cup C(X \times y)$ to ΩS^N. $X \times Y \cup C(X \times y)$ is homotopy equivalent to $X \times Y/(X \times y)$, which, in turn, is homeomorphic to $X_+ \wedge Y$, where X_+ denotes X with a disjoint basepoint added. Let $D : \Sigma X_+ \wedge Y \longrightarrow S^N$ denote the adjoint.

THEOREM 4.2.1. *D is an S-duality map.*

PROOF. For any finite subcomplex $X \subset \mathbf{R}^N$, with $Y = \mathbf{R}^N - X$, let D_X denote the map constructed above. (Here, a point y is chosen once and for all, and will be contained in the complements of all the subcomplexes we deal with.) We will show that if D_{X_1}, D_{X_2}, and $D_{X_1 \cap X_2}$ are S-duality maps for subcomplexes X_1 and X_2 of $\mathbf{R}^N$ which are contained in a ball which does not contain y, then $D_{X_1 \cup X_2}$ is also an S-duality map. Let $Y_i = \mathbf{R}^N - X_i$. Note that we have a pullback square of fibrations

$$\begin{tikzcd}
F(X_1 \cup X_{2+}, S^N) \arrow[r] \arrow[d] & F(X_{1+}, S^N) \arrow[d] \\
F(X_{2+}, S^N) \arrow[r] & F(X_1 \cap X_{2+}, S^N)
\end{tikzcd}$$

We suppose, for the moment, that N is sufficiently large that the natural maps

$$C_*(F(X_1 \cup X_{2+}, S^N)) \longrightarrow Hom(\tilde{C}_*(X_1 \cup X_{2+}), \mathbf{Z})$$

and $C_*(F(X_{i+}, S^N)) \longrightarrow Hom(\tilde{C}_*(X_{i+}), \mathbf{Z})$ induce isomorphism on homology for $* \leqslant N$. Note also that from the definitions, we have a commutative diagram

$$\begin{tikzcd}
Y_1 \cap Y_2 \arrow[rr] \arrow[dr] \arrow[dd] & & F(X_{1+} \cup X_{2+}, S^N) \arrow[dr] \arrow[dd] & \\
& Y_1 \arrow[rr] \arrow[dd] & & F(X_{1+}, S^N) \arrow[dd] \\
Y_2 \arrow[rr] \arrow[dr] & & F(X_{2+}, S^N) \arrow[dr] & \\
& Y_1 \cup Y_2 \arrow[rr] & & F(X_1 \cap X_{2+}, S^N)
\end{tikzcd}$$

and therefore a commutative diagram

$$\begin{array}{ccccc} C_*(Y_1 \cap Y_2) & \longrightarrow & C_*(Y_1) \oplus C_*(Y_2) & \longrightarrow & C_*(Y_1 \cup Y_2) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ C^*(X_1 \cup X_{2+}) & \longrightarrow & C^*(X_{1+}) \oplus C^*(X_{2+}) & \longrightarrow & C^*(X_1 \cap X_{2+}) \end{array}$$

This gives rise to a commutative diagram of Mayer–Vietoris sequences, which in the relevant range is

$$\begin{array}{ccccccc} \longrightarrow & H_{d+1}(Y_1 \cup Y_2) & \longrightarrow & H_d(Y_1 \cap Y_2) & \longrightarrow \\ & \big\downarrow & & \big\downarrow & \\ \longrightarrow & H^{N-d-1}(X_1 \cap X_{2+}) & \longrightarrow & H^{N-d}(X_1 \cup X_{2+}) & \longrightarrow \end{array}$$

$$\begin{array}{cc} H_d(Y_1) \oplus H_d(Y_2) & \longrightarrow \\ \big\downarrow & \\ H^{N-d}(X_{1+}) \oplus H^{N-d}(X_{2+}) & \longrightarrow \end{array}$$

where the vertical arrows are all adjoints to the duality maps.

Since $D_{X_1 \cap X_2}$, D_{X_1}, and D_{X_2} all induce isomorphisms on homology, so does $D_{X_1 \cup X_2}$. To obtain a proof of the required result, we must now show that the result holds for a single point. But for a single point, the complement has the homotopy type of S^{N-1}, and the map $S^0 \wedge S^{N-1} \xrightarrow{D} \Omega S^N$ is easily seen to be equal to the map $J : S^{N-1} \longrightarrow \Omega S^N$ from Section 3, whose adjoint is the identity map of S^N. This gives the result. □

If one wants to give a duality map for X itself (rather than for X_+), one must only adjoin the point at infinity to Y. More generally, let $X_1 \subseteq X_2$ be an inclusion of subcomplexes of $\mathbf{R}^N$, and let $Y_1 \supseteq Y_2$ denote the complements.

COROLLARY 4.2.1. *In the above situation, there is an S-duality map*

$$D : \Sigma(X_2/X_1 \wedge Y_1/Y_2) \longrightarrow S^N$$

When X is a compact closed manifold, we obtain the following geometric description. See [5] and [31].

COROLLARY 4.2.2 (Spanier, Atiyah). *Let X be a compact closed smooth manifold, and suppose X is smoothly embedded in $\mathbf{R}^N$. Let N denote the normal bundle to the embedding, and let $\tau(N)$ denote its Thom complex. Then there is an S-duality map $X \wedge \tau(N) \longrightarrow S^N$.*

PROOF. Let $B(X)$ denote a small tubular neighborhood of X. Via the exponential map on the normal bundle, it is homeomorphic to the open unit disc bundle of N. If $Y = \mathbf{R}^N - X$, we have the S-duality map

$$X \wedge \Sigma Y \longrightarrow \Sigma(X \wedge Y) \longrightarrow S^N$$

But, $\mathbf{R}^N/Y$ is naturally homotopy equivalent to ΣY, since it is homotopy equivalent to the mapping cone on the inclusion $Y \longrightarrow \mathbf{R}^N$, and $\mathbf{R}^N$ is contractible. On the other hand, let $\overline{B}$ denote the closure of B; then $\mathbf{R}^N/Y$ is homeomorphic to $\overline{B}/\partial\overline{B}$, which in turn is homeomorphic to the quotient of the closed unit disc bundle of N by the unit sphere bundle. This is the definition of the Thom complex of N. □

5. The construction and geometry of loop spaces

To understand stabilization a bit better it is useful to be able to compute the homology of loop spaces, and in particular loop spaces of suspensions. This was first carried out by I.M. James for the case of $\Omega\Sigma X$. Soon afterwards J.F. Adams and P. Hilton constructed a model for ΩX when X is any simply connected CW complex with one zero cell and no one cells.[2] In both cases explicit models for the loop spaces were constructed. Later developments, particularly the construction of the Eilenberg–Moore spectral sequences made these original constructions less compelling for homology calculations but nonetheless, the geometry of ΩX reveals a great deal about the structure of X, so explicit constructions still play a vital role in the theory.

Both the James and Adams–Hilton models had a multiplicative structure and were even free associative monoids with unit. In fact more was true, each was a CW complex and the multiplication was cellular, so that the cellular chain complex was a tensor algebra with one generator in each dimension $(n-1)$ for each cell in dimension n of X. However, while in the James model for $\Omega\Sigma X$, the boundary map was explicitly determined by the boundary map for ΣX, in the Adams–Hilton model the boundary map was not determined at all initially. In a following paper Adams determined the boundary map for their construction in the case where X is a *simplicial complex* with the 1-skeleton collapsed to a point.

This work was of seminal importance in the theory and, though, as indicated, we can today replace most of it using the techniques of Eilenberg–Moore and classifying space theory, in this section we will describe the techniques and results of James, Hilton and Adams, much in the spirit in which they had originally been developed.

5.1. *The space of Moore loops*

It will first be necessary to describe a space homotopy equivalent to the usual loop space, the space of "Moore loops", $\Omega^M(X,*)$. Let $F(\mathbf{R},X)$ denote the space of all maps $\phi : \mathbf{R} \longrightarrow X$, in the compact open topology. Let $\Omega^M(X,*) \subseteq F(\mathbf{R},X) \times [0,\infty)$

[2] The construction given here is first described in the proof of Theorem 2.1 of [1]. However, the actual geometric construction is secondary to their objectives there. What they do is to construct a chain map of the cellular chain complex of this model into the singular cubical complex of $\Omega^M(Y)$ and show, by chain level arguments, that the resulting embedding induces isomorphisms in homology.

In later work S.Y. Husseini directly constructs this model for $\Omega^M(Y)$ as a special case of his general notion of a "relation in r-variables, $M_r(X)$", [18].

denote the subspace of all pairs (ϕ, r) for which $\phi(0) = *$ and for which $\phi(t) = *$ for all $t \geqslant r$. Note that the standard loop space $\Omega(X, *)$ can be identified with the subspace of all pairs of the form $(\phi, 1)$ with $\phi(t) = *$ for $t \geqslant 1$.

PROPOSITION 5.1.1. *$\Omega(X, *)$ is a deformation retract of $\Omega^M(X, *)$.*

PROOF. First consider $\tilde{\Omega}(X, *) \subseteq \Omega^M(X, *)$, the subspace of all (ϕ, t) with $t \geqslant 1$. A deformation retraction, H, of $\Omega^M(X, *)$ to $\tilde{\Omega}(X, *)$ is given by the following formulae.

$$\begin{aligned} H\big(s, (\phi, r)\big) &= (\phi, r + s) \quad \text{when } r + s \leqslant 1, \\ H\big(s, (\phi, r)\big) &= (\phi, 1) \quad \text{when } r \leqslant 1 \text{ and } r + s \geqslant 1, \\ H\big(s, (\phi, r)\big) &= (\phi, r) \quad \text{when } r \geqslant 1. \end{aligned}$$

Now we give a deformation retraction G from $\tilde{\Omega}(X, *)$ to $\Omega(X, *)$ by the formula

$$G\big(s, (\phi, r)\big) = \big(\phi_s, (1 - s)r + s\big),$$

where

$$\phi_s(t) = \phi\left(\frac{r}{((1 - s)r + s)} t\right).$$

This gives the required deformation retraction. □

We now remark that $\Omega^M(X, *)$ is actually a topological monoid, where the multiplication is given by $(\phi, r) \cdot (\psi, s) = (\phi * \psi, r + s)$ and

$$\begin{cases} \phi * \psi(t) = \phi(t) & \text{when } 0 \leqslant t \leqslant r, \\ \phi * \psi(t) = \psi(t - r) & \text{when } r \leqslant t \leqslant r + s, \\ \phi * \psi(t) = * & \text{when } t \geqslant r + s. \end{cases} \tag{5.1}$$

The point $(*, 0)$, where $*$ denotes the constant loop with value 0, is the identity element.

5.2. *Free topological monoids*

We now discuss the construction of the free monoid on a based topological space. First, if we have a based set $(X, *)$, recall that the free monoid on $(X, *)$ consists of all the "words" in X, with $*$ set to the identity. Formally this can be described as

$$\coprod_{n \geqslant 0} X^n / \sim, \tag{5.2}$$

where $\sim$ is the equivalence relation generated by all relations of the form

$$(x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_n) \simeq (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n). \tag{5.3}$$

Multiplication is now just juxtaposition of words. This construction can now be applied equally well to based topological spaces, since one can construct the quotient space associated to an equivalence relation. Let the resulting construction be denoted by $M(X,*)$. It has the following universality property.

PROPOSITION 5.2.1. *Let $(X,*)$ be a based space, and let $f : X \longrightarrow M$ be any map to a topological monoid, M, with $f(*) = e$. Then there is a unique homomorphism $\bar{f} : M(X,*) \longrightarrow M$ of topological monoids so that the composite*

$$(X,*) \longrightarrow M(X,*) \xrightarrow{\bar{f}} M$$

is equal to f.

REMARK. When dealing with quotient spaces and products there is sometimes trouble, since the quotient of a product is not usually a product, even if only one of the two spaces is quotiented. However, with the compactly generated topology this difficulty is avoided, and we always assume that we are using this topology from now on. (See the remarks at the end of 2.5.)

5.3. *The James construction*

Let $(X,*)$ be any based space. Recall the definition of the "James map",

$$J : (X,*) \longrightarrow \Omega(\Sigma X,*), \qquad J(x)(t) = [t,x] \in \Sigma X.$$

If we compose this map with the inclusion into $\Omega^M(\Sigma X,*)$, we obtain a map, J, which does not carry the basepoint to the identity. Let

$$\hat{X} = X \coprod [0,1]/\simeq,$$

where $\simeq$ is generated by $1 \simeq *$, and define an extension $\hat{J}$ of J to $\hat{X}$ by $\hat{J}(s) = (*,s)$, where $0 \leqslant s \leqslant 1$, and $*$ denotes the constant map with value $*$. Of course, if X is a CW-complex, then X and $\hat{X}$ are based homotopy equivalent. This now becomes a pointed map if we let 0 be the basepoint for $\hat{X}$. Since we have a based map $\hat{J} : \hat{X} \longrightarrow \Omega^M(\Sigma X,*)$, we obtain a homomorphism $\bar{J} : M(\hat{X},0) \longrightarrow \Omega^M(\Sigma X,*)$. The theorem of James is that this map is a homotopy equivalence when X is a connected CW-complex.

Before proving this theorem we need to do some preliminary work on the homology of both spaces involved. For simplicity, we will consider homology with field coefficients ($\mathbf{F}_p$, p a prime, or $\mathbf{Q}$). For any topological monoid M, the homology groups of M form a graded, associative algebra with unit via

$$H_*(M) \otimes H_*(M) \cong H_*(M \times M) \xrightarrow{H_*(\mu)} H_*(M) \tag{5.4}$$

where $\mu : M \times M \longrightarrow M$ is the multiplication map. Thus the graded groups $H_*(M(\hat{X},0))$ and $H_*(\Omega^M(X,*))$ have the structure of graded rings, and this additional structure will be quite useful in describing the homology.

We recall the notion of the *tensor algebra* of a vector space V, $T(V)$. If V is a graded vector space, $T(V)$ obtains a natural grading where $v_1 \otimes \cdots \otimes v_n$ has grading $\sum_{i=1}^n \alpha_i$ if v_i has grading α_i. The tensor algebra has the universal property that if V is a graded vector space and $V \xrightarrow{\lambda} A$ is a map from a graded vector space into a graded algebra, then λ extends uniquely to a homomorphism of graded algebras $\Lambda : T(V) \longrightarrow A$.

Now consider $M(\hat{X},0)$; it is filtered by subspaces $M_n(\hat{X},0)$, where $M_n(\hat{X},0)$ is the image of $\hat{X}^n$ in $M_n(\hat{X},0)$. Thus $M_n(\hat{X},0)$ consists of the "words of length less than or equal to n" in the free monoid on $(\hat{X},0)$. From the definition of the equivalence relation defining $M(\hat{X},0)$ it is clear that the subquotient $M_n(\hat{X},0)/M_{n-1}(\hat{X},0)$ is homeomorphic to the smash product

$$\underbrace{\hat{X} \wedge \cdots \wedge \hat{X}}_{n \text{ times}}.$$

The Künneth formula now tells us that

$$\tilde{H}_*(\hat{X} \wedge \cdots \wedge \hat{X}) \cong \bigotimes_{i=1}^{n} \tilde{H}_*(\hat{X})$$

where the tensor product denotes tensor product of graded vector spaces. Let us now examine the collapse map

$$M_n(\hat{X},0) \longrightarrow M_n(\hat{X},0)/M_{n-1}(\hat{X},0).$$

We claim that it is surjective on homology. To see this, note that we have a map $X^n \longrightarrow M_n(\hat{X},0)$, given as the composite of the inclusion $X^n \longrightarrow \hat{X}^n$ with the identification map $\hat{X}^n \longrightarrow M_n(\hat{X},0)$. The composite

$$X^n \longrightarrow M_n(\hat{X},0) \longrightarrow M_n(\hat{X},0)/M_{n-1}(\hat{X},0)$$

is the equivalence $X^n \longrightarrow \hat{X}^n$ composed with the collapse of the product to the smash product. The Künneth formula shows that this is surjective, hence the result. We conclude that

$$H_*(M(\hat{X},0)) \cong \mathbf{F}_p \oplus \bigoplus_{i=1}^{n} \otimes_n \tilde{H}_*(X).$$

Now, the inclusion $\hat{X} \longrightarrow M(\hat{X},0)$ induces a map of graded vector spaces

$$\tilde{H}_*(\hat{X}) \rightarrow H_*(M(\hat{X},0)),$$

and hence a homomorphism of graded algebras $\Lambda : T(\tilde{H}_*(\hat{X})) \longrightarrow H_*(M(\hat{X}, 0))$.

PROPOSITION 5.3.1. *Λ is an isomorphism of graded algebras.*

PROOF. For any graded vector space V, let $T_n(V) = V \otimes \cdots \otimes V$. It now follows from the above analysis that under Λ, $T_n(\tilde{H}(X))$ has image in $H_*(M_n(\hat{X}, 0))$, and that it surjects to

$$\tilde{H}_*\big(M_n(\hat{X},0)/M_{n-1}(\hat{X},0)\big) \cong \bigotimes_{i=1}^{n} \tilde{H}_*(X).$$

Since we have a surjective map of isomorphic vector spaces, it is an isomorphism, and hence Λ is an isomorphism. □

We must now perform a similar analysis for $H_*(\Omega^M(\Sigma\hat{X}, 0)) \cong H_*(\Omega\Sigma X, *)$. Note that $\Omega\Sigma$ is equipped with its own loop sum operation μ, defined by $\mu(\phi, \psi) = \phi * \psi$, where $\phi*\psi(t) = \phi(2t)$ for $0 \leqslant t \leqslant 1/2$, and $\phi*\psi(t) = \psi(2t-1)$ for $1/2 \leqslant t \leqslant 1$. μ is not associative, but is homotopic to the restriction of the multiplication map on $\Omega^M(X, *)$ to $\Omega(X, *)$ and is therefore homotopy associative. In particular, μ gives $H_*(\Omega\Sigma X)$ the structure of an associative graded algebra. Let E denote the space of maps ϕ : $[0, 1] \longrightarrow \Sigma X$ with $\phi(0) = *$. The evaluation map $p : E \longrightarrow \Sigma X$, $p(\phi) = \phi(1)$ is a Hurewicz fibration, and the fibre over the point $*$ is clearly homeomorphic to the standard loop space $\Omega(\Sigma X, *)$. Let C_+X denote the image of $[\frac{1}{2}, 1] \times X$ in ΣX, and similarly C_-X will be the image of $[0, \frac{1}{2}] \times X$. Both these spaces are contractible, and their intersection is X. By Corollary 2.2.1, it follows that $p^{-1}(C_+X)$ (respectively $p^{-1}(C_-X)$) is homotopy equivalent as a space over C_+X (respectively C_-X) to $C_+X \times \Omega\Sigma X$ (respectively $C_-X \times \Omega\Sigma X$). We obtain explicit homotopy equivalences as follows. Let $H_\pm : C_\pm X \times I \longrightarrow C_\pm X$ be the standard deformation retraction of $C_\pm X$ to $*$. Define maps $\theta_\pm p^{-1}(C_\pm X) \longrightarrow C_\pm X \times \Omega\Sigma X$ by setting $\theta_\pm(\phi) = (p(\phi), \psi_\pm)$, where $\psi_\pm(t) = \phi(2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $\psi_\pm(t) = H_\pm(p(\phi), 2t-1)$ for $\frac{1}{2} \leqslant t \leqslant 1$. One readily checks that these are homotopy equivalences over $C_\pm X$. When we restrict $\theta_\pm$ to $X \subseteq C_\pm X$, we obtain two distinct homotopy equivalences

$$p^{-1}(X) \xrightarrow{\theta_\pm} X \times \Omega\Sigma X.$$

We also define homotopy inverses $\eta_\pm$ to $\theta_\pm$ over X as follows. $\eta_\pm(x, \phi) = (x, \xi_\pm)$, where $\xi_\pm(t) = 2t$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $\xi_\pm(t) = H(x, 2-2t)$ for $y \leqslant t \leqslant 1$. Consider the composite $\theta_- \circ \eta_+ : X \times \Omega\Sigma X \longrightarrow X \times \Omega\Sigma X$. It is given by $\theta_- \circ \eta_+(x, \phi) = (x, \zeta)$, where ζ is described by the following formulae:

$$\begin{cases} \zeta(t) = \phi(4t) & \text{for } 0 \leqslant t \leqslant \frac{1}{4}, \\ \zeta(t) = H_+(x, 2-4t) & \text{for } \frac{1}{4} \leqslant t \leqslant \frac{1}{2}, \\ \zeta(t) = H_-(x, 2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Note that after suitable reparameterization, $\theta_-\eta_+$ becomes equal to the composite

$$X \times \Omega\Sigma X \xrightarrow{\alpha} X \times \Omega\Sigma X \times \Omega\Sigma X \xrightarrow{\beta} X \times \Omega\Sigma X$$

where $\alpha(x, \phi) = (x, J(s), \phi)$ and $\beta(x, \phi_1, \phi_2) = (x, \mu \circ (\phi_1, \phi_2))$. Here $J : X \longrightarrow \Omega\Sigma X$ is the James map and μ is the loop sum multiplication on $\Omega\Sigma X$. Now consider the Mayer–Vietoris sequence for the covering of E by $p^{-1}(C_+X)$ and $p^{-1}(C_-X)$. It has the form

$$\begin{tikzcd}
 & H_*(\Omega\Sigma X) \cong H_*(p^{-1}(U_-)) \arrow[dr] & \\
\cdots \longrightarrow H_*(X \times \Omega\Sigma X) \arrow[ur, "g"] \arrow[dr, "f"'] & \bigoplus & H_*(E) \longrightarrow \cdots \\
 & H_*(\Omega\Sigma X) \cong H_*(p^{-1}(U_+)) \arrow[ur] &
\end{tikzcd}$$

If we identify $p^{-1}(X)$ with $X \times \Omega\Sigma X$ via θ_+, then f is just the homomorphism induced by projection. g, on the other hand, is given by the composite

$$H_*(X \times \Omega\Sigma X) \xrightarrow{J\times 1} H_*(\Omega\Sigma X \times \Omega\Sigma X) \xrightarrow{\mu} H_*(\Omega\Sigma X).$$

If we identify $H_*(X \times \Omega\Sigma X)$ with $H_*(X) \otimes H_*(\Omega\Sigma X)$, then the map is given by

$$H_*(X) \otimes H_*(\Omega\Sigma X) \xrightarrow{H_*(J)\otimes Id} H_*(\Omega\Sigma X) \otimes H_*(\Omega\Sigma X) \xrightarrow{H_*(\mu)} H_*(\Omega\Sigma X).$$

Since $H_*(E)$ is trivial we conclude that the map

$$\tilde{H}_*(X \times \Omega\Sigma X) \xrightarrow{(g,f)} \tilde{H}_*(\Omega\Sigma X) \oplus \tilde{H}_*(\Omega\Sigma X)$$

is an isomorphism of graded vector spaces. Further,

$$\tilde{H}_*(X \times \Omega\Sigma X) \cong \left[\tilde{H}_*(X) \otimes H_*(\Omega\Sigma X)\right] \oplus \tilde{H}_*(\Omega\Sigma X),$$

and f is just the projection on the second factor. It follows that the map $\tilde{H}_*(X) \otimes H_*(\Omega\Sigma X) \longrightarrow \tilde{H}_*(\Omega\Sigma X)$ is an isomorphism. Therefore, if we let $V_* = \tilde{H}_*(X)$ and A_* be the algebra $H_*(\Omega\Sigma X)$, and let $\bar{A}$ denote the ideal of positive dimensional elements, then $V_* \otimes A_* \longrightarrow \bar{A}_*$ is an isomorphism. We claim this characterizes A_* completely.

PROPOSITION 5.3.2. *Let A_* be a graded algebra with A_0 a field, and let $\bar{A}_*$ denote the ideal*

$$\bigotimes_{i=1}^{\infty} A_i.$$

Let $V_* \xrightarrow{i} \bar{A}_*$ be a map of graded vector spaces. Suppose the multiplication map $V_* \otimes A_* \longrightarrow \bar{A}_*$ is an isomorphism of graded vector spaces. Then the algebra homomorphism $\hat{i} : T_*(V) \longrightarrow A_*$ which restricts to i on V_* is an isomorphism.

PROOF. We first show that $\hat{i}$ is surjective. $\hat{i}$ is clearly an isomorphism in dimension 0. We now proceed by induction. Consider any $\alpha \in A_n$, and suppose it is known that all elements in A_{n-1} are in the image of $\hat{i}$. Since $V_* \otimes A_* \longrightarrow \bar{A}_*$ is an isomorphism, any homogeneous element α can be written in the form $\sum v_i \otimes \alpha_i$, where $v_i \in V_*$ and $\alpha_i \in A_*$. Since the v_i's all have grading greater than 0, the α_i's all have grading less than n and hence are in the image of $\hat{i}$. The v_i's are clearly in the image of $\hat{i}$, so therefore is α. To prove injectivity, we observe that $\hat{i}$ is an isomorphism in dimension 0. Now consider an element τ of minimal grading n on which $\hat{i}$ vanishes. Since τ is of positive grading, it lies in the image of $V \otimes T(V)$ in $T(V)$, i.e. $\tau = \Sigma v_i \otimes t_i$, where each t_i has grading less than n. Therefore, $\sum v_i \otimes \hat{i}(t_i) \neq 0$ in $V_* \otimes A_*$. But since the multiplication map $V_* \otimes A_* \longrightarrow \bar{A}_*$ is an isomorphism, we conclude that $\hat{i}(\tau) \neq 0$, which is a contradiction. □

COROLLARY 5.3.1. *Let $J : X \longrightarrow \Omega\Sigma X$ be the James map. Then the natural homomorphism $T(\bar{H}_*(X)) \longrightarrow H_*(\Omega\Sigma X)$ is an isomorphism of graded algebras.*

COROLLARY 5.3.2. *If X is a connected CW complex, the map*

$$\hat{J} : M(\hat{X}, 0) \longrightarrow \Omega^M(\Sigma X, 0)$$

induces an isomorphism on homology groups. Hence, $\hat{J}$ is a homotopy equivalence.

PROOF. The homology statement is clear since we have a commutative diagram

$$\begin{tikzcd} & T(\tilde{H}_*(\hat{X})) \arrow[dl] \arrow[dr] & \\ H_*(M(\hat{X},0)) \arrow[rr, "H_*(\hat{J})"] & & H_*(\Omega^M(\Sigma X,0)) \end{tikzcd}$$

where we have proved that both diagonal arrows are isomorphisms.

This shows that $H_*(\hat{J})$ induces isomorphism on $H_*(\ ;\mathbf{Q})$ and $H_*(\ ;\mathbf{F}_p)$. The universal coefficient theorem then gives the result for $H_*(\ ;\mathbf{Z})$. The relative Hurewicz theorem now gives the result for homotopy groups. $M(\hat{X}, 0)$ has a natural cell structure coming from the cell structures on the products $\hat{X}^n$, so $M(\hat{X}, 0)$ is a CW complex. By a theorem of Milnor, [25], $\Omega^M(\Sigma X, 0)$ has the homotopy type of a CW complex. Theorem 2.1.3 now applies. □

5.4. *The Adams–Hilton construction for ΩY*

We now build a model for $\Omega^M Y$ where Y is a simply connected CW complex but not necessarily a suspension.

The model for the construction we are about to present is James' result above. Note that $J(X) \simeq \Omega\Sigma X$ is a free, associative, unitary monoid with a natural CW decomposition provided that the base point $*$ is a vertex[3], coming from the natural decomposition of X^n as a product CW complex. Thus, $J(X)$ has the following three properties:

- every element $v \in J(X)$ has a unique expression $v = *$ or $v = x_1 x_2 \cdots x_n$, $x_i \in X - *$ for $1 \leqslant i \leqslant n$,
- $x_1 \cdots x_n$ is contained in a unique cell of $J(X)$, the cell $C_1 \times C_2 \times \cdots \times C_n$ where $x_i \in \mathrm{Int}(C_i)$, $1 \leqslant i \leqslant n$, so in particular, no indecomposable cell contains decomposable points,
- the cell complex has the form of a tensor algebra $T(C_\#(X))$, where the subcomplex $C_\#(X)$ is exactly the indecomposables, and the generating cells in dimension i are in 1–1 correspondence with the cells in dimension $i + 1$ of ΣX.

THEOREM 5.4.1 (Adams–Hilton). *Let Y be a CW complex with a single vertex and no 1-cells:*

$$Y = * \cup e_1^2 \cup e_2^2 \cup \cdots \cup e_r^2 \cup e_1^3 \cup \cdots.$$

Then there is a model for $\Omega^M(Y)$ which is a free associative CW monoid, with $*$ the only vertex, the generating cells $f_1^i, \ldots, f_r^i, \ldots$ in dimension i are in 1–1 correspondence with the $(i+1)$-dimensional cells of Y and it satisfies condition (2) above. (For (3) there is no reason to assume that ∂ of an indecomposable cell consists only of indecomposable terms.)

PROOF. The proof essentially goes by noting the way in which the loop space changes as we add cells to our space Y.

In particular, the 2-skeleton,

$$sk_2(Y) = * \cup e_1^2 \cup e_2^2 \cup \cdots \cup e_r^2 \simeq \bigvee S^2 = \Sigma \bigvee S^1,$$

is a suspension and the theorem is James' result. So what we need is a device for doing an inductive step.

DEFINITION 5.4.1. Let M be an associative, unitary monoid with base point the identity, and suppose that $f : X \longrightarrow M$ is a based map. Then the prolongation $P(M, f, cX)$ is the associative, unitary monoid

$$\coprod_{n=1}^{\infty} (M \cup_f cX)^n / \sim$$

with multiplication induced by juxtaposition, and where $\sim$ is the equivalence relation

$$(x_1, \ldots, x_n) \sim (x_1, \ldots, \widehat{x}_i, x_i x_{i+1}, \ldots, x_n)$$

if and only if both x_i and x_{i+1} are contained in M or one of x_i, x_{i+1} is the unit $*$.

[3] Using the compactly generated topology so that products behave well.

$P(M, f, cX)$ has obvious universality properties: it is universal for maps of $M \cup_f cX$ into associative unitary monoids, which are multiplicative on M. Additionally, if X is a sphere S^n, f is cellular, and M has a CW multiplication, then $P(M, f, cX)$ has a CW multiplication, and $C_\#(P(M, f, cX))$ has the form $T(A, e^{n+1})$ where A is the CW complex of M.

Now, we suppose that a principal M-quasifibering has been constructed $M \to E \to B$ with E contractible which is sufficiently structured that we can build the associated principal $P(M, f, cX)$ quasifibering over B by just replacing the fiber M by $P(M, f, cX)$, so we have, by a minor abuse of notation, the quasifibering

$$P(M, f, cX) \longrightarrow P(M, f, cX) \times_M E \longrightarrow B.$$

This extends to a quasifibering

$$\{P(M, f, cX) \times_M E\} \cup \{P(M, f, cX) \times c(cX)\}/\sim \longrightarrow B \cup c\Sigma X \tag{5.5}$$

where $\sim$ is the identification $(p, 0, \{t, x\}) \sim (p, t, f(x))$ where $(t, f(x))$ is the track of the contracting homotopy in E on the image of $f(x) \in M$.

The base of this quasifibration is $B \cup_{\Sigma f} c\Sigma X$ and it is not hard to show that the total space is again contractible if say X is a sphere S^n, $n \geqslant 1$, and f is cellular. This can be verified by using the contracting homotopy in $C_\#(E)$ together with the obvious contraction of the new cell e^{n+1} in the new part to build a contraction on the entire cellular chain complex. Moreover, in our situation it will also be direct to check that the resulting quasifibering has sufficient structure that we can again build an associated principal quasifibration from it.

We now proceed with the construction, starting with the trivial $M = *$ over $*$. The next step attaches e^1's, one for each 2-cell of Y via the unique map $f : \bigvee S^0 \to *$. The resulting quasifibering has the form $J(\bigvee S^1) \cup J(\bigvee S^1) \times c(\bigvee S^1)$ where

$$(x_1 \cdots x_r, 1, x) \sim x_1 \cdots x_r \cdot x, \quad (x_1 \cdots x_r, 0, x) \sim x_1 \cdots x_r,$$

and $(x_1 \cdots x_r, t, *) \sim x_1 \cdots x_r$ as well. The base is, of course, $sk_2(Y) \simeq \bigvee S^2$.

At each stage, the space $P(M, f, cX)$ has the homotopy type of $\Omega^M(B \cup c\Sigma X)$ where the attaching map is $\Sigma f : \Sigma X \to B$. Consequently, assuming that B is the homotopy type of $sk_i(Y)$, we can assume $\Sigma X = \bigvee S^i$, one sphere for each $(i+1)$-cell in Y, with Σf restricted to S^i_j the j-th attaching map, and the base for $P(M, f, cX)$, using the construction above has the homotopy type of $sk_{i+1}(Y)$. (It should be noted that the attaching maps in M are uniquely determined since the total space of the quasifibration at the $(i-1)^{st}$ stage is assumed to be contractible, and that the images of the traces of the contraction on f in the base will be the attaching maps for $sk_{i+1}(Y)$.) □

This is the *Adams–Hilton model* for $\Omega^M X$. Of course, since the prolongation construction is universal it is not always the most efficient way to build a model for the loop space.

EXAMPLE 5.1. $\mathbf{CP}^2 = S^2 \cup e^4$ where the attaching map is the classical Hopf map $h : S^3 \longrightarrow S^2$. There is a fibration $S^1 \longrightarrow S^5 \longrightarrow \mathbf{CP}^2$, and hence, taking loops, a fibration

$$\Omega^M S^5 \longrightarrow \Omega^M \mathbf{CP}^2 \longrightarrow S^1.$$

We claim that this fibration splits up to homotopy type as the product $\Omega^M(S^5) \times S^1$. From the long exact sequence of homotopy groups for the fibration, we see that

$$\pi_1(\Omega^M \mathbf{CP}^2) \longrightarrow \pi_1(S^1) = \mathbf{Z}$$

is an isomorphism. Consequently, mapping $S^1 \longrightarrow \Omega^M \mathbf{CP}^2$ so as to represent a generator of $\pi_1(\Omega^M \mathbf{CP}^2)$, and using the homotopy lifting property, we can map $S^1 \longrightarrow \Omega^M \mathbf{CP}^2$ so that the composite $S^1 \longrightarrow \Omega^M \mathbf{CP}^2 \longrightarrow S^1$ is the identity. Now, using the multiplication in Ω^M, we have a map of the product $(\Omega^M S^5) \times S^1 \longrightarrow \Omega^M \mathbf{CP}^2$ which gives the asserted homotopy equivalence.

This shows $\Omega^M \mathbf{CP}^n$ has the homotopy type of a CW complex with one cell in each dimension congruent 0 and 1 mod 4 and no other cells. Furthermore, the fact that the bottom circle splits off implies that the boundary map in the cellular chain complex is identically zero.

On the other hand, the Adams–Hilton theorem gives as a model for $\Omega^M \mathbf{CP}^2$ the prolongation $P(\Omega^M S^2, \Omega h, e^4)$ which has a cell decomposition given by the prolongation of

$$\left(e^1 \cup e^2 \cup e^3 \cup \cdots\right) \cup_{\Omega(h)} f^3.$$

Thus, P has cells of the form

$$e^{i_0} \times f^{3k_0} \times e^{i_1} \times f^{3k_1} \times \cdots.$$

This cellular decomposition of $\Omega^M \mathbf{CP}^2$ is much bigger than the one obtained above by splitting off the circle and therefore there must be a massive number of nontrivial boundary maps here. For example, $e^2 = e^1 * e^1$ so $\partial(e^2) = 0$, but since $H_2(\Omega^M \mathbf{CP}^2) = 0$ we must have $\partial(f^3) = e^2$. Using the multiplication in the cell complex this boundary map now determines all the boundary maps.

EXAMPLE 5.2. We know from James' construction that

$$\Omega S^{n+1} = S^n \cup e^{2n} \cup e^{3n} \cup e^{4n} \cup e^{5n} \cup \cdots.$$

Thus, the Adams–Hilton construction implies that, for $n \geqslant 2$, there is a cell decomposition

$$C_\#(\Omega^2 S^{n+1}) = T\left[e^{n-1}, e^{2n-1}, e^{3n-1}, e^{4n-1}, e^{5n-1}, \cdots\right].$$

The results in Sections 6 and 7 determine the boundary maps which are quite complex and begin to reflect some of the deeper structure of S^{n+1}. For example, it turns out that

$$\partial\left(e^{2n-1}\right) = 2\left[S^{n-1}\right] * \left[S^{n-1}\right].$$

In general the examples above show that it is quite difficult to understand the boundary maps in the cell decomposition provided by the Adams–Hilton theorem. However, in the special case that Y is given as a *simplicial complex* with no edges, and consequently only one vertex Adams built an explicit model with an explicit ∂ map and we discuss his results next.

5.5. *The Adams cobar construction*

To compute the boundary in the chain complex of $AH(X)$ for general X is a major problem in homotopy theory. (If one knows how to do this sufficiently well it gives as a special case reasonable algorithms for determining the $\pi_*^s(S^0)$ for example.) For certain special types of complexes this has been done, though, and here we follow J.F. Adams, [2], and assume that X is, in fact, an ordered simplicial complex with the 1-skeleton collapsed to the base point $*$. This is actually only a weak restriction on X since we have

LEMMA 5.5.1. *Let X be a connected, locally finite simplicial complex with $\pi_1(X) = 0$, then there is a finite 2-dimensional subcomplex, $C_2 \subset X$, containing the entire 1-skeleton, $sk_1(X)$, with $\tilde{H}_*(C_2, \mathbf{Z}) \equiv 0$ and the quotient map $p : X \longrightarrow X/C_2$ is a homotopy equivalence.*

PROOF. $sk_1(X)$ has the homotopy type of a wedge of circles, $\bigvee_1^m S^1$, and there is a cofibering

$$X \longrightarrow X/sk_1(X) \xrightarrow{w} \bigvee_1^m S^2.$$

Since $\tilde{H}_i(X) = 0$ for $i = 0, 1$, the homology long exact sequence for the cofibering implies that

$$w_* : H_2\big(X/sk_1(X); \mathbf{Z}\big) \longrightarrow H_*\big(\Sigma sk_1(X); \mathbf{Z}\big)$$

is onto. On the other hand, a basis for $H_2(X/sk_1(X); \mathbf{Z})$ can be chosen which consists only of the Hurewicz images of the fundamental classes of embeddings, $\phi(\sigma^2/\partial\sigma^2) \longrightarrow X/sk_1(X)$, where the σ^2 run over a subset of the 2-simplexes of X. Consequently the same is true for $im(w_*)$. That is to say, there are m 2-simplexes $\sigma_1^2, \ldots, \sigma_m^2$ in $sk_2(X)$ so that

$$sk_1(X) \cup \bigcup_1^m \sigma_j^2 = C_2$$

has trivial reduced homology. Now, $\pi_1(C_2)$ need not be zero, so C_2 need not be contractible. However, in the cofibering

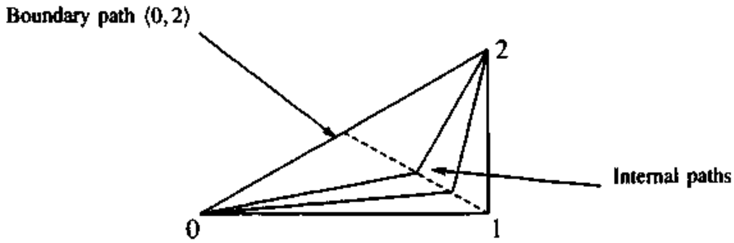
$$C_2 \longrightarrow X \xrightarrow{p} X/C_2$$

we must have $\pi_1(X/C_2) = 0$, since C_2 contains the entire 1-skeleton of X. Consequently, p is a homotopy equivalence. □

The simplicial structure of X/C_2 is sufficiently rigid to allow us to systematically compute the boundary map ∂ in $AH(X/C_2)$. Similarly, we will consider the problem when X is given as a cubical complex.

In both cases the idea is to make an explicit model consisting of paths from an initial to a final vertex in the simplex or the cube, via acyclic models types of techniques to describe each generating $n-1$ cell in $AH(X)$ for every cell $e^n_\alpha \subset X$.

To begin consider the ordered triangle $\langle 0,1,2\rangle$



(5.6)

Fig. 1

The paths we construct will start at the vertex $\langle 0\rangle$ and end at $\langle 2\rangle$. To begin we consider paths along the boundary. There are two ways of moving along edges from 0 to 2. The first path, which we denote $\langle 0,1\rangle * \langle 1,2\rangle$, moves linearly along the bottom edge from 0 to 1 and then from 1 to 2. The second path, which we denote $\langle 0,2\rangle$ moves linearly along the hypotenuse from 0 to 2. Now let l be the line connecting 1 to the midpoint[4] of the path from 0 to 2. For each $t \in l$ there is the straight line path from 0 to t to 2 and this gives a one parameter family of paths from 0 to 2 connecting the two edge paths, $\langle 0,1\rangle * \langle 1,2\rangle$ and $\langle 0,2\rangle$. If we order the vertices of X, then each 2-simplex is linearly identified with $\langle 0,1,2\rangle$ and we can use this identification to associate to each 2-simplex σ^2 a 1-simplex in the path space on σ^2.

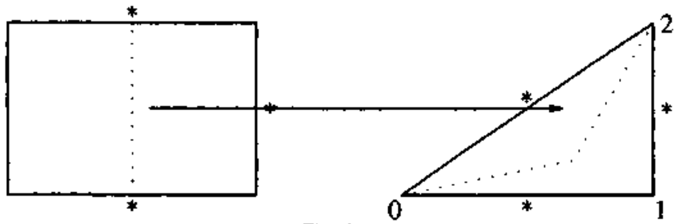
Fig. 2

Moreover, since, by assumption, X has only a single vertex, these paths actually are all in $\Omega^M X$, and we have constructed a correspondence from the 2-cells of X to 1-cells in $\Omega^M X$. In the Adams–Hilton construction, what was important to show that the cells there were "correct", was that the evaluation map

$$eval : \left(I \times e^{n-1}, \partial I \times e^{n-1}\right) \longrightarrow \left(sk_n(X), sk_{n-1}(X)\right)$$

[4] The notation is chosen to emphasize that this path is actually the composition of two paths, the first from 0 to 1 and the second from 1 to 2. Its length is 2, so we are naturally working here in the space of Moore loops.

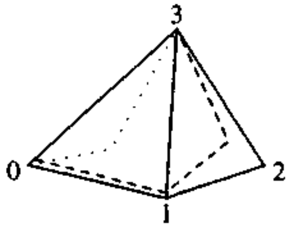
have degree one to the corresponding cell in X. In the case here the evaluation map is explicit and evidently of degree one.



(5.7)

Fig. 3

To continue we need to study the analogous construction for higher dimensional simplexes. Thus, consider the tetrahedron $\langle 0,1,2,3\rangle$



(5.8)

Fig. 4

To begin, we know how to fill in paths along the two faces $\langle 0,1,3\rangle$ and $\langle 0,2,3\rangle$ containing both vertices 0 and 3 by using the previous construction for σ^2. Moreover, along their intersection $\langle 0,3\rangle$, the paths agree. On the face $\langle 1,2,3\rangle$ we know how to construct paths from 1 to 3, and to construct paths from 0 to 3 we simply compose with the path $\langle 0,1\rangle$! Thus, here the paths are of the form $\langle 0,1\rangle * \varphi_t$. Moreover, the boundary paths are $\langle 0,1\rangle * \langle 1,3\rangle$ which is also a boundary path for the paths in $\langle 0,1,3\rangle$, and $\langle 0,1\rangle * \langle 1,2\rangle * \langle 2,3\rangle$.

Also, the paths in $\langle 0,2,3\rangle$ have boundary paths $\langle 0,3\rangle$ which is already accounted for, and $\langle 0,2\rangle * \langle 2,3\rangle$. Note that this implies that we should fill in the paths along the final face $\langle 0,1,2\rangle$ so that they have the form $\varphi_t * \langle 2,3\rangle$.

Thus we have extended the construction above to fill in paths from 0 to 3 along all four of the faces of the tetrahedron using four intervals connected together in the form of the boundary of the square, and, since the map is degree one on each face, it clearly gives a degree one map, on evaluation

$$eval : \left(I \times \partial I^2, \partial I \times \partial I^2\right) \longrightarrow \left(\partial\sigma^3, \{0,3\}\right).$$

Now, contracting σ^3 to $\langle 0,3\rangle$ extends our construction of paths to a three-dimensional analog of the previous construction,

$$h_2 : \left(I^2, \partial I^2\right) \longrightarrow \left(E_{0,3}^{\sigma^3}, E_{0,3}^{\partial\sigma^3}\right)$$

which is again degree one on evaluation,

$$eval : \left(I^3, \partial\right) \longrightarrow \left(\sigma^3, \partial\right)$$

by filling in the following diagram

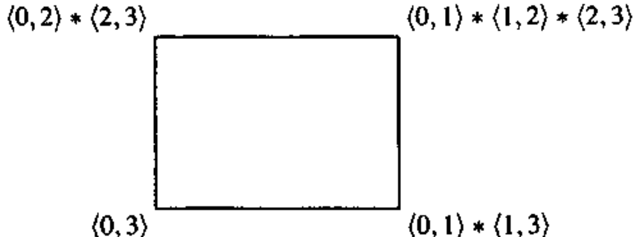


Fig. 5

(5.9)

With these preliminary constructions in mind, we can describe the general case.

THEOREM 5.5.1 (Adams). *For each positive integer n, $n = 2, 3, \ldots$, there is a map*

$$p_n : I^{n-1} \longrightarrow E_{0,n}^{\sigma^n}$$

so that $p_n | \partial I^{n-1}$ has image contained in $E_{0,n}^{\partial\sigma^n}$ and the evaluation map

$$eval(p_n) : \left(I^n, \partial I^n\right) \longrightarrow \left(\sigma^n, \partial\sigma^n\right)$$

has degree one. Moreover, the p_n fit together in the sense that p_n restricted to the boundary consists of maps of the form $p_j * p_{n-j-1}$ where $*$ represents juxtaposition of paths.

PROOF. The proof is by induction. To begin, we assume the p_j are defined for $j \leqslant n-1$, and, since we have already constructed the maps for $n = 1, 2, 3$, we might as well assume $n \geqslant 4$.

Each point in $\partial(I^{n-1})$ can be regarded as an n-tuple $(t_1, \ldots, t_n)$ where at least one of the t_i's is either zero or one. We can assign to every vertex the edge path in $\partial\sigma^n$ from 0 to $n-1$ given by

$$\left\langle 0, , \ldots, i_1 \right\rangle * \left\langle i_1, \ldots, \widehat{k_1 - 1}, \widehat{k}_1, \ldots, i_2 \right\rangle * \cdots$$

where we have cut $\langle 0, \ldots, n\rangle$ at every i_j where $t_{i_j} = 0$ by inserting a $\ldots, i_j\rangle\langle i_j, \ldots$ and dropped the vertices corresponding to every $t_{k_j} = 1$. Once again, we can fill in this map over the faces of I^{n-1}, so that, over the face $J_{j_1,\ldots,j_s}^{i_1,\ldots,i_r}$ (where $t_{i_r} = 0$ while $t_{j_s} = 1$), we have $p_{l_1} * p_{l_2} * \cdots * p_{l_{r+1}}$.

To be precise, p_{l_t} maps to the paths on the face $\langle i_t, \ldots, \widehat{j_s}, \ldots, i_{t+1}\rangle \subset \sigma^n$, from i_t to i_{t+1}, where the j_s are deleted from $\{i_t, i_t + 1, \ldots, i_{t+1} - 1, t_{t+1}\}$ for each j_s with $i_t < j_s < i_{t+1}$. Clearly, this definition is consistent and defines p_n on ∂I^{n-1}. Moreover, evaluation is degree one on each of the $n-1$ faces J_l, $1 \leqslant l \leqslant n-1$, as well as on the faces J^1 and J^{n-1} and their images lie in distinct $(n-1)$ faces of σ^n, by the inductive assumptions and the construction. Also, the images of the remaining faces all lie in the $(n-2)$ skeleton of σ^n. Hence, it follows that evaluation has degree one on ∂I^{n-1}, and hence, from the 5-lemma, also has degree one for p_n. This completes the inductive step. □

Thus, we see that the boundary of the cell I^{n-1} corresponding to the simplex $\sigma^n \subset X$ is a union of products of lower dimensional cells under the loop sum operation in the Moore loop space, as well as a piece corresponding to the original boundary of σ^n. Formally, on the complex

$$T(X) = T\left(e_1^1, \ldots, e_r^1, \ldots, e_\alpha^{n-1}, \ldots\right),$$

remembering that $\partial(\sigma^n) = \sum(-1)^i F_i(\sigma^n)$, we have

$$\partial\left(e^{n-1}\right) = \sum(-1)^i e\left(F_i(\sigma^n)\right) + \sum_{j=2}^{n-2} e\left(f_j(\sigma^n)\right) e\left(l_j(\sigma^n)\right). \tag{5.10}$$

Here

- $f_j(\sigma^n) = \langle 0, \ldots, j\rangle$ is the map on the front j face,
- $l_j(\sigma^n) = \langle j, \ldots, n\rangle$ is the map on the back $n-j$ face.

The second term in (5.7) formally corresponds to the the Alexander diagonal approximation, A, which is given on simplices as

$$A : \sigma^n \longrightarrow \sum_{j=0}^{n} f_j(\sigma^n) \otimes l_j(\sigma^n)$$

and induces a chain map on simplicial complexes and singular complexes:

$$A_* : C_*(X) \longrightarrow C_*(X) \otimes C_*(X),$$

so $A\partial = \partial^{\otimes} A$. It is also easy to check

PROPOSITION 5.5.1. *The Alexander chain map is coassociative. That is,*

$$(A \times 1) \circ A = (1 \times A) \circ A.$$

Moreover, A is chain homotopic to the diagonal map

$$C_*(X) \xrightarrow{\Delta} C_*(X) \otimes C_*(X)$$

in the singular chain complex of X.

Dualizing, we have

PROPOSITION 5.5.2. *The dual Alexander map*

$$A^* : C^*(X) \otimes C^*(X) \longrightarrow C^*(X)$$

is an associative cochain map. Moreover, the induced pairing on cohomology $H^*(X;\mathbf{F})\otimes H^*(X;\mathbf{F}) \to H^*(X;\mathbf{F})$ *is just the cup product.*

Summarizing, we have identified the second summand in 5.7,

$$\sum_{j=2}^{n-2} e\big(f_j(\sigma^n)\big)e\big(l_j(\sigma^n)\big), \tag{5.11}$$

and at the cohomology level, it is directly tied in to the cup product. In particular, if the cup product structure for $H^*(X;\mathbf{F})$ is nontrivial, then this piece *must* be present.

PROPOSITION 5.5.3. *If X is a suspension, say $X = \Sigma Z$, then all cup products in $H^*(X;\mathbf{A})$ are zero.*

PROOF. Consider the homotopy of the diagonal map

$$\Sigma X \xrightarrow{\Delta} \Sigma X \times \Sigma X$$

defined by

$$H\big(\tau, \{t,x\}\big) = \big(\{\overline{(\tau+1)t}, x\}, \{\underline{(1+\tau)t-\tau}, x\}\big)$$

where $\underline{m} = m$ if $m \geqslant 0$, and is 0 if $m \leqslant 0$, while $\overline{m} = m$ if $0 \leqslant m \leqslant 1$ and is 1 if $m \geqslant 1$. When $\tau = 1$ it has image contained in $\Sigma X \vee \Sigma X \subset \Sigma X \times \Sigma X$, and the result follows. □

This partially explains why we can replace the general Adams–Hilton model by the James model for ΩX in case X is a suspension.

To actually compute we note that given a j-cell σ^j in X we have constructed a $j-1$ cell $e(\sigma^j)$ in $AH(X)$. We denote the dual cochain by $|\sigma^j|$. Thus, given a product cell

$$I^{j_1-1} \times I^{j_2-1} \times \cdots \times I^{j_l-1}$$

in $AH(X)$ we label the dual cochain, which is of dimension $\sum j_i - l$ by

$$|\sigma^{j_1}|\sigma^{j_2}|\cdots|\sigma^{j_l}|.$$

Thus, dualizing 5.7 we can write the coboundary map

$$\begin{aligned} \delta\big(|\sigma^{j_1}|\cdots|\sigma^{j_l}|\big) &= \sum(-1)^t|\sigma^{j_1}|\cdots|\delta(\sigma^{j_t})|\cdots|\sigma^{j_l}| \\ &= \sum(-1)^s|\sigma^{j_1}|\cdots|\sigma^{j_t}\cup\sigma^{j_t}|\cdots|\sigma^{j_l}|. \end{aligned} \tag{5.12}$$

Here

- $|\delta(\sigma^{j_t})|$ is shorthand for the coboundary on $|\sigma^{j_t}|$, the dual of $e(\sigma^{j_t})$,
- $|\sigma^{j_t} \cup \sigma^{j_t}|$ is shorthand for the cup product on the obvious dual co-cells,
- the sign in the second sum is given by setting s equal to the number of bars plus the sum of the dimension of the cells that are passed over.

This is just the *Bar construction* on the associative chain algebra $C^{\#}(X)$! We can filter $C^{\#}(AH(X))$ by the number of bars describing a (dual) cell and (5.12) shows that $\delta\mathcal{F}_i(AH(X)) \subset \mathcal{F}_i(AH(X))$. Consequently, we obtain a spectral sequence for computing $H^*(\Omega X; \mathbf{F})$, with E_2-term

$$Ext_{H^*(X;\mathbf{F})}(\mathbf{F}, \mathbf{F}), \tag{5.13}$$

where the ring structure on $H^*(X; \mathbf{F})$ is obtain from the cup product. As an example $H^*(\mathbf{CP}^\infty; \mathbf{F}) = \mathbf{F}[b]$, a polynomial algebra on a two dimensional generator, and

$$Ext_{\mathbf{F}[\mathbf{p}]}(\mathbf{F}, \mathbf{F}) = E(|b|),$$

the exterior algebra on a one dimensional generator. Hence, in this case the spectral sequence collapses. But the spectral sequence does not always collapse, and the higher differentials measure the difference between the information given by the chain level Alexander diagonal approximation and the cup product.

REMARK. One other reason for the close connection between the diagonal map and ΩX is the fact that the fiber of the Serre fibration

$$X \xrightarrow{\Delta} X \times X$$

is ΩX.

6. The structure of second loop spaces

In Section 5 we showed that for a connected CW complex with no one cells one may produce a CW complex, with cell complex given as the free monoid on generating cells, each in one dimension less than the corresponding cell of X, which is homotopy equivalent to ΩX. To go further one should study similar models for double loop spaces, and more generally for iterated loop spaces.

In principle this is direct. Assume X has no i-cells for $1 \leqslant i \leqslant n$ then we can iterate the Adams–Hilton construction of Section 5 and obtain a cell complex which represents $\Omega^n X$. However, the question of determining the boundaries of the cells is very difficult as we already saw with Adams' solution of the problem in the special case that X is a simplicial complex with $sk_1(X)$ collapsed to a point. It is possible to extend Adams' analysis to $\Omega^2 X$, but as we will see there will be severe difficulties with extending it to higher loop spaces except in the case where $X = \Sigma^n Y$.

6.1. *Homotopy commutativity in second loop spaces*

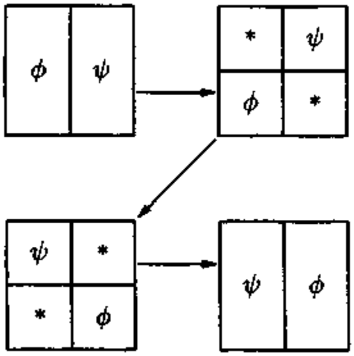
Given a based CW complex X, elements in $\Omega^2 X$ can be thought of as maps from I^2 to X, so that $\partial(I^2)$ is sent to the base point. There are two notions of loop sum in $\Omega^2 X$; we consider the one coming from the loop structure in the first variable, and call it μ; thus

$$\mu(\phi, \psi)(s,t) = \begin{cases} \phi(2s,t) & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ \psi(2s-1,t) & \text{if } \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

It is typically shown in first year topology that $\pi_2(X)$ is abelian for any complex X. From the usual adjointness considerations this is equivalent to the assertion that $\pi_0(\Omega^2 X)$ is abelian. This suggests that μ itself should, in some sense, be commutative, at least up to homotopy. The formal version of this statement is that if we let

$$T : \Omega^2 X \times \Omega^2 X \longrightarrow \Omega^2 X \times \Omega^2 X$$

be the twist map, $T(\phi_1, \phi_2) = (\phi_2, \phi_1)$, then $\mu \circ T$ is homotopic to μ. The homotopy, $\mathcal{H}$, is given by the following figure.



(6.1)

Thus, two fold loop spaces are "homotopy commutative". One might now guess that $\Omega^2 \Sigma^2 X$ should be homotopy equivalent to the free commutative monoid on X, as $\Omega \Sigma X$ is equivalent to the free monoid on X. This naive guess fails, however, as one can see from the Dold–Thom theorem, which asserts that if $SP^\infty(X)$ denotes the infinite symmetric product on X (i.e. the free abelian monoid), then $\pi_*(SP^\infty(X)) \cong H_*(X)$. Thus, $\pi_*(SP^\infty(S^2)) = 0$ for $* > 2$, while $\pi_3(\Omega^2 \Sigma^2 S^2) = \pi_5(S^4) = \mathbf{Z}/2$, generated by the double suspension of the Hopf map $\eta : S^3 \longrightarrow S^2$.

It turns out that there are "degrees" of homotopy commutativity which must be encoded in our models, and that $\Omega^2 X$ is, in a sense, minimally homotopy commutative and $\Omega^k X$ becomes more and more highly homotopy commutative as k goes to infinity. But even within the second loop space there are levels of homotopy commutativity which must be distinguished. For example there are two ways of using homotopy commutativity to pass from $a * b * c$ to $c * b * a$. We have

$$a * b * c \mapsto b * a * c \mapsto b * c * a \mapsto c * b * a,$$
$$a * b * c \mapsto a * c * b \mapsto c * a * b \mapsto c * b * a$$

corresponding to the relation $(1,2)(2,3)(1,2) = (2,3)(1,2)(2,3) = (1,3)$ in the symmetric group S_3. Gluing together the three homotopies above give two maps $\psi, \Psi : [0,3] \times (\Omega^2 X)^3 \longrightarrow \Omega^2 X$ where

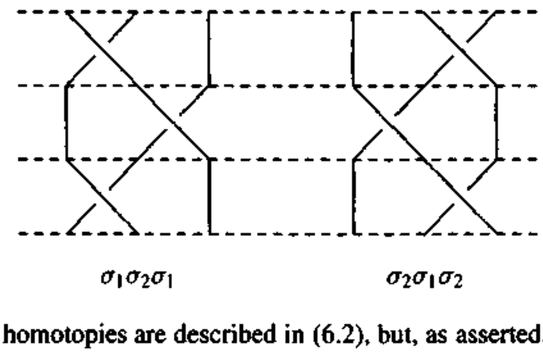
$$\psi\left(0 \times \left(\Omega^2 X\right)^3\right) = \Psi\left(0 \times \left(\Omega^2 X\right)^3\right), \quad \psi\left(3 \times \left(\Omega^2 X\right)^3\right) = \Psi\left(3 \times \left(\Omega^2 X\right)^3\right),$$

and hence a map $\Theta : D \times (\Omega^2 X)^3 \longrightarrow \Omega^2 X$ where D is the boundary of a *hexagon*, $C(2)$ and the map on each interval represents one of the homotopies.

LEMMA 6.1.1. *The map Θ may be filled in so as to give a map $A_2 : C(2) \times (\Omega^2 X)^3 \longrightarrow \Omega^2 X$ which agrees with Θ on $D \times (\Omega^2 X)^3$.*

PROOF. Note that $a * b * c$ is a map of I^2 to X with ∂I^2 mapping to $*$ and three smaller rectangles specified on which the map is, respectively a, b, and then c. What we did in the original homotopy of commutation was shrink these rectangles and move them past each other, then increase their size. So what we do is to shrink them even smaller and slide them past each other in an appropriate way so as to move from the first homotopy to the second. We can specify the motion by specifying the centers and sizes of the rectangles and then moving the centers.

The following diagram shows the movement of the respective centers in I^2 as we move from the $a * b * c$ to $c * b * a$ in the three stages indicated and in the two distinct manners indicated. The first is $\sigma_1\sigma_2\sigma_1$ and the second is $\sigma_2\sigma_1\sigma_2$ where σ_1 exchanges the first and second while σ_2 exchanges the second and third.



(6.2)

The two homotopies are described in (6.2), but, as asserted, (6.2) also makes it clear that the first can be deformed to the second without introducing any self intersections and without moving the points at the top or bottom of the two "braids". This deformation fills in the hexagon. □

In the next section we generalize this construction and extend the ideas of Section 5 to create a good model for the second loop space. Additionally, the point of view developed in the analysis here, in Section 7 becomes the key to developing good models for $\Omega^n \Sigma^n X$ for all n.

6.2. *The Zilchgon model for* $\Omega^2 X$

Adams replaced simplices, σ^n, by cubes, I^{n-1}, in building an explicit model for the Adams–Hilton construction of ΩX when X is a simplicial complex with its one skeleton collapsed to a point. It is natural to try to generalize this. Thus, suppose that Y is a *cubical* CW complex where the one skeleton has again been collapsed to a point. It is certainly possible to find combinatorial cells $C(n-1)$ which will replace each I^n in Y in building an explicit model for the Adams–Hilton construction. If this can be done in a sufficiently natural manner then, for X is a simplicial complex with $sk_2(X)$ collapsed to a point, this would give an explicit construction for $\Omega^2 X$. This, in fact turns out to be possible and we describe the construction now.

We begin by looking at the edge paths starting at $(0,\dots,0) \in I^n$ and ending at $(1,\dots,1)$. An edge has the form $(\varepsilon_1,\dots,\varepsilon_r,t,\varepsilon_{r+2},\dots,\varepsilon_n)$ where each ε_i is either a zero or a one. Then, we can specify the edge path by specifying which coordinates are moved in which order. So $E(1)E(3)E(2)$ for I^3 would mean the path which first moves the first coordinate, then goes from $(1,0,0)$ to $(1,0,1)$ by using the third coordinate, and finally goes from $(1,0,1)$ to $(1,1,1)$ using the second coordinate. It follows that these edge paths are indexed by the elements in the symmetric group S_3, and for I^n, by the symmetric group S_n. So we look for a polyhedron of dimension $n-1$ with vertices indexed by S_n to model paths in I^n.

We now introduce a family of combinatorial cells which do just this, the Zilchgons, (also called permutahedra by combinatorialists), $C(n)$. This will allow us to build explicit models for ΩX where X is a cubical complex with $sk_1(X) \sim *$ or $\Omega^2 X$ where X is a simplicial complex with $sk_2(X) \sim *$. But any attempt to continue this process will require many different combinatorial cells in each dimension $\geqslant 2$.

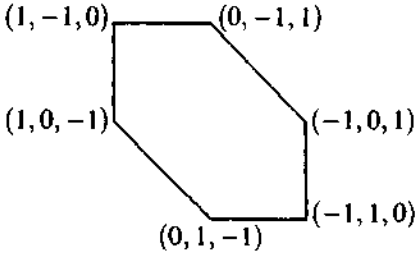
Let $e = (1,2,\dots,n) \in \mathbf{R}^n$ and let $C(n-1)$ be the convex hull of the translates of e by the usual permutation action of the symmetric group S_n on $\mathbf{R}^n$. Note that the convex hull spanned by a set S is the set of points

$$\left\{\sum t_i s_i \mid 0 \leqslant t_i,\ \sum t_i = 1,\ s_i \in S\right\}.$$

In particular $C(n-1) \subset A^{n-1}$ where A^{n-1} is the $(n-1)$ dimensional affine plane in $\mathbf{R}^n$ with equation

$$\sum_{i=1}^{n} x_i = n(n+1)/2.$$

EXAMPLE 6.1. $C(1)$ is the line segment from $(1,2)$ to $(2,1)$ in $\mathbf{R}^2$ while $C(2)$ is the convex hull spanned by the six points $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, and $(3,2,1)$, or projecting onto the plane through the origin parallel to the plane $x+y+z=6$, with coordinates $(-1,0,1)$, $(-1,1,0)$, $(0,-1,1)$, $(0,1,-1)$, $(1,-1,0)$ $(1,0,-1)$.



(6.3)

The space $C(2)$

It will turn out that $C(1)$ represents the homotopy of commutativity, while $C(2)$ represents the homotopy of $\sigma_1\sigma_2\sigma_1$ to $\sigma_2\sigma_1\sigma_2$ discussed in the last section. The higher dimensional $C(r)$'s will give all the possible ways, involving $r+1$ loops, of homotopy commuting the homotopies of commutation in the previous constructions involving fewer loops.

We now show that $C(n-1)$ is topologically a closed $(n-1)$ ball in $\mathbf{R}^{n-1}$ with boundary given as the union of products of lower dimensional $C(j)$'s.

LEMMA 6.2.1. *Let $\sigma \in S_n$ be the cycle $(1,2,3,\dots,n)$, then the n vectors $e, \sigma(e), \dots, \sigma^{n-1}(e)$ are linearly independent in A^{n-1} and consequently span an embedded $n-1$ dimensional simplex there.*

PROOF. It suffices to show that the $n-1$ vectors

$$\sigma^i(e) - e = (\underbrace{n-i,\dots,n-i}_{i \text{ times}}, \underbrace{-i,\dots,-i}_{n-i \text{ times}})$$

are linearly independent for $1 \leqslant i \leqslant n-1$. But this is clear by looking at the last $n-1$ columns of the array. $\square$

COROLLARY 6.2.1. $C(n-1)$ *is topologically a closed $n-1$ disk D^{n-1} with boundary S^{n-2}.*

PROOF. $C(n-1)$ is certainly closed and convex. It is also compact since it is contained in the cube $[0,n]^n$. The lemma above shows that it has a nonempty interior, so, by a standard result it is topologically a closed disk. $\square$

Actually more is true. $C(n-1)$ is a polyhedron with faces determined as the convex hulls of subsets of the points $\{\sigma(1,\dots,n) \mid \sigma \in S_n\}$. (This is a general property of the convex hulls of finite point sets.) We now determine these faces and show that they are closely connected with certain subgroups of $\mathcal{S}_n$.

LEMMA 6.2.2. *Let $H_r = \mathcal{S}_r \times \mathcal{S}_{n-r} \subset \mathcal{S}_n$, $1 \leqslant r \leqslant n-1$, be the subgroup preserving the first r and the last $n-r$ coordinates. Then the convex hull of the points $\sigma(e)$, $\sigma \in H_r$ is an $n-2$ dimensional face of $C(n-1)$ and, as a polyhedron, is isomorphic to $C(r-1) \times C(n-r-1)$.*

PROOF. Consider the map

$$p_r : \mathbf{R}^n \longrightarrow \mathbf{R}^+, \quad p_r(h) = \sum_1^r h_j,$$

where h_j is the j-th coordinate of h. Then for every point h of

$$C(n-1)\ p_r(h) \geqslant \frac{r(r+1)}{2}.$$

Moreover, equality occurs if and only if h is contained in the convex hull generated by the points $\sigma(e)$, $\sigma \in H_r$. It follows that this polyhedron is contained in the topological boundary of $C(n-1)$. Finally, as the two subgroups $\mathcal{S}_r$ and $\mathcal{S}_{n-r}$ act independently and on disjoint sets of coordinates the remainder of the lemma is clear. □

Note that $e = (1,2,3,\dots,n)$ is the intersection of $C(n-1)$ and the hyperplanes, $K_r = \{h \mid p_r(h) = r(r+1)/2\}$,

$$e = C(n-1) \cap K_1 \cap K_2 \cap \cdots \cap K_{n-1}$$

and, since faces of faces are faces, e is a vertex of $C(n-1)$. All the vertices of $C(n-1)$ are contained among the elements $\sigma(e)$, $\sigma \in \mathcal{S}_n$, since $C(n-1)$ is the convex hull spanned by the points $\sigma(e)$. But the symmetric group, $\mathcal{S}_n$, acts as a group of transformations on $C(n-1)$, taking faces to faces. It follows that the vertices of $C(n-1)$ are in 1–1 correspondence with the elements of $\mathcal{S}_n$ and are precisely the vectors $\sigma(e)$.

Similarly, for each r with $1 \leqslant r \leqslant n-1$ we have distinct faces of $C(n-1)$ corresponding to the cosets of $\mathcal{S}_r \times \mathcal{S}_{n-r}$ in $\mathcal{S}_n$. We now describe coset representatives for the cosets of $\mathcal{S}_r \times \mathcal{S}_{n-r} \subset \mathcal{S}_n$, which thus label the $(n-2)$ faces of $C(n-1)$ which we have found so far.

Let $(j_1, j_2, \dots, j_r)$, $j_i \geqslant 1$, $\sum j_i = n$, be an ordered partition of n. Define

$$shuff(j_1, j_2, \dots, j_r)$$

as the set of $\sigma \in \mathcal{S}_n$ so that $\sigma(i) < \sigma(j)$ whenever i and j belong to the same block in the partition, i.e. when there is a k so that

$$\sum_1^k j_s < i < j \leqslant \sum_1^{k+1} j_s, \quad 0 \leqslant k < r.$$

When $r = 2$ this corresponds to an ordinary shuffle of a deck of cards and likewise gives representatives for the cosets of $S_{j_1} \times S_{n-j_1}$ in S_n. For larger r it corresponds to breaking the deck into r pieces and then successively shuffling them together, and gives coset representatives for the cosets of $S_{j_1} \times \cdots \times S_{j_r}$ in S_n.

We note the straightforward but important

LEMMA 6.2.3. *Let $s \in shuff(j_1, j_2)$ and $s' \in shuff(j_1 + j_2, j_3, \ldots, j_r)$. Then the composite $s's \in shuff(j_1, j_2, j_3, \ldots, j_r)$ where $s \in S_{j_1+j_2}$ and $S_{j_1+j_2}$ is embedded in S_n with*

$$n = \sum_1^r j_s$$

as the subgroup fixing the last $n - (j_1 + j_2)$ points.

LEMMA 6.2.4. *The collection of all the $n - 2$ dimensional faces of $C(n-1)$ consists of those elements enumerated above in 1–1 correspondence with the union of the $(r, n-r)$ shuffles, $1 \leqslant r \leqslant n - 1$.*

PROOF. The proof is by induction. Note to begin with that the interiors of the $(n-2)$ dimensional faces in the lemma are disjoint since they lie in distinct hyperplanes. Now consider an $(n-3)$-face of one of these subcomplexes. By the inductive assumption it has the form $\sigma(C(l-1)\times C(r-l-1))\times C(n-r-1)$ or $C(r-1)\times\sigma(C(s-1)\times C(n-r-s-1))$ since $\partial A \times B = (\partial A) \times B \sqcup A \times (\partial B)$.

Assume the face is of the first type. It can be uniquely written as the face of an appropriate shuffle of $C(l-1) \times C(n-l-1)$, and in the second case it is uniquely the face of an appropriate shuffle of $C(r+s-1) \times C(n-r-s-1)$. Thus, each $n-3$ face is incident to precisely two of the $n-2$ dimensional faces listed and it follows that the sum of these faces forms a closed cycle mod(2). But this implies that we have enumerated all the $n-2$ dimensional faces and completes the proof. □

COROLLARY 6.2.2. *The complete set of faces of $C(n-1)$ is indexed by ordered pairs consisting of first an ordered partition of n $p = (j_1, \ldots, j_w)$ $(\sum j_i = n)$ and a $(j_1, \ldots, j_w)$ shuffle s. Such a face has dimension $n - w$.*

EXAMPLE 6.2. $C(3)$ has as its faces 4 copies of $C(2) \times 1$, 4 copies of $1 \times C(2)$ and 6 copies of $C(1) \times C(1)$. It has 36 edges corresponding to 12 copies each of $1 \times 1 \times C(1)$, $1 \times C(1) \times 1$, and $C(1) \times 1 \times 1$. Finally, it has 24 vertices. It can be realized by taking the tetrahedron, T, and cutting out 6 small tetrahedra about the six vertices of T.

REMARK. The lowest dimensional faces of $C(n)$ which do not have a fixed coordinate, i.e. are not translates of a face corresponding to a partition with one or more 1's in it, such as $\mathcal{S}_{n-3} \times \mathcal{S}_1 \times \mathcal{S}_2$, correspond to

$$\begin{cases} \underbrace{\mathcal{S}_2 \times \cdots \times \mathcal{S}_2}_{n/2 \text{ times}} & \text{if } n \text{ is even} \\ \underbrace{\mathcal{S}_2 \times \cdots \times \mathcal{S}_2}_{[n/2] \text{ times}} \times \mathcal{S}_3 & \text{if } n \text{ is odd.} \end{cases}$$

Hence they have the form $I^{n/2}$ or $I^{[n/2]} \times C(2)$. This leads to "stabilization" results in constructions which use Zilchgons.

Let $b_n \in C(n-1)$ be the barycenter,

$$b_n = \underbrace{\left(\frac{n+1}{2}, \ldots, \frac{n+1}{2}\right)}_{n \text{ times}}.$$

Then, for $h \in C(n-1)$, $h \neq b_n$, there are unique points $v \in \partial C(n-1)$, $t \in [0,1)$, so that $h = tb_n + (1-t)v$. Suppose that a map

$$\tilde{\phi} : \partial\big(C(n-1)\big) \longrightarrow E_{\mathbf{0},\mathbf{1}}^{\partial I^n}$$

is defined so that the image consists of linearly parameterized, piecewise linear paths. Then $\tilde{\phi}$ can be extended to $\phi : C(n-1) \to E_{\mathbf{0},\mathbf{1}}^{I^n}$ by the rule

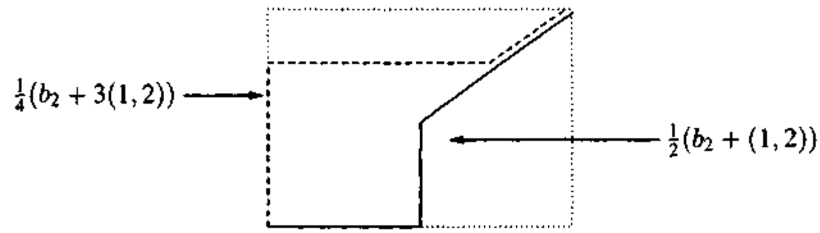
$$\begin{aligned} &\phi\big(tb_n + (1-t)v\big)(\tau) \\ &\quad = \begin{cases} (1-t)v(\frac{\tau}{1-t}) & \tau < (1-t)l(\phi(v)), \\ (\tau - (1-t)(l(\phi(v)) - 1))\mathbf{1} & (1-t)l(\phi(v)) \leqslant \tau \\ & \text{and } \tau \leqslant t + (1-t)l(\phi(v)), \\ \mathbf{1} & \tau \geqslant t + (1-t)l(\phi(v)). \end{cases} \end{aligned} \tag{6.4}$$

Note that $l(\phi((1-t)v + tb_n) = t + (1-t)l(\phi(v))$, and that the path is again linearly parameterized and piecewise linear. Indeed, it is the original path, but in the smaller cube, $[0,(1-t)]^n$, and then the diagonal path from the diagonal point $(1-t)^n$ to 1^n.

Now, let us suppose that $\phi_j : C(j-1) \longrightarrow E_{\mathbf{0},\mathbf{1}}^{I^j}$ is defined for all $j < n$. We define $\tilde{\phi}_n : \partial C(n-1) \to E_{\mathbf{0},\mathbf{1}}^{\partial I^n}$ by

$$\tilde{\phi}_n |\, \sigma\big(C(j_1-1) \times \cdots \times C(j_r-1)\big) \longrightarrow \sigma\left(\phi_{j_1} * \cdots * \phi_{j_r}\right) \tag{6.5}$$

where $*$ denotes juxtaposition of paths and $\sigma \in shuff(j_1, \ldots, j_r)$.



Sample paths in I^2

These two steps combine to define $\phi_n : C(n-1) \longrightarrow E^{I^n}_{0,1}$ for all $n \geqslant 1$ so that

(i) $eval(\phi_n) : (I \times C(n-1), \partial(I \times C(n-1))) \longrightarrow (I^n, \partial I^n)$ has degree one.

(ii) $\phi_n \mid \partial C(n-1)$ consists of two parts, the first, on the cells

$$shuff(1, n-1)C(n-2), \quad shuff(n-1, 1)C(n-2),$$

which corresponds to ∂I^n, and the second, on the

$$shuff(r, n-r)(C(r-1) \times C(n-r-1)), \quad 2 \leqslant r \leqslant n-2,$$

which corresponds to

$$\Delta(I^n) = \sum_{\sigma \in shuff(r,n-r)} \sigma I^r \times I^{n-r},$$

the usual chain approximation to the diagonal on I^n,

(iii) The paths in $\phi_n(C(n-1))$ are piecewise linear, and linearly parameterized, and have the property that over each linear segment there is a subset of $\mathcal{W} = \{1, 2, \ldots, n\}$ and the points of the segment have the form $(\varepsilon_1, \ldots, t_{w_1}, \ldots, t_{w_2}, \ldots)$. More precisely, the i^{th} coordinate is either 0 or 1 if $i \notin \mathcal{W}$, and is t if $i \in \mathcal{W}$.

This allows us to iterate the Ω construction, as promised to construct $\Omega^2 X$ when X is a simplicial complex with $sk_2(X)$ collapsed to a point.

REMARK. This was the original motivation of the second author when, in 1964, he first constructed the $C(n)$'s. When he told W. Browder about the construction, Browder suggested that it might be possible to modify it to study $\Omega^n \Sigma^n X$ since there are huge numbers of "cubes" in $J(\Sigma^{n-1} X)$, $n \geqslant 2$. (See the discussion in the next section.)

In order to push this suggestion through, the second author had to introduce degeneracies into the $C(n)$'s and construct systematic methods of reparameterizing paths to account for the effects of the *base point identifications* introduced in the James model.

In the writeup of these results in [24] only the construction of $\Omega^n\Sigma^n X$ was discussed however, and in the interim several students have written theses pointing out the connection with $\Omega^2 X$.

6.3. *The degeneracy maps for the Zilchgon models*

We now describe the degeneracy maps $d_i : C(n-1) \longrightarrow C(n-2)$. First, there are "degeneracy" maps for the symmetric groups, $d_i : S_n \longrightarrow S_{n-1}$, $1 \leqslant i \leqslant n$ defined by

$$d_i(\sigma)(j) = \begin{cases} \sigma(j) & \text{if } j < \sigma^{-1}(i),\ \sigma(j) < i, \\ \sigma(j+1) & \text{if } j \geqslant \sigma^{-1}(i),\ \sigma(j) < i, \\ \sigma(j) - 1 & \text{if } j < \sigma^{-1}(i),\ \sigma(j) > i, \\ \sigma(j+1) - 1 & \text{if } j \geqslant \sigma^{-1}(i),\ \sigma(j+1) > i. \end{cases} \tag{6.6}$$

If one writes σ as the array

$$\begin{pmatrix} 1 & 2 & \cdots & \sigma^{-1}(i) & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & i & \cdots & \sigma(n) \end{pmatrix}$$

then d_i deletes the $\sigma^{-1}(i)$ column and reindexes to get an element in S_{n-1}.

These correspond to the maps

$$p_i : I^n \longrightarrow I^{n-1}, \quad p_i(t_1, \ldots, t_n) = (t_1, \ldots, \widehat{t_i}, t_{i+1}, \ldots, t_n),$$

that deletes the i-th coordinate. The image of an edge path under p_i is an edge path in I^{n-1}, *at least as a point set*, though the parameterization is changed, since, when we come to what should have been movement along the i-th coordinate the path stays fixed in the image.

Note that if $\sigma \in shuff(j_1, \ldots, j_r)$ and i belongs to the block j_k, then $d_i(\sigma) \in shuff(j_1, \ldots, j_k - 1, \ldots, j_r)$, where, if $j_k = 1$, we simply delete that block. It follows that if $\sigma_1(e), \ldots, \sigma_r(e)$ are contained in a face, $\sigma(C(j_1 - 1) \times \cdots \times C(j_r - 1))$, of $C(n-1)$ then $d_i(\sigma_1(e)), \ldots, d_i(\sigma_r(e))$ are contained in the face

$$d_i(\sigma)\big(C(j_1 - 1) \times \cdots \times C(j_k - 2) \times \cdots \times C(j_r - 1)\big).$$

Now, by mapping b_m's to $b_{m'}$'s and extending linearly, we have geometric maps

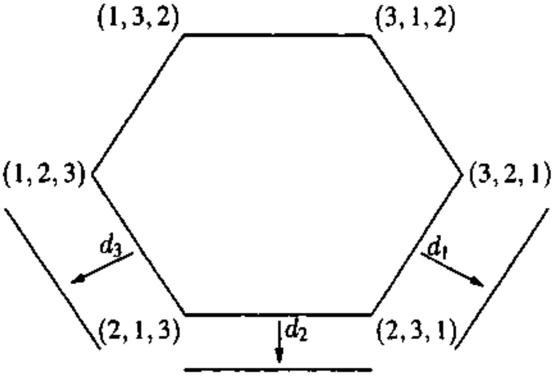
$$d_i : C(n-1) \longrightarrow C(n-2), \quad 1 \leqslant i \leqslant n,$$

which satisfy the usual condition for degeneracies:

$$d_i d_j = \begin{cases} d_j d_{i-1} & \text{if } i > j, \\ d_j d_i & \text{if } j \geqslant i. \end{cases}$$

When we compose with the ϕ_n, and use p_i, collapsing the i-th coordinate, as the corresponding degeneracy on I^n, we obtain that $\phi_{n-1} d_i(w)$ is a path which has as its image *the same point set* as $p_i\phi_n(w)$, but the parameterizations are different. However, that is easily handled since we have the following result.

LEMMA 6.3.1. *The space of nondecreasing maps of the unit interval onto itself is convex and so is the subspace of piecewise linear maps.*



(6.7)

The degeneracies for $C(2)$

6.4. *The Zilchgon models for iterated loop spaces of iterated suspensions*

To explain these models consider again the James model, $M(\Sigma X, 0)$,

$$M(\Sigma X, 0) = \coprod_{k=1}^{\infty} (\Sigma X)^k / \sim .$$

Since $\sim$ collapses the fat wedge,

$$W_n(\Sigma X) = \{(y_1, \dots, y_n) \in (\Sigma X)^n \mid y_i = * \text{ for some } i,\ 1 \leqslant i \leqslant n\},$$

onto $(\Sigma X)^{n-1}$ it follows that we have subspaces

$$M_n(\Sigma X, 0) = \coprod_{k=1}^{n} (\Sigma X)^k / \sim$$

and

$$\Omega^2 \Sigma^2 X \simeq \Omega^M M(\Sigma X, 0) = \lim_{n \to \infty} (\Omega M_n(\Sigma X, 0)).$$

On the other hand, $(\Sigma X)^n = I^n \times X^n / \mathcal{R}$ where $\mathcal{R}$ is a relation on $\partial(I^n) \times X^n \cup I^n \times W_n(X)$ which has the property that $((0, \dots, 0), (x_1, \dots, x_n))$ and $((1, \dots, 1), (x_1, \dots, x_n))$ are both identified with $(*, \dots, *)$.

From this we get a map

$$C(n-1) \times X^n \longrightarrow \Omega^2 \Sigma^2 X \tag{6.8}$$

by simply using the map $C(n-1) \longrightarrow E^{I^n}_{0,1}$ constructed in (6.4), (6.5). Inductively, we can assume that we have used this construction to build $\Omega^M M_s(\Sigma X, 0)$ for $s < n$, and we can use Lemma 6.3.1 together with the map in (6.8) to obtain the following model for $\Omega^M M_n(\Sigma X, 0)$:

$$\hat{J}_{2,n}(X) \simeq \Omega M_n(\Sigma X, 0) = P\big(\hat{J}_{2,n-1}(X), f, C(n-1) \times X^n\big) \tag{6.9}$$

where $P(-,-,-)$ is the Prolongation functor introduced in Definition 5.4.1 with the obvious modification that we are identifying a subspace of $C(n-1) \times X^n$, $\partial(C(n-1)) \times X^n \cup C(n-1) \times W_n(X)$, with a piece of $\hat{J}_{2,n-1}(X)$. The introduction here of models for the loop spaces $\Omega(M_s(\Sigma X, 0))$ is similar to some of Husseini's ideas in [18].

EXAMPLE 6.3. $\hat{J}_{2,1}(X) = M(X, 0)$, the James construction on X. Then $\hat{J}_{2,2}(X)$ is obtained by adjoining $I \times X^2$ where we have the identifications

$$\begin{aligned}
(0, x_1, x_2) &\sim (x_1, x_2) \in \hat{J}_{2,1}(X),\\
(1, x_1, x_2) &\sim (x_2, x_1) \in \hat{J}_{2,1}(X),\\
(t, x_1, *) &\sim x_1,\\
(t, *, x_2) &\sim x_2.
\end{aligned}$$

Thus we can think of $\hat{J}_{2,2}(X)$ as the free gadget which makes $M(X, 0)$ homotopy commutative.

$\hat{J}_{2,3}(X)$ is obtained from $\hat{J}_{2,2}(X)$ by adjoining $C(2) \times X^3$ where we make the identifications

$$\begin{aligned}
\big(\sigma\big(C(1) \times C(0)\big), (x_1, x_2, x_3)\big) &\sim \big\{C(1) \times \big(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}\big)\big\} * x_{\sigma^{-1}(3)},\\
\big(\sigma\big(C(0) \times C(1)\big), (x_1, x_2, x_3)\big) &\sim x_{\sigma^{-1}(1)} * \big\{C(1) \times \big(x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}\big)\big\},\\
\big(v, (*, x_2, x_3)\big) &\sim \big\{d_1(v), (x_2, x_3)\big\} \in C(1) \times X^2,
\end{aligned}$$

and similarly in the case when x_2 or x_3 is the basepoint $*$.

The general case should now be clear,

$$\Omega^2 \Sigma^2 X \simeq J_2(X) = \coprod_{k=1}^{\infty} C(k-1) \times X^k / \mathcal{R} \tag{6.10}$$

where $\mathcal{R}$ identifies points of $\partial(C(k-1)) \times X^k$ with products

$$C(j_1 - 1) \times X^{j_1} * \cdots * C(j_r - 1) \times X^{j_r}$$

where the coordinates are shuffled according to the shuffle associated with the face, and, on $W_k(X)$ makes identifications using the degeneracies on $C(k-1)$.

To go further, note that (6.10) allows us to write

$$J_2(X) = \lim_{n\to\infty} \left(J_{2,n}(X)\right)$$

where

$$J_{2,n}(X) = \coprod_{k=1}^{n} C(k-1) \times X^k/\mathcal{R}^5,$$

and

$$J_{2,n}(\Sigma X) = J_{2,n-1}(\Sigma X) \cup C(k-1) \times I^k \times X^k/\mathcal{R}'. \tag{6.11}$$

Once more we can use prolongation to iteratively build models for $\Omega^M(J_{2,n}(\Sigma X))$. Here the piece that is added at stage k is the product $C(k-1)\times C(k-1)\times X^k$. However, when we make identifications they are a bit more complex than those at the previous level: on a face in $\partial(C(k-1)) \times C(k-1) \times X^k$, we act on the second $C(k-1)$ and the X^k by the shuffle associated with the face, however, on a face $C(k-1) \times \partial(C(k-1)) \times X^k$, we must use degeneracies on the first $C(k-1)$ to project it onto an appropriate product $C(j_1-1) \times \cdots \times C(j_r-1)$. Finally, on $C(k-1) \times C(k-1) \times W_k(X)$ we use the appropriate $d_s \times d_s$ on $C(k-1) \times C(k-1)$.

At this stage we have seen all the steps needed to define the general construction

$$J_n(X) = \coprod_{k=1}^{\infty} \underbrace{C(k-1) \times \cdots \times C(k-1)}_{(n-1)\text{ times}} \times X^k/\mathcal{R} \tag{6.12}$$

which gives a model for $\Omega^n\Sigma^n X$ for any connected CW complex X.

REMARK. The explicitness of this model allows us to make chain level calculations to study the homology of $\Omega^n\Sigma^n X$. In particular, it is not hard to see that at each step passing from $\Omega^{n-k}\Sigma^n X$ to $\Omega^{n-k+1}\Sigma^n X$ the *cotor*-spectral sequence of 5.13 collapses and we obtain an effective method for determining $H_*(\Omega^n\Sigma^n X;\mathbf{F})$ for any $n > 0$ and any connected CW complex X. Further discussion of the actual results will be given in 7.3.

[5] It should be noted that the decompositions of $J_2(X)$ via the $J_{2,n}(X)$ and the $\hat{J}_{2,n}$ are quite distinct.

7. The structure of iterated loop spaces

Given any map into an iterated loop space, $f : X \longrightarrow \Omega^n(X)$, it factors through an n-fold loop map in the following way:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & \Omega^n Y \\ & \Omega^n \Sigma^n Y \arrow[ur, "\Omega^n Adj^n(f)"'] & \end{tikzcd}$$

where $i : X \longrightarrow \Omega^n \Sigma^n X$ is the usual inclusion:

$$i(x)(t_1, \ldots, t_n) = \{t_1, \ldots, t_n, x\} \in \Sigma^n X = I^n \times X/\sim .$$

Thus, the structure of the category of n-fold loop spaces and n-fold loop maps is closely reflected by the properties of the spaces $\Omega^n \Sigma^n X$, which play a role here analogous to the role of Eilenberg–MacLane spaces for ordinary spaces and maps.

It was conjectured in the 1950's that the homology of $\Omega^n \Sigma^n X$ should depend in a functorial way only on $H_*(X)$, and these homology classes will represent *homology operations* in the category. In this section we discuss the explicit construction of small models for the spaces $\Omega^n \Sigma^n X$ much as was done in Section 6, but here the models have better naturality properties which make aspects of the structure of $\Omega^n \Sigma^n X$ more transparent, in particular the proof of the conjecture above. They also allow a convenient passage to the limit, $Q(X) = \Omega^\infty \Sigma^\infty X$, under the natural inclusions

$$\begin{aligned} &i_n : \Omega^n \Sigma^n X \hookrightarrow \Omega^{n+1} \Sigma^{n+1} X, \\ &\big(i_n(f)(t_1, \ldots, t_n, t_{n+1}) = \{t_1, f(t_2, \ldots, t_{n+1})\}\big). \end{aligned}$$

An important feature of these models is that they permit the explicit description of $H_*(\Omega^k \Sigma^k X, \mathbf{F})$ as a functor of $H_*(X, \mathbf{F})$, where $\mathbf{F}$ is a field. This description is implicit in [24] but is carried out in detail in [14]. It turns out that if one considers the category of spaces which are k-fold loop spaces and maps which are k-fold loop maps, the $\mathbf{F}_p$-homology groups admit certain operations, some of which are stable and yield operations on infinite loop spaces (Dyer–Lashof operations) and some which are not (Browder operations), and the homology groups $H_*(\Omega^k \Sigma^k X; \mathbf{F}_p)$ can roughly be described as a free Hopf algebra on $H_*(X; \mathbf{F}_p)$ over an algebra involving these operations. The reader should see [14] for precise formulations and proofs of these results.

We have looked at Milgram's original Zilchgon model in Section 6.4. The models we will discuss now together with their various advantages are the May–Milgram configuration space model, Barratt–Eccles simplicial model for $Q(X)$, and J. Smith's unstable versions of the Barratt–Eccles construction.

7.1. *Boardman's little cubes*

In order to describe these models efficiently we will introduce some terminology.

DEFINITION 7.1.1. Let Γ^0 denote the category whose objects are the sets $\mathbf{n} = \{1, 2, \ldots, n\}$ for $n = 0, 1, 2 \ldots$, and where the morphisms from $\mathbf{m}$ to $\mathbf{n}$ are the injective maps.

For $n = 0$, this is understood to mean the empty set $\varnothing$, and it is understood that for every object $\mathbf{n}$ of Γ^0, there is a unique morphism from $\varnothing$ to $\mathbf{n}$ and that for every $n > 0$, the set of morphisms from $\mathbf{n}$ to $\varnothing$ is empty.

DEFINITION 7.1.2. An $\mathcal{O}$-space will be a contravariant functor from Γ^0 to the category of topological spaces.

This is the same as saying that an $\mathcal{O}$-space is a family of spaces X_n, $n \geqslant 0$, so that for each n, X_n is acted on by the symmetric group $\mathcal{S}_n$, and where for each n, we have maps $\delta_i : X_n \longrightarrow X_{n-1}$, $1 \leqslant i \leqslant n$, so that $\delta_i\delta_j = \delta_j\delta_{i+1}$ if $i \geqslant j$, and so that for any permutation $\sigma \in \mathcal{S}_{n+1}$, $\delta_i\sigma = \hat{\sigma}\delta_{\sigma^{-1}(i)}$, where $\hat{\sigma}$ is characterized by the equations in (6.6).

DEFINITION 7.1.3. If $\underline{\mathcal{C}} = \{C_n\}_{n\geqslant 0}$ is an $\mathcal{O}$-space, and X is a based CW complex, we define $\underline{\mathcal{C}}[X]$ to be

$$\coprod_{n\geqslant 0} C_n \times X^n / \simeq,$$

where $\simeq$ is the equivalence relation generated by relations of the form

$$\big(\sigma(e), x_1, \ldots, x_n\big) \simeq \big(e, x_{\sigma(1)}, \ldots, x_{\sigma(n)}\big)$$

and

$$(e, x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_n) \simeq (\delta_i e, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Next we need morphisms of $\mathcal{O}$-spaces.

DEFINITION 7.1.4. A morphism of $\mathcal{O}$-spaces is a natural transformation of functors on Γ^{op}.

A morphism $f : \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{C}}'$ induces a map $\underline{\mathcal{C}}[X] \xrightarrow{f[X]} \underline{\mathcal{C}}'[X]$.

DEFINITION 7.1.5. An $\mathcal{O}$-space $\underline{\mathcal{C}} = \{C_n\}_{n\geqslant 1}$ is said to be free if each C_n is a free $\mathcal{S}_n$-space.

$\underline{\mathcal{C}}[X]$ is also equipped with an increasing filtration $F_l\underline{\mathcal{C}}[X]$, where

$$F_l\underline{\mathcal{C}}[X] = \operatorname{image}\bigg(\coprod_{0\leqslant n\leqslant l} C_n \times X^n\bigg). \tag{7.1}$$

REMARK. The definition of an $\mathcal{O}$-space is just a part of J.P. May's definition of an operad [22]; we retain only what is needed to make the construction $\underline{C}[X]$.

EXAMPLE 7.1. Here are some examples.

(A) $\underline{C} = \{C_n\}_{n\geqslant 0}$, $C_n = *$ for all n, where all permutations and all δ_i's are identity maps. This $\mathcal{O}$-space is not free. If X is a based CW complex, $\underline{C}[X] \cong SP^\infty(X)$, the free abelian monoid on X.

(B) $\underline{F} = \{F_n\}_{n\geqslant 0}$, where F_n is the set of total orderings on the set $\mathbf{n} = \{1, 2, \dots, n\}$. S_n acts on F_n in an evident way. $\delta_i : F_n \longrightarrow F_{n-1}$ is given by restricting an ordering on $\mathbf{n}$ to an ordering on $\{1, 2, \dots, i-1, i+1, \dots, n\}$ and identifying $\{1, 2, \dots, i-1, i+1, \dots, n\}$ with $\{1, 2, \dots, n-1\}$ via the unique order preserving bijection. $\underline{F}$ is a free $\mathcal{O}$-space. In this case it follows easily from the definitions that $\underline{F}[X]$ is homeomorphic to the James construction $M(X, *)$ in Section 5.3.

(C) Fix $k \geqslant 1$ and let $\underline{C}(k) = \{C_n(k)\}_{n\geqslant 0}$ be defined as follows. $C_n(k)$ is the space of ordered n-tuples of distinct points in $\mathbf{R}^k$, i.e., $C_n(k) \subset (\mathbf{R}^k)^n$ is the set of n-tuples $(x_1, \dots, x_n)$ with $x_i \neq x_j$ if $i \neq j$. S_n acts by permuting the vectors, and δ_i deletes the i-th vector. $\underline{C}(k)$ is a free $\mathcal{O}$-space. In this case it can be shown that $\underline{C}(k)[X]$ is naturally equivalent to $\Omega^k\Sigma^k X$ for connected, based CW complexes X.

(D) Fix $k \geqslant 1$ and $d \geqslant 1$. Let $\underline{C}^d(k) = \{C_n^d(k)\}_{n\geqslant 0}$ be defined as follows. $C_n^d(k)$ will be the space of ordered n-tuples of vectors in $\mathbf{R}^k$ so that no vector occurs more than d times in the n-tuple. If $d = 1$ we are in the situation of (C). If $d > 1$, this is no longer a free operad. It is not known what $\underline{C}^d(k)[X]$ is. The case $d = 2$ has been studied by Karageuezian [20].

We record a useful technical result concerning these constructions. Both results are proved in the context of operads in [22]; the proofs in our setting are identical, and we omit them.

PROPOSITION 7.1.1 ([22, p. 14]). *Let $\underline{C}$ be an $\mathcal{O}$-space. Then the subquotients*

$$F_l\underline{C}[X]/F_{l-1}\underline{C}[X]$$

are homeomorphic to the quotients $C_l \ltimes_{S_l} X^{\wedge(l)}$. (Recall that if X and Y are spaces, and $y \in Y$ then $X \ltimes Y$ denotes the "half smash product" $X \times Y/X \times y$. If X and Y are G-spaces, where G is a group, and G fixes y, then $X \ltimes_G Y$ denotes the orbit space of the diagonal action of G on $X \ltimes Y$.)

We then have

PROPOSITION 7.1.2 ([22, p. 22]). *Let $f : \underline{C} \longrightarrow \underline{C}'$ be a map of $\mathcal{O}$-spaces. Suppose that $f_n : C_n \longrightarrow C_n'$ is a homotopy equivalence for all $n \geqslant 0$, and that both $\underline{C}$ and $\underline{C}'$ are free $\mathcal{O}$-spaces. Then $f[X] : \underline{C}[X] \longrightarrow \underline{C}'[X]$ is a weak equivalence for all based connected CW complexes, X.*

We now wish to describe the relationship between the constructions $\underline{C}(k)[X]$ and the spaces $\Omega^k\Sigma^k X$. We first define a modified version of $\underline{C}(k)$, which we denote $\underline{\hat{C}}(k)$, and call "Boardman's little cube" $\mathcal{O}$-space.

For any vector $v \in \mathbf{R}^k$ and positive real number R, let $Cu(v, R)$ denote the open k-cube centered at v, $\prod_{i=1}^{n}(v_i - R, v_i + R)$. We define $\tilde{C}_n(k)$ to be the space of ordered n-tuples $((v_1, \dots, v_n), (R_1, \dots, R_n))$, for which the cubes $Cu(v_i, R_i)$ are pairwise disjoint. Note that $\tilde{C}_n(k)$ is acted on freely by S_n, by permuting coordinates in both n-tuples.

There is an evident forgetful map $\phi_n : \tilde{C}_n(k) \longrightarrow C_n(k)$,

$$\phi_n\big((v_1, \dots, v_n), (R_1, \dots, R_n)\big) = (v_1, \dots, v_n).$$

These maps ϕ_n assemble into a map $\Phi : \tilde{\underline{C}}(k) \longrightarrow \underline{C}(k)$. Further, ϕ_n admits a section $\sigma_n : C_n(k) \longrightarrow \tilde{C}_n(k)$, defined by

$$\sigma_n(v_1, \dots, v_n) = \big((v_1, \dots, v_n), (R, \dots, R)\big)$$

where $R = R(v_1, \dots, v_n)$ is the maximal number for which the open cubes $Cu(v_i, R)$ are pairwise disjoint. (Note that R is a real valued function on $C_n(k)$.)

LEMMA 7.1.1. *The map ϕ_n is a homotopy equivalence for all n.*

PROOF. Since $\phi_n \sigma_n = id$, it will suffice to produce a homotopy from the identity map on $\tilde{C}_n(k)$ to $\sigma_n \phi_n$. We proceed as follows. For $(x, y, t) \in \mathbf{R}^3$, define

$$\begin{cases} \lambda(x, y, t) = x & \text{if } x \leqslant y, \\ \lambda(x, y, t) = (1 - t)x + ty & \text{if } x \geqslant y, \end{cases}$$

and similarly

$$\begin{cases} \mu(x, y, t) = x & \text{if } x \geqslant y, \\ \mu(x, y, t) = (1 - t)x + ty & \text{if } x \leqslant y. \end{cases}$$

The homotopy is now defined by the following formulae:

$$\begin{cases} h((v_1, \dots, v_n), & (R_1, \dots, R_n), t) = \\ & ((v_1, \dots, v_n), (\lambda(R_1, R, 2t), \dots, \lambda(R_n, R, 2t)) \\ & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ h((v_1, \dots, v_n), & (R_1, \dots, R_n), t) = \\ & ((v_1, \dots, v_n), (\mu(R_1, R, 2t - 1), \dots, \mu(R_n, R, 2t - 1)) \\ & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

□

Since each ϕ_n is a homotopy equivalence we can now record the following consequence of Proposition 7.1.2.

PROPOSITION 7.1.3. *Φ induces a homotopy equivalence $\Phi[X] : \tilde{\underline{C}}(k)[X] \longrightarrow \underline{C}(k)[X]$ for all connected, based CW complexes X.*

7.2. *The May–Milgram configuration space models for $\Omega^n\Sigma^n X$*

It is on the model $\underline{\tilde{C}}(k)[X]$ that one can define a map to $\Omega^k\Sigma^k X$. The construction goes as follows. First, for any cube $Cu(v, R_1, \ldots, R_k)$, we have a canonical identification

$$\overline{Cu(v, R_1, \ldots, R_k)} \xrightarrow{\lambda(v, R_1, \ldots, R_k)} [0,1]^k,$$

which is given by

$$(x_1, \ldots, x_k) \longrightarrow \left(\frac{x_1}{2R_1} + \frac{1}{2} - \frac{v_1}{2R_1}, \frac{x_2}{2R_2} + \frac{1}{2} - \frac{v_2}{2R_2}, \ldots, \frac{x_k}{2R_k} + \frac{1}{2} - \frac{v_k}{2R_k}\right).$$

Also, we have an identification $[0,1]^k \times X/\partial([0,1]^k) \times X \cup [0,1]^k \times * \cong \Sigma^k X$. For any

$$\big(((v_1, \ldots, v_n), (R_1, \ldots, R_n)), x_1, \ldots, x_n\big) \in \tilde{C}_n(k) \times X^n,$$

we define a map θ_n,

$$\theta_n\big((v_1, \ldots, v_n), (R_1, \ldots, R_n), (x_1, \ldots, x_n)\big) : \mathbf{R}^k \longrightarrow \Sigma^k X$$

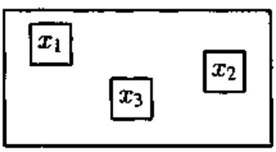
by letting

$$\theta_n \equiv * \quad \text{on } \mathbf{R}^k - \bigcup_{i=0}^{n} \overline{Cu(v_i, R_i)},$$

and on $Cu(v_i, R_i)$, we set θ_n equal to the composite

$$Cu(v_i, R_i) \xrightarrow{\lambda(v_i, R_i)} [0,1]^k \xrightarrow{id \times c_x} [0,1]^k \times X \longrightarrow \Sigma^k X$$

where c_x is the constant map with value x. This is best explained by the following picture



(7.2)

Note that since θ_n takes the value $*$ on the complement of a sufficiently large ball, θ_n extends to a map from the one point compactification of $\mathbf{R}^k$, S^k, to $\Sigma^k X$. Further, since, in this extension, ∞ is sent to $*$, we actually have an element in $\Omega^k\Sigma^k X$.

It is not hard to check that this procedure gives a map

$$\tilde{C}_n(k) \times X^n \longrightarrow \Omega^k\Sigma^k X.$$

It is also not hard to check that the θ_n's respect the equivalence relation and we obtain a map $\Theta : \bar{\underline{C}}(k)[X] \longrightarrow \Omega^k \Sigma^k X$.

THEOREM 7.2.1. *For connected X, the map Θ is a homotopy equivalence.*

PROOF. The proof of this result is too long and technical to present in its entirety here. We will, however, give a brief outline.

Sketch proof of the homotopy equivalence $\Theta : \bar{\underline{C}}(k)[X] \longrightarrow \Omega^n \Sigma^n X$

The first observation is that we have a map π of $\mathcal{O}$-spaces from $\underline{C}(1) \longrightarrow \underline{F}$, where $\underline{F}$ is the $\mathcal{O}$-space of 7.1(B). On $C_n(1)$, it is given by the observation that an $(n+1)$-tuple of distinct points in $\mathbf{R}^1$ determines an ordering on that set of points, and hence on $\{0, \ldots, n\}$. This correspondence gives a map $\pi_n : C_n(1) \longrightarrow F_n$, and it is easy to see that the π_n's give a map of $\mathcal{O}$-spaces. Further, one checks that the inverse image of the standard ordering on $\{1, \ldots, n\}$ is homeomorphic to $\mathbf{R} \times (0,1)^{n-1}$, via the map $(r_1, r_2, \ldots, r_n) \longrightarrow (r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_n)$. Since this inverse image is contractible, so is the inverse image of any other ordering, and we conclude that π_n is a homotopy equivalence. It now follows from Proposition 7.1.2 that $\pi[X] : \underline{C}(1)[X] \longrightarrow \underline{F}[X]$ is a homotopy equivalence. Since we have already observed that $\underline{F}[X]$ is homeomorphic to the James construction, we conclude that $\underline{C}(1)[X]$ is homotopy equivalent to $\Omega\Sigma X$, and it isn't hard to check that the diagram

$$\begin{tikzcd} \underline{C}(1)[X] \arrow[rr, "\Theta"] \arrow[dr, "\pi"'] & & \Omega\Sigma X \\ & \underline{F}[X] \cong M(X,*) \arrow[ur] & \end{tikzcd}$$

commutes up to homotopy, where the right hand diagonal map is the James map. The result for $k = 1$ thus follows from James' theorem.

The idea of the rest of the proof is to use induction on k. We have the loop-path fibration from Section 2.2. Furthermore, the existence of a fibration or quasifibration with contractible total space, base space X, and fibre Y shows that $Y \simeq \Omega X$. If we apply this to the space $\Omega^k \Sigma^{k+1} X \cong \Omega^k \Sigma^k(\Sigma X)$, we obtain a fibration sequence

$$\begin{tikzcd} \Omega^{k+1}\Sigma^{k+1}X \arrow[r] & E(\Omega^k \Sigma^k(\Sigma X)) \arrow[d] \\ & \Omega^k \Sigma^k(\Sigma X) \end{tikzcd} \tag{7.3}$$

Suppose we have already proved the desired result for k, and all spaces X, and wish to prove it for $k+1$. If we could construct a space $\underline{\mathcal{E}}(k)[X]$, which is contractible, and so that we have a fibration sequence

$$
\begin{array}{ccc}
\underline{C}(k+1)[X] & \longrightarrow & \underline{E}(k+1)[X] \\
& & \downarrow \\
& & \underline{C}(k)(\Sigma X)
\end{array}
\tag{7.4}
$$

which maps to the fibration sequence (7.3), with the map on base spaces being $\theta(k)$ and the map on fibres being $\theta(k+1)$, the result would be proved for $k+1$, via the long exact sequence of a fibration, and the induction could proceed. It is not possible to construct a fibration as in (7.4), but it is possible to construct a quasifibration with the desired maps on base spaces and fibres. This suffices.

We conclude our outline by describing $\underline{E}(k)[X]$. To make this definition, it is best to make a more general construction $\underline{E}'(k)[X,A]$, where $A \subseteq X$ is a based subcomplex of X. $\underline{E}'[X,A]$ will be defined as a subspace of $\underline{C}(k)[X]$. First, let $\pi : \mathbf{R}^k \longrightarrow \mathbf{R}^{k-1}$ denote projection on the first $k-1$ coordinates. For any point $(r_1,\ldots,r_k) \in \mathbf{R}^k$, let $r^+(v)$ denote the ray $\{(r_1,\ldots,r_{k-1},r_k+t) \mid t \geqslant 0\}$. For any n, let $Z_n(k) \subseteq C_n(k) \times X^n$ be the subspace of points $(v_0,\ldots,v_n,x_0,\ldots,x_n)$ so that, if $x_i \notin A$, then $v_j \notin r^+(v_i)$ for all $j \neq i$. $Z_n(k)$ is a closed subspace of $C_n(k) \times X^n$, and we define $\underline{E}'(k)[X,A]$ to be the identification space obtained by restricting the equivalence relation defining $\underline{E}(k)[X]$ to $\coprod_{n \geqslant 0} Z_n(k)$.

We will define a map

$$p : \underline{E}'(k+1)[X,A] \longrightarrow \underline{C}(k)[X/A].$$

To do this, note first that $Z_n(k)$ is the union of a family of closed subsets $Z_n^S(k)$, parameterized by subsets $S \subseteq \{1,2,\ldots,n\}$, where

$$
\begin{aligned}
Z_n^S(k) = \{(v_1,\ldots,v_n),(x_1,\ldots,x_n) \mid x_i \in A \text{ for } i \notin S \text{ and } v_i \notin r^+(v_j) \\
\text{for any } j \in S, i \neq j\}.
\end{aligned}
$$

p is now defined as follows. For a fixed $n \geqslant 0$ and $S \subseteq \{1,2,\ldots,n\}$, consider a point $(v_1,\ldots,v_n,x_1,\ldots,x_n)$ in $Z_n^S(k+1) \subseteq Z_n(k+1)$. We define $p|Z_n^S(k+1)$ by setting $p(v_1,\ldots,v_n,x_1,\ldots,x_n)$ equal to $(\pi(v_S),x_S)$ where v_S is the $\#(S)$-tuple consisting of the v_j's, $j \in S$, in increasing order, and where x_S is the $\#(S)$-tuple consisting of the x_j's, $j \in S$, also in increasing order. The fact that $\pi(v_S) \in C_{\#(S)-1}(k)$ follows from the definition of $Z_n^S(k+1)$. One now checks that the definition of P on the various $Z_n^S(k+1)$'s fit together to give a map $Z_n(k+1) \longrightarrow \underline{C}(k)[X/A]$, and that these maps respect the equivalence relation defining $\underline{E}'(k+1)[X,A]$, so we obtain a map $\underline{E}'(k+1)[X,A] \longrightarrow \underline{C}(k)[X/A]$. It is now possible to show that when applied to the pair (CX,X), $\underline{E}(k+1)[CX,X]$ is contractible, and p is a quasifibration,

$$p : \underline{E}(k+1)[cX,X] \longrightarrow \underline{C}(k)[\Sigma X].$$

Further, it also isn't hard to check that $p^{-1}(*)$ is equal to the subspace $\underline{C}(k+1)[X] \subseteq \underline{E}(k+1)[CX,X]$. We set $\underline{E}(k)[X] = \underline{E}'(k)[CX,X]$, and obtain the desired quasifibrations. To get the map to the fibration sequence (7.3), one replaces $\underline{E}(k+1)[CX,X]$

by a homotopy equivalent version $\tilde{\underline{\mathcal{E}}}(k+1)[CX, X]$, by analogy with the construction $\tilde{\underline{\mathcal{C}}}(k+1)[X]$. $\square$

7.3. *The homology of $\Omega^n\Sigma^n X$*

We now wish to discuss how these constructions can be used to obtain homological calculations. Originally, as was remarked in (6.12), the Zilchgon model was used in [24] to show that at each level $m < n$ and for any field $\mathbf{F}$ the $Cotor$-spectral sequence with E_2-term

$$Ext_{H^*(\Omega^m\Sigma^n X;\mathbf{F})}(\mathbf{F},\mathbf{F})$$

which converges to $H_*(\Omega^{m+1}\Sigma^n X;\mathbf{F})$ collapses for any connected CW complex X. Furthermore, it was shown there that $H_*(\Omega^{m+1}\Sigma^n X;\mathbf{F})$ is a primitively generated Hopf algebra as long as $m+1 < n$.

This makes the computation effective since we can start with

$$H_*(\Omega\Sigma^n X;\mathbf{F}) = T(H_*(\Sigma^{n-1}X;\mathbf{F})),$$

the primitively generated Hopf algebra. Here, using the Poincaré–Birkhoff–Witt theorem, one finds that $H^*(\Omega\Sigma^n X;\mathbf{F})$ is a tensor product of exterior algebras on (explicit) odd dimensional generators and $C(\mathbf{F})$-truncated algebras on (explicit) even dimensional generators. (Here $C(\mathbf{F})$-truncated means the free polynomial algebra on even dimensional generators b_i, subject only to the relation $b_i^p = 0$ where p is the characteristic of $\mathbf{F}$.)

Then, since

$$Ext_{A\otimes B}(\mathbf{F},\mathbf{F}) = Ext_A(\mathbf{F},\mathbf{F})\otimes Ext_B(\mathbf{F},\mathbf{F})$$

we are reduced to considering Ext for an exterior algebra $E(e_{2n+1})$ – which is $\mathbf{F}[b_{2n}]$ – and for a $C(\mathbf{F})$-truncated polynomial algebra, $\mathbf{F}[b_{2n+2}]/R$ – where it is $E(e_{2n+1})$ if R is empty and $E(e_{2n+1})\otimes\mathbf{F}[b_{2p(n+1)-2}]$ otherwise. Since these are primitively generated if $n > 1$, the dual of $\mathbf{F}[b_{2n}]$ is a tensor product of $C(\mathbf{F})$-truncated algebras and one can repeat the calculation to obtain the homology of each successive stage.

REMARK. A special case is when $\mathbf{F}$ has characteristic zero. Then, for each n there is the natural inclusion $i : \Sigma^n X \longrightarrow SP^\infty(\Sigma^n X)$. Passing to loop spaces and noting that $\Omega^n SP^\infty(\Sigma^n X) \simeq SP^\infty(X)$ by the Dold–Thom theorem, in the limit we have a map $i_\infty : Q(X) \longrightarrow SP^\infty(X)$. Then, from the discussion above it is direct to see that

$$i_{\infty *} : H^*(SP^\infty(X);\mathbf{F}) \longrightarrow H^*(Q(X);\mathbf{F})$$

is an isomorphism of rings for X a connected CW complex.

There are, of course many other paths to these results. But having the homology is not quite the same thing as understanding what it means.

To this end, initially J. Moore, then W. Browder, Araki and Kudo and finally Dyer and Lashof, [26], [10], [4], [15], constructed families of homology operations in $\Omega^n X$, $Q(X)$, and Fred Cohen showed, using the results of [24], that these operations together with loop sum, completely describe the homology of $\Omega^n \Sigma^n X$ for X a connected CW complex.

From another point of view V. Snaith proved that stably, we obtain a splitting

$$\Sigma^\infty \underline{C}(k)[X] \simeq \Sigma^\infty \bigvee_{k=1}^{\infty} C_l(k) \ltimes_{S_k} X^{\wedge n}$$

where Σ^∞ denotes "suspension spectrum"; see Section 8 for the definition of this concept. This is a direct consequence of the following result.

THEOREM 7.3.1. *There is a homotopy equivalence*

$$Q(\underline{C}(k)[X]) \simeq \prod_{k=1}^{\infty} Q(C_l(k) \ltimes_{S_k} X^{\wedge n}).$$

(See, e.g., [8] for details of a very slick proof due to F. Cohen.)

COROLLARY 7.3.1.

$$H^*(\Omega^n \Sigma^n X; \mathbf{A}) = \bigoplus_{k=1}^{\infty} H^*(C_l(k) \ltimes_{S_k} X^{\wedge n}; \mathbf{A})$$

for arbitrary untwisted coefficients $\mathbf{A}$.

Using Snaith splitting and the calculations above one can easily obtain the homology of the spaces $C_l(k) \ltimes_{S_k} X^{\wedge n}$ for any connected CW complex X and arbitrary k, l. This has had very important applications recently in many areas of mathematics. For example, in [9], it is the crucial input needed in the proof of the Atiyah–Jones conjecture.

7.4. *Barratt–Eccles simplicial model and J. Smith's unstable version*

The first two constructions we have exhibited work in the category of topological spaces. This has many advantages, for instance that the relationship of the combinatorial constructions with the iterated loop spaces is very explicit. It is also possible, as shown by Barratt and Eccles, to make the constructions for $k = \infty$ entirely inside the category of simplicial sets. Their construction has three main advantages. One is that the proofs become simpler. For instance, the analogue of the map p which could only be shown to be a quasifibration in the configuration space model is a surjective homomorphism of simplicial groups in this context, and hence automatically a Kan fibration. A second advantage is that in the Barratt–Eccles context, there is a natural extension of the result which applies to nonconnected simplicial sets. The third is that the loop sum operation arises as the multiplication operation in a simplicial group, hence is strictly associative. This is not the case for $\underline{\mathcal{C}}[X]$.

J. Smith in his thesis, [30], constructed simplicial versions of the finite stage constructions $\underline{\mathcal{C}}(k)[X]$. We will examine these at the end of this section.

The Barratt–Eccles model begins by constructing a simplicial version of the $\mathcal{O}$-space $\underline{\mathcal{C}}(\infty)$.

DEFINITION 7.4.1. An $\mathcal{O}$-simplicial set is a contravariant functor from the category Γ^0 of Definition 7.1.1 to the category of simplicial sets, which takes $\varnothing$ to the one point simplicial set $*$.

An example is the $\mathcal{O}$-space $\underline{\mathcal{F}}$ of 1.7(B), in which F_n can be viewed equally well as a discrete topological space and as a discrete simplicial set. We also note that we have a functor e from the category of sets to the category of simplicial sets, given by $e(X)_n = X^{n+1}$, where $d_i : X^{n+1} \longrightarrow X^n$ deletes the i-th coordinate for $0 \leqslant i \leqslant n$, and where s_i repeats the i-th coordinate. It is readily checked that $e(X)$ is always a contractible simplicial set. We now consider the $\mathcal{O}$-simplicial set $\underline{\mathcal{B}}$ defined as the composite

$$\Gamma^{op} \xrightarrow{\underline{\mathcal{F}}} \text{Sets} \xrightarrow{e} \text{Simplicial sets.} \tag{7.5}$$

For any $\mathcal{O}$-simplicial set $\underline{\mathcal{C}}$, and based simplicial set, X, we define the simplicial set $\underline{\mathcal{C}}[X]$ to be

$$\coprod_{n\geqslant 1} C_n \times X_.^n / \cong,$$

where $\cong$ is the equivalence relation generated by relations of the following two forms

(a) $(c, x_1, \ldots, x_n) \cong (\sigma c, x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, where $\sigma \in \mathcal{S}_n$.

(b) $(c, x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_n) \cong (\delta^i c, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Of course, these relations are precisely analogous to those used in defining the topological version.

One of the advantages of the construction is made apparent by the following proposition.

PROPOSITION 7.4.1 ([6]). *For any based, simplicial set $X_.$, $\underline{\mathcal{B}}[X]$ is a free simplicial monoid, in such a way that the natural map $\underline{\mathcal{F}}[X] \longrightarrow \underline{\mathcal{B}}[X]$ is a homomorphism of monoids, where $\underline{\mathcal{F}}[X_.]$ is identified with the free monoid on X.*

PROOF. We first observe that we have maps $\underline{\mathcal{F}}_m \times \underline{\mathcal{F}}_n \longrightarrow \underline{\mathcal{F}}_{m+n}$, by assigning to a pair of orderings $(<_m, <_n)$ on $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively, the ordering on $\{1, \ldots, m+n\}$ which we obtain by identifying $\{1, \ldots, m\} \cup \{1, \ldots, n\}$ with $\{1, \ldots, m+n\}$, where $\{1, \ldots, m\}$ is sent into $\{1, \ldots, m+n\}$ by adding m to each element of $\{1, \ldots, n\}$. Since e preserves products, we get maps $e(\{1, \ldots, m\}) \times e(\{1, \ldots, n\}) \longrightarrow e(\{1, 2, \ldots, m+n\})$. This gives a family of maps

$$B_m \times X^m \times B_n \times X^n \longrightarrow B_{m+n} \times X^{m+n},$$

and one checks that these maps respect the equivalence relations involved, to yield the required multiplication map. It is easy to check associativity, and the basepoint acts as an identity element. It is also easy to check freeness. □

This functor from simplicial sets to simplicial monoids is referred to by Barratt and Eccles as $\Gamma^+(X)$. They also compose Γ^+ with the group completion functor from simplicial monoids to simplicial groups, and call the result $\Gamma(X)$. $\Gamma(X)$ is a free simplicial group. Their main theorem now reads as follows.

THEOREM 7.4.1 ([6]). (a) *For any connected, based simplicial set $X.$, the natural inclusion $\Gamma^+(X) \longrightarrow \Gamma(X.)$ is a weak equivalence of simplicial sets.*

(b) *For any simplicial set $X.$, $|\Gamma(X.)|$ has the homotopy type of $Q(|X.|)$.*

PROOF. Part (a) is a standard fact about group completions of simplicial monoids. See [13] for details. It is essential here that $\Gamma^+(X)$ be a levelwise free simplicial monoid. To prove 7.4.1(b), one first proves that if $A. \hookrightarrow X.$ is an inclusion of simplicial sets, then the natural map $\Gamma(A.) \longrightarrow \mathrm{Ker}(\Gamma(X.) \longrightarrow \Gamma(X./A.)$, is a homotopy equivalence. This is proved in two steps. The first is to observe that Γ carries disjoint unions of based discrete simplicial sets to products, in the sense that the natural homomorphism $\Gamma(X \vee Y) \longrightarrow \Gamma(X) \times \Gamma(Y)$ is a weak equivalence of simplicial sets for all based sets X and Y. (Note that this is a special case of the required result, since it shows that $Ker(\Gamma(X \vee Y) \longrightarrow \Gamma(X))$ has the homotopy type of $\Gamma(Y)$). One first proves the analogous result for the monoid valued construction Γ^+, and concludes the result for Γ via a general comparison theorem for the homology of a simplicial monoid with that of its group completion. The second step is to prove that this special case suffices. Specifically, let T be any functor from the category of based sets to simplicial groups, and let T^s be the functor from simplicial sets to simplicial groups obtained by applying T levelwise and taking diagonal simplicial groups. Then Barratt and Eccles prove that if the natural map $T(X \vee Y) \longrightarrow T(X) \times T(Y)$ is a homotopy equivalence for all X and Y, then for all pairs of simplicial sets $(X., A.)$ the natural homomorphism $T^s(A.) \longrightarrow Ker(T^s(X.) \longrightarrow T^s(X./A.))$ is a weak equivalence of simplicial sets. Since Γ^+ is of the form T^s, this gives the result.

Since surjective homomorphisms of simplicial groups are Kan fibrations, applying the above discussion to the inclusion $X \longrightarrow CX$ shows that $|\Gamma(X)| \simeq \Omega|\Gamma\Sigma X|$, since $CX/X \simeq \Sigma X$. Iteratively, $|\Gamma(X.)| \simeq \Omega^k|\Gamma\Sigma^k X|$ for all k. On the other hand, if a simplicial set is l-connected, it is easy to check that the inclusion $X. \longrightarrow \Gamma^+(X.)$ is $(2l-1)$-connected, consequently, the map $\Omega^l|X.| \longrightarrow \Omega^l|\Gamma^+(X.)|$ is $(l-1)$-connected. Therefore the inclusion $|\Gamma^k\Sigma^k X| \longrightarrow \Omega^k|\Gamma\Sigma^k X|$ is $(k-1)$ connected. Thus, there is a map

$$\Omega^k|\Sigma^k X| \xrightarrow{\beta} \Omega^k|\Gamma(\Sigma^k X)| \xrightarrow{\theta} |\Gamma(X.)|,$$

where θ is a homotopy inverse to the inclusion $|\Gamma X.| \longrightarrow \Omega^k|\Gamma\Sigma^k X|$, where β is $(k-1)$ connected, hence $\theta \circ \beta$ is $(k-1)$ connected. Letting $k \longrightarrow \infty$ shows that $|\Gamma(X.)| \simeq Q(|X.|)$. □

This then gives a simplicial construction when $k = \infty$. For finite k, we have J. Smith's models [30]. Smith produces simplicial submonoids $\Gamma^{(n)+}(X.) \subseteq \Gamma^+(X.) \subseteq \Gamma(X.)$, whose realizations both give $\Omega^n\Sigma^n(|X.|)$ when $X.$ is connected and so that $|\Gamma^{(n)}(X.)| \cong \Omega^n\Sigma^n|X.|$ for arbitrary $X.$. First, we examine $\Gamma^{(n)+}(X.)$. $\Gamma^{(n)+}(X.)$ is constructed as $\underline{B}^{(n)}[X.]$, where $\underline{B}^{(n)}$ is a certain sub-$\mathcal{O}$-simplicial set of $\underline{B}$ above, which we now describe. Let $B = \{B_l\}_{l\geqslant 0}$, and consider the simplicial set B_l. Its k-simplices are $(k+1)$-tuples of orderings on $\{1,\dots,l\}$. For any pair (i,j), with $1 \leqslant i,j \leqslant l$, we have the restriction map from the set of orderings on $\{1,\dots,l\}$ to the set of orderings on $\{i,j\}$, which we identify with $\{1,\dots,2\}$ via $i \longrightarrow 1, j \longrightarrow 2$. This yields a simplicial map $\phi_{ij}: B_l \longrightarrow B_2$. Now, B_2 can be filtered by skeleta. It turns out that $|sk_n B_2| \cong S^n$, and the S_2-action is identified with the antipodal action on S^n. Now define $B_l^{(n)}$ to be

$$\bigcap_{1\leqslant i<j\leqslant l} \phi_{ij}^{-1}(sk_n B_2).$$

$\underline{B}^{(n)}$ becomes a sub $\mathcal{O}$-simplicial set, and $\underline{B}^{(n)}[X]$ is a subsimplicial monoid of $\underline{B}[X]$. $\Gamma^{(n)+}(X.)$ is defined to be $\underline{B}^{(n)}[X.]$, and $\Gamma^{(n)}(X.)$ is defined to be its group completion.

THEOREM 7.4.2 (Smith). *If $X.$ is connected, then $|\Gamma^{(n)+}(X.)|$ and $|\Gamma^{(n)}(X.)|$ are homotopy equivalent to $\Omega^n\Sigma^n|X.|$. In general $|\Gamma^{(n)}(X.)| \cong \Omega^n\Sigma^n|X.|$.*

It is not known that the realizations of Smith's simplicial $\mathcal{O}$-sets are equivalent to the $\mathcal{O}$-spaces $\underline{C}(n)$ although one suspects that they will be.

8. Spectra, infinite loop spaces, and category theoretic models

By the *homotopy category* Ho of based spaces, we mean the category whose objects are based spaces (X,x), and where the morphisms from (X,x) to (Y,y) are given by $[X,Y]_0$, the based homotopy classes of maps from X to Y. Similarly, one could define the *stable homotopy category* Ho^s as the category whose objects are based spaces (X,x), and where the morphisms from (X,x) to (Y,y) are given by

$$\{X,Y\} = \varinjlim_{n} [\Sigma^n X, \Sigma^n Y]_0.$$

It is proved in [3] that for any fixed X, the graded set $A_n^Y(X) = \{\Sigma^n X, Y\}$ for $n \geqslant 0$, and $A_n^Y(X) = \{X, \Sigma^{-n}Y\}$ for $n \leqslant 0$, is actually a graded abelian group, and yields a long exact sequence of graded abelian groups when applied to a cofibration sequence $X_1 \longrightarrow X_2 \longrightarrow X_2/X_1$. A_*^Y is referred to as a *generalized cohomology theory*, i.e. a graded abelian group valued functor which satisfies all the Eilenberg–Steenrod axioms except the dimension axiom which asserts that $A_n^X(S^0) = 0$ for $n \neq 0$, $A_0(S^0) = \mathbf{Z}$.

Generalized cohomology theories have turned out to be extremely useful in stable homotopy theory. K-theory and various bordism theories have been particularly so. These theories, and also singular cohomology theory are not, however of the form A_*^Y for any Y in the above Ho^s. One says they are not *representable* in Ho^s. It turns out, though, that by enlarging Ho^s a bit, one can make these theories representable. Moreover, by a theorem of E.H. Brown, [11], one can obtain a precise criterion for when a graded abelian group valued functor is representable.

To see how to construct this enlargement we consider the case of ordinary integral cohomology, $H^*(\ ;\mathbf{Z})$. In Ho the functor $X \mapsto H^n(X;\mathbf{Z})$ is representable. Let $K(\mathbf{Z},n)$ be an Eilenberg–MacLane space, i.e.

$$\pi_i\big(K(\mathbf{Z},n)\big) = \begin{cases} 0, & i \neq n, \\ \mathbf{Z}, & i = n. \end{cases}$$

Then there exists a class $\iota_n \in H^n(K(\mathbf{Z},n))$ so that the homomorphism $[X, K(\mathbf{Z},n)] \longrightarrow H^n(X;\mathbf{Z})$, given by $f \mapsto H^n(f)(\iota_n)$, is an isomorphism of functors.

Although H^n is defined on Ho^s, it is not the case that $\{X, K(\mathbf{Z},n)\} \cong H^n(X,\mathbf{Z})$, as one can easily check. The point is that, e.g., $H^{n+1}(\Sigma X;\mathbf{Z}) \not\cong [\Sigma X, \Sigma K(\mathbf{Z},n)]$ in general, since $\Sigma K(\mathbf{Z},n) \not\cong K(\mathbf{Z},n+1)$. What this suggests is that one wants to allow objects which, in a sense, contain all of the $K(\mathbf{Z},n)$'s at once. We therefore introduce the concepts of *prespectra* and *spectra*.

8.1. *Prespectra, spectra, triples, and a delooping functor*

DEFINITION 8.1.1. (a) A prespectrum $\mathbf{X}$ is a family of based spaces $\{X_i\}_{i\geqslant 0}$, together with "bonding maps" $\sigma_i : \Sigma X_i \longrightarrow X_{i+1}$.

(b) A morphism f from a prespectrum $\mathbf{X} = \{X_i\}$ to $\mathbf{Y} = \{Y_i\}$ is a family of based maps $f_i : X_i \longrightarrow Y_i$, so that $f_i\sigma^X_{i-1} = \sigma^Y_{i-1} f_{i-1}$ for all i.

(c) A prespectrum is an Ω-spectrum if, for each $i \geqslant 0$, the adjoint to σ_i,

$$Ad(\sigma_i) : X_i \longrightarrow \Omega X_{i+1}$$

is a homeomorphism.

(d) If X is any based space, the suspension prespectrum of X, $\Sigma^\infty X$, is given by $\{\Sigma^i X\}_{i\geqslant 0}$, with the evident bonding maps.

Note that given any prespectrum $\mathbf{X} = \{X_i\}_{i\geqslant 0}$, and based space Z, one can form a new prespectrum $\mathbf{X} \wedge Z = \{X_i \wedge Z\}_{i\geqslant 0}$ where the i-th bonding map is $\sigma_i \wedge id_Z$. In particular, we can let $Z = I^+$, the unit interval with a disjoint basepoint added, and declare that two maps $f, g : \mathbf{X} \longrightarrow \mathbf{Y}$ of prespectra are homotopic if there is a map $H : \mathbf{X} \wedge I^+ \longrightarrow \mathbf{Y}$ so that $H|\mathbf{X} \wedge 0^+ \longrightarrow \mathbf{Y} = f$ and $H|\mathbf{X} \wedge 1^+ \longrightarrow \mathbf{Y} = g$, where $\mathbf{X} \wedge 0^+$ and $\mathbf{X} \wedge 1^+$ are identified with $\mathbf{X}$ in the obvious way. By letting $\mathbf{X}$ be the suspension spectrum $\Sigma^\infty(S^n)$, we now obtain a definition of the homotopy groups $\pi_n(\mathbf{Y})$ for any prespectrum $\mathbf{Y}$, and of the homotopy classes of maps $[\mathbf{X}, \mathbf{Y}]$ for any pair of spectra.

EXAMPLE 8.1. Let $\mathbf{K}(\mathbf{Z},n)$ denote the prespectrum whose i-th entry is $K(\mathbf{Z}, n+i)$, and where $\sigma_i : \Sigma K(\mathbf{Z}, n+i) \longrightarrow K(\mathbf{Z}, n+i+1)$ is the map representing a generator in $H^{n+i+1}(\Sigma K(\mathbf{Z}, n+i);\mathbf{Z}) \cong \mathbf{Z}$. $\mathbf{K}(\mathbf{Z},n)$ can be taken to be an Ω-spectrum, and

$$\begin{cases} \pi_0(\mathbf{K}(\mathbf{Z},n)) = \mathbf{Z}, \\ \pi_i(\mathbf{K}(\mathbf{Z},n)) = 0 \quad \text{if } i \neq n. \end{cases}$$

More generally, $[\Sigma^\infty X, \mathbf{K}(\mathbf{Z}, n)] \cong H^n(X; \mathbf{Z})$, so, in this enlarged category $H^n(-, \mathbf{Z})$ is representable.

REMARK. The above mentioned definition of $[\mathbf{X}, \mathbf{Y}]$ does not, in fact, have good properties when $\mathbf{Y}$ is not an Ω-spectrum. The actual definition of $[\mathbf{X}, \mathbf{Y}]$ can be carried out as in [3] or by replacing $[\mathbf{X}, \mathbf{Y}]$ with $[\mathbf{X}, \omega(\mathbf{Y})]$, where ω is a functorial construction of an Ω-spectrum from $\mathbf{Y}$. We will not dwell on this point.

From the definitions, if $\mathbf{X}$ is an Ω-spectrum, it is clear that $\pi_i(\mathbf{X})$ is isomorphic to $\pi_i(X_0)$, the ordinary i-th homotopy group of the zeroth space of the spectrum $\mathbf{X}$. Consequently, the homology and other invariants of X_0 are of interest. Further, each X_i is an "i-fold delooping" of X_0 in the sense that $X_0 \simeq \Omega^i X_i$ via a composite of adjoints to the bonding maps, so X_0 is referred to as an *infinite loop space*. We also obtain maps

$$\theta_i : \Omega^i \Sigma^i(X_0) \xrightarrow{\Omega^i \sigma_i} \Omega^i X_i \simeq X_0.$$

Further, the θ_i's are compatible in the sense that $\theta_{i+1} \circ \eta_i = \theta_i$, where

$$\eta_i : \Omega^i \Sigma^i(X_0) \ \longrightarrow \ \Omega^{i+1} \Sigma^{i+1}(X_0)$$

is the inclusion, and so we obtain a map

$$Q(X_0) \xrightarrow{\nu} X_0.$$

It will turn out that this map ν will, in the case of connective Ω-spectra, determine the entire spectrum $\mathbf{X}$ up to homotopy equivalence. We will now discuss this fact.

DEFINITION 8.1.2. A triple on a category $\mathcal{C}$ is a functor $T : \mathcal{C} \longrightarrow \mathcal{C}$, together with natural transformations $\mu : T^2 \longrightarrow T$ and $\eta : Id \longrightarrow T$, so that the following diagrams commute for all $X \in \mathcal{C}$.

$$\begin{tikzcd}
TX \arrow[r, "T\eta(X)"] \arrow[dr, "id"'] & T^2X \arrow[d, "\mu(X)"] & TX \arrow[l, "\eta(TX)"'] \arrow[dl, "id"] \\
& TX &
\end{tikzcd}$$

$$\begin{tikzcd}
T^3X \arrow[r, "\mu(TX)"] \arrow[d, "T\mu(X)"] & T^2X \arrow[d, "\mu(X)"] \\
T^2X \arrow[r, "\mu(X)"] & TX
\end{tikzcd}$$

EXAMPLE 8.2. (a) Let C be the category of based sets, and let F be the functor from C to C which assigns to each based set, X, the free group on X with the basepoint set to the identity. (Note that a group is, via a forgetful functor, a set). F is a triple, since any set includes in the free group on that set as the words of length 1, and μ is obtained by evaluating a "word of words" as, simply, a word.

(b) Again, let C denote the category of based sets, and let F^{ab} denote the free abelian group functor, with basepoint set to 0. F^{ab} is also a triple on C.

(c) Let C be the category of based spaces, and let T be the functor $\Omega\Sigma$. There is the James inclusion $X \longrightarrow \Omega\Sigma X$, which is the natural transformation η. To construct μ, we first observe that there is a natural transformation $e : \Sigma\Omega \longrightarrow Id$, which is given by $e(t \wedge \phi) = \phi(t)$. $\mu(X)$ is now given by the composite

$$\Omega\Sigma\Omega\Sigma(X) \xrightarrow{\Omega(e(\Sigma X))} \Omega \circ Id \circ \Sigma(X) = \Omega\Sigma(X).$$

With this choice of μ and η, $\Omega\Sigma$ becomes a triple.

(d) Again, C will be the category of based spaces, and we let $T = \Omega^k\Sigma^k$. T becomes a triple by a construction identical to that in example (c). Even

$$Q = \varinjlim \Omega^k\Sigma^k$$

also becomes a triple on C.

DEFINITION 8.1.3. An algebra (X, ξ) over a triple T is an object $X \in C$ and a map $\xi : TX \longrightarrow X$ so that the diagrams below commute.

$$\begin{tikzcd} X \arrow[r, "\eta(X)"] \arrow[dr, "id"'] & TX \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} TTX \arrow[r, "\mu(X)"] \arrow[d, "T\xi"] & TX \arrow[d, "\xi"] \\ TX \arrow[r, "\xi"] & X \end{tikzcd}$$

Morphisms of T-algebras are defined as morphisms in C making the evident diagrams commute. Also, for any object X in C, (TX, μ) is an algebra over the triple T, to be thought of as the free T-algebra on X.

This is quite a useful notion. For instance, the reader should verify that if F is the triple in Example 8.2(a), an F-algebra structure on a based set X is the same thing as a group structure on X, where the basepoint is the identity. Similarly, if F^{ab} is as in 8.2(b), an F^{ab}-algebra structure on X is the same thing as an abelian group structure on X, where the basepoint is equal to zero. Also, any loop space has an algebra structure over the triple $\Omega\Sigma$ in 8.2(c), given by

$$\Omega\Sigma\Omega Z \xrightarrow{\Omega(e(Z))} \Omega Z,$$

and similarly, any k-fold loop space is an algebra over the triple $\Omega^k\Sigma^k$ of 8.2(d). In fact, by analogy with 8.2(a) and (b), we view $\Omega^k\Sigma^k$ as the "free k-fold loop space" functor, and it can be shown that $\Omega^k\Sigma^k$-algebra structures on a space X are the same thing (up to an obvious notion of homotopy equivalence) as k-fold deloopings Z of X, i.e. spaces, Z, together with an equivalence $X \xrightarrow{\simeq} \Omega^k Z$. This result is originally due to Beck [7]. We will indicate a proof of the $k=\infty$ version, i.e. we will show that a Q-algebra structure on a space X determines an infinite family of deloopings, with certain compatibility conditions, i.e. a spectrum with X as zeroth space.

We first discuss some generalities. Let T be a triple on a category C, and let (X,ξ) be a T-algebra. We define a simplicial object $T.(X,\xi)$ in C by setting $T_k(X,\xi)=T^{k+1}(X)$, and letting the face and degeneracies be given by the following formulae.

$$\begin{cases} d_i : T^{k+1}(X) \longrightarrow T^k(X) := T^i\mu(T^{k-i-1}X) & \text{for } 0 \leqslant i \leqslant k-1, \\ d_k : T^{k+1}(X) \longrightarrow T^k(X) := T^k(\xi), & \\ s_i : T^{k+1}(X) \longrightarrow T^{k+2}(X) := T^{i+1}(\eta(T^{k-i}X)) & \text{for } 0 \leqslant i \leqslant k. \end{cases} \tag{8.1}$$

One easily checks that $T.(X,\xi)$ is a simplicial object in the category of T-algebras. In fact $T.(X,\xi)$ should be viewed as a simplicial resolution of (X,ξ) by free T-algebras in C. Note that there is a map of simplicial objects $\alpha : T.(X,\xi) \longrightarrow X.$, where $X.$ is the constant simplicial object with value X, given in level k by the composite

$$\xi \circ T(\xi) \circ \cdots \circ T^{k-1}(\xi) \circ T^k(\xi).$$

PROPOSITION 8.1.1 ([22]). *Let C be the category of based topological spaces, and T a triple. Then the map $|T.(X,\xi)| \longrightarrow X$ induced by α is a weak equivalence.*

PROOF. This is proved in [22, Proposition 9.8, p. 90]. □

To produce deloopings we must also use the interaction of the suspension functor with the triple in question. We formalize this as follows.

DEFINITION 8.1.4. Let T be a triple on a category C. By an intertwiner Σ for T, we mean a functor $\Sigma : C \longrightarrow C$ together with a natural transformation $\zeta : \Sigma T \longrightarrow T\Sigma$, so that the following diagrams commute.

$$\begin{array}{ccccc} \Sigma T^2X & \xrightarrow{\zeta(TX)} & T\Sigma TX & \xrightarrow{T(\zeta X)} & T^2\Sigma X \\ \Big\downarrow{\scriptstyle \Sigma\mu(X)} & & & & \Big\downarrow{\scriptstyle \mu(\Sigma X)} \\ \Sigma TX & \xrightarrow{id} & \Sigma TX & \xrightarrow{\zeta(X)} & T\Sigma X \end{array}$$

$$\begin{array}{ccc}
\Sigma X & \xrightarrow{\Sigma\eta(X)} & \Sigma TX \\
\Big\downarrow{\scriptstyle \eta(\Sigma X)} & & \Big\downarrow{\scriptstyle \zeta(X)} \\
T\Sigma X & \xrightarrow{id} & T\Sigma X
\end{array}$$

DEFINITION 8.1.5. Given any intertwiner Σ for T, and T-algebra (X,ξ), we construct a simplicial object $T^{\Sigma}_{.}(X,\xi)$ by setting

$$T^{\Sigma}_{k}(X,\xi) = T\Sigma T^{k}X,$$

and declaring that the faces and degeneracies are given by the following formulae:

$$\begin{cases} d_0 : T\Sigma T^k X \longrightarrow T\Sigma T^{k-1}X := \mu(T^{k-1}X)\circ T(\zeta(T^{k-1}X)) & \\ d_i : T\Sigma T^k X \longrightarrow T\Sigma T^{k-1}X := T\Sigma T^{i-1}\mu(T^{k-i-1}X) & \text{for } i > 0. \\ s_i : T\Sigma T^k X \longrightarrow T\Sigma T^{k+1}X := T\Sigma T^{i}\eta(T^{k-i}X) & \end{cases}$$

Note that $T^{\Sigma}_{.}(X,\xi)$ is a simplicial object in the category of T-algebras. We also note that there is a morphism

$$\Sigma X. \xrightarrow{\lambda} T.^{\Sigma}(X,\xi),$$

where $\Sigma X.$ is the constant simplicial object with value ΣX. There is also a map ν : $\Sigma T.(X,\xi) \longrightarrow T.^{\Sigma}(X,\xi)$, given in level k by $\zeta(T^kX)$. $T^{\Sigma}_{.}$ is a functor from the category of T-algebras to the category of simplicial T-algebras.

We now apply this to our situation, where $T = Q$ and C is the category of based CW complexes. Σ is now ordinary suspension. To define a map $\zeta(X) : \Sigma QX \longrightarrow Q\Sigma X$, we define, for $f : S^n \longrightarrow S^n \wedge X$ and $s \in S^n$, $t \in [0,1]$,

$$\zeta(X)[t,f](s) = \big[t, f(s)\big] \tag{8.2}$$

where t is the suspension coordinate and $S^n \wedge \Sigma X$ is identified with $\Sigma(S^n \wedge X)$. It is easy to check that with this definition, the pair (Σ,ζ) forms an intertwiner for Q. Thus, for any Q-algebra (X,ξ), we obtain a simplicial Q-algebra $Q.^{\sigma}(X,\xi)$, and a map of spaces

$$\Sigma X \xrightarrow{\lambda} \big|Q.^{\Sigma}(X,\xi)\big|.$$

If we consider the adjoint $ad(\lambda) : X \longrightarrow \Omega|Q.^{\Sigma}(X,\xi)|$, then $|Q.^{\Sigma}(X,\xi)|$, is a candidate for a first delooping for X.

PROPOSITION 8.1.2. *Let X be any Q-algebra. Then $ad(\lambda)$ is a weak equivalence. Further, if X is k-connected, then $|Q.^{\Sigma}(X,\xi)|$ is $(k+1)$-connected.*

PROOF. We first observe that λ factors as

$$\Sigma X \longrightarrow \Sigma Q.(X,\xi) \stackrel{\zeta(X)}{\longrightarrow} Q.^{\Sigma}(X,\xi),$$

where the left arrow is $\Sigma\eta(X) : \Sigma X \longrightarrow \Sigma Q(X) = \Sigma Q_0(X,\xi)$. It is an equivalence by Theorem 8.1.1. It consequently suffices to show that the adjoint to $\nu(X)$, $ad(\nu(X)) : |Q.(X,\xi)| \longrightarrow \Omega|Q.^{\Sigma}(X,\xi)|$, is an equivalence. Secondly, it is standard in this case (where $\pi_0(Q_k^{\Sigma}(X,\xi))$ is a group for all k) that the natural map $|\Omega Q.^{\Sigma}(X,\xi)| \longrightarrow \Omega|Q.^{\Sigma}(X,\xi)|$ is an equivalence. See [12] for details. It therefore suffices to show that the adjoint to $\zeta(Q^k X) : \Sigma Q^{k+1}X \longrightarrow Q\Sigma Q^k X$ is an equivalence, and for this it clearly suffices to show that $ad(\zeta(X))$ is an equivalence for all X. But the adjoint of $\zeta(X)$ is the inclusion $QX \longrightarrow \Omega Q\Sigma X$, which is easily checked to be an equivalence. The connectivity statement is easy. $\square$

Let Ψ denote the functor $(X,\xi) \longrightarrow Q^{\Sigma}_{.}(X,\xi)$ from Q-algebras to simplicial Q-algebras. Applying Ψ levelwise to $Q^{\Sigma}_{.}$, we obtain a functor $\Psi[2]$ from Q-algebras to bisimplicial Q-algebras, and by iteration of this procedure functors $\Psi[k]$ to k-fold simplicial Q-algebras. By applying Proposition 8.1.2 levelwise, one obtains a natural (on the category of Q-algebras) equivalence $|\Psi[k](X,\xi)| \simeq \Omega|\Psi[k+1](X,\xi)|$. In other word, we have constructed a functor S from the category of Q-algebras to the category of Ω-spectra. It is not hard to check that the functor actually takes its values in the full subcategory of connective spectra. Further, S is homotopy invariant in the sense that if $f : (X_1,\xi_1) \longrightarrow (X_2,\xi_2)$ is a morphism of Q-algebras, so that $f : X_1 \longrightarrow X_2$ is a weak equivalence of spaces, then $S(f)$ is a weak equivalence of spectra.

8.2. *The May recognition principle for Ω-spectra*

We wish to use the $\mathcal{O}$-space constructions $\underline{\mathcal{C}}[X]$ to minimize the amount of data required to construct the deloopings. As it stands, for a general $\mathcal{O}$-space, $\underline{\mathcal{C}}$, $X \longrightarrow \underline{\mathcal{C}}[X]$ is not a triple on the category of based spaces. To have the triple structure requires that $\underline{\mathcal{C}}$ actually be an "operad" in the sense of May. We now describe this notion. In order to simplify the definition a bit, we introduce some terminology. By a *graded topological space*, we mean a space C equipped with a decomposition

$$C = \coprod_{n\geqslant 0} C_n.$$

If X is a space, we will write

$$F_C(X) = \coprod_{n\geqslant 0} C_n \times X^n.$$

This is of course functorial on the category of spaces. If

$$X = \coprod_{i \geqslant 0} X_i$$

is a graded space, then $F_C(X)$ becomes a graded space, with

$$F_C(X)_n = \coprod_{l \geqslant 0} \coprod_{j_1 + \cdots + j_l = n} C_l \times X_{j_1} \times \cdots X_{j_l} \subseteq F_C(X).$$

DEFINITION 8.2.1. Let $\underline{C}$ be an $\mathcal{O}$-space.

$$C = \coprod_{n \geqslant 0} C_n$$

is viewed as a graded set. An operad structure on $\underline{C}$ is a natural transformation of functors $\mu : F_C \circ F_C \longrightarrow F_C$, satisfying the following requirements.

(a) The diagrams

$$\begin{tikzcd} F_C \circ F_C \circ F_C(X) \arrow[r, "F_C(\mu(X))"] \arrow[d, "\mu(F_C(X))"] & F_C \circ F_C(X) \arrow[d, "\mu(X)"] \\ F_C \circ F_C(X) \arrow[r, "\mu(X)"] & F_C(X) \end{tikzcd}$$

commute for all X.

(b) $\mu(*)$ gives maps $C_k \times C_{j_1} \times \cdots \times C_{j_k} \longrightarrow C_j$, where $j = j_1 + \cdots + j_k$. Since $\underline{C}$ is an $\mathcal{O}$-space, C_j is equipped with an action by the symmetric group $\mathcal{S}_j$. On the other hand, $C_k \times C_{j_1} \times \cdots \times C_{j_k}$ is equipped with an action of $\mathcal{S}_{j_1} \times \cdots \times \mathcal{S}_{j_k}$, with each symmetric group acting on its corresponding factor, and all acting trivially on C_k. Let $\rho : \mathcal{S}_{j_1} \times \cdots \times \mathcal{S}_{j_k} \longrightarrow \mathcal{S}_j$ be the homomorphism which views $\mathcal{S}_{j_1}$ as acting on $\{1, \ldots, j_1\}$, $\mathcal{S}_{j_2}$ as acting on $\{j_1 + 1, \ldots, j_2\}$, etc. We require that $\mu(*)$ restricted to $C_k \times C_{j_1} \times \cdots \times C_{j_k}$ be equivariant with respect to ρ, i.e.

$$\mu(*)(c; \sigma_1 c_1, \ldots, \sigma_k c_k) = \rho(\sigma_1, \ldots, \sigma_k)\mu(*)(c; c_1, \ldots, c_k).$$

(c) Let $j_1, \ldots, j_k$ be given, with $j_1 + \cdots + j_k = j$. Note that $\{j_1, \ldots, j_k\}$ determines a partition

$$\{1, \ldots, j_1 | j_1 + 1, \ldots, j_1 + j_2 | j_2 + 1, \ldots, j_1 + \ldots j_{k-1} | j_1 + \ldots \\ + j_{k-1} + 1, \ldots, j_1 + \ldots + j_k\}$$

of $\{1, \ldots, j\}$, with k blocks. For any $\sigma \in \mathcal{S}_k$ let $\theta = \theta(\sigma; j_1, \ldots, j_k)$ be the unique permutation of $\{1, \ldots, j\}$ which is order preserving on each block $\{j_1 + \ldots + j_{s-1} +$

$1, \ldots, j_1 + \ldots + j_s\}$, and so that $\sigma(s) > \sigma(t)$ implies that θ carries elements of $\{j_1 + \ldots + j_{s-1} + 1, \ldots, j_1 + \ldots j_s\}$ to elements which are strictly greater than all elements in $\{j_1 + \ldots + j_{t-1} + 1, \ldots, j_1 + \ldots + j_t\}$. Then

$$\mu(*)(\sigma c; c_1, \ldots c_k) = \theta(\sigma; j_1, \ldots, j_k)\mu(*)(c; c_{\sigma(1)}, \ldots, c_{\sigma(k)}).$$

(d) There exists an element $1 \in C_1$, so that $\mu(*)(c; \underbrace{1, \ldots, 1}_{k \text{ factors}}) = c$ for all $c \in C_k$.

PROPOSITION 8.2.1. *An operad structure on an $\mathcal{O}$-space gives a triple structure on the functor $X \longrightarrow \underline{\mathcal{C}}[X]$.*

(The proof is a direct but tedious verification. See [22] for details.)

Not all $\mathcal{O}$-spaces described in Section 7.1 extend to operad structures. For instance, $\underline{\mathcal{C}}(k)$ does not, nor does the $\mathcal{O}$-space $\underline{\mathcal{C}}^d(k)$ of Example 7.1(D). However, let $\underline{\mathcal{F}}$ be the $\mathcal{O}$-space of 7.1(B). $\underline{\mathcal{F}}$ extends to an operad structure as follows. Let $j_1 + \ldots + j_k = j$, and let $B_s \subseteq \{1, \ldots, j\}$ be the subset

$$\{n | j_1 + \ldots + j_{s-1} + 1 \leqslant n \leqslant j_1 + \ldots + j_s\}.$$

The structure map $\mu(*) : F_k \times F_{j_1} \times \cdots \times F_{j_k} \longrightarrow F_j$ is given by assigning to a $(k+1)$-tuple $(\leqslant, \leqslant_1, \ldots, \leqslant_k)$ of orderings the unique ordering on $\{1, \ldots, j\}$ which restricts to the ordering $\leqslant_s$ on B_s, when B_s is identified with $\{1, \ldots, j_s\}$ in an order preserving way, and so that if $m \in B_s$ and $n \in B_t$, with $s \neq t$, $m < n$ if and only if $s < t$.

Also, recall the Barratt–Eccles $\mathcal{O}$-simplicial set $\underline{B}$ from Section 7. Here, B_n was defined as $e(F^n)$, where e was a product preserving functor from sets to simplicial sets. The above defined operad structure map for $\underline{\mathcal{F}}$ now defines similar maps $e(F_k) \times e(F_{j_1}) \times \cdots \times e(F_{j_k}) \longrightarrow e(F_j)$. Applying geometric realization gives an operad structure on the $\mathcal{O}$-space $\{|B_n|\}_{n \geqslant 0}$.

With simple modifications one can modify $\underline{\mathcal{C}}(k)$ into an $\mathcal{O}$-space with operad structure. We define $\underline{\mathcal{C}}^B(k) = \{C_n^B(k)\}_{n \geqslant 0}$ by letting $C_n^B(k)$ be the space of disjoint n-tuples of open n-cubes in $[0,1]^k$. It is understood that these are cubes with sides parallel to the coordinate axes. $\underline{\mathcal{C}}^B(k)$ now admits an operad structure. For any $(k+1)$-tuple $(c; c_1, \ldots, c_l)$ with $c_x \in C_{j_s}^B(k)$, and $c \in C_l^B(k)$, say $c = (Cu_1, \ldots, Cu_l)$, we have the identification $\lambda_i : [0,1]^k \longrightarrow Cu_i$, which is an affine linear map and carries sides parallel to a coordinate axis to sides parallel to the same coordinate axis. The j_s-tuple of cubes in $[0,1]^k$ specified by c_s is identified with a new j_s-tuple of disjoint cubes $\lambda_s(c_s)$ contained in Cu_s. The j-tuple of cubes $\{\lambda_1(c_1), \ldots, \lambda_l(c_l)\}$ consists of disjoint cubes, since the Cu_i's are disjoint. This gives the operad structure. We also remark that $\underline{\mathcal{C}}^B(k)$ includes in $\underline{\mathcal{C}}(k)$ as a sub-$\mathcal{O}$-space, and that this inclusion satisfies the hypothesis of Proposition 8.1.1. It follows that $\underline{\mathcal{C}}^B[X]$ and $\underline{\mathcal{C}}[X]$ are weakly equivalent for all based CW-complexes X.

Let T^B denote the triple $X \longrightarrow \underline{\mathcal{C}}^B[X]$. We observe that there is a functor from the category of connective spectra to Q-algebras, which assigns to each connective spectrum its zeroth space. There is a natural transformation of triples $T^B \longrightarrow Q$, which means that any Q-algebra can be viewed as a T^B-algebra, and hence we obtain a composite functor $\mathcal{U}$ from the category of connective spectra to the category of T^B-algebras. In order to state our theorem, we also define a weak natural transformation of functors to the category of spaces (or spectra, or T-algebras, where T is a triple) from a functor F^0 to a functor F^1 to be a sequence of functors $\{G_0, \dots, G_{2k}\}$, with $G_0 = F_0$ and $G_{2k} = F_1$, together with natural transformations $G_{2l+1} \longrightarrow G_{2l+2}$ and natural transformations $G_{2l+1} \longrightarrow G_{2l}$ which are weak equivalences for all objects in the domain category. A weak natural transformation is said to be a weak equivalence if in addition the natural transformations $G_{2l+1} \longrightarrow G_{2l+2}$ are weak equivalences when evaluated on any object in the domain category. Note that a morphism of T-algebras is said to be a weak equivalence if the map on spaces is a weak equivalence in the usual sense.

The May recognition principle is now stated as follows.

THEOREM 8.2.1. *There is a functor $\mathcal{S}$ from the category of T^B-algebras to the category of connective spectra, satisfying the following properties.*

(a) *If $f : (X, \xi) \longrightarrow (X', \xi')$ is a map of T^B-algebras, and the map $f : X \longrightarrow X'$ is a weak equivalence, then $\mathcal{S}(f)$ is a weak equivalence of spectra.*

(b) *There is a natural weak equivalence of functors on the category of connective spectra from $\mathcal{S} \circ \mathcal{U}$ to the identity functor.*

(c) *There is a weak natural transformation of functors on the category of T^B-algebras from the identity to $\mathcal{U} \circ \mathcal{S}$, which is a weak equivalence on T^B-algebras (X, ξ) for which $\pi_0(X)$ is a group. (Note that in general, if (X, ξ) is a T^B-algebra, we have a map*

$$C_2^B \underset{\mathcal{S}_2}{\times} X \times X \longrightarrow X,$$

and hence by choosing a point in C_2^B a map $X \times X \longrightarrow X$. Consequently, X is an H-space, and the multiplications are independent of the choice of point up to homotopy. Thus, $\pi_0(X)$ is given a well-defined monoid structure.)

We do not give a proof of this theorem, but refer to [22] or [29]. However, we do give a description of $\mathcal{S}$. We first note that the suspension functor Σ acts as an intertwiner for T^B. The map $\Sigma \underline{\mathcal{C}}^B[X] \longrightarrow \underline{\mathcal{C}}^B[\Sigma X]$ is induced by the evident maps

$$\Sigma\big(C_n(\infty) \underset{\mathcal{S}_n}{\times} X^n\big) \longrightarrow C_n(\infty) \times \mathcal{S}_n(\Sigma X)^n,$$

after factoring out the equivalence relation defining $\underline{\mathcal{C}}^B[-]$. Consequently, we may construct the simplicial T^B-algebra $\underset{\cdot}{T}^B(X, \xi)$, and iterate this construction levelwise, to obtain spaces $\mathcal{S}(X, \xi)_k$, with maps

$$\Sigma \mathcal{S}(X, \xi)_k \xrightarrow{\lambda_k} \mathcal{S}(X, \xi)_{k+1}.$$

Here, $\mathcal{S}(X, \xi)_0 = X$. One is able to show that the adjoint of λ_k is an equivalence if $k \geqslant 1$, and if X is connected so is the adjoint to λ_0. In any case, we obtain a functor to connective spectra. Requirement (a) is clearly satisfied, since T^B preserves weak equivalences.

8.3. *G. Segal's construction of Ω-spectra*

There is another point of view on these ideas, due to G.B. Segal. He enlarges the category of finite ordered sets and order preserving maps to a larger category Γ, so that (roughly) given a simplicial space $X.$, i.e. a functor $\Delta^{op} \longrightarrow X.$, an extension of $X.$ to Γ gives rise to a connective spectrum with zeroth space homotopy equivalent to X.

We outline his ideas. We first define a category Γ to have objects the finite sets $\gamma_n = \{1, \ldots, n\}$ for $n > 0$, and $\gamma_0 = \varnothing$. A morphism $\varphi : \gamma_n \longrightarrow \gamma_k$ is a function $\Phi : \mathcal{P}(\gamma_n) \longrightarrow \mathcal{P}(\gamma_k)$ (where $P(X)$ denotes the power set of X), so that $\Phi(V \cup W) = \Phi(V) \cup \Phi(W)$ and $\Phi(V - W) = \Phi(V) - \Phi(W)$. The first condition shows that Φ is determined by the sets $\Phi(\{i\})$ $1 \leqslant i \leqslant n$, and the second condition shows it is equivalent to $\Phi(\{i\}) \cap \Phi(\{j\}) = \varnothing$ if $i \neq j$.

Given morphisms $\varphi : \gamma_n \longrightarrow \gamma_m$ and $\psi : P(\gamma_m) \longrightarrow P(\gamma_k)$ corresponding to maps

$$\Phi : \mathcal{P}(\gamma_n) \longrightarrow \mathcal{P}(\gamma_m)$$

and $\Psi : P(\gamma_m) \longrightarrow P(\gamma_k)$, then $\psi \circ \varphi$ corresponds to $\Theta : P(\gamma_n) \longrightarrow P(\gamma_k)$, where $\Theta(V) = \Psi \circ \Phi$.

There is an isomorphism of categories from the opposite of the category of based finite sets $\{0, 1, \ldots, n\}$ (0 is the basepoint) and based maps to Γ given by

$$\{0, 1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$$

and $(f : \{0, 1, \ldots, n\} \longrightarrow \{0, 1, \ldots, m\}) \longrightarrow \varphi_f$. Here φ_f corresponds to the map

$$\Phi_f : P(\gamma_m) \longrightarrow P(\gamma_n), \quad \text{where } \Phi_f(V) = f^{-1}(V),$$

for any $V \subset \{1, \ldots, m\}$.

There is also a functor $i : \Delta \longrightarrow \Gamma$, where Δ is the category whose objects are the sets $\{0, 1, \ldots, n\}$, equipped with their standard ordering, and whose morphisms are the order preserving maps. To define i, we first define, for $p, q \in \{0, 1, \ldots, n\}$, $[p, q] = \{r \in \{0, 1, \ldots, n\} \mid p \leqslant r \leqslant q\}$. Note that if $q < p$, $[p, q] = \varnothing$. i is now defined on objects by $i(\{0, 1, \ldots, n\}) = \{1, \ldots, n\}$, and on morphisms by $i(f : \{0, 1, \ldots, n\} \longrightarrow \{0, 1, \ldots, m\}) = \varphi_f$, where φ_f corresponds to the function $\Phi_f : P(\gamma_n) \longrightarrow P(\gamma_m)$ defined by $\Phi_f([1, \ldots, r]) = [1, \ldots, f(r)]$. This gives, for instance,

$$\Phi_f(\{r\}) = [f(r-1) - 1, f(r)] \cap \{1, \ldots, m\}.$$

By a Γ-space, we mean a contravariant functor from Γ to topological spaces. By restriction to Δ, we obtain a simplicial topological space. In particular, we may define $|\Phi|$ for any Γ-space.

Let $\lambda_i : \{1\} \longrightarrow \{1, \ldots, n\}$ be the morphism in Γ corresponding to

$$\Lambda_i : P(\{1\}) \longrightarrow P(\{1, \ldots, n\})$$

given by $\Lambda_i(\{1\}) = \{i\}$. Then, given any Γ-space Φ we have the map

$$\Phi(\{1,\dots,n\}) \xrightarrow{\prod_{i=1}^n \Phi(\lambda_i)} \prod_{i=1}^n \Phi(\{1\}) \tag{8.3}$$

for each n.

Φ is said to be *special* if (8.3) is a weak homotopy equivalence for each n and if $\Phi(\varnothing) \simeq *$. Segal then proves the following result.

THEOREM 8.3.1. *Let X be any Γ-space. Then there is a sequence of functors B^n from the category of Γ-spaces to itself, and natural transformations*

$$|B^n\Phi| \longrightarrow \Omega|B^{n+1}\Phi|,$$

which are weak equivalences if Φ is special. In particular, the sequence

$$|\Phi|, B|\Phi|, B^2|\Phi|, \dots,$$

form an Ω-spectrum, and we obtain a functor B from special spaces to Ω-spectra. Further, there is a functor A from Ω-spectra to Γ-spaces, together with natural equivalences $BA \longrightarrow Id$ and $AB \longrightarrow Id$.

One can go a bit further. In any simplicial space $X.$ with X_0 contractible, one has a well defined homotopy class of maps from ΣX_1 to $|X.|$. Let $\mu : \{1\} \longrightarrow \{1,2\}$ be the morphism in Γ given by $\{1\} \longrightarrow \{1.2\}$. Also, let

$$\tau : \Phi(\{1\}) \times \Phi(\{1\}) \longrightarrow \Phi(\{1,2,\})$$

be the inverse to the weak equivalence occurring in the definition of the notion of a special Γ-space. Then $\Phi(\mu) \circ \tau$ gives an H-space structure on $\Phi(\{1\})$.

THEOREM 8.3.2 (Segal). *The adjoint to the inclusion $\Sigma\Phi(\{1\}) \longrightarrow |\Phi|$ is a weak equivalence if the above described H-space structure admits a homotopy inverse. In particular, this holds if $\Phi(\{1\})$ is connected.*

8.4. *The combinatorial data which build Ω-spectra*

These constructions also allow one to construct spectra from purely combinatorial data. To understand this, we recall the nerve construction, which associates to any category, C, a simplicial set, $N.C$, and hence a topological space. The k-simplices are composable k-tuples of arrows

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_k} x_k$$

in $\mathcal{C}$ if $k > 0$, and are simply objects in $\mathcal{C}$ if $k = 0$. The face maps are given by the following formulae.

$$\begin{cases} d_0(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} x_k) & = (x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_k} x_k), \\ d_i(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} x_k) & = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots x_{i-1} \xrightarrow{f_{i+1}\circ f_i} x_{i+1} \cdots \xrightarrow{f_k} x_k) \\ & \qquad \text{for } 1 \leqslant i < k, \\ d_k(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} x_k) & = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \cdots \xrightarrow{f_{k-1}} x_{k-1}), \\ s_i(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} x_k) & = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \longrightarrow x_{i-1} \xrightarrow{f_i} x_i \xrightarrow{id} x_i \\ & \qquad \xrightarrow{f_{i+1}} x_{i+1} \longrightarrow \cdots \xrightarrow{f_k} x_k). \end{cases}$$

This is often a convenient way to construct spaces and maps, since it is clear that functors induce maps of simplicial sets. Indeed, any simplicial complex is homeomorphic to the nerve of a category, hence any CW complex has the homotopy type of the nerve of a suitable category. It is reasonable to ask what additional structure on the category allows one to construct a spectrum from $N.\mathcal{C}$ in the same way as the Q or $\underline{T}^B$-algebra structures allowed one to construct spectra out of a space X. In order to describe this structure, we need a definition.

DEFINITION 8.4.1. A permutative category is a triple $(\mathcal{C}, \oplus, c)$, where $\mathcal{C}$ is a category, $\oplus : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ is a functor, and c is a natural isomorphism of functors, from $\oplus$ to $\oplus\circ\tau$, where $\tau : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}\times\mathcal{C}$ is the "reverse coordinates" map, subject to the following conditions.

(a) $\oplus$ is associative in the sense that

$$\oplus\circ(Id\times\oplus) = \oplus\circ(\oplus\times Id).$$

(b) $c(y,x)\circ c(x,y) = id_{(x,y)}$ for all $(x,y) \in \mathcal{C}\times\mathcal{C}$.

(c) The diagram

$$\begin{tikzcd} A\oplus B\oplus C \arrow[rr, "c"] \arrow[dr, "id\oplus c"'] & & C\oplus A\oplus B \arrow[dl, "c\oplus id"] \\ & A\oplus C\oplus B & \end{tikzcd}$$

commutes.

(d) $c(A\oplus *) = Id_A$.

The nerve of a permutative category becomes a simplicial monoid. Further, its realization is a $B\mathcal{E}$-algebra, where $B\mathcal{E}$ is the triple corresponding to the Barratt–Eccles $\mathcal{O}$-space $|\mathcal{B}|$. To see this, one observes that $\underline{\mathcal{B}}[|N\mathcal{C}|]$ can itself be described as the realization of the nerve of a category, what one might call the free permutative category on $\mathcal{C}$ (see [34] or or [23]). One can now use the above described space level constructions to arrive at a connective spectrum $Spt(\mathcal{C})$.

THEOREM 8.4.1. *Spt defines a functor from the category of permutative categories to the category of connective spectra. Further, the zeroth space of $Spt(\mathcal{C})$ has the homotopy type of the group completion of the monoid $N_0\mathcal{C}$.*

The last part of the statement is crucial for computations. It has as a corollary the well-known theorem of Barratt, Priddy and Quillen.

COROLLARY 8.4.1 ([28]). *Let $\mathcal{S}_\infty$ denote the infinite symmetric group, i.e.*

$$\varinjlim_n \mathcal{S}_n,$$

where $\mathcal{S}_n$ is included in $\mathcal{S}_{n+1}$ in the evident way. Let $B\mathcal{S}^+_\infty$ denote Quillen's plus construction on $BS\mathcal{S}_\infty$, which abelianizes the fundamental group without affecting homology. Then $Q(S^0) \cong B\mathcal{S}^+_\infty \times \mathbf{Z}$. In particular, if $Q(S^0)_0$ denotes the component consisting of maps of degree 0, $H_*(Q(S^0)_0;\mathbf{Z}) \cong H_*(B\mathcal{S}_\infty;\mathbf{Z})$.

PROOF. The Barratt–Eccles monoid valued construction on S^0, which is the nerve of a category with two objects $*$ and p, and only identity morphisms, is isomorphic to $\coprod_{n\geqslant 0} B\mathcal{S}_n$, equipped with an associative multiplication, carrying $B\mathcal{S}_n \times B\mathcal{S}_m$ into $B\mathcal{S}_{n+m}$. It is not hard to see that the group completion is homotopy equivalent to $B\mathcal{S}_\infty \times \mathbf{Z}$. The result now follows from the above results. □

We conclude with some examples.

(A) The category of finite sets can be given the structure of a permutative category, with the sum operation corresponding to disjoint union. The resulting spectrum is the sphere spectrum.

(B) Let G be a finite group, and consider the category of finite sets with G-action. As in (A) above, we obtain a permutative category, which corresponds to Segal's G-equivariant sphere spectrum. It is a bouquet of spectra parameterized by the conjugacy classes of subgroups K of G, where the summand corresponding to the conjugacy class of K is the suspension spectrum of the classifying space of the group $N_G(K)/K$.

(C) Let A be any abelian group. View it as a category whose objects are the elements of A, and whose only morphisms are identity maps. The addition in A makes this category into a permutative category, in which c is actually an identity map for all pairs of objects in the category. The associated spectrum is the Eilenberg–MacLane spectrum $K(A, 0)$.

(D) Let R be any ring, and consider the category of all finitely generated projective R-modules. This can be given the structure of a permutative category, where the sum operation corresponds to direct sum of modules. The corresponding spectrum is Quillen's algebraic K-theory spectrum for the ring R.

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CHAPTER 14

Stable Operations in Generalized Cohomology

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Contents

HANDBOOK OF ALGEBRAIC TOPOLOGY
Edited by I.M. James

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1. Introduction

For any space X, the Steenrod algebra $\mathcal{A}$ of stable cohomology operations acts on the ordinary cohomology $H^*(X;\mathbf{F}_p)$ to make it an $\mathcal{A}$-algebra. Milnor discovered [22] that it is useful to treat $H^*(X;\mathbf{F}_p)$ as a comodule over the dual of $\mathcal{A}$, which becomes a Hopf algebra. Adams extended this program in [1], [3] to multiplicative generalized cohomology theories $E^*(-)$, under appropriate hypotheses. The coefficient ring E^* is now graded, and $E^*(X)$ is an E^*-algebra.

Our purpose is to describe the structure of the stable operations on $E^*(-)$ in a manner that will generalize in [9] to unstable operations. Unlike some treatments, we impose no finiteness or connectedness conditions whatever on the spaces and spectra involved, only a single freeness condition on E. We emphasize universal properties as the appropriate setting for many results. An early version of some of the ideas is presented in [8], which is limited to ordinary cohomology, MU, and BP.

For general E, the stable operations form the endomorphism ring $\mathcal{A} = E^*(E, o)$ of E (in our notation). For each $x \in E^*(X)$, we have the E^*-module homomorphism $x^*: \mathcal{A} \to E^*(X)$ given by $x^*r = \pm\, rx$. The key idea is (roughly) that given an E^*-module M, we define SM as the set of all E^*-module homomorphisms $\mathcal{A} \to M$; this is to be thought of as the set of candidates for the values of all operations on a typical element of M.

Generally, we encode the action of $\mathcal{A}$ on a stable module M as the function $\rho_M: M \to SM$ given by $(\rho_M x)r = \pm\, rx$. There is an E^*-module structure on SM (different from the obvious one) that makes ρ_M a homomorphism of E^*-modules. This is not yet enough; composition of operations makes the functor S what is known as a *comonad*, and we need (M, ρ_M) to be a *coalgebra* over this comonad. When M is an E^*-algebra, so is SM, and we can similarly define stable algebras.

This work serves as more than just a pattern for the promised unstable theory of [9]. To compare unstable structures with the analogous stable structures, we shall there construct suitable natural transformations; this is far easier to do when both theories are developed in the same manner. Much of the basic category theory is the same for either case; we keep it all here for convenience. Finally, we need specific stable results for later use.

Outline. In Section 2, we introduce five assorted ring spectra E, which will serve throughout as our examples. We review some elementary category theory and set up notation.

In Sections 3 and 4, we study E-(co)homology in enough detail to suggest what categories to use. In Section 9, we consider (co)homology in the stable homotopy category of spectra. It is essential for us to work in the correct categories, in order to make our categorical machinery run smoothly; otherwise it does not run at all. We therefore take pains in Section 6 to say precisely what our categories are.

In Section 7, we discuss the various kinds of algebraic object, such as group, module, and ring, that we need in general categories. In Section 8, we rework the definition of a module over a ring until we find a way that will generalize to the unstable context.

In Section 10, we discuss stable modules from several points of view. We introduce the comonad S, and define a stable module as an S-coalgebra. Theorem 10.16 shows that $E^*(X)$ is (more or less) a stable module.

In Section 11, we make the homology $E_*(E, o)$ a coalgebra (in a sense), provided only that it is a free E^*-module. A stable module then becomes a comodule over it; indeed, Theorem 11.13 shows that the theories of stable modules and stable comodules are entirely equivalent. Theorem 11.14 provides a useful universal property of $E_*(E, o)$. Theorem 11.35 shows that our structure on $E_*(E, o)$ agrees with that introduced by Adams [1].

Everything mentioned so far works for spectra X, too. In Section 12, we take account of the multiplication present on $E^*(X)$ when X is a space by making SM an E^*-algebra whenever M is. This leads to the definition of a stable algebra. Again, there is an equivalent comodule version.

All our examples of E-cohomology come with a complex orientation. This has standard implications for the structure of $E^*(\mathbb{C}P^\infty)$ etc., which we review in Section 5. In Section 13, we study the consequences for operations.

In Section 14, we present the structure on $E_*(E, o)$ in detail for each of our five examples E. We do not actually construct the operations, which are all well known. It is clear that many other examples are available.

In Section 15, we study the special case of BP-cohomology in greater depth. For a general introduction, see Wilson [37]. Stable BP-operations are well established; a short early history would include Landweber [17], Novikov [28], Quillen [30], Adams [3], Zahler [41], [42], Miller, Ravenel and Wilson [21], and more recently, Ravenel's book [31]. We review Landweber's filtration theorem, for imitation in [9].

An index of symbols is included at the end.

Acknowledgements. We thank Dave Johnson and Steve Wilson for making this paper necessary. As noted, it serves chiefly as a platform for [9]. It incorporates several suggestions of Steve Wilson, especially the use of corepresented functors in Section 8. We also thank Nigel Ray for pointing out some useful references.

2. Notation and five examples

Our five examples of commutative ring spectra E are:

- $H(\mathbb{F}_p)$ The Eilenberg–MacLane spectrum, for a fixed prime $p \geqslant 2$, which represents ordinary cohomology $H^*(-;\mathbb{F}_p)$ and is a ring spectrum (see, e.g., Switzer [34, 13.88]);
- BP The Brown–Peterson spectrum, for a fixed prime $p \geqslant 2$ (which is suppressed from the notation), a ring spectrum by Quillen [29];
- MU The unitary (or complex) cobordism Thom spectrum, which is a ring spectrum (see, e.g., Switzer [34, 13.89]);
- KU The complex Bott spectrum (often written K), which represents topological complex K-theory and is a ring spectrum [ibid., 13.90];
- $K(n)$ The Morava K-theory spectrum, for a fixed prime $p > 2$ (again suppressed from the notation), and any $n \geqslant 0$. (We take $p > 2$ in order to ensure that the multiplication is commutative as well as associative; see Morava [26], and

especially Shimada and Yagita [33, Corollary 6.7] or Würgler [38, Theorem 2.14]. See [16] for background information.) In particular, $K(0) = H(\mathbb{Q})$ (for any p), and $K(1)$ is a summand of KU-theory mod p.

Indeed, all our ring spectra are understood to be commutative. Each E defines a multiplicative cohomology theory $E^*(X)$ and homology theory $E_*(X)$, which we discuss in Sections 3 and 4. They have the same coefficient ring E^*.

Because we deal almost exclusively in cohomology, we assign the *degree* n to cohomology classes in $E^n(X)$ and elements of E^n; this forces homology classes in $E_n(X)$ to have degree $-n$. Note that under this convention, elements of BP^* and MU^* are given *negative* degrees.

For any space X, $E^*(X)$ and $E_*(X)$ are E^*-modules. We therefore adopt E^* as our ground ring throughout, and all tensor products and groups $\mathrm{Hom}(M, N)$ are taken over E^* unless otherwise specified. Except for (co)homology, we generally follow the practice of [25] in writing a graded group with components M^n as M rather than M^*. When we do write M^* (e.g., E^* as above), we mean the whole graded group, not a typical component.

All our rings and algebras are associative and are presumed to have a unit element 1, which is to be preserved by homomorphisms. Dually, coalgebras are assumed to be coassociative.

Summations are often understood as taken over all available values of the index.

We do not attempt to give each construct a unique symbol. For example, all multiplications are named ϕ, which we decorate as ϕ_S etc. only as needed to distinguish different multiplications. All actions are named λ and all coactions are named ρ. To compensate, we generally specify where each equation takes place.

Signs. We follow the convention that a minus sign should be introduced *whenever* two symbols of odd degree become transposed for any reason. As explained in [7], this is a purely lexical convention, which depends only on the order of appearance of the various symbols, not on their meanings. The principle is that consistency will be maintained provided one starts from equations that conform and performs only "reasonable" manipulations on them. The main requirement is that each symbol having a degree should appear exactly once in every term of an equation.

Category theory. Our basic reference is MacLane's book [20], which also provides most of our notation and terminology.

In any category $\mathcal{A}$, the set of morphisms from X to Y is denoted $\mathcal{A}(X, Y)$, or occasionally $\mathrm{Mor}(X, Y)$. If $\mathcal{A}$ is a *graded* category (always assumed additive), $\mathcal{A}^n(X, Y)$ denotes the abelian group of morphisms from X to Y of degree n. Unmarked arrows are intended to be the obvious morphisms. We write

$$p_1: X \times Y \to X \quad \text{and} \quad p_2: X \times Y \to Y$$

for the projections from the product $X \times Y$ to its factors, and dually $i_1: X \to X \amalg Y$ and $i_2: Y \to X \amalg Y$ for coproducts. We also write $q: X \to T$ for the unique morphism to a terminal object T.

We denote by $I: \mathcal{A} \to \mathcal{A}$ the identity functor of $\mathcal{A}$. We sometimes find it useful to write a natural transformation α between functors $F, F': \mathcal{A} \to \mathcal{B}$ as

$$\alpha: F \to F': \mathcal{A} \to \mathcal{B}.$$

If $\mathcal{A}$ and $\mathcal{B}$ are graded, we can have $\deg(\alpha) = m \neq 0$; in this case, we require $\alpha Y \circ Ff = (-1)^{m \deg(f)} F'f \circ \alpha X$ for each morphism $f: X \to Y$. In contrast, our graded functors invariably preserve degree.

If $\alpha': F' \to F''$ is another natural transformation, we have the composite natural transformation $\alpha' \circ \alpha: F \to F''$. There is the identity natural transformation $1: F \to F$. Given also $G: \mathcal{B} \to \mathcal{C}$, we denote the composite functor as $GF: \mathcal{A} \to \mathcal{C}$ (never $G \circ F$), and define the natural transformation $G\alpha: GF \to GF': \mathcal{A} \to \mathcal{C}$ by $(G\alpha)X = G(\alpha X)$. Similarly, given $\beta: G \to G'$, we define $\beta F: GF \to G'F: \mathcal{A} \to \mathcal{C}$ by $(\beta F)X = \beta(FX)$. We also have $\beta\alpha = \beta F' \circ G\alpha = G'\alpha \circ \beta F: GF \to G'F': \mathcal{A} \to \mathcal{C}$ (or $\pm G'\alpha \circ \beta F$ in the graded case).

We make incessant use of Yoneda's Lemma [20, III.2].

Adjoint functors. It should be no surprise that we have numerous pairs of adjoint functors. Suppose given a functor $V: \mathcal{B} \to \mathcal{A}$ (which is usually, but not necessarily, some forgetful functor) and an object A in $\mathcal{A}$.

DEFINITION 2.1. We call an object M in $\mathcal{B}$ *V-free* on A, with *basis* $i: A \to VM$, a morphism in $\mathcal{A}$, if for each B in $\mathcal{B}$, any morphism $f: A \to VB$ in $\mathcal{A}$ "extends" to a unique morphism $g: M \to B$ in $\mathcal{B}$, called the *left adjunct* of f, in the sense that $Vg \circ i = f: A \to VB$.

In the language of [20, III.1], i is a *universal arrow*, which induces the bijection $\mathcal{B}(M, B) \cong \mathcal{A}(A, VB)$. The free object M is unique up to canonical isomorphism, but there is no guarantee that one exists. In the favorable case when we have a free object FA for each A in $\mathcal{A}$, with basis $\eta A: A \to VFA$, there is a unique way to define Fh for each morphism h in $\mathcal{A}$ to make η natural; then F becomes a functor and the isomorphism

$$\mathcal{B}(FA, B) \cong \mathcal{A}(A, VB) \tag{2.2}$$

is natural in both A and B. Explicitly, we recover $f: A \to VB$ from $g: FA \to B$ as

$$f = Vg \circ \eta A: A \longrightarrow VFA \longrightarrow VB \quad \text{in } \mathcal{A}. \tag{2.3}$$

For any B, we define $\varepsilon B: FVB \to B$ in $\mathcal{B}$ as extending $1: VB \to VB$. Then $\varepsilon: FV \to I$ is also natural, and we may construct the left adjunct g of f as

$$g = \varepsilon B \circ Ff: FA \longrightarrow FVB \longrightarrow B \quad \text{in } \mathcal{B}. \tag{2.4}$$

The fact that this is inverse to eq. (2.3) is neatly expressed by the pair of identities

$$\begin{aligned} &\text{(i)} \quad V\varepsilon \circ \eta V = 1: V \longrightarrow V: \mathcal{B} \longrightarrow \mathcal{A}, \\ &\text{(ii)} \quad \varepsilon F \circ F\eta = 1: F \longrightarrow F: \mathcal{A} \longrightarrow \mathcal{B}. \end{aligned} \tag{2.5}$$

We summarize the basic facts about adjoint functors from [20, Theorem IV.1.2].

THEOREM 2.6. *The following conditions on a functor $V: \mathcal{B} \to \mathcal{A}$ are equivalent:*

(i) *V has a left adjoint $F: \mathcal{A} \to \mathcal{B}$;*
(ii) *V is a right adjoint to some functor $F: \mathcal{A} \to \mathcal{B}$;*
(iii) *There is a functor $F: \mathcal{A} \to \mathcal{B}$ and an isomorphism* (2.2), *natural in A and B;*
(iv) *For all A in $\mathcal{A}$, we can choose a V-free object FA and a basis ηA of it;*
(v) *There is a functor $F: \mathcal{A} \to \mathcal{B}$ with natural transformations $\eta: I \to VF$ and $\varepsilon: FV \to I$ that satisfy eqs.* (2.5).

In view of the symmetry in (v), or between (i) and (ii), we have the dual result, which we do not state. Nevertheless, we do give the dual to Definition 2.1.

DEFINITION 2.7. The object N in $\mathcal{B}$ is *V-cofree* on A, with *cobasis* $p: VN \to A$, a morphism in $\mathcal{A}$, if for each B in $\mathcal{B}$, any $f: VB \to A$ in $\mathcal{A}$ "lifts" uniquely to a morphism $g: B \to N$ in $\mathcal{B}$, called the *right adjunct* of f, in the sense that $p \circ Vg = f$.

3. Generalized cohomology of spaces

In this section and the next, we review multiplicative cohomology theories $E^*(-)$ and their associated homology theories $E_*(-)$ in sufficient depth to decide what objects our categories should contain. We also establish much of our notation.

Spaces. We find we have to work mostly with *unbased* spaces. The most convenient spaces are CW-complexes. We denote by T the one-point space. It is sometimes useful to allow also spaces that are homotopy equivalent to CW-complexes, so that we can form products and loop spaces directly. A *pair* (X, A) of spaces is assumed to be a CW-pair (or homotopy equivalent, as a pair, to one).

Ungraded cohomology. For our purposes, an *ungraded cohomology theory* is a homotopy-invariant contravariant functor $h(-)$ that assigns to each space X an abelian group $h(X)$, and satisfies the usual two axioms:

(i) $h(-)$ is *half-exact:* If $X = A \cup B$, where A and B are well-behaved subspaces (e.g., subcomplexes of a CW-complex X), and $y \in h(A)$ and $z \in h(B)$ agree in $h(A \cap B)$, there exists $x \in h(X)$ (not in general unique) that lifts both y and z;
(ii) $h(-)$ is *strongly additive:* For any topological disjoint union $X = \coprod_\alpha X_\alpha$, the inclusions $X_\alpha \subset X$ induce $h(X) \cong \prod_\alpha h(X_\alpha)$. (3.1)

For a space X with basepoint $o \in X$, we may define the *reduced* cohomology $h(X, o)$ by the split short exact sequence

$$0 \longrightarrow h(X, o) \xrightarrow{\subset} h(X) \longrightarrow h(o) \longrightarrow 0. \tag{3.2}$$

We recover the *absolute* cohomology $h(X)$ by constructing the disjoint union X^+ of X with a (new) basepoint; by (ii), $h(X^+) \cong h(X) \oplus h(o)$ and the inclusion $X \subset X^+$ induces an isomorphism

$$h(X^+, o) \cong h(X). \tag{3.3}$$

For a good pair (X, A) of spaces, we may define the *relative* cohomology as

$$h(X, A) = h(X/A, o), \tag{3.4}$$

and these groups behave as expected. We generalize eq. (3.2).

LEMMA 3.5. *If A is a retract of X, we have the split short exact sequence*

$$0 \longrightarrow h(X, A) \longrightarrow h(X) \longrightarrow h(A) \longrightarrow 0.$$

If A has a basepoint o, we have also the split short exact sequence

$$0 \longrightarrow h(X, A) \longrightarrow h(X, o) \longrightarrow h(A, o) \longrightarrow 0.$$

With no basepoints, we have to be a little careful in representing $h(-)$. Let Ho be the homotopy category of spaces that are (homotopy equivalent to) CW-complexes.

THEOREM 3.6. *Let $h(-)$ be an ungraded cohomology theory as above. Then:*

(a) *$h(-)$ is represented in Ho by an H-space H, with a universal class $\iota \in h(H, o) \subset h(H)$ that induces an isomorphism $\mathit{Ho}(X, H) \cong h(X)$ of abelian groups by $f \mapsto h(f)\iota$ for all X;*

(b) *For any cohomology theory $k(-)$, operations $\theta: h(-) \to k(-)$ correspond to elements $\theta\iota \in k(H)$.*

PROOF. What Brown's representation theorem [10, Theorem 2.8, Example 3.1] actually provides is a based connected space (H', o), which represents $h(-, o)$ on *based connected* spaces (X, o) only. Then West [35] shows that $h(-, o)$ is represented on all *based* spaces by the product space

$$H = h(T) \times H', \tag{3.7}$$

where we treat the group $h(T)$ as a discrete space. By eq. (3.3), H also represents $h(-)$ in the unbased category Ho.

The map $\omega: T \to H$ that corresponds to $0 \in h(T)$ furnishes H with a (homotopically well-defined) basepoint, and the exact sequence (3.2) shows that $\iota \in h(H, o)$. Yoneda's Lemma represents the addition

$$\mathit{Ho}(X, H \times H) \cong h(X) \times h(X) \xrightarrow{+} h(X) \cong \mathit{Ho}(X, H)$$

by an *addition map* $\mu: H \times H \to H$ which makes H an H-space, and also gives (b). (Lemma 7.7(a) will tell us much more about H.) □

Example. KU. For finite-dimensional spaces X, the ungraded cohomology theory $KU(X)$ is defined (e.g., Husemoller [15]) as the Grothendieck group of complex vector bundles over X. The class of the vector bundle ξ is denoted $[\xi]$, and every element of $KU(X)$ has the form $[\xi] - [\eta]$. The trivial n-plane bundle is denoted simply n. Addition is defined from the Whitney sum of vector bundles, $[\xi] + [\eta] = [\xi \oplus \eta]$, and multiplication from the tensor product, $[\xi][\eta] = [\xi \otimes \eta]$. In particular, $KU(T) \cong \mathbb{Z}$, as a ring.

Let (X, o) be a based connected space, still finite-dimensional. Because any vector bundle ξ over X has a stable inverse η such that $\xi \oplus \eta$ is trivial, every element of $KU(X, o)$ can be written in the form $[\xi] - n$ for some n-plane vector bundle ξ, provided n is large enough. The bundle ξ has a classifying map $X \to BU(n) \subset BU$, which leads to the representation $Ho(X, BU) \cong KU(X, o)$. As in the proof of Theorem 3.6, this extends to an isomorphism $Ho(X, \mathbb{Z} \times BU) \cong KU(X)$, valid for all finite-dimensional spaces X.

To extend $KU(-)$ to all spaces as an ungraded cohomology theory, we must *define* $KU(X) = Ho(X, \mathbb{Z} \times BU)$. It remains true that any vector bundle ξ over X defines an element $[\xi] \in KU(X)$, but in general, not all elements of $KU(X)$ have the form $[\xi] - [\eta]$.

Splittings. All our splittings depend on the following elementary result.

LEMMA 3.8. *Assume that $\theta: h(-) \to h(-)$ is an idempotent cohomology operation, $\theta \circ \theta = \theta$. Then the image $\theta h(-)$ also satisfies the axioms* (3.1).

PROOF. For (i), given $y \in \theta h(A)$ and $z \in \theta h(B)$ that agree in $h(A \cap B)$, the half-exactness of h yields an element $x \in h(X)$ that lifts y and z. Because θ is idempotent, $\theta x \in \theta h(X)$ also lifts y and z, to show that (i) holds.

For (ii), we need only the naturality of θ. Given elements $x_\alpha = \theta x'_\alpha \in \theta h(X_\alpha)$, axiom (ii) for h provides $x' \in h(X)$ that lifts each x'_α. Then $x = \theta x' \in \theta h(X)$ lifts each x_α, and is unique because h satisfies (ii). □

We immediately deduce the standard tool for constructing splittings. Theorem 3.6(b) allows us to write the identity operation as ι.

LEMMA 3.9. *Let θ be an additive idempotent operation on the ungraded cohomology theory $h(-)$. Then:*

(a) *$\iota - \theta$ is also idempotent;*

(b) *We can define ungraded cohomology theories*

$$h'(X) = \operatorname{Ker}\left[\theta: h(X) \longrightarrow h(X)\right] = \operatorname{Im}\left[\iota - \theta: h(X) \longrightarrow h(X)\right]$$

and

$$h''(X) = \operatorname{Ker}\left[\iota - \theta: h(X) \longrightarrow h(X)\right] = \operatorname{Im}\left[\theta: h(X) \longrightarrow h(X)\right];$$

(c) *We have the natural direct sum decomposition* $h(X) = h'(X) \oplus h''(X)$;

(d) *If the H-spaces* H' *and* H'' *represent* $h'(-)$ *and* $h''(-)$ *as in Theorem* 3.6(a), *then* $H' \times H''$ *represents* $h(-)$.

For future use in [9], we extend this result to certain *nonadditive* idempotent operations. To emphasize the nonadditivity, we retain the parentheses in $\theta(-)$.

LEMMA 3.10. *Assume the* nonadditive *operation* θ *on the ungraded cohomology theory* $h(-)$ *satisfies the axioms:*

$$\begin{aligned} &\text{(i) } \theta(0) = 0;\\ &\text{(ii) } \theta(x + y - \theta(y)) = \theta(x) \text{ for any } x, y \in h(X). \end{aligned} \tag{3.11}$$

Then:

(a) θ *and* $\iota - \theta$ *are idempotent;*

(b) *We can define the kernel cohomology theory* $h'(-) = \operatorname{Ker}\theta = \operatorname{Im}(\iota-\theta)$ *as a subgroup of* $h(-)$;

(c) *We can define the coimage cohomology theory* $h''(X) = \operatorname{Coim}\theta = h(X)/h'(X)$ *as a quotient of* $h(X)$, *with projection* $\pi\colon h(X) \twoheadrightarrow h''(X)$;

(d) *We have the natural short exact sequence of ungraded cohomology theories*

$$0 \longrightarrow h'(X) \xrightarrow{\subset} h(X) \xrightarrow{\pi} h''(X) \longrightarrow 0; \tag{3.12}$$

(e) θ *induces a nonadditive operation* $\overline{\theta}\colon h''(X) \to h(X)$ *which splits* (3.12) *and induces the bijection of sets* $h''(X) = \operatorname{Coim}\theta \cong \operatorname{Im}[\theta\colon h(X) \to h(X)]$;

(f) *The short exact sequence* (3.12) *is represented by a fibration of H-spaces and H-maps*

$$H' \longrightarrow H \longrightarrow H''$$

in which $H \to H''$ *admits a section (not an H-map) and* $H \simeq H' \times H''$ *as spaces.*

REMARK. Note the asymmetry of the situation. It is necessary to distinguish (cf. [20, VIII.3]) between the *coimage* of θ, which is a quotient group of $h(X)$, and the *image* of θ, which in interesting cases is only a *subset* of $h(X)$, not a subgroup (otherwise Lemma 3.9 would be available).

PROOF. For (a), we put $x = \theta(y)$ in (ii) to see that θ is idempotent. If we put $x = 0$ instead, we see that $\theta(y - \theta(y)) = 0$, which implies that $\iota - \theta$ is idempotent.

For (b), we have just seen that $\operatorname{Im}(\iota-\theta) \subset \operatorname{Ker}\theta$. The opposite inclusion is trivial, because if $\theta(x) = 0$, we can write $x = (\iota-\theta)(x)$.

To see that $h'(X)$ is a subgroup, we first note that $0 \in h'(X)$ by (i). Take any $z \in h'(X)$, which we may write as $z = y - \theta(y)$. Then by (ii), $x + z \in h'(X)$ if and only if $x \in h'(X)$. Therefore by Lemma 3.8 (which did not require θ to be additive), $h'(-)$ is a cohomology theory.

This allows us to define the coimage $h''(X)$ in (c) as an abelian group. By (ii) and (b), $\overline{\theta}$ in (e) is well defined and provides the inverse bijection to $\operatorname{Im}\theta \subset h(X) \to h''(X)$. By Lemma 3.8, $\operatorname{Im}\theta$ and hence $h''(-)$ satisfy the axioms (3.1), and h'' is a cohomology theory. Then (d) combines (b) and (c).

For (f), we represent π by a fibration $H \to H''$, which is an H-map of H-spaces. Then $\overline{\theta}$ is represented by a section. It follows from the short exact sequence (3.12) that the fibre of π represents h'. □

Graded cohomology. A *graded cohomology theory* $E^*(-)$ consists of an ungraded cohomology theory $E^n(-)$ for each integer n, connected by natural *suspension isomorphisms*

$$\Sigma\colon E^n(X) \cong E^{n+1}(S^1 \times X, o \times X) \tag{3.13}$$

of abelian groups, much as in Conner and Floyd [12, §4]. By Lemma 3.5, there is a split short exact sequence

$$0 \longrightarrow E^{n+1}(S^1 \times X, o \times X) \longrightarrow E^{n+1}(S^1 \times X) \longrightarrow E^{n+1}(o \times X) \longrightarrow 0. \tag{3.14}$$

For a based space (X, o), Σ induces, with the help of eq. (3.4), the commutative diagram of split exact sequences

$$\begin{tikzcd} E^n(X,o) \arrow[r] \arrow[d, dotted, "\cong", "\Sigma"'] & E^n(X) \arrow[r] \arrow[d, "\cong", "\Sigma"'] & E^n(o) \arrow[d, "\cong", "\Sigma"'] \\ E^{n+1}(\Sigma X, o) \arrow[r] & E^{n+1}\left(\dfrac{S^1 \times X}{o \times X}, o\right) \arrow[r] & E^{n+1}(S^1 \times o, o) \end{tikzcd} \tag{3.15}$$

whose bottom row comes from Lemma 3.5, where the *suspension* of X is

$$\Sigma X = S^1 \wedge X = \frac{S^1 \times X}{S^1 \vee X} \cong \frac{S^1 \times X}{o \times X} \Big/ S^1 \times o.$$

We deduce the more commonly used *reduced* suspension isomorphism $\Sigma\colon E^n(X, o) \cong E^{n+1}(\Sigma X, o)$. In view of eq. (3.3), we recover eq. (3.13) as a special case.

By iteration of eq. (3.13), or analogy, there are k-fold suspension isomorphisms for all $k > 0$

$$\Sigma^k\colon E^n(X) \cong E^{n+k}(S^k \times X, o \times X). \tag{3.16}$$

THEOREM 3.17. *Any graded cohomology theory $E^*(-)$ is represented in Ho by an Ω-spectrum $n \mapsto \underline{E}_n$, consisting of H-spaces $\underline{E}_n$ equipped with universal elements $\iota_n \in E^n(\underline{E}_n, o) \subset E^n(\underline{E}_n)$ and isomorphisms (in Ho) of H-spaces $\underline{E}_n \simeq \Omega\underline{E}_{n+1}$.*

PROOF. Theorem 3.6 provides the H-spaces $\underline{E}_n$ and elements ι_n. Then as a functor of X, the sequence (3.14) is represented by the fibration of H-spaces

$$\Omega\underline{E}_{n+1} \longrightarrow \underline{E}_{n+1}^{S^1} \longrightarrow \underline{E}_{n+1}^{\{o\}}$$

(which is not to be confused with the path space fibration). In particular,

$$E^{n+1}\big(S^1 \times X, o \times X\big) \cong Ho\big(X, \Omega\underline{E}_{n+1}\big), \tag{3.18}$$

and eq. (3.13) is represented by the desired isomorphism $\underline{E}_n \simeq \Omega\underline{E}_{n+1}$. $\square$

Similarly, Σ^k in eq. (3.16) is represented by the iterated homotopy equivalence $\underline{E}_n \simeq \Omega^k\underline{E}_{n+k}$.

We find it more convenient to work with the left adjunct $\Sigma\underline{E}_n \to \underline{E}_{n+1}$ of the isomorphism. We introduce a sign, which is suggested by Section 9.

DEFINITION 3.19. For each n, we define the based structure map $f_n: \Sigma\underline{E}_n \to \underline{E}_{n+1}$ by

$$f_n^*\iota_{n+1} = (-1)^n \Sigma\iota_n \quad \text{in } E^{n+1}\big(\Sigma\underline{E}_n, o\big). \tag{3.20}$$

Theorem 3.17 gives a 1–1 correspondence between cohomology classes and maps. We suspend in both senses and compare.

LEMMA 3.21. *Given a based space X, suppose that the class $x \in E^n(X, o)$ corresponds to the based map $x_U: X \to \underline{E}_n$. Then the map $f_n \circ \Sigma x_U: \Sigma X \to \Sigma\underline{E}_n \to \underline{E}_{n+1}$ corresponds to the class $(-1)^n\Sigma x \in E^{n+1}(\Sigma X, o)$* (see diag. (3.15)).

PROOF. In $E^*(\Sigma X, o)$, we have $(\Sigma x_U)^* f_n^*\iota_{n+1} = (-1)^n(\Sigma x_U)^*\Sigma\iota_n = (-1)^n\Sigma x$. $\square$

Multiplicative graded cohomology. The cohomology theory $E^*(-)$ is *multiplicative* if $E^*(X)$ is naturally a commutative graded ring (with unit element 1_X and the customary signs) and eq. (3.13) is an isomorphism of $E^*(X)$-modules of degree 1, where we use the projection $p_2: S^1 \times X \to X$ to make (3.14) a short exact sequence of $E^*(X)$-modules. Explicitly, $\Sigma(xy) = (-1)^i(p_2^*x)\Sigma y$ for $x \in E^i(X)$ and $y \in E^*(X)$. The *coefficient ring* is defined as $E^* = E^*(T)$.

The natural ring structure on $E^*(X)$ is equivalent to having natural *cross product* pairings

$$\times: E^k(X) \times E^m(Y) \longrightarrow E^{k+m}(X \times Y)$$

that are biadditive, commutative, associative, and have $1_T \in E^*(T)$ as the unit. They may be defined in terms of the ring structure as $x \times y = (p_1^*x)(p_2^*y)$; conversely, given $x, y \in E^*(X)$, we recover $xy = \Delta^*(x \times y)$, using the diagonal map $\Delta: X \to X \times X$.

By means of $X \cong T \times X$, $E^*(X)$ becomes a module over $E^* = E^*(T)$, and we may rewrite the $\times$-product more usefully as

$$\times: E^*(X) \otimes E^*(Y) \longrightarrow E^*(X \times Y), \tag{3.22}$$

where the tensor product is taken over E^*. On the rare occasion that this is an isomorphism, it is called the *cohomology Künneth isomorphism*.

DEFINITION 3.23. We define the *canonical generator* $u_1 \in E^1(S^1, o) \subset E^1(S^1)$ as corresponding to $\Sigma 1_T \in E^1(S^1 \times T, o \times T) \cong E^1(S^1, o)$, by taking $X = T$ in eq. (3.13).

Then by naturality, for any $x \in E^n(X)$ we have

$$\Sigma x = u_1 \times x \quad \text{in } E^{n+1}\big(S^1 \times X, o \times X\big). \tag{3.24}$$

Similarly, $\Sigma^k x = u_k \times x$ in eq. (3.16), where the canonical generator $u_k \in E^k(S^k, o)$ corresponds to $\Sigma^k 1_T$.

THEOREM 3.25. *A multiplicative structure on the graded cohomology theory $E^*(-)$ is represented by multiplication maps $\phi: \underline{E}_k \times \underline{E}_m \to \underline{E}_k \wedge \underline{E}_m \to \underline{E}_{k+m}$ and a unit map $\eta: T \to \underline{E}_0$, such that:*

(a) *The cross product of $x \in E^k(X)$ and $y \in E^m(Y)$ is*

$$x \times y: X \times Y \xrightarrow{x \times y} \underline{E}_k \times \underline{E}_m \xrightarrow{\phi} \underline{E}_{k+m}; \tag{3.26}$$

(b) *The unit element of $E^*(X)$ is $1_X = \eta \circ q: X \to T \to \underline{E}_0$;*

(c) *Given $v \in E^h$, the module action $v: E^k(-) \to E^{k+h}(-)$ is represented by the map*

$$\xi v: \underline{E}_k \cong T \times \underline{E}_k \xrightarrow{v \times 1} \underline{E}_h \times \underline{E}_k \xrightarrow{\phi} \underline{E}_{k+h}; \tag{3.27}$$

(d) *The structure map $\Sigma \underline{E}_n \to \underline{E}_{n+1}$ of Definition 3.19 is*

$$f_n: \Sigma \underline{E}_n = S^1 \wedge \underline{E}_n \xrightarrow{(-1)^n u_1 \wedge 1} \underline{E}_1 \wedge \underline{E}_n \xrightarrow{\phi} \underline{E}_{n+1}. \tag{3.28}$$

PROOF. We take $\iota_k \times \iota_m \in E^{k+m}(\underline{E}_k \times \underline{E}_m)$ as ϕ and $1_T \in E^0(T)$ as η; then (a) and (b) follow by naturality. By definition, vx corresponds to $v \times x \in E^{k+h}(T \times X)$. Thus by eq. (3.26), scalar multiplication by v in $E^*(X)$ is represented by eq. (3.27); equivalently, we use the identity $vx = (v1)x$ in $E^*(X)$. By eq. (3.24), the map (3.28) takes ι_{n+1} to $(-1)^n \Sigma \iota_n$ and is therefore f_n. □

From now on, we shall assume that $E^*(-)$ is multiplicative. We shall have much more to say (in Corollary 7.8) about the spaces $\underline{E}_n$, once we have the language.

Example. KU. The key to making a graded cohomology theory out of $KU(-)$ is Bott periodicity, in the following form. (See Atiyah and Bott [6] or Husemoller [15, Chapter 10] for an elegant proof that is close to our point of view.) It gives us everything we need to build a periodic graded cohomology theory.

THEOREM 3.29 (Bott). *The Hopf line bundle ξ over $\mathbb{C}P^1 \cong S^2$ induces an isomorphism*

$$([\xi]-1)\times -: KU(X) \cong KU\big(S^2\times X, o\times X\big)$$

for any space X.

DEFINITION 3.30. We define the *graded* cohomology theory $KU^*(-)$ as having the representing spaces $\underline{KU}_{2n} = \mathbb{Z}\times BU$ and $\underline{KU}_{2n+1} = U$ for all integers n, so that $KU^{2n}(X) = Ho(X, \mathbb{Z}\times BU) = KU(X)$ and $KU^{2n+1}(X) = Ho(X, U)$.

In odd degrees, we use the suspension isomorphism

$$KU^{2n+1}(X) \cong KU^{2n+2}\big(S^1\times X, o\times X\big) \cong Ho\big(X, \Omega(\mathbb{Z}\times BU)\big) \tag{3.31}$$

represented by $U \simeq \Omega BU = \Omega(\mathbb{Z}\times BU)$. In even degrees, rather than specify $\Sigma: KU^{2n}(X) \cong KU^{2n+1}(S^1\times X, o\times X)$ directly, we use the double suspension isomorphism $\Sigma^2: KU^{2n}(X) \cong KU^{2n+2}(S^2\times X, o\times X)$ provided by Theorem 3.29.

The ring structure on $KU(X)$ makes $KU^*(X)$ multiplicative, with the help of eq. (3.31). (The only case that presents any difficulty is

$$KU^{2m+1}(X)\times KU^{2n+1}(X) \longrightarrow KU^{2(m+n+1)}(X),$$

which requires another appeal to Theorem 3.29.)

The coefficient ring is clearly $\mathbb{Z}[u, u^{-1}]$, where we *define* $u \in KU^{-2} = KU(T) = \mathbb{Z}$ as the copy of 1. To keep the degrees straight, all we have to do is insert appropriate powers u^n everywhere. (It is traditional to simplify matters by setting $u = 1$, thus making $KU^*(-)$ a $\mathbb{Z}/2$-graded cohomology theory; however, this strategy is not available to us, as it would allow only operations that preserve this identification.) For example, Theorem 3.29 provides the canonical element

$$u_2 = u^{-1}([\xi]-1) \quad \text{in } KU^2\big(S^2, o\big) \subset KU^2\big(S^2\big). \tag{3.32}$$

The skeleton filtration. The cohomology $E^*(X)$ is usually uncountable for infinite X, which makes Künneth isomorphisms (3.22) unlikely without some kind of completion. This suggests that it ought to be given a topology.

Given any space X (which we take as a CW-complex), the *skeleton filtration* of $E^*(X)$ is defined by

$$F^sE^*(X) = \operatorname{Ker}\big[E^*(X) \longrightarrow E^*\big(X^{s-1}\big)\big] = \operatorname{Im}\big[E^*\big(X, X^{s-1}\big) \longrightarrow E^*(X)\big] \tag{3.33}$$

for $s \geqslant 0$, where X^n denotes the n-skeleton of X, and this filtration is natural. It is a decreasing filtration by ideals,

$$E^*(X) = F^0E^*(X) \supset F^1E^*(X) \supset F^2E^*(X) \supset \cdots.$$

Moreover, it is multiplicative,

$$\big(F^sE^*(X)\big)\big(F^tE^*(X)\big) \subset F^{s+t}E^*(X) \quad \text{(for all } s,\ t\text{)}, \tag{3.34}$$

because $X^{s-1}\times X \cup X\times X^{t-1}$ contains the $(s+t-1)$-skeleton of $X\times X$, as in [34, Proposition 13.67].

When X is connected, with basepoint o, we recognize $F^1E^*(X)$ from the exact sequence (3.2) as the augmentation ideal

$$F^1E^*(X) = E^*(X,o) = \operatorname{Ker}\left[E^*(X) \longrightarrow E^*(o) = E^*\right]. \tag{3.35}$$

Filtered modules. We need to be somewhat more general.

DEFINITION 3.36. Given any E^*-module M filtered by submodules F^aM, the associated *filtration topology* on M has a basis consisting of the cosets $x + F^aM$, for all $x\in M$ and all indices a.

For this to be a topology, we need the *directedness* condition, that given F^aM and F^bM, there exists c such that $F^cM \subset F^aM \cap F^bM$.

We consider the projections $M \to M/F^aM$. We observe that M is Hausdorff if and only if the induced homomorphism $M \to \lim_a M/F^aM$ is monic, and that M is complete (in the sense that all Cauchy sequences $n \mapsto x_n \in M$ converge) if and only if it is epic. (A Cauchy sequence is one that satisfies $x_m - x_n \to 0$. However, its limit is unique only if M is Hausdorff.)

DEFINITION 3.37. We define the *completion* of the filtered module M as $\widehat{M} = \lim_a M/F^aM$. The projections $M \to M/F^aM$ lift to define the *completion map* $M \to \widehat{M}$.

We shall observe in Section 6 that $\widehat{M}$ has a canonical filtration that makes it complete Hausdorff.

In particular, we have the *skeleton topology* on $E^*(X)$. It is of course discrete when X is finite-dimensional. Since $E^*(X)/F^sE^*(X) \subset E^*(X^{s-1})$, Milnor's short exact sequence [24, Lemma 2]

$$0 \longrightarrow \lim_s{}^1 E^{k-1}(X^s) \longrightarrow E^k(X) \longrightarrow \lim_s E^k(X^s) \longrightarrow 0 \tag{3.38}$$

may be written in the form

$$0 \longrightarrow F^\infty E^k(X) \longrightarrow E^k(X) \longrightarrow \lim_s E^k(X)/F^sE^k(X) \longrightarrow 0, \tag{3.39}$$

where

$$F^\infty E^k(X) = \bigcap_s F^sE^k(X)$$

and we recognize the limit term as the completion of $E^k(X)$. Thus the skeleton filtration is always complete, but examples show that it need not be Hausdorff. The elements of $F^\infty E^k(X)$ are called *phantom* classes. In this case, the completion is simply the *quotient* of $E^k(X)$ by the phantom classes.

REMARK. The terminology is unfortunate, but standard. The word "complete" is sometimes understood to include "Hausdorff", which would leave us with no word to describe our situation. Here, completion is really Hausdorffification.

4. Generalized homology and duality

Associated to each of our multiplicative cohomology theories $E^*(-)$ is a multiplicative homology theory $E_*(-)$, whose coefficient ring $E_*(T)$ we can identify with $E^*(T) = E^*$. In this section, we study the relationship between them. We shall see in Section 9 that the situation is quite general. In line with a suggestion of Adams [1], we have two main tools: a Künneth isomorphism, Theorem 4.2, and a universal coefficient isomorphism, Theorem 4.14. (With our emphasis on cohomology, we *never* write E_* for E^* or E_{-n} for E^n, as is often done.)

Homology too has external cross products

$$\times: E_*(X) \otimes E_*(Y) \longrightarrow E_*(X \times Y), \tag{4.1}$$

that make $E_*(X)$ an E^*-module. This is more often than (3.22) an isomorphism.

THEOREM 4.2. *Assume that $E_*(X)$ or $E_*(Y)$ is a free or flat E^*-module. Then the pairing* (4.1) *induces the* Künneth isomorphism $E_*(X \times Y) \cong E_*(X) \otimes E_*(Y)$ *in homology.*

PROOF. See Switzer [34, Theorem 13.75]. Assume that $E_*(Y)$ is flat. The idea is that as X varies, (4.1) is then a natural transformation of homology theories, which is an isomorphism for $X = T$ and therefore generally. □

This is particularly useful for $E = K(n)$ or $H(\mathbf{F}_p)$, for then all E^*-modules are free. When $E_*(X)$ is free (or flat), we can define the comultiplication

$$\psi: E_*(X) \xrightarrow{\Delta_*} E_*(X \times X) \xleftarrow{\cong} E_*(X) \otimes E_*(X), \tag{4.3}$$

which, together with the counit $\varepsilon = q_*: E_*(X) \to E_*(T) = E^*$ induced by $q: X \to T$, makes $E_*(X)$ an E^*-coalgebra.

The homology analogue of Milnor's exact sequence (3.38) is simply [24, Lemma 1]

$$E_n(X) = \operatorname*{colim}_s E_n(X^s). \tag{4.4}$$

Duality. Our only real use of homology is the Kronecker pairing

$$\langle -, - \rangle: E^*(X) \otimes E_*(X) \longrightarrow E^*,$$

which is E^*-bilinear in the sense that $\langle vx, z \rangle = v\langle x, z \rangle = (-1)^{hi}\langle x, vz \rangle$ for $x \in E^i(X)$, $z \in E_*(X)$, and $v \in E^h$. We convert it to the right adjunct form

$$d: E^*(X) \longrightarrow DE_*(X) \tag{4.5}$$

by defining $(dx)z = \langle x, z\rangle$. Here, DM denotes the *dual module* $\mathrm{Hom}^*(M, E^*)$ of any E^*-module M, defined by $(DM)^n = \mathrm{Hom}^n(M, E^*)$. (But we still like to write the evaluation as $\langle -,-\rangle: DM \otimes M \to E^*$.) This is the correct indexing to make DM an E^*-module and d a homomorphism of E^*-modules. It is reasonable to ask whether d is an isomorphism. We shall give a useful answer in Theorem 4.14.

There is an obvious natural pairing $\zeta_D: DM \otimes DN \to D(M \otimes N)$, defined by

$$\langle \zeta_D(f \otimes g), x \otimes y\rangle = (-1)^{\deg(x)\deg(g)}\langle f, x\rangle\langle g, y\rangle \quad \text{in } E^*. \tag{4.6}$$

All these structure maps fit together in the commutative diagram

$$\begin{tikzcd}
E^*(X) \otimes E^*(Y) \arrow[r, "d\otimes d"] \arrow[d, "\times"] & DE_*(X) \otimes DE_*(Y) \arrow[r, "\zeta_D"] & D(E_*(X) \otimes E_*(Y)) \\
E^*(X \times Y) \arrow[rr, "d"] & & DE_*(X \times Y) \arrow[u, "D\times"']
\end{tikzcd} \tag{4.7}$$

which, algebraically, states that $\langle x \times y, a \times b\rangle = \pm\langle x, a\rangle\langle y, b\rangle$. Its significance is that if any four of the maps are isomorphisms, so is the fifth.

We need more. We need a topology on $DE_*(X)$ to match the topology on $E^*(X)$. There is an obvious candidate. (We stress that the homology $E_*(X)$ invariably has the discrete topology.)

DEFINITION 4.8. Given any E^*-module M, we define the *dual-finite filtration* on $DM = \mathrm{Hom}^*(M, E^*)$ as consisting of the submodules $F^L DM = \mathrm{Ker}[DM \to DL]$, where L runs through all finitely generated submodules of M. It gives rise by Definition 3.36 to the *dual-finite topology* on DM.

This filtration is obviously Hausdorff, and we see it is complete by writing $DM = \lim_L DL$, the inverse limit of discrete E^*-modules. It certainly makes d continuous, because any finitely generated $L \subset E_*(X)$ lifts to $E_*(X^s)$ for some s, by eq. (4.4).

The profinite filtration. The skeleton filtration is adequate for discussing spaces *of finite type* (those having finite skeletons), but not all our spaces have finite type. We need a somewhat coarser topology that has better properties and a better chance of making d in (4.5) a homeomorphism.

DEFINITION 4.9. Given a CW-complex X, we define the *profinite filtration* of $E^*(X)$ as consisting of all the ideals

$$F^\alpha E^*(X) = \mathrm{Ker}\left[E^*(X) \longrightarrow E^*(X_\alpha)\right] = \mathrm{Im}\left[E^*(X, X_\alpha) \longrightarrow E^*(X)\right],$$

where X_α runs through all finite subcomplexes of X. We call the resulting filtration topology (see Definition 3.36) the *profinite topology*.

The particular indexing set is not important and we rarely specify it. The ideals $F^aE^*(X)$ do form a directed system: given F^a and F^b, there exists X_c such that $F^c \subset F^a \cap F^b$, namely $X_c = X_a \cup X_b$.

This is our preferred topology on $E^*(X)$, for all spaces X. It is natural in X: given a map $f: X \to Y$, $f^*: E^*(Y) \to E^*(X)$ is continuous, because for each finite $X_a \subset X$, there is a finite $Y_b \subset Y$ for which $fX_a \subset Y_b$, so that $f^*(F^b) \subset F^a$. Indeed, it is the coarsest natural topology that makes $E^*(X)$ discrete for all finite X.

Of course, it coincides with the skeleton topology when X has finite type. However, it has one elementary property that the skeleton topology lacks.

LEMMA 4.10. *For any disjoint union $X = \coprod_\alpha X_\alpha$, the profinite topology makes $E^k(X) \cong \prod_\alpha E^k(X_\alpha)$ a homeomorphism.*

DEFINITION 4.11. For any space X, we define its *completed E-cohomology* $E^*(X)\hat{}$ as the completion of $E^*(X)$ with respect to the profinite filtration.

A result of Adams [2, Theorem 1.8] shows that the profinite topology is always complete, that

$$E^*(X) \longrightarrow \varprojlim_a E^*(X)/F^aE^*(X) \subset \varprojlim_a E^*(X_a)$$

is surjective, which allows us to identify canonically

$$E^*(X)\hat{} = E^*(X)/\bigcap_a F^aE^*(X) \cong \varprojlim_a E^*(X_a) \tag{4.12}$$

for all spaces X. This completed cohomology is not at all new; it was discussed at some length by Adams [ibid.].

As before, the topology on $E^*(X)$ need not be Hausdorff. The intersection $\bigcap_a F^aE^*(X)$ (which contains $F^\infty E^*(X)$) need not vanish, and its elements are called *weakly phantom* classes. In practice, one hopes there are none, so that $E^*(X)\hat{} = E^*(X)$.

Strong duality. We note that the morphism d in eq. (4.5) remains continuous with the profinite topology on $E^*(X)$.

DEFINITION 4.13. We say the space X *has strong duality* if $d: E^*(X) \to DE_*(X)$ in (4.5) is a homeomorphism between the profinite topology on $E^*(X)$ and the dual-finite topology on $DE_*(X)$ (see Definition 4.8).

THEOREM 4.14. *Assume that $E_*(X)$ is a free E^*-module. Then X has strong duality, i.e. $d: E^*(X) \cong DE_*(X)$ is a homeomorphism between the profinite topology on $E^*(X)$ and the dual-finite topology on $DE_*(X)$. In particular, $E^*(X)$ is complete Hausdorff.*

This is best viewed as a stable result, and will be included in Theorem 9.25.

Künneth homeomorphisms. The cohomology Künneth homomorphism (3.22) is rarely an isomorphism, but our chances improve if we complete it. Generally, given E^*-modules M and N filtered by submodules F^aM and F^bN, we filter $M \otimes N$ by the submodules

$$\begin{aligned} F^{a,b}(M \otimes N) &= \operatorname{Im}\left[(F^aM \otimes N) \oplus (M \otimes F^bN) \longrightarrow M \otimes N\right] \\ &= \operatorname{Ker}\left[M \otimes N \longrightarrow (M/F^aM) \otimes (N/F^bN)\right] \end{aligned} \tag{4.15}$$

where the second form follows from the right exactness of $\otimes$. (Often, but not always, $F^aM \otimes N$ and $M \otimes F^bN$ are submodules of $M \otimes N$.) We construct the *completed tensor product* $M \mathbin{\widehat{\otimes}} N$ as the completion of $M \otimes N$ with respect to this filtration.

The filtration makes $\times$-multiplication (3.22) continuous, because given $Z_c \subset Z = X \times Y$, the inverse image of $F^cE^*(Z)$ contains $F^{a,b}(E^*(X) \otimes E^*(Y))$, provided $Z_c \subset X_a \times Y_b$. We may therefore complete it to

$$\times : E^*(X) \mathbin{\widehat{\otimes}} E^*(Y) \longrightarrow E^*(X \times Y)\hat{} \tag{4.16}$$

and ask whether this is an isomorphism. Again, we need more than a bijection.

DEFINITION 4.17. If the pairing (4.16) is a homeomorphism and $E^*(X \times Y)\hat{} = E^*(X \times Y)$, we call the resulting homeomorphism $E^*(X \times Y) \cong E^*(X) \mathbin{\widehat{\otimes}} E^*(Y)$ a *Künneth homeomorphism*. (Note that we require $E^*(X \times Y)$ to be already Hausdorff.)

Similarly, $\zeta_D : DM \otimes DN \to D(M \otimes N)$ is continuous. We therefore complete diag. (4.7) to

$$\begin{tikzcd} E^*(X) \mathbin{\widehat{\otimes}} E^*(Y) \arrow[r, "d \otimes d"] \arrow[d, "\times"] & DE_*(X) \mathbin{\widehat{\otimes}} DE_*(Y) \arrow[r, "\zeta_D"] & D(E_*(X) \otimes E_*(Y)) \\ E^*(X \times Y) \arrow[rr, "d"] & & DE_*(X \times Y) \arrow[u, "D\times"'] \end{tikzcd} \tag{4.18}$$

THEOREM 4.19. *Assume that $E_*(X)$ and $E_*(Y)$ are free E^*-modules. Then we have the Künneth homeomorphism $E^*(X \times Y) \cong E^*(X) \mathbin{\widehat{\otimes}} E^*(Y)$ in cohomology.*

PROOF. The hypotheses, with the help of Theorems 4.2 and 4.14, make (4.18) a diagram of homeomorphisms. (For ζ_D, we may appeal to Lemma 6.15(e).) □

5. Complex orientation

All five of our examples of cohomology theories $E^*(-)$ are equipped with a complex orientation. This will provide Chern classes and a good supply of spaces with free E-homology.

The Chern class of a line bundle. Denote by $M(\xi)$ the Thom space of a vector bundle ξ. A *complex orientation* (for line bundles) assigns to each complex line bundle θ over any space X a natural Thom class $t(\theta) \in E^2(M(\theta))$, such that for the line bundle 1 over a point, $t(1) = u_2 \in E^2(S^2)$.

REMARK. We assume here a specific homeomorphism between S^2 and the one-point compactification of $\mathbb{C}$, as determined by some orientation convention. In some contexts, it is useful to allow the slightly more general normalization $t(1) = \lambda u_2$, where $\lambda \in E^*$ may be any invertible element; but then $\lambda^{-1}t(\theta)$ is a Thom class in the stricter sense. We have no need here of this extra flexibility.

For our purposes, a closely related concept is more useful.

DEFINITION 5.1. Given E, a *line bundle Chern class* assigns to each complex line bundle θ over any space X a class $x(\theta) \in E^2(X)$, called the *(first) E-Chern class* of θ, that satisfies the axioms:

(i) It is *natural*: Given a map $f: X' \to X$ and a line bundle θ over X, for the induced line bundle $f^*\theta$ over X' we have $x(f^*\theta) = f^*x(\theta)$ in $E^2(X')$;
(ii) It is *normalized*: For the Hopf line bundle ξ over $\mathbb{C}P^1 \cong S^2$, we have $x(\xi) = u_2 \in E^2(S^2)$, the canonical generator of $E^*(S^2)$.

It is easy to see that $x(\theta) = i^*t(\theta)$ satisfies the axioms, where $i: X \subset M(\theta)$ denotes the inclusion of the zero section. (Conversely, Connell [11, Theorems 4.1, 4.5] shows that every line bundle Chern class arises in this way, from a unique complex orientation.)

For $E = KU$, it is obvious from eq. (3.32) that

$$x(\theta) = u^{-1}([\theta] - 1) \in KU^2(X) \tag{5.2}$$

is a line bundle Chern class.

Complex projective spaces. Of course, Chern classes need not exist for general E. As the Hopf line bundle ξ over $\mathbb{C}P^\infty = BU(1)$ is universal, it is enough to have $x = x(\xi) \in E^2(\mathbb{C}P^\infty)$. We start with $\mathbb{C}P^n$.

LEMMA 5.3 (Dold). *Assume that the Hopf line bundle ξ over $\mathbb{C}P^n$ has the Chern class $x = x(\xi) \in E^2(\mathbb{C}P^n)$, where $n \geqslant 0$. Then:*

(a) $E^*(\mathbb{C}P^n) = E^*[x : x^{n+1} = 0]$, *a truncated polynomial algebra over E^*;*

(b) *We have the duality isomorphism $d: E^*(\mathbb{C}P^n) \cong DE_*(\mathbb{C}P^n)$;*

(c) *$E_*(\mathbb{C}P^n)$ is the free E^*-module with basis $\{\beta_0, \beta_1, \beta_2, \ldots, \beta_n\}$, where $\beta_i \in E_{2i}(\mathbb{C}P^n)$ is defined as dual to x^i.*

PROOF. See Adams [3, Lemmas II.2.5, II.2.14] or Switzer [34, Propositions 16.29, 16.30]. The idea is that the presence of x forces the Atiyah–Hirzebruch spectral sequences for both $E^*(\mathbb{C}P^n)$ and $E_*(\mathbb{C}P^n)$ to collapse. (There is of course no topology on $E^*(\mathbb{C}P^n)$ to check.) One has to verify that $x^{n+1} = 0$ exactly. In terms of the skeleton filtration, $x \in F^2E^*(\mathbb{C}P^n)$. Then by eq. (3.34), $x^{n+1} \in F^{2n+2}E^*(\mathbb{C}P^n) = 0$. □

The result for $\mathbb{C}P^\infty$ follows immediately, by eq. (4.4) and Theorem 4.14, and also clarifies exactly how nonunique a complex orientation is. Similarly named elements correspond under inclusion.

LEMMA 5.4 (Dold). *Assume that we have the Chern class $x = x(\xi) \in E^2(\mathbb{C}P^\infty)$. Then:*

(a) *$E^*(\mathbb{C}P^\infty) = E^*[[x]]$, the algebra of formal power series in x over E^*, filtered by powers of the ideal (x);*

(b) *We have strong duality $d: E^*(\mathbb{C}P^\infty) \cong DE_*(\mathbb{C}P^\infty)$;*

(c) *$E_*(\mathbb{C}P^\infty)$ is the free E^*-module with basis $\{\beta_0, \beta_1, \beta_2, \beta_3, \dots\}$, where $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ is dual to x^i for $i \geqslant 0$.*

Chern classes of a vector bundle. We proceed to BU by way of $\mathbb{C}P^\infty = BU(1) \subset BU$. A useful intermediate step is the torus group $T(n) = U(1) \times \cdots \times U(1)$, for which $BT(n) = BU(1) \times \cdots \times BU(1)$. We have Künneth isomorphisms

$$E_*(BT(n)) \cong E_*(\mathbb{C}P^\infty) \otimes E_*(\mathbb{C}P^\infty) \otimes \cdots \otimes E_*(\mathbb{C}P^\infty)$$

in homology by Theorem 4.2, and

$$E^*(BT(n)) = E^*[[x_1, x_2, \dots, x_n]] \cong E^*(\mathbb{C}P^\infty) \widehat{\otimes} \cdots \widehat{\otimes} E^*(\mathbb{C}P^\infty) \qquad (5.5)$$

in cohomology by Theorem 4.19, where $x_i = p_i^* x(\xi) = x(p_i^*\xi)$.

LEMMA 5.6. *Assume E has a line bundle Chern class. Then:*

(a) *$E^*(BU) = E^*[[c_1, c_2, c_3, \dots]]$, where $c_i \in E^{2i}(BU)$ restricts to the ith elementary symmetric function of the $x_j \in E^*(BT(n))$ for any $n \geqslant i$, and $E^*(BU(n)) = E^*[[c_1, c_2, \dots, c_n]]$ is the quotient of this with $c_i = 0$ for all $i > n$;*

(b) *We have strong duality $d: E^*(BU) \cong DE_*(BU)$ and $d: E^*(BU(n)) \cong DE_*(BU(n))$, and in particular, $E^*(BU)$ and $E^*(BU(n))$ are Hausdorff;*

(c) *$E_*(BU) = E^*[\beta_1, \beta_2, \beta_3, \dots]$, where β_i is inherited from $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ by $\mathbb{C}P^\infty = BU(1) \subset BU$ and $\beta_0 \mapsto 1$, and $E_*(BU(n)) \subset E_*(BU)$ is the E^*-free submodule spanned by all monomials of polynomial degree $\leqslant n$ in the β_i.*

PROOF. See Adams [3, Lemma II.4.1] or Switzer [34, Theorems 16.31, 16.32]. □

From this it is immediate, as in Conner and Floyd [12, Theorem 7.6], Adams [3, Lemma II.4.3], or Switzer [34, Theorem 16.2], that general Chern classes exist. The axioms determine them uniquely on $BT(n)$, and this is enough.

THEOREM 5.7. *Assume E has a complex orientation. Then there exist uniquely E-*Chern classes *$c_i(\xi) \in E^{2i}(X)$, for $i > 0$ and any complex vector bundle ξ over any space X, that satisfy the axioms:*

(i) *Naturality: $c_i(f^*\xi) = f^* c_i(\xi) \in E^{2i}(X')$ for any vector bundle ξ over X and any map $f: X' \to X$;*

(ii) *For any n-plane bundle ξ, $c_i(\xi) = 0$ for all $i > n$;*
(iii) *For any line bundle ξ, $c_1(\xi) = x(\xi)$;*
(iv) *For any vector bundles ξ and η over X, we have the* Cartan formula

$$c_k(\xi \oplus \eta) = c_k(\xi) + \sum_{i=1}^{k-1} c_{k-i}(\xi)c_i(\eta) + c_k(\eta) \quad \textit{in } E^*(X).$$

The unitary groups. We study the unitary group U by means of the Bott map $b\colon \Sigma(\mathbb{Z} \times BU) \to U$, one of the structure maps of the Ω-spectrum KU. The Hopf line bundle θ over $\mathbb{C}P^{n-1}$ defines the *unbased* inclusion

$$\mathbb{C}P^{n-1} \subset \mathbb{C}P^\infty = BU(1) \subset BU \cong 1 \times BU \subset \mathbb{Z} \times BU. \tag{5.8}$$

Its fibre over the point $A \in \mathbb{C}P^{n-1}$ is $\mathrm{Hom}_\mathbb{C}(A, \mathbb{C})$, where we also regard A as a line in $\mathbb{C}^n$.

When we apply Bott periodicity as in Theorem 3.29, we obtain the element

$$\big([\xi]-1\big) \times [\theta] = \big[(\xi\otimes\theta) \oplus \theta^\perp\big] - n \quad \text{in } KU\big(S^2\times\mathbb{C}P^{n-1}\big),$$

where $\theta^\perp$ denotes the orthogonal complement bundle having the fibre $\mathrm{Hom}_\mathbb{C}(A^\perp, \mathbb{C})$ over $A \in \mathbb{C}P^{n-1}$. The n-plane bundle $(\xi\otimes\theta) \oplus \theta^\perp$ is, by design, trivial over $D^2 \times \mathbb{C}P^{n-1}$ for any 2-disk $D^2 \subset S^2$, and its clutching function

$$h\colon S^1 \times \mathbb{C}P^{n-1} \longrightarrow U(n) \tag{5.9}$$

induces the Bott map b, restricted as in (5.8). Here, $S^1 \subset \mathbb{C}$ is to be regarded as the circle group. We read off that (for suitable choices of orientation) $h(z, A)\colon \mathbb{C}^n \to \mathbb{C}^n$ is the well-known map that preserves $A^\perp$ and on A is multiplication by z; explicitly, on any vector $Y \in \mathbb{C}^n$, it is

$$h(z, A)Y = Y + (z-1)\,\langle Y, X\rangle\, X \quad \text{in } \mathbb{C}^n, \tag{5.10}$$

where X is any unit vector in A. (From the group-theoretic point of view, the image of h is the union of all the conjugates of $U(1) \subset U(n)$.)

In [40], Yokota used (essentially) this map h and the multiplication in $U(n)$ to construct explicit cell decompositions of $SU(n)$ and hence $U(n)$, and deduce their ordinary (co)homology. The method works equally well for E-(co)homology.

LEMMA 5.11. *Assume that E has a line bundle Chern class. Then $E_*(U(n))$ is a free E^*-module with a basis consisting of all the Pontryagin products $\gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_k}$, where $n > i_1 > i_2 > \dots i_k \geqslant 0$, $k \geqslant 0$ (we allow the empty product 1), $\gamma_i = h_*(z\times\beta_i) \in E_{2i+1}(U(n))$ with h as in eq. (5.9), and $z \in E_1(S^1)$ is dual to u_1.*

PROOF. Because we are decomposing $U(n)$ rather than $SU(n)$, we use a slightly different (and simpler) decomposition. We regard $U(n)$ as a principal right $U(n-1)$-bundle over S^{2n-1}, with projection map $\pi\colon U(n) \to S^{2n-1}$ given by $\pi g = ge_n$, where

$e_n = (0, 0, \ldots, 0, 1) \in \mathbb{C}^n$ and we recognize $U(n-1)$ as the subgroup of $U(n)$ that fixes e_n. Given $g \in U(n) - U(n-1)$, so that $\pi g \neq e_n$, it is easy to solve eq. (5.10), as in [40], for a unique pair (z, A) such that $h(z, A)e_n = \pi g$, which allows us to write $g = h(z, A)g'$ for some $g' \in U(n-1)$. Moreover, $z \neq 1$ and $A \notin \mathbb{C}P^{n-2}$; in other words, $\pi \circ h$ identifies the top cell of $S^1 \times \mathbb{C}P^{n-1}$ with $S^{2n-1} - e_n$.

It follows that the map

$$S^1 \times \mathbb{C}P^{n-1} \times U(n-1) \xrightarrow{h \times 1} U(n) \times U(n-1) \xrightarrow{\mu} U(n)$$

induces the isomorphism in the commutative square

$$\begin{tikzcd}
E_*(S^1 \times \mathbb{C}P^{n-1}) \otimes E_*(U(n-1)) \arrow[r] \arrow[d] & E_*(U(n)) \arrow[d] \\
E_*(S^1 \times \mathbb{C}P^{n-1}, K) \otimes E_*(U(n-1)) \arrow[r, "\cong"] & E_*(U(n), U(n-1))
\end{tikzcd}$$

where $K = S^1 \times \mathbb{C}P^{n-2} \cup 1 \times \mathbb{C}P^{n-1}$. From Lemma 5.3, we deduce that both vertical arrows are split epic and obtain the decomposition

$$E_*\big(U(n)\big) \cong E_*\big(U(n-1)\big) \oplus \gamma_{n-1} E_*\big(U(n-1)\big)$$

of $E_*(U(n))$ as the direct sum (with a shift) of two copies of $E_*(U(n-1))$, as the multiplication by γ_{n-1} is an embedding. The result now follows by induction on n, starting from $U(1) = S^1$.

Alternatively, we apply the Atiyah–Hirzebruch homology spectral sequence to the map h, to deduce that the spectral sequence for $E_*(U(n))$ collapses whenever that for $E_*(\mathbb{C}P^{n-1})$ does. □

COROLLARY 5.12. *Assume that E has a line bundle Chern class, and that E^* has no 2-torsion. Then $E_*(U) = \Lambda(\gamma_0, \gamma_1, \gamma_2, \ldots)$, an exterior algebra on the generators $\gamma_i = b_*(z \times \beta_i)$, where $b\colon \Sigma(\mathbb{Z} \times BU) \to U$ denotes the Bott map and $\beta_i \in E_{2i}(\mathbb{Z} \times BU)$ is inherited from $\mathbb{C}P^\infty$ by the inclusion* (5.8).

PROOF. We let $n \to \infty$ in the Lemma and use eq. (4.4). The homotopy commutativity of U gives $\gamma_j \gamma_i = -\gamma_i \gamma_j$ and hence $\gamma_i^2 = 0$. □

The formal group law. Conspicuous by its absence is any formula for $c_i(\xi \otimes \eta)$. For line bundles, the universal example is $p_1^*\xi \otimes p_2^*\xi$ over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$, where ξ denotes the Hopf line bundle. In view of eq. (5.5), there must be some formula

$$x(\xi \otimes \eta) = x(\xi) + x(\eta) + \sum_{i,j} a_{i,j}\, x(\xi)^i x(\eta)^j = F\big(x(\xi), x(\eta)\big) \tag{5.13}$$

that is valid in the universal case, and therefore generally, where

$$F(x,y) = x + y + \sum_{i,j} a_{i,j}\, x^i y^j \quad \text{in } E^*[[x,y]] \tag{5.14}$$

is a well-defined formal power series with coefficients $a_{i,j} \in E^{-2i-2j+2}$ for $i > 0$ and $j > 0$. (In the common case that the series is infinite, it may be necessary to interpret eq. (5.13) in the completion $E^*(X)^\wedge$ of $E^*(X)$.) By use of the splitting principle (working in $BT(n)$) and heavy algebra, one can in principle determine formulae for $c_i(\xi \otimes \eta)$ for general complex vector bundles.

The series $F(x,y)$ is known as the *formal group law* of E (or more accurately, of its Chern class $x(-)$). It satisfies the three identities:

(i) $F(x,y) = F(y,x)$;
(ii) $F(F(x,y),z) = F(x,F(y,z))$; (5.15)
(iii) $F(x,0) = x$.

The first two reflect the commutativity and associativity of $\otimes$. The last comes from $\xi \otimes \varepsilon \cong \xi$ for a trivial line bundle ε, and shows that $F(x,y)$ has no terms of the form $a_{i,0}x^i$ other than x.

In the case $E = KU$, we can write down

$$x(\xi \otimes \eta) = x(\xi) + x(\eta) + ux(\xi)x(\eta) \quad \text{in } KU^*(X) \tag{5.16}$$

directly from eq. (5.2), since $x(\xi \otimes \eta) = u^{-1}([\xi][\eta] - 1)$; in other words, the formal group law for KU is $F(x,y) = x + y + uxy$.

6. The categories

In this section we introduce the major categories we need, based on the discussion in Section 3. We also fix some terminology and notation. Our basic reference for category theory is MacLane [20]. The ground ring throughout is our coefficient ring E^*, a commutative graded ring.

$\mathcal{A}^{\mathrm{op}}$ denotes the *dual category* of any category $\mathcal{A}$. It has a morphism $f^{\mathrm{op}}: Y \to X$ for each morphism $f: X \to Y$ in $\mathcal{A}$. If $\mathcal{A}$ is graded (and therefore additive), $\deg(f^{\mathrm{op}}) = \deg(f)$ and composition in $\mathcal{A}^{\mathrm{op}}$ is given by $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (-1)^{\deg(f)\deg(g)}(g \circ f)^{\mathrm{op}}$.

Set denotes the category of sets. Cartesian products serve as products and disjoint unions as coproducts. The one-point set T is a terminal object, and the empty set is an initial object.

Ho denotes the homotopy category of *unbased* spaces that are homotopy equivalent to a CW-complex. *This will be our main category of spaces.* Milnor proved [23, Proposition 3] that it admits products $X \times Y$, with never any need to retopologize. The one-point space T is a terminal object. Arbitrary disjoint unions serve as coproducts; in particular, any space is the disjoint union of connected spaces. We identify $E^k(X) = Ho(X, \underline{E}_k)$ according to Theorem 3.17.

Of course, any equivalent category will serve as well. We reserve the option of taking any specific space to be a CW-complex, extending constructions to the rest of *Ho* by naturality.

Ho' denotes the homotopy category of *based* spaces as in *Ho*, where the basepoint o is assumed to be nondegenerate; all maps and homotopies are to preserve the basepoint. Although this category is more common, we use it only rarely. Milnor proved [23, Corollary 3] that the loop space ΩX of such a space X again lies in the category. Finite cartesian products remain products, but the one-point space T becomes a zero object and arbitrary wedges (one-point unions) serve as coproducts. The exact sequence (3.2) identifies $E^k(X, o) = Ho'(X, \underline{E}_k)$.

Stab denotes the stable homotopy category (in any of various equivalent versions, e.g., [3]). It is an additive category, and has the point spectrum as a zero object. Arbitrary wedges of spectra serve as coproducts. It is equipped with a *stabilization* functor $Ho' \to Stab$, which we suppress from our notation. There is a biadditive smash product functor $\wedge: Stab \times Stab \to Stab$, which (up to coherent isomorphisms) is commutative and associative, has the sphere spectrum T^+ as a unit, and is compatible with the smash product in *Ho'*. We define the *suspension* $\Sigma X = S^1 \wedge X$, which is therefore compatible with $\Sigma: Ho' \to Ho'$.

*Stab** denotes the *graded* stable homotopy category; it has the same objects as *Stab*, with maps of any degree as morphisms. It is a graded additive category. We write $Stab^n(X, Y) = \{X, Y\}^n$ for the group of maps of degree n (in the conventions of Section 2). Given a fixed choice of one of the two isomorphisms $S^1 \simeq T^+$ in $Stab^*$ of degree 1, we define the canonical natural *desuspension isomorphism*

$$\Sigma X = S^1 \wedge X \simeq T^+ \wedge X \simeq X \tag{6.1}$$

of degree 1 for any spectrum X. (We do not give it a symbol.) Composition with it yields isomorphisms, for any $n \geqslant 0$:

$$\{X, \Sigma^n Y\} \cong \{X, Y\}^n; \qquad \{X, Y\}^{-n} \cong \{\Sigma^n X, Y\};$$

which express $Stab^*$ in terms of *Stab* and Σ.

However, there is a difficulty with smash products. Given maps $f: X \to X'$ and $g: Y \to Y'$ of degrees m and n, the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "f \wedge Y"] \arrow[d, "X \wedge g"'] \arrow[dr, phantom, "(-1)^{mn}"] & X' \wedge Y \arrow[d, "X' \wedge g"] \\ X \wedge Y' \arrow[r, "f \wedge Y'"] & X' \wedge Y' \end{tikzcd}$$

commutes only up to the indicated sign $(-1)^{mn}$, owing to the necessity of shuffling suspension factors. Consequently, the graded smash product is a functor defined *not* on $\mathit{Stab}^* \times \mathit{Stab}^*$, but on a new graded category (which might be called $\mathit{Stab}^* \otimes \mathit{Stab}^*$) with the biadditivity and signs built in. All we need to know is how to compose: given also $f': X' \to X''$ of degree m' and $g': Y' \to Y''$, we have

$$(g' \wedge f') \circ (g \wedge f) = (-1)^{m'n}(g' \circ g) \wedge (f' \circ f): X \wedge Y \longrightarrow X'' \wedge Y''. \quad (6.2)$$

From the topological point of view, this is the source of the principle of signs (see Section 2). For example, a map $f: X \to Y$ of degree n induces, for any W and Z, the homomorphisms of graded groups of degree n:

$$\begin{aligned} f_*: \mathit{Stab}^*(W,X) &\to \mathit{Stab}^*(W,Y) \quad \text{given by } f_* g = f \circ g; \\ f^*: \mathit{Stab}^*(Y,Z) &\to \mathit{Stab}^*(X,Z) \quad \text{given by } f^* g = (-1)^{n\deg(g)} g \circ f. \end{aligned} \quad (6.3)$$

Ab denotes the category of abelian groups. It is the prototypical abelian category and needs no review here.

*Ab** denotes the graded category of graded abelian groups, graded by all integers (positive and negative).

Mod denotes the additive category of (necessarily graded) E^*-modules, in which the morphisms are E^*-module homomorphisms of degree 0. Degreewise direct products $\prod_\alpha M_\alpha$ and sums $\bigoplus_\alpha M_\alpha$ serve as products and coproducts. It is equipped with the biadditive functor $\otimes: \mathit{Mod} \times \mathit{Mod} \to \mathit{Mod}$ (taken over E^*), which is associative, commutative, and has E^* as unit (up to coherent isomorphisms).

We note that the homology functor $E_*(-): \mathit{Ho} \to \mathit{Mod}$ preserves arbitrary coproducts, i.e. is strongly additive.

*Mod** denotes the graded category of E^*-modules, in which homomorphisms of any degree are allowed. That is, $\mathit{Mod}^*(M,N)$ is the graded group whose component $\mathit{Mod}^n(M,N)$ in degree n is the group of E^*-module homomorphisms $f: M \to N$ of degree n, with components $f^i: M^i \to N^{i+n}$ that satisfy $f^{i+h}(vx) = (-1)^{nh} v(f^i x)$ for $x \in M^i$ and $v \in E^h$. The sign must be present if the algebra is to imitate the topology.

Moreover, $\mathit{Mod}^*(M,N)$ is an E^*-module in the obvious way, with vf defined by $(vf)x = v(fx) = \pm f(vx)$ for $v \in E^*$. Given E^*-module homomorphisms $g: L' \to L$ and $h: M \to M'$, we define $\mathrm{Hom}(g,h): \mathit{Mod}^*(L,M) \to \mathit{Mod}^*(L',M')$ by

$$\mathrm{Hom}(g,h)f = \mathit{Mod}^*(g,h)f = (-1)^{\deg(g)(\deg(f)+\deg(h))} h \circ f \circ g: L' \longrightarrow M', \quad (6.4)$$

to make it a homomorphism of E^*-modules. Similarly for tensor products: given morphisms $f: L \to L'$ and $g: M \to M'$, we define the morphism $f \otimes g: L \otimes M \to L' \otimes M'$

in *Mod** by

$$(f \otimes g)(x \otimes y) = (-1)^{\deg(g)\deg(x)} fx \otimes gy.$$

If also $f': L' \to L''$ and $g': M' \to M''$, composition is given, like eq. (6.2), by

$$(g' \otimes f') \circ (g \otimes f) = (-1)^{\deg(f')\deg(g)} (g' \circ g) \otimes (f' \circ f): L \otimes M \longrightarrow L'' \otimes M''. \quad (6.5)$$

We imitate the suspension isomorphisms (3.13) and (3.16) algebraically by introducing suspension functors into *Mod* and *Mod**.

DEFINITION 6.6. Given an E^*-module M and any integer k, we define the *k-fold suspension* $\Sigma^k M$ *of* M by shifting everything up in degree by k: $(\Sigma^k M)^i$ is a formal copy of M^{i-k}, consisting of the elements $\Sigma^k x$ for $x \in M^{i-k}$.

To make the function $\Sigma^k: M \to \Sigma^k M$ an isomorphism of E^*-modules of degree k, we must define the action of $v \in E^h$ on $\Sigma^k M$ by

$$v(\Sigma^k x) = (-1)^{hk} \Sigma^k (vx) \quad \text{in } \Sigma^k M. \quad (6.7)$$

Further, $\Sigma^k: M \cong \Sigma^k M$ becomes a natural isomorphism $I \cong \Sigma^k$ of degree k of functors on *Mod** if we define $\Sigma^k f: \Sigma^k M \to \Sigma^k N$ by $(\Sigma^k f)(\Sigma^k x) = (-1)^{kn} \Sigma^k (fx)$ on a morphism $f: M \to N$ of any degree n. (Here, Σ denotes both a natural isomorphism and a functor.)

Alg denotes the category of commutative E^*-algebras. It admits arbitrary degreewise cartesian products $\prod_\alpha A_\alpha$ as products. The tensor product $A \otimes B$ of algebras serves as the coproduct of A and B, and E^* is the initial object.

Categories of filtered objects. The discussion in Sections 3 and 4 strongly suggests that for cohomology, we need filtered versions of *Mod*, *Mod**, and *Alg*.

FMod denotes the category of complete Hausdorff *filtered* E^*-modules and *continuous* E^*-module homomorphisms of degree 0. An object M is an E^*-module M, equipped with a directed system of E^*-submodules $F^a M$, and hence a topology as in Definition 3.36. (We do not require the indexing set to be the integers, or even countable.) These are required to satisfy $M = \lim_a M/F^a M$, to make the topology complete Hausdorff. The category remains an additive category.

The forgetful functor V: *FMod* $\to$ *Mod* simply discards the filtration. Conversely, any E^*-module M may be treated as a *discrete* filtered module by taking 0 as the only submodule $F^a M$; this defines an inclusion *Mod* $\subset$ *FMod*. Generally, a filtered module M is discrete if and only if some $F^a M$ is zero.

We frequently encounter filtered E^*-modules M that are not complete Hausdorff. We defined the completion $\widehat{M} = \lim_a M/F^a M$ of M in Definition 3.37. The completion map $M \to \widehat{M}$ is monic if and only if M is Hausdorff, and epic if and only if M is complete. Each $\widehat{M} \to M/F^a M$ is epic, because $M \to M/F^a M$ is.

We filter $\widehat{M}$ in the obvious way, by $F^a\widehat{M} = \text{Ker}[\widehat{M} \to M/F^aM]$. This filters the completion map and induces isomorphisms $M/F^aM \cong \widehat{M}/F^a\widehat{M}$; it is now obvious that $\widehat{M}$ is indeed complete Hausdorff (as the terminology demands) and so an object of *FMod*. If M happens to be already complete Hausdorff, $M \to \widehat{M}$ is an isomorphism in *FMod*. We make frequent use of the expected universal property: given an object N of *FMod*, any continuous E^*-module homomorphism $M \to N$ factors uniquely through a morphism $\widehat{M} \to N$ in *FMod*. In the language of Definition 2.1, $\widehat{M}$ is V-free on M, with the completion map $M \to \widehat{M}$ as a basis.

If $F^aM \subset F^bM$, we can write $F^bM/F^aM = \text{Ker}[M/F^aM \to M/F^bM]$. If we now fix F^bM and apply the left exact functor $\lim_a$, we see that the completion of F^bM, filtered by those F^aM contained in it, is just $\text{Ker}[\widehat{M} \to M/F^bM] = F^b\widehat{M}$, as expected.

None of the above facts requires the filtration to be countable.

The obvious filtration (4.15) on the tensor product $M \otimes N$ is rarely complete, even when M and N are. We therefore complete it to define the *completed tensor product* $M \widehat{\otimes} N$ in *FMod*. In view of the second form of (4.15), it may usefully be written

$$M \widehat{\otimes} N = \lim_{a,b} \left[(M/F^aM) \otimes (N/F^bN)\right]. \tag{6.8}$$

This makes it clear that $\widehat{M} \widehat{\otimes} \widehat{N} = M \widehat{\otimes} N$, that it does not matter whether we complete M and N first or not. (We continue to write $f \otimes g$ rather than $f \widehat{\otimes} g$ for the completed morphisms, leaving it to the context to indicate that completion is assumed.)

*FMod** denotes the graded category of complete Hausdorff filtered E^*-modules, in which continuous E^*-module homomorphisms of any degree are allowed.

We give the E-cohomology $E^*(X)$ of a space X the profinite topology from Definition 4.9, and complete it to $E^*(X)\hat{}$ as in Definition 4.11 if necessary; by Lemma 4.10, the functor $E^*(-)\hat{}: \mathit{Ho}^{\mathrm{op}} \to \mathit{FMod}$ takes arbitrary coproducts in *Ho* to products in *FMod*. Thus cohomology remains strongly additive in this enriched sense.

As noted in Section 4, the profinite topology on E-cohomology makes cup and cross products continuous,which suggests our other main category.

FAlg denotes the category of complete Hausdorff commutative filtered E^*-algebras A, with multiplication $\phi: A \otimes A \to A$ and unit $\eta: E^* \to A$. We filter objects as in *FMod*, except that the filtration is now by *ideals* F^aA. Then ϕ is automatically continuous, and it is sometimes useful to complete it to $A \widehat{\otimes} A \to A$. We have the forgetful functor $\mathit{FAlg} \to \mathit{FMod}$.

Degreewise cartesian products serve as products, and we note that the cohomology functor $E^*(-)\hat{}: \mathit{Ho}^{\mathrm{op}} \to \mathit{FAlg}$ takes coproducts in *Ho* to products in *FAlg*. The initial object is just E^* itself. Coproducts in *FAlg* are less obvious.

LEMMA 6.9. *The completed tensor product $A \widehat{\otimes} B$ of algebras serves as the coproduct in the category FAlg.*

PROOF. We first consider the uncompleted tensor product $A \otimes B$, made into an E^*-algebra in the standard way, filtered as in (4.15) by the ideals

$$F^{a,b}(A \otimes B) = \mathrm{Im}\left[(F^a A \otimes B) \oplus (A \otimes F^b B) \longrightarrow A \otimes B\right].$$

We define continuous injections $i: A \to A \otimes B$ and $j: B \to A \otimes B$ by $ix = x \otimes 1$ and $jy = 1 \otimes y$. Given continuous homomorphisms $f: A \to C$ and $g: B \to C$, where C is any object in *FAlg*, there is a unique homomorphism of algebras $h: A \otimes B \to C$ satisfying $h \circ i = f$ and $h \circ j = g$, defined by $h(x \otimes y) = (fx)(gy)$, thanks to the commutativity of C. It is also continuous: given $F^c C \subset C$, choose $F^a A$ and $F^b B$ such that $f(F^a A) \subset F^c C$ and $g(F^b B) \subset F^c C$; then $h(F^{a,b}(A \otimes B)) \subset F^c C$. Because $A \otimes B$ is rarely complete, we complete it, and the homomorphism h, to obtain the desired unique algebra homomorphism $\widehat{h}: A \widehat{\otimes} B \to C$ in *FAlg*. □

Although $E^*(-)^\wedge$ does not in general take products in *Ho* to coproducts in *FAlg*, it does in the favorable cases when we have the Künneth homeomorphism $E^*(X \times Y) \cong E^*(X) \widehat{\otimes} E^*(Y)$ as in Definition 4.17.

The module of indecomposables. If $(A, \phi, \eta, \varepsilon)$ is a (completed) algebra with counit (or augmentation) $\varepsilon: A \to E^*$ (which is required to be a morphism of algebras as in, e.g., a Hopf algebra), the augmentation ideal $\overline{A} = \mathrm{Ker}\, \varepsilon$ splits off as an E^*-module, $A \cong E^* \oplus \overline{A}$. One can define the *module of indecomposables* $QA = \overline{A}/\overline{A}\,\overline{A}$, i.e. $\mathrm{Coker}[\phi: \overline{A} \otimes \overline{A} \to \overline{A}]$ (or $\mathrm{Coker}[\phi: \overline{A} \widehat{\otimes} \overline{A} \to \overline{A}]$ in the completed case). A cleaner way to write this categorically is

$$QA = \mathrm{Coker}[\phi - A \otimes \varepsilon - \varepsilon \otimes A: A \otimes A \longrightarrow A] \quad \text{in } Mod, \tag{6.10}$$

as we see by using the splitting of A; the homomorphism here is zero on $\overline{A} \otimes 1$ and $1 \otimes \overline{A}$ and -1 on $E^* = E^* \otimes E^*$.

LEMMA 6.11. *The functor Q, defined on (completed) E^*-algebras with counit, preserves finite coproducts: $Q(A \otimes B) \cong QA \oplus QB$ (or $Q(A \widehat{\otimes} B) \cong QA \oplus QB$) and $QE^* = 0$.*

PROOF. For $C = A \otimes B$ (and similarly $A \widehat{\otimes} B$) we have the direct sum decomposition

$$\overline{C} = (\overline{A} \otimes 1) \oplus (1 \otimes \overline{B}) \oplus (\overline{A} \otimes \overline{B}).$$

Then $\phi(\overline{C} \otimes \overline{C})$ contains $\overline{A}\,\overline{A} \otimes 1$ from $(\overline{A} \otimes 1) \otimes (\overline{A} \otimes 1)$, $1 \otimes \overline{B}\,\overline{B}$ similarly, and $\overline{A} \otimes \overline{B}$ from $(\overline{A} \otimes 1) \otimes (1 \otimes \overline{B})$. The image is the direct sum of these, because the other six pieces of $\overline{C} \otimes \overline{C}$ give nothing new. This allows us to read off the cokernel. □

Coalg denotes the category of cocommutative E^*-coalgebras, with comultiplication $\psi: A \to A \otimes A$ and counit $\varepsilon: A \to E^*$.

When $E_*(X)$ is a free E^*-module, eq. (4.3) and $q_*: E_*(X) \to E^*$ make it an object in *Coalg*.

LEMMA 6.12. *In the category Coalg:*

(a) *The tensor product* $A \otimes B$ *of two coalgebras is again a coalgebra* (see, e.g., [25, §2]), *and serves as the product;*

(b) E^* *is the terminal object;*

(c) *Arbitrary direct sums* $\bigoplus_\alpha A_\alpha$ *of coalgebras serve as coproducts.*

There are also the slightly more general *completed coalgebras* A, where A is filtered as above and we have instead $\psi: A \to A \widehat{\otimes} A$. If A and B are completed coalgebras, so is $A \widehat{\otimes} B$.

The module of primitives. If $(A, \psi, \varepsilon, \eta)$ is a (completed) coalgebra with unit (e.g., a Hopf algebra), where $\eta: E^* \to A$ is required to be a morphism of coalgebras, we can define, dually to eq. (6.10), the *module of coalgebra primitives*

$$PA = \operatorname{Ker}[\psi - A \otimes \eta - \eta \otimes A: A \longrightarrow A \otimes A] \subset A \tag{6.13}$$

in *Mod* (or *FMod*, with $A \widehat{\otimes} A$ in place of $A \otimes A$), a submodule of A. The dual of Lemma 6.11 holds.

LEMMA 6.14. *The functor* P, *defined on (completed) coalgebras with unit, preserves finite products:* $P(A \otimes B) \cong PA \oplus PB$ *(or* $P(A \widehat{\otimes} B) \cong PA \oplus PB$*) and* $PE^* = 0$.

Dual modules. We warn that the con.pleted tensor product $\widehat{\otimes}$ does *not* make *FMod* a closed category (as $- \widehat{\otimes} M$ admits no right adjoint). Nor do we attempt to topologize $\mathit{FMod}(M, N)$ in general.

Nevertheless, we found it useful in Definition 4.8 to filter the dual $DM = \mathit{Mod}^*(M, E^*)$ of a *discrete* E^*-module M by the submodules $F^L DM = \operatorname{Ker}[DM \to DL]$, where L runs through all finitely generated submodules of M. Then $DM = \lim_L DL$ in *FMod*, where each DL is discrete; in particular, DM is automatically complete Hausdorff.

The dual $Df: DN \to DM$ of any homomorphism $f: M \to N$ is continuous, because $(Df)^{-1}(F^L DM) = F^{fL} DN$. In the important case when M is free, we obtain topologically equivalent filtrations by taking only those L that are (i) free of finite rank, or (ii) free of finite rank, and a summand of M, or (iii) generated by finite subsets of a given basis of M.

LEMMA 6.15. *Let* M, M_α, *and* N *be discrete* E^**-modules. Then:*

(a) *The canonical isomorphism* $D(M \oplus N) \cong DM \oplus DN = DM \times DN$ *is a homeomorphism;*

(b) *The canonical isomorphism* $D(\bigoplus_\alpha M_\alpha) \cong \prod_\alpha DM_\alpha$ *is a homeomorphism;*

(c) *If* $f: M \to N$ *is epic, then the dual* $Df: DN \to DM$ *is a topological embedding;*

(d) *The functor* D *takes colimits in Mod to limits in FMod;*

(e) $\zeta_D: DM \widehat{\otimes} DN \cong D(M \otimes N)$ *in FMod, if* M *or* N *is a free* E^**-module.*

PROOF. In (a), $D(M \oplus N) \to DM \times DN$ is continuous because D is a functor. Given a basic open set $F^L D(M \oplus N) \subset D(M \oplus N)$, where $L \subset M \oplus N$ is finitely generated, there are finitely generated submodules $P \subset M$ and $Q \subset N$ such that $L \subset P \oplus Q$; then $F^P DM \oplus F^Q DN \subset F^L D(M \oplus N)$ shows that we have a homeomorphism. More generally, we get (b).

In (c), we can lift any finitely generated submodule $L \subset N$ to a finitely generated submodule $K \subset M$ such that $fK = L$. Then $F^L DN = DN \cap F^K DM$ in DM.

If $C = \text{Coker}[f: M \to N]$, we have $DC = \text{Ker}[Df: DN \to DM]$ as an E^*-module. By (c), the topology on DC is correct, so that D sends cokernels to kernels. This, with (b), gives (d).

In (e), we may assume M is free. Equality is obvious for $M = E^*$ and therefore, by additivity, for M free of finite rank. By (d) and eq. (6.8), the general case is the limit in *FMod* of the isomorphisms $DL \otimes DN \cong D(L \otimes N)$ as L runs through the free submodules of M of finite rank that are summands of M. □

The evaluation $e: DL \otimes L \to E^*$, which we write as $e(r \otimes c) = \langle r, c\rangle$, is standard. The dual concept, of a homomorphism $E^* \to DL \otimes L$ for suitable L, is far less known, even for finite-dimensional vector spaces.

LEMMA 6.16. *Let L be a discrete free E^*-module. We can define the* universal element $u = u_L \in DL \widehat{\otimes} L$ *by the property that for any $r \in DL = \mathit{Mod}^*(L, E^*)$, the homomorphism*

$$DL \widehat{\otimes} \langle r, -\rangle : DL \widehat{\otimes} L \longrightarrow DL \otimes E^* \cong DL$$

takes u to r. It induces the following isomorphisms of E^*-modules:

(a) *$\mathit{Mod}^*(L, M) \cong DL \widehat{\otimes} M$ for any discrete E^*-module M, by $f \mapsto (DL \otimes f)u$, with inverse $r \otimes x \mapsto [c \mapsto (-1)^{\deg(c)\deg(x)} \langle r, c\rangle\, x]$;*

(b) *$\mathit{FMod}^*(DL, N) \cong N \widehat{\otimes} L$ for any object N of FMod, by $g \mapsto (g \otimes L)u$, with inverse $y \otimes c \mapsto [r \mapsto (-1)^e \langle r, c\rangle\, y]$, where $e = \deg(r)\deg(c) + \deg(r)\deg(y) + \deg(c)\deg(y)$;*

(c) *$\mathit{FMod}^*(DL, E^*) \cong E^* \otimes L \cong L$, by $g \leftrightarrow c$, where $c = (g \otimes L)u$ and $gr = (-1)^{\deg(r)\deg(c)} \langle r, c\rangle$.*

REMARK. We are not claiming to have isomorphisms in *FMod*. Indeed, for reasons already mentioned, we do not even topologize $\mathit{FMod}^*(DL, N)$ etc. In any case, the obvious E^*-module structures are the wrong ones for our applications.

PROOF. In terms of an E^*-basis $\{c_\alpha : \alpha \in A\}$ of L, u is given by

$$u = u_L = \sum_\alpha (-1)^{\deg(c_\alpha)} c_\alpha^* \otimes c_\alpha \in DL \widehat{\otimes} L,$$

where c_α^* denotes the linear functional dual to c_α, given by $\langle c_\alpha^*, c_\alpha\rangle = 1$ and $\langle c_\alpha^*, c_\beta\rangle = 0$ for $\beta \neq \alpha$. In effect, (a) generalizes the definition of u, and is clearly an isomorphism when L has finite rank, with inverse as stated.

For general L, we let K run through all the free submodules of L of finite rank. The functor $\mathit{Mod}^*(-, M)$ automatically takes the colimit $L = \operatorname{colim}_K K$ to a limit. On the right, the functor $-\widehat{\otimes} M$ preserves the limit $DL = \lim_K DK$ by eq. (6.8).

Similarly, (b) is obvious when L has finite rank and N is discrete. For general L and discrete N, any continuous homomorphism $DL \to N$ must factor through some DK, so that on the left, we have the colimit $\operatorname{colim}_K \mathit{Mod}^*(DK, N)$. On the right, we also have a colimit, $N \otimes L = \operatorname{colim}_K N \otimes K$ (as no completion is needed). This gives (b) for discrete N and general L. For general N, we observe that both sides preserve the limit $N = \lim_b N/F^b N$, with the help of eq. (6.8).

In the special case (c) of (b), the defining property of u implies by naturality that $gr = \pm\langle r, (g \otimes L)u\rangle$ for any $r \in DL$ and any $g: DL \to E^*$. □

It will be convenient to rearrange the signs in (b).

COROLLARY 6.17. *The general element*

$$\sum_\alpha (-1)^{\deg(y_\alpha)\deg(c_\alpha)} y_\alpha \otimes c_\alpha \in N \widehat{\otimes} L$$

of degree k corresponds to the general morphism $DL \to N$ of degree k given by

$$r \mapsto (-1)^{k\deg(r)} \sum_\alpha \langle r, c_\alpha\rangle\, y_\alpha.$$

7. Algebraic objects in categories

It has been known for a long time (e.g., Lawvere [19]) how to define algebraic objects in general categories. We are primarily interested in abelian group objects and generalizations, especially E^*-module and E^*-algebra objects, where E^* is a fixed commutative graded ring. We review the material on categories we need from MacLane's book [20, Chapters VI, VII].

Group objects. Let $\mathcal{C}$ be any category having a terminal object T and (enough) finite products. (Recall that T is the empty product.)

A *group object* in $\mathcal{C}$ is an object G equipped with a *multiplication* morphism $\mu: G \times G \to G$, a *unit* morphism $\omega: T \to G$, and an *inversion* morphism $\nu: G \to G$, that satisfy the usual axioms, expressed as well-known commutative diagrams (which may be viewed in [32, §1]). Then for any object X, $\mathcal{C}(X, G)$ becomes a group (as we see generally in Lemma 7.7), whose unit element is $\omega \circ q: X \to T \to G$. In the group $\mathcal{C}(G, G)$, ν is the inverse of 1_G.

An *abelian* group object G has μ commutative (another diagram); in this case, we call μ the *addition* and ω the *zero* morphism. Then the group $\mathcal{C}(X, G)$ is abelian.

If H is another group object in $\mathcal{C}$, a morphism $f: G \to H$ is a *morphism of group objects* if it commutes with the three structure morphisms; as is standard for sets and true generally (again by Lemma 7.7), it is enough to check μ. Thus we form the category $\mathcal{G}p(\mathcal{C})$ of all group objects in $\mathcal{C}$; one important example is $\mathcal{G}p(\mathcal{H}o)$.

EXAMPLE. In the category $\mathcal{S}et$, one writes the structure maps of an abelian group object as $\mu(x,y) = x + y$, $\omega(a) = 0$, and $\nu(x) = -x$, where $T = \{a\}$. Then the axioms take the form $(x+y)+z = x+(y+z)$, $x+0 = x$, $x+(-x) = 0$, and $x+y = y+x$, the usual axioms for an abelian group.

EXAMPLE. An (abelian) group object A in $\mathcal{C}oalg$ is a cocommutative Hopf algebra over E^*, with (commutative) multiplication $\phi: A \otimes A \to A$ and unit $\eta: E^* \to A$; the canonical antiautomorphism $\chi: A \to A$ is by [25, Definition 8.4] the inversion ν. (Recall from Lemma 6.12(a) that $A \otimes A$ is the product in $\mathcal{C}oalg$.)

Dually, a *cogroup object* in $\mathcal{C}$ is simply a group object G in the dual category $\mathcal{C}^{\mathrm{op}}$. That is, we use coproducts instead of products, an initial object I instead of T, and reverse all the arrows; so that G is equipped with a comultiplication $G \to G \amalg G$, counit $G \to I$, and inversion $G \to G$, satisfying the evident rules.

EXAMPLE. A commutative Hopf algebra A over E^* may be regarded as a cogroup object in $\mathcal{A}lg$ with comultiplication $\psi: A \to A \otimes A$, counit $\varepsilon: A \to E^*$, and inversion $\chi: A \to A$. (As in Lemma 6.9, $A \otimes A$ is the coproduct.)

EXAMPLE. In the based homotopy category $\mathcal{H}o'$, the circle S^1, and hence the suspension ΣX, are well-known cogroup objects.

In any additive category, we have abelian group objects for free.

LEMMA 7.1. *In a (graded) additive category $\mathcal{C}$:*

(a) *Every object admits a unique structure as abelian group object and as abelian cogroup object;*

(b) *Every morphism is a morphism of abelian (co)group objects;*

(c) *The (graded) abelian group structure on $\mathcal{C}(X,Y)$ resulting from the group object Y or the cogroup object X is the given one.*

PROOF. The zero object is terminal, which forces $\omega = 0$. The sum $G \oplus G$ serves as both product and coproduct. The axioms force $\mu = p_1 + p_2$ and $\nu = -1_G: G \to G$, and these choices work. The dual of an additive category is again additive. □

The *product* $G \times H$ of two group objects is another group object, with the obvious multiplication

$$\mu: G \times H \times G \times H \cong G \times G \times H \times H \xrightarrow{\mu \times \mu} G \times H, \tag{7.2}$$

unit $\omega \times \omega: T \cong T \times T \to G \times H$, and inversion $\nu \times \nu: G \times H \to G \times H$. This serves as the product in the category $Gp(\mathcal{C})$. The trivial group object T, with the unique structure morphisms, serves as the terminal object.

This allows one to define group objects in $Gp(\mathcal{C})$, as follows. To say that G is an *object* of $Gp(\mathcal{C})$ means that it is equipped with a multiplication μ_G, unit ω_G, and inversion ν_G that make it a group object in $\mathcal{C}$. In diag. (7.2) we made $G \times G$ an object of $Gp(\mathcal{C})$. Then G is a *group object* in $Gp(\mathcal{C})$ if it is equipped also with morphisms $\mu: G \times G \to G$, $\omega: T \to G$, and $\nu: G \to G$ in $Gp(\mathcal{C})$ that satisfy the axioms. The following useful result is well known.

PROPOSITION 7.3. *Let G be a group object in the category $Gp(\mathcal{C})$. Then the two group structures on G coincide and are abelian.*

PROOF. Lemma 7.7 will show that it is sufficient to consider the case $\mathcal{C} = Set$, where the result is a standard exercise (e.g., [20, Example III.6.4]). □

Module objects. A *graded group object* M in $\mathcal{C}$ is a function $n \mapsto M^n$ that assigns to each integer n (positive or negative) an abelian group object M^n in $\mathcal{C}$. (Note that the infinite product $\prod_n M^n$ and coproduct are irrelevant.)

An E^*-*module object* in a (graded) category $\mathcal{C}$ is a graded group object $n \mapsto M^n$ that is equipped with morphisms $\xi v: M^n \to M^{n+h}$ of abelian group objects (of degree h) for all $v \in E^*$ and all n, where $h = \deg(v)$, subject to the axioms:

$$\begin{aligned} &\text{(i) } \xi(v+v') = \xi v + \xi v' \text{ in the group } \mathcal{C}(M^n, M^{n+h}), \text{ for } v, v' \in E^h; \\ &\text{(ii) } \xi(vv') = \xi v \circ \xi v' \text{ for all } v, v' \in E^*; \\ &\text{(iii) } \xi 1 = 1: M^n \to M^n. \end{aligned} \tag{7.4}$$

It follows that the inversion $\nu = \xi(-1) = -1$ in $\mathcal{C}(M^n, M^n)$.

In an additive category, Lemma 7.1 shows that all we need is a graded object $n \mapsto M^n$ equipped with morphisms $\xi v: M^n \to M^{n+h}$ that satisfy the axioms (7.4). If $\mathcal{C}$ is graded, we often (but not always) have only a single object M, with $M^n = M$ for all n; then the definition reduces to a graded ring homomorphism $\xi: E^* \to \mathrm{End}^*_{\mathcal{C}}(M)$.

In a *graded* category, the concept of E^*-module object is self-dual, thanks to the commutativity of E^* (provided we watch the signs and indexing): $n \mapsto M^n$ is an E^*-module object in $\mathcal{C}$, with v acting by $\xi v: M^n \to M^{n+h}$, if and only if $n \mapsto M^{-n}$ is an E^*-module object in $\mathcal{C}^{\mathrm{op}}$, with v acting by $(\xi v)^{\mathrm{op}}: M^{n+h} \to M^n$ in $\mathcal{C}^{\mathrm{op}}$. (But we note that this observation fails in general in *ungraded* additive categories, because the required signs are absent.)

Algebra objects. A (*commutative*) *monoid object* in $\mathcal{C}$ is an object G equipped with a multiplication morphism $\phi: G \times G \to G$ and a unit morphism $\eta: T \to G$ that satisfy the axioms for associativity, (commutativity,) and unit. Apart from the lack of inverses and a change in notation, this is like a group object.

A *graded monoid object* is a graded object $n \mapsto M^n$, equipped with multiplications $\phi: M^k \times M^m \to M^{k+m}$ and a unit $\eta: T \to M^0$, that satisfy the axioms for associativity and unit. (There is a problem in defining commutativity for graded monoid objects, because extra structure is needed to handle the signs.)

An E^**-algebra object* in C is an E^*-module object that is also a graded monoid object, with the two structures related by three commutative diagrams that interpret the two distributive laws and $(vx)y = v(xy) = \pm x(vy)$. It is commutative if $yx = \pm xy$, interpreted as another diagram. Here, the sign $(-1)^n$ becomes $\xi((-1)^n)$.

It is often useful to replace the v-action $\xi v: M^n \to M^{n+h}$ in an E^*-algebra object by the simpler morphism $\eta_v = \xi v \circ \eta: T \to M^h$, so that $\eta_1 = \eta$; the diagram

$$\begin{tikzcd}
T \times M^n \arrow[r, "\eta \times M^n"] \arrow[dr, "\cong"'] & M^0 \times M^n \arrow[r, "\xi v \times M^n"] \arrow[d, "\phi"] & M^h \times M^n \arrow[d, "\phi"] \\
& M^n \arrow[r, "\xi v"] & M^{n+h}
\end{tikzcd}$$

shows that we can recover ξv from η_v as the composite

$$\xi v: M^n \cong T \times M^n \xrightarrow{\eta_v \times M^n} M^h \times M^n \xrightarrow{\phi} M^{n+h}. \tag{7.5}$$

Equivalently, we have interpreted the identity $vx = (v1)x$.

General algebraic objects. Other kinds of algebraic object can be defined similarly, provided they are (or can be) described in terms of operations $\alpha: G^{\times n(\alpha)} \to G$ subject to *universal* laws, where $G^{\times n} = G \times G \times \cdots \times G$, with n factors. Frequently, our algebraic object lies in the dual category C^{op} and is the corresponding *coalgebraic* object in C. Our general results extend without difficulty (except notationally) to the dual and graded variants, and we omit details.

The following observation is quite elementary but extremely useful.

LEMMA 7.6. *Let G be an algebraic object in C that is equipped with operations $\alpha: G^{\times n(\alpha)} \to G$, and $V: C \to \mathcal{D}$ be a functor.*

(a) *If V preserves (enough) finite powers of G, then VG is an algebraic object in $\mathcal{D}$ of the same kind, equipped with the operations*

$$\alpha: (VG)^{\times n(\alpha)} \cong V\big(G^{\times n(\alpha)}\big) \xrightarrow{V\alpha} VG;$$

(b) *If $f: G \to H$ is a morphism of algebraic objects in C, where H is another algebraic object of the same kind, and V preserves (enough) powers of G and H, then $Vf: VG \to VH$ is a morphism of algebraic objects in $\mathcal{D}$;*

(c) *If $\theta: V \to W$ is a natural transformation, where $W: C \to \mathcal{D}$ is another functor that preserves (enough) powers of G, then $\theta G: VG \to WG$ is a morphism of algebraic objects in $\mathcal{D}$.*

More precisely, V and W do not need to preserve *all* finite powers, only the powers of G and H that actually appear in the operations and laws (including the terminal object T, if used).

EXAMPLE. As S^1 is a cogroup object in Ho', (a) shows that the loop space ΩX on any based space X becomes a group object in Ho', and hence in Ho. If X is already a group object in Ho', (a) provides a second group object structure on ΩX; but by Proposition 7.3, these two group structures coincide and are abelian.

One common case where this lemma applies trivially is when V is an additive functor between additive categories. There are other functors of interest that automatically preserve products: for any object X in $\mathcal{C}$, the corepresented functor $\mathcal{C}(X,-):\mathcal{C} \to Set$ preserves products by definition, and dually, $\mathcal{C}(-,X) = \mathcal{C}^{op}(X,-):\mathcal{C}^{op} \to Set$ takes coproducts in $\mathcal{C}$ to products in Set. Then Lemma 7.6 gives parts (a), (b), and (c) of the following.

LEMMA 7.7. *Let G and H be fixed objects in the category $\mathcal{C}$, and V and W be the contravariant represented functors $\mathcal{C}(-,G), \mathcal{C}(-,H):\mathcal{C}^{op} \to Set$ (or dually, covariant corepresented functors $\mathcal{C}(G,-), \mathcal{C}(H,-):\mathcal{C} \to Set$).*

(a) *If G is a (co)algebraic object in $\mathcal{C}$, then for any object X in $\mathcal{C}$, VX is naturally an algebraic object in Set of the same kind;*

(b) *With G as in* (a), *then for any morphism $f: X \to Y$ in $\mathcal{C}$, $Vf: VY \to VX$ (or $Vf: VX \to VY$) is a morphism of algebraic objects in Set;*

(c) *Any morphism $f: G \to H$ of (co)algebraic objects in $\mathcal{C}$ induces a natural morphism $\mathcal{C}(X,f): VX \to WX$ (or $\mathcal{C}(f,X): WX \to VX$) of algebraic objects in Set;*

(d) *Conversely, if VX has a natural algebraic structure, it is induced as in* (a) *by a unique (co)algebraic structure on G of the same kind, provided the necessary (co)powers of G exist in $\mathcal{C}$;*

(e) *Any natural transformation of algebraic objects $VX \to WX$ (or $WX \to VX$) in Set is induced as in* (c) *by a unique morphism $f: G \to H$ of (co)algebraic objects in $\mathcal{C}$.*

PROOF. In (d), we may identify $\mathcal{C}(X,G)^{\times n}$ with $\mathcal{C}(X,G^{\times n})$. Then by Yoneda's Lemma, each natural transformation $\alpha: \mathcal{C}(-,G)^{\times n} \to \mathcal{C}(-,G)$ is induced by a unique morphism, which we also call $\alpha: G^{\times n} \to G$; the uniqueness shows that the same laws apply, thus making G an algebraic object. Part (e) is similar. □

This allows us to clarify Theorem 3.17.

COROLLARY 7.8. *We have the E^*-algebra object $n \mapsto \underline{E}_n$ in the category Ho; in particular, each $\underline{E}_n$ is an abelian group object in Ho. Moreover, each equivalence $\underline{E}_n \simeq \Omega\underline{E}_{n+1}$ is an isomorphism of group objects.*

PROOF. We apply (d) and (e) to the cohomology functors $E^n(-): Ho^{op} \to Set$, represented according to Theorem 3.17 by the spaces $\underline{E}_n$. Part (e) also gives the last assertion; by Proposition 7.3, the group structure on $\Omega\underline{E}_{n+1}$ is well defined. □

Symmetric monoidal categories. The theory presented so far is not general enough. In order to express the multiplicative structures, we need symmetric monoidal categories. We review the few basic facts we need from MacLane [20, Chapter VII].

A *(symmetric) monoidal category* $(\mathcal{C}, \otimes, K)$ is a category $\mathcal{C}$ equipped with a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit object $K = K_\mathcal{C}$. (But if $\mathcal{C}$ is graded, we need a more general kind of bifunctor $\otimes$ that is biadditive and includes signs, with composition as in eq. (6.5).) It is understood (but suppressed from our notation) that the specification includes [ibid.] coherent natural isomorphisms for associativity, (commutativity, with signs if $\mathcal{C}$ is graded) and $K \otimes X \cong X \cong X \otimes K$.

As examples, we have $(\mathit{Ab}, \otimes_\mathbb{Z}, \mathbb{Z})$, $(\mathit{Mod}, \otimes, E^*)$, $(\mathit{FMod}, \widehat{\otimes}, E^*)$, $(\mathit{Stab}, \wedge, T^+)$, the graded versions of all these, and the dual $(\mathcal{C}^{op}, \otimes, K)$ of any symmetric monoidal category. The original example was $(\mathcal{C}, \times, T)$, for any category $\mathcal{C}$ that admits finite products (including the empty product T).

EXAMPLE. We define the symmetric monoidal category $(\mathit{Set}^\mathbb{Z}, \times, T)$ of graded sets. For this purpose, the graded set $n \mapsto A^n$ is best treated as the disjoint union $A = \coprod_n A^n$, equipped with the degree function $A \to \mathbb{Z}$ given by $\deg(A^n) = n$. The product $A \times B$ is given the degree function $\deg((x, y)) = \deg(x) + \deg(y)$. The unit object is the set T consisting of one point in degree zero.

The purpose (for us) of a (symmetric) monoidal category is to extend the definition of monoid object. A *(commutative) monoid object* in $(\mathcal{C}, \otimes, K)$ is an object M of $\mathcal{C}$ that is equipped with a *multiplication* morphism $\phi: M \otimes M \to M$ and a *unit* morphism $\eta: K \to M$ (both of degree 0 if $\mathcal{C}$ is graded) that satisfy the usual axioms for associativity, (commutativity,) and left and right unit. In $(\mathit{Set}, \times, T)$, this reduces to the usual concept of (commutative) monoid; more generally, in $(\mathcal{C}, \times, T)$, it reduces to the concept of (commutative) monoid object as before.

A *graded monoid object* in $(\mathcal{C}, \otimes, K)$ is a graded object $n \mapsto M^n$ in $\mathcal{C}$ equipped with multiplications $\phi: M^k \otimes M^m \to M^{k+m}$ and unit $\eta: K \to M^0$ (with degree 0) that satisfy the axioms for associativity and two-sided unit. (Again, we defer the discussion of commutativity.) Morphisms of monoids are defined in the obvious way.

A *(symmetric) monoidal functor*

$$(F, \zeta_F, z_F): (\mathcal{C}, \otimes, K_\mathcal{C}) \longrightarrow (\mathcal{D}, \otimes, K_\mathcal{D})$$

between (symmetric) monoidal categories consists of a functor $F: \mathcal{C} \to \mathcal{D}$, together with a natural transformation $\zeta_F: FX \otimes FY \to F(X \otimes Y)$ and a morphism $z_F: K_\mathcal{D} \to FK_\mathcal{C}$ in $\mathcal{D}$. Of course, ζ_F and z_F are required to respect the isomorphisms for associativity, (commutativity,) and unit. If M is a (commutative) monoid object in $\mathcal{C}$, FM will be one in $\mathcal{D}$, equipped with the obvious multiplication

$$\phi: FM \otimes FM \xrightarrow{\zeta_F(M,M)} F(M \otimes M) \xrightarrow{F\phi} FM$$

and unit $F\eta \circ z_F: K_\mathcal{D} \to FK_\mathcal{C} \to FM$.

We do *not* require ζ_F and z_F to be isomorphisms (but if they are, so much the better). One example is the duality functor

$$(D, \zeta_D, z_D): (Mod^{op}, \otimes, E^*) \longrightarrow (FMod, \widehat{\otimes}, E^*)$$

defined by $DM = Mod^*(M, E^*)$ and filtered in Definition 4.8, where $z_D: E^* \cong DE^*$ is obvious and ζ_D was originally defined in eq. (4.6) and completed later for diag. (4.18). By Lemma 6.15(e), ζ_D is sometimes an isomorphism. Another example is the symmetric monoidal functor

$$(\mathcal{C}(X, -), \zeta, z): (\mathcal{C}, \times, T) \longrightarrow (Set, \times, T)$$

used in Lemma 7.7 to map an algebraic object in $\mathcal{C}$ to the corresponding algebraic object in Set; in this case, ζ and z are automatically isomorphisms.

Monoidal functors compose in the obvious way. Given another (symmetric) monoidal functor $(G, \zeta_G, z_G): (\mathcal{D}, \otimes, K_{\mathcal{D}}) \to (\mathcal{E}, \otimes, K_{\mathcal{E}})$, the composite (symmetric) monoidal functor $(GF, \zeta_{GF}, z_{GF}): (\mathcal{C}, \otimes, K_{\mathcal{C}}) \to (\mathcal{E}, \otimes, K_{\mathcal{E}})$ uses the natural transformation

$$\zeta_{GF}: GFX \otimes GFY \xrightarrow{\zeta_G} G(FX \otimes FY) \xrightarrow{G\zeta_F} GF(X \otimes Y)$$

and morphism

$$z_{GF}: K_{\mathcal{E}} \xrightarrow{z_G} GK_{\mathcal{D}} \xrightarrow{Gz_F} GFK_{\mathcal{C}}.$$

Given two (symmetric) monoidal functors

$$(F, \zeta_F, z_F), (G, \zeta_G, z_G): (\mathcal{C}, \otimes, K_{\mathcal{C}}) \longrightarrow (\mathcal{D}, \otimes, K_{\mathcal{D}}),$$

a natural transformation $\theta: F \to G$ is called *monoidal* if there are commutative diagrams

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\theta X \otimes \theta Y"] \arrow[d, "\zeta_F(X,Y)"] & GX \otimes GY \arrow[d, "\zeta_G(X,Y)"] \\ F(X \otimes Y) \arrow[r, "\theta(X \otimes Y)"] & G(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} K_{\mathcal{D}} \arrow[d, "z_F"] \arrow[dr, "z_G"] & \\ FK_{\mathcal{C}} \arrow[r, "\theta K_{\mathcal{C}}"] & GK_{\mathcal{C}} \end{tikzcd}$$

Thus if X is a monoid object in $\mathcal{C}$, $\theta X: FX \to GX$ will be a morphism of monoid objects in $\mathcal{D}$.

We adapt Lemma 7.7 to monoidal functors.

LEMMA 7.9. *Given a graded monoid object $n \mapsto C^n$ in the (graded) monoidal category $(\mathcal{C}^{op}, \otimes, K)$, write $(FM)^n = \mathcal{C}(C^n, M)$ for any object M in $\mathcal{C}$. Then:*

(a) *We can make F a monoidal functor*

$$(F, \zeta_F, z_F): (\mathcal{C}, \otimes, K) \longrightarrow (Set^{\mathbf{Z}}, \times, T); \tag{7.10}$$

(b) *If the graded monoid object $n \mapsto D^n$ defines similarly the monoidal functor G, then a morphism $h\colon C \to D$ in $\mathcal{C}^{\mathrm{op}}$ of graded monoid objects induces a monoidal natural transformation $\theta\colon F \to G$.*

PROOF. Let the multiplications and unit of C be $\phi\colon C^k \otimes C^m \to C^{k+m}$ and $\eta\colon K \to C^0$ (in $\mathcal{C}^{\mathrm{op}}$). We defined FM as a graded set. Given $f \in (FM)^k$ and $g \in (FN)^m$, we define

$$\zeta_F(f,g) \in F(M \otimes N)^{k+m} = \mathcal{C}\big(C^{k+m}, M \otimes N\big)$$

as the composite

$$C^{k+m} \xrightarrow{\phi^{\mathrm{op}}} C^k \otimes C^m \xrightarrow{f \otimes g} M \otimes N \quad \text{in } \mathcal{C}. \tag{7.11}$$

The morphism $z_F\colon T \to (FK)^0 = \mathcal{C}(C^0, K)$ has $\eta^{\mathrm{op}}\colon C^0 \to K$ as its image. In (b), we define

$$(\theta M)^n\colon (FM)^n = \mathcal{C}(C^n, M) \longrightarrow \mathcal{C}(D^n, M) = (GM)^n$$

as composition in $\mathcal{C}$ with $h^{\mathrm{op}}\colon D^n \to C^n$. The necessary verification is routine. □

Additive symmetric monoidal categories. We need a slightly more general categorical structure, arranged in two layers. If the category $\mathcal{C}$ is both monoidal and additive, it will be appropriate to use the monoidal structure $(\mathcal{C}, \otimes, K)$ to define multiplication, but to return to the additive structure of $\mathcal{C}$ to define addition. In this situation, we require the bifunctor $\otimes$ to be biadditive. Rather than strive for great generality, we limit attention to the cases we actually need. (We do not attempt to define the tensor product of E^*-module objects.)

Because $\mathcal{C}$ is additive, an E^*-module object reduces simply to a graded object $n \mapsto M^n$ equipped with morphisms $\xi v\colon M^n \to M^{n+h}$ for all $v \in E^*$ and all n (where $h = \deg(v)$) that satisfy the axioms (7.4). Further, we can now define *commutative* graded monoid objects $n \mapsto M^n$, including the expected sign.

DEFINITION 7.12. A (*commutative*) *E^*-algebra object* in the (possibly graded) additive (symmetric) monoidal category $(\mathcal{C}, \otimes, K)$ is a graded object $n \mapsto M^n$ equipped with:

(i) morphisms $\xi v\colon M^n \to M^{n+h}$, for all n, h, and $v \in E^h$, that make it an E^*-module object in $\mathcal{C}$;

(ii) morphisms (ϕ, η) that make it a graded (commutative) monoid object;

in such a way that the diagrams commute up to the indicated sign:

$$\begin{tikzcd}
M^k \otimes M^m \arrow[r, "\phi"] \arrow[d, "\xi v \otimes 1"] & M^{k+m} \arrow[d, "\xi v"] \\
M^{k+h} \otimes M^m \arrow[r, "\phi"] & M^{k+m+h}
\end{tikzcd}
\qquad
\begin{tikzcd}
M^k \otimes M^m \arrow[r, "\phi"] \arrow[d, "1 \otimes \xi v"] \arrow[dr, phantom, "(-1)^{kh}"] & M^{k+m} \arrow[d, "\xi v"] \\
M^k \otimes M^{m+h} \arrow[r, "\phi"] & M^{k+m+h}
\end{tikzcd} \tag{7.13}$$

In the commutative case, the two diagrams are equivalent.

EXAMPLE. An E^*-algebra object in $(Ab, \otimes_{\mathbb{Z}}, \mathbb{Z})$ is just an E^*-algebra.

We can again simplify the structure by replacing the v-actions ξv by the single morphism $\eta_v = \xi v \circ \eta : K \to M^h$ for each $v \in E^h$; as in eq. (7.5), we recover ξv from η_v as the composite

$$\xi v : M^n \cong K \otimes M^n \xrightarrow{\eta_v \otimes M^n} M^h \otimes M^n \xrightarrow{\phi} M^{n+h}.$$

LEMMA 7.14. *Let $n \mapsto C^n$ be a (commutative) E^*-algebra object in the (graded) additive (symmetric) monoidal category $(\mathcal{C}^{op}, \otimes, K)$. Then the functor* (7.10) *becomes a (symmetric) monoidal functor*

$$(F, \zeta_F, z_F) : (\mathcal{C}, \otimes, K) \longrightarrow (Mod, \otimes, E^*) \quad \big(\text{or } (Mod^*, \otimes, E^*)\big).$$

PROOF. For fixed L, the functor $\mathcal{C}(-, L) : \mathcal{C}^{op} \to Ab$ (or Ab^*) takes the E^*-module object C in $\mathcal{C}^{op}$ to the E^*-module FL, by Lemma 7.7(a). The action of $v \in E^h$ on FL is the composition $\mathrm{Mor}((\xi v)^{op}, L) : FL \to FL$ with $(\xi v)^{op} : C^{n+h} \to C^n$ (including signs as in eq. (6.4) if $\mathcal{C}$ is graded). As L varies, F takes values in Mod by Lemma 7.7(b); diags. (7.13) show that $\zeta_F : FL \times FN \to F(L \otimes N)$ is E^*-bilinear, allowing us to write $\zeta_F : FL \otimes FN \to F(L \otimes N)$. We define $z_F : E^* \to FK$ on $v \in E^h$ as

$$z_F v : C^h \xrightarrow{(\xi v)^{op}} C^0 \xrightarrow{\eta^{op}} K \quad \text{in } \mathcal{C}, \tag{7.15}$$

to make it an E^*-module homomorphism. □

8. What is a module?

In this section, we study the relationship between the category $R\text{-}Mod$ of left R-modules and the category Ab of abelian groups from several points of view, in order to abstract and generalize it to cover all our main objects of interest in a uniform manner. The central theme is the classical construction by Eilenberg and Moore [13] (or see MacLane [20, Chapter VI]) of a pair of adjoint functors by means of algebras in categories, except that the less familiar (but equivalent) dual formulation, in terms of comonads, turns out to be appropriate.

This will serve as a pattern for our definitions. There are of course variants for graded categories and graded objects. Graded categories can be handled by replacing the graded group $\mathcal{A}^*(X, Y)$ by the group $\bigoplus_n \mathcal{A}^n(X, Y)$, or sometimes even the disjoint union of the sets $\mathcal{A}^n(X, Y)$. Graded objects can be handled by working in the category $\mathcal{A}^{\mathbb{Z}}$ of graded objects $n \mapsto X_n$ in $\mathcal{A}$. We omit details.

The ring R is usually not commutative. Like all our rings, it is understood to have a multiplication ϕ and a unit element 1_R; we define the unit homomorphism $\eta : \mathbb{Z} \to R$ by

$\eta 1 = 1_R$. The associativity and unit axioms on R take the form of three commutative diagrams in *Ab*:

$$
\text{(i)}\ \begin{tikzcd}
R\otimes R\otimes R \arrow[r, "\phi\otimes R"] \arrow[d, "R\otimes\phi"'] & R\otimes R \arrow[d, "\phi"] \\
R\otimes R \arrow[r, "\phi"] & R
\end{tikzcd}
\qquad
\text{(ii)}\ \begin{tikzcd}
\mathbb{Z}\otimes R \arrow[r, "\eta\otimes R"] \arrow[dr, "\cong"] & R\otimes R \arrow[d, "\phi"] \\
 & R
\end{tikzcd}
\qquad
\text{(iii)}\ \begin{tikzcd}
R\otimes\mathbb{Z} \arrow[r, "R\otimes\eta"] \arrow[dr, "\cong"] & R\otimes R \arrow[d, "\phi"] \\
 & R
\end{tikzcd}
\tag{8.1}
$$

In this section (only), all tensor products $\otimes$ and Hom groups are taken over the integers $\mathbb{Z}$.

First Answer. The standard definition of a left R-module (e.g., [25, Definition 1.2]) equips an abelian group M with a *left action* $\lambda_M: R\otimes M \to M$ in *Ab*. It is required to satisfy the usual two axioms, which we express as commutative diagrams:

$$
\text{(i)}\ \begin{tikzcd}
R\otimes R\otimes M \arrow[r, "\phi\otimes M"] \arrow[d, "R\otimes\lambda_M"'] & R\otimes M \arrow[d, "\lambda_M"] \\
R\otimes M \arrow[r, "\lambda_M"] & M
\end{tikzcd}
\qquad
\text{(ii)}\ \begin{tikzcd}
\mathbb{Z}\otimes M \arrow[r, "\eta\otimes M"] \arrow[dr, "\cong"] & R\otimes M \arrow[d, "\lambda_M"] \\
 & M
\end{tikzcd}
\tag{8.2}
$$

Second Answer. We make our First Answer more functorial by introducing the functor $T = R\otimes -: Ab \to Ab$. We define natural transformations $\phi: TT \to T$ and $\eta: I \to T$ on A by $\phi A = \phi_R \otimes A: R\otimes R\otimes A \to R\otimes A$ and $(\eta A)x = 1\otimes x \in R\otimes A$. The action on M is now a morphism $\lambda_M: TM \to M$, and the axioms (8.2) take the cleaner form

$$
\text{(i)}\ \begin{tikzcd}
TTM \arrow[r, "\phi M"] \arrow[d, "T\lambda_M"'] & TM \arrow[d, "\lambda_M"] \\
TM \arrow[r, "\lambda_M"] & M
\end{tikzcd}
\qquad
\text{(ii)}\ \begin{tikzcd}
M \arrow[r, "\eta M"] \arrow[dr, "="] & TM \arrow[d, "\lambda_M"] \\
 & M
\end{tikzcd}
\tag{8.3}
$$

Third Answer. We have so far attempted to describe a module structure over a ring without first properly defining a ring structure. In particular, we have not yet mentioned the fact that R is itself an R-module, as is evident by comparing axioms (8.2) with two axioms of (8.1). The function of the other axiom (8.1)(iii) is to ensure that R is a *free* module on one generator 1_R: given $x \in M$, there is a unique module homomorphism $f: R \to M$ that satisfies $f1_R = x$, since $fr = f(r1_R) = rf1_R = rx$.

The three axioms on R translate into commutative diagrams of natural transformations in Ab:

$$\text{(i)}\ \begin{array}{ccc} TTT & \xrightarrow{\phi T} & TT \\ \downarrow{\scriptstyle T\phi} & & \downarrow{\scriptstyle \phi} \\ TT & \xrightarrow{\phi} & T \end{array} \qquad \text{(ii)}\ \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "="'] & TT \arrow[d, "\phi"] \\ & T \end{tikzcd} \qquad \text{(iii)}\ \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "="'] & TT \arrow[d, "\phi"] \\ & T \end{tikzcd} \tag{8.4}$$

Thus a ring structure on R is equivalent to what is known as a *monad* (or *triple*) structure (ϕ, η) on the functor T. By analogy, we call ϕ the *multiplication* and η the *unit* of the monad T. We recognize an R-module as being precisely what is known as a *T-algebra*, namely, an object M equipped with an action morphism $\lambda_M: TM \to M$ that satisfies the axioms (8.3).

Fourth Answer. More generally, the first two axioms of (8.4) show that for any abelian group A, the action $\phi A: TTA \to TA$ makes TA an R-module, which we call FA; this defines a functor $F: Ab \to R\text{-}Mod$. We thus have the factorization $T = VF$, where $V: R\text{-}Mod \to Ab$ denotes the forgetful functor. We similarly factor $\phi = V\varepsilon F: TT = V(FV)F \to VF = T$, where $\varepsilon: FV \to I$ is defined on the R-module M as $\varepsilon M = \lambda_M: R \otimes M \to M$; by axiom (8.3)(i), εM lies in $R\text{-}Mod$. In this formulation, axiom (8.4)(i) simply defines the natural transformation

$$V\varepsilon\epsilon F: TTT = V(FVFV)F \to VF = T,$$

while the other two reduce to the identities (2.5) relating η and ε.

All this works in any category $\mathcal{A}$, as an application of Theorem 2.6(v).

THEOREM 8.5 (Eilenberg–Moore). *Given a monad (T, ϕ, η) in $\mathcal{A}$, let $\mathcal{B}$ be the category of T-algebras, $V: \mathcal{B} \to \mathcal{A}$ the forgetful functor, and $F: \mathcal{A} \to \mathcal{B}$ the functor that assigns to each A in $\mathcal{A}$ the T-algebra $FA = (TA, \phi A)$. Then F is left adjoint to V, $\mathcal{B}(FA, M) \cong \mathcal{A}(A, VM)$ for any M in $\mathcal{B}$, and FA is V-free on A with basis $\eta A: A \to TA = VFA$* (in the language of Definition 2.1).

PROOF. We have already outlined most of the proof in the special case when $\mathcal{A} = Ab$ and $T = R \otimes -$, and can apply Theorem 2.6. For further details, see Eilenberg and Moore [13, Theorem 2.2] or MacLane [20, Theorem VI.2.1]. □

The image of F is known as the Kleisli category of all V-free objects.

Fifth Answer. The problem with our answers so far is that they rely heavily on the tensor product, which really has little to do with modules. While tensor products are (as we shall see) convenient for computation, they are simply not available in the nonadditive context of [9].

We therefore replace the functor $T = R \otimes -$ by its equivalent right adjoint $H = \mathrm{Hom}(R, -)$: $Ab \to Ab$. The right adjoint of $\phi: TT \to T$ is the *comultiplication* $\psi: H \to HH$, which is given on A as the homomorphism

$$\psi A: \mathrm{Hom}(R, A) \longrightarrow \mathrm{Hom}\big(R, \mathrm{Hom}(R, A)\big)$$

that sends $f: R \to A$ to $s \mapsto [r \mapsto f(rs)]$. The right adjoint of $\eta: I \to T$ is the *counit* $\varepsilon: H \to I$, where $\varepsilon A: \mathrm{Hom}(R, A) \to A$ is simply evaluation on 1_R. The axioms (8.4) dualize to

$$\text{(i)}\quad \begin{tikzcd} H \arrow[r, "\psi"] \arrow[d, "\psi"] & HH \arrow[d, "H\psi"] \\ HH \arrow[r, "\psi H"] & HHH \end{tikzcd} \qquad \text{(ii)}\quad \begin{tikzcd} H \arrow[r, "\psi"] \arrow[dr, "="] & HH \arrow[d, "H\varepsilon"] \\ & H \end{tikzcd} \qquad \text{(iii)}\quad \begin{tikzcd} H \arrow[r, "\psi"] \arrow[dr, "="] & HH \arrow[d, "\varepsilon H"] \\ & H \end{tikzcd} \qquad (8.6)$$

which state that (H, ψ, ε) is what is known as a *comonad* in *Ab*.

Similarly, we replace the action λ_M on a module M by the right adjunct *coaction*

$$\rho_M: M \to HM = \mathrm{Hom}(R, M).$$

This is given explicitly by $(\rho_M x)r = rx$, which also shows us how to recover the action from ρ_M. The way to think of $\mathrm{Hom}(R, M)$ is as the set of all possible candidates for the R-action on a typical element of M; then ρ_M selects for each $x \in M$ the action $r \mapsto rx$. The action axioms (8.3) become

$$\text{(i)}\quad \begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "\rho_M"] & HM \arrow[d, "\psi M"] \\ HM \arrow[r, "H\rho_M"] & HHM \end{tikzcd} \qquad \text{(ii)}\quad \begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[dr, "="] & HM \arrow[d, "\varepsilon M"] \\ & M \end{tikzcd} \qquad (8.7)$$

which state that M is what is called a *coalgebra* over the comonad H. Occasionally, it is useful to evaluate the right side of (i) on a typical $r \in R$, to yield the commutative square

$$\begin{tikzcd} M \arrow[r, "r_M"] \arrow[d, "\rho_M"] & M \arrow[d, "\rho_M"] \\ \mathrm{Hom}(R, M) \arrow[r, "\mathrm{Hom}(r^*, M)"] & \mathrm{Hom}(R, M) \end{tikzcd} \qquad (8.8)$$

where $r_M: M \to M$ denotes the action of r on M and $r^*: R \to R$ denotes right multiplication by r.

A homomorphism $f\colon M \to N$ of R-modules is now a morphism in Ab for which we have the commutative square

$$\begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "f"] & HM \arrow[d, "Hf"] \\ N \arrow[r, "\rho_N"] & HN \end{tikzcd} \tag{8.9}$$

This description successfully avoids all tensor products. It too works quite generally.

THEOREM 8.10. *Given a comonad H in $\mathcal{A}$, let $\mathcal{C}$ be the category of H-coalgebras, $V\colon \mathcal{C} \to \mathcal{A}$ the forgetful functor, and $C\colon \mathcal{A} \to \mathcal{C}$ the functor that assigns to each A in $\mathcal{A}$ the H-coalgebra HA with the coaction $\psi A\colon HA \to HHA$. Then C is right adjoint to V, $\mathcal{A}(VM, A) \cong \mathcal{C}(M, CA)$ for all M in $\mathcal{C}$, and $CA = (HA, \psi A)$ is V-cofree on A with cobasis $\varepsilon A\colon HA = VCA \to A$* (in the language of Definition 2.7).

PROOF. This is just Theorem 8.5 in the dual category $\mathcal{A}^{\mathrm{op}}$. □

Sixth Answer. The previous answer is certainly elegant, but we shall need an alternate description of R-modules that does not use ψ and ε. The key to achieving this is not to take adjuncts of everything.

Given an element $x \in M$, we put $f = \rho_M x\colon R \to M$ (given by $fr = rx$). Then commutativity of the square

$$\begin{tikzcd} R \arrow[r, "\rho_R"] \arrow[d, "f"] & HR \arrow[d, "Hf"] \\ M \arrow[r, "\rho_M"] & HM \end{tikzcd} \tag{8.11}$$

expresses the law $(sr)x = s(rx)$. In other words, $f\colon R \to M$ is a homomorphism of R-modules. The law $1_R x = x$ is expressed as $f 1_R = x$.

Seventh Answer. The first level of abstraction in category theory is to avoid dealing with the elements of a set. The next level is to avoid dealing with the objects in a category. We have not yet used the fact that H is a corepresented functor. Given any functor $F\colon Ab \to Ab$, Yoneda's Lemma (dualized) yields a 1–1 correspondence between natural transformations $\theta\colon H \to F$ and elements $(\theta R)\mathrm{id}_R \in FR$, where $\theta R\colon \mathrm{Hom}(R, R) = HR \to FR$ and $\mathrm{id}_R \in HR$ denotes the identity morphism of R. For example, $\psi\colon H \to HH$ corresponds to $\rho_R \in HHR$, the coaction on the R-module R, and $\varepsilon\colon H \to I$ corresponds to $1_R \in R = IR$. We note that $\rho_R 1_R = \mathrm{id}_R$.

To this end, we replace the object M by the corepresented functor $F_M = \mathrm{Hom}(M, -)\colon Ab \to Ab$. (We already did this for $M = R$, to get $F_R = H$.) We replace the coaction morphism $\rho_M: M \to HM$ by the equivalent natural transformation $\rho_M: F_M \to F_M H: Ab \to Ab$; explicitly, $\rho_M N: F_M N \to F_M HN$ is

$$\rho_M N: \mathrm{Hom}(M, N) \xrightarrow{H} \mathrm{Hom}(HM, HN) \xrightarrow{\mathrm{Hom}(\rho_M, 1)} \mathrm{Hom}(M, HN). \quad (8.12)$$

The axioms (8.7) translate into equivalent commutative diagrams of natural transformations

$$\text{(i)} \quad \begin{tikzcd} F_M \arrow[r, "\rho_M"] \arrow[d, "\rho_M"'] & F_M H \arrow[d, "F_M \psi"] \\ F_M H \arrow[r, "\rho_M H"] & F_M H H \end{tikzcd} \qquad \text{(ii)} \quad \begin{tikzcd} F_M \arrow[r, "\rho_M"] \arrow[dr, "="'] & F_M H \arrow[d, "F_M \epsilon"] \\ & F_M \end{tikzcd} \qquad (8.13)$$

We observe that if we take $M = R$, these reduce to axioms (8.6)(i) and (ii).

Eighth Answer. In our applications, we do not have the luxury of starting out with a comonad; we have to construct it. Consequently, we are not able to invoke Theorem 8.10 directly. Instead, we generalize our Sixth Answer. We have to treat modules and rings together.

We assume that $\mathcal{A}$ is a category of sets with structure in the sense that we are given a faithful forgetful functor $W: \mathcal{A} \to \mathcal{S}et$. We assume given:

(i) A functor $H: \mathcal{A} \to \mathcal{A}$;
(ii) An object R in $\mathcal{A}$ that corepresents H in the sense that $WHM = \mathcal{A}(R, M)$, naturally in M;
(iii) An element 1_R of the set WR; (8.14)
(iv) A morphism $\rho_R: R \to HR$ in $\mathcal{A}$, which we call the *pre-coaction* on R, such that $W\rho_R: WR \to WHR = \mathcal{A}(R, R)$ in $\mathcal{S}et$ carries $1_R \in WR$ to the identity morphism $\mathrm{id}_R: R \to R$ of R.

We impose no further axioms at this point. In fact, we call *any* morphism $\rho_M: M \to HM$ a *pre-coaction on* M, and a morphism $f: M \to N$ a *morphism of pre-coactions* if it makes diag. (8.9) commute. To see what it takes to make ρ_M a coaction, we consider the function

$$W\rho_M: WM \longrightarrow WHM = \mathcal{A}(R, M) \quad \text{in } \mathcal{S}et.$$

DEFINITION 8.15. Given an object M of $\mathcal{A}$, a *coaction* on M is a pre-coaction $\rho_M: M \to HM$ such that for any element $x \in WM$, the morphism $f = (W\rho_M)x: R \to M$ in $\mathcal{A}$ satisfies:

(i) f makes diag. (8.11) commute, i.e. is a morphism of pre-coactions;
(ii) $Wf: WR \to WM$ sends $1_R \in WR$ to $x \in WM$.

We do not assume yet that ρ_R is itself a coaction. Lemma 8.20 will show that in the presence of suitable additional structure, this definition does agree with previous notions of what a coaction should be.

Ninth Answer. We generalize our Seventh Answer to the category $\mathcal{A}$ as above. We convert everything to corepresented functors. We make no claims to elegance, only that the machinery does what we need.

We replace an object M by the corepresented functor $F_M = \mathcal{A}(M, -): \mathcal{A} \to Set$, and a pre-coaction $\rho_M: M \to HM$ by the equivalent natural transformation $\rho_M: F_M \to F_M H: \mathcal{A} \to Set$. Explicitly, $\rho_M N: F_M N \to F_M HN$ is (cf. eq. (8.12))

$$\rho_M N: \mathcal{A}(M,N) \xrightarrow{H} \mathcal{A}(HM,HN) \xrightarrow{\mathcal{A}(\rho_M,HN)} \mathcal{A}(M,HN). \tag{8.16}$$

In particular, we convert the pre-coaction ρ_R to the natural transformation $\rho_R: WH \to WHH$, where $\rho_R N: WHN \to WHHN$ is

$$\rho_R N: \mathcal{A}(R,N) \xrightarrow{H} \mathcal{A}(HR,HN) \xrightarrow{\mathcal{A}(\rho_R,HN)} \mathcal{A}(R,HN). \tag{8.17}$$

Similarly, if $g: M \to N$ is a morphism of pre-coactions, we obtain the natural transformation $F_g: F_N \to F_M$ and from diag. (8.9) the commutative square

$$\begin{tikzcd} F_N \arrow[r, "\rho_N"] \arrow[d, "F_g"] & F_N H \arrow[d, "F_g H"] \\ F_M \arrow[r, "\rho_M"] & F_M H \end{tikzcd} \tag{8.18}$$

We now assume that H is equipped with natural transformations:

(i) $\psi: H \to HH$ such that $W\psi: WH \to WHH$ is the natural transformation ρ_R of eq. (8.17);
(ii) $\varepsilon: H \to I$ such that $W\varepsilon R: WHR = \mathcal{A}(R,R) \to WR$ sends id_R to 1_R.
(8.19)

We assume *no* further properties of ψ and ε. In particular, (i) implies (and by naturality is equivalent to) the statement that

$$\mathcal{A}(R,R) = WHR \xrightarrow{W\psi R} WHHR = \mathcal{A}(R,HR)$$

takes id_R to the morphism ρ_R.

LEMMA 8.20. *Assume we have a category $\mathcal{A}$ equipped with W, H, R, ψ, and ε, satisfying the axioms* (8.14) *and* (8.19). *Then given an object M of $\mathcal{A}$, a pre-coaction $\rho_M: M \to HM$ is a coaction in the sense of Definition* 8.15 *if and only if it makes diags.* (8.7) *commute.*

PROOF. Since W is faithful, we may apply W to diags. (8.7) and work with diagrams of sets. Thus (i) becomes

$$\begin{tikzcd}
 & WM \arrow[r, "W\rho_M"] \arrow[d, "W\rho_M"] & WHM \arrow[d, "WH\rho_M"] & \mathcal{A}(R,M) \arrow[l, "\cong"'] \arrow[d, "\mathcal{A}(R,\rho_M)"] \\
\mathcal{A}(R,M) \arrow[r, "\cong"] & WHM \arrow[r, "W\psi M"] & WHHM & \mathcal{A}(R,HM) \arrow[l, "\cong"']
\end{tikzcd}$$

We evaluate on any $x \in WM$ and put $f = (W\rho_M)x\colon R \to M$. The upper route gives $\rho_M \circ f\colon R \to HM$, while the lower route gives $Hf \circ \rho_R\colon R \to HM$ by axiom (8.19)(i). These agree if and only if f is a morphism of pre-coactions as in diag. (8.11).

For diag. (8.7)(ii) we consider

$$\begin{tikzcd}
WM \arrow[r, "W\rho_M"] \arrow[dr, "="] & WHM \arrow[d, "W\varepsilon M"] & WHR \arrow[l, "WHf"'] \arrow[d, "W\varepsilon R"] \\
 & WM & WR \arrow[l, "Wf"']
\end{tikzcd}$$

The element $f \in WHM = \mathcal{A}(R,M)$ lifts to $\mathrm{id}_R \in WHR = \mathcal{A}(R,R)$, which by axiom (8.19)(ii) maps to $1_R \in WR$. Thus $(Wf)1_R = x$ is exactly what we need. $\square$

As in our Seventh Answer, we convert the objects in diags. (8.7) to corepresented functors.

COROLLARY 8.21. *The pre-coaction $\rho_M\colon M \to HM$ is a coaction* (in the sense of Definition 8.15) *if and only if the associated natural transformation $\rho_M\colon F_M \to F_M H\colon \mathcal{A} \to$ Set makes diags.* (8.13) *commute.*

Now we can recover the full strength of Theorem 8.10.

LEMMA 8.22. *Assume that $\rho_R\colon R \to HR$ is a coaction in the sense of Definition* 8.15, *and that ψ and ε satisfy axioms* (8.19). *Then:*

(a) *ψ and ε make H a comonad in $\mathcal{A}$;*

(b) *A pre-coaction $\rho_M\colon M \to HM$ makes M an H-coalgebra if and only if it is a coaction in the sense of Definition* 8.15.

PROOF. The first two axioms of (8.6) are just axioms (8.13) for $M = R$, which we have by Corollary 8.21. For the third, we have to show that $W\varepsilon HN \circ W\psi N\colon WHN \to WHN$ is the identity. We evaluate on $g \in WHN = \mathcal{A}(R,N)$. From eq. (8.17), $(W\psi N)g = Hg \circ \rho_R$. We consider the diagram in fig. 1, which commutes merely because $\varepsilon\colon H \to I$ is natural. We start from $\mathrm{id}_R \in \mathcal{A}(R,R)$, which maps to $Hg \circ \rho_R \in \mathcal{A}(R,HN)$, $1_R \in WR$, $\mathrm{id}_R \in \mathcal{A}(R,R)$ (by axiom (8.14)(iv)), and hence to $g \in \mathcal{A}(R,N)$.

Part (b) is then a restatement of Lemma 8.20. $\square$

$$\begin{tikzcd}
\mathcal{A}(R,R) \arrow[r, "="] \arrow[d, "\mathcal{A}(R,\rho_R)"] & WHR \arrow[r, "W\epsilon R"] \arrow[d, "WH\rho_R"] & WR \arrow[d, "W\rho_R"] & \\
\mathcal{A}(R,HR) \arrow[r, "="] \arrow[d, "\mathcal{A}(R,Hg)"] & WHHR \arrow[r, "W\epsilon HR"] \arrow[d, "WHHg"] & WHR \arrow[d, "WHg"] & \mathcal{A}(R,R) \arrow[l, "="'] \arrow[d, "\mathcal{A}(R,g)"] \\
\mathcal{A}(R,HN) \arrow[r, "="] & WHHN \arrow[r, "W\epsilon HN"] & WHN & \mathcal{A}(R,N) \arrow[l, "="']
\end{tikzcd}$$

Figure 1. Diagram for the comonad H.

Change of categories. Now assume $\mathcal{A}'$ is a second category, equipped similarly with W', H', ψ' etc. satisfying axioms (8.14) and (8.19). We assume that $\mathcal{A}$ and $\mathcal{A}'$ are connected by a somewhat forgetful functor $V: \mathcal{A} \to \mathcal{A}'$ such that $W'V = W$. Then given an object M of $\mathcal{A}$, there is an obvious natural transformation $\omega_V: F_M \to F_{VM}V: \mathcal{A} \to \mathit{Set}$, defined on N in $\mathcal{A}$ as $V: \mathcal{A}(M,N) \to \mathcal{A}'(VM,VN)$.

We assume that H and H' are related by a natural transformation $\theta: VH \to H'V: \mathcal{A} \to \mathcal{A}'$. If $\rho_M: M \to HM$ is a pre-coaction on M in $\mathcal{A}$, we give VM the pre-coaction

$$\rho_{VM}: VM \xrightarrow{V\rho_M} VHM \xrightarrow{\theta M} H'VM \quad \text{in } \mathcal{A}'. \tag{8.23}$$

This we convert to the commutative diagram of natural transformations

$$\begin{tikzcd}
F_M \arrow[r, "\rho_M"] \arrow[dd, "\omega_V"] & F_M H \arrow[d, "\omega_V H"] \\
 & F_{VM}VH \arrow[d, "F_{VM}\theta"] \\
F_{VM}V \arrow[r, "\rho_{VM}V"] & F_{VM}H'V
\end{tikzcd}$$

Because WH is corepresented by R, the natural transformation $W'\theta: WH = W'VH \to W'H'V$ is determined by a certain morphism $u: R' \to VR$ in $\mathcal{A}'$ (which will be obvious in applications); explicitly, given M in $\mathcal{A}$, $W'\theta M: WHM = W'VHM \to W'H'VM$ is

$$W'\theta M: \mathcal{A}(R,M) \xrightarrow{V} \mathcal{A}'(VR,VM) \xrightarrow{\mathcal{A}'(u,VM)} \mathcal{A}'(R',VM).$$

LEMMA 8.24. *Assume that u satisfies:*

(i) *$u: R' \to VR$ is a morphism of pre-coactions* (this uses eq. (8.23));
(ii) *$W'u: W'R' \to W'VR = WR$ sends $1_{R'}$ to 1_R.*

Then $\theta\colon VH \to H'V$ is a natural transformation of comonads, in the sense that we have commutative diagrams

$$\text{(i)}\quad \begin{tikzcd} VH \arrow[r, "V\psi"] \arrow[dd, "\theta"] & VHH \arrow[d, "\theta H"] \\ & H'VH \arrow[d, "H'\theta"] \\ H'V \arrow[r, "\psi' V"] & H'H'V \end{tikzcd} \qquad \text{(ii)}\quad \begin{tikzcd} VH \arrow[d, "\theta"] \arrow[dr, "V\epsilon"] & \\ H'V \arrow[r, "\epsilon' V"] & V \end{tikzcd}$$

PROOF. We apply W' and expand all the definitions. □

9. E-cohomology of spectra

In this section, we adapt the results and techniques of Sections 3 and 4 to the graded stable homotopy category Stab^* of spectra. Our general reference is Adams [3]. Many results become simpler and most are well known, apart from the topological embellishments.

Cohomology. Any *based* space (X, o) may be regarded as a spectrum, via the stabilization functor $\mathit{Ho}' \to \mathit{Stab}$. Given a spectrum E, whether X is a based space or a spectrum, we define the *reduced E-cohomology* of X as $E^*(X,o) = \{X, E\}^* = \mathit{Stab}^*(X, E)$, the graded group of morphisms in Stab^* from X to E that has the component $E^k(X,o) = \{X,E\}^k$ in degree k. The universal class $\iota \in E^0(E,o)$ is thus the identity map of E.

The suspension isomorphism $E^*(X,o) \cong E^*(\Sigma X, o)$ is that induced by the canonical desuspension map $\Sigma X \simeq X$ of degree 1 in Stab^* given by (6.1) (with signs as in eq. (6.3)). Equivalently, given $x \in E^k(X,o)$, the class $\Sigma x \in E^{k+1}(X,o)$ is the composite of the maps $\Sigma X \to \Sigma E$ and $\Sigma E \simeq E$ (with no sign).

This cohomology is the only kind available in the stable context. For compatibility with the unstable notation of Section 3, we *always* write the cohomology of a spectrum X, redundantly but unambiguously, as $E^*(X,o)$.

The *skeleton* filtration of $E^*(X,o)$ can be defined exactly as unstably, in eq. (3.33). It is quite satisfactory for spectra of finite type (those with each skeleton finite), which include many of our examples, but is wildly inappropriate for non-connective spectra such as KU. We therefore give $E^*(X,o)$ the *profinite* filtration and topology, exactly as in Definition 4.9. If necessary, we complete it as in Definition 4.11 to the *completed cohomology* $E^*(X,o)\hat{}$.

A map $r\colon E \to E$ in Stab^* of degree h induces the stable cohomology operation $r_*\colon E^k(X,o) \to E^{k+h}(X,o)$. It commutes with suspension up to the sign $(-1)^h$ as in fig. 2.

Spaces. For a *space* X, it is more useful, whether or not X is based, to work with the *absolute E-cohomology* of X defined by $E^*(X) = E^*(X^+, o)$, as suggested by eq. (3.3). The absolute theory is thereby *included* in the reduced theory. In particular, the coefficient

$$\begin{tikzcd} E^k(X,o) \arrow[r, "r_*"] \arrow[d, "\Sigma", "\cong"'] \arrow[dr, phantom, "(-1)^h"] & E^{k+h}(X,o) \arrow[d, "\Sigma", "\cong"'] \\ E^{k+1}(\Sigma X,o) \arrow[r, "r_*"] & E^{k+h+1}(\Sigma X,o) \end{tikzcd}$$

Figure 2. Operations and suspension.

group of E-cohomology is $E^* = E^*(T) = E^*(T^+,o) = \pi_*^S(E,o)$. Conversely, every graded cohomology theory on spaces has this form.

THEOREM 9.1. *Let $E^*(-)$ be a graded cohomology theory on Ho in the sense of Section 3. Then:*

(a) *There is a spectrum E, unique up to equivalence, that represents $E^*(-)$ as above;*

(b) *Any sequence of cohomology operations $r_k: E^k(X) \to E^{k+h}(X)$, that are defined and natural for spaces X and commute with suspension up to the sign $(-1)^h$ as in fig. 2, is induced by a map of spectra $r: E \to E$ of degree h.*

SKETCH PROOF. The representing spaces $\underline{E}_n$ provided by Theorem 3.17 and the structure maps $f_n: \Sigma\underline{E}_n \to \underline{E}_{n+1}$ from Definition 3.19 are used to construct the spectrum E for (a). In (b), Theorem 3.6(b) provides a representing map $r_k: \underline{E}_k \to \underline{E}_{k+h}$ for each operation r_k. We take $X = \underline{E}_k$ in fig. 2 and evaluate on the universal class ι_k. By Lemma 3.21, the class $(-1)^{k+h}\,\Sigma r_k\iota_k$ corresponds to the upper route $f_{k+h}\circ\Sigma r_k$ in the square

$$\begin{tikzcd} \Sigma\underline{E}_k \arrow[r, "\Sigma r_k"] \arrow[d, "f_k"] & \Sigma\underline{E}_{k+h} \arrow[d, "f_{k+h}"] \\ \underline{E}_{k+1} \arrow[r, "r_{k+1}"] & \underline{E}_{k+h+1} \end{tikzcd} \quad \text{in } Ho' \tag{9.2}$$

Meanwhile, by Definition 3.19, $r_{k+1}\circ f_k$ corresponds to the class $(-1)^k r_{k+1}\Sigma\iota_k$. Thus the square commutes, and we may take the maps r_k as the raw material for constructing the desired map of spectra $r: E \to E$. (However, r need not be unique.) A similar construction gives the uniqueness in (a). Further details depend on the choice of implementation of $Stab^*$. □

Stabilization. In Theorems 3.17 and 9.1 we have two ways to represent E-cohomology, in the categories Ho and $Stab^*$. Thus for any space X, we may identify:

(i) The *cohomology class* $x \in E^k(X)$;
(ii) The *map of spectra* $x_S: X^+ \to E$, of degree k, defined by $x = x_S^*\iota$;

(iii) The *map of spaces* $x_U: X \to \underline{E}_k$, defined by $x = x_U^* \iota_k$.

We compare the two maps by taking $x = \iota_k$ in (ii).

DEFINITION 9.3. For each integer k, we define the *stabilization* map of spectra $\sigma_k: \underline{E}_k \to E$ by $\sigma_k^* \iota = \iota_k \in E^k(\underline{E}_k, o) \subset E^k(\underline{E}_k)$. It has degree k.

It follows immediately that for any $x \in E^k(X)$, x_S is the composite

$$x_S: X^+ \xrightarrow{x_U^+} \underline{E}_k^+ \longrightarrow \underline{E}_k \xrightarrow{\sigma_k} E \quad \text{in } \mathit{Stab}^*. \tag{9.4}$$

If x is based, i.e. $x \in E^k(X, o)$, we can simplify this to

$$x_S: X \xrightarrow{x_U} \underline{E}_k \xrightarrow{\sigma_k} E \quad \text{in } \mathit{Stab}^*. \tag{9.5}$$

In practice, we normally omit the suffixes $_S$ and $_U$ and write x for all three. (On occasion, this can cause some difficulty with signs, as x and x_S have degree k, while x_U is a map of spaces and has no degree.)

LEMMA 9.6. *The structure maps $f_k: \Sigma \underline{E}_k \to \underline{E}_{k+1}$ and the stabilization maps Σ_k are related by the commutative square*

$$\begin{tikzcd} \Sigma \underline{E}_k \arrow[r, "f_k"] \arrow[d, "\simeq"'] & \underline{E}_{k+1} \arrow[d, "\sigma_{k+1}"] \\ \underline{E}_k \arrow[r, "\sigma_k"] & E \end{tikzcd} \quad \textit{in } \mathit{Stab}^*$$

in which we use the canonical desuspension map (6.1).

PROOF. The upper route in the square corresponds to the class $f_k^* \iota_{k+1} = (-1)^k \Sigma \iota_k \in E^*(\Sigma \underline{E}_k, o)$. If we write $g: \Sigma \underline{E}_k \simeq \underline{E}_k$ for the desuspension, the lower route corresponds to $(\sigma_k \circ g)^* \iota = (-1)^k g^* \sigma_k^* \iota = (-1)^k g^* \iota_k = (-1)^k \Sigma \iota_k$. □

These maps display E as the homotopy colimit in Stab^* of the based spaces $\underline{E}_n$. The relevant Milnor short exact sequence (cf. diag. (3.38)) is

$$0 \longrightarrow \lim_n{}^1 E^{k-1}(\underline{E}_n, o) \longrightarrow E^k(E, o) \longrightarrow \lim_n E^k(\underline{E}_n, o) \longrightarrow 0. \tag{9.7}$$

Moreover, the profinite topology makes the map from $E^k(E, o)$ an open map and therefore a homeomorphism whenever it is a bijection. (Take the basic open set $F^a E^*(E, o)$ defined by some finite subspectrum $E_a \subset E$. This inclusion lifts (up to homotopy) to a map of spectra (of degree $-n$) $E_a \to \underline{E}_{n,b} \subset \underline{E}_n$ for some n and some finite subcomplex $\underline{E}_{n,b}$ of $\underline{E}_n$. Then the image of $F^a E^*(E, o)$ contains $F^b E^*(\underline{E}_n)$.)

The maps σ_n also relate the stable and unstable operations in Theorem 9.1(b). Suppose the stable operation r of degree h is represented stably in Stab^* by a map of spectra

$r_S\colon E \to E$ of degree h, and unstably in Ho by the maps $r_k\colon \underline{E}_k \to \underline{E}_{k+h}$. These maps are related by the commutative square

$$\begin{tikzcd} \underline{E}_k \arrow[r, "r_U"] \arrow[d, "\sigma_k"] & \underline{E}_{k+h} \arrow[d, "\sigma_{k+h}"] \\ E \arrow[r, "r_S"] & E \end{tikzcd} \quad \text{in } Stab^* \tag{9.8}$$

because by the definition of σ_n, both routes represent the class $r\iota_k \in E^*(\underline{E}_k, o)$. Cohomologically,

$$\sigma_k^* r = (-1)^{kh} r \circ \sigma_k = (-1)^{kh} r_k \quad \text{in } E^{k+h}(\underline{E}_k, o). \tag{9.9}$$

(Without the sign, $r \mapsto r_k$ is *not* in general an E^*-module homomorphism.)

Ring spectra. Now let E be a ring spectrum, i.e. a commutative monoid object in the symmetric monoidal category $(Stab, \wedge, T^+)$, with multiplication $\phi\colon E \wedge E \to E$ and unit $\eta\colon T^+ \to E$. (All our ring spectra are assumed commutative.)

Given $x \in E^*(X, o)$ and $y \in E^*(Y, o)$, we define their *cross product* $x \times y \in E^*(X \wedge Y, o)$ as

$$x \times y\colon X \wedge Y \xrightarrow{x \wedge y} E \wedge E \xrightarrow{\phi} E.$$

These products are biadditive, commutative, associative, and have $\eta \in E^*(T^+, o)$ as the unit in the sense that under the isomorphism

$$E^*(T^+ \wedge X, o) \cong E^*(X, o) \tag{9.10}$$

induced by $X \simeq T^+ \wedge X$, $\eta \times x$ corresponds to x.

The coefficient group $E^* = E^*(T^+, o) = \pi_*^S(E, o)$ becomes a commutative ring, using $\times$-products and $T^+ \simeq T^+ \wedge T^+$ for multiplication; its unit element is $1_T = \eta \in E^0(T^+, o)$. Then $E^*(X, o)$ becomes a *left E^*-module* if we define $vx \in E^*(X, o)$ for $v \in E^*$ and $x \in E^*(X, o)$ as corresponding to $v \times x \in E^*(T^+ \wedge X, o)$ under the isomorphism (9.10); expanded, this is

$$vx\colon X \simeq T^+ \wedge X \xrightarrow{v \wedge x} E \wedge E \xrightarrow{\phi} E.$$

Rearranging slightly, we see that scalar multiplication by v on $E^*(-, o)$ is represented by the map

$$\xi v\colon E \simeq T^+ \wedge E \xrightarrow{v \wedge E} E \wedge E \xrightarrow{\phi} E \quad \text{in } Stab^*, \tag{9.11}$$

as in eq. (3.27). The map ξv corresponds to the class $v\iota$. We apply Lemma 7.7(d).

LEMMA 9.12. *The actions* (9.11) *make the ring spectrum E an E^*-module object in the graded category Stab*, which represents the E^*-module structure on cohomology $E^*(-,o)$.*

Now that $\times$-products are known to be E^*-bilinear, we can write them in the more familiar and useful form

$$\times : E^*(X,o) \otimes E^*(Y,o) \longrightarrow E^*(X \wedge Y,o). \tag{9.13}$$

Together with the definition $z: E^* = E^*(T^+,o)$, they make E-cohomology a symmetric monoidal functor

$$\big(E^*(-,o),\times,z\big):\big(\mathit{Stab}^{*\mathrm{op}},\wedge,T^+\big) \longrightarrow \big(\mathit{Mod}^*,\otimes,E^*\big). \tag{9.14}$$

For spaces X and Y, we have $X^+ \wedge Y^+ \cong (X \times Y)^+$, and we recover the unstable $\times$-pairing (3.22) as a special case of (9.13). The reduced diagonal map $\Delta^+ : X^+ \to (X \times X)^+ \cong X^+ \wedge X^+$ and projection $q^+ : X^+ \to T^+$ make X^+ a commutative monoid object in $\mathit{Stab}^{\mathrm{op}}$, so that $E^*(X) = E^*(X^+,o)$ becomes a commutative monoid object in Mod, i.e. a commutative E^*-algebra. We have a multiplicative graded cohomology theory in the sense of Section 3.

The stable and unstable multiplication maps are related by the commutative diagram, similar to eq. (9.5),

$$\begin{tikzcd} \underline{E}_k \times \underline{E}_m \arrow[r, "\phi_U"] \arrow[d] & \underline{E}_{k+m} \arrow[d, "="] \\ \underline{E}_k \wedge \underline{E}_m \arrow[r] \arrow[d, "\sigma_k \wedge \sigma_m"] & \underline{E}_{k+m} \arrow[d, "\sigma_{k+m}"] \\ E \wedge E \arrow[r, "\phi_S"] & E \end{tikzcd} \quad \text{in } \mathit{Stab}^*. \tag{9.15}$$

However, there is a technical difficulty in extending Theorem 9.1 to make E a ring spectrum.

THEOREM 9.16. *Assume there are no weakly phantom classes in the groups $E^0(E,o)$, $E^0(E \wedge E,o)$ and $E^0(E \wedge E \wedge E,o)$. Then any natural multiplicative structure that is defined on $E^*(X)$ for all spaces X (as in Section 3) is induced by a unique ring spectrum structure on E.*

PROOF. Theorem 3.25 provides a compatible family of unstable multiplications $\phi_U : \underline{E}_k \times \underline{E}_m \to \underline{E}_{k+m}$ and the unit $\eta_U : T \to \underline{E}_0$. We immediately recover η_S from η_U by taking $x = 1 \in E^*(T)$ in eq. (9.4), but there is a problem with ϕ_S. We may regard $E \wedge E$ as the homotopy colimit in Stab^* of the spaces $\underline{E}_n \wedge \underline{E}_n$ and obtain the Milnor short exact sequence

$$0 \longrightarrow \lim_n{}^1 E^{-1}(\underline{E}_n \wedge \underline{E}_n,o) \longrightarrow E^0(E \wedge E,o) \longrightarrow \lim_n E^0(\underline{E}_n \wedge \underline{E}_n,o) \longrightarrow 0$$

analogous to (9.7). It shows that there exists a lifting ϕ_S that makes diag. (9.15) commute for all k and m, but it is not unique in general. Our hypotheses simplify the diagrams for $E^0(E,o)$, $E^0(E\wedge E,o)$, and the analogue for $E^0(E\wedge E\wedge E,o)$ to the limit term only, to ensure respectively that ϕ_S: (i) has η_S as a unit; (ii) is unique and commutative; and (iii) is associative. □

Homology. The companion homology theory to $E^*(-)$ is easily defined (see G.W. Whitehead [36] or Adams [3]) in the stable context. The *reduced E-homology* of a spectrum or based space X is simply

$$E_*(X,o) = \{T^+, E\wedge X\}^* = \pi_*^S(E\wedge X,o), \tag{9.17}$$

the stable homotopy of $E\wedge X$. (We observe that $\pi_*^S(-,o)$ is itself the homology theory given by taking $E = T^+$, but we do not wish to write it $T_*^+(-,o)$.) It has the component $E_k(X,o) = \{T^+, E\wedge X\}^{-k} = \pi_k^S(E\wedge X,o)$ in degree $-k$. Again, we have the suspension isomorphism $E_k(X,o) \cong E_{k+1}(\Sigma X,o)$, induced by (the inverse of) the canonical desuspension (6.1).

For a space X, we have the *absolute E-homology*

$$E_*(X) = E_*(X^+,o) = \{T^+, E\wedge X^+\}^*,$$

as suggested by eq. (3.3) for cohomology, and it satisfies axioms dual to (3.1). The coefficient group is

$$E_*(T) = E_*(T^+,o) = \{T^+, E\wedge T^+\}^* \cong \{T^+, E\}^* = E^* = \pi_*^S(E,o),$$

the same as E-cohomology. (But we note that $E_k(T) \cong E^{-k}$.)

When E is a ring spectrum, it too is a symmetric monoidal functor

$$(E_*(-,o),\times,z)\colon (\mathit{Stab}^*,\wedge,T^+) \longrightarrow (\mathit{Mod}^*,\otimes,E^*), \tag{9.18}$$

with an obvious $\times$-product pairing

$$\times\colon E_*(X,o)\otimes E_*(Y,o) \longrightarrow E_*(X\wedge Y,o), \tag{9.19}$$

if we use the above identification $z\colon E^* \cong E_*(T^+,o)$. We can ask whether eq. (9.19) is an isomorphism. The following two results provide all the homology isomorphisms we need.

THEOREM 9.20. *Assume that $E_*(X,o)$ or $E_*(Y,o)$ is a free or flat E^*-module. Then the pairing* (9.19) *induces the Künneth isomorphism $E_*(X\wedge Y,o) \cong E_*(X,o)\otimes E_*(Y,o)$ in homology.*

PROOF. The proof of Theorem 4.2 works just as well for spectra. □

LEMMA 9.21. *For $E = H(\mathbf{F}_p)$, $K(n)$, MU, BP, or KU, $E_*(E, o)$ is a free E^*-module.*

REMARK. For $E = KU$, this is a substantial result of Adams and Clarke [4, Theorem 2.1].

PROOF. For $E = H(\mathbf{F}_p)$ or $K(n)$, all E^*-modules are free. For $E = MU$ or BP, the result is well known [3]. For KU, we defer the proof until we have a good description of $KU_*(KU, o)$, in Section 14. □

The homology version of the Milnor short exact sequence (9.7) is simply

$$E_*(E, o) = \operatorname*{colim}_n E_*(\underline{E}_n, o), \tag{9.22}$$

analogous to eq. (4.4). More generally, from the definition (9.17),

$$E_*(X, o) = \operatorname*{colim}_a E_*(X_a, o) \tag{9.23}$$

for any X, where X_a runs over all finite subspectra of X.

Strong duality. The Kronecker pairing $\langle -, - \rangle: E^*(X, o) \otimes E_*(X, o) \to E^*$ is easily constructed for spectra E and X, directly from the definitions. As in Section 4, it makes sense to ask whether the right adjunct form

$$d: E^*(X, o) \longrightarrow DE_*(X, o) \tag{9.24}$$

is an isomorphism, or better, a homeomorphism. Again, one theorem is all we need. It includes the unstable result Theorem 4.14.

THEOREM 9.25. *Assume that $E_*(X, o)$ is a free E^*-module. Then X* has strong duality, *i.e. d in (9.24) is a homeomorphism between the profinite topology on $E^*(X, o)$ and the dual-finite topology on $DE_*(X, o)$. In particular, $E^*(X, o)$ is complete Hausdorff.*

E-modules. To establish Theorem 9.25, we must take E-modules seriously. An *E-module* is a spectrum G equipped with an action map $\lambda_G: E \wedge G \to G$ in $\mathcal{S}tab$ that satisfies the usual two axioms (8.3), using the functor $T = E \wedge -$. Everything is formally identical to the R-module case, with the monoid object R in the symmetric monoidal category $(\mathcal{A}b, \otimes, \mathbb{Z})$ replaced by E in $(\mathcal{S}tab^*, \wedge, T^+)$. We form the category E-$\mathcal{M}od$ of E-modules, and the graded version E-$\mathcal{M}od^*$.

THEOREM 9.26. *The forgetful functor $V: E\text{-}\mathcal{M}od^* \to \mathcal{S}tab^*$ has the free functor $E \wedge -: \mathcal{S}tab^* \to E\text{-}\mathcal{M}od^*$ as a left adjoint, and for any spectrum X and E-module G, we have a natural homeomorphism*

$$G^*(X) = \mathcal{S}tab^*(X, VG) \cong E\text{-}\mathcal{M}od^*(E \wedge X, G). \tag{9.27}$$

PROOF. Theorem 8.5 provides the isomorphism. We make it trivially a homeomorphism by topologizing $E\text{-}Mod^*(E \wedge X, G)$, *not* as a subspace of $G^*(E \wedge X)$, but by filtering it by the submodules

$$F^a E\text{-}Mod^*(E\wedge X, G) = \operatorname{Ker}\left[E\text{-}Mod^*(E\wedge X, G) \longrightarrow E\text{-}Mod^*(E\wedge X_a, G)\right],$$

where X_a runs through the finite subspectra of X. □

COROLLARY 9.28. *Let $g\colon E\wedge X \to E\wedge Y$ be an E-module morphism* (not necessarily of the form $E\wedge f$). *Then for any E-module G, $g^*\colon E\text{-}Mod^*(E\wedge Y, G) \to E\text{-}Mod^*(E\wedge X, G)$ is continuous.*

PROOF. The right adjunct of g is a map $f\colon X \to E\wedge Y$ of spectra. Given a finite $X_a \subset X$, we choose a finite $Y_b \subset Y$ such that $f|X_a$ factors through $E \wedge Y_b$; then by taking left adjuncts, g restricts to a morphism of E-modules $E \wedge X_a \to E \wedge Y_b$. It follows that $g^*(F^b) \subset F^a$, in the notation of the Theorem. □

The desired theorem follows directly, as in Adams [3, Lemma II.11.1].

PROOF OF THEOREM 9.25. We choose a basis of $E_*(X, o)$ consisting of maps $S^{n_\alpha} \to E\wedge X$ of degree zero, and use them as the components of a map $f\colon W = \bigvee_\alpha S^{n_\alpha} \to E\wedge X$. By Theorem 9.26, the left adjunct of f is a morphism of E-modules $g\colon E\wedge W \to E\wedge X$. By construction, g induces an isomorphism $g_*\colon E_*(W, o) \cong E_*(X, o)$ on homotopy groups, and is therefore an isomorphism in *Stab*. It follows formally that g is also an isomorphism in E-*Mod*. We factor d to obtain the commutative diagram

$$\begin{tikzcd}
d\colon E^*(X,o) \arrow[r, "\cong"] & E\text{-}Mod^*(E\wedge X, E) \arrow[r, "\pi_*^S(-,o)"] \arrow[d, "\mathrm{Mor}(g,E)"] & DE_*(X,o) \arrow[d, "Dg_*"] \\
d\colon E^*(W,o) \arrow[r, "\cong"] & E\text{-}Mod^*(E\wedge W, E) \arrow[r, "\pi_*^S(-,o)"] & DE_*(W,o)
\end{tikzcd}$$

Theorem 9.26 provides the two marked homeomorphisms. By Corollary 9.28, $\mathrm{Mor}(g, E)$ is a homeomorphism. It is clear from Lemma 4.10 that W has strong duality. We have a diagram of homeomorphisms. □

Künneth homeomorphisms. As the Künneth pairing (9.13) is continuous, we can complete it to

$$\times\colon E^*(X,o)\,\widehat{\otimes}\,E^*(Y,o) \longrightarrow E^*(X\wedge Y, o)\hat{}, \tag{9.29}$$

and the symmetric monoidal functor (9.14) to another one,

$$\left(E^*(-,o)\hat{}, \times, z\right)\colon \left(\mathit{Stab}^{*\mathrm{op}}, \wedge, T^+\right) \longrightarrow \left(\mathit{FMod}^*, \widehat{\otimes}, E^*\right), \tag{9.30}$$

for completed cohomology. As in Theorem 4.19, we combine Theorem 9.25 with Theorem 9.20 to deduce Künneth homeomorphisms.

THEOREM 9.31. *Assume that $E_*(X,o)$ and $E_*(Y,o)$ are free E^*-modules. Then the pairing (9.29) induces the cohomology Künneth homeomorphism*

$$E^*(X \wedge Y, o) \cong E^*(X,o) \mathbin{\widehat{\otimes}} E^*(Y,o).$$

10. What is a stable module?

In this section, we give various interpretations of what it means to have a module over the stable operations on E-cohomology, with a view to future generalization in [9] to unstable operations. We are primarily interested in the absolute cohomology $E^*(X) = E^*(X^+, o)$ of a space X, and state most results for this case only. Nevertheless, we sometimes need the more general reduced cohomology $E^*(X,o)$ of a spectrum X.

An operation $r: E^*(-,o) \to E^*(-,o)$ is *stable* if it is natural on Stab^*. It is automatically additive, Stab^* being an additive category, but need not be an E^*-module homomorphism.

Recall from Section 3 (or Section 9) that the profinite filtration makes $E^*(X)$ (or $E^*(X,o)$) a filtered E^*-module. When Hausdorff, it is an object of FMod^*. We remind that all tensor products are taken over the coefficient ring $E^* = E^*(T) = E^*(T^+, o)$ unless otherwise indicated, where T denotes the one-point space and T^+ the sphere spectrum.

First Answer. Since E-cohomology $E^*(-,o)$ is represented in Stab^* by the spectrum E, Yoneda's Lemma identifies the ring $\mathcal{A}$ of all stable operations with the endomorphism ring $\mathrm{End}(E) = \{E,E\}^* = E^*(E,o)$ of E. Its unit element is ι, the universal class of E. It acts on $E^*(X) = E^*(X^+,o)$ by composition,

$$\lambda_X: \mathcal{A} \otimes E^*(X) = E^*(E,o) \otimes E^*(X) \longrightarrow E^*(X). \tag{10.1}$$

In particular, for each $v \in E^h$ we have the *scalar multiplication* operation $x \mapsto vx$ on $E^*(X)$, which by Lemma 9.12 is represented by the map of spectra $\xi v: E \to E$ of degree h in eq. (9.11) or the element $v\iota \in E^h(E,o)$. This defines an embedding of rings (usually not central)

$$\xi: E^* \longrightarrow E^*(E,o) = \mathcal{A}, \tag{10.2}$$

which we used already in eq. (10.1) to make $\mathcal{A}$ an E^*-bimodule under composition and λ_X a homomorphism of E^*-modules.

Notation. Standard notation for tensor products is ambiguous here, and will soon become hopelessly inadequate for coping with the future plethora of bimodules and multimodules.

When it is necessary to convey detailed information about the many E^*-actions involved, we rewrite λ_X as

$$\lambda_X: E1^*(E2, o) \otimes_2 E2^*(X) \longrightarrow E1^*(X), \tag{10.3}$$

which we call the *E^*-action scheme* of λ_X. Here, Ei denotes a copy of E tagged for identification, and $\otimes_i$ indicates a tensor product that is to be formed using the two E^*-actions labeled by i. If desired, we can add information about the degrees by writing

$$\lambda_X: E1^i(E2, o) \otimes_2 E2^j(X) \longrightarrow E1^{i+j}(X).$$

For example, the composition

$$\mathcal{A} \otimes \mathcal{A} = E^*(E, o) \otimes E^*(E, o) \xrightarrow{\circ} E^*(E, o) = \mathcal{A} \tag{10.4}$$

has action scheme $E1^*(E2, o) \otimes_2 E2^*(E3, o) \to E1^*(E3, o)$. We promise to use this over-elaborate notation sparingly.

The important special case $X = T$ of the action (10.1) gives

$$\lambda_T: \mathcal{A} \cong \mathcal{A} \otimes E^* \longrightarrow E^*, \tag{10.5}$$

which encodes the action of $\mathcal{A}$ on the coefficient ring $E^* = E^*(T)$.

The action (10.1) satisfies the usual two laws:

$$(sr)x = s(rx); \qquad \iota x = x; \tag{10.6}$$

for any operations s and r and any $x \in E^*(X)$. This suggests that a stable module structure on a given E^*-module M should consist of an action $\lambda_M: \mathcal{A} \otimes M \longrightarrow M$ that satisfies these laws and is a homomorphism of left E^*-modules. Because the tensor product is taken over E^*, this implies that λ_M extends the given module action of E^* on M.

Unfortunately, this description is inadequate even for finite X. In the classical case $E = H(\mathbb{F}_p)$, $\mathcal{A}$ is the Steenrod algebra over $\mathbb{F}_p$, which is generated by the Steenrod operations subject to explicitly given Adem relations. In general, $\mathcal{A}$ is uncountable, which suggests that we should make use of the profinite topology on it. We described a filtration for tensor products in eq. (4.15). However, the tensor product in the action (10.1) is formed using the *right* E^*-action on $\mathcal{A}$, for which we have not defined a filtration; worse, the usual E^*-module structure on the tensor product is not the one that makes λ_X an E^*-module homomorphism. We have to find something else.

Second Answer. In [1], [3], Adams suggested that for suitable ring spectra E, one could avoid the various limit problems and infinite products that are inherent in cohomology by replacing the action (10.1) by the dual coaction on homology. Stably, the only difference between homology operations and cohomology operations is the possibility of weakly phantom cohomology operations; in practice, these usually do not exist. Unstably, however, the difference is vast. Our ignorance of unstable homology operations in general forces us to learn to live with cohomology. We therefore dualize only partially. We defer the details until Section 11.

If $E_*(E,o)$ is a free E^*-module, we can convert the action λ_X in (10.1) into a coaction (after completion)

$$\rho_X: E^*(X) \longrightarrow E^*(X)\,\widehat{\otimes}\,E_*(E,o) \tag{10.7}$$

(whose action scheme is $E2^*(X) \to E1^*(X)\,\widehat{\otimes}_1\,E1_*(E2,o)$). There is much structure on $E_*(E,o)$, as explicated in [1], [3]. Dual to the composition (10.4) with unit (10.2) in $E^*(E,o)$, there is a coassociative comultiplication with counit

$$\psi = \psi_S: E_*(E,o) \longrightarrow E_*(E,o) \otimes E_*(E,o); \qquad \epsilon = \epsilon_S: E_*(E,o) \longrightarrow E^*;$$

on $E_*(E,o)$. The action axioms (10.6) on λ_X translate into the diagrams

$$\text{(i)}\quad \begin{array}{ccc} E^*(X) & \xrightarrow{\rho_X} & E^*(X)\,\widehat{\otimes}\,E_*(E,o) \\ \downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle 1\otimes\psi_S} \\ E^*(X)\,\widehat{\otimes}\,E_*(E,o) & \xrightarrow{\rho_X\otimes 1} & E^*(X)\,\widehat{\otimes}\,E_*(E,o)\,\widehat{\otimes}\,E_*(E,o) \end{array}$$

$$\text{(ii)}\quad \begin{array}{ccc} E^*(X) & \xrightarrow{\rho_X} & E^*(X)\,\widehat{\otimes}\,E_*(E,o) \\ \downarrow & & \downarrow{\scriptstyle 1\otimes\epsilon_S} \\ E^*(X)\hat{} & \xrightarrow{=} & E^*(X)\,\widehat{\otimes}\,E^* \end{array} \tag{10.8}$$

These are in effect the usual axioms for a comodule coaction over $E_*(E,o)$ on $E^*(X)\hat{}$, the only novelty being the two distinct E^*-actions on $E_*(E,o)$.

Historically, the original example was developed by Milnor [22] in the case $E = H(\mathbf{F}_p)$, to give a description of the Steenrod operations that is both elegant and more informative; we summarize it in Section 14. Even in this case, the completed tensor product is needed in the coaction (10.7) when X is infinite-dimensional. For finite spaces or spectra X, one can use Spanier–Whitehead duality to switch between homology and cohomology. This leads to Adams's coaction on homology [1, Lecture 3], except that he used a *left* coaction in an attempt to make the E^*-actions easier to track. It turns out that in cohomology, the right coaction, even with its notational difficulties, is both more customary and more convenient.

Third Answer. We rewrite our Second Answer in a more categorical form in order to allow generalization. We still leave the details to Section 11.

As the target of ρ_X is complete, we lose nothing if we complete the cohomology $E^*(-)$ to $E^*(-)\hat{}$ everywhere. We define the functor S': $\mathit{FMod}^* \to \mathit{FMod}^*$ by $S'M =$

$M \widehat{\otimes} E_*(E,o)$. Then we can use ψ_S and ε_S to define natural transformations

$$\psi'_S M = M \otimes \psi_S: S'M \to S'S'M, \qquad \varepsilon'_S M = M \otimes \varepsilon_S: S'M \to M.$$

The coalgebra properties of ψ_S and ε_S will supply the necessary axioms (8.6) to make S' a *comonad* in FMod^*.

We rewrite the coaction (10.7) as a morphism $\rho'_X: E^*(X)\hat{} \to S'(E^*(X)\hat{})$ in the category FMod^*. This converts the axioms (10.8) into diags. (8.7), which then state that $E^*(X)\hat{}$ is precisely an *S'-coalgebra* in FMod^*.

We have condensed our answer down to the single word *S'-coalgebra.*

Fourth Answer. We are not done rewriting yet. The problem with our Third Answer is that it still depends heavily on the tensor product, an essentially bilinear construction that is simply unavailable for operations that are not additive (not that this has stopped us from trying).

We therefore go back to our First Answer and convert λ_X to adjoint form, as suggested by Section 8. We treat $x \in E^*(X)$ as a map of spectra $x: X^+ \to E$, and note that the E^*-module homomorphism $x^*: \mathcal{A} = E^*(E,o) \to E^*(X)$ is continuous. (There is the usual sign, $x^*r = (-1)^{\deg(x)\deg(r)} r \circ x = (-1)^{\deg(x)\deg(r)} rx$, from eq. (6.3).)

For convenience, we assume that $\mathcal{A}$ is Hausdorff and work in FMod^*. Given any complete Hausdorff filtered E^*-module M (i.e. object of FMod), we define

$$SM = \mathit{FMod}^*(\mathcal{A}, M) = \mathit{FMod}^*\big(E^*(E,o), M\big). \tag{10.9}$$

Then for any space X, we define the coaction

$$\rho_X: E^*(X) \longrightarrow S\big(E^*(X)\hat{}\big) = \mathit{FMod}^*\big(\mathcal{A}, E^*(X)\hat{}\big) \tag{10.10}$$

on $x \in E^*(X)$ by $\rho_X x = x^*: \mathcal{A} = E^*(E,o) \to E^*(X)\hat{}$, completing as necessary. In the important special case $X = T$, we find

$$\rho_T: E^* \approx E^*(T) \longrightarrow \mathit{FMod}^*\big(\mathcal{A}, E^*(T)\big) = SE^*(T) = SE^*. \tag{10.11}$$

Similarly, we have $\rho_X: E^*(X,o) \to S(E^*(X,o)\hat{})$ for spectra and based spaces X.

THEOREM 10.12. *Assume that the E^*-module $\mathcal{A} = E^*(E,o)$ is Hausdorff* (as is true for $E = H(\mathbf{F}_p)$, MU, BP, KU, or $K(n)$ by Lemma 9.21 and Theorem 9.25). *Then we can make the functor S defined in eq.* (10.9) *a comonad in the category FMod of complete Hausdorff filtered E^*-modules.*

Now that we have a suitable comonad, the definition of stable module is clear. This is the answer that will generalize satisfactorily.

DEFINITION 10.13. A *stable (E-cohomology) module* is an S-coalgebra in FMod^*, i.e. a complete Hausdorff filtered E^*-module M that is equipped with a morphism

$$\rho_M: M \longrightarrow SM \quad \text{in } \mathit{FMod}^* \tag{10.14}$$

that is E^*-linear and continuous and satisfies the coaction axioms (8.7). We then *define* the action of $r \in \mathcal{A}^h = E^h(E,o)$ on $x \in M^k$ by $rx = (-1)^{kh}(\rho_M x)r \in M$.

A closed submodule $L \subset M$ is called *(stably) invariant* if ρ_M restricts to $\rho_L: L \to SL$. Then the quotient M/L also inherits a stable module structure.

The group SM may be thought of as the set of all candidates for the action of $\mathcal{A}$ on a typical element of M. Then ρ_M selects for each $x \in M$ an appropriate action on x. The axioms (8.7) translate into the usual action axioms (10.6). If we evaluate the first only partially, we obtain the commutative square

$$\begin{tikzcd} M \arrow[r, "r"] \arrow[d, "\rho_M"] & M \arrow[d, "\rho_M"] \\ SM \arrow[r, "\omega_r M"] & SM \end{tikzcd} \tag{10.15}$$

where the natural transformation ω_r is defined on $f \in SM$ as

$$(\omega_r M)f = (-1)^{\deg(r)\deg(f)} f \circ r^*: \mathcal{A} \longrightarrow M,$$

using $r^*: \mathcal{A} = E^*(E,o) \to E^*(E,o) = \mathcal{A}$. It may be viewed as the analogue of diag. (8.8).

THEOREM 10.16. *Assume that the E^*-module $\mathcal{A} = E^*(E,o)$ is Hausdorff* (as is true for $E = H(\mathbf{F}_p)$, MU, BP, KU, or $K(n)$ by Lemma 9.21 and Theorem 9.25). *Then:*

(a) *We can factor ρ_X (defined in eq.* (10.10)) *through $E^*(X)\hat{}$ as $\rho_X: E^*(X)\hat{} \to S(E^*(X)\hat{})$, to make $E^*(X)\hat{}$ a stable module for any space X (and similarly $E^*(X,o)\hat{}$ for spectra);*

(b) *ρ is universal: given an object N of FMod*, any transformation*

$$\theta X: E^*(X,o) \longrightarrow \mathit{FMod}^*\big(N, E^*(X,o)\hat{}\big)$$

(or $\widehat{\theta}X: E^*(X,o)\hat{} \to \mathit{FMod}^*(N, E^*(X,o)\hat{}))$ of any degree, that is defined for all spectra X and natural on Stab*, is induced from ρ_X by a unique morphism $f: N \to \mathcal{A}$ in FMod* as the composite

$$\begin{aligned} \theta X: E^*(X,o) &\xrightarrow{\rho_X} SE^*(X,o)\hat{} = \mathit{FMod}^*\big(\mathcal{A}, E^*(X,o)\hat{}\big) \\ &\xrightarrow{\mathrm{Hom}(f,1)} \mathit{FMod}^*\big(N, E^*(X,o)\hat{}\big). \end{aligned} \tag{10.17}$$

PROOFS OF THEOREMS 10.12 AND 10.16. The discussion in Section 8 is intended to suggest that these two proofs are interlaced. The main proof is in seven steps. Lemma 9.12 provides the E^*-module object E in *Stab**. We find it useful to write $\mathrm{id}_{\mathcal{A}}$ for the identity map $\mathcal{A} \to \mathcal{A}$, considered as an element of $S\mathcal{A}$.

Step 1. We introduce an E^*-module structure (different from the obvious one) on the graded group SM defined by eq. (10.9); by hypothesis, $\mathcal{A}$ is an object of FMod^* and S is defined. By Lemma 7.6(a), the additive functor

$$\mathsf{FMod}^*\big(E^*(-,o)\hat{}, M\big)\colon \mathsf{Stab}^* \xrightarrow{E^*(-,o)\hat{}} \mathsf{FMod}^{*\mathrm{op}} \xrightarrow{\mathrm{Mor}(-,M)} \mathsf{Ab}^*$$

takes the E^*-module object E to an E^*-module object in Ab^*, i.e. makes SM an E^*-module. (By Lemma 7.1(a), the additive structure on SM must be the obvious one.) As M varies, Lemma 7.7(b) shows that SM is functorial, and we have a functor $S\colon \mathsf{FMod}^* \to \mathsf{Mod}^*$. We enrich it later, in Step 3, to take values in FMod^*.

Step 2. We show that ρ_X is an E^*-module homomorphism. Given a spectrum (or space) X, the cohomology functor $E^*(-,o)\hat{}\colon \mathsf{Stab}^{*\mathrm{op}} \to \mathsf{FMod}^*$ induces the natural transformation of additive functors

$$\mathsf{Stab}^*(X,-) \longrightarrow \mathsf{FMod}^*\big(E^*(-,o)\hat{}, E^*(X,o)\hat{}\big)\colon \mathsf{Stab}^* \longrightarrow \mathsf{Ab}^*.$$

We apply this to the E^*-module object E in Stab^*; then Lemma 7.6(c) shows that ρ_X is a homomorphism of E^*-modules.

Step 3. In order to make S and ρ_X take values in FMod^*, we must filter SM. If M is filtered by the submodules $F^a M$, we filter SM in the obvious way by the $F^a(SM) = S(F^a M)$, which are E^*-submodules because S is a functor. We trivially have the exact sequence

$$0 \longrightarrow SF^a M \longrightarrow SM \longrightarrow S\big(M/F^a M\big),$$

which we use to rewrite the filtration in the more useful form

$$F^a SM = \mathrm{Ker}\big[SM \longrightarrow S\big(M/F^a M\big)\big]. \tag{10.18}$$

(In fact, there is a short exact sequence in all our examples. However, we do not exploit this fact because (a) it requires a stronger hypothesis on E, but more importantly, (b) it does not generalize correctly.)

It is not difficult to see directly that SM is complete Hausdorff. Because M is complete Hausdorff, we have the limit $M = \lim_a M/F^a M$, which is automatically preserved by S. This yields by eq. (10.18) the inclusion

$$\lim_a \frac{SM}{F^a SM} \cong \lim_a \mathrm{Im}\left[SM \longrightarrow S\frac{M}{F^a M}\right] \subset \lim_a S\frac{M}{F^a M} = SM$$

in Ab^*. But this inclusion is visibly epic and therefore an isomorphism, which makes SM complete Hausdorff.

We have now defined S as a functor taking values in FMod^* as required. Our choice of the profinite topology on $E^*(X,o)$ and the naturality of ρ make it clear that ρ_X is continuous and factors as asserted in Theorem 10.16(a).

Step 4. We convert the object $E^*(X)^\wedge$ of FMod^* to the corepresented functor $F_X = \mathit{FMod}^*(E^*(X)^\wedge, -): \mathit{FMod}^* \to \mathit{Ab}^*$ (and similarly $E^*(X,o)^\wedge$ for spectra X). As suggested by eq. (8.16), we also convert the coaction ρ_X to the natural transformation $\rho_X: F_X \to F_X S: \mathit{FMod}^* \to \mathit{Ab}^*$. Given M, the homomorphism

$$\rho_X M: F_X M = \mathit{FMod}^*\big(E^*(X)^\wedge, M\big) \longrightarrow \mathit{FMod}^*\big(E^*(X)^\wedge, SM\big) = F_X SM \quad (10.19)$$

is defined by $(\rho_X M)f = Sf \circ \rho_X: E^*(X)^\wedge \to S(E^*(X)^\wedge) \to SM$.

Step 5. We define the natural transformation $\psi: S \to SS$ by taking $X = E$ in eq. (10.19), so that

$$\psi M: SM = \mathit{FMod}^*(\mathcal{A}, M) \longrightarrow \mathit{FMod}^*(\mathcal{A}, SM) = SSM \quad (10.20)$$

is given on the element $f: \mathcal{A} \to M$ of SM as the composite

$$(\psi M)f: \mathcal{A} = E^*(E,o) \xrightarrow{\rho_E} SE^*(E,o) = S\mathcal{A} \xrightarrow{Sf} SM.$$

(In terms of elements, this is $r \mapsto [s \mapsto f(r^*s) = (-1)^{\deg(r)\deg(s)} f(sr)]$.) When we substitute the E^*-module object E for X in diag. (10.19), Lemma 7.6(c) shows that ψM takes values in Mod^*. Naturality in M shows that ψ is filtered and takes values in FMod^*, as required.

Step 6. The other required natural transformation,

$$\varepsilon M: SM = \mathit{FMod}^*(\mathcal{A}, M) \longrightarrow M, \quad (10.21)$$

is defined simply as evaluation on the universal class $\iota \in \mathcal{A}$, i.e. $(\varepsilon M)f = f\iota$. Once again, naturality in M shows that εM is filtered, but we have to verify that εM is an E^*-module homomorphism. (All proofs involving ε are necessarily somewhat computational, because the definition is.) Additivity is clear. Take any $v \in E^h$. By Lemma 9.12, the structure map $\xi v: E \to E$ induces $(\xi v)^*\iota = v\iota$ in $E^*(E,o)$. Given an element $f: \mathcal{A} = E^*(E,o) \to M$ of SM, we defined $vf = \pm f \circ (\xi v)^*$ in Step 1; then

$$\varepsilon(vf) = \pm\varepsilon\big(f \circ (\xi v)^*\big) = \pm f(\xi v)^*\iota = \pm f(v\iota) = vf\iota = v\varepsilon f,$$

using the given E^*-linearity of f.

Step 7. We show that S is a comonad and that $E^*(X)^\wedge$ is an S-coalgebra. Naturality of ρ with respect to the map of spectra $x: X^+ \to E$ for any $x \in E^*(X)$ shows that ρ_X is a coaction on $E^*(X)^\wedge$ in the sense of Definition 8.15, using $R = \mathcal{A} = E^*(E,o)$, $\rho_R = \rho_E$, and $1_R = \iota$. By Lemma 8.20, ρ_X makes $E^*(X)^\wedge$ (or $E^*(X,o)^\wedge$) an S-coalgebra; we constructed ψ and ε to satisfy the conditions (8.19). Finally, S is a comonad by Lemma 8.22(a).

Yoneda's Lemma gives Theorem 10.16(b) for θ. Because $E^*(-,o)$ is represented by E, θ is classified by the element $f = (\theta E)\iota \in \mathit{FMod}^*(N, \mathcal{A})$ and so given by eq. (10.17). If we are given $\widehat{\theta}$ instead, we compose with $E^*(X,o) \to E^*(X,o)^\wedge$ to obtain θ. Conversely, any θ factors through $\widehat{\theta}$ by naturality. $\square$

11. Stable comodules

Although the Fourth Answer of Section 10, in terms of stable modules, is the cleanest and most general, the Second Answer, in terms of stable comodules, is usually available and more practical in the cases of interest. (One could argue that this feature is what makes these cases interesting.) At least for $E = MU$ or BP, such comodules are called *cobordism comodules*. This is the context for Landweber theory, as developed in [17], [18] and discussed in Section 15 for BP.

Rather than develop the Second and Third Answers from scratch, we deduce them from the Fourth Answer by comparing the algebraic structures on $E^*(E, o)$ and $E_*(E, o)$. This section is entirely algebraic in the sense that the only spectrum we study in any depth is E. In Theorem 11.35 we show that the structure maps η_R, ψ_S, and ε_S on $E_*(E, o)$ agree with those of Adams.

We assume later in this section that $E_*(E, o)$ is a free E^*-module, which is true for our five examples by Lemma 9.21. The duality $d: E^*(E, o) \cong DE_*(E, o)$ in Theorem 9.25 allows us to identify the following, with only slight abuse of notation:

$$\begin{array}{rl} \text{(i)} & \text{The } \textit{cohomology operation } r \text{ on } E^*(-) \text{ (or } E^*(-, o));\\ \text{(ii)} & \text{The } \textit{class } r\iota \in E^*(E, o), \text{ which we also write simply as } r;\\ \text{(iii)} & \text{The } \textit{map of spectra } r: E \to E, \text{ a morphism in } \mathit{Stab}^*;\\ \text{(iv)} & \text{The } E^*\textit{-linear functional } \langle r, -\rangle: E_*(E, o) \to E^*. \end{array} \tag{11.1}$$

The *degree* of r is the same in any of these contexts (once we remember that $E_i(E, o)$ has degree $-i$).

The bimodule algebra $E_*(E, o)$. As $E_*(E, o)$ is better understood and smaller than $E^*(E, o)$, (iv) is the preferred choice in (11.1). There is much structure on $E_*(E, o)$. First, like all E-homology, it is a left E^*-module.

When we apply the additive functor $E_*(-, o)$ to the E^*-module object E in Lemma 9.12, we obtain by Lemma 7.6(a) the E^*-module object $E_*(E, o)$ in *Mod*, equipped with the E^*-module homomorphism $(\xi v)_*$ of degree h for each $v \in E^h$. To extract a bimodule as commonly understood, we define the right action by

$$c \cdot v = (-1)^{hm}(\xi v)_* c \quad \text{for } v \in E^h,\ c \in E_m(E, o),$$

to ensure that $v'(c \cdot v) = (v'c) \cdot v$, with no signs. Nevertheless, we find it technically convenient to keep all functions and operations on the left and work with $(\xi v)_*$.

The ring spectrum structure (ϕ, η) on E induces the *multiplication*

$$\phi = \phi_S: E_*(E, o) \otimes E_*(E, o) \xrightarrow{\times} E_*(E \wedge E, o) \xrightarrow{\phi_*} E_*(E, o)$$

and *left unit*

$$\eta = \eta_S: E^* \cong E_*(T^+, o) \xrightarrow{\eta_*} E_*(E, o)$$

for $E_*(E, o)$. In particular, we have the unit element $1 = \eta 1 \in E_0(E, o)$.

The equation $vc = v(1c) = (v1)c = (\eta v)c$ describes the left E^*-action in terms of ϕ and η, and implies that η is a ring homomorphism. We shall see presently that the right action is similarly determined by its effect on 1.

DEFINITION 11.2. We define the *right unit* function $\eta_R\colon E^* \to E_*(E,o)$ on $v \in E^* = E^*(T^+,o)$ by $\eta_R v = v_* 1$, using the homology homomorphism

$$v_*\colon E^* \cong E_*(T^+,o) \longrightarrow E_*(E,o)$$

induced by the map $v\colon T^+ \to E$ in Stab^*.

We summarize all this structure. We recall that in general, the left and right units and E^*-actions on $E_*(E,o)$ are quite different.

PROPOSITION 11.3. *In $E_*(E,o)$, for any ring spectrum E:*

(a) *$E_*(E,o)$ is an E^*-bimodule;*

(b) *The unit element $1 = \eta 1 = \eta_R 1$ is well defined;*

(c) *The multiplication ϕ makes $E_*(E,o)$ a commutative E^*-algebra with respect to the left or right E^*-action;*

(d) *$\eta\colon E^* \to E_*(E,o)$ and $\eta_R\colon E^* \to E_*(E,o)$ are ring homomorphisms;*

(e) *The left action of $v \in E^*$ is left multiplication by $v1$;*

(f) *The right action of $v \in E^*$ is right multiplication by $\eta_R v$.*

PROOF. For (c), we apply the E-homology symmetric monoidal functor (9.18) to the commutative monoid object E in Stab, to obtain the commutative monoid object $E_*(E,o)$ in Mod, i.e. commutative E^*-algebra, with respect to the left E^*-action.

We trivially have (b), because the map $\eta\colon T^+ \to E$ is $1_T \in E^0(T)$. For (f), we apply E-homology to eq. (9.11), which expresses ξv in terms of the multiplication. This implies that η_R is a ring homomorphism. □

REMARK. There is a well-known conjugation

$$\chi\colon E_*(E,o) \longrightarrow E_*(E,o)$$

which interchanges the left and right E^*-actions. We avoid it because it does not generalize to the unstable situation.

The functor S'. Duality and Lemma 6.16(b) provide the natural isomorphism

$$S'M = M \mathbin{\widehat{\otimes}} E_*(E,o) \cong \mathit{FMod}^*\big(E^*(E,o), M\big) = SM \tag{11.4}$$

for any complete Hausdorff filtered E^*-module M, with action scheme

$$(S'M)2 = M1 \mathbin{\widehat{\otimes}_1} E1_*(E2,o) \cong \mathit{FMod}_1^*\big(E1^*(E2,o), M1\big) = (SM)2.$$

The functors S and S' are those of Section 10. Moreover, this is an isomorphism of filtered E^*-modules in *FMod* if we filter $S'M$ as in eq. (4.15), which is the same as filtering it by the submodules $S'F^aM$. (We remind that $E_*(E,o)$, like all homology, invariably carries the discrete topology.) Explicitly, with the help of Proposition 11.3, the isomorphism of E^*-actions is expressed by

$$\langle r\circ(v\iota),c\rangle = \langle r,(\eta_R v)c\rangle \quad \text{for } r\in E^*(E,o),\ v\in E^*,\ c\in E_*(E,o). \tag{11.5}$$

In view of the proliferation of E^*-actions, one must be careful in applying duality; the correct way to establish all properties of S' is to deduce them from the corresponding properties of S in Section 10 by applying the isomorphism (11.4). (Once our equivalences are well established, we shall normally omit the $'$ everywhere.)

The coalgebra structure on $E_*(E,o)$. The comonad structure (ψ_S,ε_S) on S in Theorem 10.12 corresponds under eq. (11.4) to a comonad structure on S' consisting of natural transformations $\psi' M\colon S'M\to S'S'M$ and $\varepsilon' M\colon S'M\to M$. By naturality and the case $M=E^*$, $\psi' M$ must take the form $M\otimes\psi$ for a certain well-defined comultiplication

$$\psi=\psi_S\colon E_*(E,o)\longrightarrow E_*(E,o)\otimes E_*(E,o) \tag{11.6}$$

(with action scheme $E1_*(E3,o)\to E1_*(E2,o)\otimes_2 E2_*(E3,o)$). It is not cocommutative (in any ordinary sense). Similarly, $\varepsilon' M$ must have the form

$$M\otimes\varepsilon_S\colon S'M = M\mathbin{\widehat{\otimes}} E_*(E,o)\longrightarrow M\otimes E^*\cong M$$

for some well-defined counit

$$\varepsilon=\varepsilon_S\colon E_*(E,o)\longrightarrow E^*. \tag{11.7}$$

(Here and elsewhere, the isomorphism $M\otimes E^*\cong M$ always involves the usual sign, $x\otimes v\mapsto(-1)^{\deg(x)\deg(v)}vx$.) Both ψ_S and ε_S are morphisms of E^*-bimodules.

LEMMA 11.8. *Assume that $E_*(E,o)$ is a free E^*-module. Then the homomorphisms ψ_S and ε_S in diags.* (11.6) *and* (11.7) *make $E_*(E,o)$ a coalgebra over E^*.*

PROOF. By taking $M=E^*$, the comonad axioms (8.6) for S' translate into the coassociativity of ψ_S,

$$\begin{tikzcd} E_*(E,o) \arrow[r, "\psi_S"] \arrow[d, "\psi_S"] & E_*(E,o)\otimes E_*(E,o) \arrow[d, "1\otimes\psi_S"] \\ E_*(E,o)\otimes E_*(E,o) \arrow[r, "\psi_S\otimes 1"] & E_*(E,o)\otimes E_*(E,o)\otimes E_*(E,o) \end{tikzcd} \tag{11.9}$$

and the two counit axioms on ε_S:

$$\begin{tikzcd} E_*(E,o) \arrow[r, "\psi_S"] \arrow[dr, "\cong"'] & E_*(E,o)\otimes E_*(E,o) \arrow[d, "\epsilon_S\otimes 1"] \\ & E^*\otimes E_*(E,o) \end{tikzcd} \qquad \begin{tikzcd} E_*(E,o) \arrow[r, "\psi_S"] \arrow[dr, "\cong"'] & E_*(E,o)\otimes E_*(E,o) \arrow[d, "1\otimes\epsilon_S"] \\ & E_*(E,o)\otimes E^* \end{tikzcd}$$

(i) (ii)

(11.10)

These commutative diagrams express precisely what we mean by saying that $E_*(E,o)$ is a coalgebra. □

Comodules. We are now ready to convert Definition 10.13 of a stable module and Theorem 10.16, by means of the isomorphism (11.4). The coaction $\rho_M: M \to SM$ in (10.14) on a stable module M corresponds to a coaction $\rho_M: M \to S'M = M\widehat{\otimes} E_*(E,o)$.

DEFINITION 11.11. A *stable* (*E-cohomology*) *comodule* structure on a complete Hausdorff filtered E^*-module M (i.e. object of *FMod*) consists of a coaction $\rho_M: M \to M\widehat{\otimes} E_*(E,o)$ that is a continuous morphism of filtered E^*-modules (i.e. morphism in *FMod*, with action scheme $M2 \to M1\,\widehat{\otimes}_1\, E1_*(E2,o)$) and satisfies the axioms

$$\begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "\rho_M"] & M\widehat{\otimes}E_*(E,o) \arrow[d, "M\otimes\psi_S"] \\ M\widehat{\otimes}E_*(E,o) \arrow[r, "\rho_M\otimes 1"] & M\widehat{\otimes}E_*(E,o)\widehat{\otimes}E_*(E,o) \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[dr, "\cong"'] & M\widehat{\otimes}E_*(E,o) \arrow[d, "M\otimes\epsilon_S"] \\ & M\otimes E^* \end{tikzcd} \tag{11.12}$$

(i) (ii)

THEOREM 11.13. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then given a complete Hausdorff filtered E^*-module M (i.e. object of FMod), a stable module structure on M in the sense of Definition* 10.13 *is precisely equivalent under eq.* (11.4) *to a stable comodule structure on M in the sense of Definition* 11.11.

PROOF. The axioms (11.12) are just the axioms (8.7) interpreted for S'. □

THEOREM 11.14. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then:*

(a) *For any space* (*or spectrum*) X, *there is a natural coaction*

$$\rho_X: E^*(X) \longrightarrow E^*(X)\,\widehat{\otimes}\, E_*(E,o) \tag{11.15}$$

(or $\rho_X: E^*(X,o) \to E^*(X,o) \widehat{\otimes} E_*(E,o)$) in FMod that makes $E^*(X)^\wedge$ (or $E^*(X,o)^\wedge$) a stable comodule, which corresponds by Theorem 11.13 *to the coaction ρ_X of Theorem* 10.16*;*

(b) *ρ is universal: given a discrete E^*-module N, any transformation $\theta X: E^*(X,o) \to E^*(X,o) \widehat{\otimes} N$ (or $\widehat{\theta} X: E^*(X,o)^\wedge \to E^*(X,o)^\wedge \widehat{\otimes} N$), that is defined and natural for all spectra X, is induced from ρ_X by a unique morphism $f: E_*(E,o) \to N$ of E^*-modules, as*

$$\theta X: E^*(X,o) \xrightarrow{\rho_X} E^*(X,o) \widehat{\otimes} E_*(E,o) \xrightarrow{1 \otimes f} E^*(X,o) \widehat{\otimes} N. \tag{11.16}$$

PROOF. For (a), we combine Theorem 11.13 with Theorem 10.16(a).

However, (b) is *not* a translation of Theorem 10.16(b), although the proof is similar. Because $E^*(-,o)$ is represented in Stab^* by E, θ is determined by the value $(\theta E)\iota \in E^*(E,o) \widehat{\otimes} N$, which corresponds to the desired homomorphism $f: E_*(E,o) \to N$ under the isomorphism $E^*(E,o) \widehat{\otimes} N \cong \mathit{Mod}^*(E_*(E,o), N)$ of Lemma 6.16(a). □

REMARK. The universal property (b) shows that diags. (11.12), with $M = E^*(X,o)$, may be viewed as *defining* ψ_S and ε_S in terms of ρ. Three applications of the uniqueness in (b) then show that ψ_S is coassociative and has ε_S as a counit.

REMARK. From a purely theoretical point of view, one should write the coaction (11.15) as $\rho_X: E^*(X)^\wedge \to E^*(X)^\wedge \widehat{\otimes} E_*(E,o)$, using *three* completions, in order to stay inside the category *FMod* of filtered modules at all times. This seems excessive. The way we are writing ρ_X, using just the $\widehat{\otimes}$ (and that only when necessary) and leaving the other completions implicit, conveys exactly the same algebraic and topological information after completion. But we warn that in using diag. (11.12)(ii), $M \widehat{\otimes} E^* \cong M$ is valid if and only if M is complete Hausdorff. In particular, $E^*(X)$ can only be a stable comodule if it is already Hausdorff.

Linear functionals. Theorem 11.13 establishes the equivalence between stable modules and stable comodules. For applications, we need to make this correspondence explicit. Given a stable comodule M, we recover the action of $r \in E^*(E,o)$ on the stable module M from ρ_M as

$$r: M \xrightarrow{\rho_M} M \widehat{\otimes} E_*(E,o) \xrightarrow{M \otimes \langle r, -\rangle} M \otimes E^* \cong M, \tag{11.17}$$

by means of the isomorphism (11.4), whose details are supplied by Lemma 6.16(b).

To make everything explicit, we choose $x \in M^k$ and write

$$\rho_M x = \sum_\alpha (-1)^{\deg(x_\alpha)\deg(c_\alpha)} x_\alpha \otimes c_\alpha \quad \text{in } M \widehat{\otimes} E_*(E,o), \tag{11.18}$$

where the sum may be infinite. (We introduce signs here to keep other formulae cleaner. It is noteworthy that in the explicit formulae of Section 14, these signs are invariably

$+1$.) Then from Corollary 6.17, the corresponding element $x^* \in \mathit{FMod}^k(E^*(E,o), M)$ is given by

$$x^* r = (-1)^{k \deg(r)} \sum_\alpha \langle r, c_\alpha \rangle \, x_\alpha,$$

and conversely. We rewrite this more conveniently as

$$rx = \sum_\alpha \langle r, c_\alpha \rangle \, x_\alpha \quad \text{in } M, \text{ for all } r \tag{11.19}$$

(with no signs at all!), where we emphasize that the c_α and x_α depend only on x, not on r. The sums here may be infinite, but will converge because eq. (11.18) does.

The statement that ρ_M is an E^*-homomorphism may be expressed as

$$r(vx) = \sum_\alpha \langle r, (\eta_R v) c_\alpha \rangle \, x_\alpha \quad \text{in } M, \text{ for all } r, \tag{11.20}$$

for any $v \in E^*$, with the help of eq. (11.5).

It is important for our purposes *not* to require the c_α to form a basis of $E_*(E,o)$, or even be linearly independent; but if they do form a basis, the x_α are uniquely determined by eq. (11.19) as $x_\alpha = c_\alpha^* x$, where c_α^* denotes the operation dual to c_α.

As ε_S is dual to $\xi\colon E^* \to E^*(E,o)$, we see that

$$\langle \iota, - \rangle = \varepsilon_S\colon E_*(E,o) \longrightarrow E^*, \tag{11.21}$$

which is obvious by comparing diag. (11.12)(ii) with eq. (11.17). In other words, the functional ε_S corresponds to the identity operation ι in the list (11.1). In practice, ε_S is always easy to write down; it is ψ_S that causes difficulties. Of course, ψ_S is dual to composition (10.4), as we make explicit later in eq. (11.34).

The cohomology of a point. Our first test space is the one-point space T. We have enough to determine the stable structure of $E^*(T)$ completely.

PROPOSITION 11.22. *Let r be a stable operation on E-cohomology and $v \in E^*$. Assume that $E_*(E,o)$ is a free E^*-module. Then in the stable comodule $E^*(T) = E^*$:*

(a) *The action of the operation r is given by*

$$rv = \langle r, \eta_R v \rangle \quad \textit{in } E^*(T) = E^*; \tag{11.23}$$

(b) *The coaction $\rho_T\colon E^* \to E^* \otimes E_*(E,o) \cong E_*(E,o)$ coincides with the right unit $\eta_R\colon E^* \to E_*(E,o)$;*

(c) *If we write $E^* = \pi_*^S(E,o)$ and regard $r\colon E \to E$ as a map of spectra, the induced homomorphism $r_*\colon E^* \to E^*$ on stable homotopy groups is given by $r_* v = \langle r, \eta_R v \rangle$.*

PROOF. If we regard v as a map $v: T^+ \to E$ and use Definition 11.2, we find

$$rv = \pm v^* r = \pm \langle v^* r, 1\rangle = \langle r, v_* 1\rangle = \langle r, \eta_R v\rangle,$$

which is (a). We compare eq. (11.19) with eq. (11.18) to rewrite this as $\rho_T v = 1 \otimes \eta_R v$, which gives (b). Parts (a) and (c) are equivalent, because both rv and $r_* v$ are the same morphism $r \circ v: T^+ \to E \to E$ in Stab^*. □

The cohomology of spheres. Our second test space is the sphere S^k. By definition, stable operations commute up to sign with the suspension isomorphism, as in fig. 2 in Section 9. In view of the multiplicativity of E and eq. (3.24), this reduces to

$$\rho_S u_k = u_k \otimes 1 \quad \text{in } E^*(S^k, o) \otimes E_*(E, o) \tag{11.24}$$

for all integers k (positive or negative), where S^k denotes the k-sphere *spectrum* and $u_k \in E^k(S^k, o) \cong E^0$ the standard generator. Equivalently, from eqs. (11.18) and (11.19), the action of any operation r is given by

$$ru_k = \langle r, 1\rangle\, u_k \quad \text{in } E^*(S^k, o). \tag{11.25}$$

Both formulae then hold in $E^*(S^k)$ for the *space* S^k, which exists for $k \geqslant 0$. Also, eq. (11.20) gives $r(vu_k)$.

Homology homomorphisms. In some applications, it is useful to regard the element $x \in E^k(X, o)$ as a map of spectra $x: X \to E$ and compute the homomorphism induced on E-homology.

PROPOSITION 11.26. *Assume that $E_*(E, o)$ is a free E^*-module. Given $x \in E^*(X, o)$, suppose that rx is given by eq.* (11.19). *Then the E-homology homomorphism $x_*: E_*(X, o) \to E_*(E, o)$ induced by the map of spectra $x: X \to E$ is given on $z \in E_m(X, o)$ by*

$$x_* z = \sum_\alpha (-1)^{\deg(c_\alpha)(\deg(x_\alpha)+m)} \langle x_\alpha, z\rangle\, c_\alpha. \tag{11.27}$$

PROOF. For a general operation r, we have

$$\begin{aligned}\langle r, x_* z\rangle &= \pm\langle x^* r, z\rangle = \langle rx, z\rangle \\ &= \sum_\alpha \langle r, c_\alpha\rangle\, v_\alpha = \left\langle r, \sum_\alpha (-1)^{\deg(c_\alpha)\deg(v_\alpha)} v_\alpha c_\alpha \right\rangle,\end{aligned}$$

where $v_\alpha = \langle x_\alpha, z\rangle$. Since this holds for all r, eq. (11.27) follows by duality. □

Conversely, we can recover $\rho_X x$ from x_* when X is well behaved.

PROPOSITION 11.28. *Assume that $E_*(X)$ is a free E^*-module. Take $x \in E^*(X)$. Then under the isomorphism $E^*(X)\widehat{\otimes} E_*(E,o) \cong Mod^*(E_*(X), E_*(E,o))$ of Lemma* 6.16(a), *the element $\rho_X x$ corresponds to the homomorphism $x_*: E_*(X) \to E_*(E,o)$ of E^*-modules.*

PROOF. We apply Lemma 6.16(a) to eq. (11.18), using the strong duality $E^*(X) \cong DE_*(X)$ of Theorem 9.25, and compare with eq. (11.27). □

Similarly, it is important to know the E-homology homomorphism $r_*: E_*(E,o) \to E_*(E,o)$ induced by an operation r, regarded as a map $r: E \to E$ of spectra. This provides a convenient faithful representation of the operations on $E_*(E,o)$, as it is clear that $\iota_* = \mathrm{id}$ and $(sr)_* = s_* \circ r_*$. From diag. (10.15) and the isomorphism (11.4) we deduce the commutative square

$$\begin{array}{ccc} M & \xrightarrow{r} & M \\ \big\downarrow{\scriptstyle \rho_M} & & \big\downarrow{\scriptstyle \rho_M} \\ M\widehat{\otimes} E_*(E,o) & \xrightarrow{M\otimes r_*} & M\widehat{\otimes} E_*(E,o) \end{array} \tag{11.29}$$

We need to know how to pass between $\langle r, -\rangle$ and r_*. From the identity $\langle r, c\rangle = \langle r^*\iota, c\rangle = \langle \iota, r_* c\rangle$ and eq. (11.21), we easily recover the functional $\langle r, -\rangle$ from r_* as

$$\langle r, -\rangle: E_*(E,o) \xrightarrow{r_*} E_*(E,o) \xrightarrow{\epsilon_S} E^*. \tag{11.30}$$

The following result gives the reverse direction.

LEMMA 11.31. *Let $r \in E^*(E,o)$ be an operation and assume that $E_*(E,o)$ is a free E^*-module. Then:*

(a) *The diagram*

$$\begin{array}{ccc} E_*(E,o) & \xrightarrow{r_*} & E_*(E,o) \\ \big\downarrow{\scriptstyle \psi_S} & & \big\downarrow{\scriptstyle \psi_S} \\ E_*(E,o)\otimes E_*(E,o) & \xrightarrow{1\otimes r_*} & E_*(E,o)\otimes E_*(E,o) \end{array} \tag{11.32}$$

commutes; in other words, r_* is a morphism of left $E_*(E,o)$-comodules;

(b) *$r_*: E_*(E,o) \to E_*(E,o)$ is the unique homomorphism of left E^*-modules that satisfies eq.* (11.30) *and is a morphism of left $E_*(E,o)$-comodules as in* (a);

(c) *The homomorphism r_* is given in terms of the functional $\langle r, -\rangle$ as*

$$\begin{aligned} r_*: E_*(E,o) &\xrightarrow{\psi_S} E_*(E,o)\otimes E_*(E,o) \\ &\xrightarrow{1\otimes\langle r,-\rangle} E_*(E,o)\otimes E^* \cong E_*(E,o). \end{aligned} \tag{11.33}$$

PROOF. After applying $M \mathbin{\widehat{\otimes}} -$, diag. (11.32) corresponds under eq. (11.4) to the square

$$\begin{array}{ccc} SM & \xrightarrow{\theta_r M} & SM \\ \Big\downarrow{\scriptstyle \psi_S M} & & \Big\downarrow{\scriptstyle \psi_S M} \\ SSM & \xrightarrow{\theta_r SM} & SSM \end{array}$$

where $\theta_r: S \to S$ is defined on $f \in SM = \mathit{FMod}^*(\mathcal{A}, M)$ by $(\theta_r M)f = (-1)^{\deg(r)\deg(f)} f \circ r^*$. (Note that θ_r only takes values in Ab^*, because it fails to preserve the preferred E^*-module structure on SM.) Since S is corepresented by $\mathcal{A}$, commutativity of this diagram reduces to the equality

$$\rho_E \circ r^* = Sr^* \circ \rho_E: \mathcal{A} = E^*(E, o) \longrightarrow SE^*(E, o) = S\mathcal{A}$$

in $SS\mathcal{A}$, which expresses the naturality of ρ. (Explicitly, $(sr)^* = \pm\, r^* \circ s^*$ for all $s \in \mathcal{A}$, which is the associativity $t(sr) = (ts)r$ for fixed r.)

If we compose diag. (11.32) with

$$1 \otimes \varepsilon_S: E_*(E, o) \otimes E_*(E, o) \longrightarrow E_*(E, o) \otimes E^* \cong E_*(E, o)$$

and use eq. (11.30) and diag. (11.10)(ii), we obtain (c). This also establishes the uniqueness of r_* in (b). $\square$

To summarize, eqs. (11.30) and (11.33) express r_* and $\langle r, -\rangle$ in terms of each other, with the help of ψ_S and ε_S. Conversely, these equations may be viewed as characterizing ψ_S and ε_S in terms of the r_* and $\langle r, -\rangle$ for all r.

We can at last make explicit how ψ_S is dual to the composition (10.4). It is immediate from eq. (11.30) that $\langle sr, -\rangle = \langle s, -\rangle \circ r_*$. We substitute eq. (11.33) to obtain

$$\begin{aligned} \langle sr, -\rangle: E_*(E, o) &\xrightarrow{\psi_S} E_*(E, o) \otimes E_*(E, o) \\ &\xrightarrow{1 \otimes \langle r, -\rangle} E_*(E, o) \otimes E^* \cong E_*(E, o) \xrightarrow{\langle s, -\rangle} E^*. \end{aligned} \tag{11.34}$$

(Note that we cannot simply write $\langle s, -\rangle \otimes \langle r, -\rangle$ here, which is undefined unless $\langle s, -\rangle$ happens to be right E^*-linear.)

REMARK. From a more sophisticated point of view, several of our formulae may be explained by noting that ψ_S makes $E_*(E, o)$ a stable comodule, provided we use the right E^*-module action. The comodule axioms are (11.9) and (11.10)(ii). Then by comparing eq. (11.33) with eq. (11.17), we see that the action of r on $E_*(E, o)$ is just r_*, and diag. (11.32) becomes a special case of diag. (11.29), which in turn comes from diag. (8.8).

Compatibility. It is clear that eqs. (11.30) and (11.33) determine ε_S and ψ_S uniquely, and that eq. (11.23) determines η_R. We now show that they agree with the homomorphisms introduced by Adams [3, III.12].

THEOREM 11.35. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Suppose that $E_*(E,o)$ is equipped with ψ_S and ε_S that satisfy eqs.* (11.30) *and* (11.33), *and η_R as in Definition* 11.2. *Then:*

(a) η_R *must be*

$$E^* = \pi_*^S(E,o) \cong \pi_*^S(T^+ \wedge E,o) \xrightarrow{\pi_*^S(\eta\wedge E,o)} \pi_*^S(E\wedge E,o) = E_*(E,o);$$

(b) ε_S *must be*

$$\varepsilon_S\colon E_*(E,o) = \pi_*^S(E\wedge E,o) \xrightarrow{\pi_*^S(\phi,o)} \pi_*^S(E,o) = E^*;$$

(c) ψ_S *must be*

$$\psi_S\colon E_*(E,o) \cong E_*(T^+\wedge E,o) \xrightarrow{E_*(\eta\wedge E)} E_*(E\wedge E,o) \cong E_*(E,o)\otimes E_*(E,o),$$

where we use the twisted *Künneth isomorphism with action scheme*

$$E1_*(E2\wedge E3,o) \cong E1_*(E2,o)\otimes_2 E2_*(E3,o).$$

PROOF. The definition $\eta_R v = v_* 1$ expands to

$$T^+ \xrightarrow{\eta} E \simeq E\wedge T^+ \xrightarrow{E\wedge v} E\wedge E.$$

We prove (a) by rearranging this as

$$T^+ \xrightarrow{v} E \simeq T^+ \wedge E \xrightarrow{\eta\wedge E} E\wedge E.$$

Given $r \in E^*(E,o)$ and $c \in E_*(E,o)$, we may construct $\langle r,c\rangle$ as the composite

$$\langle r,c\rangle\colon T^+ \xrightarrow{c} E\wedge E \xrightarrow{r\wedge E} E\wedge E \xrightarrow{\phi} E.$$

If we take $r = \iota$ and compare with (11.30), we obtain (b).

The commutative diagram

$$\begin{tikzcd}
E_*(E,o) \arrow[r, "(\eta\wedge E)_*"] \arrow[d, "r_*"] & E_*(E\wedge E,o) \arrow[d, "(E\wedge r)_*"] & E_*(E,o)\otimes E_*(E,o) \arrow[l, "\cong"'] \arrow[d, "1\otimes r_*"] \\
E_*(E,o) \arrow[r, "(\eta\wedge E)_*"] \arrow[dr, "="] & E_*(E\wedge E,o) \arrow[d, "\phi_*"] & E_*(E,o)\otimes E_*(E,o) \arrow[l, "\cong"'] \arrow[dl, "1\otimes\epsilon_S"] \\
& E_*(E,o) &
\end{tikzcd}$$

shows, with the help of eq. (11.30), that r_* is the composite of $1\otimes\langle r,-\rangle$ with Adams's ψ, which appears as the top row. (Here, both Künneth isomorphisms are twisted.) Since this holds for all r, comparison with eq. (11.33) gives (c). □

12. What is a stable algebra?

In Section 10, we gave four answers for the structure of a module over the ring $\mathcal{A} = E^*(E,o)$ of stable operations in E-cohomology, to encode the algebraic structure present on the E^*-module $E^*(X)$ or $E^*(X,o)$ for a space or spectrum X. When X is a space, $E^*(X)$ is an E^*-algebra. In this section, we enrich each answer and theorem to include this multiplicative structure.

The organizing principle of this section is to make everything symmetric monoidal. We have three symmetric monoidal categories in view: $(\mathit{Stab}^*,\wedge,T^+)$, $(\mathit{Mod}^*,\otimes,E^*)$, and the filtered version $(\mathit{FMod}^*,\widehat{\otimes},E^*)$. We also have three symmetric monoidal functors: E-cohomology (9.14), completed E-cohomology (9.30), and E-homology (9.18).

In this section, *we generally assume that $E_*(E,o)$ is a free E^*-module;* then Theorem 9.31 provides Künneth homeomorphisms $E^*(E\wedge E,o) \cong \mathcal{A}\widehat{\otimes}\mathcal{A}$ and $E^*(E\wedge E\wedge E,o)\cong \mathcal{A}\widehat{\otimes}\mathcal{A}\widehat{\otimes}\mathcal{A}$.

First Answer. For a spectrum X, we have the action (10.1)

$$\lambda_X\colon \mathcal{A}\otimes E^*(X,o) \longrightarrow E^*(X,o).$$

Given an operation r, we would like to have an external Cartan formula

$$r(x\times y) = \sum_\alpha \pm r'_\alpha x \times r''_\alpha y \quad \text{in } E^*(X\wedge Y,o) \tag{12.1}$$

for suitable choices of operations r'_α and r''_α (and signs). For a space X, this leads to the corresponding internal Cartan formula,

$$r(xy) = \sum_\alpha \pm (r'_\alpha x)(r''_\alpha y) \quad \text{in } E^*(X). \tag{12.2}$$

For the universal example $X = Y = E$, with $x = y = \iota$, eq. (12.1) reduces to

$$\phi^* r = \sum_\alpha r'_\alpha \times r''_\alpha \quad \text{in } E^*(E \wedge E, o).$$

This requires $\phi^* r$ to lie in the image of the cross product (9.13)

$$\times : E^*(E,o) \otimes E^*(E,o) \longrightarrow E^*(E \wedge E, o),$$

which rarely happens. However, the pairing becomes an isomorphism if we use the *completed* tensor product and so allow infinite sums. This is another reason to topologize $E^*(X)$.

From this point of view, a stable algebra should consist of a filtered E^*-algebra M equipped with a continuous E^*-linear action $\lambda_M : \mathcal{A} \otimes M \to M$ that satisfies eq. (12.2) for all r. We must not forget the unit 1_M of the algebra M, for which eq. (11.23) requires $r 1_M = \langle r, 1 \rangle 1_M$.

In the classical case $E = H(\mathbb{F}_p)$, there is a good finite Cartan formula, and this description is adequate for many applications. For MU and BP, however, this approach is not very practical and must be reworked.

Second Answer. We have the coaction $\rho_X : E^*(X) \to E^*(X) \widehat{\otimes} E_*(E,o)$ from (10.7). We shall find that the rather opaque Cartan formula (12.1) translates (for spaces) into the commutative diagram

$$\begin{tikzcd}
E^*(X) \otimes E^*(Y) \arrow[r, "\rho_X \otimes \rho_Y"] \arrow[dd, "\times"] & (E^*(X) \widehat{\otimes} E_*(E,o)) \otimes (E^*(Y) \widehat{\otimes} E_*(E,o)) \arrow[d] \\
& E^*(X) \widehat{\otimes} E^*(Y) \widehat{\otimes} (E_*(E,o) \otimes E_*(E,o)) \arrow[d, "\times \otimes \phi"] \\
E^*(X \times Y) \arrow[r, "\rho_{X \times Y}"] & E^*(X \times Y) \widehat{\otimes} E_*(E,o)
\end{tikzcd} \tag{12.3}$$

By taking $Y = X$, we deduce that ρ_X is a homomorphism of E^*-algebras. This includes the units, which come from $1 \in E^0(T)$, since ρ_T is given by Proposition 11.22(b).

Explicitly, given $x \in E^*(X)$ and $y \in E^*(Y)$, assume that rx and ry are given as in eq. (11.19) by

$$rx = \sum_\alpha \langle r, c_\alpha\rangle\, x_\alpha; \quad ry = \sum_\beta \langle r, d_\beta\rangle\, y_\beta; \quad \text{(for all } r)$$

for suitable elements $x_\alpha \in E^*(X)$, $y_\beta \in E^*(Y)$, and $c_\alpha, d_\beta \in E_*(E,o)$. Evaluation of diag. (12.3) on $x \otimes y$ using eq. (11.18) yields the external Cartan formula

$$r(x\times y) = \sum_\alpha \sum_\beta (-1)^{\deg(d_\beta)\deg(x_\alpha)} \langle r, c_\alpha d_\beta\rangle\, x_\alpha \times y_\beta \quad \text{in } E^*(X\times Y)\hat{}\quad (12.4)$$

for all r. This works too for $x \in E^*(X,o)$, $y \in E^*(Y,o)$, and $x \times y \in E^*(X \wedge Y, o)$, where X and Y are based spaces or spectra. For a space X, we can take $Y = X$ and deduce the internal Cartan formula

$$r(xy) = \sum_\alpha \sum_\beta (-1)^{\deg(d_\beta)\deg(x_\alpha)} \langle r, c_\alpha d_\beta\rangle\, x_\alpha y_\beta \quad \text{in } E^*(X)\hat{}, \text{ for all } r. \quad (12.5)$$

All this makes it clear what the definition of a stable comodule algebra should be. The following lemma makes it reasonable. As in Section 10, we defer most proofs until we have our preferred definitions, at the end of the section.

LEMMA 12.6. *Assume that $E_*(E,o)$ is a free E^*-module. Then the comultiplication $\psi = \psi_S\colon E_*(E,o) \to E_*(E,o) \otimes E_*(E,o)$ and counit $\varepsilon = \varepsilon_S\colon E_*(E,o) \to E^*$ are homomorphisms of E^*-algebras.*

As an immediate corollary of $\psi 1 = 1 \otimes 1$, we have

$$\psi_S(vw) = v \otimes w \quad \text{in } E_*(E,o) \otimes E_*(E,o)$$

for any $v \in E^*$ and $w \in \eta_R E^*$. If we combine this with eq. (11.33), we obtain

$$r_*(vw) = v\, \eta_R\langle r, w\rangle \quad \text{in } E_*(E,o)$$

for any stable operation r. What makes these formulae useful is that the elements vw always span $E_*(E,o) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-module. Thus in the important case when E^* has no torsion, these innocuous equations are powerful enough to determine ψ_S and r_* completely.

DEFINITION 12.7. We call a stable comodule M in the sense of Definition 11.11 a *stable (E-cohomology) comodule algebra* if M is an object of *FAlg* and its coaction ρ_M is a morphism in *FAlg*.

In detail, M is a complete Hausdorff filtered E^*-algebra equipped with a coaction $\rho_M\colon M \to M \mathbin{\widehat{\otimes}} E_*(E,o)$ that is a continuous homomorphism of E^*-algebras and satisfies the coaction axioms (11.12), which are now diagrams in *FAlg*.

THEOREM 12.8. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbf{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then:*

(a) *For any space X, the coaction ρ_X in* (11.15) *makes $E^*(X)^\wedge$ a stable comodule algebra;*

(b) *ρ is universal: given a discrete commutative E^*-algebra B, any multiplicative transformation $\theta X: E^*(X,o) \to E^*(X,o)\widehat{\otimes} B$ (or $\widehat{\theta} X: E^*(X,o)^\wedge \to E^*(X,o)\widehat{\otimes} B$) that is defined for all spectra X and natural on $\mathcal{S}tab^*$ is induced from ρ_X by a unique E^*-algebra homomorphism $f: E_*(E,o) \to B$ as*

$$\theta X: E^*(X,o) \xrightarrow{\rho_X} E^*(X,o)\widehat{\otimes} E_*(E,o) \xrightarrow{1\otimes f} E^*(X,o)\widehat{\otimes} B.$$

Third Answer. We restate our Second Answer in terms of the functor $S'M = M\widehat{\otimes} E_*(E,o)$ introduced in Section 11. What we have really done is construct the symmetric monoidal functor

$$(S', \zeta_{S'}, z_{S'}): (\mathcal{FMod}, \widehat{\otimes}, E^*) \longrightarrow (\mathcal{FMod}, \widehat{\otimes}, E^*) \tag{12.9}$$

where $\zeta_{S'}: S'M\widehat{\otimes} S'N \to S'(M\widehat{\otimes} N)$ is given by

$$\begin{aligned} M\widehat{\otimes} E_*(E,o)\widehat{\otimes} N\widehat{\otimes} E_*(E,o) &\cong M\widehat{\otimes} N\widehat{\otimes}\big(E_*(E,o)\otimes E_*(E,o)\big) \\ &\xrightarrow{M\otimes N\otimes\phi} M\widehat{\otimes} N\widehat{\otimes} E_*(E,o) \end{aligned}$$

and $z_{S'}: E^* \to S'E^*$ is just $\eta_R: E^* \to E_*(E,o)$. We saw $\zeta_{S'}$ in diag. (12.3).

We can now reinterpret Lemma 12.6 as saying that the natural transformations $\psi': S' \to S'S'$ and $\varepsilon': S' \to I$ are monoidal, thus making S' a comonad in $\mathcal{FAlg}$. Then diag. (12.3) simply states that ρ is monoidal. Since $E^* = E^*(T)$ by definition, the other needed axiom reduces to $\rho_T = z_{S'}$, which we have by Proposition 11.22(b).

Fourth Answer. We enrich the object $SM = \mathcal{FMod}^*(A, M)$ in Section 10 to include the multiplicative structure.

THEOREM 12.10. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then we can make S a symmetric monoidal comonad in $\mathcal{FMod}$ and hence a comonad in $\mathcal{FAlg}$.*

The definition of stable algebra is now clear.

DEFINITION 12.11. A *stable* (*E-cohomology*) *algebra* is an S-coalgebra in $\mathcal{FAlg}$, i.e. a complete Hausdorff filtered E^*-algebra M equipped with a continuous homomorphism $\rho_M: M \to SM$ of E^*-algebras that satisfies the coaction axioms (8.7).

If a closed ideal $L \subset M$ is invariant (see Definition 10.13), then M/L inherits a stable algebra structure.

THEOREM 12.12. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then given a complete Hausdorff filtered E^*-algebra M (i.e. object of FAlg), a stable comodule algebra structure on M in the sense of Definition* 12.7 *is equivalent to a stable algebra structure on M in the sense of Definition* 12.11.

THEOREM 12.13. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then:*

(a) *The natural transformation $\rho\colon E^*(-)^\wedge \to S(E^*(-)^\wedge)$ defined on spaces by diag.* (10.10) (*or $\rho\colon E^*(-,o)^\wedge \to S(E^*(-,o)^\wedge)$ for spectra*) *is monoidal and makes $E^*(X)^\wedge$ a stable algebra for any space X;*

(b) *ρ is universal: given a cocommutative comonoid object C in FMod, any multiplicative transformation*

$$\theta X\colon E^*(X,o) \longrightarrow \mathit{FMod}^*\big(C, E^*(X,o)\big)$$

(*or $\widehat{\theta}X\colon E^*(X,o)^\wedge \to \mathit{FMod}^*(C, E^*(X,o)^\wedge)$*) *that is defined for all spectra X and natural on Stab is induced from ρ_X by a unique morphism $f\colon C \to \mathcal{A}$ of comonoids in FMod as*

$$\theta X\colon E^*(X,o) \xrightarrow{\rho_X} S\big(E^*(X)^\wedge\big) = \mathit{FMod}^*\big(\mathcal{A}, E^*(X)^\wedge\big) \xrightarrow{\mathrm{Hom}(f,1)} \mathit{FMod}^*\big(C, E^*(X)^\wedge\big).$$

PROOF OF THEOREMS 12.10 AND 12.13. In proving Theorem 10.12, we made $\mathcal{A} = E^*(E,o)$ an E^*-module object. We add the necessary monoidal structure to $S = \mathit{FMod}^*(\mathcal{A}, -)$ in five steps.

Step 1. We construct the symmetric monoidal functor

$$(S, \zeta_S, z_S)\colon (\mathit{FMod}^*, \widehat{\otimes}, E^*) \longrightarrow (\mathit{Mod}^*, \otimes, E^*). \tag{12.14}$$

We start from the ring spectrum E, with multiplication $\phi\colon E \wedge E \to E$, unit $\eta\colon T^+ \to E$, and v-action $\xi v\colon E \to E$, and note that it is automatically an E^*-algebra object in the symmetric monoidal category $(\mathit{Stab}^*, \wedge, T^+)$ in the sense of Definition 7.12. We apply the E-cohomology functor (9.14) to make $\mathcal{A}$ an E^*-algebra object in $\mathit{FMod}^{*\mathrm{op}}$, with the comultiplication

$$\psi_{\mathcal{A}}\colon \mathcal{A} = E^*(E,o) \xrightarrow{\phi^*} E^*(E \wedge E, o) \cong \mathcal{A} \widehat{\otimes} \mathcal{A}$$

and counit $\varepsilon_{\mathcal{A}} = \eta^*\colon \mathcal{A} = E^*(E,o) \to E^*(T^+,o) = E^*$. Then Lemma 7.14 produces the desired functor (12.14), with $z_S\colon E^* \to SE^*$ given on $v \in E^*$ by eq. (7.15) as

$$z_S v = \eta^* \circ (\xi v)^* = v^*\colon \mathcal{A} = E^*(E,o) \longrightarrow E^*(T^+,o) = E^*. \tag{12.15}$$

This identifies z_S with η_R. Then S takes monoid objects in *FMod** (i.e. objects of *FAlg*) to monoid objects in *Mod** (i.e. E^*-algebras).

Step 2. To prove that $\rho: E^*(-,o) \to S(E^*(-,o)\hat{})$ is monoidal, we need to check commutativity of the diagram in *Mod*

$$\begin{tikzcd} E^*(X,o) \otimes E^*(Y,o) \arrow[r, "\rho_X \otimes \rho_Y"] \arrow[dd, "\times"] & S(E^*(X,o)\hat{}) \otimes S(E^*(Y,o)\hat{}) \arrow[d, "\zeta_S"] \\ & S(E^*(X,o)\hat{}\,\widehat{\otimes}\, E^*(Y,o)\hat{}) \arrow[d, "S\times"] \\ E^*(X \wedge Y,o) \arrow[r, "\rho_{X\wedge Y}"] & S(E^*(X \wedge Y,o)\hat{}) \end{tikzcd} \tag{12.16}$$

By naturality, it is enough to take $X = Y = E$ and evaluate on the universal element $\iota \otimes \iota$. By construction, $\rho_E \iota = \mathrm{id}_{\mathcal{A}} \in S\mathcal{A}$. By the definition (7.11) of ζ_S, the upper route gives $\psi_{\mathcal{A}} \in S(\mathcal{A}\otimes\mathcal{A})$, which by definition corresponds under $S\times$ to $\phi^* \in SE^*(E\wedge E,o)$ as required.

Because $E^* = E^*(T^+,o)$, the other needed diagram reduces to $z_S = \rho_T$, which we have by eq. (12.15).

Step 3. For later use, we combine diag. (12.16) (still in the case $X = Y = E$) with the commutative square

$$\begin{tikzcd} E^*(E,o) \arrow[r, "\rho_E"] \arrow[d, "\phi^*"] & SE^*(E,o) \arrow[d, "S\phi^*"] \\ E^*(E \wedge E,o) \arrow[r, "\rho_{E\wedge E}"] & SE^*(E \wedge E,o) \end{tikzcd}$$

and the definition of $\psi_{\mathcal{A}}$ to obtain the following commutative diagram, which involves only $\mathcal{A}$,

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\rho_E"] \arrow[d, "\psi_{\mathcal{A}}"] & & S\mathcal{A} \arrow[d, "S\psi_{\mathcal{A}}"] \\ \mathcal{A}\widehat{\otimes}\mathcal{A} \arrow[r, "\rho_E \otimes \rho_E"] & S\mathcal{A}\widehat{\otimes}S\mathcal{A} \arrow[r, "\zeta_S"] & S(\mathcal{A}\widehat{\otimes}\mathcal{A}) \end{tikzcd} \tag{12.17}$$

Step 4. The monoidality of ψ is a formal consequence of that of ρ. The two commutative diagrams to check are

$$\begin{tikzcd} SM \otimes SN \arrow[r, "\psi M \otimes \psi N"] \arrow[dd, "\zeta_S(M,N)"] & SSM \otimes SSN \arrow[d, "\zeta_S(SM,SN)"] \\ & S(SM \widehat{\otimes} SN) \arrow[d, "S\zeta_S(M,N)"] \\ S(M \widehat{\otimes} N) \arrow[r, "\psi(M \widehat{\otimes} N)"] & SS(M \widehat{\otimes} N) \end{tikzcd} \qquad \begin{tikzcd} E^* \arrow[r, "z_S"] \arrow[d, "z_S"] & SE^* \arrow[d, "Sz_S"] \\ SE^* \arrow[r, "\psi E^*"] & SSE^* \end{tikzcd} \tag{12.18}$$

(i) (ii)

where we again leave some tensor products uncompleted.

As (i) is natural in M and N, we may work with the universal example $M = N = \mathcal{A}$ and evaluate on $\mathrm{id}_\mathcal{A} \otimes \mathrm{id}_\mathcal{A}$. The upper route gives the element

$$\mathcal{A} \xrightarrow{\psi_\mathcal{A}} \mathcal{A} \widehat{\otimes} \mathcal{A} \xrightarrow{\rho_E \otimes \rho_E} S\mathcal{A} \widehat{\otimes} S\mathcal{A} \xrightarrow{\zeta_S} S(\mathcal{A} \widehat{\otimes} \mathcal{A})$$

of $SS(\mathcal{A} \widehat{\otimes} \mathcal{A})$. The lower route gives

$$\mathcal{A} \xrightarrow{\rho_E} S\mathcal{A} \xrightarrow{S\psi_\mathcal{A}} S(\mathcal{A} \widehat{\otimes} \mathcal{A}),$$

which we just saw in diag. (12.17) is the same.

Since $z_S = \rho_T$, (ii) reduces to axiom (8.7)(i) for the S-coalgebra $E^* = E^*(T)$.

Step 5. We next check that ε is monoidal; this too is formal. As ever, there are two diagrams:

$$\text{(i)} \quad \begin{tikzcd} SM \otimes SN \arrow[d, "\zeta_S(M,N)"'] \arrow[dr, "\varepsilon \otimes \varepsilon"] & \\ S(M \widehat{\otimes} N) \arrow[r, "\varepsilon"] & M \widehat{\otimes} N \end{tikzcd} \qquad \text{(ii)} \quad \begin{tikzcd} E^* \arrow[d, "z_S"] \arrow[dr, "="] & \\ SE^* \arrow[r, "\varepsilon"] & E^* \end{tikzcd} \tag{12.19}$$

Again we take $M = N = \mathcal{A}$ in (i) and evaluate on $\mathrm{id}_\mathcal{A} \otimes \mathrm{id}_\mathcal{A}$. The lower route gives $\psi_\mathcal{A} \iota = \iota \otimes \iota$, by the definition of $\psi_\mathcal{A}$. This agrees with $\varepsilon \otimes \varepsilon$, since $\varepsilon\, \mathrm{id}_\mathcal{A} = \iota$. For (ii), it is clear from eq. (12.15) that $\varepsilon z_S v = v$.

In Theorem 12.13(b), we are given a comonoid object C, equipped with morphisms $\psi_C : C \to C \widehat{\otimes} C$ and $\varepsilon_C : C \to E^*$ in *FMod**. Let us write (V, ζ_V, z_V) for the symmetric monoidal functor with $V = \mathit{FMod}^*(C, -)$ that results from Lemma 7.9. Theorem 10.16(b) provides the unique morphism $f: C \to \mathcal{A}$ in *FMod** that induces V from S as in eq. (10.17).

We compare diag. (12.16) and a similar diagram with V in place of S. Evaluation of the universal case $X = Y = E$ on $\iota \otimes \iota$ shows that $(f \otimes f) \circ \psi_C = \psi_\mathcal{A} \circ f : C \to \mathcal{A} \widehat{\otimes} \mathcal{A}$.

Since θT^+ takes $1 \in E^* = E^*(T)$ to the unit element $z_V \in VE^*$, eq. (10.17) shows that $\varepsilon_C = \varepsilon_{\mathcal{A}} \circ f$. □

Comodule algebras. We can now fill in the missing proofs on comodule algebras. By construction, the isomorphism $S'M \cong SM$ in (11.4) transforms the symmetric monoidal structure (12.9) on S' into the symmetric monoidal structure (12.14) on S. Also, ρ' is monoidal and we have diag. (12.3).

PROOF OF LEMMA 12.6. If we replace S by S' in the four diagrams (12.18) and (12.19) for $M = N = E^*$, we obtain exactly the diagrams we need. □

PROOF OF THEOREM 12.12. The isomorphism $S'M \cong SM$ is now an isomorphism of algebras, and the two definitions agree. □

PROOF OF THEOREM 12.8. For (a), we combine Theorem 12.13(a) with Theorem 12.12.

In (b), Theorem 11.14(b) provides the unique homomorphism $f: E_*(E,o) \to B$ of E^*-modules that induces θ from ρ as in eq. (11.16); it corresponds to the element $(\theta E)\iota$ under the isomorphism $E^*(E,o)\widehat{\otimes} B \cong \mathit{Mod}^*(E_*(E,o),B)$ of Lemma 6.16(a). If we evaluate $\theta(T^+)$ on 1, we see that $f1 = 1$. The multiplicativity of θ is expressed as a diagram resembling (12.3) with B in place of $E_*(E,o)$. We evaluate it in the universal case $X = Y = E$ on $\iota \otimes \iota$ and again use Lemma 6.16(a) to convert elements of $E^*(E \wedge E, o)\widehat{\otimes} B$ to module homomorphisms $E_*(E,o) \otimes E_*(E,o) \to B$, with the help of $E^*(E \wedge E, o) \cong D(E_*(E,o) \otimes E_*(E,o))$ from Theorems 9.20 and 9.25. The upper route yields $\phi_B \circ (f \otimes f): E_*(E,o) \otimes E_*(E,o) \to B$. Since $\iota \times \iota = \phi \in E^*(E \wedge E, o)$, the lower route yields

$$E_*(E,o) \otimes E_*(E,o) \xrightarrow{\times} E_*(E \wedge E, o) \xrightarrow{\phi_*} E_*(E,o) \xrightarrow{f} B.$$

Thus f is multiplicative and so is an E^*-algebra homomorphism. □

13. Operations and complex orientation

In this section, we show how a complex orientation on E determines the elements $b_i \in E_*(E,o)$ from our point of view. *We assume that $E_*(E,o)$ is free*, so that Sections 11 and 12 apply. We pay particular attention to the p-local case, and the main relations that apply there.

Complex projective space. We recall from Definition 5.1 that a complex orientation for E yields a first Chern class $x(\theta) \in E^2(X)$ for each complex line bundle θ over any space X. As the Hopf line bundle ξ over $\mathbb{C}P^\infty$ is universal, we need only study $x = x(\xi) \in E^2(\mathbb{C}P^\infty)$. Thus $\mathbb{C}P^\infty$ is our third test space. Since $E^*(\mathbb{C}P^\infty) = E^*[[x]]$ by Lemma 5.4, the coaction ρ on $E^*(\mathbb{C}P^\infty)$ is completely determined by ρx, multiplicativity, E^*-linearity, and continuity.

Definition 13.1. Given a complex orientation for E, we define the elements $b_i \in E_{2(i-1)}(E,o)$ for all $i \geqslant 0$ by the identity

$$\rho x = b(x) = \sum_{i=0}^{\infty} x^i \otimes b_i \quad \text{in } E^*(\mathbb{C}P^\infty)\,\widehat{\otimes}\,E_*(E,o) \cong E_*(E,o)[[x]], \qquad (13.2)$$

where $b(x)$ is a convenient formal abbreviation that will rapidly become essential.

Equivalently, according to eq. (11.19), the action of any operation $r \in \mathcal{A} = E^*(E,o)$ on x is given as

$$rx = \sum_{i=0}^{\infty} \langle r, b_i \rangle\, x^i \quad \text{in } E^*(\mathbb{C}P^\infty) = E^*[[x]]. \qquad (13.3)$$

Remark. Our indexing convention is taken from [32]. We warn that b_i is often written b_{i-1} (e.g., in [3]), as its degree suggests; the latter convention is appropriate in the current stable context, where $b_0 = 0$ (see below), but less so in the unstable context of [9], where (our) b_0 *does* become nonzero.

Since the Hopf bundle is universal, eqs. (13.2) and (13.3) carry over by naturality to the Chern class $x(\theta)$ of *any* complex line bundle θ over any space X (except that when X is infinite-dimensional and $E^*(X)$ is not Hausdorff, the infinite series force us to work in the completion $E^*(X)\hat{}$).

Proposition 13.4. *The elements $b_i \in E_{2(i-1)}(E,o)$ have the following properties:*

(a) $b_0 = 0$ *and* $b_1 = 1$, *so that* $b(x) = x\otimes 1 + x^2 \otimes b_2 + x^3 \otimes b_3 + \cdots$;

(b) *The Chern class* $x \in E^2(\mathbb{C}P^\infty, o)$, *regarded as a map of spectra* $x\colon \mathbb{C}P^\infty \to E$, *induces* $x_*\beta_i = b_i \in E_*(E,o)$, *where* $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ *is dual to* x^i (as in Lemma 5.4(c));

(c) $\psi_S b_k$ *is given by*

$$\psi_S b_k = \sum_{i=1}^{k} B(i,k) \otimes b_i \quad \text{in } E_*(E,o) \otimes E_*(E,o),$$

where $B(i,k)$ *denotes the coefficient of* x^k *in* $b(x)^i$, *or, in condensed notation,* $\psi_S b(x) = \sum_i b(x)^i \otimes b_i$;

(d) $\varepsilon_S b_i = 0$ *for all* $i > 1$, *so that* $\varepsilon_S b(x) = x$.

Proof. We prove (a) by restricting to $\mathbb{C}P^1 \cong S^2$ and comparing with eq. (11.24). Part (b) is an application of Proposition 11.26, using eq. (13.3). For (c) and (d), we take $M = E^*(\mathbb{C}P^\infty)$ in diags. (11.12) and evaluate on x. □

The formal group law. Now $\mathbb{C}P^\infty = K(\mathbb{Z},2)$ is an H-space, whose multiplication map $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ may be defined by $\mu^*\xi = p_1^*\xi \otimes p_2^*\xi$ for the Hopf bundle ξ. We therefore have from eq. (5.13)

$$\mu^* x = F(x\times 1, 1\times x) = x\times 1 + 1\times x + \sum_{i,j} a_{i,j}\, x^i \times x^j, \tag{13.5}$$

where $F(x,y)$ denotes the formal group law (5.14). When we apply ρ and write x for $x\times 1$ and y for $1\times x$, we obtain from eq. (13.2) and naturality

$$b\big(F(x,y)\big) = F_R\big(b(x), b(y)\big) = b(x) + b(y) + \sum_{i,j} b(x)^i\, b(y)^j\, \eta_R a_{i,j} \tag{13.6}$$

in $E_*(E,o)[[x,y]]$, which is difficult to express without using the formal notations $b(x)$ and $F(x,y)$. On the right, $F_R(X,Y)$ is another convenient abbreviation. (In the language of formal groups, the series $b(x)$ is an isomorphism between the formal group laws F and F_R.)

The p-local case. The above rather formidable machinery does simplify in common situations. When the ring E^* is p-local, most of the b_i are redundant.

LEMMA 13.7. *Assume that E^* is p-local. Then if k is not a power of p, the element $b_k \in E_*(E,o)$ can be expressed in terms of E^*, $\eta_R E^*$, and elements of the form b_{p^i}.*

PROOF. Consider the coefficient of $x^i y^j$ in eq. (13.6), where $i+j=k$. On the left, there is a term $\binom{k}{i} b_k$ from $b_k(x+y)^k$, and all other terms involve only the lower b's. On the right, no b beyond b_i or b_j occurs. If $\binom{k}{i}$ is not divisible by p and so is a unit in E^*, we deduce an inductive reduction formula for b_k. This can be done whenever k is not a power of p, by choosing $i = p^m$ and $j = k - p^m$, where m satisfies $p^m < k < p^{m+1}$. □

We therefore reindex the b's.

DEFINITION 13.8. When E^* is p-local, we define $b_{(i)} = b_{p^i}$ for each $i \geqslant 0$.

We still need to use the internal details of Lemma 13.7 to express each $\psi b_{(k)}$ inductively in terms of the $b_{(i)}$, $a_{i,j}$, and $\eta_R a_{i,j}$.

The main relations. In the p-local case, it is appropriate to study instead of μ the much simpler p-th power map $\zeta\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ constructed from μ. In cohomology, it must induce

$$\zeta^* x = [p](x) = px + \sum_{i>0} g_i x^{i+1} \quad \text{in } E^*(\mathbb{C}P^\infty) \cong E^*[[x]] \tag{13.9}$$

for suitable coefficients $g_i \in E^{-2i}$ (which are usually written a_i; but we need to avoid conflict with certain other elements also known as a_i that appear in Section 14). The formal power series $[p](x)$ is known as the *p-series* of the formal group law. The bundle interpretation is $\zeta^*\xi = \xi^{\otimes p}$, so that

$$x(\theta^{\otimes p}) = px(\theta) + \sum_{i>0} g_i\, x(\theta)^{i+1} \quad \text{in } E^*(Z)\hat{} \tag{13.10}$$

for any line bundle θ over any space Z. (Again, completion is not necessary for finite-dimensional Z, or if the series $[p](x)$ happens to be finite.)

When we apply ρ, we obtain

$$b([p](x)) = [p]_R(b(x)) = pb(x) + \sum_{i>0} b(x)^{i+1}\, \eta_R g_i \quad \text{in } E_*(E,o)[[x]], \tag{13.11}$$

where $[p]_R(X)$ denotes the formal power series $pX + \sum_i(\eta_R g_i)X^{i+1}$. We extract the relations we need.

DEFINITION 13.12. For each $k > 0$, we define the kth *main stable relation* in $E_*(E,o)$ as

$$(\mathcal{R}_k):\quad L(k) = R(k) \quad \text{in } E_*(E,o), \tag{13.13}$$

where $L(k)$ and $R(k)$ denote the coefficient of x^{p^k} in $b([p](x))$ and $[p]_R(b(x))$ respectively.

The results of Section 14 will show that, despite appearances, the relations $(\mathcal{R}_k)$ contain all the information of eq. (13.6), with the understanding that the latter is used only to express (inductively) each redundant b_j in terms of the $b_{(i)}$, E^*, and $\eta_R E^*$, in accordance with Lemma 13.7.

14. Examples of ring spectra for stable operations

In Section 10, we developed a comonad S that, for favorable E, expresses all the structure of stable E-cohomology operations. In Section 11, we described an equivalent comonad S' in terms of structure on the algebra $E_*(E,o)$. In this section, we give the complete description of $E_*(E,o)$ for each of our five examples, namely $E = H(\mathbb{F}_p)$, MU, BP, KU, and $K(n)$. (The first splits into two, and we break out the degenerate special case $H(\mathbb{Q}) = K(0)$ merely for purposes of illustration.)

All the results here are well known, but serve as a guide for [9]. *Our purpose is to exhibit the structure of the results, not to derive them.* As Milnor discovered [22] in the case $E = H(\mathbf{F}_p)$, the most elegant and convenient formulation of stable cohomology operations is the Second Answer of Sections 10 and 12, consisting of the multiplicative (i.e. monoidal) coaction (10.7)

$$\rho_X\colon E^*(X) \longrightarrow E^*(X)\,\widehat{\otimes}\, E_*(E,o)$$

for each space X (or on $E^*(X, o)$, for a spectrum X).

The point is that the knowledge of ρ_X on a few simple test spaces and test maps is sufficient to suggest the complete structure of $E_*(E, o)$. The test spaces studied so far include the point T in Proposition 11.22, the sphere S^k in eq. (11.24), and complex projective space $\mathbb{C}P^\infty$ in eq. (13.2).

In each case, we specify (when not obvious):

(i) The coefficient ring E^*;
(ii) The E^*-algebra $E_*(E, o)$;
(iii) $\eta_R: E^* \to E_*(E, o)$, the right unit ring homomorphism;
(iv) $\psi: E_*(E, o) \to E_*(E, o) \otimes E_*(E, o)$, the comultiplication;
(v) $\varepsilon: E_*(E, o) \to E^*$, the counit.

(See Proposition 11.3 for $E_*(E, o)$ and η_R. By construction and Lemma 12.6, ψ and ε are homomorphisms of E^*-algebras and of E^*-bimodules.) In most cases, the results allow us to express the universal property of $E_*(E, o)$ very simply.

Example. $H(\mathbf{F}_2)$. We take $E = H = H(\mathbf{F}_2)$, the Eilenberg–MacLane spectrum representing ordinary cohomology with coefficients $\mathbf{F}_2$. The main reference is Milnor [22], and many of our formulae, diagrams and results can be found there. The appropriate test space is $\mathbb{R}P^\infty = K(\mathbf{F}_2, 1)$, an H-space with multiplication $\mu: \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$, and we use a mod 2 analogue of complex orientation. We have $H^*(\mathbb{R}P^\infty) = \mathbf{F}_2[t]$, with a polynomial generator $t \in H^1(\mathbb{R}P^\infty)$, and $\mu^* t = t \times 1 + 1 \times t$ is forced. By analogy with Proposition 13.4(a), we must have

$$\rho t = t \otimes 1 + \sum_{i>1} t^i \otimes c_i \quad \text{in } H^*(\mathbb{R}P^\infty) \widehat{\otimes} H_*(H, o) \cong H_*(H, o)[[t]]$$

for certain coefficients $c_i \in H_*(H, o)$. The analogue of eq. (13.6) is simply

$$(t+u) \otimes 1 + \sum_{i>1} (t+u)^i \otimes c_i = t \otimes 1 + \sum_{i>1} t^i \otimes c_i + u \otimes 1 + \sum_{i>1} u^i \otimes c_i$$

in $H_*(H, o)[[t, u]]$. Because the left side contains the terms $\binom{i}{j} t^{i-j} u^j \otimes c_i$, we must have $c_i = 0$ unless i is a power of 2. Imitating Definition 13.8, we write $\xi_i = c_{2^i} \in H_{2^i-1}(H, o)$ for $i > 0$, so that now

$$\rho t = t \otimes 1 + \sum_{i=1}^{\infty} t^{2^i} \otimes \xi_i \quad \text{in } H^*(\mathbb{R}P^\infty) \widehat{\otimes} H_*(H, o) \cong H_*(H, o)[[t]]. \tag{14.1}$$

Because $\underline{H}_1 = \mathbb{R}P^\infty$, this formula is valid for every $t \in H^1(X)$, for all spaces X. It is reasonable to define also $\xi_0 = c_1 = 1$. Milnor proved that this is all there is.

THEOREM 14.2 (Milnor). *For the Eilenberg–MacLane ring spectrum $H = H(\mathbf{F}_2)$:*

(a) $H_*(H, o) = \mathbf{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$, *a polynomial algebra over* $\mathbf{F}_2$ *on the generators* $\xi_i \in H_{2^i-1}(H, o)$ *for* $i > 0$;

(b) *In the complex orientation for* $H(\mathbb{F}_2)$, $b_{(i)} = \xi_i^2$ *for all* $i > 0$, $b_{(0)} = 1$, *and* $b_j = 0$ *if* j *is not a power of* 2;

(c) ψ *is given by*

$$\psi\xi_k = \xi_k \otimes 1 + \sum_{i=1}^{k-1} \xi_{k-i}^{2^i} \otimes \xi_i + 1 \otimes \xi_k \quad \text{in } H_*(H,o) \otimes H_*(H,o);$$

(d) $\varepsilon\xi_k = 0$ *for all* $k > 0$.

PROOF. Milnor proved (a) in [22, App. 1]. The complexified Hopf line bundle over $\mathbb{R}P^\infty$ has Chern class t^2. We compare ρt^2 with eq. (13.2) and read off (b). (For i not a power of 2, this is a stronger statement than Lemma 13.7 provides.) For (c) and (d), we substitute $M = H^*(\mathbb{R}P^\infty)$ into diags. (11.12) and evaluate on t. □

COROLLARY 14.3. *Let* B *be a discrete commutative graded* $\mathbb{F}_2$*-algebra. Assume that the operation* $\theta\colon H^*(X,o) \to H^*(X,o) \widehat{\otimes} B$ *is multiplicative* (i.e. monoidal) *and natural on* $\mathcal{S}tab^*$. *Then on* $t \in H^1(\mathbb{R}P^\infty, o) = H^1(\mathbb{R}P^\infty)$, θ *has the form*

$$\theta t = t \otimes 1 + \sum_{i=1}^{\infty} t^{2^i} \otimes \xi_i' \quad \text{in } H^*(\mathbb{R}P^\infty) \widehat{\otimes} B \cong B[[t]],$$

where the elements $\xi_i' \in B^{-(2^i-1)}$ *determine* θ *uniquely for all* X *and may be chosen arbitrarily.*

PROOF. We combine the universal property Theorem 12.8(b) of $H_*(H,o)$ with the universal property of the polynomial algebra $\mathbb{F}_2[\xi_1, \xi_2, \ldots]$. □

Example. $H(\mathbb{F}_p)$ (for p odd). We take $E = H = H(\mathbb{F}_p)$, the Eilenberg–MacLane spectrum that represents ordinary cohomology with coefficients $\mathbb{F}_p$. The main reference is still Milnor [22].

We have a complex orientation, therefore by Definition 13.8 the elements $b_{(i)} \in H_{2(p^i-1)}(H,o)$; $b_{(i)}$ is normally written ξ_i for $i \geqslant 0$, where $\xi_0 = b_{(0)} = 1$. As in the previous example, eq. (13.6) simplifies to show that $b_j = 0$ whenever j is not a power of p, so that for the Chern class $x = x(\theta) \in H^2(X)$ of any complex line bundle θ over X, eq. (13.2) reduces to

$$\rho_X x = x \otimes 1 + \sum_{i=1}^{\infty} x^{p^i} \otimes \xi_i \quad \text{in } H^*(X) \widehat{\otimes} H_*(H,o). \tag{14.4}$$

We need one more test space, the infinite-dimensional lens space $L = K(\mathbb{F}_p, 1)$, which contains S^1 and is another H-space. The cohomology $H^*(L) = \mathbb{F}_p[x] \otimes \Lambda(u)$ has an exterior generator $u \in H^1(L)$ which restricts to $u_1 \in H^1(S^1)$. As the polynomial generator $x \in H^2(L)$ is the Chern class of a certain complex line bundle, $\rho_L x$ is given by eq. (14.4). This leaves only $\rho_L u$, which must take the form

$$\rho_L u = \sum_i x^i \otimes a_i + \sum_i u x^i \otimes c_i$$

for certain well-defined coefficients $a_i, c_i \in H_*(H, o)$. By restricting to $S^1 \subset L$ and comparing with eq. (11.24), we see that $c_0 = 1$ and $a_0 = 0$.

The multiplication μ on L induces $\mu^* u = u \times 1 + 1 \times u$ and $\mu^* x = x \times 1 + 1 \times x$. Expansion of $\mu^* \rho_L u = \rho_{L \times L} \mu^* u = (\rho_L u) \times 1 + 1 \times (\rho_L u)$ yields

$$\begin{aligned}\sum_i (x \times 1 + 1 \times x)^i \otimes a_i + \sum_i (u \times 1 + 1 \times u)(x \times 1 + 1 \times x)^i \otimes c_i \\ = \sum_i (x^i \times 1) \otimes a_i + \sum_i (u x^i \times 1) \otimes c_i + \sum_i (1 \times x^i) \otimes a_i + \sum_i (1 \times u x^i) \otimes c_i.\end{aligned}$$

For $i > 0$, there is no term with $u \times x^i$ on the right, but there is on the left, which forces $c_i = 0$ for $i > 0$. When we take coefficients of $x^i \times x^j$, we find as in Lemma 13.7 that $a_i = 0$ unless i is a power of p. Again we reindex, defining $\tau_i = a_{p^i} \in H_{2p^i - 1}(H, o)$ for all $i \geqslant 0$, so that now

$$\rho_L u = u \otimes 1 + \sum_{i=0}^{\infty} x^{p^i} \otimes \tau_i \quad \text{in } H^*(L) \widehat{\otimes} H_*(H, o). \tag{14.5}$$

Again, the elements ξ_n and τ_n give everything.

THEOREM 14.6 (Milnor). *For the Eilenberg–MacLane ring spectrum $H = H(\mathbb{F}_p)$ with p odd:*

(a) *As a commutative algebra over $\mathbb{F}_p$,*

$$H_*(H, o) = \mathbb{F}_p[\xi_1, \xi_2, \xi_3, \ldots] \otimes \Lambda(\tau_0, \tau_1, \tau_2, \ldots),$$

with polynomial generators $\xi_i = b_{(i)} \in H_{2(p^i - 1)}(H, o)$ for $i \geqslant 1$ and exterior generators $\tau_i \in H_{2p^i - 1}(H, o)$ for $i \geqslant 0$;

(b) *$\psi\colon H_*(H, o) \to H_*(H, o) \otimes H_*(H, o)$ is given by*

$$\psi \xi_k = \xi_k \otimes 1 + \sum_{i=1}^{k-1} \xi_{k-i}^{p^i} \otimes \xi_i + 1 \otimes \xi_k,$$

$$\psi \tau_k = \tau_k \otimes 1 + \sum_{i=0}^{k-1} \xi_{k-i}^{p^i} \otimes \tau_i + 1 \otimes \tau_k;$$

(c) *$\varepsilon \xi_k = 0$ for all $k > 0$ and $\varepsilon \tau_k = 0$ for all $k \geqslant 0$.*

PROOF. Part (a) is Theorem 2 of Milnor [22]. Parts (b) and (c) comprise Theorem 3 [ibid.], but also follow by substituting ρ_L into diags. (11.12) and evaluating on x and u. (Proposition 13.4 also gives $\psi\xi_k$ and $\varepsilon\xi_k$.) □

We have the analogue of Corollary 14.3.

COROLLARY 14.7. *Let B be a discrete commutative graded $\mathbb{F}_p$-algebra. Assume that the operation $\theta\colon H^*(X,o) \to H^*(X,o)\widehat{\otimes} B$ is multiplicative and natural on $\mathcal{S}tab^*$. Then on $H^*(L) = \mathbb{F}_p[x] \otimes \Lambda(u)$, θ has the form*

$$\theta x = x\otimes 1 + \sum_{i=1}^{\infty} x^{p^i} \otimes \xi_i'; \qquad \theta u = u\otimes 1 + \sum_{i=0}^{\infty} x^{p^i} \otimes \tau_i';$$

where the elements $\xi_i' \in B^{-2(p^i-1)}$ and $\tau_i' \in B^{-(2p^i-1)}$ determine θ uniquely for all X and may be chosen arbitrarily.

Example. $H(\mathbb{Q})$. We take $E = H = H(\mathbb{Q})$, the Eilenberg–MacLane spectrum that represents ordinary cohomology with rational coefficients $\mathbb{Q}$. There are no interesting stable operations.

THEOREM 14.8. *For the Eilenberg–MacLane ring spectrum $H = H(\mathbb{Q})$, we have $H_*(H,o) = H_*(H(\mathbb{Q}),o) = \mathbb{Q}$.*

Example. MU. Our main reference is Adams [3, II.§11]. The coefficient ring is $MU^* = \mathbb{Z}[x_1, x_2, x_3, \ldots]$, with polynomial generators x_n in degree $-2n$ that are not canonical. We have complex orientation, almost by definition, and therefore the elements $b_n \in MU_{2n-2}(MU,o)$.

The good description of MU^* was given by Quillen [30, Theorem 6.5], as the *universal* formal group: it is generated as a ring by the coefficients $a_{i,j} \in MU^*$ that appear in the formal group law (5.14), subject to the relations (5.15). Hence the elements $\eta_R a_{i,j}$ determine η_R.

THEOREM 14.9. *For the unitary cobordism ring spectrum MU:*

(a) *As a commutative MU^*-algebra, $MU_*(MU,o) = MU^*[b_2, b_3, b_4, \ldots]$, with polynomial generators $b_i \in MU_{2(i-1)}(MU,o)$ for $i > 1$;*

(b) *$\eta_R a_{i,j} \in MU_*(MU,o)$ is determined by eq.* (13.6);

(c) *ψ is given by*

$$\psi b_k = b_k \otimes 1 + \sum_{i=2}^{k} B(i,k) \otimes b_i \quad \text{in } MU_*(MU,o) \otimes MU_*(MU,o),$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$;

(d) *$\varepsilon b_k = 0$ for all $k \geqslant 2$.*

PROOF. Part (a) is standard. In (b), the coefficient of $x^i y^j$ in eq. (13.6) provides an inductive formula for $\eta_R a_{i,j}$. Proposition 13.4 provides (c) and (d). □

As $MU_*(MU, o)$ is a polynomial algebra, Corollary 14.3 carries over to this case.

COROLLARY 14.10. *Let B be a discrete commutative MU^*-algebra. Assume that the operation $\theta\colon MU^*(X,o) \to MU^*(X,o)\widehat{\otimes} B$ is multiplicative and natural on $Stab^*$. Then on $x \in MU^2(\mathbb{C}P^\infty)$, θ has the form*

$$\theta x = x\otimes 1 + \sum_{i=2}^{\infty} x^i \otimes b'_i \quad \text{in } MU^*(\mathbb{C}P^\infty)\,\widehat{\otimes}\, B \cong B[[x]],$$

where the elements $b'_i \in B^{-2(i-1)}$ determine θ uniquely for all X and may be chosen arbitrarily.

In other words, there are no relations over MU^* between the b_i. The dual $MU^*(MU, o)$ is known as the Landweber–Novikov algebra. The results for ψ are no longer amenable to explicit expression as in Theorems 14.2 and 14.6.

Example. BP. The main reference is still Adams [3, II.§16]. The coefficient ring is now $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$, a polynomial algebra on Hazewinkel's generators v_i of degree $-2(p^i - 1)$ for $i > 0$. (One could instead use Araki's generators [5] or any other system of polynomial generators, with only slight modifications.)

We still have complex orientation, but because BP^* is p-local, we need only the generators $b_{(i)}$ from Definition 13.8, where $b_{(0)} = 1$. Moreover, it is sufficient to work with the p-series (13.9), because its coefficients g_i generate BP^* as a $\mathbb{Z}_{(p)}$-algebra (as we shall see in more detail in Section 15). We write $w_i = \eta_R v_i \in BP_*(BP, o)$.

THEOREM 14.11. *For the Brown–Peterson ring spectrum BP:*

(a) *As a commutative BP^*-algebra, $BP_*(BP,o) = BP^*[b_{(1)}, b_{(2)}, b_{(3)}, \ldots]$, with polynomial generators $b_{(i)} = b_{p^i} \in BP_{2(p^i-1)}(BP, o)$ for each $i > 0$;*

(b) *The nth main relation $(\mathcal{R}_n)$ in eq. (13.13) provides an inductive formula for $w_n = \eta_R v_n \in BP_*(BP, o)$;*

(c) *ψ is given by*

$$\psi b_{(k)} = b_{(k)}\otimes 1 + \sum_{i=2}^{p^k} B(i, p^k)\otimes b_i \quad \text{in } BP_*(BP,o) \otimes BP_*(BP,o),$$

where $B(i,p^k)$ denotes the coefficient of x^{p^k} in $b(x)^i$ (and Lemma 13.7 is used to express $b(x)$ and b_i in terms of the $b_{(j)}$ and BP^*);

(d) *$\varepsilon b_{(k)} = 0$ for all $k > 0$.*

We shall find that the generators $b_{(i)}$ are better suited to [9] than Quillen's original generators t_i, or their conjugates h_i, which were used in [8]. We have the analogue of Corollary 14.10.

COROLLARY 14.12. *Let B be a discrete commutative BP^*-algebra. Assume the operation $\theta\colon BP^*(X,o) \to BP^*(X,o)\mathbin{\widehat{\otimes}} B$ is multiplicative and natural on* Stab*. *Then on $x \in BP^2(\mathbb{C}P^\infty)$, θ has the form*

$$\theta x = x\otimes 1 + \sum_{i=2}^{\infty} x^i \otimes b_i' \quad \textit{in } BP^*(\mathbb{C}P^\infty)\mathbin{\widehat{\otimes}} B \cong B[[x]]$$

for certain elements $b_i' \in B^{-2(i-1)}$. The elements $b_{(i)}' = b_{p^i}'$ for $i \geqslant 1$ determine θ uniquely for all X and may be chosen arbitrarily.

Example. KU. We take $E = KU = K$, the complex Bott spectrum, which we constructed in Sections 3 and 9. Its coefficient ring is the ring $\mathbb{Z}[u,u^{-1}]$ of Laurent polynomials in $u \in KU^{-2}$, and one writes $v = \eta_R u$. The complex orientation (5.2) furnishes elements $b_i \in KU_*(KU,o)$, of which $b_1 = 1$. We computed its formal group law $F(x,y) = x + y + uxy$ in eq. (5.16); thus eq. (13.6) reduces to

$$b(x + y + uxy) = b(x) + b(y) + b(x)b(y)v. \tag{14.13}$$

This is small enough for explicit calculation. The coefficient of xy^i yields the relation

$$(i+1)b_{i+1} + iub_i = b_i v \tag{14.14}$$

since on the left,

$$b_j(x + y + uxy)^j \equiv b_j y^j + jb_j y^{j-1}x(1 + uy) \mod x^2.$$

(Compare [3, Lemma II.13.5].) This includes the special case $2b_2 + u = v$ for $i = 1$. Generally, for $i > 1$ and $j > 1$, the coefficient of $x^i y^j$ yields the relation

$$b_i b_j = \sum_{k=0}^{\min(i,j)} \binom{i+j-k}{i}\binom{i}{k} u^k b_{i+j-k} v^{-1}, \tag{14.15}$$

which serves to reduce any product of b's to a linear expression. Thus the general expression c in our generators may be assumed linear in the b's. Further, for large enough m, cv^m will have no negative powers of v; if we use eq. (14.14) to remove all the positive powers of v, c takes the form

$$c = u^q\bigl(\lambda_1 u^{-1} + \lambda_2 u^{-2} b_2 + \lambda_3 u^{-3} b_3 + \ldots + \lambda_n u^{-n} b_n\bigr)v^{-m} \tag{14.16}$$

for some integers λ_i, n, and q. This suggests part (a) of the following.

LEMMA 14.17. *In $KU_*(KU,o)$:*

(a) *Every element can be written in the form* (14.16);

(b) *The element c in eq.* (14.16) *is zero if and only if $\lambda_i = 0$ for all i.*

This, with eq. (14.14), is a complete description of $KU_*(KU, o)$. We shall give a proof in [9].

THEOREM 14.18. *For the complex Bott spectrum KU:*

(a) *As a commutative algebra over $KU^* = \mathbb{Z}[u, u^{-1}]$, $KU_*(KU, o)$ has the generators:*

$v = \eta_R u \in KU_2(KU, o)$;
$v^{-1} = \eta_R u^{-1} \in KU_{-2}(KU, o)$;
$b_i \in KU_{2i-2}(KU, o)$ *for* $i > 1$;

subject to the relations (14.14) *and* (14.15);

(b) *As a KU^*-module, $KU_*(KU, o)$ is spanned by the monomials v^n and $b_i v^n$, for all $i > 1$ and $n \in \mathbb{Z}$, subject to the relations* (14.14) (*multiplied by any v^n*);

(c) ψ *is given by*

$$\psi b_k = b_k \otimes 1 + \sum_{i=2}^{k} B(i, k) \otimes b_i \quad \text{in } KU_*(KU, o) \otimes KU_*(KU, o),$$

where $B(i, k)$ denotes the coefficient of x^k in $b(x)^i$;

(d) ε *is given by $\varepsilon b_i = 0$ for all $i > 1$.*

PROOF. Parts (a) and (b) follow from Lemma 14.17. Parts (c) and (d) are included in Proposition 13.4. □

Although we no longer have a polynomial algebra, we still have part of Corollary 14.10.

COROLLARY 14.19. *Let B be a discrete commutative KU^*-algebra. Then any operation $\theta: KU^*(X, o) \to KU^*(X, o) \widehat{\otimes} B$ that is multiplicative and natural on $\mathcal{S}tab^*$ is uniquely determined by its values on $KU^*(\mathbb{C}P^\infty)$.*

The module $KU_*(KU, o)$. What makes the description (14.16) unsatisfactory is that m is not unique; we can always increase m and use eq. (14.14) to remove the extra v's to obtain another expression of the same form that looks quite different. For example, $(b_3 + ub_2)/2 = (2b_4 + 3ub_3 + u^2 b_2)v^{-1} \in KU_*(KU, o)$, in spite of the denominator 2. It is notoriously difficult to write down stable operations in $KU^*(-)$ (equivalently, linear functionals $KU_*(KU, o) \to KU^*$) other than $\Psi^1 = \text{id}$ and $\Psi^{-1}[\xi] = [\overline{\xi}]$ (the complex conjugate bundle). Following Adams [3], we develop an alternate description from which the freeness of $KU_*(KU, o)$ will follow easily.

First, we note that Lemma 14.17 implies that $KU_*(KU, o)$ has no torsion, which allows us to work rationally and consider

$$KU^*[v, v^{-1}] \subset KU_*(KU, o) \subset KU^*[v, v^{-1}] \otimes \mathbb{Q}.$$

The key idea is that if we localize at a prime p, we have available (algebraically) the Adams operation Ψ^k for any invertible $k \in \mathbb{Z}_{(p)}$. Rationally, we have Ψ^k for all nonzero

$k \in \mathbb{Q}$. It is characterized by the properties that it is additive, multiplicative, and satisfies $\Psi^k[\theta] = [\theta^{\otimes k}] = [\theta]^k$ for any line bundle θ.

To compute $\Psi^k u$, we rewrite eq. (3.32) as $uu_2 = [\xi] - 1$ and apply Ψ^k. As stability requires $\Psi^k u_2 = u_2$, and $u_2^2 = 0$, we find

$$\left(\Psi^k u\right)u_2 = [\xi]^k - 1 = (1 + uu_2)^k - 1 = kuu_2.$$

Hence $\Psi^k u = ku$. Then eq. (11.23) becomes

$$\left\langle \Psi^k, v\right\rangle = \Psi^k u = ku. \tag{14.20}$$

The linear functional $\langle\Psi^k, -\rangle\colon KU_*(KU, o) \to KU^* \otimes \mathbb{Q}$ is multiplicative because Ψ^k is, as can be seen by expanding $\Psi^k(\iota\times\iota)$ by eq. (12.4). (These are precisely the multiplicative linear functionals.)

We apply $\langle\Psi^k, -\rangle$ to eq. (14.14) to obtain, by induction starting from $b_1 = 1$,

$$\left\langle \Psi^k, b_n\right\rangle = k^{-1}\binom{k}{n}u^{n-1}. \tag{14.21}$$

Alternatively, for any $n > 1$ we can write formally

$$b_n = \frac{(v-u)(v-2u)\cdots(v-(n-1)u)}{n!} \in KU^*[v, v^{-1}] \otimes \mathbb{Q} \tag{14.22}$$

and replace v by ku everywhere.

LEMMA 14.23. *An element $c \in KU^*[v, v^{-1}] \otimes \mathbb{Q}$ lies in $KU_*(KU, o)$ if and only if $\langle\Psi^k, c\rangle \in KU^* \otimes \mathbb{Z}_{(p)}$ for all primes p and integers $k > 0$ such that p does not divide k.*

From this we deduce the freeness of $KU_*(KU, o)$.

PROOF OF LEMMA 9.21 FOR $E = KU$. Denote by $F_{m,n}$ the free KU^*-module with basis $\{v^m, v^{m+1}, \ldots, v^n\}$. It is enough to show that for any m, $KU_*(KU, o) \cap (F_{-m,m} \otimes \mathbb{Q})$ is a free KU^*-module; then any basis extends to a basis of $KU_*(KU, o) \cap (F_{-m-1,m+1} \otimes \mathbb{Q})$, and thence by induction to a basis of $KU_*(KU, o)$. We may multiply by v^m and work with $F_{0,2m}$ instead.

We therefore work in degree zero and take any element

$$c = \lambda_0 + \lambda_1 w + \lambda_2 w^2 + \cdots + \lambda_{n-1} w^{n-1} \tag{14.24}$$

in $KU_0(KU, o) \cap (F_{0,n-1} \otimes \mathbb{Q})$, where each $\lambda_i \in \mathbb{Q}$ and we write $w = u^{-1}v$. We have only to find a common denominator Δ that guarantees $\Delta\lambda_i \in \mathbb{Z}$ for all i.

Given any prime p, we choose n distinct positive integers $k_1, k_2, \ldots k_n$, not divisible by p; then by eq. (14.20),

$$\left\langle \Psi^k, c\right\rangle = \sum_{i=0}^{n-1} \lambda_i k_j^i \in \mathbb{Z}_{(p)}.$$

We solve these n linear equations for the λ_i in terms of the $\langle\Psi^{k_j}, c\rangle$, which requires division by the Vandermonde determinant

$$\Delta(p) = \det_{i,j}\left(k_j^{i-1}\right) = \prod_{1\leqslant j<i\leqslant n}(k_i - k_j).$$

Then $\Delta(p)\lambda_i \in \mathbb{Z}_{(p)}$ for all i. If $p > n$, the obvious choices $k_j = j$ yield $\lambda_i \in \mathbb{Z}_{(p)}$, because then p does not divide $\Delta(p)$. We take $\Delta = \prod_{p\leqslant n}\Delta(p)$. □

Before we establish Lemma 14.23, we need a result [3, Lemma II.13.8] which explains the role of the b's.

LEMMA 14.25. *Let c be an element of $KU^*[v, v^{-1}] \otimes \mathbb{Q}$. Then c is a KU^*-linear combination of the elements* 1, $v = b_1 v$, $b_2 v$, $b_3 v, \dots$ *if and only if $\langle\Psi^k, c\rangle \in KU^*$ for all integers $k > 0$.*

PROOF. Necessity is clear from eq. (14.21). We may reduce sufficiency to the case when c has degree 0 and write c as a Laurent series in $w = u^{-1}v$. By taking k very large, it is clear that c has no negative powers of w; this allows us to write (see eq. (14.22))

$$c = \sum_{i=0}^{n}\lambda_i\binom{w}{i} = \lambda_0 + \sum_{i=1}^{n}\lambda_i b_i v$$

for some n and suitable coefficients $\lambda_i \in \mathbb{Q}$. By eq. (14.21),

$$\langle\Psi^k, c\rangle = \sum_{i=0}^{n}\lambda_i\binom{k}{i}.$$

By induction on k from 1 to $n+1$, $\langle\Psi^k, c\rangle \in \mathbb{Z}$ yields $\lambda_k \equiv (-1)^k\lambda_0 \bmod \mathbb{Z}$. But $\lambda_{n+1} = 0$. Therefore $\lambda_0 \in \mathbb{Z}$, and $\lambda_i \in \mathbb{Z}$ for all i. □

PROOF OF LEMMA 14.23. Again, necessity is clear. For sufficiency, we assume given c in the form eq. (14.24). Let m be the maximum exponent of any prime in the denominators of the λ_i, so that $p^m\lambda_i \in \mathbb{Z}_{(p)}$ for all i and all primes p. Then $p^m\langle\Psi^k, c\rangle \in \mathbb{Z}_{(p)}$ for all integers $k > 0$ and all primes p.

If p does not divide k, we have $\langle\Psi^k, cw^m\rangle = k^m\langle\Psi^k, c\rangle \in \mathbb{Z}_{(p)}$ by hypothesis. If $k = pq$, we have instead $k^m\langle\Psi^k, c\rangle = q^m p^m\langle\Psi^k, c\rangle \in \mathbb{Z}_{(p)}$, by our choice of m. Thus for each $k > 0$,

$$\langle\Psi^k, cw^m\rangle \in \bigcap_p \mathbb{Z}_{(p)} = \mathbb{Z}.$$

Then Lemma 14.25 shows that $cw^m \in KU_*(KU, o)$. □

Example. $K(n)$. The coefficient ring is now the p-local ring $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$, still with $\deg(v_n) = -2(p^n - 1)$, where p is odd. We write $w_n = \eta_R v_n$, as we did for BP. We have a complex orientation, and therefore elements $b_{(i)}$ for $i \geqslant 0$, where $b_{(0)} = 1$. Although the formal group law remains complicated, it is well known [32, Theorem 3.11(b)] that over $\mathbb{F}_p$, the p-series (13.9) reduces to exactly

$$\zeta^* x = v_n x^{p^n} \quad \text{in } K(n)^*[[x]], \tag{14.26}$$

so that eq. (13.11) simplifies drastically to $b(v_n x^{p^n}) = b(x)^{p^n} w_n$. The coefficient of x^{p^n} yields $w_n = v_n$, and the coefficient of $x^{p^{n+i}}$ then yields

$$b_{(i)}^{p^n} = v_n^{p^i - 1} b_{(i)} \quad \text{in } K(n)_*(K(n), o). \tag{14.27}$$

LEMMA 14.28. *Assume that k is not a power of p. Then:*

(a) $b_k \in K(n)_{2k-2}(K(n), o)$ *can be expressed in terms of v_n and the $b_{(i)}$;*

(b) $b_k = 0$ *if $k < p^n$.*

PROOF. Part (a) comes from Lemma 13.7. For (b), we trivially have $a_{i,j} = 0$ whenever $i + j < p^n$; in this range, eq. (13.6) behaves exactly as in Theorem 14.6 for $H(\mathbb{F}_p)$. □

We need one more test space. The infinite lens space L is *not* appropriate, as $K(n)^*(L) = K(n)^*[x : x^{p^n} = 0]$, where x is inherited from $\mathbb{C}P^\infty$. (Because ζ is trivial on L, we must have $x^{p^n} = 0$, which makes the structure of the Atiyah–Hirzebruch spectral sequence clear.) Instead, we use the finite skeleton $Y = L^{2p^n - 1}$, the orbit space of the unit sphere in $\mathbb{C}^{p^n}$ under the action of the group $\mathbb{Z}/p \subset S^1 \subset \mathbb{C}$. The spectral sequence for $K(n)^*(Y, o)$ collapses because it can support no differential, to give $K(n)^*(Y) = \Lambda(u) \otimes K(n)^*[x : x^{p^n} = 0]$, where $u \in K(n)^1(Y)$ restricts to $u_1 \in K(n)^1(S^1)$. (This fails to define u uniquely, because we can replace u by $u' = u + h v_n u x^{p^n - 1}$ for any $h \in \mathbb{F}_p$.)

We know $\rho_Y x$ is given by eq. (13.2). We write

$$\rho_Y u = \sum_{i=0}^{p^n - 1} x^i \otimes a_i + \sum_{i=0}^{p^n - 1} u x^i \otimes c_i, \tag{14.29}$$

which defines elements $a_i, c_i \in K(n)_*(K(n), o)$. (They are independent of the choice of u.) By restriction to $S^1 \subset Y$, we see that $a_0 = 0$ and $c_0 = 1$.

Unfortunately, Y is no longer an H-space. The multiplication on L restricts (after a noncanonical deformation) to a partial multiplication on skeletons $\mu: L^{2k+1} \times L^{2m} \to L^{2(k+m)+1} = Y$, whenever $k + m = p^n - 1$. Clearly,

$$K(n)^*(L^{2k+1}) = \Lambda(u) \otimes K(n)^*[x : x^{k+1} = 0],$$

with the coaction ρ obtained from ρ_Y by truncation; and similarly for $K(n)^*(L^{2m})$, except that $u x^m = 0$ also.

As x is inherited from L, we have $\mu^*x = x\times 1 + 1\times x$, for lack of any other possible terms in degree 2. For u, we must have

$$\mu^*u = u\times 1 + 1\times u + \lambda v_n u x^k \times x^m$$

for some $\lambda \in \mathbf{F}_p$. (The third term disappears if we replace u by $u + (-1)^k\lambda v_n u x^{p^n-1}$, but in any case is harmless.) We apply ρ to μ, bearing in mind that $w_n = v_n$, and carry out exactly the same algebra as for $E = H(\mathbf{F}_p)$; the coefficients of $u\times x^j$ and $x^i\times x^j$ show that $c_j = 0$ for all $j > 0$ and that $a_h = 0$ for h not a power of p. We therefore reindex, as usual.

DEFINITION 14.30. We define $a_{(i)} = a_{p^i} \in K(n)_{2p^i-1}(K(n), o)$, for $0 \leqslant i < n$.

There is no $a_{(n)}$ because u does not lift to the next skeleton L^{2p^n+1}. In the new notation, eq. (14.29) becomes

$$\rho_Y u = u\otimes 1 + \sum_{i=0}^{n-1} x^{p^i}\otimes a_{(i)} \quad \text{in } K(n)^*(Y)\otimes K(n)_*(K(n), o). \tag{14.31}$$

Having odd degree, the $a_{(i)}$ satisfy $a_{(i)}^2 = 0$.

THEOREM 14.32 (Yagita). *For the Morava K-theory ring spectrum $K(n)$:*

(a) *The commutative $K(n)^*$-algebra $K(n)_*(K(n), o)$ has the generators:*

> $a_{(i)} \in K(n)_{2p^i-1}(K(n), o)$, *for* $0 \leqslant i < n$;
> $b_{(i)} \in K(n)_{2(p^i-1)}(K(n), o)$, *for* $i > 0$;
> *subject to the relations* (14.27);

(b) *η_R is given by $\eta_R v_n = w_n = v_n \in K(n)_*(K(n), o)$;*

(c) *ψ is given by:*

$$\psi a_{(k)} = a_{(k)}\otimes 1 + \sum_{i=0}^{k-1} b_{(k-i)}^{p^i}\otimes a_{(i)} + 1\otimes a_{(k)} \quad \textit{for } 0 \leqslant k < n;$$

$$\psi b_{(k)} = b_{(k)}\otimes 1 + \sum_{i=2}^{p^k-1} B(i, p^k)\otimes b_i + 1\otimes b_{(k)} \quad \textit{for } k > 0;$$

where $B(i, p^k)$ denotes the coefficient of x^{p^k} in $b(x)^i$ (and we use Lemma 14.28 to express $b(x)$ and b_i in terms of the $b_{(i)}$ and v_n);

(d) *$\varepsilon a_{(k)} = 0$ for $0 \leqslant k < n$ and $\varepsilon b_{(k)} = 0$ for $k > 0$.*

PROOF. The whole theorem is essentially due to Yagita [39], who used different generators. We proved (b) above. For (c) and (d), we substitute ρ_Y in diags. (11.12) as usual and evaluate on u and x. □

COROLLARY 14.33. *Let B be a discrete commutative $K(n)^*$-algebra. Then any operation $\theta\colon K(n)^*(X,o) \to K(n)^*(X,o)\,\widehat{\otimes}\, B$ that is multiplicative and natural on* Stab* *is uniquely determined by its values on $K(n)^*(\mathbb{C}P^\infty)$ and $K(n)^*(Y)$.*

REMARK. For low k, the formula for $\psi b_{(k)}$ simplifies by Lemma 14.28(b) to

$$\psi b_{(k)} = b_{(k)} \otimes 1 + \sum_{i=1}^{k-1} b_{(k-i)}^{p^i} \otimes b_{(i)} + 1 \otimes b_{(k)} \quad \text{for } 0 < k \leqslant n.$$

15. Stable BP-cohomology comodules

In this section we study stable modules in the case $E = BP$ in more detail. We find it more practical to work with stable comodules, which by Theorem 11.13 are equivalent. This is the context in which Landweber showed [17], [18] that the presence of a stable comodule structure on M imposes severe constraints on its BP^*-module structure.

We recall that $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$, a polynomial ring on the Hazewinkel generators v_n of degree $-2(p^n - 1)$ (see [14]). It contains the well-known ideals

$$I_n = (p, v_1, v_2, \ldots, v_{n-1}) \subset BP^* \tag{15.1}$$

for $0 \leqslant n \leqslant \infty$ (with the convention that $I_\infty = (p, v_1, v_2, \ldots)$, $I_1 = (p)$, and $I_0 = 0$). We show in Lemma 15.8 that they are *invariant* under the action of the stable operations on $BP^*(T) = BP^*$. Indeed, Landweber [17] and Morava [27] showed that the I_n for $0 \leqslant n < \infty$ are the only finitely generated invariant prime ideals in BP^*.

Nakayama's Lemma. The fact that BP^* is a local ring with maximal ideal I_∞ is extremely useful. The advantage is that once we know certain modules are free, many questions can be answered by working over the more convenient quotient field $BP^*/I_\infty \cong \mathbb{F}_p$. We say a BP^*-module M is *of finite type* if it is bounded above and each M^k is a finitely generated $\mathbb{Z}_{(p)}$-module. (Remember that $\deg(v_i)$ is negative.)

LEMMA 15.2. *Assume that $f\colon M \to N$ is a homomorphism of BP^*-modules of finite type, with N free. Then:*

(a) *f is an isomorphism if and only if $f \otimes \mathbb{F}_p\colon M \otimes \mathbb{F}_p \to N \otimes \mathbb{F}_p$ is an isomorphism;*

(b) *f is a split monomorphism of BP^*-modules if and only if $f \otimes \mathbb{F}_p$ is monic;*

(c) *If the conditions in (b) hold, both M and* Coker f *are BP^*-free;*

(d) *f is epic if and only if $f \otimes \mathbb{F}_p$ is epic (even if N is not free).*

PROOF. The "only if" statements are obvious. For the "if" statements, we first consider $f/p\colon M/pM \to N/pN$. We filter M/pM and N/pN by powers of the ideal $(v_1, v_2, v_3, \ldots)$, so that for the associated graded groups, $\mathrm{Gr}(f/p)\colon \mathrm{Gr}(M/pM) \to \mathrm{Gr}(N/pN)$ is a module homomorphism over the bigraded ring $\mathrm{Gr}(BP^*/(p)) =$

$\mathbb{F}_p[v_1, v_2, v_3, \ldots]$, with $\mathrm{Gr}(N/pN)$ free. As M and N are bounded above, these filtrations are finite in each degree. It follows that if $f \otimes \mathbb{F}_p$ is epic (or monic), so is f/p.

Then the standard Nakayama's Lemma, applied to $\mathbb{Z}_{(p)}$-modules in each degree, gives (d). If f/p is monic and N is free, we must have $\operatorname{Ker} f \subset p^n M$ for all n; as M is of finite type, f must be monic, which gives (a) and some of (b). To see that in (c), M must be free, we lift a basis of $M \otimes \mathbb{F}_p$ to M and use the liftings to define a homomorphism of BP^*-modules $g: L \to M$, with L free, that makes $g \otimes \mathbb{F}_p$ an isomorphism. Then $f \circ g$ is monic by what we have proved so far, and g is epic by (d); therefore g must be an isomorphism.

To finish (b) and (c), we choose an $\mathbb{F}_p$-basis of $\operatorname{Coker}(f \otimes \mathbb{F}_p)$, lift it to N, and use it to define a homomorphism $h: K \to N$ of BP^*-modules with K free. We use f and h to define $M \oplus K \to N$, which by (a) is an isomorphism and identifies $\operatorname{Coker} f$ with K. □

The main relations. We need to make the structure of $BP_*(BP, o)$ more explicit than in Theorem 14.11. The first few terms of the formal group law for BP in terms of the Hazewinkel generators are easily found:

$$F(x,y) \equiv x + y - v_1 x^{p-1} y \mod \left(x^p, y^2\right). \tag{15.3}$$

Also, the p-series for BP begins with

$$[p](x) = px + \left(1 - p^{p-1}\right) v_1 x^p + \cdots . \tag{15.4}$$

All we need to know about $[p](x)$ beyond this is the standard fact (e.g., [32, Theorem 3.11(b)]) that

$$[p](x) \equiv px + \sum_{i>0} v_i x^{p^i} \mod I_\infty^2. \tag{15.5}$$

For lack of alternative, $b_i = 0$ whenever $i-1$ is not a multiple of $p-1$, so that $b(x) = x + b_{(1)} x^p + \cdots$. The first main relation is well known and readily computed from Definition 13.12, with the help of eq. (15.4), as

$$(\mathcal{R}_1): \qquad v_1 = pb_{(1)} + w_1, \tag{15.6}$$

or more easily, as the coefficient of $x^{p-1}y$ in eq. (13.6), expanded using eq. (15.3). Subsequent relations $(\mathcal{R}_k)$ are far more complicated and answers in closed form are not to be expected. To handle the right side $R(k)$, we introduce the ideal $\mathfrak{W} = (p, w_1, w_2, \ldots) \subset BP_*(BP, o)$, the analogue of I_∞ for the right BP^*-action. The right side of eq. (13.11) simplifies by eq. (15.5) to

$$pb(x) + \sum_i b(x)^{p^i} w_i \mod \mathfrak{W}^2.$$

When we expand $b(x)^{p^i}$, all cross terms may be ignored, because they contain a factor $p \in \mathfrak{W}$, and we find

$$R(k) \equiv pb_{(k)} + \sum_{i=1}^{k-1} b_{(k-i)}^{p^i} w_i + w_k \mod \mathfrak{W}^2. \tag{15.7}$$

With slightly more attention to detail, we obtain a sharper, more useful result. It also implies that $\mathfrak{W} = I_\infty BP_*(BP, o)$, so that $\mathfrak{W}$ is redundant.

LEMMA 15.8. *For any $n > 0$, we have $w_n \equiv v_n$* mod $I_n BP_*(BP, o)$.

PROOF. We show by induction on n that the relation $(\mathcal{R}_n)$ simplifies as stated, starting from eq. (15.6) for $n = 1$. If the result holds for all $i < n$, we have $w_i \equiv v_i \equiv 0$ mod I_n for $i < n$. Then $R(n) \equiv w_n$ from eq. (15.7), as $\mathfrak{W}^2$ contains nothing of interest in this degree. Meanwhile, the left side $L(n) \equiv v_n$ by eq. (15.5). □

Recall from Definition 10.13 and Theorem 11.13 that an ideal $J \subset BP^*$ is invariant if it is a stable subcomodule of $BP^* = BP^*(T)$; in view of Proposition 11.22(b), the necessary and sufficient condition for this is $\eta_R J \subset JBP_*(BP, o)$. In this case, we have the quotient stable comodule BP^*/J. For example, Lemma 15.8 shows that the ideals I_n are invariant, and we have the stable comodules $BP^*/I_n \cong \mathbf{F}_p[v_n, v_{n+1}, v_{n+2}, \ldots]$ (for $n > 0$) and $BP^*/I_0 \cong BP^*$.

Primitive elements. The key idea is to explore a general stable comodule M by looking for comodule morphisms $BP^* \to M$ from the (relatively) well understood stable comodule $BP^*(T) = BP^*$. A BP^*-module homomorphism $f: BP^* \to M$ is obviously uniquely determined by the element $x = f1 \in M$, since $fv = f(v1) = vf1 = vx$, and we can choose x arbitrarily. In BP^*, we clearly have $\rho 1 = 1 \otimes 1$, which suggests the following definition.

DEFINITION 15.9. Given a stable comodule M, we call an element $x \in M$ *stably primitive* if $\rho_M x = x \otimes 1$.

This is the necessary and sufficient condition for the above homomorphism $f: BP^* \to M$ to be a stable morphism. It then induces an isomorphism of stable comodules $BP^*/\operatorname{Ker} f \cong (BP^*)x$. In particular, $\operatorname{Ker} f = \operatorname{Ann}(x)$, the annihilator ideal of x, must be an invariant ideal. We are therefore interested in finding primitives.

The primitive elements of M^k clearly form a subgroup. Moreover, there is a good supply of primitives; if M is bounded above, axiom (11.12)(ii) forces every element $x \in M$ of top degree to be primitive. (This may be viewed as an algebraic analogue of Hopf's theorem, that for a finite-dimensional space X, $\pi^k(X) \cong H^k(X; \mathbb{Z})$ in the top degree.) If x is primitive, the BP^*-linearity of ρ_M gives $\rho_M(vx) = x \otimes \eta_R v$ for any $v \in BP^*$. It follows that the comodule structure on BP^*/J is unique if the ideal J is invariant (and none exists otherwise). Landweber [17] located all the primitive elements in the stable comodule BP^*/I_n.

THEOREM 15.10 (Landweber). *For $0 \leqslant n < \infty$, the only nonzero primitive elements in the stable comodule BP^*/I_n are those of the form:*
(i) λv_n^i, *where $i \geqslant 0$ and $\lambda \in \mathbb{F}_p$ (if $n > 0$); or*
(ii) λ, *where $\lambda \in \mathbb{Z}_{(p)}$ (if $n = 0$).*

It follows easily as in [17, Theorem 2.7] that the I_n are the *only* finitely generated invariant prime ideals in BP^*. This suggests that the BP^*/I_n should be the basic building blocks for a general stable comodule. This is the content of Landweber's filtration theorem (cf. [17, Lemma 3.3] and [18, Theorem 3.3′]).

THEOREM 15.11 (Landweber). *Let M be a stable BP-cohomology (co)module that is finitely presented as a BP^*-module (e.g., $BP^*(X)$ for any finite complex X). Then M admits a finite filtration by invariant submodules $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M$, in which each quotient M_i/M_{i-1} is generated, as a BP^*-module, by a single element x_i such that* $\operatorname{Ann}(x_i) = I_{n_i}$ *for some $n_i \geqslant 0$.*

We outline Landweber's proof [18] for reference. For nonzero M, $\operatorname{Ass}(M)$, which here may be taken as the set of all prime annihilator ideals of elements of M, is a *finite nonempty* set of invariant finitely generated prime ideals of BP^*. The recipe for constructing a filtration of M is:

(a) Let I_n be the maximal element of $\operatorname{Ass}(M)$;

(b) Construct the BP^*-submodule $N = 0{:}I_n$ of M, which is defined as $\{y \in M : I_n y = 0\}$, and prove it invariant;

(c) Take a nonzero primitive $x_1 \in N$ (e.g., any element of top degree), so that the maximality of I_n forces $\operatorname{Ann}(x_1) = I_n$;

(d) Put $M_1 = (BP^*)x_1$, so that M_1 is invariant and isomorphic to BP^*/I_n;

(e) Replace M by M/M_1 and repeat, as long as M is nonzero, making sure that the process terminates (which requires some care).

REMARKS. 1. The filtration of M is *never* a composition series. The module BP^*/I_n is not irreducible, because we have the short exact sequence

$$0 \longrightarrow BP^*/I_n \xrightarrow{v_n} BP^*/I_n \longrightarrow BP^*/I_{n+1} \longrightarrow 0$$

of stable comodules. Thus we have no uniqueness statement.

2. We cannot expect to arrange $n_1 \geqslant n_2 \geqslant \cdots$, since in (e), $\operatorname{Ass}(M/M_1)$ need not be contained in $\operatorname{Ass}(M)$.

Index of symbols

This index lists most symbols in roughly alphabetical order (English, then Greek), with brief descriptions and references. Several symbols have multiple roles.

$a_{(i)}$ stable element for $K(n)$, (14.31).
$a_{i,j}$ coefficient in formal group law, (5.14).
BG classifying space of group G.
$B(i,k)$ coefficient in $b(x)^i$, Proposition 13.4.
BP Brown–Peterson spectrum, §2.
b Bott map, Corollary 5.12.
b_i stable element, Proposition 13.4.
$b_{(i)}$ accelerated b_i, Definition 13.8.
$b(x)$ formal power series, (13.2).
C cofree functor, Theorem 8.10.
$\mathbb{C}$ the field of complex numbers.
$\mathbb{C}P^n$, $\mathbb{C}P^\infty$ complex projective space.
Coalg category of E^*-coalgebras, §6.
$c_i(\xi)$ Chern class of vector bundle ξ, Theorem 5.7.
DM dual of E^*-module M, Definition 4.8.
d duality homomorphism, (4.5), (9.24).
E generic ring spectrum.
E^* coefficient ring of E-(co)homology, §§3, 4.
$E^*(-)$ E-cohomology, Theorem 3.17.
$E^*(-)\hat{}$ completed E-cohomology, Definition 4.11.
$E_*(-)$ E-homology, (9.17).
$\underline{E}_n$ nth space of Ω-spectrum E, Theorem 3.17.
e evaluation on $DL \otimes L$, §6.
e_i basis element of $\mathbb{C}^n$.
F free functor, Theorems 2.6, 8.5.
$F(x,y)$ formal group law, (5.14).
$F^a M$ generic filtration submodule, Definition 3.36.
FAlg category of filtered E^*-algebras, §6.
$F^L DM$ filtration submodule of DM, Definition 4.8.
F_M etc. corepresented functor, §8.
FMod, *FMod** (graded) category of filtered E^*-modules, §6.
$\mathbb{F}_p$ field with p elements.
$F_R(X,Y)$ right formal group law, (13.6).
$F^s E^*(X)$ skeleton filtration, (3.33).
f etc. generic map or homomorphism.
f^*, f_* homomorphism induced by map f, (6.3).
f_n structure map of spectrum E, Definition 3.19.
G etc. generic group (object), §7.
G E-module spectrum, Theorem 9.26.
$Gp(C)$ category of group objects in C, §7.
g_i coefficient in p-series, (13.9).
H generic comonad, (8.6).
H, $H(R)$ Eilenberg–MacLane spectrum, §§2, 14.
Ho, *Ho'* homotopy category of (based) spaces, §6.
$h(-)$ generic ungraded cohomology theory, §3.
h Yokota clutching function, (5.9).
I identity functor.
I_n, I_∞ ideal in BP^*, (15.1).
i_1, i_2 injection in coproduct, §2.
id identity morphism.
K_C unit object in (symmetric) monoidal category C, §7.
$K(n)$ Morava K-theory, §2.
KU complex K-theory Bott spectrum, §2, Definition 3.30.
L infinite lens space, §14.
M etc. generic (filtered) module or algebra.
$M\hat{}$, $\widehat{M}$ completion of filtered M, Definition 3.37.
Mod, *Mod** (graded) category of E^*-modules, §6.
MU unitary Thom spectrum, §2.
o generic basepoint, point spectrum.
$-^{\mathrm{op}}$ categorical dual, §6.
PA the primitives in coalgebra A, (6.13).
p fixed prime number.
p_1, p_2 projection from product, §2.
$[p](x)$ p-series, (13.9).
$[p]_R(x)$ right p-series, (13.11).
QA the indecomposables of algebra A, (6.10).
$\mathbb{Q}$ the field of rational numbers.
q map to one-point space T, §2.
R generic ring.
R-Mod category of R-modules, §8.
$\mathbb{R}P^\infty$ real projective space.
r etc. generic cohomology operation.
$\langle r, -\rangle$ E^*-linear functional defined by operation r, (11.1).
S stable comonad, Theorem 10.12.
S' stable comonad, (11.4).
$-_S$ (subscript) stable context.
S^1 unit circle, as space or group.
S^n unit n-sphere.
Stab, *Stab** (graded) stable homotopy category, §6.
Set category of sets, §6.
Set$^{\mathbb{Z}}$ category of graded sets, §7.
T monad, (8.4).
T the one-point space.
T^+ 0-sphere, T with basepoint added.
$T(n)$ torus group.
t $\in H^1(\mathbb{R}P^\infty)$, generator of $H^*(\mathbb{R}P^\infty)$, (14.1).
U, $U(n)$ unitary group.
$-_U$ (subscript) unstable context.
u $\in KU^{-2}$, after Definition 3.30.
u $\in E^1(L)$, exterior generator of $E^*(L)$, §14.
u $\in E^1(Y)$, exterior generator of $E^*(Y)$, §14.
u universal element of $DL \widehat{\otimes} L$, Lemma 6.16.
u_1 canonical generator of $E^*(S^1)$, Definition 3.23.
u_n canonical generator of $E^*(S^n)$, §3.
V generic (often forgetful) functor.
v generic element of E^*.

v $= \eta_R u \in KU_2(KU, o)$, Theorem 14.18.
v_n Hazewinkel generator of BP^*, $K(n)^*$, §14.
W forgetful functor, §8.
$\mathfrak{W}$ ideal in $BP_*(BP, o)$, §15.
w $= u^{-1}v \in KU_0(KU, o)$, Lemma 14.23.
w_n $= \eta_R v_n$, §14.
X etc. generic space or spectrum.
X^+ space X with basepoint adjoined.
x generic cohomology class or module element.
x $\in E^*(\mathbb{C}P^\infty)$, Chern class of Hopf line bundle, Lemma 5.4.
$x(\theta)$ Chern class of line bundle θ, Definition 5.1.
Y skeleton of lens space L, §14.
$\mathbb{Z}$ the ring of integers.
$\mathbb{Z}/p$ the group of integers mod p.
$\mathbb{Z}_{(p)}$ $\mathbb{Z}$ localized at p.
z_F morphism for a (symmetric) monoidal functor F, §7.
α etc. generic index.
α generic algebraic operation, §7.
β_i $\in E_{2i}(\mathbb{C}P^n)$, Lemma 5.3.
γ_i $\in E_{2i+1}(U(n))$, Lemma 5.11.
$\Delta: X \to X \times X$ diagonal map.
ε generic counit morphism.
$\varepsilon: FV \to I$ natural transformation, §2.
ζ pth power map on $\mathbb{C}P^\infty$, (13.9).
ζ_F pairing for (symmetric) monoidal functor F, §7.
η generic monoid unit morphism.
$\eta: I \to VF$ natural transformation, §2.
η generic vector bundle.
η_R right unit, Definition 11.2.
θ generic anything.
θ complex line bundle, §5.
θ cohomology operation (usually idempotent), §3.
ι $\in h(H)$, universal class, Theorem 3.6.
ι $\in E^0(E, o)$, universal class, §9.
ι_n $\in E^n(\underline{E}_n)$, universal class, Theorem 3.17.
$\Lambda(-)$ exterior algebra.
λ generic action.
λ numerical coefficient.
μ addition or multiplication in generic group object, §7.
ν inversion morphism in generic group object, §7.
ξ Hopf line bundle over $\mathbb{C}P^n$.
ξ generic line or vector bundle.
ξ_i stable element for $H(\mathbb{F}_2)$, (14.1).
ξ_i stable element for $H(\mathbb{F}_p)$, (14.4).
ξv action of v on E^*-module, (7.4).
$\pi_*(X)$ homotopy groups of space X.
$\pi_*^S(X, o)$ stable homotopy groups of X.
ρ generic coaction.
ρ_M coaction on module M.
ρ_X coaction on $E^*(X)$ or $E^*(X)^\wedge$.
Σ, Σ^k suspension isomorphism, (3.13), Definition 6.6.
ΣX, $\Sigma^k X$ suspension of space X.
ΣM, $\Sigma^k M$ suspension of module M, Definition 6.6.
$\sigma_k: \underline{E}_k \to E$ stabilization, Definition 9.3.
τ_i stable element for $H(\mathbb{F}_p)$, (14.5).
ϕ generic monoid multiplication.
χ canonical antiautomorphism of Hopf algebra.
Ψ^k Adams operation, (14.20).
ψ generic comultiplication.
ΩX loop space on based space X.
ω zero morphism of generic group object, §7.

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CHAPTER 15

Unstable Operations in Generalized Cohomology

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1. Introduction

A multiplicative generalized cohomology theory $E^*(-)$ on spaces is represented by the spaces $\underline{E}_n$ of its Ω-spectrum, as described in detail in [8, Theorem 3.17]. We denote its coefficient ring by E^*. Our five examples are ordinary cohomology $H^*(-;\mathbf{F}_p)$, unitary cobordism $MU^*(-)$, Brown–Peterson cohomology $BP^*(-)$, complex K-theory $KU^*(-)$, and Morava K-theory $K(n)^*(-)$. (They were properly introduced in [8, §2].) Recent work [25] shows that a sixth example, the cohomology theory $P(n)^*(-)$, also satisfies our hypotheses.

We are interested in three kinds of cohomology operation: *stable* operations, which form the endomorphism ring $E^*(E,o)$ of E (in our notation) and were studied in [8]; *unstable* operations, defined on $E^n(X)$ for spaces X and fixed n, which form $E^*(\underline{E}_n)$; and *additive* unstable operations r on $E^n(-)$ (that satisfy $r(x+y) = r(x)+r(y)$), which form the subset $PE^*(\underline{E}_n)$. Since a stable operation restricts to an additive unstable operation on any degree, these are related by

$$E^*(E,o) \longrightarrow PE^*(\underline{E}_n) \subset E^*(\underline{E}_n)\ .$$

Each of these is an E^*-module in the usual way, by $(r+s)(x) = r(x) + s(x)$ and $(vr)(x) = v\,r(x)$ (for any $v \in E^*$). We can compose, $(sr)(x) = (s \circ r)(x) = s(r(x))$, whenever the sources and targets match. We can also multiply unstable operations together by $(r \smile s)(x) = r(x)s(x)$.

In the classical case $E = H(\mathbf{F}_p)$, for which $E^*(E,o)$ is the Steenrod algebra, it is true that: (a) every additive operation comes from a stable operation; (b) the additive operations generate multiplicatively all the unstable operations. Our difficulties stem from the fact that for MU and BP, both (a) and (b) are *false*. (See [27] for more discussion of the differences.) We propose to describe completely the algebraic structure that has to be present on an E^*-module or E^*-algebra to make it an unstable object, with particular attention to the case $E = BP$. Our definitions lead to structure theorems.

Stable BP-operations have been available for quite some time and are well established. Less has been done with unstable BP-operations, owing to their complexity, but we do have the work [4], [5] of Bendersky, Curtis, Davis, and Miller. The algebraic structure on an *additively* unstable module is described in [27] and (without proofs) in [6].

Our major task, therefore, is to set up precise algebraic descriptions of the unstable structures we need on modules and algebras, along the lines of the stable structures in [8]. Part of the difficulty is that one is forced to work in the unfamiliar context of nonadditive operations; but the real problem turns out to be Theorem 9.4, that unstable modules (as distinct from unstable algebras) simply *do not exist* compatibly with our other objects! When we limit attention to the less exotic *additive* operations, this difficulty does not arise and we have both modules and algebras.

In fact, there is a huge amount of data to be codified in an unstable algebra. The key idea is that given an E^*-algebra M, we define $(UM)^k$ for each k as the set of all algebra homomorphisms $E^*(\underline{E}_k) \to M$; each such homomorphism may be thought of as a candidate for the values of all operations on a typical element of M^k. Apparently merely a graded set, UM becomes an E^*-algebra for suitable E, thanks to extra structure on the spaces $\underline{E}_n$. Then an unstable structure on M is a homomorphism $\rho_M: M \to UM$ of E^*-algebras, which selects for each $x \in M^k$ the function $\rho_M(x): E^*(\underline{E}_k) \to M$; then we define $r(x) = \rho_M(x)r$. This is not enough; in order to compose operations correctly, it is necessary to know the E-cohomology homomorphism $r^*: E^*(\underline{E}_m) \to E^*(\underline{E}_k)$ induced by each operation $r: E^k(-) \to E^m(-)$. This extra structure makes the functor U a *comonad*, and (M, ρ_M) a *coalgebra* over this comonad. We have a similar construction for additive operations, and can compare with the stable constructions of [8].

This is our elegant but extremely terse answer, and we do not believe that it can be efficiently expressed without using comonads. But it does have the effect that the work consists largely of definitions. In Section 10, we translate this answer into practical language, in the context of Hopf rings, that we can use for computation. This includes Cartan formulae for $r(x+y)$ as well as $r(xy)$, and related formulae for $r_*(b*c)$ and $r_*(b\circ c)$ that we use to compute the induced E-homology homomorphism $r_*: E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ dual to r^*.

Landweber filtrations. We recall that $BP^* = BP^*(T)$, the BP-cohomology of the one-point space T, is the polynomial ring $\mathbb{Z}_{(p)}[v_1, v_2, v_3 \cdots]$, with $\deg(v_n) = -2(p^n - 1)$ (under our degree conventions). It contains the well-known ideals

$$I_n = (p, v_1, v_2, \ldots, v_{n-1}) \subset BP^* \tag{1.1}$$

for $0 \leqslant n \leqslant \infty$ (with the convention that $I_\infty = (p, v_1, v_2, \ldots)$, $I_1 = (p)$, and $I_0 = 0$).

The significance [8, Lemma 15.8] of I_n is that it is *invariant* under the action of the stable operations on $BP^*(T)$. Indeed, Landweber [15] and Morava [20] showed that the I_n for $0 \leqslant n < \infty$ are the only finitely generated invariant prime ideals in BP^*. Landweber used this fact to show (see [16, Theorem 3.3′] or [8, Theorem 15.11]) that a stable (co)module M that is finitely presented as a BP^*-module, including $BP^*(X)$ for any finite complex X, admits a finite filtration by invariant submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M \tag{1.2}$$

in which each quotient M_i/M_{i-1} is generated (as a BP^*-module) by a single element x_i whose annihilator ideal $\mathrm{Ann}(x_i) = I_{n_i}$ for some n_i. Thus $M_i/M_{i-1} \cong BP^*/I_{n_i}$.

The first unstable result on BP-cohomology, due to Quillen [22] (see Theorem 20.2), was that for a finite complex X, $BP^*(X)$ is generated, as a BP^*-module, by elements of non-negative degree. What started this project was the observation that if an unstable object M is generated by a single element x, there is an unstable operation (see Proposition 1.14 or the Remark following Corollary 20.9) that takes $v_n x$ to x, provided $\deg(x)$ is small enough; it follows that $v_n x \neq 0$ and that M cannot be isomorphic to BP^*/I_{n+1}.

The proof of Landweber's theorem depends on the concept of *primitive* element in a comodule M. Given any $x \in M$, there is the obvious homomorphism of BP^*-modules $f: BP^* \to M$, defined by $fv = vx$. It is a morphism of stable modules if and only if x is primitive, and if so, we have the isomorphism $BP^*/\mathrm{Ann}(x) \cong (BP^*)x \subset M$ of stable modules. An important example (see [8, Theorem 15.10]) is that the only nonzero primitives in BP^*/I_n, for $n > 0$, are the (images of the) elements λv_n^i, where $\lambda \in \mathbb{F}_p$,

$\lambda \neq 0$, and $i \geqslant 0$. For additive unstable operations, the appropriate definition of primitive becomes more restrictive.

THEOREM 1.3. (This is included in Theorem 20.10.) *Let M be the BP^*-module generated by a single element x with* $\mathrm{Ann}(x) = I_n$, *where $n > 0$. Then M admits an additively unstable module structure* (as defined in Section 5) *if and only if* $\deg(x) \geqslant f(n) - 2$, *and it is unique.*

The only nonzero primitive elements in M are those of the form $\lambda v_n^i x$, where $\lambda \in \mathbb{F}_p$, and $\deg(v_n^i x) \geqslant f(n)$ *if $i > 0$.*

Here, and everywhere, we need the numerical function

$$f(n) = \frac{|\deg(v_n)|}{p-1} = \frac{2(p^n-1)}{p-1} = 2\big(p^{n-1} + p^{n-2} + \cdots + p + 1\big) \tag{1.4}$$

for $n > 0$; it is reasonable to define also $f(0) = 0$.

We use this result in Theorem 20.11 to construct a Landweber filtration (1.2) of an appropriate module M, including $BP^*(X)$ for any finite complex X, in which each quotient M_i/M_{i-1} has the form in Theorem 1.3 (or is BP^*-free). Once our machinery is in working order, we are able to give a one-line proof of Theorem 20.3, the weak form of Quillen's theorem.

In our main structure theorem, we do one better by allowing all unstable operations instead of only the additive ones. One complication is that the unstable analogue of Theorem 1.3 has to be stated for algebras only, owing to the nonexistence of unstable modules.

THEOREM 1.5. (This is stated precisely as Theorem 21.12.) *Let M be an unstable BP^*-algebra such as $BP^*(X)$ for a finite complex X. Then M admits a filtration* (1.2) *by invariant ideals M_i, in which each quotient M_i/M_{i-1} is generated, as a BP^*-module, by a single element x_i such that* $\mathrm{Ann}(x_i) = I_{n_i}$ *for some $n_i \geqslant 0$, where* $\deg(x_i) \geqslant \max(f(n_i) - 1, 0)$.

Splittings of BP-cohomology. Another application of our machinery yields idempotent operations that split unstable BP-cohomology into indecomposable pieces. Such splittings were constructed in [26] by means of Postnikov systems. What is new is that explicit definitions of everything allow us to carry out computations. Our results are logically independent of [26] and rely on it only to recognize the summands as known objects; nevertheless, it is a valuable guide as to what the summands look like and where to find them. In a sequel [9], two of the authors go on to apply the structure theorems of [25] to establish analogous (but slightly different) splitting theorems for the cohomology theory $P(n)^*(-)$, whose coefficient ring is BP^*/I_n.

For each $n \geqslant 0$, we define the ideal

$$J_n = (v_{n+1}, v_{n+2}, v_{n+3}, \ldots) \subset BP^*. \tag{1.6}$$

In [26], Baas–Sullivan theory [2] was used to construct a cohomology theory $BP\langle n\rangle^*(-)$ having coefficients $BP^*/J_n \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$. In particular, $BP\langle 0\rangle^*(-) = H^*(-;\mathbb{Z}_{(p)})$. The desired splitting has the form

$$BP^k(X) \cong BP\langle n\rangle^k(X) \oplus \prod_{j>n} BP\langle j\rangle^{k+2(p^j-1)}(X). \tag{1.7}$$

The representing spectrum $BP\langle n\rangle$ is (at least) a BP-module spectrum, and comes equipped with a canonical map of BP-module spectra that we shall call $\pi\langle n\rangle\colon BP \to BP\langle n\rangle$. There is also a canonical map $\pi\colon BP\langle j\rangle \to BP\langle n\rangle$ whenever $j > n$. (Geometrically, $BP\langle n\rangle$ allows more singularities than $BP\langle j\rangle$.) Everything we need to know about $BP\langle n\rangle$ is contained in the commutative diagram

$$\begin{tikzcd}
\underline{BP}_{k+2(p^j-1)} \arrow[r, "v_j"] \arrow[d, "\pi\langle j\rangle"] & \underline{BP}_k \arrow[r, "\pi\langle n\rangle"] \arrow[d, "\pi\langle j\rangle"] & \underline{BP\langle n\rangle}_k \\
\underline{BP\langle j\rangle}_{k+2(p^j-1)} \arrow[r, "v_j"] & \underline{BP\langle j\rangle}_k \arrow[ur, "\pi"'] &
\end{tikzcd} \tag{1.8}$$

of H-spaces and H-maps, where $j > n$.

Although the cohomology theory $BP\langle n\rangle^*(-)$ may be unfamiliar, in the range of degrees of interest it is easily described in terms of BP-cohomology. It is clear by construction that $\pi\langle n\rangle_*\colon BP^*(X) \to BP\langle n\rangle^*(X)$ kills $J_nBP^*(X)$.

THEOREM 1.9. *Assume that $k \leqslant f(n+1)$, where $n \geqslant 0$, and that X is finite-dimensional. Then $\pi\langle n\rangle$ induces a natural isomorphism of BP^*-modules*

$$BP^k(X) \Big/ \sum_{j>n} v_j BP^{k+2(p^j-1)}(X) \cong BP\langle n\rangle^k(X). \tag{1.10}$$

We derive this below as an immediate consequence of Theorem 1.12. It is best possible, as [26] shows that $\pi\langle n\rangle_*$ is not surjective in general for $k > f(n+1)$.

LEMMA 1.11. (This is included in Lemma 22.1.) *Given $k < f(n+1)$, where $n \geqslant 0$, there is an H-space splitting $\overline{\theta}_n\colon \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ of $\pi\langle n\rangle\colon \underline{BP}_k \to \underline{BP\langle n\rangle}_k$ which naturally embeds $BP\langle n\rangle^k(X) \subset BP^k(X)$ as a summand* (as abelian groups).

If also $k \geqslant f(n)$, the H-space $\underline{BP\langle n\rangle}_k$ does not decompose further.

REMARK. The splittings $\overline{\theta}_n$ are not canonical or unique. The ideal J_n, unlike I_n, is in no way canonical, but depends on the choice of the polynomial generators of BP^*. Although the BP-module structure of $BP\langle n\rangle$ obviously depends on J_n, it follows from the Lemma that the resulting H-space structure on $\underline{BP\langle n\rangle}_k$ is well defined. Even for fixed J_n, we find there are many choices for $\overline{\theta}_n$, and no preferred choice is apparent.

We establish Lemma 1.11 in Section 22 by constructing a suitable idempotent operation θ_n on $BP^*(-)$. The second assertion implies that the first is best possible. We insert these splittings into diag. (1.8) to decompose BP-cohomology.

THEOREM 1.12. *Assume $n \geqslant 0$. Then:*

(a) *For $k < f(n+1)$, the injections $\overline{\theta}_n$ and $v_j \circ \overline{\theta}_j$ from Lemma* 1.11 *induce the natural abelian group decomposition* (1.7), *which is maximal if $k \geqslant f(n)$;*

(b) *For $k = f(n+1)$, we have instead the natural short exact sequence of abelian groups*

$$0 \longrightarrow \prod_{j>n} BP\langle j\rangle^{k+2(p^j-1)}(X) \longrightarrow BP^k(X) \xrightarrow{\pi\langle n\rangle_*} BP\langle n\rangle^k(X) \longrightarrow 0, \quad (1.13)$$

where none of the groups decomposes further naturally, and $\pi\langle n\rangle_*$ admits a nonadditive *natural splitting $\overline{\theta}_n$: $BP\langle n\rangle^k(X) \to BP^k(X)$, so that we have eq.* (1.7) *as a bijection of sets.*

REMARK. The simplified description of $BP\langle -\rangle$-cohomology in Theorem 1.9 applies everywhere (when X is finite-dimensional). These splittings definitely do not preserve the BP^*-module structure. We plan to return to this point in future work.

PROOF OF THEOREM 1.9. For finite-dimensional X, the sum in eq. (1.10) is in fact finite. It is clear from eq. (1.7) or (1.13) that the sum contains $\operatorname{Ker} \pi\langle n\rangle_*$. On the other hand, $\pi\langle n\rangle_*$ is a homomorphism of BP^*-modules which kills J_n. □

Projection to the first factor of the product in eq. (1.7) yields an interesting operation

$$r: BP^k(X) \longrightarrow BP\langle n+1\rangle^{k'}(X) \subset BP^{k'}(X),$$

where $k' = k + 2(p^{n+1}-1) = k + (p-1)f(n+1)$, which roughly has the effect of dividing by v_{n+1}. Precisely, $r(v_{n+1}y) = y$ whenever $y \in BP\langle n+1\rangle^{k'}(X) \subset BP^{k'}(X)$. Given any $x \in BP^{k'}(X)$, we can put $y = \theta_{n+1}x$; then by Theorem 1.12(a), applied to $BP^{k'}(X)$, we have $y \equiv x \bmod J_{n+1}$. For convenience, we reindex.

PROPOSITION 1.14. *If $k \leqslant pf(n)$, there is an operation r: $BP^{k-2(p^n-1)}(X) \to BP^k(X)$, which is additive if $k < pf(n)$, with the property that given any element $x \in BP^k(X)$, where X is finite-dimensional, there exists y such that $y \equiv x \bmod J_n BP^*(X)$ and $r(v_n y) = y$.*

Equivalently, we can represent eq. (1.7) by the decomposition of spaces

$$\underline{BP}_k \simeq \underline{BP\langle n\rangle}_k \times \prod_{j>n} \underline{BP\langle j\rangle}_{k+2(p^j-1)}. \quad (1.15)$$

THEOREM 1.16. *Assume $n \geqslant 0$. Then:*

(a) *For* $k < f(n+1)$, *we have the H-space decomposition* (1.15), *which is maximal if* $k \geqslant f(n)$;

(b) *For* $k = f(n+1)$, *we have the fibration*

$$\prod_{j>n} \underline{BP\langle j\rangle}_{k+2(p^j-1)} \longrightarrow \underline{BP}_k \xrightarrow{\pi(n)} \underline{BP\langle n\rangle}_k \tag{1.17}$$

of H-spaces and H-maps, which admits a section (not an H-map), so that eq. (1.15) *holds as an equivalence of spaces (but not as H-spaces), and none of the spaces decomposes further as a product of spaces.* (In other words, $BP^k(-)$ is represented by the right side of eq. (1.15), equipped with a different H-space multiplication.)

We use Lemma 1.11 to prove parts (a) of Theorems 1.12 and 1.16 in Section 22. For parts (b), the necessary idempotent θ_n has to be nonadditive, and we construct it in Section 23. We need the full strength of our machinery just to prove that θ_n is idempotent.

History. Our real motivation for this study is what is called the Johnson Question, which is stated in [24, p. 745]. Rephrased as a conjecture, it is:

CONJECTURE. If $x \neq 0$ in $BP_n(X)$, where X is a space, then $v_n^i x \neq 0$ for all $i > 0$.

No counter-examples are known, although examples exist [13], [14], [24] where $v_j x = 0$ for all $j < n$. It holds if x reduces nontrivially to homology, therefore for $n < 2p$. We hoped to circumvent our lack of knowledge of unstable homology operations by working instead with the rather better understood unstable BP-cohomology operations and using the (not at all unstable) duality spectral sequence

$$\mathrm{Ext}^{**}_{BP^*}\left(BP^*(X), BP^*\right) \Longrightarrow BP_*(X)$$

of Adams [1] (see also [12]). The reason for optimism is that if we substitute $\Sigma^k(BP^*/I_n)$ for $BP^*(X)$, a standard calculation shows that the only surviving Ext group is $\mathrm{Ext}^n = \Sigma^m(BP^*/I_n)$, with $m = f(n) - k - n$; so that $k \geqslant f(n) - 1$ implies $-m \geqslant n-1$, almost what we want. If we confine ourselves to additive operations, we obtain $-m \geqslant n-2$, off by one more. We can hope to work our way up from $\Sigma^k(BP^*/I_n)$ to a general $BP^*(X)$ by extension and the filtration (1.2).

This is all grounds for our suspicion that for a *geometric* unstable algebra, i.e. $M = BP^*(X)$ for some space X, the bounds in Theorem 1.5 should be one better (thus giving us $-m \geqslant n$ in the above discussion). Again, there are no known counter-examples, although spaces are known which have $\deg(x_i) = f(n_i)$, thus showing that the bounds cannot be improved by more than one.

Recently, with the help of Mike Hopkins, a new approach to the Johnson Question has been developed. It requires a much better understanding of the unstable splittings of BP. Now that we have so much explicit information on these splittings, this method of attack seems promising.

Outline. There are two main threads running through this work: the theory of additive unstable operations, which closely resembles the stable theory of [8], and the theory of all unstable operations, which is radically different. The comonad tent is big enough to accommodate both, as well as the stable theory. We have kept the additive material in separate sections so that it can be read independently.

In Section 2, we discuss several classes of cohomology operation. In Sections 3 and 4, we study the E-(co)homology of group objects, in preparation for Sections 5 and 7, where we study modules and algebras from the additive point of view. In Section 6, we consider additive operations as linear functionals. In Sections 12 and 14, we study suspensions and complex orientation. In Section 16, we present the additive structure for each of our five examples E.

It turns out that much of the stable machinery does not extend to all unstable operations, because it relies too heavily on the bilinearity of tensor products. However, the approach in terms of comonads does work, and in Section 8 we develop the requisite comonad U. We also show in Section 9 that the corresponding comonad for unstable modules does not exist and compare the various stable and unstable structures. In Section 10, we convert the categorical elegance into machinery we can use; specifically, cohomology operations become linear functionals on Hopf rings. In Theorem 10.47, we display in full detail the definition of an unstable algebra from this point of view.

In Sections 11, 13, and 15, we revisit the cohomology of a point, sphere, and complex projective space $\mathbb{C}P^\infty$ from this new Hopf ring point of view. These spaces alone yield almost enough generators and relations to specify the Hopf rings for our five examples E, as we discuss in detail in Section 17. The case $E = KU$ is used to determine the structure of $KU_*(KU, o)$, as quoted in [8, §14]. From a sufficiently elevated perspective, the results of Section 17, the additive results of Section 16, and the stable results of [8] all fit into a grand master plan.

In Section 20, we restrict attention to the case $E = BP$ and use the additive operations to recover Quillen's theorem and prove Theorem 20.11. This relies on the relations developed in Section 18. In Section 21, we use nonadditive operations to improve Theorem 20.11 by one dimension to Theorem 21.12, which is Theorem 1.5.

In Section 22, we construct additive idempotent operations θ_n which yield the desired factorizations (1.7) in all except the top degree. In Section 23, we finish off Theorems 1.12 and 1.16 by constructing nonadditive idempotent operations. To do this, it is necessary in Section 19 to develop the notion of a Hopf ring ideal.

An index of symbols is included at the end.

This work is also notable for what it does *not* contain. There are no spectral sequences, except implicitly in the references. There are no explicit Steenrod operations, except in a few examples; in our wholesale approach, most individual operations never even acquire names. There are no formal indeterminates anywhere; the elements that are sometimes treated as such are really Chern classes x; but when $x^i = 0$, we can no longer take the coefficients of x^i.

Notation. We make heavy use of the notation and machinery developed in [8]. Topologically, we generally work in the homotopy category *Ho* of *unbased* spaces. For compatibility with the unstable notation, the E-cohomology and E-homology of a *spectrum* X are written $E^*(X, o)$ and $E_*(X, o)$. Algebraically, our most important categories are the categories *FMod* and *FAlg* of filtered E^*-modules and algebras. These and the other categories we need were introduced in [8, §6]. We make frequent use of Yoneda's Lemma. All tensor products are taken over E^* unless otherwise stated.

For reasons discussed in [8], we always give cohomology $E^*(X)$ the *profinite* topology [8, Definition 4.9], and complete it as in [8, Definition 4.11] to $E^*(X)^\wedge$ as necessary. In contrast, the homology $E_*(X)$ is always discrete. Because we emphasize cohomology, we invariably assign the degree i to elements of $E^i(X)$; this forces elements of $E_i(X)$ to have degree $-i$.

One theorem provides all the duality and Künneth isomorphisms we need.

THEOREM 1.18. *Assume that $E_*(X)$ is a free E^*-module. Then we have:*

(a) $d\colon E^*(X) \cong DE_*(X)$ *in FMod, the strong duality homeomorphism;*

(b) $E_*(X \times Y) \cong E_*(X) \otimes E_*(Y)$, *the Künneth isomorphism in homology;*

(c) $E^*(X \times Y) \cong E^*(X) \widehat{\otimes} E^*(Y)$ *in FMod, the Künneth homeomorphism in cohomology, provided $E_*(Y)$ is also a free E^*-module.*

PROOF. We collect Theorems 4.2, 4.14, and 4.19 from [8]. Indeed, (c) follows from (a) and (b). □

Acknowledgements. The genesis of this paper is that the last two authors had worked out much of the unstable *BP* structure theorems, without having a precise definition of unstable algebra, when the first author supplied a suitable framework, of which [7] is an early version. In fact, this is an oversimplification: the various contributions are more intermingled than this might suggest. In the proper context, several of the proofs simplify significantly. We thank Martin Bendersky for pointing out Lemma 19.32, which is vastly simpler than our previous treatment.

The last two authors wish to thank the topologists at the University of Manchester and the Science and Engineering Research Council (SERC) for support during the summer of 1985 when this project got its start. The last author wishes to thank Miriam and Harold Levy for their hospitality and the peace they provided for the writing of the first draft.

2. Cohomology operations

In this section, we consider several kinds of unstable cohomology operation. Yoneda's Lemma allows us to identify the following:

$$\begin{aligned}
&\text{(i) The } \textit{cohomology operation } r\colon E^k(-) \to E^m(-);\\
&\text{(ii) The } \textit{cohomology class } r = r(\iota_k) \in E^m(\underline{E}_k);\\
&\text{(iii) The representing } \textit{map } r\colon \underline{E}_k \to \underline{E}_m \text{ in } Ho.
\end{aligned} \tag{2.1}$$

We write any of these more succinctly as $r\colon k \to m$. We use all three interpretations. Some care is needed with degrees and signs, as (i) has degree $m-k$ and (ii) has degree m, while (iii) has no degree at all.

Based operations. The following mild but useful condition can be interpreted many ways. The space T is the one-point space.

DEFINITION 2.2. We call the operation r *based* if $r(0) = 0$ in $E^*(T) = E^*$.

LEMMA 2.3. *The following conditions on an operation $r: k \to m$ are equivalent:*

(a) $r(0) = 0$ *in* $E^*(T)$, *i.e.* r *is a based operation;*

(b) *For any based space* (X, o), r *restricts to the* reduced *operation*

$$r: E^k(X, o) \longrightarrow E^m(X, o); \tag{2.4}$$

(c) *As a cohomology class,* $r \in E^m(\underline{E}_k, o) \subset E^m(\underline{E}_k)$*;*

(d) *The map* $r: \underline{E}_k \to \underline{E}_m$ *is (homotopically) based.*

PROOF. The short exact sequence [8, (3.2)] shows that (a) and (c) are equivalent, also that (a) implies (b); but (c) is the special case of (b) for $\iota_k \in E^k(\underline{E}_k, o)$. Part (d) is just a restatement of (a). □

Given any (good) pair of spaces (X, A), we can use (b) to make based operations $r: k \to m$ act on *relative* cohomology as in [8, (3.4)] by

$$E^k(X, A) = E^k(X/A, o) \xrightarrow{r} E^m(X/A, o) = E^m(X, A). \tag{2.5}$$

Additive operations. An *additive* operation $r: k \to m$ is one that satisfies $r(x+y) = r(x) + r(y)$ for any $x, y \in E^k(X)$. The universal example is

$$X = \underline{E}_k \times \underline{E}_k, \quad \text{with } x = \iota_k \times 1,\ y = 1 \times \iota_k,\ x + y = \mu_k, \tag{2.6}$$

which gives $r(\mu_k) = r \times 1 + 1 \times r$ in $E^*(\underline{E}_k \times \underline{E}_k)$. (The addition map $\mu_k: \underline{E}_k \times \underline{E}_k \to \underline{E}_k$ was defined in [8, Theorem 3.6].) This allows us to recognize additive operations three ways.

PROPOSITION 2.7. *The following conditions on an operation $r: k \to m$ are equivalent, and define the E^*-submodule $PE^*(\underline{E}_k) \subset E^*(\underline{E}_k, o) \subset E^*(\underline{E}_k)$:*

(a) *The operation* $r: E^k(-) \to E^m(-)$ *is additive;*

(b) *The class* $r \in E^m(\underline{E}_k)$ *satisfies* $\mu_k^* r = p_1^* r + p_2^* r$ *in* $E^m(\underline{E}_k \times \underline{E}_k)$, *i.e.*

$$PE^*(\underline{E}_k) = \operatorname{Ker}\left[\mu_k^* - p_1^* - p_2^*: E^*(\underline{E}_k) \longrightarrow E^*(\underline{E}_k \times \underline{E}_k)\right]; \tag{2.8}$$

(c) *The map* $r: \underline{E}_k \to \underline{E}_m$ *is a morphism of group objects in Ho.*

COROLLARY 2.9. *Assume that $E_*(\underline{E}_k)$ is a free E^*-module. Then $PE^*(\underline{E}_k)$ is complete Hausdorff and so an object of FMod.*

PROOF. In eq. (2.8), $E^*(\underline{E}_k)$ and $E^*(\underline{E}_k \times \underline{E}_k)$ are complete Hausdorff by Theorem 1.18. □

When $E_*(\underline{E}_k)$ is free, the Künneth homeomorphism for $E^*(\underline{E}_k \times \underline{E}_k)$ makes $E^*(\underline{E}_k)$ a completed Hopf algebra; then (b) agrees with the primitives in the sense of [8, (6.13)], completed. However, we need no hypotheses on E to define $PE^*(\underline{E}_k)$.

On some spaces, all operations are additive.

LEMMA 2.10. *On the suspension ΣX of any based space (X, o), we have $r(x+y) = r(x) + r(y)$ in $E^m(\Sigma X, o)$ for any based operation $r: k \to m$ and any elements $x, y \in E^k(\Sigma X, o)$.*

PROOF. By [8, Lemma 7.6(c)], $r: E^k(\Sigma X, o) \to E^m(\Sigma X, o)$ preserves the group structure defined from the cogroup object ΣX in Ho'. By [8, Proposition 7.3], this structure coincides with the given E-cohomology addition. □

Products of operations. Given operations $r: k \to m$ and $s: k \to n$, the *product* operation

$$r \smile s: k \to m + n,$$

defined by

$$(r \smile s)x = (rx)(sx),$$

corresponds to the cup product in $E^*(\underline{E}_k)$, which may be constructed using the diagonal map

$$\Delta: \underline{E}_k \to \underline{E}_k \times \underline{E}_k.$$

We often wish to neglect such operations; if r and s are additive, $r \smile s$ is clearly not additive, but conveys no new information.

The map Δ, together with $q: \underline{E}_k \to T$, makes $\underline{E}_k$ a monoid object in the symmetric monoidal category $(Ho^{\mathrm{op}}, \times, T)$. We therefore dualize eq. (2.8) and introduce the quotient E^*-module

$$QE^*(\underline{E}_k) = \operatorname{Coker}\left[\Delta^* - i_1^* - i_2^*: E^*(\underline{E}_k \times \underline{E}_k) \longrightarrow E^*(\underline{E}_k)\right] \tag{2.11}$$

of *"indecomposables"* of $E^*(\underline{E}_k)$, where i_1 and i_2 are the inclusions (using the basepoint). (We shall not need a topology on this module.) When $E_*(\underline{E}_k)$ is a free E^*-module, we have by Theorem 1.18(c) a Künneth homeomorphism for $E^*(\underline{E}_k \times \underline{E}_k)$, and $QE^*(\underline{E}_k)$ is the quotient of $E^*(\underline{E}_k, o)$ by all finite (or infinite) sums of products of two based operations.

Looping of operations. On restriction to spaces, a stable operation r on $E^*(-, o)$ of degree h induces a sequence of additive operations $r_k: k \to k+h$. It is clear from [8, fig. 2 in §9] that r_{k+1} determines r_k. We generalize this construction to unstable operations (but omit the sign, in order to make it a homomorphism of E^*-modules).

PROPOSITION 2.12. *Given a based unstable operation $r: k \to m$, we can define the* looped operation $\Omega r: k-1 \to m-1$ *in any of three equivalent ways:*

(a) *The operation that makes the diagram commute (with no sign),*

$$\begin{tikzcd} E^{k-1}(X) \arrow[r, "\cong", "\Sigma"'] \arrow[d, dotted, "\Omega r"] & E^k(S^1 \times X, o \times X) \arrow[d, "r"] & E^k(\Sigma(X^+), o) \arrow[l, "\cong"'] \arrow[d, "r"] \\ E^{m-1}(X) \arrow[r, "\cong", "\Sigma"'] & E^m(S^1 \times X, o \times X) & E^m(\Sigma(X^+), o) \arrow[l, "\cong"'] \end{tikzcd} \tag{2.13}$$

which we can express algebraically as

$$\Sigma(\Omega r)x = r\Sigma x; \tag{2.14}$$

(b) *The image of r under the E^*-module homomorphism*

$$\Omega: E^m(\underline{E}_k, o) \xrightarrow{(-1)^{k-1} f_{k-1}^*} E^m(\Sigma \underline{E}_{k-1}, o) \cong E^{m-1}(\underline{E}_{k-1}, o)$$

induced by the structure map $f_{k-1}: \Sigma \underline{E}_{k-1} \to \underline{E}_k$ of [8, Definition 3.19];

(c) *The map*

$$\Omega r: \underline{E}_{k-1} \simeq \Omega \underline{E}_k \xrightarrow{(-1)^{m-k}\Omega r} \Omega \underline{E}_m \simeq \underline{E}_{m-1},$$

where we use the right adjunct equivalences to f_{k-1} and f_{m-1}.

PROOF. For a based space X, diag. (2.13) simplifies by naturality to

$$\begin{tikzcd} E^{k-1}(X, o) \arrow[r, "\cong", "\Sigma"'] \arrow[d, dotted, "\Omega r"] & E^k(\Sigma X, o) \arrow[d, "r"] \\ E^{m-1}(X, o) \arrow[r, "\cong", "\Sigma"'] & E^m(\Sigma X, o) \end{tikzcd} \tag{2.15}$$

If we evaluate on the universal case $\iota_{k-1} \in E^{k-1}(\underline{E}_{k-1}, o)$ by eq. (2.14), we find

$$\Sigma(\Omega r)\iota_{k-1} = r\Sigma\iota_{k-1} = (-1)^{k-1} r f_{k-1}^* \iota_k = (-1)^{k-1} f_{k-1}^* r,$$

which gives (b). Further, by [8, Lemma 3.21], the class $\Sigma(\Omega r)\iota_{k-1} \in E^*(\Sigma \underline{E}_{k-1}, o)$

corresponds, up to the sign $(-1)^{m-1}$, to the lower route in the square

$$\begin{tikzcd} \Sigma\underline{E}_{k-1} \arrow[r, "f_{k-1}"] \arrow[d, "\Sigma(\Omega r)"'] \arrow[dr, phantom, "(-1)^{m-k}"] & \underline{E}_k \arrow[d, "r"] \\ \Sigma\underline{E}_{m-1} \arrow[r, "f_{m-1}"] & \underline{E}_m \end{tikzcd} \quad \text{in } Ho \tag{2.16}$$

which therefore commutes up to sign. We take adjuncts of this to get (c). □

We recall from [8, Definition 9.3] the stabilization map $\sigma_k: \underline{E}_k \to E$ of spectra.

COROLLARY 2.17. $\Omega \circ \sigma_k^* = \sigma_{k-1}^*: E^*(E, o) \to E^*(\underline{E}_{k-1}, o)$.

PROOF. Suppose the stable operation $r \in E^h(E, o)$ restricts to give the additive operations $r_k: k \to k + h$ and $r_{k-1}: k-1 \to k+h-1$. By [8, (9.8)], $\sigma_k^* r = (-1)^{kh} r_k$ and $\sigma_{k-1}^* r = (-1)^{(k-1)h} r_{k-1}$. We compare diag. (2.16) with [8, (9.2)] to see that $\Omega r_k = (-1)^h r_{k-1}$. □

COROLLARY 2.18. *The loop construction in Proposition* 2.12(b) *factors as*

$$\Omega: E^*(\underline{E}_k, o) \longrightarrow QE^*(\underline{E}_k) \longrightarrow PE^*(\underline{E}_{k-1}) \subset E^*(\underline{E}_{k-1}, o). \tag{2.19}$$

PROOF. It is clear from Proposition 2.12(c), or from eq. (2.14) and Lemma 2.10, that Ωr is always additive. The construction factors through $QE^*(\underline{E}_k)$ by Proposition 2.12(b) and naturality of Q, since $QE^*(\Sigma\underline{E}_{k-1}) \cong E^*(\Sigma\underline{E}_{k-1}, o)$. (Loosely, there are no products in $E^*(\Sigma X, o)$.) □

These results allow us to rewrite the Milnor short exact sequence [8, (9.7)] in the more useful form (which does not change any terms)

$$0 \longrightarrow \lim_k{}^1 PE^*(\underline{E}_k) \longrightarrow E^*(E, o) \longrightarrow \lim_k PE^*(\underline{E}_k) \longrightarrow 0. \tag{2.20}$$

It remains true that the projection from $E^*(E, o)$ is an open map, and therefore a homeomorphism whenever it is a bijection. The kth component is the E^*-module homomorphism

$$\sigma_k^*: E^*(E, o) \longrightarrow PE^*(\underline{E}_k) \subset E^*(\underline{E}_k) \tag{2.21}$$

induced by the stabilization map σ_k. It sends a stable operation r to the induced additive operation r_k on $E^k(-)$ (but with a sign; see [8, (9.9)]).

The factorization (2.19) raises two obvious questions:

(a) Can every additive operation be delooped?
(b) Does $\Omega r = 0$ imply that r decomposes? (2.22)

Both hold precisely when we have an isomorphism $\Omega: QE^*(\underline{E}_k) \cong PE^*(\underline{E}_{k-1})$. We discuss this further in Section 4.

3. Group objects and E-cohomology

Before we can discuss additive E-cohomology operations adequately, it is necessary to generalize Section 2. We extend Proposition 2.7 by defining the primitives $PE^*(X)$ for any group object X in the homotopy category *Ho*. Dually, we extend the definition of the indecomposables $QE^*(X)$ to any based space X.

Coalgebra primitives. We start from the definition (2.8) of $PE^*(\underline{E}_k)$.

DEFINITION 3.1. Given any group object (or H-space) X in *Ho*, with multiplication $\mu: X \times X \to X$, we define the E^*-submodule $PE^*(X)$ of *coalgebra primitives* in $E^*(X)$ as

$$PE^*(X) = \{x \in E^*(X) : \mu^* x = p_1^* x + p_2^* x \text{ in } E^*(X \times X)\}.$$

REMARK. As in Proposition 2.7(c), the class $x \in E^k(X)$ is primitive if and only if the associated map $x: X \to \underline{E}_k$ is a morphism of group objects in *Ho*.

We note that $PE^*(X)$ is defined even if $E^*(X)$ is not a (completed) coalgebra. Thus $PE^*(-): \mathit{Gp}(\mathit{Ho})^{\mathrm{op}} \to \mathit{Mod}$ is a functor defined on the dual of the category of group objects in *Ho*. We topologize $PE^*(X)$ as a subspace of $E^*(X)$.

If Y is another group object in *Ho*, we construct the product group object $X \times Y$ in the obvious way. The one-point space T is trivially a group object, and is terminal in $\mathit{Gp}(\mathit{Ho})$. Lemma 6.14 of [8] carries over to this situation.

LEMMA 3.2. *For the product $X \times Y$ of two group objects X and Y in Ho, we have $PE^*(X \times Y) \cong PE^*(X) \oplus PE^*(Y)$ in FMod. Also, $PE^*(T) = 0$.*

In other words, the functor $PE^*(-)$ takes finite products in $\mathit{Gp}(\mathit{Ho})$ to coproducts (direct sums) in FMod.

REMARK. No Künneth formula is needed for this result.

PROOF. We dualize the proof of [8, Lemma 6.11]. Let us write $Z = X \times Y$ for the product group object and $\omega_Y: T \to Y$ for the unit (or zero) map of Y. We note first that the maps $j_1 = 1_X \times \omega_Y: X \cong X \times T \to X \times Y = Z$, $j_2: Y \to Z$ (defined similarly), $p_1: Z = X \times Y \to X$, and $p_2: Z \to Y$ are all morphisms of group objects and therefore send primitives to primitives. Define the map

$$f: Z = X \times Y \cong (X \times T) \times (T \times Y) \longrightarrow (X \times Y) \times (X \times Y) = Z \times Z$$

using $(1_X \times \omega_Y) \times (\omega_X \times 1_Y)$. Then $\mu_Z \circ f = 1_Z$ and $P_s \circ f = j_s \circ p_s$ (for $s = 1, 2$), where $P_s: Z \times Z \to Z$ denotes the projection for Z. Any element $z \in PE^*(Z)$ satisfies

$\mu_Z^* z = P_1^* z + P_2^* z$, by definition. When we apply f^*, we obtain $z = p_1^* x + p_2^* y$, where $x = j_1^* z \in E^*(X)$ and $y = j_2^* z \in E^*(Y)$ must be primitive. Conversely, any primitives x and y determine a primitive z by this formula. We have a homeomorphism because j_s^* and p_s^* are continuous.

We compute $PE^*(T) = \{v \in E^* : v = v + v\} = 0$. □

Since the unit map $\omega: T \to X$ of X is a morphism of group objects, $PE^*(T) = 0$ implies that $PE^*(X) \subset E^*(X, o)$.

The space $\underline{E}_k$ is more than just a group object. By [8, Corollary 7.8], we have the E^*-module object $n \mapsto \underline{E}_n$ in *Ho*, on which $v \in E^h$ acts by the maps $\xi v: \underline{E}_k \to \underline{E}_{k+h}$ that represent scalar multiplication by v. Clearly, ξv is additive.

LEMMA 3.3. *Assume that $E^*(\underline{E}_k)$ is Hausdorff for all k. Then:*

(a) *We have the E^*-module object $n \mapsto PE^*(\underline{E}_n)$ in the ungraded category $FMod^{op}$, with the action of $v \in E^h$ given by $P(\xi v)^*: PE^*(\underline{E}_{k+h}) \to PE^*(\underline{E}_k)$;*

(b) *The object in (a) is related to the stable E^*-module object $E^*(E, o)$ of* [8, Proposition 11.3] *by the following diagram, which commutes up to sign for any $v \in E^h$,*

$$\begin{array}{ccc} E^*(E,o) & \xrightarrow{(\xi v)^*} & E^*(E,o) \\ \downarrow{\scriptstyle \sigma_{k+h}^*} & (-1)^{hk} & \downarrow{\scriptstyle \sigma_k^*} \\ PE^*(\underline{E}_{k+h}) & \xrightarrow{P(\xi v)^*} & PE^*(\underline{E}_k) \end{array} \tag{3.4}$$

PROOF. In (a), the object $n \mapsto \underline{E}_n$ is in fact an E^*-module object in $Gp(Ho)$. We apply [8, Lemma 7.6(a)] to the functor $PE^*(-)$; it preserves finite products by Lemma 3.2.

For (b), we apply E-cohomology to diag. [8, (9.8)], taking $r = \xi v$. □

Indecomposables. Dually, we extend eq. (2.11) to any based space X by defining the quotient E^*-module

$$QE^*(X) = \operatorname{Coker}\left[\Delta^* - i_1^* - i_2^*: E^*(X \times X) \longrightarrow E^*(X)\right] \tag{3.5}$$

of *"indecomposables"* of $E^*(X)$. (We shall not need a topology on it.)

4. Group objects and E-homology

We dualize Section 2 by defining the indecomposables $QE_*(\underline{E}_k)$ and primitives $PE_*(\underline{E}_k)$ in E-homology. This will prove useful because $E_*(\underline{E}_k)$ is usually smaller and more manageable than $E^*(\underline{E}_k)$. As in Section 3, we need to handle more general X. However, some properties that were immediate in Section 2 become less intuitive and have to be proved.

The structure map $f_k\colon \Sigma\underline{E}_k \to \underline{E}_{k+1}$ (see [8, Definition 3.19]) of the spectrum E induces the important *suspension* homomorphism

$$E_*(\underline{E}_k) \longrightarrow E_*(\underline{E}_k, o) \cong E_*(\Sigma\underline{E}_k, o) \xrightarrow{f_{k*}} E_*(\underline{E}_{k+1}, o), \tag{4.1}$$

dual (apart from sign) to the looping Ω in Proposition 2.12(b). Again, suspended elements behave better. We dualize Lemma 2.10.

LEMMA 4.2. *For any elements $x, y \in E^k(\Sigma X, o)$, the induced E-homology homomorphisms satisfy*

$$(x+y)_* = x_* + y_*\colon E_*(\Sigma X, o) \longrightarrow E_*(\underline{E}_k, o).$$

PROOF. By [8, Lemma 7.6(c)], E-homology induces a homomorphism

$$\mathit{Ho}'(\Sigma X, \underline{E}_k) \longrightarrow \mathit{Mod}\big(E_*(\Sigma X, o), E_*(\underline{E}_k, o)\big)$$

of groups, where both group structures are induced by the cogroup structure on ΣX in Ho'. By [8, Proposition 7.3], they agree with the obvious group structures. □

Indecomposables. We dualize Definition 3.1.

DEFINITION 4.3. Given any group object (or H-space) X in Ho, we define the *E^*-module $QE_*(X)$ of "indecomposables"* of $E_*(X)$ as

$$QE_*(X) = \operatorname{Coker}\big[\mu_* - p_{1*} - p_{2*}\colon E_*(X\times X) \longrightarrow E_*(X)\big].$$

It comes equipped with a canonical projection $E_*(X) \to QE_*(X)$.

When $E_*(X)$ is free, we have the Künneth isomorphism Theorem 1.18(b) for $E_*(X\times X)$ and this agrees with the usual definition for the algebra $E_*(X)$. We need one easy example.

LEMMA 4.4. *Let G be a discrete abelian group. Then $QE_*(G) \cong E^* \otimes_{\mathbb{Z}} G$ as an E^*-module.*

PROOF. We recognize $E_*(G)$ as the group algebra of G over E^*, with an E^*-basis element $[g]$ for each $g \in G$. The correspondence we seek is induced by $v[g] \leftrightarrow v \otimes g$, and is well defined in both directions. □

Lemma 3.2 dualizes without difficulty; again, no Künneth formula is needed. Then we will be able to dualize Lemma 3.3.

LEMMA 4.5. *For the product $X \times Y$ of two group objects X and Y in Ho, we have $QE_*(X\times Y) \cong QE_*(X) \oplus QE_*(Y)$. Also, $QE_*(T) = 0$. In other words, the functor $QE_*(-)\colon \mathit{Gp}(\mathit{Ho}) \to \mathit{Mod}$ preserves finite products.*

We have an immediate application to the Hopf bundle.

LEMMA 4.6. *Assume E has a complex orientation. Then the inclusion $\mathbb{C}P^\infty \to \mathbb{Z} \times BU$ (see [8, (5.8)]) defined by the Hopf line bundle ξ over $\mathbb{C}P^\infty$ induces an isomorphism of E^*-modules*

$$E_*(\mathbb{C}P^\infty) \xrightarrow{\cong} QE_*(\mathbb{Z} \times BU) \cong E^* \oplus QE_*(BU).$$

PROOF. The second isomorphism comes from Lemmas 4.5 and 4.4. We compare Lemmas 5.4 and 5.6 of [8]; the generators β_i correspond, except that $\beta_0 \mapsto (1,0)$. □

LEMMA 4.7. *For any ring spectrum E:*

(a) *$n \mapsto QE_*(\underline{E}_n)$ is an E^*-module object in the ungraded category Mod of E^*-modules;*

(b) *The suspension* (4.1) *factors through $QE_*(\underline{E}_k)$;*

(c) *The stabilization $\sigma_{k*}: E_*(\underline{E}_k, o) \to E_*(E, o)$ factors through $QE_*(\underline{E}_k)$.*

PROOF. The proof of (a) is like Lemma 3.3(a), except that we use the functor $QE_*(-)$ and Lemma 4.5.

For (c), we use $\sigma_k^* \iota = \iota_k$ to restate the universal example (2.6) as

$$\sigma_k \circ \mu_k = \sigma_k \circ p_1 + \sigma_k \circ p_2 : \underline{E}_k \times \underline{E}_k \longrightarrow E \quad \text{in } \mathit{Stab}^*.$$

We apply E-homology to see that σ_{k*} factors as desired. Similarly for (b), except that we use Lemma 4.2 with $X = \Sigma(\underline{E}_k \times \underline{E}_k)$, $x = \Sigma p_1$, and $y = \Sigma p_2$. □

Dually to the short exact sequence (2.20), we may use (b) and (c) to rewrite [8, (9.22)] in the more convenient form

$$E_*(E, o) = \operatorname*{colim}_k E_*(\underline{E}_k, o) = \operatorname*{colim}_k QE_*(\underline{E}_k). \tag{4.8}$$

There is a multiplication, analogous to the stable multiplication on $E_*(E, o)$.

LEMMA 4.9. *There is a bilinear multiplication*

$$Q\phi: QE_*(\underline{E}_k) \otimes QE_*(\underline{E}_m) \longrightarrow QE_*(\underline{E}_{k+m}),$$

which may be defined as a quotient of

$$E_*(\underline{E}_k) \otimes E_*(\underline{E}_m) \xrightarrow{\times} E_*(\underline{E}_k \times \underline{E}_m) \xrightarrow{\phi_*} E_*(\underline{E}_{k+m}).$$

PROOF. The only difficulty is to prove that $Q\phi$ is well defined. We express the distributive law for the E^*-algebra object $n \mapsto \underline{E}_n$ as the commutative square

$$\begin{array}{ccc} \underline{E}_k \times \underline{E}_k \times \underline{E}_m & \xrightarrow{\phi_L} & \underline{E}_{k+m} \times \underline{E}_{k+m} \\ \Big\downarrow{\scriptstyle f\times 1} & & \Big\downarrow{\scriptstyle g} \\ \underline{E}_k \times \underline{E}_m & \xrightarrow{\phi} & \underline{E}_{k+m} \end{array} \tag{4.10}$$

in which $f = \mu_k$, $g = \mu_{k+m}$, and ϕ_L has the components $\phi\circ(p_1 \times 1)$ and $\phi\circ(p_2 \times 1)$. (Cohomologically, ϕ_L represents the operation $(x, y, z) \mapsto (xz, yz)$.) We deduce the commutative diagram in homology

$$\begin{array}{ccccc} E_*(\underline{E}_k \times \underline{E}_k) \otimes E_*(\underline{E}_m) & \xrightarrow{\times} & E_*(\underline{E}_k \times \underline{E}_k \times \underline{E}_m) & \xrightarrow{\phi_{L*}} & E_*(\underline{E}_{k+m} \times \underline{E}_{k+m}) \\ \Big\downarrow{\scriptstyle f_*\otimes 1} & & \Big\downarrow{\scriptstyle (f\times 1)_*} & & \Big\downarrow{\scriptstyle g_*} \\ E_*(\underline{E}_k) \otimes E_*(\underline{E}_m) & \xrightarrow{\times} & E_*(\underline{E}_k \times \underline{E}_m) & \xrightarrow{\phi_*} & E_*(\underline{E}_{k+m}) \end{array} \tag{4.11}$$

By Definition 4.3, we have the exact sequence

$$E_*(\underline{E}_k \times \underline{E}_k) \xrightarrow{\mu_{k*} - p_{1*} - p_{2*}} E_*(\underline{E}_k) \longrightarrow QE_*(\underline{E}_k) \longrightarrow 0.$$

After tensoring with $E_*(\underline{E}_m)$, this remains exact. We note that diag. (4.10) and hence diag. (4.11) also commute if we take $f = p_1$ and $g = p_1$, or $f = p_2$ and $g = p_2$. Then diag. (4.11), with these three choices for f and g, shows that its bottom row induces a quotient pairing $QE_*(\underline{E}_k) \otimes E_*(\underline{E}_m) \to QE_*(\underline{E}_{k+m})$.

A second similar step, on the right, uses this pairing to produce $Q\phi$. □

Coalgebra primitives. We also dualize eq. (3.5) in the obvious way. If X is a based space, we construct the E^*-module homomorphism

$$\Delta_* - i_{1*} - i_{2*} \colon E_*(X) \longrightarrow E_*(X \times X). \tag{4.12}$$

DEFINITION 4.13. Given any based space X, we define the *E^*-submodule of coalgebra primitives* $PE_*(X) = \operatorname{Ker}[\Delta_* - i_{1*} - i_{2*}] \subset E_*(X)$.

Again, the definition is meaningful even without a Künneth formula for $E_*(X \times X)$. The companion result to Lemma 4.4 is elementary.

PROPOSITION 4.14. *For any discrete based space X, we have $PE_*(X) = 0$.*

The suspension (4.1) factors, with the help of Lemma 4.7(b), as

$$E_*(\underline{E}_k, o) \longrightarrow QE_*(\underline{E}_k) \longrightarrow PE_*(\underline{E}_{k+1}) \subset E_*(\underline{E}_{k+1}, o). \tag{4.15}$$

Again we ask whether $QE_*(\underline{E}_k) \to PE_*(\underline{E}_{k+1})$ is an isomorphism.

Duality. Under reasonable assumptions, the sequence (2.19) is dual to (4.15). One can see from Lemma 4.17 and Section 17 that this holds for each of our five examples E. Moreover, in each case there are isomorphisms $QE_*(\underline{E}_k) \cong PE_*(\underline{E}_{k+1})$ in (4.15), thus answering the questions (2.22) affirmatively.

LEMMA 4.16. *Assume that $E_*(X)$ is a free E^*-module.*

(a) *If X is a group object in Ho (or an H-space), then d induces a homeomorphism $d\colon PE^*(X) \cong DQE_*(X)$ in FMod;*

(b) *If X is a based space and the image of the homomorphism* (4.12) *splits off both $E_*(X)$ and $E_*(X\times X)$, then d induces a bijection $d\colon QE^*(X) \cong DPE_*(X)$.*

PROOF. In (a), d induces the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & PE^*(X) \arrow[r, "\subset"] \arrow[d, "d"] & E^*(X) \arrow[r, "\mu^* - p_1^* - p_2^*"] \arrow[d, "d"] & E^*(X\times X) \arrow[d, "d"] \\
0 \arrow[r] & DQE_*(X) \arrow[r] & DE_*(X) \arrow[r, "D(\mu_* - p_{1*} - p_{2*})"] & DE_*(X\times X)
\end{tikzcd}$$

whose rows are exact by Definitions 3.1 and 4.3, because D automatically takes cokernels to kernels. Strong duality for X and $X \times X$ from Theorem 1.18 provides two homeomorphisms d. The third d is therefore also a homeomorphism, because $DQE_*(X)$ has the subspace topology from $DE_*(X)$ by [8, Lemma 6.15(c)].

The proof of (b) is analogous, except that we assume the splittings to ensure that the bottom row of the relevant diagram is (split) exact, use [8, Lemma 6.15(a)] instead, and have no topology to check. □

We clearly need information on when $E_*(\underline{E}_k)$ is free.

LEMMA 4.17. *For $E = H(\mathbf{F}_p)$, BP, MU, $K(n)$, or KU:*

(a) *$E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k;*

(b) *$E^*(\underline{E}_k)$ and $PE^*(\underline{E}_k)$ are complete Hausdorff for all k.*

PROOF. For $E = H(\mathbf{F}_p)$ or $K(n)$, all E^*-modules are free and (a) is trivial.

We consider the remaining three cases together. For odd k, $E_*(\underline{E}_k)$ is an exterior algebra over E^* by [23] (for BP or MU) or [8, Corollary 5.12] (for KU, when $\underline{E}_k = U$), and (a) is clear.

For even k, we write $\underline{E}_k = E^k \times \underline{E}'_k$ as in [8, (3.7)], where $\underline{E}'_k$ denotes the zero component and E^k is treated as a discrete group. Then $E_*(\underline{E}'_k)$ is a polynomial algebra over E^*, by [23] (for BP or MU) or [8, Lemma 5.6(c)] (for KU, when $\underline{E}'_k = BU$), so that $E_*(\underline{E}'_k)$ (and hence $E_*(\underline{E}_k)$) and $QE_*(\underline{E}'_k)$ are free modules.

To finish (a), we note that by Lemmas 4.5 and 4.4,

$$QE_*(\underline{E}_k) = \left(E^* \otimes_{\mathbf{Z}} E^k\right) \oplus QE_*(\underline{E}'_k).$$

The first summand is free, because $E^k = \mathbb{Z}$ (for KU), or is $\mathbb{Z}$-free (for MU), or is $\mathbb{Z}_{(p)}$-free (for BP).

Part (b) is immediate from (a) by Theorem 1.18(a) and Corollary 2.9. □

5. What is an additively unstable module?

In this section, we give various interpretations of what it means to have a module over the additive unstable operations on E-cohomology. All four stable answers in [8] generalize.

We recall from [8, Corollary 7.8] that each $\underline{E}_k$ is an abelian group object in *Ho* and therefore also in *Gp*(*Ho*), and that $n \mapsto \underline{E}_n$ is an E^*-module object in *Ho*, with $v \in E^h$ acting by the map $\xi v: \underline{E}_k \to \underline{E}_{k+h}$. From Proposition 2.7 we have the submodule $PE^*(\underline{E}_k)$ of additive operations defined on $E^k(-)$.

We assume throughout that $E_*(\underline{E}_k)$ is a free E^*-module. Then by Corollary 2.9, $PE^*(\underline{E}_k)$ is complete Hausdorff and an object of *FMod*.

First Answer. The additive operations $r: k \to m$ act on $E^*(X)$ by composition

$$\circ: PE^m(\underline{E}_k) \times E^k(X) \longrightarrow E^m(X) \tag{5.1}$$

in *Ho*. We recover the stable action [8, (10.1)] by using $\sigma_k^*: E^*(E, o) \to PE^*(\underline{E}_k)$.

This composition is already biadditive. Given $x \in E^k(X)$ and $v \in E^h$, the commutative square

$$\begin{tikzcd} PE^m(\underline{E}_{k+h}) \times E^k(X) \arrow[r, "1\times v"] \arrow[d, "P(\xi v)^* \times 1"'] & PE^m(\underline{E}_{k+h}) \times E^{k+h}(X) \arrow[d, "\circ"] \\ PE^m(\underline{E}_k) \times E^k(X) \arrow[r, "\circ"] & E^m(X) \end{tikzcd} \tag{5.2}$$

expresses the identity $(r\cdot v)x = rvx = r(vx)$ for operations $r: k+h \to m$. It suggests that we should make the action (5.1) more closely resemble the stable action by introducing a formal shift and rewriting it with a tensor product as

$$\lambda_X: \Sigma^{-k} PE^m(\underline{E}_k) \otimes_k E^k(X) \longrightarrow E^m(X). \tag{5.3}$$

(Here, unlike [8], the action scheme is clearly visible: the notation $\otimes_k$ indicates that the tensor product is to be formed using the two E^*-actions indexed by k.)

This approach was initiated in [27, §11]. However, it presents even more problems than in the stable case, and we do not pursue it further here.

Second Answer. Our hypotheses ensure that $PE^*(\underline{E}_k)$ is dual to $QE_*(\underline{E}_k)$. We can convert the action of $PE^*(\underline{E}_k)$ into a coaction

$$E^k(X) \longrightarrow E^*(X) \widehat{\otimes} QE_*(\underline{E}_k).$$

These are clearly not the components of an E^*-module homomorphism, because the degree varies.

In Section 6, as suggested by (5.3), we shall shift degrees by introducing $Q(E)_*^k = \Sigma^k QE_*(\underline{E}_k)$, which will allow us to write the coaction as an E^*-module homomorphism with components

$$\rho_X: E^k(X) \longrightarrow E^*(X) \widehat{\otimes} Q(E)_*^k \tag{5.4}$$

and the same action scheme as stably. We shall construct a comultiplication $Q(\psi)$ and counit $Q(\varepsilon)$ that make $Q(E)_*^*$ a coalgebra and allow us to interpret $E^*(X)$ as a $Q(E)_*^*$-comodule.

Third Answer. We write our Second Answer more functorially. Given any E^*-module M, we construct the graded group $A'M$ having the component

$$\left(A'M\right)^k = M^i \widehat{\otimes}_i Q(E)_i^k = \left(M \widehat{\otimes} Q(E)_*^k\right)^k$$

in degree k. In Section 6 we shall make $A'M$ an E^*-module. Then $M \otimes Q(\psi)$ and $M \otimes Q(\varepsilon)$ define natural transformations $\psi': A' \to A'A'$ and $\varepsilon': A' \to I$, which will make A' a comonad in FMod and $E^*(X)\hat{}$ an A'-coalgebra.

Fourth Answer. Still imitating the stable case, we eliminate all tensor products by converting the First Answer to adjoint form. This will make everything very much cleaner, evidence that this is the natural answer (although the Second Answer is undeniably convenient for computation).

Any element $x \in E^k(X)$ may be regarded as a map $x: X \to \underline{E}_k$, which induces the morphism $x^*: E^*(\underline{E}_k) \to E^*(X)\hat{}$ in FMod. Generally, given any object M in FMod, we define for each integer k the abelian group

$$A^k M = \mathsf{FMod}\left(PE^*(\underline{E}_k), M\right) \tag{5.5}$$

of all continuous E^*-module homomorphisms $PE^*(\underline{E}_k) \to M$. (There is no need to shift degrees.) Then we convert the action (5.1) to the coaction

$$\rho_X: E^k(X) \longrightarrow A^k\left(E^*(X)\hat{}\right) = \mathsf{FMod}\left(PE^*(\underline{E}_k), E^*(X)\hat{}\right) \tag{5.6}$$

by defining $\rho_X x = x^* | PE^*(\underline{E}_k)$.

We assemble the $A^k M$, as k varies, to form the graded group AM with components $(AM)^k = A^k M$, and the coactions ρ_X into the single homomorphism $\rho_X: E^*(X) \to A(E^*(X)\hat{})$ of graded groups of degree zero.

The destabilization $\sigma_k^*: E^*(E, o) \to PE^*(\underline{E}_k)$ (see [8, Definition 9.3]) induces

$$A^k M = \mathsf{FMod}\left(PE^*(\underline{E}_k), M\right) \longrightarrow \mathsf{FMod}^k\left(E^*(E, o), M\right) = (SM)^k, \tag{5.7}$$

if we also assume that $E^*(E,o)$ is Hausdorff. As k varies, we take these as the components of the *stabilization* natural transformation $\sigma M: AM \to SM$, of degree zero. It allows us to compare with the stable case.

THEOREM 5.8. *Assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k* (as is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *We can make the functor A, defined in eq.* (5.5), *a comonad in the category FMod of complete Hausdorff filtered E^*-modules;*

(b) *If $E^*(E,o)$ is also Hausdorff, the stabilization $\sigma: A \to S$* (defined in eq. (5.7)) *is a morphism of comonads in FMod.*

The relevant definitions are now clear.

DEFINITION 5.9. An *additively unstable (E-cohomology) module* is an A-coalgebra in *FMod*, i.e. a complete Hausdorff filtered E^*-module M equipped with a morphism $\rho_M: M \to AM$ in *FMod* that satisfies the coaction axioms [8, (8.7)]. We then define the action of $r \in PE^m(\underline{E}_k)$ on $x \in M^k$ by $rx = \rho_M(x)r \in M$ (with no sign).

A closed submodule $L \subset M$ is called (*additively unstably*) *invariant* if ρ_M restricts to give $\rho_L: L \to AL$. Then the quotient M/L inherits an additively unstable module structure.

This is a stronger structure than a stable module (when $E^*(E,o)$ is Hausdorff, so that stable modules exist). Given a coaction ρ_M as above, Theorem 5.8(b) shows that the coaction

$$M \xrightarrow{\rho_M} AM \xrightarrow{\sigma M} SM \tag{5.10}$$

makes M a stable module.

One may think of A^kM as the set of all candidates for the action of $PE^*(\underline{E}_k)$ on a typical element of M^k, and ρ_M as the selection of a candidate for each $x \in M^k$. The coaction axioms translate into the usual action axioms $(sr)x = s(rx)$ and $\iota_k x = x$. As stably, it is sometimes useful to fix $r: k \to m$ and express the first axiom as the commutative square

$$\begin{tikzcd} M^k \arrow[r, "r"] \arrow[d, "\rho_M"] & M^m \arrow[d, "\rho_M"] \\ A^kM \arrow[r, "\omega_r M"] & A^mM \end{tikzcd} \tag{5.11}$$

where $\omega_r M$ denotes composition with $Pr^*: PE^*(\underline{E}_m) \to PE^*(\underline{E}_k)$.

THEOREM 5.12. *Assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k* (as is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) ρ_X (defined in eq. (5.6)) *factors through $E^*(X)\hat{}$ as $\rho_X: E^*(X)\hat{} \to A(E^*(X)\hat{})$ to make $E^*(X)\hat{}$ an additively unstable module for any space X;*

(b) *If $E^*(E,o)$ is Hausdorff, we recover the stable coaction in* [8, Theorem 10.16(a)] *from ρ_X by diag.* (5.10)*;*

(c) *ρ is universal: given an object N of FMod and an integer k, any additive natural transformation of abelian groups $\theta X: E^k(X) \to FMod(N, E^*(X)\hat{})$ (or $\widehat{\theta}X: E^k(X)\hat{} \to FMod(N, E^*(X)\hat{})$) that is defined on all spaces X is induced from ρ_X by a unique morphism $f: N \to PE^*(\underline{E}_k)$ in FMod, as*

$$\theta X: E^k(X) \xrightarrow{\rho_X} A^k\big(E^*(X)\hat{}\big) = FMod\big(PE^*(\underline{E}_k), E^*(X)\hat{}\big) \xrightarrow{\mathrm{Hom}(f,1)} FMod\big(N, E^*(X)\hat{}\big).$$

PROOF OF THEOREMS 5.8 AND 5.12. We prove parts (a) and (b) of both Theorems together, in the same seven steps as the stable proof of Theorems 10.12 and 10.16 of [8]. As most steps are more or less repetitions of that proof, except for the insertion of indices everywhere, we indicate only the substantive changes for (a) and the additions needed to handle σ for the (b) parts. Instead of $\iota \in E^*(E,o)$, we have $\iota_k \in PE^*(\underline{E}_k)$. Instead of id_A, we have the identity map $\mathrm{id}_k: PE^*(\underline{E}_k) = PE^*(\underline{E}_k)$, considered as an element of $A^k PE^*(\underline{E}_k)$. We write ρ_k for ρ_X when $X = \underline{E}_k$.

Step 1. We construct an E^*-module structure on the graded group AM we defined in eq. (5.5). We start with the E^*-module object $n \mapsto PE^*(\underline{E}_n)$ in $FMod^{op}$ from Lemma 3.3(a), with $v \in E^*$ acting by $P(\xi v)^*$. We apply the additive functor $\mathrm{Mor}(-, M): FMod^{op} \to Ab$ to obtain by [8, Lemma 7.6(a)] the E^*-module object $n \mapsto A^n M$ in Ab, i.e. make AM an E^*-module.

Despite appearances, the square (3.4) does commute in the dual category $FMod^{*op}$, to show that $\sigma M: AM \to SM$ is an E^*-module homomorphism.

Step 2. We have defined ρ_X as a natural transformation of sets. For fixed X, the cohomology functor $E^*(-)\hat{}: Ho \to FMod^{op}$ induces the natural transformation

$$Ho(X,-) \longrightarrow FMod\big(PE^*(-)\hat{}, E^*(X)\hat{}\big): Gp(Ho) \longrightarrow Set.$$

We apply [8, Lemma 7.6(c)] to the E^*-module object $n \mapsto \underline{E}_n$ to see that ρ_X is a morphism of E^*-module objects, i.e. takes values in Mod.

For Theorem 5.12(b), we note that given $x \in E^k(X)$, we have $(\sigma(E^*(X)\hat{}))x^*_U = x^*_U \circ \sigma^*_k = x^*_S$, by [8, (9.4)].

If X is a group object in Ho and $x \in PE^k(X)$, the associated map $x: X \to \underline{E}_k$ is a morphism of group objects (as remarked after Definition 3.1) and so induces $x^*: PE^*(\underline{E}_k) \to PE^*(X)$. If $E^*(X)$ (and hence $PE^*(X)$) is Hausdorff, ρ_X restricts to define

$$P\rho_X: PE^*(X) \longrightarrow APE^*(X). \tag{5.13}$$

Step 3. We filter AM exactly as we did SM in [8, §10], by the submodules $F^a(AM) = A(F^a M)$, using naturality. The proof that AM is complete Hausdorff is formally the same as for SM. Our choice of filtrations and the naturality of ρ clearly make ρ_X and σM continuous, so that ρ_X factors through $E^*(X)\hat{}$ and σ takes values in $FMod$.

Step 4. Whenever X is a group object in Ho and $E^*(X)$ is Hausdorff, we convert the object $PE^*(X)$ of $FMod$ to the corepresented functor

$$F_{PX} = FMod\big(PE^*(X), -\big): FMod \longrightarrow Ab$$

and the coaction $P\rho_X$ in (5.13) to a natural transformation $\rho_{PX}: F_{PX} \to F_{PX}A: FMod \to Ab$. Given M, $\rho_{PX}M: F_{PX}M \to F_{PX}AM$ is the homomorphism

$$\rho_{PX}M: FMod\big(PE^*(X), M\big) \longrightarrow FMod\big(PE^*(X), AM\big) \tag{5.14}$$

that is defined on $f: PE^*(X) \to M$ as the composite

$$(\rho_{PX}M)f: PE^*(X) \xrightarrow{P\rho_X} APE^*(X) \xrightarrow{Af} AM.$$

Step 5. To construct $\psi = \psi_A: A \to AA$, we take $X = \underline{E}_k$ in (5.14) and define

$$(\psi M)^k: FMod\big(PE^*(\underline{E}_k), M\big) \longrightarrow FMod\big(PE^*(\underline{E}_k), AM\big)$$

on the element $f: PE^*(\underline{E}_k) \to M$ of A^kM as the composite

$$(\psi M)^k f: PE^*(\underline{E}_k) \xrightarrow{P\rho_k} APE^*(\underline{E}_k) \xrightarrow{Af} AM.$$

When we substitute the E^*-module object $n \mapsto \underline{E}_n$ for X in (5.14), [8, Lemma 7.6(c)] shows that $(\psi M)^k: A^kM \to A^kAM$ lies in Mod. As k varies, we obtain the natural transformation $\psi: A \to AA$. Naturality in M also shows that ψM is filtered and so lies in $FMod$.

Step 6. The other required natural transformation, $\varepsilon: A \to I$, is defined on M simply as the evaluation

$$(\varepsilon M)^k = (\varepsilon_A M)^k: A^kM = FMod\big(PE^*(\underline{E}_k), M\big) \longrightarrow M \tag{5.15}$$

on $\iota_k \in PE^k(\underline{E}_k)$. It is continuous by naturality. It is compatible with the stable version, $\varepsilon_A = \varepsilon_S \circ \sigma: A \to I$, since given $f \in A^kM$, we have

$$(\varepsilon_S M)(\sigma M)f = \big((\sigma M)f\big)\iota = f\sigma_k^*\iota = f\iota_k = (\varepsilon_A M)f.$$

Step 7. To see that ρ_X is a coaction on $E^*(X)$, we use [8, Lemma 8.20] (adapted to graded objects). We use $R = PE^*(\underline{E}_n)$ (really, the graded object $n \mapsto PE^*(\underline{E}_n)$), $1_R = \iota_n$, and $\rho_R = P\rho_n$. By [8, Lemma 8.22], A is a comonad in $FMod$.

To see that $\sigma: A \to S$ is a morphism of comonads, we apply [8, Lemma 8.24]. The first condition on $u = \sigma_k^*: E^*(E,o) \to PE^*(\underline{E}_k)$ is the commutative diagram

$$\begin{tikzcd}
E^h(E,o) \arrow[r, "\sigma_k^*"] \arrow[dd, "\rho_E"] & PE^{k+h}(\underline{E}_k) \arrow[d, "P\rho_k"] \\
& \mathit{FMod}(PE^*(\underline{E}_{k+h}), PE^*(\underline{E}_k)) \arrow[d, "\mathrm{Hom}(\sigma_{k+h}^*,1)"] \\
\mathit{FMod}^h(E^*(E,o), E^*(E,o)) \arrow[r, "\mathrm{Hom}(1,\sigma_k^*)"] & \mathit{FMod}^{k+h}(E^*(E,o), PE^*(\underline{E}_k))
\end{tikzcd}$$

A stable operation $r_S \in E^h(E,o)$ restricts to an additive operation $r_U: k \to k+h$. On r_S, the lower route gives by diag. [8, (9.8)]

$$\sigma_k^* \circ r_S^* = (-1)^{hk}(r_S \circ \sigma_k)^* = (-1)^{hk}(\sigma_{k+h} \circ r_U)^* = (-1)^{hk} r_U^* \circ \sigma_{k+h}^*.$$

This agrees with the upper route, because $\sigma_k^* r_S = (-1)^{hk} r_U$ by [8, (9.9)]. The second condition needed is $\sigma_k^* \iota = \iota_k$, which holds by the definition of σ_k.

For Theorem 5.12(c), as in [8, Theorem 10.16(b)], it is enough to consider θX. Because $\underline{E}_k$ represents $E^k(-)$, natural transformations θ are classified by the elements $f = \theta\iota_k: N \to E^*(\underline{E}_k)$, i.e. morphisms in *FMod*. The additivity

$$(\theta X)(x+y) = (\theta X)(x) + (\theta X)(y)$$

of θX on the universal example (2.6) yields

$$\mu_k^* \circ f = p_1^* \circ f + p_2^* \circ f: N \longrightarrow E^*(\underline{E}_k \times \underline{E}_k).$$

By Proposition 2.7(b), f factors through $PE^*(\underline{E}_k)$. □

6. Unstable comodules

Although the Fourth Answer of Section 5 is the cleanest and most general, the Second Answer, in terms of unstable comodules, is usually the most practical and is available in the cases of interest. The parallel with the stable theory of [8] is extremely close, in spite of the very different provenance of the two theories. Some of the machinery was used in [6]; here we supply the missing definitions.

We assume throughout this section that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k, so that we have available all the results of Section 5.

The bigraded group $Q(E)_*^*$. As noted in Section 5, tensor products do not work correctly because the groups $QE_*(\underline{E}_k)$ have the wrong degree; we therefore shift degrees. We also adopt more efficient notation, that hides the details of construction and emphasizes the algebraic aspects and the formal similarity to stable comodules. (We remind that homology $E_i(X)$ has degree $-i$ under our conventions.)

DEFINITION 6.1. We define the bigraded group $Q(E)^*_*$ as having the components $Q(E)^k_i = QE_i(\underline{E}_k)$ (the component of $QE_*(\underline{E}_k)$ in degree $-i$), *except* that we assign the degree $k-i$ (instead of $-i$) to elements of $Q(E)^k_i$. (This is the degree that governs signs in formulae. We thus have the formal isomorphism $\Sigma^k\colon QE_*(\underline{E}_k) \cong Q(E)^k_*$ of degree k.)

We define the left action of $v \in E^h$ on $\Sigma^k c \in Q(E)^k_*$, for $c \in QE_*(\underline{E}_k)$, by $v(\Sigma^k c) = (-1)^{hk}\Sigma^k vc$, as in [8, (6.7)], to make

$$\Sigma^k\colon QE_*(\underline{E}_k) \cong Q(E)^k_*$$

an isomorphism of E^*-modules of degree k.

We equip $Q(E)^k_*$ with the projection

$$q_k\colon E_*(\underline{E}_k) \longrightarrow QE_*(\underline{E}_k) \xrightarrow{\Sigma^k} Q(E)^k_*. \tag{6.2}$$

We define the *stabilization*

$$Q(\sigma)\colon Q(E)^k_* \xrightarrow{\Sigma^{-k}} QE_*(\underline{E}_k) \xrightarrow{Q\sigma_{k*}} E_*(E,o), \tag{6.3}$$

where Lemma 4.7(c) provides the factorization $Q\sigma_{k*}$ of σ_{k*}.

We thus have the factorization into E^*-module homomorphisms

$$\sigma_{k*} = Q(\sigma)\circ q_k\colon E_*(\underline{E}_k) \longrightarrow Q(E)^k_* \longrightarrow E_*(E,o), \tag{6.4}$$

where we arranged for $Q(\sigma)$ to have degree zero and q_k to have degree k.

DEFINITION 6.5. Given an additive operation $r\colon k \to m$, i.e. an element $r_A \in PE^m(\underline{E}_k)$, we define the associated E^*-linear functional

$$\langle r_Q, -\rangle\colon Q(E)^k_* \xrightarrow{\Sigma^{-k}} QE_*(\underline{E}_k) \xrightarrow{\langle r_A,-\rangle} E^* \tag{6.6}$$

of degree $m-k$ (with no sign).

Now we can make the degree shift suggested by eq. (5.4). We have the strong duality

$$PE^*(\underline{E}_k) \cong DQE_*(\underline{E}_k)$$

from Lemma 4.16(a). Given an object M of *FMod*, we use [8, Lemma 6.16(b)] and the freeness of $QE_*(\underline{E}_k)$ to define the natural isomorphism of degree k

$$\mathit{FMod}^*\big(PE^*(\underline{E}_k), M\big) \cong M\,\widehat{\otimes}\,QE_*(\underline{E}_k) \xrightarrow{M\otimes\Sigma^k} M\,\widehat{\otimes}\,Q(E)^k_*. \tag{6.7}$$

LEMMA 6.8. *Given an additive operation $r\colon k \to m$ and an object M of FMod, the composite (formed using* (6.7))

$$\mathit{FMod}^*\big(PE^*(\underline{E}_k), M\big) \cong M \widehat{\otimes} Q(E)^k_* \xrightarrow{M\otimes\langle r_Q, -\rangle} M \otimes E^* \cong M$$

coincides with the evaluation homomorphism

$$e_r\colon \mathit{FMod}^*(PE^*(\underline{E}_k), M) \longrightarrow M$$

defined by $e_r f = (-1)^{m\deg(f)} f r_A$.

PROOF. We choose $x \in M$, $c \in QE_*(\underline{E}_k)$, and evaluate. $\square$

With Definition 6.5 in hand, we extend Proposition 2.7 and identify:

(i) the *additive operation* $r\colon E^k(-) \to E^m(-)$;
(ii) the *cohomology class* $r = r_A = r\iota_k \in PE^m(\underline{E}_k)$;
(iii) the *morphism of group objects* $r\colon \underline{E}_k \to \underline{E}_m$ in *Ho*; (6.9)
(iv) the E^**-linear functional* $\langle r, -\rangle = \langle r_Q, -\rangle\colon Q(E)^k_* \to E^*$, of degree $m-k$, defined by eq. (6.6).

(We drop the decorations $_A$ and $_Q$ on r except when we need to compare different versions.) As $Q(E)^*_*$ is smaller than $PE^*(\underline{E}_*)$, (iv) is the preferred choice. We do have to be careful with degrees, as (ii) has a different degree from (i) and (iv), while (iii) has no degree at all.

Scholium on signs. We construct the duality diagram in *FMod**

$$\begin{tikzcd}
r_S & r_A & r_U \\
E^*(E,o) \arrow[r, "\sigma_k^*"] \arrow[d, "\cong"] \arrow[dr, phantom, "(-1)^k"] & PE^*(\underline{E}_k) \arrow[r, hook] \arrow[d, "\cong"] \arrow[dr, phantom, "(-1)^k"] & E^*(\underline{E}_k) \arrow[d, "\cong"] \\
DE_*(E,o) \arrow[r, "DQ(\sigma)"] & D(Q(E)^k_*) \arrow[r, "Dq_k"] & DE_*(\underline{E}_k) \\
\langle r_S, -\rangle & \langle r_Q, -\rangle & \langle r_U, -\rangle
\end{tikzcd} \tag{6.10}$$

whose center isomorphism is taken as

$$PE^*(\underline{E}_k) \xrightarrow{d} DQE_*(\underline{E}_k) \xrightarrow{D(\Sigma^{-k})} D\big(Q(E)^k_*\big).$$

Because D is contravariant, each square commutes up to the sign $(-1)^k$.

On restriction to spaces, a stable operation r of degree h yields an additive unstable operation $r\colon k \to k+h$, and we obtain elements r_S, r_A, and r_U lying in the indicated groups. From these, we get the linear functionals $\langle r_S, -\rangle$, $\langle r_U, -\rangle$, and by eq. (6.6) also

$\langle r_Q, -\rangle$. We note that r_S and r_Q have degree h, while r_A and r_U have degree $k+h$. The algebra forces us to work with the element r_A and the functional $\langle r_Q, -\rangle$; we are not really interested in the functional $\langle r_A, -\rangle$, which appears only in the definition of $\langle r_Q, -\rangle$, and the element r_Q will occur nowhere.

The complication is that these six elements do *not* all correspond in obvious ways under the morphisms of diag. (6.10). The first surprise was [8, (9.9)], that

$$\sigma_k^* r_S = (-1)^{kh} r_A.$$

Of course, r_A and r_U do correspond, because they are the same element regarded as being in different groups. The second surprise is that r_A does not correspond to $\langle r_Q, -\rangle$, because the definition [8, (6.4)] of $D(\Sigma^k)$ requires the sign $(-1)^{k(h+k)}$, which is absent from Definition 6.5. In fact, matters are simpler if we work with elements and refrain from turning everything into E^*-module homomorphisms.

PROPOSITION 6.11. *In diag.* (6.10):

(a) *Given a stable operation r, the homomorphism $DQ(\sigma)$ takes $\langle r_S, -\rangle$ to $\langle r_Q, -\rangle$, or in elements,*

$$\langle r_Q, c\rangle = \langle r_S, Q(\sigma)c\rangle \quad \textit{for } c \in Q(E)^k_*, \tag{6.12}$$

and also

$$\langle r_U, c\rangle = \langle r_S, \sigma_{k*}c\rangle \quad \textit{for } c \in E_*(\underline{E}_k); \tag{6.13}$$

(b) *Given an additive operation $r: k \to m$, the homomorphism Dq_k takes $\langle r_Q, -\rangle$ to $(-1)^{k(m-k)}\langle r_U, -\rangle$, or equivalently, in elements,*

$$\langle r_U, c\rangle = \langle r_Q, q_k c\rangle \quad \textit{for } c \in E_*(\underline{E}_k). \tag{6.14}$$

PROOF. We just proved (a), except for eq. (6.13), which combines eqs. (6.12) and (6.14). In (b), $\langle r_A, -\rangle$ is simply the restriction of $\langle r_U, -\rangle$, so that

$$\langle r_U, c\rangle = \langle r_A, \Sigma^{-k} q_k c\rangle = \langle r_Q, q_k c\rangle.$$

But the definition of Dq_k adds the unwanted sign $(-1)^{k(m-k)}$. $\square$

$Q(E)^*_*$ as an algebra. There is much structure on $Q(E)^*_*$. First, it is by construction a left E^*-module.

PROPOSITION 6.15. *For any ring spectrum E, $Q(E)^*_*$ has the properties:*

(a) *$Q(E)^*_*$ is a bigraded E^*-algebra, with multiplication $Q(\phi)$ defined by the commutative diagram* (6.16)

$$\begin{tikzcd}
E_*(\underline{E}_k)\otimes E_*(\underline{E}_m) \arrow[r, "\times"] \arrow[d, "q_k\otimes q_m"] & E_*(\underline{E}_k\times\underline{E}_m) \arrow[r, "\phi_{U*}"] & E_*(\underline{E}_{k+m}) \arrow[d, "q_{k+m}"] \\
Q(E)^k_*\otimes Q(E)^m_* \arrow[rr, "Q(\phi)"] \arrow[d, "Q(\sigma)\otimes Q(\sigma)"] & & Q(E)^{k+m}_* \arrow[d, "Q(\sigma)"] \\
E_*(E,o)\otimes E_*(E,o) \arrow[r, "\times"] & E_*(E\wedge E,o) \arrow[r, "\phi_{S*}"] & E_*(E,o)
\end{tikzcd} \tag{6.16}$$

and unit $Q(\eta)$ defined by the commutative diagram

$$\begin{tikzcd}
E_*(T) \arrow[r, "\cong"] \arrow[d, "\eta_{U*}"] & E^* \arrow[r, "\cong"] \arrow[d, "Q(\eta)"] & E_*(T^+,o) \arrow[d, "\eta_{S*}"] \\
E_*(\underline{E}_0) \arrow[r, "q_0"] & Q(E)^0_* \arrow[r, "Q(\sigma)"] & E_*(E,o)
\end{tikzcd} \tag{6.17}$$

(b) *The stabilization $Q(\sigma)\colon Q(E)^*_* \to E_*(E,o)$ is a homomorphism of E^*-algebras.*

PROOF. $Q(\phi)$ is inherited, with a shift, from the multiplication on $QE_*(\underline{E}_*)$ constructed by Lemma 4.9. It thus fills in diag. (6.16), which is derived from [8, (9.15)] by applying E-homology and the factorization (6.4). We simply define $Q(\eta) = q_0\circ\eta_{U*}$, to fill in diag. (6.17). This comes from diag. [8, (9.4)] by taking $x = 1_T \in E^*(T)$. The algebraic properties of $Q(\phi)$ and $Q(\eta)$ are inherited from the E^*-algebra object $n \mapsto \underline{E}_n$ in *Ho*. Part (b) is clear from the diagrams. □

$Q(E)^*_*$ as a bimodule. We also need the right E^*-action. By Lemma 4.5, the functor $QE_*(-)\colon Gp(Ho) \to Mod$ preserves finite products. We apply [8, Lemma 7.6(a)] to the E^*-module object $n \mapsto \underline{E}_n$ in $Gp(Ho)$, to obtain, for each $v \in E^h$, homomorphisms $Q(\xi v)$ that fill in the commutative diagram

$$\begin{tikzcd}
E_*(\underline{E}_k) \arrow[r, "q_k"] \arrow[d, "(\xi_U v)_*"] & Q(E)^k_* \arrow[r, "Q(\sigma)"] \arrow[d, "Q(\xi v)"] & E_*(E,o) \arrow[d, "(\xi_S v)_*"] \\
E_*(\underline{E}_{k+h}) \arrow[r, "q_{k+h}"] & Q(E)^{k+h}_* \arrow[r, "Q(\sigma)"] & E_*(E,o)
\end{tikzcd} \tag{6.18}$$

and make $Q(E)^*_*$ a module object in Mod^*, i.e. an E^*-bimodule. This diagram came from diag. [8, (9.8)] by taking $r = \xi v$.

We have the additive analogue of the stable right unit.

DEFINITION 6.19. We define the *right unit* function $\eta_R: E^* \to Q(E)^*_*$ on $v \in E^h = E^h(T)$ by $\eta_R v = q_h v_* 1 \in Q(E)^h_0$, using the homology homomorphism $v_*: E^* \cong E_*(T) \to E_*(\underline{E}_h)$ induced by the map $v: T \to \underline{E}_h$.

It is clear from [8, (9.4)] and the factorization (6.4) that composition with $Q(\sigma)$ yields the stable right unit $\eta_R: E^* \to E_*(E, o)$ of [8, Definition 11.2].

PROPOSITION 6.20. *For any ring spectrum E, the algebra $Q(E)^*_*$ has the properties:*

(a) *It is a bigraded E^*-bimodule, with components $Q(E)^k_i = QE_i(\underline{E}_k)$ which are assigned the degree $k-i$;*

(b) *It has the well-defined unit element $1 = Q(\eta)1 = \eta_R 1 \in Q(E)^0_0$;*

(c) *The left action of $v \in E^h$ is left multiplication by $v1 \in Q(E)^0_{-h}$;*

(d) *The right action of $v \in E^h$ is right multiplication by $\eta_R v \in Q(E)^h_0$;*

(e) *The stabilization $Q(\sigma): Q(E)^*_* \to E_*(E, o)$ is a homomorphism of E^*-bimodules.*

REMARK. Propositions 6.15 and 6.20 are similar to [8, Proposition 11.3], except that $Q(E)^*_*$ is bigraded and the conjugation χ is conspicuous by its absence. The examples of Section 16 show that χ *does not exist*, at least, not in any obvious sense. (This is why we eschewed χ in [8].)

PROOF. Most of the proof is formally identical to the stable case [8, Proposition 11.3]. For (d), we apply E-homology to the factorization [8, (3.27)] of ξv. Part (e) is clear from diag. (6.18). □

We write the left and right E^*-actions as $\lambda_L: E^h \otimes Q(E)^k_i \to Q(E)^k_{i-h}$ and $\lambda_R: Q(E)^k_i \otimes E^h \to Q(E)^{k+h}_i$. Explicitly, the signs for λ_R are

$$\lambda_R(c \otimes v) = c \cdot v = c(\eta_R v) = (-1)^{h \deg(c)}(\eta_R v)c = (-1)^{h \deg(c)} Q(\xi v)c, \quad (6.21)$$

where $v \in E^h$ and $c \cdot v$ denotes the right action. For future use, we rewrite (d) as the commutative square

$$\begin{tikzcd} Q(E)^k_* \otimes Q(E)^m_* \arrow[r, "Q(\phi)"] \arrow[d, "Q(\xi v)\otimes 1"] & Q(E)^{k+m}_* \arrow[d, "Q(\xi v)"] \\ Q(E)^{k+h}_* \otimes Q(E)^m_* \arrow[r, "Q(\phi)"] & Q(E)^{k+m+h}_* \end{tikzcd} \quad (6.22)$$

The functor A'. Given an E^*-module M, we define (as promised in Section 5) the graded group $A'M$ as having the components

$$\left(A'M\right)^k = M^i \widehat{\otimes}_i Q(E)^k_i = \left(M \widehat{\otimes} Q(E)^k_*\right)^k \quad (6.23)$$

(where the tensor product $\widehat{\otimes}_i$ is formed using the two E^*-actions indexed by i. We have no use for the rest of $M \widehat{\otimes} Q(E)^k_*$!) We use the isomorphism (6.7) to define the isomorphism $AM \cong A'M$ as having the components

$$(AM)^k = A^k M = FMod\big(PE^*(\underline{E}_k), M\big) \cong M^i \widehat{\otimes}_i Q(E)^k_i = \big(A'M\big)^k. \quad (6.24)$$

We use *this* isomorphism to transfer all the structure of Section 5 from A to A' and make A' a comonad, just as we did stably in [8]. (We generally drop the decorations $'$ except when comparing different versions.)

In particular, we use (6.24) to convert modules to comodules. If M is an additively unstable module with coaction $\rho_M: M \to AM$ (as in Definition 5.9), we deduce the equivalent coaction $\rho'_M: M \to A'M$ with components

$$\rho'_M: M^k \longrightarrow \big(A'M\big)^k = M^i \widehat{\otimes}_i Q(E)^k_i \subset M \widehat{\otimes} Q(E)^k_*. \quad (6.25)$$

In particular, for a space X, we convert the action ρ_X in (5.6) to

$$\rho'_X: E^k(X) \longrightarrow E^i(X) \widehat{\otimes}_i Q(E)^k_i \subset E^*(X) \widehat{\otimes} Q(E)^k_*. \quad (6.26)$$

$Q(E)^*_*$ as a coalgebra. The stable discussion carries over, except that $Q(E)^*_*$ is bigraded. The comonad structure (ψ, ε) on A translates into a comonad structure (ψ', ε') on A'. By naturality and the case $M = \Sigma^i E^*$, $\psi' M: (A'M)^k \to (A'A'M)^k$ must take the form $M \widehat{\otimes} \psi$ for a certain comultiplication

$$\psi = Q(\psi): Q(E)^k_i \longrightarrow Q(E)^j_i \otimes_j Q(E)^k_j \quad (6.27)$$

(where we sum over j as in eq. (6.23)), and $\varepsilon' M: (A'M)^k \to M^k$ must take the form $M \widehat{\otimes} \varepsilon$ for a certain counit

$$\varepsilon = Q(\varepsilon): Q(E)^k_i \longrightarrow E^{k-i}. \quad (6.28)$$

By construction, these are both E^*-bimodule homomorphisms of degree zero.

PROPOSITION 6.29. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. Then:*

(a) *The homomorphisms $\psi = Q(\psi)$ and $\varepsilon = Q(\varepsilon)$ in diags.* (6.27) *and* (6.28) *make $Q(E)^*_*$ a coalgebra over E^*;*

(b) *If $E_*(E, o)$ is also free, the stabilization $Q(\sigma): Q(E)^*_* \to E_*(E, o)$ is a morphism of coalgebras* (cf. [8, Lemma 11.8]).

PROOF. By taking $M = \Sigma^i E^*$, the comonad axioms [8, (8.6)] for A' yield the coassociativity

$$\begin{tikzcd} Q(E)^k_h \arrow[r, "Q(\psi)"] \arrow[d, "Q(\psi)"] & Q(E)^j_h \otimes_j Q(E)^k_j \arrow[d, "1\otimes Q(\psi)"] \\ Q(E)^i_h \otimes_i Q(E)^k_i \arrow[r, "Q(\psi)\otimes 1"] & Q(E)^i_h \otimes_i Q(E)^j_i \otimes_j Q(E)^k_j \end{tikzcd} \tag{6.30}$$

of $Q(\psi)$ and the two counit axioms

$$\begin{tikzcd} Q(E)^k_i \arrow[r, "Q(\psi)"] \arrow[d, "="] & Q(E)^j_i \otimes_j Q(E)^k_j \arrow[d, "Q(\epsilon)\otimes 1"] & Q(E)^k_i \arrow[r, "Q(\psi)"] \arrow[d, "="] & Q(E)^j_i \otimes_j Q(E)^k_j \arrow[d, "1\otimes Q(\epsilon)"] \\ Q(E)^k_i & E^{j-i} \otimes_j Q(E)^k_j \arrow[l, "\lambda_L"'] & Q(E)^k_i & Q(E)^j_i \otimes_j E^{k-j} \arrow[l, "\lambda_R"'] \end{tikzcd} \tag{6.31}$$

Part (b) is the translation of Theorem 5.8(b). □

Comodules. Now that we have the coalgebra $Q(E)^*_*$, we can convert Definition 5.9 and Theorem 5.12.

DEFINITION 6.32. An *unstable* (*E-cohomology*) *comodule* is an A'-coalgebra in *FMod*.

In detail, given a complete Hausdorff filtered E^*-module M (i.e. object of *FMod*), an unstable comodule structure on M consists of a coaction $\rho_M: M \to A'M$, with components $M^k \to M^i \widehat{\otimes}_i Q(E)^k_i$ as in diag. (6.25), that is a continuous homomorphism of E^*-modules (i.e. morphism in *FMod*) and satisfies the axioms

$$\begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "\rho_M"] & M \widehat{\otimes} Q(E)^*_* \arrow[d, "M\otimes Q(\psi)"] \\ M \widehat{\otimes} Q(E)^*_* \arrow[r, "\rho_M \otimes 1"] & M \widehat{\otimes} Q(E)^*_* \widehat{\otimes} Q(E)^*_* \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[dr, "\cong"'] & M \widehat{\otimes} Q(E)^*_* \arrow[d, "M\otimes Q(\epsilon)"] \\ & M \otimes E^* \end{tikzcd} \tag{6.33}$$

(i) (ii)

This is a stronger structure than a stable comodule (assuming that $E_*(E, o)$ is free, so that stable comodules can be defined). Given a coaction ρ_M as above, Proposition 6.29(b) shows that the coaction

$$M \xrightarrow{\rho_M} M \widehat{\otimes} Q(E)^*_* \xrightarrow{M\otimes Q(\sigma)} M \widehat{\otimes} E_*(E, o) \tag{6.34}$$

makes M a stable comodule.

REMARK. We regard comodules as essentially additive constructs, as we find no analogue in the fully unstable context. We therefore omit the adjective "additive" from comodules.

THEOREM 6.35. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a).) *Then given a complete Hausdorff filtered E^*-module M (i.e. object of FMod), an additively unstable module structure on M in the sense of Definition* 5.9 *is equivalent to an unstable comodule structure on M in the sense of Definition* 6.32.

PROOF. We have the isomorphism $AM \cong A'M$ in eq. (6.24). The axioms (6.33) are just the general coaction axioms [8, (8.7)] interpreted for A'. □

THEOREM 6.36. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a).) *Then:*

(a) *For any space X, there is a natural coaction*

$$\rho_X : E^*(X) \longrightarrow E^*(X)\,\widehat{\otimes}\,Q(E)^*_*$$

that makes $E^*(X)^\wedge$ an unstable comodule, which corresponds by Theorem 6.35 *to the additive module structure given by Theorem* 5.12;

(b) *If also $E_*(E,o)$ is free, we recover the stable coaction* [8, (11.15)] *on $E^*(X)$ from ρ_X as in diag.* (6.34);

(c) *ρ is universal: given a discrete E^*-module N and an integer k, any additive natural transformation $\theta X : E^k(X) \to E^*(X)\,\widehat{\otimes}\,N$ (or $\widehat{\theta} X : E^k(X)^\wedge \to E^*(X)^\wedge\,\widehat{\otimes}\,N$) that is defined for all spaces X is induced from ρ_X by a unique homomorphism $f : Q(E)^k_* \to N$ of E^*-modules as*

$$\theta X : E^k(X) \xrightarrow{\rho_X} E^*(X)\,\widehat{\otimes}\,Q(E)^k_* \xrightarrow{1\otimes f} E^*(X)\,\widehat{\otimes}\,N.$$

PROOF. We deduce (a) from Theorem 5.12(a) and Theorem 6.35, just as we did stably in [8, Theorem 11.14]. In eq. (6.26), we defined the coaction ρ'_X as corresponding to ρ_X. In (b), the stabilization $Q(\sigma)$ clearly dualizes to $\sigma^*_k : E^*(E,o) \to PE^*(\underline{E}_k)$, which we used in eq. (5.7) to define the stabilization $\sigma : A \to S$ of comonads.

In (c), the natural transformation θ is classified by the element $u = \theta\iota_k \in E^*(\underline{E}_k)\,\widehat{\otimes}\,N$. Additivity of θ for the universal example (2.6) states that

$$(\mu_k^*\otimes N)u = (p_1^*\otimes N)u + (p_2^*\otimes N)u \quad \text{in } E^*(\underline{E}_k\times\underline{E}_k)\,\widehat{\otimes}\,N.$$

By [8, Lemma 6.16(a)], u corresponds to a homomorphism $f : E_*(\underline{E}_k) \to N$ of E^*-modules. The above property dualizes to

$$f\circ\mu_{k*} = f\circ p_{1*} + f\circ p_{2*} : E_*(\underline{E}_k\times\underline{E}_k) \longrightarrow N,$$

which shows that f factors through $Q(E)^k_*$ as required. □

REMARK. Just as stably, (c) allows us to use diags. (6.33) to *define* $Q(\psi)$ and $Q(\varepsilon)$ in terms of ρ. Three applications of the uniqueness in (c) show that $Q(\psi)$ is coassociative and has $Q(\varepsilon)$ as a two-sided counit.

Linear functionals. Theorem 6.35 establishes the equivalence between unstable modules and comodules. For applications, we need the details. All our formulae stabilize to the corresponding formulae of [8, §11] by applying $Q(\sigma)$, which conveniently has degree zero.

Given an unstable comodule M, we recover the action of the additive operation $r: k \to m$ on M from Lemma 6.8 as

$$r: M^k \xrightarrow{\rho_M} M \widehat{\otimes} Q(E)^k_* \xrightarrow{M\otimes\langle r,-\rangle} M \otimes E^* \cong M. \tag{6.37}$$

Because $\langle r, -\rangle$ has degree $m-k$, r takes values in M^m. To make this action explicit, let us choose $x \in M^k$ and write

$$\rho_M x = \sum_\alpha (-1)^{\deg(x_\alpha)\deg(c_\alpha)} x_\alpha \otimes c_\alpha \quad \text{in } M \widehat{\otimes} Q(E)^k_*, \tag{6.38}$$

where the sum may be infinite, and of course $\deg(x_\alpha) = k - \deg(c_\alpha)$. (As in [8], we insert signs here to keep the next formula simple.) Then

$$rx = \sum_\alpha \langle r, c_\alpha\rangle\, x_\alpha \quad \text{in } M, \text{ for all } r: k \to m, \tag{6.39}$$

where the c_α and x_α depend only on x, not on r. Because M is assumed complete, this sum converges if it is infinite. (Recall that $Q(E)^k_*$ always has the discrete topology.)

REMARK. It is important for our applications *not* to require the c_α to form a basis of $Q(E)^k_*$, or even be linearly independent; but if they do form a basis, the x_α are uniquely determined by eq. (6.39) as $x_\alpha = c^*_\alpha x$, where c^*_α denotes the operation dual to c_α.

The fact that ρ_M is an E^*-module homomorphism is expressed by

$$r(vx) = \sum_\alpha \langle r, (\eta_R v) c_\alpha\rangle x_\alpha = \sum_\alpha (-1)^{h\deg(c_\alpha)} \langle r, c_\alpha\, \eta_R v\rangle\, x_\alpha \quad \text{in } M, \tag{6.40}$$

for any $v \in E^h$ and all operations $r: k + h \to m$.

Because $Q(\varepsilon): Q(E)^k_* \to E^*$ corresponds to ε in eq. (5.15), which is evaluation on ι_k, we have immediately

$$\langle \iota_k, -\rangle = Q(\varepsilon): Q(E)^k_* \longrightarrow E^*, \tag{6.41}$$

as is obvious by comparing axiom (6.33)(ii) with eq. (6.37). In other words, in the list (6.9), the identity operation ι_k corresponds to the functional $Q(\varepsilon)$.

The cohomology of a point. Our first test space is the one-point space T.

PROPOSITION 6.42. *In the unstable comodule $E^*(T) = E^*$:*

(a) *The action of the additive operation $r: k \to m$ on $v \in E^k$ is given by*

$$rv = \langle r, \eta_R v\rangle \quad \text{in } E^*(T) = E^*; \tag{6.43}$$

(b) *The coaction $\rho_T: E^* \to E^* \otimes Q(E)^*_* \cong Q(E)^*_*$ coincides with the right unit $\eta_R: E^* \to Q(E)^*_0$* (see Definition 6.19).

PROOF. We imitate [8, Proposition 11.22]. The map $v: T \to \underline{E}_k$ yields

$$rv = \langle rv, 1\rangle = \langle v^* r_U, 1\rangle = \langle r_U, v_* 1\rangle = \langle r_Q, q_k v_* 1\rangle = \langle r_Q, \eta_R v\rangle,$$

by eq. (6.14) and Definition 6.19 of η_R. We compare eqs. (6.38) and (6.39) and rewrite this as $\rho_T v = 1 \otimes \eta_R v$, to give (b). □

Homology homomorphisms. A class $x \in E^k(X)$ may be regarded as a map $x: X \to \underline{E}_k$. We need information about the induced homology homomorphism $x_*: E_*(X) \to E_*(\underline{E}_k)$.

PROPOSITION 6.44. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. Given $x \in E^k(X)$, suppose that rx is given by eq.* (6.39). *Then the homomorphism $q_k \circ x_*: E_*(X) \to Q(E)^k_*$ induced by the map $x: X \to \underline{E}_k$ is given on $z \in E_h(X)$ by*

$$q_k x_* z = \sum_\alpha (-1)^{\deg(c_\alpha)(\deg(x_\alpha)+h)} \langle x_\alpha, z\rangle\, c_\alpha = \sum_\alpha c_\alpha \langle x_\alpha, z\rangle \quad \text{in } Q(E)^k_*. \tag{6.45}$$

PROOF. For any additive $r: k \to m$, we have $\langle r_Q, q_k x_* z\rangle = \langle r_U, x_* z\rangle$ by eq. (6.14). The rest of the proof is formally identical to the stable analogue [8, Proposition 11.26]. □

Conversely, we can recover $\rho_X x$ from x_* when X is well behaved, just as we did stably. If $E_*(X)$ is free, we have strong duality $E^*(X) \cong DE_*(X)$ by Theorem 1.18(a), and [8, Lemma 6.16(a)] supplies the isomorphism

$$E^*(X) \mathbin{\widehat{\otimes}} Q(E)^k_* \cong \mathit{Mod}^*\big(E_*(X), Q(E)^k_*\big). \tag{6.46}$$

PROPOSITION 6.47. *Assume that $E_*(X)$, $E_*(\underline{E}_k)$, and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. Take $x \in E^k(X)$. Then under the isomorphism* (6.46), *the element $\rho_X x$ corresponds to the homomorphism $q_k \circ x_*: E_*(X) \to E_*(\underline{E}_k) \to Q(E)^k_*$.*

PROOF. We apply the isomorphism to eq. (6.38) and compare with eq. (6.45). □

In particular, it is important to know the homomorphism of E^*-modules

$$Q(r): Q(E)^k_* \cong QE_*(\underline{E}_k) \xrightarrow{Qr_*} QE_*(\underline{E}_m) \cong Q(E)^m_* \tag{6.48}$$

induced by an additive operation $r: k \to m$ (which by Proposition 2.7(c) is a morphism of group objects in *Ho*). It has degree $m-k$. The $Q(r)$ provide a convenient faithful representation of the additive operations. The translation of diag. (5.11) is the commutative square

$$\begin{tikzcd} M^k \arrow[r, "r"] \arrow[d, "\rho_M"] & M^m \arrow[d, "\rho_M"] \\ M^i \otimes_i Q(E)^k_i \arrow[r, "M\otimes Q(r)"] & M^i \otimes_i Q(E)^m_i \end{tikzcd} \tag{6.49}$$

which stabilizes to diag. [8, (11.29)].

Just as stably, we easily recover the functional $\langle r, -\rangle$ from $Q(r)$ as

$$\langle r, -\rangle : Q(E)^k_* \xrightarrow{Q(r)} Q(E)^m_* \xrightarrow{Q(\varepsilon)} E^*. \tag{6.50}$$

Conversely, we have the additive analogue of [8, Lemma 11.31].

LEMMA 6.51. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. If $r: k \to m$ is an additive operation, then the homology homomorphism $Q(r): Q(E)^k_* \to Q(E)^m_*$ in eq.* (6.48) *has the properties:*

(a) *The diagram*

$$\begin{tikzcd} Q(E)^k_* \arrow[r, "Q(r)"] \arrow[d, "Q(\psi)"] & Q(E)^m_* \arrow[d, "Q(\psi)"] \\ Q(E)^*_* \otimes Q(E)^k_* \arrow[r, "1\otimes Q(r)"] & Q(E)^*_* \otimes Q(E)^m_* \end{tikzcd} \tag{6.52}$$

commutes; in other words, $Q(r)$ is a morphism of left $Q(E)^*_*$-comodules;

(b) *$Q(r): Q(E)^k_* \to Q(E)^m_*$ is the unique homomorphism of left E^*-modules that satisfies eq.* (6.50) *and is a morphism of left $Q(E)^*_*$-comodules in the sense of* (a);

(c) *$Q(r)$ is given in terms of the functional $\langle r, -\rangle$ as*

$$Q(r): Q(E)^k_i \xrightarrow{Q(\psi)} Q(E)^j_i \otimes_j Q(E)^k_j \xrightarrow{1\otimes\langle r,-\rangle} Q(E)^j_i \otimes_j E^{m-j} \xrightarrow{\lambda_R} Q(E)^m_i.$$

We deduce from (c) that the composite $sr: k \to n$ of the operations $r: k \to m$ and $s: m \to n$ corresponds to the functional

$$\langle sr, -\rangle : Q(E)^k_i \xrightarrow{Q(\psi)} Q(E)^j_i \otimes_j Q(E)^k_j \xrightarrow{1\otimes\langle r,-\rangle} Q(E)^j_i \otimes_j E^{m-j} \xrightarrow{\lambda_R} Q(E)^m_i \xrightarrow{\langle s,-\rangle} E^{n-i}. \tag{6.53}$$

REMARK. From diags. (6.30) and (6.31)(ii) we observe that for fixed h, $Q(\psi)$ makes the graded group $n \mapsto Q_h^n$ an additively unstable comodule, if we use the right E^*-module action (6.21). Then by (c), the action of $r\colon k \to m$ is just $Q(r)$, and diag. (6.52) becomes a special case of diag. (6.49).

7. What is an additively unstable algebra?

In this section, we define an additively unstable algebra by enriching each of the four Answers in Section 5 with multiplicative structure. The treatment is closely parallel to the stable case [8, §12] and we give only the significant additions. The logical sequence is made slightly complicated by the fact that the monoidal structure is most easily described in the context of the Second (or Third) Answer, while the comonad structure prefers the Fourth Answer.

In Definition 7.13 we introduce the collapse operation, which detects the connectedness of a space.

We assume throughout this section that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k, which is true for our five examples by Lemma 4.17(a). Then by Corollary 2.9, $PE^*(\underline{E}_k)$ is an object of *FMod*.

First Answer. We have, for any space X, the additively unstable action (5.1)

$$\circ\colon PE^m(\underline{E}_k) \times E^k(X) \longrightarrow E^m(X).$$

Given $x \in E^k(X)$, $y \in E^m(X)$, and $r \in PE^*(\underline{E}_{k+m})$, we would like to have a Cartan formula

$$r(xy) = \sum_\alpha (r'_\alpha x)(r''_\alpha y) \quad \text{in } E^*(X), \tag{7.1}$$

for suitably chosen operations r'_α and r''_α (depending on k and m as well as r). For the universal example

$$X = \underline{E}_k \times \underline{E}_m, \quad \text{with } x = \iota_k \times 1,\ y = 1 \times \iota_m,\ xy = \phi = \iota_k \times \iota_m, \tag{7.2}$$

where $\phi\colon \underline{E}_k \times \underline{E}_m \to \underline{E}_{k+m}$ denotes the multiplication map of [8, Theorem 3.25], eq. (7.1) reduces to

$$\phi^* r = \sum_\alpha r'_\alpha \times r''_\alpha \quad \text{in } E^*(\underline{E}_k \times \underline{E}_m).$$

To ensure that $\phi^* r$ is expressible in this form, we need to allow infinite sums and use the Künneth homeomorphism $E^*(\underline{E}_k \times \underline{E}_m) \cong E^*(\underline{E}_k) \mathbin{\widehat{\otimes}} E^*(\underline{E}_m)$ from Theorem 1.18(c).

We need to know more, that $r'_\alpha, r''_\alpha \in PE^*(\underline{E}_*)$. We have enough duality isomorphisms to dualize the multiplication in Lemma 4.9 and define a comultiplication ψ_P by the commutative diagram

$$\begin{array}{ccccc} PE^*(\underline{E}_{k+m}) & & \xrightarrow{\quad\psi_P\quad} & & PE^*(\underline{E}_k)\,\widehat{\otimes}\,PE^*(\underline{E}_m) \\ \Big\downarrow\subset & & & & \Big\downarrow\subset \\ E^*(\underline{E}_{k+m}) & \xrightarrow{\phi^*} & E^*(\underline{E}_k\times\underline{E}_m) & \xleftarrow{\cong} & E^*(\underline{E}_k)\,\widehat{\otimes}\,E^*(\underline{E}_m) \end{array} \tag{7.3}$$

Then we write $\psi_P r = \sum_\alpha r'_\alpha \otimes r''_\alpha$, as required.

We must not forget the unit element $1_X \in E^*(X)$. We define the counit $\varepsilon_P \colon PE^*(\underline{E}_0) \to E^*$ as the restriction of $\eta^*\colon E^*(\underline{E}_0) \to E^*(T) = E^*$, so that $r1_X = (\varepsilon_P r)1_X$ in $E^*(X)$.

It is now clear what an additively unstable algebra should be. Given an E^*-algebra M, we need actions $PE^m(\underline{E}_k) \times M^k \to M^m$ that compose correctly, are biadditive and E^*-bilinear in the sense of diag. (5.2), satisfy the Cartan formula (7.1), and respect the unit in the sense that $r1_M = (\varepsilon_P r)1_M$. In the classical case $E = H(\mathbb{F}_p)$, there is a good Cartan formula and this approach is useful. For more general E, such as MU and BP, this structure seems even more impractical than it was stably.

Second Answer. We have the coaction (6.26),

$$\rho_X\colon E^k(X) \longrightarrow E^i(X)\,\widehat{\otimes}_i\,Q(E)^k_i.$$

In contrast to the Cartan formula of the First Answer, and just as stably in [8], all we have to do is observe that as k varies, ρ_X is a homomorphism of E^*-algebras, where we use the bigraded algebra structure on $Q^*_* = Q(E)^*_*$ from Proposition 6.15.

Explicitly, if for particular $x, y \in E^*(X)$ we have, as in eq. (6.39),

$$rx = \sum_\alpha \langle r, c_\alpha\rangle\, x_\alpha; \quad ry = \sum_\beta \langle r, d_\beta\rangle\, y_\beta; \quad \text{for all } r,$$

the Cartan formula (7.1) becomes (cf. the stable analogue [8, (12.5)])

$$r(xy) = \sum_\alpha\sum_\beta (-1)^{\deg(d_\beta)\deg(x_\alpha)}\langle r, c_\alpha d_\beta\rangle\, x_\alpha y_\beta \quad \text{in } E^*(X)\hat{}\,, \text{ for all } r. \tag{7.4}$$

LEMMA 7.5. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k. Then the homomorphisms $Q(\psi)$ and $Q(\varepsilon)$ in (6.27) and (6.28) are multiplicative and respect the unit element.*

We defer the proofs until after Theorem 7.9, as the coalgebra structure on $Q(E)^*_*$ is not easily handled directly. The Lemma makes the following definition reasonable.

DEFINITION 7.6. We call an unstable comodule M in the sense of Definition 6.32 an *unstable* (E-*cohomology*) *comodule algebra* if M is a filtered algebra (i.e. object of *FAlg*) and its coaction $\rho_M: M \to M \widehat{\otimes} Q(E)^*_*$ is a homomorphism of E^*-algebras.

In detail, M is a complete Hausdorff commutative filtered E^*-algebra, equipped with a structure map ρ_M that is a continuous homomorphism of E^*-algebras and makes diags. (6.33) commute.

THEOREM 7.7. *Assume that* $E_*(\underline{E}_k)$ *and* $QE_*(\underline{E}_k)$ *are free* E^*-*modules for all* k (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *For any space* X, ρ_X *makes* $E^*(X)^\wedge$ *an unstable comodule algebra in the sense of Definition* 7.6;

(b) ρ *is universal: given a (possibly bigraded) discrete* E^*-*algebra* B, *any natural transformation of rings* $\theta X: E^*(X) \to E^*(X) \widehat{\otimes} B$ (*or* $\widehat{\theta} X: E^*(X)^\wedge \to E^*(X)^\wedge \widehat{\otimes} B$) *that is defined for all spaces* X *is induced from* ρ_X *by a unique homomorphism* $f: Q(E)^*_* \to B$ *of left* E^*-*algebras as*

$$\theta X: E^*(X) \xrightarrow{\rho_X} E^*(X) \widehat{\otimes} Q(E)^*_* \xrightarrow{1 \otimes f} E^*(X) \widehat{\otimes} B.$$

PROOF. This will follow from Theorem 7.9 in the same way that the stable result Theorem 12.8 followed from Theorem 12.10 in [8]. □

Third Answer. We use the multiplication $Q(\phi): Q^k_* \otimes Q^m_* \to Q^{k+m}_*$ from Proposition 6.15 to make A' a symmetric monoidal functor $(A', \zeta_{A'}, z_{A'})$ in *FMod*, with

$$\zeta_{A'}(M,N): (A'M)^k \widehat{\otimes} (A'N)^m \longrightarrow (A'(M \widehat{\otimes} N))^{k+m}$$

given by

$$\begin{aligned} \zeta_{A'}(M,N): M \widehat{\otimes} Q^k_* \widehat{\otimes} N \widehat{\otimes} Q^m_* &\cong M \widehat{\otimes} N \widehat{\otimes} (Q^k_* \otimes Q^m_*) \\ &\longrightarrow M \widehat{\otimes} N \widehat{\otimes} Q^{k+m}_* \end{aligned} \tag{7.8}$$

and $z_{A'} = \eta_R: E^h \to E^* \otimes Q^h_* \cong Q^h_*$. Thus when M is an E^*-algebra, so is $A'M$. We see that A', equipped with natural transformations $\psi': A' \to A'A'$ and $\varepsilon': A' \to I$ constructed from $Q(\psi)$ and $Q(\varepsilon)$, becomes a symmetric monoidal comonad in *FMod* and therefore a comonad in *FAlg*.

Fourth Answer. For suitable E, we can make A a comonad in *FAlg*.

THEOREM 7.9. *Assume that* $E_*(\underline{E}_k)$ *and* $QE_*(\underline{E}_k)$ *are free* E^*-*modules for all* k (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *We can enrich A to make it a symmetric monoidal comonad in FMod and therefore a comonad in FAlg;*

(b) *If also $E_*(E,o)$ is free, the stabilization $\sigma: A \to S$ is a monoidal natural transformation in FMod.*

The relevant definition is now clear.

DEFINITION 7.10. An *additively unstable (E-cohomology) algebra* is an A-coalgebra in *FAlg*, i.e. a complete Hausdorff commutative filtered E^*-algebra M equipped with a morphism $\rho_M: M \to AM$ in *FAlg* that satisfies the coaction axioms [8, (8.7)].

If the closed ideal $L \subset M$ is invariant, the quotient algebra M/L inherits a well-defined A-coalgebra structure.

THEOREM 7.11. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then given a complete Hausdorff commutative filtered E^*-algebra M* (i.e. object of *FAlg*), *an unstable comodule algebra structure on M in the sense of Definition 7.6 is equivalent to an additively unstable algebra structure on M in the sense of Definition 7.10.*

THEOREM 7.12. *Assume that $E_*(\underline{E}_k)$ and $QE_*(\underline{E}_k)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *For any space X, the coaction $\rho_X: E^*(X) \to A(E^*(X)\hat{})$ in diag. (5.6) is a homomorphism of E^*-algebras and makes $E^*(X)\hat{}$ an additively unstable algebra;*

(b) *ρ is universal: given a graded monoid object $n \mapsto C^n$ in $FMod^{op}$, so that (by [8, Lemma 7.9]) $n \mapsto G^n(X) = FMod(C^n, E^*(X)\hat{})$ is a graded ring, any natural transformation of graded rings $\theta X: E^*(X) \to G^*(X)$ (or $\hat{\theta}X: E^*(X)\hat{} \to G^*(X)$), that is defined for all spaces X, is induced from ρ_X by a unique morphism in $FMod^{op}$ of graded monoid objects with components $f^n: C^n \to PE^*(\underline{E}_n)$ in FMod, as*

$$\theta X: E^n(X) \xrightarrow{\rho_X} FMod(PE^*(\underline{E}_n), E^*(X)\hat{}) \xrightarrow{\mathrm{Hom}(f^n,1)} FMod(C^n, E^*(X)\hat{}).$$

PROOF OF THEOREMS 7.9 AND 7.12. The main proof proceeds by the same five steps as stably for [8, Theorems 12.10, 12.13], except based on Theorem 5.8 instead of [8, Theorem 10.12]. We give only the major changes. We recall the universal class $\iota_k \in E^k(\underline{E}_k)$, element $\mathrm{id}_k \in A^k PE^*(\underline{E}_k)$, and ρ_k from the proof of Theorem 5.8.

Step 1. We construct the symmetric monoidal functor

$$(A, \zeta_A, z_A): (FMod, \widehat{\otimes}, E^*) \longrightarrow (Mod, \otimes, E^*).$$

Then A will take monoid objects in *FMod* (i.e. objects of *FAlg*) to monoid objects in *Mod* (i.e. E^*-algebras).

By Lemma 4.16(a), we can construct the diagram (7.3) that defines ψ_P and verify its properties, which are dual to those of $Q(\phi)$ in Propositions 6.15 and 6.20. The counit $\varepsilon_P: PE^*(\underline{E}_0) \to E^*$ is the restriction of $\eta^*: E^*(\underline{E}_0) \to E^*(T) = E^*$. These make

$n \mapsto PE^*(\underline{E}_n)$ an E^*-algebra object in $FMod^{*\mathrm{op}}$, to which we apply [8, Lemma 7.14]. The necessary compatibility axiom [8, (7.13)] is the dual of diag. (6.22). As stably, we use [8, (7.15)] to identify z_A with $\rho_T \colon E^*(T) \to AE^*(T)$.

If $E_*(E, o)$ is also free, we can dualize Proposition 6.15(b) to see that the destabilizations $\sigma_n^* \colon E^*(E, o) \to PE^*(\underline{E}_n)$ form a morphism of graded monoid objects in $(FMod^{*\mathrm{op}}, \widehat{\otimes}, E^*)$. Then [8, Lemma 7.9(b)] shows that $\sigma \colon A \to S$ is monoidal.

Step 2. The proof that ρ is monoidal is similar to the stable case. Here, the universal example is $X = \underline{E}_k$ and $Y = \underline{E}_m$, with the element $\iota_k \otimes \iota_m$. The two elements of $A^{k+m}E^*(\underline{E}_k \times \underline{E}_m)$ to be compared are

$$
\begin{aligned}
PE^*(\underline{E}_{k+m}) \xrightarrow{\psi_P} PE^*(\underline{E}_k) \widehat{\otimes} PE^*(\underline{E}_m) \subset E^*(\underline{E}_k) \widehat{\otimes} E^*(\underline{E}_m)& \\
\xrightarrow{\times} E^*(\underline{E}_k \times \underline{E}_m)&
\end{aligned}
$$

and

$$PE^*(\underline{E}_{k+m}) \subset E^*(\underline{E}_{k+m}) \xrightarrow{\phi^*} E^*(\underline{E}_k \times \underline{E}_m).$$

These agree by diag. (7.3). The second condition needed is just $z_A = \rho_T$.

$$
\begin{tikzcd}
PE^*(\underline{E}_{k+m}) \arrow[r, "\psi_P"] \arrow[dd, "P\rho_{k+m}"] & PE^*(\underline{E}_k) \widehat{\otimes} PE^*(\underline{E}_m) \arrow[d, "P\rho_k \otimes P\rho_m"] \\
& APE^*(\underline{E}_k) \widehat{\otimes} APE^*(\underline{E}_m) \arrow[d, "\zeta_A"] \\
APE^*(\underline{E}_{k+m}) \arrow[r, "A\psi_P"] & A(PE^*(\underline{E}_k) \widehat{\otimes} PE^*(\underline{E}_m))
\end{tikzcd}
$$

Figure 1. Additive operations and comultiplication.

Step 3. The analogue of diag. [8, (12.17)] for this situation is fig. 1. To establish this, we proceed as in [8, Theorem 12.10]. Because ρ is monoidal and natural, we have the commutative diagram fig. 2 (cf. diag. [8, (12.16)]) which includes an isomorphism from Theorem 1.18(c). Figure 1 is obtained from this by restriction, using the coaction (5.13) and diag. (7.3).

Step 4. The monoidality of ψ follows formally from that of ρ, just as stably (cf. diags. [8, (12.18)]). The universal example is $M = PE^*(\underline{E}_m)$ and $N = PE^*(\underline{E}_n)$, with element $\mathrm{id}_m \otimes \mathrm{id}_n$. We use fig. 1 instead of diag. [8, (12.17)].

Step 5. The proof that ε is monoidal is formally the same as stably, except for the insertion of indices.

$$
\begin{tikzcd}
E^*(\underline{E}_k)\widehat{\otimes} E^*(\underline{E}_m) \arrow[r, "\rho_k\otimes\rho_m"] \arrow[dd, "\times", "\cong"'] & AE^*(\underline{E}_k)\widehat{\otimes} AE^*(\underline{E}_m) \arrow[d, "\zeta_A"] \\
 & A(E^*(\underline{E}_k)\widehat{\otimes} E^*(\underline{E}_m)) \arrow[d, "A\times", "\cong"'] \\
E^*(\underline{E}_k\times\underline{E}_m) \arrow[r, "\rho"] & AE^*(\underline{E}_k\times\underline{E}_m) \\
E^*(\underline{E}_{k+m}) \arrow[r, "\rho_{k+m}"] \arrow[u, "\phi^*"] & AE^*(\underline{E}_{k+m}) \arrow[u, "A\phi^*"]
\end{tikzcd}
$$

Figure 2. The monoidality of ρ.

In Theorem 7.12(b), C has comultiplications $\psi_C\colon C^{k+m} \to C^k\widehat{\otimes}C^m$ and a counit $\varepsilon_C\colon C^0 \to E^*$ which make $n \mapsto \mathit{FMod}(C^n, E^*(X)^\wedge)$ a graded ring. For each n, Theorem 5.12(c) provides a morphism $f^n\colon C^n \to PE^*(\underline{E}_n)$ in FMod. For the universal example (7.2), the multiplicativity $(\theta X)(xy) = ((\theta X)x)((\theta X)y)$ reduces to the commutativity of the outside of the diagram in fig. 3. The lower rectangle is diag. (7.3). It follows that the upper square commutes, so that f preserves the comultiplication. Similarly, $(\theta T)1 = 1$ yields fig. 4, which shows that f preserves the counit. □

$$
\begin{tikzcd}
C^{k+m} \arrow[r, "\psi_C"] \arrow[d, "f^{k+m}"] & C^k\widehat{\otimes}C^m \arrow[d, "f^k\otimes f^m"] \\
PE^*(\underline{E}_{k+m}) \arrow[r, "\psi_P"] \arrow[ddd, "\subset"] & PE^*(\underline{E}_k)\widehat{\otimes}PE^*(\underline{E}_m) \arrow[d, "\subset"] \\
 & E^*(\underline{E}_k)\widehat{\otimes}E^*(\underline{E}_m) \arrow[dd, "\cong"] \\
 & \\
E^*(\underline{E}_{k+m}) \arrow[r, "\phi^*"] & E^*(\underline{E}_k\times\underline{E}_m)
\end{tikzcd}
$$

Figure 3. Comparison of comultiplications.

PROOF OF THEOREM 7.11. We use the isomorphism (6.7) to translate the monoidal structure of A to A'. From ζ_A, which is given by [8, (7.11)], we obtain eq. (7.8). We have identified both z_A and $z_{A'}$ with the coaction ρ_T. □

$$\begin{tikzcd}
C^0 \arrow[r, "f^0"] \arrow[dr, "\varepsilon_C"'] & PE^*(\underline{E}_0) \arrow[r, "\subset"] \arrow[d, "\varepsilon_P"] & E^*(\underline{E}_0) \arrow[dl, "\eta^*"] \\
& E^* &
\end{tikzcd}$$

Figure 4. Comparison of counits.

PROOF OF LEMMA 7.5. Theorem 7.9(a) shows in particular that $\psi: A \to AA$ and $\varepsilon: A \to I$ are monoidal natural transformations. By the isomorphism (6.24), so are $\psi': A' \to A'A'$ and $\varepsilon': A' \to I$. Evaluation of the relevant diagrams involving ζ for $M = N = E^*$ show precisely that $Q(\psi)$ and $Q(\varepsilon)$ are multiplicative. Since $z_{A'} = \eta_R: E^h \to E^* \otimes Q(E)^h_* \cong Q(E)^h_*$, the two diagrams involving z show that $\psi 1 = 1 \otimes 1$ and $\varepsilon 1 = 1$, simply because $\eta_R 1$ is the unit element of $Q(E)^*_*$. □

PROOF OF THEOREM 7.7. Part (a) follows from Theorem 7.12(a). In (b), Theorem 6.36(c) provides for each n the E^*-module homomorphism $f^n: Q(E)^n_* \to B$ that induces $\theta X: E^n(X) \to E^*(X) \widehat{\otimes} B$. As in the proof of [8, Theorem 12.8(b)], the resulting $f: Q(E)^*_* \to B$ is an E^*-algebra homomorphism. □

Connectedness. There is a particular operation that is useful for expressing the concept of connectedness in a cohomology algebra. It sees only the path components of a space.

DEFINITION 7.13. For each n, we define the *collapse* operation $\kappa_n: n \to n$ as the map $\kappa_n: \underline{E}_n \to \underline{E}_n$ (well defined up to homotopy) that sends each path component of $\underline{E}_n$ to one point in that path component.

It is clearly additive, multiplicative ($\kappa(xy) = (\kappa x)(\kappa y)$), unital ($\kappa_0 1_X = 1_X$), and idempotent. It commutes with all operations in the sense that $\kappa_m \circ r = r \circ \kappa_k: k \to m$ for all $r: k \to m$; in particular, κ is E^*-linear. It is zero in any degree n for which $E^n = 0$. In spite of being defined in all degrees, it is not at all stable, as $\Omega\kappa_n = 0$. All these properties carry over to any additively unstable algebra M; in particular, we always have the E^*-module decomposition $M = \operatorname{Im}\kappa \oplus \operatorname{Ker}\kappa$, with $(E^*)1_M \subset \operatorname{Im}\kappa$.

For a connected space X with basepoint o, it is clear that the augmentation ideal $E^*(X, o) \subset E^*(X)$ is precisely $\operatorname{Ker}\kappa$. In general, $\operatorname{Ker}\kappa = F^1E^*(X)$ for any space X, the first stage of the skeleton filtration. This suggests the following definition.

DEFINITION 7.14. We call the additively unstable algebra M *connected* if $\operatorname{Im}\kappa = (E^*)1_M$. We call M *spacelike* if it is a product (in *FAlg*) of connected algebras.

In particular, for a space X, $E^*(X)\hat{}$ is always spacelike, and is connected if and only if X is connected.

8. What is an unstable object?

In this section, we interpret what it means to have an algebra over all the unstable operations on E-cohomology. Tensor products rapidly become unworkable for nonadditive operations, with the effect that only the First and Fourth Answers from Section 5 survive intact.

We generally assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k. Then Theorem 1.18 provides all the Künneth and duality isomorphisms and homeomorphisms we need. Of course, when we compare with the additive or stable theory, we impose the appropriate extra conditions.

As in (2.1), we identify:

(i) The *cohomology operation* $r: E^k(-) \to E^m(-)$;
(ii) The *class* $r = r(\iota_k) \in E^m(\underline{E}_k)$;
(iii) The representing *map* $r: \underline{E}_k \to \underline{E}_m$;

and write any of these as $r: k \to m$. (We shall retain the parentheses in $r(x)$ whenever r is nonadditive.)

We first deal with the *constant* operations $r: k \to m$, those of the form $r(x) = v1_X \in E^m(X)$ for all $x \in E^k(X)$ and all spaces X, where $v \in E^m$.

LEMMA 8.1. *Any operation $r: k \to m$ decomposes uniquely as the sum of a based operation $s: k \to m$ and a constant operation.*

PROOF. We set $v = r(0) \in E^m(T) = E^m$ and define the operation s by $s(x) = r(x) - v1_X$ in $E^*(X)$, to make $s(0) = 0$. □

First Answer. Since $E^k(-)$ is represented in *Ho* by $\underline{E}_k$, we have as in (5.1) the actions

$$\circ: E^m(\underline{E}_k) \times E^k(X) \longrightarrow E^m(X), \tag{8.2}$$

except that we cannot write them using tensor products. Instead, we need a Cartan formula for $r(x+y)$ as well as for $r(xy)$.

To find $r(x+y)$, we consider the abelian group object $\underline{E}_k$ of *Ho* provided by [8, Corollary 7.8], which is equipped with the addition map $\mu_k: \underline{E}_k \times \underline{E}_k \to \underline{E}_k$ and zero map $\omega_k: T \to \underline{E}_k$. By Lemma 8.1, we may restrict attention to based operations r. The group axioms on $\underline{E}_k$ lead (as in any Hopf algebra) to a formula of the form

$$\mu_k^* r = r \times 1 + \sum_\alpha r'_\alpha \times r''_\alpha + 1 \times r \quad \text{in } E^*(\underline{E}_k \times \underline{E}_k) \cong E^*(\underline{E}_k) \widehat{\otimes} E^*(\underline{E}_k),$$

where the r'_α and r''_α are also based. The only novelty is that the sum may be infinite. This translates into the desired Cartan formula

$$r(x+y) = r(x) + \sum_\alpha r'_\alpha(x) r''_\alpha(y) + r(y) \quad \text{in } E^*(X) \tag{8.3}$$

for any $x, y \in E^k(X)$.

There is a similar Cartan formula for multiplication, given $x \in E^k(X)$ and $y \in E^m(X)$, of the form

$$r(xy) = \sum_\alpha r'_\alpha(x)\, r''_\alpha(y) \quad \text{in } E^*(X), \tag{8.4}$$

for certain (other) based operations r'_α and r''_α (which depend on k and m).

This suggests that an unstable algebra should consist of an E^*-algebra M equipped with operations r that compose correctly and satisfy both Cartan formulae. This requires knowing the operations r'_α and r''_α in eqs. (8.3) and (8.4) for all r. In Section 10, we shall in effect expand both Cartan formulae explicitly.

Second Answer. We convert the First Answer to adjoint form, corresponding to the Fourth Answer in Section 5. (We skip the Second and Third Answers.) Everything becomes far cleaner, more evidence that this is the natural answer.

Any element $x \in E^k(X)$, regarded as a map $x: X \to \underline{E}_k$, induces the continuous homomorphism $x^*: E^*(\underline{E}_k) \to E^*(X)$ of E^*-algebras. By Theorem 1.18(a), $E^*(\underline{E}_k)$ is Hausdorff and so in FAlg; we may therefore define, for any object M of FAlg,

$$U^k M = \mathit{FAlg}\big(E^*(\underline{E}_k), M\big), \tag{8.5}$$

the set of all continuous E^*-algebra homomorphisms $E^*(\underline{E}_k) \to M$. This encodes the set of all possible actions on a typical element of degree k. We convert the action (8.2) to what we continue to call a coaction,

$$\rho_X: E^k(X) \longrightarrow U^k\big(E^*(X)\hat{}\big) = \mathit{FAlg}\big(E^*(\underline{E}_k), E^*(X)\hat{}\big), \tag{8.6}$$

by defining $\rho_X x = x^*$, completing $E^*(X)$ if necessary to get it into FAlg. We assemble the sets $U^k M$ to form the graded set UM, which has the component $(UM)^k = U^k M$ in degree k, and obtain $\rho_X: E^*(X) \to U(E^*(X)\hat{})$.

We compare UM with the stable and additive versions. Restriction to $PE^*(\underline{E}_k)$ induces the natural transformation

$$(\tau M)^k: U^k M = \mathit{FAlg}\big(E^*(\underline{E}_k), M\big) \longrightarrow \mathit{FMod}\big(PE^*(\underline{E}_k), M\big) = A^k M. \tag{8.7}$$

These form $\tau M: UM \to AM$. Composition with $\sigma M: AM \to SM$ (see eq. (5.7)) yields

$$U^k M = \mathit{FAlg}\big(E^*(\underline{E}_k), M\big) \longrightarrow \mathit{FMod}^k\big(E^*(E, o), M\big) = (SM)^k,$$

which is induced by the destabilization $\sigma_k^*: E^*(E, o) \to PE^*(\underline{E}_k) \subset E^*(\underline{E}_k)$.

Apparently only a morphism of graded sets, ρ_X has far more structure, thanks to the rich structure on the spaces $\underline{E}_k$.

THEOREM 8.8. *Assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *We can make the functor* U, *defined in eq.* (8.5), *a comonad in the category FAlg of filtered* E^**-algebras;*

(b) *If* $QE_*(\underline{E}_k)$ *is a free* E^**-module for all* k, $\tau: U \to A$ (see (8.7)) *is a morphism of comonads in FAlg;*

(c) *If* $E_*(E, o)$ *is a free* E^**-module,* $\sigma \circ \tau: U \to S$ (see (8.7) and (5.7)) *is a morphism of comonads in FAlg.*

Our main definition is now clear.

DEFINITION 8.9. An *unstable (E-cohomology) algebra* is just a U-coalgebra in *FAlg*, i.e. a complete Hausdorff filtered E^*-algebra M equipped with a continuous morphism $\rho_M: M \to UM$ of E^*-algebras that satisfies the coaction axioms [8, (8.7)]. We then *define* the action of $r \in E^*(\underline{E}_k)$ on $x \in M^k$ by $r(x) = \rho_M(x)r \in M$.

A closed ideal $J \subset M$ is called *(unstably) invariant* if the quotient algebra M/J inherits a well-defined unstable algebra structure from M.

It follows that the Cartan formulae (8.3) and (8.4) hold in M. The constant operations behave correctly because $\rho_M(x)$ is required to be a morphism of E^*-algebras. We need to be able to recognize invariant ideals.

LEMMA 8.10. *Given an unstable algebra* M, *a closed ideal* $J \subset M$ *is unstably invariant if and only if* $r(y) \in J$ *for all* $y \in J$ *and all based operations* r.

PROOF. To make $\rho_{M/J}$ well defined, we need $r(x+y) \equiv r(x) \bmod J$, for all $x \in M$ and $y \in J$. This is trivial for constant operations r, and so by Lemma 8.1, we need only check for based r. The stated condition is obviously necessary, by taking $x = 0$. It is also sufficient, by eq. (8.3). □

THEOREM 8.11. *Assume that* $E_*(\underline{E}_k)$ *is a free* E^**-module for all* k (which is true for $E = H(\mathbf{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then:*

(a) *For any space* X, *the coaction* (8.6) *factors through* $E^*(X)\hat{}$ *to make* $E^*(X)\hat{}$ *an unstable E-cohomology algebra;*

(b) *We recover the additively unstable coaction* (5.6) *from* ρ_X *as*

$$E^*(X) \xrightarrow{\rho_X} U(E^*(X)\hat{}) \xrightarrow{\tau} A(E^*(X)\hat{});$$

(c) *If* $E^*(E, o)$ *is Hausdorff, we recover the stable coaction* [8, (10.10)] *from* ρ_X *as*

$$E^*(X) \xrightarrow{\rho_X} U(E^*(X)\hat{}) \xrightarrow{\tau} A(E^*(X)\hat{}) \xrightarrow{\sigma} S(E^*(X)\hat{});$$

(d) ρ *is universal: given an object* B *of FAlg and an integer* k, *any natural transformation of sets* $\theta X: E^k(X) \to FAlg(B, E^*(X)\hat{})$ *(or* $\widehat{\theta}X: E^k(X)\hat{} \to FAlg(B, E^*(X)\hat{})$*), that* *is defined for all spaces X, is induced from ρ_X by a unique morphism $f\colon B \to E^*(\underline{E}_k)$ in FAlg as*

$$\theta X\colon E^k(X) \xrightarrow{\rho_X} U\big(E^*(X)\hat{}\big) = \mathit{FAlg}\big(E^*(\underline{E}_k), E^*(X)\hat{}\big) \xrightarrow{\mathrm{Mor}(f,1)} \mathit{FAlg}\big(B, E^*(X)\hat{}\big)$$

Proof of Theorems 8.8 and 8.11. The proof breaks up into the same seven steps as additively (and stably), in Theorems 5.8 and 5.12. However, it is far simpler than Theorems 7.9 and 7.12 on algebras, because we are able to treat the multiplicative and module structures together. At each step, we also discuss τ and $\tau \circ \sigma$, assuming the extra conditions hold.

Corollary 7.8 of [8] provides the E^*-algebra object $n \mapsto \underline{E}_n$ in *Ho*. We again write ρ_k for ρ_X when $X = \underline{E}_k$.

Step 1. We endow the functor U with an E^*-algebra structure. For each object M of *FAlg*, we observe that according to [8, Lemma 6.9], the functor

$$\mathit{FAlg}\big(E^*(-)\hat{}, M\big)\colon \mathit{Ho} \xrightarrow{E^*(-)\hat{}} \mathit{FAlg}^{\mathrm{op}} \xrightarrow{\mathrm{Mor}(-,M)} \mathit{Set}$$

preserves enough products that by [8, Lemmas 7.6(a), 7.7(a)] it takes the E^*-algebra object $n \mapsto \underline{E}_n$ to the E^*-algebra object UM in *Set*; i.e. UM is an E^*-algebra. It is clear that UM is functorial in M. We shall filter it in Step 3.

To see that τM is a homomorphism of E^*-modules, we apply [8, Lemma 7.6(c)] to the E^*-module object $n \mapsto \underline{E}_n$ in $Gp(Ho)$, using the natural transformation

$$\mathit{FAlg}\big(E^*(-)\hat{}, M\big) \longrightarrow \mathit{FMod}\big(PE^*(-)\hat{}, M\big)$$

defined by restriction. To see that τ is monoidal, we apply [8, Lemma 7.9(b)]. The monoidal structure of U is simply the multiplicative part of the algebra structure, and diag. (7.3) shows that the inclusions $PE^*(\underline{E}_n) \subset E^*(\underline{E}_n)$ form a morphism of graded monoid objects in $\mathit{FMod}^{\mathrm{op}}$. The units are correct by definition. For $\tau \circ \sigma$, we bypass $PE^*(\underline{E}_n)$ and use the duals of diags. (6.16) and (6.17) instead.

Step 2. In order to define ρ_X (in (8.6)) as a morphism of E^*-algebras, we consider the *Set*-valued natural transformation

$$\mathit{Ho}(X, -) \longrightarrow \mathit{FAlg}\big(E^*(-)\hat{}, E^*(X)\hat{}\big)$$

induced by $E^*(-)\hat{}\colon \mathit{Ho}^{\mathrm{op}} \to \mathit{FAlg}$. We apply [8, Lemma 7.6(c)] to the E^*-algebra object $n \mapsto \underline{E}_n$, to obtain Theorem 8.11(a). Then Theorem 8.11(b) is clear by comparing with the additive coaction (5.6), and for Theorem 8.11(c), we combine with Theorem 5.12(b).

Step 3. For U to take values in *FAlg*, we must filter UM. If M is filtered by the ideals $F^a M$, we filter UM by the ideals

$$F^a(UM) = \mathrm{Ker}\left[UM \longrightarrow U\left(\frac{M}{F^a M}\right)\right]$$

Just as stably, this filtration is complete Hausdorff and makes ρ_X continuous by naturality. This allows us to factor ρ_X through $E^*(X)\hat{}$. Similarly, τM and $\sigma M \circ \tau M$ are also filtered and therefore continuous.

Step 4. We convert the object $E^*(X)\hat{}$ of *FAlg* to the corepresented functor $F_X = \mathit{FAlg}(E^*(X)\hat{}, -): \mathit{FAlg} \to \mathit{Set}$. For example, when $X = \underline{E}_k$, $F_X = U^k$. As suggested by [8, (8.16)], we also convert the coaction ρ_X to the natural transformation $\rho_X: F_X \to F_X U: \mathit{FAlg} \to \mathit{Set}$. Given M in *FAlg*, $\rho_X M: F_X M \to F_X UM$ is thus defined on $f \in F_X M = \mathit{FAlg}(E^*(X)\hat{}, M)$ as

$$(\rho_X M)f = Uf \circ \rho_X: E^*(X)\hat{} \longrightarrow U\big(E^*(X)\hat{}\big) \longrightarrow UM, \tag{8.12}$$

an element of $F_X UM$.

Step 5. We define the natural transformation

$$\psi M: U^k M = \mathit{FAlg}\big(E^*(\underline{E}_k), M\big) \longrightarrow \mathit{FAlg}\big(E^*(\underline{E}_k), UM\big) = U^k UM \tag{8.13}$$

by taking $X = \underline{E}_k$ in eq. (8.12). On the element $f: E^*(\underline{E}_k) \to M$ of $U^k M$, it is

$$(\psi M)f: E^*(\underline{E}_k) \xrightarrow{\rho_k} UE^*(\underline{E}_k) \xrightarrow{Uf} UM.$$

(In terms of elements, this is $r \mapsto [s \mapsto f(r^* s) = f(sr)]$.) If we substitute the E^*-algebra object $n \mapsto \underline{E}_n$ for X in eq. (8.12), [8, Lemma 7.6(c)] shows that ψM takes values in *Alg*. Naturality in M shows that ψM is filtered and so takes values in *FAlg* as required.

Step 6. The other required natural transformation,

$$\varepsilon M: U^k M = \mathit{FAlg}\big(E^*(\underline{E}_k), M\big) \longrightarrow M,$$

is defined simply as evaluation on $\iota_k \in E^*(\underline{E}_k)$. As before, naturality in M shows that εM is filtered, but we have to calculate that ε is an E^*-algebra homomorphism.

Take any binary operation $s(-,-)$ in E^*-algebras (addition, multiplication, or any other), represented in *Ho* by the map $s: \underline{E}_k \times \underline{E}_m \to \underline{E}_q$, which therefore induces $s^*\iota_q = s(p_1^*\iota_k, p_2^*\iota_m)$. We need to show that the square

$$\begin{tikzcd} U^k M \times U^m M \arrow[r, "s"] \arrow[d, "\varepsilon\times\varepsilon"] & U^q M \arrow[d, "\varepsilon"] \\ M^k \times M^m \arrow[r, "s"] & M^q \end{tikzcd}$$

commutes. We evaluate on $f \in U^k M$ and $g \in U^m M$. Because $E^*(\underline{E}_k \times \underline{E}_m)$ is by [8, Lemma 6.9] the coproduct in *FAlg*, there is a unique $h: E^*(\underline{E}_k \times \underline{E}_m) \to M$ in *FAlg* such that $h \circ p_1^* = f$ and $h \circ p_2^* = g$. Then by definition of the algebra structure of UM, $s(f,g) = h \circ s^*: E^*(\underline{E}_q) \to M$. Since h is an algebra homomorphism,

$$\varepsilon s(f,g) = hs^*\iota_n = hs\big(p_1^*\iota_k, p_2^*\iota_m\big) = s\big(hp_1^*\iota_k, hp_2^*\iota_m\big) = s\big(f\iota_k, g\iota_m\big) = s\big(\varepsilon f, \varepsilon g\big).$$

For unary and 0-ary operations, we may adapt the above proof, or simply throw away any unwanted arguments. (For example, given $v \in E^*$, we could define the constant binary operation $s(x,y) = v1$ in any E^*-algebra, to deduce that $\varepsilon v = v$.)

Step 7. The proof that $E^*(X)^\circ$ is a U-coalgebra and that U is a comonad is formally identical to the stable case, except that we need versions of [8, Lemmas 8.20, 8.22] for graded objects.

We use [8, Lemma 8.24] to show that $\tau: U \to A$ is a natural transformation of comonads. We take R as $n \mapsto E^*(\underline{E}_n)$, R' as $n \mapsto PE^*(\underline{E}_n)$, $1_R = 1'_R$ as $n \mapsto \iota_n$, and $u: PE^*(\underline{E}_k) \subset E^*(\underline{E}_k)$ as the inclusion. The first hypothesis on u is the commutativity of the diagram

$$\begin{tikzcd}
PE^*(\underline{E}_k) \arrow[rr, "\subset"] \arrow[dd, "P\rho_k"] & & E^*(\underline{E}_k) \arrow[d, "\rho_k"] \\
& & FAlg(E^*(\underline{E}_k), E^*(\underline{E}_k)) \arrow[d] \\
FMod(PE^*(\underline{E}_k), PE^*(\underline{E}_k)) \arrow[rr, "\subset"] & & FMod(PE^*(\underline{E}_k), E^*(\underline{E}_k))
\end{tikzcd}$$

which is obvious by construction, as $r \in PE^*(\underline{E}_k)$ yields $r^*|PE^*(\underline{E}_k)$.

The proof of Theorem 8.11(d) is formally the same as stably. Since $E^k(-)$ is represented by $\iota_k \in E^k(\underline{E}_k)$, θ is classified by $f = (\theta\underline{E}_k)\iota_k \in FAlg(B, E^*(\underline{E}_k))$. □

9. Unstable, additive, and stable objects

In previous sections and [8], we constructed five different kinds of object: stable modules and algebras, additively unstable modules and algebras, and unstable algebras. In this section we compare them all. Unstable modules are conspicuous by their absence; Theorem 9.4 will show that they cannot be defined compatibly with our other objects.

Each kind of object is defined by a comonad. Theorems 8.8(b) and 7.9(b) provide natural transformations

$$U \xrightarrow{\tau} A \xrightarrow{\sigma} S \quad \text{in } FAlg \tag{9.1}$$

between the comonads that define unstable, additively unstable, and stable algebras. Theorem 5.8(b) provides the natural transformation

$$\overline{A} \xrightarrow{\overline{\sigma}} \overline{S} \quad \text{in } FMod \tag{9.2}$$

between the comonads that define additively unstable and stable modules (where we temporarily rename the module versions of A and S to $\overline{A}$ and $\overline{S}$). They are related to the algebra versions by the forgetful functor $V: FAlg \to FMod$, so that $VA = \overline{A}V$ and $VS = \overline{S}V$.

$$\begin{tikzcd}
\text{unstable algebras} \arrow[r, "\tau"] & \text{additively unstable algebras} \arrow[r, "\sigma"] \arrow[d, "V"] & \text{stable algebras} \arrow[d, "V"] \\
& \text{additively unstable modules} \arrow[r, "\overline{\sigma}"] & \text{stable modules}
\end{tikzcd}$$

Figure 5. Five kinds of object.

We have the category, e.g., U-coalgebras, of each kind of object. We consider the diagram of categories and functors in fig. 5. For example, a stable algebra B with coaction $\rho_B: B \to SB$ in *FAlg* yields the stable module VB with coaction $V\rho_B: VB \to VSB = \overline{S}VB$ in *FMod*.

THEOREM 9.3. *Assume that $E_*(\underline{E}_k)$, $QE_*(\underline{E}_k)$, and $E_*(E, o)$ are free E^*-modules for all k* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a) and [8, Lemma 9.21]). *Then we have the diagram fig. 5 of categories and functors.*

For any space X, $E^*(X)\hat{}$ is an object in each of the five categories, related by these functors.

PROOF. The last assertion combines Theorems 5.12, 7.12, and 8.11 with Theorems 10.16 and 12.13 of [8]. □

There is a glaring gap: we have not defined unstable modules. We now show that this gap cannot be filled, for rather silly reasons. In fact, the three most natural definitions are for stable modules, additively unstable modules, and unstable algebras. We can enrich the two kinds of module with multiplicative structure, but it is not possible to remove the multiplicative structure from the definition of unstable algebra. This is already strongly suggested by the appearance of multiplication in the Cartan formula (8.3) for $\tau(x+y)$.

We ignore most of the structure and the topology, fix k, and restrict attention to the two functors $U^k: \mathit{FAlg} \to \mathit{Ab}$ and $A^k: \mathit{FMod} \to \mathit{Ab}$ and the natural transformation $\tau^k: U^kV \to A^k$.

THEOREM 9.4. *Even in the classical case $E = H(\mathbb{F}_p)$, unstable modules do not exist in the sense that we cannot insert a suitable comonad $\overline{U}$ into diag.* (9.2). *Specifically, for fixed $k > 0$ there do not exist:*

(i) *a functor $\overline{U}^k: \mathit{FMod} \to \mathit{Ab}$;*

(ii) *a natural isomorphism $\overline{U}^k V \cong U^k: \mathit{FAlg} \to \mathit{Ab}$;*

(iii) *a natural transformation $\overline{\tau}^k: \overline{U}^k \to A^k$ of functors $\mathit{FMod} \to \mathit{Ab}$;*

such that on FAlg, $\overline{\tau}^k V: \overline{U}^k V \to A^k$ agrees with $\tau^k: U^k \to A^k$.

PROOF. We assume that $\overline{U}^k$ and $\overline{\tau}^k$ exist as stated and derive a contradiction. Given any (filtered) graded $\mathbb{F}_p$-module M, we construct the $\mathbb{F}_p$-algebra $M^+ = \mathbf{F}_p \oplus M$ with the unit element $1 \in \mathbf{F}_p$ and $xy = 0$ for all $x, y \in M$. Then M is a retract in *FMod* of VM^+ and we can compute $\overline{\tau}^k M$ from the commutative diagram

$$\begin{tikzcd}
 & & \mathit{FAlg}(\mathcal{A}_k, M^+) \arrow[d, "="] \\
\overline{U}^k M \arrow[r] \arrow[d, "\overline{\tau}^k M"] & \overline{U}^k VM^+ \arrow[r, "\cong"] \arrow[d, "\overline{\tau}^k VM^+"] & U^k M^+ \arrow[d, "\tau^k M^+"] \\
\overline{A}^k M \arrow[d, "="] & \overline{A}^k VM^+ \arrow[l] & A^k M^+ \arrow[l, "\cong"'] \arrow[d, "="] \\
\mathit{FMod}(P\mathcal{A}_k, M) & & \mathit{FMod}(P\mathcal{A}_k, M^+)
\end{tikzcd}$$

where $\mathcal{A}_k = H^*(\underline{H}_k)$. Because M^+ has no decomposables, every homomorphism $P\mathcal{A}_k \to M$ in the image of $\overline{\tau}^k M$ kills the decomposable elements of $P\mathcal{A}_k$ (of which there are many).

But for a general algebra B, $\tau^k B \colon U^k B \to A^k B$ does not have this property, e.g., $(\tau^k \mathcal{A}_k)\mathrm{id}_k \in A^k \mathcal{A}_k$ is the inclusion $P\mathcal{A}_k \subset \mathcal{A}_k$. Taking $M = VB$ shows that $\overline{\tau}^k VB$ does not agree with $\tau^k B$. □

Objects in ordinary cohomology. Theorem 9.4 demands an immediate explanation of our terminology even in the case of ordinary cohomology. We give details for $E = H(\mathbb{F}_2)$; the case $E = H(\mathbb{F}_p)$ for odd p is similar, with the usual changes.

The *Steenrod algebra* $\mathcal{A} = E^*(E, o)$ is exactly as expected: it is the $\mathbf{F}_2$-algebra generated by the Steenrod squares Sq^i for $i > 0$, subject to the standard Adem relations. It is useful to write $\mathrm{Sq}^0 = \iota$. We note that for $E = H(\mathbf{F}_2)$:

(i) σ_k^* makes $PE^*(\underline{E}_k)$ a *quotient* of $E^*(E, o)$;
(ii) $E^*(\underline{E}_k)$ is a *primitively generated* Hopf algebra.

Below, M is to be an object of *FMod* (or *FAlg*), i.e. a complete Hausdorff filtered graded $\mathbf{F}_2$-module (or commutative $\mathbf{F}_2$-algebra). Topological conditions apply (which we ignore for now). We list the five kinds of object we have defined, under our names for them:

(i) A *stable module* M is an $\mathcal{A}$-module.
(ii) A *stable algebra* M is both an $\mathbf{F}_2$-algebra and an $\mathcal{A}$-module that satisfies the Cartan formula

$$\mathrm{Sq}^k(xy) = \sum_{i=0}^{k} \left(\mathrm{Sq}^i x\right)\left(\mathrm{Sq}^{k-i} y\right) \quad \text{for } k > 0.$$

It follows by induction that $\mathrm{Sq}^k 1_M = 0$ for all $k > 0$.

(iii) An *additively unstable module* M is an $\mathcal{A}$-module that satisfies the extra condition

$$\mathrm{Sq}^i x = 0 \quad \text{for all } x \in M \text{ and all } i > \deg(x). \tag{9.5}$$

Since $\mathrm{Sq}^0 x = x$, it follows that $M^n = 0$ for all $n < 0$.

(iv) An *additively unstable algebra* is a stable algebra that satisfies (9.5).

(v) An *unstable algebra* M is a stable algebra that satisfies (9.5) as well as the extra condition

$$\mathrm{Sq}^k x = x^2 \quad \text{for } x \in M \text{ and } k = \deg(x).$$

The objects normally known as unstable modules appear here as *additively* unstable modules (although the word "additively" could well be omitted, there being no danger of confusion with something that does not exist).

However, we do have two kinds of unstable algebra. We emphasize that in (iv), the squaring operation $M^k \to M^{2k}$ given by $x \mapsto x^2$ (which looks additive but from our point of view is not, because it is defined only when M is an algebra) is unrelated to Sq^k.

We have equivalent comodule descriptions in terms of $E_*(E,o) = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots]$ and the corresponding bigraded algebra $Q(E)^*_* = \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \dots]$, which has polynomial generators $\xi_i \in Q(E)^1_{2^i}$ (as we shall see in Theorem 16.2):

(i) A *stable comodule* M has a coaction

$$\rho_M \colon M \longrightarrow M \mathbin{\widehat{\otimes}} E_*(E,o) = M \mathbin{\widehat{\otimes}} \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots]$$

that satisfies the usual axioms [8, (8.7)]. Then Sq^k is dual to ξ_1^k.

(ii) A *stable comodule algebra* M is both a stable comodule and a commutative $\mathbb{F}_2$-algebra, in such a way that ρ_M is an algebra homomorphism.

(iii) An *unstable comodule* M has coactions

$$\rho_M \colon M^k \longrightarrow M^i \mathbin{\widehat{\otimes}}_i Q(E)^k_i \subset M \mathbin{\widehat{\otimes}} \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \dots]$$

that satisfy the coaction axioms (6.33). The unstable operation $\mathrm{Sq}^i \colon k \to k+i$ is now dual to $\xi_0^{k-i}\xi_1^i$ for $i \leqslant k$, or is zero if $i > k$.

(iv) An *unstable comodule algebra* M is an unstable comodule that is also a commutative $\mathbb{F}_2$-algebra, in such a way that ρ_M is an algebra homomorphism.

The special features of $H(\mathbb{F}_2)$ allow us to handle unstable algebras too:

(v) For any $x \in M^k$, $\rho_M x$ contains the term $x^2 \otimes \xi_1^k$.

REMARK. There is one candidate for an unstable module, but it does not work. One could try defining $G^k M = \mathit{FMod}(E^*(\underline{E}_k), M)$ for any object M of FMod, with $\rho_X \colon E^k(X) \to G^k E^*(X)$ defined as usual, by $\rho_X x = x^*$. We would like ρ_X to be at least additive, but the standard additive structure on FMod does not give this.

Indeed, it is easy to see that in general *no* abelian group structure on $G^k M$ makes ρ_X additive (not even for $E = H(\mathbb{F}_p)$). By [8, Lemma 7.7(d)], such a structure would have to be induced by some morphism $\psi: E^*(\underline{E}_k) \to E^*(\underline{E}_k) \oplus E^*(\underline{E}_k)$ in *FMod*. Take any $r \in E^*(\underline{E}_k)$ and write $\psi r = (r', r'')$. Then additivity of ρ_X translates into $r(x+y) = r'x + r''y$ for all $x, y \in E^k(X)$, which is absurd unless r happens to be additive.

In fact, these objects appear to be particularly devoid of interest. In the case $E = H(\mathbf{F}_2)$, for example, they are modules equipped not only with Steenrod squares Sq^i that behave as expected, but also operations such as $x \mapsto (\mathrm{Sq}^2 x)(\mathrm{Sq}^3 x)$, without having cup products.

10. Enriched Hopf rings

In Definition 8.9 we condensed all the structure of an unstable algebra down to the single word *U-coalgebra*. In this section, we unpack the information again to give a complete description of an unstable algebra in the language of Hopf rings, enriched with certain additional structure. This description is summarized in Theorem 10.47, which may be regarded as the unstable analogue of Theorem 11.14 of [8] and Theorem 6.36. Indeed, we find a whole new paradigm for handling unstable operations, making computations with them reasonably practical and efficient. It serves as the true successor to the Second Answer of Section 5 and [8, §10].

We assume in this section that $E_*(\underline{E}_k)$ is a free E^*-module for all k, which is true for our five examples by Lemma 4.17(a). Thus all the results of Section 8 are available, and by [8, Lemma 6.16(c)], the topological dual $FMod^*(E^*(\underline{E}_k), E^*)$ of $E^*(\underline{E}_k)$ is $E_*(\underline{E}_k)$.

We shall consistently identify (with some abuse of notation):

(i) the *cohomology operation* $r: E^k(-) \to E^m(-)$;
(ii) the *cohomology class* $r(\iota_k) \in E^m(\underline{E}_k)$, which we often write simply as $r \in E^m(\underline{E}_k)$; (10.1)
(iii) the representing *map of spaces* $r: \underline{E}_k \to \underline{E}_m$;
(iv) the *E^*-linear functional* $\langle r, - \rangle: E_*(\underline{E}_k) \to E^*$ of degree m.

REMARK. In some situations, these identifications can obscure the correct signs in formulae. Considered as a cohomology class or functional, r has degree m, while its degree as an operation is $m-k$, and as a map of spaces, r has no degree at all.

In any unstable algebra M, including $E^*(X)\hat{}$ for any space X, Definition 8.9 gives, for each $x \in M^k$, the homomorphism $\rho_M(x): E^*(\underline{E}_k) \to M$. Then we defined $r(x) = \rho_M(x)r \in M$ for any operation (i.e. class) $r \in E^*(\underline{E}_k)$. In practice, we find it more convenient to revert to the First Answer $r(x)$ of Section 8, although the Second Answer, in terms of ρ_M, will continue to inform us as to what to do, even when only implicit. Classically, one investigates cohomology operations by studying what happens to $r(x)$ when r is fixed and x varies; but it is clear from Section 8 that what we should do is fix x and allow r to vary.

Linear functionals. We need to develop a *computational* description of ρ_M in an unstable algebra M. We start from the fact that $\rho_M(x)$ is E^*-linear, i.e. $r(x)$ is E^*-linear in r.

DEFINITION 10.2. Let M be an unstable algebra, and fix an element $x \in M^k$. We say $r(x)$ is written *in standard form* if

$$r(x) = \sum_\alpha \langle r, c_\alpha \rangle\, x_\alpha \quad \text{in } M \text{ (for all } r), \tag{10.3}$$

for suitable choices $c_\alpha \in E_*(\underline{E}_k)$ and $x_\alpha \in M$, where $\deg(x_\alpha) = -\deg(c_\alpha)$. If the sum is infinite, we require each ideal $F^a M$ in the filtration of M to contain all except finitely many of the x_α.

This is the closest we will come to an unstable replacement for the tensor products and homomorphisms of Section 6 *and* [8, §11]. Our convention here and in all similar formulae is that r runs through all unstable cohomology operations having the correct domain degree (different in nearly every formula, and rarely specified) but arbitrary target degree. The indexing set for α is often left implicit.

It is easy to achieve eq. (10.3) in the universal form

$$r(x) = \sum_\alpha \langle r, c_\alpha \rangle\, r_\alpha(x) \quad \text{in } M \text{ (for all } r), \tag{10.4}$$

by allowing c_α to run through some basis of $E_*(\underline{E}_k)$, which forces us to take $x_\alpha = r_\alpha(x)$, where r_α denotes the operation (linear functional) dual to c_α. Continuity of $\rho_M(x)\colon E^*(\underline{E}_k) \to M$ assures the finiteness condition in Definition 10.2. We may therefore always assume that $r(x)$ is written in standard form.

Where we depart from tradition is in *not* picking a definite basis of $E_*(\underline{E}_k)$ in advance. We do not even insist on the c_α being linearly independent. Nor do we require the c_α to span; we may obviously omit zero terms. This does not affect the linearity of eq. (10.3) and allows the flexibility that our formulae require. One consequence is that *most cohomology operations will never acquire names.*

We have the analogue of Proposition 6.44.

PROPOSITION 10.5. *Given $x \in E^k(X)$, regarded as a map of spaces $x\colon X \to \underline{E}_k$, assume that $r(x)$ is given by eq.* (10.3). *Then $x_*\colon E_*(X) \to E_*(\underline{E}_k)$ is given by*

$$x_* z = \sum_\alpha (-1)^{\deg(c_\alpha)(\deg(x_\alpha)+\deg(z))} \langle x_\alpha, z\rangle\, c_\alpha = \sum_\alpha c_\alpha \langle x_\alpha, z\rangle.$$

The nonuniqueness in eq. (10.3) is really not a problem because we are using it to *describe*, not *define* the structure on M. The real definitions are all in Section 8; here, we are only reinterpreting them. Nevertheless, it is easy to convert one standard form to another.

LEMMA 10.6. *Any standard form* (10.3) *can be transformed into the universal form* (10.4), *and hence into any other standard form, by iterating three kinds of replacement* (*in either direction*):

$$\text{(i)} \qquad \langle r, c + c' \rangle\, x' = \langle r, c \rangle\, x' + \langle r, c' \rangle\, x';$$

$$\text{(ii)} \qquad \langle r, vc \rangle\, x' = (-1)^{\deg(c)\deg(v)} \langle r, c \rangle\, vx';$$

$$\text{(iii)} \quad \langle r, c \rangle\, x' + \langle r, c \rangle\, x'' = \langle r, c \rangle\, (x' + x'').$$

(*Infinitely many replacements may be needed; however, each $F^a M$ contains x' for all except finitely many of them.*)

Stabilization. We need to record how eq. (10.3) behaves when we restrict the operation r to be additive or stable. We recall from [8, Definition 9.3] the stabilization homomorphism $\sigma_{k*}: E_*(\underline{E}_k) \to E_*(E, o)$ and from eq. (6.2) the algebraic homomorphism $q_k: E_*(\underline{E}_k) \to Q(E)^k_*$, both of which have degree k under our conventions.

LEMMA 10.7. *Let M be an unstable algebra, and assume that $r(x)$ is expressed in the standard form* (10.3), *where $x \in M^k$. Then:*

(a) *The unstable comodule coaction $\rho_M: M^k \to M \widehat{\otimes} Q(E)^k_*$ is given by*

$$\rho_M x = \sum_\alpha (-1)^{\deg(x_\alpha)(k - \deg(x_\alpha))} x_\alpha \otimes q_k c_\alpha \quad \text{in } M \widehat{\otimes} Q(E)^k_*,$$

provided $QE_*(\underline{E}_k)$ is a free E^*-module;

(b) *The stable comodule coaction $\rho_M: M \to M \widehat{\otimes} E_*(E, o)$ is given by*

$$\rho_M x = \sum_\alpha (-1)^{\deg(x_\alpha)(k - \deg(x_\alpha))} x_\alpha \otimes \sigma_{k*} c_\alpha \quad \text{in } M \widehat{\otimes} E_*(E, o),$$

provided $E_*(E, o)$ is a free E^*-module.

The signs are as expected, once we remember that if $\deg(x_\alpha) = i$, then $\deg(c_\alpha) = -i$ and $\deg(q_k c_\alpha) = \deg(\sigma_{k*} c_\alpha) = k - i$.

PROOF. For additive r, Proposition 6.11 converts eq. (10.3) to $r_A x = \sum_\alpha \langle r_Q, q_k c_\alpha \rangle x_\alpha$. We deduce $\rho_M x$ in (a) by comparing eqs. (6.38) and (6.39). Part (b) is similar, using [8, (11.18), (11.19)] instead. □

Unstable algebra structure. Our task is to convert all the algebraic structure of an unstable algebra M in Definition 8.9 into the current context. There are in effect four pairs of axioms:

(a) Two axioms to make $\rho_M(x): E^*(\underline{E}_k) \to M$ an E^*-algebra homomorphism, rather than merely E^*-linear: $(r \smile s)(x) = r(x)s(x)$ and $1(x) = 1_M$, which will become eqs. (10.14) and (10.15);

(b) Two axioms to make $\rho_M: M \to UM$ E^*-linear: $\rho_M(x+y) = \rho_M(x) + \rho_M(y)$ and $\rho_M(vx) = v\rho_M(x)$, which will become eqs. (10.20) and (10.16);
(c) Two axioms to make $\rho_M: M \to UM$ multiplicative: $\rho_M(1_M) = 1_{UM}$ and $\rho_M(xy) = \rho_M(x)\rho_M(y)$, which will become eqs. (10.41) and (10.34);
(d) Two axioms to make M a U-coalgebra: $(sr)(x) = s(r(x))$ and $\iota_k x = x$, which will become eqs. (10.45) and (10.43).

The natural language for expressing the first three pairs is that of Hopf rings, while the last requires some additional structure.

Hopf rings. We recall from [8, Lemma 6.12] that in *Coalg*, tensor products of coalgebras serve as products and E^* is the terminal object. A commutative (graded) ring object in *Coalg* is called a *Hopf ring over* E^*. (The terminology and some of the notation were suggested by Milgram [17]; see [23, §1] for a detailed exposition.)

We start from the E^*-algebra object $n \mapsto \underline{E}_n$ in *Ho* provided by [8, Corollary 7.8]. We apply [8, Lemma 7.6(a)], using the homology functor $E_*(-)$, which takes values in *Coalg* on the spaces we need and preserves enough products to make $n \mapsto E_*(\underline{E}_n)$ an E^*-algebra object in *Coalg*. In particular, this is an E^*-module object, and each $E_*(\underline{E}_k)$ is an abelian group object in *Coalg* and thus a Hopf algebra.

There are seven parts to the Hopf ring structure of $n \mapsto E_*(\underline{E}_n)$: two from the coalgebra, three from the abelian group object $\underline{E}_k$, and two from the multiplicative monoid object, in addition to the underlying E^*-module structure on E-homology. They are as follows (for each k and m, where relevant):

(i) $\psi: E_*(\underline{E}_k) \to E_*(\underline{E}_k) \otimes E_*(\underline{E}_k)$, the *comultiplication* induced by the diagonal map $\Delta: \underline{E}_k \to \underline{E}_k \times \underline{E}_k$;
(ii) $\varepsilon: E_*(\underline{E}_k) \to E^*$, the *counit* for ψ, induced by the map $q: \underline{E}_k \to T$;
(iii) $*: E_*(\underline{E}_k) \otimes E_*(\underline{E}_k) \to E_*(\underline{E}_k)$, a *multiplication*, induced by the addition map $\mu_k: \underline{E}_k \times \underline{E}_k \to \underline{E}_k$;
(iv) $1_k = \omega_{k*}1 \in E_0(\underline{E}_k)$, the $*$-*unit element*, induced by the zero map $\omega_k: T \to \underline{E}_k$;
(v) $\chi: E_*(\underline{E}_k) \to E_*(\underline{E}_k)$, the *canonical (anti)automorphism* of the Hopf algebra $E_*(\underline{E}_k)$, induced by the inversion map $\nu_k: \underline{E}_k \to \underline{E}_k$;
(vi) $\circ: E_*(\underline{E}_k) \otimes E_*(\underline{E}_m) \to E_*(\underline{E}_{k+m})$, another *multiplication*, induced by the multiplication map $\phi: \underline{E}_k \times \underline{E}_m \to \underline{E}_{k+m}$;
(vii) $[1] = \eta_*1 \in E_0(\underline{E}_0)$, the $\circ$-*unit element*, induced by the algebra unit map $\eta: T \to \underline{E}_0$.

Because $n \mapsto E_*(\underline{E}_n)$ is an E^*-algebra object rather than merely a ring object, we have, for each $v \in E^h$, the actions $(\xi v)_*: E_*(\underline{E}_k) \to E_*(\underline{E}_{k+h})$. As in Section 6, this reduces to a simpler structure.

DEFINITION 10.8. We define the *right unit* function $\eta_R: E^* \to E_*(\underline{E}_*)$. We regard $v \in E^h = E^h(T)$ as a map $v: T \to \underline{E}_h$, and use the induced homomorphism $v_*: E^* \cong E_*(T) \to E_*(\underline{E}_h)$ to define $[v] = v_*1 \in E_0(\underline{E}_h)$ and $\eta_R(v) = [v]$.

In particular, this includes $[1] = \eta_*1$ as in (vii), and $[0_k] = \omega_{k*}1 = 1_k$ as in (iv). It is clear from Definition 6.19 and [8, Definition 11.2] that $q_h[v]$ and $\sigma_{h*}[v]$ are the additive and stable versions of $\eta_R v$. The elements $[v]$ determine the E^*-module object structure completely, because when we apply E-homology to [8, (7.5)], we obtain

$$(\xi v)_* c = [v] \circ c \quad \text{for all } c \in E_*(\underline{E}_*). \tag{10.9}$$

For the sake of completeness, we list all 33 laws that a Hopf ring satisfies, beyond the usual axioms for an E^*-module. (Your count may vary.) Most need no comment. They are as follows, where in several we write $\psi c = \sum_i c_i' \otimes c_i''$:

(i) The five operations are (bi)additive: $\psi(b+c) = \psi b + \psi c$, $\varepsilon(b+c) = \varepsilon b + \varepsilon c$, $(a+b) * c = a*c + b*c$, $\chi(b+c) = \chi b + \chi c$, and $(a+b) \circ c = a \circ c + b \circ c$;

(ii) The five operations are E^*-linear: $\psi(vc) = \sum_i v c_i' \otimes c_i''$, $\varepsilon(vc) = v\varepsilon c$, $(vb) * c = v(b * c)$, $\chi(vc) = v\chi c$, and $(vb) \circ c = v(b \circ c)$, for all $v \in E^*$;

(iii) Three coalgebra axioms: ψ is coassociative and cocommutative (with the standard sign), and ε is a counit: $\sum_i (\varepsilon c_i') c_i'' = c$;

(iv) The five parts of the ring object structure respect ψ: $\psi(b * c) = (\psi b) * (\psi c)$ (where we give $E_*(\underline{E}_k) \otimes E_*(\underline{E}_k)$ the obvious $*$-multiplication, with signs), $\psi(b \circ c) = (\psi b) \circ (\psi c)$ (similarly), $\psi 1_k = 1_k \otimes 1_k$, $\psi \chi c = \sum_i \chi c_i' \otimes \chi c_i''$, and $\psi[1] = [1] \otimes [1]$;

(v) The five parts of the ring object structure respect ε: $\varepsilon(b * c) = (\varepsilon b)(\varepsilon c)$, $\varepsilon 1_k = 1$, $\varepsilon \chi c = \varepsilon c$, $\varepsilon(b \circ c) = (\varepsilon b)(\varepsilon c)$, and $\varepsilon[1] = 1$;

(vi) Four abelian group object axioms: associativity $(a * b) * c = a * (b * c)$, commutativity $b * c = (-1)^{ij} c * b$ (where $i = \deg(b)$, $j = \deg(c)$), unit $1_k * c = c$, and inverse $\sum_i \chi c_i' * c_i'' = (\varepsilon c) 1_k$;

(vii) Three axioms for a commutative monoid: associativity $(a \circ b) \circ c = a \circ (b \circ c)$, commutativity, which takes the somewhat complicated form (see [23, Lemma 1.12(c)(v)])

$$b \circ c = (-1)^{ij} \chi^{km} c \circ b \tag{10.10}$$

for $b \in E_i(\underline{E}_k)$ and $c \in E_j(\underline{E}_m)$ (where $\chi^{km} = \chi$ if k and m are odd, and is the identity otherwise, as in Proposition 10.12(b) below), and $[1] \circ c = c$;

(viii) Three ring object axioms to state that $- \circ c$ respects the abelian group object structure: for addition, which yields the distributive law, in the complicated form [ibid. (vi)]

$$(a * b) \circ c = \sum_i (-1)^{\deg(c_i') \deg(b)} a \circ c_i' * b \circ c_i''; \tag{10.11}$$

for the zero, $1_m \circ c = (\varepsilon c) 1_{m+k}$ [ibid. (ii)]; and for the inverse, $\chi(b \circ c) = (\chi b) \circ c$.

Many standard laws follow from these axioms. In order to simplify notation in eq. (10.11) and elsewhere, we give $\circ$-multiplication greater binding strength than $*$-multiplication, so that $a * b \circ c$ always means $a * (b \circ c)$, never $(a * b) \circ c$. In all our Hopf rings, Proposition 11.2 will provide the laws relating the added elements $[v]$ and identify the useful element $\chi[1]$ with $[-1]$.

Proposition 10.12. *In any Hopf ring, the operation χ has the following properties:*

(a) $\chi c = \chi[1] \circ c$, *so that* $\chi[1]$ *determines* χ;

(b) $\chi\chi c = c$;

(c) $\chi(a * b) = \chi a * \chi b$;

(d) $\chi[1] \circ \chi[1] = [1]$.

Proof. For (a), $\chi c = \chi([1] \circ c) = \chi[1] \circ c$. Since $\psi[1] = [1] \otimes [1]$ and hence $\psi\chi[1] = \chi[1] \otimes \chi[1]$, the distributive law gives (c), by

$$\chi(a * b) = \chi[1] \circ (a * b) = \chi[1] \circ a * \chi[1] \circ b = \chi a * \chi b.$$

Also, we have $\chi[1] * [1] = 1_0$ and similarly $\chi\chi[1] * \chi[1] = 1_0$, which yield

$$\chi\chi[1] = \chi\chi[1] * 1_0 = \chi\chi[1] * \chi[1] * [1] = 1_0 * [1] = [1].$$

But (a) gives $\chi\chi[1] = \chi[1] \circ \chi[1]$, and hence (d) and the general case of (b). □

Generators. We wish to use the laws to reduce any element of a Hopf ring to some standard form. The distributive law (10.11) plays a key role. We shall describe our Hopf rings H by specifying two sets of elements:

(i) the $\circ$-*generators* of H;
(ii) the $*$-*generators* of H, each of which is a $\circ$-product of $\circ$-generators and possibly $\chi[1]$, where we allow the empty $\circ$-product $[1]$.

We require every element of H to be an E^*-linear combination of $*$-products of the $*$-generators of H; in other words, the $*$-generators generate H as an E^*-algebra. For each $\circ$-generator g, we need formulae for ψg (so we can expand eq. (10.11)), εg, and χg. Although Hopf rings tend to be huge, each of our examples (see Section 17) has a conveniently small set of $\circ$-generators.

Hopf rings over $\mathbb{F}_p$. One can define the *Frobenius* operator $Fc = c^{*p}$ in any algebra with multiplication $*$, and it is multiplicative if $*$ is commutative. It is additive if also the ground ring has characteristic p. It is most useful when the ground ring is $\mathbb{F}_p$, because it is then automatically $\mathbb{F}_p$-linear. Commutativity of $*$-multiplication implies that $Fc = 0$ whenever c has odd degree (unless $p = 2$).

Moreover, in a Hopf ring (or cocommutative Hopf algebra) H over $\mathbb{F}_p$, one has dually the *Verschiebung* operator $V: H \to H$, defined so that $DV = F: DH \to DH$ in the dual Hopf algebra. It divides degrees by p. Then $Vc = 0$ unless $\deg(c)$ is divisible by $2p$ (if $p \neq 2$). Both F and V preserve all the Hopf algebra structure: $F(a * c) = Fa * Fc$, $F1_k = 1_k$, $\psi Fc = (F \otimes F)\psi c$, $\varepsilon Fc = \varepsilon c$, and dually $V(a * c) = Va * Vc$, $V1_k = 1_k$, $\psi Vc = (V \otimes V)\psi c$, and $\varepsilon Vc = \varepsilon c$. For $\circ$-products, we can iterate eq. (10.11) and obtain the identity

$$a \circ (Fc) = F(Va \circ c), \tag{10.13}$$

which is useful for reducing elements of the Hopf ring to standard form. (Normally, a and c both have even degree.)

Multiplication of operations. The first pair of axioms on M we listed earlier, that for fixed $x \in M$, $\rho_M(x)$ is a homomorphism of E^*-algebras, is easily translated into Hopf rings. Because the diagonal map in $\underline{E}_k$ induces both the cup product $r \smile s$ and the comultiplication ψ on $E_*(\underline{E}_k)$, we can write down the cup product from eq. (10.3) as

$$(r \smile s)(x) = \sum_\gamma \langle r \smile s, c_\gamma\rangle\, x_\gamma = \sum_\gamma \langle r \otimes s, \psi c_\gamma\rangle\, x_\gamma \quad \text{in } M.$$

The product $r(x)s(x)$ becomes, after some shuffling,

$$r(x)s(x) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(x_\beta)} \langle r \otimes s, c_\alpha \otimes c_\beta\rangle\, x_\alpha x_\beta.$$

Since $(r \smile s)(x) = r(x)s(x)$ has to hold for all r and s, we deduce the identity

$$\sum_\gamma \psi c_\gamma \otimes x_\gamma = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(x_\beta)} c_\alpha \otimes c_\beta \otimes x_\alpha x_\beta \tag{10.14}$$

in $(E_*(\underline{E}_*) \otimes E_*(\underline{E}_*)) \widehat{\otimes} M$, where the tensor products are formed using only the usual left E^*-actions.

The identity element $1_k \in E^0(\underline{E}_k)$ is the constant operation $E^k(X) \to E^0(X)$ that sends everything to 1_X; regarded as a linear functional, it is simply ε. In terms of eq. (10.3), the axiom $1_k(x) = 1_M$ becomes

$$\sum_\alpha (\varepsilon c_\alpha)\, x_\alpha = 1_M \quad \text{in } M. \tag{10.15}$$

Linear structure. We next decode the statement that $\rho_M: M \to UM$ is linear, namely that $\rho_M(x+y) = \rho_M(x) + \rho_M(y)$ and $\rho_M(vx) = v\rho_M(x)$. Related to the first is the formula for $r_*(b * c)$, which can be shown to be the translation of the statement that $\psi M: UM \to UUM$ is additive. We assume that $r(x)$ is given by eq. (10.3), where $x \in M^k$.

The v-action $U^k M \to U^{k+h} M$ was given by composing with $(\xi v)^*: E^*(\underline{E}_{k+h}) \to E^*(\underline{E}_k)$; dually, we use eq. (10.9) to translate $\rho_M(vx) = v\rho_M(x)$ into

$$r(vx) = \sum_\alpha \langle r, [v] \circ c_\alpha\rangle x_\alpha \quad \text{in } M \text{ (for all } r). \tag{10.16}$$

For addition, the idea is that $\mu_k: \underline{E}_k \times \underline{E}_k \to \underline{E}_k$ induces both the additive structure in UM and the $*$-multiplication in $E_*(\underline{E}_k)$. Of course, $r_* c$ is not additive in r. Given two operations $r, s: k \to m$, their sum may be constructed as

$$r + s: \underline{E}_k \xrightarrow{\Delta} \underline{E}_k \times \underline{E}_k \xrightarrow{r \times s} \underline{E}_m \times \underline{E}_m \xrightarrow{\mu_m} \underline{E}_m,$$

as we can check by composing with $x: X \to \underline{E}_k$. When we apply E-homology, we find

$$(r+s)_*c = \sum_i r_*c_i' * s_*c_i'' \quad \text{in } E_*(\underline{E}_m), \tag{10.17}$$

if we write $\psi c = \sum_i c_i' \otimes c_i''$ for $c \in E_*(\underline{E}_k)$. (In other words, we add r_* and s_* according to the group structure on $Mod(E_*(\underline{E}_k), E_*(\underline{E}_m))$ described by Milnor and Moore in [19, Definition 8.1], which makes use of the coalgebra structure of $E_*(\underline{E}_k)$ and the algebra structure of $E_*(\underline{E}_m)$.) To add more than two operations, we need *iterated* coproducts: given any finite indexing set Λ, we write the iterated comultiplication $\Psi: E_*(\underline{E}_k) \to \bigotimes_{\alpha\in\Lambda} E_*(\underline{E}_k)$ in the form

$$\Psi c = \sum_i \bigotimes_\alpha c_{i,\alpha} \quad \text{in } \bigotimes_{\alpha\in\Lambda} E_*(\underline{E}_k) \tag{10.18}$$

for suitable elements $c_{i,\alpha} \in E_*(\underline{E}_k)$. We can of course replace $\underline{E}_k$ by any space for which we have the necessary Künneth formulae.

THEOREM 10.19. *Let M be an unstable algebra and assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k. Take $x, y \in M^k$ and assume that $r(x)$ is in the standard form* (10.3). *Then:*

(a) *We have the* Cartan formula for addition

$$r(x+y) = \sum_\alpha x_\alpha r_\alpha''(y) \quad \textit{for all } r: k \to m, \tag{10.20}$$

where for each α, the operation $r_\alpha'': k \to m + \deg(c_\alpha)$ is defined as having the functional

$$\langle r_\alpha'', c\rangle = (-1)^{\deg(c_\alpha)(m+\deg(c_\alpha))}\langle r, c_\alpha * c\rangle \quad \textit{for all } c \in E_*(\underline{E}_k); \tag{10.21}$$

(b) *If, similarly, $r(y)$ has the standard form*

$$r(y) = \sum_\beta \langle r, d_\beta\rangle\, y_\beta, \tag{10.22}$$

then we have the full Cartan formula for addition,

$$r(x+y) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)}\langle r, c_\alpha * d_\beta\rangle\, x_\alpha y_\beta \tag{10.23}$$

for all $r: k \to m$;

(c) *Assume $a, b \in E_*(\underline{E}_k)$. Let c_α run through a basis of $E_*(\underline{E}_k)$, and denote by r_α' the operation dual to c_α. Let $\Psi a = \sum_i \bigotimes_\alpha a_{i,\alpha}$ and $\Psi b = \sum_j \bigotimes_\alpha b_{j,\alpha}$ be the iterated coproducts of a and b as in eq.* (10.18), *where in both cases, we ignore those α for which*

$$r_{\alpha *}' a_{i,\alpha} = (\varepsilon a_{i,\alpha})1 \quad \textit{for all } i \tag{10.24}$$

(see the Remark following). *Then the homology homomorphism* $r_*: E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ *satisfies*

$$r_*(a * b) = \sum_i \sum_j \pm \mathop{\Huge *}_\alpha r'_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} \quad \textit{in } E_*(\underline{E}_m), \tag{10.25}$$

where r''_α *is defined by eq.* (10.21) *and the only signs come from shuffling the factors to form* $\Psi(a \times b)$.

REMARK. The formula (10.25) demands some explanation. The proof will show that the relevant set of α is in fact finite, so that the iterated coproducts Ψa and Ψb are defined.

If α satisfies eq. (10.24), we have

$$r'_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} = (\varepsilon a_{i,\alpha}) 1 \circ r''_{\alpha *} b_{j,\alpha} = (\varepsilon a_{i,\alpha}) \varepsilon r''_{\alpha *} b_{j,\alpha} = (\varepsilon a_{i,\alpha})(\varepsilon b_{j,\alpha}) 1_m.$$

In the usual (and sufficient) case when $\varepsilon a = \varepsilon b = 0$, we can easily arrange for each $a_{i,\alpha}$ and $b_{j,\alpha}$ to be 1 or lie in $\operatorname{Ker} \varepsilon$, by breaking up terms and shuffling as necessary. Then the ij-term contributes nothing to $r_*(a * b)$ unless $a_{i,\alpha} = 1$ and $b_{j,\alpha} = 1$ for all $\alpha \in \Lambda$ that satisfy eq. (10.24). Such an index α may be omitted from the $*$-product in eq. (10.25) and the iterated coproducts Ψa and Ψb.

PROOF. We first assume that the c_α form a basis of $E_*(\underline{E}_k)$, so that $x_\alpha = r'_\alpha(x)$ as in eq. (10.4). By the Künneth homeomorphism, we can write

$$\mu_k^* r = \sum_\alpha r'_\alpha \times r''_\alpha \quad \text{in } E^*(\underline{E}_k \times \underline{E}_k), \tag{10.26}$$

for uniquely determined elements $r''_\alpha \in E^*(\underline{E}_k)$. In other words, in the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "x+y"'] & \underline{E}_k \times \underline{E}_k \arrow[r, "r'_\alpha \times r''_\alpha"] \arrow[d, "\mu_k"] & \underline{E}_? \times \underline{E}_? \arrow[d, "\phi"] \\ & \underline{E}_k \arrow[r, "r"] & \underline{E}_m \end{tikzcd} \tag{10.27}$$

the map $r \circ \mu_k$ is expressed as the sum of the maps $g_\alpha = \phi \circ (r'_\alpha \times r''_\alpha)$, and is the universal example for computing $r(x+y)$, where $u: X \to \underline{E}_k \times \underline{E}_k$ has coordinates $x: X \to \underline{E}_k$ and $y: X \to \underline{E}_k$. Evaluation on $c_\alpha \times c$ identifies r''_α as in eq. (10.21), with the help of

$$\langle \mu_k^* r, c_\alpha \times c \rangle = \langle r, \mu_{k*}(c_\alpha \times c) \rangle = \langle r, c_\alpha * c \rangle.$$

Then eq. (10.20) is induced from eq. (10.26). To deduce (b), we substitute eq. (10.22) in eq. (10.20) and watch the signs.

To remove the requirement that the c_α form a basis, we note that by linearity, eq. (10.20) is preserved by each of the replacements listed in Lemma 10.6. (The operation r'_α is no longer defined, but appears only in (c).)

For (c), we apply homology everywhere. We have to add the homomorphisms $g_{\alpha *}$ in the sense of eq. (10.17), using the iterated coproduct $\Psi(a \times b)$, which is obtained from $\Psi a \times \Psi b$ by shuffling. We note that any $a \in E_*(\underline{E}_k)$ comes from some finite subcomplex Y of $\underline{E}_k$. All but finitely many of the r'_α vanish on Y, by the strong duality for $\underline{E}_k$; these α satisfy eq. (10.24), as we see by computing the iterated coproduct Ψa first in Y, since the zero operation $0: k \to m$ induces $0_* c = (\epsilon c) 1_m$. □

Similarly, the zero map $\omega_k: T \to \underline{E}_k$ and inversion map $\nu_k: \underline{E}_k \to \underline{E}_k$ of $\underline{E}_k$ yield the useful formulae

$$r(0_k) = \langle r, 1_k \rangle\, 1_M \quad \text{in } M \text{ (for all } r) \tag{10.28}$$

and

$$r(-x) = \sum_\alpha \langle r, \chi c_\alpha \rangle\, x_\alpha \quad \text{in } M \text{ (for all } r). \tag{10.29}$$

For some applications, it is useful to cut out the finiteness argument in the proof of Theorem 10.19(c) and work directly in a finite space Y.

Proposition 10.30. *Let $f: Y \to \underline{E}_k$ be a map, where $E_*(Y)$ is a free E^*-module of finite rank, with basis elements z_α. Let $y_\alpha \in E^*(Y)$ be dual to z_α. Then for any $a \in E_*(Y)$, $b \in E_*(\underline{E}_k)$, and operation $r: k \to m$,*

$$r_*(f_* a * b) = \sum_i \sum_j \pm \mathop{\ast}_\alpha y_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} \quad \text{in } E_*(\underline{E}_m),$$

where $r''_\alpha: k \to m + \deg(z_\alpha)$ denotes the operation having the functional

$$\langle r''_\alpha, c \rangle = (-1)^{\deg(z_\alpha)(m+\deg(z_\alpha))} \langle r, f_* z_\alpha * c \rangle,$$

Ψa and Ψb are computed as in eq. (10.18), and we use $y_{\alpha *}: E_*(Y) \to E_*(\underline{E}_?)$.

Proof. By Theorem 1.18(a), $E^*(Y)$ is dual to $E_*(Y)$ and y_α is defined. We modify the proof of the Theorem by composing the square in diag. (10.27) with $f \times 1: Y \times \underline{E}_k \to \underline{E}_k \times \underline{E}_k$. We work in $E^*(Y \times \underline{E}_k)$ instead of $E^*(\underline{E}_k \times \underline{E}_k)$ and write

$$(f \times 1)^* \mu_k^* r = \sum_\alpha y_\alpha \times r''_\alpha.$$

We evaluate this on $z_\alpha \times c$ to determine r''_α. □

Remark. The commutativity of $*$-multiplication ensures that $r(x+y) = r(y+x)$. Conversely, one could say that $x + y = y + x$ in M *requires* $*$-multiplication to be commutative. The universal example has $M = E^*(\underline{E}_k \times \underline{E}_k)$, $x = \iota_k \times 1$, and $y = 1 \times \iota_k$, and c_α and d_β run through bases of $E_*(\underline{E}_k)$. Then $r(x+y) = r(y+x)$ for all r implies that $c_\alpha * d_\beta = \pm\, d_\beta * c_\alpha$ for all α and β. The commutativity of $*$ in general follows by linearity.

Similar discussions hold for other laws in a ring. In particular, $x+0=x$ corresponds in this way to $c * 1_k = c$, $-(-x) = x$ to $\chi\chi c = c$, $-(x+y) = (-x)+(-y)$ to $\chi(a*b) = \chi a * \chi b$, and the associativity of $+$ to the associativity of $*$.

Given a prime p, we can iterate eq. (10.23) to get

$$r(px) = r(x+x+\cdots+x) = \sum \pm \langle r, c_{\alpha_1} * c_{\alpha_2} * \cdots * c_{\alpha_p}\rangle\, x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_p}.$$

If the indices α_i are not all the same, we can permute them cyclically and obtain p distinct terms which by commutativity are all equal, with the same sign. This leaves only the terms with $\alpha_i = \alpha$ for all i, and we find

$$r(px) \equiv \sum_\alpha \langle r, Fc_\alpha\rangle\, Fx_\alpha \bmod p. \tag{10.31}$$

This is particularly useful when E^* has characteristic p, so that $px = 0$, because comparison with eq. (10.28) then yields

$$\sum_\alpha \langle r, Fc_\alpha\rangle\, Fx_\alpha = \langle r, 1_k\rangle\, 1_M \quad \text{in } M \text{ (for all } r). \tag{10.32}$$

Multiplicative structure. The multiplication maps $\phi\colon \underline{E}_k \times \underline{E}_m \to \underline{E}_{k+m}$ induce both the multiplication in UM and the $\circ$-multiplication in $E_*(\underline{E}_*)$. This allows us to translate the axiom that ρ_M is multiplicative, $\rho_M(xy) = \rho_M(x)\rho_M(y)$ in UM.

THEOREM 10.33. *Let M be an unstable algebra, and assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k. Take $x \in M^k$ and $y \in M^m$ and assume that $r(x)$ is in the standard form* (10.3). *Then:*

(a) *We have the* Cartan formula for multiplication

$$r(xy) = \sum_\alpha x_\alpha r''_\alpha(y) \quad \textit{for all } r\colon k+m \to h, \tag{10.34}$$

where for each α, the operation $r''_\alpha\colon m \to h + \deg(c_\alpha)$ is defined as having the functional

$$\langle r''_\alpha, c\rangle = (-1)^{\deg(c_\alpha)(h+\deg(c_\alpha))} \langle r, c_\alpha \circ c\rangle \quad \textit{for all } c \in E_*(\underline{E}_m); \tag{10.35}$$

(b) *If, similarly, $r(y)$ is given by eq.* (10.22), *we have the* full Cartan formula for multiplication,

$$r(xy) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)} \langle r, c_\alpha \circ d_\beta\rangle\, x_\alpha y_\beta \tag{10.36}$$

for all $r\colon k+m \to h$;

(c) *Take* $a \in E_*(\underline{E}_k)$ *and* $b \in E_*(\underline{E}_m)$. *Assume that* c_α *runs through a basis of* $E_*(\underline{E}_k)$, *and denote by* r'_α *the operation dual to* c_α. *Let* $\Psi a = \sum_i \otimes_\alpha a_{i,\alpha}$ *and* $\Psi b = \sum_j \otimes_\alpha b_{j,\alpha}$ *be the iterated coproducts of* a *and* b *as in eq.* (10.18), *where in both cases, we ignore all* α *that satisfy eq.* (10.24). *Then the homology homomorphism* $r_*: E_*(\underline{E}_{k+m}) \to E_*(\underline{E}_h)$ *satisfies*

$$r_*(a \circ b) = \sum_i \sum_j \pm \mathop{\ast}_\alpha r'_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} \quad \text{in } E_*(\underline{E}_m), \tag{10.37}$$

where r''_α *is defined by eq.* (10.35) *and the only signs come from shuffling the factors to form* $\Psi(a \times b)$.

The Remark following Theorem 10.19 applies.

PROOF. This is formally identical to the proof of Theorem 10.19, with $\mu_k: \underline{E}_k \times \underline{E}_k \to \underline{E}_k$ replaced everywhere by $\phi: \underline{E}_k \times \underline{E}_m \to \underline{E}_{k+m}$. □

By naturality, we can adapt eq. (10.36) to $\times$-products.

COROLLARY 10.38. *Given spaces* X *and* Y *and elements* $x \in E^k(X)$ *and* $y \in E^m(Y)$, *assume that* $r(x)$ *and* $r(y)$ *are given by eqs.* (10.3) *and* (10.22). *Then we have the Cartan formula*

$$r(x \times y) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)} \langle r, c_\alpha \circ d_\beta \rangle\, x_\alpha \times y_\beta \tag{10.39}$$

in $E^*(X \times Y)$, *for any operation* $r: k + m \to h$.

We have also the analogue of Proposition 10.30.

PROPOSITION 10.40. *Let* $f: Y \to \underline{E}_k$ *be a map as in Proposition* 10.30. *Then for any* $a \in E_*(Y)$, $b \in E_*(\underline{E}_k)$, *and operation* $r: k + m \to h$,

$$r_*(f_* a \circ b) = \sum_i \sum_j \pm \mathop{\ast}_\alpha y_{\alpha *} a_{i,\alpha} \circ r''_{\alpha *} b_{j,\alpha} \quad \text{in } E_*(\underline{E}_h),$$

where $r''_\alpha: m \to h + \deg(z_\alpha)$ *denotes the operation having the functional*

$$\langle r''_\alpha, c \rangle = (-1)^{\deg(z_\alpha)(m+\deg(z_\alpha))} \langle r, f_* z_\alpha \circ c \rangle$$

and Ψa *and* Ψb *are computed as in eq.* (10.18).

Since the unit element of UM is

$$\eta_M \circ \eta^*: E^*(\underline{E}_0) \to E^* \to M,$$

the axiom $\rho_M(1_M) = 1_{UM}$ translates easily into

$$r(1_M) = \langle \eta^* r, 1 \rangle\, 1_M = \langle r, \eta_* 1 \rangle\, 1_M = \big\langle r, [1] \big\rangle\, 1_M \quad \text{in } M \tag{10.41}$$

for all r.

Just as with addition, certain laws in the Hopf ring correspond to laws in an E^*-algebra M. For example, associativity of $\circ$-multiplication corresponds to associativity of multiplication in M. Commutativity is slightly trickier: given $x \in M^k$ and $y \in M^m$, $r(yx) = r((-1)^{km} xy)$ leads to the identity (10.10), thus explaining the signs and the appearance of χ.

The comonad structure. Finally, we translate the two axioms which state that ρ_M makes M a U-coalgebra. Since we have in effect returned to the First Answer of Section 8, these are the usual axioms for an action, $(sr)(x) = s(r(x))$ and $\iota_k x = x$.

The second is easily handled. From Proposition 6.11, we can use (6.41) to express the identity operation ι_k as the functional

$$\langle \iota_k, - \rangle = Q(\varepsilon) \circ q_k \colon E_*(\underline{E}_k) \longrightarrow Q(E)^k_* \longrightarrow E_*(E, o) \longrightarrow E^*. \tag{10.42}$$

When we put $r = \iota_k$, eq. (10.3) expands easily to yield the axiom

$$\sum_\alpha \big(Q(\varepsilon) q_k c_\alpha\big)\, x_\alpha = x \quad \text{in } M \tag{10.43}$$

for $x \in M^k$. We have thus interpreted the counit natural transformation $\varepsilon M \colon UM \to M$ of the comonad U, which was defined by $(\varepsilon M) f = f \iota_k$. The functional $\varepsilon_S \circ \sigma_{k*}$ is *not* part of the Hopf ring structure as given so far, so we add it. (It is unrelated to the counit $\epsilon \colon E_*(\underline{E}_k) \to E^*$ of the Hopf algebra $E_*(\underline{E}_k)$.)

It is easy to recover the functional $\langle r, - \rangle$ from r_*, as in eq. (6.50), in the form

$$\langle r, - \rangle \colon E_*(\underline{E}_k) \xrightarrow{r_*} E_*(\underline{E}_m) \xrightarrow{\sigma_{m*}} E_*(E, o) \xrightarrow{\varepsilon_S} E^*, \tag{10.44}$$

by writing $\langle r, c \rangle = \langle r^* \iota_m, c \rangle = \langle \iota_m, r_* c \rangle$ and using eq. (10.42). In the additive context, the reverse construction of r_* from $\langle r, - \rangle$ was neatly encoded in the comultiplication $Q(\psi)$ on $Q(E)^*_*$. Here, we have no such map and must rely on the definition of ψM, which explicitly uses r^*. In effect, we dualize and use r_* instead.

The first is the most complicated of all the axioms. When we substitute sr and r in eq. (10.3) and use

$$\langle sr, c_\alpha \rangle = \langle r^* s, c_\alpha \rangle = \langle s, r_* c_\alpha \rangle,$$

the axiom $(sr)(x) = s(r(x))$ expands to

$$\sum_\alpha \langle s, r_* c_\alpha \rangle\, x_\alpha = s\big(r(x)\big) = s\Big(\sum_\alpha \langle r, c_\alpha \rangle\, x_\alpha \Big) \quad \text{in } M, \tag{10.45}$$

for all r, s. The right side is to be expanded using eqs. (10.20) and (10.16), and in general is extremely complicated.

Our conclusion is that we need to know the induced homology homomorphism $r_*: E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ for every operation $r: E^k(-) \to E^m(-)$. This is the final piece of structure to add to the Hopf ring. To compute it successfully, we need r_*c for each $\circ$-generator c of $E_*(\underline{E}_*)$, and then use formulae (10.25) and (10.37) for $r_*(a * b)$ and $r_*(a \circ b)$.

Summary. We collect the various formulae to form the main theorem of this section. In addition to the Hopf ring structure on $E_*(\underline{E}_*)$, we need:

(i) The element $[v] \in E_0(\underline{E}_*)$ for each $v \in E^*$ (see Definition 10.8);
(ii) The augmentation (see eq. (10.42))

$$Q(\varepsilon) \circ q_k: E_*(\underline{E}_k) \longrightarrow Q(E)^k_* \longrightarrow E_*(E, o) \longrightarrow E^* \tag{10.46}$$

which may be written $\varepsilon_S \circ \sigma_{k*}$;
(iii) The homomorphism $r_*: E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ induced by each operation $r: k \to m$.

These constitute what we mean by an *enriched* Hopf ring structure.

THEOREM 10.47. *Let M be an object of FAlg, i.e. a complete Hausdorff filtered E^*-algebra, and assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k* (which is true for $E = H(\mathbf{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 4.17(a)). *Then an unstable algebra structure on M consists of a value $r(x) \in M$ for all $x \in M$ and all $r \in E^*(\underline{E}_k)$ (where $k = \deg(x)$ and $r(x) \in M^m$ if $r \in E^m(\underline{E}_k)$), which is E^*-linear in r and therefore (for fixed x) expressible in the standard form* (10.3)

$$r(x) = \sum_\alpha \langle r, c_\alpha \rangle x_\alpha \quad \textit{in } M \textit{ (for all } r).$$

These values are subject to the following axioms:

(a) *For fixed $x \in M^k$, $r(x)$ satisfies the three consistency conditions:*

$$\text{(i)} \qquad \sum_\gamma \psi c_\gamma \otimes x_\gamma = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(x_\beta)} c_\alpha \otimes c_\beta \otimes x_\alpha x_\beta$$
$$\textit{in } (E_*(\underline{E}_k) \otimes E_*(\underline{E}_k)) \mathbin{\widehat{\otimes}} M;$$

$$\text{(ii)} \qquad \sum_\alpha (\varepsilon c_\alpha) x_\alpha = 1_M \quad \textit{in } M;$$

$$\text{(iii)} \qquad \sum_\alpha (\varepsilon_S \sigma_{k*} c_\alpha) x_\alpha = x \quad \textit{in } M;$$

(b) *As x varies, $r(x)$ satisfies the following identities in M for all r, where we assume similarly* (as in eq. (10.22)) *that* $r(y) = \sum_\beta \langle r, d_\beta \rangle\, y_\beta$:

$$\text{(i)}\quad r(x+y) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)} \langle r, c_\alpha * d_\beta \rangle\, x_\alpha y_\beta;$$

$$\text{(ii)}\quad r(vx) = \sum_\alpha \big\langle r, [v] \circ c_\alpha \big\rangle\, x_\alpha;$$

$$\text{(iii)}\quad r(xy) = \sum_\alpha \sum_\beta (-1)^{\deg(x_\alpha)\deg(y_\beta)} \langle r, c_\alpha \circ d_\beta \rangle\, x_\alpha y_\beta;$$

$$\text{(iv)}\quad r(1_M) = \big\langle r, [1] \big\rangle\, 1_M;$$

(c) *The composition law*

$$\sum_\alpha \langle s, r_* c_\alpha \rangle\, x_\alpha = s\big(r(x)\big) = s\Big(\sum_\alpha \langle r, c_\alpha \rangle\, x_\alpha\Big) \quad \textit{in } M$$

holds for all r, s, and all $x \in M$;

(d) *For each of the ideals $F^a M$ in the filtration of M:*

(i) *For fixed $x \in M$, all except finitely many of the x_α lie in $F^a M$;*
(ii) *There exists $F^b M$ such that $r(x) \in F^a M$ for all $x \in F^b M$ and all based operations r.*

PROOF. The equations in (a) are (10.14), (10.15), and (10.43). Those in (b) are (10.23), (10.16), (10.36), and (10.41). The equation in (c) is (10.45). In (d), (i) states that $\rho_M(x): E^*(\underline{E}_k) \to M$ is continuous for each x, while (ii) states that $\rho_M: M \to UM$ is continuous. □

REMARK. By (b), an unstable algebra structure on M is determined by the values $r(x)$ on a set of (topological) E^*-algebra generators x of M. Moreover, the Hopf ring laws imply that it is sufficient to verify axioms (a) and (d)(i) on these generators. In practice, the topological conditions (d) rarely cause us any distress.

11. The E-cohomology of a point

In this section, we study the first of our test spaces, the one-point space T, for which $E^*(T)$ is by definition the coefficient ring E^*. Its unstable structure is completely determined by eqs. (10.41) and (10.16) as

$$r(v) = \big\langle r, [v] \big\rangle \quad \text{in } E^* = E^*(T) \text{ (for all } r), \tag{11.1}$$

which may be taken as an alternate definition of the Hopf ring elements $[v]$, instead of Definition 10.8.

It is easy to deduce how $[v]$ interacts with each piece of the structure on $E_*(\underline{E}_*)$. Much of this can be read off from the Hopf ring structure in Section 10. In particular, η_R is still in some sense a ring homomorphism.

PROPOSITION 11.2. *The Hopf ring elements $[v] \in E_0(\underline{E}_h)$ for each $v \in E^h$ have the properties:*

(a) $\psi[v] = [v] \otimes [v]$;

(b) $\varepsilon[v] = 1$;

(c) $[v + v'] = [v] * [v']$ *for* $v' \in E^h$;

(d) $[-v] = \chi[v]$;

(e) $[vv'] = [v] \circ [v']$ *for* $v' \in E^k$;

(f) $1_m \circ [v] = 1_{m+h}$;

(g) $r_*[v] = [\langle r, [v]\rangle]$ *(for all r)*;

(h) $r_* 1_h = [\langle r, 1_h\rangle]$;

(i) $q_h[v] = \eta_R v$ *in* $Q(E)^h_0$;

(j) $\sigma_{h*}[v] = \eta_R v$ *in* $E_{-h}(E, o)$, *under stabilization.*

PROOF. For (a) and (b) we substitute eq. (11.1) in eqs. (10.14) and (10.15). For (c) and (e), we write down the Cartan formulae (10.23) and (10.36) for $r(v+v')$ and $r(vv')$ and compare with eq. (11.1). For (d), we write down $r(-v)$ from eq. (10.29) and compare with eq. (11.1). For (g), we use eq. (11.1) to compute $s(r(v)) = \langle s, [\langle r, [v]\rangle]\rangle$; by eq. (10.45), this must agree with $\langle s, r_*[v]\rangle$ for all s. Since $[0_n] = 1_n$, (f) and (h) are special cases of (e) and (g). For (i) and (j), we compare eq. (11.1) with eq. (6.43) and [8, (11.23)], and use eqs. (6.14) and (6.13). □

Constant operations. Constant operations were introduced briefly in Section 8. Although they are of no real interest and contain no information, they are undeniably natural and we have to be able to recognize them in their several disguises.

PROPOSITION 11.3. *Let $r: k \to m$ be the constant operation defined by $r(x) = v1_X$ for all $x \in E^k(X)$, where $v \in E^m$. Then:*

(a) *As a class,* $r = v1_k \in E^*(\underline{E}_k)$;

(b) *As a map, r is the composite* $v \circ q: \underline{E}_k \to T \to \underline{E}_m$;

(c) *As a functional,* $\langle r, c\rangle = (\varepsilon c)v$ *in* E^* *for all* $c \in E_*(\underline{E}_k)$;

(d) $r_*: E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ *is given by* $r_* c = (\varepsilon c)[v]$ *for all* $c \in E_*(\underline{E}_k)$.

Based operations. Given a based space (X, o), we consider the naturality of an operation $r: k \to m$ with respect to the inclusion of the basepoint. We augment Lemma 2.3.

PROPOSITION 11.4. *The following conditions on an operation $r: k \to m$ are equivalent:*

(a) $r(0) = 0$ *in* $E^*(T) = E^*$, *i.e. r is* based *in the sense of Definition 2.2;*

(b) $r(0) = 0$ *in* $E^*(X)$ *for all spaces X;*

(c) *The operation* r *induces* $r\colon E^k(X,o) \to E^m(X,o)$ *for all* X;

(d) *The class* r *lies in* $E^m(\underline{E}_k, o) \subset E^m(\underline{E}_k)$;

(e) *The map* $r\colon \underline{E}_k \to \underline{E}_m$ *is based (up to homotopy)*;

(f) *The linear functional* $\langle r, -\rangle$ *satisfies* $\langle r, 1_k\rangle = 0$;

(g) *The homomorphism* $r_*\colon E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ *satisfies* $r_* 1_k = 1_m$.

PROOF. Part (b) is equivalent to (a) by naturality. Because $r(0) = \langle r, 1_k\rangle\, 1_X$ by eq. (10.28), (f) is equivalent to (b), and with the help of Proposition 11.2(h), to (g). □

We can generalize (f).

LEMMA 11.5. *Let* (X,o) *be a based space. Then for any* $x \in E^k(X,o)$ *and any operation* $r\colon k \to m$, *we have* $r(x) \equiv \langle r, 1_k\rangle 1_X \bmod E^*(X,o)$.

PROOF. We use eq. (10.28) and the naturality of r in diag. [8, (3.2)]. □

It is sometimes useful to be more specific. If we choose a basis of $E_*(\underline{E}_k)$ consisting of 1_k and elements $c_\alpha \in \operatorname{Ker}\varepsilon$, then for any $x \in E^k(X,o)$, eq. (10.4) takes the form

$$r(x) = \langle r, 1_k\rangle\, 1_X + \sum_\alpha \langle r, c_\alpha\rangle\, x_\alpha \quad \text{in } E^*(X)\hat{}\ \text{(for all } r\text{)}, \tag{11.6}$$

where the elements $x_\alpha \in E^*(X,o)\hat{}$.

Formulae are often simpler for based operations, but the case of general r can be recovered easily enough by decomposing as in Lemma 8.1.

LEMMA 11.7. *If we write* $r(x) = s(x) + v 1_X$, *where* s *is a based operation and* $v \in E^m$, *the homology homomorphism* $r_*\colon E_*(\underline{E}_k) \to E_*(\underline{E}_m)$ *is given by* $r_* c = s_* c * [v]$, *where we recognize* $v = r(0) = \langle r, 1_k\rangle$.

PROOF. We write r as the composite

$$\underline{E}_k \xrightarrow{\Delta} \underline{E}_k \times \underline{E}_k \xrightarrow{1\times q} \underline{E}_k \times T \xrightarrow{s\times v} \underline{E}_m \times \underline{E}_m \xrightarrow{\mu_m} \underline{E}_m$$

and take E-homology. The first two maps just give $\underline{E}_k \cong \underline{E}_k \times T$. □

12. Spheres, suspensions, and additive operations

So far, except for adding an extra grading, our additive results are formally very similar to the stable case discussed in [8]. What is new is that suspension is no longer an isomorphism, but defines a new element e. The stable results can all be recovered by stabilizing, which consists merely of setting $e = 1$.

We assume throughout that $E_*(\underline{E}_k)$, $QE_*(\underline{E}_k)$, *and* $E_*(E,o)$ *are free* E^**-modules*, so that we have available the machinery of comodule algebras of Sections 6 and 7 as well as the stable results of [8]. In particular, the coaction $\rho_X: E^*(X) \to E^*(X) \widehat{\otimes} Q(E)^*_*$ is a homomorphism of E^*-algebras for any X.

Spheres. Our second test space, after the one-point space T, is the circle S^1. Its cohomology $E^*(S^1, o)$ is a free E^*-module with the basis $\{1_S, u_1\}$, where the canonical generator $u_1 \in E^1(S^1, o)$ is provided by [8, Definition 3.23]. Thus $\rho_S: E^*(S^1) \to E^*(S^1) \otimes Q(E)^*_*$ is determined by $\rho_S u_1$.

DEFINITION 12.1. We define the *suspension element* $e = e_Q \in Q(E)^1_1$ by the identity

$$\rho_S u_1 = u_1 \otimes e \quad \text{in } E^*(S^1, o) \otimes Q(E)^1_* \cong Q(E)^1_*. \tag{12.2}$$

It has degree zero.

More generally, for the k-sphere S^k, $E^*(S^k)$ is free on the basis $\{u_k, 1_S\}$, where $u_k \in E^k(S^k, o)$.

PROPOSITION 12.3. *The suspension element $e \in Q(E)^1_1$ has the following properties, where $k \geqslant 0$:*

(a) $\rho_S u_k = u_k \otimes e^k$ *in* $E^*(S^k) \otimes Q(E)^k_*$;

(b) $r u_k = \langle r, e^k \rangle u_k$ *in* $E^*(S^k)$ *for any additive operation* $r: k \to m$;

(c) *The class* $u_k \in E^k(S^k)$, *regarded as a map* $u_k: S^k \to \underline{E}_k$, *induces* $q_k u_{k*} z = e^k \in Q(E)^k_k$, *where* $z \in E_k(S^k)$ *is dual to* u_k;

(d) *In the coalgebra structure on* $Q(E)^*_*$, $Q(\psi)e = e \otimes e$ *and* $Q(\varepsilon)e = 1$;

(e) $Q(\psi)(ve^k w) = ve^k \otimes e^k w$ *in* $Q(E)^*_* \otimes Q(E)^*_*$, *for any* $v \in E^*$ *and* $w \in \eta_R E^*$;

(f) *Given* $v \in E^*$ *and* $w \in \eta_R E^h$, *the homomorphism* $Q(r): Q(E)^{k+h}_* \to Q(E)^m_*$ *induced on homology by any operation* $r: k + h \to m$ *satisfies*

$$Q(r)(ve^k w) = ve^k \eta_R \langle r, e^k w \rangle \quad \text{in } Q(E)^m_*;$$

(g) *Under stabilization,* $Q(\sigma)e = 1$ *in* $E_*(E, o)$.

PROOF. We prove (a) for $k > 0$ by induction on k, starting from eq. (12.2). If it holds for k and m, the multiplicativity of ρ gives

$$\rho(u_k \times u_m) = (u_k \times u_m) \otimes e^{k+m} \quad \text{in } E^*(S^k \times S^m).$$

The projection map $q: S^k \times S^m \to S^{k+m}$ induces $q^* u_{k+m} = u_k \times u_m$, which gives (a) for $k+m$. The result is true also for $k = 0$, if we make the obvious identification $e^0 = 1$. Then (b) follows by eq. (6.39) and (c) is an application of Proposition 6.44.

To prove (d), we evaluate both axioms (6.33) for $M = E^*(S^1)$ on u_1. Part (e) follows immediately from (d) and the fact that $Q(\psi)$ is a homomorphism of algebras and of E^*-bimodules. Then (f) follows from (e) and Lemma 6.51(c). For (g), we apply $1 \otimes Q(\sigma)$ to eq. (12.2) and compare with the stable coaction $\rho_S u_1 = u_1 \otimes 1$ in [8, (11.24)]. □

REMARK. As v, k, and w vary, the elements ve^kw span $Q(E)^*_* \otimes \mathbb{Q}$ as a $\mathbb{Q}$-module. (In fact, $Q(\sigma)$ induces $Q(E)^k_* \otimes \mathbb{Q} \cong E_*(E,o) \otimes \mathbb{Q}$ if E is $(-k-1)$-connected.) Thus in the important case when $Q(E)^k_*$ has no torsion, the innocuous formulae in (e) and (f) are powerful enough to determine $Q(\psi)$ and $Q(r)$ completely.

COROLLARY 12.4. *Let $r: k \to m$ be an additive operation, regarded as a map of H-spaces $r: \underline{E}_k \to \underline{E}_m$. Then the induced homomorphism on homotopy groups*

$$E^* \cong \pi_*(\underline{E}_k, o) \xrightarrow{r_*} \pi_*(\underline{E}_m, o) \cong E^*$$

is given on $v \in E^{-h}$ by $r_*v = \langle r, e^{k+h}\eta_R v\rangle$.

PROOF. We reinterpret r_* as the action of the operation r on $E^k(S^{k+h}, o)$. The element v corresponds to the class vu_{k+h}. From Proposition 12.3(b) and eq. (6.40),

$$r(vu_{k+h}) = \langle r, e^{k+h}\eta_R v\rangle\, u_{k+h} \quad \text{in } E^*(S^{k+h}, o). \qquad \square$$

Suspensions. More generally, the action of the operations on the suspension ΣX of a based space X is easily deduced from the action on X.

LEMMA 12.5. *Given a based space (X, o) and $x \in E^k(X, o)$, the coaction $\rho_{\Sigma X}\Sigma x$ is the image of $\rho_X x$ under*

$$\Sigma \otimes e: E^*(X,o)\,\widehat{\otimes}\, Q(E)^k_* \longrightarrow E^*(\Sigma X, o)\,\widehat{\otimes}\, Q(E)^{k+1}_*,$$

where e denotes multiplication by the element $e \in Q(E)^1_*$.

PROOF. The projection map $S^1 \times X \to \Sigma X$ embeds $E^*(\Sigma X, o)$ in $E^*(S^1 \times X, S^1 \times o)$. Here, Σx corresponds to $u_1 \times x$, whose coaction is known. $\square$

We can mimic this algebraically. We defined the formal suspension ΣM of any E^*-module M in [8, Definition 6.6], merely by shifting all the degrees up one.

DEFINITION 12.6. Given any unstable comodule M, we make the *suspension* ΣM of M an unstable comodule by equipping it with the coaction $\rho_{\Sigma M}$ defined by the commutative square

$$\begin{array}{ccc} M^k & \xrightarrow{\rho_M} & M\,\widehat{\otimes}\, Q(E)^k_* \\ \cong\Big\downarrow \Sigma & & \Big\downarrow \Sigma\otimes e \\ (\Sigma M)^{k+1} & \xrightarrow{\rho_{\Sigma M}} & \Sigma M\,\widehat{\otimes}\, Q(E)^{k+1}_* \end{array}$$

The axioms on $\rho_{\Sigma M}$ are readily verified.

13. Spheres, suspensions, and unstable operations

In this section, we continue Section 12 by computing all the unstable operations on $E^*(S^k)$ for the spheres S^k, which requires one new Hopf ring element, the suspension element e. This leads to the unstable structure of $E^*(\Sigma X)$ in terms of $E^*(X)$.

We recall that $E^*(S^k)$ is a free E^*-module with basis $\{1_S, u_k\}$, where u_k is the standard generator. The algebra structure is given by $u_k^2 = 0$, except that of course $u_0^2 = u_0$. By the Remark after Theorem 10.47, we have only to find $r(u_k)$. Lemma 11.5 gives partial information.

We assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k.

DEFINITION 13.1. We define the *suspension element* $e = e_U \in E_1(\underline{E}_1)$ by the identity

$$r(u_1) = \langle r, 1_1\rangle\, 1_S + \langle r, e\rangle\, u_1 \quad \text{in } E^*(S^1) \text{ (for all } r). \tag{13.2}$$

Here and in similar definitions, we use the freeness of $E^*(S^1)$ and the duality $FMod^*(E^*(\underline{E}_k), E^*) \cong E_*(\underline{E}_k)$ to ensure that e exists and is well defined. We note that $\varepsilon e = 0$ from eq. (10.15). Rather than develop all the properties of e now, we include them below in Proposition 13.7 as the special case $e_1 = e$.

Suspensions. We deduce from eq. (13.2) the behavior of unstable operations under the suspension isomorphism $\Sigma\colon E^*(X,o) \cong E^*(\Sigma X, o)$. We take an element $x \in E^k(X,o) \subset E^k(X)$ and assume that $r(x)$ is given by eq. (11.6), so that $\varepsilon c_\alpha = 0$. The quotient map $q\colon S^1 \times X \to \Sigma X$ embeds $E^*(\Sigma X)$ in $E^*(S^1 \times X) \cong E^*(S^1) \otimes E^*(X)$; under this embedding, Σx corresponds to $u_1 \times x$. We compute $r(u_1 \times x)$ from the Cartan formula (10.39) and find

$$r(\Sigma x) = \langle r, 1_{k+1}\rangle\, 1_{\Sigma X} + \sum_\alpha (-1)^{\deg(x_\alpha)} \langle r, e \circ c_\alpha\rangle\, \Sigma x_\alpha \tag{13.3}$$

for all r. The other terms drop out because $1_1 \circ c_\alpha = \varepsilon c_\alpha = 0$ and $e \circ 1_k = \varepsilon e = 0$.

This suggests how the suspension of an unstable algebra should be defined. The treatment is slightly different from the additive version in Section 12. First, we need a basepoint.

DEFINITION 13.4. We call the unstable algebra M *based* if we are given an augmentation $M \to E^*$ of unstable algebras. Then the kernel $\overline{M}$ is an invariant ideal, and we have the splitting $M = E^* \oplus \overline{M}$ as E^*-modules.

We define the *unstable suspension* $\Sigma_U M$ of M as the subalgebra

$$\Sigma_U M = \left(1_S \otimes E^*\right) \oplus \left(u_1 \otimes \overline{M}\right) \subset E^*(S^1) \otimes M. \tag{13.5}$$

The action of r is given on $u_1 \otimes \overline{M}$ by eq. (13.3) and on $1_S \otimes E^*$ by eq. (11.1).

For example, if (X, o) is a based space, we have the augmentation $E^*(X) \to E^*(o) = E^*$, with kernel $E^*(X, o)$ (as in [8, (3.2)]). Inspection shows that much of the structure on M is not used. The multiplication on M is totally ignored. Indeed, we do not need an unstable structure on M at all.

THEOREM 13.6. *Given an additively unstable module $\overline{M}$, we can make $E^* \oplus \Sigma\overline{M}$ an unstable algebra, with $1 \in E^*$ as the unit element and trivial multiplication on $\Sigma\overline{M}$, as follows. If $x \in \overline{M}^k$ and $r(x) = \sum_\alpha \langle r_Q, c_\alpha\rangle x_\alpha$ for additive operations r, where $c_\alpha \in Q(E)^k_*$, we lift each c_α to $\tilde{c}_\alpha \in E_*(\underline{E}_k)$ such that $q_k\tilde{c}_\alpha = c_\alpha$, and define the action of unstable operations r on Σx by*

$$r(\Sigma x) = \langle r_U, 1_{k+1}\rangle\, 1 + \sum_\alpha (-1)^{\deg(x_\alpha)} \langle r_U, e\circ\tilde{c}_\alpha\rangle\, \Sigma x_\alpha.$$

PROOF. Because $e \circ 1 = 0$ and $e\circ(b*c) = 0$ whenever $\varepsilon b = 0$ and $\varepsilon c = 0$, $r(\Sigma x)$ is independent of the choices of the $\tilde{c}_\alpha$. The definition (with signs) has been chosen so that: (a) the additive unstable structure on $E^* \oplus \Sigma\overline{M}$ restricts to that on $\Sigma\overline{M}$ given by Definition 12.6, and (b) it includes eq. (13.5) for a based unstable algebra M. (For (a), we note that diag. (6.16) gives $q_{k+1}(e_U \circ \tilde{c}_\alpha) = (-1)^k e_Q c_\alpha$.) Verification of the axioms of Theorem 10.47 is tedious but routine. □

The elements e_k. It is convenient to use eq. (13.3) to find the structure of $E^*(S^k)$. We deduce the fundamental properties of the Hopf ring element e.

PROPOSITION 13.7. *We define the Hopf ring elements $e_k \in E_k(\underline{E}_k)$ for $k \geqslant 0$ in terms of $e \in E_1(\underline{E}_1)$ by $e_k = (-1)^{k(k-1)/2} e^{\circ k}$ for $k > 0$ (so that $e_1 = e$) and $e_0 = [1] - 1_0$. They have the following properties:*

(a) *In $E^*(S^k)$ we have, for any $k \geqslant 0$:*

$$r(u_k) = \langle r, 1_k\rangle\, 1_S + \langle r, e_k\rangle\, u_k \quad (\textit{for all } r); \tag{13.8}$$

(b) *The class u_k, regarded as a map $u_k\colon S^k \to \underline{E}_k$, induces $u_{k*}z = e_k \in E_k(\underline{E}_k)$ in homology, where $z \in E_k(S^k)$ is dual to u_k;*

(c) *$e_k \circ e_m = (-1)^{km} e_{k+m}$ if $k > 0$ or $m > 0$;*

(d) *$\psi e_k = e_k \otimes 1 + 1 \otimes e_k$ for all $k > 0$;*

(e) *$\varepsilon e_k = 0 \in E^*$ for all $k \geqslant 0$;*

(f) *$\chi e_k = -e_k$ for all $k > 0$;*

(g) *$e_k \circ [\lambda] = \lambda e_k$ for all rational numbers $\lambda \in E^0$ and all $k > 0$;*

(h) *$r_* e_k = [\langle r, 1_k\rangle] * [\langle r, e_k\rangle] \circ e_k$ for all $k \geqslant 0$ and all $r\colon k \to m$;*

(i) *$q_k e_k = e^k_Q = e^k$ in $Q(E)^k_*$, for all $k \geqslant 0$, for additive operations;*

(j) *$\sigma_{k*} e_k = 1$ in $E_*(E, o)$, for all $k \geqslant 0$, under stabilization.*

REMARK. The results make it clear that the correct interpretation of $e^{\circ 0}$ is $[1] - 1_0 = [1] - [0_0]$, as in [28] and elsewhere, rather than just the element $[1]$.

Proof. We give extensive details of this proof (only), as a good example of our machinery in action.

We establish eq. (13.8) for $k > 0$, and thus (a), by induction on k. It holds for $k = 1$ by definition. We recognize ΣS^k as S^{k+1} and Σu_k as u_{k+1}; then by eq. (13.3), eq. (13.8) holds for $k+1$ if it holds for k, provided that $e_{k+1} = (-1)^k e \circ e_k$. Our definition of e_k is designed to do exactly this. More generally, we have (c).

For $k = 0$, we write $E^*(S^0) = E^* \oplus E^*$. In *Alg*, this is a product of algebras, with the projections induced respectively by the inclusions of the basepoint and the other point. In this presentation, $u_0 = (0, 1)$, and of course $1_S = (1, 1)$. By eq. (11.1), the action on u_0 is

$$r(u_0) = r\big((0,1)\big) = \big(\langle r, 1_0\rangle, \langle r, [1]\rangle\big) = \langle r, 1_0\rangle(1_S - u_0) + \langle r, [1]\rangle u_0,$$

which gives (a) if we define $e_0 = [1] - 1_0$.

Then (b) is an application of Proposition 10.5. When we substitute eq. (13.8) into eq. (10.14), we find, for $k > 0$,

$$\psi 1_k \otimes 1_S + \psi e_k \otimes u_k = 1_k \otimes 1_k \otimes 1_S + 1_k \otimes e_k \otimes u_k + e_k \otimes 1_k \otimes u_k,$$

since $u_k^2 = 0$. This gives (d). (But ψe_0 acquires the extra term $e_0 \otimes e_0$, because $u_0^2 \neq 0$; this is obvious anyway from Proposition 11.2. Also, (c), (d), and (g) are clearly false for $k = m = 0$.) Similarly, eq. (10.15) yields $1_S + (\varepsilon e_k)u_k = 1_S$ (even for $k = 0$), which gives (e).

For (g), which includes (f) as the special case $\lambda = -1$ (by Proposition 10.12(a) and Proposition 11.2(d)), the distributive law (10.11) and (d) yield $e_k \circ [\lambda + \mu] = e_k \circ [\lambda] + e_k \circ [\mu]$ for all $\lambda, \mu \in E^0$. Since $e_k \circ [1] = e_k$, (g) follows. (We are in effect expanding $r(\lambda u_k)$.)

For (h), we substitute eq. (13.8) into eq. (10.45). On the left, we have

$$(sr)(u_k) = \langle s, r_* 1_k\rangle\, 1_S + \langle s, r_* e_k\rangle\, u_k,$$

while on the right, iteration of eq. (13.8) yields, after simplification,

$$s\big(r(u_k)\big) = \big\langle s, [\langle r, 1_k\rangle]\big\rangle\, 1_S + \big\langle s, [\langle r, 1_k\rangle] * [\langle r, e_k\rangle] \circ e_k\big\rangle\, u_k,$$

with the help of eqs. (10.16) and (10.23). Comparison of these gives $r_* e_k$.

For $k = 1$ in (i) and (j), we stabilize eq. (13.2) by Lemma 10.7 and compare with Definition 12.1 and [8, (11.24)]. For general k, we use the multiplicative properties in diag. (6.16) of q_k and σ_{k*}. $\square$

We have the analogue of Corollary 12.4. By Lemma 2.3(d), a based operation $r: k \to m$ is represented by a based map $r: (\underline{E}_k, o) \to (\underline{E}_m, o)$. We need to know its effect on homotopy groups.

LEMMA 13.9. *Given a based operation $r: k \to m$, the induced homomorphism on homotopy groups*

$$E^{k-h} \cong \pi_h(\underline{E}_k, o) \xrightarrow{r_*} \pi_h(\underline{E}_m, o) \cong E^{m-h}$$

is given on $v \in E^{k-h}$ for any $h \geqslant 0$ by

$$r_* v = \langle r, [v] \circ e_h \rangle \quad \text{in } E^{m-h}.$$

PROOF. Viewed cohomologically, the element $v \in E^{k-h}$ or map $v: S^h \to \underline{E}_k$ corresponds to $vu_h \in E^k(S^h, o)$. From eqs. (10.16) and (13.8), we compute $r(vu_h) = \langle r, [v] \circ e_h \rangle \, u_h$, which simplified because r is based, so that $\langle r, 1_k \rangle = 0$. □

14. Complex orientation and additive operations

In this section, we study the effect of a complex orientation on additive operations. The relevant test space is $\mathbb{C}P^\infty$, for which $E^*(\mathbb{C}P^\infty) = E^*[[x]]$ by [8, Lemma 5.4], where $x = x(\xi)$ is the Chern class of the Hopf line bundle ξ. All the stable results carry over, almost without change, except that now $b_1 = e^2$ instead of 1.

We assume that $E_*(\underline{E}_k)$, $Q(E)^k_*$, and $E_*(E, o)$ are free E^*-modules.

DEFINITION 14.1. We define elements $b_i \in Q(E)^2_{2i}$ for all $i \geqslant 0$ by the identity

$$\rho x = b(x) = \sum_{i=0}^{\infty} x^i \otimes b_i \quad \text{in } E^*(\mathbb{C}P^\infty) \,\widehat{\otimes}\, Q(E)^2_* \cong Q(E)^2_*[[x]], \tag{14.2}$$

where $b(x)$ is a convenient formal abbreviation that rapidly becomes essential.

We use eq. (6.39) to convert eq. (14.2) to the equivalent form

$$rx = \sum_{i=0}^{\infty} \langle r, b_i \rangle \, x^i \quad \text{in } E^*(\mathbb{C}P^\infty) = E^*[[x]], \text{ for all } r. \tag{14.3}$$

Since the Hopf bundle is universal, eqs. (14.2) and (14.3) hold for the Chern class $x = x(\theta)$ of any complex line bundle θ over any space X (after completion, if necessary).

PROPOSITION 14.4. *The elements $b_i \in Q(E)^2_{2i}$ have the following properties:*

(a) $b_0 = 0$ *and* $b_1 = e^2$, *so that* $b(x) = x \otimes e^2 + x^2 \otimes b_2 + x^3 \otimes b_3 + \cdots$;

(b) *The Chern class $x \in E^2(\mathbb{C}P^\infty)$, regarded as a map of spaces $x: \mathbb{C}P^\infty \to \underline{E}_2$, induces $q_2 x_* \beta_i = b_i \in Q(E)^2_{2i}$, where $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ is dual to x^i;*

(c) *$Q(\psi) b_k$ is given by*

$$Q(\psi) b_k = \sum_{i=1}^{k} B(i,k) \otimes b_i \quad \text{in } Q(E)^*_* \otimes Q(E)^2_*,$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$, or formally,

$$Q(\psi)b(x) = \sum_{i=1}^{\infty} b(x)^i \otimes b_i;$$

(d) *$Q(\epsilon)b_k = 0$ for $k > 1$, or formally, $Q(\epsilon)b(x) = x$;*

(e) *The stabilization $Q(\sigma): Q(E)^2_* \to E_*(E,o)$ sends the element $b_i \in Q(E)^2_{2i}$ to the stable element $b_i \in E_{2i-2}(E,o)$ of* [8, Definition 13.1].

PROOF. For (a), we restrict eq. (14.2) to $\mathbb{C}P^1 \cong S^2$ and compare with eq. (12.2). For (b), we apply Proposition 6.44. For (c) and (d), we substitute ρ into diags. (6.33) and evaluate on x. For (e), we compare Definition 14.1 with [8, Definition 13.1]. □

Still following the stable strategy, we next apply ρ to the multiplication map $\mu: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, to obtain the formal identity

$$b\big(F(x,y)\big) = F_R\big(b(x), b(y)\big) = b(x) + b(y) + \sum_{i,j} b(x)^i\, b(y)^j\, \eta_R a_{i,j} \tag{14.5}$$

in $Q(E)^*_*[[x,y]]$, which looks exactly like the stable version [8, (13.6)]. Again, $F_R(X,Y)$ is a convenient abbreviation. The consequences are the same.

The p-local case.

LEMMA 14.6. *Assume that E^* is a p-local ring. Then the generator b_k of $Q(E)^*_*$ is redundant unless k is a power of p.*

PROOF. The proof of [8, Lemma 13.7] applies without change. □

We therefore reindex the b's.

DEFINITION 14.7. When E^* is a p-local ring, we define $b_{(i)} = b_{p^i}$ for each $i \geqslant 0$.

As in [8, §13], we obtain

$$b\big([p](x)\big) = [p]_R\big(b(x)\big) = pb(x) + \sum_{i>0} b(x)^{i+1} \eta_R g_i \tag{14.8}$$

in $Q(E)^2_*[[x]]$, which looks exactly like the stable version [8, (13.11)] but is in a different place. Again, we extract the coefficient of x^{p^k}.

DEFINITION 14.9. For each $k > 0$, we define the kth *main* (*additively unstable*) *relation* as

$$(\mathcal{R}_k): \qquad L(k) = R(k) \quad \text{in } Q(E)^2_*, \tag{14.10}$$

where $L(k)$ and $R(k)$ denote the coefficient of x^{p^k} in the left and right sides of eq. (14.8) respectively.

15. Complex orientation and unstable operations

In this section, we extend our study of the test space $\mathbb{C}P^\infty$ to all unstable operations. Everything we did in Section 14 carries over, with a lot more complication but no essential difficulty. Again, it is enough to know $r(x)$ for all operations r, where $x = x(\xi) \in E^2(\mathbb{C}P^\infty)$ is the Chern class.

We assume that $E_*(\underline{E}_k)$ is a free E^*-module for all k.

DEFINITION 15.1. We define elements $b_i \in E_{2i}(\underline{E}_2)$ for $i \geqslant 0$ by the identity

$$r(x) = \sum_{i=0}^{\infty} \langle r, b_i \rangle\, x^i = \langle r, b(x) \rangle \quad \text{in } E^*(\mathbb{C}P^\infty) = E^*[[x]] \tag{15.2}$$

for all r, where we take x^i inside the $\langle\ ,\ \rangle$ and write formally $b(x) = \sum_i b_i x^i$.

We first determine how the elements b_k interact with the Hopf ring structure.

PROPOSITION 15.3. *The elements $b_k \in E_{2k}(\underline{E}_2)$ of the Hopf ring $E_*(\underline{E}_*)$ have the properties:*

(a) $b_0 = 1_2$ *and* $b_1 = e_2 = -e^{\circ 2}$, *so that* $b(x) = 1_2 + \bar{b}(x)$ *if we define*

$$\bar{b}(x) = \sum_{i=1}^{\infty} b_i x^i \quad \textit{in } E_*(\underline{E}_2)[[x]]; \tag{15.4}$$

(b) *The universal Chern class $x \in E^2(\mathbb{C}P^\infty)$, regarded as a map $x\colon \mathbb{C}P^\infty \to \underline{E}_2$, induces $x_*\beta_k = b_k \in E_{2k}(\underline{E}_2)$, where $\beta_k \in E_{2k}(\mathbb{C}P^\infty)$ is dual to x^k* (as in [8, Lemma 5.4])*;*

(c) $\psi b_k = \sum_{i+j=k} b_i \otimes b_j$, *or formally,* $\psi b(x) = b(x) \otimes b(x)$*;*

(d) $\varepsilon b_k = 0$ *if* $k > 0$, *and* $\varepsilon b_0 = 1$, *or formally,* $\varepsilon b(x) = 1$*;*

(e) $\chi b(x) = (1_2 + \bar{b}(x))^{*(-1)} = 1_2 - \bar{b}(x) + \bar{b}(x)^{*2} - \bar{b}(x)^{*3} + \cdots$*;*

(f) *For all rational numbers $\lambda \in E^0$,*

$$b(x) \circ [\lambda] = \left(1_2 + \bar{b}(x)\right)^{*\lambda} = 1_2 + \lambda \bar{b}(x) + \frac{\lambda(\lambda-1)}{2} \bar{b}(x)^{*2} + \cdots; \tag{15.5}$$

(g) *For all r, $r_* b_k$ is given as the coefficient of x^k in the formal identity*

$$r_* b(x) = \left[\langle r, 1_2 \rangle\right] * \mathop{\Large *}_{j=1}^{\infty} b(x)^{\circ j} \circ \left[\langle r, b_j \rangle\right] \quad \textit{in } E_*(\underline{E}_*)[[x]];$$

(h) $q_2 b_k = b_k \in Q(E)^2_{2k}$, *the additively unstable element in Definition* 14.1;
(i) $\sigma_{2*} b_k = b_k \in E_{2k-2}(E, o)$, *the stable element in* [8, Definition 13.1].

REMARK. The sign in (a) is absent from [23, Proposition 2.4]. The commutativity of diag. (6.16) requires

$$Q(\phi)(q_1 \otimes q_1)(e \otimes e) = -(q_1 e)(q_1 e) = -q_2 b_{(0)} = -q_2 e_2 = q_2(e \circ e),$$

bearing in mind that $\deg(q_1) = 1$. The unexpected sign first appeared in Proposition 13.7(c).

PROOF. Naturality for the inclusion $S^2 \cong \mathbb{C}P^1 \subset \mathbb{C}P^\infty$ gives (a), by comparison with Proposition 13.7. Part (b) comes from Proposition 10.5. We read off (c) and (d) from eqs. (10.14) and (10.15). Part (e) is the special case $\lambda = -1$ of (f). For (f), eq. (10.11) and (c) give $b(x) \circ [\lambda + \mu] = b(x) \circ [\lambda] * b(x) \circ [\mu]$ for all $\lambda, \mu \in E^*$. Since $b(x) \circ [1] = b(x)$ and we are working in the $*$-multiplicative group of formal power series over $E_*(\underline{E}_2)$ of the form $1 + \cdots$, which has no n-torsion if $1/n \in E^*$, the result follows. (We are in effect expanding $r(\lambda x)$; cf. eq. (10.16).) For (g), we apply eq. (10.45) to $x \in E^2(\mathbb{C}P^\infty)$ and expand. For (h) and (i), we stabilize eq. (15.2) by Proposition 6.11 and compare with the additive and stable versions, eq. (14.3) and [8, (13.3)]. □

From (c) and eq. (10.11), we deduce the convenient distributive law

$$(a * c) \circ b(x) = a \circ b(x) * c \circ b(x), \tag{15.6}$$

where a and c are allowed to involve x. This formal device will prove extremely useful for computations in Hopf rings. We have one immediate application to the Frobenius operator F defined by $Fc = c^{*p}$.

COROLLARY 15.7. *For any element c in the Hopf ring $E_*(\underline{E}_*)$,*

$$(Fc) \circ b_k \equiv \begin{cases} F(c \circ b_n) \bmod p, & \text{if } k = pn; \\ 0 \bmod p, & \text{if } k \text{ is not divisible by } p. \end{cases}$$

PROOF. By iterating eq. (15.6) we have $(Fc) \circ b(x) = F(c \circ b(x))$. We pick out the coefficient of x^k, working mod p. □

We next study the naturality of operations with respect to the multiplication $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. We expand $\mu^* r(x) = r(\mu^* x)$ by the formal group law [8, (13.5)] and the Cartan formulae, to obtain the analogue of eq. (14.5). The complicated result is best expressed formally as

$$b(F(x,y)) = F_R(b(x), b(y)) = b(x) * b(y) * \mathop{\Large *}_{i,j} \left(b(x)^{\circ i} \circ b(y)^{\circ j} \circ [a_{i,j}]\right) \tag{15.8}$$

as in [23, Theorem 3.8(i)], where $F_R(X,Y) = X * Y * \mathop{*}_{i,j} X^{\circ i} \circ Y^{\circ j} \circ [a_{i,j}]$, in the sense that the $\circ$- and $*$-multiplications apply only to Hopf ring elements, not to x and y.

The p-local case. Lemma 14.6 carries over.

LEMMA 15.9. *Assume that E^* is a p-local ring. Then the $\circ$-generator b_k of the Hopf ring $E_*(\underline{E}_*)$ is redundant unless k is a power of p.*

PROOF. As before, we take the coefficient of $x^i y^j$ in eq. (15.8), where $i+j=k$. On the left, there is a term $\binom{k}{i} b_k$, from $b_k(x+y)^k$, and this is the highest b that occurs; on the right, no b beyond b_i or b_j occurs. We choose i and j as in [8, Lemma 13.7], to make $\binom{k}{i}$ not divisible by p and therefore invertible, which shows that b_k is redundant. □

We therefore reindex the b's as usual.

DEFINITION 15.10. When E^* is a p-local ring, we define $b_{(i)} = b_{p^i}$ for each $i \geqslant 0$.

We extend standard multi-index notation slightly by defining

$$b^{\circ I} = b_{(0)}^{\circ i_0} \circ b_{(1)}^{\circ i_1} \circ b_{(2)}^{\circ i_2} \circ b_{(3)}^{\circ i_3} \circ \cdots \tag{15.11}$$

for any multi-index $I = (i_0, i_1, i_2, \ldots)$. We also need a shift operation.

DEFINITION 15.12. Given a multi-index $I = (i_0, i_1, i_2, \ldots)$, we define the *shifted* multi-index $s(I) = (0, i_0, i_1, i_2, \ldots)$. We iterate this process h times, for any $h \geqslant 0$, to form $s^h(I) = (0, \ldots, 0, i_0, i_1, i_2, \ldots)$. We even undo it, by defining $s^{-1}(I) = (i_1, i_2, i_3, \ldots)$, provided $i_0 = 0$; our convention is that this is *undefined* if $i_0 \neq 0$.

This notation allows us to iterate Corollary 15.7 neatly in the form

$$(Fc) \circ b^{\circ I} \equiv \begin{cases} F(c \circ b^{\circ s^{-1}(I)}) \bmod p & \text{if } i_0 = 0; \\ 0 \bmod p & \text{if } i_0 \neq 0. \end{cases} \tag{15.13}$$

We follow the stable plan and study instead of μ the much simpler p-th power map $\zeta: \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. Naturality of the general operation r is expressed by $\zeta^* r(x) = r(\zeta^* x)$. When we substitute the p-series [8, (13.9)] and expand, we obtain, as in [23, Theorem 3.8(ii)],

$$b\Bigl(px + \sum_i g_i x^{i+1}\Bigr) = b(x)^{*p} * \mathop{\ast}_i b(x)^{\circ\, i+1} \circ [g_i] \tag{15.14}$$

in $E_*(\underline{E}_*)[[x]]$, or, in condensed notation, $b([p](x)) = [p]_R(b(x))$.

DEFINITION 15.15. For each $k > 0$, we define the kth *main unstable relation* as

$$(\mathcal{R}_k): \qquad L(k) = R(k) \quad \text{in } E_*(\underline{E}_2), \tag{15.16}$$

where $L(k)$ and $R(k)$ denote respectively the coefficient of x^{p^k} in the left and right sides of eq. (15.14).

Thus $L(k)$ is the coefficient of x^{p^k} in $b([p](x))$, exactly as in Definition 14.9. However, $R(k)$ is vastly more complicated than before, and we study it in more detail in Section 19 in the case $E = BP$. The work of Ravenel and Wilson [23], which we review in Section 17, implies that, despite appearances, the relations $(\mathcal{R}_n)$ contain all the information present in eq. (15.8), with the understanding that we use eq. (15.8), by way of Lemma 15.9, only to express the redundant b_j's (which still appear in $\psi b_{(k)}$, $b_{(k)} \circ [\lambda]$, and $r_* b_{(k)}$) in terms of the $b_{(i)}$.

16. Examples for additive operations

In Section 5, we developed a comonad to express all the structure of additive unstable E-cohomology operations, for favorable E. In Section 6, we developed a bigraded algebra $Q(E)^*_*$ that contains equivalent information, where $Q(E)^k_i$ has degree $k - i$. In this section, we describe $Q(E)^*_*$ for each of our five cohomology theories $E^*(-)$, namely $E = H(\mathbf{F}_p)$, MU, BP, KU, and $K(n)$. (The first example splits into two, and we break out the degenerate special case $K(0) = H(\mathbf{Q})$.) As stably in [8], our purpose is to exhibit the structure of the results, not to derive them.

All the results here are formally very close to the stable results. By Proposition 12.3(g), $Q(\sigma)e = 1$. As $E_*(E, o) = \operatorname{colim}_k Q(E)^k_*$ by eq. (4.8), where the suspensions $Q(E)^k_* \to Q(E)^{k+1}_*$ have been revealed in Lemma 12.5 as simply multiplication by e, we stabilize everything merely by setting the suspension element $e = 1$. In this way, we recover all the corresponding stable results. Indeed, in the case $E = KU$, we have to obtain the stable structure this way.

All four answers of Section 5 are of course available, but the Second Answer remains the most practical, consisting as in Theorem 7.7 of the coactions

$$\rho_X : E^k(X) \longrightarrow E^*(X) \widehat{\otimes} Q(E)^k_*.$$

These coactions are automatically additive, multiplicative (for cup products and $\times$-products), and unital ($\rho_X 1_X = 1_X \otimes 1$). (We simplify notation by suppressing redundant completions and suffixes.)

We use exactly the same test spaces and test maps as we did stably. The point remains that complete knowledge of the behavior of $E^*(-)$ on these is sufficient to suggest the correct structure of $Q(E)^*_*$ (except that the case $E = K(n)$ requires some extra work). By Proposition 6.42(b), the one-point space T in effect defines the right unit η_R, and the circle S^1 defines $e \in Q(E)^1_1$ by eq. (12.2). As all our examples have complex orientation, we have available the elements b_i of Definition 14.1.

In each case, we list the generators and relations for the bigraded E^*-algebra $Q(E)^*_*$, describe the right unit η_R, and give the values of the algebra homomorphisms $\psi = Q(\psi): Q(E)^*_* \to Q(E)^*_* \otimes Q(E)^*_*$ and $\varepsilon = Q(\varepsilon): Q(E)^*_* \to E^*$ on each generator. In some cases, we can express the universal property of $Q(E)^*_*$ very simply. The stabilization $Q(\sigma)$ maps each generator to its stable namesake, except that of course $Q(\sigma)e = 1$.

Example. $H(\mathbb{F}_2)$. We take $E = H = H(\mathbb{F}_2)$, the Eilenberg–MacLane spectrum. Our test space is $\mathbb{R}P^\infty$, for which $H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[t]$, a polynomial algebra on the generator $t \in H^1(\mathbb{R}P^\infty)$. We define elements $c_i \in Q(H)^1_*$ by the identity

$$\rho t = \sum_{i=0}^{\infty} t^i \otimes c_i \quad \text{in } H^*(\mathbb{R}P^\infty)\,\widehat{\otimes}\,Q(H)^1_* \cong Q(H)^1_*[[t]].$$

Restriction to $S^1 = \mathbb{R}P^1$ shows that $c_0 = 0$ and $c_1 = e$. As stably, the multiplication $\mu\colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$ implies that $c_i = 0$ unless i is a power of 2. We therefore write $\xi_i = c_{2^i} \in Q(H)^1_{2^i}$ for each $i \geqslant 0$, so that

$$\rho t = \sum_{i=0}^{\infty} t^{2^i} \otimes \xi_i \quad \text{in } H^*(\mathbb{R}P^\infty)\,\widehat{\otimes}\,Q(H)^1_* \cong Q(H)^1_*[[t]], \tag{16.1}$$

which looks just like the stable version [8, (14.1)], except that now $\xi_0 = e$.

THEOREM 16.2. *For the Eilenberg–MacLane ring spectrum $H = H(\mathbb{F}_2)$:*

(a) $Q(H)^*_* = \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \xi_3, \ldots]$, *a polynomial algebra over $\mathbb{F}_2$ on generators $\xi_i \in Q(H)^1_{2^i}$ for $i \geqslant 0$, where $\xi_0 = e$;*

(b) *In the complex orientation for $H(\mathbb{F}_2)$, $b_{(i)} = \xi_i^2$ for all $i \geqslant 0$, and $b_j = 0$ if j is not a power of 2;*

(c) *ψ is given by*

$$\psi \xi_n = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i \quad \text{in } Q(H)^*_* \otimes Q(H)^1_*;$$

(d) *ε is given by $\varepsilon \xi_n = 0$ for $n > 0$ and $\varepsilon \xi_0 = 1$.*

PROOF. Part (a) is of course a reformulation of classical results. For fixed k, the stabilization $Q(\sigma)\colon Q(H)^k_* \to H_*(H, o)$ is the monomorphism that is dual (with a shift in degree) to the well-known epimorphism $\sigma_k^*\colon H^*(H, o) \to PH^*(\underline{H}_k)$ that tells which Steenrod operations can act nontrivially on $H^k(-)$. The proof of (b) is the same as stably. We prove (c) and (d) by taking $M = H^*(\mathbb{R}P^\infty)$ in diags. (6.33) and evaluating on t. $\square$

As stably in [8, §14], we combine the universal property of the polynomial algebra $\mathbb{F}_2[\xi_0, \xi_1, \xi_2, \ldots]$ with Theorem 7.7(b).

COROLLARY 16.3. *Let B be a discrete commutative graded $\mathbb{F}_2$-algebra. Assume that the ring homomorphism $\theta\colon H^*(X) \to H^*(X)\,\widehat{\otimes}\,B$ is natural for spaces X. Then on $t \in H^1(\mathbb{R}P^\infty)$, θ has the form*

$$\theta t = \sum_{i=0}^{\infty} t^{2^i} \otimes \xi'_i \quad \textit{in } H^*(\mathbb{R}P^\infty)\,\widehat{\otimes}\,B \cong B[[t]],$$

where the elements $\xi'_i \in B^{-(2^i-1)}$ determine θ uniquely for all X and may be chosen arbitrarily.

Example. $H(\mathbb{F}_p)$ (for p odd). We write $H = H(\mathbb{F}_p)$, the Eilenberg–MacLane spectrum. The complex orientation defines elements $\xi_i = b_{(i)}$ for $i \geqslant 0$, and, just as stably, $b_j = 0$ whenever j is not a power of p. The only difference now is that $\xi_0 = b_1 = e^2$ instead of 1.

The other test space is the lens space $L = K(\mathbb{F}_p, 1)$, for which $H^*(L) = \mathbb{F}_p[x] \otimes \Lambda(u)$. As x is a Chern class, $\rho_L x$ is given by eq. (14.2). This leaves only $\rho_L u$, which reduces (as stably) to

$$\rho_L u = u \otimes e + \sum_{i=0}^{\infty} x^{p^i} \otimes \tau_i \quad \text{in } H^*(L) \widehat{\otimes} Q(H)^1_*, \tag{16.4}$$

for certain elements τ_i that it defines.

THEOREM 16.5. *For the Eilenberg–MacLane ring spectrum $H = H(\mathbb{F}_p)$, with p odd:*

(a) *$Q(H)^*_*$ is the commutative algebra over $\mathbb{F}_p$ with generators:*

$e \in Q(H)^1_1$, *a polynomial generator;*
$\xi_i \in Q(H)^2_{2p^i}$ *for all $i \geqslant 0$, a polynomial generator for $i > 0$;*
$\tau_i \in Q(H)^1_{2p^i}$ *for all $i \geqslant 0$, an exterior generator;*

subject to the relation $\xi_0 = e^2$;

(b) *ψ is given by $\psi e = e \otimes e$,*

$$\psi \xi_k = \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \xi_i \quad \text{in } Q(H)^*_* \otimes Q(H)^2_*, \tag{16.6}$$

and

$$\psi \tau_k = \tau_k \otimes e + \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \tau_i \quad \text{in } Q(H)^*_* \widehat{\otimes} Q(H)^1_*;$$

(c) *ε is given by $\varepsilon e = 1$, $\varepsilon \xi_i = 0$ for $i > 0$, and $\varepsilon \tau_i = 0$ for all i.*

PROOF. Part (a) is again a reformulation of classical results, which may be recovered in this form from [27, Theorem 8.5], in somewhat different notation, by taking the indecomposables. We obtain (b) and (c) by substituting ρ_L in diags. (6.33) and evaluating on x and u. $\square$

We have the analogue of Corollary 16.3.

COROLLARY 16.7. *Let B be a discrete commutative graded $\mathbb{F}_p$-algebra. Assume that the ring homomorphism $\theta: H^*(X) \to H^*(X)\widehat{\otimes} B$ is natural for spaces X. Then on $H^*(L) = \mathbb{F}_p[x] \otimes \Lambda(u)$, θ has the form*

$$\theta x = x \otimes e'^2 + \sum_{i=1}^{\infty} x^{p^i} \otimes \xi'_i,$$

$$\theta u = u \otimes e' + \sum_{i=0}^{\infty} x^{p^i} \otimes \tau'_i,$$

where the elements $e' \in B^0$, $\xi'_i \in B^{-2(p^i-1)}$, and $\tau'_i \in B^{-(2p^i-1)}$ determine θ uniquely for all X and may be chosen arbitrarily.

Example. $H(\mathbb{Q})$. We write $E = H = H(\mathbb{Q})$, the Eilenberg–MacLane spectrum. As always, there is the suspension element $e \in Q(H(\mathbb{Q}))^1_1$, whose properties we know from Proposition 12.3. There is nothing else.

THEOREM 16.8. *For the ring spectrum $H = H(\mathbb{Q})$:*

(a) $Q(H)^*_* = \mathbb{Q}[e]$, *a polynomial algebra on $e \in Q(H)^1_1$;*

(b) *The coalgebra structure is given by $\psi e = e \otimes e$ and $\varepsilon e = 1$.*

Example. MU. The coefficient ring is $MU^* = \mathbb{Z}[x_1, x_2, x_3, \ldots]$, with a polynomial generator x_i in degree $-2i$ for each i. These give rise to the elements $\eta_R x_i \in Q(MU)^{-2i}_0$. We have complex orientation, almost by definition, and therefore the elements $b_i \in Q(MU)^2_{2i}$, with $b_0 = 0$ and $b_1 = e^2$. We have the relations (14.5) between the b's and the $\eta_R v$, but unlike the stable case, because e is no longer invertible, they do *not* render the generators $\eta_R x_i$ redundant. Implicit in [23, Corollary 4.6(a)] is that this is the whole story.

THEOREM 16.9 (Ravenel–Wilson). *For the unitary Thom ring spectrum MU:*

(a) *$Q(MU)^*_*$ is the commutative algebra over MU^* with generators:*

$\eta_R x_i \in Q(MU)^{-2i}_0$ *(for $i > 0$);*
$e \in Q(MU)^1_1$;
$b_i \in Q(MU)^2_{2i}$ *(for $i \geqslant 1$);*

all of even degree, subject to the relations (14.5) *and $b_1 = e^2$;*

(b) *ψ is given by $\psi e = e \otimes e$ and*

$$\psi b_k = \sum_{i=1}^{k} B(i,k) \otimes b_i \quad \textit{in } Q(MU)^*_* \otimes Q(MU)^2_*\,,$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$;

(c) *ε is given by $\varepsilon e = 1$ and $\varepsilon b_k = 0$ for $k > 1$.*

Although we no longer have a polynomial algebra, part of Corollary 16.3 carries over. It applies equally well to the two following cases, which we include here.

COROLLARY 16.10. *Let B be a discrete commutative E^*-algebra, where $E = MU$, BP, or KU. Then a ring homomorphism $\theta: E^*(X) \to E^*(X) \widehat{\otimes} B$ that is natural for spaces X is uniquely determined by its values on $E^*(S^1)$ and $E^*(\mathbb{C}P^\infty)$.*

Example. BP. The coefficient ring is now $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$, with polynomial generators v_n in degree $-2(p^n - 1)$. We have complex orientation, but because BP^* is p-local, we need only the generators $b_{(i)} \in Q(BP)^2_{2p^i}$, where $b_{(0)} = e^2$. Again, [23, Corollary 4.6(b)] implies that this is all there is; in particular, eq. (14.5) is redundant, except to express the other b_j in terms of the $b_{(i)}$ and the elements v_i and $w_i = \eta_R v_i$.

THEOREM 16.11 (Ravenel–Wilson). *For the Brown–Peterson ring spectrum BP:*

(a) *$Q(BP)^*_*$ is the commutative algebra over BP^* with generators:*

$w_i = \eta_R v_i \in Q(BP)_0^{-2(p^i-1)}$ *(for $i > 0$);*
$e \in Q(BP)^1_1$;
$b_{(i)} \in Q(BP)^2_{2p^i}$ *(for $i \geqslant 0$);*

subject to the main relations $(\mathcal{R}_k)$ (from eq. (14.10)) for $k > 0$ and $b_{(0)} = e^2$;

(b) *ψ is given by $\psi e = e \otimes e$ and*

$$\psi b_{(k)} = \sum_{i=1}^{p^k} B(i, p^k) \otimes b_i \quad \text{in } Q(BP)^*_* \otimes Q(BP)^2_*,$$

where $B(i, p^k)$ denotes the coefficient of x^{p^k} in $b(x)^i$;

(c) *ε is given by $\varepsilon e = 1$ and $\varepsilon b_{(k)} = 0$ (for $k > 0$).*

We discuss the structure of $Q(BP)^*_*$ in more detail in Section 18.

REMARK. Alternatively, we could use the generator h_i instead of $b_{(i)}$ as in [6]; however, Quillen's element t_i (see [21] or Adams [1, II.16]) *does not exist* in this context for $i > 1$, for lack of conjugation in $Q(BP)^*_*$.

Example. KU. We take $E = KU$, the complex Bott spectrum, with the coefficient ring $KU^* = \mathbb{Z}[u, u^{-1}]$ (where $u \in KU^{-2}$), right unit $\eta_R: KU^* \to Q(KU)^*_*$ given by $\eta_R u = v$, and Chern class x given by [8, (5.2)]. The simple form [8, (5.16)] of the formal group law reduces eq. (14.5) to

$$b(x + y + uxy) = b(x) + b(y) + b(x)b(y)v, \tag{16.12}$$

which looks like the stable version [8, (14.13)], with $b(x) = b_1x + b_2x^2 + b_3x^3 + \cdots$, except that now $b_1 = e^2 \neq 1$. The coefficient of x^iy^j yields the relation

$$b_ib_j = \sum_{k=0}^{\min(i,j)} \binom{i+j-k}{i}\binom{i}{k} u^k b_{i+j-k}v^{-1}, \tag{16.13}$$

like [8, (14.15)], except that the case $i = 1$ now gives the reduction formula

$$b_1b_i = (i+1)b_{i+1}v^{-1} + iub_iv^{-1} \quad \text{for } i > 0. \tag{16.14}$$

The results here are much clearer than in the stable case, and there is some overlap with the work of tom Dieck [10].

THEOREM 16.15. *For the complex Bott spectrum KU:*

(a) *$Q(KU)^*_*$ is generated as an algebra over $KU^* = \mathbb{Z}[u, u^{-1}]$ by the elements:*

$v = \eta_R u \in Q(KU)^{-2}_0$;
$v^{-1} = \eta_R u^{-1} \in Q(KU)^2_0$;
$e \in Q(KU)^1_1$, *the suspension element;*
$b_i \in Q(KU)^2_{2i}$ *for* $i > 0$;

subject to the relations $b_1 = e^2$ and (16.13) for $i > 0$, $j > 0$;

(b) *$Q(KU)^*_*$ is a free KU^*-module, with a basis consisting of all monomials of the forms v^n, b_iv^n, ev^n, and eb_iv^n, for $i > 0$ and $n \in \mathbb{Z}$;*

(c) *ψ is given by $\psi e = e \otimes e$ and*

$$\psi b_k = \sum_{i=1}^{k} B(i,k) \otimes b_i \quad \text{in } Q(KU)^*_* \otimes Q(KU)^2_*,$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$;

(d) *ε is given by $\varepsilon e = 1$ and $\varepsilon b_k = 0$ for all $k > 1$.*

PROOF. We start with (b). We take the Hopf line bundle ξ over $\mathbb{C}P^\infty$ and regard the element $u^{-1}[\xi] \in KU^2(\mathbb{C}P^\infty)$ as a map $f: \mathbb{C}P^\infty \to \underline{KU}_2 = \mathbb{Z} \times BU$. By Lemma 4.6, f induces an isomorphism of KU^*-modules

$$KU_*(\mathbb{C}P^\infty) \longrightarrow QKU_*(\mathbb{Z} \times BU) \cong KU^* \oplus QKU_*(BU),$$

which we compute. By the definition [8, (5.2)] of the Chern class x, $u^{-1}[\xi] = u^{-1} + x$ in $KU^2(\mathbb{C}P^\infty)$; geometrically, the components of f are the map $\mathbb{C}P^\infty \to \mathbb{Z}$ with image 1, and $x: \mathbb{C}P^\infty \to BU$.

Thus $q_2f_*\beta_0 = v^{-1}$ and $q_2f_*\beta_i = q_2x_*\beta_i = b_i$ for $i > 0$, with the help of Proposition 14.4(b); we have the desired basis of $Q(KU)^2_*$. For $Q(KU)^{2n}_*$, we multiply by v^{-n+1}, an isomorphism.

For the odd case, the description of $KU_*(U)$ in [8, Corollary 5.12] in terms of the Bott map $b\colon \Sigma(\mathbb{Z}\times BU) \to U$ shows that multiplication by e induces an isomorphism $Q(KU)^{2n}_* \cong Q(KU)^{2n+1}_*$.

We have specified enough relations to reduce any monomial in the b's, e, v, and v^{-1} to a linear combination of the elements in (b), which proves (a). Parts (c) and (d) are included in Propositions 14.4 and 12.3. □

Now that we know the additive situation, we return to finish off the stable case. We may discard the odd spaces in eq. (4.8) and write

$$KU_*(KU, o) = \operatorname*{colim}_n Q(KU)^{2n}_*.$$

COROLLARY 16.16. *In the stable algebra $KU_*(KU,o)$:*

(a) *Every element of $KU_*(KU,o)$ of even degree can be written in the form*

$$c = u^q\big(\lambda_1 u^{-1} + \lambda_2 u^{-2} b_2 + \cdots + \lambda_n u^{-n} b_n\big)v^{-m}$$

for some integers q, m, n, and λ_i;

(b) *This element $c = 0$ if and only if $\lambda_i = 0$ for all i.*

PROOF. By Theorem 16.15(b), we can write the general element of $Q(KU)^{2m+2}_{2q}$ uniquely in the form

$$c = u^q\left(\lambda_0 v^{-1} + \sum_{i=1}^n \lambda_i u^{-i} b_i\right)v^{-m}$$

with integer coefficients. Since $e^2 = b_1$, eq. (16.14) yields

$$e^2 c = u^{q+1}\left(\lambda_0 u^{-1} b_1 + \sum_{i=1}^n (i+1)\lambda_i u^{-i-1} b_{i+1} + \sum_{i=1}^n i\lambda_i u^{-i} b_i\right)v^{-m-1}$$

in $Q(KU)^{2m+4}_{2q+2}$, which gives (a). Further, $e^2 c = 0$ only if $c = 0$, which implies (b). □

Example. $K(n)$. The coefficient ring is $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$, where $v_n \in K(n)^{-2(p^n-1)}$. We write $w_n = \eta_R v_n$, as we did for *BP*. Obviously, w_n and v_n are no longer equal as they were stably, because they lie in different groups.

We have a complex orientation, and therefore the usual elements b_j. Because $K(n)^*$ is p-local, we need only the $b_{(i)}$ for $i \geqslant 0$. (In fact, $b_j = 0$ if j is not a power of p and $j < p^n$, for dimensional reasons, but not in general if $j > p^n$.) When we apply ρ to the p-th power map $\zeta\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, which induces $\zeta^* x = v_n x^{p^n}$ as in [8, (14.26)], we obtain $b_j^{p^n} w_n = v_n^j b_j$, and therefore

$$b_{(i)}^{p^n} = v_n^{p^i} b_{(i)} w_n^{-1} \quad \text{in } Q\big(K(n)\big)^{2p^n}_* \tag{16.17}$$

for $i \geqslant 0$. This stabilizes to [8, (14.27)].

In particular, $b_{(0)}^{p^n} = v_n b_{(0)} w_n^{-1}$. As always, $b_{(0)} = e^2$. A more sophisticated analysis, involving other cohomology theories as in [28, Proposition 1.1(j)], shows that this relation can be desuspended once to give

$$eb_{(0)}^{p^n-1} = v_n e w_n^{-1} \quad \text{in } Q\big(K(n)\big)^{2p^n-1}. \tag{16.18}$$

The other test space is the skeleton $Y = L^{2p^n-1}$ of the lens space L, for which $K(n)^*(Y) = \Lambda(u) \otimes K(n)^*[x: x^{p^n} = 0]$. We know $\rho_Y x$, because x is inherited from $\mathbb{C}P^\infty$. As stably, we define elements $a_i, c_i \in Q(K(n))_*^1$ by the coaction

$$\rho_Y u = \sum_{i=0}^{p^n-1} x^i \otimes a_i + \sum_{i=0}^{p^n-1} ux^i \otimes c_i. \tag{16.19}$$

By restriction to $S^1 \subset Y$, we see that $a_0 = 0$ and $c_0 = e$. Then eq. (16.18) is equivalent to the statement $\rho_Y y = y \otimes e$, where $y = v_n u x^{p^n-1} \in K(n)^*(Y)$; in other words, y behaves like $u_1 \in K(n)^1(S^1)$. The same partial multiplications $\mu: L^{2k+1} \times L^{2m} \to Y$ as in [8, §14] show that $c_i = 0$ for all $i > 0$ and that $a_i = 0$ for i not a power of p. We therefore reindex, as usual.

DEFINITION 16.20. We define $a_{(i)} = a_{p^i} \in Q(K)_{2p^i}^1$, for $0 \leqslant i < n$.

In the new notation,

$$\rho_Y u = u \otimes e + \sum_{i=0}^{n-1} x^{p^i} \otimes a_{(i)} \quad \text{in } K(n)^*(Y) \otimes Q\big(K(n)\big)_*^1. \tag{16.21}$$

Having odd degree, the $a_{(i)}$ are exterior generators of $Q(K(n))_*^*$. This is not all; we again appeal to [28, Proposition 1.1(i)] to find that one more factor e can be squeezed out of eq. (16.18) if we first multiply by $a_{(0)}$, to give the relation

$$a_{(0)} b_{(0)}^{p^n-1} = v_n a_{(0)} w_n^{-1} \quad \text{in } Q\big(K(n)\big)_*^{2p^n-1}. \tag{16.22}$$

THEOREM 16.23. *For the Morava K-theory ring spectrum $K(n)$:*

(a) *$Q(K(n))_*^*$ is the commutative bigraded algebra over $K(n)^* = \mathbf{F}_p[v_n, v_n^{-1}]$, where $v_n \in K(n)^{-2(p^n-1)}$, with generators:*

$w_n = \eta_R v_n \in Q(K(n))_0^{-2(p^n-1)}$;
$w_n^{-1} = \eta_R v_n^{-1}$;
$e \in Q(K(n))_1^1$;
$a_{(i)} \in Q(K(n))_{2p^i}^1$ *(for $0 \leqslant i < n$);*
$b_{(i)} \in Q(K(n))_{2p^i}^2$ *(for $i \geqslant 0$);*

subject to the relations $b_{(0)} = e^2$, (16.17), (16.18), *and* (16.22);

(b) ψ *is given by* $\psi e = e \otimes e$,

$$\psi a_{(k)} = a_{(k)} \otimes e + \sum_{i=0}^{k} b_{(k-i)}^{p^i} \otimes a_{(i)} \quad \textit{in } Q\big(K(n)\big)_*^* \otimes Q\big(K(n)\big)_*^1, \tag{16.24}$$

and

$$\psi b_{(k)} = \sum_{i=1}^{p^k} B(i, p^k) \otimes b_i \quad \textit{in } Q\big(K(n)\big)_*^* \otimes Q\big(K(n)\big)_*^2, \tag{16.25}$$

where $B(i, p^k)$ *denotes the coefficient of* x^{p^k} *in* $b(x)^i$ *(and Lemma* 14.6 *is used to express* $b(x)$ *in terms of the* $b_{(j)}$, v_n, *and* w_n*);*

(c) ε *is given by* $\varepsilon e = 1$, $\varepsilon a_{(k)} = 0$ *(for* $k \geqslant 0$*), and* $\varepsilon b_{(k)} = 0$ *(for* $k > 0$*).*

PROOF. The algebra structure (a) is implicit in the main theorem of [28], by taking indecomposables. As always, we obtain $\psi a_{(i)}$ and $\varepsilon a_{(i)}$ by evaluating the coaction axioms (6.33) on $u \in K(n)^*(Y)$. The rest of (b) and (c) can be obtained similarly, or by appealing to Propositions 12.3 and 14.4. □

COROLLARY 16.26. *Let* B *be a discrete commutative* $K(n)^*$*-algebra. Then a ring homomorphism* $\theta: K(n)^*(X) \to K(n)^*(X) \widehat{\otimes} B$ *that is natural for spaces* X *is uniquely determined by its values on* $K(n)^*(\mathbb{C}P^\infty)$ *and* $K(n)^*(Y)$.

REMARK. If $k \leqslant n$, eq. (16.25) simplifies just as in [8, Theorem 14.32] to

$$\psi b_{(k)} = \sum_{i=1}^{k} b_{(k-i)}^{p^i} \otimes b_{(i)} \quad \text{in } Q\big(K(n)\big)_*^* \otimes Q\big(K(n)\big)_*^2,$$

which resembles eq. (16.6).

17. Examples for unstable operations

In this section, we discuss the enriched Hopf ring for each of our five cohomology theories $E^*(-)$, namely for $E = H(\mathbb{F}_p)$, MU, BP, KU, and $K(n)$. According to Section 10, this is what we need to handle general unstable operations. As in Section 16, we divide the case $H(\mathbf{F}_p)$ in two and treat $K(0) = H(\mathbb{Q})$ separately. Even more than before, our intent is to exhibit the structure of the results, not to reestablish them.

Our strategy is the same as in the stable and additive contexts, using exactly the same test spaces and test maps. Each E has a complex orientation, which provides by Definition 15.1 the elements b_i of the Hopf ring, in addition to e and the $[v]$. We have $\chi[1] = [-1]$ by Proposition 11.2(d), and its properties were listed in Proposition 10.12.

As pointed out in (10.46), we need more than just the Hopf ring and the elements $[v]$. The elements $Q(\varepsilon)q_k c = \varepsilon_S \sigma_{k*} c$ are given by Section 16. We also need $r_* c$ for each operation r; by Theorems 10.19(c) and 10.33(c), it is in principle enough to know these for each $\circ$-generator c.

Our presentation changes somewhat from Section 16. Each family of $\circ$-generators has its own Proposition, which lists all the pertinent information. It is therefore sufficient to describe each Hopf ring by listing its $\circ$-generators and the defining relations, and to refer to these propositions for further details. We recover all the results for additive operations merely by taking the indecomposables.

Example. MU. We recall that $MU^* = \mathbb{Z}[x_1, x_2, x_3, \ldots]$, where $\deg(x_i) = -2i$, is better described as generated by the elements $a_{i,j}$, as in [8, §14]. We have the elements b_i, as well as e and $[v] = \eta_R(v)$. Stably, [8, (13.6)] gave an inductive formula for $\eta_R a_{i,j}$ in terms of MU^* and the b_i. Unstably, eq. (15.8) is only a relation between these elements. Corollary 4.6(a) of [23] says in effect that this is all there is.

THEOREM 17.1 (Ravenel–Wilson). *For the unitary cobordism ring spectrum MU, $MU_*(\underline{MU}_*)$ is the Hopf ring over $MU^* = \mathbb{Z}[x_1, x_2, x_3, \ldots]$ with $\circ$-generators:*

$[x_i] \in MU_0(\underline{MU}_{-2i})$ *for each* $i > 0$ (see Proposition 11.2);
$e \in MU_1(\underline{MU}_1)$ (see Proposition 13.7);
$b_i \in MU_{2i}(\underline{MU}_2)$ *for* $i \geqslant 1$ (see Proposition 15.3);

subject to the relations $e^{\circ 2} = -b_1$ *and eq.* (15.8).

Example. BP. The main reference is still [23]. As BP^* is p-local, Lemma 15.9 and Definition 15.10 apply, to define the elements $b_{(i)}$ of the Hopf ring. We have as always e and the elements $[v]$ for each $v \in BP^*$.

THEOREM 17.2 (Ravenel–Wilson). *For the Brown–Peterson ring spectrum BP, $BP_*(\underline{BP}_*)$ is the Hopf ring over $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$ with the $\circ$-generators:*

$[\lambda] \in BP_0(\underline{BP}_0)$, *for each* $\lambda \in \mathbb{Z}_{(p)}$ (see Proposition 11.2);
$[v_i] \in BP_0(\underline{BP}_{-2(p^n-1)})$, *for* $i > 0$ (see Proposition 11.2);
$e \in BP_1(\underline{BP}_1)$ (see Proposition 13.7);
$b_{(i)} \in BP_{2p^i}(\underline{BP}_2)$ *for* $i \geqslant 0$ (see Proposition 15.3);

subject to the relations $[\lambda] \circ [\lambda'] = [\lambda\lambda']$, $[\lambda] * [\lambda'] = [\lambda + \lambda']$, $e \circ [\lambda] = \lambda e$, $b_{(i)} \circ [\lambda] = \ldots$ (see Proposition 15.3(f)), $e^{\circ 2} = -b_{(0)}$, *and the main relations* $(\mathcal{R}_n)$ *for* $n > 0$ *as in eq.* (15.16).

We implicitly use eq. (15.8), but only to express inductively the b_j, for j not a power of p, in terms of the $b_{(i)}$, v, and $[v]$; this is needed for computing $\psi b_{(i)}$, $\chi b_{(i)}$, $b_{(i)} \circ [\lambda]$, and $r_* b_{(i)}$.

PROOF. This is the content of [23, Corollary 4.6(b)]. By Proposition 11.2, each $[v]$ for $v \in BP^*$ can be expressed in terms of the $[\lambda]$ and $[v_i]$; we have enough generators. The listed relations come from Propositions 11.2, 13.7, and 15.3, and eq. (15.16). This reduces the $*$-generators (see Section 10) to three types:

(i) $b^{\circ I} \circ [v^J]$;
(ii) $e \circ b^{\circ I} \circ [v^J]$; (17.3)
(iii) $[\lambda v^J]$;

in terms of the multi-index notation $b^{\circ I}$ introduced in eq. (15.11).

For each k, the $*$-generators that lie in $BP_*(\underline{BP}_k)$ generate it as a BP^*-algebra. Assume first that k is even, so that we have only types (i) and (iii). We write $\underline{BP}_k = BP^k \times \underline{BP}'_k$ as in Lemma 4.17; then

$$BP_*(\underline{BP}_k) \cong BP_*(BP^k) \otimes BP_*(\underline{BP}'_k), \tag{17.4}$$

where we recognize the first factor as the group ring over BP^* of the abelian group BP^k with basis elements $[v]$ for $v \in BP^k$. The type (i) generators lie in $BP_*(\underline{BP}'_k)$ and the type (iii) in $BP_*(BP^k)$, which is described by Lemma 4.4. Because $[\lambda v^J] * [\lambda' v^J] = [(\lambda+\lambda')v^J]$, we have enough relations for the type (iii) generators. The work of [23] reduces the type (i) generators to certain *allowable* generators $b^{\circ I} \circ [v^J]$, which form a system of polynomial generators $BP_*(\underline{BP}'_k)$. Since this reduction (see Section 19) uses only the relations $(\mathcal{R}_n)$, we have enough relations.

If k is odd, only generators of type (ii) occur. These reduce similarly to the allowable generators of type (ii), which are exterior generators of $BP_*(\underline{BP}_k)$. □

Example. $H(\mathbb{Q})$. This example is of course classical.

THEOREM 17.5. *For the ring spectrum $H = H(\mathbb{Q})$, $H_*(\underline{H}_*)$ is the Hopf ring over $\mathbb{Q}$ with generators:*

$[\lambda] \in H_0(\underline{H}_0)$ *for each* $\lambda \in \mathbb{Q}$ (see Proposition 11.2);
$e \in H_1(\underline{H}_1)$ (see Proposition 13.7);

subject to the relations $[\lambda] \circ [\lambda'] = [\lambda\lambda']$, $[\lambda] * [\lambda'] = [\lambda+\lambda']$, *and* $e \circ [\lambda] = \lambda e$.

PROOF. For $k < 0$, $\underline{H}_k = T$, and we have only the $\mathbb{Q}$-basis element 1_k.

For $k = 0$, $\underline{H}_0 = \mathbb{Q}$, regarded as a discrete group, and the group ring $H_*(\underline{H}_0) = \mathbb{Q}[\mathbb{Q}]$ has a basis consisting of the elements $[\lambda]$. The first two relations, from Proposition 11.2, show how these multiply.

For $k > 0$, the third relation, from Proposition 13.7(g), reduces us to the single $*$-generator $e^{\circ k} \in H_k(\underline{H}_k)$ of $H_*(\underline{H}_k)$. We have the polynomial algebra $\mathbb{Q}[e^{\circ k}]$ if k is even, or the exterior algebra $\Lambda(e^{\circ k})$ if k is odd. □

Example. $H(\mathbf{F}_2)$. We write $H = H(\mathbb{F}_2)$. As $H_*(\underline{H}_*)$ is a Hopf ring over $\mathbb{F}_2$, we have the Frobenius operator F and the Verschiebung V.

We imitate Definitions 15.1 and 15.10 in a mod 2 version, using the same test space $\mathbb{R}P^\infty = K(\mathbf{F}_2, 1)$ as before, for which $H^*(\mathbb{R}P^\infty) = \mathbf{F}_2[t]$. We define $c_i \in H_i(\underline{H}_1) = H_i(\mathbb{R}P^\infty)$ for $i \geqslant 0$ by the identity

$$r(t) = \sum_{i=0}^{\infty} \langle r, c_i \rangle\, t^i = \langle r, c(t) \rangle \quad \text{in } H^*(\mathbb{R}P^\infty) \text{ (for all } r), \tag{17.6}$$

where we write formally $c(t) = \sum_i c_i t^i$ as in Definition 15.1. In other words, c_i is dual to t^i and the elements c_i form an $\mathbb{F}_2$-basis of $H_*(\underline{H}_1)$.

We are primarily interested in the accelerated elements $c_{(i)} = c_{2^i}$. As before, we have the suspension element e. The complex orientation provides elements b_i which are redundant, as in Section 16.

PROPOSITION 17.7. *The Hopf ring elements $c_i \in H_i(\underline{H}_1)$ (for $i \geqslant 0$) and $c_{(i)} = c_{2^i} \in H_{2^i}(\underline{H}_1)$ (for $i \geqslant 0$) have the following properties:*

(a) $c_0 = 1_1$ *and* $c_{(0)} = c_1 = e$;

(b) $\psi c_k = \sum_{i+j=k} c_i \otimes c_j$, *or formally,* $\psi c(t) = c(t) \otimes c(t)$;

(c) $V c_{(i)} = c_{(i-1)}$ *for* $i > 0$, *and* $V c_{(0)} = 0$;

(d) $\varepsilon c_k = 0$ *if* $k > 0$, *and* $\varepsilon c_0 = 1$, *or formally,* $\varepsilon c(t) = 1$;

(e) $\chi c(t) = c(t)^{*(-1)}$, *expanded as in Proposition* 15.3(e);

(f) $c_i * c_j = \binom{i+j}{i} c_{i+j}$;

(g) $F c_{(i)} = c_{(i)} * c_{(i)} = 0$;

(h) $b_i = c_i \circ c_i$ *in* $H_{2i}(\underline{H}_2)$;

(i) *For all r, $r_* c_k$ is the coefficient of t^k in the formal identity*

$$r_* c(t) = \mathop{\huge *}_{j=0}^{\infty} c(t)^{\circ j} \circ \left[\langle r, c_j\rangle\right] \quad in\ H_*(\underline{H}_*)[[t]];$$

(j) $q_1 c_{(i)} = \xi_i$ *in* $Q(H)^1_*$, *and* $q_1 c_j = 0$ *if j is not a power of* 2;

(k) $\sigma_{1*} c_{(i)} = \xi_i$ *in* $H_*(H, o)$, *and* $\sigma_{1*} c_j = 0$ *if j is not a power of 2.*

PROOF. The naturality of r for the multiplication $\mu\colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$, which induces $\mu^* t = t \times 1 + 1 \times t$, yields the identity

$$\sum_k \langle r, c_k\rangle\, (t \times 1 + 1 \times t)^k = \sum_i \sum_j \langle r, c_i * c_j\rangle\, t^i \times t^j$$

in $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) = \mathbb{F}_2[t \times 1, 1 \times t]$, with the help of the Cartan formula (10.23). The coefficient of $t^i \times t^j$ gives (f). The special case (g) of (f) also follows from eq. (10.32). We expand $r(t^2)$ for the Chern class t^2 by eqs. (17.6) and (10.36) and compare with eq. (15.2); most terms cancel, to give (h).

The other parts are formally as in Proposition 15.3, with all degrees halved, except that (c) is immediate from (b). □

Just as in Lemma 15.9, except that everything is now explicit in (f), c_j is redundant unless j is a power of 2. This leads to the following elegant description of the Hopf ring, which is a reformulation of classical results.

THEOREM 17.8. *For the Eilenberg–MacLane ring spectrum $H = H(\mathbb{F}_2)$, $H_*(\underline{H}_*)$ is the Hopf ring over $\mathbb{F}_2$ with generators $c_{(i)} \in H_{2^i}(\underline{H}_1)$ for $i \geqslant 0$* (see Proposition 17.7), *subject to the relation* $[1]^{*2} = 1_0$.

PROOF. By Proposition 17.7(c), we can write $c^{\circ I} = Vc^{\circ s(I)}$ for any multi-index $I = (i_0, i_1, i_2, \ldots)$. Then $Fc^{\circ I} = F([1] \circ c^{\circ I}) = F[1] \circ c^{\circ s(I)} = 0$ by eq. (10.13), as in eq. (15.13), and $H_*(\underline{H}_k)$ is an exterior algebra on those generators $c^{\circ I}$ for which $\sum_t i_t = k$. Here, $c^{\circ I}$ is dual to the primitive element $\mathrm{Sq}^{i_1, i_2, \ldots}\iota_k$ in cohomology (in terms of the Milnor basis [18] of $H^*(H, o)$). (The index i_0 serves only as padding, to ensure that $i_1 + i_2 + i_3 + \cdots \leqslant k$.) □

Example. $H(\mathbb{F}_p)$ (for p odd). We write $H = H(\mathbb{F}_p)$. We have, as always, the suspension element e. The complex orientation defines elements b_i for all $i \geqslant 0$; but Lemma 15.9 shows that only the $b_{(i)} = b_{p^i}$ for $i \geqslant 0$ are needed. Also, $b_0 = 1_2$ and $b_1 = e_2 = -e^{\circ 2}$. However, the b_j for j not a power of p do not vanish, but satisfy $b_i * b_j = \binom{i+j}{i} b_{i+j}$, which is all that survives from eq. (15.8). In particular, $b_{(i)}^{*p} = 0$ for all $i > 0$, as is also clear from eq. (10.32) applied to x.

For the other test space $L = K(\mathbb{F}_p, 1)$, we have $H^*(L) = \mathbb{F}_p[x] \otimes \Lambda(u)$. We only need to know $r(u)$. We define elements $a_i \in H_{2i}(\underline{H}_1)$ and $c_i \in H_{2i+1}(\underline{H}_1)$ by

$$r(u) = \sum_{i=0}^{\infty} \langle r, a_i \rangle\, x^i + \sum_{i=0}^{\infty} \langle r, c_i \rangle\, ux^i \quad \text{in } H^*(L),$$

which we condense formally to $\langle r, a(x) \rangle + \langle r, c(x) \rangle\, u$ by writing $a(x) = \sum_i a_i x^i$ and $c(x) = \sum_i c_i x^i$. Thus a_i is dual to x^i, c_i is dual to ux^i, and the a_i and c_i form a basis of $H_*(\underline{H}_1)$.

Again, we accelerate the indexing by defining $a_{(i)} = a_{p^i}$ for $i \geqslant 0$.

PROPOSITION 17.9. *The Hopf ring elements $a_i \in H_{2i}(\underline{H}_1)$, $a_{(i)} = a_{p^i} \in H_{2p^i}(\underline{H}_1)$, and $c_i \in H_{2i+1}(\underline{H}_1)$, (for $i \geqslant 0$), have the following properties:*

(a) $a_0 = 1_1$ *and* $c_0 = e$;

(b) $\psi a_k = \sum_{i+j=k} a_i \otimes a_j$;

(c) $Va_{(i)} = a_{(i-1)}$ *for* $i > 0$, *and* $Va_{(0)} = 0$;

(d) $\varepsilon a_k = 0$ *for all* $k > 0$;

(e) $\chi a(x) = a(x)^{*(-1)}$, *expanded as in Proposition* 15.3(e);

(f) $a_i * a_j = \binom{i+j}{i} a_{i+j}$;

(g) $Fa_{(i)} = a_{(i)}^{*p} = 0$;

(h) $c_i = e * a_i$;

(i) *For all* r, $r_* a_k$ *is the coefficient of* x^k *in the formal identity*

$$r_* a(x) = \mathop{\Large *}_{i=0}^{\infty} b(x)^{\circ i} \circ [\langle r, a_i \rangle] * \mathop{\Large *}_{i=0}^{\infty} a(x) \circ b(x)^{\circ i} \circ [\langle r, c_i \rangle] \quad \text{in } H_*(\underline{H}_*)[[x]];$$

(j) $q_1 a_{(i)} = \tau_i$ *in* $Q(H)^1_*$, *and* $q_1 a_j = 0$ *if* j *is not a power of* p;

(k) $\sigma_{1*} a_{(i)} = \tau_i$ *in* $H_*(H, o)$, *and* $\sigma_{1*} a_j = 0$ *if* j *is not a power of* p.

PROOF. We consider naturality of operations with respect to the multiplication $\mu\colon L\times L\to L$, for which $\mu^*u = u\times 1 + 1\times u$. In condensed notation, we compare

$$\mu^*r(u) = \langle r, a(x\times 1 + 1\times x)\rangle + \langle r, c(x\times 1 + 1\times x)\rangle(u\times 1 + 1\times u)$$

with $r(\mu^*u)$, which we expand by eq. (10.23) as

$$\begin{aligned} r(\mu^*u) = {} & \langle r, a(x\times 1) * a(1\times x)\rangle + \langle r, c(x\times 1) * a(1\times x)\rangle u\times 1 \\ & + \langle a(x\times 1) * c(1\times x)\rangle 1\times u + \langle r, c(x\times 1) * c(1\times x)\rangle u\times u. \end{aligned}$$

The coefficient of $x^i \times x^j$ gives (f), which implies (g). (Alternatively, (g) follows from eq. (10.32) applied to u.) The coefficient of $u \times x^i$ gives (h). The other parts require no new ideas. □

In particular, all the c_i and most of the a_i are redundant. We trivially have the relation $[1]^{*p} = [p] = [0] = 1$, from which it follows, as in the previous example, that $(a^{\circ I})^{*p} = 0$ and $(b^{\circ I})^{*p} = 0$ for all I. Once again, this is the whole story. A detailed exposition by Ravenel and Wilson from this point of view is presented in [27, Theorem 8.5] (with slightly different notation: $a_{(i)}$ is written $\alpha_{(i)}$, and $b_{(i)}$ is written $\beta_{(i)}$).

THEOREM 17.10 (Ravenel–Wilson). *For the Eilenberg–MacLane ring spectrum $H = H(\mathbb{F}_p)$, $H_*(\underline{H}_*)$ is the Hopf ring over $\mathbf{F}_p$ with the $\circ$-generators:*

$e \in H_1(\underline{H}_1)$ (see Proposition 13.7);
$a_{(i)} \in H_{2p^i}(\underline{H}_1)$, *for* $i \geqslant 0$ (see Proposition 17.9);
$b_{(i)} \in H_{2p^i}(\underline{H}_2)$, *for* $i \geqslant 0$ (see Proposition 15.3);

subject to the relations $[1]^{*p} = 1_0$ *and* $e^{\circ 2} = -b_{(0)}$.

Example. KU. We recall that $KU^* = \mathbb{Z}[u, u^{-1}]$. The complex orientation defines elements b_i for $i > 0$. As before, these, along with elements $[\lambda u^n] = [\lambda]\circ[u^n]$ and e, are all we need.

In view of the formal group law

$$F(x,y) = x + y + uxy,$$

the relation (15.8) becomes

$$1 + \bar{b}(x+y+uxy) = \big(1 + \bar{b}(x)\big) * \big(1 + \bar{b}(y)\big) * \big(1 + \bar{b}(x)\circ\bar{b}(y)\circ[u]\big), \tag{17.11}$$

which is more complicated than the additive analogue (16.12), but still manageable. Again, we take the coefficient of x^iy^j. The left side is the same as before. On the right, we may choose x^sy^t with $s > 0$ and $t > 0$ from the third factor, which forces us to take x^{i-s} from the first factor and y^{j-t} from the second; or we can take all of x^iy^j from the first two factors. The result, after some rearranging, is

$$
\begin{aligned}
b_i \circ b_j = & \sum_{k=0}^{\min(i,j)} \binom{i+j-k}{i} \binom{i}{k} u^k b_{i+j-k} \circ [u^{-1}] \\
& - \sum_{s=1}^{i-1} \sum_{t=1}^{j-1} b_{i-s} \circ [u^{-1}] * b_{j-t} \circ [u^{-1}] * b_s \circ b_t \\
& - \sum_{s=1}^{i-1} b_{i-s} \circ [u^{-1}] * b_s \circ b_j - \sum_{t=1}^{j-1} b_{j-t} \circ [u^{-1}] * b_i \circ b_t \\
& - b_i \circ [u^{-1}] * b_j \circ [u^{-1}].
\end{aligned}
\tag{17.12}
$$

This serves as an inductive reduction formula for $b_i \circ b_j$, for any $i > 0$ and $j > 0$. In particular, the suspension formula becomes

$$
\begin{aligned}
b_1 \circ b_j = & (j+1) b_{j+1} \circ [u^{-1}] + j u b_j \circ [u^{-1}] \\
& - \sum_{k=1}^{j-1} b_{j-k} \circ [u^{-1}] * b_1 \circ b_k - b_1 \circ [u^{-1}] * b_j \circ [u^{-1}].
\end{aligned}
\tag{17.13}
$$

THEOREM 17.14. *For the complex K-theory ring spectrum KU, $KU_*(\underline{KU}_*)$ is the Hopf ring over $KU^* = \mathbb{Z}[u, u^{-1}]$ with the $\circ$-generators:*

> $[u] \in KU_0(\underline{KU}_{-2})$ (see Proposition 11.2);
> $[u^{-1}] \in KU_0(\underline{KU}_2)$ (see Proposition 11.2);
> $e \in KU_1(\underline{KU}_1)$ (see Proposition 13.7);
> $b_i \in KU_{2i}(\underline{KU}_2)$ *for* $i > 0$ (see Proposition 15.3);

subject to the relations $[u] \circ [u^{-1}] = [1]$, $\chi e = -e$, $\chi b_i = \ldots$ (see Proposition 15.3(e)), $e^{\circ 2} = -b_1$, *and eq.* (17.12).

Explicitly, for the even spaces we have

$$KU_*(\underline{KU}_{2n}) = \bigoplus_{m \in \mathbb{Z}} [mu^{-n}] * KU^*[b_1 \circ [u^{-n+1}], b_2 \circ [u^{-n+1}], b_3 \circ [u^{-n+1}], \ldots],$$

a direct sum (over m) of polynomial algebras, and for the odd spaces

$$KU_*(\underline{KU}_{2n+1}) = \Lambda\big(e \circ [u^{-n}], e \circ b_1 \circ [u^{-n+1}], e \circ b_2 \circ [u^{-n+1}], \ldots\big),$$

an exterior algebra over KU^* (where we use $[mu^{-n}] = [m] \circ [u^{-n}]$, $[u^n] = [u]^{\circ n}$, $[u^{-n}] = [u^{-1}]^{\circ n}$, $[u^0] = [1]$, $[n] = [1]^{*n}$, and $[-n] = [-1]^{*n} = (\chi[1])^{*n}$).

PROOF. We computed $KU_*(BU)$ in [8, Lemma 5.6]. By Proposition 15.3(b), the Chern class $x \colon \mathbb{C}P^\infty \to \underline{KU}_2$ induces $x_* \beta_i = b_i$, so that we may write $KU_*(0 \times BU) =$

$KU^*[b_1, b_2, \ldots]$. For the copy $KU_*(m \times BU)$, we $*$-multiply this by $[m]$. This gives $KU_*(\underline{KU}_2)$. For other even spaces, we apply the $*$-isomorphism $-\circ[u^n]$.

For the odd spaces, we quote [8, Corollary 5.12].

To see that we have specified enough relations, we note that every $*$-generator reduces to $e\circ[u^n]$ or $e\circ b_i\circ[u^n]$ on the odd spaces, or $b_i\circ[u^n]$ or $[\lambda]\circ[u^n]$ on the even spaces, where $\lambda \in \mathbb{Z}$. We allow $n = 0$ and $\lambda = 1$ and use $[u^m]\circ[u^n] = [u^{m+n}]$ and $[\lambda]\circ[\lambda'] = [\lambda\lambda']$. In the even case, we need at most one $*$-factor of the form $[\lambda]\circ[u^n]$, and we may always insert the redundant factor $[0]\circ[u^n] = 1$. Thus we can reduce any expression in the generators to standard form. $\square$

Example. $K(n)$. We use the same test spaces as before, $\mathbb{C}P^\infty$ and the finite lens space L^{2p^n-1}, and follow the same strategy. The main reference is [28]. Some of the algebra resembles the case $E = H(\mathbf{F}_p)$.

As usual, the complex orientation determines Hopf ring elements b_i, where $b_0 = 1_2$ and $b_1 = e_2 = -e^{\circ 2}$. As $K(n)$ is p-local, Lemma 15.9 shows that the b_j other than the $b_{(i)} = b_{p^i}$ are redundant. If we apply eq. (10.32) to the Chern class x, we obtain the identity $\sum_j \langle r, Fb_j\rangle\, x^{pj} = \langle r, 1_2\rangle\, 1$. This shows that $Fb_j = 0$ for all $j > 0$; in particular, $b_{(i)}^{*p} = 0$.

Next, we apply the general operation r to $\zeta^* x = v_n x^{p^n}$ by eq. (15.2) to obtain $b(v_n x^{p^n}) = b(x)^{\circ p^n}\circ[v_n]$. Equating coefficients of $x^{p^{n+i}}$ yields the relation

$$b_{(i)}^{\circ p^n} = v_n^{p^i} b_{(i)} \circ [v_n^{-1}], \tag{17.15}$$

the obvious analogue of eq. (16.17).

For the other test space $Y = L^{2p^n-1}$, we have

$$K(n)^*(Y) = \Lambda(u) \otimes K(n)^*[x : x^{p^n} = 0].$$

The class x is a Chern class, which we know all about. Parallel to eq. (16.19), we use $u \in K(n)_1(Y)$ to define elements $a_i, c_i \in K(n)_*(\underline{K(n)}_1)$ for $0 \leqslant i < p^n$ by the identity

$$r(u) = \sum_{i=0}^{p^n-1} \langle r, a_i\rangle\, x^i + \sum_{i=0}^{p^n-1} \langle r, c_i\rangle\, u x^i \quad \text{in } K(n)^*(Y) \text{ (for all } r).$$

PROPOSITION 17.16. *The Hopf ring elements $a_i \in K(n)_{2i}(\underline{K(n)}_1)$ (for $0 \leqslant i < p^n$), $a_{(i)} = a_{p^i} \in K(n)_{2p^i}(\underline{K(n)}_1)$ (for $0 \leqslant i < n$), and $c_i \in K(n)_{2i+1}(\underline{K(n)}_1)$ (for $0 \leqslant i < p^n$) have the following properties:*

(a) $a_0 = 1_1$ *and* $c_0 = e$;

(b) $\psi a_k = \sum_{i+j=k} a_i \otimes a_j$;

(c) $Va_{(i)} = a_{(i-1)}$ *for* $0 < i < n$, *and* $Va_{(0)} = 0$;

(d) $\varepsilon a_k = 0$ *for all* $k > 0$;

(e) χa_k *is the coefficient of* x^k *in* $a(x)^{*(-1)}$, *expanded as in Proposition* 15.3(e);

(f) $a_i * a_j = \binom{i+j}{i} a_{i+j}$ *if* $i + j < p^n$;

(g) $Fa_{(i)} = a_{(i)}^{*p} = 0$ *for* $0 \leqslant i < n-1$;

(h) $c_i = e * a_i$;

(i) *For all* r, $r_* a_k$ *is the coefficient of* x^k *in the formal identity*

$$r_* a(x) = \mathop{\LARGE *}_{i=0}^{p^n-1} b(x)^{\circ i} \circ \big[\langle r, a_i\rangle\big] * \mathop{\LARGE *}_{i=0}^{p^n-1} a(x) \circ b(x)^{\circ i} \circ \big[\langle r, c_i\rangle\big]$$

in $K(n)_*(\underline{K(n)}_*)[x : x^{p^n} = 0]$;

(j) $q_1 a_{(i)} = a_{(i)} \in Q(K(n))_*^1$, *and* $q_1 a_j = 0$ *if* j *is not a power of* p;

(k) $\sigma_{1*} a_{(i)} = a_{(i)} \in K(n)_*(K(n), o)$, *and* $\sigma_{1*} a_j = 0$ *if* j *is not a power of* p.

PROOF. All the proofs are formally identical to those of Proposition 17.9, except that we use the space Y instead of L. As in Section 16, the partial multiplications $\mu: L^{2k+1} \times L^{2m} \to Y$ yield (f) and (h).

For (g), we apply eq. (10.32) to u and obtain

$$\sum_{i>0} \langle r, Fa_i\rangle\, x^{pi} = 0.$$

But because $x^{p^n} = 0$ already, we are able to deduce that $a_i^{*p} = 0$ only for $0 < i < p^{n-1}$. (We shall see in a moment that $a_{(n-1)}^{*p} \neq 0$.) □

We have to rely on [28, Proposition 1.1] for two facts, just as in Section 16. The first is that when $i = 0$, eq. (17.15) desuspends once, exactly as eq. (16.18) suggests, to

$$e \circ b_{(0)}^{\circ p^n - 1} = v_n e \circ \big[v_n^{-1}\big]. \tag{17.17}$$

In other words, the class $y = v_n u x^{p^n - 1} \in K(n)^1(Y)$ still behaves like $u_1 \in K(n)^1(S^1)$ and satisfies eq. (13.2). The second is that when we take account of decomposables, eq. (16.22) acquires an extra term,

$$a_{(n-1)}^{*p} = v_n a_{(0)} - a_{(0)} \circ b_{(0)}^{\circ p^n - 1} \circ [v_n]. \tag{17.18}$$

This complements (g). We have the material for the main theorem of [28].

THEOREM 17.19. *For the Morava* K*-theory ring spectrum* $K(n)$, $K(n)_*(\underline{K(n)}_*)$ *is the Hopf ring over* $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$ *with the* $\circ$*-generators:*

$[v_n] \in K(n)_0(\underline{K(n)}_{-2(p^n-1)})$ (see Proposition 11.2);
$[v_n^{-1}] \in K(n)_0(\underline{K(n)}_{2(p^n-1)})$ (see Proposition 11.2);
$e \in K(n)_1(\underline{K(n)}_1)$ (see Proposition 13.7);
$a_{(i)} \in K(n)_{2p^i}(\underline{K(n)}_1)$, *for* $0 \leqslant i < n$ (see Proposition 17.16);
$b_{(i)} \in K(n)_{2p^i}(\underline{K(n)}_2)$, *for* $i \geqslant 0$ (see Proposition 15.3);

subject to the relations $[1]^{*p} = 1_0$, $[v_n]\circ[v_n^{-1}] = [1]$, $e^{\circ 2} = -b_{(1)}$, (17.15), (17.17), *and* (17.18). □

Thus we have the $*$-generators:

(i) $a^{\circ I}\circ b^{\circ J}\circ[v_n^k]$ in even degrees;
(ii) $e\circ a^{\circ I}\circ b^{\circ J}\circ[v_n^k]$ in odd degrees;

where $I = (i_0, i_1, \ldots, i_{n-1})$, with each $i_r = 0$ or 1, and $J = (j_0, j_1, j_2, \ldots)$, with $0 \leqslant j < p^n$, and $k \in \mathbb{Z}$. In (ii), we may assume $j_0 < p^n - 1$ by eq. (17.17). The relations $a_{(i)}^{*p} = 0$ (for $i < n-1$) and $b_{(i)}^{*p} = 0$ (for all i) follow from $[1]^{*p} = 1_0$ by eq. (10.13), as in Theorem 17.10.

18. Relations for additive BP-operations

In this section, we discuss relations in the bigraded algebra $Q_*^* = Q(BP)_*^*$, following [23], in preparation for discussing additive unstable operations in BP-cohomology. In view of Theorem 16.11(a), Q_*^* is spanned as a BP^*-module by the monomials

$$e^\varepsilon b^I w^J = e^\varepsilon b_{(0)}^{i_0} b_{(1)}^{i_1} \cdots w_1^{j_1} w_2^{j_2} \cdots, \tag{18.1}$$

where $\varepsilon \leqslant 1$ and we use standard notation with multi-indices $I = (i_0, i_1, i_2, \ldots)$ and $J = (j_1, j_2, \ldots)$. We define the *length* of I as $|I| = \sum_t i_t$, and similarly $|J| = \sum_t j_t$. We also need the special multi-index $\Delta_0 = (1, 0, 0, \ldots)$.

The main relations. For $E = BP$, we easily compute the first main relation from Definition 14.9 and [8, (15.4)] (or equivalently from eqs. (14.5) and [8, (15.3)]) as

$$(\mathcal{R}_1):\quad v_1 b_{(0)} = p b_{(1)} + b_{(0)}^p w_1 \quad \text{in } Q(BP)_*^2. \tag{18.2}$$

(Indeed, this is the only candidate that stabilizes correctly to [8, (15.6)].) We still have $b_i = 0$ whenever $i-1$ is not a multiple of $p-1$. We can use the p-series [8, (15.5)], just as stably, to simplify the higher relations $(\mathcal{R}_k)$ by neglecting enough. Denote by $\mathfrak{V}$ and $\mathfrak{W}$ respectively the ideals $(p, v_1, v_2, \ldots)$ and $(p, w_1, w_2, \ldots)$ in Q_*^*, which correspond to the left and right actions of the ideal I_∞. We also need the ideal $\mathfrak{M} = (e, b_{(0)}, b_{(1)}, b_{(2)}, \ldots) \subset Q_*^*$, so that $\mathfrak{M} + \mathfrak{V}$ is the obvious augmentation ideal consisting of all the Q_i^k for $i > 0$. In particular, $b_i \in \mathfrak{M} + \mathfrak{V}$ for all i. From Definition 14.9 and [8, (15.7)], the right side of $(\mathcal{R}_k)$ has the form

$$R(k) \equiv \sum_{i=1}^{k-1} b_{(k-i)}^{p^i} w_i + b_{(0)}^{p^k} w_k \mod \mathfrak{V} + \mathfrak{M}\mathfrak{W}^2, \tag{18.3}$$

while the left side $L(k) \in \mathfrak{V}$ and will not much concern us here. The new feature is that because w_k appears in the form $b_{(0)}^{p^k} w_k$, where $b_{(0)} = e^2$ is no longer 1, $(\mathcal{R}_k)$ fails to express w_k in terms of the other generators, and $\mathfrak{W} \neq \mathfrak{V}$; this made it necessary to add w_k as a new generator of Q^*_* in Theorem 16.11.

The Ravenel–Wilson basis. The relations $(\mathcal{R}_k)$ show that many of the monomials (18.1) are redundant. In defining the basis, it is easier to specify which monomials are not wanted.

DEFINITION 18.4. We *disallow* all monomials of the form

$$b^p_{(i_1)} b^{p^2}_{(i_2)} \cdots b^{p^n}_{(i_n)} w_n c \quad (i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n,\ n > 0), \tag{18.5}$$

where c stands for any monomial in the $b_{(i)}$, w_i, and e ($c = 1$ is permitted). All monomials (18.1) *not* of this form are declared to be *allowable*.

Nevertheless, we need a positive construction of the allowable monomials, and we need to know how they behave under suspension. Given any indices

$$0 = k_0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n, \quad \text{where } n \geqslant 0, \tag{18.6}$$

we define the monomial

$$b^L = b_{(k_0)} b^p_{(k_1)} b^{p^2}_{(k_2)} \cdots b^{p^n}_{(k_n)} = b_{(0)} b^{L-\Delta_0}. \tag{18.7}$$

It is easy to see that every allowable monomial can be written *uniquely* in the canonical form

$$c = e^\varepsilon b^{L-\Delta_0} b^M w^J = e^\varepsilon b^p_{(k_1)} b^{p^2}_{(k_2)} \cdots b^{p^n}_{(k_n)} b^M w^J, \tag{18.8}$$

where $\varepsilon = 0$ or 1 and M and J satisfy the conditions:

$$\begin{array}{ll} \text{(i)} & t < k_u \text{ implies } m_t < p^u, \text{ for } 0 < u \leqslant n; \\ \text{(ii)} & t \geqslant k_n \text{ implies } m_t < p^{n+1}; \\ \text{(iii)} & j_t = 0 \text{ for all } t \leqslant n; \end{array} \tag{18.9}$$

as well as (18.6). In detail, we choose, by induction on u, the smallest k_u such that $b^p_{(k_1)} b^{p^2}_{(k_2)} \cdots b^{p^u}_{(k_u)}$ divides c, to make (i) hold for u. If no such k_u exists, we set $n = u - 1$ and have (ii). Since c is allowable, it can have no factor w_u, which gives (iii) for $t = u$. (In case $n = 0$, we have merely $c = e^\varepsilon b^M w^J$, (i) and (iii) are vacuous, and (ii) says only that $m_t < p$ for all t.)

The main technical result is that there is only one way the suspension ec of c can fail to be allowable. (This is in effect equivalent to the discussion in [23, §5].) We recall from Definition 15.12 the shifted multi-index $s(I)$.

LEMMA 18.10. *Assume that the monomial* $b^H = b^p_{(i_0)} b^{p^2}_{(i_1)} \cdots b^{p^{n+1}}_{(i_n)}$ *divides* $b^L b^M$, *where* b^L *(with the same* n*) is as in eq.* (18.7), $i_0 \leqslant i_1 \leqslant \cdots \leqslant i_n$, *and* M *satisfies conditions* (*i*) *and* (*ii*) *of* (18.9). *Then:*

(a) $i_u = k_u$ *for* $0 \leqslant u \leqslant n$, *so that* $H = pL$;

(b) *We can write* $M = (p-1)L + s(M')$, *where* M' *again satisfies* (*i*) *and* (*ii*).

PROOF. We show first that $i_u \geqslant k_u$ for all u. For any $t < k_u$, we have $m_t < p^u$ by (i). Then the exponent of $b_{(t)}$ in $b^L b^M$ is at most

$$\left(1 + p + p^2 + \cdots + p^{u-1}\right) + \left(p^u - 1\right) < p^{u+1},$$

which shows that $t \neq i_u$.

We proceed by induction on n. For $n = 0$, $b^p_{(i_0)}$ divides $b_{(0)}b^M$, where $m_t < p$ for all t. We must have $i_0 = 0$ and $m_0 = p - 1$, which gives M the required form.

For $n > 0$ we must have $i_n = k_n$, since $i_n > k_n$ is forbidden by (ii). Let $\alpha \geqslant 0$ be the smallest index such that $k_\alpha = k_n$; then we must have $i_\alpha = i_{\alpha+1} = \ldots = i_n = k_n$. From $l_{k_n} = p^\alpha + p^{\alpha+1} + \ldots + p^n$ and $m_{k_n} < p^{n+1}$ we deduce

$$m_{k_n} - (p-1)l_{k_n} < p^{n+1} - (p-1)\left(p^\alpha + \cdots + p^n\right) = p^{n+1} - \left(p^{n+1} - p^\alpha\right) = p^\alpha. \quad (18.11)$$

If $k_n = 0$ we clearly have $\alpha = 0$ and hence $m_0 = (p-1)l_0$, and can write $M = (p-1)L + s(M')$. If $k_n > 0$, we have $\alpha > 0$. We delete the factors $b^{p^{u+1}}_{(i_u)}$ for $\alpha \leqslant u \leqslant n$ from both sides of our hypothesis, as well as any factors $b_{(t)}$ for $t > k_n$, to deduce that $b^p_{(i_0)} b^{p^2}_{(i_1)} \cdots b^{p^\alpha}_{(i_\alpha - 1)}$ divides $b_{(k_0)} b^p_{(k_1)} \cdots b^{p^{\alpha-1}}_{(k_{\alpha-1})} b^{M''}$, where M'' satisfies (i) and (ii) for the sequence $(k_0, k_1, \cdots, k_{\alpha-1})$. By induction, we deduce that $H = pL$ and that M has the form $(p-1)L + s(M')$.

If $t \geqslant k_n$, we have $m'_t = m_{t+1} < p^{n+1}$, which gives (ii) for M'. To establish (i), assume that $t < k_u$. If also $t + 1 < k_u$, we have $m'_t \leqslant m_{t+1} < p^u$, as desired. Otherwise, $k_u = t + 1$. Let β be the smallest index such that $k_\beta > t + 1$, so that $k_u = k_{u+1} = \cdots = k_{\beta-1} = t + 1$. Then $m_{t+1} < p^\beta$ and $l_{t+1} \geqslant p^u + p^{u+1} + \cdots + p^{\beta-1}$. As in eq. (18.11), we find $m'_t = m_{t+1} - (p-1)l_{t+1} < p^u$. □

LEMMA 18.12. *In the bigraded algebra* $Q^*_* = Q(BP)^*_*$:

(a) *Every allowable monomial can be written uniquely in the form* (18.8), *subject to the conditions* (18.6) *and* (18.9), *and conversely, every monomial of this form is allowable;*

(b) *The suspended monomial from eq.* (18.8)

$$b_{(0)}c = e^\varepsilon b^L b^M w^J = e^\varepsilon b_{(0)} b^p_{(k_1)} \cdots b^{p^n}_{(k_n)} b^M w^J$$

is disallowed if and only if $j_{n+1} > 0$ *and* $b^{(p-1)L}$ *divides* b^M, *in which case we can write* $w^J = w_{n+1}w^{J'}$ *and* $b^M = b^{(p-1)L}b^{s(M')}$, *with* $b^{L-\Delta_0}b^{M'}w^{J'}$ *allowable;*

(c) *Every allowable monomial can be written uniquely in the* extended canonical form

$$c = e^\varepsilon\, b^{L-\Delta_0}\, b^{(p-1)(L+s(L)+s^2(L)+\cdots+s^{h-1}(L))}\, b^{s^h(M)}\, w^h_{n+1} w^J \qquad (18.13)$$

with L *as in eq.* (18.7), *where* $h \geqslant 0$, *either* $b^{(p-1)L}$ *does not divide* b^M *or* $j_{n+1} = 0$ (*or both*), *and conditions* (18.6) *and* (18.9) *hold;*

(d) *In* (c), *the monomial* $b^L b^M w^J$ *is allowable.*

PROOF. In (a), we need to establish the converse. If c is disallowed, so is $b_{(0)}c$. By Lemma 18.10(a), $b_{(0)}c$ can be disallowed only if $H = pL$; but by Lemma 18.10(b), the necessary factors $b_{(0)}$ are not present in c.

Moreover, $b_{(0)}c$ is disallowed if and only if it contains $b^{pL}w_{n+1}$ as a factor (using the same n). If so, we write $b^M = b^{(p-1)L}b^{s(M')}$, where $b^{L-\Delta_0}b^{M'}w^{J'}$ is allowable by (a). This proves (b).

Parts (c) and (d) follow by induction on h. We take h maximal. □

LEMMA 18.14. *In the stable range defined by* $i \leqslant pk$, *every allowable monomial in* $Q_i^k = Q(BP)_i^k$ *has the form* $e^\epsilon b_{(0)}^{i_0} b_{(1)}^{i_1} \cdots$, *with no factors of the form* w_i^j.

PROOF. For each monomial $c \in Q_*^*$, we define $g(c) = i - pk$, where $c \in Q_i^k$. We compute g from $g(b_{(n)}) \geqslant 0$ if $n > 0$, $g(w_n) = 2p(p^n - 1)$, $g(e) = -(p-1)$, and $g(b_{(0)}) = -2(p-1)$, using $g(ac) = g(a) + g(c)$. Thus if c contains w_n as a factor, $g(c) > 0$ unless c contains at least $\{2p(p^n-1) - (p-1)\}/2(p-1)$ factors $b_{(0)}$, which disallows it. □

THEOREM 18.15 (Ravenel–Wilson). *The allowable monomials* (18.1) *form a basis of the free* BP^**-module* $Q_*^* = Q(BP)_*^*$.

This is proved in [23, Theorem 5.3, Proposition 5.1]. We content ourselves with showing, as part of Theorem 18.16, that the allowable monomials *span* Q_*^*, assuming that it is spanned by all the monomials (18.1). We shall obtain for each disallowed monomial (18.5) a reduction formula that expresses it in terms of other monomials. A finiteness argument then implies that the allowable monomials must span. A counting argument is needed to show they in fact form a basis. As only the relations $(\mathcal{R}_k)$ and $e^2 = b_{(0)}$ are used in the reduction, they constitute sufficient relations in Theorem 16.11.

Knowing that the allowable monomials form a basis of Q_*^* is not enough. In order to work with this basis, we need to know how the ideal $\mathfrak{W}$ looks in terms of the basis. We therefore define $\mathfrak{A}_m$ for any $m \geqslant 0$ as the BP^**-submodule* of Q_*^* spanned by all the *allowable* monomials $e^\epsilon b^I w^J$ that have $I \neq 0$ and $|J| \geqslant m$. Although $\mathfrak{A}_m$ is not an ideal for $m > 0$, it is convenient for computation, because when an element $c \in Q_*^*$ is expressed in terms of the basis, it is obvious whether or not it lies in $\mathfrak{A}_m$. We shall prove the following parts of the structure of Q_*^*, after developing the necessary reduction formula.

THEOREM 18.16. *In the bigraded algebra* $Q_*^* = Q(BP)_*^*$:

(a) $\mathfrak{A}_m + \mathfrak{V} = \mathfrak{M}\mathfrak{W}^m + \mathfrak{V}$ *for any* $m > 0$ (*or* $\mathfrak{A}_0 + \mathfrak{V} = \mathfrak{M} + \mathfrak{V}$ *if* $m = 0$), *so that the image of* $\mathfrak{A}_m$ *in the quotient algebra* $\overline{Q}_*^*$ (see eq. (18.17)) *is an ideal;*

(b) *The allowable monomials span* $Q_*^* = Q(BP)_*^*$ *as a* BP^**-module.*

Lemma 15.2 of [8] allows us to work mod $\mathfrak{V}$, in the quotient $\mathbb{F}_p$-algebra

$$\overline{Q}_*^* = Q_*^*/\mathfrak{V} \cong QH_*(\underline{BP}_*; \mathbb{F}_p). \tag{18.17}$$

Better yet, we may ignore e and work in the subalgebra $\overline{Q_*}^{\text{even}}$.

Higher order relations. As they stand, the relations $(\mathcal{R}_k)$ are not very practical. We derive a more useful relation by eliminating the terms that come from $b(x)^{p^j}w_j$ for $j < n$ in eq. (18.3) from the n relations $(\mathcal{R}_{k_1})$, $(\mathcal{R}_{k_2}), \ldots, (\mathcal{R}_{k_n})$, as in [23, Lemma 5.13]. The result is of course a determinant. For ulterior purposes, we make the elimination totally explicit.

DEFINITION 18.18. Given any positive integers $i_1, i_2, \ldots, i_n$, where $n \geqslant 1$, we define $L(i_1, i_2, \ldots, i_n)$ and $R(i_1, i_2, \ldots, i_n)$ as the coefficient of $x_1^{p^{i_1}} x_2^{p^{i_2}} \cdots x_{n-1}^{p^{i_{n-1}}} x_n^{p^{i_n}}$ in

$$b(x_1)^p b(x_2)^{p^2} \cdots b(x_{n-1})^{p^{n-1}} b\big([p](x_n)\big)$$

and

$$b(x_1)^p b(x_2)^{p^2} \cdots b(x_{n-1})^{p^{n-1}} [p]_R\big(b(x_n)\big)$$

respectively. By eq. (14.8), these are equal in Q_*^*.

Then given any integers $0 < k_1 < k_2 < \cdots < k_n$, where $n > 1$, we deduce the *nth order derived relation*

$$(\mathcal{R}_{k_1,k_2,\ldots,k_n}): \quad \sum_\pi \varepsilon_\pi L(i_1, i_2, \ldots, i_n) = \sum_\pi \varepsilon_\pi R(i_1, i_2, \ldots, i_n) \qquad (18.19)$$

in Q_*^* by summing over all permutations $\pi \in \Sigma_n$, where ε_π denotes the sign of π and we permute the n entries in $(i_1, i_2, \ldots, i_n) = \pi(k_1, k_2, \ldots, k_n)$. (For $n = 1$, it reduces to $L(k_1) = R(k_1)$, which is just $(\mathcal{R}_{k_1})$.)

We note that this relation lies in $Q_*^{f(n)}$, where the numerical function

$$f(n) = 2\big(1 + p + p^2 + \cdots + p^{n-1}\big) = \frac{2(p^n - 1)}{p - 1} = \frac{|\deg(v_n)|}{p - 1}$$

was introduced in eq. (1.4).

The left side of $(\mathcal{R}_{k_1,k_2,\ldots,k_n})$ lies in $\mathfrak{W}$ and will be of little interest here. By eq. (18.3), the right side reduces to

$$\sum_{\pi,j} \varepsilon_\pi b^p_{(i_1-1)} b^{p^2}_{(i_2-2)} \cdots b^{p^{n-1}}_{(i_{n-1}-n+1)} b^{p^j}_{(i_n-j)} w_j \bmod \mathfrak{W} + \mathfrak{M}\mathfrak{W}^2, \qquad (18.20)$$

where we sum over all permutations π and all $j > 0$, and adopt the convention that $b_{(i)} = 0$ for $i < 0$. However, we have arranged matters so that no (explicit) terms in w_j with $j < n$ survive; when we interchange i_j and i_n, we find identical terms having opposite signs. The term of most interest is the *leading* term with $\pi = \text{id}$,

$$b^L w_n = b^p_{(k_1-1)} b^{p^2}_{(k_2-2)} \cdots b^{p^{n-1}}_{(k_{n-1}-n+1)} b^{p^n}_{(k_n-n)} w_n, \qquad (18.21)$$

which is thereby expressed in terms of other monomials and hence redundant. (The multi-index L serves only as a convenient abbreviation, unrelated to eq. (18.7). The indices k_u are different, too.)

To make this more precise, we note that all terms $b^I w_j$ in the sum (18.20) have $|I| = |L| = p+p^2+\ldots+p^n$ if $j = n$, or $|I| > |L|$ if $j > n$. We order the terms that contain w_n by defining the *weight* of any multi-index $I = (i_0, i_1, i_2, \ldots)$ as $\mathrm{wt}(I) = \sum_t t i_t$ (which is not the weight used in [23]). This makes $b^L w_n$ the heaviest term with its length, because if we improve the ordering of the indices of any other term in (18.20) by interchanging i_r and i_s, where $r < s$ and $i_r > i_s$, we increase its weight by

$$(i_s - r)p^r + (i_r - s)p^s - (i_r - r)p^r - (i_s - s)p^s = (i_r - i_s)(p^s - p^r) > 0.$$

Thus $(\mathcal{R}_{k_1,k_2,\ldots,k_n})$ provides a reduction formula

$$b^L w_n = b^p_{(k_1-1)} b^{p^2}_{(k_2-2)} \cdots b^{p^n}_{(k_n-n)} w_n \equiv \sum_{I,j} \pm b^I w_j \tag{18.22}$$

in $\overline{Q}^*_*$ mod $\mathfrak{M}\mathfrak{W}^2$, where the sum is taken over certain pairs (I, j) with $j \geqslant n$, for which $|I| > |L|$, or $|I| = |L|$ and $\mathrm{wt}(I) < \mathrm{wt}(L)$.

The first nth order relation $(\mathcal{R}_{1,2,\ldots,n})$ is particularly important, as only one term of the sum (18.20) is meaningful, namely $b^{pm}_{(0)} w_n$, where $m = f(n)/2$. We observe that this monomial lies just inside the stable range of Lemma 18.14. In this simple case, we can do better with a little more attention to detail, to obtain the direct analogue of [8, Lemma 15.8].

LEMMA 18.23. *In $Q^{f(n)}_* = Q(BP)^{f(n)}_*$ we have the relation*

$$b^{pm}_{(0)} w_n \equiv v_n b^m_{(0)} \bmod I_n Q(BP)^{f(n)}_*$$

for each $n > 0$, where $m = f(n)/2 = 1 + p + p^2 + \cdots + p^{n-1}$.

PROOF. We proceed by induction on n, starting from eq. (18.2), and work throughout mod $I_n Q^*_*$. On the left side of eq. (14.8) we have $b([p](x)) \equiv b(v_n x^{p^n} + \cdots)$, by [8, (15.5)]. Then $R(j) = L(j) \equiv 0$ for all $j < n$, and the only surviving terms in $(\mathcal{R}_{1,2,\ldots,n})$ are $b^h_{(0)} L(n) \equiv b^h_{(0)} R(n)$, where $h = p + p^2 + \ldots + p^{n-1}$. On the left, we clearly have $L(n) \equiv v_n b_{(0)}$. On the right, $b^h_{(0)} w_j \equiv 0$ for all $j < n$, by the induction hypothesis; by eq. (18.3) and dimensional reasons, the only surviving term in $R(n)$ is $b^{p^n}_{(0)} w_n$. □

PROOF OF THEOREM 18.16. We work entirely in the quotient algebra $\overline{Q}^*_*$ defined by eq. (18.17). We first generalize (18.22) to show that

$$\mathfrak{M}\mathfrak{W}^m \subset \mathfrak{A}_m + \mathfrak{M}\mathfrak{W}^{m+1} \tag{18.24}$$

for any $m \geqslant 1$. As an $\mathbf{F}_p$-module, $\mathfrak{M}\mathfrak{W}^m$ is generated by those monomials $e^\varepsilon b^I w^J$ that have $|J| \geqslant m$. These lie in $\mathfrak{A}_m$ or $\mathfrak{M}\mathfrak{W}^{m+1}$ except for the disallowed monomials that have $|J| = m$. On comparing the monomial (18.5) with eq. (18.21), we see that each such monomial has the form $b^L w_n c$, where L is given by eq. (18.21) and $c = e^\varepsilon b^L w^N$, with $|N| = m - 1$. When we multiply eq. (18.22) by c, both orderings are preserved, and we express the general disallowed monomial $b^L w_n c$ as a signed sum of monomials with greater length, or the same length and lower weight, mod $\mathfrak{M}\mathfrak{W}^{m+1}$. Because there are only finitely many monomials in each bidegree, eq. (18.24) follows by induction.

For any $i > m$, eq. (18.24) gives

$$\mathfrak{A}_m + \mathfrak{M}\mathfrak{W}^i \subset \mathfrak{A}_m + \mathfrak{A}_i + \mathfrak{M}\mathfrak{W}^{i+1} = \mathfrak{A}_m + \mathfrak{M}\mathfrak{W}^{i+1}.$$

Then by induction on i, starting from eq. (18.24),

$$\mathfrak{M}\mathfrak{W}^m \subset \mathfrak{A}_m + \mathfrak{M}\mathfrak{W}^i$$

for all $i > m$. In any fixed bigrading, $\mathfrak{M}\mathfrak{W}^i$ is zero for large i. Thus $\mathfrak{M}\mathfrak{W}^m \subset \mathfrak{A}_m$ and we have (a) for $m > 0$. For $m = 0$, we note that every monomial in $\mathfrak{M}$ either lies in $\mathfrak{M}\mathfrak{W} \subset \mathfrak{A}_1$ or is automatically allowable and so lies in $\mathfrak{A}_0$.

On reinstating the monomials of the form w^J, which are all allowable, we see that the allowable monomials span $\overline{Q^*_*}$. Then (b) follows by Nakayama's Lemma in the form [8, Lemma 15.2(d)]. □

The ideals $\mathfrak{J}_n$. Just as the ideal $I_\infty \subset BP^*$ led to the introduction of the ideal $\mathfrak{W} \subset Q^*_*$, the ideal J_n, needed for our splitting theorems, leads to an ideal in Q^*_*.

DEFINITION 18.25. We define the ideal $\mathfrak{J}_n = (w_{n+1}, w_{n+2}, w_{n+3}, \ldots) \subset Q^*_*$.

We need to know how $\mathfrak{J}_n$ sits inside Q^*_*. The answer is remarkably clean, in a certain range.

LEMMA 18.26. *Assume $n \geqslant 0$. Then:*

(a) *If $k < f(n+1)$, $Q^k_* \cap \mathfrak{J}_n$ is the left BP^*-submodule of Q^k_* spanned by all the allowable monomials $e^\varepsilon b^I w^J \in Q^k_*$ that contain an explicit factor w_t for some $t > n$;*

(b) *If $k = f(n+1)$, $Q^k_* \cap \mathfrak{J}_n$ is the left BP^*-submodule of Q^k_* spanned by all the allowable monomials as in (a), together with all disallowed monomials of the form*

$$b^p_{(i_1)} b^{p^2}_{(i_2)} \cdots b^{p^{n+1}}_{(i_{n+1})} w_{n+1},$$

where $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{n+1}$.

REMARK. The first disallowed monomial in (b) is $b^{pm}_{(0)} w_{n+1}$, where $m = f(n+1)/2$. Lemma 18.23 shows it definitely does not lie in the submodule described in (a).

PROOF. The stated elements obviously lie in $\mathfrak{J}_n$. To show the converse, we fix k and a large integer m, and prove by *downward* induction on h that all elements in Q^k_i of the form cw_h lie in the indicated submodule whenever $i < m$. This statement is vacuous for sufficiently large h (depending on m and k). We therefore fix $t > n$, assume the statement for all $h > t$, and prove it for $h = t$. We ignore e^ϵ throughout and assume k is even.

Case 1: $c = b^I$. The number $|I|$ of b-factors in c is $k/2+p^t-1$. In (a), as $k < f(n+1)$ and $t > n$, this is always less than $p+p^2+\cdots+p^t$, which makes $cw_t = b^I w_t$ automatically allowable. The same holds in (b), except in the extreme case $b^I w_{n+1}$, which may be allowable or disallowed; either way, it is in.

Case 2: $c = b^I w_h w^J$ *allowable, with* $h \leqslant t$. Then $cw_t = b^I w_h w_t w^J$ remains allowable, by the form of Definition 18.4.

Case 3: $c = aw_h$, *with* $h > t$, *any* a. Then $cw_t = (aw_t)w_h$ is in by induction, provided $i < m$.

By Theorem 18.15, these c generate $Q_*^{k+2(p^t-1)}$ as a BP^*-module. □

19. Relations in the Hopf ring for BP

In this section, we develop the unstable analogues of the results of Section 18, working in the Hopf ring $BP_*(\underline{BP}_*)$ for BP. By taking account of $*$-decomposable elements, we can improve many of these results by one. The structure of the Hopf ring was described briefly in Section 17. Before we can even state some of our results precisely, it is necessary to clarify the concept of *ideal* in a Hopf ring.

Hopf ring ideals. As it is obviously impractical to retain everything in typical Hopf ring calculations (the preceding sections should convince), we need to control carefully what is thrown away. There is an obvious relevant concept, valid in any Hopf ring H. We concentrate on the structure of H as a $*$-algebra, treating $\circ$-multiplication chiefly as a means of creating new $*$-generators from old.

DEFINITION 19.1. We call a bigraded R-submodule $\mathfrak{I}$ of any Hopf ring H over R a *Hopf ring ideal* if the quotient $H/\mathfrak{I}$ inherits a well-defined Hopf ring structure from H (over the possibly smaller ground ring $R/\varepsilon\mathfrak{I}$).

If we ignore the $\circ$-multiplication and coalgebra structure, $\mathfrak{I}$ must obviously be a $*$-ideal in the ordinary sense, i.e. an R-submodule for which $b * c \in \mathfrak{I}$ whenever $b \in H$ and $c \in \mathfrak{I}$.

LEMMA 19.2. *Let H be a Hopf ring over R and $I \subset R$ an ideal. Let $\mathfrak{I}$ be the $*$-ideal in H generated by the elements c_α. Then $\mathfrak{I}$ is a Hopf ring ideal, with quotient a Hopf ring over R/I, if and only if:*

(i) $\psi c_\alpha \in \mathfrak{I}\otimes H + H\otimes\mathfrak{I}$ *for all* α;
(ii) $\varepsilon c_\alpha \in I$ *for all* α;
(iii) $a \circ c_\alpha \in \mathfrak{I}$ *for all* $a \in H$ *and all* α;
(iv) $IH \subset \mathfrak{I}$.

PROOF. The conditions are evidently necessary. Conditions (i) and (ii) ensure that $H/\mathfrak{I}$ inherits a comultiplication ψ and counit ε. Condition (iv) shows that $H/\mathfrak{I}$ is defined over

R/I. For any $a, b \in H$, eq. (10.11) and (iii) show that $a \circ (b * c_\alpha) \in \mathfrak{I}$; this is enough to furnish $H/\mathfrak{I}$ with a $\circ$-multiplication. All the necessary identities in $H/\mathfrak{I}$ (see Section 10) are inherited from H. $\square$

REMARK. It is clear from the Lemma that the sum $\mathfrak{I} + \mathfrak{J}$ of two Hopf ring ideals is another Hopf ring ideal. However, their $*$-product ideal $\mathfrak{I} * \mathfrak{J}$ (defined as the usual product of ideals) *need not* be a Hopf ring ideal, as (i) can fail. We note that (ii) and (iii) nevertheless continue to hold for $\mathfrak{I} * \mathfrak{J}$, with the help of eq. (10.11).

When $R = \mathbb{F}_p$, we can define a rather more useful ideal.

DEFINITION 19.3. Given an ideal $\mathfrak{I}$ in a Hopf ring over $\mathbb{F}_p$, we define $F\mathfrak{I}$ as the $*$-ideal generated by $\{Fx : x \in \mathfrak{I}\}$.

The ideal $F\mathfrak{I}$ is far smaller than $\mathfrak{I}^{*p}$, and clearly *is* a Hopf ring ideal by Lemma 19.2 whenever $\mathfrak{I}$ is. (We use eq. (10.13) to verify (iii).)

The redundant generators. We proved in Lemma 15.9 that the generator b_i is redundant unless i is a power of p. As in (17.3), this implies that $BP_*(\underline{BP}_*)$ is $*$-generated as a BP^*-algebra by $\circ$-monomials of the forms (cf. eq. (18.1))

$$
\begin{aligned}
&\text{(i)} && b^{\circ I} \circ [v^J] = b_{(0)}^{\circ i_0} \circ b_{(1)}^{\circ i_1} \circ b_{(2)}^{\circ i_2} \circ \cdots \circ [v_1^{j_1} v_2^{j_2} \cdots] \\
& && \qquad = b_{(0)}^{\circ i_0} \circ b_{(1)}^{\circ i_1} \circ b_{(2)}^{\circ i_2} \circ \cdots \circ [v_1]^{\circ j_1} \circ [v_2]^{\circ j_2} \circ \cdots, \\
&\text{(ii)} && e \circ b^{\circ I} \circ [v^J], \\
&\text{(iii)} && [\lambda v^J] = [\lambda] \circ [v_1]^{\circ j_1} \circ [v_2]^{\circ j_2} \circ \cdots,
\end{aligned}
\tag{19.4}
$$

in the notation of eq. (15.11). To carry out computations, we need to express the redundant b_i in terms of these $*$-generators.

In order to make the finiteness of our computations apparent, we write $b(x) = 1_2 + \bar{b}(x)$ as in eq. (15.4) and use $1 \circ \bar{b}(x) = 0$. Then eq. (15.8) expands to

$$
\begin{aligned}
&1_2 + \bar{b}\Big(x + y + \sum_{i,j} a_{i,j} x^i y^j\Big) \\
&\quad = \big(1_2 + \bar{b}(x)\big) * \big(1_2 + \bar{b}(y)\big) * \mathop{\Large *}_{i,j} \big\{1_2 + \bar{b}(x)^{\circ i} \circ \bar{b}(y)^{\circ j} \circ [a_{i,j}]\big\}.
\end{aligned}
\tag{19.5}
$$

As in Lemma 15.9, if n is not a power of p, we take s as the largest power of p less than n, and the coefficient of $x^s y^{n-s}$ then yields a reduction formula for b_n. For the low b_n's we can be explicit; they are no longer trivially zero, as in Section 18.

LEMMA 19.6. *For $1 \leqslant i < p$ we have $b_i = b_{(0)}^{*i}/i!$.*

PROOF. All that is left of eq. (19.5) in this range is $b(x+y) = b(x) * b(y)$. Hence $b(x)$ must be the exponential series $\exp(b_1 x)$, expanded using $*$-multiplication. $\square$

Beyond this range, we must settle for inductive formulae in terms of $\circ$-monomials of the form

$$b_{i_1} \circ b_{i_2} \circ \cdots \circ b_{i_r} \circ [v^J]. \tag{19.7}$$

We expand the formal group law $F(x,y)$ *fully*, in the form

$$F(x,y) = x + y + \sum_{\lambda,I,i,j} \lambda v^I x^i y^j,$$

summing over appropriate quadruples (λ, I, i, j) consisting of a coefficient $\lambda \in \mathbb{Z}_{(p)}$, a multi-index I, and exponents i and j. The right side of eq. (19.5) becomes

$$\left(1_2 + \bar{b}(x)\right) * \left(1_2 + \bar{b}(y)\right) * \mathop{\ast}_{\lambda,I,i,j} \left\{1_2 + \bar{b}(x)^{\circ i} \circ \bar{b}(y)^{\circ j} \circ \left[v^I\right]\right\}^{*\lambda},$$

where $\{1 + \ldots\}^{*\lambda}$ is expanded by the binomial series as in eq. (15.5). Every element of the Hopf ring that appears here is a $*$-product of elements of the form (19.7).

This is still not enough! To make the induction succeed, we really need a reduction formula for *every* $\circ$-monomial (19.7) that contains a $\circ$-factor b_n with n not a power of p, without relying on iterated appeals to the distributive law (10.11). A reduction formula for $b_n \circ b_{h_1} \circ b_{h_2} \circ \cdots \circ b_{h_q}$, whenever n is not a power of p and the h_i are any positive integers, will suffice, as $- \circ [v^I]$ is a $*$-homomorphism and $[v^I] \circ [v^J] = [v^{I+J}]$.

We therefore $\circ$-multiply eq. (15.8) by $b(z_1) \circ b(z_2) \circ \cdots \circ b(z_q)$ (and thus work in the $(q+2)$-fold product $(\mathbb{C}P^\infty)^{q+2}$). On the right, we use the distributive law (15.6) to move all the $b(-)$'s inside the $*$-factors, to obtain

$$\begin{aligned}
&1_{2q+2} + \bar{b}\Big(x + y + \sum_{\lambda,I,i,j} \lambda v^I x^i y^j\Big) \circ \bar{b}(z_1) \circ \cdots \circ \bar{b}(z_q) \\
&\quad = \left\{1_{2q+2} + \bar{b}(x) \circ \bar{b}(z_1) \circ \cdots \circ \bar{b}(z_q)\right\} \\
&\qquad * \left\{1_{2q+2} + \bar{b}(y) \circ \bar{b}(z_1) \circ \cdots \circ \bar{b}(z_q)\right\} \\
&\qquad * \mathop{\ast}_{\lambda,I,i,j} \left\{1_{2q+2} + \bar{b}(x)^{\circ i} \circ \bar{b}(y)^{\circ j} \circ \bar{b}(z_1) \circ \cdots \circ \bar{b}(z_q) \circ \left[v^I\right]\right\}^{*\lambda}
\end{aligned} \tag{19.8}$$

The coefficient of $x^s y^{n-s} z_1^{h_1} \ldots z_q^{h_q}$ yields the desired reduction formula. Inspection of the $\circ$-monomials that appear on the right shows that they are all simpler, so that the induction makes progress. (In detail, they all have lower height, or the same height but more b-factors, if we define the *height* of the monomial (19.7) as $\sum_r i_r$.)

None of this is necessary for the other generators, (19.4)(ii). For these it is far simpler to start from Lemma 14.6, work in $Q(BP)^*_*$, and suspend by applying $e \circ -$.

The main relations. As given in Definition 15.15, the main relations are particularly opaque. We make eq. (15.14) more useful in our situation by first expanding the p-series

[8, (13.9)] for BP *in full* as

$$[p](x) = px + \sum_{\lambda,I,m} \lambda v^I x^m, \tag{19.9}$$

much as we just did for $F(x,y)$, and summing over appropriate combinations of coefficient $\lambda \in \mathbb{Z}_{(p)}$, multi-index I, and exponent m. Then eq. (15.14) becomes

$$1_2 + \bar{b}\Big(px + \sum_{\lambda,I,m} \lambda v^I x^m\Big) = \big\{1_2 + \bar{b}(x)\big\}^{*p} * \mathop{\ast}_{\lambda,I,m} \big\{1_2 + \bar{b}(x)^{\circ m} \circ [v^I]\big\}^{*\lambda} \tag{19.10}$$

where we again expand $\{1_2 + \cdots\}^{*\lambda}$ by the binomial series as in eq. (15.5).

The first main relation, the coefficient of x^p, simplifies (with the help of [8, (15.4)] and Lemma 19.6) to

$$(\mathcal{R}_1): \quad v_1 b_{(0)} = p b_{(1)} + b_{(0)}^{\circ p} \circ [v_1] - \frac{b_{(0)}^{*p}}{(p-1)!} \quad \text{in } BP_*(\underline{BP}_2) \tag{19.11}$$

(although it is far easier to extract this as the coefficient of $x^{p-1}y$ in eq. (15.8), using [8, (15.3)]). Subsequent relations rapidly become extremely complicated and can be handled only by neglecting terms wholesale. We need some ideals.

Let $\mathfrak{V}$ be the ideal $(p, v_1, v_2, \ldots)$ in $BP_*(\underline{BP}_*)$ (more accurately, generated as a graded $*$-ideal by all the elements $p1_k$ and $v_n 1_k$ for each k). We need the unstable analogue of the ideals $\mathfrak{M}\mathfrak{W}^m$ of Section 18, coming from the right action of I_∞ on Q_*^*. It is obvious how to handle the generators v_i of I_∞. For the generator p, eq. (10.13) shows that in the quotient Hopf ring $BP_*(\underline{BP}_*)/\mathfrak{V}$ over $\mathbf{F}_p = BP^*/I_\infty$, we may write $c\circ[p] = c\circ(F[1]) = F(Vc\circ[1]) = FVc$. Indeed, it is even more convenient to ignore e and work in the Hopf subring

$$\overline{H} = BP_*(\underline{BP}_{\text{even}})/\mathfrak{V} \cong H_*(\underline{BP}_{\text{even}}; \mathbf{F}_p), \tag{19.12}$$

using only those elements that do not involve the $\circ$-generator e (though of course we keep $b_{(0)} = -e^{\circ 2}$).

DEFINITION 19.13. We define $\mathfrak{M}_0$ as the $*$-ideal in $\overline{H}$ generated by all the elements $b^{\circ I} \circ [v^J]$ with $I \neq 0$, whether allowable or not. For $m > 0$, we define $\mathfrak{M}_m$ inductively as the $*$-ideal generated by $F\mathfrak{M}_{m-1}$ and all elements $b^{\circ I} \circ [v^J]$ with $I \neq 0$, whether allowable or not, that have $|J| \geqslant m$.

Equivalently, $\mathfrak{M}_m$ is the $*$-ideal generated by all elements $F^h(b^{\circ I} \circ [v^J])$ with $I \neq 0$ and $h + |J| \geqslant m$. (Thus $\mathfrak{M}_m$ is roughly, but not quite, the Hopf ring analogue of the right BP^*-action of the ideal I_∞^m.) We thus have the decreasing sequence of ideals

$$\overline{H} \supset \mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \cdots.$$

We note that $\mathfrak{M}_0$ is just the obvious augmentation ideal in $\overline{H}$ consisting of all the $H_i(\underline{BP}_{\text{even}};\mathbf{F}_p)$ with $i > 0$.

LEMMA 19.14. *For all $m \geqslant 0$:*

(a) *$\mathfrak{M}_m$ is a Hopf ring ideal in the Hopf ring $\overline{H} = BP_*(\underline{BP}_{\text{even}})/\mathfrak{V}$;*

(b) *$\mathfrak{M}_m \circ [v_n] \subset \mathfrak{M}_{m+1}$ for all $n > 0$;*

(c) *$\mathfrak{M}_m \circ [v^J] \subset \mathfrak{M}_{m+|J|}$;*

(d) *$\mathfrak{M}_m \circ [p] \subset \mathfrak{M}_{m+1}$.*

PROOF. We first prove (b), from which (c) follows by induction. As $-\circ[v_n]$ is a $*$-homomorphism, it is enough to check that $c \circ [v_n] \in \mathfrak{M}_{m+1}$ for the generators c of $\mathfrak{M}_m$. For $c = b^{\circ I} \circ [v^J]$, we use $[v^J] \circ [v_n] = [v^J v_n]$. For $c = Fa = a^{*p}$, where $a \in \mathfrak{M}_{m-1}$, we have $c \circ [v_n] = F(a \circ [v_n])$ by eq. (10.13). This lies in $F\mathfrak{M}_m$, by induction on m.

We next apply Lemma 19.2 to prove (a). Clearly, $\varepsilon\mathfrak{M}_m = 0$. For a generator of the form $c = b^{\circ I} \circ [v^J]$, with $|J| \geqslant m$, we have $a \circ c = (a \circ b^{\circ I}) \circ [v^J] \in \mathfrak{M}_m$ by (c), since $a \circ b^{\circ I} \in \mathfrak{M}_0$. Similarly, if we write $\psi b^{\circ I} = \sum_i B'_i \otimes B''_i$, we find that $\psi c = \sum_i B'_i \circ [v^J] \otimes B''_i \circ [v^J]$ has the required form, because for each i, either $B'_i \in \mathfrak{M}_0$ or $B''_i \in \mathfrak{M}_0$ for reasons of degree.

For a generator Fc with $c \in \mathfrak{M}_{m-1}$, we use induction on m. By eq. (10.13), $a \circ (Fc) = F(Va \circ c) \in F\mathfrak{M}_{m-1}$. Also, $\psi Fc = (F \otimes F)\psi c$ has the required form.

Because $\mathfrak{M}_m$ is now known to be a Hopf ring ideal, we have $Va \in \mathfrak{M}_m$ for any $a \in \mathfrak{M}_m$. Then (d) is immediate from eq. (10.13), using $[p] = F[1]$. □

We now have the tools to handle eq. (19.10). We work entirely in $\overline{H}$, so that by [8, (15.5)], the left side is trivial. By Lemma 19.14, $\bar{b}(x)^{\circ m} \circ [v^I] \in \mathfrak{M}_{|I|}$. Most $*$-factors on the right side of eq. (19.10) are trivial mod $\mathfrak{M}_2$ and we are left with only

$$\{1_2 + F\bar{b}(x)\} * \mathop{\ast}_{j>0} \left\{1_2 + \bar{b}(x)^{\circ p^j} \circ [v_j]\right\} \quad \text{in } \overline{H}[[x]] \bmod \mathfrak{M}_2.$$

When we pick out the coefficient of x^{p^k} and neglect also certain products, we obtain

$$R(k) \equiv Fb_{(k-1)} + \sum_{j=1}^{k} b_{(k-j)}^{\circ p^j} \circ [v_j] \quad \text{in } \overline{H} \bmod \mathfrak{M}_2 + \mathfrak{M}_1 * \mathfrak{M}_1, \tag{19.15}$$

analogous to eq. (18.3). Although the ideal here is *not* a Hopf ring ideal, (ii) and (iii) of Lemma 19.2 still hold, according to the Remark following that lemma.

The Ravenel–Wilson generators. We lift the allowable monomials of Section 18 via the canonical projections $q_k\colon BP_*(\underline{BP}_k) \to Q(BP)^k_*$, so that multiplication is now to be interpreted as $\circ$-multiplication.

DEFINITION 19.16. We *disallow* all $\circ$-monomials of the form

$$b_{(i_1)}^{\circ p} \circ b_{(i_2)}^{\circ p^2} \circ \cdots \circ b_{(i_n)}^{\circ p^n} \circ [v_n] \circ c \quad (i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n,\ n > 0), \tag{19.17}$$

where c stands for any $\circ$-monomial in the $b_{(i)}$, $[v_j]$, and e ($c = [1]$ is permitted). All $\circ$-monomials (19.4)(i) and (ii) not of this form are declared to be *allowable*.

It follows from Theorem 18.15 (and local finiteness) that the allowable $\circ$-monomials generate $BP_*(\underline{BP}_*)$, but far more is true, by [23, Theorem 5.3, Remark 4.9].

THEOREM 19.18 (Ravenel–Wilson). *In the Hopf ring for BP:*

(a) *If k is even, denote by $\underline{BP}'_k$ the zero component of the space $\underline{BP}_k$ (so that $\underline{BP}'_k = \underline{BP}_k$ if $k > 0$). Then $BP_*(\underline{BP}'_k)$ is a polynomial algebra over BP^* on those allowable $\circ$-monomials $b^{\circ I} \circ [v^J]$ with $I \neq 0$ that lie in it. If $k \leqslant 0$, $BP_*(\underline{BP}_k) = BP_*(BP^k) \otimes BP_*(\underline{BP}'_k)$ as in eq.* (17.4).

(b) *If k is odd, $BP_*(\underline{BP}_k)$ is an exterior algebra over BP^* on those allowable $\circ$-monomials $e \circ b^{\circ I} \circ [v^J]$ that lie in it.*

As in Section 18, we need information on where the disallowed monomials lie. The difficulty with eq. (19.15) is that it is hard to tell whether a given element lies in $\mathfrak{M}_2$. We therefore define analogous ideals in terms of the polynomial generators in Theorem 19.18 for which this problem does not exist. Again, we ignore e and neglect $\mathfrak{V}$ by working in the Hopf ring $\overline{H}$ over $\mathbf{F}_p$ (see eq. (19.12)).

DEFINITION 19.19. We define $\mathfrak{A}_0$ as the $*$-ideal in $\overline{H}$ generated by all the *allowable* $\circ$-monomials $b^{\circ I} \circ [v^J]$ that have $I \neq 0$. For $m > 0$, we define $\mathfrak{A}_m$ inductively as the $*$-ideal generated by $F\mathfrak{A}_{m-1}$ and all the *allowable* $\circ$-monomials $b^{\circ I} \circ [v^J]$ for which $I \neq 0$ and $|J| \geqslant m$.

In other words, $\mathfrak{A}_m$ is the $*$-ideal generated by all the elements $F^h(b^{\circ I} \circ [v^J])$, where $b^{\circ I} \circ [v^J]$ is allowable, $I \neq 0$, and $h + |J| \geqslant m$.

THEOREM 19.20. *For all $m \geqslant 0$, $\mathfrak{A}_m = \mathfrak{M}_m$ and is therefore a Hopf ring ideal in $\overline{H} = BP_*(\underline{BP}_{\text{even}})/\mathfrak{V} \cong H_*(\underline{BP}_{\text{even}}; \mathbf{F}_p)$.*

This result we shall prove in full. For $m = 0$, it is part of Theorem 19.18.

Higher order relations. As in Section 18, we derive a more useful relation by elimination from the n relations $(\mathcal{R}_{k_1}), (\mathcal{R}_{k_2}), \ldots, (\mathcal{R}_{k_n})$, with multiplication now interpreted as $\circ$-multiplication. We find it simpler to return to eq. (19.10) rather than try to deal directly with eq. (19.15).

DEFINITION 19.21. Given any positive integers $i_1, i_2, \ldots, i_n$, where $n \geqslant 1$, we define $L(i_1, i_2, \ldots, i_n)$ and $R(i_1, i_2, \cdots, i_n)$ as the coefficient of $x_1^{p^{i_1}} x_2^{p^{i_2}} \cdots x_n^{p^{i_n}}$ in

$$b(x_1)^{\circ p} \circ b(x_2)^{\circ p^2} \circ \cdots \circ b(x_{n-1})^{\circ p^{n-1}} \circ b\big([p](x_n)\big)$$

and

$$b(x_1)^{\circ p} \circ b(x_2)^{\circ p^2} \circ \cdots \circ b(x_{n-1})^{\circ p^{n-1}} \circ P(x_n) \tag{19.22}$$

respectively, where $P(x)$ denotes the right side of eq. (19.10).

Then given any integers $0 < k_1 < k_2 < \cdots < k_n$, where $n > 1$, we define the *n-th order derived relation*

$$(\mathcal{R}_{k_1,k_2,\dots,k_n}): \qquad \sum_\pi \varepsilon_\pi\, L(i_1, i_2, \dots, i_n) = \sum_\pi \varepsilon_\pi\, R(i_1, i_2, \dots, i_n)$$

by summing over all permutations $\pi \in \Sigma_n$, where $(i_1, i_2, \dots, i_n) = \pi(k_1, k_2, \dots, k_n)$. (For $n = 1$, we recover $(\mathcal{R}_{k_1})$.)

This relation lies in $BP_*(\underline{BP}_{f(n)})$, where $f(n)$ denotes the usual numerical function (1.4). To study it, we work in $\overline{H}$. The left side of $(\mathcal{R}_{k_1,k_2,\dots,k_n})$ vanishes, as before. To handle the right side, we first rewrite (19.22) just as we did eq. (19.8), by using eq. (15.6) to move all the $\circ$-factors $b(-)$ inside the $*$-factors. The term px of $[p](x)$ produces the $*$-factor

$$\left\{1 + \bar{b}(x_1)^{\circ p} \circ \bar{b}(x_2)^{\circ p^2} \circ \cdots \circ \bar{b}(x_{n-1})^{\circ p^{n-1}} \circ \bar{b}(x_n)\right\}^{*p}, \tag{19.23}$$

and the general term $\lambda v^I x^m$ produces the $*$-factor

$$\left\{1 + \bar{b}(x_1)^{\circ p} \circ \bar{b}(x_2)^{\circ p^2} \circ \cdots \circ \bar{b}(x_{n-1})^{\circ p^{n-1}} \circ \bar{b}(x_n)^{\circ m} \circ \left[v^I\right]\right\}^{*\lambda},$$

to be expanded as in eq. (15.5). By the form [8, (15.5)] of the p-series, the only $*$-factors of the latter kind that are not trivial mod $\mathfrak{M}_2$ are

$$1 + \bar{b}(x_1)^{\circ p} \circ \bar{b}(x_2)^{\circ p^2} \circ \cdots \circ \bar{b}(x_{n-1})^{\circ p^{n-1}} \circ \bar{b}(x_n)^{\circ p^j} \circ [v_j] \tag{19.24}$$

for $j > 0$. We can now efficiently extract the coefficient $R(i_1, i_2, \cdots, i_n)$ of $x_1^{p^{i_1}} x_2^{p^{i_2}} \cdots x_n^{p^{i_n}}$. From the factor (19.23) we have the term

$$F\left(b_{(i_n-1)} \circ b^{\circ p}_{(i_1-2)} \circ b^{\circ p^2}_{(i_2-3)} \circ \cdots \circ b^{\circ p^{n-1}}_{(i_{n-1}-n)}\right),$$

after some shuffling, while the factor (19.24) yields

$$b^{\circ p}_{(i_1-1)} \circ b^{\circ p^2}_{(i_2-2)} \circ \cdots \circ b^{\circ p^{n-1}}_{(i_{n-1}-n+1)} \circ b^{\circ p^j}_{(i_n-j)} \circ [v_j].$$

(We continue the convention of Section 18 that meaningless terms, those involving any $b_{(i)}$ with $i < 0$, are treated as zero.) We now sum over π and j, taking the opportunity to permute the i_r in the terms with F (which introduces a sign), to obtain $(\mathcal{R}_{k_1,k_2,\dots,k_n})$ in the desired form

$$\begin{aligned}&(-1)^{n-1} \sum_\pi \varepsilon_\pi\, F\left(b_{(i_1-1)} \circ b^{\circ p}_{(i_2-2)} \circ \cdots \circ b^{\circ p^{n-1}}_{(i_n-n)}\right) \\ &\quad + \sum_{\pi,j} \varepsilon_\pi\, b^{\circ p}_{(i_1-1)} \circ b^{\circ p^2}_{(i_2-2)} \circ \cdots \circ b^{\circ p^{n-1}}_{(i_{n-1}-n+1)} \circ b^{\circ p^j}_{(i_n-j)} \circ [v_j] \equiv 0\end{aligned} \tag{19.25}$$

in $\overline{H}$ mod $\mathfrak{M}_2 + \mathfrak{M}_1 * \mathfrak{M}_1$. As before, the terms involving $[v_j]$ for $j < n$ cancel: when we interchange i_j and i_n, we obtain two identical terms having opposite signs. We therefore sum only over $j \geqslant n$. The terms of most interest are the two *leading* terms with $\pi = \mathrm{id}$:

$$(-1)^{n-1} F\left(b^{\circ L}\right) = (-1)^{n-1} F\left(b_{(k_1-1)} \circ b^{\circ p}_{(k_2-2)} \circ \cdots \circ b^{\circ p^{n-1}}_{(k_n-n)}\right) \tag{19.26}$$

and

$$b^{\circ pL} \circ [v_n] = b^{\circ p}_{(k_1-1)} \circ b^{\circ p^2}_{(k_2-2)} \circ \cdots \circ b^{\circ p^n}_{(k_n-n)} \circ [v_n], \tag{19.27}$$

for a certain multi-index L (different from Section 18).

The reduction formula. We obtain a reduction formula for the general disallowed $\circ$-monomial (19.17) in $BP_*(\underline{BP}_k)$. First, we assume k is even. For any $n > 0$, $0 < k_1 < k_2 < \cdots < k_n$, and multi-indices M and J, the desired formula is:

$$\begin{aligned} & b^{\circ p}_{(k_1-1)} \circ b^{\circ p^2}_{(k_2-2)} \circ \cdots \circ b^{\circ p^n}_{(k_n-n)} \circ b^{\circ M} \circ [v_n v^J] \\ & \quad \equiv -\sum_{\pi \neq \mathrm{id}} \varepsilon_\pi \, b^{\circ p}_{(i_1-1)} \circ \cdots \circ b^{\circ p^n}_{(i_n-n)} \circ b^{\circ M} \circ [v_n v^J] \\ & \qquad + (-1)^n F\left(b_{(k_1-1)} \circ b^{\circ p}_{(k_2-2)} \circ \cdots \circ b^{\circ p^{n-1}}_{(k_n-n)} \circ b^{\circ s^{-1}(M)} \circ [v^J]\right) \\ & \qquad + (-1)^n \sum_{\pi \neq \mathrm{id}} \varepsilon_\pi \, F\left(b_{(i_1-1)} \circ b^{\circ p}_{(i_2-2)} \circ \cdots \circ b^{\circ p^{n-1}}_{(i_n-n)} \circ b^{\circ s^{-1}(M)} \circ [v^J]\right) \\ & \qquad \text{in } \overline{H} \bmod \mathfrak{M}_{h+2} + \mathfrak{M}_{h+1} * \mathfrak{M}_{h+1}, \end{aligned} \tag{19.28}$$

where we sum over permutations $\pi \in \Sigma_n$, $(i_1, i_2, \ldots, i_n) = \pi(k_1, k_2, \ldots, k_n)$, and $h = |J|$. (Terms involving $s^{-1}(M)$ with $m_0 \neq 0$ are to be omitted.) To obtain this, we first apply $- \circ b^{\circ M}$ to eq. (19.25), using eq. (15.13) to rewrite the terms involving F. The suppressed terms lie in $\mathfrak{M}_2 \circ b^{\circ M} \subset \mathfrak{M}_2$ and

$$(\mathfrak{M}_1 * \mathfrak{M}_1) \circ b^{\circ M} \subset \mathfrak{M}_1 * \mathfrak{M}_1,$$

as we know from Lemma 19.14(a) that $\mathfrak{M}_2$ and $\mathfrak{M}_1$ are Hopf ring ideals. Then we apply the $*$-homomorphism $- \circ [v^J]$ and use Lemma 19.14(c).

REMARK. Strictly speaking, this is only a reduction formula mod $\mathfrak{V}$, but it meets our present needs. One can work modulo the slightly smaller ideal $(v_1, v_2, \ldots)$ instead and extract a more complicated reduction formula that is valid in $BP_*(\underline{BP}_*)$ itself, without recourse to Nakayama's Lemma.

For odd k, the reduction formula takes the far simpler form

$$\begin{aligned} e\circ b^{\circ p}_{(k_1-1)}\circ b^{\circ p^2}_{(k_2-2)}\circ\cdots\circ b^{\circ p^n}_{(k_n-n)}\circ b^{\circ M}\circ[v_n v^J] \\ \equiv -\sum_{\pi\neq\mathrm{id}}\varepsilon_\pi\, e\circ b^{\circ p}_{(i_1-1)}\circ b^{\circ p^2}_{(i_2-2)}\circ\cdots\circ b^{\circ p^n}_{(i_n-n)}\circ b^{\circ M}\circ[v_n v^J] \end{aligned}$$

mod $\mathfrak{M}_{h+2}$. To see this, one can suspend eq. (19.28) by applying $e\circ -$, which kills all $*$-products, including Fc; but it is far simpler to suspend eq. (18.22) instead.

PROOF OF THEOREM 19.20. For $m>0$, it follows from eq. (19.28) that

$$\mathfrak{M}_m\subset\mathfrak{A}_m+\mathfrak{M}_{m+1}+\mathfrak{M}_m*\mathfrak{M}_m+F\mathfrak{M}_{m-1}\quad\text{in } \overline{H}, \tag{19.29}$$

by using exactly the same orderings of monomials (reinterpreted) as in the proof of Theorem 18.16. For $m=0$, we clearly have $\mathfrak{M}_0=\mathfrak{A}_0+\mathfrak{M}_1$ because the generators of $\mathfrak{M}_0$ that are not in $\mathfrak{M}_1$ are all allowable.

We show by induction on m that the term $F\mathfrak{M}_{m-1}$ is not needed, that

$$\mathfrak{M}_m\subset\mathfrak{A}_m+\mathfrak{M}_{m+1}+\mathfrak{M}_m*\mathfrak{M}_m \tag{19.30}$$

for all $m\geqslant 1$. This is clear for $m=0$. If it holds for $m-1$, applying F yields

$$F\mathfrak{M}_{m-1}\subset F\mathfrak{A}_{m-1}+F\mathfrak{M}_m+F\mathfrak{M}_{m-1}*F\mathfrak{M}_{m-1}.$$

Each term on the right is already included in the other terms of eq. (19.29) and may be omitted.

Next, we dispose of $\mathfrak{M}_m*\mathfrak{M}_m$. On $*$-multiplying eq. (19.30) by $\mathfrak{M}_1^{*i}$ we have

$$\begin{aligned} \mathfrak{M}_m*\mathfrak{M}_1^{*i} &\subset \mathfrak{A}_m*\mathfrak{M}_1^i+\mathfrak{M}_{m+1}*\mathfrak{M}_1^i+\mathfrak{M}_m*\mathfrak{M}_m*\mathfrak{M}_1^i \\ &\subset \mathfrak{A}_m+\mathfrak{M}_{m+1}+\mathfrak{M}_m*\mathfrak{M}_1^{i+1}. \end{aligned}$$

It follows by induction on i that

$$\mathfrak{M}_m\subset\mathfrak{A}_m+\mathfrak{M}_{m+1}+\mathfrak{M}_m*\mathfrak{M}_1^{*i}$$

for all i. Since $\mathfrak{M}_1^{*i}$ is zero in each bigrading for large enough i, we must have $\mathfrak{M}_m\subset\mathfrak{A}_m+\mathfrak{M}_{m+1}$. As in the proof of Theorem 18.16, this implies $\mathfrak{M}_m=\mathfrak{A}_m$. □

The suspension. We can use eq. (19.28) to extract detailed information about the suspension homomorphism $e\circ -\colon Q^k_*\to PBP_*(\underline{BP}_{k+1})$ when k is odd. (When k is even, there is nothing to discuss: the allowable monomial $b^I w^J\in Q^k_*$ suspends to the allowable $\circ$-monomial $e\circ b^{\circ I}\circ[v^J]\in PBP_*(\underline{BP}_{k+1})$.)

By Lemma 18.12(c), we can write every allowable monomial in Q_*^k uniquely in the extended canonical form

$$c = e\, b^{L-\Delta(0)}\, b^{(p-1)(L+s(L)+\cdots+s^{h-1}(L))}\, b^{s^h(M)}\, w_{n+1}^h w^J,$$

where $0 = k_0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$, $n \geqslant 0$, $b^L = b_{(k_0)} b^p_{(k_1)} \cdots b^{p^n}_{(k_n)}$, M and J satisfy the conditions (18.9), and $h \geqslant 0$ is maximal. What happens to $e \circ c$ is that if $h > 0$, it is disallowed, as the derived relation $(\mathcal{R}_{k_0+1,k_1+2,\ldots,k_n+n+1})$ applies, and we pick out the leading term (19.26) mod $\mathfrak{V}$. If $h > 1$, we can repeat this cycle h times (always with the same indices k_u). In all cases, $e \circ c$ has the leading term

$$F^h\big(b^{\circ L} \circ b^{\circ M} \circ [v^J]\big), \tag{19.31}$$

where $b^{\circ L} \circ b^{\circ M} \circ [v^J]$ is allowable by Lemma 18.12(d) and primitive in $\overline{H}$ because $b^{\circ L}$ contains the factor $b_{(0)}$.

In fact, one can show that every primitive allowable $\circ$-monomial in $BP_*(\underline{BP}_{k+1})$ can be written uniquely in the form $b^{\circ L} \circ b^{\circ M} \circ [v^J]$, subject to the conditions (18.9). We have a computational verification mod $\mathfrak{V}$ of the isomorphism $Q_*^k \cong PBP_*(\underline{BP}_{k+1})$ induced by suspension.

The first nth order relation. The relation $(\mathcal{R}_{1,2,\ldots,n})$ is particularly important, as only the two leading terms are meaningful. Bendersky has pointed out (during the proof of [3, Theorem 6.2]) that with a little more attention to detail, one obtains a sharper version, the unstable analogue of Lemma 18.23.

LEMMA 19.32 (Bendersky). *In $BP_*(\underline{BP}_{f(n)})$ we have the relation*

$$b_{(0)}^{\circ pm} \circ [v_n] \equiv v_n b_{(0)}^{\circ m} + (-1)^n \big(b_{(0)}^{\circ m}\big)^{*p} \bmod I_n BP_*\big(\underline{BP}_{f(n)}\big), \tag{19.33}$$

for each $n > 0$, where $m = f(n)/2 = 1 + p + p^2 + \cdots + p^{n-1}$.

PROOF. Although this result can be extracted from $(\mathcal{R}_{1,2,\ldots,n})$ by detailed examination, it is far simpler to return to $(\mathcal{R}_n)$. We proceed by induction on n, starting from eq. (19.11) for $n = 1$. For $n > 1$, we assume the result for all smaller n, and obtain it for n by evaluating $b_{(0)}^{\circ ph} \circ (\mathcal{R}_n) \bmod I_n$, where $h = f(n-1)/2 = 1 + p + p^2 + \cdots + p^{n-2}$.

We recall that $(\mathcal{R}_n)$ is defined as the coefficient of x^{p^n} in eq. (19.10). On the left, we have $b_{(0)}^{\circ ph} \circ b(v_n x^{p^n} + \cdots)$ by [8, (15.5)], which provides only the term $v_n b_{(0)}^{\circ ph+1}$. The right side simplifies enormously, because $h > 0$ and $b_{(0)} \circ -$ kills $*$-decomposables; we obtain

$$b_{(0)}^{\circ ph} \circ P(x) = p b_{(0)}^{\circ ph} \circ \overline{b}(x) + \sum_{\lambda,I,m} \lambda b_{(0)}^{\circ ph} \circ \overline{b}(x)^{\circ m} \circ [v^I].$$

By induction, $b_{(0)}^{\circ ph} \circ [v_j] \equiv 0 \bmod I_n$ for all $j < n-1$, since $h = f(n-1)/2 > f(j)/2$.

Thus the only terms of interest in $[p](x)$ in our range of degrees are $v_n x^{p^n}$ and $v_{n-1}x^{p^{n-1}}$, as it follows from [8, (14.26)] and the map $BP \to K(n-1)$ of ring spectra that any terms in eq. (19.9) of the form $\lambda v_{n-1}^i x^m$ with $i > 1$ have λ divisible by p. The term $v_n x^{p^n}$ yields $b_{(0)}^{\circ ph} \circ b_{(0)}^{\circ p^n} \circ [v_n]$, which is the leading term (19.27). By induction and eq. (15.13), $v_{n-1}x^{p^{n-1}}$ yields

$$b_{(0)}^{\circ ph} \circ b_{(1)}^{\circ p^{n-1}} \circ [v_{n-1}] \equiv (-1)^{n-1} F\big(b_{(0)}^{\circ h}\big) \circ b_{(1)}^{\circ p^{n-1}} \equiv (-1)^{n-1} F\big(b_{(0)}^{\circ h} \circ b_{(0)}^{\circ p^{n-1}}\big),$$

which is the other leading term, (19.26). □

The ideals $\mathfrak{J}_n$. For the unstable version of our splitting theorems we need the unstable analogue of the ideal $\mathfrak{J}_n$ of Definition 18.25.

DEFINITION 19.34. For $n \geqslant 0$, we define $\mathfrak{J}_n \subset BP_*(\underline{BP}_*)$ as the $*$-ideal generated by all elements of the form $c \circ([v_j] - 1)$, where $j > n$.

LEMMA 19.35. *$\mathfrak{J}_n$ is a Hopf ring ideal in $BP_*(\underline{BP}_*)$.*

PROOF. We apply Lemma 19.2; only (i) requires any comment. It holds for $[v_j] - 1$, by the identity

$$\psi([v] - 1) = ([v]-1)\otimes[v] + 1\otimes([v]-1), \tag{19.36}$$

which is valid for any $v \in BP^*$ by Proposition 11.2(a). We combine this with $\psi c = \sum_i c_i' \otimes c_i''$ to obtain

$$\psi\big(c\circ([v]-1)\big) = \sum_i c_i'\circ([v]-1) \otimes c_i''\circ[v] + \sum_i c_i'\circ 1 \otimes c_i''\circ([v]-1), \tag{19.37}$$

which shows that (i) holds for the typical $*$-generator of $\mathfrak{J}_n$. □

LEMMA 19.38. $[v] \equiv 1 \bmod \mathfrak{J}_n$ *for all* $v \in J_n$.

PROOF. Suppose $v = v' + \lambda v_j v^K$ with $j > n$. As $\mathfrak{J}_n$ is a Hopf ring ideal, we have

$$[v] = [v'] * [\lambda v^K]\circ[v_j] \equiv [v'] * [\lambda v^K]\circ 1 = [v'] \bmod \mathfrak{J}_n.$$

The result follows by induction on the number of terms in v. □

The unstable analogue of Lemma 18.26 requires more detail but no new ideas.

LEMMA 19.39. *For $k \leqslant f(n+1)$, $\mathfrak{J}_n \cap BP_*(\underline{BP}_k)$ is the $*$-ideal in $BP_*(\underline{BP}_k)$ generated by all elements that lie in $BP_*(\underline{BP}_k)$ and have any of the following forms, where v^J contains a factor v_j with $j > n$:*

(i) *(if k is even) an allowable monomial $b^{\circ I} \circ [v^J]$;*

(ii) *(if k is odd) an allowable monomial* $e \circ b^{\circ I} \circ [v^J]$;
(iii) *(if $k \leqslant 0$ and is even)* $[\lambda v^J] - 1_k$, *with* $\lambda \in \mathbb{Z}_{(p)}$;
(iv) *(if $k = f(n+1)$) a disallowed monomial*

$$b^{\circ p}_{(k_1-1)} \circ b^{\circ p^2}_{(k_2-2)} \circ \cdots \circ b^{\circ p^{n+1}}_{(k_{n+1}-n-1)} \circ [v_{n+1}]$$

with $0 < k_1 < k_2 < \cdots < k_{n+1}$.

REMARK. To make (i) correct for $I = 0$, it is necessary to define $b^{\circ 0} = e^{\circ 0} = [1] - 1$ as in Proposition 13.7, so that $b^{\circ 0} \circ [v^J] = [v^J] - 1$.

PROOF. Denote by $\mathfrak{J}$ the $*$-ideal in $BP_*(\underline{BP}_k)$ generated by the stated elements. It is clear from Lemma 19.38 that $\mathfrak{J} \subset \mathfrak{J}_n$.

To show the converse, we fix k and a large m, and prove by *downward* induction on h that all elements in $BP_i(\underline{BP}_k)$ of the form $c \circ ([v_h] - 1)$ lie in $\mathfrak{J}$ whenever $i < m$. This statement is vacuous for sufficiently large h (depending on m and k). We therefore fix $t > n$ and assume the statement holds for all $h > t$.

Case 1: $c = [\lambda v^J]$. (This includes the degenerate cases $[1]$ and $1_k = [0_k]$.) Then $c \circ ([v_t] - 1) = [\lambda v^J v_t] - 1$ is listed in (iii).

Case 2: $c = e^\epsilon \circ b^{\circ I}$. As in Lemma 18.26, $c \circ ([v_t] - 1) = e^\epsilon \circ b^{\circ I} \circ [v_t]$ has to be allowable, except in the extreme case when $k = f(n+1)$ and $j = n+1$; either way, it is a listed generator of $\mathfrak{J}$.

Case 3: $c = e^\epsilon \circ b^{\circ I} \circ [v_h v^J]$ *allowable, where* $h \leqslant t$. From the form of Definition 19.16, $c \circ ([v_t] - 1) = e^\epsilon \circ b^{\circ I} \circ [v_h v_t v^J]$ remains allowable and is thus a listed generator of $\mathfrak{J}$.

Case 4: $c = e^\epsilon \circ b^{\circ I} \circ [v_h v^J]$, *with* $h > t$. We can write $c \circ ([v_t] - 1) = e^\epsilon \circ b^{\circ I} \circ [v_t v^J] \circ ([v_h] - 1)$, which lies in $\mathfrak{J}$ by induction, provided $i < m$.

By Theorem 19.18, we have enough $*$-generators c. If $c = a * d$, eqs. (10.11) and (19.36) give

$$c * ([v_t] - 1) = a \circ ([v_t] - 1) * d \circ [v_t] + a \circ 1 * d \circ ([v_t] - 1),$$

which shows that the statement holds for $c = a * d$ whenever it holds for a and d. □

20. Additively unstable BP-objects

In this section, we discuss the additively unstable structures developed in Sections 5 and 7 in the case $E = BP$, with particular attention to what becomes of the stable results of [8, §15]. We easily recover Quillen's theorem, that for any space X, the generators of $BP^*(X)$ all lie in non-negative degrees. Our main result Theorem 20.11 says in effect that there are no relations there either; more precisely, all relations follow from relations in non-negative degrees. We apply the theory to Landweber filtrations of an additively unstable module or algebra M, and find that the presence of additive unstable operations implies severe constraints on the degrees of the generators of M; this may be viewed as a better version of Quillen's theorem.

By Theorems 6.35 and 7.11, module and comodule structures are equivalent, with or without multiplication. The most convenient context remains the Second Answer of Section 5, that an additively unstable BP-cohomology module (algebra) consists of a BP^*-module (BP^*-algebra) M equipped with coactions

$$\rho_M: M^k \longrightarrow M \widehat{\otimes} Q(BP)^k_* \tag{20.1}$$

that (as k varies) form a homomorphism of BP^*-modules (BP^*-algebras) and satisfy the usual coaction axioms (6.33). We continue to abbreviate $Q(BP)^*_*$ to Q^*_*. The bigraded algebra Q^*_* was discussed in detail in Section 18.

Connectedness. The principle is that nothing interesting ever happens in negative degrees. The first result in this direction is due to Quillen [22, Theorem 5.1].

THEOREM 20.2 (Quillen). *For any space X, $BP^*(X)\hat{}$ is generated, as a BP^*-module, (topologically if X is infinite) by elements of positive degree and exactly one element of degree 0 for each component of X.*

This will be an immediate consequence of Lemmas 4.10 of [8] and 20.5 (below). Quillen's proof is geometric; in contrast, Section 6 provides a global algebraic proof of the weak form of Quillen's theorem.

THEOREM 20.3. *Given any integer $k < 0$, there exist for $n \geqslant 1$:*

(i) *additive unstable BP-operations r_n defined on $BP^k(-)$, with $\deg(r_n) \to \infty$ and $\deg(r_n) \geqslant |k|$ for all n;*
(ii) *elements $v(n) \in BP^*$;*

such that in any additively unstable BP-cohomology module M (e.g., $BP^*(X)\hat{}$ for any space X), any $x \in M^k$ decomposes as the (topological infinite) sum $x = \sum_n v(n) r_n x$, with $\deg(r_n x) \geqslant 0$ for all n.

In particular, M is generated (topologically) by elements of degree $\geqslant 0$.

PROOF. Let $\{c_1, c_2, c_3, \dots\}$ be the Ravenel–Wilson (or any other) basis of the free BP^*-module Q^k_*. By eq. (6.39) and the following Remark, we can write

$$x = \iota_k x = \sum_n \langle \iota_k, c_n \rangle \, x_n \tag{20.4}$$

with $x_n = r_n x$, where r_n denotes the operation dual to c_n. If $c_n \in Q^k_j$, we must have $j \geqslant 0$; then $\deg(r_n) = -\deg(c_n) = j - k \geqslant -k$ gives (i). We put $v(n) = \langle \iota_k, c_n \rangle$ and note that $\deg(x_n) = \deg(r_n) + \deg(x) = j \geqslant 0$. □

REMARK. The coefficients in eq. (20.4) are readily computed from eq. (6.41) as $v(n) = Q(\varepsilon) c_n$. Thus $v(n) = v^J$ if $c_n = e^\varepsilon b^m_{(0)} w^J$, and vanishes for monomials c_n not of this form, so that many terms in eq. (20.4) are zero.

If M is bounded above or X is finite-dimensional, the sum is finite and no topology on M is needed.

To handle the generators in degree 0, we need a stronger hypothesis.

LEMMA 20.5. *Let M be a connected* (see Definition 7.14) *additively unstable algebra (e.g., $BP^*(X)\hat{}$ for any connected space X). Then as a topological BP^*-module, M is generated by $1_M \in M^0$ and elements of strictly positive degree. The generator 1_M is never redundant.*

REMARK. Again, we may ignore the topology on M if M is bounded above or X is finite-dimensional.

PROOF. We choose a basis $\{c_1, c_2, c_3, \ldots\}$ of Q_*^0 with $c_1 = 1$; then given $x \in M^0$, we have eq. (20.4) with $\deg(x_n) = -\deg(c_n) > 0$ for all $n > 1$. Thus $x \equiv \langle \iota_0, 1\rangle\, x_1 \bmod L$, where L denotes the BP^*-submodule of M generated (topologically) by the elements of positive degree.

For the collapse operation κ_0 introduced in Definition 7.13, we similarly have $\kappa_0 x \equiv \langle \kappa_0, 1\rangle \bmod L$. But $\langle \iota_0, 1\rangle = \langle \kappa_0, 1\rangle = 1$. As M is connected, $\kappa_0 x = \lambda 1_M$ for some $\lambda \in \mathbb{Z}_{(p)}$, by Definition 7.14. We deduce from Theorem 20.3 that $M = L + (BP^*)1_M$. Since $\kappa L = 0$ and $\kappa(v 1_M) = v 1_M$ for any $v \in BP^*$, this is a direct sum decomposition. □

Primitive elements. We generalize the theory of Landweber filtrations to the additive unstable context by following the same strategy as stably. We explore a general unstable comodule M by looking for morphisms $f\colon BP^*(S^k, o) \to M$, for any $k \geqslant 0$. As a BP^*-module, $BP^*(S^k, o)$ is free on the canonical generator u_k. Thus f is determined, as a homomorphism of BP^*-modules, by the element $x = f u_k \in M$. Since $\rho_S u_k = u_k \otimes e^k$ by Proposition 12.3(a), the condition we need is clear.

DEFINITION 20.6. Let M be any unstable comodule. If $k \geqslant 0$, we call $x \in M^k$ *additively unstably primitive* if $\rho_M x = x \otimes e^k$ in $M \widehat{\otimes} Q_*^k$.

This obviously stabilizes to [8, Definition 15.9], so that the additively unstable primitives of M form a subgroup of the stable primitives of M. *We do not define primitives in negative degrees*, for lack of a space S^k, and because e^k is meaningless. In fact, for $k < 0$, $x \otimes 1$ does not in general lie in the image of the stabilization

$$M \otimes Q(\sigma)\colon M \widehat{\otimes} Q_*^k \longrightarrow M \widehat{\otimes} BP_*(BP, o).$$

(Perhaps it *never* does?)

REMARK. One might object that we have abolished primitives in negative degrees by simply defining them away, while some alternate definition might work. However, no such definition can be satisfactory.

It is obvious from Definition 12.6 that if $x \in M$ is primitive, so is $\Sigma x \in \Sigma M$. On the other hand, we shall find (nontrivially) in Corollary 20.12 that the only primitive in

ΣM of degree zero is 0 (at least, for the kind of comodule we discuss). It follows, by suspending enough, that no definition of primitive can have both these properties and produce anything interesting in negative degrees.

It is immediate from the definition that if $x \in M^k$ is primitive,

$$\rho_M(vx) = x \otimes e^k \eta_R v \quad \text{in } M \mathbin{\widehat{\otimes}} Q_*^* \text{ (for } v \in BP^*). \tag{20.7}$$

We again recall from eq. (1.4) the numerical function

$$f(n) = \frac{2(p^n - 1)}{p - 1} = 2\big(p^{n-1} + p^{n-2} + \cdots + p + 1\big)$$

and remind that $\deg(v_n) = -(p-1)f(n)$ for $n > 0$.

LEMMA 20.8. *Let $x \in M^k$ be a nonzero primitive element of the unstable BP-cohomology comodule M, and take $n > 0$.*

(a) *If $k < pf(n)$, then $v_n^i x \neq 0$ for all $i > 0$ and is not additively unstably primitive;*

(b) *If $k \geqslant pf(n)$ and $I_n x = 0$, then $v_n x$ is additively unstably primitive.*

COROLLARY 20.9. *If the additively unstably primitive element $x \in M$ satisfies $I_n x = 0$ and is a v_n-torsion element, then:*

(a) $\deg(v_n^i x) \geqslant pf(n)$ *whenever* $v_n^i x \neq 0$;

(b) $v_n^i x$ *is additively unstably primitive or zero for all* i.

PROOF. We apply the Lemma to $v_n^i x$ by induction on i. Part (a) never applies (unless $v_n^i x = 0$); hence (b) must apply, to show that $v_n^{i+1} x$ is primitive. □

All this follows easily from Lemma 18.23.

PROOF OF LEMMA 20.8. From eq. (20.7) we have

$$\rho_M(v_n^i x) = x \otimes e^k w_n^i.$$

In case (a), we note that by Definition 18.4, $e^k w_n^i$ is a basis element of Q_*^*, so that $\rho_M(v_n^i x)$ is clearly nonzero. Even if $k \geqslant 2(p^n - 1)i$, $v_n^i x$ is not primitive because $\rho_M(v_n^i x)$ is different from

$$v_n^i x \otimes e^{k-2(p^n-1)i} = x \otimes v_n^i e^{k-2(p^n-1)i}.$$

In case (b), we use the same formulae, with $i = 1$. The difference is that by Lemma 18.23, they now coincide, since $e^2 = b_{(0)}$ and $I_n x = 0$. □

REMARK. For any $x \in M^k$, where $k \geqslant 0$, the coaction axiom (ii) of [8, (8.7)] forces $\rho_M x$ to have the form

$$\rho_M x = x \otimes e^k + \sum_\alpha x_\alpha \otimes c_\alpha,$$

where the c_α are other Ravenel–Wilson basis elements and $\deg(x_\alpha) > k$. Assuming that $k < pf(n)$, so that $e^k w_n$ is a basis element, let r be the operation (or functional) dual to it. Proceeding as in the proof of the Lemma, we obtain

$$r(v_n x) = x + \sum_\alpha \langle r, c_\alpha w_n \rangle\, x_\alpha,$$

which shows that $v_n x \neq 0$ if (for example) x is a module generator of M.

Landweber filtrations. The preceding results allow us to sharpen Theorems 15.10 and 15.11 of [8].

THEOREM 20.10. *Let M be the BP^*-module with the single generator $x \in M^k$ and* $\mathrm{Ann}(x) = I_n$, *so that $M \cong \Sigma^k(BP^*/I_n)$.*

(a) *If $n > 0$, M admits an unstable comodule structure if and only if $k \geqslant f(n) - 2$, and it is unique. The additively unstably primitive elements are those of the form $\lambda v_n^i x$, where $\lambda \in \mathbb{F}_p$, and $k + \deg(v_n^i) \geqslant f(n)$ if $i > 0$.*

(b) *If $n = 0$, $M \cong \Sigma^k BP^*$ admits an unstable comodule structure if and only if $k \geqslant 0$, and it is unique. The additively unstably primitive elements are those of the form λx, with $\lambda \in \mathbb{Z}_{(p)}$.*

REMARK. Unlike the stable case, there are only finitely many primitives for $n > 0$. Of course, our definition forces this by requiring the degree of a primitive element to be non-negative. However, the theorem gives a much stronger condition.

PROOF. By Theorem 20.3, we must have $k \geqslant 0$, the canonical generator x is necessarily primitive, and ρ must be given by eq. (20.7). Thus in (a), ρ will be well defined if and only if $e^k(\eta_R v) \in I_n Q_*^*$ whenever $v \in I_n$. Lemma 18.23 shows that this holds for $v = v_i$ for all $i < n$, since $k \geqslant f(n) - 2 \geqslant pf(i)$; this is sufficient. On the other hand, if $k < f(n) - 2 = pf(n-1)$, Lemma 20.8(a) (with n replaced by $n-1$) would contradict $v_{n-1}x = 0$.

Because ρ is a BP^*-module homomorphism (when it exists), the coaction axioms [8, (8.7)] need only be checked on x, where they are obvious. (Alternatively, $\Sigma^k(BP^*/I_n)$ is a quotient of the geometric comodule $BP^*(S^k, o)$.)

Since any additively unstably primitive element is also by design stably primitive, [8, Theorem 15.10] restricts the candidates for primitives to $\lambda v_n^i x$. Lemma 20.8 shows, by induction on $i \geqslant 0$, that $v_n^{i+1}x$ is additively unstably primitive if and only if $\deg(v_n^i x) \geqslant pf(n)$. This is what we want, since $|\deg(v_n)| = (p-1)f(n)$.

The proof of (b) is similar, but far simpler. □

With this restriction on the basic building blocks for an unstable module, we obtain the expected improvement in [8, Theorem 15.11].

THEOREM 20.11. *Let M be an unstable BP-cohomology comodule that is finitely presented as a BP^*-module and has the discrete topology. Then there exists a filtration by subcomodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M,$$

where each M_i/M_{i-1} is generated, as a BP^*-module, by a single element x_i, whose annihilator ideal $\mathrm{Ann}(x_i) = I_{n_i}$ *for some n_i, and* $\deg(x_i) \geqslant f(n_i) - 2$ *(if $n_i > 0$), or* $\deg(x_i) \geqslant 0$ *(if $n_i = 0$).*

If, further, M is a spacelike BP^*-algebra (see Definition 7.14), *for example $BP^*(X)$ for any finite complex X, we can take each M_i to be an invariant ideal in M. At the last stage, we may take $x_m = 1$ and $n_m = 0$ or 1.*

Unfortunately, although the statement of the Theorem is exactly as expected, Landweber's method *fails*; Lemma 2.3 of [16] does not appear to be available here. (The BP^*-submodule $0{:}\,I_n = \{y \in M{:}\, I_n y = 0\}$ of M is defined but does not appear to be unstably invariant, owing to the dimensional restriction in Lemma 18.23.) Instead, we are forced to construct a suitable primitive $x_1 \in M$ directly. We would have preferred Landweber's construction because it guarantees that $\mathrm{Ann}(x_1)$ is maximal, which is useful in applications.

PROOF. We start with a nonzero element $x \in M^k$ of top degree; by Theorem 20.2, $k \geqslant 0$ and x is automatically primitive. We construct a sequence of nonzero primitive elements $y_s \in M$ such that $I_s y_s = 0$, starting with $y_0 = x$. (Here, it is convenient to write $v_0 = p$.) We stop when we reach an element y_n that is v_n-torsion-free ($v_n^i y_n \neq 0$ for all $i > 0$) and put $x_1 = y_n$ and $n_1 = n$; this must occur eventually, by Lemma 20.8(a) (e.g., when $2p^n > k$). Assume we have y_s, where $s \geqslant 0$. If it is v_s-torsion-free, we stop; this is y_n. Otherwise, take the smallest exponent q such that $v_s^q y_s = 0$ and put $y_{s+1} = v_s^{q-1} y_s$, to get $I_{s+1} y_{s+1} = 0$. By Corollary 20.9 (with s in place of n), y_{s+1} is primitive and $\deg(y_{s+1}) \geqslant pf(s)$.

We have found a primitive x_1 such that $I_n x_1 = 0$, x_1 is v_n-torsion-free, and $\deg(x_1) \geqslant pf(n-1) = f(n) - 2$. (If $n = 0$, there was no induction, and $\deg(y_0) = k \geqslant 0$.) As $\mathrm{Ann}(x_1)$ is an invariant ideal (in the stable sense), its radical ideal must be a finite intersection of invariant prime ideals in BP^*, therefore be I_m for some m. That is,

$$I_n \subset \mathrm{Ann}(x_1) \subset \sqrt{\mathrm{Ann}(x_1)} = I_m.$$

Since $v_n \notin \sqrt{\mathrm{Ann}(x_1)}$, we conclude that $m = n$ and $\mathrm{Ann}(x_1) = I_n$.

We finish as in the stable case, by setting $M_1 = (BP^*)x_1$, observing that this submodule is invariant by eq. (20.7), and replacing M by M/M_1. The induction continues until $M = 0$, and must terminate (easily, unlike the stable case), because each M^k is a finitely generated module over the Noetherian ring $\mathbb{Z}_{(p)}$ and we need consider only $k \geqslant 0$.

Now assume that M is a spacelike algebra, i.e. a product of connected algebras. This product is evidently finite, otherwise M would be uncountable. We easily reduce to the case when M is connected, which includes the case when $M = BP^*(X)$ for a connected finite complex X. By Lemma 20.5, the module $(BP^*)x$ is automatically an ideal in M; by induction, so is $(BP^*)y_s$ for each s, in particular M_1. At the last step, the module M/M_{m-1} is also an algebra; we therefore have $1 = vx_m$ and $x_m^2 = v'x_m$ for some $v, v' \in BP^*$. Then $x_m = 1x_m = vx_m^2 = vv'x_m = v'1$, which shows that $\mathrm{Ann}(1) = \mathrm{Ann}(x_m) = I_{n_m}$, and we may replace the generator x_m by 1. This implies $n_m \leqslant 1$, since $f(n) > 2$ for $n \geqslant 2$. □

COROLLARY 20.12. *For M as in Theorem 20.11, the suspension ΣM contains no nonzero additively unstably primitive elements in degree zero.*

PROOF. We observe that

$$0 = \Sigma M_0 \subset \Sigma M_1 \subset \cdots \subset \Sigma M_m = \Sigma M$$

is a Landweber filtration of ΣM. By Theorem 20.10, the only unstable comodule of the form $\Sigma^k(BP^*/I_n)$ that has a nonzero primitive in degree zero is BP^*, which does not occur as a Landweber factor $\Sigma M_i/\Sigma M_{i-1}$ of ΣM. □

21. Unstable BP-algebras

In this section, we apply the theory of Sections 10 and 19 to an unstable BP-cohomology algebra M. Our main application is Theorem 21.12 on Landweber filtrations of M, which contains Theorem 1.5 and improves on Theorem 20.11 by one degree.

Of course, we can always recover an additively unstable algebra from an unstable algebra simply by discarding the nonadditive operations. As a general rule, we can improve our results by one degree (but never more than one, in view of Theorem 13.6) by retaining all operations, at the cost of working in a far more complicated and unfamiliar environment. We developed the necessary machinery in Section 10.

Primitive elements. It is clear from Section 20 that the way to study a general unstable algebra M is to look for unstable morphisms $f\colon BP^*(S^k) \to M$ from the (relatively) well understood object $BP^*(S^k)$. Since $BP^*(S^k)$ is a free BP^*-module with basis $\{1_S, u_k\}$, f is uniquely determined, as a homomorphism of BP^*-modules, by $f1_S = 1_M$ and the element $x = fu_k \in M^k$. We extend the concept of primitive element to the unstable context, using Proposition 13.7 as a guide.

DEFINITION 21.1. We call $x \in M^k$ (where $k \geqslant 0$) *unstably primitive* if

$$r(x) = \langle r, 1_k \rangle\, 1_M + \langle r, e_k \rangle\, x \quad \text{for all } r, \tag{21.2}$$

where we interpret $e_0 = [1] - 1_0$ (as in Proposition 13.7).

This is a necessary and sufficient condition for f to be a morphism of unstable algebras, by eqs. (10.41), (10.16), and the Cartan formula (10.23). Among the unstable operations is the squaring operation, defined by $r(y) = y^2$ for all y, which implies that f is a homomorphism of BP^*-algebras (even if $k = 0$). When we restrict to additive operations, x is automatically additively primitive, and we have available all the results of Section 20.

Many elementary properties of primitives follow directly from the definition.

PROPOSITION 21.3. *Let M be an unstable algebra. Then:*

(a) *Unstable primitives are natural: if $x \in M$ is unstably primitive and $f\colon M \to N$ is a morphism of unstable algebras, then $fx \in N$ is also unstably primitive;*

(b) *The elements $0 \in M^k$ (for any $k \geqslant 0$) and 1_M are unstably primitive;*

(c) *If $x \in M^k$ is unstably primitive, where $k > 0$, then $x^2 = 0$;*

(d) *If $x \in M^k$ is unstably primitive, where $k > 0$, then λx is unstably primitive for any $\lambda \in \mathbb{Z}_{(p)}$;*

(e) *If $k > 0$ is odd, the unstable primitives in M^k form a $\mathbb{Z}_{(p)}$-submodule;*

(f) *If $k > 0$ is even and $x, y \in M^k$ are unstably primitive, then $x + y$ is unstably primitive if and only if $xy = 0$;*

(g) *The only nonzero unstable primitive in $BP^* = BP^*(T)$ is 1;*

(h) *Any unstable primitive $x \in M^0$ is idempotent, $x^2 = x$;*

(i) *If $x \in M^0$ is unstably primitive (and therefore idempotent), then the conjugate idempotent $1_M - x$ is also unstably primitive, but $-x$ is never unstably primitive (unless $-x = x$).*

PROOF. Part (a) is trivial. Part (b) is clear from eqs. (10.41) and (10.28). As noted above, f is an algebra homomorphism, which gives (c) and (h). Then (g) follows from (b) and (h).

In (d), eq. (10.16) gives

$$r(\lambda x) = \langle r, 1_k \rangle \, 1_M + \langle r, [\lambda] \circ e_k \rangle \, x.$$

Since $k > 0$, Proposition 13.7(g) gives $[\lambda] \circ e_k = \lambda e_k$, which shows that λx is primitive.

We prove (e) and (f) together. If $x, y \in M^k$ are primitive, the Cartan formula (10.23) yields

$$r(x+y) = \langle r, 1_k \rangle \, 1_M + \langle r, e_k \rangle \, x + \langle r, e_k \rangle \, y + (-1)^k \langle r, e_k * e_k \rangle \, xy,$$

which is to be compared with eq. (21.2). The unwanted last term vanishes if k is odd, because e_k is then an exterior generator; but if k is even, $e_k * e_k$ is a basis element of $BP_*(\underline{BP}_k)$. For (e), we combine this with (d).

For (i), we first use eq. (10.29) to compute $r(-x) = \langle r, 1_0 \rangle \, 1_M + \langle r, [-1] - 1_0 \rangle \, x$, which shows that $-x$ is not primitive. We then use eqs. (10.23) and (10.41) to compute $r(1_M - x) = \langle r, [1] \rangle \, 1_M + \langle r, 1_0 - [1] \rangle \, x$, which shows that $1_M - x$ is primitive. $\square$

We deduce that the Remark following Definition 20.6 extends to show that unstable primitives cannot usefully be defined in negative degrees, even though the unstable suspension (see Definition 13.4) had to be defined somewhat differently.

COROLLARY 21.4. *Let $M = BP^* \oplus \overline{M}$ be a based unstable BP-algebra.*

(a) *If $x \in \overline{M}$ is unstably primitive, so is $\Sigma x \in BP^* \oplus \Sigma\overline{M}$;*

(b) *If M is the kind of algebra considered in Theorem 20.11, there are no unstable primitives of degree zero in $BP^* \oplus \Sigma\overline{M}$ other than 0 and $1 \in BP^*$.*

PROOF. Part (a) is clear from eq. (13.3). For (b), take any primitive $y \in BP^* \oplus \Sigma\overline{M}$ in degree 0. By Proposition 21.3(g), its augmentation in BP^* must be 0 or 1; if 1, we use Proposition 21.3(i) to replace y by $1 - y$. Then $y = \Sigma x$ for some $x \in \overline{M}$. As $y \in \Sigma\overline{M}$ is also additively primitive, Corollary 20.12 shows that $y = 0$. □

If X is the disjoint union $X_1 \amalg X_2$ of two spaces, we have $BP^*(X) = BP^*(X_1) \oplus BP^*(X_2)$, a product of unstable algebras. By Proposition 21.3, the elements $(1,0)$ and $(0,1)$ are primitive idempotents in $BP^*(X)$. The converse is also true, algebraically.

THEOREM 21.5. *If $x \in M^0$ is an unstably primitive element in the unstable algebra M, other than 0 and 1_M, so that x and $1_M - x$ are idempotents, we have the splitting $M \cong xM \oplus (1_M - x)M$ of M as a product of unstable algebras.*

PROOF. By Proposition 21.3(i), both x and $1_M - x$ are primitive and idempotent. We define the first projection $p_K \colon M \to K = xM$ by $p_K y = xy$; since x is idempotent, p_K is a homomorphism of BP^*-algebras. We define $p_L \colon M \to L = (1_M - x)M$ similarly, by $p_L y = (1_M - x)y$. These will give the desired splitting of M.

Given $y \in M$, we assume that $r_M(y)$ is in the standard form (10.22), where r_M denotes the operation of r on M. By the Cartan formula (10.36),

$$\begin{aligned} r_M(xy) &= \sum_\beta \langle r, 1_0 \circ d_\beta\rangle\, y_\beta + \sum_\beta \langle r, d_\beta - 1_0 \circ d_\beta\rangle\, xy_\beta \\ &= x r_M(y) + \sum_\beta \langle r, 1_0 \circ d_\beta\rangle\, (1_M - x) y_\beta. \end{aligned}$$

Hence $x r_M(xy) = x r_M(y)$, which shows that p_K is an unstable morphism, provided we define the action $r_K \colon K \to K$ of r on K by $r_K(z) = x r_M(z)$ for $z \in K \subset M$. All the necessary laws are inherited from M. We treat p_L similarly. □

Landweber filtrations. We repeat the theory of Section 20, with an improvement of one in degree. If $x \in M^k$ is primitive in the unstable algebra M, where $k > 0$, we compute from eq. (10.16) that

$$r(vx) = \langle r, 1_{k-h}\rangle\, 1_M + \langle r, e_k \circ [v]\rangle\, x \tag{21.6}$$

for any $v \in BP^{-h}$.

Lemma 21.7. *Let M be an unstable algebra, and $x \in M^k$ an unstably primitive element, where $k > 0$. Then the BP^*-submodule $(BP^*)x$ generated by x is an unstably invariant ideal in M, provided it is an ideal.*

Proof. We apply Lemma 8.10, with the help of eq. (21.6). □

It is still true that an element of positive top degree in M is automatically primitive, for lack of any other possible terms in $r(x)$.

We now use the additional structure of the unstable operations to sharpen Lemma 20.8. We recall once more from eq. (1.4) the numerical function

$$f(n) = \frac{2(p^n - 1)}{p - 1} = 2\big(p^{n-1} + p^{n-2} + \cdots + 1\big).$$

Lemma 21.8. *Let $x \in M^k$ be a nonzero unstably primitive element of the unstable algebra M, and $n > 0$.*

(a) *If $k \leqslant pf(n)$, then $v_n^i x \neq 0$ for all $i > 0$ and is not unstably primitive;*

(b) *If $k > pf(n)$ and $I_n x = 0$, then $v_n x$ is unstably primitive.*

Corollary 21.9. *If the unstably primitive element $x \in M$ satisfies $I_n x = 0$ and is a v_n-torsion element, where $n > 0$, then:*

(a) $\deg(v_n^i x) > pf(n)$ *whenever* $v_n^i x \neq 0$.

(b) $v_n^i x$ *is unstably primitive or zero for all* i.

Proof. This is formally the same as for Corollary 20.9.

Proof of Lemma. Part (a) adds nothing to Lemma 20.8(a) unless $k = pf(n)$, in which case we must take $i = 1$ if we are to have $\deg(v_n^i x) \geqslant 0$.

To test whether or not $v_n x$ is primitive, we have to compare

$$r(v_n x) = \langle r, 1_{k-d}\rangle\, 1_M + \big\langle r, e_k \circ [v_n]\big\rangle\, x$$

from eq. (21.6) with

$$\langle r, 1_{k-d}\rangle\, 1_M + \langle r, e_{k-d}\rangle\, v_n x = \langle r, 1_{k-d}\rangle\, 1_M + \langle r, v_n e_{k-d}\rangle\, x,$$

where we write $\deg(v_n) = -d$. For (a), we take $k = 2pm$, where $m = f(n)/2$. Lemma 19.32 expands $e_{2pm} \circ [v_n]$, to show that $r(v_n x)$ has the term $\pm\langle r, (b_{(0)}^{\circ m})^{*p}\rangle\, x$. As $b_{(0)}^{\circ m}$ is a $*$-polynomial generator of $BP_*(\underline{BP}_{2m})$, we deduce that $v_n x$ cannot be primitive or zero, whatever $\operatorname{Ann}(x)$ is. Similarly, for $i > 1$, $r(v_n^i x)$ has the term

$$\pm\big\langle r, \big(b_{(0)}^{\circ m}\big)^{*p} \circ \big[v_n^{i-1}\big]\big\rangle x = \pm\big\langle r, \big(b_{(0)}^{\circ m} \circ \big[v_n^{i-1}\big]\big)^{*p}\big\rangle x,$$

which shows that $v_n^i x \neq 0$.

For (b), we apply a further suspension $e_{k-2pm} \circ -$ to eq. (19.33), which kills decomposables, to yield

$$e_{k-2pm} \circ e_{2pm} \circ [v_n] \equiv v_n e_{k-2pm} \circ e_{2m} = v_n e_k \bmod I_n.$$

This shows that $v_n x$ is unstably primitive, a stronger statement than Lemma 20.8 provides. □

As promised, these two results improve on Lemma 20.8 and Corollary 20.9 by one degree. We use them to deduce the main theorems, which likewise improve on Theorems 20.10 and 20.11 by one.

THEOREM 21.10. *Let M be the BP^*-module $BP^* \oplus (BP^*)x$, where the annihilator ideal* $\mathrm{Ann}(x) = I_n$ *and* $\deg(x) = k > 0$. *If M is made an algebra by taking $1 \in BP^*$ as the unit element and setting $x^2 = 0$, then:*

(a) *If $n > 0$, M admits an unstable algebra structure if and only if $k \geqslant f(n) - 1$, and it is unique. The nonzero unstably primitive elements in M are 1_M and the elements $\lambda v_n^i x$, where $\lambda \in \mathbb{F}_p$ $(\lambda \neq 0)$ and i satisfies $i = 0$ or* $\deg(v_n^i x) > f(n)$.

(b) *If $n = 0$, M admits a unique unstable structure. The nonzero unstably primitive elements in M are 1_M and the elements λx with $\lambda \in \mathbb{Z}_{(p)}$ $(\lambda \neq 0)$.*

PROOF. In (a), we regard M as the quotient of the geometric unstable algebra $BP^*(S^k)$ with BP^*-basis $\{1_S, u_k\}$ by the ideal $I_n u_k$. The proof is formally the same as Theorem 20.10, except that we use Lemma 21.8 instead of Lemma 20.8, Corollary 21.9 instead of Corollary 20.9, and eq. (21.6) instead of eq. (20.7).

To determine the primitives in positive degrees, we first note that λx is primitive by Proposition 21.3(d) and apply Lemma 21.8 to $\lambda v_n^i x$, by induction on i. The primitives in degree zero are given already by Proposition 21.3. □

For completeness, we mention the analogous results for $k = 0$.

PROPOSITION 21.11. *For the unstable algebra $BP^*(T) = BP^*$:*

(a) *BP^* has no proper nonzero invariant ideals;*

(b) *The unstable algebra $BP^*(S^0) \cong BP^* \oplus BP^*$ has the two copies of BP^* as its only proper nonzero invariant ideals.*

PROOF. In (a), assume J is a nonzero ideal, and take $v \neq 0$ in J. As the elements $[v]$ are linearly independent in the Hopf ring, we see from eq. (11.1) that there is an operation r such that $r(v) = 1$ and $r(0) = 0$. Thus if J is invariant, we must have $1 \in J$, and therefore $J = BP^*$.

In (b), the operations are given similarly by

$$r\big((v, v')\big) = \big(\langle r, [v]\rangle, \langle r, [v']\rangle\big) \in BP^* \oplus BP^*,$$

from which it is easy to see that any invariant ideal J that contains an element (v, v') with both v and v' nonzero must contain $1_S = (1,1)$ and therefore everything. For other ideals J, we can apply (a). □

THEOREM 21.12. *Given any spacelike* (see Definition 7.14) *discrete unstable BP-cohomology algebra M that is finitely presented as a BP^*-module (e.g., $BP^*(X)$ for any finite complex X), there is a filtration by unstably invariant ideals*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

in which each quotient M_i/M_{i-1} is generated, as a BP^*-module, by a single element x_i, whose annihilator ideal $\operatorname{Ann}(x_i) = I_{n_i}$ *for some* $0 \leqslant n_i < \infty$, *and* $\deg(x_i) \geqslant \max(f(n_i)-1, 0)$. *At the last step, $n_m = 0$ and we may take $x_m = 1_M$.*

PROOF. This is formally identical to the algebra case of the proof of Theorem 20.11, except that we use the corresponding results from this section instead of Section 20. Lemma 21.7 shows that $M_1 = (BP^*)x_1$ is indeed an invariant ideal. □

22. Additive splittings of *BP*-cohomology

Lemma 22.1 will construct idempotent operations θ_n in *BP*-cohomology, from which Parts (a) of our splitting Theorems 1.12 and 1.16 will follow. In fact, we find a large class of θ_n, among which none seems to be preferred. At the end of the section, we give an example where *no* choice of θ_n has the obvious image $\mathbb{Z}_{(p)}[v_1, \ldots, v_n]$ on homotopy groups.

LEMMA 22.1. *Assume that $k < f(n+1)$, where $n \geqslant 0$. Then there exists an additive idempotent operation $\theta_n : k \to k$ having the following properties:*

(i) *The image of $\theta_n : \underline{BP}_k \to \underline{BP}_k$ can be canonically identified with $\underline{BP\langle n\rangle}_k$;*
(ii) *The map θ_n factors to yield an H-space splitting $\overline{\theta}_n : \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ of the canonical H-map $\pi\langle n\rangle : \underline{BP}_k \to \underline{BP\langle n\rangle}_k$;*
(iii) *For all spaces X, $\overline{\theta}_n$ naturally embeds $BP\langle n\rangle^k(X) \subset BP^k(X)$ as a summand, in the sense of abelian groups* (but not as BP^*-modules)*;*
(iv) *If also $k \geqslant f(n)$, the H-space $\underline{BP\langle n\rangle}_k$ does not decompose further.*

REMARK. This result is best possible, in the sense that no additive θ_n exists when $k \geqslant f(n+1)$. (In more detail, choose m so that $f(m) \leqslant k < f(m+1)$; then $m > n$ and θ_m exists. Lemma 22.2 will show that if θ_n exists, we automatically have $\theta_n \circ \theta_m = \theta_n$. The modified idempotent $\theta'_n = \theta_m \circ \theta_n$ satisfies

$$\theta'_n \circ \theta_m = \theta'_n = \theta_m \circ \theta'_n$$

and therefore decomposes $\underline{BP\langle m\rangle}_k$ further, contrary to (iv).) For $k > f(n+1)$ this is obvious, because $H_*(\underline{BP\langle n\rangle}_k)$ then has torsion [26]. The borderline case $k = f(n+1)$ will be discussed in Section 23, where we find that a nonadditive θ_n *does* exist.

PROOF OF THEOREM 1.12(a) AND THEOREM 1.16(a) (*assuming Lemma* 22.1). The two Theorems are equivalent by [8, Theorem 3.6(a)]. As indicated, we use the splittings provided by Lemma 22.1, namely $\bar{\theta}_n : \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ and, for each $j > n$, the map

$$f_j : \underline{BP\langle j\rangle}_{k+2(p^j-1)} \xrightarrow{\bar{\theta}_j} \underline{BP}_{k+2(p^j-1)} \xrightarrow{v_j} \underline{BP}_k.$$

This $\bar{\theta}_j$ exists because

$$k + 2(p^j - 1) < f(n+1) + (p-1)f(j) \leqslant pf(j) < f(j+1).$$

On homotopy groups, $\bar{\theta}_n$ induces a splitting of $BP^* \to BP^*/J_n$, while f_j induces a splitting of $J_{j-1} \to J_{j-1}/J_j$, in view of the commutative diagram

$$\begin{tikzcd}
BP^*/J_j \arrow[r, "\bar{\theta}_j"] \arrow[dr, "="'] & BP^* \arrow[r, "v_j"] \arrow[d] & J_{j-1} \arrow[d] \\
& BP^*/J_j \arrow[r, "\cong"] & J_{j-1}/J_j
\end{tikzcd}$$

in which multiplication by v_j induces the isomorphism.

We use the H-space structure of $\underline{BP}_k$ to multiply the maps θ_n and the f_j together to form a map $f : W \to \underline{BP}_k$ from the *restricted* product W (the union of the finite subproducts) of $\underline{BP\langle n\rangle}_k$ and the spaces $\underline{BP\langle j\rangle}_{k+2(p^j-1)}$. The homotopy groups of W are the direct *sums*

$$\pi_s(W) = \pi_s\left(\underline{BP\langle n\rangle}_k\right) \oplus \bigoplus_{j>n} \pi_s\left(\underline{BP\langle j\rangle}_{k+2(p^j-1)}\right).$$

We have enough information to conclude that f induces an isomorphism of filtered groups $f_* : \pi_*(W) \cong \pi_*(\underline{BP}_k)$. For connectedness reasons, the above sum is in fact a product of graded groups, which makes W homotopy equivalent to the desired *product* of spaces. Finally, Lemma 22.1 shows that all factors of W after the first are indecomposable, since

$$k + 2\left(p^j - 1\right) \geqslant 2\left(p^j - 1\right) = (p-1)f(j) \geqslant f(j).$$

If $k \geqslant f(n)$, so is the first. □

Construction of idempotent operations. To complete the proof, we need an idempotent operation θ_n. We actually construct the BP^*-linear functional $\langle\theta_n, -\rangle : Q^k_* = Q(BP)^k_* \to BP^*$ that corresponds to it in the list (6.9). We recall the coalgebra structure $(Q(\psi), Q(\varepsilon))$ on Q^*_* and the ideal $\mathfrak{J}_n$ introduced in Definition 18.25.

LEMMA 22.2. *Assume the linear functional $\langle\theta_n,-\rangle\colon Q^k_* \to BP^*$ defined by the additive operation $\theta_n\colon k \to k$ satisfies the conditions:*

$$\begin{aligned} &\text{(i)}\quad \langle\theta_n, Q^k_* \cap \mathfrak{J}_n\rangle = 0;\\ &\text{(ii)}\quad \langle\theta_n, c\rangle \equiv Q(\varepsilon)c \bmod J_n \quad \textit{for all } c \in Q^k_*. \end{aligned} \tag{22.3}$$

Then:

(a) *The homology homomorphism $Q(\theta_n)\colon Q^k_* \to Q^k_*$ satisfies*

$$\begin{aligned} &\text{(i)}\quad Q(\theta_n)\mathfrak{J}_n = 0;\\ &\text{(ii)}\quad Q(\theta_n) \equiv \mathrm{id}\colon Q^k_* \to Q^k_* \bmod \mathfrak{J}_n; \end{aligned}$$

(b) *$Q(\theta_n)$ induces a splitting of the short exact sequence*

$$0 \longrightarrow Q^k_* \cap \mathfrak{J}_n \longrightarrow Q^k_* \longrightarrow Q^k_*/(Q^k_* \cap \mathfrak{J}_n) \longrightarrow 0$$

of left BP^*-modules;

(c) $\pi\langle n\rangle \circ \theta_n = \pi\langle n\rangle\colon \underline{BP}_k \to \underline{BP\langle n\rangle}_k$;

(d) *The operation θ_n is idempotent and has the properties listed in Lemma* 22.1.

We shall write $Q^k_*/\mathfrak{J}_n$ for the tedious but more accurate expression $Q^k_*/(Q^k_* \cap \mathfrak{J}_n)$.

REMARK. From a more invariant point of view, $Q(\varepsilon)$ induces the quotient augmentation $\overline{Q(\varepsilon)}\colon Q^k_*/\mathfrak{J}_n \to BP^*/J_n$. The conditions (22.3) on $\langle\theta_n,-\rangle$ are conveniently expressed by the commutative diagram

$$\begin{tikzcd} Q^k_* \arrow[r, dotted, "\langle\theta_n,-\rangle"] \arrow[d, "\pi"] & BP^* \arrow[d] \\ Q^k_*/\mathfrak{J}_n \arrow[r, "\overline{Q(\varepsilon)}"] \arrow[ur, dotted] & BP^*/J_n \end{tikzcd} \tag{22.4}$$

in which the vertical arrows are the obvious projections. In words, we plan to lift $\overline{Q(\varepsilon)}$ to a homomorphism of BP^*-modules $Q^k_*/\mathfrak{J}_n \to BP^*$ and define $\langle\theta_n,-\rangle$ as the composite. This is easy if $Q^k_*/\mathfrak{J}_n$ is a free BP^*-module (and in view of (b), impossible otherwise).

PROOF. We enlarge diag. (22.4) to the commutative diagram

$$\begin{tikzcd} Q^k_* \arrow[r, "Q(\psi)"] \arrow[d, "\pi"] & Q^*_* \otimes Q^k_* \arrow[r, dotted, "1\otimes\langle\theta_n,-\rangle"] \arrow[d, "1\otimes\pi"] & Q^*_* \otimes BP^* \arrow[r, "\lambda_R"] \arrow[d] & Q^k_* \arrow[d, "\pi"] \\ Q^k_*/\mathfrak{J}_n \arrow[r, "\overline{Q(\psi)}"] & Q^*_* \otimes Q^k_*/\mathfrak{J}_n \arrow[r, "1\otimes\overline{Q(\varepsilon)}"] \arrow[ur, dotted] & Q^*_* \otimes BP^*/J_n \arrow[r, "\overline{\lambda}_R"] & Q^*_*/\mathfrak{J}_n \end{tikzcd}$$

of BP^*-module homomorphisms, where $\overline{Q(\psi)}$ and $\widetilde{\lambda}_R$ are quotients of $Q(\psi)$ and λ_R. By Lemma 6.51(c), we recover $Q(\theta_n)$ as the top row, while the bottom row reduces by diag. (6.31) to the identity homomorphism of $Q^k_*/\mathfrak{J}_n$. Thus the diagonal provides a splitting $j: Q^k_*/\mathfrak{J}_n \to Q^k_*$ such that $j \circ \pi = Q(\theta_n)$ and $\pi \circ j = 1$.

This is enough to establish (a), that $Q(\theta_n)$ is idempotent with kernel exactly $Q^k_* \cap \mathfrak{J}_n$. Part (b) is merely a restatement of (a). It follows that θ_n also is idempotent.

By [8, Lemma 3.9], the idempotent operation θ_n is represented in *Ho* by the idempotent map $\theta_n = i_2 \circ p_2$ on the product $W = W_1 \times W_2$ of H-spaces, where $i_2: W_2 \to W$ and $p_2: W \to W_2$. Corollary 12.4 gives the effect of θ_n on homotopy groups: eq. (22.3)(i) shows that $\theta_{n*}v = 0$ if $v \in J_n$, while (ii) shows that

$$\theta_{n*}v \equiv Q(\varepsilon)(e^{k+h}\eta_R v) = v \bmod J_n \quad \text{in } \pi_*(\underline{BP}_k) \cong BP^*$$

for all $v \in BP^{-h}$. These two statements identify $\pi_*(W_2)$ with BP^*/J_n; more precisely, the composite $f = \pi\langle n\rangle \circ i_2: W_2 \to \underline{BP}_k \to \underline{BP\langle n\rangle}_k$ induces the desired isomorphism on homotopy groups and is thus an isomorphism of abelian group objects in *Ho*.

We need (c) to be sure our identifications are correct. Now that we know $\underline{BP\langle n\rangle}_k$ is a summand of $\underline{BP}_k$, it is enough to work in $QBP_*(-)$. By construction, $Q\pi\langle n\rangle_*$ kills $\mathfrak{J}_n$; this, with (a)(ii), gives $Q\pi\langle n\rangle_* \circ Q\theta_{n*} = Q\pi\langle n\rangle_*$.

We can now define the splitting $\overline{\theta}_n = i_2 \circ f^{-1}: \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ of $\pi\langle n\rangle$, so that $\pi\langle n\rangle \circ \overline{\theta}_n = 1$. From (c), we have $\pi\langle n\rangle = \pi\langle n\rangle \circ \theta_n = \pi\langle n\rangle \circ i_2 \circ p_2 = f \circ p_2$, which shows that the idempotent $\overline{\theta}_n \circ \pi\langle n\rangle = \overline{\theta}_n \circ f \circ p_2 = i_2 \circ p_2 = \theta_n$ is as expected. Now we can read off properties (i), (ii), and (iii) of Lemma 22.1.

Property (iv) was proved in [26], but also follows from Corollary 12.4. Suppose there is a splitting

$$\underline{BP}_k \simeq W_1 \times \underline{BP\langle n\rangle}_k \simeq W_1 \times W \times W'$$

of H-spaces that induces the decomposition $BP^* = J_n \oplus G \oplus G'$ on homotopy groups, where $1 \in G$, and let r be the idempotent that splits off W', so that $\langle r, 1\rangle = 0$ and $\langle r, Q^k_* \cap \mathfrak{J}_n\rangle = 0$. Suppose that W' is $(k+h-1)$-connected, where we must have $h > 0$. Then $\langle r, c\rangle = 0$ for all $c \in Q^k_i$ whenever $i < k+h$.

Choose a nonzero element $v \in BP^{-h}$ that lies in G' and is not divisible by p. Then $r_*v = v$ in homotopy and $v \notin I_1 + J_n$ (recall that $I_1 = (p)$). Obviously, $v \in I_\infty = I_{n+1} + J_n$. There must be some integer m, satisfying $1 \leqslant m \leqslant n$, such that $v \in I_{m+1} + J_n$ but $v \notin I_m + J_n$. We write

$$v = py_0 + \sum_{j=1}^{m} v_j y_j + z,$$

with $z \in J_n$. Since

$$k + h \geqslant f(n) + 2(p^j - 1) = f(n) + (p-1)f(j) \geqslant pf(j),$$

we have enough factors e to apply Lemma 18.23 for each $j \leqslant m$, in the form

$$\begin{aligned}\langle r, e^{k+h} w_j \eta_R y_j \rangle &\equiv \langle r, v_j e^{k+h-2(p^j-1)} \eta_R y_j \rangle \bmod I_j \\ &= v_j \langle r, e^{k+h-2(p^j-1)} \eta_R y_j \rangle = 0.\end{aligned}$$

By Corollary 12.4, $r_* v \equiv 0 \bmod I_m$, which contradicts our choices of v and m. □

PROOF OF LEMMA 22.1. Lemma 18.26(a) makes it obvious that linear functionals $\langle \theta_n, - \rangle$ exist as in diag. (22.4), so that Lemma 22.2 applies. □

EXAMPLE. Even in the simplest case, namely $\theta_1 : \underline{BP}_2 \to \underline{BP}_2$ for $p = 2$, θ_{1*} *never* induces the obvious splitting on homotopy groups. (Presumably, this failure is completely general.) We compute $\theta_{1*} v_1^3$ in terms of the Hazewinkel generators [11]. The element $b_{(0)}^4 w_1^3 \in Q_*^2$ is not allowable; instead,

$$b_{(0)}^4 w_1^3 = -\frac{12}{7} v_1 b_{(0)} b_{(1)} w_1 + \left(v_1^3 + \frac{4}{7} v_2 \right) b_{(0)} - \frac{10}{7} v_1^2 b_{(1)} - \frac{4}{7} b_{(0)}^4 w_2 - \frac{8}{7} b_{(2)},$$

as can be checked by stabilizing and working in $BP_*(BP, o)$. By construction, $\langle \theta_1, - \rangle$ takes the values 1 on $b_{(0)}$, λv_2 on $b_{(2)}$ for some $\lambda \in \mathbb{Z}_{(2)}$, and zero on the other allowable monomials that appear. Thus by Corollary 12.4,

$$\theta_{1*} v_1^3 = v_1^3 + \frac{4 - 8\lambda}{7} v_2,$$

which always contains a term in v_2.

REMARK. It is often useful to arrange the operations $\theta_n : k \to k$ compatibly as n and k vary. However, we emphasize that Theorem 1.12 as stated requires no compatibility conditions whatever.

For fixed n, compatibility in k is easily arranged. Given $\theta_n : k \to k$ that satisfies conditions (22.3), the looped operation $\Omega\theta_n : k-1 \to k-1$ has the functional

$$Q_*^{k-1} \xrightarrow{e} Q_*^k \xrightarrow{\langle \theta_n, - \rangle} BP^*$$

and clearly again satisfies (22.3). We may choose $\theta_n : k \to k$ arbitrarily for $k = f(n+1) - 1$ and use this approach for all lower k.

For fixed k, we have θ_n for all sufficiently large n. The compatibility condition $\theta_n \circ \theta_{n+1} = \theta_n$ (equivalently, $\operatorname{Ker} \theta_{n+1} \subset \operatorname{Ker} \theta_n$) is automatic, from Lemma 22.2. The other condition, $\theta_{n+1} \circ \theta_n = \theta_n$ (equivalently, $\operatorname{Im} \theta_n \subset \operatorname{Im} \theta_{n+1}$), does *not* hold in general, but can be arranged for all n simultaneously by replacing each θ_n by $\theta'_n = \ldots \circ \theta_{n+2} \circ \theta_{n+1} \circ \theta_n$. (The infinite composite presents no difficulty, as $Q(\theta_n) = \mathrm{id} : Q_i^k \to Q_i^k$ for $i < k + 2(p^{n+1} - 1)$.) This results in a sequence of commuting idempotents θ_n that satisfy $\theta_n \circ \theta_m = \theta_m \circ \theta_n = \theta_n$ whenever $n < m$.

23. Unstable splittings of BP-cohomology

In this section, we improve the splitting in Lemma 22.1 by one by allowing the idempotent operation θ_n to be nonadditive. We defer the proof until after stating Lemma 23.5. For this, we need the more detailed relations in the Hopf ring developed in Section 19.

LEMMA 23.1. *Assume that $k = f(n+1)$, where $n \geqslant 0$. Then there is a nonadditive operation $\theta_n: k \to k$ having the following properties:*

(a) *It satisfies the axioms* [8, (3.11)] *and so is idempotent;*

(b) *It has a coimage* $\operatorname{Coim}\theta_n$ *which is represented by the H-space $\underline{BP\langle n\rangle}_k$;*

(c) *Its representing map $\theta_n: \underline{BP}_k \to \underline{BP}_k$ factors to yield a section $\overline{\theta}_n: \underline{BP\langle n\rangle}_k \to \underline{BP}_k$ (not an H-map) of the canonical H-map $\pi(n): \underline{BP}_k \to \underline{BP\langle n\rangle}_k$.*

PROOF OF THEOREMS 1.12 AND 1.16, FOR $k = f(n+1)$ (*assuming Lemma* 23.1). This is almost identical to the proof given in Section 22 for $k < f(n+1)$, except that we apply [8, Lemma 3.10] instead of [8, Lemma 3.9]. The maps f_j appearing there are still H-maps; only $\overline{\theta}_n$ is not. We can still represent $\operatorname{Ker}\theta_n$ by

$$\prod_{j>n} \underline{BP\langle j\rangle}_{k+2(p^j-1)}.$$

If any of the spaces decomposed as a product, we could apply the loop space functor Ω to obtain an H-space decomposition of $\underline{BP}_{k-1}$, using additive operations, which would contradict the part of Theorem 1.12 already proved. □

Of course, we know from Lemma 22.1 that for $k = f(n+1)$, $\theta_n: k \to k$ can never be additive and that $\overline{\theta}_n$ is never an H-map. However, looping gives an *additive* idempotent operation $\Omega\theta_n: k-1 \to k-1$, which will be one of those provided by Lemma 22.1. We have the converse, which we prove after stating Lemma 23.5.

THEOREM 23.2. *Let $\theta_n: k-1 \to k-1$ be any of the additive idempotent operations provided by Lemma* 22.1. *Then:*

(a) *If $k-1$ is even, θ_n can be delooped uniquely to an additive idempotent operation $k \to k$ as in Lemma* 22.1;

(b) *If $k-1$ is odd, θ_n can be delooped (not uniquely) to a* nonadditive *idempotent operation $k \to k$ as in Lemma* 23.1.

The next two lemmas constitute the unstable analogue of Lemma 22.2. They are far more complicated, because instead of $Q(\psi)$, we have only the natural transformation $\psi: U \to UU$. This requires knowledge of the homology homomorphisms r_* induced by each operation r, which is provided by Theorems 10.19 and 10.33 and the properties of each $\circ$-generator of $BP_*(\underline{BP}_*)$. *We warn that as a consequence, the form of the proofs runs totally counter to traditional proofs involving cohomology operations.* We abbreviate $\langle r, \mathfrak{J}_n \cap BP_*(\underline{BP}_k)\rangle$ to $\langle r, \mathfrak{J}_n\rangle$, etc.

LEMMA 23.3. *If the unstable operation $r\colon k \to m$ satisfies $\langle r, \mathfrak{J}_n\rangle = 0$, then the homology homomorphism $r_*\colon BP_*(\underline{BP}_k) \to BP_*(\underline{BP}_m)$ satisfies $r_*\mathfrak{J}_n = 0$.*

PROOF. Our plan is to show that $r_*c = 0$ in three steps, depending on the form of $c \in \mathfrak{J}_n$, simultaneously for *all* operations $r\colon k \to m$ that satisfy $\langle r, \mathfrak{J}_n\rangle = 0$, where $c \in BP_*(\underline{BP}_k)$ determines k and m is arbitrary.

Case 1: $c = [v_j] - 1$, where $j > n$. By hypothesis, $\langle r, [v_j]\rangle = \langle r, 1\rangle$. Then by Proposition 11.2(g),

$$r_*\big([v_j] - 1\big) = \big[\langle r, [v_j]\rangle\big] - \big[\langle r, 1\rangle\big] = 0.$$

Case 2: $c = a\circ([v_j]-1)$, where $j > n$. Thus c is a $*$-generator of $\mathfrak{J}_n$. We apply Theorem 10.33(c); the operations r''_α defined by eq. (10.35) satisfy our hypothesis

$$\langle r''_\alpha, d\rangle = \pm\langle r, c_\alpha \circ d\rangle = 0 \quad \text{for all } d \in \mathfrak{J}_n$$

because $c_\alpha \circ d \in \mathfrak{J}_n$, $\mathfrak{J}_n$ being a Hopf ring ideal by Lemma 19.35. Using eq. (19.36) to compute the iterated coproduct $\Psi([v_j]-1)$, we see that every term of r_*c in eq. (10.37) contains a factor $r''_{\alpha*}([v_j]-1)$, which vanishes by Case 1.

Case 3: $c = a * b$, with b as in Case 2. Since such elements span $\mathfrak{J}_n$ as a BP^*-module, this will complete the proof. We apply Theorem 10.19(c); the operations r''_α defined by eq. (10.21) satisfy our hypothesis

$$\langle r''_\alpha, d\rangle = \pm\langle r, c_\alpha * d\rangle = 0 \quad \text{for all } d \in \mathfrak{J}_n$$

because $\mathfrak{J}_n$ is a $*$-ideal. Using eq. (19.37) to compute the iterated coproduct Ψb, we see that every term of r_*c in eq. (10.25) contains a factor of the form $r''_{\alpha*}(b'\circ([v_j]-1))$, which vanishes by Case 2. □

LEMMA 23.4. *Let $r\colon k \to m$ be an unstable operation.*

(a) *If r satisfies $\langle r, c\rangle \in J_n$ for all $c \in BP_*(\underline{BP}_k)$, then $r_*c \equiv (\varepsilon c)1_m$ mod $\mathfrak{J}_n$ for all $c \in BP_*(\underline{BP}_k)$;*

(b) *If r satisfies $\langle r, c\rangle \equiv Q(\varepsilon)q_k c$ mod J_n for all $c \in BP_*(\underline{BP}_k)$, then $r_*c \equiv c$ mod $\mathfrak{J}_n$ for all $c \in BP_*(\underline{BP}_k)$.*

PROOF. We prove (a) in five steps, depending on the form of c, simultaneously for all $r\colon k \to m$ that satisfy the hypothesis, where $c \in BP_*(\underline{BP}_k)$ determines k and m is arbitrary. We work throughout mod $\mathfrak{J}_n$, which is a Hopf ring ideal by Lemma 19.35.

Case 1: $c = [v]$, for any $v \in BP^*$. By Proposition 11.2(g) and Lemma 19.38, $r_*[v] = [\langle r, [v]\rangle] \equiv 1$. This includes the special case $c = 1 = [0]$.

Case 2: $c = e$. By Proposition 13.7(h) and Lemma 19.38, $r_*e \equiv 1 * 1\circ e = 1 * 0 = 0$.

Case 3: $c = b_i$, where $i > 0$. By Proposition 15.3, working formally in $BP_*(\underline{BP}_m)[[x]]$,

$$r_*b(x) = [\langle r, 1_2\rangle] * \mathop{\Large *}_{j>0} b(x)^{\circ j}\circ[\langle r, b_j\rangle] \equiv \mathop{\Large *}_{j>0} b(x)^{\circ j}\circ 1 = \mathop{\Large *}_{j>0} \varepsilon b(x)^{\circ j} = 1.$$

The coefficient of x^i gives $r_* b_i \equiv 0$.

Case 4: $c = a \circ b$, where $b = e$ or $b = b_i$ for some $i > 0$. We apply Theorem 10.33(c); the operations r''_α defined by eq. (10.35) satisfy the hypothesis $\langle r''_\alpha, d\rangle = \pm\langle r, c_\alpha \circ d\rangle \in J_n$ for all d. Then using Proposition 13.7(d) or Proposition 15.3(c) to compute the iterated coproduct Ψb, we see that every term of $r_* c$ in eq. (10.37) contains a factor $r''_{\alpha *} e$ or $r''_{\alpha *} b_j$ with $j > 0$, which lies in $\mathfrak{J}_n$ by Case 2 or Case 3. This, with Case 1, takes care of all the $*$-generators (19.4) of $BP_*(\underline{BP}_*)$.

Case 5: $c = a * d$, with d as in Case 4. We apply Theorem 10.19(c) and again find that each r''_α satisfies our hypothesis $\langle r''_\alpha, g\rangle = \pm\langle r, c_\alpha * g\rangle \in J_n$ for all g. In the iterated coproduct

$$\Psi d = \sum_j \bigotimes_\alpha d_{j,\alpha},$$

every term contains a factor $d_{j,\alpha}$ to which Case 4 applies. Thus every term of $r_* c$ in eq. (10.25) has a factor $r''_{\alpha *} d_{j,\alpha} \equiv 0$.

As every $*$-monomial in the $\circ$-generators of $BP_*(\underline{BP}_*)$ is included in Cases 1 and 5 (by writing $[v] * [v'] = [v+v']$), this completes the proof of (a).

For (b), we recall from eq. (10.42) that $\langle \iota_k, c\rangle = Q(\varepsilon) q_k c$, so that (a) applies to $r - \iota_k$. We apply eq. (10.17) to $r = (r - \iota_k) + \iota_k$ to deduce that for any

$$c \in E_*(\underline{E}_k), \quad r_* c \equiv \sum_i (\varepsilon c'_i) c''_i = c,$$

where as usual we write $\psi c = \sum_i c'_i \otimes c''_i$. □

We need one more result before we prove Lemma 23.1 and Theorem 23.2. The structure of $BP_*(\underline{BP}_k)/\mathfrak{J}_n$ is much more opaque when $k = f(n+1)$. We defer the proof until after Lemma 23.12.

LEMMA 23.5. *For $k \leqslant f(n+1)$, where $n \geqslant 0$:*

(a) *$BP_*(\underline{BP}_k)/\mathfrak{J}_n$ is a free BP^*-module;*

(b) *The homomorphism $Q(BP)^{k-1}_*/\mathfrak{J}_n \to BP_*(\underline{BP}_k)/\mathfrak{J}_n$ induced by suspension is a split monomorphism of BP^*-modules.*

Note that we have two different ideals $\mathfrak{J}_n$ here. One is an ideal in the algebra Q^*_* in the ordinary sense, while the other is a Hopf ring ideal in $BP_*(\underline{BP}_*)$.

PROOF OF LEMMA 23.1 (*assuming Lemma* 23.5). To apply the method of Lemma 22.1, we need an operation $\theta_n \colon k \to k$ that satisfies $\theta_* \mathfrak{J}_n = 0$ and $\theta_* \equiv \mathrm{id} \bmod \mathfrak{J}_n$. In view of Lemma 23.3 and Lemma 23.4(b), these conditions are ensured by (and in fact equivalent to) the following conditions on the linear functional $\langle \theta_n, -\rangle$:

$$\begin{aligned} &\text{(i) } \langle \theta_n, \mathfrak{J}_n\rangle = 0; \\ &\text{(ii) } \langle \theta_n, c\rangle \equiv Q(\varepsilon) q_k c \bmod J_n \text{ for all } c \in BP_*(\underline{BP}_k). \end{aligned} \tag{23.6}$$

Therefore we need to fill in the diagram

$$\begin{tikzcd} Q_*^{k-1} \arrow[r] \arrow[d] & BP_*(\underline{BP}_k) \arrow[r, dotted, "\langle\theta_n,-\rangle"] \arrow[d] & BP^* \arrow[d] \\ Q_*^{k-1}/\mathfrak{J}_n \arrow[r] & BP_*(\underline{BP}_k)/\mathfrak{J}_n \arrow[r] \arrow[ur, dotted] & BP^*/J_n \end{tikzcd} \tag{23.7}$$

analogous to diag. (22.4) with a lifting $BP_*(\underline{BP}_k)/\mathfrak{J}_n \to BP^*$ of the homomorphism $BP_*(\underline{BP}_k)/\mathfrak{J}_n \to BP^*/J_n$ induced by $Q(\varepsilon)\circ q_k$, which then defines $\langle\theta_n,-\rangle$. Lemma 23.5(a) makes this easy to do.

For (a), we must verify the axioms [8, (3.11)] on θ_n. The first holds trivially, for dimensional reasons. The second is the identity $\theta_n(x+z)=\theta_n(x)$, for $z=y-\theta_n(y)$. We assume that the standard form $r(x)=\sum_\alpha\langle r,c_\alpha\rangle\, x_\alpha$ holds for all r, as in eq. (10.3). Then by eq. (10.20), $\theta_n(x+z)=\sum_\alpha x_\alpha\theta''_\alpha(z)$, where the operation θ''_α is defined as having the functional $\langle\theta''_\alpha,c\rangle=\langle\theta_n,c_\alpha * c\rangle$. Because $z=(\iota_k-\theta_n)(y)$, we have only to prove that $(\iota_k-\theta_n)^*\theta''_\alpha=\langle\theta_n,c_\alpha\rangle\,1$ in $BP^*(\underline{BP}_k)$ for each α. We compute the associated linear functional as

$$\big\langle(\iota_k-\theta_n)^*\theta''_\alpha,c\big\rangle=\big\langle\theta''_\alpha,(\iota_k-\theta_n)_*c\big\rangle=\big\langle\theta_n,c_\alpha*(\iota_k-\theta_n)_*c\big\rangle.$$

By Lemma 23.4(a), $(\iota_k-\theta_n)_*c\equiv(\varepsilon c)1 \bmod \mathfrak{J}_n$. As $\langle\theta_n,-\rangle$ kills $\mathfrak{J}_n$ by Lemma 23.3 and $\mathfrak{J}_n$ is an ideal, this agrees with $\langle\theta_n,(\varepsilon c)c_\alpha\rangle=\langle\theta_n,c_\alpha\rangle\varepsilon c$. Now we can apply [8, Lemma 3.10] to construct the coimage of θ_n.

For (b) and (c), we have to check that θ_n acts as desired on homotopy groups. By Lemma 13.9, θ_{n*} is given on $v\in BP^{-h}\cong\pi_{k+h}(\underline{BP}_k)$ by $\theta_{n*}v=\langle\theta_n,e^{\circ\,k+h}\circ[v]\rangle$. For $v\in J_n$, we have $[v]\equiv 1 \bmod \mathfrak{J}_n$ by Lemma 19.38, so that $\theta_{n*}v=0$ by (i). For any v, (ii) gives $\theta_{n*}v\equiv Q(\varepsilon)q_k(e^{\circ\,k+h}\circ[v])=v \bmod J_n$. □

Proof of Theorem 23.2 (*assuming Lemma* 23.5). Part (a) is trivial and belongs in Section 22, as suspension induces an isomorphism $Q_*^{k-1}\cong Q_*^k$ and preserves the conditions (22.3).

In (b), we must have $k\leqslant f(n+1)$ for θ_n to exist. In effect, the lifting $BP_*(\underline{BP}_k)/\mathfrak{J}_n\to BP^*$ in diag. (23.7) is prescribed on $Q_*^{k-1}/\mathfrak{J}_n$. As we have by Lemma 23.5(b) a split monomorphism with free cokernel, it is easy to extend the given lifting over $BP_*(\underline{BP}_k)/\mathfrak{J}_n$. □

Resolutions. Lemma 23.5 is easy to prove when $k<f(n+1)$. In the borderline case $k=f(n+1)$, the presence of the extra disallowed monomials in Lemma 19.39 makes it necessary to do some homological algebra.

Lemma 23.8. *In the sequence of homomorphisms of BP^*-modules*

$$C_2\xrightarrow{\partial_2}C_1\xrightarrow{\partial_1}C_0\xrightarrow{\varepsilon}M\longrightarrow 0, \tag{23.9}$$

assume that:

(i) *Each C_i is free of finite type;*
(ii) *We have exactness at C_0 and M;*
(iii) $\partial_1 \circ \partial_2 = 0$ (we do not assume exactness at C_1);
(iv) *The sequence*

$$C_2 \otimes \mathbb{F}_p \xrightarrow{\partial_2 \otimes \mathbb{F}_p} C_1 \otimes \mathbb{F}_p \xrightarrow{\partial_1 \otimes \mathbb{F}_p} C_0 \otimes \mathbb{F}_p \xrightarrow{\varepsilon \otimes \mathbb{F}_p} M \otimes \mathbb{F}_p \quad (23.10)$$

is exact at $C_1 \otimes \mathbb{F}_p$ (as well as at $C_0 \otimes \mathbb{F}_p$).

Then:

(a) *The sequence (23.9) is split exact in the sense that:*

(i) *C_0 splits as $C_0 \cong M \oplus \partial_1 C_1$;*
(ii) *C_1 splits as $C_1 \cong \partial_1 C_1 \oplus \partial_2 C_2$;*

(b) *M is a free BP^*-module; explicitly, if L_0 is a free module and the module homomorphism $g_0: L_0 \to C_0$ induces an isomorphism*

$$L_0 \otimes \mathbb{F}_p \xrightarrow{g_0 \otimes \mathbb{F}_p} C_0 \otimes \mathbb{F}_p \longrightarrow \operatorname{Coker}(\partial_1 \otimes \mathbb{F}_p) \cong M \otimes \mathbb{F}_p,$$

the composite $\varepsilon \circ g_0: L_0 \to M$ is an isomorphism.

PROOF. We build the following commutative diagram, which includes the projections from diag. (23.9) to diag. (23.10),

$$\begin{array}{ccccccc}
L_2 & & L_1 & & L_0 & & \\
\downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \\
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\varepsilon} & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_2 \otimes \mathbb{F}_p & \xrightarrow{\partial_2 \otimes \mathbb{F}_p} & C_1 \otimes \mathbb{F}_p & \xrightarrow{\partial_1 \otimes \mathbb{F}_p} & C_0 \otimes \mathbb{F}_p & \xrightarrow{\varepsilon \otimes \mathbb{F}_p} & M \otimes \mathbb{F}_p
\end{array}$$

It is easy to construct g_0 as in (b), by lifting a basis of $\operatorname{Coker}(\partial_1 \otimes \mathbb{F}_p)$ to C_0. Similarly, we construct $g_1: L_1 \to C_1$, with L_1 free, that induces an isomorphism

$$L_1 \otimes \mathbb{F}_p \cong \operatorname{Coker}(\partial_2 \otimes \mathbb{F}_p) \cong \operatorname{Im}(\partial_1 \otimes \mathbb{F}_p),$$

and again $g_2: L_2 \to C_2$, with L_2 free, that induces $L_2 \otimes \mathbb{F}_p \cong \operatorname{Im}(\partial_2 \otimes \mathbb{F}_p)$.

Then by Nakayama's Lemma in the form [8, Lemma 15.2(a)], the homomorphism $L_0 \oplus L_1 \to C_0$ with components g_0 and $\partial_1 \circ g_1$ is an isomorphism, and similarly $L_1 \oplus L_2 \cong C_1$. These allow us to write $g_i: L_i \subset C_i$ for $i = 0, 1, 2$, and the isomorphisms simplify to $C_0 = L_0 \oplus \partial_1 L_1$ and $C_1 = L_1 \oplus \partial_2 L_2$. The latter gives $\partial_1 C_1 = \partial_1 L_1$, which shows that $M \cong \operatorname{Coker}[\partial_1 \otimes \mathbb{F}_p] \cong L_0$ is free. Moreover, because $\partial_1 | L_1$ is monic, $\partial_2 C_2 = \partial_2 L_2$ and we have split exactness at C_1. □

For our application, we take a polynomial algebra

$$R = BP^*[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]$$

on generators of negative degree, with $\deg(x_i) \to -\infty$ and $\deg(y_i) \to -\infty$ as $i \to \infty$, to make R a BP^*-module of finite type. We consider the quotient ring $M = R/\mathfrak{J}$ as a BP^*-module, where the ideal $\mathfrak{J} = (x_1^p - c_1, x_2^p - c_2, \ldots)$. The elements c_i are to be in some sense negligible. We construct what we hope is the beginning (or end) of an R-free resolution of M,

$$C_2 = \bigoplus_{i<j} Ru_iu_j \xrightarrow{\partial_2} C_1 = \bigoplus_i Ru_i \xrightarrow{\partial_1} C_0 = R \longrightarrow M \longrightarrow 0, \quad (23.11)$$

with R-linear differentials given by $\partial_1 u_i = x_i^p - c_i$ and $\partial_2 u_i u_j = (x_i^p - c_i)u_j - (x_j^p - c_j)u_i$. This is part of a familiar Koszul-type resolution if the c_i are in fact zero, and the structure of M is then clear. Lemma 23.8 supplies conditions under which M has the expected size, even when $c_i \neq 0$.

LEMMA 23.12. *Assume that the sequence* (23.11) *induces an exact sequence* (23.10), *and that the set B of monomials in R of the form*

$$x^I y^J = x_1^{i_1} x_2^{i_2} \cdots y_1^{j_1} y_2^{j_2} \cdots,$$

with $i_t < p$ for all t, yields an $\mathbb{F}_p$-basis of $\operatorname{Coker}(\partial_1 \otimes \mathbb{F}_p)$. *Then the sequence* (23.11) *is exact, $M = R/\mathfrak{J}$ is a free BP^*-module, and B yields a BP^*-basis of it.*

PROOF OF LEMMA 23.5. Nakayama's Lemma [8, Lemma 15.2] and Lemma 23.12 allow us to work mod $\mathfrak{V}$ everywhere.

For (a), we apply Lemma 23.12 to $BP_*(\underline{BP}_k)/\mathfrak{J}_n$, using the detailed information on $\mathfrak{J}_n$ provided by Lemma 19.39. The $*$-ideal $\mathfrak{J}_n \cap BP_*(\underline{BP}_k) \subset BP_*(\underline{BP}_k)$ has two kinds of generator: the first kind are standard polynomial generators, but the second kind (which occur only if $k = f(n+1)$) are disallowed; we express them in terms of allowable monomials by means of eq. (19.28), of which the leading term (19.26) is of most interest.

We therefore classify the Ravenel–Wilson polynomial generators of $BP_*(\underline{BP}_k)$ into three types:

(i) The allowable $b^{\circ I} \circ [v^J]$ in which v^J contains some factor v_j with $j > n$;

(ii) Monomials of the form $b_{(k_0)} \circ b_{(k_1)}^{\circ p} \circ \ldots \circ b_{(k_n)}^{\circ p^n}$, where $0 \leqslant k_0 \leqslant k_1 \leqslant \cdots \leqslant k_n$; (23.13)

(iii) All other allowable monomials $b^{\circ I} \circ [v^J]$.

The first type visibly lie in $\mathfrak{J}_n$, and we ignore them, by taking R in Lemma 23.12 as the quotient polynomial ring (using $*$-multiplication, of course) on the second and third types, which serve as the x_i and y_i respectively. The interesting generators of $\mathfrak{J}_n$ then have the form $x_i^p - c_i$.

There are five types of term in the reduction formula (19.28) for the monomial $b^{\circ p}_{(k_1-1)} \circ b^{\circ p^2}_{(k_2-2)} \circ \cdots \circ b^{\circ p^{n+1}}_{(k_{n+1}-n-1)} \circ [v_{n+1}]$:

(i) $b^{\circ p}_{(i_1-1)} \circ b^{\circ p^2}_{(i_2-2)} \circ \cdots \circ b^{\circ p^{n+1}}_{(i_{n+1}-n-1)} \circ [v_{n+1}]$;
(ii) $F(b_{(k_1-1)} \circ b^{\circ p}_{(k_2-2)} \circ \cdots \circ b^{\circ p^n}_{(k_{n+1}-n-1)})$;
(iii) $F(b_{(i_1-1)} \circ b^{\circ p}_{(i_2-2)} \circ \cdots \circ b^{\circ p^n}_{(i_{n+1}-n-1)})$;
(iv) Terms in $\mathfrak{A}_2$;
(v) Terms in $\mathfrak{A}_1 * \mathfrak{A}_1$;

where $(i_1, i_2, \ldots, i_{n+1})$ denotes any nontrivial permutation of $(k_1, k_2, \ldots, k_{n+1})$.

Because the suffixes in (i) are out of order, (i) is an example of a type (i) generator in (23.13), which has been discarded. The term we want is (ii), which is x_i^p. We can take care of (iii) and (iv) by filtering R by powers of the ideal $(y_1, y_2, \ldots)$ and working with the associated graded groups; if we have exactness in diag. (23.10) after filtering, we had exactness before. In effect, we may ignore the y_i's. We take care of (v) by filtering again, this time by powers of the ideal $\mathfrak{A}_1 + (u_1, u_2, \ldots)$ in (23.10). This done, we have effectively reduced c_i to zero, when we have exactness. Thus $BP_*(\underline{BP}_k)/\mathfrak{J}_n$ is a free BP^*-module, and we have constructed a basis.

For (b), we have only to show that we have a monomorphism mod $\mathfrak{V}$. By Lemma 18.26(a) and Lemma 18.12(c), $Q_*^{k-1}/\mathfrak{J}_n$ is a free BP^*-module with a basis consisting of the monomials of the extended canonical form (18.13)

$$e\, b^{L-\Delta_0}\, b^{L+s(L)+\cdots+s^{h-1}(L)}\, b^{s^h(M)}\, w_m^h w^J,$$

that lie in Q_*^{k-1} and have no factor w_j with $j > n$, where

$$b^L = b_{(k_0)} b^p_{(k_1)} \cdots b^{p^m}_{(k_m)}, \quad 0 = k_0 \leqslant k_1 \leqslant \cdots \leqslant k_m,\ m \geqslant 0,\ h \geqslant 0,$$

and the conditions (18.9) on M and J hold. After suspension, we find the leading term (19.31), namely $F^h(b^{\circ L} \circ b^{\circ M} \circ [v^J])$, which by Lemma 18.12(d) is the p^hth power of an allowable monomial.

There are two cases:

Case $m < n$. The element $b^{\circ L} \circ b^{\circ M} \circ [v^J]$ is a generator y_i of type (iii) in (23.13), and therefore harmless.

Case $m \geqslant n$. Since $j_t = 0$ for all $t \leqslant m$ and $t > n$, we must have $J = 0$. Also, $h = 0$. We must have $m = n$, otherwise we would have $k > f(n+1)$. We have a generator x_i of type (ii), but it is not raised to a power.

By Lemma 23.12, the elements $F^h y_i$ and x_i (for certain i) map to part of a basis of $\overline{H}_*^k/\mathfrak{J}_n$, which is sufficient. (Because $k_0 = 0$, it is clear that these elements lie in $P\overline{H}_*^k$. In view of the suspension isomorphism $Q_*^{k-1}/\mathfrak{V} \cong P\overline{H}_*^k$ in [23, Theorem 5.3], all we really need to know is that enough basis elements of $P\overline{H}_*^k$ in each degree remain linearly independent in $\overline{H}_*^k$ mod $\mathfrak{J}_n$.) $\square$

Index of symbols

This index lists most symbols in roughly alphabetical order (English, then Greek), with brief descriptions and references. Several symbols have multiple roles.

L infinite lens space.
$L(k)$ left side of main relation $(\mathcal{R}_k)$.
$L(i_1, \dots, i_n)$ coefficient, Definitions 18.18, 19.21.
M etc. generic (filtered) module or algebra.
$M^\wedge$, $\widehat{M}$ completion of filtered M, [8, Definition 3.37].
$\mathfrak{M}$ ideal in $Q(BP)^*_*$, §18.
$\mathfrak{M}_n$ Hopf ring ideal, Definition 19.13.
Mod, *Mod** (graded) category of E^*-modules, [8, §6].
MU unitary Thom spectrum, [8, §2].
o generic basepoint, point spectrum.
PA the primitives in coalgebra A, [8, (6.13)].
$PE^*(\underline{E}_k)$ the additive operations, Proposition 2.7.
$PE_*(X)$ the primitives in homology of space X, Definition 4.13.
$PE^*(X)$ the primitives in cohomology of H-space X, Definition 3.1.
$P(n)$ modified BP spectrum, §1.
p fixed prime number.
p_1, p_2 projection from product, [8, §2].
$[p](x)$ p-series, [8, (13.9)].
$[p]_R(x)$ right p-series, (14.8), (15.14).
$-_Q$ (subscript) additive unstable context, shifted degree.
QA the indecomposables of algebra A, [8, (6.10)].
$QE^*(X)$ the indecomposables of cohomology of space X, (3.5).
$QE_*(X)$ the indecomposables of homology of H-space X, Definition 4.3.
$Q(E)^*_*$ bigraded algebra, Definition 6.1.
$Q(r)$ homology homomorphism induced by operation r, (6.48).
$\underline{Q}^*_*$ $= Q(BP)^*_*$, abbreviation.
$\overline{Q}^*_*$ quotient algebra of Q^*_*, (18.17).
$Q(\varepsilon)$ counit of $Q(E)^*_*$, (6.28).
$Q(\eta)$ unit morphism of $Q(E)^*_*$, (6.17).
$Q(\sigma)$ stabilization on $Q(E)^*_*$, (6.3).
$Q(\phi)$ multiplication in $Q(E)^*_*$, (6.16).
$Q(\psi)$ comultiplication on $Q(E)^*_*$, (6.27).
$\mathbb{Q}$ field of rational numbers.
q map to one-point space T.
q_k projection to $Q(E)^k_*$, (6.2).
$\mathbb{R}P^\infty$ real projective space.
$R(k)$ right side of main relation $(\mathcal{R}_k)$.
$R(i_1, \dots, i_n)$ coefficient, Definitions 18.18, 19.21.
$(\mathcal{R}_k)$ kth main relation, (14.10), (15.16).
$(\mathcal{R}_{k_1,\dots,k_n})$ nth order relation, Definitions 18.18, 19.21.
r generic cohomology operation.
$\langle r, - \rangle$ E^*-linear functional defined by operation r, (6.9), (10.1).
S stable comonad, [8, Theorem 10.12].
$-_S$ (subscript) stable context.
S^1 unit circle, as space or group.
S^n unit n-sphere.
$\overline{S}$ comonad S on modules, §9 (only).
Stab, *Stab** (graded) stable homotopy category, [8, §6].
Set category of sets, [8, §6].
Set$^{\mathbb{Z}}$ category of graded sets, [8, §7].
$s(I)$, $s^h(I)$ shifted multi-index I, Definition 15.12.
T the one-point space.
T^+ 0-sphere, T with basepoint added.
$T(n)$ torus group.
t $\in H^1(\mathbb{R}P^\infty)$, generator of $H^*(\mathbb{R}P^\infty)$, (16.1).
U unstable comonad, Theorem 8.8.
$-_U$ (subscript) unstable context.
U, $U(n)$ unitary group.
u $\in KU^{-2}$, generator.
u $\in E^1(L)$, exterior generator of $E^*(L)$, §§16, 17.
u_1 canonical generator of $E^*(S^1)$, [8, Definition 3.23].
u_n canonical generator of $E^*(S^n)$, [8, §3].
V generic (often forgetful) functor.
V Verschiebung operator, §10.
$\mathfrak{V}$ ideal in $Q(BP)^*_*$, §18.
$\mathfrak{V}$ ideal in $BP_*(\underline{BP}_*)$, §19.
v generic element of E^*.
v $= \eta_R u \in KU_2(KU, o)$, Theorem 16.15.
$[v]$ $\in E_0(\underline{E}_*)$, Definition 10.8.
v_n Hazewinkel generator of BP^*, $K(n)^*$, [11].
$\mathfrak{W}$ ideal in $Q(BP)^*_*$, §18.
w generic element of $\eta_R E^*$, Proposition 12.3.
w_n $= \eta_R v_n$, §16.
$\mathrm{wt}(I)$ weight of multi-index I, §18.
X etc. generic space.
X^+ space X with basepoint adjoined.
x generic cohomology class or module element.
x $\in E^*(\mathbb{C}P^\infty)$, Chern class of Hopf line bundle, [8, Lemma 5.4].
$x(\theta)$ Chern class of line bundle θ, [8, Definition 5.1].
Y skeleton of lens space L, [8, §14].
$\mathbb{Z}$ the ring of integers.
$\mathbb{Z}/p$ the group of integers mod p.
$\mathbb{Z}_{(p)}$ $\mathbb{Z}$ localized at p.
z_F morphism for a (symmetric) monoidal functor F, [8, §7].
α etc. generic index.
β_i $\in E_{2i}(\mathbb{C}P^n)$, [8, Lemma 5.3].
γ_i $\in E_{2i+1}(U(n))$, [8, Lemma 5.11].

$\Delta: X \to X \times X$ diagonal map.
Δ_0 $= (1, 0, 0, \ldots)$, multi-index, §18.
ϵ generic counit morphism.
ζ pth power map on $\mathbb{C}P^\infty$, [8, (13.9)].
ζ_F pairing for (symmetric) monoidal functor F, [8, §7].
η generic unit morphism.
η_R right unit, Definitions 6.19, 10.8.
θ generic anything.
θ_n idempotent cohomology operation on BP, Lemmas 22.1, 23.1.
$\overline{\theta}_n$ splitting of $\pi(n)$, Lemmas 22.1, 23.1.
ι $\in E^0(E, o)$, universal class, [8, §9].
ι_n $\in E^n(\underline{E}_n)$, universal class, [8, Theorem 3.17].
κ_n collapse operation, Definition 7.13.
$\Lambda(-)$ exterior algebra.
λ generic action.
λ numerical coefficient.
λ_L left E^*-action on $Q(E)^*_*$, §6.
λ_R right E^*-action on $Q(E)^*_*$, (6.21).
μ addition or multiplication in generic group object, [8, §7].
ν inversion morphism in generic group object, [8, §7].
ξ Hopf line bundle over $\mathbb{C}P^n$.
ξ generic line or vector bundle.
ξ_i element for $H(\mathbf{F}_2)$, (16.1).
ξ_i element for $H(\mathbf{F}_p)$, Theorem 16.5.
ξv action of v on E^*-module, [8, (7.4)].
π generic permutation in Σ_n.
$\pi_*(X)$ homotopy groups of space X.
$\pi(n): BP \to BP(n)$ projection, (1.8).
ρ generic coaction.
ρ_M coaction on module M.
ρ_X coaction on $E^*(X)$ or $E^*(X)^\wedge$.
Σ, Σ^k suspension isomorphism, [8, (3.13), Definition 6.6].
ΣX, $\Sigma^k X$ suspension of space X.
ΣM, $\Sigma^k M$ suspension of module M, [8, Definition 6.6].
Σ_n permutation group on $\{1, 2, \ldots, n\}$.
$\sigma: A \to S$ natural transformation of comonads, Theorem 5.8.
$\overline{\sigma}: \overline{A} \to \overline{S}$ natural transformation of comonads, on modules, §9 (only).
$\sigma_k: \underline{E}_k \to E$ stabilization map, [8, Definition 9.3].
$\tau: U \to A$ natural transformation of comonads, Theorem 8.8.
τ_i element for $H(\mathbf{F}_p)$, (16.4).
ϕ generic multiplication.
χ canonical antiautomorphism of Hopf algebra.
Ψ iterated coproduct, (10.18).
ψ generic comultiplication.
ΩX loop space on based space X.
Ωr looped operation, Proposition 2.12.
ω zero morphism of generic group object, [8, §7].

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CHAPTER 16

Differential Graded Algebras in Topology

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Contents

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HANDBOOK OF ALGEBRAIC TOPOLOGY
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1. Introduction

The homotopy theory of topological spaces attempts to classify weak homotopy types of spaces and homotopy classes of maps. For a given geometric problem, however, it is often sufficient to replace the homotopy theory of spaces by a weaker 'algebraic' homotopy theory as described, for instance, in Baues' book [3]. This article will consider the specific case when spaces are modeled by the homotopy theory of differential graded algebras.

To be more explicit we require some definitions. We work over a commutative ground ring $\mathbb{k}$. Thus linear, bilinear, etc. means with respect to $\mathbb{k}$, and we write $-\otimes-$ for $-\otimes_{\mathbb{k}}-$ and $\mathrm{Hom}(-,-)$ for $\mathrm{Hom}_{\mathbb{k}}(-,-)$. A graded module is a family $M=\{M_i\}_{i\in\mathbb{Z}}$ of $\mathbb{k}$-modules; by abuse of language an *element* $x\in M$ is an element in some M_i and x has degree i ($\deg x=i$). A linear map $f:M\to N$ of degree k is a family of linear maps $f_i:M_i\to N_{i+k}$.

A *differential* in M is a linear map $d:M\to M$ of degree -1 such that $d^2=0$; and the quotient graded module $H(M,d)=\ker d/\operatorname{Im} d$ is the *homology* of M; we often simplify the notation to $H(M)$. The *suspension* of (M,d) is the differential graded module $s(M,d)$ defined by $(sM)_i=M_{i-1}$ and $s(dx)=-d(sx)$; here $x\mapsto sx$ denotes the identifications $M_{i-1}\xrightarrow{=}(sM)_i$. A morphism $\phi:(M,d)\to(N,d)$ is a family of linear maps $\phi_i:M_i\to N_i$ commuting with the differentials. It induces $H(\phi):H(M)\to H(N)$. If $H(\phi)$ is an isomorphism ϕ is called a *quasi-isomorphism* and we write $\phi:M\xrightarrow{\simeq}N$. A *chain equivalence* $\phi:(M,d)\longrightarrow(N,d)$ is a morphism such that for a second morphism $\psi:(N,d)\longrightarrow(M,d)$, $\phi\psi-id=hd+dh$ and $\psi\phi-id=h'd+dh'$; here $h:N\to N$ and $h':M\to M$ are linear maps of degree 1. A chain equivalence is clearly a quasi-isomorphism.

We shall occasionally use the convention $M^i=M_{-i}$ to write $M=\{M^i\}_{i\in\mathbb{Z}}$; then $d:M^i\to M^{i+1}$. It will be clear from the context whether 'degree' means 'upper degree' or 'lower degree'; note however that $(-1)^{\deg x}$ is unambiguously defined.

The tensor product of graded modules is defined by

$$(M\otimes N)_n=\bigoplus_{i+j=n}M_i\otimes N_j.$$

If $f:M\to M'$ and $g:N\to N'$ are linear maps of degrees k and ℓ then $f\otimes g:M\otimes N\to M'\otimes N'$ is the linear map of degree $k+\ell$ defined by

$$(f\otimes g)(x\otimes y)=(-1)^{\deg g\cdot\deg x}f(x)\otimes g(y).$$

In particular $(M,d)\otimes(N,d)$ is the graded module $M\otimes N$ with differential $d\otimes 1+1\otimes d$.

A *differential graded algebra*, or DGA, is a graded (associative) algebra $A=\{A_i\}_{i\in\mathbb{Z}}$ with an identity $1\in A_0$, together with a differential in A satisfying $d(xy)=(dx)y+(-1)^{\deg x}x(dy)$. It follows that $\operatorname{Im} d$ is an ideal in the graded subalgebra $\ker d\subset A$. Hence $H(A)$ inherits a graded algebra structure: the *homology algebra* of A. A morphism $\phi:(A,d)\to(B,d)$ of DGA's is a morphism of differential graded modules that preserves products and the identity; thus $H(\phi)$ is a morphism of graded algebras. If $H(\phi)$ is an isomorphism, then ϕ is a *quasi-isomorphism of DGA's*. We identify two important subclasses of DGA's: a *cochain algebra* is a DGA of the form (A, d) with $A = \{A^i\}_{i\geqslant 0}$ and a *chain algebra* is a DGA of the form (A, d) with $A = \{A_i\}_{i\geqslant 0}$.

A *(left) module* over a DGA, (A, d), is a graded differential module (M, d) together with a linear map of degree zero $A \otimes M \to A$, $a \otimes m \mapsto a \cdot m$, such that

$$(aa') \cdot m = a \cdot (a' \cdot m), \quad 1 \cdot m = m$$

and

$$d(a \cdot m) = da \cdot m + (-1)^{\deg a} a \cdot dm.$$

A *morphism of* (A, d)*-modules* is a linear map $\phi : (M, d) \to (N, d)$ such that $\phi(a \cdot m) = a \cdot \phi(m)$.

The elements of the theory of DGA's are reviewed in §2 and §3. In §2 we consider modules over a DGA and their 'homological algebra' as introduced by Eilenberg and Moore ([10], [14], [16]). Our approach is via the semifree resolutions described in [2] and [13]. In §3 we recall the homotopy theory of DGA's.

In this homotopy theory there is a notion of weak equivalence class, which plays the same role as that of weak homotopy type in topological spaces. We define this notion more generally in any category, $\mathcal{C}$, of differential graded modules:

DEFINITION. Two objects A and B in $\mathcal{C}$ are *weakly* $\mathcal{C}$*-equivalent* if they are connected by a chain of $\mathcal{C}$-quasi-isomorphisms of the form

$$A \xleftarrow{\simeq} A(1) \xrightarrow{\simeq} \cdots\cdots \xleftarrow{\simeq} A(n) \xrightarrow{\simeq} B.$$

Note that in the case of DGA's the homology algebra is an invariant of the weak equivalence class. However, it is easy to construct two DGA's that are NOT weakly equivalent, but DO have isomorphic homology algebras. In fact, the weak equivalence class of (A, d) is a much stronger invariant than the algebra $H(A)$.

We turn next to modeling the homotopy theory of topological spaces by that of DGA's. This will convert weak homotopy types to weak equivalence classes, thereby exhibiting weak equivalence classes of DGA's as homotopy invariants of spaces. It will be convenient to use $C_*(X) = C_*(X; \Bbbk)$ to denote the quotient chain complex obtained by dividing the singular chains on X by the degenerate simplices: thus $C_k(X)$ is free on the nondegenerate k-simplices. It is classical (and straightforward) that $H(C_*(X)) = H_*(X)$ and $C_*(pt) = C_0(pt) = \Bbbk$.

In passing from spaces to algebras we need to convert products of spaces to tensor products of chain complexes, and for this we use the standard chain equivalences of Eilenberg–Zilber and Alexander–Whitney:

$$C_*(X) \otimes C_*(Y) \xrightarrow[EZ]{\simeq} C_*(X \times Y) \xrightarrow[AW]{\simeq} C_*(X) \otimes C_*(Y)$$

(cf. [17, Chapter VIII]). These provide two classical functors from topological categories to DGA's. The first functor is the singular cochain algebra $C^*(X)$ dual to $C_*(X)$, with multiplication (cup product) dual to the comultiplication

$$AW \circ C_*(\Delta) : C_*(X) \longrightarrow C_*(X) \otimes C_*(X),$$

where $\Delta : X \to X \times X$ is the diagonal. The second functor is defined for topological monoids G: if $\mu : G \times G \to G$ is the multiplication, then

$$C_*(\mu) \circ EZ : C_*(G) \otimes C_*(G) \longrightarrow C_*(G)$$

makes $C_*(G)$ into a chain algebra. (Note that in each case these assertions require some easily established properties – like associativity – for AW or EZ.)

Clearly both these functors carry weak homotopy equivalences to DGA-quasi-isomorphisms, so that the weak equivalence class of $C^*(X)$ (or $C_*(G)$) depends only on the weak homotopy type of X (or G). We shall see in §3 that they also convert homotopy classes of maps to homotopy classes of DGA morphisms.

Just as the computation of the homology of a fibration is an essential tool in homotopy theory, so it is important to understand how these invariants behave with respect to fibrations. To make this more precise we introduce what we shall call a *G-fibration*. Indeed, for any fibration $\pi : E \to X$ let $E_x = \pi^{-1}(x)$ be the fibre at x. Then we make the

DEFINITION. Let G be a topological monoid. A *G-fibration* consists of a (surjective) fibration $\pi : E \to X$ and a continuous right action $\mu_E : E \times G \to E$ satisfying the following conditions:

(i) $E_x \cdot G \subset E_x$, $x \in X$.
(ii) For each $z \in E$ the map $a \mapsto z \cdot a$ is a weak homotopy equivalence from G to $E_{\pi(z)}$.

One important class of such fibrations is that of the Moore path space fibrations $\pi : PX \to X$ [21, p. 111], of a pointed space (X, x_0). The fibre of π at x_0 is the Moore loop space ΩX, and it is a topological monoid acting from the right on PX. (We recall the definitions in §6.)

This gives a second DGA associated with X: the chain algebra $C_*(\Omega X)$. As we shall see in Theorem 6.3, the weak equivalence class of the DGC $C_*(X)$ can be recovered from that of the DGA $C_*(\Omega X)$ – cf. also [12] and [16]. An important result of Adams [1] asserts that when X is 1-connected the converse is also true, and in fact $C_*(\Omega X)$ is a DGA quasi-isomorphic to the cobar construction on the differential graded coalgebra $C_*(X)$ (cf. §4 for the definition of a differential graded coalgebra).

Now consider a general G-fibration

$$\pi : E \longrightarrow X, \qquad \mu_E : E \times G \longrightarrow E.$$

The principal original contribution of this paper is to show how the weak equivalence class of the differential graded coalgebra $C_*(X)$ can be computed from algebraic data on $C_*(G)$ and $C_*(E)$ reflecting the action μ_E.

Indeed, just as the multiplication in G makes $C_*(G)$ into a DGA, so the action μ_E makes $C_*(E)$ into a $C_*(G)$-module. More is true, however. As noted by Eilenberg and Moore, the differential graded coalgebra structures in $C_*(G)$ and $C_*(E)$ are compatible with these algebraic structures, so that $C_*(G)$ is a *differential graded Hopf algebra* (DGH) and $C_*(E)$ is a *differential graded coalgebra* (DGC) over this DGH. These terms are defined in §4, where we show that for any DGC, (C, d), over a DGH, (K, d), the bar construction $B(C; K)$ has a natural differential graded coalgebra structure. Using this, in §5 we prove our main theorem:

THEOREM 1.1. *In the case of a G-fibration the DGC,* $B(C_*(E); C_*(G))$, *is quasi-isomorphic to* $C_*(X)$.

REMARKS.

1. Theorem 5.1 holds for *any* commutative coefficient ring $\Bbbk$ and for any G-fibration. In particular there are *no* restrictions on the connectivity or the fundamental groups of G, E and X.

2. This theorem strengthens the classical Eilenberg–Moore formula, which identifies

$$H_*(X) = Tor^{C_*(G)}(\Bbbk, C_*(E)),$$

as graded $\Bbbk$-modules.

3. In the case of a compact connected Lie group G and a principal G-bundle $E \to X$, Cartan [7] provides a DGA quasi-isomorphism of the form

$$([S(g^*) \otimes A(E)]^g, D) \xleftarrow{\simeq} A(X),$$

where $A(-)$ denotes the commutative DGA of C^∞-differential forms on a smooth manifold. The DGA $([S(g^*) \otimes A(E)]^g, D)$ now plays a role in theoretical physics, where its cohomology is called the BRST cohomology.

This construction permits one to recover the weak equivalence class of $A(X)$ from the action of g on $A(E)$, and it raises the question of whether there is a topological analogue for G-fibrations. Theorem 5.1 states that the dual DGA, $\mathrm{Hom}(B(C_*(E); C_*(G)), \Bbbk)$, is precisely such an analogue.

When G is a torus, Cartan's DGA has the form

$$\left(S(g^*) \otimes A(E)^g,\ \mathrm{id} \otimes d + \sum \mu_i \otimes \theta_i\right)$$

where θ_i is the substitution operator for the ith fundamental vector field h_i and μ_i is multiplication by the dual element $b_i \in g^*$. As we shall show in §7, when G is a torus $\mathrm{Hom}(B(C_*(E); C_*(G)), \Bbbk)$ is naturally quasi-isomorphic with a DGA of the form

$$\left(S(g^*) \otimes C^*(E),\ \mathrm{id} \otimes d + \sum \mu_i \otimes \theta_i\right)$$

where now θ_i is a degree -1 derivation in $(C^*(E), d)$ arising out of the action of G. Thus in this case the two constructions are essentially identical.

In §6 we give a variety of applications. In §7 we consider DGA's (A, d) over (K, d) and the DGA $B_K(A) = \mathrm{Hom}_K(B(K; K), A)$. In §8 we show how to construct a DGA, $\tilde{B}_K(A)$, starting from any acyclic resolution P of $\Bbbk$ as a (K, d)-module. The construction of $\tilde{B}_K(A)$ is inspired from the A_∞-algebras of Stasheff and ideas in the equivariant DGA homotopy theory of one of the authors and Hess. We end by showing that $\tilde{B}_K(A)$ and $B_K(A)$ are weakly DGA equivalent.

2. Semifree modules

Suppose (A, d) is a DGA and let (M, d) and (N, d) be (A, d)-modules. Then an *A-linear map* (of degree i) is a linear map $f : N \to M$ (of degree i) such that $f(a \cdot n) = (-1)^{i \deg a} a \cdot (f(n))$. These A-linear maps form a graded differential module $(\mathrm{Hom}_A(N, M), d)$ with differential given by $df = d \circ f + (-1)^{\deg f} f \circ d$. Similarly if (Q, d) is a right (A, d)-module then $(Q \otimes_A M, d)$ is the quotient module $(Q, d) \otimes (M, d)/(q \cdot a \otimes m - q \otimes a \cdot m)$.

In studying ordinary modules over ordinary rings R one uses free (or projective) resolutions to compensate for the fact that the functors $- \bigotimes_R -$ and $\mathrm{Hom}_R(-,-)$ are not exact. Analogously, it turns out that the functors $\mathrm{Hom}_A(-,-)$ and $- \bigotimes_A -$ above do not always preserve quasi-isomorphisms. We compensate for this by using the semifree resolutions of [2] and [13].

First we note that a graded module V is *free* if each V_i is a $\Bbbk$-free module; the disjoint union of bases of the V_i is called a *basis for* V. Similarly a graded module M over a graded algebra A is called *A-free* if it has the form $M \cong A \otimes V$ with V a free graded module; in this case a basis of V is called a *basis of the A-module* M.

DEFINITION. (1) An (A, d)-module (P, d) is *a semifree extension of an* (A, d)*-module* (M, d) if it is the union of an increasing family of (A, d)-submodules

$$P(-1) \subset P(0) \subset \cdots,$$

such that $P(-1) = (M, d)$ and each $P(k)/P(k-1)$, $k \geqslant 0$, is A-free on a basis of cycles. If $M = 0$ we say (P, d) is an (A, d)*-semifree module.*

(2) Let $f : (M, d) \longrightarrow (N, d)$ be a morphism of (A, d)-modules. A *semifree resolution* of f is a semifree extension (P, d) of (M, d) together with a quasi-isomorphism of (A, d)-modules $(P, d) \xrightarrow{\simeq} (N, d)$ restricting to f in (M, d).

(3) A *semifree resolution* of an (A, d)-module (N, d) is a semifree resolution of $0 \to (N, d)$.

REMARK. We may consider any differential graded module (M, d) as a module over the DGA, $(\Bbbk, 0)$. If it is a semifree $(\Bbbk, 0)$-module we shall say it is a $\Bbbk$-semifree. If M is $\Bbbk$-free and either $M = \{M_i\}_{i \geqslant 0}$ or else $\Bbbk$ is a principal ideal domain then (M, d) is automatically $\Bbbk$-semifree.

The results in this section are well known, and we sketch the proofs only for the convenience of the reader.

PROPOSITION 2.1. (i) *Any morphism* $f : (M,d) \to (N,d)$ *of* (A,d)*-modules has a semifree resolution* $(Q,d) \xrightarrow{\simeq} (N,d)$. *In particular, every* (A,d)*-module has a semifree resolution.*

(ii) *If* (P,d) *is* (A,d)*-semifree then* $\mathrm{Hom}_A(P,-)$ *preserves quasi-isomorphisms.*

PROOF. (i) Let $V(0)$ and $V(1)$ be the free graded $\Bbbk$-modules whose bases are respectively the cycles of N and the elements of N. Set $d = 0$ in $V(0)$ and extend the differential in N to a linear map $d : V(1) \to V(0)$.

Now set

$$Q(0) = (M,d) \oplus \left[(A,d) \otimes \left(V(0) \oplus V(1), d\right)\right].$$

The obvious maps $V(0)$, $V(1) \to N$, together with f define a morphism $g(0)$: $(Q(0),d) \to (N,d)$. Clearly $H(g(0))$ is surjective. We now construct an increasing sequence of morphisms $g(k) : (Q(k),d) \to (N,d)$. If $g(k-1)$ is defined, let $V(k)$ be the free $\Bbbk$-module whose basis $v_{\alpha i}$ in degree i is in $1-1$ correspondence with the cycles in $[\ker(g(k-1))]_{i-1}$. Set $Q(k) = Q(k-1) \oplus (A \otimes V(k))$, define $dv_{\alpha i}$ to be the element in $[\ker g(k-1)]_{i-1}$ corresponding to $v_{\alpha i}$ and put $g(k)v_{\alpha i} = 0$. Finally, set

$$Q = \bigcup_k Q(k) \quad \text{and} \quad g = \varinjlim_k g(k).$$

(ii) Suppose $\eta : (M,d) \xrightarrow{\simeq} (N,d)$ is a quasi-isomorphism of (A,d)-modules. Assume $f : P \to M$ and $g : P \to N$ are A-linear maps of degrees j and $j+1$ such that $df = 0$ and $\eta \circ f = dg$, We shall construct A-linear maps $\alpha : P \to M$ and $\beta : P \to N$ such that $d\alpha = f$ and $\eta\alpha - g = d\beta$. This implies that $\mathrm{Hom}_A(P,\eta)$ is a quasi-isomorphism (take the case $f = 0$ and g is a cycle to see that $H(\mathrm{Hom}_A(P,\eta))$ is surjective).

The actual construction of α and β is by induction: we assume them constructed in $P(k-1)$ and extend to $P(k)$. Now $P(k) = P(k-1) \oplus (A \otimes V(k))$ with $V(k)$ free on a basis v_i such that $dv_i \in P(k-1)$. Thus we need to find elements $\alpha(v_i) \in M$ and $\beta(v_i) \in N$ such that

$$d(\alpha v_i) = f(v_i) + (-1)^{\deg \alpha} \alpha(dv_i)$$

and

$$d(\beta v_i) = \eta\alpha(v_i) - g(v_i) + (-1)^{\deg \beta} \beta(dv_i).$$

Put $u_i = f(v_i) + (-1)^{\deg \alpha}\alpha(dv_i)$ and $w_i = g(v_i) - (-1)^{\deg \beta}\beta(dv_i)$. Then $du_i = 0$ and $\eta u_i = dw_i$. Since η is a quasi-isomorphism the equations $d(\alpha v_i) = u_i$ and $d(\beta v_i) = \eta\alpha(v_i) - w_i$ have a solution. $\square$

An *equivalence* of (A,d)-modules is a morphism $f : (M,d) \longrightarrow (N,d)$ such that for some second morphism $f' : (N,d) \longrightarrow (M,d)$ there are A-linear maps $h : M \to M$, $h' : N \to N$ of degree 1 satisfying

$$f'f - id = dh + hd \quad \text{and} \quad ff' - id = dh' + h'd. \tag{2.2}$$

It follows from Proposition 2.1 that a quasi-isomorphism between semifree modules is an equivalence.

PROPOSITION 2.3. *Suppose $f : (P,d) \xrightarrow{\simeq} (Q,d)$ is a quasi-isomorphism between (left) semifree (A,d)-modules.*

(i) *If $g : (M,d) \xrightarrow{\simeq} (N,d)$ is a quasi-isomorphism between right (A,d)-modules then*

$$g \otimes f : M \otimes_A P \longrightarrow N \otimes_A Q$$

is a quasi-isomorphism.

(ii) *If $g : (M,d) \xrightarrow{\simeq} (N,d)$ is a quasi-isomorphism between left (A,d)-modules then*

$$\mathrm{Hom}(f,g) : \mathrm{Hom}_A(Q,M) \longrightarrow \mathrm{Hom}_A(P,N)$$

is a quasi-isomorphism.

PROOF. (i) Use $- \otimes -$ to denote $- \otimes_A -$, and write $g \otimes f = (id_N \otimes f) \circ (g \otimes id_P)$. Since P, Q are semifree, f is an equivalence (Proposition 2.1). Let f', h, h' be as in (2.2). Since these are A-linear we can form $id_N \otimes f'$, $id_N \otimes h$, $id_N \otimes h'$ and these exhibit $id_N \otimes f$ as an equivalence. Hence it is a quasi-isomorphism.

Write $P = \bigcup_k P(k)$ as in the definition of a semifree module. Then $g \otimes \mathrm{id}_P : M \otimes_A P(k) \to N \otimes_A P(k)$. Consider the quotient map

$$M \otimes_A P(k)/P(k-1) \longrightarrow N \otimes_A P(k)/P(k-1).$$

Since $P(k)/P(k-1) = \bigoplus_\alpha [(A,d) \otimes \mathbb{k}v_\alpha]$ this map can be identified as

$$\bigoplus_\alpha \left[(M,d) \otimes \mathbb{k}v_\alpha \xrightarrow{g\otimes\mathrm{id}} (N,d) \otimes \mathbb{k}v_\alpha\right].$$

Thus each quotient map is a quasi-isomorphism, and hence so is $g \otimes \mathrm{id}_P$.

(ii) Write $\mathrm{Hom}(f,g) = \mathrm{Hom}(f, \mathrm{id}_N) \circ \mathrm{Hom}(\mathrm{id}_Q, g)$. It follows exactly as in (i) that since f is an equivalence so is $\mathrm{Hom}(f, \mathrm{id}_N)$. Proposition 2.1 asserts that $\mathrm{Hom}(\mathrm{id}_Q, g)$ is a quasi-isomorphism. □

We shall need the following extension of Proposition 2.3 in which:

(i) $\phi : (B,d) \to (A,d)$ is a DGA morphism;

(ii) $f : (P, d) \to (Q, d)$ is a morphism from a (B, d)-semifree module to an (A, d)-semifree module satisfying $f(b \cdot x) = \phi(b) \cdot f(x)$;
(iii) $g : (M, d) \to (N, d)$ is a morphism from an (A, d)-module to a (B, d)-module satisfying $g(\phi(b) \cdot y) = b \cdot g(y)$, and
(iv) $h : (S, d) \to (T, d)$ is a morphism from a (B, d)-module to (A, d)-module satisfying $h(b \cdot x) = \phi(b) \cdot h(x)$.

PROPOSITION 2.4. *With the notation above, if ϕ, f, g, h are all quasi-isomorphisms so are*

$$\mathrm{Hom}_\phi(f, g) : \mathrm{Hom}_A(Q, M) \longrightarrow \mathrm{Hom}_B(P, N), \quad \alpha \mapsto g \circ \alpha \circ f,$$

and

$$h \otimes_\phi f : S \otimes_B P \longrightarrow T \otimes_A Q.$$

PROOF. Put $P' = A \otimes_B P$. It is clear that (P', d) is a semifree (A, d)-module, and that f factors as the composite

$$B \otimes_B P \xrightarrow{\phi \otimes_B \mathrm{id}} A \otimes_B P \xrightarrow{f'} Q,$$

with $f'(a \otimes x) = a \cdot f(x)$. Proposition 2.3(i) asserts that $\phi \otimes \mathrm{id}$ is a quasi-isomorphism, because ϕ is and P is semifree. Hence f' is a quasi-isomorphism.

Now write

$$\mathrm{Hom}_\phi(f, g) = \mathrm{Hom}_B(\mathrm{id}_P, g) \circ \mathrm{Hom}_\phi(\phi \otimes \mathrm{id}_P, \mathrm{id}_M) \circ \mathrm{Hom}_A(f', \mathrm{id}_M).$$

Note that $\mathrm{Hom}_\phi(\phi \otimes \mathrm{id}_P, \mathrm{id}_M) : \mathrm{Hom}_B(P, M) \to \mathrm{Hom}_A(P', M)$ is an isomorphism, while $\mathrm{Hom}_B(\mathrm{id}_P, g)$ and $\mathrm{Hom}_A(f', \mathrm{id}_M)$ are quasi-isomorphisms by Proposition 2.3(ii).

On the other hand, $h \otimes_\phi f$ is the composite

$$S \otimes_B P \xrightarrow{h \otimes_B \mathrm{id}} T \otimes_B P = T \otimes_A P' \xrightarrow{\mathrm{id} \otimes_A f'} T \otimes_A Q.$$

Since $h \otimes_B \mathrm{id}$ and $\mathrm{id} \otimes_A f'$ are quasi-isomorphisms (Proposition 2.3(i)) so is $h \otimes_\phi f$. $\square$

3. Homotopy theory of DGA's

The homotopy theory of DGA's is a 'nonlinear' analogue of the 'linear homotopy theory' of modules over a DGA described in §2. We shall sketch it here; for details and proofs the reader is referred to [5] and [3, I§7]. These references limit themselves to chain algebras and coalgebras, but the arguments and ideas for general DGA's are identical.

It is straightforward but tedious to construct coproducts

$$A \xrightarrow{\alpha} A \amalg B \xleftarrow{\beta} B$$

in the category of graded algebras. However, when B is a tensor algebra TV,

$$A \amalg TV = \bigoplus_{k=0}^{\infty} A \otimes (V \otimes A)^{\otimes k}$$

with the obvious multiplication. If (A, d) and (B, d) are DGA's then there is a unique DGA of the form $(A \amalg B, d)$ such that α and β are DGA morphisms, and $(A \amalg B, d)$ is then the DGA coproduct of (A, d) and (B, d).

In §2 we considered semifree modules over a DGA. The nonlinear analogue for DGA's is equally important:

DEFINITION. A *free extension* is a DGA morphism of the form

$$(A, d) \xrightarrow{i} (A \amalg TV, d)$$

in which

(i) i is the obvious inclusion.
(ii) V can be written as the union $V = \bigcup_{k=0}^{\infty} V(k)$ of an increasing family $V(0) \subset V(1) \subset \cdots$ of graded submodules such that $V(0)$ and each $V(k)/V(k-1)$ are $\Bbbk$-free.
(iii) $d : V(0) \to A$ and $d : V(k) \to A \amalg TV(k-1)$, $k \geqslant 1$.

Essentially the same proof as that of Proposition 2.1 (i) gives

PROPOSITION 3.1. *Any DGA morphism* $(A, d) \longrightarrow (B, d)$ *factors as*

$$(A, d) \xrightarrow{i} (A \amalg TV, d) \xrightarrow[\simeq]{m} (B, d)$$

in which i is a free extension and m is a surjective DGA quasi-isomorphism.

Applying Proposition 3.1 to the morphism $\Bbbk \to \Bbbk \cdot 1 \subset (A, d)$ gives a quasi-isomorphism

$$m : (TV, d) \xrightarrow{\simeq} (A, d).$$

Any such quasi-isomorphism is called a *free model* for (A, d).

Now consider a commutative diagram of DGA morphisms of the form

$$\begin{tikzcd}
(A, d) \arrow[r, "\alpha"] \arrow[d, "i"'] & (B, d) \arrow[d, "\eta", "\simeq"'] \\
(A \amalg TV, d) \arrow[r, "\phi"'] & (E, d)
\end{tikzcd}$$

in which η is a surjective quasi-isomorphism and i is the inclusion of a free extension. Write $V(k) = V(k-1) \oplus W(k)$, with $W(k)$ $\Bbbk$-free. Then

$$A \amalg TV(k) = \left[A \amalg T\big(V(k-1)\big)\right] \amalg TW(k).$$

Thus an obvious modification of the proof of Proposition 2.1(ii) gives

PROPOSITION 3.2. *There is a DGA morphism* $\psi : (A \amalg TV, d) \longrightarrow (B, d)$ *such that* $\eta\psi = \phi$ *and* $\psi i = \alpha$.

It is now possible to describe homotopy theory in the category of DGA's. Given a DGA (A, d) let (A', d) and (A'', d) be two isomorphic copies and denote by $j' : a \mapsto a'$, $j'' : a \mapsto a''$ the obvious morphisms of (A, d) into

$$(A' \amalg A'', d) = (A', d) \amalg (A'', d).$$

By Proposition 3.1 the morphism

$$(\mathrm{id}, \mathrm{id}) : (A' \amalg A'', d) \to (A, d)$$

factors as a free extension followed by a quasi-isomorphism:

$$(A' \amalg A'', d) \longrightarrow (A' \amalg A'' \amalg TV, d) \xrightarrow[\simeq]{\rho} (A, d).$$

Any free extension $(A' \amalg A'', d) \longrightarrow (A' \amalg A'' \amalg TV, d)$ obtained this way is called a *cylinder object* for (A, d).

DEFINITION. Two DGA morphisms $\phi', \phi'' : (A, d) \longrightarrow (B, d)$ are *homotopic* ($\phi' \sim \phi''$) if for some cylinder object for (A, d) the morphism

$$(\phi', \phi'') : (A' \amalg A'', d) \longrightarrow (B, d)$$

extends to a morphism

$$\Phi : (A' \amalg A'' \amalg TV, d) \longrightarrow (B, d).$$

In this case Φ is called a *DGA homotopy* from ϕ' to ϕ''. Note that if $\phi' \sim \phi''$ and $\psi : (B, d) \to (C, d)$ is any DGA morphism then obviously $\psi\phi' \sim \psi\phi''$.

LEMMA 3.3. *Suppose given DGA morphisms*

$$(A, d) \underset{\phi_1}{\overset{\phi_0}{\rightrightarrows}} (B, d) \xrightarrow{\psi} (A, d)$$

such that ψ *is a quasi-isomorphism and* $\psi\phi_0 = id = \psi\phi_1$. *Then* $\phi_0 \sim \phi_1$.

PROOF. As in Proposition 3.1, factor

$$(\phi_0, \phi_1) : (A', d) \amalg (A'', d) \to (B, d)$$

in the form

$$(A', d) \amalg (A'', d) \xrightarrow{i} (A' \amalg A'' \amalg TV, d) \xrightarrow[\simeq]{m} (B, d),$$

with i a free extension. Then ψm restricts to (id, id) in $A' \amalg A''$. Thus i is a cylinder object for (A, d) and $\psi \circ m$ is a DGA homotopy from ϕ_0 to ϕ_1. □

PROPOSITION 3.4. *Fix DGA morphisms $\phi', \phi'', \phi''' : (A, d) \to (E, d)$.*

(i) *If $\phi' \sim \phi''$ with respect to some cylinder object then they are DGA homotopic with respect to every cylinder object.*
(ii) *If the inclusion $\Bbbk \to A$ is a $\Bbbk$-semifree extension then DGA-homotopy is an equivalence relation on morphisms $(A, d) \to (E, d)$.*
(iii) *If $\phi' \sim \phi''$ then $H(\phi') = H(\phi'')$.*

PROOF. (i) Proposition 3.2 implies that any two cylinder objects for (A, d) are connected by a quasi-isomorphism extending the identity in $A' \amalg A''$. This gives (i).

(ii) Reflexivity and symmetry are obvious. Let

$$\Phi : (A' \amalg A'' \amalg TV, d) \to E$$

be a DGA homotopy from ϕ' to ϕ'' and let

$$\Psi : (A'' \amalg A''' \amalg TW, d) \to E$$

be a DGA homotopy from ϕ'' to ϕ'''. The morphisms Φ and Ψ define in an obvious way a morphism of the form

$$(\Phi, \Psi) : (A' \amalg A'' \amalg A''' \amalg TV \amalg TW, d) \longrightarrow (E, d).$$

Denote this simply by $(\Phi, \Psi) : (B, d) \to (E, d)$.

Now $(A' \amalg A'' \amalg TV, d)$ is a cylinder object, and so there is a quasi-isomorphism

$$\rho : (A' \amalg A'' \amalg TV, d) \longrightarrow (A'', d)$$

such that $\rho a' = a''$ and $\rho a'' = a''$. Since $(A''', d) \cong (A, d)$ is a $\Bbbk$-semifree extension of $\Bbbk$, it is easy to see that there is a filtration by graded differential submodules of the form $\Bbbk \subset A'''(1) \subset \cdots$ with each $A'''(k)/A'''(k-1)$ free on a basis of cycles. Using this a straightforward argument shows first that

$$(\rho, \mathrm{id}) : (A' \amalg A'' \amalg TV \amalg A''', d) \xrightarrow{\simeq} (A'' \amalg A''', d)$$

and then that

$$(\rho, \mathrm{id}, \mathrm{id}) : (A' \amalg A'' \amalg TV \amalg A''' \amalg TW, d) \xrightarrow{\simeq} (A'' \amalg A''' \amalg TW, d) \xrightarrow{\simeq} (A, d).$$

It follows from Lemma 3.3 that the inclusions $i' : A = A' \subset B$ and $i''' : A = A''' \subset B$ are homotopic. Hence $(\Phi, \Psi) \circ i' \sim (\Phi, \Psi) \circ i'''$; i.e. $\phi' \sim \phi'''$.

(iii) This is obvious, since $H(j') = H(\rho)^{-1} = H(j'')$, and so $H(\phi') = H(\Phi) \circ H(j') = H(\phi'')$. □

In the special case of a free model (TV, d) there is a particularly useful cylinder object, introduced by Baues and Lemaire in [4] using the suspension sV defined in the introduction. This is the DGA $(TV' \amalg TV'' \amalg TsV, d)$ where d is defined as follows: Extend s to a map

$$S : TV \longrightarrow TV' \amalg TV'' \amalg TsV$$

by the conditions

$$Sv = sv, \quad v \in V,$$
$$S(xy) = Sx \cdot y'' + (-1)^{\deg x} x' \cdot Sy, \quad x, y \in TV.$$

Then d is defined by the formulae:

$$dx' = (dx)', \quad dx'' = (dx)'' \quad \text{and} \quad dsv = v'' - v' - Sdv.$$

A straightforward calculation (cf. [4]) shows that a quasi-isomorphism $TV' \amalg TV'' \amalg TsV \to TV$ is given by $v' \mapsto v$, $v'' \mapsto v$ and $sv \mapsto 0$. Thus this is indeed a cylinder object: it permits a useful, second description of DGA homotopy.

First we note that given graded algebra morphisms $\phi', \phi'' : A \longrightarrow B$, a (ϕ', ϕ'')-*derivation* of degree r is a degree r linear map $F : A \longrightarrow B$ satisfying

$$F(x \cdot y) = F(x) \cdot \phi''(y) + (-1)^{r \cdot \deg x} \phi'(x) \cdot F(y).$$

If (A, d) and (B, d) are DGA's and ϕ' and ϕ'' are DGA morphisms then the (ϕ', ϕ'')-derivations form a sub differential graded module of $\mathrm{Hom}((A, d), (B, d))$. Clearly $\phi' - \phi''$ is a (ϕ', ϕ'')-derivation that is a cycle. The next proposition asserts that if (A, d) is a free model then $\phi' - \phi''$ is a boundary if and only $\phi' \sim \phi''$:

PROPOSITION 3.5. *Let $\phi', \phi'' : (TV, d) \to (B, d)$ be DGA morphisms from a free model. Then $\phi' \sim \phi''$ if and only if there is a (ϕ', ϕ'')-derivation F of degree* 1 *such that*

$$\phi' - \phi'' = Fd + dF.$$

PROOF. The derivation F and the homotopy

$$\Phi : (TV' \amalg TV'' \amalg TsV, d) \longrightarrow (B, d)$$

determine each other by the rule $Fv = -\Phi sv,\ v \in V$. □

The interpretation of homotopy in terms of derivations is useful with regards to lifting properties up to homotopy. Consider a homotopy commutative diagram of the form

$$\begin{tikzcd} (TW,d) \arrow[r, "\alpha"] \arrow[d, "i"] & (A,d) \arrow[d, "\eta", "\simeq"'] \\ (TW \amalg TV, d) \arrow[r, "\phi"] & (B,d) \end{tikzcd},$$

where η is a quasi-isomorphism, (TW,d) is a free model and i is a free extension.

LEMMA 3.6. *α extends to a morphism $\beta : (TW \amalg TV, d) \longrightarrow (A,d)$ such that $\eta\beta \sim \phi$.*

PROOF. Denote by $\gamma : TW \longrightarrow B$ an $(\eta\alpha, \phi i)$-derivation such that $d\gamma + \gamma d = \eta\alpha - \phi i$; it exists by Proposition 3.5. Let $V(0) \subset \cdots \subset V(n) \subset \cdots$ exhibit i as a free extension.

We define β and an $(\eta\beta, \phi)$-derivation γ' extending γ by induction in each $V(n)$. Assume β and γ' have been constructed in $V(n-1)$. Let v be a basis element of $V(n)/V(n-1)$, then $\beta(dv)$ and $\gamma'(dv)$ are already defined and

$$\eta\beta(dv) = d(\phi v + \gamma' dv).$$

Since η is a quasi-isomorphism we can find $x \in A$ and $y \in B$ such that $dx = \beta dv$ and $\eta x = \phi v + \gamma' dv + dy$.

Extend β and γ' to $TW \amalg TV(n)$ by setting

$$\beta(v) = x \quad \text{and} \quad \gamma'(v) = y.$$

The maps $\eta\beta - \phi$ and $d\gamma' + \gamma' d$ are $(\eta\beta, \phi)$-derivations. Since they agree on $V(n)$, they coincide on $TW \amalg TV(n)$. This extends α and γ to all of $TW \amalg TV$ and a second application of Proposition 3.5 shows that $\eta\beta \sim \phi$. □

THEOREM 3.7. *Let $\eta : (A,d) \longrightarrow (B,d)$ be a DGA quasi-isomorphism and let $(T(V),d)$ be a free model. Then η induces a bijection between homotopy classes of morphisms*

$$\eta_* : \big[(TV,d),(A,d)\big] \longrightarrow \big[(T(V),d),(B,d)\big].$$

PROOF. Lemma 3.6, applied with $W = 0$ and $U = V$, shows that η_* is surjective.

Now, if $f', f'' : (TV,d) \longrightarrow (A,d)$ are DGA morphisms such that $\eta f' \sim \eta f''$, then Lemma 3.6 (applied with i the inclusion

$$TV' \amalg TV'' \to TV' \amalg TV'' \amalg TsV, \quad \alpha = f \amalg f',$$

and ϕ the homotopy between $\eta f'$ and $\eta f''$) shows that $f' \sim f''$. □

Let $\mathcal{T}$ denote the category of topological spaces and $\mathcal{T}_*$ the category of topological spaces pointed by a nondegenerate basepoint (i.e. the inclusion of the base point is a cofibration). Recall from the introduction the functors $C^* : \mathcal{T} \rightsquigarrow \mathrm{DGA}$ and $C_*\Omega : \mathcal{T}_* \to \mathrm{DGA}$ assigning to a space X the singular cochain algebra $C^*(X)$ and the chain algebra $C_*(\Omega X)$.

THEOREM 3.8. *The functors C^* and $C_*\Omega$ convert weak homotopy equivalences to quasi-isomorphisms and homotopy classes of maps to homotopy classes of DGA morphisms.*

PROOF. The first assertion simply restates that a weak homotopy equivalence induces an isomorphism of singular homology. For the second, consider first $C_*\Omega$ and let $H : (X, x_0) \times I \to (Y, y_0)$ be a based homotopy between maps $f, g : (X, x_0) \to (Y, y_0)$.

Let $i_0, i_1 : (X, x_0) \to (X \times I/x_0 \times I, [x_0 \times I])$ be the inclusions at 0 and 1, and $p : X \times I/x_0 \times I \to X$ be the projection on the first factor. Since $p_\circ i_0 = \mathrm{id} = p_\circ i_1$ and p is a weak homotopy equivalence Lemma 3.3 asserts that the DGA morphisms $C_*(\Omega i_0)$ and $C_*(\Omega i_1)$ are homotopic. The DGA morphisms $C_*(\Omega f)$ and $C_*(\Omega g)$ are therefore homotopic because $f = Hi_0$ and $g = Hi_1$.

The corresponding property for C^* follows in the same way. Let $H : X \times [0,1] \to Y$ be a homotopy between maps $f, g : X \to Y$. Denote by $k : X \to Y^{[0,1]}$ the map deduced from H by adjunction, by $p_0, p_1 : Y^{[0,1]} \to Y$ the evaluation maps at 0 and 1 and by $s : Y \to Y^{[0,1]}$ the map sending a point y to the constant path at y. Then $p_0 s = \mathrm{id}_Y = p_1 s$ and $C^*(s)$ is a quasi-isomorphism. Thus by Lemma 3.3, the DGA morphisms $C^*(p_0)$ and $C^*(p_1)$ are homotopic. The DGA morphisms $C^*(f)$ and $C^*(g)$ are therefore homotopic because $C^*(f) = C^*(k) \circ C^*(p_0)$ and $C^*(g) = C^*(k) \circ C^*(p_1)$. □

4. Differential graded Hopf algebras

Suppose a topological monoid G acts continuously from the right on a space E. Multiplication in C makes $C_*(G)$ into an augmented DGA and the action in E makes $C_*(E)$ into a $C_*(G)$-module. Now the classical bar construction [17] (reviewed below) associates a different graded module $B(M; A)$ with any right module (M, d) over any augmented DGA, (A, d). In the case of G-action we can form $B(C_*(E); C_*(G))$.

As pointed out in the introduction, however, in this 'topological situation' we have additional structure: $C_*(G)$ is a differential graded Hopf algebra and $C_*(E)$ is a differential graded coalgebra over $C_*(G)$. We shall define these terms below and prove

THEOREM 4.1. *If (C, d) is a differential graded coalgebra over the differential graded Hopf algebra (K, d) there is a natural comultiplication in $B(C; K)$ which makes this into a differential graded coalgebra.*

We first review the bar construction on a DGA. Then we define differential graded coalgebras and differential graded Hopf algebras and recall (Example 4.4) why $C_*(E)$ is a coalgebra over the Hopf algebra $C_*(G)$. Finally, we prove Theorem 4.1.

An *augmented DGA* is a DGA, (A, d) equipped with a morphism $\varepsilon_A : (A, d) \to \Bbbk$ (the *augmentation*). The ideal $\overline{A} = \ker \varepsilon_A$ is the *augmentation ideal*.

Denote $(s\overline{A})^{\otimes k}$ by $T^k(s\overline{A})$ and denote the tensor product of elements $sa_i \in s\overline{A}$ by $[sa_1|\cdots|sa_k] \in T^k(s\overline{A})$.

The *acyclic bar construction* on the augmented DGA (A,d) is then the left (A,d)-module $B(A;A) = (A \otimes T(s\overline{A}), D)$ in which

(i) $T(s\overline{A}) = \bigoplus_{k=0}^{\infty} T^k(s\overline{A})$.

(ii) $D = D_1 + D_2$, with D_2 the A-linear map defined by

$$D_2[sa_1|\cdots|sa_k] = a_1[sa_2|\cdots|sa_k] + \sum_{i=2}^{k}(-1)^{\varepsilon_i}[sa_1|\cdots|sa_{i-1}a_i|\cdots|sa_k],$$

and

$$\begin{aligned} D_1 a[sa_1|\cdots|sa_k] &= (da)[sa_1|\cdots|sa_k] \\ &\quad + \sum_{i=1}^{k}(-1)^{\varepsilon_i+\deg a} a[sa_1|\cdots|sda_i|\cdots|sa_k]. \end{aligned}$$

(Here $\varepsilon_i = \sum_{j<i} \deg sa_j$.)

The acyclic bar construction has an augmentation

$$\varepsilon : B(A;A) \longrightarrow \Bbbk \tag{4.2}$$

defined by $\varepsilon(a \otimes 1) = \varepsilon_A(a)$ and

$$\varepsilon\big(A \otimes T^k(s\overline{A})\big) = 0, \quad k \geqslant 1.$$

More generally, if (M,d) is any right (A,d)-module, the differential graded module

$$B(M;A) = (M,d) \otimes_A B(A;A) = \big(M \otimes T(s\overline{A}), D\big)$$

is called the *bar construction with coefficients in* (M,d).

Suppose $\phi : (A,d) \to (A',d)$ is a morphism of augmented DGA's and $\psi : (M,d) \to (M',d)$ is a map of differential graded modules. Assume (M,d) and (M',d) are respectively a right (A,d)-module and a right (A',d)-module and that

$$\psi(ma) = \psi(m)\phi(a), \quad m \in M,\ a \in A.$$

Use ϕ also to denote the map $sa \mapsto s\phi a$ from $s\overline{A}$ to $s\overline{A}'$. Then

$$B(\psi;\phi) = \psi \otimes \left(\bigoplus_{k=0}^{\infty} \otimes^k \phi\right) : B(M;A) \to B(M';A')$$

is a morphism of differential graded modules.

LEMMA 4.3. *With the notation above,*

(i) ε *is a quasi-isomorphism.*

(ii) *If $(\overline{A}, d)$ is $\Bbbk$-semifree then $B(A;A)$ is (A,d)-semifree.*

(iii) *If ϕ and ψ are quasi-isomorphisms, and if $(\overline{A}, d)$ and $(\overline{A}', d)$ are $\Bbbk$-semifree, then $B(\psi;\phi)$ is a quasi-isomorphism.*

PROOF. (i) is straightforward (cf. [12, Proposition 2.4]). For (ii) we need only show that $(B(A;A), D)$ is (A,d)-semifree. Since $\overline{A}$ is $\Bbbk$-semifree so is $s(\overline{A})$ with differential $dsa = -sda$. It follows that $(s\overline{A}, d)^{\otimes k}$ is $\Bbbk$-semifree and so $(A,d) \otimes (s\overline{A}, d)^{\otimes k}$ is (A,d)-semifree. This identifies the quotients

$$\left(A \otimes T^{\leqslant k}(s\overline{A}), D\right) / \left(A \otimes T^{<k}(s\overline{A}), D\right)$$

as (A,d)-semifree – whence the desired conclusion.

Finally, to prove (iii), we note (by (i) and (ii)) that

$$B(\phi;\phi) : B(A;A) \to B(A';A')$$

is a quasi-isomorphism over ϕ between semi-free modules. It follows (Proposition 2.4) that $B(\psi;\phi) = \psi \otimes_\phi B(\phi;\phi)$ is a quasi-isomorphism. □

We turn next to the definition of a differential graded coalgebra (DGC) and a differential graded Hopf algebra (DGH). A *differential graded coalgebra* is a differential graded module (C,d) equipped with two morphisms, the *comultiplication* $\Delta : (C,d) \to (C,d) \otimes (C,d)$ and the *augmentation* $\varepsilon : (C,d) \to \Bbbk$. These are required to satisfy $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ (associativity) and $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$.

For the definition of Hopf algebras we need to recall that the tensor product of DGA's (A,d) and (A',d) is the DGA $(A,d) \otimes (A'd)$ with multiplication given by

$$(a \otimes a')(b \otimes b') = (-1)^{\deg a' \cdot \deg b} ab \otimes a'b'.$$

Similarly the tensor product of DGC's (C,d) and (C',d) is the DGC $(C,d) \otimes (C',d)$ with comultiplication $(\mathrm{id} \otimes \omega \otimes \mathrm{id}) \circ (\Delta \otimes \Delta')$, where

$$\omega(x \otimes x') = (-1)^{\deg x \cdot \deg x'} x' \otimes x.$$

A *differential graded Hopf algebra* is a DGA (K,d) together with DGA morphisms $\Delta : (K,d) \to (K,d) \otimes (K,d)$ and $\varepsilon : (K,d) \to \Bbbk$ that make (K,d) into a DGC (i.e. it is a coalgebra in the category of DGA's). If (M,d) and (N,d) are (K,d)-modules then the comultiplication in K makes $(M,d) \otimes (N,d)$ into a (K,d)-module as well:

$$g \cdot (m \otimes n) = \Delta g \cdot (m \otimes n), \quad g \in K,$$

where

$$(g_1 \otimes g_2) \cdot (m \otimes n) = (-1)^{\deg g_2 \cdot \deg m} g_1 \cdot m \otimes g_2 \cdot n.$$

We say K *acts diagonally* in $M \otimes N$.

DEFINITION. Let (K,d) be a DGH. A (K,d)*-algebra* is a DGA, (A,d), with a (K,d)-module structure such that multiplication $A \otimes A \to A$ and the map $\Bbbk \to \Bbbk . 1 \subset A$ are morphisms of (K,d)-modules.

A (K,d)*-coalgebra* is a DGC with a (K,d)-module structure for which the comultiplication and augmentation are morphisms of (K,d)-modules. Thus a coalgebra (C,d) with a (K,d)-module structure is a (K,d)-coalgebra if and only if the action, $C \otimes K \to C$, of K is a DGC morphism.

EXAMPLE 4.4. For any topological space X, $C_*(X)$ is a DGC with comultiplication $AW \circ C_*(\Delta) : C_*(X) \longrightarrow C_*(X) \otimes C_*(X)$ and augmentation C_* (constant map): $C_*(X) \longrightarrow C_*(pt) = \Bbbk$. Moreover it follows from [9, §17] that the Eilenberg–Zilber map $EZ : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$ is a morphism of DGC's.

Now suppose a topological monoid, G, acts from the right on a space E. The multiplication $\mu : G \times G \to G$ defines a multiplication

$$C_*(\mu) \circ EZ : C_*(G) \otimes C_*(G) \longrightarrow C_*(G).$$

Since this is also a morphism of DGC's (because EZ is), this multiplication makes $C_*(G)$ into a differential graded Hopf algebra. Similarly, the action $\mu_E : E \times G \to E$ induces

$$C_*(\mu_E) \circ EZ : C_*(E) \otimes C_*(G) \longrightarrow C_*(E),$$

which makes $C_*(E)$ into a right DGC over the DGH, $C_*(G)$.

Note that a DGH (K,d) is, in particular, an augmented DGA. Thus for any (K,d)-module we can form the bar constructions $B(M;K)$. We are ready to prove our main theorem:

PROOF OF 4.1. We begin by considering the acyclic bar construction $B(K;K)$. Recall that a permutation $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ of elements in a graded module is assigned the sign $\varepsilon(\sigma, x_1, \dots, x_n) = (-1)^k$, where $k = \sum \deg x_{\sigma(j)} \deg x_i$ and the sum is over all (j,i) with $\sigma(j) > i$. In particular, we can define linear maps of degree zero,

$$\begin{aligned}(s\overline{K} \otimes K)^{\otimes k} \otimes B(K;K) &\longrightarrow T^k(s\overline{K}) \otimes B(K;K),\\ \Phi_1 \otimes \cdots \otimes \Phi_k \otimes \Psi &\mapsto \Phi_1 \bullet \cdots \bullet \Phi_k \bullet \Psi,\end{aligned}$$

by

$$(sx_1 \otimes y_1) \bullet \cdots \bullet (sx_k \otimes y_k) \bullet \Psi = \pm [sx_1 | \cdots | sx_k] \otimes y_1 \cdot \cdots \cdot y_k \Psi,$$

where $\pm$ is the sign of the obvious permutation of the elements sx_i, y_i.

Next note that the comultiplication Δ_K in K determines a linear map

$$\alpha : s\overline{K} \longrightarrow s\overline{K} \otimes K$$

as follows: if $x \in \overline{K}$ then $\Delta'_K x = \Delta_K x - 1 \otimes x \in \overline{K} \otimes K$. Put $\alpha(sx) = (s \otimes \mathrm{id})(\Delta'_K x)$, and define Δ in $T(s\overline{K})$ by

$$\Delta(1) = 1 \otimes 1,$$

and

$$\begin{aligned}\Delta[sx_1|\cdots|sx_k] = 1 \otimes [sx_1|\cdots|sx_k] \\ + \sum_{r=1}^{k-1} \alpha(sx_1) \bullet \cdots \bullet \alpha(sx_r) \bullet [sx_{r+1}|\cdots|sx_k] \\ + \alpha(sx_1) \bullet \cdots \bullet \alpha(sx_k) \bullet 1.\end{aligned}$$

Thus if $\Delta'_K x_i = \sum x_{ij} \otimes x'_{ij}$ then

$$\Delta[sx_1|\cdots|sx_k] = \sum_{r=0}^{k} \sum_j \pm [sx_{ij}|\cdots|sx_{rj}] \otimes x'_{ij} \cdots x'_{rj}[sx_{r+1}|\cdots|sx_k].$$

Finally, extend Δ to $K \otimes T(s\overline{K})$ by the requirement that $\Delta(a\Phi) = \Delta_K a \Delta\Phi$, $a \in K$, $\Phi \in T(s\overline{K})$.

The associativity of Δ_K, together with the fact that Δ_K is an algebra morphism, implies (after a tedious computation) that Δ is associative. The augmentation ε (4.2) is K-linear and clearly satisfies $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$. Thus $(B(K;K), \Delta, \varepsilon)$ is a graded coalgebra over K.

Since the differential in K is compatible with both multiplication and comultiplication, a short calculation shows that $\Delta \circ D_1 = (D_1 \otimes \mathrm{id} + \mathrm{id} \otimes D_1) \circ \Delta$. A long tedious calculation shows that $\Delta \circ D_2 = (D_2 \otimes \mathrm{id} + \mathrm{id} \otimes D_2) \circ \Delta$. Thus $(B(K;K), D)$ is a DGC over the DGH, (K, d).

Now suppose (C, d) is any right DGC over the DGH, (K, d). Then we have the tensor product DGC, $(C, d) \otimes B(K;K)$. Because C and $B(K;K)$ are both coalgebras over K, the comultiplication and augmentation in $C \otimes B(K;K)$ factor over the surjection

$$C \otimes B(K;K) \longrightarrow C \otimes_K B(K;K) = B(C;K)$$

to define the desired graded coalgebra structure in $B(C;K)$. □

Suppose next that

$$\phi : (K, d) \longrightarrow (K', d) \quad \text{and} \quad \psi : (C, d) \longrightarrow (C', d)$$

are respectively a morphism of DGH's and a morphism of DGC's. Assume (C, d) and (C', d) are respectively right DGC's over (K, d) and (K', d), and that

$$\psi(c \cdot x) = \psi(c) \cdot \phi x, \quad c \in C,\ x \in K.$$

PROPOSITION 4.5. *With the hypotheses above,*

(i) $B(\psi;\phi)$ *is a morphism of DGC's.*
(ii) *If* $(\overline{K},d)$ *is* $\mathbb{k}$*-semifree and* ψ *and* ϕ *are quasi-isomorphisms, so is* $B(\psi,\phi)$.

PROOF. (i) is immediate from the construction of Theorem 4.1, and (ii) follows from Lemma 4.3. □

EXAMPLE 4.6. Let (A,d) be an augmented DGA. The augmentation makes $\mathbb{k}$ into an (A,d)-module, and so we can form $B(\mathbb{k};A) = (T(s\overline{A}),D)$. This is naturally a DGC, with comultiplication

$$\Delta : [sa_1|\cdots|sa_k] \longrightarrow \sum_{i=0}^{k}[sa_1|\cdots|sa_i]\otimes[sa_{i+1}|\cdots|sa_k].$$

This DGC is called the (reduced) *bar construction* on (A,d_A) and is denoted by BA.

In the case of a DGH, (K,d), $\mathbb{k}$ is a differential graded coalgebra over (K,d). Thus Theorem 4.1 appears to provide a second DGC structure in $T(s\overline{K})$; however it is immediate from the definitions that these two structures coincide.

5. The base of a G-fibration

Fix a G-fibration

$$\pi : E \longrightarrow X, \quad \mu_E : E\times G \longrightarrow E.$$

As observed in Example 4.4, $C_*(X)$ is then a DGC, $C_*(G)$ is a DGH and $C_*(E)$ is a DGC over the DGH $C_*(G)$. In particular we may form the differential graded coalgebra $B(C_*(E);C_*(G))$ of Theorem 4.1. Here we establish

THEOREM 5.1. *For any G-fibration as above there is a natural quasi-isomorphism*

$$B\big(C_*(E);C_*(G)\big) \xrightarrow{\simeq} C_*(X)$$

of differential graded coalgebras.

We begin with an observation and a standard proposition.

LEMMA 5.2. $C_*(\pi)$ *factors as*

$$C_*(E) \longrightarrow C_*(E)\otimes_{C_*(G)}\mathbb{k} \xrightarrow{\overline{C_*(\pi)}} C_*(X).$$

PROOF. This is a straightforward computation using the fact that $C_*(pt)=\mathbb{k}$ (cf. introduction) and that $\pi(e\cdot g)=\pi e,\ e\in E,\ g\in G$. □

Lemma 5.2 implies that for any morphism $m : M \to C_*(E)$ of $C_*(G)$-modules, $C_*(\pi)m$ factors to yield $\overline{m} : M \otimes_{C_*(G)} \Bbbk \to C_*(X)$.

PROPOSITION 5.3. *There is a $C_*(G)$-semifree resolution $m : M \xrightarrow{\simeq} C_*(E)$ such that $\overline{m} : M \otimes_{C_*(G)} \Bbbk \longrightarrow C_*(X)$ is also a quasi-isomorphism.*

(This follows easily from a theorem of Brown [6]. We include a short proof for the convenience of the reader.)

PROOF OF 5.3. Let $Y \xrightarrow{\simeq} X$ be a weak homotopy equivalence from a CW complex and let $E_Y \to Y$ be the pull back fibration. Since $E_Y \to E$ is also a weak homotopy equivalence, it is sufficient to prove the lemma for $C_*(E_Y)$; i.e. we may assume X itself is a CW complex.

Denote the n-skeleton of X by X_n and let $E_n \xrightarrow{\pi} X_n$ be the restriction of the fibration to X_n. We shall construct a quasi-isomorphism of $C_*(G)$-modules of the form

$$m : \big(V \otimes C_*(G), d\big) \xrightarrow{\simeq} C_*(E)$$

satisfying the following four conditions:

(i) $V = \{V_n\}_{n \geqslant 0}$ and $V_n = H(X_n, X_{n-1})$ is the free $\Bbbk$-module on a basis v_α indexed by the n-cells D^n_α of X.

(ii) $C_*(G)$ acts by right multiplication in $V \otimes C_*(G)$.

(iii) m restricts to quasi-isomorphisms

$$m_n : \big(V_{\leqslant n} \otimes C_*(G), d\big) \xrightarrow{\simeq} C_*(E_n), \quad n \geqslant 0.$$

(iv) Identify $[V \otimes C_*(G), d] \otimes_{C_*(G)} \Bbbk = (V, \overline{d})$. Then the morphisms

$$\overline{m}_n : \big(V_{\leqslant n}, \overline{d}\big) \longrightarrow C_*(X_n)$$

induce the identity maps $V_n \xrightarrow{=} H(X_n, X_{n-1})$.

Note that $M = (V \otimes C_*(G), d)$ is obviously $C_*(G)$-semifree. Since (iv) implies that $\overline{m}$ is a quasi-isomorphism, this construction will establish the lemma.

The construction itself is a simple adaptation of the construction of the cellular chain complex for X. Since the existence of m_0 is obvious it is sufficient to show that a morphism m_{n-1} as above can be extended to a suitable m_n. Let $D = \coprod_\alpha D^n_\alpha$ be the disjoint union of the n-cells of X and put $S = \coprod_\alpha S^{n-1}_\alpha$. Use the characteristic map $f : (D, S) \longrightarrow (X_n, X_{n-1})$ to pull the fibration back to a G-fibration $\pi : (E_D, E_S) \longrightarrow (D, S)$. Since the D^n_α are contractible this fibration has a cross section $\sigma : D \longrightarrow E_D$. Thus a weak homotopy equivalence $(D, S) \times G \longrightarrow (E_D, E_S)$ is given by $(x, g) \mapsto \sigma(x) \cdot g$. Composing this with the pullback map defines $\Phi : (D, S) \times G \longrightarrow (E_n, E_{n-1})$ covering f.

Now consider the commutative diagram

$$\begin{tikzcd}
C_*(D,S)\otimes C_*(G) \arrow[r, "C_*(\Phi)\circ EZ"] \arrow[d, "-\otimes_{C_*(G)}\mathbb{k}"'] & C_*(E_n,E_{n-1}) \arrow[d, "C_*(\pi)"] \\
C_*(D,S) \arrow[r, "C_*(f)"'] & C_*(X_n,X_{n-1})
\end{tikzcd}$$

Both $C_*(\Phi)$ and $C_*(f)$ are quasi-isomorphisms, by excision. Identify

$$H(D,S)=\bigoplus_\alpha H_n(D^n_\alpha,S^{n-1}_\alpha)=\bigoplus_\alpha \mathbb{k}v_\alpha=V_n.$$

Since m_{n-1} is a quasi-isomorphism there are cycles $z_\alpha\in[V_{\leqslant n-1}\otimes C_*(G)]_{n-1}$ and elements $w_\alpha\in C_n(E_n)$ such that

$$w_\alpha \text{ projects to } \big(C_*(\Phi)\circ EZ\big)(v_\alpha\otimes 1) \text{ in } C_*(E_n,E_{n-1}),$$

and

$$dw_\alpha=m_{n-1}(z_\alpha).$$

Define m_n to be the unique extension of m_{n-1} such that $d(v_\alpha\otimes 1)=z_\alpha$ and $m_n(v_\alpha\otimes 1)=w_\alpha$. Conditions (i), (ii) and (iv) hold by definition. The quotient morphism $V_n\otimes C_*(G)\longrightarrow C_*(E_n,E_{n-1})$ is equivalent to $C_*(\Phi)\circ EZ$ and hence a quasi-isomorphism. Thus, by the 5-lemma, m_n is a quasi-isomorphism too. □

PROOF OF 5.1. We first construct the DGC morphism $B(C_*(E);C_*(G))\xrightarrow{\simeq}C_*(X)$. Note that $\mathbb{k}$ is (trivially) a DGC over $C_*(G)$. Hence the map of Lemma 5.2,

$$\overline{C_*(\pi)}:C_*(E)\otimes_{C_*(G)}\mathbb{k}\longrightarrow C_*(X),$$

is in fact a morphism of DGC's. Thus the composite

$$\begin{aligned}B\big(C_*(E);C_*(G)\big)&=C_*(E)\otimes_{C_*(G)}B\big(C_*(G);C_*(G)\big)\\&\xrightarrow{\mathrm{id}\otimes\varepsilon}C_*(E)\otimes_{C_*(G)}\mathbb{k}\xrightarrow{\overline{C_*(\pi)}}C_*(X)\end{aligned}$$

is a morphism of DGC's. We shall show it is a quasi-isomorphism.

For this, let $m : M \xrightarrow{\simeq} C_*(E)$ be a $C_*(G)$-semifree resolution as in Proposition 5.3. Consider the commutative diagram

$$\begin{tikzcd}
M \otimes_{C_*(G)} B(C_*(G); C_*(G)) \arrow[r, "m \otimes \mathrm{id}"] \arrow[dd, "\mathrm{id} \otimes \varepsilon"'] & C_*(E) \otimes_{C_*(G)} B(C_*(G); C_*(G)) \arrow[d, "\mathrm{id} \otimes \varepsilon"] \\
 & C_*(E) \otimes_{C_*(G)} \boldsymbol{k} \arrow[d, "\overline{C_*(\pi)}"] \\
M \otimes_{C_*(G)} \boldsymbol{k} \arrow[r, "\overline{m}"'] & C_*(X)
\end{tikzcd}$$

Because $\overline{C_*(G)}$ is $\boldsymbol{k}$-free and concentrated in degrees $\geqslant 0$, it is $\boldsymbol{k}$-semifree. Hence, by Lemma 4.3(ii), $B(C_*(G); C_*(G))$ is $C_*(G)$-semifree. Thus Proposition 2.3(i) asserts that $m \otimes \mathrm{id}$ is a quasi-isomorphism, because m is.

On the other hand, since M is $C_*(G)$-semifree and ε is a quasi-isomorphism (Lemma 4.3(i)), a second application of Proposition 2.3(i) shows that $\mathrm{id} \otimes \varepsilon$ is a quasi-isomorphism. Finally, $\overline{m}$ is a quasi-isomorphism by hypothesis. Hence so is $\overline{C(\pi)} \circ (\mathrm{id} \otimes \varepsilon)$. □

6. Examples and applications

6.1. $C_*(X) \simeq B(C_*(\Omega X))$

Let X be any path connected space. The space of *free Moore paths* on X is the subspace $MX \subset X^{[0,\infty)} \times [0,\infty)$ of pairs (α, r) such that $\alpha(t) = \alpha(r)$, $t \geqslant r$. The choice of a basepoint $x_0 \in X$ determines the subspaces $\Omega X \subset PX \subset MX$ by the conditions:

$$PX = \{(\alpha, r) | \alpha(r) = x_0\} \quad \text{and} \quad \Omega X = \{(\alpha, r) | \alpha(0) = \alpha(r) = x_0\}.$$

These are respectively the *Moore loop space* and the *Moore path space* and have the homotopy type [21] of the standard loop and path spaces. A continuous map $PX \times \Omega X \to PX$ is then given by $(\alpha, r) \times (\beta, s) \mapsto (\alpha * \beta, r + s)$ where

$$(\alpha * \beta)(t) = \begin{cases} \alpha(t), & t \leqslant r, \\ \beta(t - r), & t \geqslant r. \end{cases}$$

This map when restricted to $\Omega X \times \Omega X$ makes ΩX into a topological monoid, and the map itself is an action of ΩX on PX ([21]). Since $\pi : PX \to X$, $(\alpha, r) \mapsto \alpha(0)$, is easily seen to be a fibration we have

LEMMA 6.2. *For any path connected space X,*

$$\pi : PX \longrightarrow X, \quad PX \times \Omega X \longrightarrow PX$$

is an ΩX-fibration.

Now, as promised in the introduction, we establish

THEOREM 6.3. *For any path connected space X, the DGC $C_*(X)$ is weakly DGC-equivalent to the bar construction $B(C_*(\Omega X))$.*

PROOF. As noted in the proof of Theorem 5.1, $B(C_*(\Omega X); C_*(\Omega X))$ is $C_*(\Omega X)$-semifree. Moreover, PX is contractible, and so the constant map $PX \to pt$ induces a quasi-isomorphism $\varepsilon_P : C_*(PX) \to \boldsymbol{k}$ of $C_*(\Omega X)$-coalgebras. Thus, by Proposition 2.3(i),

$$\varepsilon_P \otimes \mathrm{id} : C_*(PX) \otimes_{C_*(\Omega X)} B\big(C_*(\Omega X); C_*(\Omega X)\big) \xrightarrow{\simeq} B\big(C_*(\Omega X)\big)$$

is a DGC quasi-isomorphism. On the other hand, Theorem 5.1 gives a DGC quasi-isomorphism

$$C_*(PX) \otimes_{C_*(\Omega X)} B\big(C_*(\Omega X); C_*(\Omega X)\big) \xrightarrow{\simeq} C_*(X).$$

□

REMARK 6.4. As noted in Example 4.6, the DGC

$$B\big(C_*(\Omega X)\big) = \big(T\big(s\overline{C_*(\Omega X)}\big), d\big)$$

depends only on the augmented DGA structure of $C_*(\Omega X)$. Thus (cf. Proposition 4.5) the weak DGC equivalence class of $C_*(X)$ is determined by the weak augmented DGA equivalence class of $C_*(\Omega X)$.

For simply connected spaces this correspondence is a bijection: the weak equivalence DGC class of $C_*(X)$ determines the weak augmented DGA equivalence class of $C_*(\Omega X)$, as follows from Adams' theorem [1] (cf. also [16] and [12]).

6.5. *Homotopy fibres and models of fibrations*

Let $\phi : X \to Y$ be any continuous map between path connected spaces. It determines the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "\phi"'] & & X \times_Y MY \arrow[dl, "p"] \\ & Y & \end{tikzcd} \tag{6.6}$$

in which $p(x,(\alpha,r)) = \alpha(r)$ and $i(x) = (x,(c_{\phi x},0))$, $c_{\phi x}$ denoting the constant path at ϕx. In this diagram i is a homotopy equivalence and p is a fibration ([21]); we say i *converts* ϕ *to the fibration* p. The fibre, F, of p is called the *homotopy fibre of* ϕ.

As in 6.1, composition of paths defines a right action of ΩY on F. This identifies the restriction of the projection $X \times_Y MY \to X$ as an ΩY-fibration $\rho : F \to X$. Thus Theorem 5.1 yields

PROPOSITION 6.7. *With the notation above* $C_*(X)$ *is weakly DGC equivalent to* $B(C_*(F); C_*(\Omega Y))$.

Now consider the special case that

$$Z \longrightarrow X \xrightarrow{\phi} Y$$

is a fibration with fibre Z. Then the map i (in (6.6)) restricts to a weak homotopy equivalence $Z \longrightarrow F$, hence inducing a DGC quasi-isomorphism $C_*(Z) \xrightarrow{\phi} C_*(F)$. Thus Proposition 6.7 provides a DGC quasi-isomorphism

$$C_*(F) \text{“}\otimes\text{”} B\big(C_*(\Omega Y)\big) \xrightarrow{\simeq} C_*(X), \tag{6.8}$$

where "$\otimes$" indicates a "twisted" DGC structure and (cf. Theorem 6.3) there are DGC quasi-isomorphisms

$$C_*(F) \xleftarrow{\simeq} C_*(Z) \quad \text{and} \quad B\big(C_*(\Omega Y)\big) \xrightarrow{\simeq} C_*(Y).$$

This is a DGC analogue of Brown's Theorem [6] and reminiscent of the Dupont–Hess models introduced in [9].

6.9. *Universal enveloping algebras*

Suppose $\frac{1}{2} \in \Bbbk$ and (L, d) is a $\Bbbk$-free differential graded Lie algebra. We suppose that either $\Bbbk$ is a principal ideal domain or else that $L = \{L_i\}_{i \geqslant r}$ for some $r \in \mathbb{Z}$.

The universal enveloping algebra of (L, d), $U(L, d)$, is a DGH. Regard UL as a left UL-module and consider the chain complex $C_*(UL; L)$ of chains on L with coefficients in UL as defined in [8] in the ungraded case, in [18, Appendix B] for the graded case where $\Bbbk$ is a field of characteristic zero and in [11] or [15, §1] in general.

Now $C_*(L) = C_*(\Bbbk; L)$ is a DGC (the divided powers coalgebra $\Gamma(sL)$) and $C_*(UL; L) = UL \otimes C_*(L)$ can be given the tensor product coalgebra structure. This makes $C_*(UL; L)$ into a DGC, and left multiplication by UL identifies $C_*(UL, L)$ as a left UL-coalgebra. Moreover $C_*(UL; L)$ is UL-semifree and the augmentation to $\Bbbk$ is a quasi-isomorphism.

For any free graded module W, $PW \subset TW$ is the subcoalgebra of completely symmetric elements. In particular, the inclusion $sL \subset s\overline{UL}$ defines a coalgebra inclusion $\Gamma(sL) \subset T(s\overline{UL})$. This extends to a morphism of UL-coalgebras

$$\alpha : C_*(UL; L) = UL \otimes \Gamma(sL) \longrightarrow UL \otimes T(s\overline{UL}) = B(UL; UL).$$

It is well known (e.g., [15, Theorem 1.5]) that α is a quasi-isomorphism of UL-modules, whence

PROPOSITION 6.10. *The injection* $\alpha : C_*(UL; L) \to B(UL; UL)$ *is a DGC quasi-isomorphism.*

Now let (C, d) be a right DGC over UL. We give to $C_*(C; L)$ the tensor product coalgebra structure, and we consider the induced map

$$\alpha_C : C_*(C; L) = C \otimes_{UL} C_*(UL; L) \xrightarrow{\mathrm{id} \otimes \alpha} C \otimes_{UL} B(UL; UL) = B(C; UL).$$

PROPOSITION 6.11. α_C *is a DGC quasi-isomorphism.*

PROOF. The hypothesis on L implies that $B(UL; UL)$ and $C_*(UL : L)$ are UL-semifree modules (cf. Lemma 4.3). The morphism α_L is therefore a quasi-isomorphism (Proposition 2.3(i)). □

THE BOREL CONSTRUCTION 6.12. *Let* $EG \to BG$ *be the universal bundle of a topological group. If* G *acts on the right on some space* F *then the space*

$$F_G = F \times_G EG$$

is called the Borel construction on F.

Now F_G is the base of the principal G-bundle $F \times EG \to F_G$. Since principal bundles are (trivially) G-fibrations, Theorem 5.1 provides a DGC quasi-isomorphism

$$B(C_*(F \times EG); C_*(G)) \xrightarrow{\simeq} C_*(F_G).$$

On the other hand, the projection $F \times EG \to F$ is a G-equivariant weak homotopy equivalence. Thus by Proposition 4.5 it induces a DGC quasi-isomorphism

$$B\big(C_*(F \times EG); C_*(G)\big) \xrightarrow{\simeq} B\big(C_*(F); C_*(G)\big).$$

This proves:

PROPOSITION 6.13. *The DGC, $C_*(F_G)$ is weakly DGC equivalent to $B(C_*F; C_*G)$.*

7. Cochain algebras

Let (K,d) be a DGH, and suppose (A,d) is a left DGA over (K,d). Then we can form the DGA

$$\mathcal{B}_K(A) = \mathrm{Hom}_K\big(B(K;K), A\big),$$

with multiplication given by $(f \cdot g)(\Phi) = \mu_A \circ (f \otimes g) \circ \Delta\Phi$, and identity $1 : \Phi \mapsto \varepsilon(\Phi)1_A$. This construction is functorial in A and contrafunctorial in K: suppose (A',d) is a second DGA over a second DGH, (K',d) and suppose $\phi : (K',d) \to (K,d)$ and $\psi : (A,d) \to (A',d)$ are respectively a DGH morphism and a DGA morphism such that $\psi(\phi x \cdot a) = x \cdot \psi(a)$, $x \in K'$, $a \in A$. Then we put

$$\mathcal{B}_\phi(\psi) = \mathrm{Hom}_\phi\big(B(\phi;\phi), \psi\big) : \mathcal{B}_K(A) \to \mathcal{B}_{K'}(A').$$

From Proposition 2.4 we deduce

PROPOSITION 7.1. *If $(\overline{K}, d)$ is $\Bbbk$-semifree and ϕ and ψ are quasi-isomorphisms, then so is $\mathcal{B}_\phi(\psi)$.*

For any graded module W, write $W^\vee = \mathrm{Hom}(W, \Bbbk)$. Then, as above, the dual of a DGC is a DGA, with multiplication given by $(f \cdot g)(x) = (f \otimes g)(\Delta x)$. If (C,d) is a right DGC over a DGH, (K,d), then $(C,d)^\vee$ is a left DGA over (K,d) in the obvious way. From Theorem 5.1, Theorem 6.3 and Proposition 6.7, and from the natural isomorphism $\mathrm{Hom}(M \otimes N, -) = \mathrm{Hom}(M, \mathrm{Hom}(N, -))$ we deduce

THEOREM 7.2. (i) *If $\pi : E \to X$ is a G-fibration, then there is a natural DGA-quasi-isomorphism*

$$C^*(X) \xrightarrow{\simeq} \mathcal{B}_{C_*(G)}\big(C^*(E)\big).$$

(ii) *For any path connected, pointed space X, the DGA, $C^*(X)$, is weakly DGA-equivalent to $B(C_*\Omega X)^\vee$.*

(iii) *Let $\phi : X \to Y$ be any continuous map between path connected spaces, with homotopy fibre F. Then $C^*(X)$ is weakly DGA-equivalent to $\mathcal{B}_{C_*(G)}(C^*(F))$.*

(iv) *For any action of a topological group G on a space F, $C^*(F_G)$ is naturally weakly DGA equivalent to $\mathcal{B}_{C_*(G)}(C^*(F))$.*

7.3. *Torus actions*

In the case of a torus $T = S^1 \times \cdots \times S^1$ there is a very simple model for $C_*(T)$. Indeed, the 1-simplex $\sigma : t \longrightarrow e^{2\pi i t}$ is a cycle in $C_*(S^1)$ satisfying $\Delta\sigma = \sigma \otimes 1 + 1 \otimes \sigma$ and $\sigma^2 = 0$. Thus a DGH quasi-isomorphism $(\Lambda a, 0) \xrightarrow{\simeq} C_*(S^1)$ is given by $a \mapsto \sigma$, where Λa is the exterior algebra on an element a of degree one.

Next, a straightforward calculation shows that if $\tau : X \times Y \longrightarrow Y \times X$ is the map $(x, y) \to (y, x)$, then

$$C_*(\tau) \circ EZ(u \otimes v) = (-1)^{\deg u \cdot \deg v} EZ(v \otimes u).$$

Using this it is easy to see that if K and G are topological monoids then

$$EZ : C_*(K) \otimes C_*(G) \longrightarrow C_*(K \times G)$$

is a DGH quasi-isomorphism. In this way we obtain a DGH-quasi-isomorphism

$$\big(\Lambda(a_1, \ldots, a_n), 0\big) = \bigotimes_{j=1}^{n} (\Lambda a_j, 0) \xrightarrow{\simeq} C_*(T). \tag{7.4}$$

In particular, assume T acts from the right on a topological space F. Then $C_*(F)$ inherits the structure of a $\Lambda(a_1, \ldots, a_n)$-DGC from the quasi-isomorphism (7.4), and dually, $C^*(F)$ is a left DGA over $\Lambda(a_1, \ldots, a_n)$. Explicitly, this means that a_j acts by a degree -1 derivation θ_j in $C^*(F)$ satisfying

$$\theta_j\theta_i = -\theta_i\theta_j, \quad \theta_j^2 = 0 \quad \text{and} \quad d\theta_j + \theta_j d = 0.$$

Let μ_j denote multiplication by b_j in the polynomial algebra $\Bbbk[b_1, \ldots, b_n]$ in which each b_j has (upper) degree 2.

THEOREM 7.5. *The DGA,* $\big(\Bbbk[b_1, \ldots, b_n] \otimes C^*(F), \mathrm{id} \otimes d + \sum_j \mu_j \otimes \theta_j\big)$ *is weakly DGA equivalent to* $C^*(F_T)$.

PROOF. Identify $(\Lambda(a_1, \ldots, a_n), 0)$ as the universal enveloping algebra on the submodule $L = \bigoplus_j \Bbbk a_j$. Proposition 6.11 yields a DGC quasi-isomorphism

$$C(UL; L) \xrightarrow{\simeq} B(UL; UL) \tag{7.6}$$

and in this case $C(UL; L) = \Lambda(a_1, \ldots, a_n) \otimes \Gamma(b_1^*, \ldots, b_n^*)$ where, in particular, $\Gamma(b_1^*, \ldots, b_n^*)$ is the graded coalgebra dual to $\Bbbk[b_1, \ldots, b_n]$. Since (7.6) is a quasi-isomorphism of semifree UL-modules, we apply Proposition 2.3(ii) to obtain a dual DGA quasi-isomorphism

$$\mathrm{Hom}_{UL}\big(C(UL; L), C^*(F)\big) \xleftarrow{\simeq} \mathrm{Hom}_{UL}\big(B(UL; UL), C^*(F)\big).$$

The left hand side is precisely

$$\left(\mathbb{k}[b_1, \ldots, b_n] \otimes C^*(F); \mathrm{id} \otimes d + \sum \mu_j \otimes \theta_j\right)$$

and Theorem 7.2(iv) gives a DGA quasi-isomorphism from the right hand side to $C^*(F_T)$. □

8. Stasheff structures and the DGA $\tilde{B}_K(A)$

An *acyclic closure* for a DGH, (K, d) is a quasi-isomorphism $\gamma : P \xrightarrow{\simeq} \mathbb{k}$ of (K, d)-modules in which

$$P = \bigcup_{k \geqslant 0} P(k),$$

as in the definition of semifree in §2, and γ restricted to $P(0)$ is identified with $\varepsilon_K : K \to \mathbb{k}$. Theorem 4.1 shows that (K, d) admits an acyclic closure $\gamma : P \xrightarrow{\simeq} \mathbb{k}$ that is a (K, d)-DGC (cf. §4). However, in any acyclic closure, it is possible to construct a strongly homotopy associative comultiplication in the sense of Stasheff [20], and this turns out to be sufficient to construct a DGA, $\tilde{B}_K(A)$, in the weak DGA equivalence class of $B_K(A)$, for any (K, d)-DGA, (A, d). In particular, in a G-fibration $\pi : E \to X$, we can recover the class of $C^*(X)$ this way from the $C_*(G)$-DGA, $C^*(E)$. Henceforth we shall restrict ourselves to DGH's (K, d) such that $(\overline{K}, d)$ is $\mathbb{k}$-semifree.

Fix an acyclic closure,

$$\gamma : P \xrightarrow{\simeq} \mathbb{k}.$$

The identity element of $K (= P(0))$ is then a cycle $1_P \in P$ such that $\gamma(1_P) = 1$. Now consider the diagram of (K, d) modules (with K acting diagonally in $P \otimes P$)

$$\begin{tikzcd} & P \otimes P \arrow[d, "(\gamma \otimes \mathrm{id}, \mathrm{id} \otimes \gamma) = \eta"] \\ P \arrow[r, "(\mathrm{id}, \mathrm{id})"'] & P \times_{\mathbb{k}} P \end{tikzcd} \tag{8.1}$$

The vertical arrow, η, is a surjective quasi-isomorphism. Thus, since P is (K, d) semifree, Proposition 2.1 implies that $\mathrm{Hom}(P, \eta)$ is also a surjective quasi-isomorphism. Thus we may lift $(\mathrm{id}, \mathrm{id})$ through the vertical arrow to a morphism $\alpha : P \longrightarrow P \otimes P$. Moreover, because $P(0) = K$ and $\Delta : K \longrightarrow K \otimes K$ does lift $(\mathrm{id}, \mathrm{id})$, we may suppose

$\alpha|_K = \Delta$. Altogether then we obtain a morphism $\alpha : P \longrightarrow P \otimes P$ of (K, d)-modules satisfying

$$\left.\begin{array}{c}(\gamma \otimes \mathrm{id}) \circ \alpha = \mathrm{id} = (\mathrm{id} \otimes \gamma) \circ \alpha \\ \text{and} \\ \alpha(1_P) = 1_P \otimes 1_P\end{array}\right\}. \tag{8.2}$$

Such a morphism will be called a *weak comultiplication* in P.

In fact if α were strictly associative then (P, α, γ) would be a left (K, d)-coalgebra with comultiplication α. There is, however, no a priori reason for this to be true. What is true is that α is homotopy associative. In fact it follows from (8.2) that

$$(\alpha \otimes \mathrm{id}) \circ \alpha - (\mathrm{id} \otimes \alpha) \circ \alpha : P \longrightarrow \ker(\gamma \otimes \gamma \otimes \gamma).$$

By hypothesis (K, d) is $\Bbbk$-semifree. Hence (P, d) is also $\Bbbk$-semifree, which implies (Proposition 2.3(i)) that $H(\ker(\gamma \otimes \gamma \otimes \gamma)) = 0$. Since P is (K, d)-semifree it follows (Proposition 2.1(ii)) that $(\alpha \otimes \mathrm{id}) \circ \alpha - (\mathrm{id} \otimes \alpha) \circ \alpha \sim 0$ as a map of (K, d)-modules.

Indeed it turns out that there is an infinite sequence of 'higher homotopies' that exhibits α as strongly homotopy associative in the sense of Stasheff [20]. It is this sequence that forms what we call a *Stasheff structure* in P.

More precisely, define the (K, d)-module $s^{-1}P$ by $(s^{-1}P)_i = P_{i+1}$, $d(s^{-1}x) = -s^{-1}dx$ and $g \cdot s^{-1}x = (-1)^{\deg g} s^{-1}(g \cdot x)$, $g \in K$. Let K act diagonally on $(s^{-1}P)^{\otimes k}$ and let $I(k)$ be the kernel of the K-linear map

$$\gamma_k : \left(s^{-1}P\right)^{\otimes k} \longrightarrow \Bbbk, \quad s^{-1}x_1 \otimes \cdots s^{-1}x_k \mapsto \gamma(x_1) \cdots \gamma(x_k).$$

Since P is $\Bbbk$-semifree, $H(\gamma_k)$ is an isomorphism of degree k (Proposition 2.3(i)) and $H(I(k)) = 0$.

DEFINITION. A *Stasheff structure* in the acyclic closure $P \xrightarrow{\gamma} \Bbbk$ of a DGH, (K, d), is a sequence of derivations $\delta_1, \delta_2, \ldots$ of degree -1 in the tensor algebra $T(s^{-1}P)$ satisfying:

(i) $\delta_1 = d : s^{-1}P \longrightarrow s^{-1}P$.

(ii) $\delta_2 : s^{-1}P \longrightarrow s^{-1}P \otimes s^{-1}P$ and is obtained from a weak comultiplication α as follows: if

$$\alpha x = \sum_i x_{i1} \otimes x_{i2}$$

then

$$\delta_2 s^{-1}x = \sum_i (-1)^{\deg x_{i1}} s^{-1}x_{i1} \otimes s^{-1}x_{i2}.$$

In particular, δ_2 is K-linear.

(iii) For $k \geqslant 3$, δ_k is a K-linear derivation satisfying $\delta_k(s^{-1}1_P) = 0, \delta_k : s^{-1}P \longrightarrow I(k)$, and

$$\left(\sum_{i=1}^{k}\delta_i\right)^2 : s^{-1}P \longrightarrow \bigoplus_{q\geqslant k+1}\left(s^{-1}P\right)^{\otimes q}.$$

PROPOSITION 8.3. *If $(\overline{K}, d)$ is $\Bbbk$-semifree then any acyclic closure has a Stasheff structure.*

PROOF. Use d and an arbitrary weak comultiplication α in P to define δ_1 and δ_2. Suppose $\delta_1, \ldots, \delta_k$ are constructed, $k \geqslant 2$. Put $\beta = \delta_2\delta_k + \delta_3\delta_{k-1} + \cdots + \delta_k\delta_2$. Thus

$$\beta : s^{-1}P \longrightarrow \left(s^{-1}P\right)^{\otimes k+1}$$

and

$$\left(\sum\delta_i\right)^2 - \beta : s^{-1}P \longrightarrow \bigoplus_{q\geqslant k+2}\left(s^{-1}P\right)^{\otimes q}.$$

The following observations are straightforward: β is K-linear, $\beta\delta_1 = \delta_1\beta, \beta(1_P) = 0$ and $\operatorname{Im}\beta \subset I(k+1)$.

Because $H(I(k+1)) = 0$, and P is (K,d)-semifree there is a (K,d)-linear map $\delta_{k+1} : s^{-1}P \longrightarrow I(k+1)$ such that $\delta_{k+1}\delta_1 + \delta_1\delta_{k+1} = -\beta$. (cf. Proposition 2.1(ii)). Since P begins with $P(0) = K$ and since $\beta(1_P) = 0$ we may construct δ_{k+1} with $\delta_{k+1}(s^{-1}1_P) = 0$. Extend δ_{k+1} (uniquely) to a derivation in $T(s^{-1}P)$. □

Next, suppose $\phi : (K,d) \longrightarrow (K',d)$ is a DGH morphism with both $(\overline{K}, d)$ and $(\overline{K}', d)$ $\Bbbk$-semifree. Equip acyclic closures $P \xrightarrow{\simeq} \Bbbk$, $P' \xrightarrow{\simeq} \Bbbk$ with Stasheff structures. Regard P' as a (K,d)-module via ϕ and lift $\mathrm{id}_{\Bbbk}$ to a morphism

$$\psi : P \longrightarrow P', \quad \psi(1_P) = (1_{P'})$$

of (K,d)-modules. An argument completely analogous to that of Proposition 8.3 establishes

PROPOSITION 8.4. *There is a sequence of degree zero K-linear maps $\psi_k : s^{-1}P \longrightarrow (s^{-1}P')^{\otimes k}$, $k \geqslant 1$, such that*

(i) $\psi_1(s^{-1}x) = s^{-1}\psi x$.
(ii) *For* $k \geqslant 2$, $\psi_k(s^{-1}1_P) = 0$ *and* $\operatorname{Im}\psi_k \subset I(k)'$.
(iii) *If $\Psi_k : T(s^{-1}P) \to T(s^{-1}P')$, is the unique algebra morphism extending $\psi_1 + \cdots + \psi_k$ then*

$$\Psi_k(\delta_1 + \cdots + \delta_k) - (\delta_1 + \cdots + \delta_k)\Psi_k : s^{-1}P \longrightarrow \bigoplus_{q\geqslant k+1}\left(s^{-1}P'\right)^{\otimes q}.$$

Now define a cycle $\iota \in \mathrm{Hom}_K(P, A)_0$ by setting $\iota(x) = \gamma(x)1_A$. Define a map $\bar{\varepsilon} : (\mathrm{Hom}_K(P, A), d) \longrightarrow \Bbbk$ by $\bar{\varepsilon} f = \varepsilon_A(f(1_P))$. Then $\bar{\varepsilon}(\iota) = 1$, so $\bar{\varepsilon}$ is surjective. As in §4, if $f_1, \ldots, f_k \in \ker\bar{\varepsilon}$ we denote by $[sf_1|\cdots|sf_k]$ the element $sf_1 \otimes \cdots \otimes sf_k$ in $T(s(\ker\bar{\varepsilon}))$. We regard $T(s(\ker\bar{\varepsilon}))$ as a graded coalgebra with comultiplication given by

$$\Delta[sf_1|\cdots|sf_k] = \sum_{i=0}^{k}[sf_1|\cdots|sf_i] \otimes [sf_{i+1}|\cdots|sf_k].$$

The main step in the construction of $\tilde{B}_K(A, d)$ is the definition of a DGC, $(T(s(\ker\bar{\varepsilon})), \partial)$, with the aid of a Stasheff structure in P.

Recall that a coderivation $\theta : C \to C$ in a graded coalgebra is a linear map such that $(\theta \otimes \mathrm{id} + \mathrm{id} \otimes \theta) \circ \Delta = \Delta \circ \theta$. Now some notation. Define pairings

$$\langle\ ,\ \rangle : \left[s\mathrm{Hom}_K(P, A)\right]^{\otimes k} \times \left[s^{-1}P\right]^{\otimes k} \to A$$

by

$$\left\langle [sf_1|\cdots|sf_k],\ s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_k \right\rangle = (-1)^{\sigma} f_1(x_1) \cdot \cdots \cdot f_k(x_k),$$

where

$$\sigma = \sum_{i>j} \deg sf_i \cdot \deg s^{-1}x_j + \sum_i \deg f_i.$$

Then a K-linear map $\gamma : s^{-1}P \longrightarrow (s^{-1}P)^{\otimes k}$ dualizes to the linear map

$$\tilde{\gamma} : \left[s\mathrm{Hom}_K(P, A)\right]^{\otimes k} \longrightarrow s\mathrm{Hom}_K(P, A)$$

given by

$$\left\langle \tilde{\gamma}\Phi, s^{-1}x \right\rangle = (-1)^{\deg\gamma \cdot \deg\Phi} \left\langle \Phi, \gamma s^{-1}x \right\rangle.$$

If θ is the unique derivation in $T(s^{-1}P)$ extending γ and if $\tilde{\theta}$ is the unique coderivation in $T(s\mathrm{Hom}_K(P, A))$ lifting $\tilde{\gamma}$ then clearly

$$\langle \tilde{\theta}\Phi, z \rangle = (-1)^{\deg\tilde{\theta} \cdot \deg\Phi} \langle \Phi, \theta z \rangle, \quad z \in T\left(s^{-1}P\right),\ \Phi \in T\left(s\mathrm{Hom}_K(P, A)\right).$$

We say $\tilde{\theta}$ is the coderivation dual to θ.

Now choose a Stasheff structure, $\{\delta_k\}_{k \geqslant 1}$ in P as described above. Since $-\delta_k$ is a K-linear derivation for $k \geqslant 2$ it dualizes to a coderivation ∂_k in $T(s\mathrm{Hom}_K(P, A))$ and ∂_k satisfies

$$\langle \partial_k\Phi, z \rangle + (-1)^{\deg\Phi} \langle \Phi, \delta_k z \rangle = 0, \quad \Phi \in T\left(s\mathrm{Hom}_K(P, A)\right),\ z \in T\left(s^{-1}P\right).$$

Let ∂_1 be the unique coderivation that preserves tensor length and satisfies $\partial_1 sf = -sdf$.

Because ∂_k *decreases* tensor length by $k-1$, the infinite sum

$$\partial = \sum_{k=1}^{\infty} \partial_k$$

is a well defined coderivation in $T(s\mathrm{Hom}_K(P,A))$. It follows easily from the defining properties of a Stasheff structure that $\partial^2 = 0$ and that ∂ restricts to a coderivation in $T(s\ker\bar{\varepsilon})$. Thus we have constructed a DGC, $(T(s\ker\bar{\varepsilon}), \partial)$ as desired.

This is a *supplemented* DGC, that is a DGC of the form $(C,\partial) = \ker\varepsilon_C \oplus \boldsymbol{k}$ with $\Delta 1 = 1\otimes 1$ and $\partial(1) = 0$. The reduced diagonal for such a DGC is the map $\overline{\Delta}$: $\ker\varepsilon_C \to \ker\varepsilon_C \otimes \ker\varepsilon_C$ given by $\Delta x = x\otimes 1 + 1\otimes x + \overline{\Delta}x$. The *cobar construction* on (C,∂) is the DGA

$$\Omega(C,\partial) = \big(T(s^{-1}\ker\varepsilon_C), D\big)$$

defined by

$$Ds^{-1}x = -s^{-1}\partial x + \sum (-1)^{\deg x_{i1}} s^{-1}x_{i1} \otimes s^{-1}x_{i2},$$

where $\overline{\Delta}x = \sum x_{i1}\otimes x_{i2}$.

Given a Stasheff structure in an acyclic closure of (K,d) and given a (K,d)-algebra (A,d) we write

$$\tilde{B}_K(A) = \Omega\big(T(s\ker\bar{\varepsilon}), \partial\big),$$

where $(T(s\ker\bar{\varepsilon}),\partial)$ is the DGC defined above. Clearly $\tilde{B}_K(A)$ depends on the choice of acyclic closure and Stasheff structure. We shall see below that its weak equivalence class is independent of these choices, and contains the DGA $B_K(A)$ of §7.

First we make the important observation that, as a differential graded module, $\tilde{B}_K(A)$ is naturally equivalent with $\mathrm{Hom}_K(P,A)$. In fact, given any supplemented DGC of the form (TV,∂) with the tensor coalgebra structure, ∂ restricts to a differential ∂_1 and V and the composite

$$(\boldsymbol{k},0)\oplus s^{-1}(V,\partial_1) \to s^{-1}(T^{+}V,\partial) \to \Omega(TV,d)$$

is a quasi-isomorphism by [12, Proposition 2.8(ii)]. Thus writing

$$\mathrm{Hom}_K(P,A) = \ker\bar{\varepsilon}\oplus \boldsymbol{k}\cdot\iota$$

we have:

LEMMA 8.5. *A quasi-isomorphism* $\lambda : \mathrm{Hom}_K(P,A) \xrightarrow{\simeq} \tilde{B}_K(A)$ *is given by* $\lambda(f) = s^{-1}[sf]$, $f\in\ker\bar{\varepsilon}$ *and* $\lambda(\iota) = 1$.

Proof. This is precisely [12, Proposition 2.8(ii)]. □

Next, suppose

$$\phi : (K', d) \longrightarrow (K, d) \quad \text{and} \quad \tau : (A, d) \longrightarrow (A', d)$$

are respectively a DGH morphism and a DGA morphism. Assume (A, d) is a (K, d)-algebra, (A', d) is a (K', d) algebra and that

$$\tau(\phi(g) \cdot a) = g \cdot \tau(a), \quad a \in A,\ g \in K'. \tag{8.6}$$

Assume also that $(\overline{K}, d)$ and $(\overline{K}', d)$ are $\Bbbk$-semifree. Choose acyclic closures P and P' for (K, d) and (K', d) and equip each with a Stasheff structure. Let

$$\bar{\varepsilon} : \mathrm{Hom}_K(P, A) \longrightarrow \Bbbk \quad \text{and} \quad \bar{\varepsilon}' : \mathrm{Hom}_{K'}(P', A') \longrightarrow \Bbbk$$

be the augmentations. Let $\psi : P' \longrightarrow P$ be a K-linear lift of $\mathrm{id}_{\Bbbk}$ and recall the K-linear maps

$$\psi_k : s^{-1}P' \longrightarrow \left(s^{-1}P\right)^{\otimes k}$$

of Proposition 8.4. As above these determine linear maps

$$\xi_k : (s \ker \bar{\varepsilon})^{\otimes k} \longrightarrow s \ker \bar{\varepsilon}', \quad k \geqslant 1,$$

characterized by

$$\left\langle \xi_k \Phi, s^{-1}x \right\rangle = \tau\left\langle \Phi, \psi_k\left(s^{-1}x\right)\right\rangle, \quad x \in P,\ \Psi \in (s \ker \bar{\varepsilon})^{\otimes k}.$$

This sequence lifts to a unique morphism $\xi : T(s \ker \bar{\varepsilon}) \longrightarrow T(s \ker \bar{\varepsilon}')$ of supplemented graded coalgebras and it follows from Proposition 8.4(iii) that $\xi\partial = \partial\xi$; i.e. ξ is a DGC-morphism. Thus $\Omega\xi$ is a DGA morphism, which we denote by

$$\tilde{B}_\phi(\tau) : \tilde{B}_{K'}(A') \longrightarrow \tilde{B}_K(A).$$

Note the commutative diagram

$$\begin{tikzcd} \tilde{B}_K(A) \arrow[r, "B_\phi(\tau)"] & \tilde{B}_{K'}(A') \\ \mathrm{Hom}_K(P, A) \arrow[u, "\lambda", "\simeq"'] \arrow[r, "\mathrm{Hom}(\psi, \tau)"] & \mathrm{Hom}_{K'}(P', A') \arrow[u, "\simeq", "\lambda'"'] \end{tikzcd} \tag{8.7}$$

PROPOSITION 8.8.

(i) *If ϕ and τ are quasi-isomorphisms so is the DGA-morphism $\tilde{B}_\phi(\tau)$.*

(ii) *In particular, the weak equivalence class of $\tilde{B}_K(A)$ is independent of the choice of acyclic closure and its Stasheff structure.*

PROOF. (i) In view of (8.7) we have only to show that $\mathrm{Hom}(\psi, \tau)$ is a quasi-isomorphism. Since ϕ, ψ and τ are quasi-isomorphisms and P, P' are semifree, this is Proposition 2.4.

(ii) Simply apply (i) with $\phi = \mathrm{id}$, $\tau = \mathrm{id}$ but with two different acyclic closures and Stasheff structures. □

Suppose P is a (K, d)-acyclic closure with a weak comultiplication α that is strictly associative. Then P has a Stasheff structure in which δ_1 and δ_2 are derived from d and α and $\delta_k = 0,\ k \geqslant 3$.

On the other hand, $\mathrm{Hom}_K(P, A)$ is itself an augmented DGA with multiplication given by $f \cdot g = \mu \circ (f \otimes g) \circ \alpha$, μ denoting the multiplication in A. Moreover $(T(s \ker \bar{\varepsilon}), \partial)$ is simply the bar construction on the augmented DGA $\mathrm{Hom}_K(P, A)$. Hence $\tilde{B}_K(A) = \Omega B(\mathrm{Hom}_K(P, A))$.

We can now apply [12, Proposition 2.14] to obtain a DGA quasi-isomorphism

$$\sigma : \tilde{B}_K(A) \xrightarrow{\simeq} \mathrm{Hom}_K(P, A)$$

such that $\sigma \circ \lambda = \mathrm{id}$. If we apply this to the case $P = B(K; K)$ we obtain $\tilde{B}_K(A) \simeq B_K(A)$.

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CHAPTER 17

Real and Rational Homotopy Theory

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1. Introduction

In this chapter, we give an exposition of the Quillan, Sullivan rational homotopy theory ([14], [18]) and the authors extension of this theory to real homotopy theory ([4], [5], [6]). The treatment is via the Sullivan approach emphasizing differential forms. We assume the reader is familiar with the standard material in algebraic topology: homology, cohomology, Serre spectral sequence, homotopy groups ([16]) and with differential forms on manifolds ([22]). In addition, a knowledge of simplicial sets ([13]) would be very helpful.

We give in this introductory section, a quite detailed overall picture of real-rational homotopy theory recalling along the way classical deRham cohomology, simplicial sets and, in the context of simplicial sets, function spaces, fibrations, the Kan extension condition and twisted cartesian products. To fully deal with real homotopy theory, it is essential that one uses simplicial *spaces* and continuous cohomology. We take up these issues very briefly at the end of this section and seriously in Section 2. We begin this section with deRham cohomology with real coefficients and carry this as far as we can without continuous cohomology. To simplify the exposition and still capture the main ideas, we then shift to rational coefficients and nilpotent simplicial sets. In this context, we develop four theorems which in our view form the foundation of real and rational theory. We conclude this section by setting forth our most general setting for this foundational material.

We begin by recalling the deRham cohomology groups of a manifold and their relation with singular cohomology ([22]). Suppose M is a smooth manifold and $\Omega^*(M)$ is the differential graded algebra of smooth differential forms on M. On a neighborhood of M with coordinates $x_1, x_2, \dots, x_n$ an element $\omega \in \Omega^p(M)$ is given by

$$\sum a_{i_1 i_2 \dots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

where the a's are C^∞ functions of $x_1, \dots, x_n$ and $d\omega \in \Omega^{p+1}(M)$ is given by

$$d\omega = \sum \frac{\partial a_{i_1 i_2 \dots i_p}}{\partial x_j} \, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Then $d^2 = 0$ and the deRham cohomology group, $H^p_{\mathrm{dR}}(M)$, is defined by

$$H^p_{\mathrm{dR}}(M) = \frac{\{\omega \in \Omega^p(M) \mid d\omega = 0\}}{\{d\alpha \mid \alpha \in \Omega^{p-1}(M)\}},$$

that is, the closed forms modulo the exact forms. The product operation on differential forms induces a graded algebra structure on $\Omega^*(M)$ and on $H^*_{\mathrm{dR}}(M)$ and, according to the classical deRham Theorem, this latter algebra is isomorphic to the singular cohomology algebra, $H^*(M; R)$. We next describe a map giving this isomorphism.

Let Δ^q be the standard q-simplex:

$$\Delta^q = \left\{(t_0, t_1, \ldots, t_q) \in R^{q+1} \mid t_i \geqslant 0,\ \sum t_i = 1\right\}.$$

Let $\Delta_q^\infty(M)$ be the set of C^∞ maps of Δ^q into M and $C^q(M;R)$ the vector space of all real valued functions on $\Delta_q^\infty(M)$ with the usual cup product and coboundary operator δ. Using $C^q(M;R)$ and δ to form cohomology gives the usual singular cohomology. Define

$$\Psi : \Omega^p(M) \to C^p(M;R)$$

by

$$\Psi(\omega)(T) = \int_{\Delta^p} T^*\omega.$$

Stokes theorem states that $\Psi d = \delta\Psi$ and hence Ψ induces a mapping

$$\Psi_* : H^p_{\mathrm{dR}}(M) \to H^p(M;R)$$

which the deRham Theorem asserts is an algebra isomorphism.

In a broad sense, the study of real and rational homotopy theory generalizes the application of differential forms in two directions. First of all, real and rational versions of $\Omega^*(X)$ are defined for arbitrary topological spaces X. The important features of these extended deRham complexes is that they have a commutative multiplication (in the graded sense) and that the corresponding cohomology algebras are isomorphic to the singular cohomology algebras of the space X. The second direction is the study of other topological invariants such as homotopy groups and homotopy type using differential forms.

Defining a deRham complex for topological spaces is quite straightforward. By way of motivation, we first describe a variant of $H^p_{\mathrm{dR}}(M)$ utilizing a triangulation of M. Suppose K is an oriented simplicial complex (the vertices of each simplex are ordered) and $t : |K| \approx M$ a homeomorphism giving a smooth triangulation of M (t is C^∞ on each simplex $|s| \subset |K|$, $s \in K$). Let $\Omega_t^p(M)$ be the set of all functions ω which assign to each simplex $t(|s|) \subset M$ a differential p-form $\omega(s)$ on $t(|s|)$ such that if s' is a face of s and $i : t(|s'|) \subset t(|s|)$ is the inclusion, then $\omega(s') = i^*\omega(s)$. The differential and product operations carry over to $\Omega_t^*(M)$ making it into a differential graded algebra and its homology (homology always means the quotient of the kernel of the differential by the image of the differential) into a graded commutative algebra. Let $C^q(K;R)$ be the cochain complex of the oriented simplicial complex K. Then we define

$$\Psi : \Omega_t^p(M) \to C^p(K;R)$$

by

$$\Psi(\omega)(s) = \int_{t|s|} \omega(t|s|)$$

where s is an oriented p-simplex of K. Not surprisingly, Ψ can be shown to induce an isomorphism in homology.

Note that $\Omega_t^p(M)$ really depends only on K and could be so expressed by defining $\Omega^p(K)$ to be all functions assigning to each simplex $s \in K$ a p-form $w(s)$ on $|s|$ such that if $s' \subset s$, then $\omega(s')$ is the restriction of $\omega(s)$ to s'. This definition does not depend on $|K|$ being a manifold. Indeed, Ψ makes sense for any simplicial complex, and as we will see, induces an isomorphism in homology. We now generalize this idea to arbitrary spaces via singular cohomology.

For any space X, let $\Delta_q(X)$ be the set of all singular q-simplices $T : \Delta^q \to X$ of X. Let $e_i : \Delta^{q-1} \to \Delta^q$ and $d_i : \Delta^{q+1} \to \Delta^q$, $i = 0, 1, \ldots, q$, be the usual face inclusions and degeneracy projections,

$$e_i(t_0, \ldots, t_{q-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{q-1}),$$

$$d_i(t_0, \ldots, t_{q+1}) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{q+1})$$

and define face and degeneracy mappings

$$\partial_i : \Delta_q(X) \to \Delta_{q-1}(X),$$

$$s_i : \Delta_q(X) \to \Delta_{q+1}(X),$$

$i = 0, 1, \ldots, q$, by $\partial_i T = T \circ e_i$, $s_i T = T \circ d_i$. For G an abelian group, let

$$C^q(X; G) = \{u : \Delta_q(X) \to G \mid u \circ s_i = 0\}$$

be the singular cochain complex with the usual coboundary and cup product operation when G is a ring. Let $\Omega^p(X)$ be the set of all functions ω which assign to each q-simplex $T \in \Delta_q(X)$ an element $\omega(T)$ of $\Omega^p(\Delta^q)$, satisfying $\omega(\partial_i T) = e_i^* \omega(T)$ and $\omega(s_i T) = d_i^* \omega(T)$. The differential and product on $\Omega^*(\Delta^q)$ give a differential and product on $\Omega^*(X)$ making it into a differential graded commutative algebra over R. We define the deRham cohomology of X by

$$H^p_{\mathrm{dR}}(X) = H_p\big(\Omega^*(X), d\big).$$

Let $\Psi : \Omega^p(X) \to C^p(X; R)$ be given by

$$\Psi(\omega)(T) = \int_{\Delta^p} \omega(T)$$

for $T \in \Delta_p(X)$. Then $\Psi d = \delta\Psi$ and we have

THEOREM 1.1. *The mapping Ψ induces an algebra isomorphism*

$$H^*_{\mathrm{dR}}(X) \approx H^*(X; R).$$

This result follows from Theorem 1.3 below. Thus we have attained our first objective, namely, generalizing deRham cohomology to arbitrary spaces. We next transform the notions defined above into simplicial set language. The collection of sets, $\Delta_q(X)$, $q = 0, 1, 2, \ldots$, and operations ∂_i, s_i is the prototype of a simplicial set. More generally, we have the following.

DEFINITION 1.2. For any category $\mathcal{C}$, a simplicial $\mathcal{C}$ is a collection

$$N = \{N_q, \partial_i, s_i,\ q = 0, 1, \ldots\}$$

where each N_q is an object in $\mathcal{C}$ and $\partial_i : N_q \to N_{q-1}$ and $s_i : N_q \to N_{q+1}$, $0 \leqslant i \leqslant q$, are mappings in $\mathcal{C}$, the face and degeneracy mappings. These mappings satisfy

$$\begin{aligned} \partial_i\partial_j &= \partial_{j-1}\partial_i, \quad i < j, \\ s_i s_j &= s_{j+1} s_i, \quad i \leqslant j, \\ \partial_i s_j &= s_{j-1}\partial_i, \quad i < j, \\ &= \text{identity}, \quad i = j, j-1, \\ &= s_j \partial_{i-1}, \quad i > j+1. \end{aligned}$$

(These are just the relations ∂_i, s_i satisfy on $\Delta_q(X)$.) A simplicial map $f : N \to M$ is a collection $f_q : N_q \to M_q$ commuting with ∂_i and s_i.

We denote the category of simplicial $\mathcal{C}$'s by $\Delta\mathcal{C}$. For any $\mathcal{C}$, $A \in \mathcal{C}$ means A is an object of $\mathcal{C}$ and, for $A, B \in \mathcal{C}$, (A, B) is the set of morphisms in $\mathcal{C}$. We let $\mathcal{S}$ denote the category of sets and hence $\Delta\mathcal{S}$ denotes the category of simplicial sets.

Let $\mathcal{A}$ be the category of differential non-negatively graded commutative algebras with unit over R. Thus if $A \in \mathcal{A}$, $A = \{A^p, d\}$, $A^p = 0$, $p < 0$, the product $A^p \otimes A^q \to A^{p+q}$ satisfies $ab = (-1)^{pq}ba$ and $d : A^p \to A^{p+1}$ is a derivation in the graded sense. Central to this theory are the vector spaces $\Omega^p_q = \Omega^p(\Delta^q)$, the smooth differential p-forms on the standard q-simplex Δ^q. The maps $e_i : \Delta^{q-1} \to \Delta^q$ and $d_i : \Delta^{q+1} \to \Delta^q$ define operations

$$\partial_i = e_i^* : \Omega^p_q \to \Omega^p_{q-1},$$

$$s_i = d_i^* : \Omega^p_q \to \Omega^p_{q+1}$$

making $\Omega^p = \{\Omega^p_q, \partial_i, s_i\}$ into a simplicial vector space. On the other hand, $\Omega_q = \{\Omega^p_q, d\} \in \mathcal{A}$ and hence $\Omega \in \Delta\mathcal{A}$. With this notation, our previous definition of $\Omega^*(U)$ for a space U becomes

$$\Omega^p(U) = (\Delta(U), \Omega^p),$$

the set of simplicial maps of $\Delta(U)$ into Ω^p, where $\Delta(U) = \{\Delta_q(U), \partial_i, s_i\}$ is the singular simplicial set of U. The $\mathcal{A}$ structure on Ω_q carries over to $\Omega^*(U)$ making it an object of $\mathcal{A}$. For any $X \in \Delta S$, we define $\Omega^*(X) \in \mathcal{A}$ by

$$\Omega^*(X) = (X, \Omega^*)$$

and $H^*_{\mathrm{dR}}(X)$ by

$$H^*_{\mathrm{dR}}(X) = H_*\big(\Omega^*(X)\big).$$

Just as above, we obtain a map $\Psi : \Omega^*(X) \to C^*(X; R)$ where

$$C^q(X; R) = \{u : X_q \to R \mid u \circ s_i = 0 \text{ all } i\}$$

and $\delta : C^q(X; R) \to C^{q+1}(X; R)$ is defined in the usual way using the face mappings.

THEOREM 1.3. *The mapping $\Psi : H^*_{\mathrm{dR}}(X) \to H^*(X; R)$ is an algebra isomorphism.*

This result follows from Lemma 1.4 below.

As we will see, the most important feature of this result is that it gives a functorial way of defining $H^*(X; R)$ using a cochain complex which has a graded commutative product, namely $\Omega^*(X)$. In the remainder of this paper, we will be using simplicial methods and $X, Y, \dots$ will denote simplicial objects. In preparation for our main constructions, we give some of the machinery used to prove Theorem 1.3 above.

Let $\Delta[q] \in \Delta S$ be defined by

$$\Delta[q]_p = \{(i_0, i_1, \dots, i_p) \mid 0 \leqslant i_0 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant q\},$$

$$\partial_j(i_0, \dots, i_p) = (i_0, \dots, \hat{i}_j, \dots, i_p),$$

$$s_j(i_0, \dots, i_p) = (i_0, \dots, i_j, i_j, \dots, i_p),$$

and define $e_i : \Delta[q-1] \to \Delta[q]$, $d_i : \Delta[q+1] \to \Delta[q]$ to be the unique simplicial mapping with

$$e_i(0, 1, \dots, q-1) = (0, 1, \dots, i-1, i+1, \dots, q),$$

$$d_i(0, 1, \dots, q+1) = (0, 1, \dots, i, i, \dots, q).$$

For any group (algebra) G, let $C^*(G)$ be the simplicial differential group (algebra) given by

$$C^*(G) = \big\{C^*(\Delta[q]; G), \delta\big\}$$

with face and degeneracy operations defined using the mappings e_i, d_i above. Then, for $X \in \Delta S$,

$$C^*(X;G) = \bigl(X, C^*(G)\bigr)$$

and in Section 3 we show that Ψ above gives a mapping $\psi : \Omega^* \to C^*(R)$ and prove:

LEMMA 1.4. *The mapping $\psi : \Omega^* \to C^*(R)$ is a simplicial mapping which preserves the grading and commutes with differentials (not multiplicative). Furthermore, there is a simplicial mapping $\varphi : C^*(R) \to \Omega^*$ which preserves grading and commutes with differentials and simplicial linear mappings $\gamma : \Omega^p \to \Omega^{p-1}$ satisfying*

$$\psi\varphi = \mathrm{id}, \quad d\gamma + \gamma d = \varphi\psi - \mathrm{id}.$$

The map $\psi : \Omega^* \to C^*(R)$ induces a map

$$\Psi : \Omega^*(X) = \bigl(X, \Omega^*\bigr) \to \bigl(X, C^*(R)\bigr) = C^*(X;R)$$

by composition, $\Psi(\omega) = \psi \circ \omega$, similarly for φ and γ. It then follows that these mappings also satisfy the identities in Lemma 1.4 and hence induce isomorphisms between $H^p(X;R)$ and $H^p_{\mathrm{dR}}(X)$, which proves Theorem 1.3.

We next present a detailed outline of the entire theory in the simplest case, namely, for nilpotent simplicial sets of finite type and we work over Q, the rational numbers instead of over R. We define $\Omega = \Omega_Q$ to be the simplicial subalgebra of Ω consisting of all differential forms

$$\sum a_{i_1,\ldots,i_p} dt_{i_1} \wedge \cdots \wedge dt_{i_p}$$

where the a's are polynomials in the barycentric coordinates $t_0, t_1, \ldots, t_q$ of Δ_q with coefficients in Q. The category $\mathcal{A}$ consists of differential, graded algebras as before, but over Q instead of R. Then everything we have said about Ω, $\mathcal{A}$, Ψ, φ, γ, C^*, H^* carries over for Q. For example:

THEOREM 1.5. *For any simplicial set X, the mapping $\psi : \Omega^*_Q \to C^*(Q)$ induces a map $\Psi : \Omega^*_Q(X) \to C^*(X;Q)$ which in turn induces an algebra isomorphism*

$$\Psi_* : H_*\bigl(\Omega_Q(X)\bigr) \to H^*(X;Q).$$

Note that ψ is defined over Q because the integral of a rational polynomial is a rational number. For example, for $t_1^k dt_1 \in \Omega_1^1$,

$$\psi\bigl(t_1^k dt_1\bigr)\bigl((0,1)\bigr) = \int_{\Delta_1} t_1^k \, dt_1 = \frac{t_1^{k+1}}{k+1}\bigg|_0^1 = \frac{1}{k+1}.$$

We next review the results that we will need about simplicial sets.

An analogue of a space of functions in ΔS can be defined as follows: Suppose $X, Y \in \Delta S$. Then $\mathcal{F}(X,Y) \in \Delta S$ is defined by

$$\mathcal{F}(X,Y)_q = (\Delta[q] \times X, Y)$$

where ∂_i and s_i are defined to be composition with the maps e_i and d_i on $\Delta[q]$. The notion of a fibration is defined by the following version of the homotopy lifting property: a mapping $p : E \to B$ in ΔS is a *fibration* if for $b \in B_q$ and $e_0, \dots, e_{j-1}, e_{j+1}, \dots, e_q \in E_{q-1}$ such that

$$pe_i = \partial_i b, \quad i \neq j,$$

$$\partial_k e_i = \partial_{i-1} e_k, \quad k < i, j, \ k \neq j,$$

there exists an $e \in E_q$ such that $\partial_i e = e_i$ and $pe = b$. A simplicial set X is said to be Kan (or to satisfy the Kan extension condition) if $X \to \mathrm{pt}$ is a fibration. (We later slightly modify this definition in order to extend it to simplicial spaces.)

A particularly useful class of fibrations is the twisted Cartesian products (TCP). Let B and F be simplicial sets and G a simplicial group acting on F. A twisting function τ is a sequence of mappings $\tau = \tau_q : B_q \to G_{q-1}$ satisfying the following identities:

$$\tau(\partial_1 b) = \tau(\partial_0 b)\partial_0\tau(b),$$

$$\tau(\partial_i b) = \partial_{i-1}\tau(b), \quad i > 1,$$

$$\tau(s_0 b) = 1_{q-1},$$

$$\tau(s_i b) = s_{i-1}\tau(b), \quad i > 0.$$

The twisted Cartesian product $B \times_\tau F$ is the simplicial set with

$$(B \times_\tau F)_q = B_q \times F_q$$

and whose face and degeneracy mappings are the product of those in B and F except that

$$\partial_0(b, f) = (\partial_0 b, \tau(b)\partial_0 f)$$

(see [13]). The theorems about fibrations which we will have to prove in each of our variations are:

THEOREM 1.6. *If $p : E \to B$ is a fibration then $p_\# : \mathcal{F}(X, E) \to \mathcal{F}(X; B)$ is a fibration. Hence, if Y is Kan, so is $\mathcal{F}(X,Y)$.*

THEOREM 1.7. *If G is a simplicial group then G is Kan.*

THEOREM 1.8. *If F is Kan, then the projection $B \times_\tau F \to B$ is a fibration.*

A homotopy between two maps $f, g : X \to Y$ is a map $F : X \times \Delta[1] \to Y$ satisfying

$$F(x, s_0^q 0) = f(x),$$

$$F(x, s_0^q 1) = g(x)$$

for $x \in X_q$ or equivalently, $F \in \mathcal{F}(X,Y)_1$ with $\partial_0 F = g$ and $\partial_1 F = f$. It follows from Theorem 1.6 that homotopy between mappings is an equivalence relation when Y is Kan. If $Y \in \Delta S$ is Kan, one can also define the homotopy groups of Y as follows. For $y_0 \in Y_0$,

$$\pi_n(Y, y_0) = \{y \in Y_n : \partial_i y = s_0^{n-1} y_0,\ i = 0, \ldots, n\} / \sim$$

where $y \sim y'$ if there is a $z \in Y_{n+1}$ with $\partial_0 z = y$, $\partial_1 z = y'$, and $\partial_i z = s_0^n y_0$, $i > 1$. In particular, the set of homotopy classes of mappings from X to Y is given by

$$[X, Y] = \pi_0(\mathcal{F}(X, Y)).$$

We can also define simplicial set analogues of the Eilenberg–MacLane spaces and the contractible fibrations over them as follows:

$$K(G,n)_q = Z^n(\Delta[q]; G) = \{u \in C^n(G)_q \mid \delta u = 0\},$$

$$E(G,n) = C^n(G),$$

$$p : E(G,n) \to K(G,n+1), \quad pu = \delta u.$$

Since $K(G,n)$ and $E(G,n)$ are simplicial groups they are Kan. In addition, it is easy to check that

$$\begin{aligned} \pi_i(K(G,n)) &= G, \quad i = n, \\ &= 0, \quad i \neq n, \\ \pi_i(E(G,n)) &= 0, \quad \text{all } i. \end{aligned}$$

In Section 4 we prove:

LEMMA 1.9. *The mapping $p : E(G,n) \to K(G,n+1)$ is a principal TCP with group and fibre $K(G,n)$.*

If $X \in \Delta S$ and $k : X \to K(G,n)$, then k may be viewed as a cocycle on X, that is, $k \in (X, C^n(G)) = C^n(X; G)$ with $\delta k(x) = 0$ all $x \in X$. One easily sees that this correspondence gives the Hopf–Whitney theorem:

THEOREM 1.10. *For $X \in \Delta S$, we have $[X, K(G,n)] \approx H^n(X; G)$.*

For $k : X \to K(G, n+1)$, let $p : X_k \to X$ be the induced fibration with fibre $K(G, n)$:

$$\begin{array}{ccc} X_k & \longrightarrow & E(G,n) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ X & \xrightarrow{\ k\ } & K(G,n+1) \end{array}$$

That is,

$$(X_k)_q = \{(x,u) \in X \times C^n(\Delta[q], G) \mid \delta u = k(x)\}.$$

DEFINITION 1.11. We say that $X \in \Delta S$ is nilpotent and finite type if there is a sequence $\{X_n\}$, $X_n \in (\Delta T_\pi)_{X_0}$, $X_0 = \mathrm{pt}$, a sequence $1 \leqslant m_1 \leqslant m_2 \leqslant \cdots$ of integers with $m_n \to \infty$ as $n \to \infty$, a sequence $\{G_n\}$ of finitely generated abelian groups, and a sequence of mappings $k^{n+1} : X_{n-1} \to K(G_n, m_n + 1)$ such that $X_n = (X_{n-1})_{k^{n+1}}$ and $\varprojlim X_n$ is homotopy equivalent to X.

Let ΔS_{NF} denote the subcategory of ΔS of such simplicial sets.

REMARK 1.12. If U is a path connected topological space, $\Delta(U)$ is nilpotent and finite type if and only if there are $\pi_1(U)$ submodules π_n^ℓ of $\pi_n(U)$ such that $\pi_n^1 = \pi_n(U)$, $\pi_n^\ell \supset \pi_n^{\ell+1}$, $\pi_n^\ell = 0$ for $\ell > N_n$ and $\pi_n^\ell/\pi_n^{\ell+1}$ are finitely generated trivial $\pi_1(U)$ modules for all ℓ and $n \geqslant 1$. In particular this is true if $\pi_1(U) = 0$ and $\pi_n(U)$ is finitely generated for all n.

DEFINITION 1.13. A map $g : A \to B$ in $\mathcal{A}$ is a weak equivalence if $g_* : H_*(A) \approx H_*(B)$. A map $f : X \to Y$ in ΔS is a weak Q-equivalence if $\Omega f : \Omega Y \to \Omega X$ is a weak equivalence, or equivalently, $f^* : H^*(Y; Q) \approx H^*(X; Q)$.

If X is connected, nilpotent and finite type, a Q localization of X consists of a weak Q-equivalence $h : X \to X_Q$ where $\pi_n(X_Q)$ is uniquely divisably for all $n \geqslant 1$ [11]. If $\varprojlim X_n$ is as in Definition 1.11, then we can define $X_Q = \varprojlim X_{n,Q}$ where $h_n : X_n \to X_{n,Q}$ is a weak equivalence defined by induction on n satisfying $X_{0,Q} = \mathrm{pt}$ and $X_{n,Q} = X_{k_Q^n}$ where k_Q^n is such that the diagram

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{\ k^n\ } & K(G_n, m_n) \\ \Big\downarrow{\scriptstyle h_{n-1}} & & \Big\downarrow \\ X_{n-1,Q} & \xrightarrow{\ k_Q^n\ } & K(G_n \otimes Q, m_n) \end{array}$$

is homotopy commutative. The map h_{n-1} being a weak equivalence implies k_Q^n exists, the commutativity of the above diagram gives the existence of h_n and a Serre spectral sequence comparison argument shows that h_n is a weak equivalence.

To each $A \in \mathcal{A}$, we associate a simplicial set $\Delta(A) \in \Delta S$ given by

$$\Delta(A)_q = (A, \Omega_q) = \mathrm{morph}_{\mathcal{A}}(A, \Omega_q).$$

Composition with ∂_i and s_i on Ω_q defines the face and degeneracy mappings in $\Delta(A)_q$. In general, $\Delta(A)$ is quite mysterious. However, if A is freely generated, then $\Delta(A)$ is well understood ([4], [5]). Indeed, a central feature of rational and real homotopy theory is that the Q and R-localizations above can be realized as $\Delta(A)$ where A is free. Furthermore, one can choose A so that the differentials of elements of A are decomposable, in which case A is said to be *minimal* ([18]). In this case, A is uniquely determined up to isomorphism by X_Q, i.e. the set of homotopy types of Q-local nilpotent, finite type simplicial sets is in one to one correspondence with the set of isomorphism classes of minimal finite type algebras.

We have defined two contravariant functors $\Omega : \Delta S \to \mathcal{A}$ and $\Delta : \mathcal{A} \to \Delta S$ which are, in a sense, adjoints of one another. Let i and j be the mappings

$$i : A \to \Omega\big(\Delta(A)\big), \quad i(a)(\omega) = \omega(a),$$

$$j : X \to \Delta\big(\Omega(X)\big), \quad j(x)(u) = u(x).$$

and let α and β be the mappings

$$\big(A, \Omega(X)\big) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \big(X, \Delta(A)\big)$$

where $\alpha(h) = \Delta(h)j$ and $\beta(f) = \Omega(f)i$. Trivial manipulations yield

THEOREM 1.14. *For α, β as above, we have $\alpha\beta = \mathrm{id}$ and $\beta\alpha = \mathrm{id}$.*

Denote $\alpha(h)$ by $\hat{h}$ and $\beta(f)$ by $\hat{f}$. Let $\mathcal{V}$, $\mathcal{V}_{\mathrm{G}}$, $\mathcal{V}_{\mathrm{DG}}$ and $\mathcal{V}_{\mathrm{F}}$ denote the categories of vector spaces over Q, graded vector spaces over Q, differential graded vector spaces over Q, and finite dimensional vector spaces over Q respectively. If $V \in \mathcal{V}$, let $V(n) \in \mathcal{V}_{\mathrm{G}}$ be defined by

$$\begin{aligned} V(n)_q &= V \quad \text{if } q = n, \\ &= 0 \quad \text{if } q \neq n. \end{aligned}$$

If $V \in \mathcal{V}_{\mathrm{G}}$, let $S(V)$ and $V^* \in \mathcal{V}_{\mathrm{G}}$ be defined by $S(V)_q = V_{q-1}$ and $V^q = Hom(V_q, Q)$. Let $Q(V) \in \mathcal{V}_{\mathrm{G}}$ be defined as the quotient of the tensor algebra generated by V^* modulo the ideal generated by elements of the form $a \otimes b - (-1)^{pq} b \otimes a$ for $a \in V^p$ and $b \in V^q$. We also denote $Q(V)$ by $Q[V^*]$. If $\{v_i^q\}$ is a basis for V^*, we will write $Q[\{v_i^q\}]$ for $Q[V^*]$ in which case $Q[V^*]$ is the tensor product of the polynomial algebra generated by v_i^q, q even, with the exterior algebra on v_i^q, q odd. If $V \in \mathcal{V}$, $A \in \mathcal{A}$, and $\lambda : V^* \to A^{n+1}$ is a linear map with $d \circ \lambda = 0$, then $A(V, \lambda) \in \mathcal{A}$ is defined to be the algebra $A \otimes Q(V(n))$ with derivation d defined by $da = da$, $a \in A$, and $dv^* = \lambda(v^*)$, $v^* \in V^*$. Let $Q(V, n) = Q(V, \lambda)$ where $\lambda : V \to Q(0)^{n+1} = 0$, that is $Q(V, n) = Q(V(n))$ with $d = 0$. Note that $\lambda^* : V \to A^{n+1}$ extends uniquely to an $\mathcal{A}$ map $\lambda : Q(V, n+1) \to A$, giving $\Delta(\lambda) : \Delta(A) \to \Delta(Q(V, n+1))$ and

$$\Delta\big(Q(V, n+1)\big)_q = \big(Q(V, n+1), \Omega_q\big)$$

$$= \{u : V^* \to \Omega_q^{n+1} \mid du = 0\}$$
$$= \{\omega \in \Omega_q^{n+1} \mid d\omega = 0\} \otimes V.$$

Let $Z_n(\Omega) \in \Delta S$ be defined by $Z_n(\Omega)_q = \{\omega \in \Omega_q^n \mid d\omega = 0\}$ and similarly for $Z_n(C^*(R))$. Then

$$Z_n(\Omega) \otimes V = \Delta(Q(V,n)),$$
$$Z_n(C^*(Q)) \otimes V = K(V,n).$$

The mapping ψ and φ from Lemma 1.4 define mapping $\bar{\psi} : Z_n(\Omega) \otimes V \to Z_n(C^*(R) \otimes V$ and $\bar{\varphi}$ in the opposite direction such that $\bar{\psi}\bar{\varphi} =$ identity. In Section 3, we show that the mapping γ of Lemma 1.4 yields a homotopy of $\bar{\varphi}\bar{\psi}$ to the identity. Thus $\Delta(Q(V,n))$ is another model for $K(V,n)$. We also prove the following.

LEMMA 1.15. *The mapping $d : \Omega_n \otimes V \to Z_{n+1}(\Omega) \otimes V$ is a principal TCP with group and fibre $Z_n(\Omega) \otimes V$.*

Since

$$\Delta(A(V,\lambda)) = (A(V,\lambda), \Omega_q)$$
$$= \{(u,w) \in \Delta(A) \times (V^*, \Omega_q^n) \mid dw = u\lambda\}$$

we have:

THEOREM 1.16. *The simplicial set $\Delta(A(V,\lambda))$ is the total space of the induced fibration*

$$\begin{array}{ccc} \Delta(A(V,\lambda)) & \longrightarrow & \Omega_n \otimes V \\ \downarrow p & & \downarrow \\ \Delta(A) & \xrightarrow{\Delta(\lambda)} & \Delta(Q(V,n+1)) = Z_{n+1}(\Omega) \otimes V \end{array}$$

where p is induced by the inclusion of A in $A(V,\lambda)$ and $Z_n(\Omega) \otimes V$ is its fibre. Hence, $\Delta(A(V,\lambda))$ is a TCP,

$$\Delta(A(V,\lambda)) \approx \Delta(A) \times_\tau (Z_n(\Omega) \otimes V).$$

DEFINITION 1.17. We say that $A \in \mathcal{A}$ is free, nilpotent and of finite type (FNF) if A is the union of subalgebras $\{A_n\}$, $A_n \subset A_{n+1}$ with $A_0 = Q$ and $A_n = A_{n-1}(V_n(m_n), \lambda_n)$ where $V_n \in \mathcal{V}_{\mathrm{F}}$, $m_1 \leqslant m_2 \leqslant \cdots$ is a sequence of integers with $m_n \to \infty$ as $n \to \infty$, and $\lambda_n : V_n^* \to A_{n-1}^{m_n+1}$ with $d \circ \lambda = 0$. We say that A is minimal if, in addition, da is decomposable for all $a \in A$.

A corollary of Theorem 1.16 and Lemma 1.15 is:

THEOREM 1.18. *If A is FNF, then $\Delta(A)$ is Kan, nilpotent and finite type.*

The following is the main theorem in this subject:

THEOREM 1.19. *If A is FNF, then*

$$i_* : H_*(A) \approx H_*\big(\Omega(\Delta A)\big) = H^*\big(\Delta(A); Q\big).$$

The proof proceeds as follows: The standard Serre spectral sequence argument for computing $H^*(K(Q,n);Q)$ ([4, Section 8]) applied to the fibration

$$Z_n(\Omega) \otimes V \to \Omega_n \otimes V \to Z_{n+1}(\Omega) \otimes V$$

yields Theorem 1.19 when $A = Q(V,n)$ by induction on n. The general case then follows by proving it for A_n by induction on n. The inductive step follows from:

LEMMA 1.20. *If $i : A \to \Omega\Delta(A)$ is a weak equivalence, then the same is true for $i : A(V,\lambda) \to \Omega\Delta A(V,\lambda)$.*

To prove Lemma 1.20, filter $A(V,\lambda)$ by letting

$$F^p = \left\{ \sum a_i u_i \,\middle|\, a_i \in A, u_i \in Q(V) \text{ and } \dim u_i \geqslant p \right\}.$$

Note that $dF^p \subset F^p$ and $F^p \subset F^{p-1}$. Let $\tilde{F}^p \subset C^*(\Delta(A(V,\lambda)); Q)$ be the Serre filtration for the fibration in Theorem 1.16, that is

$$\tilde{F}^p = \left\{ u \in C^*\big(\Delta\big(A(V,\lambda); Q\big)\big) \middle| u\big(\pi^{-1}\big(\Delta(A)^{p-1}\big)\big) = 0 \right\}$$

where $\pi : \Delta(A(V,\lambda)) \to \Delta(A)$ is the projection and $\Delta(A)^{p-1}$ is the $(p-1)$-skeleton of $\Delta(A)$. A simple calculation yields:

LEMMA 1.21. *The map*

$$A(V,\lambda) \xrightarrow{i} \Omega\big(\Delta A(V,\lambda)\big) \xrightarrow{\psi} C^*\big(\Delta\big(A(V,\lambda)\big); Q\big)$$

is filtration preserving and the induced mapping on spectral sequences is an isomorphism at the E_2 level.

Lemma 1.20 now follows from Lemma 1.21. Theorem 1.19 then gives:

COROLLARY 1.22. *If A is FNF, then $h : A \to \Omega(X)$ is a weak equivalence if and only if $\hat{h} : X \to \Delta(A)$ is a weak Q-equivalence.*

Theorem 1.16 and the exact sequence of homotopy groups for a fibration yield

THEOREM 1.23. *If $A = (Q(V), d)$ is FNF and minimal, then $\pi_n(\Delta(A)) = V_n$ for $n > 1$ and*

$$V_1 = \sum \Gamma^i / \Gamma^{i+1}$$

where $\{\Gamma^i\}$ is the lower central series for $\pi_1(\Delta(A))$ [5].

In Section 5, Theorem 5.9, we prove

LEMMA 1.24. *Suppose $X \in \Delta\mathcal{S}, A \in \mathcal{A}$, $h : A \to \Omega(X)$ is a weak equivalence and $k : X \to K(V, n+1)$, where $V \in \mathcal{V}_{\mathbb{F}}$. Let $\lambda : V^* \to A^{n+1}$ be a map which corresponds to $[k]$ under the composite*

$$\begin{aligned}(V^*, Z_{n+1}(A)) &\to H_{n+1}(A \otimes V) \to H_{n+1}(\Omega(X) \otimes V) \\ &\to H_{n+1}(C^*(X) \otimes V) = H^{n+1}(X; V) = [X, K(V, n+1)].\end{aligned}$$

Then there exists a weak equivalence $\bar{h}$ yielding a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d] & \Omega(X) \arrow[d] \\ A(V,\lambda) \arrow[r, "\bar{h}"] & \Omega(X_k) \end{tikzcd}$$

where X_k is defined just after the statement of Theorem 1.10.

Applying Lemma 1.24 as above gives the first part of:

COROLLARY 1.25. *If $X \in \Delta\mathcal{S}$ is nilpotent and finite type, then there exists a minimal FNF $A \in \mathcal{A}$ and a weak equivalence $h : A \to \Omega(X)$ and hence a weak Q-equivalence $\hat{h} : X \to \Delta(A)$. Furthermore A is unique up to an isomorphism.*

The uniqueness of A follows from:

REMARK 1.26. If $A \in \mathcal{A}$, $H_0(A) = Q$, $H_1(A) = 0$ and $H_n(A)$ is finite dimensional for all n, it is easy to construct a FNF minimal B with a weak equivalence $h : B \to A$. One can also show that if $A \in \mathcal{A}$, $h_i : B_i \to A$, $i = 1, 2$, are weak equivalences with B_1 and B_2 minimal and FNF, then there is an isomorphism $g : B_1 \to B_2$ such that $h_2 g$ and h_1 are homotopic ([18]). Thus, in the simply connected case, one does not need Lemma 1.24. However, we do need Lemma 1.24 to deal with the nilpotent case. In later sections, we will have a group acting on everything in sight in which case Lemma 1.24 still holds but the algebraic argument above is not available.

We conclude this description of rational homotopy theory with a discussion of how mappings behave under the Δ functor. For $A, B \in \mathcal{A}$, define $\mathcal{F}(A, B) \in \Delta\mathcal{S}$ by $\mathcal{F}(A, B)_q = (A, \Omega_q \otimes B)$. For $A, B, C \in \mathcal{A}$ define the composition mapping

$$c : \mathcal{F}(A, B) \times \mathcal{F}(B, C) \to \mathcal{F}(A, C)$$

to be the composite

$$\begin{aligned}(A, \Omega_q \otimes B) \times (B, \Omega_q \otimes C) &\xrightarrow{\mathrm{id} \times b} (A, \Omega_q \otimes B) \times (\Omega_q \otimes B, \Omega_q \otimes C) \\ &\xrightarrow{c_0} (A, \Omega_q \otimes C)\end{aligned}$$

where $b(u)(\omega\otimes b) = (\omega\otimes 1)(u(b))$ and c_0 is the usual composition. Taking $C = Q$, then $\mathcal{F}(A,C) = \Delta(A)$ and the adjoint of c gives

$$\Delta : \mathcal{F}(A,B) \to \mathcal{F}\bigl(\Delta(B),\Delta(A)\bigr).$$

Let $\alpha : \mathcal{F}(A,\Omega(X)) \to \mathcal{F}(X,\Delta(A))$ be the adjoint of the composite

$$\mathcal{F}\bigl(A,\Omega(X)\bigr)\times X \xrightarrow{\mathrm{id}\times j} \mathcal{F}\bigl(A,\Omega(X)\bigr)\times\mathcal{F}\bigl(\Omega(X),Q\bigr) \xrightarrow{c} \mathcal{F}(A,Q).$$

DEFINITION 1.27. If $X, Y \in \Delta S$ are Kan, the mapping $f : X \to Y$ is a weak equivalence if $f_* : \pi_n(X,x_0) \to \pi_n(Y,f(x_0))$ is an isomorphism for all $x_0 \in X_0$ and all $n \geqslant 0$.

Theorems 1.16 and 1.17 yield

THEOREM 1.28. *If $A, B \in \mathcal{A}$ and A is FNF, then $\mathcal{F}(A,B)$ is Kan and hence $[A,B] = \pi_0(\mathcal{F}(A,B))$ is defined.*

In Section 6, we prove:

THEOREM 1.29. *If $A,B,C,D \in \mathcal{A}$, A and B are FNF, $X \in \Delta S$ is path connected and Kan and $g : C \to D$ is a weak equivalence, then the following are weak equivalences:*

(i) $\Delta : \mathcal{F}(A,B) \to \mathcal{F}(\Delta(B),\Delta(A))$,
(ii) $\alpha : \mathcal{F}(A,\Omega(X)) \to \mathcal{F}(X,\Delta(A))$,
(iii) $g_* : \mathcal{F}(A,C) \to \mathcal{F}(A,D)$.

The proof of Theorem 1.29 has an easy part and a hard part. The easy part is proving that

$$\mathcal{F}\bigl(C(V,\lambda),D\bigr) \to \mathcal{F}(C,D)$$

is a fibration with fibre $\mathcal{F}(Q(V,n),D)$. Taking $A = \varprojlim A_n$, one proves the theorem for $A = A_n$ by induction on n and the usual five lemma argument on homotopy groups. The hard part is proving the maps Δ, α and c induce isomorphism on π_0. Interestingly, for Δ one needs Theorem 1.19 for B as well as A.

We conclude this introduction by giving our most general setting for these discussions and indicate how it specializes to various special cases of interest.

We view Theorems 1.19, 1.23, 1.25, and 1.29 as constituting the foundation of rational homotopy theory for finite type nilpotent simplicial sets. We extend these results in two directions. One direction consists in replacing Q by R, simplicial sets by simplicial spaces, algebras by topological algebras (R with its usual topology) and requiring all maps to be continuous. This yields continuous cohomology and the theorems cited above carry over more or less unchanged to this new context. Simplicial spaces and continuous cohomology naturally arise in various situations some of which we describe in Section 8 including characteristic classes of foliations. The most convincing motivation for this setting is that, when one includes topologies and continuity, a generalized Van

Est Theorem holds, namely, $H^*(K(R,n);R)$ is a polynomial or exterior algebra on one generator in dimension n, according as n is even or odd. The rational and real theories are then so similar that we can do them simultaneously. Hereafter, we let $\mathbf{R}$ denote R or Q. When $\mathbf{R} = Q$ our theorems apply to ΔS and when $\mathbf{R} = R$ to $\Delta\mathcal{T}$ where $\mathcal{T}$ is the category of topological spaces with compactly generated topologies. (See Section 2.) We also view simplicial sets as simplicial spaces with the discrete topology so the real theory also applies to ΔS. In Section 8, we compare the real and rational theories on ΔS. They turn out to be substantially different.

Our second direction for extending this theory is to eliminate the nilpotent requirement. We do this by fixing a group π, considering path connected $X \in \Delta\mathcal{T}$ with $\varepsilon : \pi_1(X) \approx \pi$ and localizing X by fibrewise localizing the map X to $B\pi$ defined by ε. The problem is then to make sense out of minimal models in this context. A first approximation would be to take a minimal model for $\tilde{X}$, the universal covering of X. However, this is too crude. For example, one loses the action of π on the higher homotopy groups so that X and $\tilde{X} \times B\pi$ would have the same minimal model. Including a π action on the minimal model for $\tilde{X}$ will give a satisfactory definition of a minimal model for X when π is finite. However, this model will not in general, contain enough information to include all possible k invariants, for example, for adding Z as π_2 to $B\pi$ when $\pi = Z$. A strategy that works for all π is to replace $\mathbf{R}$ by a DG algebra A_0 with a π action which models $\Omega(E\pi)$ (for all local coefficients). One can then define Δ_π, Ω_π and minimal models so that the foundational theorems referred to above hold. (See Section 5.)

The notion of localization being considered here may also be viewed as localizing a category with respect to a set of weak equivalences [14]. For the Quillen–Sullivan rational homotopy theory one considers the category ΔS_{NF} of nilpotent simplicial sets of finite type and as weak equivalences, mappings $f : X \longrightarrow Y$ inducing an isomorphism on rational cohomology. For real homotopy theory one enlarges ΔS_{NF} to $\Delta\mathcal{T}_{\mathrm{NF}}$, the category of nilpotent simplicial spaces of finite type and as weak equivalences, maps which induce an isomorphism on continuous cohomology with coefficients in the reals. In this paper, we in effect consider $\Delta S_{0,\mathrm{F}}$, the category of connected simplicial sets with base point and finitely generated homotopy groups and as weak equivalences mappings $f : X \longrightarrow Y$ which induce isomorphisms on fundamental groups and on cohomology with local coefficients in Q vector spaces. We also consider $\Delta\mathcal{T}_{0,\mathrm{F}}$, the category of connected simplicial spaces with base point and locally Euclidean homotopy groups and as weak equivalences mappings $f : X \longrightarrow Y$ which induce isomorphisms on fundamental groups and on continuous cohomology with local coefficients in R vector spaces.

2. The categories $\Delta\mathcal{T}_\pi$ and $\mathcal{A}_\pi$

In this section, we introduce the important categories $\Delta\mathcal{T}_\pi$ and $\mathcal{A}_\pi$. We also prove a basic result (Theorem 2.2) relating function spaces and fibrations in these categories.

Recall that for a category $\mathcal{C}$, $A, B \in \mathcal{C}$ means A and B are objects of $\mathcal{C}$ and (A, B) denotes morph(A, B). We denote by $\mathcal{C}_\pi$ the category of $\mathcal{C}$ objects with π actions and π-equivariant maps as morphisms. An object of $\mathcal{C}_\pi$ is a pair (A, ρ) where $A \in \mathcal{C}$ and ρ is a homomorphism of π into (A, A), required to be continuous when the morphisms of

$\mathcal{C}$ are topologized. If $A \in \mathcal{C}_\pi$, $\bar{A} \in \mathcal{C}$ forgets the π action and, if A has elements, A^π denotes the set of elements fixed under the π action. If $A, B \in \mathcal{C}_\pi$ then $(\bar{A}, \bar{B})$ has the π action given by $gu(a) = g(u(g^{-1}a))$, $g \in \pi$, $u \in (\bar{A}, \bar{B})$. If $A \in \mathcal{C}$, then $\mathcal{C}_A$ and ${}_A\mathcal{C}$ denote the categories of pairs (B, f) where $f : B \longrightarrow A$ and $f : A \longrightarrow B$, respectively, the morphisms being the obvious commutative diagrams.

Let $E\pi \longrightarrow B\pi$ be the usual model for the universal simplicial π-bundle. That is, $B\pi$ the bar construction on π and $E\pi_q = \pi^{q+1}$ with

$$\partial_i(g_0, \dots, g_q) = (g_0, \dots, \hat{g}_i, \dots, g_q),$$

$$s_i(g_0, \dots, g_q) = (g_0, \dots, g_i, g_i, \dots, g_q).$$

It is easy to prove that $B\pi = E\pi/\pi$ where π acts diagonally on $E\pi$.

For a topological space Z, let $k(Z)$ denote Z with its compactly generated topology which by definition means that U is open in $k(Z)$ if and only if $U \cap C$ is open in C for each compact subspace C of Z ([17]). Let $\mathcal{T}$ denote the category of Hausdorff spaces with compactly generated topology ($Z = k(Z)$) and continuous maps. Hereafter, we assume that π as a space is in $\mathcal{T}$. We also view the category $\mathcal{S}$ of sets as a subcategory of $\mathcal{T}$, namely spaces with discrete topology. For spaces X and Y, $\langle X, Y\rangle$ will denote the set of continuous maps of X into Y with the compact open topology. Furthermore, we let $(X, Y) = k\langle kX, Y\rangle = k\langle kX, kY\rangle$. (See [17].) For X and $Y \in \mathcal{T}$, the $\mathcal{T}$ morphisms will be (X, Y). One defines products in $\mathcal{T}$ by $\Pi X_\alpha = k(\Pi_c X_\alpha)$ where Π_c denotes the usual cartesian product. If $X, Y \in \Delta\mathcal{T}$, we topologize (X,Y) as a subspace of $\Pi(X_q, Y_q)$.

We thus have the category $\Delta\mathcal{T}_\pi$. When $\mathbf{R} = R$, $\mathcal{A}$ will denote the category of differential graded algebras over R as in Section 1 which in addition have a compactly generated topology making them topological algebras over R. Thus, if $A \in \mathcal{A}$, $A^p \in \mathcal{T}$. Morphisms in $\mathcal{A}$ are topologized by viewing (A, B) as a subspace of $\Pi(A^p, B^p)$. The tensor product of algebras, $A \otimes B$ is topologized with the strongest compactly generated topology making it a topological algebra and $A \times B$ a subspace. When $\mathbf{R} = Q$, $\mathcal{A}$ is as in Section 1.

We view $\Delta[q]$ and Ω as not having π actions. If $X \in \Delta\mathcal{T}_\pi$ and $A \in \mathcal{A}_\pi$, π acts on $\Delta[q] \times X$ and $\Omega_q \otimes A$ by acting on the second factor. Then, as in Section 1,

$$\mathcal{F}(X, Y)_q = \big(\Delta[q] \times X, Y\big),$$

$$\mathcal{F}(A, B)_q = (A, \Omega_q \otimes B),$$

$\mathcal{F}(A, B) \in \Delta\mathcal{T}$, $\mathcal{F}(\bar{A}, \bar{B}) \in \Delta\mathcal{T}_\pi$ and similarly for $\mathcal{F}(X, Y)$. We define $\Omega(X) \in \mathcal{A}_\pi$ and $\Delta(A) \in \Delta\mathcal{T}_\pi$ by

$$\Delta(A) = (\bar{A}, \Omega),$$

$$\Omega(X) = (\bar{X}, \Omega).$$

The π actions are given by $g\omega(x) = \omega(g^{-1}x)$. It then follows that the maps $i : A \to \Omega(\Delta A))$ and $j : X \to \Delta(\Omega(X))$ are well defined continuous π-equivariant mappings and

Theorem 1.14 holds, that is, $(A, \Omega(X)) = (X, \Delta(A))$. Theorem 1.29 has two versions in this context, one involving function spaces of the form $\mathcal{F}(A, B)$ and the other involving function spaces of the form $\mathcal{F}(\bar{A}, \bar{B})$. In the second of these, the mappings Δ, α, and g_* are equivariant.

If $(X, f), (Y, g) \in (\Delta\mathcal{T}_\pi)_{X_0}$ we define $\mathcal{F}((X, f), (Y, g))_q \subset \mathcal{F}(X, Y)_q$ to consist in all maps $h : \Delta[q] \times X \to Y$ such that the diagram

$$\begin{tikzcd} \Delta[q] \times X \arrow[r, "h"] \arrow[d, "p_2"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & X_0 \end{tikzcd}$$

is commutative. A similar definition holds for $\mathcal{F}$ in ${}_{A_0}\mathcal{A}_\pi$. Again, the maps Δ, α and g_* are well defined and equivariant when bars are put over the objects.

The main goal in this section is to define "fibration" in $\Delta\mathcal{T}_\pi$ so that Theorems 1.6, 1.7 and 1.8 hold. For the remainder of this section, let $\mathcal{K} = \Delta\mathcal{T}_\pi$.

The following is an improved version of the definition of fibration in the category $\mathcal{K}$. (Compare [4, Definition 6.1].) Suppose $p : E \to B$ is a mapping in $\mathcal{K}$. Let $E_{qi} \subset E^q_{q-1} \times B_q$ consist of all elements

$$(e_0, e_1, \ldots, \hat{e}_i, \ldots, e_q, b) \in E^q_{q-1} \times B_q$$

satisfying $pe_k = \partial_k b$, $\partial_j e_k = \partial_{k-1} e_j$, $j < k$, $j, k \neq i$, and let $p_{qi} : E_q \to E_{qi}$ be the obvious map. Saying p_{qi} has a section for all q and i is the fibration version of the Kan extension condition, as described in Section 1.

DEFINITION 2.1. The mapping $p : E \to B$ in $\mathcal{K}$ is a fibration if, for each q and i, p_{qi} has a continuous π-equivariant section λ_{qi} which satisfies

$$\lambda_{qi}(p_{qi} s_j e) = s_j e$$

for all $e \in E_{q-1}$ and all j. A simplicial space $(X, f) \in (\Delta\mathcal{T}_\pi)_{X_0}$ is Kan if $f : X \longrightarrow X_0$ is a fibration in $\Delta\mathcal{T}_\pi$. In particular, $X \in \Delta\mathcal{T} = \Delta\mathcal{T}_{\mathrm{pt}}$ is Kan if $X \longrightarrow \mathrm{pt}$ is a fibration.

An easy argument shows that the degeneracy requirement is superfluous in $\Delta\mathcal{S}$.

Theorem 1.6 is now a special case of:

THEOREM 2.2. *If $p : E \longrightarrow B$ is a fibration in $\Delta\mathcal{T}_\pi$ and $X \in \Delta\mathcal{T}_\pi$, then $\mathcal{F}(\bar{X}, \bar{E}) \longrightarrow \Delta\mathcal{F}(\bar{X}, \bar{B})$ is a fibration in $\Delta\mathcal{T}_\pi$ and $\mathcal{F}(X, E) \longrightarrow \mathcal{F}(X, B)$ is a fibration in $\Delta\mathcal{T}$. Furthermore, if $f : E \longrightarrow X_0$, $g : B \longrightarrow X_0$, $h : X \longrightarrow X_0$ satisfy $gp = f$, then $\mathcal{F}((X, h), (E, f)) \longrightarrow \mathcal{F}((X, h), (B, g))$ is a fibration in $\Delta\mathcal{T}$.*

PROOF. We first give a proof of the first part of the theorem when $\pi = 1$ and then describe the modifications required to prove the general case. The remaining parts of the theorem then follow easily.

Suppose $p : E \to B$ is a fibration with sections λ_{qi} as above and $X \in \mathcal{K}$. We wish to show that

$$\mathcal{F}(X,p) : \mathcal{F}(X,E) \to \mathcal{F}(X,B)$$

is a fibration. Let $\Delta[q,i]$ be the simplicial subset of $\Delta[q]$ consisting of all simplices not having $(0,1,\ldots,\hat{i},\ldots,q)$ as a face. For $A \subset X \in \mathcal{K}$, let $\mathcal{F}(X,A,p) \in \mathcal{K}$ have as q-simplices all pairs (f,g) of mapping in $\mathcal{K}$ making a commutative diagram:

$$\begin{tikzcd} \Delta[q] \times A \arrow[r, "f"'] \arrow[d] & E \arrow[d, "p"] \\ \Delta[q] \times X \arrow[r, "g"'] & B \end{tikzcd}$$

Let $(X,A,p) = \mathcal{F}(X,A,p)_0$. Then p is a fibration if and only if the mappings

$$\big(\Delta[q],E\big) \to \big(\Delta[q],\Delta[q,i],p\big)$$

have sections λ_{qi} satisfying $\lambda_{qi}p_{qi}h = h$ if $h = h'd_j$. The manipulation

$$\big(X,\mathcal{F}(Y,Z)\big) = \mathcal{F}\big(X,\mathcal{F}(Y,Z)\big)_0 = \mathcal{F}\big(Y,\mathcal{F}(X,Z)\big)_0 = \big(Y,\mathcal{F}(X,Z)\big)$$

transforms the mapping

$$\big(\Delta[q],\mathcal{F}(X,E)\big) \to \big(\Delta[q],\Delta[q,i],\mathcal{F}(X,p)\big)$$

into

$$\big(X,\mathcal{F}(\Delta[q],E)\big) \to \big(X,\mathcal{F}\big(\Delta[q],\Delta[q,i],p\big)\big).$$

Hence, it is sufficient to find sections μ_{qi} of

$$\mathcal{F}\big(\Delta[q],E\big) \to \mathcal{F}\big(\Delta[q],\Delta[q,i],p\big)$$

satisfying $\mu_{qi}ph = h$ if $h' : \Delta[r] \times \Delta[q] \to E$ and $h = h'(\mathrm{id}_{\Delta[p]} \times d_i)$.

Suppose $U,V \subset X \in \Delta S$ and $(U, V \cap U) \approx (\Delta[q],\Delta[q,i])$. Then the λ_{qi} give a continuous section g which exists and is uniquely defined by the commutative diagram

$$\begin{tikzcd} (X,V,p) \arrow[r] \arrow[d, "g"] & (X,V,p) \times (U,U \cap V,p) \arrow[d] \\ (X,U \cup V,p) \arrow[r] & (X,V) \times (U,E) \end{tikzcd}$$

We build $\Delta[r] \times I$ from $\Delta[r] \times \{0\} \cup \Delta[r]^\bullet \times I$ ($\Delta[r]^\bullet$ denotes the boundary of $\Delta[r]$) by adding one $(r+1)$ simplex at a time which meets the previously constructed subsets in all but one r-face as follows. Let $I = \Delta[1]$ and $\sigma_0, \ldots, \sigma_q \in \Delta[r] \times I$ be the simplices defined by

$$\sigma_j = \left(s_{q-j}(0, \ldots, r), s_1^j s_0^{r-j}(0, 1)\right)$$

and let

$$U_j = \Delta[r]^\bullet \times I \cup \Delta[q] \times 0 \cup \bigcup_{k \leqslant j} \bar{\sigma}_j$$

where $\bar{\sigma} = \sigma$ and all its faces and degeneracies. Then $U_j = U_{j-1} \cup \bar{\sigma}_j$ and

$$\left(\bar{\sigma}_j, U_{j-1} \cap \bar{\sigma}_j\right) \approx \left(\Delta[r+1], \Delta[r+1, r-i]\right).$$

Define mappings

$$g_i : \left(\Delta[q] \times I, \Delta[q]^\bullet \times I \cup \Delta[q] \times 0\right) \to \left(\Delta[q+1], \Delta[q+1, i]\right)$$

by $g_i(j, 0) = i$ and $g_i(j, 1) = e_i(j)$.

Using induction on $|\sigma|$, $\sigma \in \Delta[m] \times \Delta[q] - \Delta[m] \times \Delta[q]^\bullet$ and moving up $\sigma \times I$ using $\sigma_0, \sigma_1, \ldots$ as above, we first construct the section μ_q in the diagram below and then note there are unique sections μ_{qi} fitting into a commutative diagram

$$\begin{array}{ccc}
(\Delta[m] \times \Delta[q] \times I, \Delta[m] \times (\Delta[q]^\bullet \times I \cup \Delta[q] \times 0), p) & \xrightarrow{\mu_q} & (\Delta[m] \times \Delta[q] \times I, E) \\
\big\downarrow{\scriptstyle g_i^*} & & \big\downarrow{\scriptstyle g_i^*} \\
(\Delta[m] \times \Delta[q+1], \Delta[m] \times \Delta[q+1, i], p) & \xrightarrow{\mu_{qi}} & (\Delta[m] \times \Delta[q+1], E)
\end{array}$$

By the uniqueness of all the maps involved, it is immediate that the μ_{qi} commute with the maps induced by e_j and d_j and hence define a section of

$$\mathcal{F}\left(\Delta[q+1], E\right) \to \mathcal{F}\left(\Delta[q+1], \Delta[q+1, i], p\right).$$

The degeneracy requirement follows from the fact that if $\sigma \in \Delta[m] \times \Delta[q] - \Delta[m] \times \Delta[q]^\bullet$, $\sigma = (\sigma', \sigma'')$, where $\sigma'' = s_I(0, \ldots, q)$. Hence the degeneracy map $d_i : \Delta[q+1] \to \Delta[q]$ will correspond to the degeneracy projection $\Delta[q] \times I \to \Delta[q-1] \times I$ or the projection $\Delta[q] \times I \to \Delta[q]$ in which cases μ_q will yield the desired map.

Now suppose π is not trivial. In the above argument replace all occurrences of "$\Delta[q]$" by "$\pi \times \Delta[q]$" (but not "$\Delta[m]$") . Note that if $x \in X \in \mathcal{K}$, there is a unique map $t_x : \pi \times \Delta[q] \to X$ such that $t_x(e, (0, \ldots, q)) = x$. □

Theorem 2.2 immediately gives:

COROLLARY 2.3. *If $X, Y \in \Delta\mathcal{T}_\pi$ and Y is Kan, then $\mathcal{F}(\bar{X}, \bar{Y})$ and $\mathcal{F}(X, Y)$ are Kan. If $(X, f), (Y, g) \in (\Delta\mathcal{T}_\pi)_{X_0}$ and (Y, g) is Kan then $\mathcal{F}((X, f), (Y, g))$ is Kan.*

The standard argument gives:

COROLLARY 2.4. *If $X, Y \in \Delta\mathcal{T}_\pi$ and Y is Kan, then homotopy is an equivalence relation on the set of maps of X into Y. This also holds in $(\Delta\mathcal{T}_\pi)_{X_0}$.*

As in Section 1, let $[X,Y] = \pi_0(\mathcal{F}(X,Y))$ denote the homotopy classes of mappings from X to Y. Then Theorem 1.7 is a special case of:

THEOREM 2.5. *If $G \in \mathcal{K}$ is a $\mathcal{K}$ group (the multiplication map is in $\mathcal{K}$), then G is Kan. That is $G \to \mathrm{pt}$ is a fibration.*

PROOF. Suppose $g = (g_0, \ldots, \hat{g}_i, \ldots, g_q)$ satisfies $\partial_k g_j = \partial_{j-1} g_k$, $k < j$, $k, j \neq i$. Define h_j, $j = 0, \ldots, q$, by

$$h_0 = e_q,$$
$$h_j = h_{j-1}\big((s_r \partial_t h_{j-1})^{-1} s_r g_j\big), \quad j > 0,$$

where $r = t = j - 1$ if $j \leqslant i$ and $t - 1 = r = q - j + i$ if $j > i$. Induction on j yields $\partial_k h_j = g_k$ for $k < j \leqslant i$ and $\partial_k h_j = g_k$ for $j > i$, $k < i$ or $k \geqslant q - j + i$. If $g_j = \partial_j s_k g$, then $h_j = s_k h'_j$, some h'_j where $h'_j = g$ for $j < i$, $j - 1 \geqslant k$ or $j > i$, $j \geqslant k$. Then $\lambda_{q,i}(g) = h_q$ has the desired properties. □

The definition of twisted cartesian product given in Section 1 carries over to $\mathcal{K}$ without change where all the objects are required to be in $\mathcal{K}$ and the maps are required to be continuous and π-equivariant. Theorem 1.8 is a special case of:

THEOREM 2.6. *If F is Kan and $E = B \times_\tau F$ is a TCP, all in $\mathcal{K}$, then $E \to B$ is a fibration in $\mathcal{K}$.*

PROOF. Suppose $\bar{\lambda}_{q,i}$ are the sections of c_{qi}, where $c : F \to \mathrm{pt}$. Suppose $(e_0, \ldots, \hat{e}_i, \ldots, e_q, b)$ satisfy $pe_j = \partial_j b, \partial_j e_k = \partial_{k-1} e_j$, $j < k$, $j, k \neq i$, and $e_j = (b_j, y_j) \in B_{q-1} \times F_{q-1}$. Let $x_j \in F_{q-1}$, $j \neq i$, be given by

$$\begin{aligned} x_j &= s_0 \tau(b_j) y_j, & j > 1,\\ &= \tau(b) y_1, & j = 1,\\ &= y_0, & j = 0. \end{aligned}$$

Then $\partial_k x_j = \partial_{j-1} x_k$, $k < j$, $k, j \neq i$. Let

$$\lambda_{q,i}(e_0, \ldots, \hat{e}_1, \ldots, e_q, b) = \big(b, s_0 \tau(b)^{-1} \bar{\lambda}(x_0, \ldots, \hat{x}_i, \ldots x_q)\big).$$

Then a straightforward calculation shows that $\lambda_{q,i}(g)$ has the desired properties. □

3. The deRham theorem in $\Delta\mathcal{T}_\pi$

Throughout this paper we will be dealing with cochains and differential forms on simplicial spaces with π actions taking values in π-modules; these will always be ordinary, in contrast to equivariant, so that cochain groups are π-modules. On the other hand, cohomology and homology will always be equivariant. Recall that equivariant cohomology is defined as follows. Suppose M is a topological π-module. If $X \in \Delta\mathcal{T}_\pi$, we define $C^q(X;M)$ to be the continuous, normalized cochains on X with coefficients in M, that is, all continuous maps $u : X_q \longrightarrow M$ such that $us_i = 0$, all i. We topologize $C^q(X;M)$ as a subset of (X_q, M) and make $C^q(X;M)$ into a π-module by defining $(\alpha u)(x) = \alpha(u(\alpha^{-1}x))$, $\alpha \in \pi$, $u \in C^q(X;M)$. The coboundary operator δ is given by the usual formula and we define the (π-equivariant) cohomology of X by

$$H^q(X;M) = H_q\big(C^*(X;M)^\pi, \delta\big).$$

As in Section 1, for $\mathbf{R} = R$, let $\Omega^p_q = \Omega^p_q(R)$ be the C^∞ differential p-forms on the standard geometric simplex Δ^q with the C^∞ topology. Note that this topology is compactly generated since it can be defined from a metric ([17]). When $\mathbf{R} = Q$, $\Omega^p_q = \Omega^p_q(Q)$ as in Section 1 with the discrete topology. In both cases, the algebra $\Omega(X)$ of differential forms on $X \in \Delta S$ is defined by

$$\Omega(X) = (X, \Omega).$$

One can define cochains on X in an analogous fashion. Namely, if M is a topological π-module, let $C^p_q(M) = C^p(\Delta[q]; M)$ where π acts on $C^p_q(M)$ in the obvious way. Then there is a π-isomorphism:

$$C^p(X;M) = \big(\bar{X}, C^p(\bar{M})\big).$$

When $\mathbf{R} = R$ let $\mathcal{V}$ be the category of topological vector spaces over R and when $\mathbf{R} = Q$, $\mathcal{V}$ is the category of vector spaces over Q with the discrete topology. As in Section 1, $\mathcal{V}_{\mathrm{F}}$, $\mathcal{V}_{\mathrm{G}}$, and $\mathcal{V}_{\mathrm{DG}}$ denote the categories of finite dimensional, graded, and differential graded vector spaces. For $X \in \Delta\mathcal{T}_\pi$ and $V \in \mathcal{V}_{\mathrm{F},\pi}$ we define $\Omega(X;V) = \Omega(\bar{X}) \otimes V$ and the deRham (π-equivariant) cohomology of X with coefficients in V by

$$H^q_{\mathrm{dR}}(X;V) = H_q\big(\Omega(X;V)^\pi, d\big).$$

We next define and develop the properties of the mapping

$$\Psi : \Omega(X;V) \to C^*(X;V).$$

The proofs of most of the lemmas below are trivial and omitted or presented very briefly. We begin with some preliminaries about the simplicial differential graded algebra Ω. (Compare Dupont [10].)

Let $\Omega(\Delta^q \times I)$ be the C^∞ differential forms on $\Delta^q \times I$. If t denotes the coordinate function on I, then $\omega \in \Omega(\Delta^q \times I)^p$ can be written as $\omega = \omega_1(t) + \omega_2(t)dt$, where $\omega_1(t) \in \Omega^p_q$ and $\omega_2(t) \in \Omega^{p-1}_q$. Define

$$\mu : \Omega\big(\Delta^q \times I\big)^p \to \Omega^{p-1}_q$$

to be the usual "integration along fibres" mapping,

$$\mu(\omega) = (-1)^{p-1} \int_0^1 \omega_2(t)\, dt.$$

A straightforward computation proves the following.

LEMMA 3.1. *Let $i_0, i_1 : \Delta^q \to \Delta^q \times I$ be given by $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$. Then*

$$d\mu + \mu d = i_1^* - i_0^*, \quad \partial_i \mu = \mu(e_i \times \mathrm{id})^*, \quad s_i \mu = \mu(d_i \times \mathrm{id})^*.$$

Here, d is the exterior differential, ∂_i, s_i are the face and degeneracy mappings, and $e_i : \Delta^{q-1} \to \Delta^q$, $d_i : \Delta^{q+1} \to \Delta^q$ are the usual face inclusions and degeneracy projections defined in Section 1.

For the remainder of this section only, we define $\Omega_q^{-1} = \mathbf{R}$ with $\partial_i = s_i = \mathrm{id}$ and $d : \Omega_q^{-1} \to \Omega_q^0$ by $d(r) = r$, the constant function. Let $b_i : \Delta^q \times \Delta^1 \to \Delta^q$ be given by

$$b_i(x, t) = tx + (1 - t)v_i,$$

where v_i is the i-th vertex of Δ^q and let $\mu_i : \Omega_q^p \to \Omega_q^{p-1}$ be given by

$$\mu_i(\omega) = \begin{cases} \mu b_i^* \omega, & p > 0, \\ \omega(v_i), & p = 0. \end{cases}$$

LEMMA 3.2. *The functions μ_i satisfy the following*

$$d\mu_i + \mu_i d = \mathrm{id},$$

$$\partial_j \mu_i = \begin{cases} \mu_i \partial_j, & i < j, \\ \mu_{i-1} \partial_j, & i > j, \end{cases}$$

$$\mu_i s_j = \begin{cases} s_j \mu_i, & i \leqslant j, \\ s_j \mu_{i-1}, & i > j. \end{cases}$$

The proof is trivial.

For $I = (i_0, \ldots, i_p)$, let μ_I be defined by

$$\mu_I = \mu_{i_p} \mu_{i_{p-1}} \cdots \mu_{i_0}.$$

An easy induction proves the following.

LEMMA 3.3. *For $I = (i_0, \ldots, i_p) \in \Delta[q]_p$, we have*

$$d\mu_I + (-1)^p \mu_I d = (-1)^p \sum_{j=0}^{p} (-1)^j \mu_{\partial_j I}$$

where ∂_j denotes the j-th face operator in $\Delta[q]$.

If $I = (i_0, \ldots, i_p) \in \Delta[p]_q$, let $\beta_1 \in \Omega_q^q$ be defined by

$$\beta_1 = \sum_{j=0}^{p} (-1)^j t_{i_j} dt_{i_0} \ldots \hat{dt}_{i_j} \ldots dt_{i_p},$$

where $t_0, \ldots, t_q$ are the barycentric coordinates in Δ^q.

LEMMA 3.4. *For $I \in \Delta[q]_p$ and $0 \leqslant j \leqslant p$, we have*

$$\partial_i \beta_I = \sum_{e_i J = I} \beta_J, \qquad s_i \beta_I = \sum_{d_i J = I} \beta_J.$$

The proof is straightforward.

Define mappings

$$\psi : \Omega_q^p \to C_q^p(R), \quad \varphi : C_q^p(R) \to \Omega_q^p, \quad \gamma : \Omega_q^p \to \Omega_q^{p-1},$$

by

$$\psi(\omega)(I) = \mu_I(\omega), \quad \varphi(u) = \sum_{I \in \Delta[q]_p} p! u(I) \beta_I,$$

$$\gamma(\omega) = \sum_{p \leqslant q} p! \sum_{I \in \Delta[q]_p} \mu_I(\omega) \beta_I.$$

LEMMA 3.5. *The mappings ψ, φ, and γ define simplicial mappings which satisfy the following:*

$$\psi d = \delta \psi, \quad \varphi \delta = d \varphi,$$

$$\psi \varphi = \mathrm{id}, \quad d\gamma + \gamma d = \varphi \psi - \mathrm{id}.$$

PROOF. The fact that ψ is simplicial follows immediately from Lemma 3.2. To see that φ and γ are simplicial, one uses Lemma 3.2 and 3.4. The equation $\psi d = \delta \psi$ is an easy consequence of Lemma 3.4.

To prove that $\psi \varphi = \mathrm{id}$, we first note that in terms of coordinates, $b_j : \Delta_q \times I \to \Delta_q$ is given by

$$b_j(t_0, \ldots, t_q, t)_k = t t_k + \delta_{jk}(1 - t),$$

where $\delta_{jk} = 1$ if $j = k$ and zero otherwise. Hence

$$b_j^* \beta_I = t^{p+1} \beta_I + t^p (1-t) dt_{i_0} \ldots d\hat{t}_{i_j} \ldots dt_{i_p} + \left(\sum \delta_{j i_k} (-1)^{p+k+1} t^{p-1} \beta_{\partial_k I} \right) dt$$

and

$$\mu_j\beta_I = \begin{cases} \frac{(-1)^k}{p}\beta_{\partial_k I}, & \text{if for some } k,\ j = i_k, \\ 0, & \text{if } i \notin I. \end{cases}$$

Thus $\mu_J\beta_I = 1/p!$ if $J = I$ and is zero if $J \neq I$. Therefore, $\psi\varphi = \mathrm{id}$.

We next show that $\varphi\delta = d\varphi$. For any tuple J of integers between 0 and q, $J = (j_0, j_1, \ldots, j_p)$, let $\beta_J = (\operatorname{sign}\alpha)\beta_I$ where $I = (j_{\alpha(0)}, \ldots, j_{\alpha(p)})$, $j_{\alpha(0)} \leqslant j_{\alpha(1)} \leqslant \cdots \leqslant j_{\alpha(0)}$. Note that the ambiguity of α does not matter because $\beta_I = 0$ if the entries of I are not distinct. Then

$$\begin{aligned}\sum_{k\notin I}\beta_{[k,I]} &= \sum_k \beta_{[k,I]} = \sum t_k dt_{i_0}\ldots dt_{i_p} + dt_k\beta_I \\ &= \Big(\sum t_k\Big)dt_{i_0}\ldots dt_{i_p} + \Big(\sum dt_k\Big)\beta_I \\ &= dt_{i_0}\ldots dt_{i_p} = \frac{1}{p+1}d\beta_I\end{aligned}$$

and

$$\begin{aligned}d\varphi u &= d\sum p!u(I)\beta_I = (p+1)\sum_{I,\ k\notin I} u(I)\beta_{[k,I]} \\ &= \sum u(\partial_i J)(-1)^i(p+1)!\beta_J = \varphi\delta u.\end{aligned}$$

Using Lemma 3.3 and the above argument, one can show that $d\gamma + \gamma d = \varphi\psi - \mathrm{id}$. □

The mapping ψ then defines a mapping

$$\Psi = \Psi_X^V : \Omega^p(X) \otimes V \to C^p(X; \mathbf{R}) \otimes V$$

for $V \in \mathcal{V}_{\mathrm{F}}$ and hence a mapping

$$\Psi_* : H^q_{\mathrm{dR}}(X; V) \longrightarrow H^q(X; V).$$

It follows immediately from Lemma 3.5 that Ψ_* is an isomorphism. Thus we have

THEOREM 3.6. *The map* $(\Psi_X^V)_* : H^*_{\mathrm{dR}}(X, V) \longrightarrow H^*(X, V)$ *is an isomorphism for all* $X \in \Delta\mathcal{T}_\pi$ *and* $V \in \mathcal{V}_{\mathrm{F},\pi}$.

Again standard arguments ([4, Theorem 2.2]) yield:

THEOREM 3.7. *If* $X \in \Delta\mathcal{T}_\pi$ *the usual map gives a natural equivalence*

$$[X, K(M,n)] \approx H^n(X; M).$$

Thus we have natural equivalences

$$H^*_{\mathrm{dR}}(X; V) \approx H^n(X; V) \approx [X, K(V,n)].$$

We now show that $\Psi_* : H^*_{\mathrm{dR}}(X) \to H^*(X, \mathbf{R})$ is an algebra isomorphism for $X \in \Delta T$. Let $C^{r,s} \in \Delta T$ be defined by

$$C_q^{r,s} = \begin{cases} C^r(\Delta[q]; \Omega_q^s), & s \geqslant -1,\ r \geqslant 0, \\ \Omega_q^s, & r = -1,\ s \geqslant 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$d_1 : C_q^{r,s} \to C_q^{r+1,s}, \quad d_2 : C_q^{r,s} \to C_q^{r,s+1}$$

by

$$d_1 u(I) = \sum_{j=0}^{r} (-1)^j u(\partial_j I), \quad d_2 u(I) = (-1)^r d(u(I)).$$

Define mappings

$$\gamma_1 : C_q^{r,s} \to C_q^{r-1,s}, \quad \gamma_2 : C_q^{r-s} \to C_q^{r,s-1},$$

$$\psi : C_q^{r,s} \to C_q^{r+s}(R), \quad \rho : C_q^{r,s} \otimes C_q^{1,m} \to C_q^{r+1,s+m}$$

by

$$\begin{aligned} \gamma_1(u)(I) &= \sum_{j=0}^{q} t_j u(j, I), \\ \gamma_2(u)(I) &= \mu_{i_0} u(i_1, \ldots, i_r), \\ \psi(u)(I) &= \mu_{(i_0,\ldots,i_q)} u(i_r, \ldots, i_{r+s}), \\ \rho(u \otimes v)(I) &= u(i_0, \ldots, i_r) v(i_r, \ldots, i_{r+1}). \end{aligned}$$

LEMMA 3.8. *The mappings d_1, d_2, γ_1, γ_2, ψ and ρ defined above are simplicial and satisfy the identities*

$$d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0,$$

$$d_1 \gamma_1 + \gamma_1 d_1 = \mathrm{id}, \qquad d_2 \gamma_2 + \gamma_2 d_2 = \mathrm{id},$$

$$\psi(d_1 + d_2) = \delta \psi.$$

Furthermore, $d_1 + d_2$ is a derivation in the graded sense with respect to the multiplication ρ.

The proof is tedious but straightforward.

It follows from Lemma 3.8 that $\{C^{r,s}; d_1, d_2\}$ is a simplicial double complex. Note that $C(R)^p_q = C^{p,-1}_q$ and $\Omega^p_q = C^{-1,p}_q$. Let $\bar{C}^n_q$ and $\bar{d} : \bar{C}^n_q \to \bar{C}^{n+1}_q$ be the simplicial chain complex defined by

$$\bar{C}^n_q = \sum_{0 \leqslant r \leqslant n} C^{r,n-r}_q, \qquad \bar{d} = d_1 + d_2$$

and define mappings $\alpha : \Delta \to \bar{C}$, $\beta : C(R) \to \bar{C}$ by

$$\alpha = d_1 : C^{-1,p} \to C^{0,p}, \qquad \beta = d_2 : C^{p,-1} \to C^{p,0}.$$

For $X \in \mathcal{T}$, let $\bar{C}(X) = (X, \bar{C}) \in \mathcal{A}$. Composition with the mappings in Lemma 3.8 gives mappings on $\bar{C}(X)$ satisfying the same identities.

PROPOSITION 3.9. *In the diagram*

$$\begin{tikzcd} \Omega(X) \arrow[r, "\alpha"] \arrow[dr, "\psi"'] & \bar{C}(X) \arrow[d, shift left, "\psi"] \\ & C(X;R) \arrow[u, shift left, "\beta"] \end{tikzcd}$$

one has $\psi\alpha = \psi$, $\psi\beta = \mathrm{id}$, and α and β are ring homomorphisms inducing isomorphisms on homology.

PROOF. Using the chain homotopies γ_1 and γ_2, we see that, for any $p \geqslant 0$, the sequences

$$0 \to \Omega^p(X) \xrightarrow{\ \alpha\ } C^{0,p} \xrightarrow{\ d_1\ } C^{1,p} \to \cdots,$$

$$0 \to C^p(X, R) \xrightarrow{\ \beta\ } C^{p,0} \xrightarrow{\ d_2\ } C^{p,1} \to$$

are exact. Standard results about double complexes now show that

$$\alpha_* : H_*\big(\Omega(X)\big) \to H^*(\bar{C}), \qquad \beta_* : H_*\big(C(X;R)\big) \to H_*(\bar{C})$$

are isomorphisms. □

COROLLARY 3.10. *The mapping $\Psi : \Omega(X) \to C(X; R)$ induces an algebra isomorphism*

$$\Psi_* : H^*_{\mathrm{dR}}(X) \to H^*(X; R).$$

The computations in cohomology which we need are derived from Serre's computation of $H^*(K(Q,n), Q)$ ([15]), Van Est's computation of $H^*(K(R,1); R)$ ([21]) and the Serre spectral sequence in $\Delta\mathcal{T}_\pi$. The Serre spectral sequence in $\Delta\mathcal{T}_\pi$ is developed in [3]. The results are stated in the next section.

4. Postnikov systems and the Serre spectral sequence in ΔT_π

If M is a topological π module, we form $p : E(M,n) \longrightarrow K(M,n+1)$ in ΔT_π just as in Section 1.

THEOREM 4.1. *The mapping $p : E(M,n) \longrightarrow K(M,n+1)$ is a principal TCP in ΔT_π with group and fibre $K(M,n)$. Furthermore, $E(M,n)$ is contractible.*

PROOF. To prove the first part, let $\mu : C_q^{n+1}(M) \to C_q^n(M)$ be given by

$$\mu(u)(i_0,\dots,i_n) = u(0,i_0,\dots,i_n)$$

and let $\tau : K(M,n+1)_{n+1} \to K(M,n)_n$ be defined by

$$\tau(\alpha) = \partial_0\mu(\alpha) - \mu(\partial_0\alpha).$$

Then τ is a twisting function and

$$L : K(M,n+1) \times_\tau K(M,n) \longrightarrow E(M,n)$$

given by $L(\alpha,\beta) = \mu(\alpha) + \beta$ is an isomorphism commuting with the appropriate projections.

For the second part, a contracting homotopy $F : E(M,n) \times I \to E(M,n)$ is given by $F(u,\sigma) = (s^*v)\cup u$ where $u \in E(M,n)_q$, $v \in C^0(I;Z)$ is given by $v(0) = 0, v(1) = 1$, and $s : \Delta[q] \to I$ is the unique simplicial mapping with $s(0,1,\dots,q) = \sigma$. □

The same proof yields the following result. Suppose $V \in \mathcal{V}_\pi$ so that $(\Omega^*\otimes V)^n \in \Delta T_\pi$. Let $Z_n(\Omega\otimes V)_q = \{u \in (\Omega_q\otimes V)^n | du = 0\}$.

THEOREM 4.2. *The mapping $d : (\Omega\otimes V)^n \to Z_{n+1}(\Omega\otimes V)$ is a TCP with group and fibre $Z_n(\Omega\otimes V)$. Furthermore, $(\Omega\otimes V)^n$ is contractible.*

The proof of the first part of Theorem 4.2 is the same as the proof of the first part of Theorem 4.1 replacing μ by $\mu_0\otimes \mathrm{id}_V$ where μ_0 is as in Section 3. To prove $(\Omega\otimes V)^n$ contractible, let $F : (\Omega\otimes V)^n \times I \to (\Omega\otimes V)^n$ be the contracting homotopy given by $F(w,s) = (s^*(t_1))w$.

If $k : X \longrightarrow K(M,n+1)$, define X_k to be the total space of the induced TCP,

$$\begin{array}{ccc} X_k & \longrightarrow & E(M,n) \\ \downarrow & & \downarrow \\ X & \xrightarrow{k} & K(M,n+1) \end{array}$$

where all of the above takes place in ΔT_π.

DEFINITION 4.3. We say that $X \in (\Delta T_\pi)_{X_0}$ has a nilpotent Postnikov system if there is a sequence $\{X_n\}$, $X_n \in (\Delta T_\pi)_{X_0}, X_0 = X_0$, a sequence $1 \leqslant m_1 \leqslant m_2 \leqslant \cdots$ of integers with $m_n \to \infty$ as $n \to \infty$, a sequence $\{M_n\}$ of topological π-modules in $\mathcal{T}$, and a sequence of mappings $k^{n+1} : X_{n-1} \to K(M_n, m_{n+1})$ such that $X_n = (X_{n-1})_{k^{n+1}}$ and $\varprojlim X_n$ is homotopy equivalent to X. We say X has a simple Postnikov system if $M_1 = 0$ and $m_n = n$ for all n.

Suppose X is a 0-connected Kan simplicial space with $\pi_1(X) = \pi$ and suppose $\rho : X \longrightarrow B\pi = K(\pi, 1)$ induces an isomorphism on fundamental groups. Let $\tilde{\rho} : \tilde{X} \longrightarrow E(\pi, 0) = E\pi$ be the induced fibration:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\rho}"] \arrow[d] & E\pi \arrow[d] \\ X \arrow[r, "\rho"] & B\pi \end{tikzcd}$$

The following is well known:

THEOREM 4.4. *If π is discrete and $(\tilde{X}, \tilde{\rho}) \in (\Delta\mathcal{T}_\pi)_{E\pi}$ is a simplicial set, then $(\tilde{X}, \tilde{\rho})$ has a simple Postnikov system.*

Within the context of our machinery, one can study all simplicial spaces (sets) through the following approach. Let $\Delta\mathcal{T}_\pi$ be the category of pairs (X, ρ) where X is a 0-connected Kan simplicial space (set) with base point and $\rho : X \longrightarrow B\pi$ is a fibration such that $\rho_* : \pi_1(X) \approx \pi_1(B\pi)$. The category $\Delta\mathcal{T}_\pi$ embeds in $(\Delta\mathcal{T}_\pi)_{E\pi}$ by sending (X, ρ) to $(\tilde{X}, \tilde{\rho})$.

DEFINITION 4.5. We say that $(X, \rho) \in \Delta\mathcal{T}_\pi$ has a Postnikov system if $(\tilde{X}, \tilde{\rho})$ has a simple Postnikov system.

THEOREM 4.6. *If $(X, \rho) \in \Delta\mathcal{T}_\pi$ and X is a 0-connected simplicial set (discrete topology) then it has a Postnikov system.*

Suppose $X = B \times_\tau F$ is a twisted Cartesian product of B and F with structural group G, all in $\Delta\mathcal{T}_\pi$. Let $B^{(p)}$ be the p skeleton of B, that is, the smallest simplicial subspace of B containing all B_q, $q \leqslant p$. Filter $C^*(X; M)^\pi$ by

$$F^{p,q} = \left\{u \in C^{p+q}(X; M)^\pi \,\middle|\, u\left(B^{(p-1)}_{p+q} \times F_{p+q}\right) = 0\right\}.$$

The usual definitions ([15]) then yield the Serre spectral sequence $\{E_r^{p,q}\}$ with its usual relation to $H^*(X; M)$. Let

$$\theta : C^p\left(B; C^q(F; M)\right) \longrightarrow C^{p+q}(X; M)$$

be given by

$$\theta(u)(b, f) = u\left(\partial^q_{p+1} b, \partial^p_0 f\right)$$

for $(b, f) \in B_{p+q} \times F_{p+q}$. Then θ induces a map:

$$\theta_0 : C^p\big(B; C^q(F; M)\big)^\pi \longrightarrow E_0^{p,q}$$

where π acts on $C^*(F; M)$ by $gu(f) = g(u(g^{-1}f))$. Furthermore, $d_0\theta_0 = \theta_0\delta_2$ where δ_2 is the differential coming from $\delta : C^q(F; M) \longrightarrow C^{q+1}(F; M)$. To compute E_1 one needs to show that (kernel δ_2)/(image δ_2) is isomorphic to $C^p(B; H^q(\bar{F}; \bar{M}))^\pi$. When $\mathbf{R} = Q$, this is immediate. When $\mathbf{R} = R$, the following two conditions insure that this is true.

CONDITION 4.7. As a π-space, B_q is homeomorphic to $(B_q/\pi) \times \pi$ for $q \geqslant 0$.

CONDITION 4.8. The cochain complex $C^*(F; M)$ is splittable ([4]), that is the maps $Z^q(F; M) \longrightarrow H^q(F, M)$ and $C^{q-1}(F; M) \longrightarrow B^q(F; M)$ have continuous sections.

THEOREM 4.9. *If X, B and F are as above and satisfy Conditions* 4.7 *and* 4.8, *then θ induces an isomorphism*

$$E_2^{p,q} \approx H^p\big(B; H^q(\bar{F}; \bar{M})\big)$$

in the Serre spectral sequence for $H^*(X; M)$, where $H^q(\bar{F}; \bar{M})$ has the π action induced by the π action on $C^*(F; M)$.

In our applications of this theorem, Condition 4.7 will be true by inspection. For Condition 4.8, we will use:

THEOREM 4.10. *The algebra $C^*(K(R, n), R)$ is splittable and $H^*(K(R, n); R) \simeq R[x]$, degree $x = n$. Hence, if $U, V \in \mathcal{V}_{\mathbf{F}}$, then $C^*(K(V, n), U)$ is splittable.*

PROOF. We proceed by induction on n. The usual Serre spectral sequence argument applied to the TCP

$$K(R, n) \subset C^n(R) \to K(R, n + 1)$$

(see Theorem 4.1) yields the desired result if we can show that the Serre spectral sequence is applicable. In going from n to $n + 1$, we need to know that $C^*(K(R, n))$ is splittable. Assume Theorem 4.10 is true for $n - 1$ and hence, by the Serre spectral sequence, $H^*(K(R, n))$ is isomorphic to $R[x]$. We show $B^q(K(R, n)) \subset C^q(K(R, n))$ is closed. By hypothesis, the inclusion $K(Z, n) \subset K(R, n)$ induces an isomorphism,

$$H^q\big(K(R, n)\big) \to H^q\big(K(Z, n)\big) \xrightarrow{\cong} Hom\big(H_q\big(K(Z, n), R\big), R\big).$$

Hence for any q for which these groups are nonzero, there is a chain $c_q \in C_*(K(Z, n))$ such that evaluation on c_q gives a continuous isomorphism $H^q(K(R, n)) \to R$. Then B^q is closed because it is the kernel of the continuous map of $Z^n(K(R, n))$ to R given by evaluation on c_q.

A topological vector space is said to be Frechet if it is complete, locally convex and metrizable. (See [20].) If U and V are Frechet spaces, and W is a locally compact topological space countable at infinity, then (W, V) is Frechet. If $S \subset V$ is closed, then S is Frechet, and if $f : U \to V$ is an epimorphism, then it has a continuous section. Thus $((K(R,n)_q, R))$ is Frechet and hence $B^q \subset Z^q \subset C^q \subset (K(R,n)_q, R)$ and Z^q/B^q are all Frechet. Thus $C^q \to B^{(q+1)}$ and $Z^q \to H^q$ have continuous sections, showing that $C^*(K(R,n))$ is splittable. $\square$

5. The main theorems in ΔT_π

Recall in Section 1 we suggested that Theorems 1.8, 1.23, 1.25, and 1.29 formed the foundations of rational homotopy theory. In this section we formulate the analogues of Theorems 1.19, 1.25 and 1.29, namely, Theorems 5.4, 5.9, and 5.6 respectively and prove them in this section and in Section 6. The analogue of Theorem 1.23 is immediate.

Let $\mathbf{T} = (\Delta T_\pi)_{E\pi}$, $A_0 \in \mathcal{A}_\pi$, and suppose that $h : A_0 \to \Omega(E\pi)$ induces an isomorphism on $H_*(\ ; V)$ for all $V \in \mathcal{V}_{F\pi}$. For example, one can take $A_0 = \Omega(E\pi)$. However, more economical choices can be made in some cases as we demonstrate later in this section. Let $\mathbf{A} =_{A_0} \mathcal{A}_\pi$ and define functors $\Omega_\pi : \mathbf{T} \longrightarrow \mathbf{A}$ and $\Delta_\pi : \mathbf{A} \longrightarrow \mathbf{T}$ as follows: If $(X, f) \in \mathbf{T}$, then

$$\Omega_\pi(X, f) = \big(\Omega(X), \Omega(f)h\big).$$

If $(A, g) \in \mathbf{A}$, then

$$\Delta_\pi(A, g) = \big(\Delta_\pi(A), f\big)$$

where $\Delta_\pi(A)$ is the pull back:

$$\begin{tikzcd}
\Delta_\pi(A) \arrow[rr] \arrow[d, "f"] & & \Delta(A) \arrow[d, "\Delta(g)"] \\
E\pi \arrow[r, "i"] & \Delta\Omega(E\pi) \arrow[r, "\Delta(h)"] & \Delta(A_0)
\end{tikzcd}$$

Just as in the discussion preceding Theorem 1.13, the identification $(W, (Y, Z)) = (Y, (W, Z))$ in $\mathcal{T}$ gives an adjoint isomorphism: $\eta : (A, \Omega_\pi(X)) \approx (X, \Delta_\pi(A))$ for $A \in \mathbf{A}$ and $X \in \mathbf{T}$. Furthermore, η gives mappings

$$i : A \longrightarrow \Omega_\pi\big(\Delta_\pi(A)\big),$$

$$j : X \longrightarrow \Delta_\pi\big(\Omega_\pi(X)\big),$$

$$\eta : \mathcal{F}\big(A, \Omega_\pi(X)\big) \longrightarrow \mathcal{F}\big(X, \Delta_\pi(A)\big),$$

$$\Delta : \mathcal{F}(A, B) \longrightarrow \mathcal{F}\big(\Delta_\pi(B), \Delta_\pi(A)\big).$$

For $(A,g) \in \mathbf{A}$ and $V \in \mathcal{V}_{\mathrm{F}\pi}$, we define $H_*(A,g;V) = H_*(A;V)$. For $W \in \mathcal{V}_{\mathrm{DG}\pi}$ and $V \in \mathcal{V}_{\mathrm{F}\pi}$ define $H_*(W;V) = H_*((W \otimes V)^\pi)$. Then a mapping $f : W_1 \to W_2$ in $\mathcal{V}_{\mathrm{DG}\pi}$ is a weak equivalence if $f_* : H_*(W_1;V) \to H_*(W_2;V)$ is an isomorphism for all $V \in \mathcal{V}_{\mathrm{F}\pi}$.

In Section 1, we described the construction $A(V,\lambda)$. This construction, for $A \in \mathcal{A}_\pi$, $V \in \mathcal{V}_{\mathrm{F}\pi}$, $\lambda : V^* \to A^{n+1}$ an equivariant map, and Q replaced by $\mathbf{R}$ is defined in exactly the same way with diagonal group actions on $A(V,\lambda) = A \otimes \mathbf{R}(V,n)$.

DEFINITION 5.1. The algebra $A \in {}_{A_0}\mathcal{A}_\pi$ is said to be free, nilpotent of finite type over A_0 if A is the union of subalgebras $\{A_n\}$, $A_n \subset A_{n+1}$, with $A_0 = A_0$ and $A_n = A_{n-1}(V_n,\lambda_n)$ where $V_n \in \mathcal{V}_{\mathrm{F}\pi}$, $m_1 \leqslant m_2 \leqslant \cdots$ is a sequence of integers with $m_n \to \infty$ as $n \to \infty$, and $\lambda_n : V_n^* \to A_{n-1}^{m_n+1}$ with $d \circ \lambda = 0$. In addition, A is minimal if its differential d is decomposable.

Suppose A is FNF. The following provides an inductive procedure for dealing with $\pi_*(\Delta(A)), H^*(\Delta(A);V)$, Postnikov systems for $\Delta(A), \mathcal{F}(A,B)$ and for showing that $\pi_*(\mathcal{F}(A,B)) \to \pi_*\mathcal{F}(\Delta(B),\Delta(A))$ is an isomorphism. Recall that $\Delta(A) = \mathcal{F}(\tilde{A};\mathbf{R})$.

THEOREM 5.2. *If $A,B \in \mathcal{A}_\pi$ and $V \in \mathcal{V}_{\mathrm{F}\pi}$, then the fibration $\mathcal{F}(\bar{A}(\bar{V},\lambda),\bar{B}) \to \mathcal{F}(\bar{A},\bar{B})$ is a TCP,*

$$\mathcal{F}\big(\bar{A}(\bar{V},\lambda),B\big) = \mathcal{F}(\bar{A},\bar{B}) \times_\tau \mathcal{F}\big(\mathbf{R}(\bar{V},n),\bar{B}\big)$$

in $\Delta\mathcal{T}_\pi$. Thus the fibration $\mathcal{F}(A(V,\lambda),B) \to \mathcal{F}(A,B)$ is a TCP in $\Delta\mathcal{T}$,

$$\mathcal{F}\big(A(V,\lambda),B\big) = \mathcal{F}(A,B) \times_\tau \mathcal{F}\big(\mathbf{R}(V,n),B\big).$$

The proof is exactly the same as the proof of Theorem 1.15 with the construction A into $\Delta(A) = \mathcal{F}(A;\mathbf{R})$ replaced by the construction A into $\mathcal{F}(A,B)$.

COROLLARY 5.3. *If $A \in {}_{A_0}\mathcal{A}_\pi$ is FNF, then $\Delta(A)$ in $(\Delta\mathcal{T}_\pi)_{\Delta(A_0)}$ and $\mathcal{F}(A,B)$ in $(\Delta\mathcal{T}_\pi)_{\mathcal{F}(A_0,B)}$ are Kan.*

THEOREM 5.4. *If $A \in \mathbf{A}$ is FNF over A_0, then $i : A \longrightarrow \Omega_\pi(\Delta_\pi(A))$ is a weak equivalence.*

PROOF. Let $A = \bigcup A_n$, $A_n = A_{n-1}[V_n,\lambda_n]$ as above. We prove Theorem 5.4 for $A = A_n$ by induction on n. When $n = 0$, $\Delta_\pi(A_0) = E\pi$ and $A_0 \longrightarrow \mathcal{A}(\Delta_\pi(A_0))$ is a weak equivalence by construction. The inductive step follows from:

LEMMA 5.5. *If $A \to \Omega_\pi(\Delta_\pi A)$ is a weak equivalence, then the same is true for $A[V,\lambda] \to \Omega_\pi(\Delta_\pi(A[V,\pi]))$.*

PROOF. The proof of this lemma goes through exactly as in the proof of Lemma 1.19 using the splittability of $K(V,n)$. $\square$

THEOREM 5.6. *If $A, B \in \mathbf{A}$ are FNF over A_0, then*

$$\Delta_* : \pi_*\big(\mathcal{F}(A,B)\big) \longrightarrow \pi_*\big(\mathcal{F}\big(\Delta_\pi(B), \Delta_\pi(A)\big)\big)$$

is an isomorphism. In particular, in $\mathbf{A}$ *and* $\mathbf{T}$, $\Delta_* : [A,B] \longrightarrow [\Delta_\pi B, \Delta_\pi A]$ *is an isomorphism. If in addition,* $X \in \mathbf{T}$ *is zero connected, then*

$$\eta_* : \pi_*\big(\mathcal{F}\big(A, \Omega_\pi(X)\big)\big) \longrightarrow \pi_*\big(\mathcal{F}\big(X, \Delta_\pi(A)\big)\big)$$

is an isomorphism. Finally, if $f : B \longrightarrow C$ *is a weak equivalence in* $\mathbf{A}$, *then the induced map*

$$\pi_*\big(\mathcal{F}(A,B)\big) \longrightarrow \pi_*\big(\mathcal{F}(A,C)\big)$$

is an isomorphism.

The proof of this theorem is given in Section 6.

DEFINITION 5.7. Let G be a topological π module and let $G^* = Hom(G,R) \in \mathcal{V}_\pi$ be the vector space of all continuous homomorphisms of G into R with the compact open topology. We define the R-completion $\hat{G}$ of G by $\hat{G} = (G^*)^*$.

THEOREM 5.8. *If G is locally Euclidean, that is, a finite sum of cyclic groups, copies of S^1, and copies of R, then the inclusion $G \longrightarrow \hat{G}$ induces an isomorphism*

$$H^*\big(\bar{K}(G,n);R\big) \longrightarrow H^*\big(\bar{K}(\hat{G},n);R\big).$$

Furthermore, $C^*(\bar{K}(G,n);V)$ and $C^*(\bar{K}(\hat{G},n);V)$ are splittable for all finite dimensional $V \in \mathcal{V}$. (When $\mathbf{R} = Q$, the corresponding results are well known.)

PROOF. If G is finite, the result is trivial. If $G = S^1$, then $\hat{G}$ is trivial and $H^*(K(G,n);V)$ by the Van Est Theorem ([21]). The cases $G = Z$ and $G = R$ follow from Theorem 4.10. □

We next show that a large class of $X \in T$ have minimal models.

THEOREM 5.9. *Suppose $X \in \mathbf{T}$ has a nilpotent Postnikov system as in Definition* 4.3 *with M_n locally Euclidean for each $n > 1$. Then there is a minimal algebra A and a weak R-equivalence $f : A \to \Omega_\pi(X)$ in $\mathbf{A}$ and hence a weak equivalence $\hat{f} : X \to \Delta_\pi(A)$. Furthermore, if $g : B \to \Omega_\pi(X)$ is another such map, then there is a weak equivalence $\gamma : A \to B$ such that $g\gamma$ and f are homotopic.*

PROOF. In Section 3, we defined ΔDGA mappings $\psi : \Omega \to C^*(\mathbf{R})$ and $\varphi : C^*(\mathbf{R}) \to \Omega$ with $\psi\varphi = \mathrm{id}$ and $\varphi\psi$ homotopic to the identity via a homotopy γ. These mappings define, for any $V \in \mathcal{V}_{\mathrm{F},\pi}$, homotopy equivalences in $\Delta\mathcal{T}_\pi$,

$$K(V,n) = Z_n\big(C^*(\mathbf{R})\big) \otimes V \rightleftarrows Z_n(\Omega) \otimes V = \Delta\big(\mathbf{R}(V,n)\big)$$

where the homotopy

$$K : Z_n(\Omega) \otimes V \times I \to Z_n(\Omega) \otimes V$$

between $\varphi\psi$ and the identity is defined as follows: For $\omega \in Z_n(\Omega)$, $v \in V$ and $s \in \Delta[1]_q$, viewed as $s : \Delta[q] \to \Delta[1]$, we set

$$K(\omega \otimes v, s) = \big((\varphi\psi(\omega) - \omega)s^*t_1 + \omega + \gamma\omega^* dt_1\big) \otimes v.$$

Suppose X has a nilpotent Postnikov system $X_n = (X_{n-1})_{k^{n+1}}$, where

$$k^{n+1} \in Z^{n+1}(X_{n-1}; M_n).$$

Using induction on n we construct a minimal $A_n \in \mathbf{A}$ and a weak equivalence $f_n : A_n \to \Omega_\pi(X_n)$ such that $A_n = A_{n-1}(\hat{M}_n, l^{n+1})$ for some ℓ^{n+1} and f_n extends f_{n-1}. The mappings f_n then gives the desired map f. For $n = 0$, take $A_0 = A_0$ and $f_0 = h : A_0 \to \Omega(E\pi) = \Omega(X_0)$. Suppose f_{n-1} has been constructed and let $\rho : M_n \subset \hat{M}_n$. The mapping

$$A_{n-1} \xrightarrow{f_{n-1}} \Omega(X_{n-1}) \xrightarrow{\psi} C^*(X_{n-1})$$

is a weak equivalence and hence there are elements

$$\ell \in \big(\hat{M}_n^*, A_{n-1}^{n+1}\big)^\pi = \big(A_{n-1}^{n+1} \otimes \hat{M}_n\big)^\pi,$$

$$v \in \big(C^n(X_{n-1}) \otimes \hat{M}_N\big)^\pi = C^n\big(X_{n-1}; A_n\big)^\pi$$

such that

$$\big(\psi f_{n-1} \otimes \mathrm{id}(\hat{M}_n)\big)(\ell) = \rho k^{n+1} + \delta v.$$

Let $\ell^{n+1} = \ell$ and $A_n = A_{n-1}(\hat{M}_n, \ell^{n+1})$. Viewing ℓ and ρk^{n+1} as mappings, v is a homotopy and $\varphi\psi$ is homotopic to the identity. Thus the above equation gives a homotopy commutative diagram

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{\rho k^{n+1}} & K(\hat{M}_n, n+1) \\ \big\downarrow{\scriptstyle \hat{f}_{n-1}} & & \big\downarrow{\scriptstyle \bar{\varphi}} \\ \Delta(A_{n-1}) & \xrightarrow{\ell} & (\mathbf{R}(\hat{M}_n, n+1)) \end{array}$$

where $\bar{\varphi}$ is induced by $\varphi \otimes \mathrm{id}(\hat{M}_n)$. Since $\hat{f}_{n-1}$ is a weak equivalence and $\bar{\varphi}$ is a homotopy equivalence, the Serre spectral sequence implies that $\hat{f}_{n-1}$ lifts to a weak equivalence

$$\bar{f} : (X_{n-1})_{\rho k^{n+1}} \to \Delta(A_{n-1})_\ell = \Delta(A_n).$$

By Theorem 5.6, ρ induces a weak equivalence

$$\bar{\rho} : X_n = (X_{n-1})_{k^{n+1}} \to (X_{n-1})_{\rho k^{n+1}}$$

over the identity map on X_{n-1}. Hence $\bar{f}\bar{\rho} : X_n \to \Delta(A_n)$ has an adjoint $f_n : A_n \to \Omega(X_n)$ giving the desired mapping and completing the inductive step.

Suppose $g : B \to \Omega(X)$ is another such map. The last part of Theorem 5.6 implies that $f_* : [B, A] \to [B, \Omega(X)]$ is an isomorphism and hence $\gamma \in f_*^{-1}[g]$ gives the desired mapping. □

COROLLARY 5.10. *Suppose that X is a 0-connected, finite type Kan simplicial set with universal cover $\tilde{X}$ and $\pi_1(X)$ isomorphic to π. Then there is a minimal $A \in \mathbf{A}$ and a map $g : \tilde{X} \to \Delta_\pi(A)$ such that*

$$g^* : H^*(\Delta_\pi(A); V) \longrightarrow H^*(\tilde{X}; V)$$

is an isomorphism for all $V \in \mathcal{V}_{\mathbf{F}}$.

REMARK 5.11. If π is finite, we can strengthen the conclusion of Theorem 5.9. Suppose A and B are minimal and $\gamma : B \to A$ is a weak equivalence. If we could show that

$$\gamma_* : H_*(\bar{B}) = H_*(B; \mathbf{R}[\pi]) \longrightarrow H_*(A; \mathbf{R}[\pi]) = H_*(\bar{A})$$

is an isomorphism, we could conclude that γ is an isomorphism by [1], Proposition 7.6. Thus, when π is finite, $\mathbf{R}[\pi]$ is finite dimensional and hence γ is an isomorphism.

The next two theorems give economical choices for A_0 when π is finite or infinite cyclic.

THEOREM 5.12. *If π is finite, $\mathbf{R} \longrightarrow \Omega(E_\pi)$ is a weak equivalence.*

PROOF. Let j be the inclusion of $(\Omega(E\pi) \otimes V)^\pi$ into $\Omega(E\pi) \otimes V$ and let k be the retraction of $\Omega(E\pi) \otimes V$ on $(\Omega(E\pi) \otimes V)^\pi$ given by

$$k(x) = (1/m)\sum_g gx$$

where m is the order of π. Then in the sequence

$$H_*(\mathbf{R}\otimes V)^\pi \xrightarrow{i_*} H_*\big((\Omega(E\pi)\otimes V)^\pi\big) \xrightarrow{j_*} H_*\big((\Omega(E\pi)\otimes V)^\pi\big) \xrightarrow{k_*} H_*\big((\Omega(E\pi)\otimes V)^\pi\big)$$

j_*i_* is an isomorphism and k_*j_* is the identity map. Thus i_* is an isomorphism. □

THEOREM 5.13. *Let Z be the group of integers. Then there is a weak equivalence $R[t, dt] \to \Omega(EZ)$ where $\dim(t) = 0$ and Z acts on $R[t, dt]$ by $n * t = t + n$, $n * dt = dt$.*

PROOF. Embed EZ in $\Delta(R)$, the singular complex of R, by sending $(n_0, \ldots, n_q)$ to the linear simplex sending the i-th vertex of Δ_q to $n_i \in R$. Let $\Omega(R)$ be the algebra C^∞ differential forms on R. We then have mappings

$$R[t, dt] \subset \Omega(R) \xrightarrow{\eta} \Omega(\Delta(R)) \longrightarrow \Omega(EZ) \longrightarrow C^*(EZ; R)$$

where $\eta(w)(T) = T^* w$. One may verify by inspection that all of these maps except η are weak equivalences and one may verify that η_* is an isomorphism by computing the groups $H^p(R; V)$, for $p = 0, 1$. □

6. The proof of Theorem 5.6

The proof of each of the statements in Theorem 5.6 follows the same course. We carry out in detail the proof that

$$\Delta_\pi : \mathcal{F}(A, B) \to \mathcal{F}(\Delta_\pi B, \Delta_\pi A).$$

is a weak equivalence, where $A, B \in \mathbf{A}$ are FNF, dealing with the remaining two statements only when differences in the proofs require us to do so. (Recall that, as in the previous section, $\mathbf{A}$ denotes the category ${}_{A_0}\mathcal{A}_\pi$ and $\mathbf{T}$ denotes the category $(\Delta\mathcal{T}_\pi)_{E\pi}$.) The idea of the proof is to show, by a sequence of reductions, that the theorem is true for general A if

$$\mathcal{F}(\mathbf{R}(V, n), B) \to \mathcal{F}(\Delta_\pi B, \Delta_\pi(\mathbf{R}(V, n)))$$

is a weak equivalence for all $V \in \mathcal{V}_{\mathrm{F}\pi}$, $n \geqslant 0$, and $B \in \mathcal{A}_\pi$.

REMARK. Each of the three statements of Theorem 5.6 involves proving that a continuous simplicial mapping induces isomorphisms on homotopy groups. Since the homotopy groups $\pi_q(X)$ of a simplicial space X are defined to be the homotopy groups of the underlying simplicial set X^δ (with topology on $\pi_q(X)$ induced from the topology on X), we can ignore the topology on the simplicial spaces that occur and work in the category of simplicial sets. We will do so for the remainder of the section without further mention.

Suppose now that A and B in $\mathbf{A}$ are FNF and consider the diagram

$$\begin{tikzcd}
\mathcal{F}(\Delta_\pi B, \Delta_\pi A) \arrow[r] \arrow[d, "\Delta"] & \cdots \arrow[r] & \mathcal{F}(\Delta_\pi B, \Delta_\pi A_n) \arrow[r] \arrow[d, "\Delta_n"'] & \mathcal{F}(\Delta_\pi B, \Delta_\pi A_{n-1}) \arrow[r] \arrow[d, "\Delta_{n-1}"] & \cdots \\
\mathcal{F}(\Delta_\pi B, \Delta_\pi A) \arrow[r] & \cdots \arrow[r] & \mathcal{F}(\Delta_\pi B, \Delta_\pi A_n) \arrow[r] & \mathcal{F}(\Delta_\pi B, \Delta_\pi A_{n-1}) \arrow[r] & \cdots
\end{tikzcd}$$

where $A = \varprojlim A_n$, $A_n = A_{n-1}(V_n, m_n)$ as in Definition 5.1. Here, A and A_n are considered to be in $\mathbf{A}$ using the inclusions $A_0 \subset A$, $A_0 \subset A_n$. According to Theorem 5.2 each of the mappings

$$\pi : \mathcal{F}(A_n, B) \to \mathcal{F}(A_{n-1}, B)$$

is a fibration with fibre $\mathcal{F}(\mathbf{R}(V_n, m_n), B)$, B viewed in $\mathcal{A}_\pi$. Since $\Delta A_n \to \Delta A_{n-1}$ is a fibration with fibre $\Delta(\mathbf{R}(V_n, m_n))$, so is $\Delta_\pi(A_n) \to \Delta_\pi(A_{n-1})$. Therefore, using Theorem 2.2 we see that

$$\pi' : \mathcal{F}(\Delta_\pi B, \Delta_\pi A_n) \to \mathcal{F}(\Delta_\pi B, \Delta_\pi A_{n-1})$$

is a fibration with fibre $\mathcal{F}(\Delta B, \Delta\mathbf{R}(V_n, m_n))$. Thus, according to [2, Theorem 3.1, p. 254], we have a commutative diagram

$$\begin{array}{ccccccccc}
0 \to \varprojlim^1 \pi_{i+1}(\mathcal{F}(A_n, B), \alpha_n) & \xrightarrow{i_*} & \pi_i(\mathcal{F}(A, B), \alpha) & \xrightarrow{j_*} & \varprojlim \pi_i(\mathcal{F}(A_n, B), \alpha_n) \to 0 \\
\downarrow \tilde{\Delta}^1 & & \downarrow \Delta_* & & \downarrow \tilde{\Delta} \\
0 \to \varprojlim^1 \pi_{i+1}(\mathcal{F}(\Delta_\pi B, \Delta_\pi A_n), \alpha'_n) & \xrightarrow{i'_*} & \pi_i(\mathcal{F}(\Delta_\pi B, \Delta_\pi A), \alpha') & \xrightarrow{j'_*} & \varprojlim \pi_i(\mathcal{F}(\Delta_\pi B, \Delta_\pi A_n), \alpha'_n) \to 0
\end{array} \tag{6.1}$$

for each $i > 0$, where each of the horizontal sequences is exact and $\tilde{\Delta}^1, \tilde{\Delta}$ are induced by the Δ_*. When $i = 0$, each term in the diagram is a set with distinguished base point (determined by the compatible sequence of base points $\{\alpha_n\}, \{\alpha'_n\}$) and exactness is defined in the usual way.

LEMMA 6.2. *If, for each n, the mapping $\Delta_n : \mathcal{F}(A_n, B) \to \mathcal{F}(\Delta_\pi B, \Delta_\pi A_n)$ is a weak equivalence, then the mapping*

$$\Delta : \mathcal{F}(A, B) \to \mathcal{F}(\Delta_\pi B, \Delta_\pi A)$$

is a weak equivalence.

PROOF. If $i > 0$, it follows immediately from (6.1) that the mapping

$$\Delta_* : \pi_i\big(\mathcal{F}(A, B), \alpha\big) \to \pi_i\big(\mathcal{F}(\Delta_\pi B, \Delta_\pi A), \alpha'\big)$$

is an isomorphism. Suppose that $i = 0$ and that we have $b_1, b_2 \in \pi_0(\mathcal{F}(A, B))$ with $\Delta_* b_1 = \Delta_* b_2$. It then follows from diagram (6.1) that $j_* b_1 = j_* b_2 = \{\alpha_n\} \in \varprojlim \pi_0(\mathcal{F}(A_n, B))$. We use this sequence of base points as our distinguished points in the top row of (6.1) (with $i = 0$) and $\Delta_n \alpha_n = \alpha'_n$ as the distinguished base points in the bottom row. By exactness, we can find $c_1, c_2 \in \varprojlim^1 \pi_1(\mathcal{F}(A_n, B), \alpha_n)$ with $j_* c_1 = b_1$, $j_* c_2 = b_2$. But then $i'_* \tilde{\Delta}^1 c_1 = i'_* \tilde{\Delta}^1 c_2$ which implies $c_1 = c_2$. Thus $b_1 = b_2$ and

$$\Delta_* : \pi_0\big(\mathcal{F}(A, B)\big) \to \pi_0\big(\mathcal{F}(\Delta_\pi B, \Delta_\pi A)\big)$$

is injective. The proof of surjectivity is similar and is left to the reader. $\square$

In the following, if $K \in \mathcal{A}_\pi$ and $B \in \mathbf{A}$, we form $\mathcal{F}(K, B)$ by viewing B as contained in $\mathcal{A}_\pi$. For $V \in \mathcal{V}_{\mathrm{F}\pi}$, let $\tilde{V}$ be a copy of V and define

$$\mathbf{R}E(V, n) = \mathbf{R}(\tilde{V}, n + 1)(V, \lambda)$$

where $\lambda : V^* \simeq \tilde{V}^*$. If $\{v_i\}$ is a basis for V and $u_i = \lambda v_i$, then

$$RE(V,n) = \mathbf{R}[v_1, \dots, v_\ell, u_1, \dots, u_\ell]$$

with $\dim v_i = n, \dim u_i = u + 1$ and $dv_i = u_i$. Note that

$$\mathcal{F}(\mathbf{R}E(V,n), B) = ((\Omega \otimes B \otimes V)^n)^\pi,$$
$$\Delta(\mathbf{R}E(V,n)) = \Omega^n \otimes V$$

both of which are contractible.

We now proceed with the proof that $\Delta_n : \mathcal{F}(A_n, B) \to \mathcal{F}(\Delta_\pi B, \Delta_\pi A_n)$ is a weak equivalence. We know by Theorem 5.2 that the fibration

$$\pi : \mathcal{F}(A_n, B) \to \mathcal{F}(A_{n-1}, B) \tag{6.3}$$

is induced from the fibration

$$p : \mathcal{F}(\mathbf{R}E(V_n, m_n), B) \to \mathcal{F}(\mathbf{R}(V_n, m_n + 1), B) \tag{6.4}$$

by a mapping $f : \mathcal{F}(A_{n-1}, B) \to \mathcal{F}(\mathbf{R}(V_n, m_n + 1), B)$. Since $\mathcal{F}(\mathbf{R}E(V_n, m_n), B)$ is contractible, the image of the mapping (6.4) is contained in a single component X' of $\mathcal{F}(\mathbf{R}(V_n, m_n+1), B)$. Now, if Z is a component of $\mathcal{F}(A_{n-1}, B)$, then $\pi^{-1}Z \subset \mathcal{F}(A_n, B)$ is empty unless

$$fZ \subset X'. \tag{6.5}$$

It follows that $\mathcal{F}(A_n, B)$ is the disjoint union of the $\pi^{-1}Z$ as Z ranges over those components of $\mathcal{F}(A_{n-1}, B)$ satisfying (6.5).

In the same way, we see that the fibration

$$\pi' : \mathcal{F}(\Delta_\pi B, \Delta_\pi A_n) \to \mathcal{F}(\Delta_\pi B, \Delta_\pi A_{n-1}) \tag{6.6}$$

is induced by the mapping $\Delta f : \mathcal{F}(\Delta_\pi B, \Delta_\pi A_{n-1}) \to \mathcal{F}(\Delta B, \Delta\mathbf{R}(V_n, m_n + 1))$ from the fibration

$$\mathcal{F}(\Delta B, \Delta \mathbf{R}E(V_n, m_n)) \to \mathcal{F}(\Delta B, \Delta\mathbf{R}(V_n, m_n + 1)). \tag{6.7}$$

Thus, if X' is the component of $\mathcal{F}(\Delta B, \Delta\mathbf{R}(V_n, m_n + 1))$ containing the image of the mapping (6.7), then $\mathcal{F}(\Delta_\pi B, \Delta_\pi A_n)$ is the disjoint union of $(\pi')^{-1}Z$ where Z ranges over the components of $\mathcal{F}(\Delta_\pi B, \Delta_\pi A_{n-1})$ satisfying $(\Delta_\pi f)Z \subset X$.

We now need the following.

PROPOSITION 6.8. *Let $p' : Y' \to X'$ be a fibration with X' connected and Y' contractible. Let $\hat{f} : \tilde{X} \to X'$ be a mapping and $\tilde{p} : \tilde{Y} \to \tilde{X}$ the induced fibration. Then, for any component X of $\tilde{X}$, we have an exact sequence*

$$\dots \to \pi_q(Y, y_0) \xrightarrow{p_*} \pi_q(X, x_0) \xrightarrow{f_*} \pi_q(X', x_0') \to \dots \to \pi_1(X', x_0') \xrightarrow{\partial_*} \pi_0 Y \to *,$$

where $Y = \tilde{p}^{-1}X$, $p = \tilde{p} \mid Y$, $f = \tilde{f} \mid X$.

PROOF. If we replace $f : X \to X'$ by a fibration, one sees easily that the fibre of this fibration has the homotopy type of Y (since Y' is contractible). The sequence above is the exact homotopy sequence of this fibration. $\square$

We can now apply Proposition 6.8 to the fibrations (6.3) and (6.6) obtaining the diagram of exact sequences

$$\begin{array}{ccccccccccc} \cdots \to & \pi_q(Y, y_0) \to & \pi_q(Z, z_0) \to & \pi_q(X, x_0) \to \cdots \to & \pi_1(X, x_0) \to & \pi_0(Y) \to * \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \cdots \to & \pi_q(Y', y_0') \to & \pi_q(Z', z_0') \to & \pi_q(X', x_0') \to \cdots \to & \pi_1(X', x_0') \to & \pi_0'(Y) \to * \end{array}$$

Here X' is the image of the mapping (6.4), Z is a component of $\mathcal{F}(A_{n-1}, B)$ mapping into X' under the mapping f and $Y = \pi^{-1}Z \subset \mathcal{F}(A_n, B)$. Similarly, X' is the image of (6.7), Z is any component of $\mathcal{F}(\Delta_\pi B, \Delta_\pi A_{n-1})$ mapping into X' under the mapping (Δf), and $Y' = (\pi^1)^{-1}Z \subset \mathcal{F}(\Delta_\pi B, \Delta_\pi A_n)$. Note that $\mathcal{F}(A_0, B) = \mathcal{F}(\Delta_\pi(B), \Delta_\pi(A_0)) = *$.

Using Lemma 6.2, the 5-Lemma, and induction on n, we have the following.

LEMMA 6.9. *The mapping* $\Delta : \mathcal{F}(A, B) \to \mathcal{F}(\Delta_\pi B, \Delta_\pi A)$ *is a weak equivalence if* $\mathcal{F}(\mathbf{R}(V, n), B) \to \mathcal{F}(\Delta B, \Delta\mathbf{R}(V, n))$ *is a weak equivalence for any* $A \in \mathcal{A}_\pi$ *which is FNF,* $V \in \mathcal{V}_{\mathrm{F}\pi}$ *and* $n \geqslant 0$.

To prove that $\Delta : \mathcal{F}(\mathbf{R}(V, n), B) \to \mathcal{F}(\Delta_\pi B, \Delta_\pi \mathbf{R}(V, n))$ is a weak equivalence, we need the following.

LEMMA 6.10. *The mapping* $\Delta_* : \pi_0(\mathcal{F}(\mathbf{R}(V, n), B)) \to \pi_0(\mathcal{F}(\Delta B, \Delta\mathbf{R}(V, n))$ *is a bijection for* $n \geqslant 0$. *Furthermore, if* $n = 0$, *then*

$$\Delta_* : \pi_j\big(\mathcal{F}\big(\mathbf{R}(V, n), B\big), \alpha\big) \to \pi_j\big(\mathcal{F}\big(\Delta B, \Delta\mathbf{R}(V, n)\big), \alpha'\big)$$

is an isomorphism for all $j > 0$.

COROLLARY 6.11. *The mapping* $\Delta : \mathcal{F}(\mathbf{R}(V, n), B) \to \mathcal{F}(\Delta B, \Delta\mathbf{R}(V, n))$ *is a weak equivalence.*

PROOF of Corollary 6.11. Consider the diagram of fibrations

$$\begin{array}{ccccc} \mathcal{F}(\mathbf{R}(V, n-1), B) & \longrightarrow & \mathcal{F}(\mathbf{R}E(V, n-1), B) & \longrightarrow & \mathcal{F}(\mathbf{R}(V, n), B) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(\Delta B, \Delta\mathbf{R}(V, n-1)) & \longrightarrow & \mathcal{F}(\Delta B, \Delta\mathbf{R}E(V, n-1)) & \longrightarrow & \mathcal{F}(\Delta B, \Delta\mathbf{R}(V, n)) \end{array}.$$

Since the total space of each of the above fibrations is contractible (by Theorem 4.1), the homotopy exact sequences of these fibrations reduce to

$$\begin{array}{ccc} \pi_{j+1}(\mathcal{F}(\mathbf{R}(V,n),B)) & \longrightarrow & \pi_j(\mathcal{F}(\mathbf{R}(V,n-1),B)) \\ \downarrow & & \downarrow \\ \pi_{j+1}(\mathcal{F}(\Delta B,\Delta\mathbf{R}(V,n))) & \longrightarrow & \pi_j(\mathcal{F}(\Delta B,\Delta\mathbf{R}(V,n-1))) \end{array}$$

An easy induction using Lemma 6.10 gives the required result □

It now follows that the first assertion of Theorem 5.6 will be proved once we prove Lemma 6.10. Indeed, the arguments that we have developed in this section up to this point can be modified by making the obvious substitutions so as to deal with the second and third assertions of Theorem 5.6. As a result, Theorem 5.6 will be proved once we prove Lemma 6.10 and the following lemma.

LEMMA 6.12. *Let $f : B \to C$ be a weak equivalence in $\mathbf{A}$ and suppose $X \in \mathbf{T}$ is 0-connected. Then the mappings*

$$\gamma_* : \pi_0\big(\mathcal{F}\big(\mathbf{R}(V,n),\Omega(X)\big)\big) \to \pi_0\big(\mathcal{F}\big(X,\Delta\mathbf{R}(V,n)\big)\big),$$

$$f_* : \pi_0\big(\mathcal{F}\big(\mathbf{R}(V,n),B\big)\big) \to \pi_0\big(\mathcal{F}\big(\mathbf{R}(V,n-1),C\big)\big)$$

are bijections for $n \geqslant 0$. Furthermore, if $n = 0$, then

$$\gamma_* : \pi_j\big(\mathcal{F}\big(\mathbf{R}(V,n),\Omega(X)\big),\alpha\big) \to \pi_j\big(\mathcal{F}\big(X,\Delta\mathbf{R}(V,n)\big),\alpha'\big)$$

$$f_* : \pi_j\big(\mathcal{F}\big(\mathbf{R}(V,n),B\big),\alpha\big) \to \pi_j\big(\mathcal{F}\big(\mathbf{R}(V,n-1),C\big),\alpha'\big)$$

are isomorphisms for all $j > 0$.

The proofs of Lemma 6.10 and 6.12 involve the same ideas so we present them together. We begin by defining an isomorphism

$$\xi : \pi_0\big(\mathcal{F}\big(\mathbf{R}(V,n),B\big)\big) \to H_n(B;V).$$

(Note: Both $\mathcal{F}(\mathbf{R}(V,n),B)$ and $\mathcal{F}(\Delta B,\Delta\mathbf{R}(V,n))$ are simplicial groups so $\pi_0(\mathcal{F}(\mathbf{R}(V,n), B))$ and $\pi_0(\mathcal{F}(\Delta B,\Delta\mathbf{R}(V,n)))$ are groups.) By definition,

$$\mathcal{F}\big(\mathbf{R}(V,n),B\big)_0 = \big(\mathbf{R}(V,n),B\big) = Z_n\big((V^*\otimes B)^\pi\big) = Z_n\big((B\otimes V)^\pi\big),$$

where Z_n denotes the group of n-cycles. It follows that we have an epimorphism

$$\xi : \mathcal{F}\big(\mathbf{R}(V,n),B\big)_0 \to H_n(B;V).$$

Suppose $z \in \mathcal{F}(\mathbf{R}(V,n),B)_0$ is in the same component as 0. Then there is an element $w \in \mathcal{F}(\mathbf{R}(V,n),B)_1 = Z_n((\Omega_1 \otimes B \otimes V)^\pi)$ with $\partial_0 w = z$, $\partial_1 w = 0$. If we define

$$r : \left((\Omega_1 \otimes B \otimes V)^\pi\right)^n \to \left((B \otimes V)^\pi\right)^{n-1}$$

by $r(a \otimes b \otimes v) = 0$ if $a \in \Omega_1^0$, $b \in B^n$ and

$$r(a dt_1 \otimes b \otimes v) = \left(\int_0^1 a(t_1)\,dt\right) b \otimes v$$

if $b \in B^{n-1}$, then $dr + rd = \partial_0 - \partial_1$ so that $dr(v) = z$. Hence ξ induces an epimorphism

$$\xi : \pi_0\big(\mathcal{F}\big(\mathbf{R}(V,n),B\big)\big) \to H_n(B;V).$$

To show that ξ is a monomorphism, suppose $z = du$ for some $u \in B^{n-1} \otimes V$. Then, if $\bar{u} \in (\Omega_1 \otimes B \otimes V)^n$ is given by

$$\bar{u} = t_1 \otimes z + dt_1 \otimes u,$$

we have $\partial_0 \bar{u} = z$, $\partial_1 \bar{u} = 0$ and $z = 0$ in $\pi_0(\mathcal{F}(\mathbf{R}(V,n),B))$. It follows that ξ is an isomorphism.

Now, if $f : B \to C$ is a DG algebra mapping and $A = \mathbf{R}(V,n)$, we have a commutative diagram

$$\begin{array}{ccc}
\pi_0(\mathcal{F}(A,B)) & \xrightarrow{f_*} & \pi_0(\mathcal{F}(A,C)) \\
\xi \downarrow \simeq & & \xi \downarrow \simeq \\
H_n(B;V) & \xrightarrow{f_*} & H_n(C;V)
\end{array}$$

If $f_* : H_*(B;V) \to H_*(C;V)$ is an isomorphism, then

$$f_* : \pi_0\big(\mathcal{F}(A,B)\big) \to \pi_0\big(\mathcal{F}(A,C)\big)$$

is an isomorphism and the second of the four assertions of Lemma 6.12 is proved.

We next define an isomorphism

$$\xi' : \pi_0\big(\mathcal{F}\big(X, \Delta\mathbf{R}(V,n)\big)\big) \to H_n\big(\Omega(X);V\big)$$

for any simplicial space X. By definition, $\mathcal{F}(X, \Delta(\mathbf{R}(V,n)))$ can be identified with $Z_n(\Omega(X;V)^\pi)$. Using this identification we let

$$\xi' : \pi_0\big(\mathcal{F}\big(X, \Delta\big(\mathbf{R}(V,n)\big)\big)\big) \to H_n\big(\Omega(X;V)\big)$$

be given by

$$\xi'\{z\} = z + B_n\big(\Omega(X;V)^\pi\big).$$

We show that ξ' is well defined and an isomorphism.

Recall that in Section 3, we defined $\Omega^p(\Delta^q \times I)$ to be the p forms on $\Delta^q \times I$ and we defined

$$\mu : \Omega^p(\Delta^p \times I) \to \Omega_q^{p-1}$$

by

$$\mu(w) = (-1)^{p-1} \int_0^1 w_2(t)\, dt,$$

where $w = w_1(t) + w_2(t)dt$. Note that we may view the elements of $\Omega^p(\Delta[q] \times \Delta[1])$ as continuous, piecewise smooth p-forms on $\Delta^q \times I$ and the formula for μ makes sense on such forms and gives a mapping

$$\mu : \Omega^p(\Delta[q] \times \Delta[1]) \to \Omega^{p-1}(\Delta[q]).$$

Furthermore, we have $d\mu + \mu d = i_1^* - i_0^*$, where $i_0, i_1 : \Delta[q] \to \Delta[q] \times \Delta[1]$ are the inclusions.

To see that ξ' is well defined, suppose z_0 and z_1 are elements of $Z_n(\Omega(X;V)^\pi)$ and $z_i = \partial_i u$, where $u \in Z_n(\Omega(X \times \Delta[1];V)^\pi)$. Let $\bar{u} \in \Omega^{n-1}(X)$ be given by

$$\bar{u}(x) = \mu(t_x \times \mathrm{id}_{\Delta[1]})^* u$$

for $x \in X_q$, where $t_x : \Delta[q] \to X$ with $t_x(0, \ldots, q) = x$. Then

$$\begin{aligned} d\bar{v}(x) &= (i_1^* - i_0^* - \mu d)(t_x \times \mathrm{id}_{\Delta[1]})^* u \\ &= (i_1^* - i_0^*)(t_x \times \mathrm{id}_{\Delta[1]})^* u \\ &= (\partial_0 u - \partial_1 u)(x) = z_0(x) - z_1(x) \end{aligned}$$

so ξ' is well defined.

Clearly ξ' is an epimorphism; we show it is a monomorphism. Suppose $z = du, u \in \Omega^{n-1}((\Delta(B;V)^\pi)$. Let $w \in Z_n(\Omega(X \times \Delta[1];V)^\pi)$ be given as follows: If p_1 and p_2 are the projections of $X \times \Delta[1]$ onto the factors, then

$$w(x) = \big(d(p_2^* t_1)\big) p_1^* u(x) + (p_2^* t_1)(p_1^* z)(x).$$

It is easily checked that $dw = 0$, $\partial_0 w = 0$, and $\partial_1 w = z$. Thus ξ' is an isomorphism.

It follows directly from the definitions that the diagrams

$$\begin{array}{ccc} \pi_0(\mathcal{F}(\mathbf{R}(V,n), B)) & \xrightarrow{\Delta_0} & \pi_0(\mathcal{F}(\Delta B, \Delta\mathbf{R}(V,n))) \\ \xi \downarrow \simeq & & \xi' \downarrow \simeq \\ H_n(B;V) & \xrightarrow{i_*} & H_n(\Omega\Delta B;V) \end{array}$$

and

$$\begin{array}{ccc}
\pi_0(\mathcal{F}(\mathbf{R}(V,n),\Omega(X))) & \xrightarrow{\gamma} & \pi_0(\mathcal{F}(X,\Delta\mathbf{R}(V,n))) \\
\epsilon \downarrow \simeq & & \epsilon' \downarrow \simeq \\
H_n(\Omega(X);V) & \xrightarrow{id} & H_n(\Omega(X);V)
\end{array}$$

are commutative, where $i_* : H_n(B) \to H_n(\Omega(\Delta B))$ is the canonical mapping. Since B is FNF, i_* is an isomorphism by Theorem 5.4. The first assertions of both Lemmas 6.10 and 6.12 are an immediate consequence of these diagrams.

We now prove the remaining assertions of Lemmas 6.10 and 6.12, namely that certain mappings between simplicial spaces defined in terms of $\mathbf{R}(V,n)$ induce isomorphisms on homotopy in positive dimensions if $n = 0$. In fact, we prove that all of these homotopy groups vanish.

Assume $n = 0$. By definition,

$$\mathcal{F}\big(\mathbf{R}(V,n),B\big)_q = \big(\mathbf{R}(V,n),\Omega_q \otimes B \otimes V\big)^{\pi} = Z_0\big((\Omega_q \otimes B \otimes V)^{\pi}\big).$$

It is easy to see that $d(f \otimes b) = 0$ for $f \in \Omega_q^0$, $b \in B^0 \otimes V$ if and only if $df = 0$ and $db = 0$. Thus,

$$\mathcal{F}\big(\mathbf{R}(V,n),B\big)_q = H_0(B;V)$$

for all $q \geqslant 0$ and all face and degeneracy mappings are the identity. Therefore, $\pi_j(\mathcal{F}(\mathbf{R}(V,n),B),\alpha) = 0$ for all $j > 0$.

Similarly, $\Delta\mathbf{R}(V,n)_q = V$ for all $q \geqslant 0$ and all face and degeneracy mappings in $\Delta\mathbf{R}(V,n)$ are the identity. It follows that, for any $Z \in \Delta\mathcal{T}$,

$$\mathcal{F}\big(Z,\Delta\mathbf{R}(V,n)\big)_q = \big(\Delta[q] \times Z,\Delta\mathbf{R}(V,n)\big)^{\pi}$$

consists of mappings that are constant on components; that is, all continuous equivariant mappings of $\pi_0(Z)$ into V and all face and degeneracy mappings the identity. Again, $\pi_j(\mathcal{F}(Z,\Delta\mathbf{R}(V,n))) = 0$ for $j > 0$ and Theorem 5.6 is proved.

7. Comparison of real and rational homotopy theory

Let $\Delta\mathcal{T}_0$ (respectively, $\Delta\mathcal{S}_0$) be the full subcategory of $\Delta\mathcal{T}$ (respectively, $\Delta\mathcal{S}$) consisting of those X such that $\Omega(X)$ is FNF. Denote by $\Delta\mathcal{S}_{0Q}$ the category $\Delta\mathcal{S}_0$ localized with respect to Q-equivalence and $\Delta\mathcal{T}_{0R}$ the category $\Delta\mathcal{T}_0$ localized with respect to R-equivalence. Finally, let $\alpha : \Delta\mathcal{S} \to \Delta\mathcal{T}$ be the functor which assigns to any simplicial set X the simplicial space X in the discrete topology and

$$\bar{\alpha} : \Delta\mathcal{S}_{0Q} \to \Delta\mathcal{T}_{0R}$$

the induced functor. The next result shows that this functor is neither injective or surjective.

THEOREM 7.1. *There are simply connected simplicial sets X_1 and X_2 each of finite type such that X_1 and X_2 are not isomorphic in ΔS_{0Q} but $\bar{\alpha}(X_1)$ and $\bar{\alpha}(X_2)$ are isomorphic in ΔT_{0R}. In addition, there is an FNF algebra $A \in \mathcal{A}$ such that $\Delta(A)$ is not isomorphic to anything in the image of $\bar{\alpha}$.*

PROOF. We begin with the construction of simplicial sets X_1 and X_2 satisfying the conditions of the theorem. Let $\iota_q \in H^q(K(Z,q);Z)$ be the generator and let $X_{n,m}$ be the fibration over $K(Z,2) \vee K(Z,2)$ induced from the contractible fibration by

$$f_{n,m} : K(Z,2) \vee K(Z,2) \to K(Z,4),$$

where $f^*_{n,m}(\iota_4) = n(\iota_2 \otimes 1)^2 - m(1 \otimes \iota_2)^2$. Then

$$H^*(X_{n,m};Q) = Q[x,y]/\{nx^2 - my^2, xy\},$$

where x and y have degree 2. If $z = x^2/m$, then $H^2(X_{n,m};Q) \simeq Q \oplus Q$ with basis $\{x,y\}$ and $H^4(X_{n,m};Q) \simeq Q$ with basis $\{z\}$. The matrix of the quadratic form $H^2(X_{n,m}) \to H^4(X_{n,m})$ in this basis is

$$\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}.$$

Let $M_{n,m} = R[x,y,u,v] \in \mathcal{A}$, where $dx = dy = 0$, $du = nx^2 - my^2$ and $dv = xy$. In [1, Section 16], it is shown that since $mx^2 - my^2, xy$ is an ESP sequence, $M_{n,m}$ is a minimal model for $\Omega(X_{n,m})$.

Let $X_1 = X_{1,15}$ and $X_2 = X_{3,5}$. Then the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 15 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \tag{7.2}$$

are not equivalent over Q. For if they were equivalent, one would have rational solutions to the equation $3a^2 + 5b^2 = 1$, or, equivalently, integer solutions to the equations

$$3a^2 + 5b^2 = c^2. \tag{7.3}$$

The fact that this is not possible is proved by working mod 3 and using the notion of "infinite descent"; the existence of solutions a, b, c for (7.3) imply the existence of solutions a', b', c' with $a' < a$, $b' < b$, $c' < c$.

It follows that $H^*(X_1;Q)$ and $H^*(X_2;Q)$ are not isomorphic. However, the matrices (7.2) are equivalent over R (since they have the same signatures). Hence $M_{1,15}$ and $M_{3,5}$ are isomorphic and then X_1 and X_2 are isomorphic in ΔT_{0R}.

We now prove the second assertion of Theorem 7.1. We are indebted to Tsuneo Tamagawa for his assistance with this proof.

Let A be the free DG algebra over R generated by $x_1, \ldots, x_n \in A^2$, $y_1, \ldots, y_m \in A^3$ and with

$$dx_i = 0, \quad 1 \leqslant i \leqslant n,$$
$$dy_i = \sum_{j \leqslant k} b_i^{jk} x_j x_k, \quad 1 \leqslant i \leqslant m.$$

We say that A has a *rational form* if there is a DG algebra A' over Q and an isomorphism $A \simeq A' \otimes_Q R$ of DG algebras. Equivalently, A has a rational form if we can choose bases $\tilde{x}_1, \ldots, \tilde{x}_n$ for $A^2, \tilde{y}_1, \ldots, \tilde{y}_m$ for A^3 such that

$$d\tilde{y}_i = \sum_{j \leqslant k} \tilde{b}^{jk} \tilde{x}_j \tilde{x}_k$$

with $\tilde{b}_i^{jk} \in Q$.

Suppose A has a rational form. Let $P = (p_{ij})$ be an invertible $n \times n$ matrix, $Q = (q_{ij})$ an invertible $m \times m$ matrix with

$$x_i = \sum p_{ij} \tilde{x}_j, \qquad \tilde{y} = \sum q_{ij} y_j.$$

Then

$$d\tilde{y}_i = \sum q_{ij} dy_j = \sum q_{ij} b_j^{k\ell} p_{kr} p_{ls} \tilde{x}_r \tilde{x}_s$$

so that

$$\tilde{b}_i^{rs} = \sum_{j,k,l} q_{ij} b_j^{kl} p_{kr} p_{ls}.$$

Of course, b_i^{jk} can be expressed in terms of $\tilde{b}_i^{rs}$ in a similar way.

Let $N = n(n+1)m/2$. Then there are N of the b_i^{jk} and $n^2 + m^2$ of the p_{ij}, q_{ij}. We can think of the passage from $\tilde{b}_i^{rs}$ to b_i^{jk} as defining a mapping $\varphi_{P,Q} : R^N \to R^N$ depending on the particular choice of P and Q. If we fix a rational point $\tilde{b} \in Q^N \subset R^N$, then the set

$$A'(\tilde{b}) = \{\varphi_{P,Q}(\tilde{b}) : P \text{ invertible } n \times n,\ Q \text{ invertible } m \times m\}$$

corresponds to all DG algebras A (which of course depend on b_i^{jk}) with A' defined by $\tilde{b}$ as their rational form. Thus, the set $\bigcup\{A'(\tilde{b}) \mid \tilde{b} \in Q^N\}$ corresponds to the set of all A as above which have a rational form.

Now, if $N > n^2 + m^2$ (for example, $n = 5,\ m = 2$), then $A'(\tilde{b})$ is the image of an open set in $R^{n^2+m^2}$ under a differentiable mapping $R^{n^2+m^2} \to R^N$. Thus, the set $\bigcup\{A'(\tilde{b}) \mid \tilde{b} \in Q^N\}$ cannot be all of R^N and any point in its complement corresponds to a real form of A with no rational form. $\square$

8. Applications

We now give some applications of the ideas developed in this paper. We begin with a formulation of the main result of [9] in our context.

If B is a commutative graded algebra of finite type, we can view it as being in the category $\mathcal{A}$ by taking $d = 0$. Let $M(B)$ be a minimal algebra and $\gamma : M(B) \to B$ a map such that γ induces an isomorphism

$$H_*(M(B)) \to H_*(B) = B.$$

DEFINITION 8.1. A simplicial set $X \in \Delta\mathcal{T}$ is *R-formal* if there is a mapping $g : X \to \Delta(M(H^*(X))$ inducing an isomorphism in cohomology.

The main result of [9] can now be stated as follows.

THEOREM 8.2. *If N is a compact nilpotent Kähler manifold, then the total singular complex $\Delta(N)$ in the discrete topology is R-formal. Hence, there is a minimal algebra $M(H^*(N)) \in \mathcal{A}$ and a map $g : \Delta(N) \to \Delta(M(H^*(N)))$ inducing isomorphisms on H^* and on π^*.*

We next describe a construction of the continuous characteristic classes of foliations in our context.

Let $\zeta_q \to BGL_q$ be the universal real q-plane bundle and let $B(q)$ be the simplicial set consisting of pairs $(T, \mathcal{F})$, where $T : \Delta^p \to BGL_q$ is smooth and $\mathcal{F}$ is a smooth foliation on $T^*\zeta_q$ transverse to fibres. For fixed T, it is easy to define a sensible topology on $\{(T, \mathcal{F})\}$, making $B(q)$ into a simplicial space. Then, the geometric realization $|B(q)^\delta|$ is a model for $B\Gamma_q$ and $H^*(B(q))$ is a plausible definition of the continuous cohomology of $B\Gamma_q$.

Let $\overline{B(q)}$ be the fibre of the natural mapping $B(q) \to \Delta(BGL_q)^\delta$, L_q the Lie algebra of C^∞ vector fields on R^q in the C^∞ topology, and let $C^*(L_q) \in \mathcal{A}$ be the algebra of continuous cochains on L_q (continuous skew forms on L_q with differential defined by the Lie bracket). Then $\overline{B\Gamma_q} = |\overline{B(q)^\delta}|$ is the classifying space for foliations with trivialized normal bundle. In [11], it is shown that $\overline{B(q)} = \Delta(C^*(L_q))$ and

$$C^*(L_q) \xrightarrow{\ i\ } \Omega(\Delta(C^*(L_q))) = \Omega\overline{B(q)} \to \Omega\overline{B(q)^\delta}$$

gives the characteristic mapping

$$H^*(L_q) \xrightarrow{\ i_*\ } H^*(\Delta(C^*(L_q)) = H^*(\overline{B(q)}) \to H^*(\overline{B\Gamma_q})$$

which takes the secondary characteristic classes into $H^*(\overline{B\Gamma_q})$.

A fundamental question in the study of characteristic classes of foliations asks if this mapping is injective. In [5], we answered this question in the affirmative for G-foliations, ([11]), G a compact Lie group. In addition, we proved in [8] that

$$i : C^*(L) \to \Omega\Delta C^*(L)$$

is a homology isomorphism for a large class of Lie algebras L including the Lie algebras of vector fields on a compact manifold.

A similar construction can be used to obtain a mapping

$$H^*(L_q, O_q) \to H^*(B\Gamma_q).$$

Just as above, the injectivity of this mapping is an important open question.

The next two results deal with the rational homotopy type of function spaces. The first describes a minimal model of a function space and the second gives an explicit description of the Q-localization of a function space. (See [7] for details.)

THEOREM 8.3. *Let Y be a nilpotent space of finite type and let X be a space of finite type with $H^q(X;Q) = 0$ for $q > N$, some N. For $f : X \to Y$, let $\mathcal{F}(X,Y,f)$ be the component of the function space $\mathcal{F}(X,Y)$ containing f. Then $\mathcal{F}(X,Y,f)$ has a minimal model $A \simeq Q[W]$ where W^q, $q > 0$, is isomorphic to a subspace of*

$$\sum_n \pi^n(Y) \otimes H_{n-q}(X;Q).$$

If f is the constant map, this inclusion is equality.

For any simplicial set Z, the geometric realization of Z is denoted by $|Z|$.

THEOREM 8.4. *Let X and Y be CW complexes with Y nilpotent and finite type, X formal, and $H_q(X;Q) = 0$ for $q > N$, some N. Then the space*

$$\left|\Delta\left(Q\left[\pi^*(Y) \otimes H_*(X;Q)\right], d_1\right)\right|$$

is a Q localization of $\mathcal{F}(X,Y)$.

The differential d_1 in Theorem 8.4 can be calculated as follows: If $c \in H_*(X;Q)$ and $v \in \pi^*(Y)$, then $dv \in Q[\pi^*(Y)]$ and $d_1(v \otimes c) = (dv) \otimes c$ where $(dv) \otimes c$ is expanded by

$$(u_1 + u_2) \otimes c = u_1 \otimes c + u_1 \otimes c,$$
$$(u_1 u_2) \otimes c = \sum (-1)^{|u_2||c_j'|}\left(u_1 \otimes c_j'\right)\left(u_2 \otimes c_j''\right),$$

where $D : H_*(X;Q) \to H_*(X;Q) \otimes H_*(X;Q)$ is the coproduct induced by the diagonal mapping, $Dc = \sum c_j' \otimes c_j''$.

For example, if $c \in H_*(X;Q)$ and $v, v_1, v_2 \in V$ with $dv = v_1 v_2$, then

$$d_1(v \otimes c) = \sum (-1)^{|v_2| \cdot |c_j|}\left(v_1 \otimes c_j'\right)\left(v_1 \otimes c_j''\right).$$

Note that, in this case, d_1 depends only on the differential in A and the coproduct in $H_*(X;Q)$.

The special case of Theorem 8.4 when $X = S^1$ was proved in [19].

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CHAPTER 18

Cohomology of Groups

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HANDBOOK OF ALGEBRAIC TOPOLOGY
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1. Introduction

The cohomology of groups is one of the crossroads of mathematics. It has its origins in representation theory, class field theory and algebraic topology, and thanks to the extraordinary work of Quillen, it has led to the modern development of algebraic K-theory. It is therefore not surprising that wherever one looks in pure mathematics and mathematical physics, one cannot escape its influence. In this survey, we can only hope to give a brief glimpse of the power and beauty of the subject.

For an account of the origins of the subject, we cannot do better than refer the reader to the excellent 1978 survey article of MacLane [88]. If we fall short of a full historical account, it is because we cannot hope to compete. We feel obliged, nonetheless, to begin with a few words on the beginnings of the cohomology of groups.

The first appearances only feature the low degree cohomology. Schur [102], [103] (1904, 1907), in his studies of projective representations (i.e. homomorphisms to a projective general linear group, not to be confused with the theory of projective modules), studied what we now write as $H^2(G, \mathbf{C}^\times)$. This parameterizes the central extensions of the group by $\mathbf{C}^\times$. For a finite group, the dual object $H_2(G, \mathbf{Z})$ is called the Schur multiplier. In the case of a perfect group, this appears as the extending central subgroup of the universal central extension of G. Schreier [101] (1926) and Baer [12] (1934) generalized this notion to group extensions which are not necessarily central. The same idea appears in the theory of crossed product algebras, as developed by Brauer, Hasse and Noether [37] (1932). Here, the appropriate group is $H^2(\mathrm{Gal}(L/K), L^\times)$. These authors proved that all central simple algebras can be described as crossed products. This fact is central to the development of class field theory.

2. Algebraic topology

The theory of cohomology of groups in degrees higher than two really begins with a theorem in algebraic topology. Hurewicz [75] (1936), having just defined the higher homotopy groups of a topological space, proved that for an aspherical space (namely, a path connected space X for which $\pi_n(X) = 0$ for all $n \geqslant 2$), the fundamental group determines all the homology groups. Although Hurewicz works with homology with integer coefficients, the same methods work in homology or cohomology, and with arbitrary nontwisted or twisted coefficients. One possible approach to group cohomology is to define $H_n(\pi_1(X), A) = H_n(X, A)$ and $H^n(\pi_1(X), A) = H^n(X, A)$, for coefficients A.

Hopf [73] (1942) found an algebraic description of the second homology group of an aspherical space as

$$H_2(X, \mathbf{Z}) = R \cap [F, F]/[F, R]$$

where F is a free group mapping onto $\pi_1(X)$ and R is the subgroup of relations. This is exactly the same formula as Schur had written down for his multiplier nearly forty years earlier.

Based on the work of Hurewicz and Hopf, Eilenberg and MacLane [56] (1943) constructed an algebraically defined chain complex $K(G)$ whose homology groups are exactly the homology groups of an aspherical space X with $\pi_1(X) = G$. The degree n term in this complex is the free $\mathbf{Z}$-module with symbols $[g_1|g_2|\cdots|g_n]$ with $g_i \in G$, and the boundary homomorphism is given by

$$\partial[g_1|g_2|\cdots|g_n] = [g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ + (-1)^n[g_1|\cdots|g_{n-1}].$$

This complex may be viewed as the simplicial chains on an "Eilenberg–MacLane space" $K(G,1)$ whose simplices correspond to these symbols, and where the ith face of such a simplex is given by the ith term in the above formula. The universal cover of this space has simplices corresponding to symbols $g_0[g_1|g_2|\cdots|g_n]$, and with boundary homomorphism given by

$$\partial g_0[g_1|g_2|\cdots|g_n] = g_0g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i g_0[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ + (-1)^n g_0[g_1|\cdots|g_{n-1}].$$

This complex forms a *free resolution* of $\mathbf{Z}$ as a $\mathbf{Z}G$-module. Namely, it is an exact sequence

$$\cdots \to F_n \to \cdots \to F_1 \to F_0 \to \mathbf{Z} \to 0$$

with each F_i a free $\mathbf{Z}G$-module.

Hopf [74] (1945) noticed the algebraic analogue of the theorem of Hurewicz, and used it to give the definition of homology of a group G with coefficients in an arbitrary module, by taking a free resolution and forming the quotient by the augmentation ideal of the group algebra. He proved a comparison theorem which showed that the resulting definition is independent of the choice of resolution. In modern language, he described group homology in terms of Tor. Cohomology is similarly described in terms of Ext. The appropriate definitions are as follows. If A is a ring and N and M are A-modules, we form a projective resolution of M as an A-module

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

A projective module is simply a direct summand of a free module, and it is an easy step to generalize from free to projective modules. The characterization in terms of the usual lifting property is all that is needed for the comparison theorem to work; this theorem states that any map of modules extends to a chain map between projective resolutions. Furthermore, any two such chain maps are chain homotopic. Then $\mathrm{Tor}_n^A(N,M)$ is defined to be the nth homology group of the chain complex

$$\cdots \to N \otimes_A P_n \to \cdots \to N \otimes_A P_1 \to N \otimes_A P_0 \to 0.$$

In particular, we have $\mathrm{Tor}_0^A(N, M) = \mathrm{Hom}_A(N, M)$. Applying the comparison theorem to the identity map on M shows that these groups $\mathrm{Tor}_n^A(N, M)$ are independent (up to natural isomorphism) of the choice of projective resolution. It is this statement which may be interpreted as the algebraic analogue of Hurewicz's statement that the homology groups of an aspherical space are determined by the fundamental group. In these terms, Hopf's definition amounts to

$$H_n(G, M) = \mathrm{Tor}_n^{\mathbf{Z}G}(\mathbf{Z}, M).$$

In particular, we have $H_0(G, M) = M^G$, the fixed points of G on M.

The definition of Tor is symmetric, in the sense that we get the same answer if we form a projective resolution of N:

$$\cdots \to P'_n \to \cdots \to P'_1 \to P'_0 \to N \to 0$$

and take the homology of the chain complex

$$\cdots \to P'_n \otimes_A M \to \cdots \to P'_1 \otimes_A M \to P'_0 \otimes M \to 0.$$

The definition of Ext is similar. We may either form an injective resolution of M:

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to \cdots$$

and define $\mathrm{Ext}_A^n(N, M)$ to be the nth cohomology group of the cochain complex

$$0 \to \mathrm{Hom}_A(N, I_0) \to \mathrm{Hom}_A(N, I_1) \to \cdots \to \mathrm{Hom}_A(N, I_n) \to \cdots$$

or we may form a projective resolution of N as above and define $\mathrm{Ext}_A^n(N, M)$ to be the nth cohomology group of the cochain complex

$$0 \to \mathrm{Hom}_A(P'_0, M) \to \mathrm{Hom}_A(P'_1, M) \to \cdots \to \mathrm{Hom}_A(P'_n, M) \to \cdots$$

In these terms, group cohomology is defined by

$$H^n(G, M) = \mathrm{Ext}_{\mathbf{Z}G}^n(\mathbf{Z}, M).$$

Since a short exact sequence of chain complexes induces a long exact sequence of (co)homology groups, we get long exact sequences in (co)homology from short exact sequences of modules.

It is also worth remarking at this stage that if R is a ring of coefficients, then a projective resolution of R as an RG-module may be obtained by tensoring R with a projective resolution of $\mathbf{Z}$ as a $\mathbf{Z}G$-module. It follows that if M is an RG-module then

$$\mathrm{Ext}_{RG}^n(R, M) \cong \mathrm{Ext}_{\mathbf{Z}G}^n(\mathbf{Z}, M).$$

Returning to the algebraic topology, we say that a space X is an Eilenberg–MacLane space $K(G,1)$ if X is an aspherical space with the homotopy type of a connected CW-complex with $\pi_1(X) \cong G$. Such a space is unique up to homotopy equivalence, and the simplicial chains $C_*(\tilde{K}(G,1);R)$ on its universal cover $\tilde{K}(G,1)$ form a free resolution of the coefficient ring R over the group ring RG. Since

$$\begin{aligned} C^*\big(K(G,1);R\big) &= \operatorname{Hom}_R\big(C_*\big(K(G,1);R\big),R\big) \\ &\cong \operatorname{Hom}_{RG}\big(C_*\big(\tilde{K}(G,1);R\big),R\big), \end{aligned}$$

we have $H^*(K(G,1);R) \cong \operatorname{Ext}^*_{RG}(R,R) = H^*(G,R)$.

There are two useful generalizations of this. The first is that if $n \geqslant 1$ and A is an abelian group then a space X is an Eilenberg–MacLane space $K(A,n)$ if X has the homotopy type of a connected CW-complex with $\pi_n(X) \cong A$ and $\pi_i(X) = 0$ for $i \neq n$. It turns out that such a space "represents" cohomology in the sense that there is a natural one-one correspondence between homotopy classes of maps from a CW-complex Y to X and elements of $H^n(Y;A)$.

Secondly, if G is a topological group, one can form a classifying space BG for principal G-bundles over a paracompact base space. The total space of the universal principal G-bundle over BG is written EG. It is characterized by the property that it is a contractible space with a free G-action. Thus the loop space ΩBG is homotopy equivalent to the group G. In case G happens to be discrete, BG is an Eilenberg–MacLane space $K(G,1)$, and EG is its universal cover.

3. Cohomological finiteness conditions

The links between algebraic topology and group theory lead naturally to the idea of cohomological finiteness conditions. It is useful to compare this with the idea of a finiteness condition in abstract group theory. The latter notion, very prevalent in the work of Philip Hall and other influential groups theorists some thirty years ago, has had a powerful influence on the study of abstract infinite groups. As a definition, we say that a *finiteness condition* is any group theoretic property which holds for all finite groups. For example, the properties of being finitely generated or finitely presented are finiteness conditions. Further important examples include residual finiteness, Hopficity, and linearity.

By analogy with this, we can define a *cohomological finiteness condition* to be any group theoretic property which holds for all those groups G which admit a finite Eilenberg–MacLane space, $K(G,1)$. Finite generation and finite presentation are both examples of cohomological finiteness conditions, but residual finiteness, Hopficity and linearity are not cohomological finiteness conditions.

A group is said to be of type (F) if it satisfies the strongest possible cohomological finiteness condition, namely that there is a finite Eilenberg–MacLane space. Examples of such groups include torsion-free arithmetic groups and torsion-free polycyclic-by-finite groups. The case of arithmetic groups was studied in detail by Borel and Serre, [35]. A further very important source of examples is given by torsion-free subgroups of finite index in Coxeter groups. All these examples are also residually finite, Hopfian and linear. But in addition, all finitely generated torsion-free one-relator groups are of type (F), having an Eilenberg–MacLane space formed by adjoining a single 2-cell to a finite bouquet of circles, and amongst these there are examples of groups which are non-Hopfian, and in particular not residually finite, and far from linear.

On the other hand there are some cohomological finiteness conditions which are also abstract finiteness conditions. Perhaps the most important of these is the property that the cohomology ring $H^*(G, \mathbf{Z})$ is finitely generated. This plays a major role in the study of cohomology of finite groups, as we shall see subsequently.

There is an important family of cohomological finiteness conditions which are best defined in terms of projective resolutions, and which make sense for modules over any associative ring. Let R be a ring and let M be an R-module. We say that M is of type $(\mathrm{FP})_n$ if there is a projective resolution

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

in which the P_i are finitely generated for all $i \leqslant n$. The module M is said to be of type $(\mathrm{FP})_\infty$ if the projective resolution can be chosen so that every P_i is finitely generated, or in other words, if there is a projective resolution of *finite type*. The module M is said to have finite projective dimension if the projective resolution can be chosen of *finite length*, that is, with $P_i = 0$ for all sufficiently large i. In this case the projective dimension of a nonzero module M is defined to be the least n such that there is a projective resolution terminating with P_n. The module M is said to be of type (FP) if there is a *finite* projective resolution, that is a resolution which is simultaneously of finite type and of finite length; and finally, M is said to be of type (FL) if there is a finite free resolution, that is a finite projective resolution in which each P_i is free.

It is natural to ask how sensitive these definitions are to the choice of projective resolution, because one imagines choosing a projective resolution step by step. For example, if a module is of type $(\mathrm{FP})_\infty$ and one has made a choice of the first few steps in a projective resolution so that the projectives so far chosen are finitely generated, is it possible to continue the resolution so that it realizes the $(\mathrm{FP})_\infty$ property, or is it conceivable that some early choice that was made renders it impossible to maintain finitely generated projectives indefinitely even in the presence of the property $(\mathrm{FP})_\infty$? The answer in all cases is that projective resolutions behave as well as one could wish them to. With the exception of type (FL), where the story involves an extra twist, there are two important ways of understanding the stability of these definitions.

The first of these depends on Schanuel's lemma which states that given two short exact sequences of R-modules

$$0 \to K \to P \to M \to 0$$

and

$$0 \to L \to Q \to M \to 0$$

which both end with the same module M, and in which P and Q are projective, then

$P \oplus L$ is isomorphic to $Q \oplus K$. An easy induction yields the following embellishment: if

$$0 \to K \to P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

and

$$0 \to L \to Q_i \to Q_{i-1} \to \cdots \to Q_0 \to M \to 0$$

are two partial projective resolutions of M then

$$K \oplus Q_i \oplus P_{i-1} \oplus Q_{i-2} \oplus P_{i-3} \oplus \cdots \cong L \oplus P_i \oplus Q_{i-1} \oplus P_{i-2} \oplus Q_{i-3} \oplus \cdots.$$

From this it is easy to see that our homological finiteness conditions for projective resolutions behave well. For example, if M is of type $(\mathrm{FP})_\infty$ and the first partial resolution is taken from a projective resolution of finite type, then any partial resolution $0 \to L \to Q_i \to Q_{i-1} \to \cdots \to Q_0 \to M \to 0$ in which the Q_i are finitely generated automatically has L finitely generated, and hence any partial projective resolution by finitely generated projectives can always be continued so that all projectives are finitely generated. Except for type (FL), similar remarks can be made of the other finiteness conditions. The embellished Schanuel's Lemma also shows that M has type $(\mathrm{FP})_\infty$ if and only if it has type $(\mathrm{FP})_n$ for all n, and that M has type (FP) if and only if has type $(\mathrm{FP})_\infty$ and finite projective dimension.

The second way of understanding the invariance of these finiteness conditions is to interpret them as properties of the cohomology functors $\mathrm{Ext}^*_R(M,-)$. Since these functors can be defined by using projective resolutions of M, but are at the same time independent of the particular choice of resolution, their properties on the one hand reflect the nature of projective resolutions of M but are, on the other hand, invariant. Projective dimension is the simplest to interpret this way: M has finite projective dimension if and only if the functors $\mathrm{Ext}^n_R(M,-)$ are zero for all sufficiently large n, and if M is a nonzero module of finite projective dimension then its projective dimension is the largest n for which the functor $\mathrm{Ext}^n_R(M,-)$ is nonzero.

To interpret other properties through Ext one needs to consider filtered colimit systems. A filtered colimit system $(N_\lambda \mid \lambda \in \Lambda)$ consists of a directed partially ordered set Λ together with a family of R-modules N_λ and a compatible system of maps $N_\lambda \to N_\mu$ for each pair $\lambda \leqslant \mu$. We write $\varinjlim N_\lambda$ for the colimit of such a system. Now it can be shown that an R-module M is of type $(\mathrm{FP})_n$ (resp. $(\mathrm{FP})_\infty$) if and only if

$$\varinjlim \mathrm{Ext}^i_R(M, N_\lambda) = 0$$

for all $i \leqslant n$ (resp. for all i) and all (N_λ) filtered colimit systems satisfying $\varinjlim N_\lambda = 0$. Notice that in view of this it becomes transparent that M is of type $(\mathrm{FP})_\infty$ if and only if it is of type $(\mathrm{FP})_n$ for all n. Another very important consequence is that for M of type $(\mathrm{FP})_n$, the functors $\mathrm{Ext}^i_R(M,-)$ commute with arbitrary direct sums when $i \leqslant n$.

The story for the property (FL) is more subtle and we postpone this to the next section.

Now suppose that G is a group and that X is the universal cover of a cellular $K(G,1)$. Then G acts freely on X by covering transformations, and X is contractible, so that the associated augmented chain complex

$$\to C_i(X) \to \cdots \to C_1(X) \to C_0(X) \to \mathbf{Z} \to 0$$

is a free resolution of the trivial module $\mathbf{Z}$ over the group ring $\mathbf{Z}G$. Finiteness properties of the chosen $K(G,1)$ are reflected in this chain complex, so that for example, if one begins with a finite $K(G,1)$ then the chain complex is a finite free resolution of $\mathbf{Z}$ and hence $\mathbf{Z}$ is a $\mathbf{Z}G$-module of type (FL). In general, one says that the group G is of type (FL) when this conclusion holds. If one begins with any finite dimensional $K(G,1)$ then the chain complex has finite length and $\mathbf{Z}$ has finite projective dimension. In this case one says that G has finite cohomological dimension, and its dimension can be defined to be the projective dimension of $\mathbf{Z}$ over $\mathbf{Z}G$. It is now natural to define the properties $(\mathrm{FP})_n$, $(\mathrm{FP})_\infty$ and (FP) for groups simply in terms of the corresponding property for the trivial module $\mathbf{Z}$ over the group ring.

For groups G these properties are interrelated in the following way:

$$(\mathrm{F}) \Rightarrow (\mathrm{FL}) \Rightarrow (\mathrm{FP}) \Rightarrow (\mathrm{FP})_\infty \Rightarrow \cdots \Rightarrow (\mathrm{FP})_2 \Rightarrow (\mathrm{FP})_1.$$

Moreover, a group is of type $(\mathrm{FP})_1$ if and only if it is finitely generated, and every finitely presented group is of type $(\mathrm{FP})_2$. There is no known example of a group of type $(\mathrm{FP})_2$ which is not finitely presented. In general, a group is finitely presented if and only if there is a cellular $K(G,1)$ with finite 2-skeleton, and so in particular, every group of type (F) is finitely presented as well as being of type (FL). One of the most important facts here is the theorem of Wall [122] which states that the converse holds: every finitely presented group of type (FL) has a finite $K(G,1)$. This allows one to translate topological questions into algebra, although there remains one major difficulty. From the algebraic point of view, the property (FP) is much easier to grasp than (FL), being equivalent to type $(\mathrm{FP})_\infty$ together with finite cohomological dimension. However there are few theorems in algebra which enable one to establish that a group has type (FL), and usually this requires powerful geometric methods. The work of Borel and Serre on arithmetic groups is a good example of this: in [34] they use the natural action of the arithmetic group on the symmetric space for the ambient Lie group, and although this provides a fast proof that arithmetic groups have finite cohomological dimension, it takes considerably more work to establish that they have type (F). At present there is no known example of a group of type (FP) which is not of type (FL).

We conclude with a discussion of cohomological dimension, which has proved to be a remarkably subtle group theoretic invariant. Of all the cohomological finiteness conditions introduced in this section, this is the only one which is always inherited by subgroups. Indeed, if H is a subgroup of G then

$$\mathrm{cd}(H) \leqslant \mathrm{cd}(G)$$

because the trivial module $\mathbf{Z}$ has a projective resolution of length $\mathrm{cd}(G)$ as a $\mathbf{Z}G$-module and this serves equally as a projective resolution over $\mathbf{Z}H$. From another point of view, a group G has finite cohomological dimension if and only if it admits a finite dimensional $K(G,1)$. Nontrivial finite groups have infinite cohomological dimension, and consequently all groups of finite cohomological dimension are torsion-free. There is a beautiful and fundamental theorem of Serre which states that if H is a subgroup of finite index in a torsion-free group G and H has finite cohomological dimension then so does G, and moreover, in these circumstances $\mathrm{cd}(H) = \mathrm{cd}(G)$. A good account of this result can be found in Brown's book, [38]. We can give only a brief outline of the proof of this result here. First we may assume that H is normal in G. Now G can be identified with a subgroup of the wreath product W of H by the finite quotient $Q := G/H$. Since H has finite cohomological dimension there is a finite dimensional $K(H,1)$, and its universal cover X is a space on which H acts freely by covering translations. The wreath product W now acts naturally on the cartesian product $Y := X \times \cdots \times X$ of $|Q|$ copies of X with finite isotropy, and through its identification as a subgroup of W, G also acts on Y. The fact that W acts with finite isotropy means that G acts *freely*, because G is torsion-free. The conclusion is that Y/G is a finite dimensional $K(G,1)$ and so G has finite cohomological dimension. The fact that $\mathrm{cd}(G) = \mathrm{cd}(H)$ can deduced with techniques of homological algebra.

Since many groups which arise in nature, such as arithmetic groups and Coxeter groups, are not torsion-free, it may seem unlikely that cohomological dimension could be a useful invariant. However, Serre's theorem provides a way around this because it leads to the notion of *virtual cohomological dimension*. A group G is said to have finite virtual cohomological dimension if it has a subgroup of finite index which has finite cohomological dimension. If G is such a group, then $\mathrm{vcd}(G)$ is defined to be the cohomological dimension of a torsion-free subgroup of finite index. The point here is that Serre's theorem guarantees vcd to be well defined: it makes no difference which torsion-free subgroup of finite index is chosen. Arithmetic groups, Coxeter groups, and polycyclic-by-finite groups are amongst many naturally occurring families of groups of finite vcd.

4. Euler characteristics and K-theory

If G is a group of type (F) then its Euler characteristic $\chi(G)$ can be defined as the Euler characteristic of any finite $K(G,1)$. This is an invariant of the group because one has the formula

$$\chi(G) = \sum_i (-1)^i \dim H_i(G, \mathbf{Q}),$$

in terms of the rational homology groups. This formula makes sense more generally, provided the rational homology groups are finite dimensional and only finitely many are nonzero. Based on this, K.S. Brown has developed a theory of Euler characteristics for a much larger class of groups. Care has to be taken: it is not simply a matter of adopting the alternating sum formula whenever it makes sense because one wants to preserve the following property which holds for groups of type (F).

If G is a group of type (F) and H is a subgroup of finite index then H is also of type (F) and

$$\chi(H) = |G : H|\chi(G).$$

In view of this, one can define $\chi(G)$ for any group G which is virtually of type (F) by the formula

$$\chi(G) := \frac{1}{|G : H|}\chi(H),$$

where H is any subgroup of finite index which is of type (F). In general, this formula yields a rational number, not necessarily an integer. For example, when G is finite, $\chi(G) = \frac{1}{|G|}$. Since arithmetic groups are virtually of type (F), these have well defined Euler characteristics and many interesting calculations have been carried out.

Using Swan's work on the ranks of projective modules over integral group rings of finite groups, Brown showed that if G is torsion-free and virtually of type (F) then $\chi(G)$ is an integer. Now a torsion-free group which is virtually of type (F) is of type (FP), and looking at this from a different point of view, one can use the alternating sum formula to define an integral Euler characteristic for any group of type (FP) so that, because of Swan's theorem, the finite index formula holds. Hence there is a well behaved rational Euler characteristic which can be defined for any group which is virtually of type (FP). Brown's theory is more general even than this, although at present the Euler characteristic remains of most interest for groups which are virtually of type (F). We refer the reader to Brown's book, [38], for a more detailed account of this topic.

Here, we turn briefly to K-theory. For any ring R, let $K_0(R)$ denote the Grothendieck group of finitely generated projective R-modules. We write $[P]$ for the class in $K_0(R)$ of a projective module P. If M is any R-module of type (FP) then we write $[M]$ for the class

$$[M] := \sum_i (-1)^i [P_i],$$

where $P_* \to M$ is a finite projective resolution of M. The generalized form of Schanuel's Lemma shows at once that this does not depend on the choice of finite resolution, and hence every module of type (FP) has a well defined *Euler class* in $K_0(R)$. Given an additive homomorphism, or *rank function* $\rho : K_0(R) \to \mathbf{Z}$, we can associate the integer $\rho([M])$ to M, which can be viewed as a kind of Euler characteristic.

Suppose that G is a group of type (FP). Then the trivial module $\mathbf{Z}$ is of type (FP) over $\mathbf{Z}G$, and the Euler class $\mathcal{E}(G)$ can be defined to be the class $[\mathbf{Z}]$ in $K_0(\mathbf{Z}G)$. Over the integral group ring, the simplest rank function on projective modules is given by $\rho([P]) := \mathrm{rank}_{\mathbf{Z}}(P \otimes_{\mathbf{Z}G} \mathbf{Z})$, and using this one obtains the formula

$$\rho(\mathcal{E}(G)) = \chi(G).$$

If G is of type (F), or more generally of type (FL) then the Euler class is simply a multiple of the class $[\mathbf{Z}G]$:

$$\mathcal{E}(G) = \chi(G)[\mathbf{Z}G].$$

This formula is very significant, because it is not hard to show that when G is of type (FP) then this formula holds if and only if G is of type (FL). In this way the Euler class of a group of type (FP) carries crucial information.

At present, in all known cases, the Grothendieck group $K_0(\mathbf{Z}G)$ is generated by the class $[\mathbf{Z}G]$ when G is torsion-free. If $K_0(\mathbf{Z}G)$ were always trivial for torsion-free group then it would follow that every group of type (FP) is in fact of type (FL). Thus the question of whether there are groups of type (FP) which are not of type (FL) is really K-theoretic. A great deal of research has been carried out into K-theory. The work of Adem, [1], [2], [3] is especially relevant to our discussion here.

5. Ends and cohomological dimension one

There is an attractive quality to any result in abstract group theory whose proof depends on a cohomological insight. One such result, suggested by Serre, and later proved by Stallings and Swan [112], [117] states that if G is a torsion-free group which has a free subgroup of finite index then G is free. Serre proved an analogous result for pro-p-groups, but using very different techniques.

To understand how Stallings and Swan proved this result, and the way in which cohomological techniques are involved, we need to briefly recount the basis of the Bass–Serre theory of group actions on trees. At a combinatorial level a graph Γ is a quadruple (V, E, ι, τ) comprising a set of vertices V, a set of edges E and two functions $\iota, \tau : E \to V$ which indicate the initial and terminal vertices of each edge. Bass and Serre prefer to work with unoriented graphs, but for this account it seems convenient to work with directed graphs as we have defined them. Any graph $\Gamma = (V, E, \iota, \tau)$ can be realized geometrically as a 1-dimensional CW-complex: if I denotes the unit interval then one first forms the disjoint union of V and $E \times I$, and then one identifies each $(e, 0)$ with ιe and each $(e, 1)$ with τe. The graph has an associated chain complex with the group of 1-chains being $\mathbf{Z}E$, the free abelian group on E and the group of 0-chains being $\mathbf{Z}V$, the free abelian group on V. It is useful to consider the augmented chain complex:

$$0 \to \mathbf{Z}E \to \mathbf{Z}V \to \mathbf{Z} \to 0,$$

because this carries the crucial homological data for the graph. In particular, this sequence is exact at $\mathbf{Z}$ if and only if the graph is nonempty, it is exact at $\mathbf{Z}V$ if and only if the graph, or rather its realization as a space, is *connected*, and it is exact at $\mathbf{Z}E$ if and only if the graph has no loops, which is to say that the graph is a *forest*. Thus the chain complex is a short exact sequence if and only if the graph is a nonempty connected forest, or more simply, a *tree*.

An action of a group G on a graph comprises an action on the vertex and edge sets which is compatible with the initial and terminal vertex maps. If Γ is a G-graph then the associated chain complex is naturally a chain complex of G-modules. One of the most important examples of a G-graph is the Cayley graph of G with respect to a subset X of G. This is defined by setting $V := G$, $E := X \times G$, $\iota(x,g) := g$, and $\tau(x,g) := xg$, and G acts by right multiplication. It is connected if and only if G is generated by X, and one of the fundamental cornerstones of the Bass–Serre theory states that it is a tree if and only if G is the free group on X. Since the group G always acts freely on its Cayley graphs, then in the case when G is the free group on X, the chain complex is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}G$ of length one, and this gives an algebraic proof that free groups have cohomological dimension one. Geometrically, one sees this, because a free group has a bouquet of circles as an Eilenberg–MacLane space.

A remarkable theorem proved by Stallings [112] for finitely generated groups and generalized by Swan [117] to arbitrary groups states that the converse holds: every group of cohomological dimension one is free. This is a very deep result, but notice that the group theoretic application with which we opened this section follows immediately from it. If G is torsion-free and has a free subgroup of finite index then Serre's theorem on cohomological dimension together with the basic results from Bass–Serre theory, shows that G has cohomological dimension one, and hence G is free by Stallings and Swan.

In the remainder of this section we discuss Stallings contribution to this area and some of the ideas it has led to. Let G be a finitely generated group and let Γ be the Cayley graph of G with respect to some finite generating set. Then Γ is connected, but it is possible that, if a finite set of edges of Γ are removed then Γ will become disconnected and even that it may have more than one infinite component. The number of ends of G is defined to be the supremum of the number of infinite components of $\Gamma \setminus F$ as F runs through all finite sets of edges. It can be shown that this number $e(G)$ is an invariant independent of the choice of finite generating set for G. Moreover, it is known that $e(G)$ can take only one of four possible values: 0, 1, 2 or ∞. It is very easy to see from the definition that $e(G) = 0$ if and only if G is finite. If G is infinite then there is the classical formula of Hopf:

$$e(G) = 1 + \dim_{\mathbf{F}} H^1(G, \mathbf{F}G),$$

where $\mathbf{F}$ denotes the field of two elements.

Motivated by his work on 3-dimensional manifolds, Stallings proved a remarkable theorem about groups G with $e(G) > 1$. The theorem concerns *splittings* of groups. Here, we say that a group G splits over a subgroup H if and only if there is an action of G on a tree with one orbit of edges so that H is an edge stabilizer and so that all vertex stabilizers are proper subgroups of G. Let T be a G-tree with one orbit of edges, and let e be an edge with stabilizer $H := G_e$. There can be at most two orbits of vertices. If ιe and τe are in different orbits then G is isomorphic to the free product of the vertex stabilizers $G_{\iota e}$ and $G_{\tau e}$, amalgamated over H, and if ιe and τe belong to the same orbit, so that there is a $g \in G$ with $(\tau e)g = \iota e$, then there is only one orbit of vertices and G is isomorphic to the HNN-extension $G_{\iota e} *_{H,g}$. Stallings theorem [112] asserts that if G is a finitely generated group then $e(G) > 1$ if and only if G splits over a finite subgroup. One direction of this theorem is not hard. If G does split over a finite subgroup then one can use the associated G-tree to construct *almost invariant* subsets of G in the following way: let e be an edge of T and let B be the set of $g \in G$ such that e points towards $(\iota e)g$. It turns out that B is almost invariant in the sense that for all $g \in G$, the symmetric difference $B + Bg$ is finite. Finite subsets of G naturally correspond to elements of $\mathbf{F}G$, and so the function $g \mapsto B + Bg$ can be regarded as a derivation from G to $\mathbf{F}G$ which in turn gives rise to a nontrivial element of $H^1(G, \mathbf{F}G)$. Hopf's formula now shows that $e(G) > 1$. The other direction of Stallings theorem is a real *tour de force* and we cannot give a detailed account here.

Stallings theorem can be used to show that finitely generated groups of cohomological dimension one are free. If G is such a group, then one shows first that $e(G) > 1$ by means of Hopf's formula. Therefore G splits over a finite subgroup. Being torsion-free, G is therefore a nontrivial free product of two subgroups. Since these subgroups still have cohomological dimension one, the argument can be repeated, until one concludes that G is free. For infinitely generated groups, additional arguments are needed. Swan solved this problem [117], showing that all groups of cohomological dimension one are free.

Subsequently much further work has been done on the theory of ends. Holt's paper [72] in which he showed that uncountable locally finite groups have one end is an important landmark. Much more has been discovered by Dunwoody [51], [52]. Dunwoody's work has led to a theory of accessibility for groups. A group G is called accessible if there is a G-tree with finite edge stabilizers, finitely many orbits of edges, and with all vertex stabilizers having at most one end. Dunwoody proved that all finitely presented groups are accessible, and more recently he has shown that there exist finitely generated groups which are not accessible and that such groups always contain an infinite locally finite subgroup. This last result is a far reaching generalization of a theorem of Linnell asserting that a finitely generated group is accessible if there is a bound on the orders of the finite subgroups.

6. Duality groups

If M is a closed orientable manifold of dimension n then there are isomorphisms between the ith homology and the $(n-i)$th cohomology for each i. This is the familiar Poincaré duality for manifolds and the isomorphisms can be defined as cap products with the fundamental class of the manifold. In this article we discuss products in cohomology in the next section, and the reader may ask why it is possible to discuss duality between homology and cohomology of groups before products are introduced. While it is true that duality in group cohomology can be described in terms of cap products, it is possible to develop a workable theory which avoids any discussion of products. This approach was first taken by Bieri and Eckmann, [19], [22], [27], [29], [28].

A group G is called an *n-dimensional duality group* if there is a module D such that for each i there is a natural isomorphism

$$H^i(G, M) \cong H_{n-i}(G, D \otimes M),$$

where $D \otimes M$ is made into a G-module with the diagonal action.

Let G be such a group. Bieri and Eckmann reformulated the definition of duality in the following way. First, it is an immediate consequence of the definition that the cohomology of G vanishes in dimensions greater than n, and that if M is a free $\mathbf{Z}G$-module, then the cohomology groups $H^i(G, M)$ are zero except possibly in dimension n. Hence G is a group of cohomological dimension n. Secondly, since homology always commutes with direct limits, it follows that the cohomology functors also commute with direct limits and hence that G is of type $(\mathrm{FP})_\infty$. Thus G is a group of type (FP). Thirdly, it can be shown that D is necessarily torsion-free as an additive group and that it is isomorphic to $H^n(G, \mathbf{Z}G)^{\mathrm{op}}$. (Here we are viewing cohomology and homology as functors from right $\mathbf{Z}G$-modules to abelian groups, but since $\mathbf{Z}G$ is a bimodule, the cohomology group $H^n(G, \mathbf{Z}G)$ inherits a left module structure. If L is any left $\mathbf{Z}G$-module we write L^{op} for the right module with the same additive group and with G action defined by $\ell \cdot g := g^{-1}\ell$.) Thus an n-dimensional duality group G satisfies three properties:

- G is of type (FP);
- $H^i(G, \mathbf{Z}G) = 0$ for $i \neq n$; and
- $D := H^n(G, \mathbf{Z}G)^{\mathrm{op}}$ is torsion-free as additive group.

Bieri and Eckmann then showed that these three properties actually imply that G is a duality group with dualizing module D. The Bieri–Eckmann approach avoids using cup or cap products, and has proved invaluable for further study of duality groups.

In the special case when G is a duality group with dualizing module D being infinite cyclic, one says that G is a Poincaré duality group. Here there is a very close link with manifold theory. A Poincaré duality group is called orientable if its action on the dualizing module is trivial, and it is called nonorientable if the action is nontrivial. Moreover, no example is known of a Poincaré duality group which is not the fundamental group of a closed aspherical manifold. In a series of papers, Bieri, Eckmann, Müller and Linnell proved that every 2-dimensional Poincaré duality group is the fundamental group of a closed 2-manifold. Subsequently Thomas, Hillman, Kropholler and Roller [120], [70], [71], [80], [82], [83], [84], [85] have developed the theory of 3-dimensional Poincaré duality groups and there are indications of a close link with Thurston's geometrization programme for 3-manifolds.

The key idea behind the study of Poincaré duality groups is a generalization of the theory of ends. Scott introduced a notion of ends of a pair (G, H), where G is a group and H is a subgroup. If G is finitely generated then $e(G, H)$ can be defined to be the number of ends of the quotient graph Γ/H where Γ is a Cayley graph for G. He showed that if G splits over H then $e(G, H) > 1$. However, the converse does not hold. For example, the group G generated by x, y, z subject to relations

$$x^2 = y^3 = z^7 = xyz$$

has many free abelian subgroups A such that $e(G, A) = 2$ but it admits no splitting over any subgroup. This is an interesting example because it is the fundamental group of a closed Seifert fibred 3-manifold, and in particular, it is a Poincaré duality group. Nevertheless, some splitting theorems have been proved in terms of the new end invariant. The first of these is Scott's theorem that if $H \leqslant G$ are finitely generated groups and

H is an intersection of subgroups of finite index in G then $e(G, H) > 1$ if and only if some subgroup of finite index in G splits over H. A second theorem of this kind, due to Kropholler [80] states that if $H \leqslant G$ are groups which both have one end $[e(G) = e(H) = 1]$ and if $H \cap H^g = 1$ for all $g \notin H$, then $e(G, H) > 1$ if and only if G splits over H. There are more delicate versions of these results, and these play a role in understanding 3-dimensional Poincaré duality groups.

A further key ingredient is the beautiful theorem of Strebel [115] which asserts that if G is an n-dimensional Poincaré duality group then every subgroup of infinite index in G has cohomological dimension strictly less than n. This is an important ingredient in the study of 2-dimensional Poincaré duality groups where it shows that every subgroup of infinite index has cohomological dimension $\leqslant 1$ and hence is free by the Stallings–Swan theorem.

Finally we should mention Davis' paper [50]. Davis provides a way of constructing many examples of closed aspherical manifolds using Coxeter groups. Not only has this led to examples of aspherical manifolds not covered by Euclidean space, but it has also provided many other interesting examples of Poincaré duality groups. Mess has shown in [89] how one can use Davis' construction to embed certain groups into Poincaré duality groups. In particular, every group of type (F) and of dimension n can be embedded into a $(2n + 1)$-dimensional Poincaré duality group.

7. Products

One advantage of cohomology over homology is the existence of cup products and Yoneda products. Yoneda products are defined in terms of composition of maps between projective resolutions. This gives a map

$$\mathrm{Ext}^m_A(M_2, M_3) \otimes \mathrm{Ext}^n_A(M_1, M_2) \to \mathrm{Ext}^{m+n}_R(M_1, M_3).$$

Cup products are defined in terms of tensor products of resolutions. They are not defined in the same generality as Yoneda products, but they are defined for modules over group rings, or more generally over Hopf algebras. The cup product is a map

$$\mathrm{Ext}^m_{RG}(N_1, M_1) \otimes \mathrm{Ext}^n_{RG}(N_2, M_2) \to \mathrm{Ext}^{m+n}_{RG}(N_1 \otimes N_2, M_1 \otimes M_2),$$

and hence in particular

$$H^m(G, M_1) \otimes H^n(G, M_2) \to H^{m+n}(G, M_1 \otimes M_2),$$

so that $H^\bullet(G, R)$ becomes a ring.[1] This ring is graded commutative, in the sense that $xy = (-1)^{|x||y|}yx$, where $|x|$ denotes the degree of an element x.

[1] Here and elsewhere, unless otherwise stated, tensor products of modules are taken to be over the coefficient ring, with diagonal G-action.

Any cup product may be written as a Yoneda product as follows:

$$\begin{tikzcd}
\mathrm{Ext}^m_{RG}(N_1, M_1) \otimes \mathrm{Ext}^m_{RG}(N_2, M_2) \arrow[r, "\cup"] \arrow[d, "{(-\otimes M_2, N_1 \otimes -)}"] & \mathrm{Ext}^{n+m}_{RG}(N_1 \otimes N_2, M_1 \otimes M_2) \\
\mathrm{Ext}^m_{RG}(N_1 \otimes M_2, M_1 \otimes M_2) \otimes \mathrm{Ext}^n_{RG}(N_1 \otimes N_2, N_1 \otimes M_2) \arrow[ur, "\text{Yoneda}"'] &
\end{tikzcd}$$

However, not every Yoneda product may be written as a cup product. In particular, Yoneda and cup products agree on $\mathrm{Ext}^*_{RG}(R, R)$, but even if M is a simple RG-module, $\mathrm{Ext}^*_{RG}(M, M)$ need not be graded commutative. Carlson [43] has constructed examples with G finite, R a field of characteristic p and M a simple RG-module, where the rings $\mathrm{Ext}^*_{RG}(M, M)$ have complete matrix rings as quotients.

8. Tate cohomology

Motivated by the development of class field theory, Tate introduced a variation on the cohomology of finite groups, $\hat{H}^n(G, M)$ where n runs over the positive and negative integers, which agrees with ordinary cohomology for $n \geqslant 1$ and is reindexed ordinary homology for $n \leqslant -2$. Long exact sequences for this theory extend in both directions.

If G is a finite group, we define a complete resolution of $\mathbf{Z}$ as a $\mathbf{Z}G$-module to be a doubly infinite exact sequence of free modules

$$\cdots \to F_{-n} \to \cdots \to F_{-1} \to F_0 \to F_1 \to \cdots \to F_n \to \cdots$$

with the property that the non-negative part forms a free resolution of $\mathbf{Z}$ as a $\mathbf{Z}G$-module. For example, one may form the $\mathbf{Z}$-dual of a resolution of $\mathbf{Z}$ by finitely generated free $\mathbf{Z}G$-modules, shift in degree by one, and splice to the original resolution:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \mathrm{Hom}_{\mathbf{Z}}(F_{n-1}, \mathbf{Z}) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_{\mathbf{Z}}(F_0, \mathbf{Z}) \arrow[rr] \arrow[dr] & & F_0 \arrow[r] & F_1 \arrow[r] & \cdots \arrow[r] & F_n \arrow[r] & \cdots \\
& & & & \mathbf{Z} \arrow[ur] \arrow[dr] & & & & & \\
& & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

Tate cohomology $\hat{H}^n(G, M)$ $(-\infty \leqslant n \leqslant \infty)$ with coefficients in a $\mathbf{Z}G$-module M is then defined to be the cohomology of the doubly infinite complex $\mathrm{Hom}_{\mathbf{Z}G}(F_n, M)$. So for $n > 0$ this agrees with ordinary group cohomology, while for $n < -1$, we have $\hat{H}^n(G, M) \cong H_{-n-1}(G, M)$.

If H is a subgroup of G, then the inclusion of H in G induces a restriction map in (ordinary or) Tate cohomology

$$\mathrm{res}_{G,H} : \hat{H}^*(G, M) \to \hat{H}^*(H, M)$$

and a corestriction map

$$\mathrm{cor}_{H,G} : \hat{H}^*(H, M) \to \hat{H}^*(G, M).$$

In positive degrees, these are the usual restriction and transfer maps, while in negative degrees (regarded as homology groups) the roles are reversed. This has the interesting consequence that transfer may be defined in terms of restriction by dimension shifting past zero.

If H has index n in G, then $\mathrm{cor}_{H,G} \circ \mathrm{res}_{G,H}$ is equal to multiplication by n. In particular, this implies that $\hat{H}^*(G, M)$ is killed by multiplication by $|G|$, so that it is entirely torsion. Furthermore, it implies that if P is a Sylow p-subgroup of G, then $\mathrm{res}_{G,P}$ injects the p-primary component of $\hat{H}^*(G, M)$ into $\hat{H}^*(P, M)$. If M is killed by multiplication by p, and M is not a projective $\mathbf{F}_pP$-module, then for every value of $n \in \mathbf{Z}$, we have $\hat{H}^n(P, M) \neq 0$. This follows from the fact that $\mathbf{F}_pP$ is self-injective and the fact that every nontrivial $\mathbf{F}_pP$-module has a nonzero fixed point. If, on the other hand, M is $\mathbf{Z}$-torsion-free, then the long exact sequence in Tate cohomology arising from the short exact sequence

$$0 \to M \xrightarrow{p} M \to M/pM \to 0$$

shows that if $\hat{H}^n(P, M)$ is zero for two consecutive values of n then M/pM is a projective $\mathbf{F}_pP$-module, hence a projective $\mathbf{F}_pG$-module, and so the p-primary part of $\hat{H}^*(H, M)$ is zero for every subgroup H of G. If this holds for all primes p, we say that M is cohomologically trivial. By dimension shifting, the assumption that M is $\mathbf{Z}$-torsion-free may be dropped.

9. Class field theory

In Helmut Hasse's article [69] on the history of class field theory, he states the view that "the sharply profiled lines and individual features of this magnificent edifice seem to me to have lost somewhat of their original splendor and plasticity by the penetration of class field theory with cohomological concepts and methods, which set in so powerfully after the [second world] war". It is our hope to convince the reader that in fact the introduction of methods from the cohomology of groups only served to enhance the splendor of this subject.

The principal goal of class field theory is to describe the abelian extensions of a number field in terms of the internal structure of the field. This is usually described in terms of reciprocity maps. We shall limit ourselves to discussing that part of class field theory which is most easily dealt with in cohomological terms. For a fuller account of the local case see Serre [110], [111], and for the global theory see Tate [119].

The subject begins with the computation of the Brauer group of a local field. For any field K, the Brauer group $\mathrm{Br}(K)$ has as its elements the isomorphism classes of division rings Δ which are finite dimensional over K and which have K as their center. If Δ_1 and Δ_2 are two such, then $\Delta_1 \otimes_K \Delta_2$ is a full matrix ring over another such division algebra Δ_3. This gives $\mathrm{Br}(K)$ its group structure, with identity element represented by K itself and with Δ^{op} playing the role of the inverse of Δ. If L is a finite extension of K, we obtain a map $\mathrm{Br}(K) \to \mathrm{Br}(L)$ by tensoring over K with L, and the kernel of this map is written $\mathrm{Br}(L/K)$. It consists of the division rings over K split by L. Every such division ring can be written as a crossed product algebra, so that there is an isomorphism between $\mathrm{Br}(L/K)$ and $H^2(G, L^\times)$, where $G = \mathrm{Gal}(L/K)$.

In the case where K is a local field, it turns out that $H^2(G, L^\times) \cong \mathbf{Z}/|G|$. Moreover, generators for these cyclic groups can be chosen consistently, in the sense that if $H \leqslant G$ then the restriction of the generator of $H^2(G, L^\times)$ to $H^2(H, L^\times)$ is the generator for the latter. We write $u \in H^2(G, L^\times)$ for the generator.

Now write $\Omega^2\mathbf{Z}$ for the second kernel in a projective resolution of $\mathbf{Z}$ as a $\mathbf{Z}G$-module. This is not uniquely defined, but by Schanuel's lemma it is uniquely defined modulo adding projective summands. Then we can represent $u \in H^2(G, L^\times)$ by a $\mathbf{Z}G$-module homomorphism $\hat{u} : \Omega^2\mathbf{Z} \to L^\times$. Adding projective summands to $\Omega^2\mathbf{Z}$ if necessary, we may assume that this map is surjective, say with kernel M. For any $H \leqslant G$, the short exact sequence

$$0 \to M \to \Omega^2\mathbf{Z} \xrightarrow{\hat{u}} L^\times \to 0$$

of $\mathbf{Z}H$-modules gives rise to a long exact sequence in Tate cohomology, a portion of which is as follows:

$$\begin{aligned} \hat{H}^1(H, L^\times) \to \hat{H}^2(H, M) \to \hat{H}^2(H, \Omega^2\mathbf{Z}) &\xrightarrow{u_*} \hat{H}^2(H, L^\times) \\ \to \hat{H}^3(H, M) &\to \hat{H}^3(H, \Omega^2\mathbf{Z}). \end{aligned}$$

Now by Hilbert's Theorem 90, $\hat{H}^1(H, L^\times) = 0$. By the above calculation of the Brauer group, the map u_* from $\hat{H}^2(H, \Omega^2\mathbf{Z}) \cong \hat{H}^0(H, \mathbf{Z}) \cong \mathbf{Z}/|H|$ to $\hat{H}^2(H, L^\times)$ is an isomorphism. Also, $\hat{H}^3(H, \Omega^2\mathbf{Z}) \cong \hat{H}^1(H, \mathbf{Z}) \cong \mathrm{Hom}(H, \mathbf{Z}) = 0$ since H is finite. So by the above long exact sequence, we deduce that for all subgroups H of G, $\hat{H}^2(H, M)$ and $\hat{H}^3(H, M)$ are zero. Thus M is cohomologically trivial, and so the map u_* from $\hat{H}^n(G, \Omega^2\mathbf{Z}) \cong \hat{H}^{n-2}(G, \mathbf{Z})$ to $\hat{H}^n(G, L^\times)$ (i.e. cup product with u) is an isomorphism for all $n \in \mathbf{Z}$.

Let us apply this in the case $n = 0$. The group $\hat{H}^{-2}(G, \mathbf{Z}) = H_1(G, \mathbf{Z})$ is just the abelianization G_{ab}, while $\hat{H}^0(G, L^\times)$ is fixed points modulo transfers. Since we are working multiplicatively, transfer here really means the usual number theoretic norm map, and so we have

$$G_{\mathrm{ab}} \cong K^\times / N_{L/K}(L^\times).$$

The inverse map to this cup product isomorphism is called the local reciprocity map, or norm residue symbol, and it has an explicit character theoretic characterization which may be found in Serre [110]. This allows the explicit construction of all abelian extensions of K.

In the global case, we must replace the multiplicative group $L^\times$ by the idèle class group $C_L = J_L/L^\times$. The properties of this module are analogous to the properties of $L^\times$ in the local case. Namely $\hat{H}^1(G, C_L) = 0$, and there are compatible isomorphisms $\hat{H}^2(G, C_L) \cong \mathbf{Z}/|G|$. Thus the generator $u \in \hat{H}^2(G, C_L)$ gives rise in the same way as before to a map $\hat{u} : \Omega^2\mathbf{Z} \to C_L$ which induces isomorphisms

$$\hat{H}^{n-2}(G, \mathbf{Z}) \to \hat{H}^n(G, C_L).$$

In particular, the corresponding reciprocity law is the inverse of the isomorphism

$$G_{\mathrm{ab}} \cong C_K / N_{L/K}(C_L)$$

arising from the case $n = 0$. Details may be found in Tate [119].

Finally, we mention that the notion of a class formation is designed to abstract what makes the above arguments work. See the notes of Artin and Tate [8].

10. The complete cohomology of Mislin and Vogel

Although Tate cohomology was introduced only for finite groups, it was subsequently generalized by Farrell G with $\operatorname{vcd} G < \infty$. As well as Farrell's paper [61], there is also a useful account in [38]. Farrell's approach was similar to Tate's and involved the use of complete projective resolutions.

More recently, Vogel and Mislin have independently discovered a generalized Tate cohomology theory, [67], [90] which works for any group. This theory does not depend on complete resolutions, and has many exciting applications. Mislin was strongly influenced by the paper [63] of Gedrich and Gruenberg in which the authors develop a theory of *terminal completions* of cohomological functors for certain classes of groups. A different approach to generalized Tate cohomology was discovered by Vogel [67] and by Benson and Carlson, [16]. The paper by Benson and Carlson is ostensibly about Tate cohomology (of finite groups), but they work with definitions which make sense for arbitrary groups and which turn out to yield a theory isomorphic to that of Vogel and Mislin. This theory, which we shall call the *complete cohomology*, turns out to be ideally suited for proving that certain groups have finite cohomological dimension. We shall denote the cohomology groups by $\widehat{H}^j(G, M)$. Like ordinary cohomology, one can have coefficients in any G-module M, and each cohomology group is functorial in M. The theory shares basic properties with ordinary cohomology, but it also enjoys some distinctive features. One point to emphasize at once is that $\widehat{H}^j(G, M)$ is defined and can be nonzero for all integers j, positive and negative, as indeed must be the case because complete cohomology coincides with Tate cohomology for finite groups.

Some properties of complete cohomology are just as for ordinary cohomology. First there are natural long exact sequences of complete cohomology associated to short exact sequences of coefficient modules.

Thus, for any short exact sequence $0 \to A \to B \to C \to 0$ of G-modules, there are natural connecting homomorphisms $\delta : \widehat{H}^j(G, C) \to \widehat{H}^{j+1}(G, A)$ which, together with functorially induced maps, give rise to a long exact sequence:

$$\cdots \to \widehat{H}^j(G, A) \to \widehat{H}^j(G, B) \to \widehat{H}^j(G, C) \to \widehat{H}^{j+1}(G, A) \to \cdots$$

Secondly, if G is of type $(\mathrm{FP})_\infty$ then the functors $\widehat{H}^j(G, \quad)$ commute with filtered colimits, just as the ordinary cohomology functors do.

There are, however, two properties which distinguish complete cohomology from ordinary cohomology. The first of these is very striking, and of great importance:

$$\widehat{H}^0(G,\mathbf{Z}) = 0 \quad \text{if and only if} \quad \operatorname{cd} G < \infty.$$

This stands out because it ensures that complete cohomology need not be identically zero. It is wonderfully convenient that this zeroth cohomology group with trivial coefficients carries so much information. In general, not surprisingly, $\widehat{H}^0(G,\mathbf{Z})$ is very hard to compute for groups of infinite cohomological dimension. If G is finite then we know that

$$\widehat{H}^0(G,\mathbf{Z}) = \mathbf{Z}/|G|\mathbf{Z}$$

but even for polycyclic-by-finite groups, no general formula is known.

The last property we mention, like the first, is really axiomatic. This is built in to complete cohomology when it is defined, and the theory satisfies a universal property in relation to ordinary cohomology subject to this condition: For all integers j, and all projective modules P, the complete cohomology groups $\widehat{H}^j(G,P)$ are zero.

This property does not usually hold for ordinary cohomology. There are exceptions: for example if G is a free abelian group of infinite rank then $H^j(G,P)$ is zero for all j (including $j = 0$) and all projective modules P. For such groups, the complete cohomology and the ordinary cohomology coincide. More generally, if there is an integer j such that the ordinary cohomology vanishes on projectives from dimension j onwards, then the complete cohomology and the ordinary cohomology coincide from that point on.

At the time of writing, there have been only a few applications for complete cohomology, but those that have been discovered strike us as very exciting. One of the best results to emerge is Kropholler's theorem [81] that soluble and linear groups of type $(\mathrm{FP})_\infty$ are virtually of type (FP). The proof uses the fact that complete cohomology provides a criterion for a group to have finite cohomological dimension. But in order to prove such a result, one also needs a method of computing $\widehat{H}^0(G,\mathbf{Z})$. The method depends on the fact that both soluble and linear groups admit useful actions on finite dimensional contractible complexes. For linear groups, one needs the methods introduced by Alperin and Shalen when they established their criterion for finite dimensionality. For soluble groups one uses the simpler fact that every infinite finitely generated soluble group has an infinite quotient which is a crystallographic group, and which thus acts on a Euclidean space.

These results can be generalized in various ways. For example, in [81] Kropholler also proves that if G is a torsion-free soluble or linear group then every $\mathbf{Z}G$-module of type $(\mathrm{FP})_\infty$ has finite projective dimension. It seems likely that complete cohomology will have many further applications in the near future.

On the other hand, these results do not hold for arbitrary groups. Brown and Geoghegan [42], [41] have shown that there exist torsion-free groups of type $(\mathrm{FP})_\infty$ which have infinite cohomological dimension.

11. Spectral sequences

There are several spectral sequences of importance in group cohomology. Perhaps the most commonly used is the Lyndon–Hochschild–Serre spectral sequence of a group extension. If

$$1 \to N \to G \to G/N \to 1$$

is a short exact sequence of groups and group homomorphisms, and R is a commutative ring of coefficients, then for any RG-module M there are spectral sequences in homology and cohomology:

$$E^2_{pq} = H_p\big(G/N, H_q(N,M)\big) \Rightarrow H_{p+q}(G,M),$$

$$E_2^{pq} = H^p\big(G/N, H^q(N,M)\big) \Rightarrow H^{p+q}(G,M).$$

To construct these spectral sequences, we take a projective resolution of R as an $R(G/N)$-module

$$\cdots \to P_1 \to P_0 \to R \to 0$$

and as an RG-module

$$\cdots \to Q_1 \to Q_0 \to R \to 0$$

and we form the double complexes

$$E^0_{pq} = P_p \otimes_{R(G/N)} (Q_q \otimes_R NM) \cong (P_p \otimes Q_q) \otimes_{RG} M,$$

$$E_0^{pq} = \mathrm{Hom}_{R(G/N)}\big(P_p, \mathrm{Hom}_{RN}(Q_q, M)\big) \cong \mathrm{Hom}_{RG}(P_p \otimes_R Q_q, M).$$

In the spectral sequences of these double complexes, we have

$$E^1_{pq} = P_p \otimes_{R(G/N)} H_q(N,M),$$

$$E_1^{pq} = \mathrm{Hom}_{R(G/N)}\big(P_p, H^q(N,M)\big),$$

and so the E^2_{pq}, respectively E_2^{pq} terms are as given above. Since $P_* \otimes_R Q_*$ is a projective resolution of R as an RG-module, the spectral sequences converge to $H_*(G,M)$, respectively $H^*(G,M)$.

Calculations involving the Lyndon–Hochschild–Serre spectral sequence tend to be intricate, and to depend on having available as much extra structure as possible. This involves products, Steenrod operations, restrictions, transfers, and so on. Particularly useful is the fact (Serre [109]) that the Steenrod operations commute with transgressions (i.e.

the differentials going from the vertical edge to the horizontal edge) in the cohomology spectral sequence with $\mathbf{F}_p$ coefficients.

In the case of a central extension of groups, or more generally a central product of groups, there is the Eilenberg–Moore spectral sequence. Namely, if G_1 and G_2 are groups and $\phi_1 : Z \hookrightarrow Z(G_1)$, $\phi_2 : Z \hookrightarrow Z(G_2)$ are inclusions of a common central subgroup, then we may form the central product

$$G_1 \circ G_2 = G_1 \times G_2 / \{(\phi_1(z), \phi_2(z)),\ z \in Z\}.$$

In this situation, there is a pullback diagram of fibrations of Eilenberg–MacLane spaces

$$\begin{array}{ccc} K(G_1 \circ G_2, 1) & \rightarrow & K(G_1/\phi_1(Z), 1) \\ \downarrow & & \downarrow \\ K(G_2/\phi_2(Z), 1) & \rightarrow & K(Z, 2). \end{array}$$

The Eilenberg–Moore spectral sequence of this pullback square is

$$\operatorname{Tor}^{**}_{H^*(K(Z,2);R)}(H^*(G_1/\phi_1(Z), R), H^*(G_2/\phi_2(Z), R)) \Rightarrow H^*(G_1 \circ G_2, R).$$

The case where $G_1 = \phi_1(Z)$ applies to any central extension, and is useful for example in calculating the cohomology of finite p-groups. Rusin [99] calculated the mod two cohomology of the groups of order 32 this way.

Another spectral sequence which is of occasional use in finite group cohomology is the Atiyah spectral sequence [9]. This has as its E_2 term the cohomology ring $H^*(G, \mathbf{Z})$, and converges to the completion $\mathcal{R}(\mathbf{C}G)^\wedge$ of the complex representation ring $\mathcal{R}(\mathbf{C}G)$ at the augmentation ideal.

12. Cohomological dimension

We have already discussed groups of low cohomological dimension, and especially aspects of the theory of cohomological dimension one. In higher dimensions many interesting calculations have been carried out, but calculations are often very hard. Infinite soluble groups provide a good source of examples. If G is a soluble group, one says that it has finite Hirsch length if and only if there is a subnormal series $1 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ in which the factors G_i/G_{i-1} are either infinite cyclic or locally finite. For such groups, the Hirsch length $h(G)$ is defined to be the number of infinite cyclic factors in any such series. This is easily seen to be an invariant of the group. Stammbach, [113] established that a soluble group has finite cohomological dimension if and only if it is torsion-free of finite Hirsch length. The simplest instance of this is a free abelian group of finite rank, in which case the cohomological dimension is equal to the rank. Gruenberg took this further by showing that if G is a torsion free nilpotent group with finite Hirsch length then $\operatorname{cd} G$ is equal to either $h(G)$ or $h(G) + 1$, and that $\operatorname{cd} G = h(G)$ if and only if G is finitely generated. Bieri [20] studied this question for soluble groups in [20], and it became clear that the cohomological dimension of a soluble group was an interesting invariant. In general, for any torsion-free soluble group G of finite Hirsch length, it is the case that either $\operatorname{cd} G = h(G)$ or $\operatorname{cd} G = h(G)+1$. After this had been established, it was some years before progress was made. Then Gildenhuys [65] solved the question in the case of Hirsch length 2, coming up with a complete list of the soluble groups G with $\operatorname{cd} G = h(G) = 2$. These are precisely the groups with a presentation on two generators x and y subject to a single relation

$$y^{-1}xy = x^m,$$

where m is a nonzero integer. Gildenhuys and Strebel studied the question further in [66], and finally a complete story for soluble groups emerged in Kropholler's paper [79]. The result is that if G is a soluble group then the following statements are equivalent

- $\operatorname{cd} G = h(G) < \infty$;
- G is of type (FP);
- G is a duality group;
- G is torsion-free and can be built up from the trivial group by a finite sequence of HNN-extensions and finite extensions.

As in the case of the Stallings–Swan theorem, one sees that the cohomological dimension is intimately related to the structure of the group, and it remains mysterious that this single invariant should carry so much information.

Another important theorem concerns the cohomological dimension of linear groups. Here, in general there is no exact formula for the dimension. Alperin and Shalen [7] proved that a finitely generated linear group (meaning a subgroup of $GL_n(\mathbf{C})$) has finite cohomological dimension if and only if there is a bound on the ranks of the unipotent subgroups. To prove this, they depend on two techniques. The first is a generalized version of the method Borel and Serre used to study arithmetic groups. Secondly, they develop the theory of valuations and use the fact that if A is a finitely generated commutative ring then, associated to each discrete valuation on A one can construct an $(n-1)$-dimensional affine building on which $SL_n(A)$ acts. Their method draws attention to an important principle in studying cohomological dimension. In general, to show that a group G has finite cohomological dimension it suffices to find a free action of G on a finite dimensional contractible cell complex. In practice, one cannot always easily find free actions, but if G acts on an r-dimensional contractible complex in such a way that all isotropy groups have dimension $\leqslant n$, then $\operatorname{cd} G \leqslant n + r$.

13. Elementary abelian subgroups

Evens [58] and Venkov [121] (1961) proved independently that the cohomology ring $H^*(G,R)$ of a finite group G with coefficients in a commutative Noetherian ring R is finitely generated. The two proofs are quite different in nature. Venkov's is topological, and uses the theory of Chern classes. Evens' is purely algebraic, and in fact he proves the stronger theorem that for any commutative coefficient ring R, if M is an RG-module which is Noetherian over R, then $H^*(G, M)$ is Noetherian as a module over $H^*(G, R)$. The ideas in Evens' proof gave rise to his formulation of the Evens norm map [59] (of course, he didn't call it that), which is a multiplicative analogue of the usual additive transfer map.

In the case where $R = k$ is a field of characteristic p dividing $|G|$, Quillen [93] (1971) gave an explicit description of the homogeneous affine variety $V_G(k)$ obtained by looking at the maximal ideal spectrum of $H^*(G, k)$. His description involves the structure of the set of elementary abelian subgroups of G.

We begin by explaining that since $H^*(G, k)$ is graded commutative, if p is odd then elements of odd degree square to zero and elements of even degree commute, while if $p = 2$ then all elements commute. In either case, $H^*(G, k)$ modulo its nil radical is commutative. We write $H^{\cdot}(G, k)$ to denote the even cohomology ring if p is odd, and the whole of $H^*(G, k)$ if $p = 2$. Then $H^*(G, k)$ and $H^{\cdot}(G, k)$ have "the same" maximal ideals.

If $\zeta_1, \ldots, \zeta_s$ are homogeneous elements generating $H^{\cdot}(G, k)$ then the map

$$k[\zeta_1, \ldots, \zeta_s] \to H^{\cdot}(G, k)$$

gives rise to a map of maximal ideal spectra (by the weak Nullstellensatz)

$$\max H^{\cdot}(G, k) \to \mathbf{A}^s(k)$$

which embeds $\max H^{\cdot}(G, k)$ as a closed homogeneous subvariety of affine space $\mathbf{A}^s$. Of course, this embedding depends on the choice of generators, but the abstract homogeneous affine variety $\max H^{\cdot}(G, k)$ does not.

If $\phi : A \to B$ is a morphism of finitely generated commutative k-algebras, with the property that the kernel is nilpotent and for some value of t, the p^tth power of every element of B lies in $\mathrm{Im}(\phi)$, then $\phi^* : \max B \to \max A$ is bijective. Such a map is called an "inseparable isogeny" or an "F-isomorphism" (F for Frobenius). Quillen's theorem states that the restriction maps give rise to an inseparable isogeny

$$H^*(G, k) \to \varprojlim_E H^*(E, k),$$

so that

$$\varinjlim_E V_E(k) \to V_G(k)$$

is bijective. Here, the limit is taken over the category whose objects are the elementary abelian p-subgroups E of G, and whose morphisms are generated by the conjugations and inclusions in G. This theorem says that an element of $H^*(G, k)$ is nilpotent if and only if its restriction to every elementary abelian subgroup is nilpotent; and moreover, given elements $x_E \in H^n(E, k)$ for each elementary abelian subgroup E, consistent under conjugations and restrictions, there is an element $x \in H^{np^a}(G, k)$ for some $a \geqslant 0$ such that for each E we have $\mathrm{res}_{G,E}(x) = X_E^{p^a}$.

Five years later, Chouinard [48] (1976) proved that a kG-module is projective if and only if its restriction to every elementary abelian p-subgroup E of G is projective. The connection between this and Quillen's theorem was not apparent until Alperin and Evens [6] (1981) formulated the notion of the complexity $c_G(M)$ of a finitely generated kG-module M, and proved that it is equal to the maximal complexity of a restriction of M to an elementary abelian p-subgroup E of G:

$$c_G(M) = \max_{E \leqslant G} c_E(M \downarrow_E).$$

The definition of complexity is that if $\cdots \to P_1 \to P_0 \to M \to 0$ is a minimal projective resolution of M, then $c_G(M)$ is the smallest integer $c \geqslant 0$ such that

$$\lim_{n \to \infty} \dim_k P_n / n^c = 0.$$

The fact that there is such a value of c follows from Evens' finite generation theorem. It turns out that M is projective if and only if $c_G(M) = 0$, and M is a direct sum of a projective and a periodic module (a module is said to be periodic if its minimal resolution repeats) if and only if $c_G(M) = 1$ (Eisenbud [57]; see also the beginning of the next section).

Another way to interpret the complexity is as follows. By Evens' finite generation theorem, $\mathrm{Ext}^*_{kG}(M, M)$ is a finitely generated module over the image of the map

$$H^*(G, k) = \mathrm{Ext}^*_{kG}(k, k) \xrightarrow{\otimes M} \mathrm{Ext}^*_{kG}(M, M)$$

given by tensoring exact sequences with M. We write $I_G(M)$ for the kernel of this map. Then since $\mathrm{Ext}^*_{kG}(M, S)$ is finitely generated as a module over $\mathrm{Ext}^*_{kG}(M, M)$ for each simple module S, it follows that $c_G(M)$ is the least integer c such that

$$\lim_{n \to \infty} \dim_k \mathrm{Ext}^n_{kG}(M, M) / n^c = 0,$$

and is therefore equal to the Krull dimension of $\mathrm{Ext}^*_{kG}(M, M)$, or equivalently the Krull dimension of $H^*(G, k)/I_G(M)$. So the case $M = k$ of the Alperin–Evens theorem may be interpreted as Quillen's statement that the Krull dimension of $H^*(G, k)$ is equal to $r_p(G)$.

The work of Alperin and Evens, together with some work of Carlson [44], [45] on rank varieties of modules for elementary abelian groups, led to the formulation of the notion of varieties for modules. If M is a finitely generated kG-module, we define $V_G(M)$ to be the closed homogeneous subvariety of $V_G(k)$ determined by the ideal $I_G(M)$; namely the subset consisting of the maximal ideals containing $I_G(M)$. The following is a list of properties of the varieties $V_G(M)$:

(i) The dimension of $V_G(M)$ is equal to the complexity of M. In particular, $V_G(M) = \{0\}$ if and only if M is projective, and $V_G(M)$ is a finite union of lines through the origin if and only if M is a direct sum of a projective and a periodic module.

(ii) $V_G(M_1 \oplus M_2) = V_G(M_1) \cup V_G(M_2)$.
(iii) $V_G(M_1 \otimes M_2) = V_G(M_1) \cap V_G(M_2)$.
(iv) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of finitely generated kG-modules, then $V_G(M_i) \subseteq V_G(M_j) \cup V_G(M_k)$ whenever $\{i, j, k\} = \{1, 2, 3\}$.
(v) Write $\Omega(M)$ for the kernel of the projective cover of M. Then $V_G(\Omega(M)) = V_G(M)$.
(vi) (Avrunin and Scott [11]) The map $\varinjlim_E V_E(M) \to V_G(M)$ induced by Quillen's map $\varinjlim_E V_E(k) \to V_G(k)$ is an inseparable isogeny.
(vii) If $0 \neq \zeta \in H^n(G, k)$ is represented by a cocycle $\hat{\zeta} : \Omega^n(k) \to k$, write L_ζ for the kernel of $\hat{\zeta}$. Then $V_G(L_\zeta)$ is the closed homogeneous hypersurface $V_G\langle\zeta\rangle$ given by regarding ζ as a polynomial function on $V_G(k)$.
(viii) (Carlson's connectedness theorem [46]) If $V_G(M) = V_1 \cup V_2$, where V_1 and V_2 are closed homogeneous subvarieties intersecting in the origin, then M decomposes as $M_1 \oplus M_2$ with $V_G(M_1) = V_1$ and $V_G(M_2) = V_2$. In particular, if M is indecomposable then $V_G(M)$ is (projectively) connected.

It is worth remarking that it follows from (ii) and (vii) that every closed homogeneous subvariety of $V_G(k)$ is the variety of some finitely generated module; namely a tensor product of suitable L_ζ's.

Recently (Benson, Carlson and Rickard [17], [18]), it has been realized that much of what is described in this section can be extended to infinitely generated modules. A kG-module M is said to have complexity at most c if every map from a finitely generated kG-module to M factors through a finitely generated module of complexity at most c. The complexity of a module is equal to the maximal complexity of a restriction to an elementary abelian subgroup, just as in the finitely generated case.

Instead of a single variety, an infinitely generated module has a collection of varieties. If M itself is finitely generated, then $\mathcal{V}_G(M)$ consists of the closed homogeneous subvarieties of $V_G(M)$, but for an infinitely generated module, $\mathcal{V}_G(M)$ need not have a unique maximal element. Nonetheless, elements of $\mathcal{V}_G(M)$ certainly have a maximal dimension, and this is equal to the complexity of M.

14. Multiple complexes

If M is a finitely generated kG-module with the property that $V_G(M)$ is one dimensional, then we may choose an element $0 \neq \zeta \in H^n(G, k)$ (for some $n > 0$) satisfying $V_G\langle\zeta\rangle \cap V_G(M) = \{0\}$. Then $L_\zeta \otimes M$ is projective, and so the exact sequence

$$0 \to L_\zeta \otimes M \to \Omega^n(k) \otimes M \to M \to 0$$

shows that $\Omega^n(k) \otimes M$ is isomorphic to a direct sum of M with a projective module. It follows easily from Schanuel's lemma that $\Omega^n(k) \otimes M$ is also isomorphic to a direct sum of $\Omega^n(M)$ with a projective module, and so M is a direct sum of a periodic and a projective module. It should be mentioned that this is not the original proof of Eisenbud's theorem [57], but it is the one which generalizes to a "multiple periodicity" theorem for arbitrary finitely generated modules (Benson and Carlson [14]).

We begin with a construction. If $\zeta \in H^n(G,k)$, we form the pushout

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & & & & \\
 & \downarrow & & \downarrow & & & & & \\
 & L_\zeta & = & L_\zeta & & & & & \\
 & \downarrow & & \downarrow & & & & & \\
0 \to & \Omega^n(k) & \to & P_{n-1} & \to \cdots \to & P_0 & \to & k & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & k & \to & P_{n-1}/L_\zeta & \to \cdots \to & P_0 & \to & k & \to 0 \\
 & \downarrow & & \downarrow & & & & & \\
 & 0 & & 0 & & & & &
\end{array}$$

Truncate the bottom row of this diagram by removing the copies of k at the beginning and the end. The resulting complex $\mathbf{C}_\zeta$ has length $n-1$. It has one dimension of homology in degrees zero and $n-1$, and is otherwise exact. This complex may be spliced to itself infinitely often to form a periodic complex

$$\cdots \to P_{n-1}/L_\zeta \to \cdots \to P_0 \to P_{n-1}/L_\zeta \to \cdots \to P_0 \to 0$$

resolving the trivial module. All the modules except the copies of P_{n-1}/L_ζ are projective. We write $\mathbf{C}_\zeta^{(\infty)}$ for this complex.

If M is a finitely generated kG-module of complexity c, then the Noether normalization lemma implies that we can find a polynomial subring $k[\zeta_1,\ldots,\zeta_c] \subseteq H^*(G,k)$ generated by homogeneous elements $\zeta_i \in H^{n_i}(G,k)$, which injects into $\mathrm{Ext}^*_{kG}(M,M)$, and over which the latter is finitely generated as a module. Under these conditions, the module

$$P_{n_1-1}/L_{\zeta_1} \otimes \cdots \otimes P_{n_c-1}/L_{\zeta_c} \otimes M$$

is projective, and so the complex

$$\mathbf{C}_{\zeta_1}^{(\infty)} \otimes \cdots \otimes \mathbf{C}_{\zeta_c}^{(\infty)} \otimes M$$

is a "c-fold multiply periodic" projective resolution of M. It has the same polynomial rate of growth as the minimal resolution, though it is not in general minimal.

In case $M = k$, the trivial module, the complexity is $r = r_p(G)$, the maximal rank of an elementary abelian p-subgroup of G. In this case, the elements $\zeta_1,\ldots,\zeta_r$ are a homogeneous system of parameters for the cohomology ring. The complex

$$\mathbf{C} = \mathbf{C}_{\zeta_1} \otimes \cdots \otimes \mathbf{C}_{\zeta_r}$$

may be thought of as a period hypercube for the above constructed resolution. It is a finite Poincaré duality complex of projective modules, in the sense that it is homotopy equivalent to its graded dual, shifted in degree by $s = \sum_{i=1}^{r}(n_i - 1)$.

A great deal of information about group cohomology may be obtained from the equivariant cohomology spectral sequence of the complex $\mathbf{C}$:

$$E_2^{pq} = \mathrm{Ext}^p_{kG}\big(H_q(\mathbf{C}), k\big) \Rightarrow H^{p+q}\big(\mathrm{Hom}_{kG}(\mathbf{C}, k)\big).$$

For example, if the cohomology ring happens to be Cohen–Macaulay, then this spectral sequence is effectively the Koszul complex, and it converges to $H^*(G,k)/(\zeta_1, \dots, \zeta_r)$. It follows that this finite quotient ring satisfies Poincaré duality with dualizing degree s, which means that the Poincaré series

$$p_G(t) = \sum_{j=0}^{\infty} \dim_k H^j(G,k)$$

satisfies the functional equation

$$p_G(1/t) = (-t)^{r_p(G)} p_G(t).$$

15. Calculations

In this section, we describe some of the calculations which have been carried out in finite group cohomology, and their theoretical impact. We begin with Quillen's calculation [95] of the cohomology of the finite general linear groups $H^*(GL(n, \mathbf{F}_q), \mathbf{F}_l)$, where $q = p^a$ is a prime power, $l \neq p$ is a prime. He showed that there is a map from $BGL(n, \mathbf{F}_q)$ to the homotopy fixed point set $F\Psi^q$ of the Adams operation $\Psi^q : BU \to BU$ which induces an epimorphism in cohomology away from the prime p, and this enabled him to give explicit generators and relations for the mod l cohomology. He was not able to calculate the cohomology at the prime p (nobody since has managed this either), but he was able to show that as n gets larger, the cohomology in any particular degree is eventually zero.

This calculation led him to the definition of the plus construction (see, for example, Gersten [64]), which is a procedure for killing a perfect normal subgroup of the fundamental group of a space by adding 2-cells and 3-cells, without altering the cohomology. He observed that there is a homotopy equivalence between $BGL(\infty, \mathbf{F}_q)^+$ (the perfect normal subgroup in this case is the derived subgroup of $GL(\infty, \mathbf{F}_q) = \lim_{n\to\infty} GL(n, \mathbf{F}_q)$) and the homotopy fixed point set of Ψ^q on BU. He used this as one of the motivations for his definition of algebraic K-theory: if Λ is a ring then the algebraic K-groups of Λ are defined by

$$K_i(\Lambda) = \pi_i\big(BGL(\infty, \Lambda)^+\big) \quad (i \geqslant 1).$$

Again, the plus construction is with respect to the derived subgroup of $GL(\infty, \Lambda)$, which is the perfect subgroup $E(\Lambda)$ generated by the elementary matrices, which differ from the identity matrix in a single off-diagonal entry. The above calculation of the cohomology of the finite general linear groups then proves that

$$K_{2j-1}(\mathbf{F}_q) = \mathbf{Z}/\left(q^{2j} - 1\right), \qquad K_{2j}(\mathbf{F}_q) = 0.$$

In general, the group $K_1(\Lambda)$ is isomorphic to $GL(\infty, \Lambda)/E(\Lambda)$, and $K_2(\Lambda)$ is isomorphic to $H_2(E(\Lambda), \mathbf{Z})$.

Analogous calculations have been made for the finite unitary, orthogonal and symplectic groups by Fiedorowicz and Priddy [62], and for the Chevalley groups of exceptional Lie type by Kleinerman [78].

Another example of the plus construction comes from the cohomology of the symmetric groups. The homology of the infinite symmetric group $\Sigma_\infty = \varinjlim \Sigma_n$ was calculated by Nakaoka [91]. Based on this, together with work of Dyer and Lashof [53], Priddy proved [92] that there is a canonical map $B\Sigma_\infty \times \mathbf{Z} \to \Omega^\infty S^\infty = \varinjlim \Omega^n S^n$, inducing isomorphisms in mod p and integral homology. The induced map $(B\Sigma_\infty)^+ \times \mathbf{Z} \to \Omega^\infty S^\infty$ is then a homotopy equivalence. Here, the plus construction is performed with respect to the infinite alternating group, which is a perfect subgroup of index two. The example of Quillen above splits off this one, essentially as the image of the J-homomorphism (away from p).

In the case of a group G with perfect derived group, the plus construction BG^+ with respect to the derived group is homotopy equivalent to the Bousfield–Kan $\mathbf{Z}$-completion [36] $\mathbf{Z}_\infty BG$.

If G is a finite group, then the Bousfield–Kan $\mathbf{Z}$-completion of BG is homotopy equivalent to the product of the $\mathbf{F}_p$-completions, as p runs over the prime divisors of $|G|$. The fundamental group $\pi_1(\mathbf{F}_p)_\infty BG$ is equal to $G/O^p(G)$, the largest p-factor group of G, so $\pi_1\mathbf{Z}_\infty BG$ is the largest nilpotent quotient of G. It is an interesting question in general to ask what sort of spaces one obtains by looking at the loop spaces $\Omega(\mathbf{F}_p)_\infty BG$. For example, when G is a perfect group with a cyclic Sylow p-subgroup P of order p^r, then this space is the homotopy fiber of a self-map of degree p^r of a $2e-1$-sphere (Cohen [49]). Here, $e = |N_G(P) : C_G(P)|$.

A theorem of Kan and Thurston [77] says that given any topological space X, one can find a group G and a homology equivalence $\phi : BG \to X$ (with arbitrary coefficients). The proof is constructive, but produces very large groups in general. Letting N denote the kernel of the epimorphism $\phi_* : G \to \pi_1(X)$, it follows that N is a perfect normal subgroup of G, and there is an induced weak homotopy equivalence from BG^+ (with respect to N) to X.

Various other calculations should also be mentioned. Quillen [94] calculated the mod two cohomology of the extraspecial 2-groups. The analogous computation for odd primes still eludes us, though some progress has been made. A survey of work on the cohomology of extraspecial groups can be found in [15]. Rusin [99] has computed the mod two cohomology of all the groups of order 32. The cohomology of a number of the sporadic simple Mathieu groups at the prime two has been calculated: that of M_{11} by Benson and Carlson [13], of M_{12} by Adem, Maginnis and Milgram [4], and of M_{22} by Adem and Milgram [5].

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CHAPTER 19

Homotopy Theory of Lie Groups

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HANDBOOK OF ALGEBRAIC TOPOLOGY
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1. Introduction

It was around 1880 that M.S. Lie created the notion of Lie group, then called topological group. One of the motivations was to consider various geometries from the group-theoretic view point. Roughly speaking, a Lie group is a manifold with group structure and locally it corresponds to a Lie algebra. At the beginning of this century E. Cartan and H. Weyl classified completely semi-simple Lie algebras and studied the properties of Lie groups in the large. In 1952 D. Montgomery and L. Zippin solved Hilbert's Fifth Problem affirmatively, namely that any locally Euclidean group is a Lie group. Thus one can define a Lie group as follows:

DEFINITION. A *Lie group* G is a group which is also a smooth manifold such that the maps

$$G \times G \to G,\ (g,h) \mapsto gh \quad \text{and} \quad G \to G,\ g \mapsto g^{-1}$$

are smooth.

A Lie group is compact or connected if the underlying manifold is compact or connected. Two Lie groups are locally isomorphic if there exists a homeomorphism between two neighborhoods of the identities compatible with the product.

A Lie group G is orientable as a manifold. In fact, an orientation at the identity can be translated to an arbitrary point by left translation. Quite similarly one can show that G is parallelizable ($\Leftrightarrow$ the tangent bundle of G is trivial). Therefore all the Stiefel–Whitney characteristic classes are trivial and in particular the Euler–Poincaré characteristic $\chi(G)$ (= the alternating sum of the Betti numbers) is zero.

Now we recall the following

THEOREM 1.1 (Cartan, Malcev and Iwasawa [109]). *Any connected Lie group G is homeomorphic to the Cartesian product of a compact subgroup K and a subset which is homeomorphic with a Euclidean space $\mathbb{R}^n$:*

$$G \approx K \times \mathbb{R}^n, \quad \textit{where } \dim G - \dim K = n.$$

Moreover the group K is a maximal compact subgroup, which is essentially unique; that is, all maximal compact subgroups are conjugate.

REMARK. The theorem holds even when the number of connected components is finite.

Thus from the homotopy theoretic view point it is sufficient to consider a compact Lie group. For example, G is connected if and only if K is connected; G is simply connected if and only if K is simply connected.

Any abelian compact connected Lie group of dimension n is isomorphic to a torus $T^n = S^1 \times \cdots \times S^1$ (n copies).

DEFINITION. A subgroup T of G is a *maximal torus*, if it is

(1) a subgroup which is a torus, such that

(2) if $T \subset U \subset G$ and U is a torus then $T = U$.

It is known that the maximal tori of a compact Lie group G are conjugate to each other by inner automorphisms.

DEFINITION. The dimension of a maximal torus is the *rank* of G.

NOTATION. $N(T)$ = normalizer of a maximal torus T in G.

The normalizer of T determines G itself; that is,

THEOREM 1.2 (Curtis, Wiederhold and Williams [64]). *Let G_1, G_2 be compact connected semi-simple Lie groups and let N_1, N_2 be normalizers of maximal tori in them. Then $G_1 \cong G_2$ if and only if $N_1 \cong N_2$.*

Let T be a maximal torus of G.

DEFINITION. The *Weyl group* $W(G)$ of G is the group of automorphisms of T which are the restrictions of inner automorphisms of G. (This is independent of the choice of T.)

Note that the representation of $W(G)$ as an automorphism group of T is faithful when G is connected.

A maximal torus T has finite index in its normalizer $N(T)$ and the quotient $N(T)/T$ is a finite group.

THEOREM 1.3. $N(T)/T \cong W(G)$.

THEOREM 1.4. $|W(G)| = \chi(G/T)$.

DEFINITION. A compact connected Lie group is called *simple* if it is non-abelian and has no proper closed normal subgroups of dimension > 0; it is called *semi-simple* if its center is finite.

Compact connected Lie groups are locally isomorphic to direct products of tori and simple non-abelian Lie groups. Thus the classification problem of such groups reduces to that of simple groups.

CLASSIFICATION THEOREM. *The connected compact simple Lie groups are exactly the following:*

	dim	linear group	universal cover	center
$\mathrm{A}_n\ (n \geqslant 1)$	$n(n+2)$	$\mathrm{SU}(n+1)$		$\mathbb{Z}_{n+1}$
$\mathrm{B}_n\ (n \geqslant 2)$	$n(2n+1)$	$\mathrm{SO}(2n+1)$	$\mathrm{Spin}(2n+1)$	$\mathbb{Z}_2$
$\mathrm{C}_n\ (n \geqslant 3)$	$n(2n+1)$	$\mathrm{Sp}(n)$		$\mathbb{Z}_2$
$\mathrm{D}_n\ (n \geqslant 4)$	$n(2n-1)$	$\mathrm{SO}(2n)$	$\mathrm{Spin}(2n)$	$\mathbb{Z}_2 \cdot \mathbb{Z}_2$
G_2	14			1
F_4	52			1
E_6	78			$\mathbb{Z}_3$
E_7	133			$\mathbb{Z}_2$
E_8	248			1

where $\mathbb{Z}_2 \cdot \mathbb{Z}_2 = \mathbb{Z}_4$ if n is odd, $= \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if n is even. The first four are called the classical groups and the last five are called the exceptional groups.

REMARK. $\mathrm{A}_1 \cong \mathrm{B}_1 \cong \mathrm{C}_1$, $\mathrm{B}_2 \cong \mathrm{C}_2$, $\mathrm{A}_3 \cong \mathrm{D}_3$ ($\mathrm{D}_2 \cong \mathrm{A}_1 \oplus \mathrm{A}_1$).

NOTATION. $\mathrm{PU}(n+1)$, $\mathrm{PO}(2n+1)$, $\mathrm{PSp}(n)$, $\mathrm{PO}(2n)$ are the quotient groups of the linear groups by the centers, called the *projective classical groups*; similarly for PE_6 and PE_7.

When n is even, the centre of $\mathrm{Spin}(2n)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus there are three central subgroups of order 2; the one gives $\mathrm{SO}(2n)$ and the other two give the two "semi-spinor" groups, which are isomorphic to each other for all n, denoted $\mathrm{Ss}(2n)$. Note that $\mathrm{Ss}(4) = \mathrm{SO}(3) \times \mathrm{SU}(2) \not\cong \mathrm{SO}(4)$ and that $\mathrm{Ss}(8) \cong \mathrm{SO}(8)$ by the triality principle. Let $n > 4$ be an even number. Baum and Browder [22] proved the following by showing that $H^*(\mathrm{SO}(2n);\mathbb{Z}_2)$ and $H^*(\mathrm{Ss}(2n);\mathbb{Z}_2)$ are not isomorphic as algebras over the Steenrod algebra $\mathcal{A}_2$, although they are isomorphic as algebras.

THEOREM 1.5. *Let $n > 4$ be an even number. Then $\mathrm{SO}(2n)$ and $\mathrm{Ss}(2n)$ are of different homotopy type.*

Further, using the classification and the results of Borel [30] they deduced the following from the cohomology structure of simple Lie groups.

THEOREM 1.6. *Let G and G' be connected simple Lie groups. Then G and G' have the same homotopy type if and only if G and G' are isomorphic.*

REMARK. If the word "simple" is deleted, the theorem is false, as the examples $\mathrm{SO}(4)$ and $\mathrm{Ss}(4) = \mathrm{SO}(3) \times \mathrm{SU}(2)$ show.

In fact, they give many examples of this type; compact semi-simple Lie groups which are homeomorphic but are not isomorphic.

On the other hand Scheerer [230] showed

THEOREM 1.7. *If two compact, connected Lie groups are homotopy equivalent, then they are locally isomorphic.*

COROLLARY 1.8. *Two simply connected compact Lie groups are isomorphic if they are homotopy equivalent.*

Another observation is given by Toda [254]:

THEOREM 1.9. *Two simply connected compact semi-simple Lie groups are isomorphic to each other if and only if they have isomorphic homotopy groups for each dimension.*

One cannot replace the word "isomorphic homotopy groups" by "isomorphic cohomology groups over the Steenrod algebra $\mathcal{A}_p$". In fact, the 6 dimensional homotopy groups of $\mathrm{G}_2 \times \mathrm{Sp}(2)$ and $\mathrm{Spin}(7) \times \mathrm{SU}(2)$ are $\mathbb{Z}_3$ and $\mathbb{Z}_{12}$ respectively, although they have isomorphic mod p cohomology over $\mathcal{A}_p$.

In concluding this section, we note that Hilbert's Fifth Problem has a negative solution in the homotopy category: there is a topological group of the homotopy type of the compact manifold E_5 which is not of the homotopy type of a Lie group. This example is constructed by Hilton and Roitberg [92]: E_5 is an S^3-bundle over S^7 induced from $\mathrm{Sp}(2)$ by a map of degree 5 of S^7 onto itself.

2. Cohomology

In this section we collect cohomological results on compact Lie groups.

2.1. *Rational cohomology*

First we recall the following

THEOREM 2.1 (Hopf [97]). *For a compact, connected simple Lie group G, we have*

$$H^*(G;\mathbb{Q}) \cong \Lambda(x_1,\ldots,x_\ell), \quad \deg x_i = 2n_i - 1,$$

where $\ell = \operatorname{rank} G$ *and* $\dim G = \sum_i \deg x_i$.

DEFINITION. $(n_1,\ldots,n_\ell)$ is called the *type* of G.

The importance of the type of G is given by the following

THEOREM 2.2. $|W(G)| = n_1 \cdots n_\ell$.

The degrees of the simple groups are summarized below:

A_n (3, 5, ..., $2n+1$),
B_n (3, 7, ..., $4n-1$),
C_n (3, 7, ..., $4n-1$),
D_n (3, 7, ..., $4n-5$, $2n-1$),
G_2 (3, 11),
F_4 (3, 11, 15, 23),
E_6 (3, 9, 11, 15, 17, 23),
E_7 (3, 11, 15, 19, 23, 27, 35),
E_8 (3, 15, 23, 27, 35, 39, 47, 59).

2.2. *Integral cohomology and torsion*

The classical Lie groups $U(n)$, $SU(n)$, $Sp(n)$ have no torsion and so

$$(2.3)\qquad \begin{aligned} H^*(\mathrm{U}(n);\mathbb{Z}) &= \Lambda(x_1,x_3,\ldots,x_{2n-1}),\\ H^*(\mathrm{SU}(n);\mathbb{Z}) &= \Lambda(x_3,x_5,\ldots,x_{2n-1}),\\ H^*(\mathrm{Sp}(n);\mathbb{Z}) &= \Lambda(x_3,x_7,\ldots,x_{4n-1}); \end{aligned}$$

in particular

$$(2.3)_p\qquad \begin{aligned} H^*(\mathrm{U}(n);\mathbb{Z}_p) &= \Lambda(x_1,x_3,\ldots,x_{2n-1}),\\ H^*(\mathrm{SU}(n);\mathbb{Z}_p) &= \Lambda(x_3,x_5,\ldots,x_{2n-1}),\\ H^*(\mathrm{Sp}(n);\mathbb{Z}_p) &= \Lambda(x_3,x_7,\ldots,x_{4n-1}), \end{aligned}$$

where the generators are chosen to be primitive, and for $p > 2$

$$\wp^a(x_{2i-1}) = (i-1-a, a)x_{2i-1+2a(p-1)} \text{ in } H^*(\mathrm{SU}(n);\mathbb{Z}_p),$$
$$\wp^a(x_{4i-1}) = (-1)^{a(p-1)/2}(2i-1-a, a)x_{4i-1+2a(p-1)} \text{ in } H^*(\mathrm{Sp}(n);\mathbb{Z}_p).$$

Here $(a,b) = [(a+b)!]/(a!b!)$ with $0! = 1$ for non-negative integers a, b, and $= 0$ with either a or $b < 0$.

The spinor group $\mathrm{Spin}(n)$ has no torsion for $n \leqslant 6$ and has only 2-torsion for $n \geqslant 7$. The following results are due to Baum and Browder ([22]).

THEOREM 2.4. (1) *Let* $\mathbb{Z}_\ell$ *be a subgroup of the center* $\mathbb{Z}_n$ *of* $\mathrm{SU}(n)$, *and let* p *be a prime dividing* ℓ. *Let* $n = p^r n'$, $\ell = p^s \ell'$, *where* n' *and* ℓ' *are not divisible by* p. *Set* $G = \mathrm{SU}(n)/\mathbb{Z}_\ell$. *If* $p \neq 2$, *or* $p = 2$ *and* $s > 1$, *then there exist generators* $z_i \in H^{2i-1}(G;\mathbb{Z}_p)$, $1 \leqslant i \leqslant n$, $i \neq p^r$ *and* $y \in H^2(G;\mathbb{Z}_p)$ *such that*:

(i) *As an algebra*

$$H^*(G;\mathbb{Z}_p) = \mathbb{Z}_p[y]/\big(y^{p^r}\big) \otimes \Lambda(z_1, \dots, \hat{z}_{p^r}, \dots, z_n),$$

(ii) $\overline{\phi}(z_i) = \delta_{rs} z_1 \otimes y^{i-1} + \sum_{j=2}^{i-1}(j-1, i-j)z_j \otimes y^{i-j}$ *for* $i \geqslant 2$, *where* δ_{rs} *is the Kronecker delta*, $\overline{\phi}(z_1) = 0$, $\overline{\phi}(y) = 0$,

(iii) $\wp^k z_i = (k, i-k-1)z_{i+k(p-1)}$, $\beta_1 z_q = y^q$, *where* $q = p^{r-1}$.

If $p = 2$ *and* $s = 1$, *then we must modify the above by*:

(i′) *In* (i), $y = z_1^2$,

(ii′) = (ii),

(iii′) = (iii).

(iv) *In addition we have that* $\mathrm{Sq}^{2k+1} z_i = 0$ *unless* $k = 0$, $r \geqslant 2$, $i = 2^{r-1} = q$, *and* $\mathrm{Sq}^1 z_q = y^q = z_1^{2q} = z_1^{2^r}$.

(2) *Let* n *be even*, $q = 2^r$ *the largest power of* 2 *dividing* n. *Then in* $H^*(\mathrm{PO}(n);\mathbb{Z}_2)$ *we may find generators* $v, u_1, \dots, \hat{u}_{q-1}, \dots, u_{n-1}$, $\deg v = 1$, $\deg u_i = i$, *such that*:

(i) *As an algebra*

$$H^*(\mathrm{PO}(n);\mathbb{Z}_2) = \mathbb{Z}_2[v]/(v^q) \otimes \Delta(u_1, \dots, \hat{u}_{q-1}, \dots, u_{n-1}),$$

where Δ *indicates the simple system of generators*,

(ii) $\overline{\phi}(u_k) = \sum_{i=1}^{k-1}(k-i, i)u_i \otimes v^{k-i}$, $k \geqslant 2$,

(iii) $\mathrm{Sq}^j u_k = (k-j, j)u_{k+j}$ *except when* $r \geqslant 3$, $j = 1$, $k = 2^{r-1} - 1$ *in which case* $\mathrm{Sq}^1 u_{m-1} = u_m + v^m$, *where* $m = 2^{r-1}$, $r \geqslant 3$.

(3) *Let* n *be a positive integer*, $q = 2^r$ *the largest power of* 2 *dividing* n. *Then in* $H^*(\mathrm{PSp}(n);\mathbb{Z}_2)$ *there are generators* $v, b_3, b_7, \dots, \hat{b}_{4q-1}, \dots, b_{4n-1}$, $\deg v = 1$, $\deg b_i = i$, *such that*:

(i) *As an algebra*

$$H^*(\mathrm{PSp}(n);\mathbb{Z}_2) = \mathbb{Z}_2[v]/\big(v^{4q}\big) \otimes \Lambda(b_3, \dots, \hat{b}_{4q-1}, \dots, b_{4n-1}),$$

(ii) $\overline{\phi}(b_{4k+3}) = \sum_{i=1}^{k-1}(k-i,i)b_{4i+3}\otimes v^{4k-4i}$ *for* $k \geqslant 2$, $\overline{\phi}(b_7) = b_3\otimes v^4$, $\overline{\phi}(b_3) = 0$,

(iii) $\mathrm{Sq}^{4j}b_{4k+3} = (k-j,j)b_{4k+4j+3}$, $\mathrm{Sq}^j b_{4k+3} = 0$ *if* $j \not\equiv 0 \bmod 4$, *unless* $r \geqslant 1$, $j=1$ *and* $4k+3 = 2q-1$, *in which case* $\mathrm{Sq}^1 u_{2q-1} = v^{2q}$, *if* $r \geqslant 1$.

In the above, $\wp^k$ is the Steenrod reduced power if $p \neq 2$, while if $p = 2$, $\wp^k = Sq^{2k}$.

THEOREM 2.5 ([108]). (1)

(i) $H^*(\mathrm{Spin}(n);\mathbb{Z}_2) = \Delta(x_i, z; 3 \leqslant i < n, i \neq 4, 8, \ldots, 2^{s-1})$, *where* $2^{s-1} < n \leqslant 2^s$, $\deg x_i = i$, $\deg z = 2^s - 1$, $x_i = 0$ *if* $i = 2^t$ *or* $i \geqslant n$,

(ii) $\overline{\phi}(x_i) = 0$, $\overline{\phi}(z) = \sum_{i+j=2^{s-1}} x_{2i}\otimes x_{2j-1}$,

(iii) $\mathrm{Sq}^1 z = \sum_{1<i<2^{s-1}} x_{2i}x_{2^s-2i}$;

(2)

(i) $H^*(\mathrm{Ss}(4m);\mathbb{Z}_2) = \Delta(x_i, z; 3 \leqslant i < 4m, i \neq 4, 8, \cdots, 2^{s-1}, 2^r - 1) \otimes \mathbb{Z}_2[y]/(y^{2^r})$, *where* $4m = 2^r{\cdot}\mathit{odd}$, $2^{s-1} < 4m \leqslant 2^s, \deg x_i = i, \deg z = 2^s - 1, \deg y = 1, x_i = 0$ *if* $i = 2^t$ *or* $i \geqslant 4m$,

(ii) $\overline{\phi}(y) = 0$, $\overline{\phi}(x_i) = \sum_{1<j<i/2}(i-2j,2j)y^{2j}\otimes x_{i-2j} + i\cdot x_i\otimes y$ $(i \neq 2^r - 1)$,

$$\overline{\phi}(z) = \sum_{\substack{i+j+k=2^{s-1}\\ 0<i<j}} (i,j)y^{2i}x_{2j}\otimes x_{2k-1} + \sum_{\substack{i+j=2^{s-1}-1\\ 0<i<j}} x_{2i}x_{2j}\otimes y,$$

(iii) $\mathrm{Sq}^1 z = \sum_{1<i<2^{s-1}-1} x_{2i}x_{2^s-2i} + \sum_{1<i<2^{s-1}-1} y^2 x_{2i}x_{2^s-2i-2}$.

The special orthogonal group $\mathrm{SO}(n)$ has only 2-torsion for $n \geqslant 3$;

$$\begin{aligned} H^*(\mathrm{SO}(n);\mathbb{Z}_2) &= \Delta(x_1, x_2, \ldots, x_{n-1}) \\ &\cong \mathbb{Z}_2[x_1, x_3, \ldots, x_{2m-1}]/\left(x_i^{2^{s(i)}} \mid i = 1, \ldots, m\right), \end{aligned}$$

where $m = \left[\frac{n}{2}\right]$ and $s(i)$ is the smallest number such that $2^{s(i)}(2i-1) \geqslant n$. One has

$$\mathrm{Sq}^a(x_i) = (i-a,a)x_{i+a}.$$

For $p > 2$,

$$\begin{aligned} H^*(\mathrm{SO}(2n);\mathbb{Z}_p) &= \Lambda(x_3, x_7, \ldots, x_{4n-5}, x_{2n-1}), \\ H^*(\mathrm{SO}(2n-1);\mathbb{Z}_p) &= \Lambda(x_3, x_7, \ldots, x_{4n-5}). \end{aligned}$$

If we put

$$\mathrm{SO}^-(n) = \{A \in \mathrm{O}(n) \mid \det A = -1\}$$

which is homeomorphic to $\mathrm{SO}(n)$, then for any p

$$H^*(\mathrm{O}(n);\mathbb{Z}_p) \cong H^*(\mathrm{SO}(n);\mathbb{Z}_p) \oplus H^*(\mathrm{SO}^-(n);\mathbb{Z}_p).$$

The exceptional Lie groups have p-torsions only for the following cases:
G_2 for $p = 2$; F_4, E_6, E_7 for $p = 2, 3$; E_8 for $p = 2, 3, 5$.

(2.6) $H^*(G_2; \mathbb{Z}_2) = \mathbb{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5)$
with all x_i primitive, where $\mathrm{Sq}^2 x_3 = x_5$.

(2.6)′ $H^*(G_2; \mathbb{Z}_p) = \Lambda(x_3, x_{11})$ *for* $p > 2$
with all x_i primitive, where $\wp^1 x_3 = x_{11}$ *for* $p = 5$.

(2.7) $H^*(F_4; \mathbb{Z}_2) = \mathbb{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_{15}, x_{23})$
with all x_i primitive, where $\mathrm{Sq}^2 x_3 = x_5$ *and* $\mathrm{Sq}^8 x_{15} = x_{23}$.

(2.7)′ $H^*(F_4; \mathbb{Z}_3) = \mathbb{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15})$
with x_i primitive for $i = 3, 7, 8$ *and*

$$\overline{\phi}(x_j) = x_8 \otimes x_{j-8} \text{ for } j = 11, 15,$$

where $\wp^1 x_3 = x_7, \beta x_7 = x_8$, $\wp^1 x_{11} = x_{15}$.

(2.7)″ $H^*(F_4; \mathbb{Z}_p) = \Lambda(x_3, x_{11}, x_{15}, x_{23})$ *for* $p > 3$
with all x_i primitive, where $\wp^1 x_3 = x_{11}$, $\wp^1 x_{15} = x_{23}$ *for* $p = 5$; $\wp^1 x_3 = x_{15}$, $\wp^1 x_{11} = x_{23}$ *for* $p = 7$; $\wp^1 x_3 = x_{23}$ *for* $p = 11$.

(2.8) $H^*(E_6; \mathbb{Z}_2) = \mathbb{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_9, x_{15}, x_{17}, x_{23})$
with x_i primitive for $i = 3, 5, 9, 17$, *and*

$$\overline{\phi}(x_j) = x_3^2 \otimes x_{j-6} \text{ for } j = 15, 23,$$

where $\mathrm{Sq}^2 x_3 = x_5$, $\mathrm{Sq}^4 x_5 = x_9$, $\mathrm{Sq}^8 x_9 = x_{17}$, $\mathrm{Sq}^8 x_{15} = x_{23}$.

(2.8)′ $H^*(E_6; \mathbb{Z}_3) = \mathbb{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17})$
with x_i primitive for $i = 3, 7, 8, 9$, *and*

$$\overline{\phi}(x_j) = x_8 \otimes x_{j-8} \text{ for } j = 11, 15, 17,$$

where $\wp^1 x_3 = x_7$, $\beta x_7 = x_8$, $\wp^1 x_{11} = x_{15}$.

(2.8)″ $H^*(E_6; \mathbb{Z}_p) = \Lambda(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23})$ *for* $p > 3$
with all x_i primitive, where $\wp^1 x_3 = x_{11}$, $\wp^1 x_9 = x_{17}$, $\wp^1 x_{15} = x_{23}$ *for* $p = 5$; $\wp^1 x_3 = x_{15}$, $\wp^1 x_{11} = x_{23}$ *for* $p = 7$; $\wp^1 x_3 = x_{23}$ *for* $p = 11$.

(2.9) $H^*(E_7; \mathbb{Z}_2) = \mathbb{Z}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda(x_{15}, x_{17}, x_{23}, x_{27})$,
with x_i primitive for $i = 3, 5, 9, 17$ *and*

$$\overline{\phi}(x_{15}) = x_5 \otimes x_5^2 + x_9 \otimes x_3^2,$$

$$\overline{\phi}(x_{23}) = x_5 \otimes x_9^2 + x_{17} \otimes x_3^2,$$
$$\overline{\phi}(x_{27}) = x_9 \otimes x_9^2 + x_{17} \otimes x_5^2,$$

where $\mathrm{Sq}^2 x_3 = x_5$, $\mathrm{Sq}^4 x_5 = x_9$, $\mathrm{Sq}^8 x_9 = x_{17}$, $\mathrm{Sq}^8 x_{15} = x_{23}$, $\mathrm{Sq}^4 x_{23} = x_{27}$.

(2.9)′ $H^*(\mathrm{E}_7; \mathbb{Z}_3) = \mathbb{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35})$
with x_i *primitive for* $i = 3, 7, 8, 19$, *and*

$$\overline{\phi}(x_j) = x_8 \otimes x_{j-8} \quad \textit{for } j = 11, 15, 27,$$
$$\overline{\phi}(x_{35}) = x_8 \otimes x_{27} - x_8^2 \otimes x_{19},$$

where $\wp^1 x_3 = x_7$, $\beta x_7 = x_8$, $\wp^1 x_{11} = x_{15}$, $\wp^3 x_7 = x_{19}$, $\wp^3 x_{15} = x_{27}$, $\wp^1 x_{15} = \varepsilon x_{19}$ $(\varepsilon = \pm 1)$.

(2.9)″ $H^*(\mathrm{E}_7; \mathbb{Z}_p) = \Lambda(x_3, x_{11}, x_{15}, x_{19}, x_{23}, x_{27}, x_{35})$ *for* $p > 3$
with all x_i *primitive, where* $\wp^1 x_3 = x_{11}$, $\wp^1 x_{15} = x_{23}$, $\wp^1 x_{19} = x_{27}$, $\wp^1 x_{27} = x_{35}$ *for* $p = 5$; $\wp^1 x_3 = x_{15}$, $\wp^1 x_{11} = x_{23}$, $\wp^1 x_{23} = x_{35}$ *for* $p = 7$; $\wp^1 x_3 = x_{23}$, $\wp^1 x_{15} = x_{35}$ *for* $p = 11$; $\wp^1 x_3 = x_{27}$, $\wp^1 x_{11} = x_{35}$ *for* $p = 13$; $\wp^1 x_3 = x_{35}$ *for* $p = 17$.

(2.10) $H^*(\mathrm{E}_8; \mathbb{Z}_2) = \mathbb{Z}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Lambda(x_{17}, x_{23}, x_{27}, x_{29})$
with x_i *primitive for* $i = 3, 5, 9, 17$, *and*

$$\overline{\phi}(x_{15}) = x_3 \otimes x_3^4 + x_5 \otimes x_5^2 + x_3^2 \otimes x_9,$$
$$\overline{\phi}(x_{23}) = x_3 \otimes x_5^4 + x_5 \otimes x_9^2 + x_3^2 \otimes x_{17},$$
$$\overline{\phi}(x_{27}) = x_3 \otimes x_3^8 + x_9 \otimes x_9^2 + x_5^2 \otimes x_{17},$$
$$\overline{\phi}(x_{29}) = x_5 \otimes x_3^8 + x_9 \otimes x_5^4 + x_3^4 \otimes x_{17},$$
$$\overline{\phi}(x_{30}) = x_3^2 \otimes x_3^8 + x_5^2 \otimes x_5^4 + x_3^4 \otimes x_9^2,$$

where $\mathrm{Sq}^2 x_3 = x_5$, $\mathrm{Sq}^4 x_5 = x_9$, $\mathrm{Sq}^8 x_9 = x_{17}$, $\mathrm{Sq}^8 x_{15} = x_{23}$, $\mathrm{Sq}^4 x_{23} = x_{27}$, $\mathrm{Sq}^2 x_{27} = x_{29}$.

(2.10)′ $H^*(\mathrm{E}_8; \mathbb{Z}_3) = \mathbb{Z}_3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$
with x_i *primitive for* $i = 3, 7, 8, 19, 20$, *and*

$$\overline{\phi}(x_{15}) = x_8 \otimes x_7,$$
$$\overline{\phi}(x_{27}) = x_8 \otimes x_{19} + x_{20} \otimes x_7,$$
$$\overline{\phi}(x_{35}) = x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_{20} x_8 \otimes x_7,$$
$$\overline{\phi}(x_{39}) = x_{20} \otimes x_{19},$$
$$\overline{\phi}(x_{47}) = x_8 \otimes x_{39} + x_{20} \otimes x_{27} + x_{20} x_8 \otimes x_{19} - x_{20}^2 \otimes x_7,$$

where $\wp^1 x_3 = x_7$, $\beta x_7 = x_8$, $\wp^1 x_{15} = \varepsilon x_{19}$, $\beta x_{19} = x_{20}$, $\wp^1 x_{35} = \varepsilon x_{39}$, $\wp^3 x_7 = x_{19}$, $\wp^3 x_8 = x_{20}$, $\wp^3 x_{15} = x_{27}$, $\wp^3 x_{27} = -x_{39}$, $\wp^3 x_{35} = x_{47}$ *with* $\varepsilon = \pm 1$ *simultaneously.*

(2.10)″ $H^*(\mathrm{E}_8;\mathbb{Z}_5) = \mathbb{Z}_5[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})$
with x_i primitive for $i = 3, 11, 12$, and

$$\begin{aligned}
&\overline{\phi}(x_j) = x_{12} \otimes x_{j-12} && \textit{for } j = 15, 23,\\
&\overline{\phi}(x_k) = 2x_{12} \otimes x_{k-12} + x_{12}^2 \otimes x_{k-24} && \textit{for } k = 27, 35,\\
&\overline{\phi}(x_\ell) = 3x_{12} \otimes x_{\ell-12} + 3x_{12}^2 \otimes x_{\ell-24} + x_{12}^3 \otimes x_{\ell-36} && \textit{for } \ell = 39, 47,
\end{aligned}$$

where $\wp^1 x_3 = x_{11}$, $\beta x_{11} = x_{12}$, $\wp^1 x_{15} = x_{23}$, $\wp^1 x_{27} = x_{35}$, $\wp^1 x_{39} = x_{47}$.

(2.10)‴ $H^*(\mathrm{E}_8;\mathbb{Z}_p) = \Lambda(x_3, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}, x_{59})$ *for $p > 5$ with all x_i primitive, where $\wp^1 x_3 = x_{15}$, $\wp^1 x_{23} = x_{35}$, $\wp^1 x_{27} = x_{39}$, $\wp^1 x_{35} = x_{47}$, $\wp^1 x_{47} = x_{59}$ for $p = 7$; $\wp^1 x_3 = x_{23}$, $\wp^1 x_{15} = x_{35}$, $\wp^1 x_{27} = x_{47}$, $\wp^1 x_{39} = x_{59}$ for $p = 11$; $\wp^1 x_3 = x_{27}$, $\wp^1 x_{15} = x_{39}$, $\wp^1 x_{23} = x_{47}$, $\wp^1 x_{35} = x_{59}$ for $p = 13$; $\wp^1 x_3 = x_{35}$, $\wp^1 x_{15} = x_{47}$, $\wp^1 x_{27} = x_{59}$ for $p = 17$; $\wp^1 x_3 = x_{39}$, $\wp^1 x_{23} = x_{59}$ for $p = 19$; $\wp^1 x_3 = x_{47}$, $\wp^1 x_{15} = x_{59}$ for $p = 23$; $\wp^1 x_3 = x_{59}$ for $p = 29$.*

REMARK. For each pair of the inclusions $\mathrm{G}_2 \subset \mathrm{F}_4 \subset \mathrm{E}_6 \subset \mathrm{E}_7 \subset \mathrm{E}_8$ the smaller group is totally nonhomologous to zero, mod 2, in the larger group.

As is well known, among the exceptional groups, only E_6 and E_7 have the nontrivial centers ($\mathbb{Z}_3$ and $\mathbb{Z}_2$ respectively); the quotient groups are denoted by PE_6 and PE_7.

(2.8)′ $H^*(\mathrm{PE}_6;\mathbb{Z}_3) = \mathbb{Z}_3[x_2, x_8]/(x_2^9, x_8^3) \otimes \Lambda(x_1, x_3, x_7, x_9, x_{11}, x_{15})$,
with x_i primitive for $i = 1, 2$, and

$$\begin{aligned}
\overline{\phi}(x_3) &= x_2 \otimes x_1,\\
\overline{\phi}(x_7) &= x_2^3 \otimes x_1,\\
\overline{\phi}(x_8) &= x_2^3 \otimes x_2,\\
\overline{\phi}(x_9) &= x_2 \otimes x_7 - x_2^3 \otimes x_3 + x_8 \otimes x_1 + x_2^4 \otimes x_1,\\
\overline{\phi}(x_{11}) &= x_2 \otimes x_9 - x_2^2 \otimes x_7 + x_8 \otimes x_3 - x_2^4 \otimes x_3 + x_8 x_2 \otimes x_1 - x_2^5 \otimes x_1,\\
\overline{\phi}(x_{15}) &= x_2^3 \otimes x_9 + x_8 \otimes x_7 + x_2^6 \otimes x_3 + x_8 x_2^3 \otimes x_1,
\end{aligned}$$

where $x_2 = \beta x_1$, $x_7 = \wp^1 x_3$, $x_8 = \beta x_7$, $x_{15} = \wp^1 x_{11}$, $\wp^1 x_8 = -x_2^6$.

(2.8)″ $H^*(\mathrm{PE}_6;\mathbb{Z}_p) = H^*(\mathrm{E}_6;\mathbb{Z}_p)$ *as Hopf algebras over $\mathcal{A}_p$ for $p \neq 3$.*

(2.9)′ $H^*(\mathrm{PE}_7;\mathbb{Z}_2) = \mathbb{Z}_2[x_1, x_5, x_9]/(x_1^4, x_5^4, x_9^4) \otimes \Lambda(x_6, x_{15}, x_{17}, x_{23}, x_{27})$,
with x_i primitive for $i = 1, 3, 5, 6, 9, 17$, and

$$\begin{aligned}
\overline{\phi}(x_{15}) &= x_6 \otimes x_9 + x_5^2 \otimes x_5,\\
\overline{\phi}(x_{23}) &= x_9^2 \otimes x_5 + x_6 \otimes x_{17},\\
\overline{\phi}(x_{27}) &= x_9^2 \otimes x_9 + x_5^2 \otimes x_{17}.
\end{aligned}$$

(2.9)″ $H^*(\mathrm{PE}_7;\mathbb{Z}_p) = H^*(\mathrm{E}_7;\mathbb{Z}_p)$ *as Hopf algebras over $\mathcal{A}_p$ for $p \neq 2$.*

2.3. *K-theory*

First we recall Hodgkin's theorem on the unitary K-theory of G with $\pi_1(G)$ torsion-free.

THEOREM 2.11 (Hodgkin [93]). *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free. Then*

(1) *$K^*(G)$ is torsion free;*

(2) *$K^*(G)$ can be given the structure of a Hopf algebra over the integers, graded by $\mathbb{Z}_2$;*

(3) *Regarded as a Hopf algebra, $K^*(G)$ is the exterior algebra on the module of primitive elements, which are of degree* 1;

(4) *A unitary representation $\rho : G \to \mathrm{U}(n)$, by composition with the inclusion $\mathrm{U}(n) \subset \mathrm{U}$, defines a homotopy class $\beta(\rho)$ in $[G, \mathrm{U}] = K^1(G)$. The module of primitive elements in $K^1(G)$ is exactly the module generated by all classes $\beta(\rho)$ of this type;*

(5) *If G is semi-simple of rank ℓ, and the ℓ basic representations are denoted ρ_i $(1 \leqslant i \leqslant \ell)$, then the classes $\beta(\rho_i)$ form a basis for the above set of primitive elements; we can write*

$$K^*(G) = \Lambda\big(\beta(\rho_1), \ldots, \beta(\rho_\ell)\big).$$

The assumption in the above theorem covers the case that G is semi-simple and simply connected. The proof makes use of the Atiyah–Hirzebruch spectral sequence, appealing to the classification of Lie groups.

The proof without using classification is due to Araki [14] and Atiyah [19].

Now we consider the case when G is a compact, connected Lie group of rank ℓ with finite fundamental group $\mathbb{Z}_p = \pi_1(G)$ (p: prime). The Lie group G may be considered as the quotient group G_0/π, where $\pi \to G_0 \xrightarrow{u} G$ is the universal covering of G. The inclusion $i : \pi \to G_0$ induces a homomorphism $i^* : R(G_0) \to R(\pi)$ of complex representation rings. Then Held and Suter observed the following

PROPOSITION 2.12. *There are generators $\lambda_1, \ldots, \lambda_{\ell-1}, \lambda_\ell$ of the exterior algebra $K^*(G_0)$ and elements $\nu_1, \ldots, \nu_{\ell-1}, \varepsilon_\ell$ in $K^1(G)$ such that*

(1) *$u^*(\nu_i) = \lambda_i$ for $1 \leqslant i \leqslant \ell - 1$ and $\varepsilon_\ell = u_*(\lambda_\ell)$ with $u^*(\varepsilon_\ell) = p\lambda_\ell$;*

(2) *$K^*(G)/\mathrm{Tors}K^*(G) = \Lambda(u'(\nu_1), \ldots, u'(\nu_{\ell-1}), u'(\varepsilon_\ell))$;*

where $u' : K^*(G) \to K^*(G)/\mathrm{Tors}K^*(G)$ is the natural projection;

(3) *the elements $\nu_1, \ldots, \nu_{\ell-1}, \varepsilon_\ell$ generate an exterior algebra $\Lambda(\nu_1, \ldots, \nu_{\ell-1}, \varepsilon_\ell)$ in $K^*(G)$.*

Based on this they proved

THEOREM 2.13 (Held and Suter [88]). *Let $\pi_1(G) = \mathbb{Z}_p$ (p a prime) and $\ell = \mathrm{rank}\, G$. Then there exist elements $\nu_1, \cdots, \nu_{\ell-1}, \varepsilon_\ell$ in $K^1(G)$ such that*

$$K^*(G) \cong \big(\Lambda(\nu_1, \ldots, \nu_{\ell-1}, \varepsilon_\ell) \otimes T\big)/\big(\varepsilon_\ell \otimes \tilde{T}\big)$$

as rings, where $T \cong R(\mathbb{Z}_p)/i^*(I(G_0))$ with $I(G_0)$ the augmentation ideal of $R(G_0)$ and $\tilde{T}$ is the direct summand in T such that $T \cong \mathbb{Z} \oplus \tilde{T}$.

As examples they computed explicitly $K^*(G)$ for $G = \mathrm{PSp}(n)$, $\mathrm{PU}(p)$, $\mathrm{SO}(n)$, $\mathrm{Ss}(4)$, PE_6, PE_7.

Along the same line of proof, it is possible to compute $K^*(\mathrm{PU}(n))$ for an arbitrary n (see also [212]). These results were also obtained by Hodgkin [94].

As for the orthogonal K-theory $KO(G)$ of a compact connected Lie group, we have Seymour's result [233] in which he determined the module structure of $KO(G)$ with $\pi_1(G)$ torsion free by first calculating Real K-theory $KR^*(G)$. The ring structure is mentioned in Crabb [62].

Among the cases where $\pi_1(G)$ is finite, Minami determined $KO^*(\mathrm{SO}(n))$ in [186], [187], [188], $KO^*(\mathrm{PE}_6)$ in [190] and $KO^*(\mathrm{PE}_7)$ in [189].

2.4. *Morava K-theory*

Let $K(n)^*(-)$ be Morava K-theory with coefficients $K(n)^* = \mathbb{Z}_p[v_n, v_n^{-1}]$. Note that $K(1)^*(-)$ is the $(p-1)$ component of the mod p K-theory $K^*(-;\mathbb{Z}_p)$.

The key results needed to determine this are the following

(2.14) *Let X be a finite CW-complex. Then*

(1) *if $H^*(X;\mathbb{Z})$ has no p-torsion, then the Atiyah–Hirzebruch spectral sequence collapses for all $n \geqslant 0$;*

(2) *if the Atiyah–Hirzebruch spectral sequence for $K(n)^*(X)$ collapses for some $n \geqslant 1$, so does that for $K(n+1)^*(X)$.*

We recall that the following is a list of the compact, simply connected, simple groups with p-torsion:

(2.15)
$$\begin{aligned} p &= 2 \quad G = \mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{Spin}(m) \text{ for } m \geqslant 7; \\ p &= 3 \quad G = \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8; \\ p &= 5 \quad G = \mathrm{E}_8. \end{aligned}$$

We consider the odd and even cases separately:

a) p: odd prime

THEOREM 2.16 (Hunton [103] and Yagita [272]). *Let p be an odd prime and $n \geqslant 2$. The Atiyah–Hirzebruch spectral sequence for $K(n)^*(G)$ collapses and there is a $K(n)^*$-module isomorphism*

$$K(n)^*(G) \cong K(n)^* \otimes H^*(G;\mathbb{Z}_p)$$

except for the case $K(2)^*(G)$ with $(G,p) = (\mathrm{E}_8, 3)$.

Here there is a $K(2)^*$-module isomorphism

$$K(2)^*(\mathrm{E}_8) \cong K(2)^* \otimes \Lambda\big(\{x_3 x_{20}^2\}\big) \otimes \mathbb{Z}_3[x_8]/\big(x_8^3\big) \otimes \Lambda(x_{15}) \otimes B,$$

with $B = \Lambda(x_7, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$.

b) $p = 2$.

THEOREM 2.17 (Hunton [103]). *Let* $n \geqslant 2$. *At* $p = 2$, *the Atiyah–Hirzebruch spectral sequence collapses for the following cases*: $n \geqslant 2$ *for* G_2, F_4, E_6; $n \geqslant 4$ *for* E_7, E_8.

The reader is referred to [103] for $n = 2, 3$ with E_7, E_8 and for small n with $\mathrm{Spin}(m)$.

2.5. *Brown–Peterson cohomology*

Let $BP^*(-)$ be Brown–Peterson cohomology theory with coefficients $BP^* = \mathbb{Z}_{(p)}[v_1, \dots]$ at a prime p.

The following results are due to Yagita.

THEOREM 2.18 ([274]). *There are the following* BP^**-module isomorphisms for* $p = 2$:

$$(1)\qquad BP^*(G_2) \cong BP^*\{1, 2x_3, x_3^3x_5\} \oplus BP^*\{x_3^3, x_3^2x_5\}/(2x_3^3 + v_1x_3^2x_5) \oplus BP^*/(2, v_1)\{x_3^2\},$$

$$(2)\qquad BP^*(F_4) \cong BP^*(G_2) \otimes \Lambda(x_{15}, x_{23}),$$

$$(3)\qquad BP^*(E_6) \cong BP^*(F_4) \otimes \Lambda(x_9, x_{17}).$$

THEOREM 2.19 ([271]). *There are the following* BP^**-algebra isomorphisms*:
(1) *For* $p = 3$,

$$BP^*(F_4) \cong \bigl(BP^*\{1, y_3, y_{26}\} \oplus BP^*\{y_{19}, y_{23}\}/(3y_{19} = v_1y_{23}) \oplus BP^*/(3, v_1) \otimes \mathbb{Z}_3[x_8]/(x_8^3)\bigr) \otimes \Lambda(x_{11}, x_{15}),$$

and $y_3y_{23} = 3y_{26}$, $y_3y_{19} = v_1y_{26}$, $y_iy_j = 0$ *for* $i \neq 3$, $j \neq 19, 23$, $x_8y_i = 0$.
(2) *For* $p = 3$,

$$BP^*(E_6) \cong BP^*(F_4) \otimes \Lambda(x_9, x_{17}).$$

(3) *For* $p = 5$,

$$BP^*(E_8) \cong (BP^*\{1, y_3, y_{62}\} \oplus BP^*\{y_{51}, y_{59}\}/(5y_{51} = v_1y_{59}) \oplus BP^*/(5, v_1) \otimes \mathbb{Z}_5[x_{12}]/(x_{12}^5)) \otimes \Lambda(x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}),$$

and $y_3y_{59} = 5y_{62}$, $y_3y_{51} = v_1y_{62}$, $y_iy_j = 0$ *for* $i \neq 3$, $j \neq 51, 59$, $x_{12}y_i = 0$.

THEOREM 2.20 ([273]). (1) *There is the following* BP^**-module isomorphism for* $p = 3$:

$$\begin{aligned} BP^*(E_7) &\cong BP^*(F_4) \otimes \Lambda(x_{19}, x_{27}, x_{35}) \\ &\cong \bigl[BP^*\{1, y_3, y_{26}\} \oplus BP^*\{y_{19}, y_{23}\}/(3y_{19} = v_1y_{23}) \\ &\quad \oplus BP^*/(3, v_1)[x_8]/(x_8^3)\bigr] \otimes \Lambda(x_{11}, x_{15}, x_{19}, x_{27}, x_{35}). \end{aligned}$$

(2) *There is the following BP^*-module isomorphism for $p = 3$:*

$$BP^*(\mathrm{E}_8) \cong (T/R_1 \oplus F/R_2) \otimes \Lambda(x_{27}, x_{35}, x_{39}, x_{47}),$$

where

(1) $T = BP^*/(3) \otimes [(\mathbb{Z}_3[x_8]/(x_8^3) \otimes \mathbb{Z}_3[x_{20}]/(x_{20}^3) \otimes \Lambda(u_{27}) - \{1\} - \{u_{27}x_8^2x_{20}^2\}) \oplus \mathbb{Z}_3\{(x_8, x_8^2, u_{27}, u_{27}x_8) \otimes (w_{43}, w_{55})\}]$,

(2) $R_1 = \mathrm{Ideal}(v_1x_8 - v_2x_{20}, v_1w_{43} - v_2w_{55}, v_1x_{20}, v_2abc,$ *where* $a, b, c \in \{x_8, x_{20}, u_{27}\})$,

(3) $F = BP^*\{1, y_{23}, w_{15}, y_{59}, w_{55}, w_{43}, y_{23}, y_{59}, y_{23}w_{55}, y_{23}w_{43}, s_{74}, y_3, y_3y_{23}, w_{22}, y_{62}, y_{85}, y_{81}, y_{15}, y_{38}, w_{34}, y_{74}, y_{97}, s_{93}, y_3y_{15}, y_{41}, y_{77}, y_{100}\}$,

(4) $R_2 = \mathrm{Ideal}(v_1^2y_{23} - 3w_{15}, v_1y_{59} - 3w_{55}, v_2y_{59} - 3w_{43}, v_1w_{43} - v_2w_{55}, v_1y_{23}w_{55} - 3s_{74}, v_1y_3y_{23} - 3w_{22}, v_1y_{85} - 3s_{81}, v_1y_{38} - 3w_{34}, v_1y_{97} - 3s_{93})$.

Let p be an odd prime and $BP^*(-;\mathbb{Z}_p)$ be mod p Brown–Peterson cohomology theory with coefficients $BP^*/(p) = \mathbb{Z}_p[v_1, v_2, \ldots]$. Observe that $BP^*(-;\mathbb{Z}_p)$ has a commutative associative multiplication.

First note that if $H^*(G;\mathbb{Z})$ is p-torsion free, then there is a BP^*-algebra isomorphism

$$BP^*(G;\mathbb{Z}_p) \cong BP^* \otimes H^*(G;\mathbb{Z}_p).$$

Therefore, by (2.14), it suffices to determine $BP^*(G;\mathbb{Z}_p)$ only for the following cases: $p = 3$ for $G = \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$; $p = 5$ for $G = \mathrm{E}_8$.

The following results are also due to Yagita.

THEOREM 2.21 ([272]). (1) *There are the following BP^*-algebra isomorphisms*:

(i) $BP^*(\mathrm{F}_4;\mathbb{Z}_3) \cong (BP^*/(3) \otimes \Lambda(w_{19}) \oplus BP^*/(3, v_1) \otimes (\mathbb{Z}_3[x_8]/(x_8^3) - \{1\})) \otimes \Lambda(x_7, x_{11}, x_{15})$,

(ii) $BP^*(\mathrm{E}_6;\mathbb{Z}_3) \cong BP^*(\mathrm{F}_4;\mathbb{Z}_3) \otimes \Lambda(x_9, x_{17})$,

(iii) $BP^*(\mathrm{E}_7;\mathbb{Z}_3) \cong BP^*(\mathrm{F}_4;\mathbb{Z}_3) \otimes \Lambda(x_{19}, x_{27}, x_{35})$,

(iv) $BP^*(\mathrm{E}_8;\mathbb{Z}_5) \cong (BP^*/(5) \otimes \Lambda(w_{51}) \oplus BP^*/(5, v_1) \otimes (\mathbb{Z}_5[x_{12}]/(x_{12}^5) - \{1\})) \otimes \Lambda(x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})$,

where $w_{19}x_8 = 0$, $w_{51}x_{12} = 0$.

(2) *There is the following BP^*-module isomorphism*:

$$\text{(v) } BP^*(\mathrm{E}_8;\mathbb{Z}_3) \cong (BP^*/(3)\{1, w_{15}, w_{74}\} \oplus BP^*/(3)[\mathbb{Z}_3\{w_{55}, w_{43}, x_{20}, x_{20}^2\} \otimes \mathbb{Z}_3\{1, x_8, x_8^2\}]/(v_1w_{43} = v_2w_{55}, v_1x_8 = v_2x_{20}, v_1x_{20} = 0)) \otimes \Lambda(x_7, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}).$$

Moreover $x_{20}^3 = 0$, $w_{43}x_{20} = w_{55}x_8$, $w_{15}w_{55} = v_1w_{74}$, $w_{55}w_{43} = 0$, $w_{15}x_{20} = 0$ *and* $w_{55}x_{20} = 0$.

3. Homotopy groups

3.1. *Stable homotopy groups: Bott periodicity*

We denote by G one of U, SU, O, SO, Sp so that

$$\mathrm{G}(n) = \mathrm{U}(n), \mathrm{SU}(n), \mathrm{O}(n), \mathrm{SO}(n), \mathrm{Sp}(n).$$

By the correspondence $\mathrm{G}(n) \ni A \mapsto A \oplus I_1 \in \mathrm{G}(n+1)$, we obtain a sequence of compact Hausdorff spaces

$$\mathrm{G}(1) \subset \mathrm{G}(2) \subset \cdots \subset \mathrm{G}(n) \subset \mathrm{G}(n+1) \subset \cdots,$$

where I_n is the identity matrix of $\mathrm{G}(n)$.

DEFINITION. The group $\mathbf{G} = \bigcup_n \mathrm{G}(n)$ with the weak topology is called the *infinite dimensional classical group* (*infinite dimensional unitary, special unitary, orthogonal, special orthogonal, symplectic group*, respectively).

The inclusion $\mathrm{H}(n) \subset \mathrm{G}(n)$ means one of the following cases:

$$\mathrm{SO}(n) \subset \mathrm{SU}(n) \subset \mathrm{U}(n) \subset \mathrm{Sp}(n); \quad \mathrm{SO}(n) \subset \mathrm{O}(n) \subset \mathrm{U}(n).$$

Then we have $\mathrm{H}(n) = \mathrm{G}(n) \cap \mathrm{H}(n+1)$ for the natural inclusions

$$\mathrm{H}(n) = \mathrm{H}(n) \times 1 \subset \mathrm{H}(n+1), \quad \mathrm{G}(n) = \mathrm{G}(n) \times 1 \subset \mathrm{G}(n+1),$$

and hence the natural maps $\mathrm{G}(n)/\mathrm{H}(n) \to \mathrm{G}(n+1)/\mathrm{H}(n+1)$ are injections. Thus $\mathrm{G}(n)/\mathrm{H}(n)$ can be regarded as a closed subspace of $\mathrm{G}(n+1)/\mathrm{H}(n+1)$ through the above map, since it is a compact Hausdorff space.

DEFINITION. The space $\bigcup_n \mathrm{G}(n)/\mathrm{H}(n)$ with the weak topology is called the *quotient space* of $\mathbf{G}$ by $\mathbf{H}$, denoted $\mathbf{G}/\mathbf{H} = \bigcup_n \mathrm{G}(n)/\mathrm{H}(n)$.

Then the natural inclusion $\mathrm{H}(n) \to \mathrm{G}(n)$ and projection $\mathrm{G}(n) \to \mathrm{G}(n)/\mathrm{H}(n)$ induce respectively the inclusion $\boldsymbol{i} : \mathbf{H} \to \mathbf{G}$ and the projection $\boldsymbol{p} : \mathbf{G} \to \mathbf{G}/\mathbf{H}$.

Let τ be the permutation of the set $\{1, 2, \ldots, 2m\}$ defined by $\tau(i) = 2i - 1$ and $\tau(i+m) = 2i$ for $1 \leqslant i \leqslant m$ and consider the permutation matrix $P_\tau = (\delta_{i\tau(j)})$ defined by it. Consider the following subgroups $\mathrm{K}(m)$ of $\mathrm{G}(2m)$:

$$\begin{aligned}
&\mathrm{K}(m) = \tau_m(\mathrm{G}(m) \times \mathrm{G}(m)) && \text{for } \mathrm{G} = \mathrm{SO, O, SU, U, Sp};\\
&\mathrm{K}(m) = \tau_m(r(\mathrm{H}(m))) && \text{for } \mathrm{G} = \mathrm{O, SO} \text{ with } \mathrm{H} = \mathrm{U, SU};\\
&\mathrm{K}(m) = \tau_m(c'(\mathrm{H}(m))) && \text{for } \mathrm{G} = \mathrm{U, SU} \text{ with } \mathrm{H} = \mathrm{Sp},
\end{aligned}$$

where $r : \mathrm{U}(m) \to \mathrm{SO}(2m)$ and $c' : \mathrm{Sp}(m) \to \mathrm{SU}(2m)$ are the natural inclusions. Then we have $\mathrm{K}(m) = \mathrm{G}(2m) \cap \mathrm{K}(m+1)$, and hence

$$\mathrm{G}(2m)/\mathrm{K}(m) \subset \mathrm{G}(2m+2)/\mathrm{K}(m+1).$$

DEFINITION. The space $\bigcup_m \mathrm{G}(2m)/\mathrm{K}(m)$ with the weak topology is called the *quotient space* of $\mathbf{G}$ by $\mathbf{G} \times \mathbf{G}$ or $\mathbf{H}$ and denoted as follows:

$$\begin{aligned}
&B\mathbf{G} = \mathbf{G}/\mathbf{H} = \textstyle\bigcup_m G(2m)/\tau_m\big(\mathrm{G}(m) \times \mathrm{G}(m)\big), \text{ where } \mathbf{H} \cong \mathbf{G} \times \mathbf{G};\\
&\mathbf{G}/\mathbf{H} = \textstyle\bigcup_m \mathrm{G}(2m)/\tau_m\big(r\big(\mathrm{H}(m)\big)\big) \text{ for G = O, SO, where } \mathbf{H} \cong \mathbf{U}, \mathbf{SU};\\
&\mathbf{G}/\mathbf{H} = \textstyle\bigcup_m \mathrm{G}(2m)/\tau_m\big(c'\big(\mathrm{H}(m)\big)\big) \text{ for G = U, SU, where } \mathbf{H} \cong \mathbf{Sp}.
\end{aligned}$$

NOTATION. $R_t(n) = e^{\pi i t} I_n \oplus e^{-\pi i t} I_n \in \mathrm{SU}(2n)$,

$$S_t(n) = r\left(e^{\pi i t} I_n\right) = \begin{pmatrix} \cos \pi t I_n & -\sin \pi t I_n \\ \sin \pi t I_n & \cos \pi t I_n \end{pmatrix} \in \mathrm{SO}(2n).$$

DEFINITION. The three maps defined below are called *Bott maps*.

(1) $\beta(n) : \mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n) \to \Omega\mathrm{SU}(2n)$ defined by
$\beta(n)\{A\}(t) = AR_t(n)A^{-1}R_t(n)^{-1}$, $A \in \mathrm{U}(n)$;

(2) $\beta_{\mathrm{Sp}}(n) : \mathrm{Sp}(n)/\mathrm{U}(n) \to \Omega\mathrm{Sp}(n)$ defined by
$\beta_{\mathrm{Sp}}(n)\{A\}(t) = A(e^{\pi i t} I_n)A^{-1}(e^{\pi i t} I_n)^{-1}$, $A \in \mathrm{Sp}(n)$;

(3) $\beta_{\mathrm{O}}(n) : \mathrm{O}(2n)/\mathrm{U}(n) \to \Omega\mathrm{SO}(2n)$ defined by
$\beta_{\mathrm{O}}(n)\{A\}(t) = AS_t(n)A^{-1}S_t(n)^{-1}$, $A \in \mathrm{O}(2n)$.

They induce Hopf maps $\beta : B\mathbf{U} \to \Omega\mathbf{SU}$, $\beta_{\mathrm{Sp}} : \mathbf{Sp}/\mathbf{U} \to \Omega\mathbf{Sp}$, $\beta_{\mathrm{O}} : \mathbf{O}/\mathbf{U} \to \Omega\mathbf{SO}$.

THEOREM 3.1. *β, β_{Sp}, β_{O} are (weak) homotopy equivalences.*

DEFINITION. The four maps defined below are called *Bott maps*.

(1) $\beta_{\mathrm{O/U}}(n) : \mathrm{U}(2n)/\mathrm{Sp}(n) \to \Omega(\mathrm{SO}(4n)/\mathrm{U}(2n))$ defined by
$\beta_{\mathrm{O/U}}(n)\{A\}(t) = AT_t(n)A^{-1}T_t(n)^{-1}$, $A \in \mathrm{U}(2n)$;

(2) $\beta_{\mathrm{U/O}}(n) : \mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n) \to \Omega(\mathrm{SU}(2n)/\mathrm{SO}(2n))$ defined by
$\beta_{\mathrm{U/O}}(n)\{A\}(t) = AR_t(n)A^{-1}R_t(n)^{-1}$, $A \in \mathrm{O}(2n)$;

(3) $\beta_{\mathrm{Sp/U}}(n) = \mathrm{U}(n)/\mathrm{SO}(n) \to \Omega(\mathrm{Sp}(n)/\mathrm{U}(n))$ defined by
$\beta_{\mathrm{Sp/U}}(n)\{A\}(t) = A(e^{\pi j t/2} I_n)A^{-1}(e^{-\pi j t/2} I_n)$, $A \in \mathrm{U}(n)$;

(4) $\beta_{\mathrm{U/Sp}}(n) : \mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n) \to \Omega(\mathrm{SU}(4n)/\mathrm{Sp}(2n))$ defined by
$\beta_{\mathrm{U/Sp}}(n)\{A\}(t) = AR_t^2(n)A^{-1}R_t^2(n)^{-1}$, $A \in \mathrm{Sp}(2n)$,

where $T_t(n) = S_{t/2}(n) \oplus S_{t/2}(n)$ and $R_t^2(n) = R_t(n) \oplus R_t(n)$.

They induce the following Hopf maps:

$$\beta_{\mathrm{O/U}} : \mathbf{U}/\mathbf{Sp} \to \Omega(\mathbf{SO}/\mathbf{U}), \quad \beta_{\mathrm{U/O}} : B\mathbf{O} \to \Omega(\mathbf{SU}/\mathbf{SO}),$$

$$\beta_{\mathbf{Sp}/\mathbf{U}} : \mathbf{U}/\mathbf{SO} \to \Omega(\mathbf{Sp}/\mathbf{U}), \quad \beta_{\mathbf{U}/\mathbf{Sp}} : B\mathbf{Sp} \to \Omega(\mathbf{SU}/\mathbf{Sp}).$$

THEOREM 3.2. $\beta_{\mathbf{O}/\mathbf{U}}, \beta_{\mathbf{U}/\mathbf{O}}, \beta_{\mathbf{Sp}/\mathbf{U}}, \beta_{\mathbf{U}/\mathbf{Sp}}$ *are (weak) homotopy equivalences.*

COROLLARY 3.3 (Bott periodicity). $B\mathbf{O} \simeq \Omega^7\mathbf{SO}$, $B\mathbf{Sp} \simeq \Omega^7\mathbf{Sp}$.

From the fibrations $\mathrm{G}(n+1)/\mathrm{G}(n) = S^{d(n+1)-1}$ one has

$$\begin{aligned} \pi_k(\mathbf{Sp}) &= \pi_k(\mathrm{Sp}(n)) \quad \text{for } n \geqslant (k-1)/4; \\ \pi_k(\mathbf{U}) &= \pi_k(\mathrm{U}(n)) \quad \text{for } n \geqslant (k+1)/2; \\ \pi_k(\mathbf{O}) &= \pi_k(\mathrm{O}(n)) \quad \text{for } n \geqslant k+2. \end{aligned}$$

Their values are given as follows:

$$\pi_k(\mathbf{Sp}) \cong \begin{cases} \mathbb{Z} & (k \equiv 3, 7 \bmod 8), \\ \mathbb{Z}_2 & (k \equiv 4, 5 \bmod 8), \\ 0 & (k \equiv 0, 1, 2, 6 \bmod 8), \end{cases}$$

$$\pi_k(\mathbf{O}) \cong \begin{cases} \mathbb{Z} & (k \equiv 3, 7 \bmod 8), \\ \mathbb{Z}_2 & (k \equiv 0, 1 \bmod 8), \\ 0 & (k \equiv 2, 4, 5, 6 \bmod 8), \end{cases}$$

$$\pi_k(\mathbf{U}) \cong \begin{cases} \mathbb{Z} & (k \equiv 1 \bmod 2), \\ 0 & (k \equiv 0 \bmod 2). \end{cases}$$

The original proof of the periodicity by Bott made use of Morse theory. The proof using homotopy and cohomology groups was first given by Toda [249] for **SU** and then by Dyer and Lashof [68] for **O** and **Sp**.

In fact, Toda discussed as follows. The CW-complex $X = \mathbf{SU}$ has the following properties:

(1) it is simply connected,

(2) it is a homotopy associative Hopf space,

(3) its integral cohomology ring is an exterior algebra $\Lambda(e_1, e_2, \ldots)$, $e_i \in H^{2i+1}$,

(4) there is a map $f : S\mathbb{C}P^\infty \to X$ such that the induced homomorphism f^* of integral cohomology is epimorphic.

Then he proved the following

THEOREM 3.4. *If a space X satisfies* (1) $\sim$ (4), *then so does $X' = \Omega((\Omega X, 3))$, where $(\Omega X, 3)$ is a 2-connective fibre space over ΩX.*

As an immediate corollary he obtained

COROLLARY 3.5 (Borel–Hirzebruch). $\pi_{2n}(\mathrm{SU}(n)) \cong \mathbb{Z}_{n!}$ *for* $n \geqslant 2$.

3.2. *Unstable homotopy groups*

The only results which can be proved without appealing to the classification are Theorems 3.6–3.10 below.

THEOREM 3.6 (Weyl). *The fundamental group $\pi_1(G)$ is a finite abelian group for G compact, connected, and semi-simple.*

THEOREM 3.7 (Cartan). $\pi_2(G) = 0$.

This follows also from the following

THEOREM 3.8 (Bott [41]). *The integral cohomology of ΩG has no torsion for simply connected G.*

THEOREM 3.9 (Bott). $\pi_3(G) = \mathbb{Z}$ *for G compact, connected, simple, and non-abelian.*

THEOREM 3.10 (Bott and Samelson [47]). *Let G be a compact, connected, simply connected, simple Lie group, T a maximal torus of G, $\mathbb{R}^r$ its universal covering and Γ the inverse image of the identity of T in $\mathbb{R}^r$. Let a be the dominant root with respect to some lexicographic order of the roots of G. Then*

$$\pi_4(G) \cong \begin{cases} \mathbb{Z} & \textit{if the hyperplane } a = 1 \textit{ contains a point of } \Gamma, \\ 0 & \textit{if the hyperplane } a = 1 \textit{ contains no point of } \Gamma. \end{cases}$$

The higher homotopy groups of G can be obtained by appealing to the classification and using the homotopy exact sequence associated with the appropriate bundles involving G.

We list some of the results:

$$\pi_1(G) = \begin{cases} \mathbb{Z} & (G = \mathrm{U}(n)\ \textit{for}\ n \geqslant 1, \mathrm{SO}(2)), \\ \mathbb{Z}_2 & (G = \mathrm{SO}(n)\ \textit{for}\ n \geqslant 3), \\ 0 & (\textit{the other } G), \end{cases}$$

$$\pi_2(G) = 0,$$

$$\pi_3(G) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & (G = \mathrm{SO}(4)), \\ \mathbb{Z} & (G \neq \mathrm{SO}(4)), \end{cases}$$

$$\pi_4(G) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & (G = \mathrm{SO}(4), \mathrm{Spin}(4)), \\ \mathbb{Z}_2 & (G = \mathrm{Sp}(n), \mathrm{SU}(2), \mathrm{SO}(3), \mathrm{SO}(5), \mathrm{Spin}(3), \mathrm{Spin}(5)), \\ 0 & (G = \mathrm{SU}(n)\ \textit{for}\ n \geqslant 3, \mathrm{SO}(n)\ \textit{for}\ n \geqslant 6), \\ 0 & (G = \mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8), \end{cases}$$

$$\pi_5(G) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & (G = \mathrm{SO}(4), \mathrm{Spin}(4)), \\ \mathbb{Z}_2 & (G = \mathrm{Sp}(n), SU(2), \mathrm{SO}(3), \mathrm{SO}(5), \mathrm{Spin}(3), \mathrm{Spin}(5)), \\ \mathbb{Z} & (G = \mathrm{SU}(n)\, \textit{for}\ n \geqslant 3, \mathrm{SO}(6), \mathrm{Spin}(6)), \\ 0 & (G = \mathrm{SO}(n), \mathrm{Spin}(n)\, \textit{for}\ n \geqslant 7, \mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8). \end{cases}$$

$\pi_k(G)$ for $k \geqslant 6$

$G\backslash k$	6	7	8	9	10	11	12	13	14	15
Sp(1)	12	2	2	3	15	2	2^2	$12+3$	$84+2^2$	2^2
Sp(2)	0	∞	0	0	120	2	2^2	$4+2$	1680	2
Sp(3)	0	∞	0	0	0	∞	2	2	10080	2
Sp(4)	0	∞	0	0	0	∞	2	2	0	∞
SU(2)	12	2	2	3	15	2	2^2	$12+3$	$84+2^2$	2^2
SU(3)	6	0	12	3	30	4	60	6	$84+2$	36
SU(4)	0	∞	24	2	$120+2$	4	60	4	$1680+2$	$72+2$
SU(5)	0	∞	0	∞	120	0	360	4	1680	6
SU(6)	0	∞	0	∞	0	∞	720	2	$5040+2$	6
SU(7)	0	∞	0	∞	0	∞	0	∞	5040	0
SU(8)	0	∞	0	∞	0	∞	0	∞	0	∞
SO(5)	0	∞	0	0	120	2	2^2	$4+2$	1680	2
SO(6)	0	∞	24	2	$120+2$	4	60	4	$1680+2$	$72+2$
SO(7)	0	∞	2^2	2^2	8	$\infty+2$	0	2	$2520+8+2$	2^4
SO(8)	0	$\infty+\infty$	2^3	2^3	$24+8$	$\infty+2$	0	2^2	$2520+120+8+2$	2^7
SO(9)	0	∞	2^2	2^2	8	$\infty+2$	0	2	$8+2$	$\infty+2^3$
SO(10)	0	∞	2	$\infty+2$	4	∞	12	2	8	$\infty+2^2$
SO(11)	0	∞	2	2	2	∞	2	2^2	8	$\infty+2$
SO(12)	0	∞	2	2	0	$\infty+\infty$	2^2	2^2	$24+4$	$\infty+2$
SO(13)	0	∞	2	2	0	∞	2	2	8	$\infty+2$
SO(14)	0	∞	2	2	0	∞	0	∞	4	∞
SO(15)	0	∞	2	2	0	∞	0	0	2	∞
SO(16)	0	∞	2	2	0	∞	0	0	0	$\infty+\infty$
SO(17)	0	∞	2	2	0	∞	0	0	0	∞
G_2	3	0	2	6	0	$\infty+2$	0	0	$168+2$	2
F_4	0	0	2	2	0	$\infty+2$	0	0	2	∞
E_6	0	0	2	∞	0	∞	12	0	0	∞
E_7	0	0	0	0	0	∞	2	2	0	∞
E_8	0	0	0	0	0	0	0	0	0	∞

Here ∞ denotes $\mathbb{Z}$, the integer n means $\mathbb{Z}_n$, and 2^n means $2 + \cdots + 2$ (n times).

For the metastable case one has ([161], [163], [174], [249]):

$$\pi_{2n}(\mathrm{SU}(n)) \cong n!.$$

$$\pi_{2n+1}(\mathrm{SU}(n)) \cong \begin{cases} 2 & (n \text{ is even}), \\ 0 & (n \text{ is odd}). \end{cases}$$

$$\pi_{2n+2}(\mathrm{SU}(n)) \cong \begin{cases} (n+1)!+2 & (n \text{ is even }, n \geqslant 4), \\ (n+1)!/2 & (n \text{ is odd}). \end{cases}$$

$$\pi_{2n+3}(\mathrm{SU}(n)) \cong \begin{cases} (24,n) & (n \text{ is even}), \\ (24,n+3)/2 & (n \text{ is odd}). \end{cases}$$

$$\pi_{2n+4}(\mathrm{SU}(n)) \cong \begin{cases} (n+2)!(24,n)/48 & (n \text{ is even}, n \geqslant 4), \\ (n+2)!(24,n+3)/24 & (n \text{ is odd}). \end{cases}$$

$$\pi_{2n+5}(\mathrm{SU}(n)) \cong \pi_{2n+5}(\mathrm{U}(n+1)).$$

$$\pi_{2n+6}(\mathrm{SU}(n)) \cong \begin{cases} \pi_{2n+6}(\mathrm{U}(n+1)) & (n \equiv 2,3 \bmod 4, n \geqslant 3), \\ \pi_{2n+6}(\mathrm{U}(n+1))+2 & (n \equiv 0,1 \bmod 4). \end{cases}$$

$$\pi_{4n+2}(\mathrm{Sp}(n)) \cong \begin{cases} (2n+1)! & (n \text{ is even}), \\ 2(2n+1)! & (n \text{ is odd}). \end{cases}$$

$$\pi_{4n+3}(\mathrm{Sp}(n)) \cong 2.$$

$$\pi_{4n+4}(\mathrm{Sp}(n)) \cong \begin{cases} 2+2 & (n \text{ is even}), \\ 2 & (n \text{ is odd}). \end{cases}$$

$$\pi_{4n+5}(\mathrm{Sp}(n)) \cong \begin{cases} (24,n+2)+2 & (n \quad \text{is even}), \\ (24,n+2) & (n \quad \text{is odd}). \end{cases}$$

$$\pi_{4n+6}(\mathrm{Sp}(n)) \cong \begin{cases} (2n+3)!(24,n+2)/12 & (n \text{ is even}), \\ (2n+3)!(24,n+2)/24 & (n \text{ is odd}). \end{cases}$$

$$\pi_{4n+7}(\mathrm{Sp}(n)) \cong 2.$$

$$\pi_{4n+8}(\mathrm{Sp}(n)) \cong 2+2.$$

For $\pi_{n+i}(\mathrm{SO}(n))$ we use the following isomorphism for $n \geqslant 16$ and $3 \geqslant i \geqslant -1$ due to Barratt and Mahowald [21]:

$$\pi_{n+i}(\mathrm{SO}(n)) \cong \pi_{n+i}(\mathrm{O}) \oplus \pi_{n+i+1}(V_{i+3+n,i+3}(\mathbb{R})).$$

For further results on the unstable homotopy groups of G we refer to [95], [105], [106], [123], [130], [155], [175], [178], [192], [199], [200], [207].

4. Localization and mod p decomposition

Localization is a very strong tool in homotopy theory. In particular, it is quite effective in pursuing the problem of the mod p decomposability of simple Lie groups.

To some extent this problem has been apparent since Serre introduced the class $\mathcal{C}$ theory of abelian groups. For example, Hopf's Theorem 2.1 can be interpreted as

$$G \underset{p}{\simeq} \prod_i S^{2n_i - 1} \tag{4.1}$$

where $p = 0$, that is, G is rationally equivalent to the product of odd dimensional spheres. When G is of low rank, Serre [232] (for classical groups) and Kumpel [148] (for exceptional groups) proved that it is valid for p a prime. Then Harris [82], [83] and Kumpel [149] gave mod p decompositions of Lie groups of a somewhat different type. Mimura and Toda [180] and Oka [204] obtained mod p decompositions of Lie groups of moderate rank. But the ultimate result was Nishida's mod p decompositions of $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ ([196] and [184]). He constructed spaces which are components in a mod p decomposition of $\mathrm{U}(n)$ by making use of two maps, one a loop product of $\Omega B\mathrm{U}(n) = \mathrm{U}(n)$ and the other an unstable Adams operation $\psi^q : B\mathrm{U}(n) \to B\mathrm{U}(n)$ defined by Sullivan [244]. Using them he decomposed $\mathrm{U}(n)$ into the product of $p-1$ spaces in the mod p sense. The corresponding results for exceptional Lie groups, when they are p-torsion free, were obtained by Mimura and Toda [184] by an *ad-hoc* method using obstruction theory based on a hard calculation of homotopy groups. Wilkerson [267] gave a universal result which extends Nishida's method so that it includes the results of Mimura and Toda. By making use of algebraic geometry he constructed a map $\psi^q : BG \to BG$ for any compact connected semi-simple Lie group G and proved the following

THEOREM 4.1 (Wilkerson [267]). *If G is a compact connected semi-simple Lie group, there exists an "unstable" Adams operation $\psi^p : BG_{P-p} \to BG_{P-p}$ with the property that $\psi^{p*} | H^{2n}(BG_{P-p}; \mathbb{Q}) = p^n \cdot \mathrm{Id}$. Here BG_{P-p} denotes the localization of BG away from the prime p.*

COROLLARY 4.2. *If $W(G)$ is the Weyl group of G and p does not divide the order of $W(G)$, then there exists $\psi^p : BG \to BG$ with the above property.*

The following is a generalization of Nishida's result:

THEOREM 4.3. *Let G be a finite Hopf space. Suppose that there exists a map $\Phi : G \to G$ such that $\Phi^* | QH^{2n-1}(G; \mathbb{Q}) = q^n \cdot \mathrm{Id}$ for all $n > 0$.*

If q is a primitive $(p-1)$-st root of unity mod p, *and $H_*(G; \mathbb{Z})$ has no p-torsion, then G is p-equivalent to $\prod_i X_i(G)$ where the type $(2i_1 - 1, \ldots, 2i_j - 1)$ of $X_i(G)$ has the* *property that* $i_1 \equiv \cdots \equiv i_j \equiv i \bmod (p-1)$ *and the product is taken over all residue classes* $\bmod\ (p-1)$.

For a Lie group G, one can take $\Phi = \Omega\psi^q$ by Theorem 4.1 and so one has

COROLLARY 4.4. *Let G be a compact connected simple Lie group such that $H_*(G;\mathbb{Z})$ has no p-torsion. Then G is p-equivalent to a product of Hopf spaces $X_i(G)$ where each $X_i(G)$ is indecomposable* mod p *and the type* $(2i_1-1,\ldots,2i_j-1)$ *of* $X_i(G)$ *has the property that* $i_1 \equiv \cdots \equiv i_j \equiv i \bmod (p-1)$.

EXAMPLE. The exceptional Lie groups when localized at p split as indicated below.

G_2	$p=3$	$B_2(3,11)$,
	$p=5$	$B(3,11)$,
	$p>5$	$S^3 \times S^{11}$.
F_4	$p=5$	$B(3,11)\times B(15,23)$,
	$p=7$	$B(3,15)\times B(11,23)$,
	$p=11$	$B(3,23)\times S^{11}\times S^{15}$,
	$p>11$	$S^3\times S^{11}\times S^{15}\times S^{23}$.
E_6	$p=5$	$F_4\times B(9,17)$,
	$p>5$	$F_4\times S^9\times S^{17}$.
E_7	$p=5$	$B(3,11,19,27,35)\times B(15,23)$,
	$p=7$	$B(3,15,27)\times B(11,23,35)\times S^{19}$,
	$p=11$	$B(3,23)\times B(15,35)\times S^{11}\times S^{19}\times S^{27}$,
	$p=13$	$B(3,27)\times B(11,35)\times S^{15}\times S^{19}\times S^{23}$,
	$p=17$	$B(3,35)\times S^{11}\times S^{15}\times S^{19}\times S^{23}\times S^{27}$,
	$p>17$	$S^3\times S^{11}\times S^{15}\times S^{19}\times S^{23}\times S^{27}\times S^{35}$.
E_8	$p=7$	$B(3,15,27,39)\times B(23,35,47,59)$,
	$p=11$	$B(3,23)\times B(15,35)\times B(27,47)\times B(39,59)$,
	$p=13$	$B(3,27)\times B(15,39)\times B(23,47)\times B(35,59)$,
	$p=17$	$B(3,35)\times B(15,47)\times B(27,59)\times S^{23}\times S^{39}$,
	$p=19$	$B(3,39)\times B(23,59)\times S^{15}\times S^{27}\times S^{35}\times S^{47}$,
	$p=23$	$B(3,47)\times B(15,59)\times S^{23}\times S^{27}\times S^{35}\times S^{39}$,
	$p=29$	$B(3,59)\times S^{15}\times S^{23}\times S^{27}\times S^{35}\times S^{39}\times S^{47}$,
	$p>29$	$S^3\times S^{15}\times S^{23}\times S^{27}\times S^{35}\times S^{39}\times S^{47}\times S^{59}$.

Here each space $B(2n_1+1,\ldots,2n_r+1)$ is built up from fibrations involving p-local spheres of the indicated dimensions, and is equivalent to a direct factor of the p-localization of $\mathrm{SU}(n_r+1)/\mathrm{SU}(n_1)$. The attaching map of the middle cell of $B(2n+1,2n+2p-1)$ is $\alpha_1 \in \pi_{2n+2p-2}(S^{2n+1})$. The only factor not of this type, labeled $B_2(3,11)$, is a sphere bundle with attaching map α_2, and is not a direct factor in the 3-localization of a quotient of SU's. Here the α_t are the standard elements of order p in $\pi_{2(p-1)t-1}(S^0)$.

One advantage of Wilkerson's argument is that Theorem 4.3 is still applicable, even when $H_*(G;\mathbb{Z})$ has p-torsion, if it can be verified that

$$H^*(G;\mathbb{Z}_p) \cong \bigotimes_i H^*(X_i(G);\mathbb{Z}_p).$$

He obtained

PROPOSITION 4.5. (1) $\mathrm{E}_6 \underset{3}{\simeq} X_1(\mathrm{E}_6) \times X_2(\mathrm{E}_6)$,

where $H^*(X_1(\mathrm{E}_6);\mathbb{Z}_3) = \Lambda(x_9, x_{17})$,

$H^*(X_2(\mathrm{E}_6);\mathbb{Z}_3) = \mathbb{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15})$.

(2) $\mathrm{E}_8 \underset{5}{\simeq} X_0(\mathrm{E}_8) \times X_2(\mathrm{E}_8)$,

where $H^*(X_0(\mathrm{E}_8);\mathbb{Z}_5) = \Lambda(x_{15}, x_{23}, x_{39}, x_{47})$,

$H^*(X_2(\mathrm{E}_8);\mathbb{Z}_5) = \mathbb{Z}_5[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{27}, x_{35})$.

REMARK. (1) Statement (1) was previously obtained by Harris [83], who showed

$$\mathrm{E}_6 \underset{3}{\simeq} \mathrm{F}_4 \times \mathrm{E}_6/\mathrm{F}_4.$$

(2) Gonçalves [79] showed that $X_0(\mathrm{E}_8)$ is indecomposable.

5. Homotopy commutativity, normality and nilpotency

When X is a finite homotopy associative Hopf space, say a topological group, the functor $[-, X]$ takes its values in the category of groups. So we are interested in when this functor takes its values in various subcategories of groups.

EXAMPLE. X is homotopy commutative if and only if $[Y, X]$ is abelian for any Y.

5.1. *Homotopy commutativity*

Let G be a topological group and H a subgroup of G.

DEFINITION. H is *homotopy commutative* in G if $f \simeq f' : H \times H \to G$, where f and f' are defined by

$$f(x,y) = xy = f'(y,x), \quad x, y \in H.$$

This is the case when G is pathwise connected and H is conjugate to a subgroup whose elements commute with those of H.

When considering the standard embeddings $G(n) \subset G(m)$, $m > n$, where G = SO for $n \geqslant 3$, G = U for $n \geqslant 2$, G = Sp for $n \geqslant 1$, the elements of $G(n)$ commute with those of appropriate conjugate subgroups and hence $G(n)$ is homotopy commutative in $G(2n)$.

On the other hand, James and Thomas [118] showed

(5.1) $G(n)$ *is not homotopy commutative in* $G(2n-1)$ *for* G = U *and* Sp.

The analogous statement is not true for G = SO, since SO(4) is homotopy commutative in SO(7). However they showed (see also [53])

(5.2) $SO(n)$ *is not homotopy commutative in* $SO(2n-r)$, *where* $r = 2$ *for* n *odd and* $r = 4$ *for* n *even.*

Furthermore they showed

(5.3) *There exists no classical Lie group which is homotopy commutative but not commutative.*

DEFINITION. A topological group is said to be *homotopy commutative* if it is homotopy commutative in itself.

EXAMPLE. There exist homotopy commutative groups, such as the stable (infinite dimensional) classical groups, which are not commutative.

The question of the homotopy commutativity of a compact Lie group has a long history. First, Araki, James and Thomas ([15]) showed

(5.4) *A compact connected Lie group is homotopy commutative only if it is commutative.*

(5.5) *If a Lie group is homotopy commutative, then its maximal compact connected subgroup is commutative.*

On the other hand, Browder ([53]) showed

(5.6) *A Lie group which has 2-torsion in its homology (such as* $SO(n)$, $Spin(n)$, *the exceptional groups and all projective groups except* $PU(n)$, n *odd) does not admit a homotopy commutative Hopf structure.*

Then James and Thomas ([121]) showed

(5.7) *Let G be a countable connected CW-complex with finitely generated total singular homology. If G is a homotopy commutative topological group, then G has the homotopy type of a torus. In particular, if G is simply connected, as well as homotopy commutative, then G is contractible.*

On the other hand James ([112]) obtained

(5.8) *The underlying space of a compact connected Lie group cannot support a homotopy commutative Hopf structures unless it is a torus.*

However a stronger result is proved by Hubbuck:

THEOREM 5.9 ([100]). *Let X be a noncontractible, connected finite complex which is a homotopy commutative Hopf space. Then X has the homotopy type of a torus.*

5.2. *Homotopy normality*

Let H be a subgroup of a topological group G. Consider the commutator map

$$c : G \wedge H \to G \Leftrightarrow c(g,h) = ghg^{-1}h^{-1},$$

where $G \wedge H = G \times H / G \vee H$. The condition for H to be normal in G is that the image of c lies in H. So we make the following definition:

DEFINITION (James). H is said to be *homotopy normal* in G if c can be deformed into H ($\Leftrightarrow$ there exists a map $f_t : G \wedge H \to G$ such that $f_0 = c$ and $f_1(G \wedge H) \subset H$).

An alternative definition is:

DEFINITION (McCarty). H is said to be *homotopy normal* in G if there exists a homotopy $f_t : (G \times H, H \times H) \to (G, H)$ such that $f_0(g,h) = ghg^{-1}$ for $g \in G$, $h \in H$ and $f_1(G \times H) \subset H$.

REMARK. If H is homotopy normal in the sense of McCarty, then H is homotopy normal in the sense of James, that is, if the subgroup H of G is not homotopy normal in the sense of James, then H cannot be homotopy normal in the sense of McCarty.

EXAMPLE. S^1 is homotopy normal in S^3 in the sense of James but not in the sense of McCarty.

Obviously, if G is homotopy commutative, then every subgroup is homotopy normal in both senses. So the stable classical groups **O**, **U**, **Sp** contain examples of subgroups which are homotopy normal but not normal.

If every inner automorphism of G is homotopic to the identity, then every finite subgroup of G is homotopy normal.

EXAMPLE. $G = \mathrm{O}(n)$, n: odd.

As is easily seen, $\mathrm{SU}(n)$ is homotopy normal in $\mathrm{U}(n)$ for $n \geqslant 2$. In particular, $\mathrm{Sp}(1) = \mathrm{SU}(2)$ is homotopy normal in $\mathrm{U}(2)$.

NOTATION. $\mathrm{G}(n) = \mathrm{O}(n), \mathrm{U}(n), \mathrm{Sp}(n)$.

Consider the standard inclusions: $\mathrm{G}(n) \subset \mathrm{G}(n+1) \subset \cdots$.

(5.10) ([113]) *Let $n \geqslant 1$ and $r \geqslant 1$. Exclude the real orthogonal case when $n = 1$ and r is even. Then $\mathrm{G}(n)$ is not homotopy normal in $\mathrm{G}(n+r)$ in the sense of James.*

The exclusion is necessary since $\mathrm{O}(1)$ is a finite subgroup of $\mathrm{O}(n+1)$.

James' method of proving this yields a similar result for $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{Spin}(n)$.

(5.11) ([116]) (1) *If $n = 2$ or $n \geqslant 4$, then $\mathrm{U}(n)$ is not homotopy normal in $\mathrm{SO}(2n)$ in the sense of McCarty.*

(2) *If* $n \geqslant 2$, *then* $\mathrm{Sp}(n)$ *is not homotopy normal in* $\mathrm{U}(2n)$ *in the sense of McCarty.*

(5.12) ([77]) $\mathrm{U}(3)$ *is not homotopy normal in* $\mathrm{SO}(6)$ *in the sense of James.*

(5.13) ([166]) $\mathrm{G}(n)$ *is not homotopy normal in* $\mathrm{G}(n+1)$ *in the sense of McCarty, if* $n \geqslant 2$ *for* $\mathrm{G} = \mathrm{O}, \mathrm{Sp}$ *and if* $n = 2$ *or* $n \geqslant 4$ *for* $\mathrm{G} = \mathrm{U}$.

(5.14) ([125]) *If* $n \geqslant 2$, *then* $\mathrm{U}(n)$ *is not homotopy normal in* $\mathrm{Sp}(n)$ *in the sense of McCarty.*

Consider the following chain of compact, 1-connected, simple Lie groups:

$$\mathrm{SU}(3) \subset \mathrm{G}_2 \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9) \subset \mathrm{F}_4 \subset \mathrm{E}_6 \subset \mathrm{E}_7 \subset \mathrm{E}_8.$$

(5.15) ([77]; see also [53]) *Let* G, H *be any subgroups in this chain, with* $G \supset H$. *Then* H *is not homotopy normal in* G *in the sense of McCarty.*

5.3. *Homotopy nilpotency*

Now we define the analogous notions of homotopy nilpotency and homotopy solvability of a finite homotopy associative Hopf space X in the obvious way. These properties can also be expressed in terms of the structure map of X. Let μ and σ be the Hopf structure (the multiplication) and the inverse map of X respectively.

DEFINITION. The *commutator map* c_2 is defined by the composite $(\mu(\mu \times \mu))(1_X \times 1_X \times \sigma \times \sigma)(\Delta_{X \times X})$: $X \times X \longrightarrow X \times X \times X \times X \longrightarrow X \times X \times X \times X \longrightarrow X$.

The *iterated commutator maps* $s_n : X^{2^n} \to X$ and $c_n : X \to X$ are defined inductively by $s_n = c_2(s_{n-1} \times s_{n-1})$ and $c_n = c_2(c_{n-1} \times 1_X)$ respectively.

Then Zabrodsky showed

THEOREM 5.16 ([280], Lemma 2.6.1). *A finite homotopy associative Hopf space* X *is*
(1) *homotopy solvable if and only if* s_n *is null homotopic for sufficiently large* n;
(2) *homotopy nilpotent if and only if* c_n *is null homotopic for sufficiently large* n.

He also showed that the classical Lie groups $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n+1)$ are homotopy solvable.
Recently Hopkins gave cohomological criteria:

THEOREM 5.17 ([98]; see also [222]). *Let* X *be a finite homotopy associative Hopf space. Then the following conditions are equivalent:*
(1) X *is homotopy nilpotent;*
(2) $\widetilde{MU}^* c_n = 0$ *for sufficiently large* n;
(3) *For every prime* p, $\widetilde{BP}^* c_n = 0$ *for sufficiently large* n;
(4) *For every prime* p *and positive integer* ℓ, $\widetilde{K(\ell)}_*(c_n) = 0$ *for sufficiently large* n. *Here* $K(\ell)$ *is the* ℓ*-th Morava* K*-theory at the prime* p.

Using this he showed that torsion free homotopy associative Hopf spaces are homotopy nilpotent; for example $U(n)$ and $Sp(n)$ are homotopy nilpotent.

He even conjectured that all finite connected homotopy associative Hopf spaces are homotopy nilpotent. But Rao gave counter-examples by proving

THEOREM 5.18 ([222]). $SO(n)$ *and* $Spin(n)$ *are not homotopy nilpotent if* $n \geqslant 7$. ($SO(3)$ *and* $SO(4)$ *are not homotopy nilpotent.*)

This is proved by showing the iterated commutator maps are nontrivial in a suitable Morava K-theory.

At the same time Yagita showed

THEOREM 5.19 ([275]). *Let G be a simply connected Lie group. Then for each prime p, the p-localization $G_{(p)}$ is homotopy nilpotent if and only if $H^*(G;\mathbb{Z})$ has no p-torsion.*

The proof consists of case by case analysis.

6. Lusternik–Schnirelmann category

Let X be a topological space and A a subspace of X.

DEFINITION. The *relative category* of A in X, denoted $\mathrm{cat}_X A$, is the smallest number n such that A can be covered by n open subsets each of which is contractible in X. If $X = A$, we simply denote $\mathrm{cat}\, X = \mathrm{cat}_X A$. This is the so called *Lusternik–Schnirelmann* category of X.

THEOREM 6.1. *When X is a compact differentiable manifold and f is a smooth real valued function X, we have $\sharp\{$critical points of $f\} \geqslant \mathrm{cat}\, X$.*

More precisely, let A_k be the family of all subsets A of X such that $\mathrm{cat}_X A \geqslant k$. Then $\inf_{A\in A_k} \sup_{x\in A} f(x)$ is a critical value of f, if A_k is not empty.

To enumerate $\mathrm{cat}\, G$ for a Lie group G, we introduce a notion of the cup-length.

DEFINITION. The *cup-length* of X is the largest number n such that there are cohomology classes $x_i \in \tilde{H}^*(X;R)$ satisfying $x_1 \cdots x_n \neq 0$, where R is some coefficient ring.

Then the following is due to Berstein and Ganea:

PROPOSITION 6.2. *If n is the cup-length, then* $\mathrm{cat}\, X \geqslant n+1$.

Of course the cohomology structure of a compact Lie group G gives the lower bound of $\mathrm{cat}\, G$; for example, $\mathrm{cat}\, SU(n) \geqslant n$. In particular we immediately see that $\mathrm{cat}\, SU(2) = 2$, since $SU(2) = S^3$.

By making use of secondary cohomology operations induced by the diagonal mapping, Schweitzer showed that $\mathrm{cat}\, Sp(2) \geqslant 4$ and hence

THEOREM 6.3 ([231]). $\mathrm{cat}\, Sp(2) = 4$.

Then Singhof showed cat $\mathrm{SU}(n) \leqslant n$ and hence

THEOREM 6.4 ([234]). cat $\mathrm{SU}(n) = n$ *and* cat $\mathrm{U}(n) = n + 1$.

In fact, he constructs n open contractible subsets A_i covering $\mathrm{SU}(n)$, namely

$$A_i = \{X \in \mathrm{SU}(n) | \text{ no eigenvalues of } X \text{ is } \xi_i\},$$

where ξ_i, $1 \leqslant i \leqslant n$, are different complex numbers with absolute value 1 such that $\xi_1\xi_2 \cdots \xi_n \neq 1$.

He also showed the following by using Schweitzer's method.

THEOREM 6.5 ([234]). cat $\mathrm{Sp}(n) \geqslant n + 2$.

7. The number of multiplications

The underlying space of a compact Lie group has many multiplications (Hopf structures). In this section we will discuss such a multiplication.

First we recall a definition due to Zassenhaus:

DEFINITION. A group A is said to be *of finite rank* if there exists a chain of normal subgroups $N_i : A = N_0 \supset \cdots \supset N_i \supset N_{i+1} \supset \cdots \supset N_k = 1$, for which each factor N_i/N_{i+1} is either a periodic group or an infinite cyclic group.

Note that the number $r(A)$ of infinite cyclic factor groups N_i/N_{i+1} is an invariant of the group A.

DEFINITION. $r(A)$ = the *rank* of A.

When A is an abelian group, $r(A)$ is the rank of A in the usual sense.

It follows immediately from the definition of rank that $r(A) = 0$ if and only if A is periodic ($\Leftrightarrow$ every element has finite order).

NOTATION. $\beta_n(X)$ is the n-th Betti number of the space X. $\gamma_n(X)$ is the rank of the homotopy group $\pi_n(X)$.

Consider the group $[SY, X]$. The following proposition due to Arkowitz and Curjel [17] determines the rank of $[SY, X]$ in terms of the n-th Betti number of SY and the rank $\gamma_n(X)$ of $\pi_n(X)$.

PROPOSITION 7.1. *Let Y be a* 1*-connected finite complex. Then*

$$\rho([SY, X]) = \sum_n \beta_n(SY)\gamma_n(X).$$

Let X be a homotopy associative Hopf space. Then the following lemma is a key result which provides a link between Proposition 7.1 and the number of multiplications of a Hopf space.

LEMMA 7.2. *The homotopy set* $[X \wedge X, X]$ *is in one to one correspondence with the set* $M(X)$ *of homotopy classes of multiplications of* X.

Further we need

DEFINITION. A positive integer n is called a *cup number* relative to a sequence $(n_1, \dots, n_q)$ of positive integers if $H^n(K(\mathbb{Z}, n_1) \times \cdots \times K(\mathbb{Z}, n_q); \mathbb{Q})$ contains a non-trivial cup product of two positive dimensional elements.

Now suppose that X is an associative finite Hopf space such that

$$H^*(X; \mathbb{Q}) \cong \Lambda(x_1, \dots, x_q), \quad |x_i| = n_i.$$

Then we have

THEOREM 7.3 (Arkowitz and Curjel [17]). *X has an infinite number of nonhomotopic multiplications if and only if some n_j is a cup number relative to*

$$(n_1, \dots, n_q, n_1, \dots, n_q).$$

The proof is as follows. First we have

$$[X \wedge X, X] \cong [X \wedge X, \Omega BX] \cong [SX \wedge X, BX],$$

and hence

$$\begin{aligned} \rho([X \wedge X, X]) &= \rho([SX \wedge X, BX]) \\ &= \sum \beta_n(SX \wedge X)\gamma_n(BX) \\ &= \sum \beta_{n-1}(X \wedge X)\gamma_{n-1}(X). \end{aligned}$$

By combining this with Lemma 7.2 we see that $M(X)$ is infinite if and only if $\beta_n(X \wedge X)\gamma_n(X) > 0$ for some n.

By (4.1) we know the integers $n_1, \dots, n_q$ for a compact simple Lie group G and that $G \underset{0}{\simeq} S = S^{n_1} \times \cdots \times S^{n_q}$. It follows from the definition that n is a cup number relative to $(n_1, \dots, n_q, n_1, \dots, n_q)$ if and only if $H^n(S \wedge S; \mathbb{Q}) \neq 0$. Thus we have

THEOREM 7.4 ([17]). *The following Lie groups have an infinite number of nonhomotopic multiplications*:

$$\mathrm{SO}(10), \mathrm{SO}(14), \mathrm{SO}(n) \textit{ for } n \geqslant 17, \ \mathrm{SU}(n) \textit{ for } n \geqslant 6, \ \mathrm{Sp}(n) \textit{ for } n \geqslant 8, \ \mathrm{E}_6, \ \mathrm{E}_8.$$

All other groups have a finite number of multiplications.

We list some concrete examples of the enumeration of $M(G)$:

EXAMPLE. (1) $M(S^1) = [S^1 \wedge S^1, S^1] = \pi_2(S^1) = 0$.

(2) $M(S^3) = [S^3 \wedge S^3, S^3] = \pi_6(S^3) \cong \mathbb{Z}_{12}$.

(3) (Naylor [195].) *There exist precisely* 768 *distinct homotopy classes of multiplications on* SO(3).

(4) (Mimura [176].) *There exist precisely* $2^{15} \cdot 3^9 \cdot 5 \cdot 7$ *and* $2^{20} \cdot 3 \cdot 5^5 \cdot 7$ *distinct homotopy classes of multiplications on* SU(3) *and* Sp(2) *respectively.*

8. Lie groups as framed manifolds

Let G be a compact connected Lie group of dimension n. The tangent bundle of G can be trivialized by choosing a basis of the Lie algebra, which is the tangent space at the identity $L(G)$, and by using left translation to give an isomorphism of the tangent space at any point with the tangent space $L(G)$. Any trivialization of the tangent bundle induces a trivialization of the stable normal bundle (unique up to homotopy) and hence an element of the n-th framed cobordism group Ω^n_{fr}. We apply the Pontryagin–Thom construction to the element to obtain an element of π^s_n depending only on the orientation of the basis and denote it by $[G, \alpha, L]$, where α is the orientation of G and L indicates that we have used left translation. Replacing it by right translation we obtain another element $[G, \alpha, R]$. Observe that

$$[G, \alpha, R] = [G, -\alpha, L] = -[G, \alpha, L].$$

From now on we fix the orientation and denote the element simply by $[G, L]$. It is straightforward to see that the elements $[G, L]$ behave well with respect to product:

$$[G, L_G] \times [H, L_H] = [G \times H, L_{G\times H}].$$

It is folklore that $[S^1, L] = \eta \in \pi^s_1$ and $[S^3, L] = \nu \in \pi^s_3$, where η and ν represent the Hopf elements. Hence we know $[G, L]$ for any abelian group G. If G is non-abelian and contains a torus in the center, then there is a diffeomorphism of framed manifolds between (G, L) and $(T, L) \times (G/T, L)$ and thus the problem to determine the element $[G, L]$, which was firstly proposed by Gershenson [78], may be reduced to the case where G is semi-simple.

There are some results of a general nature:

(8.1) (Becker and Schultz [23].) $2[\mathrm{SO}(2n), L] = 0$ *and* $\eta[\mathrm{SU}(2n), L] = 0$.

(8.2) (Knapp [131], [133].) *The p-primary component of* $[G, L]$ *has* BP*-Adams filtration at least* n, *and at least* $n + 2(p-1)$ *if* $p > 3$.

(8.3) (Atiyah and Smith [20].) *Let* G *be a non-abelian compact connected Lie group of rank* > 1 *and of dimension* $4k - 1$. *Assume further that the adjoint representation of*

G lifts to Spin. Then $e[G, L] = 0$, *where* $e : \pi_n^s \to \mathbb{Q}/\mathbb{Z}$ *is the Adams e-invariant. In particular, this holds for simply connected Lie groups.*

Some other results are summarized in the following table ([20], [23], [131], [133], [240], [270].

rank	G	dim	$[G, L]$	π_n^s
1	SU(2)	3	ν	$\mathbb{Z}_{24}$
	SO(3)		2ν	
2	SU(3)	8	$\overline{\nu}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
	Sp(2)	10	$\beta_1(3)$	$\mathbb{Z}_6$
	SO(5)		$-\beta_1(3)$	
	G_2	14	κ	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
	SO(4)	6	0	$\mathbb{Z}_2$
3	SU(4)	15	$\kappa\eta$	$\mathbb{Z}_{480} \oplus \mathbb{Z}_2$
	SO(6)		0	
	Spin(7)	21	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
	SO(7)		0	
	Sp(3)		$\sigma^3 + \overline{\kappa}\eta$	
4	SU(5)	24	$\eta^*\sigma\eta$ or 0	$\mathbb{Z}_6 \oplus \mathbb{Z}_2$
	Spin(9)	36	0	$\mathbb{Z}_6$
	SO(9)		0	
	Sp(4)			
	Spin(8)	28	0	$\mathbb{Z}_2$
	SO(8)		0	
	F_4	52		$\mathbb{Z}_3 \oplus$ 2-primary

(8.4)

The best result so far is due to Ossa [211]:

THEOREM 8.5. *Let G be a compact connected Lie group. Then*

$$72[G, L] = 0.$$

Moreover, if G is not locally isomorphic to a product of E_6, E_7, E_8, *then*

$$24[G, L] = 0.$$

In fact he shows more:

(8.6) *In Theorem* 8.5, *the framing on G need not be left invariant: it is sufficient that the framing be invariant under a suitable subgroup S^1.*

(8.7) *The order of the homotopy element $G \to G/T$ in $\pi_d^s(G/T)$ is*
24 *if the Lie algebra of G contains a simple factor of type* $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n, \mathrm{G}_2$;
72 *if it contains a factor* $\mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7$.

(In the case where G is a product of groups E_8, he has only the estimate 360.)

The idea of the proof is based on Knapp's approach [131], [133] via the S^1-transfer, using the classification theorem.

The following is a list of books and papers related to the topology of Lie groups and their homotopy-theoretic study. It is by no means complete. Extensive bibliographies relating to the study of Hopf spaces may be found in [115], [238] and [280]. As for those relating to characteristic classes see, for example, [31] and [177].

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CHAPTER 20

Computing v_1-periodic Homotopy Groups of Spheres and some Compact Lie Groups

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Contents

HANDBOOK OF ALGEBRAIC TOPOLOGY
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1. Introduction

In this paper, we present an account of the principal methods which have been used to compute the v_1-periodic homotopy groups of spheres and many compact simple Lie groups. The two main tools have been J-homology and the unstable Novikov spectral sequence (UNSS), and we shall strive to present all requisite background on both of these.

The v_1-periodic homotopy groups of a space X, denoted $v_1^{-1}\pi_*(X;p)$, are a certain localization of the actual p-local homotopy groups $\pi_*(X)_{(p)}$. We shall drop the p from the notation except where it seems necessary. Very roughly, $v_1^{-1}\pi_*(X)$ is a periodic version of the portion of $\pi_*(X)$ detectable by real and complex K-theory and their operations. It is the first of a hierarchy of theories, $v_n^{-1}\pi_*(X)$, which should account for all of $\pi_*(X)$. Each group $v_1^{-1}\pi_i(X)$ is a direct summand of some actual group $\pi_{i+L}(X)$, at least if X has an H-space exponent and $v_1^{-1}\pi_i(X)$ is a finitely-generated abelian group, which is the case in all examples discussed here.

The v_1-periodic homotopy groups are important because for spaces such as spheres and compact simple Lie groups they give a significant portion of the actual homotopy groups, and yet are often completely calculable. The goal of this paper is to explain how those calculations can be made. One might hope that these methods can be adapted to learning about v_n-periodic homotopy groups for $n > 1$.

One application of v_1-periodic homotopy groups is to obtain lower bounds for the exponents of spaces. The p-exponent of X is the largest e such that some homotopy group of X contains an element of order p^e. We can frequently determine the largest p-torsion summand in $v_1^{-1}\pi_*(X)$. Such a summand must also exist in some $\pi_i(X)$, although we cannot usually specify *which* $\pi_i(X)$. Thus we obtain lower bounds for the p-exponents of spaces, which we conjecture to be sharp in many of the cases studied here. It is known to be sharp for S^{2n+1} if p is odd. See Corollaries 7.8 and 8.9 for estimates of the p-exponent of $\mathrm{SU}(n)$ when p is odd.

Another application, which we will not discuss in this paper, is to James numbers, which are an outgrowth of work on vector fields in the 1950's. It is proved in [23] that, for sufficiently large values of the parameters, the unstable James numbers equal the stable James numbers, and these equal a certain value which had been conjectured by a number of workers.

In Section 2, we present the definition and basic properties of the v_1-periodic homotopy groups. In Section 3, we describe the reduction of the calculation of the unstable groups $v_1^{-1}\pi_*(S^{2n+1})$ to the calculation of stable groups $v_1^{-1}\pi_*^s(B^{qn})$. Here B^{qn} is a space which will be defined in Section 3; when $p = 2$, it is the real projective space P^{2n}. Here we have begun the use, continued throughout the paper, of q as $2p-2$. The original proof of this result, 3.1, appeared in [40] and [56]; it involved delicate arguments involving the lambda algebra. We present a new proof, due to Langsetmo and Thompson, which involves completely different techniques, primarily K-theoretic.

In Section 4, we explain how to compute $J_*(B^{qn})$, while in Section 5, we sketch the proof that if X is a spectrum, then $v_1^{-1}\pi_*(X) \approx v_1^{-1}J_*(X)$. Combining the results of Sections 3, 4, and 5 yields a nice complete result for $v_1^{-1}\pi_*(S^{2n+1})$, which can be summarized as

$$v_1^{-1}\pi_{2n+1+i}(S^{2n+1}) \approx v_1^{-1}\pi_i^s(B^{qn}) \approx v_1^{-1}J_i(B^{qn})$$

and, if p is odd,

$$\approx \begin{cases} \mathbf{Z}/p^{\min(n,\nu_p(a)+1)} & \text{if } i = qa-2 \text{ or } qa-1, \\ 0 & \text{if } i \not\equiv -1 \text{ or } -2 \bmod q. \end{cases}$$

We will use $\mathbf{Z}/n$ and $\mathbf{Z}_n$ interchangeably, and let $\nu_p(n)$ denote the exponent of p in n. The subscript p of ν will sometimes be omitted if it is clear from the context. The final result for $v_1^{-1}\pi_*(S^{2n+1})$ when $p=2$ is more complicated; see Theorem 4.2.

In Section 6, we sketch the formation of the UNSS and its v_1-localization, and for S^{2n+1} we compute the entire v_1-localized UNSS and part of the unlocalized UNSS. In Section 7, we discuss the computation of the v_1-periodic UNSS and v_1-periodic homotopy groups in general for spherically resolved spaces and specifically for the special unitary groups $\mathrm{SU}(n)$. This is considerably easier at the odd primes than at the prime 2. The following key result of Bendersky ([4]) will be proved by observing that the homotopy-theoretic calculation and UNSS calculation agree for S^{2n+1}.

THEOREM 1.1. *If p is odd, and X is built by fibrations from finitely many odd-dimensional spheres, then $v_1^{-1}E_2^{s,t}(X) = 0$ in the v_1-periodic UNSS unless $s = 1$ or 2 and t is odd, in which case*

$$v_1^{-1}\pi_i(X;p) \approx \begin{cases} v_1^{-1}E_2^{1,i+1}(X) & \textit{if } i \textit{ is even}, \\ v_1^{-1}E_2^{2,i+2}(X) & \textit{if } i \textit{ is odd}. \end{cases}$$

In Section 7 we also review the computation of $E_2^1(\mathrm{SU}(n))$ in [6] and combine it with Theorem 1.1 to obtain the following result, which was the main result of [23].

DEFINITION 1.2. Let $\nu_p(m)$ denote the exponent of p in m, and define integers $a(k,j)$ and $e_p(k,n)$ by

$$(\mathrm{e}^x - 1)^j = \sum_{k \geqslant j} a(k,j)\frac{x^k}{k!},$$

and $e_p(k,n) = \min\{\nu_p(a(k,j)) : n \leqslant j \leqslant k\}$.

THEOREM 1.3. *If p is odd, then $v_1^{-1}\pi_{2k}(\mathrm{SU}(n)) \approx \mathbf{Z}/p^{e_p(k,n)}$, and $v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n))$ is an abelian group of order $p^{e_p(k,n)}$, although not always cyclic.*

In Section 8, we illustrate the two principal methods used in computing v_1-periodic homotopy groups of the exceptional Lie groups, focusing on $v_1^{-1}\pi_*(G_2;5)$ for UNSS methods, and on $v_1^{-1}\pi_*(F_4/G_2;2)$ for homotopy (J-homology) methods. We also discuss the recent thesis of Yang ([58]), which gives formulas more tractable than that of Definition 1.2 for the numbers $e_p(k,n)$ which appear in Theorem 1.3, provided $n \leqslant p^2 - p$.

2. Definition of v_1-periodic homotopy groups

In this section, we present the definition and basic properties of the v_1-periodic homotopy groups. We work toward the definition of $v_1^{-1}\pi_*(X)$ by recalling the definition of $v_1^{-1}\pi_*(X;\mathbf{Z}/p^e)$. Let $M^n(k)$ denote the Moore space $S^{n-1}\cup_k e^n$. The mod k homotopy group $\pi_n(X;\mathbf{Z}/k)$ is defined to be the set of homotopy classes $[M^n(k),X]$. With the prime p implicit, and $q=2(p-1)$, let

$$s(e)=\begin{cases} p^{e-1}q & \text{if } p \text{ is odd,}\\ \max(8,2^{e-1}) & \text{if } p=2.\end{cases} \tag{2.1}$$

Let $A: M^{n+s(e)}(p^e)\to M^n(p^e)$ denote a map, as introduced by Adams in [1], which induces an isomorphism in K-theory. Such a map exists provided $n\geqslant 2e+3$ ([28, 2.11]). Then $v_1^{-1}\pi_i(X;\mathbf{Z}/p^e)$ is defined to be

$$\operatorname*{dirlim}_N\,\left[M^{i+Ns(e)}(p^e),X\right],$$

where the maps A are used to define the direct system. The map is what Hopkins and Smith would call a v_1-map, and they showed in [35] that any two v_n-maps of a finite complex which admits such maps become homotopic after a finite number of iterations (of suspensions of the same map), and hence $v_1^{-1}\pi_*(X;\mathbf{Z}/p^e)$ does not depend upon the choice of the map A. Note that although $v_1^{-1}\pi_*(X;\mathbf{Z}/p^e)$ is a theory yielding information about the unstable homotopy groups of X, the maps A which define the direct system may be assumed to be stable maps, since the direct limit only cares about large values of $i+Ns(e)$. Note also that the groups $v_1^{-1}\pi_i(X;\mathbf{Z}/p^e)$ are defined for all integers i and satisfy $v_1^{-1}\pi_i(X;\mathbf{Z}/p^e)\approx v_1^{-1}\pi_{i+s(e)}(X;\mathbf{Z}/p^e)$.

There is a canonical map $\rho: M^n(p^{e+1})\to M^n(p^e)$ which has degree p on the top cell, and degree 1 on the bottom cell. It satisfies the following compatibility with Adams maps.

LEMMA 2.1 ([34, p. 633]). *If $A: M^{n+s(e)}(p^e)\to M^n(p^e)$ and*

$$A': M^{n+s(e+1)}\left(p^{e+1}\right)\to M^n\left(p^{e+1}\right)$$

are v_1-maps, then there exists k so that the following diagram commutes.

$$\begin{tikzcd} M^{n+ks(e+1)}(p^{e+1}) \arrow[r, "\rho"] \arrow[d, "A'^k"] & M^{n+ks(e+1)}(p^e) \arrow[d, "A^{kp'}"] \\ M^n(p^{e+1}) \arrow[r, "\rho"] & M^n(p^e) \end{tikzcd}$$

Here $p'=p$ unless $p=2$ and $e<4$, in which case $p'=1$.

Thus, after sufficient iteration of the Adams maps, there are morphisms ρ^* between the direct systems used in defining $v_1^{-1}\pi_*(X;\mathbf{Z}/p^e)$ for varying e, and passing to direct limits, we obtain a direct system

$$v_1^{-1}\pi_*(X;\mathbf{Z}/p^e) \xrightarrow{\rho^*} v_1^{-1}\pi_*(X;\mathbf{Z}/p^{e+1}) \xrightarrow{\rho^*} \cdots. \tag{2.2}$$

The following definition was given in [28], following less satisfactory definitions in [30] and [23].

DEFINITION 2.2. For any space X and any integer i,

$$v_1^{-1}\pi_i(X) = \operatorname*{dirlim}_e v_1^{-1}\pi_{i+1}(X;\mathbf{Z}/p^e),$$

using the direct system in (2.2).

The reason for the use of the $(i+1)$st mod-p^e periodic homotopy groups in defining the i-th (integral) periodic groups is that the maps ρ of Moore spaces have degree 1 on the bottom cells, but mod-p^e homotopy groups are indexed by the dimension of the top cell.

The mod-p^e periodic homotopy groups have received more attention in the literature, especially when $e = 1$. For spaces with H-space exponents, there is a close relationship between the integral periodic groups and the mod-p^e groups, which we recall after giving the relevant definition.

DEFINITION 2.3. A space X has H-space exponent p^e if for some positive integer L the p^e-power map $\Omega^L X \to \Omega^L X$ is null-homotopic.

By [20] and [36], spheres and compact Lie groups have H-space exponents.

PROPOSITION 2.4. (i) [28, 1.7] *If X has H-space exponent p^e, then there is a split short exact sequence*

$$0 \to v_1^{-1}\pi_i(X) \to v_1^{-1}\pi_i(X;\mathbf{Z}/p^e) \to v_1^{-1}\pi_{i-1}(X) \to 0.$$

(ii) *On the category of spaces with H-space exponents, there is a natural transformation $\pi_*(-)_{(p)} \to v_1^{-1}\pi_*(-;p)$.*

(iii) *If X has an H-space exponent, and $v_1^{-1}\pi_i(X)$ is a finitely generated abelian group, then $v_1^{-1}\pi_i(X)$ is a direct summand of $\pi_{i+L}(X)$ for some non-negative integer L.*

The proof of part ii utilizes the fibration

$$\mathrm{map}_*\big(M^{n+1}(p^e), X\big) \to \Omega^n X \xrightarrow{p^e} \Omega^n X,$$

where $\mathrm{map}_*(-,-)$ denotes the space of pointed maps. If the second map is null-homotopic, then the first map admits a section s. The natural transformation is induced by

$$\Omega^n X \xrightarrow{s} \mathrm{map}_*\big(M^{n+1}(p^e), X\big) \to \operatorname*{dirlim}_k \mathrm{map}_*\big(M^{n+1+ks(e)}(p^e), X\big).$$

Techniques of [28] imply naturality of this construction. To prove part (iii), we note that the map s allows $v_1^{-1}\pi_i(X)$ to be written as $\operatorname{dirlim}_k \pi_{i+ks(e)}(X)$, which is a direct summand of one of the groups in the direct system, provided the direct limit is finitely generated.

If X is a spectrum, then $v_1^{-1}\pi_*(X)$ is defined in exactly the same way as for spaces, that is, as in Definition 2.2. If X is a space, then stable groups, $v_1^{-1}\pi_*^s(X)$, can be defined either as $v_1^{-1}\pi_*(\Sigma^\infty X)$, where $\Sigma^\infty X$ denotes the suspension spectrum of X, or as $v_1^{-1}\pi_*(QX)$, where $QX = \Omega^\infty\Sigma^\infty X$ is the associated infinite loop space.

3. The isomorphism $v_1^{-1}\pi_*(S^{2n+1}) \approx v_1^{-1}\pi^s_{*-2n-1}(B^{qn})$

In this section, we sketch a new proof, due to Thompson and Langsetmo ([39, 4.2]), of the following crucial result.

THEOREM 3.1. *There is a map*

$$\Omega^{2n+1}S^{2n+1} \to QB^{qn} \tag{3.1}$$

which induces an isomorphism in $v_1^{-1}\pi_*(-)$.

Here $QX = \Omega^\infty\Sigma^\infty X$, and B^{qn} is the qn-skeleton of the p-localization of the classifying space $B\Sigma_p$ of the symmetric group Σ_p on p letters. Note that if $p = 2$, then B^{qn} is the real projective space RP^{2n}. The original proof, from [40] when $p = 2$ and [56] when p is odd, involved delicate arguments involving the lambda algebra and unstable Adams spectral sequences. We feel that the following argument, primarily K-theoretic, will speak to a broader cross-section of readers. The following elementary result shows that it is enough to show that the map (3.1) induces an iso in v_1-periodic mod p homotopy.

LEMMA 3.2. *If a map $X \to Y$ induces an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$, then it induces an isomorphism in $v_1^{-1}\pi_*(-;p)$.*

PROOF. There are cofibrations of Moore spaces which induce natural exact sequences

$$\begin{aligned} \to v_1^{-1}\pi_n(X;\mathbf{Z}/p^e) &\xrightarrow{\rho^*} v_1^{-1}\pi_n(X;\mathbf{Z}/p^{e+1}) \to \\ v_1^{-1}\pi_n(X;\mathbf{Z}/p) &\to v_1^{-1}\pi_{n-1}(X;\mathbf{Z}/p^e) \to . \end{aligned}$$

Induction on e using the 5-lemma implies that there are isomorphisms

$$v_1^{-1}\pi_*(X;\mathbf{Z}/p^e) \to v_1^{-1}\pi_*(Y;\mathbf{Z}/p^e)$$

for all e, compatible with the maps ρ^* which define the direct system (2.2). The desired isomorphism of the direct limits is immediate. $\square$

The construction of the map (3.1) takes us far afield, and is not used elsewhere in the computations. For completeness, we wish to say something about it, but we will be extremely sketchy. The map is due to Snaith. Work of many mathematicians, especially Peter May, is important in the construction. However, we shall just refer the reader to [37], where the proof of naturality of these maps is given, along with references to the earlier work.

THEOREM 3.3. (i) *There are maps*

$$s_n : \Omega^{2n+1}S^{2n+1} \to QB^{qn}$$

which are compatible with respect to inclusion maps as n increases, and such that the adjoint map

$$\Sigma^\infty \Omega^{2n+1}S^{2n+1} \to \Sigma^\infty B^{qn}$$

is the projection onto a summand in a decomposition of $\Sigma^\infty\Omega^{2n+1}S^{2n+1}$ as a wedge of spectra.

(ii) *There is a map $QS^{2n+1} \xrightarrow{g} Q\Sigma^{2n+1}B_{(n+1)q-1}$ whose fiber, $\mathcal{F}$, satisfies*

$$v_1^{-1}\pi_*\big(\Omega^{2n+1}\mathcal{F};\mathbf{Z}/p\big) \approx v_1^{-1}\pi_*\big(QB^{qn};\mathbf{Z}/p\big).$$

SKETCH OF PROOF. Let $C_N(k)$ denote the space of ordered k-tuples of disjoint little cubes in I^N. If X is a based space, let C_NX denote the space of finite collections of disjoint little cubes of I^N labeled with points of X. More formally,

$$C_NX = \coprod_{k\geqslant 1} C_N(k) \times_{\Sigma_k} X^k/\sim,$$

where

$$\big[(c_1,\ldots,c_k),(x_1,\ldots,x_{k-1},*)\big] \sim \big[(c_1,\ldots,c_{k-1}),(x_1,\ldots,x_{k-1})\big].$$

There are natural maps, due to May,

$$C_NX \to \Omega^N\Sigma^NX, \tag{3.2}$$

which are weak equivalences if X is connected.

The space C_NX is filtered by defining $\mathcal{F}_m(C_NX)$ to be the subspace of m or fewer little labeled cubes. The successive quotients are defined by

$$D_{N,m}X = \mathcal{F}_m(C_NX)/\mathcal{F}_{m-1}(C_NX) \approx C_N(m)^+ \wedge_{\Sigma_m} X^{[m]},$$

where $X^{[m]}$ denotes the m-fold smash product. Snaith proved that if X is path-connected, there is a weak equivalence of suspension spectra

$$\Sigma^\infty C_NX \simeq \bigvee_{m\geqslant 1} \Sigma^\infty D_{N,m}X. \tag{3.3}$$

Let $N = 2n + 1$ and $X = S^0$. Stabilize the equivalence of (3.2), and project onto the summand of (3.3) with $m = p$ to get

$$\Sigma^\infty \Omega^{2n+1} S^{2n+1} \to \Sigma^\infty C_{2n+1}(p)/\Sigma_p.$$

The identification of $C_{2n+1}(p)/\Sigma_p$ as the nq-skeleton of $B\Sigma_p$ after localization at p was obtained by Fred Cohen in [19, p. 246].

Note that (3.2) and (3.3) yield a map $\Omega^N \Sigma^N X \to QD_{N,m}X$. The map g of ii) is obtained from the case $N = \infty$ and $X = S^{2n+1}$ as the composite

$$QS^{2n+1} \to QD_{\infty,p}S^{2n+1} = Q\big(B\Sigma_p^+ \wedge_{\Sigma_p} (S^{2n+1})^{[p]}\big) \simeq Q\Sigma^{2n+1}B_{(n+1)q-1}.$$

Here we have used results of [45] for the last equivalence.

The compatible maps s_n of 3.3(i) combine to yield a map $s' : QS^0 \to QB^\infty$, and one can show that the right square commutes in the diagram of fibrations below.

$$\begin{tikzcd}
\Omega^{2n+1}\mathcal{F} \arrow[r] \arrow[d, "t"] & QS^0 \arrow[r, "\Omega^{2n+1}g"] \arrow[d, "s'"] & QB_{(n+1)q-1} \arrow[d, "="] \\
QB^{qn} \arrow[r] & QB^\infty \arrow[r] & QB_{(n+1)q-1}
\end{tikzcd}$$

The map t then follows, and will induce an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$, as asserted in Theorem 3.3(ii), once we know that s' does. Kahn and Priddy showed that there is an infinite loop map $\lambda : QB^\infty \to QS^0$ such that $\lambda \circ s'$ induces an isomorphism in $\pi_j(-)$ for $j > 0$. It is easily verified using methods of the next two sections that the associated stable map $B^\infty \to S^0$ induces an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$. Hence so does s'. □

Throughout this section, let $K_*(-)$ denote mod p K-homology, and M^k denote the mod p Moore space $M^k(p)$. The following two theorems, whose proofs occupy most of the rest of this section, imply that $j : S^{2n+1} \to \mathcal{F}$ induces an iso in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$.

THEOREM 3.4 ([16, 14.4]). *If $k \geqslant 2$, and $\phi : X \to Y$ is a map of k-connected spaces such that $\Omega^k\phi$ is a K_*-equivalence, then ϕ induces an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$.*

THEOREM 3.5 ([39]). *Let $\mathcal{F}$ be as in Theorem 3.3(ii). The map $S^{2n+1} \to QS^{2n+1}$ lifts to a map $j : S^{2n+1} \to \mathcal{F}$ such that $\Omega^2 j$ is a K_*-equivalence.*

The map j of Theorem 3.5 can be chosen so that $t \circ \Omega^{2n+1} j = s_n$, where t is the map in the proof of Theorem 3.3(ii) which induces the isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$. Theorem 3.1 is now proved by applying Lemma 3.2 to s_n.

Bousfield localization is involved in the proof of Theorem 3.4 and several other topics later in the paper, and so we review the necessary material. If E is a spectrum, then a space (or spectrum) X is *E_*-local* if every E_*-equivalence $Y \to W$ induces a bijection $[W, X] \to [Y, X]$. The *E_*-localization* of X is an E_*-local space (or spectrum) X_E together with an E_*-equivalence $X \to X_E$. The existence and uniqueness of these localizations were established in [18] and [17]. We will need the following result of

Bousfield, the proof of which is easily obtained using the ideas at the beginning of Section 5.

THEOREM 3.6 ([18, 4.8]). *A spectrum is K_*-local if and only if its mod p homotopy groups are periodic under the action of the Adams map.*

We will omit the proof of Theorem 3.4, as it requires many peripheral ideas. Instead, we will sketch a proof of the following result, which, although weaker than 3.4, has the same flavor, and predated it. An alternate proof of Theorem 3.1 can be given by using Theorem 3.7 and strengthening Theorem 3.5 to show that $K_*(\Omega^3 j)$ is bijective. As the calculations for Ω^3 seem significantly more difficult than for Ω^2, we omit that approach.

THEOREM 3.7 ([57]). *Let p be an odd prime, and let X and Y be 3-connected spaces. Suppose that $f : X \to Y$ is a map such that $K_*(\Omega^k f)$ is an isomorphism for $k = 0$, 1, 2, and 3. Then f induces an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$.*

SKETCH OF PROOF. If $F \to E \to B$ is a principal fibration, there is a bar spectral sequence converging to $K_*(B)$ with $E^2_{s,t} \approx \operatorname{Tor}^{K_*F}_{s,t}(K_*E, K_*)$. (See [55] for a discussion of this spectral sequence.)

Applied to the commutative diagram of principal fiber sequences

$$\begin{array}{ccccc} \Omega^3 X & \xrightarrow{p} & \Omega^3 X & \to & \operatorname{map}_*(M^3, X) \\ \downarrow{\scriptstyle \Omega^3 f} & & \downarrow{\scriptstyle \Omega^3 f} & & \downarrow{\scriptstyle f'} \\ \Omega^3 Y & \xrightarrow{p} & \Omega^3 Y & \to & \operatorname{map}_*(M^3, Y) \end{array},$$

the spectral sequence and the hypothesis of the theorem imply that f' induces an isomorphism in $K_*(-)$. Hence there is an equivalence of the K_*-localizations

$$\left(\operatorname{map}_*(M^3, X)\right)_K \xrightarrow{f'_K} \left(\operatorname{map}_*(M^3, Y)\right)_K.$$

Let $V(X)$ denote the mapping telescope of

$$\operatorname{map}_*(M^3, X) \xrightarrow{A^*} \operatorname{map}_*(M^{3+q}, X) \xrightarrow{A^*} \cdots.$$

Then $V(X)$ is K_*-local, since it is Ω^∞ of a periodic spectrum which is K_*-local by Theorem 3.6. This implies that there are maps i' making the following diagram commute.

$$\begin{array}{ccccc} \operatorname{map}_*(M^3, X) & \to & (\operatorname{map}_*(M^3, X))_K & \xrightarrow{i'} & V(X) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f'_K} & & \downarrow{\scriptstyle V(f)} \\ \operatorname{map}_*(M^3, Y) & \to & (\operatorname{map}_*(M^3, Y))_K & \xrightarrow{i'} & V(Y) \end{array}$$

Since $v_1^{-1}\pi_*(X;\mathbf{Z}/p) \approx \pi_*(V(X);\mathbf{Z}/p)$, the desired isomorphism is a consequence of the following construction of an inverse to

$$V(f)_* : \pi_*(V(X);\mathbf{Z}/p) \to \pi_*(V(Y);\mathbf{Z}/p).$$

An element $\alpha \in \pi_k(V(Y); \mathbf{Z}/p)$ can be represented by a map

$$M^k \to \mathrm{map}_*(M^{3+qpj}, Y),$$

or, adjointing, by a map $M^{k+qpj} \to \mathrm{map}_*(M^3, Y)$. Here some care is required to see that we can switch the Moore space factor on which the map A is performed. The element which corresponds to α is the composite

$$\begin{aligned} M^{k+qpj} &\to \mathrm{map}_*(M^3, Y) \to (\mathrm{map}_*(M^3, Y))_K \\ &\xrightarrow{(f'_K)^{-1}} (\mathrm{map}_*(M^3, X))_K \xrightarrow{i'} V(X). \end{aligned}$$

□

The proof of Theorem 3.5 involves a good bit of delicate computation. The hardest part is the determination of $K_*(\Omega^2\mathcal{F})$ as a Hopf algebra. In order to conveniently obtain the coalgebra structure of $K_*(\Omega^2\mathcal{F})$, we proceed in two steps. We first calculate the algebra $K_*(\Omega^3\mathcal{F})$, using the bar spectral sequence associated to the principal fibration

$$\Omega^4 QS^{2n+1} \to \Omega^4 Q\Sigma^{2n+1} B_{q(n+1)-1} \to \Omega^3\mathcal{F}.$$

This spectral sequence is calculated in [38], obtaining an algebra isomorphism

$$K_*(\Omega^3\mathcal{F}) \approx P[y_1, y_2] \otimes E[z]. \tag{3.4}$$

Here y_i (resp. z) has bidegree $(1,1)$ (resp. $(0,1)$) in the spectral sequence, and hence even (resp. odd) degree in $K_*(\Omega^3\mathcal{F})$. The calculation of this spectral sequence requires some preliminary computation regarding the algebra structure of $K_*(\Omega^2\mathcal{F})$, and this requires major input from [44].

Now we calculate the bar spectral sequence associated to the principal fibration

$$\Omega^3\mathcal{F} \to * \to \Omega^2\mathcal{F}.$$

This spectral sequence, with $E_2 \approx \mathrm{Tor}^{P[y_1,y_2]\otimes E[z]}(K_*, K_*)$, collapses to yield an isomorphism of Hopf algebras

$$K_*(\Omega^2\mathcal{F}) \approx E[a_1, a_2] \otimes \Gamma[b], \tag{3.5}$$

where Γ denotes the divided polynomial algebra over $\mathbf{Z}_p$. The coproduct has a_1, a_2, and $\gamma_1(b)$ as the primitives, and

$$\psi(\gamma_i(b)) = \sum \gamma_j(b) \otimes \gamma_{i-j}(b).$$

Dualizing eq. (3.5) yields an isomorphism of algebras

$$K^*(\Omega^2\mathcal{F}) \approx E[\alpha_1, \alpha_2] \otimes P[\beta],$$

with $|\alpha_i|$ odd and $|\beta|$ even. This matches nicely with the following result from [52, 3.8].

PROPOSITION 3.8. *For any prime p, there is an isomorphism of algebras*

$$K^*(\Omega^2 S^{2n+1}) \approx E[u_0, u_1] \otimes P[w],$$

where E and P denote exterior and polynomial algebras over K_*.

We will use the Atiyah–Hirzebruch spectral sequence to show that the map

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega^2 j} \Omega^2 \mathcal{F}$$

of Theorem 3.5 sends the generators of the isomorphic $K^*(-)$-algebras across. This will imply the second half of Theorem 3.5.

In [52], it is shown how the generators u_0, w, and u_1 of $K^*(\Omega^2 S^{2n+1})$ arise in the Atiyah–Hirzebruch spectral sequence whose E_2-term is $H^*(\Omega^2 S^{2n+1}; K_*)$. Indeed, they arise from the bottom three cohomology classes, of grading $2n-1$, $2pn-2$, and $2pn-1$, respectively. The map $\Omega^2 j$ induces an isomorphism in $H^i(-;\mathbf{Z}_p)$ for $i < 2pn - 1 + \min(q, 2n-2)$, and so it maps onto the three generators of $K^*(\Omega^2 S^{2n+1})$.

4. J-homology

In this section, we show how to compute $J_*(B^{qn})$. When combined with Theorems 3.1 and 5.1, this gives an explicit computation of $v_1^{-1}\pi_*(S^{2n+1})$, which we state at the end of this section as Theorem 4.2. This will be extremely important in our calculation of $v_1^{-1}\pi_*(Y)$ for other spaces Y. We begin with the case p odd, where the results are somewhat simpler to state. Historically it worked in the other order, with Mahowald's 2-primary results in [40] preceding Thompson's odd-primary work in [56].

Let p be an odd prime. We follow quite closely the exposition in [24] and [56, §3]. The spectrum $bu_{(p)}$ splits as a wedge of spectra $\Sigma^{2i}\ell$ satisfying $H^*(\ell;\mathbf{Z}_p) \approx A//E$, where A is the mod p Steenrod algebra, and E is the exterior subalgebra generated by $Q_0 = \beta$ and $Q_1 = P^1\beta - \beta P^1$. The spectrum ℓ is sometimes written $BP\langle 1\rangle$. Then $\ell_* = \pi_*(\ell)$ is calculated from the Adams spectral sequence (ASS) with

$$E_2^{s,t} \approx \operatorname{Ext}_A^{s,t}(H^*\ell, \mathbf{Z}_p) \approx \operatorname{Ext}_E^{s,t}(\mathbf{Z}_p, \mathbf{Z}_p) \approx \mathbf{Z}_p[a_0, a_1], \tag{4.1}$$

where a_i has bigrading $(1, iq+1)$. Here we have used the change-of-rings theorem in the middle step. There are no possible differentials in the spectral sequence, and since multiplication by a_0 corresponds to multiplication by p in homotopy, we find that $\pi_*(\ell)$ is a polynomial algebra over $\mathbf{Z}_{(p)}$ on a class of grading q. Using the ring structure of ℓ, one easily sees that there is a cofibration

$$\Sigma^q\ell \to \ell \to H\mathbf{Z}_{(p)}.$$

Let k be a $(p-1)$st root of unity mod p but not mod p^2, and let ψ^k denote the Adams operation. The map $\psi^k - 1 : \ell \to \ell$ lifts to a map $\theta : \ell \to \Sigma^q\ell$. The connective

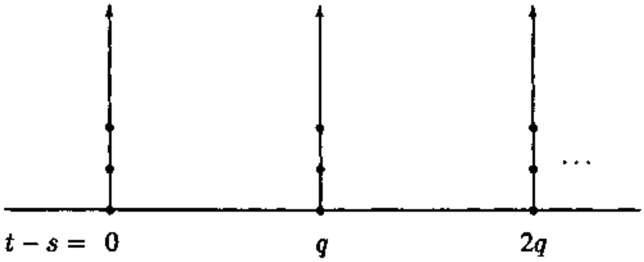

Figure 1. ASS for $\ell_*(R)$.

J-spectrum, J, is defined to be the fiber of θ. The homotopy exact sequence of θ easily implies that

$$\pi_i(J) \approx \begin{cases} \mathbf{Z}_{(p)} & \text{if } i = 0, \\ \mathbf{Z}/p^{\nu_p(j)+1} & \text{if } i = qj-1 \text{ with } j > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The image of the classical J-homomorphism is mapped isomorphically onto these groups by the map $S^0 \to J$; this is the reason for the name of the spectrum.

We now proceed toward the calculation of $J_*(B^{qn})$. We let B be the p-localization of the suspension spectrum of $B\Sigma_p$. Then, with coefficients always in $\mathbf{Z}_p$, the only nonzero groups $H^i(B)$ occur when $i \equiv 0$ or $-1 \bmod q$, and $i > 0$. These groups are cyclic of order p with generator x_i satisfying $Q_0 x_{aq-1} = x_{aq}$ and $Q_1 x_{aq-1} = x_{(a+1)q}$. We will work with the skeleta B^{qn} and the quotients $B_{q(n+1)-1} = B/B^{qn}$; these are suspension spectra of the spaces which appeared in Theorem 3.3.

There is a p-local map $B \to S^0$ constructed by Kahn and Priddy. If R denotes its cofiber, there is a filtration of the E-module H^*R with subquotients $\Sigma^{qi}E//E_0$ for $i \geqslant 0$. Here E_0 is the exterior subalgebra of E generated by Q_0. Since $\mathrm{Ext}_E(E//E_0) \approx \mathrm{Ext}_{E_0}(\mathbf{Z}_p) \approx \mathbf{Z}_p[a_0]$, we find that $\mathrm{Ext}_E(H^*R)$ has a "spike", consisting of the powers of a_0, for each non-negative value of $t-s$ which is a multiple of q. Here we have begun a practice of omitting $\mathbf{Z}_p$ from the second variable of $\mathrm{Ext}_B(-,-)$ if B is a subalgebra of the Steenrod algebra. The action of $\mathrm{Ext}_E(\mathbf{Z}_p)$ on $\mathrm{Ext}_E(H^*R)$ has a_1 always acting nontrivially.

We draw ASS pictures with coordinates $(t-s, s)$, so that horizontal component refers to homotopy group. A chart for the ASS of $R \wedge \ell$ is given in fig. 1.

The short exact sequence

$$0 \to H^*(\Sigma B) \to H^*(R) \to H^*(S^0) \to 0$$

induces an exact sequence

$$\begin{aligned} &\to \mathrm{Ext}_E^{s,t}(H^*S^0) \xrightarrow{i_*} \mathrm{Ext}_E^{s,t}(H^*R) \to \mathrm{Ext}_E^{s,t}\big(H^*(\Sigma B)\big) \\ &\to \mathrm{Ext}_E^{s+1,t}(H^*S^0) \to . \end{aligned} \tag{4.2}$$

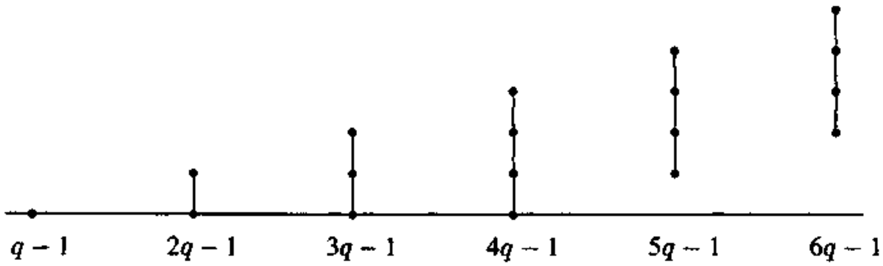

Figure 2. ASS for $\ell_*(B^{4q})$ for $t-s<6q$.

These morphisms are $\mathrm{Ext}_E(\mathbf{Z}_p)$-module maps, and the action of a_1 implies that i_* is injective. Thus there are elements $x_{iq-1} \in \mathrm{Ext}_E^{0,iq-1}(H^*B)$ for $i>0$ such that

$$\mathrm{Ext}_E^{s,t}(H^*B) = \begin{cases} \mathbf{Z}_p & \text{if } t-s=iq-1,\ i>0,\ 0 \leqslant s < i, \\ 0 & \text{otherwise}, \end{cases}$$

with generators $a_0^s x_{iq-1}$. Hence

$$\ell_i(B) \approx \begin{cases} \mathbf{Z}/p^{(i+1)/q} & \text{if } i \equiv -1 \bmod q, \text{ and } i>0, \\ 0 & \text{otherwise.} \end{cases}$$

There is an isomorphism of E-modules $H^*(B_{q(n+1)-1}) \approx \Sigma^{qn} H^*(B)$, and so

$$\ell_*(B_{q(n+1)-1}) \approx \ell_*(\Sigma^{qn} B).$$

The Ext calculation easily implies that the morphism $\ell_*(B) \to \ell_*(B_{q(n+1)-1})$ induced by the collapse map is surjective, and so the exact sequence of the cofibration $B^{qn} \to B \to B_{q(n+1)-1}$ implies that

$$\ell_i(B^{qn}) \approx \begin{cases} \mathbf{Z}/p^{\min((i+1)/q,n)} & \text{if } i \equiv -1 \bmod q, \text{ and } i>0, \\ 0 & \text{otherwise.} \end{cases}$$

The ASS chart for B^{4q} is illustrated in fig. 2.

The map $S^0 \to R$ implies that $\theta_* : \ell_{qj}(X) \to \ell_{(q-1)j}(X)$ is multiplication by the same number for $X=R$ as it was for $X=S^0$. Thus it is multiplication by $p^{\nu_p(j)+1}$. Now the map $R \to \Sigma B$ implies that $\theta_* : \ell_{qj-1}(B) \to \ell_{(q-1)j-1}(B)$ is multiplication by $p^{\nu_p(j)+1}$. The maps $B^{qn} \to B \to B_{q(n+1)-1}$ imply that the same is true in B^{qn} and $B_{q(n+1)-1}$. We obtain

$$J_i(B^{qn}) \approx \begin{cases} \mathbf{Z}/p^{\min(n,\nu_p(j)+1)} & \text{if } i=jq-1,\ j>0, \\ \mathbf{Z}/p^{\min(n,\nu_p(j)+1)} & \text{if } i=jq-2,\ j>n, \\ \mathbf{Z}/p^{\min(n-1,\nu_p(j))} & \text{if } i=jq-2,\ 0<j \leqslant n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

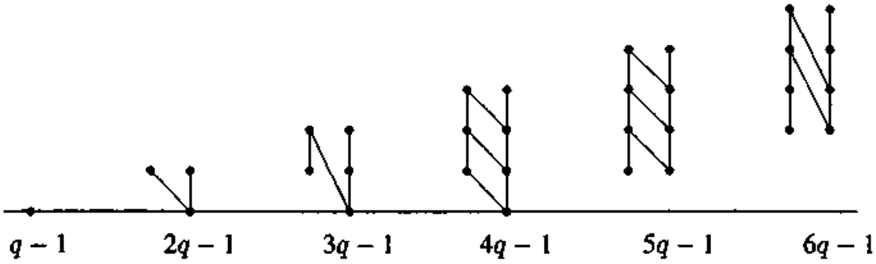

Figure 3. Beginning of chart for $J_*(B^{4q})$.

This is illustrated in fig. 3, which is not quite an ASS chart. It is a combination of the charts for $\ell_*(B^{qn})$ and $\ell_{*-q+1}(B^{qn})$ and the homomorphism θ_* between them, which is represented by lines of negative slope. The exact sequence

$$0 \to \operatorname{coker}(\theta_{*+1}) \to J_*(B^{qn}) \to \ker(\theta_*) \to 0$$

says that elements which are not involved in these boundary morphisms comprise $J_*(B^{qn})$. There are several reasons for our having elevated the filtrations of $\ell_{*-q+1}(B^{qn})$ by 1 in this chart. One is that it makes all the boundary morphisms go up, so that it looks like an ASS chart. Another is that (by [40]) there is a resolution of $B^{qn} \wedge J$ (which is not an Adams resolution) for which the homotopy exact couple is depicted by this chart. A third is that if J_1 is defined to be the fiber of $J \to H\mathbf{Z}_2$, then the ASS chart for $B^{qn} \wedge J_1$ will agree with this chart in filtration greater than 1. See [13, §6] for an elaboration on this.

If X is a space or spectrum, then $v_1^{-1}J_i(X)$ is defined analogously to Definition 2.2 to be

$$\operatorname*{dirlim}_{e,k} \left[M^{i+1+ks(e)}(p^e), X \wedge J\right].$$

Since J is a stable object, we can S-dualize the Moore space, obtaining

$$\operatorname*{dirlim}_{e,k} J_{i+1+ks(e)}\left(X \wedge M(p^e)\right),$$

where the Moore spectrum $M(n)$ has cells of degree 0 and 1. The "+1" in this J-group is present due to the maps $M(p^e) \to M(p^{e+1})$ having degree 1 on the 1-cell.

One can often compute $v_1^{-1}J_*(X)$ directly from $J_*(X)$ without having to worry about the "$\wedge M(p^e)$". This can be done by extending the periodic behavior which occurs in positive filtration down into negative filtrations and negative stems. For example (cf. fig. 3), a chart for $v_1^{-1}J_*(B^{nq})$ has, for all integers a, adjacent towers of height n in $aq-2$ and $aq-1$ with $d_{\nu(a)+1}$-differential. If $\nu(a)+1 \geqslant n$, then the differential is 0. This interpretation of $v_1^{-1}J_*(-)$ can be justified using the following result.

PROPOSITION 4.1. *Let p be an odd prime, and let KU be the spectrum for periodic K-theory localized at p. Let k be a $(p-1)$st root of unity mod p but not mod p^2, and let* Ad *denote the fiber of $KU \xrightarrow{\psi^k - 1} KU$. Then $v_1^{-1}J_*(-) \approx \mathrm{Ad}_*(-)$.*

This follows from the fact that if $v_1^{-1}\ell_*(-)$ is defined as

$$\operatorname*{dirlim}_{e,k}\ \ell_{i+1+ks(e)}\big(X \wedge M(p^e)\big),$$

then $v_1^{-1}\ell_*(-) \approx KU_*(-)$, which is a consequence of the fact that $A \wedge \iota = v \wedge 1_M : \Sigma^q M \to M \wedge \ell$, where $A : \Sigma^q M \to M$ and $v : S^q \to \ell$.

When $p = 2$, the results are a bit messier to state and picture. If bsp denotes the 2-local connected Ω-spectrum whose $(8k)$th space is $BSp[8k]$, then $\Sigma^4 bsp \simeq bo[4]$, the spectrum formed from bo by killing $\pi_i(-)$ for $i < 4$. The map $\psi^3 - 1 : bo \to bo$ lifts to a map $\theta : bo \to \Sigma^4 bsp$, and J is defined to be the fiber of θ.

Let A_1 denote the subalgebra of the mod 2 Steenrod algebra A generated by Sq^1 and Sq^2. Then $H^*bo \approx A//A_1$ and $H^*bsp \approx A \otimes_{A_1} N$, where $N = \langle 1, \mathrm{Sq}^2, \mathrm{Sq}^3 \rangle$. Hence, using the change of rings theorem, the E_2-term of the ASS converging to $\pi_*(bo)$ is $\mathrm{Ext}_{A_1}(\mathbf{Z}_2)$, while that for bsp is $\mathrm{Ext}_{A_1}(N)$. These are easily computed to begin as in fig. 4, with each chart acted on freely by an element in $(t-s, s) = (8, 4)$. Positively sloping diagonal lines indicate the action of $h_1 \in \mathrm{Ext}_{A_1}^{1,2}(\mathbf{Z}_2)$. It corresponds to the Hopf map η in homotopy.

There are no possible differentials in these ASS's, and so we obtain

$$\pi_i(bo) \approx \begin{cases} \mathbf{Z}_{(2)} & \text{if } i \equiv 0 \bmod 4,\ i \geqslant 0, \\ \mathbf{Z}_2 & \text{if } i \equiv 1, 2 \bmod 8,\ i > 0, \\ 0 & \text{otherwise,} \end{cases}$$

in accordance with Bott periodicity.

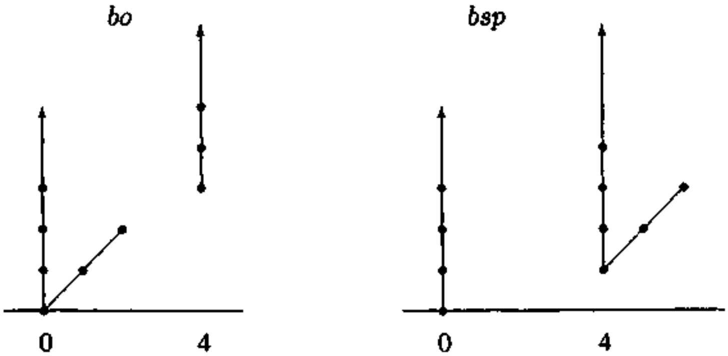

Figure 4. Part of ASS for bo and bsp.

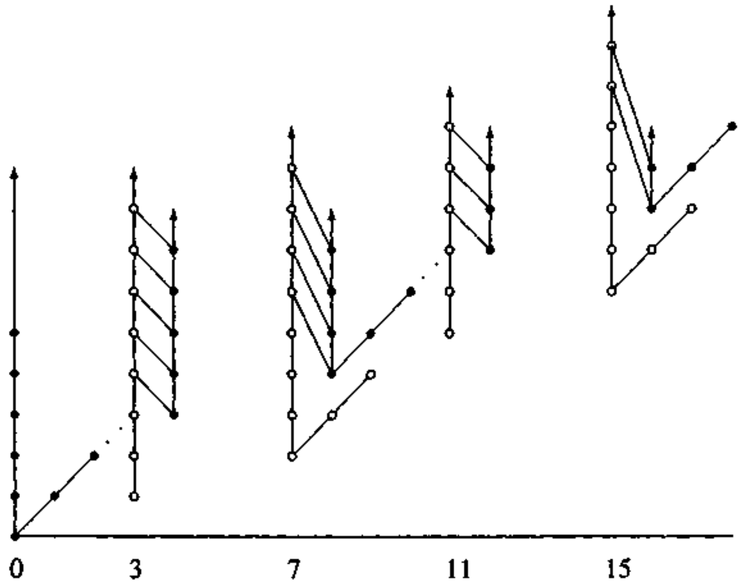

Figure 5. 2-primary $\pi_i(J)$, $i \leqslant 18$.

From Adams' work, we have $\theta_* : \pi_{4j}(bo) \to \pi_{4j}(\Sigma^4 bsp)$ hitting all multiples of $2^{\nu_2(j)+3}$, while θ_* is 0 on the $\mathbf{Z}_2$'s. This yields $\pi_i(J) = 0$ if $i < 0$, while for $i \geqslant 0$

$$\pi_i(J) \approx \begin{cases} \mathbf{Z}_{(2)} & \text{if } i = 0, \\ \mathbf{Z}/2^{\nu_2(i)+1} & \text{if } i \equiv 3 \bmod 4, \\ \mathbf{Z}_2 & \text{if } i \equiv 0, 2 \bmod 8,\ i > 0, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } i \equiv 1 \bmod 8,\ i > 1, \\ 0 & \text{if } i \equiv 4, 5, 6 \bmod 8. \end{cases} \tag{4.4}$$

From Adams' work and the confirmation of the Adams Conjecture, it is known that $\pi_*(S^0) \to \pi_*(J)$ sends the image of the classical J-homomorphism plus Adams' elements μ_j and $\eta\mu_j$ isomorphically onto $\pi_*(J)$. A chart for $\pi_i(J)$ with $i \leqslant 18$ is given in fig. 5. Here the elements coming from bo are indicated by $\bullet$'s, while those from $\Sigma^4 bsp$ are indicated by $\circ$'s.

The first dotted η-extension can be deduced from the fact that $\theta^* : H^4(\Sigma^4 bsp) \to H^4(bo)$ hits Sq^4, together with the relation $\eta^3 = 4\nu$. This η-action is then pushed along by periodicity. Another argument which is frequently useful for deducing η-extensions such as this involves Toda brackets. The generator of $\pi_{8i+4}(bo)$ is obtained from the element $\alpha \in \pi_{8i+2}(bo)$ as $\langle \alpha, \eta, 2\rangle$. Clearly α pulls back to $\pi_*(J)$, and if $\alpha\eta$ were 0 here, then the bracket could also be formed in J. However, the boundary morphism on $\pi_{8i+4}(bo)$ implies that this bracket cannot be formed, and so $\alpha\eta$ must be nonzero in J. Moreover, it must be the element β such that 2β is $\delta(\langle \alpha, \eta, 2\rangle)$.

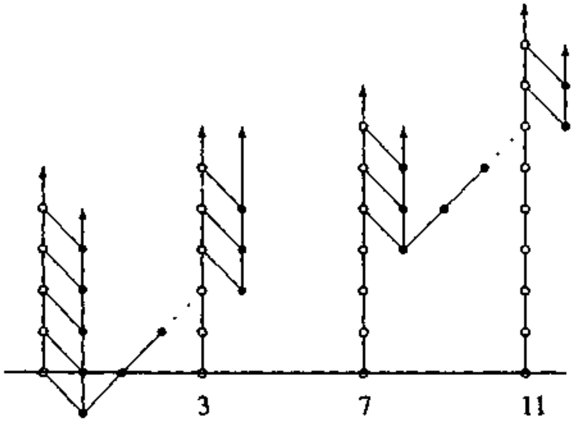

Figure 6. $\mathrm{Ext}_{A_1}(H^*P)$, from $\mathrm{Ext}_{A_1}(H^*S^0)$ and $\mathrm{Ext}_{A_1}(H^*R)$, $t-s<15$.

We will rename B^{qn} as P^{2n} when $p=2$, with P denoting the suspension spectrum of RP^∞. As in the odd primary case, there is a map $\lambda : P \to S^0$ with nontrivial cohomology operations in its mapping cone R. This 2-primary map λ can be viewed more geometrically than its odd-primary analogue, as an amalgamation of composites

$$P^n \to \mathrm{SO}(n+1) \xrightarrow{J} \Omega^n S^n.$$

With A_0 denoting the exterior subalgebra of A generated by Sq^1, H^*R can be filtered as an A_1-module with subquotients $\Sigma^{4i}A_1//A_0$ for $i \geqslant 0$, and so $\mathrm{Ext}_A(H^*(R \wedge bo))$ consists of h_0-spikes rising from each position $(t-s,s)=(4i,0)$ for $i \geqslant 0$. Here h_0 is the element of $\mathrm{Ext}_A^{1,1}(\mathbf{Z}_2)$ or $\mathrm{Ext}_{A_1}^{1,1}(\mathbf{Z}_2)$ corresponding to Sq^1 and to multiplication by 2 in homotopy. Also, we begin a practice of using without comment the relation

$$\mathrm{Ext}_A\big(H^*(X \wedge bo)\big) \approx \mathrm{Ext}_A\big(H^*X \otimes A//A_1\big) \approx \mathrm{Ext}_{A_1}\big(H^*X\big).$$

Thus the nonzero groups $bo_i(R)$ occur only when $i \equiv 0 \bmod 4$ and $i \geqslant 0$, and these groups are $\mathbf{Z}_{(2)}$. We also need

$$bsp_i(R) \approx bo_i\big(R \wedge (S^0 \cup_\eta e^2 \cup_2 e^3)\big) \approx \begin{cases} \mathbf{Z}_{(2)} & \text{if } i \equiv 0 \bmod 4,\ i \geqslant 0, \\ \mathbf{Z}_2 & \text{if } i \equiv 2 \bmod 4,\ i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The $\mathbf{Z}_2$'s are obtained from the exact sequence in $bo_*(-)$ associated to the cofibration

$$R \wedge S^2 \to R \wedge \big(S^2 \cup_2 e^3\big) \to R \wedge S^3 \xrightarrow{2} .$$

Analogous to (4.2) is an exact sequence which allows us to compute $\mathrm{Ext}_{A_1}(H^*P)$ from $\mathrm{Ext}_{A_1}(H^*S^0)$ and $\mathrm{Ext}_{A_1}(H^*R)$. This is most easily seen in the chart of fig. 6, in

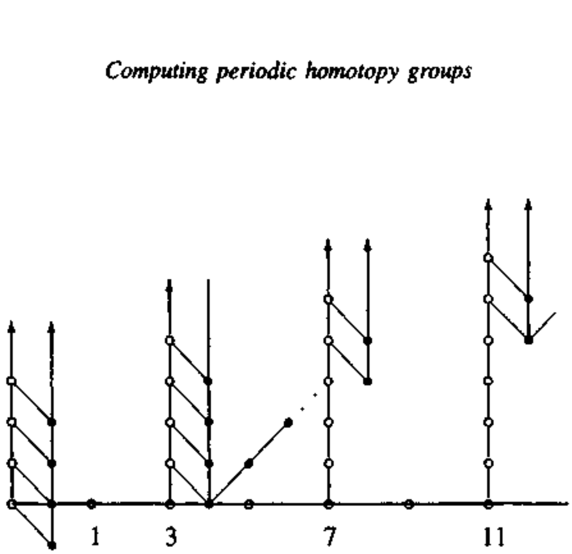

Figure 7. $bsp_*(P)$, from $bsp_*(S^0)$ and $bsp_*(R)$, $* \leqslant 11$.

which $\bullet$'s are from $\mathrm{Ext}_{A_1}(H^*S^0)$, and $\circ$'s are from $\mathrm{Ext}_{A_1}(H^*R)$.

The groups are read off from this as

$$bo_i(P) \approx \begin{cases} \mathbf{Z}/2^{4j+3}, & i = 8j+3,\ j \geqslant 0, \\ \mathbf{Z}/2^{4j}, & i = 8j-1,\ j > 0, \\ \mathbf{Z}_2, & i = 8j+1 \text{ or } 8j+2,\ j \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Next on the agenda is $bsp_*(P)$, which is computed from $bsp_*(S^0)$ (in $\bullet$) and $bsp_*(R)$ (in $\circ$) as in fig. 7.

Note that in positive filtration $bsp_*(P)$ looks like $bo_*(\Sigma^4 P)$ pushed up by 1 filtration. The explanation for this is the short exact sequence of A_1-modules

$$0 \to \Sigma^5 \mathbf{Z}_2 \to A_1//A_0 \to N \to 0, \tag{4.5}$$

where $N = \langle 1, \mathrm{Sq}^2, \mathrm{Sq}^3 \rangle$, as before. If this is tensored with any A_0-free A_1-module M, such as P, then the exact Ext_{A_1}-sequence reduces to isomorphisms

$$\mathrm{Ext}_{A_1}^{s-1,t}(\Sigma^5 M) \to \mathrm{Ext}_{A_1}^{s,t}(N \otimes M)$$

when $s > 1$. When $M = P$, an iso is also obtained when $s = 1$.

The isomorphism of A_1-modules $H^*(P_{4n+1}) \approx H^*(\Sigma^{4n} P)$ allows one to immediately obtain $bo_*(P_{4n+1})$ and $bsp_*(P_{4n+1})$ from the above calculations. One way of determining $bo_*(P_{4n+3})$ and $bsp_*(P_{4n+3})$ is from the short exact sequence of A_1-modules

$$0 \to H^*(\Sigma^{4n+4}\mathbf{Z}_2) \to H^*(P_{4n+3}) \to H^*(\Sigma^{4n+3} R) \to 0.$$

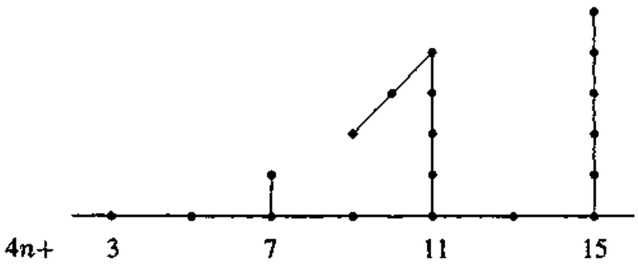

Figure 8. $bsp_*(P_{4n+3})$, $* \leqslant 15$.

This yields as $bsp_*(P_{4n+3})$ a chart which begins as in fig. 8, while $bo_*(P_{4n+7}) \approx bo_*(\Sigma^4 P_{4n+3})$ is obtained from this chart by deleting all classes in filtration 0.

Next we compute $bo_*(P^{2m})$ and $bsp_*(P^{2m})$ using the exact Ext_{A_1}-sequence corresponding to the short exact sequence

$$0 \to H^*(P_{2m+1}) \to H^*(P) \to H^*(P^{2m}) \to 0.$$

For example, this yields the calculation of $bo_*(P^{8n})$ indicated in fig. 9, where $bo_*(P)$ is in $\bullet$'s, while $bo_*(P_{8n+1})$ is in $\circ$'s.

Next we form $J_*(P^{2m})$ from $bo_*(P^{2m})$ and $(\Sigma^3 bsp)_*(P^{2m})$, with filtrations of the latter pushed up by 1, similarly to the odd primary case. The boundary morphism $bo_{4i-1}(P^{2m}) \to (\Sigma^4 bsp)_{4i-1}(P^{2m})$ is pictured by a differential in the chart, and, for the same reason as in the odd-primary case, its value is the same as in $bo_{4i-1}(S^0) \to (\Sigma^4 bsp)_{4i-1}(S^0)$, namely a nonzero $d_{\nu(i)+1}$ wherever possible. If $m \geqslant k$, then the charts for $J_*(P^{2m})$ and $J_*(P^{2k})$ are isomorphic through dimension $2k-1$. This is illustrated in fig. 10 for $k = 8$.

For $* \geqslant 2m-1$, the form of the chart for $J_*(P^{2m})$ depends upon the mod 4 value of m. The last of the filtration-1 $\mathbf{Z}_2$'s occurs in $* = 2m$ or $2m+2$. The chart near $* = 2m$ is indicated in fig. 11. Note how the bsp-part is like the bo-part shifted one unit left and two units down. Differentials d_r with $r > 1$ are omitted from this chart; they occur on the towers in $8n-1$ and $8n+7$.

To obtain $v_1^{-1}J_*(P^{2m})$ from $J_*(P^{2m})$, one removes the filtration-1 $\mathbf{Z}_2$'s, and extends into negative filtration the periodic behavior which is present in the towers to the right of dimension $2m$. The justification for this is similar to that in the odd-primary case, namely Proposition 4.1. For example, if i is any integer, $v_1^{-1}J_*(P^{8n+4})$ for $8i+6 \leqslant * \leqslant 8i+13$ looks like the portion of fig. 11 for P^{8n+4} between $8n+6$ and $8n+13$, with a $d_{\nu(4i+4)}$-differential on the tower in $8i+7$. One might find it easier to compute $v_1^{-1}J_*(P^{2m})$ directly without bothering to first compute the nonperiodic J; however, one sometimes needs the nonperiodic J-groups.

We combine the results of this section with Theorem 3.1 and Theorem 5.1 to obtain the following extremely important result, Theorem 4.2.

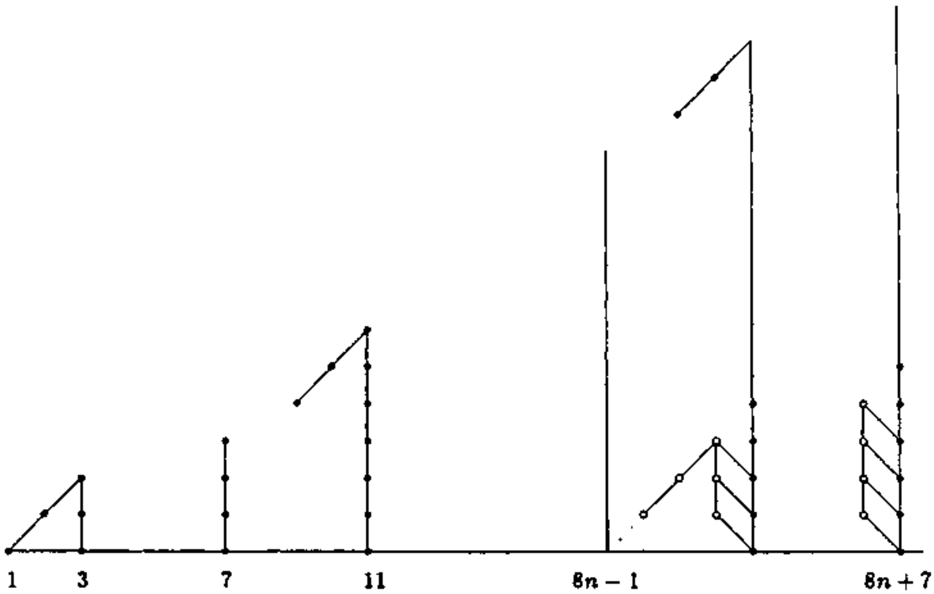

Figure 9. $bo_*(P^{8n})$, from $bo_*(P)$ and $bo_*(P_{8n+1})$.

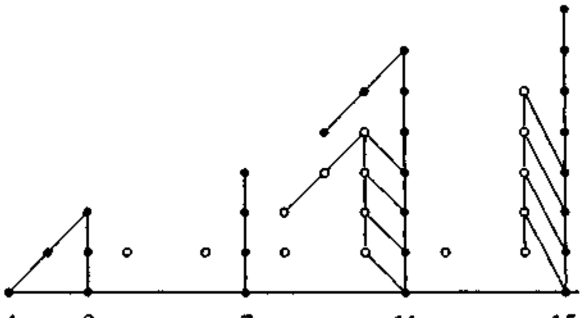

Figure 10. $J_*(P^{2m})$ in $* \leqslant 15$, provided $m \geqslant 8$.

THEOREM 4.2. *If p is odd, then*

$$v_1^{-1}\pi_{2n+1+i}\left(S^{2n+1};p\right) \approx \begin{cases} \mathbf{Z}/p^{\min(n,\nu_p(a)+1)} & \text{if } i = qa-2 \text{ or } qa-1, \\ 0 & \text{if } i \not\equiv -1 \text{ or } -2 \bmod q. \end{cases}$$

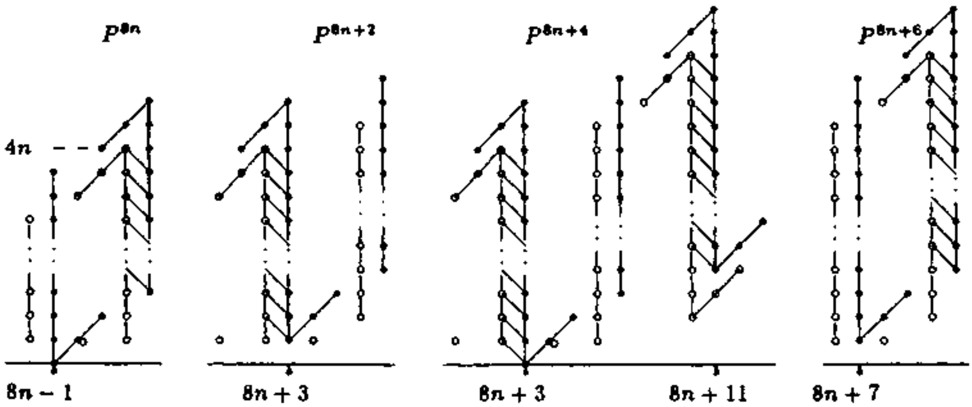

Figure 11. $J_*(P^{2m})$ where it starts to ascend.

If $n \equiv 1$ *or* $2 \bmod 4$, *then*

$$v_1^{-1}\pi_{2n+1+i}(S^{2n+1};2) \approx \begin{cases} \mathbf{Z}_2 & \text{if } i \equiv 0,5 \bmod 8, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } i \equiv 1,4 \bmod 8, \\ \mathbf{Z}_2 \oplus \mathbf{Z}/2^{\min(3,n+1)} & \text{if } i \equiv 2,3 \bmod 8, \\ \mathbf{Z}/2^{\min(n-1,\nu_2(j)+4)} & \text{if } i = 8j-2 \text{ or } 8j-1. \end{cases}$$

If $n \equiv 0$ *or* $3 \bmod 4$, *then*

$$v_1^{-1}\pi_{2n+1+i}(S^{2n+1};2) \approx \begin{cases} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } i \equiv 0,1 \bmod 8, \\ \mathbf{Z}_8 \oplus \mathbf{Z}_2 & \text{if } i \equiv 2 \bmod 8, \\ \mathbf{Z}_8 & \text{if } i \equiv 3 \bmod 8, \\ 0 & \text{if } i \equiv 4,5 \bmod 8, \\ \mathbf{Z}/2^{\min(n,\nu_2(j)+4)} & \text{if } i = 8j-2, \\ \mathbf{Z}_2 \oplus \mathbf{Z}/2^{\min(n,\nu_2(j)+4)} & \text{if } i = 8j-1. \end{cases}$$

5. The v_1-periodic homotopy groups of spectra

In this section, we sketch three proofs of the following central result.

THEOREM 5.1. *If X is a spectrum, then* $v_1^{-1}\pi_*(X) \approx v_1^{-1}J_*(X)$.

This result was first stated, at least for mod p v_1-periodic homotopy groups, in [56].

Theorem 5.1 is a consequence of the following result, which is the special case where X is the mod p Moore spectrum $M = S^0 \cup_p e^1$.

THEOREM 5.2. *Let $v_1^{-1}M$ denote the mapping telescope of*

$$M \to \Sigma^{-s}M \to \Sigma^{-2s}M \to \cdots,$$

where $s = 8$ if $p = 2$ and $s = q$ if p is odd, and the maps are all suspensions of an Adams map A. Then the Hurewicz morphism

$$\pi_*(v_1^{-1}M) \to J_*(v_1^{-1}M)$$

is an isomorphism.

This theorem implies that for any spectrum X, the map

$$X \wedge v_1^{-1}M \to X \wedge v_1^{-1}M \wedge J$$

is an equivalence, which, after dualizing the Moore spectra, implies that Theorem 5.1 is true with mod p coefficients. The general case of the theorem then follows from Lemma 3.2.

As an aside, we note that these results are equivalent to the validity of Ravenel's Telescope Conjecture ([53]) when $n = 1$. This result, for which the analogue with $n = 2$ has been shown to be false, can be stated in the following way.

COROLLARY 5.3. *The v_1-telescope equals the K_*-localization, i.e. $v_1^{-1}M = M_K$.*

PROOF. Since the Adams maps induce isomorphisms in $K_*(-)$, the inclusion $M \to v_1^{-1}M$ is a K_*-equivalence. Since $v_1^{-1}M \simeq v_1^{-1}M \wedge J \simeq M \wedge v_1^{-1}J$, and, similarly to Proposition 4.1, there is a cofibration

$$v_1^{-1}J \to KO \to KO,$$

it follows readily that $v_1^{-1}M$ is K_*-local. □

The remainder of this section is concerned with proofs of Theorem 5.2. Three distinct proofs have been given, although each is too complicated to present in detail here. We sketch each, relegating details to the original papers.

The first proof, when $p = 2$, was given by Mahowald in [40], although he was offering sketches of this proof as early as 1970. The odd-primary analogue was given in [24]. A sketch of Mahowald's proof, involving *bo*-resolutions, follows.

Using self-duality of M, it suffices to show that

$$\operatorname*{dirlim}_i \left[\Sigma^{k+8i}M, S^0\right] \to \operatorname*{dirlim}_i \left[\Sigma^{k+8i}M, J\right] \tag{5.1}$$

is an isomorphism. The target groups are easily determined by the methods of the preceding section to be given by two sequences of "lightning flashes" as in fig. 12. It is easily seen, for example from the upper edge of Adams spectral sequence, that these elements all come from actual stable homotopy classes, i.e. the morphism (5.1) is surjective.

The injectivity of the morphism (5.1) will be proved by showing that if $\Sigma^k M \xrightarrow{f} S^0 \to J$ is trivial, then for i sufficiently large, $\Sigma^{k+8i}M \xrightarrow{A^i} \Sigma^k M \xrightarrow{f} S^0$ is trivial. This will be done using *bo*-resolutions.

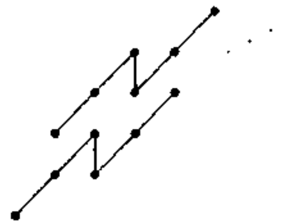

Figure 12. $\operatorname{dirlim}_i[\Sigma^{k+8i}M, J]$.

Let $\overline{bo}$ denote the cofiber of the inclusion $S^0 \to bo$. There is a tower of (co)fibrations

$$\begin{array}{ccccc} S^0 & \longleftarrow & \Sigma^{-1}\overline{bo} & \longleftarrow & \Sigma^{-2}\overline{bo}\wedge\overline{bo} \longleftarrow \\ \downarrow & & \downarrow & & \downarrow \\ bo & & \Sigma^{-1}\overline{bo}\wedge bo & & \Sigma^{-2}\overline{bo}\wedge\overline{bo}\wedge bo \end{array}$$

The homotopy exact couple of this tower gives the bo-ASS for S^0. It was proved in [25], following [41], that the E_2-term of this spectral sequence vanishes above a line of slope 1/5. That is, $E_2^{s,t}(S^0) = 0$ if $s > \frac{1}{5}(t-s)+3$. One can show that $\overline{bo}\wedge bo \simeq \Sigma^4 bsp \vee W$, where W can be written explicitly, and the map

$$\Sigma^{-1}bo \xrightarrow{\delta} \Sigma^{-1}\overline{bo}\wedge bo \to \Sigma^3 bsp$$

may be used as the map whose fiber is J. Here δ induces the lowest d_1 in the bo-ASS, and the second map collapses W.

Let $E_s = (\Sigma^{-1}\overline{bo})^{\wedge s}$, the sth stage of the tower. By explicit calculation it can be shown that, if $s > 1$ or if $s = 1$ and the map is detected entirely in the W-part, a map $X \to E_s$ of Adams ($H\mathbf{Z}/2$) filtration greater than 1 can be varied so that its projection to E_{s-1} is unchanged, while the new map lifts to E_{s+1}. Originally it was thought that this was true for maps of Adams filtration greater than 0, but a complication was noted in [27].

Now suppose that $\Sigma^k M \xrightarrow{f} S^0 \to J$ is trivial. Then f lifts to a map into E_1 whose projection into $\Sigma^3 bsp$ is trivial. The Adams map A can be written as the composite of two maps, each of $H\mathbf{Z}/2$-Adams filtration greater than 1. Thus, by the result of the previous paragraph, $f\circ A^i$ lifts to E_{2i+1}. If i is chosen large enough that $2i+1 > \frac{1}{5}(k+8i+1)+3$, then this map $\Sigma^{k+8i}M \to S^0$ will have bo-filtration so large that all such maps are trivial by the vanishing line result, completing the proof.

The first proof of Theorem 5.2 for p odd was given by Haynes Miller. A proof analogous to his for $p = 2$ has not been achieved; [32] was a step in that direction. Miller's work did not involve the spectrum J. Instead, in [46], he defined a localized

ASS for M, and computed its E_2-term. Then, in [47], using a clever comparison with the BP-based Novikov spectral sequence, he computed the differentials in the ASS, obtaining the following result.

PROPOSITION 5.4. *If p is odd, then $\pi_*(v_1^{-1}M)$ is free over $\mathbf{Z}_p[v_1^{\pm 1}]$ on two classes, namely $[S^0 \hookrightarrow M \to v_1^{-1}M]$ and $[S^{q-1} \xrightarrow{\alpha_1} M \to v_1^{-1}M]$.*

The methods of Section 4 show easily that these map isomorphically to $v_1^{-1}J_*(M)$.

We provide a little more detail about Miller's calculations. In [46] he obtained as an E_2-term for the localized ASS

$$v_1^{-1}E[h_{i,0} : i \geqslant 1] \otimes P[b_{i,0} : i \geqslant 1], \tag{5.2}$$

where $h_{i,0}$ corresponds to $[\xi_i]$ and has bigrading $(1, 2(p^i - 1))$, while $b_{i,0}$ corresponds to

$$\sum \frac{1}{p}\binom{p}{j}[\xi_i^j|\xi_i^{p-j}]$$

and has bigrading $(2, 2p(p^i - 1))$. The first step in obtaining this is to use a change-of-rings theorem to write the E_2-term as $v_1^{-1}\mathrm{Cotor}_{A(1)_*}(\mathbf{Z}_p, \mathbf{Z}_p)$, where $A(1)_*$ is the quotient $A_*/(\tau_0)$. This is then shown to be isomorphic to

$$\mathbf{Z}_p\left[v_1^{\pm 1}\right] \otimes \mathrm{Cotor}_{P(1)}(\mathbf{Z}_p, \mathbf{Z}_p),$$

where $P(1) = \mathbf{Z}_p[\xi_1, \xi_2, \ldots]/(\xi_1^p, \xi_2^p, \ldots)$, and this yields eq. (5.2).

In [47], the differential $d_2(h_{i,0}) = v_1 b_{i-1,0}$ is established in the localized ASS. This leaves $\mathbf{Z}_p[v_1^{\pm 1}] \otimes E[h_{1,0}]$ as $E_3 = E_\infty$, and this is easily translated into Proposition 5.4. Miller first established this differential in an algebraic spectral sequence converging to the E_2-term of the BP-based Novikov spectral sequence, and then showed that this implies the desired differential in the ASS by a comparison theorem.

Somewhat later, Crabb and Knapp ([21]) gave a proof of Theorem 5.1 for finite spectra X which was much less computational than those just discussed. Their proof utilized the solution of the Adams conjecture, and some refinements thereof. They let $\mathrm{Ad}^*(-)$ be the generalized cohomology theory corresponding to the fiber of $\psi^k - 1 : KO \to KO$. By Proposition 4.1, this is just our $v_1^{-1}J^*$. They prove the following result about stable cohomotopy, which by S-duality is equivalent to Theorem 5.1 for finite spectra.

THEOREM 5.5. *If X is a finite spectrum, then the Hurewicz morphism*

$$v_1^{-1}\pi_s^*(X; \mathbf{Z}/p^e) \xrightarrow{h} \mathrm{Ad}^*(X; \mathbf{Z}/p^e)$$

is bijective.

Their main weapon is a result of May and Tornehave which says that if $A^*(-)$ is the connective theory associated to $\mathrm{Ad}^*(-)$, and j is the morphism given by a solution of the Adams conjecture, then the composite

$$A^0(X) \xrightarrow{j} \pi_s^0(X) \xrightarrow{h} A^0(X)$$

is bijective for a connected space X. This is used to show that, for k sufficiently large, there is a stable Adams map $\Sigma^{ks(e)}M(p^e) \xrightarrow{A_e} M(p^e)$ which is in the image under j from $A^{-ks(e)}(M(p^e);\mathbf{Z}/p^e)$. This is then used to show that for any element x of $\pi_s^n(X;\mathbf{Z}/p^e)$, for L sufficiently large, $A_e^L x$ is in the image of the morphism j, and this easily implies injectivity in Theorem 5.5. Care is required throughout in distinguishing stable maps from actual maps.

6. The v_1-periodic unstable Novikov spectral sequence for spheres

In this section, we review the basic properties of the unstable Novikov spectral sequence (UNSS) based on the Brown–Peterson spectrum BP, and sketch the determination of the 1- and 2-lines of this spectral sequence when applied to S^{2n+1}. Then we show how the v_1-periodic UNSS is defined, and compute it completely for S^{2n+1}.

The spectrum BP associated to the prime p is a commutative ring spectrum satisfying $BP_* = \pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ and $BP_*(BP) = BP_*[h_1, h_2, \ldots]$, with $|v_i| = |h_i| = 2p^i - 2$. The generators v_i are those of Hazewinkel, while h_i is conjugate to Quillen's generator t_i. We shall often abbreviate BP_*BP as Γ.

We will make frequent use of the right unit $\eta_R : BP_* \to BP_*BP$.

PROPOSITION 6.1. *$\eta_R(v_1) = v_1 - ph_1$, and*

$$\eta_R(v_2) = v_2 - ph_2 + (p^{p-1} - 1)h_1^p v_1 + (p+1)v_1^p h_1 + \sum_{i=2}^{p} a_i v_1^{p+1-i} p^i h_1^i,$$

where $a_i \in \mathbf{Z}$.

In writing $h_1^p v_1$ here, we have begun the practice of writing $\eta_R(v)h$ as hv. Thus $h_1^p v_1 \neq v_1 h_1^p$. Proposition 6.1 is easily derived from formulas relating v_i to m_i, and for $\eta_R(m_i)$. See [14, 2.6], where the following formula for the comultiplication $\Delta : BP_*BP \to BP_*BP \otimes BP_*BP$ is also computed. All tensor products in this and subsequent sections are over BP_*.

PROPOSITION 6.2. *$\Delta(h_1) = h_1 \otimes 1 + 1 \otimes h_1$, and*

$$\Delta(h_2) = h_2 \otimes 1 + 1 \otimes h_2 + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} h_1^i \otimes h_1^{p-i} v_1 + h_1^p \otimes h_1.$$

Let $\mathbf{BP}_n$ denote the nth space in the Ω-spectrum for BP. If X is a space, then a space $BP(X)$ is defined as $\lim_n \Omega^n(\mathbf{BP}_n \wedge X)$. Define $D^1(X)$ to be the fiber of the unit map $X \to BP(X)$, and inductively define $D^s(X)$ to be the fiber of $D^{s-1}(X) \to D^{s-1}(BP(X))$. This gives rise to a tower of fibrations

$$\cdots \to D^2(X) \to D^1(X) \to X.$$

The homotopy exact couple of this tower is the UNSS of X; if X is simply connected, it converges to the localization at p of $\pi_*(X)$.

In general, computing this spectral sequence can be extremely difficult, but if BP_*X is free as a BP_*-module, and cofree as a coalgebra, then it becomes somewhat tractable. Indeed, in such a case

$$E_2^{s,t}(X) \approx \mathrm{Ext}^s_{\mathcal{U}}\big(A_t, P(BP_*X)\big), \tag{6.1}$$

where A_t denotes a free BP_*-module on a generator of degree t, $P(-)$ denotes the primitives in a coalgebra, and $\mathcal{U}$ denotes the category of unstable Γ-comodules. We sketch a definition of the category $\mathcal{U}$ and the proof of eq. (6.1), referring the reader to [6, p. 744] or [8, §7] for more details. If M is a free BP_*-module, then $U(M)$ is defined to be the BP_*-submodule of $\Gamma \otimes M$ spanned by all elements of the form $h^I \otimes m$ satisfying the unstable condition

$$2(i_1 + i_2 + \cdots) < |m|, \tag{6.2}$$

where $h^I = h_1^{i_1} h_2^{i_2} \cdots$. If M is not BP_*-free, then $U(M)$ is defined as $\mathrm{coker}(U(F_1) \to U(F_0))$, where F_0 and F_1 are free BP_*-modules with $M = \mathrm{coker}(F_1 \to F_0)$. We define $U^s(M)$ by iterating $U(-)$. The category $\mathcal{U}$ consists of BP_*-modules equipped with morphisms $M \xrightarrow{\psi} U(M)$, $U(M) \xrightarrow{\delta} U^2(M)$, and $U(M) \xrightarrow{\varepsilon} M$ satisfying certain properties. The unstable condition (6.2) is analogous to the one for unstable right modules over the Steenrod algebra, but its proof relies on deep work of Ravenel and Wilson in [54].

The category $\mathcal{U}$ is abelian. We abbreviate $\mathrm{Ext}^s_{\mathcal{U}}(A_t, N)$ to $\mathrm{Ext}^{s,t}_{\mathcal{U}}(N)$. These groups may be calculated as the homology groups of the unstable cobar complex $\overline{C}^{*,*}(N)$, defined by $\overline{C}^{s,t}(N) = U^s(N)_t$, with boundary $\overline{C}^s \xrightarrow{d} \overline{C}^{s+1}$ defined by

$$\begin{aligned} d[\gamma_1|\cdots|\gamma_s]m &= [1|\gamma_1|\cdots|\gamma_s]m \\ &\quad + \sum(-1)^j[\gamma_1|\cdots|\gamma_j'|\gamma_j''|\cdots|\gamma_s]m \\ &\quad + \sum(-1)^{s+1}[\gamma_1|\cdots|\gamma_s|\gamma']m'', \end{aligned}$$

where $\Delta(\gamma_j) = \sum \gamma_j' \otimes \gamma_j''$ and $\psi(m) = \sum \gamma' \otimes m''$.

We will use a reduced complex $C^{*,*}(N)$, which is chain equivalent to $\overline{C}^{*,*}(N)$. This is obtained from $\widetilde{U}(N) = \ker(U(N) \xrightarrow{\varepsilon} N)$ and its iterates $\widetilde{U}^s$ by $C^{s,t}(N) = \widetilde{U}^s(N)_t$. Finally, we illustrate how $d(v) = \eta_R(v) - v$ for $v \in BP_*$ comes into play. Suppose $\Delta(h) = h \otimes 1 + 1 \otimes h + \sum h' \otimes h''$ and $\psi(I) = 1 \otimes I$, and let $v, v' \in BP_*$. Then

$$\begin{aligned} d\big([vh]v'I\big) &= [1|vh]v'I - [vh|1]v'I - [v|h]v'I - \sum[vh'|h'']v'I + [vh|v']I \\ &= [\eta_R(v) - v|h]v'I - \sum[vh'|h'']v'I - [vh|\eta_R(v') - v']I. \end{aligned}$$

We abbreviate $C^{*,*}(BP_*(X))$ to $C^{*,*}(X)$.

The first result about the UNSS, both historically and pedagogically, is the following, which appeared in [8, 9.12]. We repeat their proof because it gives a good first example of working with the unstable cobar complex. Recall that $q = 2(p-1)$.

Theorem 6.3. *Let p be an odd prime. If $k > 0$, then*

$$E_2^{1,2n+1+kq}(S^{2n+1}) \approx \mathbf{Z}/p^{\min(n,\nu_p(k)+1)}.$$

If $t \not\equiv 2n+1 \bmod q$, or if $t < 2n+1$, then $E_2^{s,t}(S^{2n+1}) = 0$.

Proof. Since $|v_i|$ and $|h_i|$ are divisible by q, all nonzero elements in $BP_*(S^{2n+1})$ have degree congruent to $2n+1 \bmod q$, and so the only possible nonzero elements in $E_2^{s,t}(S^{2n+1})$ occur when $t \equiv 2n+1 \bmod q$, and $t \geqslant 2n+1$. There is an injective chain map

$$C^{*,*}(S^{2n-1}) \to C^{*,*+2}(S^{2n+1})$$

defined by $A \otimes \iota_{2n-1} \mapsto A \otimes \iota_{2n+1}$, corresponding to the double suspension homomorphism of homotopy groups. Since the boundaries in $C^1(S^{2n-1})$ are sent bijectively to those in $C^1(S^{2n+1})$, the morphism $E_2^{1,t}(S^{2n-1}) \to E_2^{1,t+2}(S^{2n+1})$ is injective.

We quote a result, originally due to Novikov (but see [51, §5.3] for the proof), about the stable groups: if n is sufficiently large, then

$$E_2^{1,2n+1+kq}(S^{2n+1}) \approx \mathbf{Z}/p^{\nu_p(k)+1}$$

with generator $d(v_1^k)\iota_{2n+1}/p^{\nu_p(k)+1}$. We will prove Theorem 6.3 by showing that if $n \leqslant \nu_p(k)+1$, then $d(v_1^k)/p^n$ is defined on S^{2n+1}, but not on S^{2n-1}.

We begin by observing

$$\begin{aligned} d(v_1^k)/p^n &= \left(\left(\eta_R(v_1)\right)^k - v_1^k\right)/p^n = \left((v_1 - ph_1)^k - v_1^k\right)/p^n \\ &= \sum_{j=1}^{k}(-1)^j \binom{k}{j} p^{j-n} v_1^{k-j} h_1^j. \end{aligned} \tag{6.3}$$

Note that the coefficients $\binom{k}{j}p^{j-n}$ have non-negative powers of p, since

$$\nu_p\left(\binom{k}{j}\right) + j \geqslant \nu_p(k) + 1 \geqslant n$$

for $j \geqslant 1$. Now we work mod terms that are defined on S^{2n-1}. This allows us to ignore terms in the sum (6.3) for $j < n$. For the other terms, we write $p^{j-n}h_1^j$ as $(v_1 - \eta_R v_1)^{j-n} h_1^n$, and note that when this is expanded by the binomial theorem, all terms except $v_1^{j-n}h_1^n$ may be ignored, since $(\eta_R v_1)^i h_1^n \iota_{2n-1} = h_1^n v_1^i \iota_{2n-1}$ satisfies eq. (6.2) when $i > 0$. Thus the sum (6.3) reduces to

$$\sum_{j=n}^{k}(-1)^j \binom{k}{j} v_1^{k-n} h_1^n = -\sum_{j=0}^{n-1}(-1)^j \binom{k}{j} v_1^{k-n} h_1^n,$$

since

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}=0.$$

If $j>0$ in the right-hand sum, then $\binom{k}{j}$ is divisible by p, and then ph_1^n can be written as $v_1h_1^{n-1}-h_1^{n-1}v_1$, so that the term is defined on S^{2n-1}. Thus, mod S^{2n-1}, (6.3) reduces to $-v_1^{k-n}h_1^n$. This class is not defined on S^{2n-1}. □

If $p=2$, a similar argument establishes the following result.

THEOREM 6.4. *If $p=2$, then, for $u>0$,*

$$E_2^{1,2n+1+u}(S^{2n+1})\approx\begin{cases}0, & u \text{ odd},\\ \mathbf{Z}/2, & \nu_2(u)=1,\\ \mathbf{Z}/4, & u=4,\\ \mathbf{Z}/2^{\min(n,\nu_2(u)+1)}, & u\equiv 0 \bmod 4, \text{ and } u>4.\end{cases}$$

If $u=2k$ in the three nonzero cases, then the generators are, respectively, $d(v_1^k)/2$, $d(v_1^2)/4$, and $d(v_1^k+2^{\nu_2(k)+1}v_1^{k-3}v_2)/2^{\nu_2(k)+2}$.

When p is odd, the element

$$-\big(d(v_1^k)/p^j\big)\iota_{2n+1}\in E_2^{1,2n+1+kq}(S^{2n+1})$$

is denoted $\alpha_{k/j}$. If $j=1$, this will frequently be shortened to α_k. We note the following from the proof of Theorem 6.3.

PROPOSITION 6.5. *If $n\geqslant j$, then $\alpha_{k/j}\iota_{2n+1}=v_1^{k-j}h_1^j\iota_{2n+1}$ mod terms defined on S^{2j-1}.*

Next we cull from [5] information about unstable elements in $E_2^{2,*}(S^{2n+1})$, which form a subgroup which we shall denote by $\widetilde{E}_2^{2,*}(S^{2n+1})$. By "unstable", we mean an element in the kernel of the iterated suspension. The main theorem of [5] is the following.

THEOREM 6.6. *Let p be an odd prime, and let $t=\nu_p(a)$. Then*

$$\widetilde{E}_2^{2,qa+2n+1}(S^{2n+1})\approx\begin{cases}\mathbf{Z}/p^n & \text{if } n\leqslant t+1,\\ \mathbf{Z}/p^{t+1} & \text{if } t+1\leqslant n<a-t,\\ \mathbf{Z}/p^{a-n} & \text{if } a-t-1\leqslant n<a.\end{cases}$$

The homomorphism $\widetilde{E}_2^{2,qa+2n-1}(S^{2n-1})\xrightarrow{E^2}\widetilde{E}_2^{2,qa+2n+1}(S^{2n+1})$ is

$$\begin{cases}\text{injective} & \text{if } n\leqslant t+1,\\ \cdot p & \text{if } t+1<n<a-t,\\ \text{surjective} & \text{if } a-t\leqslant n<a.\end{cases}$$

Let $m = \min(n, a - t - 1)$. Then p^j times the generator of $\widetilde{E}_2^{2,qa+2n+1}(S^{2n+1})$ is $h_1 \otimes v_1^{a-m+j-1} h_1^{m-j} \iota_{2n+1}$ mod terms defined on $S^{2(m-j)-1}$.

This is illustrated in the chart below, where we list just leading terms, an element connected to one just below it by a vertical line is p times that element, and elements at the same horizontal level are related by the iterated double suspension homomorphism. We omit the subscript from h_1 and v_1, and the $\otimes$.

$$
\begin{array}{ccccc}
S^3 & S^5 & & S^{2t+3} & S^{2t+5} \\
\hline
hv^{a-2}h & hv^{a-2}h & \cdots & hv^{a-2}h & \\
 & | & & | & \\
 & hv^{a-3}h^2 & & hv^{a-3}h^2 & hv^{a-3}h^2 \\
 & & & | & | \\
 & & & \vdots & \vdots \\
 & & & | & | \\
 & & & hv^{a-t-2}h^{t+1} & hv^{a-t-2}h^{t+1} \\
 & & & & | \\
 & & & & hv^{a-t-3}h^{t+2} \\
 & & & & & \ddots
\end{array}
$$

$$
\begin{array}{cccc}
S^{2(a-t)-1} & S^{2(a-t)+1} & & S^{2a-1} \\
\hline
hv^{2t}h^{a-2t-1} & & & \\
| & & & \\
hv^{2t-1}h^{a-2t} & hv^{2t-1}h^{a-2t} & & \\
| & | & & \\
\vdots & \vdots & \ddots & \\
hv^t h^{a-t-1} & hv^t h^{a-t-1} & \cdots & hv^t h^{a-t-1}
\end{array}
$$

The proof of Theorem 6.6 requires results about Hopf invariants which we will address shortly. We begin by describing a plausibility argument for it using only elementary ideas about the unstable cobar complex. We continue to omit the subscript of h_1 and v_1.

(i) The lead term $h \otimes v^{a-n+j-1} h^{n-j}$ does not pull back to $S^{2(n-j)-1}$ because $h^{n-j}\iota_{2(n-j)-1}$ does not satisfy the unstable condition.

(ii) By Proposition 6.1, $ph = v - \eta_R(v)$, a fact which we will begin to use frequently. It implies that

$$p \cdot h \otimes v^{a-n+j-1} h^{n-j} = h \otimes v^{a-n+j} h^{n-j-1} - h \otimes v^{a-n+j-1} h^{n-j-1} v.$$

The second term desuspends below $S^{2(n-j)-1}$, and the first term is the next term up the unstable tower.

(iii) We show that Σ^2 applied to the element of order p on S^{2n+1} is a boundary when $n \geqslant t+1$. To do this, we give a more precise description of this element of order p as $d(v^{a-n-1}h^{n+1})\iota_{2n+1}$. This clearly double suspends to the boundary $d(v^{a-n-1}h^{n+1}\iota_{2n+3})$. Note how we had to wait until S^{2n+3} in order to put the ι inside the $d(-)$, since $h_{n+1}\iota_{2n+1}$ does not satisfy the unstable condition. It remains to show that the lead term is correct, which is the content of the following proposition.

PROPOSITION 6.7. *If* $t = \nu_p(a)$, *then*

$$d\big(v^{a-n-1}h^{n+1}\big) \equiv h \otimes v^{a+t-n-1}h^{n-t}$$

mod terms defined on $S^{2(n-t-1)+1}$.

PROOF. Replacing v by $ph + \eta_R(v)$ implies

$$v^{a-n-1}h^{n+1} \equiv p^{a-n-1}h^a \quad \big(\operatorname{mod} S^{2n+1}\big).$$

Since boundaries on S^{2n+1} desuspend to $S^{2(n-t)-1}$, we obtain

$$d\big(v^{a-n-1}h^{n+1}\big) \equiv p^{a-n-1}d\big(h^a\big)$$

mod the indeterminacy stated in the proposition. Now

$$d(h^a) = \sum \binom{a}{j} h^j \otimes h^{a-j},$$

and, since

$$\nu_p\left(\binom{a}{j}\right) \geqslant t+2-j \quad \text{for } j>1,$$

we find

$$p^{a-n-1}d\big(h^a\big) \equiv sp^{a-n-1+t}h \otimes h^{a-1} \equiv sh \otimes v^{a+t-n-1}h^{n-t}$$

mod terms defined on $S^{2(n-t-1)+1}$. Here $a = sp^t$ with s not a multiple of p, and we have freely replaced ph by $v - \eta_R(v)$. $\square$

The main detail in the proof of Theorem 6.6 which is lacking in the plausibility argument above is an argument for why these are the only unstable elements on the 2-line. For this, we need the following result, which will also be useful in other contexts.

THEOREM 6.8. (i) *There is an unstable Γ-comodule $W(n)$ and an exact sequence*

$$\xrightarrow{P_2} E_2^{s,t-1}(S^{2n-1}) \xrightarrow{\Sigma^2} E_2^{s,t+1}(S^{2n+1}) \xrightarrow{H_2} \mathrm{Ext}_{\mathcal{U}}^{s-1,t-1}(W(n)) \xrightarrow{P_2} E_2^{s+1,t-1}(S^{2n-1}). \tag{6.4}$$

(ii) *$W(n)$ is a free module over BP_*/p on classes x_{2p^in-1} for $i > 0$ with coaction*

$$\psi(x_{2p^kn-1}) = \sum_i p^{k-i} h_{k-i}^{np^i} \otimes x_{2p^in-1}.$$

(iii) $\mathrm{Ext}_{\mathcal{U}}^0(W(n)) \approx \mathbf{Z}_p[v_1]x_{2pn-1}$.

(iv) *If $z \in \mathrm{Ext}_{\mathcal{U}}(W(n))$ is represented by $\sum \gamma_k \otimes x_{2p^kn-1}$, then*

$$P_2(z) = d\Big(\sum \gamma_k \otimes p^{k-1} h_k^n\Big) \otimes \iota_{2n-1}.$$

(v) *Every element $x \in E_2^s(S^{2n+1})$ may be represented, mod terms which desuspend to S^{2n-1}, by a cycle of the form*

$$\sum \gamma_k \otimes p^{k-1} h_k^n \otimes \iota_{2n+1},$$

with $\gamma_k \in C^*(A_{2p^kn-1} \otimes \mathbf{Z}_p)$. Then

$$H_2(x) = \sum \gamma_k \otimes x_{2p^kn-1}.$$

Recall in (v) that A_t is the free BP_*-module on a generator of degree t, and that $C^*(-)$ denotes the reduced unstable cobar complex.

We provide a bare outline of the proof, beginning with the construction from [9]. There is a nonabelian category G of unstable Γ-coalgebras, and a notion of Ext_G such that if BP_*X is a free BP_*-module of finite type, then $E_2(X)$ (of the UNSS) is $\mathrm{Ext}_G(BP_*X)$. Letting $P_G(-)$ denote the primitives in G, one finds that if M is an object of G, then $P_G(M)$ is in the category $\mathcal{U}$. By considering an appropriate double complex, one can construct a composite functor spectral sequence converging to $\mathrm{Ext}_G(M)$ with

$$E_2^{p,q} \approx \mathrm{Ext}_{\mathcal{U}}^p\big(R^qP_G(M)\big).$$

Here R^qP_G denotes the qth right derived functor of P_G. If M satisfies $R^qP_GM = 0$ for $q > 1$, then the spectral sequence has only two nonzero columns, and reduces to an exact sequence

$$\to \mathrm{Ext}_{\mathcal{U}}^s(P_GM) \to \mathrm{Ext}_G^s(M) \to \mathrm{Ext}_{\mathcal{U}}^{s-1}\big(R^1P_GM\big) \to . \tag{6.5}$$

This will be the case when $M = BP_*(\Omega S^{2n+1})$. One verifies that $P_GBP_*(\Omega S^{2n+1}) \approx A_{2n}$, and that $R^1P_GBP_*(\Omega S^{2n+1})$ is the comodule $W(n)$ described in Theorem 6.8, and the exact sequence (6.5) reduces to the sequence (6.4) in this case. The descriptions of the morphisms in (iv) and (v) are obtained in [3] using an alternate construction of the exact sequence. Part (iii) is proved in [5, p. 535] by studying explicit cycles.

Now we complete our observations on the proof of Theorem 6.6. If x is any nonzero unstable element on the 2-line, then there must be k and n so that

$$\Sigma^{2k}x \neq 0 \in \ker\big(E_2^{2,*}(S^{2n-1}) \xrightarrow{\Sigma^2} E_2^{2,*+2}(S^{2n+1})\big).$$

Then by Theorem 6.8, there must be an element $v_1^e x_{2pn-1} \in \mathrm{Ext}^0_{\mathcal{U}}(W(n))$ such that

$$P_2(v_1^e x_{2pn-1}) = d(v_1^e h_1^n)\iota_{2n-1} = \Sigma^{2k}x.$$

But this is exactly the description of the unstable elements on the 2-line which was given in the third part of our plausibility argument for Theorem 6.6.

When $p = 2$, the discussion above about the unstable elements on the 2-line goes through almost without change, as described in [10, pp. 482–484]. The result is that if $n \leqslant a - \nu_2(a) - 2$, then

$$\widetilde{E}_2^{2,2n+1+2a}(S^{2n+1}) \approx \begin{cases} \mathbf{Z}/2 & \text{if } a \text{ is odd,} \\ \mathbf{Z}/2^{\min(\nu_2(a)+2,n)} & \text{if } a \text{ is even.} \end{cases}$$

The orders when a is even are 1 larger than in the odd primary case because v_2 can be used to obtain 1 additional desuspension.

Now we construct the v_1-periodic UNSS, following [4] for the most part. In [7] a UNSS converging to $\mathrm{map}_*(Y, X)$ was constructed. If $Y = M^n(p^e)$, the E_2-term is the homology of $C^*(P(BP_*(X))) \otimes \mathbf{Z}/p^e$. The Adams map A induces

$$\mathrm{UNSS}\big(\mathrm{map}_*(M^n(p^e), X)\big) \xrightarrow{A^*} \mathrm{UNSS}\big(\mathrm{map}_*(M^{n+s(e)}(p^e), X)\big),$$

where $s(e)$ is as in eq. (2.1). By [35], on E_2 this is just multiplication by a power of v_1 after iterating sufficiently. As in our definition of $v_1^{-1}\pi_*(-)$, we define the v_1-periodic UNSS of X by

$$v_1^{-1}E_r^{*,*}(X) = \operatorname*{dirlim}_{e,k} E_r^{*,*+1}\big(\mathrm{map}_*(M^{ks(e)}(p^e), X)\big).$$

The direct system over e utilizes the maps $\rho : M^n(p^{e+1}) \to M^n(p^e)$ used in Section 1, and the shift of one dimension is done for the same reason as in our definition of $v_1^{-1}\pi_*(-)$. Similarly to Proposition 2.4, we have

PROPOSITION 6.9. *On the category of spaces with H-space exponents, there is a natural transformation from the UNSS to the v_1-periodic UNSS.*

One of the main theorems of [4] is the following determination of the v_1-periodic UNSS of S^{2n+1}.

THEOREM 6.10. *Let p be an odd prime. The v_1-periodic UNSS of S^{2n+1} collapses from E_2 and satisfies*

$$v_1^{-1}E_2^{s,2n+1+u}\left(S^{2n+1}\right) \approx \begin{cases} \mathbf{Z}/p^{\min(n,\nu_p(a)+1)} & \text{if } s = 1 \text{ or } 2, \text{ and } u = qa, \\ 0 & \text{otherwise.} \end{cases}$$

The morphism $E_2^{s,t}(S^{2n+1}) \to v_1^{-1}E_2^{s,t}(S^{2n+1})$ is an isomorphism if $s = 1$ and $t > 2n + 1$, while for $s = 2$ it sends the unstable towers injectively, and bijectively unless $n \geqslant a - \nu_p(a) - 1$, where $t = 2n + 1 + qa$.

PROOF. It is readily verified that the elements on the 1- and 2-lines described in Theorems 6.3 and 6.6 form v_1-periodic families. The main content of this theorem is that there is nothing else which is v_1-periodic.

In order to prove this, we use a v_1-periodic version of the double suspension sequence (6.4). It is proved in [4] that the morphisms of (6.4) behave nicely with respect to v_1-action, yielding an exact sequence

$$\begin{aligned} &\xrightarrow{P_2} v_1^{-1}E_2^{s,t-1}\left(S^{2n-1}\right) \xrightarrow{\Sigma^2} v_1^{-1}E_2^{s,t+1}\left(S^{2n+1}\right) \\ &\xrightarrow{H_2} v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s-1,t-1}\left(W(n)\right) \xrightarrow{P_2}. \end{aligned} \tag{6.6}$$

By Theorem 6.8(ii), there is a spectral sequence converging to $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(W(n))$ with

$$E_1^{s,t} \approx \bigoplus_{i\geqslant 1} v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t}(A_{2p^in-1} \otimes \mathbf{Z}_p). \tag{6.7}$$

There is a short exact sequence given by the universal coefficient theorem

$$\begin{aligned} 0 &\to v_1^{-1}E_2^{s,t}\left(S^n\right) \otimes \mathbf{Z}_p \to v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t}(A_n \otimes \mathbf{Z}_p) \\ &\to \mathrm{Tor}\left(v_1^{-1}E_2^{s-1,t}\left(S^n\right), \mathbf{Z}_p\right) \to 0. \end{aligned} \tag{6.8}$$

We will use eqs. (6.6), (6.7), and (6.8) to show inductively that there are no unexpected elements in $v_1^{-1}E_2(S^{2n+1})$. But first we show how the known elements fit into this framework. By (6.8), each summand in $v_1^{-1}E_2^1(S^n)$ gives two $\mathbf{Z}_p$'s, called stable, in $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(A_n \otimes \mathbf{Z}_p)$, and similarly each summand in $v_1^{-1}E_2^2(S^n)$ gives two summands in $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(A_n \otimes \mathbf{Z}_p)$, called unstable. We claim that

$$v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t}\left(W(n)\right) \approx \begin{cases} \mathbf{Z}_p & \text{if } s = 0 \text{ or } 1, \text{ and } t \equiv 2n - 1 \bmod q, \\ 0 & \text{otherwise.} \end{cases} \tag{6.9}$$

In [4, p. 57], the relationship between (6.9) and (6.7) is discussed: in the spectral sequence (6.7), stable classes from the $(i + 1)$-summand hit unstable classes from the i-summand, yielding in $v_1^{-1}E_\infty$ only the stable classes from the 1-summand. These are the elements described in eq. (6.9). On the other hand, in the exact sequence (6.6), let $t = 2n + kq$ with $e = \nu_p(k)$. If $e < n - 1$, then Σ^2 is $\mathbf{Z}/p^e \xrightarrow{\cdot p} \mathbf{Z}/p^e$ when $s = 2$, yielding the elements in $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t-1}(W(n))$ for $s = 0$ and 1, while if $e \geqslant n - 1$, then for $s = 1$ and 2, Σ^2 is $\mathbf{Z}/p^{n-1} \hookrightarrow \mathbf{Z}/p^n$, also yielding elements in $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t-1}(W(n))$ for $s = 0$ and 1.

It seems useful for this proof to have one more bit of input, namely the result for the v_1-periodic stable Novikov spectral sequence, which can be defined as a direct limit over e of stable Novikov spectral sequences of $M(p^e)$.

THEOREM 6.11 ([13, §2]). *There is a v_1-periodic stable Novikov spectral sequence for S^0, satisfying*

$$v_1^{-1}E_r^{s,t}(S^0) = \operatorname*{dirlim}_n v_1^{-1}E_r^{s,t+2n+1}(S^{2n+1}),$$

and

$$v_1^{-1}E_2^{s,t}(S^0) \approx \begin{cases} \mathbf{Z}/p^{\nu_p(t)+1} & \text{if } s = 1 \text{ and } t \equiv 0 \bmod q, \\ 0 & \text{otherwise.} \end{cases}$$

We prove by induction on s that, for all n, $v_1^{-1}E_2^s(S^{2n+1})$, $v_1^{-1}E_2^{s+1}(S^{2n+1})$, and $v_1^{-1}\mathrm{Ext}^s_{\mathcal{U}}(W(n))$ contain only the elements described in Theorem 6.10 and eq. (6.9). This is easily seen to be true when $s = 0$, where we know all groups completely. Assume it is true for all $s < \sigma$. Then $v_1^{-1}E_2^\sigma(S^{2n+1})$ contains no unexpected elements because $\sigma = (\sigma - 1) + 1$. If $v_1^{-1}E_2^{\sigma+1}(S^{2n+1})$ contains an unexpected element, then some $v_1^{-1}E_2^{\sigma+1}(S^{2L+1})$ must contain one in $\ker(\Sigma^2)$ by Theorem 6.11. Such an element must be $P_2(x)$, where x is an unexpected element of $v_1^{-1}\mathrm{Ext}^{\sigma-1}_{\mathcal{U}}(W(n))$, but no such element exists by our induction hypothesis. Finally, $v_1^{-1}\mathrm{Ext}^\sigma_{\mathcal{U}}(W(n))$ contains no unexpected elements by (6.7) and (6.8) since, as just established, $v_1^{-1}E_2^\sigma(S^{2m+1})$ and $v_1^{-1}E_2^{\sigma+1}(S^{2m+1})$ contain no unexpected elements for any m. □

The UNSS and v_1-periodic UNSS are considerably more complicated at the prime 2 than at the odd primes, but the v_1-periodic UNSS of S^{2n+1} is still completely understood. We shall not discuss it in detail because most of our applications in this paper will be at the odd primes. The reader desiring more detail is referred to [4], which gives a chart with UNSS names of the elements.

We reproduce in figs. 13 and 14 the charts from [10, p. 488] of the v_1-periodic UNSS of S^{2n+1} at the prime 2. Here "3" means $\mathbf{Z}/2^3$, and "ν" means $\mathbf{Z}/2^\nu$, where

$$\nu = \min\left(\nu_2(8k+8) + 1, n\right).$$

Differentials emanating from a summand of order greater than 2 are nonzero only on a generator of the summand. Note how $\mathbf{Z}/8$ in periodic homotopy is obtained as an extension by the $\mathbf{Z}_2$ in filtration 3 of the elements in the 1-line group which are divisible by 2 in a $\mathbf{Z}/8$.

Figure 14 applies to S^{2n+1} when $n \equiv 1$ or 2 mod 4, with $n > 2$. The reader is referred to [10, p. 487] for the minor changes required when $n \leqslant 2$. In fig. 14, the dotted differential is present if and only if $\nu = n$. In both charts, the left η-action on the $\mathbf{Z}/2^\nu$ on line 1, which is usually indicated by positively sloping solid lines, is indicated by the dotted line if $n < \nu(8k+8) + 1$.

The argument establishing these charts appears in [4]. Note that $v_1^{-1}E_4^{s,*}(S^{2n+1}) = 0$ if $s > 4$, and hence no higher differentials are possible.

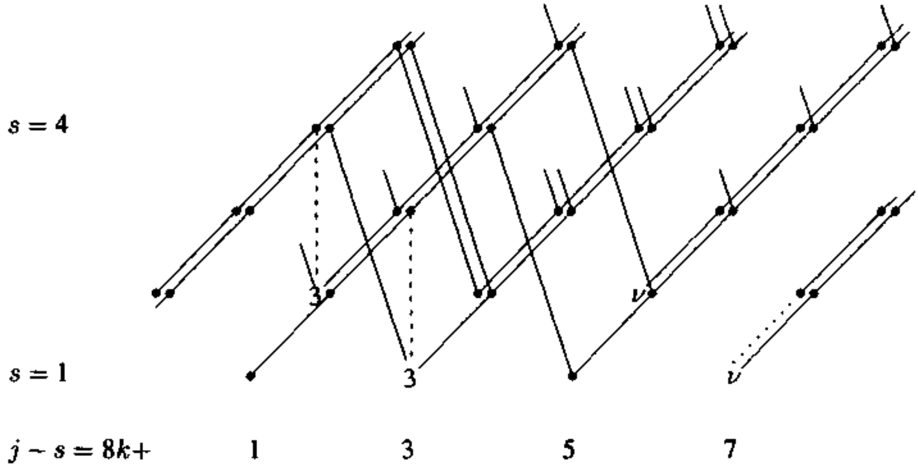

Figure 13. $v_1^{-1}E_2^{s,2n+1+j}(S^{2n+1})$, $n \equiv 0, 3 \bmod 4$, $p=2$.

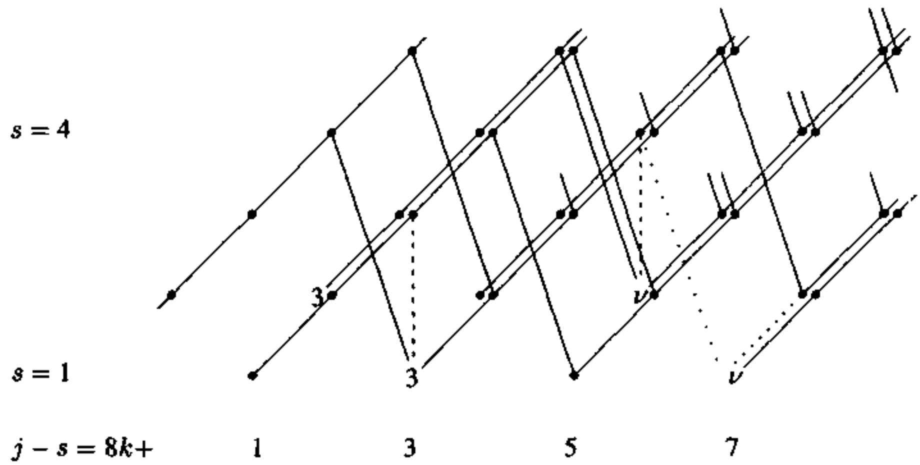

Figure 14. $v_1^{-1}E_2^{s,2n+1+j}(S^{2n+1})$, $n \equiv 1, 2 \bmod 4$, $n>2$, $p=2$.

7. v_1-periodic homotopy groups of $\mathrm{SU}(n)$

In this section we show how the v_1-periodic UNSS determines the v_1-periodic homotopy groups of spherically resolved spaces. This relationship is particular nice when localized at an odd prime, and it is this case on which we focus most of our attention. At the end of the section, we discuss the changes required when $p=2$. After proving the general result for spherically resolved spaces, we specialize to $\mathrm{SU}(n)$, where the result is, in some sense, explicit.

It is not clear that the v_1-periodic UNSS of a space X must converge to the p-primary v_1-periodic homotopy groups of X. There might be periodic homotopy classes which are not detected in the periodic UNSS because multiplication by v_1 repeatedly increases BP filtration. It is also possible that a v_1-periodic family in E_2 might support arbitrarily long differentials in the unlocalized spectral sequence, in which case it would exist in all $v_1^{-1}E_r$, but would not represent an element of periodic homotopy. We now show that neither of these anomalies can occur for a spherically resolved space, essentially because they cannot happen for an odd sphere, where the v_1-periodic homotopy groups are known to agree with the $v_1^{-1}E_2$-term.

DEFINITION 7.1. A space X is *spherically resolved* if there are spaces $X_0, \dots, X_L$, with $X_0 = *$, $X_L = X$, and fibrations

$$X_{i-1} \to X_i \to S^{n_i} \tag{7.1}$$

with n_i odd, and algebra isomorphisms

$$H^*(X_i) \approx H^*(X_{i-1}) \otimes H^*(S^{n_i}).$$

The following result was stated as Theorem 1.1. It was proved in [4].

THEOREM 7.2. *If p is odd, and X is spherically resolved, then $v_1^{-1}E_2^{s,t}(X) = 0$ unless $s = 1$ or 2, and t is odd. The v_1-periodic UNSS collapses to the isomorphisms*

$$v_1^{-1}\pi_i(X) \approx \begin{cases} v_1^{-1}E_2^{1,i+1}(X) & \text{if } i \text{ is even,} \\ v_1^{-1}E_2^{2,i+2}(X) & \text{if } i \text{ is odd.} \end{cases}$$

PROOF. Each algebra $BP^*(X_i)$ is free, and so eq. (6.1) applies to give $E_2(X_i) \approx \mathrm{Ext}_{\mathcal{U}}(M(x_{n_1}, \dots, x_{n_i}))$, where $M(-)$ denotes a free BP_*-module on the indicated generators. There are short exact sequences in $\mathcal{U}$

$$0 \to M(x_{n_1}, \dots, x_{n_{i-1}}) \to M(x_{n_1}, \dots, x_{n_i}) \to M(x_{n_i}) \to 0,$$

and hence long exact sequences

$$\to E_2^{s,t}(X_{i-1}) \to E_2^{s,t}(X_i) \to E_2^{s,t}(S^{n_i}) \to E_2^{s+1,t}(X_{i-1}) \to . \tag{7.2}$$

These exact sequences are compatible with the direct system of v_1-maps whose limit is the v_1-periodic groups. Thus there is a v_1-periodic version of (7.2), and hence by Theorem 6.10 and induction on i, $v_1^{-1}E_2^{s,t}(X) = 0$ unless $s = 1$ or 2, and t is odd. Thus $v_1^{-1}E_2(X) \approx v_1^{-1}E_\infty(X)$.

If $u > \max\{n_i\}$, $s = 1$ or 2, and $s + u$ is odd, there are natural edge morphisms $\pi_u(X) \to E_2^{s,s+u}(X)$, and these are compatible with the direct system of v_1-maps, giving morphisms $v_1^{-1}\pi_u(X) \to v_1^{-1}E_2^{s,s+u}(X)$. These yield a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & v_1^{-1}\pi_{2k}(X_{i-1}) \arrow[r] \arrow[d, "\phi_{i-1}"] & v_1^{-1}\pi_{2k}(X_i) \arrow[r] \arrow[d, "\phi_i"] & v_1^{-1}\pi_{2k}(S^{n_i}) \arrow[r] \arrow[d, "\psi"] & {} \\
0 \arrow[r] & v_1^{-1}E_2^{1,2k+1}(X_{i-1}) \arrow[r] & v_1^{-1}E_2^{1,2k+1}(X_i) \arrow[r] & v_1^{-1}E_2^{1,2k+1}(S^{n_i}) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & v_1^{-1}\pi_{2k-1}(X_{i-1}) \arrow[r] \arrow[d, "\phi'_{i-1}"] & v_1^{-1}\pi_{2k-1}(X_i) \arrow[r] \arrow[d, "\phi'_i"] & v_1^{-1}\pi_{2k-1}(S^{n_i}) \arrow[r] \arrow[d, "\psi'"] & 0 \\
{} \arrow[r] & v_1^{-1}E_2^{2,2k+1}(X_{i-1}) \arrow[r] & v_1^{-1}E_2^{2,2k+1}(X_i) \arrow[r] & v_1^{-1}E_2^{2,2k+1}(S^{n_i}) \arrow[r] & 0.
\end{tikzcd} \tag{7.3}$$

The zeros at the ends of the $v_1^{-1}E_2$-sequence follow from the previous paragraph. The zero morphism coming into $v_1^{-1}\pi_{2k}(X_{i-1})$ follows from $v_1^{-1}E_2^0(S^{n_i}) = 0$ and ϕ_{i-1} being an isomorphism, which is inductively known. The zero morphism coming out from $v_1^{-1}\pi_{2k-1}(S^{n_i})$ follows since anything in the image must have filtration $\geqslant 2$, but, by induction, $v_1^{-1}\pi_{2k-2}(X_{i-1})$ is 0 above filtration 1.

Comparison of Theorems 4.2 and 6.10 shows that the groups related by ψ and by ψ' are isomorphic, and it is easy to see that ψ and ψ' induce the isomorphisms. (See [23] for reasons.) Since X_1 is a sphere, this comparison also shows that ϕ_1 and ϕ'_1 are isomorphisms, which starts the induction. Thus all ϕ_i and ϕ'_i are isomorphisms by induction and the 5-lemma. □

Theorem 7.2 is a nice result, but it still leaves the formidable task of calculating $v_1^{-1}E_2(X)$. In [6], Bendersky proved the following seminal result, whose proof we will discuss throughout much of the remainder of this section.

THEOREM 7.3. *If $k \geqslant n$, then in the UNSS $E_2^{1,2k+1}(\mathrm{SU}(n)) \approx \mathbf{Z}/p^{e_p(k,n)}$, where $e_p(k,n)$ is as defined in Definition* 1.2, *and p is any prime.*

This allows us to easily deduce Theorem 1.3, now demoted to corollary status, which we restate for the convenience of the reader.

COROLLARY 7.4. *If p is odd, then $v_1^{-1}\pi_{2k}(\mathrm{SU}(n)) \approx \mathbf{Z}/p^{e_p(k,n)}$, and $v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n))$ is an abelian group of the same order.*

PROOF OF COROLLARY. The first part of the corollary is a straightforward application of Theorems 7.2 and 7.3, once we know that the groups in Theorem 7.3 are v_1-periodic. This can be seen by observing how they arise, from exact sequences built from spheres, where the classes are all v_1-periodic. In [23], a slightly different proof of this part of the corollary was given, before the v_1-periodic UNSS had been hatched.

The exact sequence like the top row of (7.3) for the fibration

$$\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1} \tag{7.4}$$

implies, by induction on n, that $|v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n))| = |v_1^{-1}\pi_{2k}(\mathrm{SU}(n))|$. Indeed, the orders are equal for S^{2n-1} by Theorem 6.10, and so if they are equal for $\mathrm{SU}(n-1)$, then they will be equal for $\mathrm{SU}(n)$, since the alternating sum of the exponents of p in an exact sequence is 0. The fact that $\mathrm{SU}(2) = S^3$ starts the induction. □

In [23], an example of a noncyclic group $v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n); 3)$ was given, and in [13] it was shown that $v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n); 2)$ will often have many summands (in addition to a regular pattern of $\mathbf{Z}_2$'s).

In order to prove Theorem 7.3, it is convenient to work with the UNSS based on MU, rather than BP. This allows us to work with the ordinary exponential series, rather than its p-typical analogue. The facts about MU that we need are summarized in the following result.

PROPOSITION 7.5. (i) *$MU_*(MU)$ is a polynomial algebra over MU_* with generators B_i of grading $2i$ for $i > 0$. There are elements $\beta_i \in MU_{2i}(CP^\infty)$ which form a basis for $MU_*(CP^\infty)$ as an MU_*-module.*

(ii) *Let B denote the formal sum $1 + \sum_{i>0} B_i$. The coaction*

$$MU_*(CP^\infty) \xrightarrow{\psi} MU_*MU \otimes_{MU_*} MU_*(CP^\infty)$$

satisfies

$$\psi(\beta_n) = \sum_j (B^j)_{n-j} \otimes \beta_j.$$

Here $(B^j)_{n-j}$ denotes the component in grading $2(n-j)$ of the jth power of the formal sum B.

(iii) *There is a ring homomorphism $\bar{e} : MU_*(MU) \otimes \mathbf{Q} \to \mathbf{Q}$ satisfying*

- $\bar{e}(B_i) = 1/(i+1)$.
- $\bar{e}(\eta_R(a)) = 0$ *if $a \in MU_i$ with $i > 0$.*
- $\bar{e}$ *induces an injection $E_2^{1,2n+1+2k}(S^{2n+1}) \to \mathbf{Q}/\mathbf{Z}$.*

(iv) *The BP-based UNSS is the p-localization of the MU-based UNSS.*

PROOF. Part (i) is standard (e.g., [2]), while part (ii) is [2, 11.4]. Part (iii) is from [6]. There are elements $m_i \in MU_{2i} \otimes \mathbf{Q}$ such that $MU_*MU \otimes \mathbf{Q}$ is a polynomial algebra over $\mathbf{Q}$ on all m_i and $\eta_R(m_i)$. One defines $\bar{e}$ to be the ring homomorphism which sends m_i to $1/(i+1)$ and $\eta_R(m_i)$ to 0. The second property in (iii) is clear, and the first follows by conjugating [2, 9.4] to obtain

$$m_n = \sum \eta_R(m_i)\left(B^{i+1}\right)_{n-i},$$

and then applying $\bar{e}$ to obtain $\bar{e}(m_n) = \bar{e}(B_n)$.

One way to see the third property is to localize at p and pass to BP. Then $m_{p^i-1}^{MU}$ passes to m_i^{BP}, and so our $\bar{e}$ passes to that of [6, 4.3]. It is shown on [6, p. 751] that $\bar{e}_{BP}$ sends the p-local 1-line injectively, and this works for all p. □

Now we can state a general theorem which incorporates most of the work in proving Theorem 7.3. This theorem was stated without proof in [11, 3.10], where it was applied to $X = Sp(n)$, $p = 2$. We will outline the proof, which is a direct generalization of [6], later in this section.

THEOREM 7.6. *Suppose X is spherically resolved as in Definition 7.1 with $n_1 < n_2 < \cdots$, and L possibly infinite. Then $MU_*(X_k)$ is an exterior algebra over MU_* on classes $y_1, \ldots, y_k$, with $|y_i| = n_i$. Let*

$$\overline{E}^{1,t}(X_k) = \ker\bigl(E_2^{1,t}(X_k) \to E_2^{1,t}(X_L)\bigr).$$

Let $\gamma_{k,j} \in MU_*(MU)$ be defined in terms of the coaction in $MU_*(X_i)$ by

$$\psi(y_k) = \sum_{j=1}^{k} \gamma_{k,j} \otimes y_j, \tag{7.5}$$

and let $b_{k,j} = \overline{e}(\gamma_{k,j}) \in \mathbf{Q}$. Then the matrix $B = (b_{k,j})$ is lower triangular with 1's on the diagonal. Let $C = (c_{k,j})$ be the inverse of B, and let

$$\omega_k(m) = \text{l.c.m.}\bigl\{\text{den}(c_{k,j}) :\ m \leqslant j \leqslant k\bigr\}.$$

Then $\operatorname{coker}(\overline{E}^{1,n_k}(X_{m-1}) \to \overline{E}^{1,n_k}(X_{k-1}))$ *is cyclic of order $\omega_k(m)$.*

Now we specialize to $\mathrm{SU}(n)$, where we need the following result.

PROPOSITION 7.7. *In the UNSS*

(i) $E_2^{s,t}(\mathrm{SU}) = 0$ *if* $s > 0$.
(ii) $E_2^{1,2k+1}(\mathrm{SU}(n)) = 0$ *if* $n > k$.
(iii) $E_2^{1,2k+1}(\mathrm{SU}(k)) \approx \mathbf{Z}/k!$.
(iv) *If $i < j \leqslant k$, the inclusion $\mathrm{SU}(i) \to \mathrm{SU}(j)$ induces an injection in $E_2^{1,2k+1}$.*
(v) SU *is spherically resolved as in Theorem 7.6 with $X_k = \mathrm{SU}(k+1)$, $n_k = 2k+1$, and if the MU-coaction on* SU *is as in eq. (7.5), then*

$$\sum_{k \geqslant j} \overline{e}(\gamma_{k,j}) x^k = \bigl(-\log(1-x)\bigr)^j. \tag{7.6}$$

PROOF. Part (i) is a nontrivial consequence of the fact that SU is an H-space with torsion-free homology and homotopy. See [6, 3.1]. The coalgebra $MU_*(\mathrm{SU}(n))$ is cofree with primitives isomorphic to $MU_*(\Sigma CP^{n-1})$. Hence by (6.1)

$$E_2(\mathrm{SU}(n)) \approx \mathrm{Ext}_{\mathcal{U}}\bigl(MU_*(\Sigma CP^{n-1})\bigr),$$

and so the fibrations (7.4) induce exact sequences in E_2. Part ii follows from part i and the exact sequence, and part iv is also immediate from the exact sequence.

Let $y_{2k+1} \in MU_{2k+1}(\mathrm{SU}(n))$ be the generator corresponding to

$$\Sigma\beta_k \in MU_{2k+1}\left(\Sigma CP^{n-1}\right)$$

for $k < n$. Part (iii) is proved on [6, p. 748] by showing that the generator of $E_2^{0,2k+1}(\mathrm{SU})$ is of the form $k!y_{2k+1}+$ lower terms, so that

$$E_2^{0,2k+1}\left(\mathrm{SU}(k+1)\right) \to E_2^{0,2k+1}\left(S^{2k+1}\right)$$

sends the generator of one $\mathbf{Z}$ to $k!$ times the generator of the other $\mathbf{Z}$. This implies part (iii).

By Proposition 7.5(ii) and the relationship of $MU_*(\mathrm{SU}(n))$ with $MU_*(\Sigma CP^{n-1})$ noted above, we find

$$\psi(y_{2k+1}) = \sum \left(B^j\right)_{k-j} \otimes y_{2j+1}.$$

By Proposition 7.5(iii), we have $\sum \overline{e}(B_i)x^{i+1} = -\log(1-x)$. These facts yield part (v). $\square$

Now we can prove Theorem 7.3. If $f(x)$ is a power series with constant term 0, let $[f(x)]$ denote the infinite matrix whose entries $a_{k,j}$ satisfy

$$f(x)^j = \sum_k a_{k,j}x^k.$$

One easily verifies that $[g(x)][f(x)] = [f(g(x))]$. Hence the inverse of the matrix $[-\log(1-x)]$ is $[1-e^{-x}]$. This observation, with Theorem 7.6 and Proposition 7.7, implies that there is a short exact sequence

$$0 \to E_2^{1,2k+1}\left(\mathrm{SU}(n)\right) \to E_2^{1,2k+1}\left(\mathrm{SU}(k)\right) \to \mathbf{Z}/\omega_k(n) \to 0$$

with middle group $\mathbf{Z}/k!$ and

$$\omega_k(n) = \mathrm{l.c.m.}\left\{\mathrm{den}\left(\mathrm{coef}\left(x^k, (1-\mathrm{e}^{-x})^j\right)\right) \colon n \leqslant j \leqslant k\right\}.$$

Thus $E_2^{1,2k+1}(\mathrm{SU}(n))$ is cyclic of order

$$\begin{aligned} &k! \;/\; \mathrm{l.c.m.}\left\{\mathrm{den}\left(\mathrm{coef}\left(x^k, (\mathrm{e}^x - 1)^j\right)\right) \colon n \leqslant j \leqslant k\right\} \\ &\quad = \gcd\left\{\mathrm{coef}\left(\frac{x^k}{k!}, (\mathrm{e}^x - 1)^j\right) \colon n \leqslant j \leqslant k\right\}. \end{aligned}$$

Looking at exponents of p yields Theorem 7.3 by Proposition 7.5(iv).

It remains to prove Theorem 7.6, the notation of which we employ without comment. Define $b_{k,j}(m)$ for $m \leqslant k$ recursively by $b_{k,j}(k) = b_{k,j}$ and

$$b_{k,j}(m) = b_{k,j}(m+1) - b_{k,m}(m+1)b_{m,j}. \tag{7.7}$$

We begin by noting that if row reduction is performed on $(B|I)$ so as to get at each step one more diagonal of 0's below the main diagonal of B, we find that the entries $c_{k,j}$ of B^{-1} satisfy

$$c_{k,j} = \begin{cases} 0 & \text{if } j > k, \\ 1 & \text{if } j = k, \\ -b_{k,j}(j+1) & \text{if } j < k. \end{cases}$$

Then

$$\omega_k(m) = \max\big(\operatorname{ord}\big(b_{k,j}(j+1)\big):\ m \leqslant\ j \leqslant k\big). \tag{7.8}$$

Here and throughout this proof, $\operatorname{ord}(-)$ refers to order in $\mathbf{Q}/\mathbf{Z}$.

Fix k, and let

$$\tau(m) = \big|\operatorname{coker}\big(\overline{E}^{1,n_k}(X_{m-1}) \to \overline{E}^{1,n_k}(X_m)\big)\big|.$$

We drop the subscript k from eqs. (7.7) and (7.8). The fibration $X_{k-1} \to X_k \to S^{n_k}$ implies that the generator $g(k-1)$ of $\overline{E}^{1,n_k}(X_{k-1})$ is

$$d(y_k) = \sum_{j<k} \gamma_{k,j} \otimes y_j.$$

Let $a_j(k-1) = \overline{e}(\gamma_{k,j}) = b_{k,j}$. By Proposition 7.5(iii), $\tau(k-1) = \operatorname{ord}(a_{k-1}(k-1))$. Then there is $\alpha \in MU_*$ so that $\tau(k-1)g(k-1) + d(\alpha y_{k-1})$ pulls back to a generator of $\overline{E}^{1,n_k}(X_{k-2})$. Write this generator as $\sum \gamma_j(k-2)y_j$. Then

$$\begin{aligned} a_j(k-2) := \overline{e}\big(\gamma_j(k-2)\big) &= \tau(k-1)a_j(k-1) + \overline{e}(\alpha)b_{k-1,j} \\ &= \tau(k-1)\big(a_j(k-1) - a_{k-1}(k-1)b_{k-1,j}\big). \end{aligned}$$

Here we have used that $\overline{e}(\alpha) = -\overline{e}(d(\alpha))$, which follows from Proposition 7.5(iii).

This procedure can be continued until we obtain a generator $\sum \gamma_j(m-1)y_j$ of $\overline{E}^{1,n_k}(X_{m-1})$ with $a_j(m-1) := \overline{e}(\gamma_j(m-1))$ satisfying

$$a_j(m-1) = \tau(m)\big(a_j(m) - a_m(m)b_{m,j}\big)$$

and $\tau(m) = \operatorname{ord}(a_m(m))$. Now we prove by downward induction on m that $a_j(m-1) = \omega(m)b_j(m)$, the case $m = k$ being trivial.

$$\begin{aligned} a_j(m-1) &= \tau(m)\ \big(a_j(m) - a_m(m)b_{m,j}\big) \\ &= \tau(m)\ \omega(m+1)\big(b_j(m+1) - b_m(m+1)b_{m,j}\big) \\ &= \operatorname{ord}(a_m(m))\omega(m+1)b_j(m) \\ &= \operatorname{ord}\big(\omega(m+1)b_m(m+1)\big)\omega(m+1)b_j(m) \\ &= \max\big(\omega(m+1), \operatorname{ord}\big(b_m(m+1)\big)\big)b_j(m) \\ &= \omega(m)b_j(m) \end{aligned}$$

The portion of this string after the first line and not including the last factor shows that $\tau(m)\omega(m+1) = \omega(m)$, and hence $\omega(m) = \tau(m)\cdots\tau(k-1)$, which is a restatement of the desired conclusion of Theorem 7.6. □

Since each $v_1^{-1}\pi_i(\mathrm{SU}(n))$ occurs as a direct summand of some actual homotopy group of $\mathrm{SU}(n)$, the following result about the p-exponent of $\mathrm{SU}(n)$ is immediate from Theorem 1.3.

COROLLARY 7.8. *Let $\exp_p(X)$ denote the largest e such that, for some i, $\pi_i(X)$ has an element of order p^e. Then, if p is odd,*

$$\exp_p(\mathrm{SU}(n)) \geqslant e_p(n),$$

where $e_p(n) = \max\{e_p(k,n) : k \geqslant n\}$.

We conjecture that this bound is sharp, the main evidence being that the analogous statement is true for odd spheres, by [20]. The numbers $e_p(k,n)$ are explicit in Definition 1.2, and this formula can be used with a computer for specific calculations. However, this formula is not very tractable. In Section 8, we sketch how simple formulas for $v_1^{-1}\pi_{2k}(\mathrm{SU}(n))$ can be obtained for $n \leqslant p^2 - p$ without using Theorem 1.3, but rather by studying the exact sequences of UNSS E_2-terms.

Theorem 7.6 is valid when $p = 2$, but Theorems 7.2 and 1.3 must be modified, due to the more complicated form of the v_1-periodic UNSS for S^{2n+1} when $p = 2$, as discussed at the end of Section 6. We state below the 2-primary version of Theorem 7.2 which was proved in [4], but we refer the reader to [10] for the 2-primary analogue of Theorem 1.3.

THEOREM 7.9 ([6]). *Suppose X is spherically resolved and $p = 2$. Then*

- *$v_1^{-1}E_2(X)$ is generated as an η-module by elements with $s = 1$ or 2. Here η has $(s,t) = (1,2)$.*
- *η acts freely on elements with $s > 2$.*
- *$v_1^{-1}E_4(X) = v_1^{-1}E_\infty(X)$, and $v_1^{-1}E_4^s(X) = 0$ if $s > 4$.*
- *If the groups $v_1^{-1}E_2^{1,t}(X)$ are cyclic, then the v_1-periodic UNSS converges to $v_1^{-1}\pi_*(X)$.*

8. v_1-periodic homotopy groups of some Lie groups

In this section we focus on two examples. One uses UNSS methods to determine $v_1^{-1}\pi_*(G_2;5)$, while the other uses ASS methods to determine $v_1^{-1}\pi_*(F_4/G_2;2)$. Here G_2 and F_4 are the two simplest exceptional Lie groups. The first example is just one of many discussed in [14]. We also discuss how these UNSS methods can be used to give tractable formulas for $v_1^{-1}\pi_*(\mathrm{SU}(n);p)$ when p is odd and $n \leqslant p^2 - p$. We close by summarizing the status of the program, initially proposed by Mimura, of computing the v_1-periodic homotopy groups of all compact simple Lie groups.

Our first theorem concerns the v_1-periodic homotopy groups of certain sphere bundles over spheres, which appear frequently as direct factors of compact simple Lie groups localized at p, according to the decompositions given in [49].

THEOREM 8.1. *Let p be an odd prime, and let $B_1(p)$ denote an S^3-bundle over S^{2p+1} with attaching map α_1. Then the only nonzero v_1-periodic homotopy groups of $B_1(p)$ are*

$$v_1^{-1}\pi_{2p+qm-1}\big(B_1(p)\big) \approx v_1^{-1}\pi_{2p+qm}\big(B_1(p)\big) \approx \mathbf{Z}/p^{\min(p+1,1+\nu_p(m-p^{p-1}))}.$$

This is the case $k=1$ of [14, 2.1]. The spaces $B_1(p)$ were called $B(3,2p+1)$ in [14]. The following result follows immediately from the 5-local equivalence $G_2 \simeq B_1(5)$.

COROLLARY 8.2.

$$v_1^{-1}\pi_i(G_2;5) \approx \begin{cases} \mathbf{Z}/5^{\min(6,1+\nu_5(i-5009))} & \text{if } i \equiv 1 \bmod 8,\\ \mathbf{Z}/5^{\min(6,1+\nu_5(i-5010))} & \text{if } i \equiv 2 \bmod 8,\\ 0 & \text{otherwise.}\end{cases}$$

The following result is the central part of the proof of Theorem 8.1. Indeed, this theorem, 8.3, along with Theorem 6.3, gives the order of each group $E_2^{1,t}(B_1(p))$, and Theorem 8.5 shows the group is cyclic. Then Theorem 7.2 shows that this gives $v_1^{-1}\pi_*(B_1(p))$ when $*$ is even, and the proof of Corollary 7.4 shows that

$$\big|v_1^{-1}\pi_{2k-1}\big(B_1(p)\big)\big| = \big|v_1^{-1}\pi_{2k}\big(B_1(p)\big)\big|.$$

Finally, $v_1^{-1}\pi_*(B_1(p))$ is shown to be cyclic when $*$ is odd in Theorem 8.6.

THEOREM 8.3. *In the exact sequence*

$$\begin{aligned} 0 \to E_2^{1,qm+2p+1}(S^3) &\xrightarrow{i_*} E_2^{1,qm+2p+1}\big(B_1(p)\big)\\ &\xrightarrow{j_*} E_2^{1,qm+2p+1}\big(S^{2p+1}\big) \xrightarrow{\partial} E_2^{2,qm+2p+1}\big(S^3\big),\end{aligned}$$

the morphism ∂ is a surjection to $\mathbf{Z}/p$ unless

$$\nu_p(m) \geqslant p-1 \quad \textit{and} \quad m/p^{p-1} \equiv 1 \bmod p,$$

in which case it is 0.

We will use the double suspension Hopf invariant H_2 discussed in Theorem 6.8. We denote by H' the morphism

$$H' : E_2^2\big(S^{2n+1}\big) \xrightarrow{H_2} \mathrm{Ext}^1_{\mathcal{U}}\big(W(n)\big) \to E_2^1(M),$$

obtained by following H_2 by the stabilization. Here

$$E_2^s(M) \approx \mathrm{Ext}^s_{BP_*BP}(BP_*, BP_*/p)$$

denotes the E_2-term of the stable NSS for the mod p Moore spectrum M. We will eventually need the following facts about $E_2(M)$.

LEMMA 8.4. (i) $E_2(M)$ *is commutative.*

(ii) $v_1^k h_1 \neq 0 \in E_2^1(M)$.

(iii) *If* $x \in E_2(M)$, *then* $v_2 x = x v_2 = 0$.

(iv) $v_1 h_1^{k(p-1)+1} = v_1^{k(p-1)+1} h_1$ *in* $E_2(M)$.

(v) *If* $s \not\equiv 0 \bmod p$, *then* $\alpha_{sp^{e-1}/e} = -s v_1^{sp^{e-1}-1} h_1$ *in* $E_2(M)$.

PROOF. Part (i) is well known, part ii is [51, p 157], and part (iii) follows from [48, 2.10]. Part (iv) follows from Proposition 6.1, part (iii) of this lemma, and the fact that $ph_1 = 0 \in E_2(M)$. To prove part (v), we use 6.1 to expand $v_1^{sp^{e-1}}$, obtaining

$$\begin{aligned}\alpha_{sp^{e-1}/e} &= \frac{1}{p^e}\Big(\eta_R\big(v_1^{sp^{e-1}}\big) - \big(ph_1 + \eta_R(v_1)\big)^{sp^{e-1}}\Big)\\ &= -\sum_{j=1}^{sp^{e-1}} \binom{sp^{e-1}}{j} p^{j-e} h_1^j v_1^{sp^{e-1}-j}.\end{aligned}$$

All terms except $j = 1$ are divisible by p, and hence are 0. To insure that terms with j large are p times an admissible element, write $p^{j-e}h_1^j$ as $p(v_1 - \eta_R(v_1))^{j-e-1}h^{e+1}$. □

Now we begin the proof of Theorem 8.3. We begin with the case $\nu(m) < p - 1$. In this case,

$$\partial(\text{gen}) = \alpha_{m/\nu(m)+1} \otimes \alpha_1 \iota_3 \equiv -\alpha_{m/\nu(m)+1} \otimes h_1 \iota_3, \tag{8.1}$$

mod terms that desuspend to S^1. Here we have used [6, 4.9] and Proposition 6.5. By Proposition 6.5, the assumption that $\nu(m) < p - 1$ implies that $\alpha_{m/\nu(m)+1}$ is defined on S^{2p-1}, and hence Theorem 6.8(v) implies that

$$H'\big(\partial(\text{gen})\big) = -\alpha_{m/\nu(m)+1} \neq 0,$$

where the last step uses parts v and ii of Lemma 8.4. Thus $\partial \neq 0$ in this case, as claimed.

Now we complete the proof of Theorem 8.3 by considering the case $\nu(m) \geqslant p - 1$. We let $s = m/p^{\nu(m)}$ and

$$\varepsilon = \begin{cases} 0 & \text{if } \nu(m) > p-1,\\ 1 & \text{if } \nu(m) = p-1.\end{cases}$$

We will establish the following string of equations in the next paragraph, and then we will further analyze whether these terms are 0 by studying their Hopf invariant. The following string is valid mod terms which desuspend to S^1.

$$\begin{aligned}\partial(\text{gen}) &= \alpha_{m/p} \otimes \alpha_1 \iota_3\\ &= p^{-p}\big(\eta_R(v_1^m) - \big(ph_1 + \eta_R(v_1)\big)^m\big) \otimes \alpha_1 \iota_3 \qquad (8.2)\\ &= -\sum_{j=1}^{m} \binom{m}{j} p^{j-p} h_1^j \otimes v_1^{m-j} \alpha_1 \iota_3\end{aligned}$$

$$= -p^{m-p}h_1^m \otimes \alpha_1\iota_3 - \varepsilon s h_1 \otimes v_1^{m-1}\alpha_1\iota_3 \tag{8.3}$$

$$= -v_1^{m-p}h_1^p \otimes \alpha_1\iota_3 - v_1^{m-p-1}h_1^p \otimes v_1 h_1\iota_3 + \varepsilon s h_1 \otimes v_1^{m-1}h_1\iota_3 \tag{8.4}$$

$$= A + B + C, \tag{8.5}$$

where A, B, and C denote the three terms in the preceding line.

Line (8.2) follows from Propositions 6.5 and 6.1. Line (8.3) has been obtained by observing that in the sum all terms desuspend to S^1 except $j = m$ and, if $\nu(m) = p-1$, $j = 1$. To see this, we observe that we need to have a p to make $\alpha_1\iota_3$ desuspend. This factor will be present unless $j = 1$ and $\nu(m) = p-1$. The requirement that j be $\leqslant \frac{1}{2}$ times the degree of the symbols following h_1^j will only be a problem for large values of j. When j is large, write the term as

$$\binom{m}{j} p\bigl(v_1 - \eta_R(v_1)\bigr)^{j-p-1} h_1^{p+1} \otimes v_1^{m-j}\alpha_1\iota_3.$$

Since p times anything which is defined on S^3 desuspends to S^1, this desuspends to S^1 provided $p+1 \leqslant (p-1)(m-j+1)+1$, which simplifies to $1 \leqslant (p-1)(m-j)$, i.e. $j < m$. To obtain (8.4), we have rewritten the first term of eq. (8.3) as

$$-p\bigl(v_1 - \eta_R(v_1)\bigr)^{m-p-1} h_1^{p+1} \otimes \alpha_1\iota_3,$$

observed that when this is expanded, all terms except the first desuspend, and in that first term we write $ph_1 = v_1 - \eta_R(v_1)$.

We note first that, by Proposition 6.2, A is $d(h_2)$ mod S^1, and so $H'(A) = 0$. We can evaluate the Hopf invariant of B and C by Theorem 6.8(v); using Lemma 8.4, we obtain

$$H'(B+C) = (-1+\varepsilon s)v_1^{m-1}h_1.$$

Hence $H'(\partial(\text{gen})) = 0$ if and only if $-1+\varepsilon s \equiv 0 \bmod p$. Since H' is injective on $E_2^2(S^3)$, this completes the proof of Theorem 8.3. □

Now we settle the extension in the exact sequence of Theorem 8.3.

THEOREM 8.5. *The groups* $E_2^{1,qm+2p+1}(B_1(p))$ *in Theorem* 8.3 *are cyclic.*

PROOF. We will show that whenever $\ker(\partial) \neq 0$ in the exact sequence of 8.3, there is an element $z \in E_2^{1,qm+2p+1}(B_1(p))$ such that $j_*(z) = \alpha_m\iota_{2p+1}$, the element of order p, and $pz = i_*(\text{gen})$. Since $\partial(\alpha_m\iota_{2p+1}) = 0$ in these cases, there is $w\iota_3 \in C^{1,qm+2p+1}(S^3)$ such that $d(w\iota_3) = \alpha_m \otimes \alpha_1\iota_3$. Let

$$z = \alpha_m\iota_{2p+1} - w\iota_3.$$

Then z is a cycle, since $d(z) = \alpha_m \otimes \alpha_1\iota_3 - \alpha_m \otimes \alpha_1\iota_3$, and clearly $j_*(z)$ is as required. Since $p\alpha_m = d(v_1^m)$, we have

$$\begin{aligned} pz - d\bigl(v_1^m\iota_{2p+1}\bigr) &= d\bigl(v_1^m\bigr)\iota_{2p+1} - pw\iota_3 - d\bigl(v_1^m\bigr)\iota_{2p+1} + v_1^m\alpha_1\iota_3 \\ &= i_*\bigl((v_1^m\alpha_1 - pw)\iota_3\bigr). \end{aligned}$$

We show $(v_1^m\alpha_1 - pw)\iota_3 \neq 0 \in E_2^{1,qm+2p+1}(S^3)$ by noting from [7, §7] that the Hopf invariant

$$H_2 : E_2^1(S^3) \to \operatorname{Ext}^0(W(1))$$

factors through the mod p reduction of the unstable cobar complex. Thus

$$H_2\big((v_1^m\alpha_1 - pw)\iota_3\big) = H_2\big(v_1^m h_1\iota_3\big) = v_1^m \neq 0.$$

The second "=" uses Theorem 6.8(v), and the "$\neq$" uses 6.8(iii). □

The following result completes the proof of Theorem 8.1 according to the outline given after Corollary 8.2.

THEOREM 8.6. *In Theorem* 8.1, *the group* $v_1^{-1}\pi_{qm+2p-1}(B_1(p))$ *is cyclic.*

PROOF. We use the exact sequence in $v_1^{-1}\pi_*(-)$ of the fibration which defines $B_1(p)$. The cyclicity follows from that of $v_1^{-1}\pi_{qm+2p-1}(S^{2p+1})$ unless

$$\partial = 0 : v_1^{-1}\pi_{qm+2p}\big(S^{2p+1}\big) \to v_1^{-1}\pi_{qm+2p-1}\big(S^3\big). \tag{8.6}$$

If eq. (8.6) is satisfied, then

$$\circ\alpha_1 \neq 0 : v_1^{-1}\pi_{qm+2}\big(S^{2p+1}\big) \to v_1^{-1}\pi_{qm+2p-1}\big(S^{2p+1}\big)$$

by [23, 6.2], and

$$\partial\colon\ v_1^{-1}\pi_{qm+2}\big(S^{2p+1}\big) \to v_1^{-1}\pi_{qm+1}\big(S^3\big)$$

is an isomorphism of $\mathbf{Z}/p$'s by Theorem 8.3. Let G denote a generator of $v_1^{-1}\pi_{qm+2}(S^{2p+1})$, and let

$$Y \in v_1^{-1}\pi_{qm+2p-1}\big(B_1(p)\big)$$

project to $G \circ \alpha_1$. By [50, 2.1],

$$pY = i_*\big(\langle\partial G, \alpha_1, p\rangle\big) = \partial(G) \circ v_1 \neq 0.$$

□

It is shown in [49] that $B_1(p)$ is a direct factor of $\mathrm{SU}(n)_{(p)}$ if $p < n < 2p$, and hence $v_1^{-1}\pi_i(\mathrm{SU}(n);p)$ is given by Theorem 8.1 if $p < n < 2p$ and $i \equiv 1$ or $2 \bmod q$. This yields the following number theoretic result.

COROLLARY 8.7. *If* p *is an odd prime,* $k \equiv 1 \bmod p-1$, *and* $p < n < 2p$, *then the number* $e_p(k,n)$ *defined in* 1.2 *equals* $\min(p, \nu_p(k - p - p^p + p^{p-1})) + 1$.

The author has been unable to prove this result without the UNSS. In fact, the only tractable result for $e_p(k,n)$ which follows easily from Definition 1.2 seems to be that if

$n \leqslant p$ and $k \equiv n-1 \bmod p-1$, then $e_p(k,n) \geqslant \min(n-1, \nu_p(k-n+1)+1)$, which is proved using the Little Fermat Theorem as in [22, p. 792].

Using methods similar to those in our proof of Theorem 8.1, Yang ([58]) has proved the following tractable result for $v_1^{-1}\pi_*(\mathrm{SU}(n);p)$ when p is odd and $n \leqslant p^2-p$. Of course this can also be interpreted as a theorem about the numbers $e_p(k,n)$. We emphasize that the proof of Theorem 8.8 does not involve the use of Theorem 1.3.

THEOREM 8.8. *Suppose p is odd, $k = c+(p-1)d$ with $1 \leqslant c < p$, and*

$$c+(p-1)b+1 \leqslant n \leqslant c+(p-1)(b+1)$$

with $0 \leqslant b \leqslant p-1$. Define j by $1 \leqslant j \leqslant p$ and $d \equiv j \bmod p$. Then $v_1^{-1}\pi_{2k}(\mathrm{SU}(n);p)$ is cyclic of order p^e, with

$$e = \begin{cases} \min(c+(p-1)j,\ b+\nu(d-j)+1) \\ \qquad\qquad \text{if } b < c \text{ and } 1 \leqslant j \leqslant b, \\ \min(c+(p-1)j+1,\ b+\nu(d-j+(-1)^j j\binom{c}{j}p^{j(p-1)})) \\ \qquad\qquad \text{if } c \leqslant b \text{ and } 1 \leqslant j \leqslant b, \\ \min(c,\ b+1+\nu(d)) \\ \qquad\qquad \text{if } b < c \text{ and } b < j \leqslant p, \\ b \qquad\qquad \text{if } c \leqslant b \text{ and } b < j \leqslant p. \end{cases}$$

One can easily read off from Theorem 8.8 the precise value of the numbers $e_p(n)$ which appeared in Corollary 7.8, yielding the following result for the p-exponent of the space $\mathrm{SU}(n)$.

COROLLARY 8.9. *If p is odd and $n \leqslant p^2-p$, then*

$$\exp_p(\mathrm{SU}(n)) \geqslant e_p(n) = \begin{cases} n & \textit{if } i(p-1)+2 \leqslant n \leqslant ip+1 \textit{ for some } i, \\ n-1 & \textit{otherwise.} \end{cases}$$

When $p=2$, UNSS methods of computing $v_1^{-1}\pi_*(-;p)$ become more complicated because of the η-towers. Then ASS techniques become more useful, as they did for $v_1^{-1}\pi_*(G_2;2)$ in [29]. Here we show how to use ASS methods to determine $v_1^{-1}\pi_*(F_4/G_2;2)$. It is hoped that the result of the calculations of $v_1^{-1}\pi_*(G_2;2)$ and $v_1^{-1}\pi_*(F_4/G_2;2)$ might be combined to yield $v_1^{-1}\pi_*(F_4;2)$, but this involves one difficulty not yet resolved. Our main reason for including this example is to give a new illustration of this method.

The proof of the following theorem will consume most of the remainder of this paper. If G denotes an abelian group, then mG denotes the direct sum of m copies of G.

THEOREM 8.10. *Let $G(n)$ denote some group of order n.*

$$v_1^{-1}\pi_i(F_4/G_2;2) \approx \begin{cases} 4\mathbf{Z}_2 & i \equiv 0 \bmod 8, \\ \mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 & i \equiv 1 \bmod 8, \\ \mathbf{Z}/64 & i \equiv 2 \bmod 8, \\ 0 & i \equiv 3 \text{ or } 4 \bmod 8, \\ G(2^{\min(15,\nu(i-21)+4)}) & i \equiv 5 \bmod 8, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}/2^{\min(15,\nu(i-22)+4)} & i \equiv 6 \bmod 8, \\ 5\mathbf{Z}_2 & i \equiv 7 \bmod 8. \end{cases}$$

There is a fibration

$$S^{15} \xrightarrow{i} F_4/G_2 \xrightarrow{p} S^{23}, \tag{8.7}$$

derived in [29, 1.1]. Here and throughout this proof all spaces and spectra are localized at 2.

In [31], it was shown that for any spherically resolved space Y, there is a finite torsion spectrum X satisfying $v_1^{-1}\pi_*(Y) \approx v_1^{-1}J_*(X)$. In our case, we have

PROPOSITION 8.11. *There is a spectrum X such that*

(i) $v_1^{-1}J_*(X) \approx v_1^{-1}\pi_*(F_4/G_2)$, *and*
(ii) *there is a cofibration*

$$\Sigma^{15}P^{14} \to X \to \Sigma^{23+L}P^{22}, \tag{8.8}$$

where L equals 0 or a large 2-power.

We present in fig. 15 a chart which depicts an initial part of the ASS for $v_1^{-1}J_*(X)$ if X is as in Proposition 8.11 and $L = 0$. It depicts the direct sum of the spectral sequences for $v_1^{-1}J_*(\Sigma^{15}P^{14})$ and $v_1^{-1}J_*(\Sigma^{23}P^{22})$, together with one differential, which will be established in Proposition 8.13. The •'s are elements from P^{14}, while the ○'s are from P^{22}. Charts such as these for $v_1^{-1}J_*(P^n)$ were derived in Section 4.

To see that fig. 15 is also valid when L in Proposition 8.11 is a large 2-power, we use the following result.

LEMMA 8.12. *If L is a large 2-power, then the attaching map*

$$\Sigma^{22+L}P^{22} \to v_1^{-1}\Sigma^{15}P^{14}$$

in Proposition 8.11 *has filtration $L/2 + 1$.*

Using results of [40], this implies that a resolution of $v_1^{-1}X$ can be formed from $v_1^{-1}\Sigma^{15}P^{14}$ and $\phi^{L/2}v_1^{-1}\Sigma^{23+L}P^{22}$, where ϕ^j increases filtrations by j, and this yields fig. 15.

PROOF OF LEMMA 8.12. Under S-duality, the generator corresponds to an element of $v_1^{-1}J_{L+7}(P^{14} \wedge D(P^{22}))$. This group is isomorphic to

$$v_1^{-1}J_{L+6}(P^{14} \wedge P_{-23}^{-2}). \tag{8.9}$$

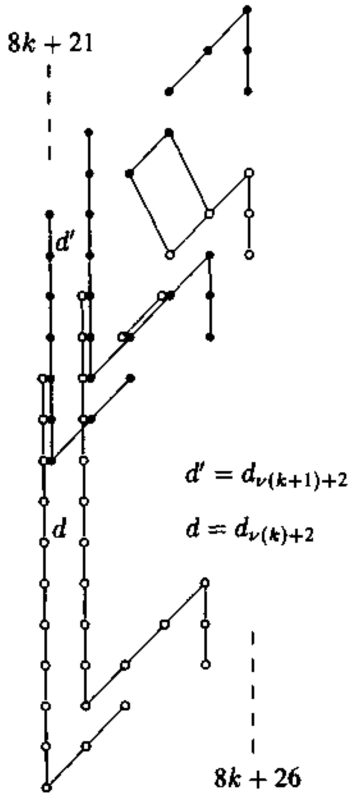

Figure 15. Initial chart for $v_1^{-1}\pi_*(F_4/G_2)$.

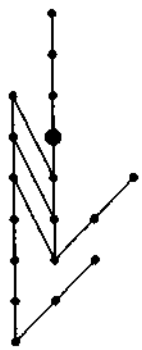

Figure 16. The generator of $v_1^{-1}J_{L+6}(P^{14} \wedge P_{-23}^{-2})$.

Using methods of [26], one can show that the relevant chart is as in fig. 16, where the class indicated with a bigger • is the generator of (8.9), and has filtration $L/2+1$.

Borel ([15]) showed that $\mathrm{Sq}^8(x_{15}) = x_{23}$ in $H^*(F_4;\mathbf{Z}_2)$. This implies that the attaching map in F_4/G_2 and in X is the Hopf map σ, and hence corresponds to the generator of (8.9). Thus the attaching map has filtration $L/2+1$. $\square$

We can now establish the d_2-differentials in fig. 15.

PROPOSITION 8.13. *There are d_2-differentials as indicated in fig.* 15.

PROOF. This follows from the σ attaching map just observed, together with the observation that the first of the pair of elements that are related by the differential in fig. 15 are v_1-periodic versions of η_{23} and $\eta\sigma_{15}$. $\square$

This d_2-differential implies there is a nontrivial extension in $v_1^{-1}\pi_{8k+26}(F_4/G_2)$ as follows.

PROPOSITION 8.14. $v_1^{-1}\pi_{8k+26}(F_4/G_2) \approx \mathbf{Z}/64$.

PROOF. This follows from fig. 15 and a standard Toda bracket argument ([50, 2.1]), which in this situation says the following. Let A be the element supporting the higher of the two d_2-differentials, and let D be the lowest • in $8k+26$. Let ∂ denote the boundary morphism in the exact sequence in $v_1^{-1}\pi_*(-)$ associated to the fibration (8.7). Then D lies in the Toda bracket $\langle\partial(A),\eta,2\rangle$, and so there exists an element $E \in v_1^{-1}\pi_{8k+26}(F_4/G_2)$ such that $p_*(E) = A\circ\eta$ and $i_*(D) = 2E$. $\square$

As indicated in fig. 15, there are $d_{\nu(k)+2}$-differentials between ∘-towers in $8k+22$ and $8k+21$, and there are $d_{\nu(k+1)+2}$-differentials between •-towers in $8k+22$ and $8k+21$. This follows just from standard $J_*(-)$-considerations. But there may also be differentials from the ∘-tower in $8k+22$ to the •-tower in $8k+21$. These differentials from ∘ to • are determined from the homomorphism

$$v_1^{-1}\pi_{8k+22}(S^{23}) \xrightarrow{\partial} v_1^{-1}\pi_{8k+21}(S^{15}), \tag{8.10}$$

which is evaluated in the following result.

PROPOSITION 8.15. *The image of the homomorphism* (8.10) *consists of all multiples of* 8 *if $\nu(k) \leqslant 7$, and is* 0 *if $\nu(k) > 7$.*

The following result plays a central role in the proof of Proposition 8.15.

PROPOSITION 8.16. *Let $(S^{15})_K$ denote the K_*-localization as constructed in* [42]. *There is a commutative diagram*

$$\begin{tikzcd}
\Omega S^{23} \arrow[r, "\partial"] \arrow[d, "\ell"] & S^{15} \arrow[d, "e"] \\
\Omega^\infty(\Sigma^{15}P^{14}\wedge J) \arrow[r, "h"] & (S^{15})_K
\end{tikzcd}$$

in which ∂ is obtained from the fiber sequence (8.7), *e is the localization, and h induces an isomorphism in $\pi_j(-)$ for $j = 22$ and $j \geqslant 28$.*

PROOF. The map h is constructed as in [29, pp. 669–670], using results of [42]. It induces an isomorphism in $\pi_j(-)$ for many other small values of j, but we only care about values of j which are positive multiples of 22. The obstructions to its being an isomorphism for all small values of j are $\mathbf{Z}_2$-classes in filtration 1 in $J_j(\Sigma^{15}P^{14})$ for $j = 19, 23$, and 27. The map ℓ is obtained by obstruction theory, since ΩS^{23} has cells only in dimensions which are positive multiples of 22. □

Now we prove Proposition 8.15. Let ℓ be as in Proposition 8.16. The morphism $\pi_*(\ell)$ can be factored as

$$\pi_*(\Omega S^{23}) \to \pi_*^s(\Omega S^{23}) \to J_*(\Sigma^{15}P^{14}).$$

There is a splitting

$$\pi_*^s(\Omega S^{23}) \approx \bigoplus_{i>0} \pi_*^s(S^{22i}).$$

We will use the method of [43] to deduce that

$$\pi_{8k+21}^s(S^{22}) \to J_{8k+21}(\Sigma^{15}P^{14})$$

sends the v_1-periodic generator ρ_k to 8 times the generator. Indeed, the stable map $S^{22} \to \Sigma^{15}P^{14} \wedge J$ which induces the morphism factors through $\Sigma^{15}P^8 \wedge J$, from which it projects nontrivially to $\Sigma^{15}P_7^8 \wedge J$. We then use [43, 2.8] to deduce that ρ_k goes to the nonzero element of $J_{8k+21}(\Sigma^{15}P_7^8)$. This implies that its image in $J_{8k+21}(\Sigma^{15}P^8)$ is the generator, and this maps to 8 times the generator of $J_{8k+21}(\Sigma^{15}P^{14})$.

The composite of v_1-periodic summands of

$$\pi_{8k+21}(\Omega S^{23}) \to \pi_{8k+21}^s(\Omega S^{23}) \to \pi_{8k+21}^s(S^{22}) \tag{8.11}$$

is bijective if $\nu(k) \leqslant 7$, but is not surjective if $\nu(k) > 7$. Thus when the composite (8.11) is followed into $J_{8k+21}(\Sigma^{15}P^{14})$, the image of a v_1-periodic generator is 8 times the generator if $\nu(k) \leqslant 7$ and $0 \in \mathbf{Z}/16$ if $\nu(k) > 7$. Once we observe that, in the diagram of Proposition 8.16, h induces an isomorphism in $\pi_{8k+21}(-)$ and e sends the v_1-periodic summand isomorphically, we obtain the desired conclusion of Proposition 8.15. □

The differentials implied by Proposition 8.15 have an interesting and unexpected implication about fig. 15. Since d_r from the $\circ$-tower in $8k + 22$ hits the top $\circ$ in $8k + 21$ and the $\bullet$ just above it with the same r, and since d_r respects the action of h_0, there must be an h_0-extension between these classes in $8k + 21$. If $L = 0$ in (8.8), then this extension can only be accounted for by a failure of the map (8.8) to induce a split short exact sequence of A_1-modules in cohomology. Indeed, we have

PROPOSITION 8.17. *If $L = 0$ in* (8.8), *then there is a splitting of A_1-modules*

$$H^*X \approx H^*(\Sigma^{15}P^{30}) \oplus H^*(\Sigma^{23}P^6).$$

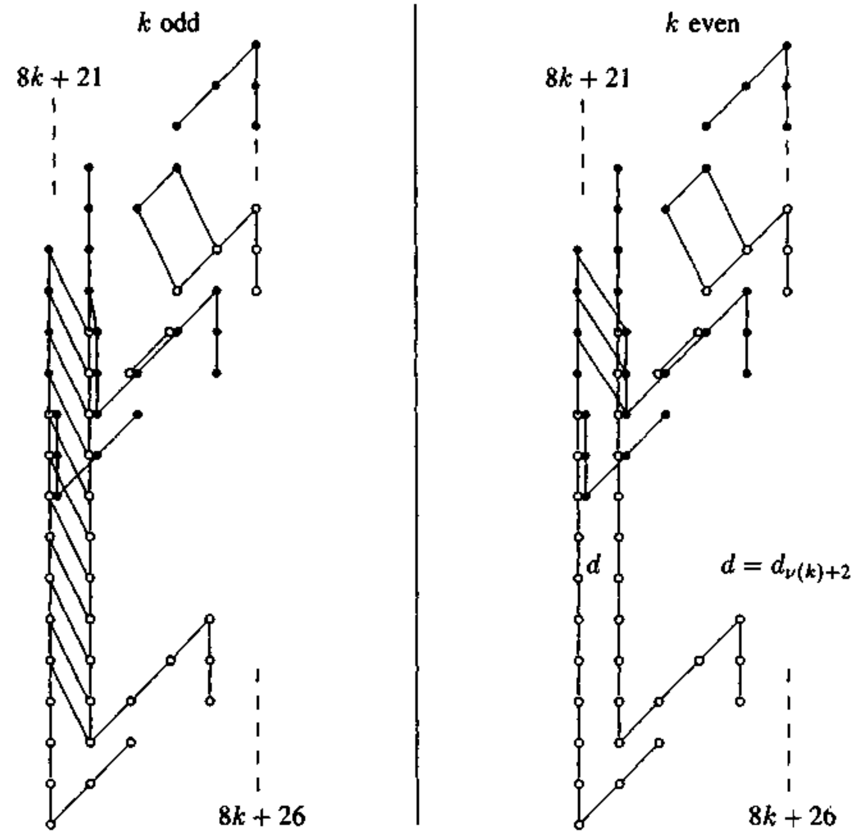

Figure 17. Final chart for $v_1^{-1}\pi_*(F_4/G_2)$.

This splitting is caused by having $\mathrm{Sq}^2 \neq 0$ on the class in H^*X corresponding to the top cell of $H^*(\Sigma^{15}P^{14})$, i.e.

$$\mathrm{Sq}^2 : H^{29}X \to H^{31}X$$

is an isomorphism. This is the only way to account for the h_0-extension in fig. 15. Proposition 8.17 implies that the h_0-extensions are present in the chart for values of k ($\nu(k) > 7$) where they cannot be deduced from differentials. It also implies that there is an h_0-extension on the top $\circ$ in $8k+22$. If $L > 0$ in (8.8), the same conclusion about the charts can be deduced from a more complicated analysis.

It causes fig. 15 to take the form of fig. 17.

We can read off almost all of Theorem 8.10 from fig. 17. We must show that d_6 is 0 on the $\circ$'s near the bottom in $8k+23$ and $8k+24$. This is done by the argument used to prove Proposition 8.15. The classes involved are present in all spaces in the diagram in

Proposition 8.16, but they are not mapped across by ℓ_*, since it factors through $\pi_*^s(\Omega S^{23})$.

All that remains is the verification of the abelian group structure. Most of the extensions are trivial due to the relation $2\eta = 0$. The extension in $8k+22$ when k is odd was present before the exotic extension was deduced, and remains true. The cyclicity of this 2^7 summand can also be deduced by consideration of the kernel of the homomorphism in the fibration which defines J, but that seems unnecessary. Note that no claim is made about the group structure in $8k+21$. □

In 1989, Mimura suggested to the author that he try to calculate $v_1^{-1}\pi_*(G)$ for all compact simple Lie groups G. If p is odd, and $G = \mathrm{Sp}(n)$ or $\mathrm{SO}(n)$, then the result follows from Theorem 1.3 and [33]. With great effort, $v_1^{-1}\pi_*(\mathrm{Sp}(n);2)$ was calculated in [13]. The result involves a surprising pattern of differentials among $\mathbf{Z}_2$'s from the various spheres which build $\mathrm{Sp}(n)$, resulting in $[\log_2(4n/3)]$ copies of $\mathbf{Z}_2$ in certain $v_1^{-1}\pi_i(\mathrm{Sp}(n))$. Of the classical groups, only $v_1^{-1}\pi_*(\mathrm{SO}(n);2)$ remains. All torsion-free exceptional Lie groups were handled in [14], using the UNSS. In [29] and [12], the torsion cases $(G_2,2)$, $(F_4,3)$, and $(E_6,3)$ were handled. Remaining then are seven cases of (G,p) yet to be calculated. At least a few of these should lend themselves to the methods of this paper.

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CHAPTER 21

Classifying Spaces of Compact Lie Groups and Finite Loop Spaces

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Edited by I.M. James

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0. Introduction

The basic problem of homotopy theory is to classify spaces and maps between spaces up to homotopy by means of algebraic invariants such as homology or cohomology. Since their discovery, classifying spaces of compact Lie groups G, denoted by BG, have been a very important part in homotopy theory. For example, they appeared as target in the set of homotopy classes of maps $[X, Y]$, because of their central role in bundle theory. In the last decade, some striking progress was made in the understanding of the homotopy theory of classifying spaces of compact Lie groups. We mention some aspects:

- It has been shown that, for a simple connected compact Lie group G, two self maps of BG are homotopic if and only if they induce the same map in rational cohomology.
- It also has been proved that for a large class of simply connected compact Lie groups G the mod-p cohomology with cup products and Steenrod operations completely determines the homotopy type of the p-adic completion $BG_p^\wedge$ of BG (for odd primes this contains all classical matrix groups).
- Similar methods have also been used to obtain new results for Steenrod's problem: which polynomial algebras can be realized as the mod-p cohomology of a space?
- The program of understanding 'classical' Lie group theory from the homotopy point of view, i.e. to express Lie group theory in terms of classifying spaces, is developed to a large extent and might lead to a complete classification of finite loop spaces.

The study of maps between classifying spaces goes back to Hurewicz. For aspherical spaces X and Y he showed that

$$[X, Y] \to Hom(\pi_1(X), \pi_1(Y))/Inn(\pi_1(Y))$$

is a bijection. In particular this applies to classifying spaces of finite or more generally of discrete groups. Here, $Hom(\ ,\ )$ denotes the set of homomorphisms between groups and $Inn(\ )$ the group of inner automorphisms. Moreover, the homotopy type of an aspherical space is determined by the fundamental group. This fed the hope that, up to homotopy, every map between the classifying spaces of any pair of compact Lie groups is induced by a homomorphism. However, in 1970, the first counterexamples were constructed by Sullivan, namely self maps of $BU(n)$, which even in rational cohomology do not look like a map coming from a homomorphism. Inspired by Sullivan's work, a careful investigation of Hubbuck, Mahmud and Adams gave necessary criteria for the effect that maps between classifying spaces of compact Lie groups could have in rational cohomology.

The idea of developing Lie group theory in terms of homotopy theory goes back to Rector. In his study of loop structures on S^3 and sub-loop spaces of finite loop spaces first definitions of basic notions of Lie group theory appeared in terms of classifying spaces, such as homomorphisms, subgroups, maximal tori and Weyl groups.

The proof of the Sullivan conjecture by Miller and Carlsson and subsequent work of Lannes was the break through for the recent fast development in this area. The Sullivan conjecture states as follows:

0.1. THEOREM (Sullivan conjecture, [59]). *Let π be a locally finite group and K a finite CW-complex. Then, the evaluation at a basepoint*

$$map(B\pi, K) \to K$$

is a weak equivalence.

Lannes developed machinery for a purely algebraic calculation of the mod-p cohomology of mapping spaces of the form $map(B\mathbb{Z}/p, X)$. Under some mild assumptions his T-functor calculates $H^*(map(B\mathbb{Z}/p, K); \mathbf{F}_p)$ as an algebra over the Steenrod algebra only using the mod-p cohomology $H^*(X, \mathbf{F}_p)$ as input. For example, this led to a complete description of the mapping space $map(BP, BG)$ for any p-toral groups P and any compact Lie group G, due to Dwyer and Zabrodsky [37] and the author [69].

Based on this and a decomposition of BG into a homotopy direct limit of classifying spaces of certain p-toral groups [50], Jackowski, McClure and Oliver set up a program for studying maps $BG \to BH$ for any pair of compact Lie groups G and H. In the case of $G = H$ being a simple connected compact Lie group the program went through and led to:

0.2. THEOREM ([50]). *Let G be a simple connected compact Lie group. Then two self maps $f, g : BG \to BG$ are homotopic if and only if $f^* = g^* : H^*(BG; \mathbb{Q}) \to H^*(BG; \mathbb{Q})$.*

Lannes' theory and the Jackowski–McClure–Oliver approach also allowed the homotopy type of the classifying space for a large class of compact Lie groups to be characterized.

0.3. THEOREM ([72]). *Let p be an odd prime. Let G be a simply connected compact Lie group such that $H^*(G; \mathbb{Z})$ has no p-torsion, and let X be a space. Then, the p-adic completion $X_p^\wedge$ and $BG_p^\wedge$ are homotopy equivalent if and only if $H^*(X; \mathbb{F}_p)$ and $H^*(BG; \mathbb{F}_p)$ are isomorphic as algebras over the Steenrod algebra.*

The same result for $G = SU(2)$ and $G = SO(3)$ was proved by Dwyer, Miller and Wilkerson for all primes [26], which was the first homotopy uniqueness theorem. The same authors also proved Theorem 0.3 for primes not dividing the order of the Weyl group W_G of G without any extra assumption on G beside being connected [27].

From the homotopy point of view the essential property of a compact Lie group G is the existence of a classifying space BG and a finiteness condition on G, namely that G is a finite CW-complex or a little weaker that $H^*(G; \mathbb{Z})$ is a finitely generated module over $\mathbb{Z}$. Because completion always makes life easier in homotopy theory, these facts led Dwyer and Wilkerson to the definition of p-compact groups. For p-compact groups, the classifying space has to be p-complete, and the finiteness condition is expressed in terms of mod-p cohomology. The main examples are given by completing a connected compact Lie group and the associated classifying space. A generalization of Smith theory to actions of finite p-groups on $\mathbf{F}_p$-finite p-complete spaces allowed Dwyer and Wilkerson to achieve the following fundamental result in the Lie group theory of p-compact groups.

It generalizes well known facts about compact Lie groups. Here, a space is called $\mathbb{F}_p$-finite if the mod-p cohomology is finite.

0.4. THEOREM ([33]). (1) *For any p-compact group X, there exists a maximal torus T_X of X and a Weyl group W_X.*

(2) *If X is connected, the inclusion of the maximal torus induces an isomorphism*

$$H^*(BX;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q}\cong\left(H^*(BT_X;\mathbb{Z})\otimes\mathbb{Q}\right)^{W_X}.$$

The representation $W_X\to Gl(H^2(BT_X;\mathbb{Z})\otimes\mathbb{Q})$ is faithful and represents W_X as a pseudo reflection group.

We think, that these are some of the high lights of the recent achievements in the study of classifying spaces. The idea of this article is to describe developments after the proof of the Sullivan conjecture.

We strongly encourage the reader to take a look at the very nice survey article of Jackowski, McClure and Oliver [53] on a very similar topic. Parts of this article are covered in their paper, in particular the discussion about decomposition and maps between classifying spaces of compact Lie groups. Because this is needed for an understanding of the later development, and because we like to keep this article self contained, we also present this part of the homotopy theory of classifying spaces of compact Lie groups, but much more briefly. For example, we omit completely the discussion about the computation of higher inverse limits.

At the end we add as appendix some remarks and facts about homotopy colimits, Lannes' theory and Smith theory for homotopy fixed-points, which we feel is necessary for an understanding of this article by nonexperts.

1. Decompositions of classifying spaces

In the analysis of maps between classifying spaces, decompositions into simpler pieces have proved to be quite a powerful tool. By simpler pieces, we mean classifying spaces of subgroups. There are two different types of such decompositions. One uses centralizers of elementary abelian subgroups. The other is based on p-toral subgroups. Both types are useful for different problems as we will show later.

The idea of decompositions or approximations of classifying spaces goes back to Adams [1]. In his analysis of the effect, which maps between classifying spaces may have in complex K-theory, he approximated p-toral subgroups by their finite p-subgroups. This construction was extended by Feshbach to the case of finite extensions of tori [38]. For any such extension N, he showed that there exists a sequence

$$M_1\subset M_2\subset M_3\subset\cdots\subset N$$

of finite subgroups such that

$$\operatorname*{hocolim}_{\mathbb{N}} BM_i\simeq Tel(BM_i)\simeq BM_\infty\to BN$$

is a homology equivalence for any kind of coefficients taken in a finite group. Here, the group $M_\infty := \bigcup_i M_i$ is a locally finite group and gives an approximation of BN at any prime p.

This approach was extended further by Friedlander and Mislin [40], [41]. For a compact Lie group G they showed the existence of a locally finite group with similar properties. A locally finite group is the union of finite groups.

1.1. THEOREM ([40], [41]). *Let G be a compact Lie group, and let q be a prime not dividing the order of $\pi_0(G)$. Then there exists a locally finite group γ and a map $B\gamma \to BG$ which is a mod-p equivalence for any prime p different from q.*

In general, the finite subgroups of γ may not be subgroups of G and the restriction of the map to the classifying space of a finite subgroup may not be induced by a homomorphism. For $G = U(n)$, and a prime p the approximation is given by a map $BGL(n, \bar{\mathbf{F}}_q) \to BU(n)$, where $(q,p) = 1$ and where $\bar{\mathbf{F}}_q$ is the algebraic closure of the field $\mathbf{F}_q$ of q elements.

As a consequence of this approximation theorem, Friedlander and Mislin generalized the Sullivan conjecture to the case of compact Lie groups. For a $\mathbf{F}_p$-finite p-complete space X, they showed that the evaluation at a basepoint $map(BG, X) \to X$ is an equivalence [41]. The approximation was also used by Mislin to get a complete classification of self maps of $BSU(2)$ up to homotopy [60]. This was the first case of such analysis beyond the "simple" case of finite groups or tori.

Although decompositions via centralizers of elementary abelian subgroups or via p-toral subgroups seem to be more useful in the study of maps between classifying spaces, it would be of great interest to have an analogue of Theorem 1.1 for p-compact groups. Examples of this form are given by calculations of Quillen on the cohomology of general linear groups of finite fields [83].

Decompositions other than telescope constructions were first introduced by Dwyer, Miller and Wilkerson. The pushout of the diagrams

$$\begin{array}{ccc} SO(3)/D(8) & \longrightarrow & SO(3)/O(2) \\ \downarrow & & \\ SO(3)/\Sigma_4 & & \end{array} \qquad \begin{array}{ccc} BD(8) & \longrightarrow & BO(2) \\ \downarrow & & \\ B\Sigma_4 & & \end{array} \tag{1}$$

is $\mathbf{F}_2$-acyclic for the left side, and mod-2 equivalent to $BSO(3)$ for the right side. For the left side, this is not to hard to check by explicit calculations, and for the right side, it follows because the Borel construction $EG \times_G -$, as a homotopy colimit, commutes with pushouts. Here, $D(8)$ denotes the dihedral subgroup of $SO(3)$, $O(2)$ the normalizer of the maximal torus and Σ_4 the octohedral subgroup.

There exists a closely related decomposition of $BSO(3)$ at the prime 2, given by the diagram

$$\Sigma_3 \circlearrowright ESO(3)/(\mathbb{Z}/2)^2 \overset{\rightarrow}{\rightrightarrows} ESO(3)/O(2). \tag{2}$$

That is that the underlying category has two objects 0 and 1 and the morphisms sets are given by $End(0) = \Sigma_3$, $End(1) = \{id\}$, $Hom(0,1) = \Sigma_3/\Sigma_2$ and $Hom(1,0) = \varnothing$. Decomposition in this case means that the homotopy colimit of the diagram is mod 2 equivalent to $BSO(3)$. Notice that the left space is equivalent to $B(\mathbb{Z}/2)^2$and the right to $BO(2)$. A cohomological calculation based on the spectral sequence of Theorem A.1 gives a proof. It also can be shown directly that the homotopy colimit of diagram (2) is equivalent to the pushout of diagram (1).

Unlike the pushout diagram, the decomposition of $BSO(3)$ via diagram (2) can be generalized to compact Lie groups in essentially two different ways, which we discuss next.

Decompositions via p-toral subgroups

A p-toral group is a compact Lie group P whose component of the unit P_0 is a torus and whose group of components P/P_0 is a finite p-group. p-toral groups play the same role for compact Lie groups as finite p-groups do for finite groups. For example, every compact Lie group G has a p-toral Sylow subgroup $S_pG \subset G$. It has the same properties as a p-Sylow subgroup of a finite group; e.g., the group S_pG is maximal in the sense that every p-toral subgroup of G is subconjugate to S_pG. It is also characterized by the condition that the Euler characteristic of G/S_pG is coprime to p. Let $T_G \subset G$ be a maximal torus of G and let $N(T_G) \to W_G$ be the projection of the normalizer of T_G onto the Weyl group of G. Then, the counter image of a p-Sylow subgroup S_pW_G of W_G is a p-toral Sylow subgroup of G.

For any compact Lie group G, let $\mathcal{O}(G)$ denote the (topological) orbit category, whose objects are homogeneous spaces G/H with $H \subset G$ being a closed subgroup and whose morphisms are given by G-equivariant maps. Let $\mathcal{O}_p(G) \subset \mathcal{O}(G)$ denote the full subcategory of all objects G/P, where $P \subset G$ is a p-toral subgroup. Let $\mathcal{I} : \mathcal{O}_p(G) \to \mathcal{T}op$ be the inclusion functor. Then, the Borel construction defines a (continuous) functor

$$EG \times_G \mathcal{I} : \mathcal{O}_p(G) \to \mathcal{T}op\ .$$

Notice that the $EG \times_G G/P \simeq BP$.

If G is a finite group, the category $\mathcal{O}_p(G)$ is finite (and so is $\mathcal{O}$). In this case, the map $hocolim_{\mathcal{O}_p(G)} EG \times_G \mathcal{I} \to BG$ is a mod-p equivalence, since all higher inverse limits in the associated spectral sequence of Theorem A.1 vanish [60] and since the inverse limit involved equals the mod-p cohomology of BG [17, XII, 10.1].

For compact Lie groups, the category $\mathcal{O}_p(G)$ is not finite and not even discrete in general. For a generalization of the above result, the question comes up, which of the p-toral subgroups cannot be got rid of in a decomposition of BG. This motivates the notion of p-stubborn subgroups. More concretely, a p-toral subgroup is called p-stubborn if the quotient $N(P)/P$ of the normalizer of P by P is finite and does not contain any nontrivial normal p-subgroup. Let $\mathcal{R}_p(G) \subset \mathcal{O}_p(G)$ denote the full subcategory of all objects G/P where P is p-stubborn. This turned out to be a finite category [50]. Restricting the above functor to this subcategory, Jackowski, McClure and Oliver proved the following decomposition theorem using techniques from the theory of transformation groups.

1.2. THEOREM ([50]). *For any compact Lie group G, the map*

$$\operatorname*{hocolim}_{\mathcal{R}_p(G)} EG \times_G \mathcal{I} \to BG$$

is a p-local equivalence, i.e. induces an isomorphism in $H^*(-;\mathbb{Z}_{(p)})$ cohomology.

Decompositions via centralizers of elementary abelian subgroups

For any compact Lie group G, let $A_p(G)$ denote the Quillen category [82] whose objects are nontrivial elementary abelian p-subgroups $V \subset G$ and whose morphisms are given by restrictions of conjugations by elements of G. Actually, Quillen also allowed the trivial group to be an object of $A_pP(G)$ but we exclude it. Let

$$\beta : A_p^{op}(G) \to \mathcal{T}op$$

be the functor given by the Borel construction $\beta(V) := EG \times_G G/C_G(V)$, where $C_G(V)$ denotes the centralizer of V in G and where G acts on $G/C_G(V)$ via left translation. Starting from the opposite category of $A_p(G)$ makes β into a covariant functor.

The projection $G/C(V) \to *$ to a point establishes a natural transformation from β to the constant functor with value BG and a map $hocolim_{A_p(G)} \beta \to BG$. These constructions were used by Jackowski and McClure to get a decomposition of BG into simpler pieces.

1.3. THEOREM ([49]). *Let G be a compact Lie group. Then the map*

$$\operatorname*{hocolim}_{A_p(G)} \beta \to BG$$

is a mod-p equivalence.

The proof is based on the spectral sequence of Theorem A.1. Using transfers and Feshbach's double coset formula [38], one first proves that

$$H^*(BG;\mathbb{F}_p) \cong \varprojlim_{A_p(G)} H^*(\beta(-);\mathbb{F}_p)$$

is an isomorphism. The proof of the vanishing of the higher derived functors of the inverse limit functor is also based on the existence of a transfer for the functor $H^*(\beta(-);\mathbb{F}_p)$. This functor turns out to be a Mackey functor in a sense closely related to the definition given in [21]. This extra structure allows the proof to be completed.

This geometric decomposition was generalized by Dwyer and Wilkerson [31]. They formulated Theorem 5.1 in purely algebraic terms using mod-p cohomology and also gave an algebraic proof of this theorem based on Lannes' T-functor.

Let $\mathcal{K}$ denote the category of unstable algebras over the Steenrod algebra. For any object $R \in \mathcal{K}$, Rector defined a category $A(R)$ [86]. The objects are given by morphisms $\phi_V : R \to H^*(BV;\mathbb{F}_p)$ such that $H^*(BV;\mathbb{F}_p)$ is a finitely generated module over R

and such that V is a nontrivial elementary abelian group. The morphisms are given by commutative triangles

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ H^*(BV;\mathbb{F}_p) \arrow[rr] & & H^*(BW;\mathbb{F}_p) \end{tikzcd}$$

Lannes' T-functor defines a functor

$$\tau : A(R) \to \mathcal{K}$$

given by $\tau(\phi_V) := T_V(R, \phi_V)$.

As a consequence of Lannes' theory (see Appendix B), of a theorem of Dwyer and Zabrodsky [37] (see Theorem 2.1) and of a result of Quillen [82] (see 5.2), passing to mod-p cohomology establishes an isomorphism $A_p(G) \cong A(H^*(BG;\mathbb{F}_p)$ of categories and a natural equivalence

$$H^*(\beta(-);\mathbb{F}_p) \xrightarrow{\cong} \tau.$$

To reprove Theorem 5.1 in algebraic terms, Dwyer and Wilkerson also used the existence of a p-toral Sylow subgroup $P \subset G$, which, by analogy to the p-Sylow group of a finite group, has the properties that $H^*(BG;\mathbb{F}_p) \to H^*(BP;\mathbb{F}_p)$ is a monomorphism, that this map has a left inverse as $H^*(BG;\mathbb{F}_p)$-module homomorphism given by the transfer, and that P has a central subgroup. Translating group theory notions into mod-p cohomology we say that a morphism $\phi_V : R \to H^*(BV;\mathbb{F}_p)$ of $A(R)$ is central if $T_V(R,\phi_V) \cong R$ and that R has a nontrivial center if there exists a central morphism in $A(R)$. Now, Theorem 5.1 can be reformulated to

1.4. THEOREM ([31]). *Let $i : R \to S$ be a morphism in $\mathcal{K}$ such that the following holds:*

(1) *R and S are Noetherian algebras.*

(2) *There exists a left inverse $S \to R$ of i which is both a map of R-modules and a map over the Steenrod algebra.*

(3) *The algebra S has a nontrivial center.*

Then $\varprojlim_{A(R)} \tau \cong R$ and $\varprojlim_{A(R)}{}^i \tau$ vanishes for $i \geqslant 1$.

For any compact Lie group G, the mod-p cohomology $H^*(BG;\mathbb{F}_p)$ is Noetherian [90]. Hence, the first condition is also satisfied in the case of $R = H^*(BG;\mathbb{F}_p)$.

For objects of $\mathcal{K}$ with nontrivial center, it turns out that the center (which is fixed under "conjugations") plays the role of an initial element in the category $A(R)$. This makes the proof in this case possible. For the general case, the higher inverse limits of τ defined on $A(R)$ and on $A(S)$ are compared. Using the exactness of the T-functor, it can be shown that the higher derived inverse limits taken over $A(R)$ are a direct summand of the ones over $A(S)$.

This algebraic proof of the geometric decomposition theorem allows a generalization to a much larger class of spaces than just compact Lie groups. One only needs that passing to mod-p cohomology controls a sufficient part of the homotopy theory of a given space (for explicit conditions see [31]). In particular, decompositions of this type exist for p-compact groups (see Theorem 5.14).

The important role which the Quillen and the Rector categories play in the homotopy theory of classifying spaces inspired Oliver to analyze inverse limits of general functors from the Rector category into the category of abelian groups [80]. He set up a spectral sequence converging to the higher limits and computed the E^2-term in terms of the endomorphism sets of single objects. Because every endomorphism of the Rector category is an isomorphism, the objects form a poset. This gives rise to a filtration of the functor such that the quotients are nontrivial only on one particular object. Then, the E^2-term is given by the higher limits of these atomic functors which turn out only to depend on the automorphism set of this object. In particular, Oliver showed that, for any Noetherian algebra K over the Steenrod algebra and any functor $F : \mathcal{A}(K) \to \mathcal{A}b$, all higher limits vanish above a certain degree.

2. Maps between classifying spaces

Sullivan [89] (for $BU(n)$) and later Wilkerson [91] (in the general case) constructed self maps of classifying spaces of connected compact Lie groups, which are called unstable Adams operations. That is a self map $f : BG \to BG$ which, for a suitable $k \in \mathbb{N}$, induces multiplication by k^i in the rational cohomology group $H^{2i}(BG; \mathbb{Q})$. In this case we say that f has degree k. The name comes from the fact that f induces in complex K-theory a map which looks like an Adams operation of degree k. These examples destroyed the hope that, up to homotopy, all maps between classifying spaces are induced by homomorphisms.

These examples also motivated Adams, Mahmud and Hubbuck [2], [3], [1], [44], [45] to study carefully the effect maps between classifying space could have in rational cohomology. Methods and results, which are available today and which are consequences of the generalized Sullivan conjecture (Theorem C.1), allow a more precise analysis of such maps. Results of great importance are those of Dwyer and Zabrodsky [37] and of the author [69].

In contrast to the above mentioned examples of Sullivan and Wilkerson, maps $BP \to BG$ are always induced by homomorphisms if P is a p-toral group and G a compact Lie group. To be more explicit, let $Rep(P, G) := Hom(P, G)/Inn(G)$ denote the set of representations $P \to G$, i.e. the set of all homomorphisms $P \to G$ modulo inner automorphisms of G.

2.1. THEOREM ([37], [99], [69]). *Let P be a p-toral group and G a compact Lie group. Then, passing to classifying spaces induces a bijection*

$$Rep(P, G) \to [BP, BG].$$

Moreover, for any homomorphisms $\rho : P \to G$, there exist equivalences

$$BC_G(\rho(P))^\wedge_p \xrightarrow{\simeq} \left(map(BP, BG)_{B\rho}\right)^\wedge_p \simeq map\left(BP, BG^\wedge_p\right)_{B\rho^\wedge_p}.$$

The map is given by the adjoint of

$$BC_G(\rho(P)) \times BP \simeq B(C_G(P)) \times P) \to BG$$

induced by the homomorphism ρ. The second equivalence comes from the technical dainty that in this case passing to completion and taking mapping spaces commute (see [50, Theorem 3.2] and for a general statement [11]). In the first part of the statement we have to divide out conjugations by elements of G, because for any compact Lie group inner automorphisms induce self maps on the classifying space homotopic to the identity [87]. For finite p-groups Theorem 2.1 was proved by Dwyer and Zabrodsky, and in the general case by the author. Zabrodsky also found a proof for tori.

For an outline of the proof let us assume that P is a finite p-group. The second part of the theorem is a consequence of the generalized Sullivan conjecture (Theorem C.1). Taking loops in the map $BC_G(\rho(p)) \to map(BP, BG_p^\wedge)_{B\rho}$ gives the fixed-point set G^P for the source and the homotopy fixed-point set $G_p^{\wedge hP}$ for the target. Here, the group P acts on G via the homomorphism ρ and conjugation. For G^P this is obvious, and for $G_p^{\wedge hP}$ this follows from the observation that the loop space $\Omega map(BP, BG)_{B\rho}$ is equivalent to the space of sections of the pull back fibration of the free loop space fibration $\Lambda BG \to BG$ along the map $BP \to BG$ and the fact that the free loop space fibration is fiber homotopy equivalent to the fibration $EG \times_G G \to BG$ where G acts via conjugation on itself. The proof of the first part goes by an induction over the order of P. The starting point is given by the case $P = \mathbb{Z}/p$. In this case Lannes' theory is available and gives a way to calculate $[B\mathbb{Z}/p, BG]$ in terms of representations. In the induction step it only remains left to calculate the set of homotopy classes $[BP, BG]$. This is done using obstruction theory and by describing

$$\coprod_{Rep(P,G)} BC_G(\rho(P))$$

and $map(BP, BG)$ as homotopy fixed-point sets of $\mathbb{Z}/p$-actions on suitable spaces, which we can apply to the induction hypothesis. The step is based on the observation that any finite p-group P fits into a short exact sequence $1 \to P_0 \to P \to \mathbb{Z}/p \to 1$.

In the case of P being a p-toral group, one uses the mod-p approximation of P by its finite p-subgroups to achieve a generalization of the generalized Sullivan conjecture [68] and a proof of Theorem 2.1.

Theorem 2.1 also allows the following corollary.

2.2. COROLLARY ([69]). *Let T be a torus and G a compact Lie group. Then, passing to rational cohomology induces an injection*

$$[BT, BG] \to Hom(H^*(BG; \mathbb{Q}), H^*(BT; \mathbb{Q})).$$

And using this corollary we can reformulate a theorem of Adams and Mahmud as

2.3. THEOREM ([2]). *Let G and H be two connected compact Lie groups with maximal tori T_G and T_H. Then, for every map $BG \to BH$, there exists a homomorphism $\alpha : T_G \to T_H$ such that the diagram*

$$\begin{tikzcd} BT_G \arrow[r, "B\alpha"] \arrow[d] & BT_H \arrow[d] \\ BG \arrow[r, "f"] & BH \end{tikzcd}$$

commutes up to homotopy. Moreover, if $\beta : T_G \to T_H$ is another homomorphism with this property, then we have $\beta = w' \circ \alpha$ for some element $w' \in W_H$.

Adams and Mahmud actually proved that the diagram commutes in rational cohomology, and that the last identity also holds in rational cohomology. But by Corollary 2.2 this is an equivalent statement. This theorem also led to the notion of admissible homomorphisms. A homomorphism $\alpha : T_G \to T_H$ is called admissible if for each $w \in W_G$ there exists a $w' \in W_H$ such that $w' \circ \alpha = \alpha \circ w$ (notice that for every $w \in W_G$ the composition $\alpha \circ w$ satisfies Theorem 2.3 if α does).

Actually, this is a stronger definition than the one of Adams and Mahmud. They were interested in maps which exist after localization at a set of primes and compared them with linear maps between the universal covers of the maximal tori.

Based on Theorem 2.1 and Theorem 2.3, Jackowski, McClure and Oliver set up a program to attack the classification of homotopy classes of maps between classifying spaces [50]. This program splits into several steps, which we explain next. For a much more detailed survey of this program we refer the reader to [53]. For the following G and H denote connected compact Lie groups.

Step 1: Admissible homomorphisms

By Theorem 2.3 every map $BG \to BH$ gives rise to a W_H-conjugacy class of an admissible homomorphism. For an admissible homomorphism $\alpha : T_G \to T_H$, let $[BG, BH]_\alpha$ denote the set of homotopy classes of maps $BG \to BH$ which all give rise to the W_H-conjugacy class of α. Then we have

$$[BG, BH] = \coprod_{\alpha} [BG, BH]_\alpha$$

where we take the union over all admissible homomorphisms α. The question comes down to a classification of all admissible homomorphisms and a study of the sets $[BG, BH]_\alpha$.

Adams and Mahmud proved that a homomorphism $\alpha : T_G \to T_H$ is admissible if and only if

$$B\alpha^* : H^*(BT_H; \mathbb{Q}) \to H^*(BT_G; \mathbb{Q})$$

maps the W_H-invariants $H^*(BT_H;\mathbb{Q})^{W_H} \cong H^*(BH;\mathbb{Q})$ into the W_G-invariants $H^*(BT_G;\mathbb{Q})^{W_G} \cong H^*(BG;\mathbb{Q})$ [2], which gives a necessary and sufficient condition to check the first part. The following steps deal with the second part of the problem.

Step 2: Passing to completions
If we want to use the mod-p decomposition of BG given by Theorem 1.3, we can only analyze maps $BG \to BH_p^\wedge$ into the p-adic completion. Sullivan's arithmetic square [89], [15] gives a way to pass forward and back between global data on the one side and mod-p and rational data on the other side. Because BH is simply connected and because BH is rationally a product of Eilenberg–MacLane spaces of even degrees, these techniques allow a proof of

2.4. PROPOSITION ([50, Theorem 3.1]). *Let G and H be connected compact Lie groups. For each admissible homomorphism $\alpha : T_G \to T_H$, the map*

$$[BG, BH]_\alpha \to \prod_{p \mid |W_H|} [BG, BH_p^\wedge]_\alpha$$

is a bijection.

If p does not divide the order $|W_H|$ of the Weyl group W_H, then there always exists an extension of α to a map $BG \to BH_p^\wedge$, unique up to homotopy (see Theorem 2.6). This is the reason why one only has to take into account those primes which divide $|W_H|$.

So we are left with the problem of calculating the sets $[BG, BH_p^\wedge]_\alpha$.

Step 3: $\mathcal{R}_p(G)$-invariant representations
For this step we fix a prime p. Let $N_p(T_G) \subset N(T_G) \subset G$ be a p-toral Sylow subgroup of G; i.e. $N_p(T_G)/T_G \subset W_G$ is a p-Sylow subgroup of W_G.

For a map $f : BG \to BH$, Theorem 2.1 provides more information than just the existence of an admissible homomorphism. The restriction $f|_{BN_p(T_G)} \simeq B\rho$ is homotopic to a map induced by a homomorphism $\rho : N_p(T_G) \to H$ which is unique up to conjugation in H (and this also is true for any p-toral subgroup). In particular, for any pair $P_1, P_2 \subset G$ of p-toral subgroups and subconjugations $c_{g_i} : P_i \to N_p(T_G)$ and any subconjugation $c_g : P_1 \to P_2$, the compositions $\rho \circ c_{g_2} \circ c_g$ and $\rho \circ c_{g_1}$ are conjugate in H. That is to say that the homomorphism ρ establishes an element

$$\hat{\rho} := (\rho_P) \in \lim_{\substack{\longleftarrow \\ G/P \in \mathcal{R}_p(G)}} Rep(P, H).$$

Homomorphisms $N_p(T_G) \to H$ with this property are called $\mathcal{R}_p(G)$-invariant representations of $N_p(T_G)$. Every map $BG \to BH_p^\wedge$ which comes from an integral map gives rise to such an $\mathcal{R}_p(G)$-invariant representation. Thus the problem is now: given an admissible map $\alpha : T_G \to T_H$, does there exist an extension to an $\mathcal{R}_p(G)$-invariant representation $\rho : N_p(T_G) \to H$? And if so, how many conjugacy classes are there?

There is a lack of general techniques for doing this. But in the case of H being a classical matrix group like $U(n)$, $SU(n)$, $O(n)$ or $Sp(n)$, character theory is sufficient to check if two homomorphisms are conjugate and therefore, if a homomorphisms

$N_p(T_G) \to H$ is $\mathcal{R}_p(G)$-invariant. If G is connected, every element of G is subconjugate to T_G. Thus, if H is one of the above mentioned groups character theory also tells us that there is at most one $\mathcal{R}_p(G)$-invariant extension of a given admissible homomorphism.

Finally we have to pass from $\mathcal{R}_p(G)$-invariant representations to actual maps.

Step 4: From $\mathcal{R}_p(G)$-invariant representations to actual maps
For this step we fix a prime p and a $\mathcal{R}_p(G)$-invariant representation $\rho : N_p(T_G) \to H$ respectively an element

$$\hat{\rho} = (\rho_P)_{G/P} \in \lim_{\substack{\longleftarrow \\ G/P \in \mathcal{R}_p(G)}} Rep(P, G).$$

Let $map(BG, BH_p^{\wedge})_\rho$ denote the union of all components given by the counterimage of ρ under the obvious map

$$[BG, BH] \to \lim_{\substack{\longleftarrow \\ G/P \in \mathcal{R}_p(G)}} Rep(G).$$

Using the decomposition of BG of Theorem 1.3, we get

$$map\big(BG, BH_p^{\wedge}\big)_\rho \simeq map\Big(\operatorname*{hocolim}_{\mathcal{R}_p(G)} EG \times_G \mathcal{I}, BH_p^{\wedge}\Big)_\rho,$$

and applying Theorem A.2 establishes a spectral sequence calculating the homotopy of $map(BG, BH_p^{\wedge})_\rho$. Let

$$\Pi_1^\rho : \mathcal{R}_p(G) \to p\text{-groups} \quad \text{and} \quad \Pi_n^\rho : \mathcal{R}_p(G) \to \mathcal{A}b$$

denote the functors given by

$$\pi_n^\rho(G/P) := \pi_n\big(map(BP, BH_p^{\wedge})_{B\rho_P}\big) \cong \pi_n\big(BC_H(\rho_P)_p^{\wedge}\big).$$

(Note that for any p-toral subgroup $P \subset G$ the group of components of the centralizer $C_H(\rho_P)$ is a finite p-group if $\pi_0(G)$ is one [50, Proposition A.4].) Now Theorem A.1 and Corollary A.2 take the form

2.5. THEOREM ([50]). *Let $\rho : N_p(T_G) \to H$ be a $\mathcal{R}_p(G)$-invariant representation. Then, there exists a spectral sequence*

$$E_2^{p,q} := \lim_{\substack{\longleftarrow \\ G/P \in \mathcal{R}_p(G)}}{}^{p} \; \Pi_q^\rho \Longrightarrow \pi_{q-p}\big(map\big(BG, BH_p^{\wedge}\big)\big)_\rho$$

which strongly converges. In particular, the map $B\rho$ has an extension $f : BG \to BH_p^{\wedge}$ if

$$\lim_{\substack{\longleftarrow \\ G/P \in \mathcal{R}_p(G)}}{}^{n+1} \; \Pi_n^\rho$$

vanishes for all $n \geqslant 1$, *and there exists at most one extension if*

$$\lim_{\substack{\longleftarrow \\ G/P \in \mathcal{R}_p(G)}}{}^{n} \quad \Pi_n^{\rho}$$

vanishes for all $n \geqslant 1$.

The strong convergence follows from the fact that there exists an N, depending only on G, such that the n-th higher limit of any functor defined on $\mathcal{R}_p(G)$ vanishes for $n \geqslant N$ [50].

Now one has to face the analysis of the spectral sequences, i.e. in the first place the calculation of the higher limits. Although this looks like a very difficult and hard question, Jackowski, McClure and Oliver developed techniques to attack this problem successfully in many interesting cases (see [50], [51], [52]).

We demonstrate the power of this machinery in two cases.

2.6. THEOREM ([53]). *If G is connected and if $(p, |W_G|) = 1$, then any admissible homomorphism $\alpha : T_G \to T_H$ has an extension $f : BG \to BH_p^\wedge$, unique up to homotopy.*

PROOF. Actually this was already proved by Adams and Mahmud [2] with different methods. For a proof, the present theory can be used as follows. Because $(p, |W_G|) = 1$, we have $T_G = N_p(T_G)$, and the category $\mathcal{R}_p(G)$ consists only of the object G/T_G. The set of endomorphisms is given by W_G. Hence, for any admissible homomorphism $\alpha : T_G \to T_H$, there exists a unique $\mathcal{R}_p(G)$-invariant representation $\rho = \alpha : N_p(T_G) = T_G \to H$. The higher limits are isomorphic to the cohomology groups $H^p(W_G; \pi_q(BT_{G_p}^\wedge))$ and vanish for $p \geqslant 1$. The associated spectral sequence of Theorem A.1 collapses, which finishes the proof. $\square$

The other case, which is much more difficult and much deeper concerns integral self maps. Using their machinery, Jackowski, McClure and Oliver proved the following beautiful classification theorem for self maps of classifying spaces of simple connected compact Lie groups.

2.7. THEOREM ([50]). *Let G be a simple connected compact Lie group. Then there exists a bijection*

$$\Phi : [BG, BG] \to \{0\} \coprod \big(Out(G) \times \{k \geqslant 1 : (k, |W_G|) = 1\}\big).$$

For two self maps $f, g : BG \to BG$, the following conditions are equivalent:

(1) *f and g are homotopic.*

(2) *The restrictions $f|_{BT_G}$ and $g|_{BT_G}$ are homotopic.*

(3) *The induced maps $H^*(f; \mathbb{Q})$ and $H^*(g; \mathbb{Q})$ in rational cohomology are equal.*

Moreover, for each map $f : BG \to BG$ and each prime there exist equivalences

$$BZ(G)_p^\wedge \simeq map(BG, BG)_{f_p^\wedge} \simeq map\big(BG, BG_p^\wedge\big)_{f_p^\wedge}.$$

OUTLINE OF PROOF. The first step consists of a characterization of all admissible homomorphisms $T_G \to T_G$, which is due to Hubbuck [45] and Ishiguro [47]. Hubbuck showed that, for G simple, every self map of BG looks in rational cohomology like a composition of a map induced by an outer automorphism and an unstable Adams operation of degree $\geqslant 0$. Ishiguro proved that, for a connected compact Lie group G, every unstable Adams operation $BG \to BG$ has a degree coprime to the order of the Weyl group. Putting these facts together and passing to rational cohomology establishes the map Φ of the statement.

The existence of unstable Adams operations of any degree coprime to $|W_G|$ was shown by Sullivan [89], Wilkerson [91] and Friedlander [39]. Hence, the map Φ is surjective, and it only remains to show that rational cohomology detects homotopy classes of self maps, or, what is sufficient, that rational cohomology detects homotopy classes of unstable Adams operations. One has to distinguish between the degree 0 and degrees $\geqslant 1$. We only consider the latter case; the former one can be treated similarly.

The associated admissible homomorphism of an unstable Adams operation of degree $k \geqslant 1$ is given by $\alpha_k : T_G \to T_G$, $t \mapsto t^k$. In the next step one has to show that up to conjugation there exists at most one $\mathcal{R}_p(G)$-invariant representation $\rho_k : N_p(T_G) \to G$. For most of the classical matrix groups this follows from character theory and in general this is done in [50, Proposition 3.5].

After passing to completions as described in Step 2, one finally has to calculate the higher limits

$$\lim_{\substack{\longleftarrow \\ G/P\in\mathcal{R}_p(G)}}{}^{p} \ \pi_q\Big(map\big(BP, BG_p^\wedge\big)_{B(\rho_k|_P)}\Big) \cong \lim_{\substack{\longleftarrow \\ G/P\in\mathcal{R}_p(G)}}{}^{p} \ \pi_q\big(BC_G\big(\rho_k(P)\big)_p^\wedge\big).$$

The equivalence is a consequence of Theorem 2.1. Using the fact that, for p-stubborn subgroups $P \subset G$, the centralizer $C_G(\rho_k(P))$ is equal to the center $Z(P)$ of P [50, Lemma 1.5] and using general techniques developed for the calculation of higher limits of functors on $\mathcal{R}_p(G)$, Jackowski, McClure and Oliver were able to prove the vanishing of all higher limits under consideration and to show that

$$\lim_{\substack{\longleftarrow \\ G/P\in\mathcal{R}_p(G)}} \ \pi_q\big(BC_G\big(\rho_k(P)\big)_p^\wedge\big) \cong \pi_q\big(BZ(G)_p^\wedge\big).$$

Hence, the associated spectral sequence of Theorem A.1 collapses. This proves the statement. (For more details see [50], [53].) □

For any connected compact Lie group G, Jackowski, McClure and Oliver applied the same method to self maps $BG \to BG$ which induce an isomorphism in rational cohomology. There also exists a complete characterization of all admissible maps in terms of Dynkin diagram symmetries or outer automorphisms and unstable Adams operations [52], [61], [71]. The analogous statement as in Theorem 2.7 is true [52]. In particular, homotopy classes of rational self equivalences are detected by rational cohomology as well as by the restriction to the maximal torus.

Based on Theorem 2.7, different proofs for the same results about rational self equivalences of classifying spaces of connected compact Lie groups are given by Møller [62] and by the author [71].

For general self maps, the classification problem is much harder, because that involves the classification of all maps between classifying spaces of connected compact Lie groups. In their work, Jackowski, McClure and Oliver constructed examples contradicting all reasonable conjectures one could make; e.g., they found a pair of nonhomotopic maps $f, g : B(SO(3) \times SO(3)) \to BSO(25)$, whose restrictions to $BN(T_G)$ are homotopic. Hence, homotopy classes of maps between classifying spaces cannot be detected by any cohomology theory in general.

What is known beyond this point? One can look at self maps $BG_p^\wedge \to BG_p^\wedge$ of completed classifying spaces. For connected compact Lie groups similar results are obtained. The "completion" of Corollary 2.2 and of the Adams–Mahmud theorem (Theorem 2.3) are proved by Adams and Wojtkowiak [5] and by Smith and the author [76], the completion of Theorem 2.6 by Wojtkowiak [95] and the completion of Theorem 2.7 is due to Jackowski, McClure and Oliver [52].

Møller studied self maps of classifying spaces of nonconnected compact Lie groups [63]. Every compact Lie group G gives rise to a fibration $Fib(G) : BG_0 \to BG \to B\pi := B\pi_0(G)$. Every self map $f : BG \to BG$ establishes a self map $B\rho : B\pi \to B\pi$ which is induced by a homomorphism. A rational self equivalence of BG is defined to be a fiber self map $(f, B\rho_f)$, such that $f|_{BG_0}; BG_0 \to BG_0$ induces an isomorphism in rational cohomology. Let $\varepsilon_{\mathbf{Q}}(BG)$ denote the monoid of all vertical homotopy classes of rational self equivalences of BG. By definition there is a monoid map

$$\varepsilon_{\mathbf{Q}}(BG) \to \varepsilon_{\mathbf{Q}}(BG_0) \times End(\pi)$$

where $End(-)$ denotes the set of endomorphisms of a group, and where $\varepsilon_{\mathbf{Q}}(BG_0)$ is actually the monoid of ordinary homotopy classes of rational self equivalences. The second coordinate of the map takes image among homomorphisms because a vertical homotopy induces a pointed homotopy on the base.

Let $g := f|_{BG_0}$ be the restriction on the fiber. The pull back via $B\rho$ establishes an induced fibration $B\rho^* Fib(G)$. Møller also constructed a fibration $g_* Fib(G)$ by imitating the push out for groups. Let $\varepsilon_{\mathbf{Q},G}(BG_0, B\pi) \subset \varepsilon_{\mathbf{Q}}(BG_0) \times End(\pi)$ be the subset of all pairs (g, ρ) such that $B\rho^* Fib(G)$ and $g_* Fib(G)$ are fiber homotopy equivalent fibrations. Møller proved the following classification result for rational self equivalences of nonconnected compact Lie groups.

2.8. THEOREM ([63]). *For every compact Lie group G, there exists a short exact sequence of monoids*

$$1 \to H^1(\pi_0(G); Z(G)) \to \varepsilon_{\mathbf{Q}}(BG) \to \varepsilon_{\mathbf{Q},G}(BG_0, B\pi_0(G)).$$

The action of $\pi_0(G)$ on the center $Z(G)$ is induced by conjugation. Using the analogue of Theorem 2.7 for connected compact Lie groups, this gives a complete classification of homotopy classes of rational self equivalences of classifying spaces of compact Lie groups.

More recently, Jackowski and Oliver used the method described to analyze maps $BG \to BU(n)$. In fact, they stabilized such maps in the same sense as is done for honest representations in $U(n)$. They defined

$$\mathbb{K}(BG) := Gr\Big(\coprod_{n\geqslant 0} [BG, BU(n)]\Big)$$

as the Grothendieck group of the monoid $\coprod_n [BG, BU(n)]$. The monoid structure is inherited from the Whitney sum of vector bundles. There is also a multiplication which comes from the tensor product of vector bundles. Of course, this definition makes perfect sense for topological spaces, and, if X is a finite CW-complex, then we have $\mathbb{K}(X) \cong K(X) = [X, \mathbb{Z} \times BU]$.

For every prime, restriction to a p-toral subgroup P establishes a map

$$R : \mathbb{K}(BG) \to \prod_{\substack{P\subseteq G \\ \text{all primes}}} \mathbb{K}(BP).$$

Using Theorem 2.1, for p-toral groups, we can identify $\mathbb{K}(BP)$ with the representation ring $R(P)$. Let $\mathcal{O}_{\mathcal{P}}(G) \subset \mathcal{O}(G)$ denote the full subcategory of the objects G/P, where P is a p-toral group for some prime. So, what is the image and the kernel of R?

Passing from homomorphisms to maps between classifying spaces and from $BU(n)$ to BU gives rise to a sequence of maps

$$R(G) \to \mathbb{K}(BG) \to K(BG)$$

from classical representation theory to $\mathbb{K}(BG)$ and to complex K-theory.

Let $I_R(G)$ denote the augmentation ideal of $R(G)$. Then, Atiyah [7] (for finite groups) and Atiyah and Segal [9] (for general compact Lie groups) showed that $I_R(G)$-adic completion induces an isomorphism

$$R(G)^\wedge_{I_R(G)} \xrightarrow{\cong} K(BG).$$

How does the group $\mathbb{K}(BG)$ fit into this picture? Functorial properties establish a commutative diagram

$$\begin{tikzcd} R(G) \arrow[r, "\lambda_G"] \arrow[d, "\bar{\alpha}_G"', "\cong"] \arrow[dr, "\alpha_G"] & R(G)^\wedge_{I_R(G)} \arrow[d, "\alpha_G{}^\wedge_I"] \\ \mathbb{K}(BG) \arrow[r, "\beta_G"] & K(BG) \end{tikzcd} \quad . \tag{3}$$

The following statement, due to Jackowski and Oliver, answers the above questions.

2.9. THEOREM ([54]). *For every compact Lie group G, the map R induces an isomorphism*

$$\mathbb{K}(BG) \xrightarrow{\cong} \varprojlim_{\mathcal{O}_{\mathcal{P}}(G)} R(P).$$

Furthermore, in diagram (3), *we have $ker(\lambda_G) = ker(\overline{\alpha}_G)$ and β_G is a monomorphism.*

The kernel of λ_G is known. It is given by all representations of G whose restriction to p-th power elements of G is trivial. For example, for connected compact Lie groups, λ_G as well as $\overline{\alpha}_G$ are monomorphisms. Jackowski and Oliver also computed the image of β_G and identified it with the formally finite elements of $K(BG)$ (see [1]). These are elements of $K(BG)$ which are mapped on 0 by λ-operations of large degree.

As usual, when proving a statement about homotopy classes of maps, the full mapping space has to be considered. There is a parallel construction leading to a Grothendieck group

$$\mathcal{K}(BG) := Gr\bigg(\coprod_n map(BG, BU(n))\bigg).$$

The disjoint union $\coprod_n map(BG, BU(n)$ gets a monoid structure from the map $BU(n)\times BU(m) \to BU(n+m)$ which is induced from the Whitney sum of vector bundles. Then Jackowski and Oliver used a refinement of the above general approach to calculate the homotopy groups of $\mathcal{K}(BG)$. These calculations give a proof of Theorem 2.9.

Jackowski and Oliver also looked at real vector bundles over classifying spaces of compact Lie groups and got similar results as in Theorem 2.9 [54].

3. The Steenrod problem: Realizations of polynomial algebras

Steenrod posed the question, which polynomial algebras over $\mathbb{F}_p$ appear as the mod-p cohomology of a topological space [88]? Examples are provided by classifying spaces of connected compact Lie groups. For a connected compact Lie group G the mod-p cohomology $H^*(BG;\mathbb{F}_p)$ is polynomial for almost all primes (in particular for primes coprime to the order of the Weyl group and in several cases for all primes). If this is the case, a result of Borel [12] tells us that, at least for odd primes, the inclusion of the maximal torus $T_G \to G$ induces an isomorphism $H^*(BG;\mathbb{F}_p) \cong H^*(BT_G;\mathbb{F}_p)^{W_G}$. For primes not dividing the order of W_G, a straightforward calculation of a Serre spectral sequence shows that the map $BN(T_G) \to BG$ is a mod-p equivalence. This observation led Clark and Ewing to a construction of several exotic examples of spaces with polynomial cohomology [19]. They considered finite pseudo reflection groups $W \to Gl(n, \mathbb{Z}_p^\wedge)$, whose order is coprime to p. That is the map is a monomorphism and W is generated by pseudo reflections. And a pseudo reflection is a linear map of finite order, which fixes a hyperplane of codimension 1. Every such group induces an action on the Eilenberg–MacLane space $K := K(\mathbb{Z}_p^{\wedge n}, 2)$. Then, the mod-$p$ cohomology of the Borel construction

$EW \times_W K$ is given by the invariants and is polynomial by a theorem of Chevalley [18]. Here, exotic means that these spaces are not equivalent to the completion of a classifying space of a compact Lie group.

Later Adams and Wilkerson gave criteria which ensure that a polynomial algebra on n generators over the Steenrod algebra is isomorphic to the invariants of a pseudo reflection group $W \to Gl(n, \mathbb{F}_p)$ acting on the polynomial part P_V of $H^*((B\mathbb{Z}/p)^n; \mathbb{F}_p)$ [4]. For odd primes, Dwyer, Miller and Wilkerson showed that, for a polynomial algebra $P \cong H^*(X; \mathbb{F}_p)$ given by the mod-p cohomology of a space, these conditions are always satisfied and that the associated pseudo reflection group $W \to Gl(n; \mathbb{F}_p)$ always lifts to $Gl(n; \mathbb{Z}_p^\wedge)$ [27].

3.1. THEOREM ([4], [27]). *Let p be an odd prime. Let P b a polynomial algebra over the Steenrod algebra. If $P \cong H^*(X; \mathbb{F}_p)$, then there exists a pseudo reflection group $W \to Gl(n; \mathbb{Z}_p^\wedge)$ such that $P \cong (P_V)^W$.*

Based on this result, Dwyer and Wilkerson proved the following realization and uniqueness theorem for polynomial algebras.

3.2. THEOREM ([27]). *Let P be a polynomial algebra over the Steenrod algebra generated by elements of degree coprime to p. Then there exists a p-complete space X, unique up to homotopy, with $H^*(X; \mathbb{F}_p) \cong P$.*

All these algebras are realized by the examples of Clark and Ewing. For a proof of the uniqueness see the proof of Theorem 4.2.

Theorem 3.1 also shows that, for odd primes, a solution of Steenrod's problem asks for a classification of pseudo reflection groups over the p-adic integers. Clark and Ewing gave a complete list of all p-adic rational irreducible reflection groups $W \to Gl(U)$ where U is a vector space over the p-adic rationals. So, slightly changing the problem, one might ask for a realization of these irreducible p-adic rational pseudo reflection groups. Notice that the classifying space of every simple connected compact Lie group realizes one of the irreducible pseudo reflection groups, but not every of these spaces has polynomial mod-p cohomology.

Besides the Clark–Ewing spaces, computations of Quillen on the mod-p group cohomology of general linear groups over finite fields of characteristic coprime to p [83] and ad hoc constructions of Zabrodsky [96] gave further spaces whose mod-p cohomology is polynomial. In these cases the order of the associated pseudo reflection group is not coprime to p, which makes constructions much more difficult. The examples of Quillen and Zabrodsky as well as the examples we discuss next realize irreducible pseudo reflection groups.

More recently, Aguadé [6] and Dwyer and Wilkerson [32] approached the Steenrod question using ideas from the decomposition theorems for classifying spaces. Aguadé looked at diagrams similar to the decomposition diagram of $BSO(3)$ (see Section 1, diagram (2)). For a pair of groups $H \subset G$ he considered the category $C(G, H)$ with two objects 0 and 1 and morphism sets given by $End(0) = G$, $End(1) = \{1\}$, and $Hom(0, 1) = G/H$ and $Hom(1, 0) = \emptyset$, constructed a "nice" functor $C(G, H) \to Top$

into the category of topological spaces and took the homotopy limit of F. The Bousfield–Kan spectral sequence of Theorem A.1 gives a tool to compute the mod-p cohomology of the homotopy colimit. Hopefully all higher limits involved vanish.

3.3. THEOREM ([6]). *Let $H \subset G$ be a pair of finite groups. If, for any $\mathbb{F}_p[G]$-module M, restriction induces an isomorphism $H^*(G,M) \cong H^*(H,M)$, then there exists a functor $F : \mathcal{C}(G,H) \to Top$ such that $H^*(F(0);\mathbb{F}_p) =: P$ is a polynomial algebra generated by elements of degree 2, such that $H^*(F(1);\mathbb{F}_p) \cong P^H$ and such that*

$$H^*\Big(\underset{\mathcal{C}(G,H)}{hocolim} F;\mathbb{F}_p\Big) \cong P^G.$$

Aguadé applied his result to several cases of the Clark–Ewing list, covering the examples of Zabrodsky and producing some new spaces with polynomial mod-p cohomology. He also reconstructed the classifying spaces of the exceptional Lie groups E_6, E_7, E_8 at those primes which do not appear as torsion primes in the integral homology of the particular group.

Dwyer and Wilkerson chose an approach based on the algebraic decomposition via centralizers of elementary abelian subgroups (Theorem 1.3 and Theorem 1.4). They found a space whose mod-2 cohomology is given by the Dickson invariants in dimension 4.

3.4. THEOREM ([32]). *There exists a space X with*

$$H^*(X;\mathbb{F}_2) \cong H^*\big((B\mathbb{Z}/2)^4;\mathbb{F}_2\big)^{Gl(4,\mathbb{F}_2)} =: D(4).$$

OUTLINE OF PROOF. The idea of the proof comes from the fact, that there exists an algebraic decomposition of $D(4)$ over the Rector category of $D(4)$ (Theorem 1.4) and that the topological realization should give a topological decomposition of such a space. For each object $\phi : D(4) \to H^*(BV;\mathbb{F}_2)$ of the Rector category $A_2(D(4))$, a calculation of the pieces $T_V(D(4);\phi)$ of the algebraic decomposition shows that these algebras are given by the mod-2 cohomology of $BC_{Spin(7)}(V)$ for a suitable inclusion $V \subset Spin(7)$. Here, V is an elementary abelian 2-group of dimension $\leqslant 4$. In the next step Dwyer and Wilkerson constructed a functor $F : A_2(D(4)) \to HoTop$ into the homotopy category of topological spaces, which realizes the algebraic data. This is the hard part of the matter. Beside the solution in this special case, Dwyer and Wilkerson approached such questions in a more general context, including algebraic decomposition of spaces whose mod-p cohomology satisfies the assumptions of Theorem 1.4 [36].

Because homotopy colimits do not exist in the homotopy category, one has finally to find a lift $\tilde{F} : A_2(D(4)) \to Top$ of F. For such a need, Dwyer and Kan had developed an obstruction theory [23], [24]. In the case under consideration the obstruction groups are given by some higher limits of a functor on $A_2(D(4))$ with the homotopy groups of $map(\widetilde{F}(\phi), \widetilde{F}(\phi))_{id}$ as values. The functor F takes image among the classifying spaces of certain subgroups of $Spin(7)$ and, passing everywhere to completions, the mapping spaces can be identified with centers (see Theorem 2.7 and extensions). This allows a proof of the vanishing of the obstruction groups and of the existence of $\widetilde{F}$. The mod-2 cohomology of the homotopy colimit $hocolim_{A_2(D(4))}\widetilde{F}$ can be computed using the spectral sequence of Theorem A.1. Theorem 1.4 proves the vanishing of all higher limits involved. This finishes the proof. □

An Eilenberg–Moore spectral sequence argument shows that, for every space X with polynomial mod-p cohomology, the cohomology $H^*(\Omega X;\mathbf{F}_p)$ of the loop space is finite. That is that X is the classifying space of a connected p-compact group. In Section 5, we show that every connected p-compact group comes with an associated pseudo reflection group $W \to Gl(n;\mathbb{Q}_p^\wedge)$. This group W plays the same role as Weyl groups do for connected compact Lie groups. In this sense the example of Dwyer and Wilkerson gives a realization of another irreducible pseudo reflection group at the prime 2, although the group $Gl(4,\mathbb{F}_2)$ does not lift to $\mathbb{Z}_2^\wedge$. ($Gl(4,\mathbb{F}_2)$ is not the Weyl group of the Dwyer–Wilkerson example.)

The Eilenberg–Moore spectral sequence argument also shows that the solution of Steenrod's problem is closely related to the classification of all connected p-compact groups, which we will discuss in Section 5.

4. Homotopy uniqueness of classifying spaces

Connected compact Lie groups are very rigid objects. A few combinatorial data are sufficient to distinguish between two connected compact Lie groups; e.g., Dynkin diagrams classify the local isomorphism types of semi-simple connected compact Lie groups, or the isomorphism types of simply connected compact Lie groups.

Surprisingly, classifying spaces of connected compact Lie groups also seem to be very rigid objects. The algebra $H^*(BG;\mathbf{F}_p)$ considered as an algebra over the Steenrod algebra determines the homotopy type of the p-adic completion $BG_p^\wedge$ in a large number of cases. This is what we mean by the homotopy uniqueness of the classifying spaces of connected compact Lie groups. The first results of this type were proved by Dwyer, Miller and Wilkerson [26], [27]. We say that two spaces X and Y have the same mod-p type if $H^*(X;\mathbf{F}_p)\cong H^*(Y;\mathbf{F}_p)$ as algebras over the Steenrod algebra.

4.1. THEOREM ([26]). *Let $G = SU(2)$ or $G = SO(3)$ and let X be a p-complete space. Then the spaces X and BG have the same mod-p type if and only if they are homotopy equivalent.*

4.2. THEOREM ([27]). *Let G be a connected compact Lie group and let X be a p-complete space. Assume that $(p,|W_G|) = 1$. Then the two spaces X and BG have the same mod-p type if and only if they are homotopy equivalent.*

For $U(2)$, McClure and Smith proved the analogous result of Theorem 4.1 [57]. The second theorem is a special case of Theorem 3.2 and also covers Theorem 4.1 for odd primes. To give the reader an idea of the techniques used for the proof we outline the proof of the second theorem. The main idea is to combine Lannes' theory (see Appendix B) and the Dwyer–Zabrodsky theorem (Theorem 2.1) into a powerful tool.

PROOF of Theorem 4.2. By assumption, p is an odd prime not dividing the order of W_G. By [12], this implies that $H^*(BG;\mathbb{F}_p) \cong H^*(BT_G;\mathbb{F}_p)^{W_G}$. We fix such an isomorphism and try to realize it by a topological map. As a first step we construct a map $f : BT_G \to X$ which looks in mod-p cohomology like the map $Bi : BT_G \to BG$. By Theorem B.1, the composition

$$H^*(X;\mathbb{F}_p) \cong H^*(BG;\mathbb{F}_p) \to H^*(BV;\mathbb{F}_p)$$

has a topological realization $f_V : BV \to X$ where $V \subset T_G$ is a maximal elementary abelian subgroup. Because p is odd, we have $C_G(V) = T_G$ for the centralizer of V in G [27]. The application of the T-functor and Theorem 2.1 establish a diagram

$$\begin{tikzcd}
T_V(H^*(X;\mathbb{F}_p), f^*) \arrow[r] \arrow[d, "\cong"'] & H^*(map(BV,X)_f;\mathbb{F}_p) & \\
T_V(H^*(BG;\mathbb{F}_p), Bi^*) \arrow[r] & H^*(map(BV,BG_p^\wedge)_{f_V};\mathbb{F}_p) \arrow[r, "\cong"] & H^*(BT_G;\mathbb{F}_p)
\end{tikzcd}$$

Because $BT_{G}{}_p^\wedge$ is simply connected, the lower left arrow is an isomorphism (Theorem B.2) and so is the upper arrow, since $T_V(H^*(X;\mathbb{F}_p), f^*)$ vanishes in degree 1 (Theorem B.2). The mod-p cohomology determines the homotopy type of $BT_{G}{}_p^\wedge$. Hence, the mapping space $map(BV,X)_f$ and $BT_{G}{}_p^\wedge$ are equivalent. Again by Theorem B.2, the action of W_G on V fixes the component of f, for it does so cohomologically, and establishes a second action of W_G on $BT_{G}{}_p^\wedge$. mod-p, both actions are equivalent, which follows from the above sequence of isomorphisms. For an appropriate basepoint of BV, the evaluation $map(BV,X)_f \to X$ induces the desired map $f : BT_{G}{}_p^\wedge \to X$, which is W_G-equivariant as well as Bi, where W_G acts trivially on BG and on X.

Because p is coprime to $|W_G|$, both actions of W_G on $BT_{G}{}_p^\wedge$ are equivalent over the p-adic integers. Passing to the Borel construction yields a map $EW_G \times_{W_G} BT_G \to X$. Again, because $(p, |W_G|) = 1$ we have a sequence of maps

$$BG_p^\wedge \leftarrow BN(T_G)_p^\wedge \simeq (EW_G \times_{W_G} BT_G)_p^\wedge \to X.$$

A straight forward calculation of the mod-p cohomology shows that both arrows are homotopy equivalences, which finishes the proof. □

For $p = 2$ and $G = SO(3)$, Dwyer, Miller and Wilkerson used the pushout diagram of $BSO(3)$ described in Section 1 (diagram (1)). Given a space Y with the same mod-2 cohomology, they constructed maps from all pieces of this diagram into Y and showed that the associated diagram

$$\begin{tikzcd}
BD(8) \arrow[r] \arrow[d] & BO(2) \arrow[d] \\
B\Sigma_4 \arrow[r] & Y
\end{tikzcd}$$

commutes up to homotopy. This is the hard part of the proof and again based on combining Lannes' theory with the Dwyer–Zabrodsky theorem (notice that $O(2) = C_{SO(3)}(\mathbb{Z}/2)$). The above diagram establishes a homotopy equivalence $BSO(3)^\wedge_2 \to Y$. For $G = SU(2)$, the homotopy uniqueness is proved with the help of the sequence of fibrations $B\mathbb{Z}/2 \to BSU(2) \to BSO(3) \to B^2\mathbb{Z}/2$.

In general, mod-p cohomology is not sufficient to characterize the homotopy type of BG for connected compact Lie groups G; e.g., the spaces $B(SU(p^2) \times S^1)$ and $B(SU(p^2) \times_{\mathbb{Z}/p} S^1)$ have isomorphic mod-p cohomology, but are not homotopy equivalent [72, 9.6]. For nonsimply connected compact Lie groups one needs a little extra information. Let X be a p-complete space with the same mod-p type as BG. As shown above there exists a map $BT_G{}^\wedge_p \to X$ and another W_G-action on $BT_G{}^\wedge_p$ making the map equivariant (for $p = 2$ one needs an extra assumption (see [72]). We say that X has the p-adic type of BG if it has the same mod-p type and if both actions of W_G on $BT_G{}^\wedge_p$ are p-adically equivalent. Actually, this is a rough version of the technical definition given in [72] but hits the heart of the matter. We say that BG is p-torsion free if $H^*(BG;\mathbb{Z})$ has no p-torsion. In [72], the following homotopy uniqueness result is proved.

4.3. THEOREM ([72]). *Let p be an odd prime. Let G be a connected compact Lie group such that BG is p-torsion free. Let X be a p-complete space.*

(1) *If X has the mod-p type of BG, then there exists a connected compact Lie group H such that X and BH have the same p-adic type.*

(2) *The space X has the p-adic type of BG, if and only if X and $BG^\wedge_p$ are homotopy equivalent.*

(3) *If G is simply connected, if G is a product of unitary groups, or if $(p, |W_G|) = 1$, then X has the mod-p type of BG if and only if X and $BG^\wedge_p$ are homotopy equivalent.*

For $p = 2$ similar results are true for quotients of products of unitary and special unitary groups, but one has to exclude $SU(2) = Sp(1)$ as factor. For odd primes, this covers all classical matrix groups and among the exceptional Lie groups only a few cases are missed (for a complete list see [72]).

The proof of Theorem 4.3 is heavily based on the work of Jackowski, McClure and Oliver, their decomposition of BG via p-stubborn subgroups and their analysis of self maps of BG. The idea is to construct the identity $id : BG^\wedge_p \to BG^\wedge_p$ purely by algebraic means. The assumption that BG is p-torsionfree is essential for the proof. In particular, it implies that $H^*(BG;\mathbb{F}_p) \cong H^*(BT_G;\mathbb{F}_p)^{W_G}$ [12] which makes calculation with the Lannes T-functor easier. Furthermore, Oliver's computation of the p-stubborn subgroups of the classical matrix groups [79] allows a sufficient understanding of the category $\mathcal{R}_p(G)$ in these cases. The proof also uses the classification of connected compact Lie groups. For simple simply connected Lie groups, the proof is done by a case by case checking and differs only in details from the one for $U(n)$.

To demonstrate the ideas we consider the case of $G = U(n)$ and an odd prime. We fix an isomorphism $H^*(X;\mathbb{F}_p) \cong H^*(BU(n);\mathbb{F}_p)$. As in the proof of Theorem 4.2, we construct a "maximal torus" $f_T : BT_{U(n)}{}^\wedge_p \to X$ and another action of $W_{U(n)} = \Sigma_n$ on $BT_{U(n)}{}^\wedge_p$. By construction, the two representations $\rho_{U(n)}, \rho_X : W_{U(n)} \to Gl(n;\mathbb{Z}^\wedge_p)$, associated to the two actions of $W_{U(n)}$ on $BT_{U(n)}{}^\wedge_p$, are equivalent mod-p. Because the standard permutation representation over $\mathbf{F}_p$ has only one p-adic lift [72, Proposition 11.1], the two representations are p-adically conjugate, and we assume they are equal. That is to say that $id : BT_{U(n)}{}^\wedge_p \to BT_{U(n)}{}^\wedge_p$ is an "admissible" map. (For a general connected compact Lie group there may exist several, but finitely many p-adic liftings, and each of them is associated with a possibly different connected compact Lie group.)

The classifying space $BN(T_{U(n)})$ of the normalizer of $T_{U(n)}$ can be thought of as the homotopy colimit of the diagram $\mathcal{D}$ given by the action of $W_{U(n)}$ on $EN(T_{U(n)})/T_{U(n)} \simeq BT_{U(n)}$. Then, by Corollary A.2, the obstruction for extending f_T to a map $f_N : BN(T_{U(n)}) \to X$ lie in the groups

$$\begin{aligned} H^{*+1}\big(W_{U(n)}; \pi_*\big(map(BT_{U(n)}, X)_{f_T}\big)\big) &\cong H^{*+1}\big(W_{U(n)}, \pi_*(BT_{U(n)}{}^\wedge_p)\big) \\ &\cong H^3\big(\Sigma_n; (\mathbb{Z}_p^\wedge)^n\big) \end{aligned}$$

which vanish for odd primes. In this case, the higher limits are given by group cohomology. The first isomorphism follows from a lemma we will mention in a moment.

Thus, the extension $f_N : BN(T_{U(n)}) \to X$ exists and for every object $U(n)/P \in \mathcal{R}_p(U(n))$ we define a map $f_P := f_N|_P : BP \to X$. We have to show that this gives rise to an $\mathcal{R}_p(U(n))$-invariant representation. First one shows that the triangle

$$\begin{tikzcd} & H^*(BN(T_{U(n)}); \mathbb{F}_p) \arrow[dl] \arrow[dr] & \\ H^*(X; \mathbf{F}_p) \arrow[rr, "\cong"] & & H^*(BU(n); \mathbb{F}_p) \end{tikzcd}$$

commutes. This is based upon proving that the map

$$H^*\big(BU(n); \mathbf{F}_p\big) \to H^*\big(BT_{U(n)}; \mathbf{F}_p\big)$$

has only one lift to $H^*(BN(T_{U(n)}); \mathbf{F}_p)$. Using this mod-$p$ information one shows that all triangles

$$\begin{tikzcd} BP \arrow[rr] \arrow[dr, "f_P"] & & BP' \arrow[dl, "f_{P'}"'] \\ & X & \end{tikzcd}$$

given by morphisms in $\mathcal{R}_p(U(n))$, commute up to homotopy. This is the trickiest part of the proof and uses Oliver's explicit description of p-stubborn subgroups of $U(n)$ [79] and a lemma which, for any abelian p-toral group A, calculates the mod-p cohomology of the mapping space $map(BA, X)$ [72, Theorem 10.1]. That tells us that the maps f_P define an $\mathcal{R}_p(U(n))$-invariant representation.

Finally, one has to show that this $\mathcal{R}_p(U(n))$-invariant representation extends to a map

$$f : \underset{\mathcal{R}_p(U(n))}{hocolim}\, EU(n) \times_{U(n)} \mathcal{I} \to X,$$

which, because X is p-complete, establishes a homotopy equivalence $BU(n)^\wedge_p \to X$. The obstruction groups for this extension are given by higher limits of the functor

$$\Pi_i^X\big(U(n)/P\big) := \pi_i\big(map(BP, X)_{f_P}\big)$$

(Corollary A.3 or Theorem 2.5). Fortunately, the mapping spaces $map(BP, X)_{f_P}$ are computable and there exists a natural equivalence

$$\Pi_i^X \xrightarrow{\cong} \Pi_i^{U(n)},$$

where $\Pi_i^{U(n)}$ is the functor given by replacing X by $BU(n)^\wedge_p$. As Jackowski, McClure and Oliver showed [50], all higher limits of $\Pi_i^{U(n)}$ vanish and so do all obstruction groups involved. This finishes the proof of Theorem 4.3.

We finish this section with

4.4. CONJECTURE. *Theorem* 4.3 *holds for every connected compact Lie group.*

5. Lie group theory for finite loop spaces and p-compact groups

The starting point of this theory was an idea of Rector [84], [85], who suggested studying a compact Lie group G (as Lie group) by looking at its classifying space BG and expressing classical Lie group notions in terms of classifying spaces. This would allow Lie group theory to be applied to a much larger class of spaces, namely finite loop spaces.

A loop space $L := (L, BL, e)$ consists of a pair of spaces L and BL, BL pointed, and a homotopy equivalence $e : \Omega BL \simeq L$ defining a loop structure on L. The space BL is called the classifying space of L. A loop space L inherits properties from the space L, e.g., a loop space is called finite, if $H^*(L;\mathbb{Z})$ is a finitely generated as graded abelian group (usually, one asks for an equivalence between L and a finite CW-complex, but the homological condition is sufficient for most of the results about finite loop spaces). Examples of finite loop spaces are given by compact Lie groups. For every compact Lie group G, there exists a canonical equivalence $e : \Omega BG \simeq G$ which establishes a finite loop space structure (G, BG, e) on G.

Rector gave definitions for subgroups, maximal tori and Weyl groups of a finite loop space [84], [85] and used this "Lie group theory" for a study of loop space structures on S^3. In particular, he showed that there exist uncountable many loop structures on S^3 (compare this with Theorem 4.1) and, with the help of McGibbon at the prime 2 [58], that the property of admitting a maximal torus distinguishes the genuine loop space structure of all the others [84].

The real break through in this theory was by Dwyer and Wilkerson [33]. Instead of looking at finite loop spaces, they passed to p-adic completions and called a loop space $X := (X, BX, e)$ a p-compact group if X is $\mathbf{F}_p$-finite and if BX is a p-complete space. The latter if-part is equivalent to the condition that X is p-complete and that $\pi_0(X)$ is a finite p-group. Again, the main examples are given by compact Lie groups. But the triple $(G^\wedge_p, BG^\wedge_p, e)$ is only a p-compact group if $\pi_0(G)$ is a finite p-group. As already mentioned in Section 3, further examples are given by pairs $(\Omega BX, BX)$, where BX has polynomial mod-p cohomology.

In contrast to finite loop spaces, Dwyer and Wilkerson showed that every p-compact group has a maximal torus and a Weyl group with similar properties known for classical Lie group theory [33]. This astonishing similarity was extended by the same people and by Møller and the author [66] to the philosophical theorem that p-compact groups enjoy almost every property of compact Lie groups.

If you believe in this similarity, then the game goes as follows: You take your favorite theorem about compact Lie groups, you translate it into the language of classifying spaces or p-compact groups and you try to find a "new" proof in these terms. If you are successful, you get a new and interesting result about classifying spaces of p-compact groups. There is a lack of a notion of the Lie algebra. But, when proving conjectures suggested by classical Lie group theory, the existence of maximal tori and Weyl groups and an induction principle, due to Dwyer and Wilkerson [34], are a good replacement for the Lie algebra.

Next we set up part of the dictionary (for p-compact groups) and try to explain what the new techniques in the proofs are. Several of the notions have also a straightforward translation into the category of finite loop spaces.

5.1. *Special p-compact groups.* The component X_0 of the unit of a p-compact group X is given by one component of X or by the universal cover of BX. A p-compact torus is a triple (T, BT, e) where $T \simeq K(\mathbb{Z}_p^{\wedge n}, 1)$ is an Eilenberg–MacLane space of degree 1. A p-compact group X is called toral if X_0 is a p-compact torus, finite if X is homotopically discrete, and abelian if $map(BX, BX)_{id} \simeq BX$. For honest abelian compact Lie groups, the last definition is actually a theorem.

5.2. *Homomorphisms.* A homomorphism $f : X \to Y$ is a pointed map $Bf : BX \to BY$. The homomorphism f is an isomorphism if Bf is a homotopy equivalence. It is a monomorphism if the homotopy fiber Y/X of Bf is $\mathbb{F}_p$-finite or equivalently if $H^*(BX; \mathbb{F}_p)$ is a finitely generated module over $H^*(BY; \mathbb{F}_p)$ ([33, Proposition 9.11]). This also defines subgroups. These definitions are motivated by the fact that every monomorphism $\rho : G \to H$ of compact Lie groups establishes a fibration $H/G \to BG \to BH$ and by a theorem of Quillen saying that ρ has a finite p-torsion free kernel if and only if $H^*(BG; \mathbb{F}_p)$ is finitely generated over $H^*(BH; \mathbb{F}_p)$.

A short exact sequence $X \to Y \to Z$ of p-compact groups is a fibration $BX \to BY \to BZ$.

Two homomorphisms $f_1, f_2 : X \to Y$ are conjugate if Bf_1 and Bf_2 are freely homotopic. A subgroup $i_1 : X_1 \hookrightarrow Y$ is subconjugate to another subgroup $i_2 : X_2 \hookrightarrow Y$ if there exists a homomorphism $j : X_1 \to X_2$ such that $i_2 j$ and i_1 are conjugate.

5.3. *Elements of p-compact groups.* An element of a p-compact group X of order p^n is a monomorphism $\mathbb{Z}/p^n \to X$.

5.4. Proposition ([33, Proposition 5.4]). *Every p-compact group X has an element of order p.*

For a proof of such a statement in classical Lie group theory, usually a 1-dimensional parameterized subgroup is constructed with the help of the tangent bundle, i.e. a subgroup isomorphic to S^1. So we need a "new" proof.

PROOF OF 5.4. For a compact Lie group G let $G \to G^p$ be the diagonal embedding. Then, the group $\mathbb{Z}/p$ acts on G^p via cyclic permutations and also on the quotient G^p/G. Each element of the fixed-point set $(G^p/G)^{\mathbb{Z}/p}$ can be represented by a tuple of the form $(1, h, \ldots, h^{p-1})$ with $h^p = 1$ and equals therefore the set of all elements of G of order p.

Now we can argue for p-compact groups. The diagonal $\Delta : X \to X^p$ is a $\mathbb{Z}/p$-equivariant homomorphism and establishes a $\mathbb{Z}/p$-equivariant fibration $X^p/X \to BX \to BX^p$. Taking homotopy fixed-points yields a fibration

$$\left(X^p/X\right)^{h\mathbb{Z}/p} \to BX^{h\mathbb{Z}/p} \simeq map(B\mathbb{Z}/p, BX) \xrightarrow{\Delta^{h\mathbb{Z}/p}} \left(BX^p\right)^{h\mathbb{Z}/p} \simeq BX.$$

The equivalence follows from the identities $map(\mathbb{Z}/p, BX) = BX^{\mathbb{Z}/p}$ of $\mathbb{Z}/p$-equivariant spaces and

$$\begin{aligned} BX &\simeq map(E\mathbb{Z}/p, BX) \cong map(E\mathbb{Z}/p \times_{\mathbb{Z}/p} \mathbb{Z}/p, BX) \\ &\cong map(E\mathbb{Z}/p \times \mathbb{Z}/p, BX)^{\mathbb{Z}/p} \cong map\big(E\mathbb{Z}/p, map(\mathbb{Z}/p, BX)\big)^{\mathbb{Z}/p} \\ &\simeq (BX^{\mathbb{Z}/p})^{h\mathbb{Z}/p}. \end{aligned}$$

This argument also shows that the map $\Delta^{h\mathbb{Z}/p}$ is given by the evaluation at the basepoint.

Because X is a loop space with $H^*(X; \mathbb{F}_p)$ being finite, the Euler characteristic $\chi(X^{p-1}) = \chi(X^p/X)$ vanishes. Therefore, by Smith theory for homotopy fixed-points (Theorem C.3), we have $\chi(X^p/X)^{h\mathbb{Z}/p}) \equiv 0 \bmod p$. The constant map $const : B\mathbb{Z}/p \to BX$ gives rise to one homotopy fixed point of X^p/X (for compact Lie groups this is given by the unit) which belongs to a contractible component as the above fibration shows. Hence, this component has Euler characteristic 1. Thus, there must be another one which gives rise to a nontrivial map $B\mathbb{Z}/p \to BX$. Because of the structure of $H^*(B\mathbb{Z}/p; \mathbb{F}_p)$ this has to be a monomorphism. □

With similar methods Dwyer and Wilkerson showed that, if X is connected, every element of order p^n has a p-th root, i.e. every monomorphism $\mathbb{Z}/p^n \to X$ extends to $\mathbb{Z}/p^{n+1}$. Taking p-th roots up to infinity defines a map $BS^1{}^\wedge_p \simeq (B\mathbb{Z}/p^\infty)^\wedge_p \to BX$, which establishes a monomorphism $S^1{}^\wedge_p \to X$ of p-compact groups.

5.5. *Centralizers.* For a homomorphism $f : Y \to X$ of p-compact groups we define the centralizer $C_X(f(Y))$ by the equation $BC_X(f(Y)) := map(BY, BX)_{Bf}$. If Y is abelian, i.e. $C_Y(Y) \cong Y$, the homomorphism f factors over $C_X(f(Y))$. Evaluation induces a map $BY \times BC_X(f(Y)) \to BX$ and therefore a homomorphism $Y \times C_X(f(Y)) \to X$ of p-compact groups. If Y is a p-compact toral group, the centralizer is again a p-compact group [33, Propositions 5.1 and 6.1]. The motivation for this definition comes from Theorem 2.1, which says that for a homomorphism $\rho : P \to G$

from a p-toral group into a compact Lie group the above defining equation is actually a homotopy equivalence.

For finite loop spaces this definition does not make that much sense, because Theorem 2.1 is not true integrally (see [69] or [99]).

5.6. *Maximal tori.* A monomorphism $T \to X$ of a p-compact torus T into a p-compact group X is a maximal torus if $C_X(T)$ is a p-compact toral group and if $C_X(T)/T$ (T is abelian) is homotopically discrete.

For a finite loop space L, we call a monomorphism $T \to L$ of an honest torus considered as finite loop space into L a maximal torus if $rk(T) = rk(L)$. Here, following a result of Hopf [43], the rank is given by the transcendence degree of $H^*(BL, \mathbb{Q})$ over $\mathbb{Q}$. This definition is the original one of Rector [84], [85]. It can be pushed forward to p-compact groups by completion and is then equivalent to the above one [67].

The first definition is motivated by the fact that, for connected compact Lie groups, the maximal torus is self centralizing, and the second by the fact that the rank of a connected compact Lie group, defined as above, equals the dimension of the maximal torus.

5.7. THEOREM ([33, 8.11, 8.13 and 9.1]). *Let X be a p-compact group. Then, X has a maximal torus $T_X \to X$ and any two maximal tori are conjugate.*

In general, finite loop spaces do not enjoy this property, as the examples of Rector show [84].

OUTLINE OF PROOF. Without loss of generality we can assume that X is connected. Then, by 5.3, there exists a monomorphism $S^1 \to X$. If the centralizer $C := C_X(S^1)$ is smaller than X, it has a maximal torus $T \to C$ by induction hypothesis. And the composition $T \to C \to X$ is a maximal torus of X. Here, the size of a p-compact group is given by the cohomological dimension, i.e. the highest degree of a nonvanishing mod-p cohomology class, and the number of components. If C and X have the same size, one can show that $C \cong X$, that $S^1 \subset X$ is a central subgroup, that there exists a short exact sequence $S^1 \to X \to \overline{X} := X/S^1$ of p-compact groups and that $\overline{X}$ is smaller than X. By induction hypothesis, there exists a maximal torus $\overline{T} \to \overline{X}$. Because every extension of a torus by a torus is again a torus, a pull back yields a maximal torus of X. The induction starts from finite groups or from p-compact toral groups, for which the first part is obvious.

For a classical proof of the second part, usually the fixed-point set $G/T_1)^{T_2}$ is analyzed, where $T_1, T_2 \subset G$ are two different maximal tori of G. Every fixed-point conjugates T_2 into T_1. By the general philosophy, fixed-points are replaced by homotopy fixed-points and Smith theory is still available. In a little more detail, the pull back diagram

$$\begin{tikzcd} X/T_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & BT_2 \arrow[d] \\ X/T_1 \arrow[r] & BT_1 \arrow[r] & BX \end{tikzcd}$$

given by two maximal tori $T_1, T_2 \subset X$, establishes a T_2-proxy action on X/T_1. (For proxy actions see Appendix C.) Every homotopy fixed-point is a section in the upper row and defines a lift from BT_2 into BT_1 and therefore conjugates T_2 into T_1. The set $(X/T_1)^{hT_2}$ is $\mathbb{F}_p$-finite and for the Euler characteristics we have the identity $\chi((X/T_1)^{hT_2}) = \chi(X/T_1) \neq 0$ (Theorem C.5 and C.6). The inequality is shown in [33, 9.5] (see also Theorem 5.9). This shows that there exists at least one homotopy fixed-point. □

5.8. *Weyl spaces and Weyl groups.* Let $T_X \to X$ be a maximal torus of a p-compact group X. We think of $BT_X \to BX$ as being a fibration. Then, the Weyl space $\mathcal{W}_X$ is defined to be the space of all fiber maps over the identity. By the arguments in the proof of the second part of Theorem 5.7 we have a proxy action of T_X on X/T_X and an equivalence $\mathcal{W}_X \simeq (X/T_X)^{hT_X}$. This fact was used by Dwyer and Wilkerson to show that $\mathcal{W}_X$ is homotopically discrete and that $W_X := \pi_0(\mathcal{W}_X)$ is a finite group under composition. Because all maximal tori are conjugate, the definition of W_X does not depend essentially on the chosen maximal torus. Dwyer and Wilkerson also proved the following analogues of well known results about compact Lie groups.

5.9. THEOREM ([29, 9.5 and 9.7]). *Let $T_X \to X$ be a maximal torus of a connected p-compact group X of rank n. Then the following holds:*

(1) *The order of W_X is equal to the Euler characteristic of X/T_X.*

(2) *The action of W_X on BT_X induces a faithful representation*

$$W_X \to Gl\big(H^*\big(BT_X;\mathbb{Z}_p^\wedge\big)\otimes\mathbb{Q}\big) \cong Gl\big(n;\mathbb{Q}_p^\wedge\big)$$

whose image is generated by pseudo reflections, i.e. W_X is a pseudo reflection group.

(3) *The map $H^*(BX;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q} \to (H^*(BT_X;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q})^{W_X}$ is an isomorphism.*

One cannot expect that the Weyl group is always generated by honest reflections, as examples of Clark and Ewing show.

The proof of the first part follows from the above equivalence between the Weyl space and the homotopy fixed-point set and because $(X/T_X)^{hT_X}$ and X/T_X have the same Euler characteristic (Theorem C.5 and formula C.6). The second part is a consequence of the third which is the difficult part of the proof.

5.10. *Normalizers and p-normalizers of the maximal torus.* Again we think of a maximal torus as being a fibration $BT_X \to BX$. The Weyl space $\mathcal{W}_X$ acts on BT_X via fiber maps. This establishes a monoid homomorphism $\mathcal{W}_X \to aut(BT_X)$ where $aut(BT_X)$ denotes the monoid of all self equivalences of BT_X. Passing to classifying spaces establishes a map $B\mathcal{W}_X \to Baut(BT_X)$ which can be thought of as being a classifying map of a fibration $BT_X \to BN(T_X) \to B\mathcal{W}_X$. The total space gives the classifying space of the normalizer $N(T_X)$ of T_X. This construction is nothing but the Borel construction.

In general $BN(T_X)$ is not a p-compact group, because W_X is not a finite p-group. Let $\mathcal{W}_p$ be the union of those components of $\mathcal{W}_X$ corresponding to a p-Sylow subgroup W_p of W_X. The restriction of the above construction to $\mathcal{W}_p$ gives the classifying space of the p-normalizer $N_p(T_X)$, which is a p-compact group. Since the action of $\mathcal{W}_X$ respects the map $BT_X \to BX$, the monomorphism $T_X \to X$ extends to a loop map $N(T_X) \to X$. The restriction $N_p(T_X) \to X$ is a monomorphism and the Euler characteristic $\chi(X/N_p(T_X))$ is coprime to p [29, Proof of 2.3]. A homotopy fixed-point argument, similar to that in the proof of Theorem 5.7, shows that p-compact toral subgroups of X are subconjugate to $N_p(T_X)$. That is to say that $N_p(T_X)$ is a p-toral Sylow subgroup of X.

This gives the basis of a Lie group theory for p-compact groups. Now we can look at the wide and rich field of classical Lie group theory and try to rediscover it in this spirit through results about p-compact groups. So, let us continue.

5.11. *Centers.* A subgroup $Z \subset X$ of a p-compact group X is called central, if evaluation induces an isomorphism $C_X(Z) \cong X$ of p-compact groups. A subgroup $Z(X) \subset X$ is called the center of X if it is central and if every central subgroup $Z \subset X$ is subconjugate to $Z(X)$. That is to say the center is the maximal central subgroup. This already gives an idea how the center can be constructed. The "union" of two central subgroups should be again a central subgroup. The following theorem was proved independently by Dwyer and Wilkerson and by Møller and the author.

5.12. THEOREM ([34], [66]). *Every p-compact group X has a center $Z(X) \subset X$. There exists a short exact sequence*

$$Z(X) \to X \to \overline{X} := X/Z(X).$$

If X is connected, then $\overline{X}$ is centerfree.

5.13. *An induction principle.* In [22], Dwyer showed that a transfer for "nice" cohomology theories (including mod-p cohomology) exists if the fiber satisfies some finiteness conditions expressed in terms of the cohomology theory. In general, these conditions are slightly weaker as being equivalent to a finite CW-complex. For example, for mod-p cohomology, the fiber only has to be $\mathbb{F}_p$-finite. Applying this to the homomorphism $N_p(T_X) \to X$ of the p-normalizer of the maximal torus of a p-compact group X, Dwyer and Wilkerson showed that $H^*(BX;\mathbb{F}_p)$ is a Noetherian algebra and that the map $H^*(BX;\mathbb{F}_p) \to H^*(BN_p(T_X);\mathbb{F}_p)$ satisfies the assumption of Theorem 1.4. That is to say there exists an algebraic decomposition of BX via centralizers of elementary abelian subgroups which can be realized on the geometric level. Let $\mathcal{A}_p(X)$ denote the Quillen category of X, omitting the trivial subgroup. This makes perfect sense, because we already translated all necessary notions, involved in the definition, into the language of p-compact groups. By Lannes' theory, the Quillen category is equivalent to the Rector category of $H^*(BX;\mathbb{F}_p)$. For elementary abelian p-subgroups $V \subset X$, the maps $BC_X(V) \to BX$ induced by evaluation at basepoints are compatible with the morphisms of $\mathcal{A}_p(X)$. Hence the following theorem is a consequence of Theorem 1.4 and Theorem B.2. It also generalizes the decomposition of classifying spaces of compact Lie groups via centralizers of elementary abelian subgroups by Jackowski and McClure (Theorem 1.3).

5.14. THEOREM. *The natural map $\mathrm{hocolim}_{\mathcal{A}_p(X)} BC_X(V) \to BX$ is a homotopy equivalence.*

This is the key for the following induction principle of Dwyer and Wilkerson.

5.15. PROPOSITION. *Let Cl be a class of p-compact groups satisfying the following conditions:*

(1) *If $X \in Cl$ and $Y \cong X$, then $Y \in Cl$.*

(2) *The trivial group belongs to Cl.*

(3) *If the component X_0 of the unit is contained in Cl, then $X \in Cl$.*

(4) *If X is connected and if $X/Z(X) \in Cl$, then $X \in Cl$.*

(5) *If X is connected and centerfree and $Y \in Cl$ for every p-compact group Y with smaller cohomological dimension than X, then $X \in Cl$.*

Then, the class Cl contains every p-compact group.

The proof is nothing but the observation that for a centerfree connected p-compact group X, the centralizer of any subgroup has smaller cohomological dimension than X. The cohomological dimension is defined to be the maximal degree of the nonvanishing mod-p cohomology classes.

To demonstrate the induction principle we want to prove the Sullivan conjecture for p-compact groups.

5.16. THEOREM ([33]). *Let X be a p-compact group, and let K be a p-complete $\mathbf{F}_p$-finite space. Then evaluation induces an equivalence $ev : map(BX, K) \xrightarrow{\cong} K$.*

PROOF. Let Cl denote the class of all p-compact groups which satisfy the statement. Then, the first two conditions are obviously satisfied.

Any p-compact group X fits into a short exact sequence $X_0 \to X \to \pi := \pi_0(X)$ of p-compact groups. If $X_0 \in Cl$, one can show that $map(B\pi, K) \to map(BX, K)$ is an equivalence. Hence, the third condition is satisfied by the Sullivan conjecture for finite groups [59].

For any connected p-compact group X, Theorem 5.12 establishes a fibration $BZ(X) \to BX \to B\overline{X} := B(X/Z(X))$. Actually, as for compact Lie groups, this is a principal fibration with classifying map $B\overline{X} \to B^2(Z(X))$. The center is a product of a p-compact torus and a finite abelian p-group and therefore satisfies the Sullivan conjecture. In this situation we can apply a lemma of Zabrodsky [98] to show that $map(B\overline{X}, K) \to map(BX, K)$ is an equivalence. Hence, the fourth condition is satisfied.

To prove the fifth condition, we use the decomposition theorem (Theorem 5.14). Let X be connected and centerfree. We have

$$map(BX, K) \simeq map\Big(\operatorname*{hocolim}_{\mathcal{A}_p(X)} BC_X(V), K\Big).$$

The centralizers are smaller than X. Thus if they satisfy the theorem, the higher limits in the spectral sequence of Theorem A.2 for calculating the homotopy groups of the latter mapping space have to be taken over the constant functors with $\pi_*(K)$ as value. But Theorem 1.4 implies that all higher limits of a constant functor on $\mathcal{A}_p(X)$ vanish. Hence, we have $map(BX, K) \simeq K$ and $X \in Cl$. □

This induction principle is quite a powerful tool. For example, along the same lines, Møller proved that every homomorphism $f : X \to Y$ between connected p-compact groups is trivial if and only if the restriction to a maximal torus is trivial if and only if $H^*(Bf;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q}$ is trivial [63]. And Dwyer and Wilkerson showed that there exists an equivalence $BZ(X) \simeq map(BX, BX)_{id}$ for any p-compact group X. The map is the adjoint of a multiplication $Z(X) \times X \to X$ on the level of classifying spaces [34]. This generalizes parts of the results of Jackowski, McClure and Oliver (Theorem 2.7 for connected compact Lie groups). This equivalence will become an important fact in the further study of p-compact groups; in particular in the analysis of homomorphisms between p-compact groups and proofs of homotopy uniqueness properties using the decomposition of BX via centralizers of elementary abelian subgroups. For example, the equivalence allows it to be shown that $BZ(C_X(V)) \to map(BC_X(V), BX)_{Bi}$ is an equivalence for any elementary abelian subgroup $V \subset X$.

The final goal of the theory of p-compact groups is a complete classification. Again, classical Lie group theory serves as a guide. On the one hand every connected compact Lie group has a finite cover which is a product of a simply connected compact Lie group and a torus and every simply connected compact Lie group splits into a product of simple simply connected pieces. On the other hand, two connected compact Lie groups are isomorphic if the normalizers of the maximal tori are. (This is implicitly contained in Bourbaki [14]. For an explicit proof see [20] (only for semi simple Lie groups), [62], [81] or [73]. The first statements can completely reproved for p-compact groups due to work of Dwyer and Wilkerson [35] and Møller and the author [66], [75]. For the second statement there exist partial results of the latter group of authors [67].

5.17. THEOREM ([66]). *Let X be a connected p-compact group. Then there exists a short exact sequence*

$$K \to X_s \times T \to X$$

of p-compact groups, where K is a finite group, where T is a p-compact torus and where X_s is a simply connected p-compact group. The group K is a central subgroup of $X_s \times T$.

The simply connected part X_s is given by the 2-connected cover of BX which also is a p-compact group. The toral part T is given by the component of the unit of the center of X. Because $C_X(T) \cong X$, there exists a map $X_s \times T \to X$. To show that this is a finite covering comes down to a proof that the center of a p-compact group is finite if and only if the fundamental group is finite, a well-known fact for connected compact Lie groups.

For every simple connected compact Lie group G, the associated representation $W_G \to Gl(H^2(BT_G;\mathbb{Q})$, is irreducible. This property is used for the definition of a simple p-compact group, i.e. X is simple if the associated representation $W_X \to Gl(H^2(BT_X;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q}$ is irreducible.

5.18. THEOREM ([35], [75]). *Let X be a simply connected p-compact group. Then there exists a splitting $X \cong \prod_i X_i$ into a product of simple simply connected p-compact groups X_i.*

Dwyer and Wilkerson proved this for all primes and independently the author for odd primes. Actually, the statement is first proved for centerfree p-compact groups and then for simply connected p-compact groups by passing to the universal cover. For centerfree p-compact groups the integral representation $W_X \to Gl(H^2(BT_X;\mathbb{Z}_p^\wedge)$ is under control and splits into a direct sum, where each piece belongs to an irreducible factor of the associated rational representation of W_X [35], [74]. This splitting gives rise to a splitting $W_X \cong \prod_i W_i$ of W_X and $T_X \cong \prod T_i$ such that W_i acts only on T_i nontrivially. The centralizer $C_X(T_i)$ for fixed i splits into a product $X_i \times \prod_{i\neq j} T_j$. This part of the proof goes the same way as for simply connected compact Lie groups. The construction of a homomorphism $\prod_i X_i \to X$ is the difficult part of the proof of Theorem 5.18. Basically, the centralizer $C_X(X_i)$ has to be computed.

Theorem 5.9 connects connected p-compact groups with pseudo reflection groups. The list of Clark and Ewing gives a complete classification of such irreducible gadgets. By the above two theorems, a complete classification of connected p-compact groups consists of the construction of a simple simply connected p-compact group and a homotopy uniqueness result for each irreducible pseudo reflection group of the list. For most of the irreducible pseudo reflection groups examples are constructed (see Section 3). Homotopy uniqueness results, in particular if p-torsion in the cohomology is around, are the main missing link for a complete classification of connected p-compact groups.

The most general homotopy uniqueness results in terms of normalizers of maximal tori so far are proved by Møller and the author. We say that two p-compact groups X and Y have the same (p-adic) Weyl group type if X and Y have the same rank n and if the two associated integral representations $W_X, W_Y \to Gl(n,\mathbb{Z}_p^\wedge)$ are equivalent.

5.19. THEOREM ([67]). *Let p be an odd prime. Let G be a connected compact Lie group such that $H^*(G;\mathbb{Z})$ has no p-torsion. Let X be a connected p-compact group with the same p-adic Weyl group type as G. Then X and $G_p^\wedge$ are isomorphic as p-compact groups.*

The proof is based on Theorem 4.3, which states a homotopy uniqueness result based on mod-p cohomology. The main part of the proof is to show that the p-compact groups under consideration have torsion free p-adic cohomology. Again, this is first proved for unitary groups and then extended to the other cases. For $p = 2$ a similar result is true for quotients of products of unitary and special unitary groups, but again, $SU(2)$ is excluded.

For $p = 2$ the Weyl group data are not sufficient to distinguish between connected p-compact groups as a comparison of $SO(2n+1)$ and $Sp(n)$ shows. In general the following conjecture should be true. As usual, it generalizes a known statement about connected compact Lie groups.

5.20. CONJECTURE. *Two connected p-compact groups X and Y are isomorphic if and only if the normalizers of the maximal tori are isomorphic (as loop spaces). At odd primes the normalizer splits and the Weyl group data are sufficient to distinguish between connected p-compact groups.*

There has also been some work done on the analysis of endomorphisms of p-compact groups. Møller studied rational self equivalences. These are endomorphisms $f : X \to X$ such that $H^*(Bf;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is an isomorphism. He was able to generalize parts of the Jackowski–McClure–Oliver theorem (Theorem 2.7) and reduced the homotopy classification of rational self equivalences to the case of endomorphisms of simple simply connected p-compact groups [64], [65]. For endomorphisms of nonconnected p-compact groups, he proved similar results to those he proved for nonconnected compact Lie groups (see Section 2).

6. Finite loop spaces and integral questions

As we already mentioned in the previous section, Rector suggested studying compact Lie groups from the homotopy point of view, i.e. passing to classifying spaces [84], [85]. If we consider a compact Lie group just as a topological space, we would lose too much information. Although the homotopy type of G distinguishes between two simple connected compact Lie groups [10], this is not true in general, even not for semi simple groups. Counterexamples may also be found in [10]. The concept of Rector is based on the hope that the classifying space BG contains all the information about the compact Lie group G. This was independently proved by Møller [62] and Osse [81] for connected compact Lie groups and in general by the author [73].

6.1. THEOREM ([62], [81], [73]). *Two compact Lie groups G and H are isomorphic if and only if the classifying spaces BG and BH are homotopy equivalent.*

The p-adic completion of a connected finite loop space gives a p-compact group, and the rationalization is a product of rational Eilenberg–MacLane spaces. Hence, via Sullivan's arithmetic square [89], finite loop spaces are built out of p-compact groups. This gives a way to transport p-compact group results to theorems about finite loop spaces.

Most of the recent research on finite loop spaces can be understood as an attempt to solve the following conjecture, which we could trace back to Wilkerson [92].

6.2. CONJECTURE ([92]). Every finite loop space with maximal torus is equivalent to a compact Lie group (as finite loop space).

In a first approach, finite loop spaces L were considered whose classifying space BL has the same (adic) genus as BG for a given connected compact Lie group G. That is that all p-adic completions $BG^\wedge_p$ and $BL^\wedge_p$ are homotopy equivalent. Because BG as well as BL is rationally a product of Eilenberg–MacLane spaces, this also implies that the rationalizations are equivalent. Also, by passing to loops, the space L has the same genus as G, if G is connected. Actually, the genus of a space was defined via localization, but using completion is more appropriate to this kind of question. In particular, if BL is in the genus of BG, we can think of BL as built out of the p-adic completions of BG, the rationalization of BG and some gluing data, encoded in a self map of the adele type of BG. Actually, the adele type of BG is a product of Eilenberg–MacLane spaces. Hence, the gluing code is completely contained in cohomological information. If, for a finite loop space L, the classifying space BL has the same genus as BG, we say that L is a fake Lie group of type G.

Rector solved the above conjecture for S^3 under this genus assumption by showing (with a little help from McGibbon at the prime 2 [58]) that the property of admitting a maximal torus distinguishes the genuine group among the fakes [84]. Actually, because of the homotopy uniqueness property of $BS^3{}^\wedge_p$ (Theorem 4.1), every loop space structure on S^3 is a fake Lie group of type S^3. The genus assumption is superfluous in this case.

Rector's result was extended by Smith and the author to the case of simply connected compact Lie groups [76], [77], [78], [70]. In a series of papers, they analyzed the genus of a classifying space of a connected compact Lie group G and arrived at the following result:

6.3. THEOREM ([76], [77], [78], [70]). *Let G be a connected compact Lie group and L a fake Lie group of type G.*

(1) *If L admits a maximal torus $T_L \to L$, then there exists a connected compact Lie group H such that L and H are isomorphic as finite loop spaces.*

(2) *If G is simply connected in addition, and L admits a maximal torus, then L and G are isomorphic.*

This solves the above conjecture under the genus assumption. The proof is based on an identification of several Weyl groups connected with L, the Weyl group of L as finite loop space, of $L^\wedge_p$ as p-compact group (for all primes) and the rational Weyl group of L, which can be defined to be the Galois group of the integral ring extension $H^*(BL;\mathbb{Q}) \to H^*(BT_L;\mathbb{Q})$. Moreover, even together with the action on the maximal torus, all these Weyl groups turn out to be isomorphic to W_G. These identifications allow the gluing data for BL to be decoded out of the existence of a maximal torus.

By a theorem of Atiyah [8], complex K-theory together with the Adams operations determines the mod-p cohomology of simply connected spaces as algebras over the Steenrod algebra, if the integral cohomology is torsionfree. The Chern character connects complex K-theory with rational cohomology, and complex K-theory determines the rational cohomology ring. Hence, for nice spaces, complex K-theory contains all information about the gluing data of an arithmetic square. It has turned out to be quite a useful tool in the theory of finite loop spaces. For example, a fake Lie group L of type G has a maximal torus if and only if $K(BG) \cong K(BL)$ as λ-rings [76], and two fake Lie groups L_1 and L_2 of the same type are isomorphic if $K(BL_1) \cong K(BL_2)$ as λ-rings [71].

The following theorem is a consequence of the mod-p homotopy uniqueness properties of connected compact Lie groups (Theorem 4.3).

6.4. THEOREM ([72]). *Let G be a quotient of a product of unitary and special unitary groups, different from $SU(2)$, such that $H^*(G;\mathbb{Z})$ is torsionfree. Let X be a simply connected CW-complex of finite type such that $H^*(X;\mathbb{Z})$ is torsionfree. Then, BG and X are homotopy equivalent if and only if $K(BG) \cong K(X)$ as λ-rings.*

By Atiyah's result it follows that BG and X have isomorphic mod-p cohomology, and by Theorem 4.3 it follows that the p-adic completions of BG and X are homotopy equivalent. The rationalizations are equivalent because both have isomorphic complex

K-theory. Hence, X has the same genus as BG. In fact both spaces are equivalent because of the K-theory isomorphism. This outlines the proof of Theorem 6.4.

Recently, Møller and the author used the ideas of the proof of Theorem 6.3 to study the Weyl group of a general connected finite loop space with maximal torus [67]. They showed that at least the Weyl group action is the expected one.

6.5. THEOREM ([67]). *Let L be a finite loop space with maximal torus $T_L \to L$. Then, the representation $W_L \to Gl(H^2(BT_L;\mathbb{Q})$ is faithful and represents W_L as a crystallographic group, and we have $H^*(BL;\mathbb{Q}) \cong H^*(BT_L;\mathbb{Q})^{W_L}$.*

This is an integral version of Theorem 5.9, which the proof is very much based on. It is also one step forward to a proof of Conjecture 6.2. To complete the proof, a complete classification of connected p-compact groups is necessary in terms of the Weyl group action on the maximal torus or in terms of the normalizer of the maximal torus.

Appendix

A. *Homotopy colimits*

In this section we recall a construction for homotopy colimits which goes back to Segal [87] and some spectral sequences related to this construction. Let $\mathcal{C}$ be a small (topological) category (as defined in [87]), and let $F : \mathcal{C} \to \mathcal{T}op$ be a covariant (continuous) functor into the category of (compactly generated) topological spaces. The homotopy colimit $hocolim_{\mathcal{C}} F$ can be thought of as a kind of bar construction. It can be constructed as the quotient space

$$hocolim_{\mathcal{C}} F := \Big(\coprod_{n \geqslant 0} \coprod_{c_0 \to \cdots \to c_n} F(c_0) \times \Delta^n \Big) \Big/ \sim$$

where c_i is an object of $\mathcal{C}$, where Δ^n is the n-simplex and where each face or degeneracy map between the sequences $c_0 \to \cdots \to c_n$ gives rise to the obvious identification. This construction is obviously functorial with respect to (continuous) natural transformations. The natural transformation from F to the constant functor $* : \mathcal{C} \to \mathcal{T}op$ taking a point as value induces a map $hocolim_{\mathcal{C}} F \to hocolim_{\mathcal{C}} * =: B\mathcal{C}$. The target is called the classifying space of the category $\mathcal{C}$. In the above construction this map is given by the projection onto the second factor. The above construction also allows an obvious filtration of the homotopy colimit given by taking the coproduct for $0 \leqslant n \leqslant N$. This filtration gives rise to a first quadrant cohomological spectral sequence which calculates the cohomology of the homotopy colimit. The E^2-term is formed by higher derived functors of the inverse limit functor of the contravariant functors $H^q(F(-)) : \mathcal{C} \to \mathcal{A}b$ into the category of abelian groups, established by taking cohomology groups of $F(c)$.

A.1. THEOREM. *For any covariant functor $F : \mathcal{C} \to \mathcal{T}op$ there exists a spectral sequence*

$$E^2_{p,q} = \varprojlim_{\mathcal{C}}{}^{p} H^q(F(-)) \Longrightarrow H^{p+q}\Big(hocolim_{\mathcal{C}} F \Big).$$

For a proof see [15], where a different approach for the construction of the homotopy colimit is used (see also [87]).

Important examples of the homotopy colimit construction are given by:

a) *Pushout diagrams.* Let $\mathcal{C}$ be the category $\mathcal{C} := \{c_1 \longleftarrow c_0 \to c_2\}$. Then, for any functor $F : \mathcal{C} \to \mathcal{T}op$, the homotopy colimit is homotopy equivalent to the pushout of the diagram $F(c_1) \longleftarrow F(c_0) \to F(c_2)$. The spectral sequence reduces to the Mayer–Vietoris sequence.

b) *Mapping telescopes.* let $\mathcal{C} := \mathbb{N}$ be the category given by the totally ordered set of the natural numbers. Then, for any functor $F : \mathbb{N} \to \mathcal{T}op$, the homotopy colimit is equivalent to the mapping telescope of the sequence

$$F(0) \to F(1) \to \cdots \to F(n) \to \cdots .$$

The spectral sequence reduces to the Milnor sequence for the cohomology of mapping telescopes.

c) *Borel constructions.* To any topological group G we can associate a category βG with one object and whose endomorphisms are given by G. In this case, a functor $F : \beta G \to \mathcal{T}op$ is nothing but a G-space X. The homotopy colimit of F is equivalent to the Borel construction $EG \times_G X$. If X is a point, then $hocolim_{\mathcal{C}}\, F \simeq EG/G = BG$ is just the classifying space of the group G (for details see [87]).

Let $\mathcal{C}$ be a discrete category, and let $F : \mathcal{C} \to \mathcal{T}op$ be a functor. For any space X, the filtration of the homotopy colimit $hocolim_{\mathcal{C}}\, F$ establishes a tower of fibrations under $map(hocolim_{\mathcal{C}}, X)$. This tower gives rise to a spectral sequence calculating the homotopy groups of $map(hocolim_{\mathcal{C}}, X)$. In more detail, the restrictions to $F(c)$ for each $c \in \mathcal{C}$ establish a map

$$R : \Big[\operatorname*{hocolim}_{\mathcal{C}} F, X\Big] \to \varprojlim_{\mathcal{C}} \big[F(-), X\big].$$

Let

$$\hat{f} = (f_c)_{c \in \mathcal{C}} \in \varprojlim_{\mathcal{C}} \big[F(-), X\big].$$

Let $\phi_n : \mathcal{C} \to \mathcal{A}b$ be the contravariant functor given by $\phi_n(c) := \pi_n(map(F(c), X)_{f_c}$.

A.2. Theorem. *Under the same assumptions as above there exists a spectral sequence*

$$E_2^{p,q} = \varprojlim_{\mathcal{C}}{}^p\, \phi_q \Longrightarrow \pi_{q-p}\Big(map\Big(\operatorname*{hocolim}_{F}, X\Big)_{R^{-1}(\hat{f})}\Big),$$

which converges strongly if

$$\varprojlim_{\mathcal{C}}{}^p\, \phi_q = 0$$

for all $p \geqslant N$ and all q.

Here, $map(hocolim_{\mathcal{F}}, X)_{R^{-1}(\hat{f})}$ means the union of all components of maps f : $BG \to X$ such that $R(f) = \hat{f}$. For a proof see [15] and [93]. Bousfield and Kan arrived at this result by constructing a spectral sequence for homotopy inverse limits and Wojtkowiak discusses carefully all questions related to choosing basepoints and what happens when the fundamental group is nonabelian.

If we are only interested in the set of components of the mapping space we have the following corollary.

A.3. COROLLARY. *Under the above assumptions, the set $R^{-1}(\hat{f})$ is nonempty if*

$$\varprojlim_{\mathcal{C}}{}^{n+1} \phi_n = 0 \quad \textit{for all } n \geqslant 1,$$

and contains at most one element if

$$\varprojlim_{\mathcal{C}}{}^{n} \phi_n = 0 \quad \textit{for all } n \geqslant 1.$$

B. *Lannes' theory*

In this section we recall some results and basic definitions of Lannes' theory. Proofs and ideas of proofs are completely omitted. The material is taken from [55], [56].

For the following we fixed a prime p. The cohomology groups are always taken with coefficients in $\mathbb{Z}/p$ and $H^*(\ )$ always means $H^*(\ ;\mathbb{Z}/p)$. We denote by $\mathcal{K}$ the category of unstable algebras over the Steenrod algebra $\mathcal{A}_p$.

Let V be an elementary abelian p-group. An algebra A over $\mathcal{A}_p$ is called of finite type if A is finite in each dimension.

B.1. THEOREM ([56]). *If X is a p-complete space and $H^*(X)$ is of finite type, then the canonical map*

$$[BV, X] \to Hom_{\mathcal{A}_p}\big(H^*(X), H^*(BV)\big)$$

is an isomorphism.

The evaluation map

$$BV \times map(BV, X) \to X$$

induces a cohomological map

$$H^*(X) \to H^*(BV) \otimes H^*\big(map(BV, X)\big).$$

Lannes studied the functor $T_V : \mathcal{K} \to \mathcal{K}$ which is the left adjoint of the functor $H^*(BV) \otimes_{\mathbb{Z}/p} -$. Taking the adjoint of the evaluation map yields a map

$$T_V H^*(X) \to H^*\big(map(BV, X)\big).$$

For any map $g : BV \to X$, there is an associated direct summand $T_V(H^*(X), g^*)$ of $T^V H^*(X)$ which corresponds to the summand $H^*(map(BV, X)_g)$ of $H^*(map(BV, X))$. With respect to this splitting the above map is a direct sum of maps with coordinates

$$T_V\big(H^*(X), g^*\big) \;\to\; H^*\big(map(BV, X)_g\big).$$

B.2. THEOREM ([56]). *Let X be a space, such that $H^*(X)$ is of finite type. Let $g : BV \to X$ be a map. The map*

$$T^V\big(H^*(X), g^*\big) \;\to\; H^*\big(map(BV, X_p^\wedge)_g\big)$$

is an isomorphism if $T^V(H^*(X), g^*)$ is of finite type and one of the following three conditions is satisfied:

(1) *$T^V(H^*(X), g^*)$ is zero in degree* 1.

(2) *$map(BV, X_p^\wedge)_g$ is* 1*-connected.*

(3) *There is a connected space Z with the property that $H^*(Z)$ is of finite type and a map*

$$Z \;\to\; map(BV, X)_g,$$

such that the associated map

$$T^V\big(H^*(X), g^*\big) \;\to\; H^*(Z)$$

is an isomorphism.

B.3. THEOREM ([56, 3.4.3]). *In addition to the assumptions of Theorem* B.2, *let X be p-complete. Then the following conditions are equivalent:*

(1) *$map(BV, X)_g$ is p-complete.*

(2) *$T^V(H^*(X), g^*) \to H^*(map(BV, X)_g)$ is an isomorphism.*

If we consider a collection S of maps $BV \to X$, then of course we get a direct summand $T_V(H^*(X), S^*)$, where S^* is the collection of the associated cohomological maps. The theorems B.2 and B.3 are still true in this situation [56].

C. *Homotopy fixed-points and Smith theory*

Let G be a group acting on a space X. Then the fixed-point set can be described as the mapping space $X^G = map_G(*, X)$ of G-equivariant maps from a point into X. The homotopy fixed-point set is defined as the mapping space $X^{hG} := map_G(EG, X)$, where EG is a contractible free G-space. The G-equivariant projection $EG \to *$ induces a map $X^G \to X^{hG}$. As one easily sees, the homotopy fixed-point is equivalent to the space of sections $\Gamma(X_{hG} \to BG)$ of the bundle $X_{hG} \to BG$ given by the Borel construction $X_{hG} := EG \times_G X$. Compared with fixed-point sets, homotopy fixed-point sets have the very nice advantage of only depending on the homotopy type of X; i.e. any G-equivariant map $X \to Y$, which is also a homotopy equivalence (nonequivariant) induces an equivalence $X^{hG} \to Y^{hG}$. This follows straightforwardly from the description in terms of section spaces. In general this is not true for fixed-point sets. But for G a finite p-group, the generalized Sullivan conjecture says that there is a close connection between fixed-point and homotopy fixed-point sets. By functoriality there is a natural map $X^{G\wedge}_p \to X_p^{\wedge hG}$.

C.1. THEOREM ([25], [16], [55], [56]). *Let G be a finite p-group acting on a finite G-complex. Then the map $X^{G\wedge}_p \to X_p^{\wedge hG}$ is a weak equivalence.*

In the case of $G = V$ an elementary abelian p-group, there are two ways to get a hold of the cohomology of the homotopy fixed-point set. The first is an output of Lannes' theory and his T-functor. The second is a functorial recipe of calculating the cohomology of X^{hV} on the lines of the localization theorem, a reformulation of classical Smith theory (e.g., see [42]).

For a space X with a V-action there exists a fibration

$$X^{hV} \simeq \Gamma(X_{hV} \to BV) \to map(BV, X_{hV})_{sec} \to map(BV, BV)_{id} \qquad (**)$$

where the total space is the union of the components given by sections in the fibration $X_{hV} \to BV$. Because $map(BV, BV)_{id} \simeq BV$ [46] (see the introduction), composition defines a map $X^{hV} \times BV \to map(BV, X_{hV})_{sec}$ which establishes a fiber homotopy trivialization of $(**)$. This was used by Lannes to establish a functor

$$Fix_V(X) := T_V\big(H_V^*(X;\mathbb{F}_p), sec^*\big) \otimes_{H^*V} \mathbb{F}_p.$$

Here, sec^* denotes the set of all sections on the cohomological level, and $H_V^*(X) := H^*(X_{hV})$ denotes equivariant cohomology. We also defined $H_V^* := H^*(BV;\mathbb{F}_p)$. As in the case of the T-functor, there is a natural map

$$Fix_V(X) \to H^*\big(X^{hV};\mathbb{F}_p\big)$$

which turns out to be an isomorphism in several cases.

C.2. THEOREM ([56]). *Let V act on a p-complete space X whose cohomology $H^*(X;\mathbb{F}_p)$ is of finite type. Then, the natural map*

$$Fix_V(X) := T_V\big(H_V^*(X;\mathbb{F}_p), sec^*\big) \otimes_{H^*V} \mathbb{F}_p \to H^*\big(X^{hV};\mathbb{F}_p\big)$$

is an equivalence if and only if the homotopy fixed-point set X^{hV} is p-complete.

Analogously as in Theorem B.2, there are several other conditions which ensure that $Fix_V(X) \to H^*(X^{hT};\mathbb{F}_p)$ is an isomorphism (see [56]). The condition of X^{hV} being p-complete seems hard to check, but fortunately in all cases relevant for the proofs of the results in the previous sections, it turns out that this condition is always satisfied.

For the following we assume that X is $\mathbb{F}_p$-finite. Let $S \subset H^*(BV;\mathbb{F}_p)$ be the multiplicative subset of all elements of strictly positive degree in the image of $H^*(BV;\mathbb{Z}) \to$

$H^*(BV;\mathbb{F}_p)$. This is isomorphic to the submodule of positive elements of the torsionfree quotient of H_V^* considered as a module over itself. For a nice action of V on X, the above mentioned localization theorem reads

$$S^{-1}H^*(X^V;\mathbb{F}_p) \cong S^{-1}H^*(X_{hV};\mathbb{F}_p).$$

The localized algebra $S^{-1}H^*(X_{hV};\mathbb{F}_p)$ is still an algebra over the Steenrod algebra but may not satisfy the unstability conditions. Let $Un(S^{-1}H^*(X_{hV};\mathbb{F}_p)$ be the unstable part of $S^{-1}H^*(X_{hV};\mathbb{F}_p)$. Dwyer and Wilkerson proved [30] that

$$H_V^* \otimes Fix_V(X) \cong Un\big(S^{-1}H^*(X_{hV};\mathbb{F}_p)\big).$$

Because $H^*(X_{hV};\mathbb{F}_p)$ is a finitely generated module over H_V^*, a Serre spectral sequence argument shows that the $\mathbb{F}_p$-module $Fix_V(X)$ is also finite. Using Theorem C.2 in addition, the same authors proved

C.3. Theorem ([33]). *Let $V = \mathbb{Z}/p$ be the cyclic group of order p acting on a $\mathbb{F}_p$-finite p-complete space X. If the homotopy fixed-point set X^{hV} is also p-complete, then the following holds:*

(1) *The homotopy fixed-point set X^{hV} and the pair $(X_{hV}, BV \times X^{hV})$ are $\mathbb{F}_p$-finite.*
(2) *For the Euler characteristics we have $\chi(X) \equiv \chi(X^{hV})$ mod p.*

This parallels the classical facts that, for $V = \mathbb{Z}/p$ acting nicely on a finite V-complex X, the analogous statements for the actual fixed-point set are true.

The generalization of these results to the case of finite p-groups goes by an induction over the group order. One uses the fact that any finite p-group π fits into a short exact sequence $1 \to \pi_0 \to \pi \to \mathbb{Z}/p \to 1$, that $\mathbb{Z}/p$ acts on the homotopy fixed-point set $X^{h\pi_0} \simeq map_{\pi_0}(E\pi, X)$ and that $X^{h\pi} \simeq (X^{h\pi_0})^{h\mathbb{Z}/p}$. Assuming that for any subgroup $\pi' \subset \pi$ the homotopy-fixed point set $X^{h\pi'}$ is p-complete gives the desired Euler characteristic formula

$$\chi(X) \equiv \chi(X^{h\pi}) \bmod p. \tag{C.4}$$

The arguments involved in the proof of Theorem C.3 do not care about the explicit action of the finite p group π on the space X. The fibration $X \to X_{h\pi} \to B\pi$ contains all the necessary information. For example, the homotopy fixed-point set $X^{h\pi}$ is equivalent to the space of sections in the above bundle. This motivates the following definition of Dwyer and Wilkerson. A proxy action of π on X is a fibration $X \to E \to B\pi$. The homotopy fixed-point set $X^{h\pi}$ is given by the space of sections of this fibration. Then, Theorem C.3 is still true for proxy actions of finite p groups on $\mathbb{F}_p$-finite spaces.

Using the mod-p approximation $B\mathbb{Z}/p^\infty{}^\wedge_p \to BS^1{}^\wedge_p$ of a torus, Dwyer and Wilkerson extended Theorem C.3 to proxy actions of tori.

C.5. Theorem ([33]). *Let $X \to E \to BT$ be a proxy action of a torus on a $\mathbb{F}_p$-finite space X. Assume that for every finite p-subgroup $\pi \subset T$ the homotopy fixed-point set $X^{h\pi}$ is p-complete. Then, for every finite p-subgroup $\pi \subset T$, the space $X^{h\pi}$ is $\mathbb{F}_p$-finite and $\chi(X^{h\pi}) = \chi(X)$.*

In general it is not known if X^{hT} is $\mathbf{F}_p$-finite or if $\chi(X^{hT}) = \chi)X)$. But in all cases, appearing in the proofs of the above theorems, the homotopy fixed-point set X^{hT} is equivalent to $X^{h\pi}$ for $\pi \subset T$ big enough. This establishes the formula

$$\chi(X^{hT}) = \chi(X). \tag{C.6}$$

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CHAPTER 22

H-spaces with Finiteness Conditions

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HANDBOOK OF ALGEBRAIC TOPOLOGY
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0. Introduction

Recall an H-space is a pointed space X, together with a binary operation $X \times X \xrightarrow{m} X$ called multiplication such that the base point acts as a left and right unit. H-spaces occur all the time in mathematics. We list some examples here:

(1) Take any space Y. Let ΩY be the space of all loops that begin and end at a fixed point y_0 in Y.

(2) Any topological group.

(3) Eilenberg–MacLane complexes.

(4) The 7 sphere, or various Lie groups, are examples of "finite H-spaces;" i.e. H-spaces that have the homotopy type of finite complexes.

(5) Applying localization at a prime, we have that any odd sphere localized at an odd prime is an H-space.

(6) Applying "mixing techniques" of Zabrodsky [25] one can obtain other non-Lie finite H-spaces.

(7) An omega spectrum can be thought of as an infinite sequence of loop spaces.

(8) The fibre of an H-map.

Clearly we are carving out an extremely large area of topology. This paper will not attempt to survey all these examples. Instead, we will offer useful references and comment on some open problems. Further, for the most part we will restrict ourselves to the homology of H-spaces, and their interplay with various algebraic invariants such as the Steenrod algebra, the Dyer Lashof algebra, and various higher order cohomology operations. This exposition is intended to be accessible rather than exhaustive. The examples are chosen for their simplicity rather than generality.

An important class of examples are the "finite H-spaces." A finite H-space is an H-space that has the homotopy type of a finite complex. A theorem in Lie groups states that any Lie group is a product of a compact Lie group and Euclidean space, so all Lie groups are finite H-spaces. For purposes of this exposition, we would like to extend our definition of finite to include those H-spaces whose mod p cohomology is finite dimensional. This allows us to also consider p-localizations which have become a large part of the subject.

It may be helpful to consider the following sequence of inclusions

$$\begin{aligned}\text{Lie groups} &\subset \text{finite topological groups} \subset \text{finite loop spaces}\\ &\subset \text{finite } A_n\text{-spaces} \subset \text{finite} H\text{-spaces}.\end{aligned}$$

After describing the cohomology of an H-space mod p as a Hopf algebra over the Steenrod algebra, we will proceed to describe various constructions such as the projective spaces of Stasheff [83] and the interplay between cup products in the cohomology of the projective space and obstructions to higher homotopy associativity in the H-space.

The following is an outline of the paper. Let X be an H-space, p a prime.

I (Sections 1, 2) $H^*(X;\mathbb{Z}_p)$ is a (graded) commutative Hopf algebra over the Steenrod Algebra.

- (a) As an algebra, $H^*(X;\mathbb{Z}_p)$ is a tensor product of exterior algebras, truncated polynomial algebras, and free polynomial algebras.

(b) If X is a finite H-space, the Steenrod action is very restrictive. The starting point is Browder's Theorem which states $\beta_1 H^{\text{even}}(X;\mathbb{Z}_p)$ consists of decomposable elements and the pth powers of even homology primitives vanish.

II (Sections 3, 4, 5) Projective Spaces Associated to H-spaces

(a) If X is an H-space, there is a twofold projective space P_2X and a cofibration sequence $\Sigma(X \wedge X) \longrightarrow \Sigma X \longrightarrow P_2X$. The cup product structure in $H^*(P_2X;\mathbb{Z}_p)$ is related to the coproduct in $H^*(X;\mathbb{Z}_p)$.
(b) If X is an H-space with higher homotopy associative structures called A_n-structures, there exist a sequence of spaces $\Sigma X \subset P_2X \subset \cdots \subset P_nX$ and cofibration sequences $P_\ell X \longrightarrow P_{\ell+1}X \longrightarrow (\Sigma X)^{\wedge^{\ell+1}}$
(c) Given a map $f : X \longrightarrow Y$ where X and Y are A_n-spaces and f is an A_{n-1} map, the A_n-deviation can be defined in terms of the projective spaces.

III (Sections 6, 7) Mapping A_n-Spaces into Postnikov Systems

(a) If X is an A_n-space, the theory of lifting an A_n-map into a 2-stage Postnikov System by an A_n-map can be described via obstruction theory. The obstructions lie in $Cotor_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$
(b) A 2-stage Postnikov System is constructed which has a nonzero homology pth power.
(c) Using this Postnikov System, we apply the obstruction theory to several examples to show how the action of the Steenrod algebra on certain H-spaces is restricted. In particular, there are several results about atomic spaces.

IV (Section 8) Homotopy Commutative Spaces

(a) If X is homotopy commutative one can construct a projective space P_2^2X.
(b) If $f : X \longrightarrow Y$ is an H-map between homotopy commutative spaces, a homotopy commutative obstruction $c(f)$ can be defined.

In many of the sections there will be some overlap with excellent books on the subject. We recommend that the reader look at Kane's book [48], *Homology of Hopf Spaces*, and Zabrodsky's book [99], *Hopf Spaces*. Other survey papers we recommend are [7], [64]. My main contribution is the open problems and a certain personal preference of presentation. In the later chapters there will be new material that addresses the questions of nonfinite H-spaces, and H-spaces with added structures such as higher homotopy commutativity or homotopy associativity.

Throughout, all spaces will be connected and base pointed. All homotopies will respect the basepoint. All homologies and cohomologies will be of finite type.

1. Cup products and Hopf algebras

In a basic course in algebraic topology one usually calculates cup products on various spaces toward the end of the course. The computations are often not easy – they require a piece of complicated machinery such as Poincaré duality and cup products are usually computed only for orientable manifolds. One leaves with the illusion that the cup product structure is not always easy to compute.

In contrast, the cohomology of an H-space with coefficients in a field has lots of nontrivial cup products and it is not difficult to prove. Let k be a field. Recall the Künneth theorem implies

$$H^*(X \times X; k) \cong H^*(X; k) \otimes H^*(X; k),$$

$$H_*(X \times X; k) \cong H_*(X; k) \otimes H_*(X; k).$$

Given an H-space X with multiplication

$$m\colon X \times X \longrightarrow X,$$

we get two induced maps

$$m^*\colon H^*(X \times X; k) \longleftarrow H^*(X; k),$$

$$m_*\colon H_*(X \times X; k) \longrightarrow H_*(X; k).$$

Combining with Künneth isomorphisms, we get maps

$$H^*(X; k) \otimes H^*(X; k) \xleftarrow{\Delta} H^*(X; k),$$
$$H_*(X; k) \otimes H_*(X; k) \longrightarrow H_*(X; k).$$

$H^*(X; k)$ is already an algebra with respect to cup product. Further this makes $H^*(X; k) \otimes H^*(X; k)$ an algebra. In fact if A, B are graded algebras, $A \otimes B$ is an algebra with $(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg b \deg c}(ac \otimes bd)$. It follows that Δ is a map of algebras since m^* is and the Künneth isomorphism is also. The fact that the base point acts as a left and right unit implies if $x \in H^*(X; k)$ then

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x_i' \otimes x_i'', \quad \deg x_i' > 0,\ \deg x_i'' > 0 \tag{1.1}$$

where $\deg x_i' + \deg x_i'' = \deg x$. This is all discussed in detail in a number of places [76], [49], [94], [70].

Suppose the first positive nonvanishing cohomology group of $H^*(X; \mathbb{Z}_2)$ occurs in an even degree n. Then if $x \in H^n(X; \mathbb{Z}_2)$,

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

by (1.1). Since Δ is a map of algebras, and degree of x is even

$$\Delta(x^n) = (\Delta x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i} \in H^*(X; \mathbb{Z}_2) \otimes H^*(X; \mathbb{Z}_2).$$

Now if $x^{n-1} \neq 0$ and $\binom{n}{i} \not\equiv 0 \bmod 2$, for some i between 0 and n, then $\Delta(x^n) \neq 0$ so $x^n \neq 0$. Checking binomial coefficients, $\binom{n}{i} \equiv 0 \bmod 2$ for all $0 < i < n$ only if $n = 2^j$ for some $j > 0$. So we have if $\Delta(x) = x \otimes 1 + 1 \otimes x$ and deg x even, then the lowest power of x that is trivial is 2^j for some $j > 0$.

As a corollary we see that $\mathbb{RP}^n$ cannot be an H-space if $n \neq 2^j - 1$ for some $j > 0$. This follows since

$$H^*(\mathbb{RP}^n; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[x_1]}{(x_1)^{n+1}}.$$

Similarly, $\mathbb{CP}^n$ cannot be an H-space for any n since there exists a prime p such that

$$H^*(\mathbb{CP}^n; \mathbb{Z}_p) = \frac{\mathbb{Z}_p[x_2]}{x_2^{n+1}}, \qquad \Delta(x_2) = x_2 \otimes 1 + 1 \otimes x_2,$$

and $n + 1 \neq p^j$ for some p.

If one has an odd degree generator x in $H^*(X; \mathbb{Z}_p)$ for p odd, we know that $x^2 = -x^2$ by the anticommutativity of cup product, so $2x^2 = 0$ which implies $x^2 = 0$. One might ask what goes wrong with the above argument if we apply it to a generator of odd degree? What happens is this:

$\Delta x = x \otimes 1 + 1 \otimes x$ so the signed commutativity implies

$$(1 \otimes x) \cdot (x \otimes 1) = (-1)^{(\deg x)^2} x \otimes x = -x \otimes x.$$

It follows that

$$\begin{aligned}\Delta(x^2) = (\Delta(x))^2 &= x^2 \otimes 1 + x \otimes x - x \otimes x + 1 \otimes x^2 \\ &= x^2 \otimes 1 + 1 \otimes x^2.\end{aligned}$$

Thus, the binomial coefficient argument does not apply.

Now that we have a general idea of the restrictions the Hopf algebra structure places on $H^*(X; k)$ for X an H-space, we should give the general theorems due to Hopf and Borel:

THEOREM 1.1 (Hopf [36]). *Let X be an H-space. Then*

$$H^*(X; \mathbb{Q}) \cong \wedge(x_1, \ldots, x_\ell, \ldots) \otimes \mathbb{Q}[y_1, \ldots y_m, \ldots]$$

where degree x_i are odd, degree y_j are even. If X is finite, $H^*(X; \mathbb{Q})$ has only a finite number of odd degree generators. This is an isomorphism as algebras.

THEOREM 1.2 (Borel [7]). *Let X be an H-space. Then*

$$H^*(X; \mathbb{Z}_2) \cong \wedge(x_1, \ldots x_\ell, \ldots) \bigotimes_i \frac{\mathbb{Z}_2[y_i]}{(y_i)^{2^{f_i}}} \bigotimes_j \mathbb{Z}_2[z_j].$$

For p odd:

$$H^*(X;\mathbb{Z}_p) \cong \wedge(x_1, \dots x_\ell, \dots) \otimes \frac{\mathbb{Z}_p[y_i]}{(y_i)^{p^{f_i}}} \otimes \mathbb{Z}_p[z_j]$$

degree x_i odd, degree y_i and degree z_j are even.

These are algebra isomorphisms. We make no statement about the coproduct structure.

If A is a Hopf algebra, the "module of primitives" is denoted by $P(A)$ and is defined by

$$P(A) = \{x \in A \mid \Delta x = x \otimes 1 + 1 \otimes x\}.$$

Note that a map of Hopf algebras $f : A \to B$ that preserves products and coproducts induces a map $P(f) : P(A) \to P(B)$. We also define "the module of indecomposables."

$Q(A) = I(A)/I(A)^2$ where $I(A)$ is the augmentation ideal of A.

A is *primitively generated* if the natural map

$$P(A) \longrightarrow IA \longrightarrow Q(A)$$

is onto. If A is a Hopf algebra over $\mathbb{Z}_p$, p a prime, A is primitively generated if and only if its dual Hopf algebra A_* is commutative, associative and has no pth powers. This result appears in [70].

There are other properties about H-spaces that are reflected in the Hopf algebra structure. We say X, m is "homotopy associative" if the diagram

$$\begin{tikzcd} X \times X \times X \arrow[r, "m \times 1"] \arrow[d, "1 \times m"'] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "m"] & X \end{tikzcd}$$

commutes up to homotopy. Hence applying homology with field coefficients we obtain a commutative diagram

$$\begin{tikzcd} H_*(X) \otimes H_*(X) \otimes H_*(X) \arrow[r, "m_* \otimes 1"] \arrow[d, "1 \otimes m_*"'] & H_*(X) \otimes H_*(X) \arrow[d, "m_*"] \\ H_*(X) \otimes H_*(X) \arrow[r, "m_*"'] & H_*(X) \end{tikzcd}$$

It follows that $H_*(X)$ is an associative ring. Similarly X, m is "homotopy commutative" if the following diagram homotopy commutes

$$\begin{tikzcd} X \times X \arrow[r, "m"] \arrow[d, "T"] & X \\ X \times X \arrow[ur, "m"'] & \end{tikzcd}$$

where $T(x_1, x_2) = (x_2, x_1)$. It follows if X, m is homotopy commutative, then $H_*(X)$ is a (graded) commutative ring.

The converse is not true. There are many H-spaces whose homology rings are commutative (resp. associative), but as H-spaces the multiplications are not homotopy commutative (resp. homotopy associative). Topologists have studied this extensively and they have invented projective spaces associated to the H-space that exist in the presence of homotopy commutativity or homotopy associativity. As we will see in later chapters, these projective spaces place further restrictions on an H-space that is homotopy associative or commutative.

For more information about Hopf algebras that can occur as the cohomology of H-spaces, we refer the reader to [48], [59], [60].

Notice that Hopf's theorem and Borel's theorem only specify the cup product structure of the cohomology rings. One might ask about the map

$$H_*(X;k) \otimes H_*(X;k) \longrightarrow H_*(X \times X;k) \xrightarrow{m_*} H_*(X;k).$$

This makes $H_*(X;k)$ into a ring, but there is no guarantee that the ring structure is associative or commutative. In fact, we have the following questions and conjectures.

CONJECTURE 1.1 (Arkowitz, Lin, McGibbon). Let X, m be a finite H-space. There is a multiplication on X such that $H_*(X;\mathbb{Q})$ is an associative ring. There is a multiplication on X such that $H_*(X;\mathbb{Z}_p)$ is associative, for p a prime.

QUESTION 1.2. Investigate the algebra structure of $H_*(X;\mathbb{Z}_p)$ for X a finite H-space. In particular, if $H_*(X;\mathbb{Z}_p)$ is associative, what are all the possible commutators? This is closely related to a question posed by Kane [48].

QUESTION 1.3. Let p be an odd prime. If X is a finite H-space, are all pth powers of elements of $H^*(X;\mathbb{Z}_p)$ trivial? This is equivalent to choosing generators of $H_*(X;\mathbb{Z}_p)$ to be primitive.

The following references address Questions 2 and 3 [47], [48]. Lin [60] has shown all p^2 powers are trivial in $H^*(X;\mathbb{Z}_p)$ for p an odd prime, X a simply connected H-space.

QUESTION 1.4. Describe $H^*(X;\mathbb{Z}_{(p)})$ as a ring for X a finite H-space with p-torsion.

QUESTION 1.5. Let p be an odd prime. Suppose X is a finite H-space, simply connected. Do the even generators always lie in degrees $2p+2$ and $2p^2+2$?

QUESTION 1.6. Let X be a simply connected, finite H-space and suppose $H^*(X;\mathbb{Z})$ has 2-torsion. Is there always a 3 or 7 dimensional class in $H^*(X;\mathbb{Z}_2)$ whose cup product square is nonzero?

QUESTION 1.7 (Jeanneret). Let X be a simply connected finite H-space with p-torsion in $H^*(X;\mathbb{Z}_p)$. Does this imply there is an even generator of degree $2p+2$ in $H^*(X;\mathbb{Z}_p)$ for p an odd prime?

QUESTION 1.8 (Jeanneret). Does a 6-connected 2-torsion free finite H-space have the homotopy type of a product of 7 spheres?

2. Action of the Steenrod algebra

There is an extensive theory of graded connected Hopf algebras. See [70]. However, $H^*(X;\mathbb{Z}_p)$ for X an H-space is more than just a graded connected Hopf algebra. It supports the action of the Steenrod algebra. Recall the Steenrod algebra $\mathcal{A}(p)$ can be described as all linear natural transformations of the mod p cohomology functor. Thus if $\theta \in \mathcal{A}(p)$, $f : X \to Y$ one has a commutative diagram of linear transformations

$$\begin{array}{ccc} H^*(Y) & \xrightarrow{f^*} & H^*(X) \\ \theta\downarrow & & \downarrow\theta \\ H^*(Y) & \xrightarrow{f^*} & H^*(X) \end{array} .$$

We refer the reader to [71], [86] for several basic properties of $\mathcal{A}(p)$.

For p odd $\mathcal{A}(p)$ is generated by symbols $\beta_1, \mathcal{P}^n$ for $n \geqslant 1$, subject to the "Adem relations." That is, if one takes the tensor algebra on these symbols and quotients by the ideal generated by the Adem relations one obtains $\mathcal{A}(p)$. We have $\deg \beta_1 = 1$, $\deg \mathcal{P}^n = 2n(p-1)$. Further if $\deg x = 2n$, $\mathcal{P}^n x = x^p$. For $p = 2$, $\mathcal{A}(2)$ is generated by symbols Sq^n, $n \geqslant 1$. $\deg Sq^n = n$.

Some of the special properties that apply to H-spaces are:

(1) Cartan formulae:

$$\mathcal{P}^i(x \otimes y) = \sum_s (\mathcal{P}^s x) \otimes (\mathcal{P}^{i-s} y).$$

(2) Steenrod operations commute with coproduct $\mathcal{P}^i(\Delta x) = \Delta \mathcal{P}^i x$, so Steenrod operations preserve the module of primitives and the module of indecomposables.

Thus, one can begin to study $QH^*(X;\mathbb{Z}_p)$ or $PH^*(X;\mathbb{Z}_p)$ as modules over $\mathcal{A}(p)$. If one can describe their structure, much can be learned about $H^*(X;\mathbb{Z}_p)$. This is exploited in [58], [59], [60], for finite H-spaces.

A second area of study is the p-torsion in $H^*(X;\mathbb{Z})$. This can be investigated via the Bockstein spectral sequence.

THEOREM 2.1 (Browder [9]). *If X is a finite H-space, $x \in H^{2n}(X;\mathbb{Z}_p)$ then $\beta_i x$ is decomposable for $i \geqslant 1$.*

This was one of the key results that started topologists thinking about the action of the Steenrod algebra determining the structure of $H^*(X;\mathbb{Z}_p)$. Browder uses a number of very interesting techniques. In particular, he notices that the Bockstein spectral sequence is a spectral sequence of differential Hopf algebras. Further, he maps a space X into a $K(\mathbb{Z}_{p^r}, 2n)$ and uses the Bockstein sequence in $K(\mathbb{Z}_{p^r}, 2n)$ to derive information about the Bockstein sequence for $H^*(X;\mathbb{Z}_p)$. These ideas are all generalized to prove

THEOREM (a) (Kane [50]). *$QH^{\text{even}}(X;\mathbb{Z}_2) = 0$ for X, a finite simply connected H-space.*

THEOREM (b) (Lin [60]). *$QH^{\text{even}}(X;\mathbb{Z}_p) = \sum \beta_1 \mathcal{P}^n QH^{2n+1}(X;\mathbb{Z}_p)$ for p odd, X, a finite H-space.*

THEOREM (c) (Kane, Lin [51], [60], [59]). *The Bockstein spectral sequence B_r collapses and $B_2 = B_\infty$ for X, a simply connected finite H-space. $H^*(X;\mathbb{Z})$ has no p^2 torsion.*

THEOREM (d) (Lin [58]). *$Sq^{2^r} QH^{2^r+2^{r+1}k-1}(X;\mathbb{Z}_2) = 0$ and*

$$QH^{2^r+2^{r+1}k-1}(X;\mathbb{Z}_2) \subseteq Sq^{2^r k} QH^*(X;\mathbb{Z}_2)$$

for X a simply connected finite H-space with $H_*(X;\mathbb{Z}_2)$ associative, $k > 0$.

THEOREM (e) (Jeanneret, Lin [61]). *The first nonvanishing homotopy group of X, a finite H-space, lies in degrees 1, 3 or 7 if $H_*(X;\mathbb{Z}_2)$ is associative.*

REMARKS. All these theorems rely on a careful analysis of the action of the Steenrod algebra. They use methods we will describe in Section 7.

QUESTION 2.1. Determine the action of the Steenrod algebra on $H^*(\Omega X;\mathbb{Z}_p)$ for X a Lie group or finite H-space with torsion. In particular what is the action of the Steenrod algebra on the various divided powers that occur in a coalgebra decomposition of $H^*(\Omega X;\mathbb{Z}_p)$?

QUESTION 2.2. In the above Theorems (d) and (e) can the assumption $H_*(X;\mathbb{Z}_2)$ associative be removed? Or is it true that a finite H-space always has a multiplication that makes $H_*(X;\mathbb{Z}_2)$ associative?

QUESTION 2.3. Is there some general formula similar to Theorem (d) for the action of the Steenrod algebra on $QH^{\text{odd}}(X;\mathbb{Z}_p)$ for p an odd prime and X having sufficient associativity hypotheses for example X, a loop space? Note we have to assume some associativity since otherwise an odd sphere localized to an odd prime is an H-space.

Hemmi has done some work on Question 2.3, at the prime 3 [33]. In particular, he notes there are Lie groups with odd generators in any given degree that are not in the image of any mod 3 Steenrod operation. So whatever formula we get must depend on more than the p-adic expansion of the degree.

Browder's work [9] can be looked at from a different viewpoint. Essentially, he proves the following.

Let B_i be the ith term of the Bockstein spectral sequence. If $\bar{x} \in QB_i^{2n}$ with $\beta_i\bar{x} = \bar{y} \neq 0$, then either $x^p \neq 0$ with $\beta_{i+1}\{x^p\} = \{x^{p-1}y\} \neq 0$ in B_{i+1} or there is an $\bar{x}_1 \in QB_i^{2np}$ with $\beta_i\bar{x}_1 = \bar{y}_1 \neq 0$. Thus, either a new algebra generator in B_i is created or a new pth power is created.

Wilkerson proves if B_i is finitely generated as an algebra, then so is B_{i+1}. In fact he proves the following stronger theorem:

THEOREM 2.2 (Wilkerson). *Let A be a differential commutative algebra mod p and suppose A is finitely generated as an algebra. Then the homology of A is also finitely generated as an algebra.*

PROOF. Consider A as a left module over the subring ξA of pth powers. Then the differential d is an ξA-module map and A is finitely generated as a ξA-module. ξA is Noetherian, so A is a Noetherian ξA-module. Hence the cycles $Z(A)$ and boundaries $B(A)$ are also finitely generated ξA modules. Hence the homology $H(A)$ is a finitely generated ξA-module and algebra generators of $H(A)$ may be chosen from $Q(\xi A)$ and the module generators of $Z(A)$. $\square$

THEOREM 2.3. *Suppose $H^*(X;\mathbb{Z}_p)$ is finitely generated as an algebra, and $H^*(X;\mathbb{Z})$ has bounded p-torsion. Then if $x \in H^{2n}(X;\mathbb{Z}_p)$, then $\beta_\ell x$ is decomposable for $\ell \geqslant 1$.*

PROOF. Suppose $\beta_\ell x$ is indecomposable. Since $H^*(X;\mathbb{Z}_p)$ has bounded p-torsion, $B_i = B_\infty$ for some i. By the above observation, there is an element $\bar{z} \in QB_i^{2np^j}$ with $\beta_i\bar{z} = \bar{y} \neq 0$. This implies there exists an infinite sequence of generators $\bar{z}_k \in QB_i^{2np^{j+k}}$ with $\beta_i\bar{z}_k = \bar{y}_k \neq 0$. Hence B_i is not finitely generated as an algebra. But this contradicts Theorem 2.2. $\square$

Lin [56] proves that actually the hypotheses of the Theorem 2.3 imply $H^*(X;\mathbb{Z}_p)$ is finite dimensional.

QUESTION 2.4. Do all simply connected H-spaces whose mod p cohomology is finitely generated as an algebra have mod p cohomology isomorphic to a product of $K(\mathbb{Z},2)s$ with 3-connective covers of finite H-spaces and finite H-spaces?

Most of the exposition that follows can be used to study H-spaces whose mod p cohomology is finitely generated as an algebra. Recently, in the homotopy commutative case, new theorems have been proven.

THEOREM (Slack [79]). *Let X be a homotopy commutative, homotopy associative H-space whose* mod 2 *cohomology is finitely generated as an algebra. Then X has the* mod 2 *homotopy type of an Eilenberg–MacLane space in degrees* 1 *and* 2.

THEOREM (Lin [57]). *Let X be a double loop space whose* mod *p cohomology is finitely generated as an algebra. Then X has the* mod *p homotopy type of an Eilenberg–MacLane space in degrees* 1 *and* 2.

The second theorem is actually true under the weaker hypothesis that X is the loop on an "A_p space" in the sense of Stasheff [83].

THEOREM (Lin [57]). *Let X be an H-space whose mod p cohomology is finitely generated as an algebra for p odd. Then all generators of infinite height lie in degrees $2p^j$ for $j \geqslant 0$.*

THEOREM (Slack [81]). *Let X be a homotopy commutative, homotopy associative H-space whose* mod *p cohomology is finitely generated as an algebra for p odd. Then all even generators lie in degrees $2p^j$ for $j \geqslant 0$.*

3. Projective planes

Given an H-space X, μ, one can build a "projective plane" P_2X. For the H-spaces $\mathbb{Z}_2$, S^1, S^3, $P_2\mathbb{Z}_2$, P_2S^1 and P_2S^3 have the homotopy type of $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{HP}^2$, respectively. We know from a basic graduate topology course that

$$H^*(\mathbb{RP}^2; \mathbb{Z}_2) = \mathbb{Z}_2[x_1]/x_1^3,$$
$$H^*(\mathbb{CP}^2; \mathbb{Z}) = \mathbb{Z}[x_2]/x_2^3,$$
$$H^*(\mathbb{HP}^2; \mathbb{Z}) = \mathbb{Z}[x_4]/x_4^3.$$

Thus, in a different way, the existence of an H-space structure on X produces nonzero cup products in another space P_2X. This is one of the key ideas in the theory of H-spaces. Thomas and Sugawara and Toda [90], [91], [87] exploit this idea to determine most of the action of the Steenrod algebra on finite H-spaces whose mod 2 cohomology is primitively generated. We describe the construction here.

Given spaces X, Y, the join $X * Y$ is the quotient of the space $X \times I \times Y$ with the identifications $(x, 0, y) \sim (x, 0, y')$ and $(x, 1, y) \sim (x', 1, y)$. If X is an H-space, there is a well-defined map

$$X * X \xrightarrow{h} \Sigma X$$

with $h[x, t, y] = (\mu(x, y), t)$. The cofibre of this map is P_2X, the projective plane. Sometimes it is useful to replace $X * X$ with $\Sigma(X \wedge X)$ via the homotopy equivalence

$$X * X \xrightarrow{k} \Sigma(X \times X) \xrightarrow{\Sigma q} \Sigma(X \wedge X)$$

$k[x, t, y] = (x, y, t)$, here q is the quotient map. If we do this, then there is a map

$$\Sigma(X \wedge X) \xrightarrow{\theta_X} \Sigma X$$

with $\theta_X = -\Sigma\pi_1 + \Sigma\mu - \Sigma\pi_2$. Here we are using the fact that maps of a suspension into another space form a group [76]. The cofibre of θ_X can also be considered to be P_2X up to homotopy. For our purposes it is helpful to know θ_X^* induces the reduced coproduct

$$H^*(X) \xrightarrow{\bar{\Delta}} H^*(X \times X) \cong \overline{H}^*(X) \otimes \overline{H}^*(X),$$

$$\bar{\Delta}x = \Delta x - x \otimes 1 - 1 \otimes x.$$

Thus we have a long exact cofibration sequence

$$\overline{H}^*(X) \xrightarrow{\bar{\Delta}} \overline{H}^*(X) \otimes \overline{H}^*(X) \xrightarrow{\lambda} H^*(P_2X) \xrightarrow{i} \overline{H}^*(X) \longrightarrow \tag{3.1}$$

Work of Thomas [89], or also in [94, p. 502] shows if

$$i(a) = x, \ i(b) = y \quad \text{then} \quad \lambda(x \otimes y) = ab.$$

(Here ab denotes the cup product of a and b.) Thus if $ab = 0$ there exists a z with

$$\bar{\Delta} z = x \otimes y.$$

So the vanishing of cup products in $H^*(P_2X)$ implies the existence of nonzero coproducts in $H^*(X)$. Conversely, if $H^*(X)$ is primitively generated, very few cup products vanish in $H^*(P_2X; \mathbb{Z}_2)$. See [10]. We know that two-fold cup product squares mod 2 are obtained by applying a Steenrod square to an element.

If $n \neq 2^j$, S^{n-1} is not a mod 2 H-space. To see this,

$$H^*\left(P_2S^{n-1}; \mathbb{Z}_2\right) = \mathbb{Z}_2\frac{[x_n]}{x_n^3}$$

but $x_n^2 = Sq^n x_n = \sum a_i b_i x_n$, $a_i, b_i \in \mathcal{A}(2)$ and $b_i x_n = 0$.

Here $Sq^n = \sum a_i b_i$ factors through other operations as long as n is not 1, 2, 4 or 8. This is proved by Adams in [1].

We note here that if X, u_X and Y, u_Y are H-spaces and $f : X \to Y$ is an H-map, there is an induced map of projective planes

$$P_2 f : P_2 X \to P_2 Y.$$

In fact we have a commutative diagram

$$\begin{tikzcd}
\Sigma(X \wedge X) \arrow[r, "\Sigma(f\wedge f)"] \arrow[d, "\theta_X"'] & \Sigma(Y \wedge Y) \arrow[d, "\theta_Y"] \\
\Sigma X \arrow[r, "\Sigma(f)"] \arrow[d] & \Sigma Y \arrow[d] \\
P_2 X \arrow[r, dotted, "P_2 f"] & P_2 Y
\end{tikzcd}$$

The cofibration property implies there is a map $P_2 f : P_2X \to P_2Y$ that extends Σf. Stasheff [83] proves, given a commutative diagram

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma f"] \arrow[d] & \Sigma Y \arrow[d] \\
P_2 X \arrow[r, "P_2 f"'] & P_2 Y
\end{tikzcd}$$

that f is an H-map.

The main advantage of looking at P_2X is that its cohomology supports 2-fold cup products. The fact that the cup product square can be defined mod 2 as a Steenrod operation $Sq^n x = x^2$ if degree $x = n$ allows one to derive information about the mod 2 cohomology of an H-space.

Hemmi [32] proves at an odd prime p, if $\deg a = 2n+1$ and $i(a) = u$, then up to a constant,

$$\lambda\left(\sum_{i=1}^{p-1}\binom{p}{i}\frac{1}{p}u^i \otimes u^{p-1}\right) = \beta_1\mathcal{P}^n(a) \quad \text{if } p \text{ odd.} \tag{3.2}$$

These two facts show that the coproduct structure of $H^*(X)$ is linked to the cup product and Steenrod algebra structure of $H^*(P_2X)$. In fact since the sequence (3.1) is exact, we conclude

PROPERTY 1. *If $ab = 0$ in $H^*(P_2X)$, there exists a $c \in \overline{H}^*(X)$ with $\bar{\Delta}c = a \otimes b$.*

PROPERTY 2. *If for an odd prime $\beta_1\mathcal{P}^n a = 0$ there exists a $d \in \overline{H}^*(X)$ with*

$$\bar{\Delta}(d) = \sum\binom{p}{i}\frac{1}{p}u^i \otimes u^{p-i}.$$

Property 1 corresponds to the existence of a commutator in $H_*(X)$. In fact if $\bar{a}, \bar{b} \in H_*(X)$ are dual to $a, b \in H^*(X)$, $a \neq b$, $\langle\bar{a}, a\rangle \neq 0\langle\bar{b}, b\rangle \neq 0$ then $\langle[\bar{a}, \bar{b}], c\rangle \neq 0$.

Property 2 corresponds to the existence of a pth power in homology. If $\langle\bar{u}, u\rangle \neq 0$ then $\langle\bar{u}^p, d\rangle \neq 0$.

Recall a Hopf algebra is primitively generated if and only if its dual Hopf algebra is commutative, associative and has no pth powers (see [70]). So the action of $\beta_1\mathcal{P}^n$ together with the cup product structure in $H^*(P_2X)$ provides key information about how $H^*(X)$ differs from being primitively generated. Browder and Thomas exploit this fact in [10] to prove if $H^*(X)$ is primitively generated, $H^*(P_2X)$ contains a polynomial algebra truncated at height three.

Thomas [90] then uses this polynomial algebra at the prime 2 to determine most of the action of the Steenrod algebra on the cohomology of finite H-spaces that are primitively generated.

Other authors have used variations on this technique. Hubbuck [37] uses the projective plane together with the squaring map to show any finite H-space that is homotopy commutative is homotopy equivalent to a torus.

QUESTION 3.1. What can be said about $H^*(P_2X; \mathbb{Z}_p)$ for X a finite H-space whose cohomology is not primitively generated?

QUESTION 3.2 (Arkowitz and Silberbush). There are different maps $\Sigma(X \wedge X) \to \Sigma X$ obtained by permuting $-\Sigma\pi_1$, $\Sigma\mu$ and $-\Sigma\pi_2$. How are these maps related?

4. Higher projective spaces

Sugawara, Stasheff and Milnor [83], [69] have constructed higher projective spaces P_nX, that are related to the higher homotopy associativity of the H-space X. Milnor constructed projective spaces when X is a topological group; Sugawara and Stasheff constructed projective spaces under much weaker hypotheses. Recall an H-space is homotopy associative if the diagram below commutes up to homotopy

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

$\mu(\mu \times 1) \simeq \mu(1 \times \mu)$. Attached to any homotopy associative H-space X is its 3-fold projective space P_3X. We leave it to the reader to look at [83] for an explicit construction. We have

$$\Sigma X \overset{i_1}{\hookrightarrow} P_2X \overset{i_2}{\hookrightarrow} P_3X.$$

Further $P_2X \xrightarrow{i_2} P_3X \to (\Sigma X)^{\wedge^3}$ is a cofibration sequence. Hence we get an exact triangle

$$\begin{tikzcd} H^*(P_3X) \arrow[rr, "i_2^*"] & & H^*(P_2X) \arrow[dl] \\ & \overline{H}^*(X)^{\otimes^3} \arrow[ul, "\lambda"] & \end{tikzcd}$$

If

$$x = (i_1i_2)^*(a), \quad y = (i_1i_2)^*(b), \quad z = (i_1i_2)^*(c),$$

then

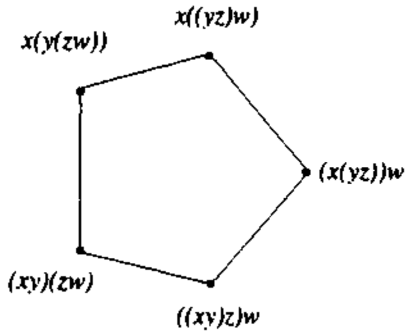
$$abc = \lambda(x \otimes y \otimes z).$$

From this, we can see S^7 is not a mod 3 homotopy associative H-space. The cofibration sequence implies

$$H^*(P_3S^7; \mathbb{Z}_3) = \frac{\mathbb{Z}_3[x_8]}{x_8^4}.$$

But $x_8^3 = \mathcal{P}^4x_8 = \mathcal{P}^1\mathcal{P}^3x_8$ and for degree reasons, $\mathcal{P}^3x_8 = 0$. Once again we are using the factorization of a cup product through primary operations. This will be a recurring theme.

Stasheff [83] defines an A_4-structure on an H-space to be a map $I^2 \times X^4 \longrightarrow X$ that allows one to "fill in" the following pentagon



An A_4-structure yields a projective space P_4X and a cofibration sequence

$$P_3X \longrightarrow P_4X \longrightarrow (\Sigma X)^{\wedge^4}.$$

This yields analogous relations with 4-fold cup products in $H^*(P_4X)$.

In general, an A_n-structure yields inclusions

$$\Sigma X \longrightarrow P_2X \longrightarrow P_3X \longrightarrow \cdots \longrightarrow P_nX$$

with

$$P_\ell X \longrightarrow P_{\ell+1}X \longrightarrow (\Sigma X)^{\wedge^{\ell+1}}$$

a cofibration sequence. An A_∞-structure implies $X \simeq \Omega B$ and

$$\Sigma X \subseteq P_2X \subseteq \cdots \subseteq P_\ell X \subseteq \cdots \subseteq B.$$

This filtration produces a spectral sequence whose exact couple can be described by

$$\begin{tikzcd}
H^*(\Sigma X) \arrow[dr, "\alpha_1"'] & & H^*(P_2X) \arrow[ll, "i_1"'] \arrow[dr, "\alpha_2"] & & H^*(P_3X) \arrow[ll, "i_2"'] \arrow[dr, "\alpha_3"] & & H^*(P_4X) \arrow[ll, "i_3"'] & \cdots \arrow[l] & H^*(P_nX) \arrow[l] \\
& H^*(X^{\wedge^2}) \arrow[ur, "\lambda_2"'] & & H^*(X^{\wedge^3}) \arrow[ur, "\lambda_3"'] & & H^*(X^{\wedge^4}) \arrow[ur, "\lambda_4"'] & & &
\end{tikzcd}$$

The map

$$\begin{tikzcd}
& H^*(P_\ell X) \arrow[dr] & \\
H^*(X^{\wedge^\ell}) \arrow[ur] \arrow[rr, "\alpha_\ell\lambda_\ell"] & & H^*(X^{\wedge^{\ell+1}})
\end{tikzcd}$$

is the "cobar differential"

$$d_1 = \bar{\Delta} \otimes 1 \cdots \otimes 1 - 1 \otimes \bar{\Delta} \otimes \cdots \otimes 1 + \cdots 1 \otimes 1 \otimes \cdots \otimes \bar{\Delta}.$$

By definition, $\ker d_1 / \operatorname{im} d_1$ is

$$Cotor_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p).$$

So the E_2 term is $Cotor_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$ (or $Ext_{H_*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$).

In particular if

$$x_i = i_1 i_2 \cdots i_{n-1}(a_i).$$

Then

$$\lambda_n(x_1 \otimes \cdots \otimes x_n) = a_1 a_2 \cdots a_n \in H^*(P_n X).$$

QUESTION 4.1. Compute $H^*(P_n X; \mathbb{Z}_p)$ for X a finite A_n-space or Lie group with p-torsion in its cohomology. For X p-torsion free, and not generated by A_n-classes, $H^*(P_n X; \mathbb{Z}_p)$ is also unknown.

QUESTION 4.2. Compute $H^*(P_n \Omega X; \mathbb{Z}_p)$ for X an H-space. Some calculations have been made by Hemmi for $n = 2$ and $p = 2$ [32].

QUESTION 4.3. If X is an H-space, what conditions on $H^*(X; \mathbb{Z}_p)$ imply $H^*(P_n \Omega X; \mathbb{Z}_p)$ contains a polynomial algebra truncated at height $n + 1$? See [33], [34].

QUESTION 4.4 (McGibbon [67]). Does there exist a p local finite homotopy type that admits an A_p-structure, but admits no A_∞-structure?

QUESTION 4.5. Suppose $\Sigma X \xrightarrow{i_n} P_n X$ has $i_n^*(y_{2k+1}) = s x_{2k}$ $i_n^*(z) = sw$. Is

$$\lambda_n\left(\sum \left(\frac{1}{p}\right)\binom{p}{i} x^i \otimes x^{p-i} \otimes w \otimes \cdots \otimes w\right) = (\beta_1 \mathcal{P}^k y) \cdot z \cdots z?$$

There are analogous cup product questions for more factors of the form $\sum \frac{1}{p}\binom{p}{i} x^i \otimes x^{p-i}$.

QUESTION 4.6 (Arkowitz). Let X be a finite associative H-space and $\phi : X \times X \to X$ the commutator map. It is known that ϕ has finite order in the group $[X \times X, X]$. See [42]. If ϕ has order N, then $N\langle \alpha, \beta \rangle = 0$ for every Samuelson product $\langle \alpha, \beta \rangle \in \pi_*(X)$.

PROBLEM. What is the order of ϕ, at least in the case when X is a low-dimensional Lie group? For $X = S^3$, ϕ has order 12. See [41]. For $X = \mathbb{RP}^3 = SO(3)$ ϕ has order 12. See [75].

The following problems are due to Cornea, University of Toronto:

The main relation between H-spaces and the Lusternik–Schnirelmann category emerges from the work of Ganea [22]: If G is a topological group the L.S.-category for BG, $cat\,BG$, can be defined as the least n for which the map $B_nG \xrightarrow{i} BG$ admits a homotopy section; that is, there exists a map $r : BG \to B_nG$ such that ri is homotopic to the identity. Here B_nG is the nth order Milnor approximation of the classifying space BG, [69]. If X is a CW-complex, we can replace ΩX with a topological group, hence, the above definition may be applied to ΩX giving $cat\,X$.

Notice that, in Milnor's construction, we have homotopy cofibration sequences: $E_kG \to B_kG \to B_{k+1}G$, with $B_0 = *$. Whenever a homotopy type of a space X can be obtained from a point, by iteratively attaching cones, in n steps, we say that the strong L.S.-category, $Cat\,X$, is $\leqslant n$. In particular $Cat(B_kG) \leqslant k$. The strong Lusternik–Schnirelmann category was introduced by Ganea in [22]; it is a reasonable approximation of the usual one as shown by the inequality $cat\,X \leqslant Cat\,X \leqslant cat\,X + 1$.

It was remarked in [14] that: $Cat(B_kG) = cat(B_kG) = k$ when $k \leqslant cat\,BG$ and $cat(B_kG) = cat\,BG$ when $k > cat\,BG$. Also $Cat(B_kG) \leqslant Cat\,BG$.

QUESTION 4.7. Suppose $Cat\,BG \neq cat\,BG$. Describe precisely how does $Cat\,(B_kG)$ change with respect to k?

REMARK. This is closely related to the problem of finding examples of spaces X such that $Cat\,X \neq cat\,X$. Any co-H-space which is not a suspension is such an example but no examples are known when $cat\,X > 1$. Rationally a conjecture of Lemaire and Sigrist [55], if true, would imply (and, as shown in [15], is equivalent to) $cat\,X_0 = Cat\,X_0$.

It is natural to try to understand the behaviour of $B_k(G \times H)$ with respect to that of $B_i(G)$ and $B_j(H)$. In particular there is the conjecture of Ganea:

QUESTION 4.8 (Ganea). Is it true that $cat(X \times S^n) = cat(X) + 1$.

REMARK. It is trivial that $cat\,(X \times S^n) \geqslant cat\,(X)$. Recently K. Hess, based on work of Barry Jessup [46], has proved, [35], the rational version of the conjecture. The equality $cat\,(X_{(p)} \times Y_{(p)}) = cat(X_{(p)} \times cat\,Y_{(p)})$ is also conjectured; $(-)_{(p)}$ being localization at p. Another, somewhat more general, problem consists in trying to identify restrictions on the homology or homotopy structure of G imposed by $cat\,BG \leqslant n$. In particular see the classical work of Ginsburg [23], Toomer [93], and the recent and very fruitful approach of Felix, Halperin, Lemaire and Thomas [18] and Felix, Halperin and Thomas [19]. The fundamental reference for the main properties of the L.S.-category and the related invariants is the survey paper of James [43]. For the rational results see the book of Felix [17]. In closing let's mention the existence of a very useful list of problems compiled by J.C. Thomas [92].

5. Maps between H-spaces with higher homotopy associativity

It is natural to try to approach the theory of H-spaces like the theory of groups. In group theory, groups are studied by mapping in or out of them via homomorphisms. We then have isomorphism theorems. In H-space theory there are several kinds of maps between H-spaces that preserve certain structures. For example, if X, μ_X and Y, μ_Y are two homotopy commutative H-spaces, an H-map may or may not preserve the homotopy commutativity. In group theory any homomorphism between abelian groups preserves the commutativity. So in H-space theory the homotopies provide an *extra element* that does not exist in group theory. Many of these properties can be viewed alternatively in terms of the projective spaces. We describe some of these properties now.

Given $f : X \to Y$, where X and Y are H-spaces, we say f is an H-map if the following diagram commutes up to homotopy.

$$\begin{array}{ccc} X \times X & \xrightarrow{f \times f} & Y \times Y \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_Y} \\ X & \xrightarrow[f]{} & Y \end{array}.$$

If Y has inverses, we can consider the map

$$X \times X \xrightarrow{D_f} Y,$$

$$D_f(x_1, x_2) = \mu_Y\big(f\mu_X(x_1, x_2), \mu_Y\big(f(x_2)^{-1}, f(x_1)^{-1}\big)\big).$$

If the basepoint acts as the identity, D_f vanishes on $X \vee X$, so it induces a map $\widehat{D}_f : X \wedge X \to Y$. Clearly f is an H-map if and only if $\widehat{D}_f$ is null homotopic. We call $\widehat{D}_f$ the "H-deviation of f." We can reformulate $\widehat{D}_f$ in terms of the projective planes. There exists the following (possibly not commutative) diagram

$$\begin{array}{ccc} \Sigma(X \wedge X) & \xrightarrow{\Sigma(f \wedge f)} & \Sigma(Y \wedge Y) \\ {\scriptstyle \theta_X}\downarrow & & \downarrow{\scriptstyle \theta_Y} \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \\ {\scriptstyle i_X}\downarrow & & {\scriptstyle i_Y}\downarrow \\ P_2X & & P_2Y \end{array}$$

The columns are cofibrations.

We define the H-deviation $\widetilde{D}_f : \Sigma(X \wedge X) \to \Sigma Y$ by $\Sigma f\theta_X - \theta_Y\Sigma(f \wedge f)$. The minus sign uses the cogroup structure in $\Sigma(X \wedge X)$.

LEMMA 5.1. *$i_Y\widetilde{D}_f \simeq *$ if and only if f is an H-map.*

PROOF. If $i_Y\widetilde{D}_f \simeq *$ then $* \simeq i_Y\widetilde{D}_f \simeq i_Y\Sigma f\theta_X$ since $i_Y\theta_Y \simeq *$ by the cofibration property of the right column.

By the cofibre property of the left column, there exists a map $P_2f : P_2X \longrightarrow P_2Y$ such that

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ P_2X & \xrightarrow{P_2 f} & P_2Y \end{array}$$

commutes. This is known to imply f is an H-map. See [83]. Conversely if f is an H-map, P_2f exists. Hence $i_Y\Sigma f\theta_X \simeq *$. So $i_Y\widetilde{D}_f \simeq *$. □

Note P_2f is determined by the choice of an H-structure $F : f\mu \simeq \mu(f\times f)$. Different choices of H-structures yield possibly different maps from P_2X to P_2Y.

Given three H-spaces X, Y, Z and maps

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

we have diagrams (not necessarily commutative)

$$\begin{array}{ccccc} \Sigma(X\wedge X) & \xrightarrow{\Sigma(f\wedge f)} & \Sigma(Y\wedge Y) & \xrightarrow{\Sigma(g\wedge g)} & \Sigma(Z\wedge Z) \\ {\scriptstyle \theta_X}\downarrow & & {\scriptstyle \theta_Y}\downarrow & & {\scriptstyle \theta_Z}\downarrow \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \xrightarrow{\Sigma g} & \Sigma Z \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle i_Z} \\ P_2X & & P_2Y & & P_2Z \end{array}$$

LEMMA 5.2. $\widetilde{D}_{gf} \simeq \Sigma g\widetilde{D}_f + \widetilde{D}_g\Sigma(f\wedge f)$.

PROOF. Using the functorial property of Σ, we have

$$\begin{aligned} \widetilde{D}_{gf} &= \Sigma(gf)\theta_X - \theta_Z\Sigma(gf\wedge gf) \\ &\simeq \Sigma g\big(\Sigma f\theta_X - \theta_Y\Sigma(f\wedge f)\big) + \big(\Sigma g\theta_Y - \theta_Z\Sigma(g\wedge g)\big)\Sigma(f\wedge f) \\ &= \Sigma g\widetilde{D}_f + \widetilde{D}_g\Sigma(f\wedge f). \end{aligned}$$

□

Recall that $H^n(X;k)$ is homotopy classes of maps $X \xrightarrow{f} K(k,n)$. See [71]. $K(k,n) = \Omega K(k,n+1)$ is an H-space so if $[f] \in H^n(X,k)$, f is an H-map iff the diagram

$$\begin{array}{ccc} X\times X & \xrightarrow{f\times f} & K(k,n)\times K(k,n) \\ \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_K} \\ X & \xrightarrow[f]{} & K(k,n) \end{array}$$

commutes up to homotopy. But $[\mu_K(f\times f)]$, $[f\mu_X]\in H^n(X\times X;k)$. So if k is a field,

$$\begin{aligned}[\mu_K(f\times f)-f\mu_X] &= (f^*\otimes f^*)\Delta(i_n)-\Delta f^*(i_n)\\ &= (f^*\otimes f^*)(i_n\otimes 1+1\otimes i_n)-f^*(i_n)\otimes 1-1\otimes f^*(i_n)\\ &\quad -\bar{\Delta}f^*(i_n)=-\bar{\Delta}f^*(i_n).\end{aligned}$$

So up to sign, $D_f^*(i_n)=\bar{\Delta}f^*(i_n)=\bar{\Delta}[f]$. We have

LEMMA 5.3. *$f: X\to K(k,n)$ is an H-map* (a) *if and only if $[f]\in H^n(X;k)$ is primitive,* (b) *if and only if f is adjoint to a map $f':\Sigma X\to K(k,n+1)$ that extends to P_2X*

$$\begin{tikzcd}\Sigma X \arrow[rr, "f'"] \arrow[dr, "i_1'"'] & & K(k,n+1) \\ & P_2X \arrow[ur, dotted, "\bar{f}"'] & \end{tikzcd}.$$

PROOF. To prove (b) note if $\bar{f}:P_2X\to K(k,n+1)$ exists, then taking adjoints

$$\begin{tikzcd}X \arrow[rr, "f"] \arrow[dr, "i_1"'] & & K(k,n)=\Omega K(k,n+1) \\ & \Omega P_2X \arrow[ur, "\Omega\bar{f}"'] & \end{tikzcd}$$

Stasheff proves i_1 is an H-map [84], so f is also. Conversely if $f: X\to K(k,n)$ is an H-map, there exists a commutative diagram

$$\begin{tikzcd}\Sigma X \arrow[r, "\Sigma f"] \arrow[d, "i_1(X)"'] & \Sigma K(k,n) \arrow[d, "i_1(K)"] & \\ P_2X \arrow[r, "P_2f"] & P_2K(k,n) \arrow[r, "\eta"] & K(k,n+1)\end{tikzcd}$$

with $\eta i_1(K)\Sigma f$ adjoint to f, so we can define $\bar{f}=\eta P_2 f$. □

LEMMA 5.4. *An H-map induces a Hopf algebra map on cohomology with field coefficients.*

These ideas extend to the higher projective spaces. Stasheff [83] introduces the idea of "A_n map". Given A_n-spaces X, Y, a map $f: X\to Y$ is an A_n map if there is a commutative ladder

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma f"] \arrow[d] & \Sigma Y \arrow[d] \\ P_2X \arrow[r, "P_2 f"] \arrow[d] & P_2Y \arrow[d] \\ \vdots & \vdots \\ P_nX \arrow[r, "P_n f"] & P_nY \end{tikzcd}$$

An A_2 map is simply an H-map. One can talk about "A_n-deviations". Given an A_{n-1} map $f : X \to Y$ where X, Y are A_n-spaces, consider the diagram

$$\begin{tikzcd} P_{n-1}X \arrow[r, "P_{n-1}f"] \arrow[d] & P_{n-1}Y \arrow[d] \\ P_nX \arrow[d] & P_nY \arrow[d] \\ (\Sigma X)^{\wedge^n} \arrow[r, "(\Sigma f)^{\wedge^n}"] \arrow[d, "\alpha_X"'] & (\Sigma Y)^{\wedge^n} \arrow[d, "\alpha_Y"] \\ \Sigma P_{n-1}X \arrow[r, "\Sigma P_{n-1}f"] & \Sigma P_{n-1}Y \end{tikzcd}$$

The bottom square may not commute. We define the a_n-deviation of f to be

$$a_n(f) = (\Sigma P_{n-1}f)\alpha_X - \alpha_Y(\Sigma f)^{\wedge^n} : (\Sigma X)^{\wedge^n} \longrightarrow \Sigma P_{n-1}Y$$

using the cogroup structure of $(\Sigma X)^{\wedge^n}$. As before, we obtain if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps between A_n-spaces and g, f are a_{n-1} maps, then

$$a_n(gf) \simeq a_n(g)(\Sigma f)^{\wedge^n} + (\Sigma P_{n-1}g)a_n(f).$$

Now we should clarify the relation between $a_n(f)$ and $A_n(f)$. Suppose X is an A_n-space. Let $f : X \to K(\mathbb{Z}_p, \ell)$ be an A_{n-1} map in the sense of Stasheff. Then we have for $K = K(\mathbb{Z}_p, \ell)$

$$\begin{tikzcd}
P_{n-1}X \arrow[r, "P_{n-1}f"] \arrow[d] & P_{n-1}K \arrow[d] & \\
P_nX \arrow[d] & P_nK \arrow[d] & \\
(\Sigma X)^{\wedge^n} \arrow[r, "(\Sigma f)^{\wedge^n}"] \arrow[d, "\alpha_X"'] & (\Sigma K)^{\wedge^n} \arrow[d, "\alpha_K"] & \\
\Sigma P_{n-1}X \arrow[r, "\Sigma P_{n-1}f"] & \Sigma P_{n-1}K \arrow[r, "i(K)"'] & B^2K
\end{tikzcd}$$

Note that $(\alpha_X)^*$ induces d_{n-1} in the Stasheff spectral sequence so

$$(\alpha_X)^*\{i(K)(\Sigma P_{n-1}f)\} = d_{n-1}[f].$$

Further $i(K)\alpha_K$ factors

$$(\Sigma K)^{\wedge^n} \xrightarrow{\alpha_K} \Sigma P_{n-1}K \xrightarrow{\Sigma i_{n-1}} \Sigma P_nK \longrightarrow B^2K$$

and the composition of the first two maps is part of a cofibration sequence and hence is null homotopic.

Therefore we have

LEMMA 5.5.

$$\begin{aligned} i(K)a_n(f) &= i(K)\left[(\Sigma P_{n-1}f)\alpha_X - \alpha_K(\Sigma f)^{\wedge^n}\right] \\ &= d_{n-1}[f] = A_{n-1}(f) \end{aligned}$$

by Stasheff [83].

LEMMA 5.6. *Given X an A_n-space, $f : X \to K(k, \ell)$.*

If f is an A_{n-1} map, then f is an A_n-map (a) if and only if the map

$$(\Sigma X)^{\wedge^n} \xrightarrow{a_n(f)} \Sigma P_{n-1}K(k,\ell) \longrightarrow K(k,\ell+2)$$

is trivial; (b) if and only if f is adjoint to a map f' that extends to $\bar{f}$

$$\begin{tikzcd}
\Sigma X \arrow[rr, "f'"] \arrow[dr, "i"'] & & K(k,\ell+1) \\
& P_nX \arrow[ur, dotted, "\bar{f}"'] &
\end{tikzcd}$$

PROOF. If f is an A_n map, there exists a commutative diagram

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma f"] \arrow[d] & \Sigma K(k,\ell) \arrow[d] \\
P_{n-1}X \arrow[r, "P_{n-1}f"] \arrow[d] & P_{n-1}K(k,\ell) \arrow[d] \\
P_nX \arrow[r, "P_nf"] \arrow[d] & P_nK(k,\ell) \arrow[d] \\
(\Sigma X)^{\wedge^n} \arrow[r, "(\Sigma f)^{\wedge^n}"] & (\Sigma K(k,\ell))^{\wedge^n}
\end{tikzcd}$$

Therefore $a_n(f)$ is trivial. Conversely if

$$(\Sigma X)^{\wedge^n} \xrightarrow{a_n(f)} \Sigma P_{n-1}K(k,\ell) \to K(k,\ell+2)$$

is trivial, then the diagram

$$\begin{tikzcd}
(\Sigma X)^{\wedge^n} \arrow[r, "(\Sigma f)^{\wedge^n}"] \arrow[d] & (\Sigma K)^{\wedge^n} \arrow[d] & \\
\Sigma P_{n-1}X \arrow[r, "\Sigma P_{n-1}f"'] \arrow[d] & \Sigma P_{n-1}K \arrow[r] \arrow[d] & B^2K = K(k,\ell+2) \\
\Sigma P_nX & \Sigma P_nK \arrow[ur] &
\end{tikzcd}$$

implies there exists a map $\Sigma P_nX \to K(k,\ell+2)$ that makes

$$\begin{tikzcd}
\Sigma^2 X \arrow[d] \arrow[dr, "f''"] & \\
\Sigma P_{n-1}X \arrow[r] \arrow[d] & K(k,\ell+2) \\
\Sigma P_nX \arrow[ur] &
\end{tikzcd}$$

commute, where f'' is the double adjoint of f. Hence, taking double adjoints,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & K(k,\ell) \\ \Omega P_n X \arrow[ur, "f_n"'] & \end{tikzcd}$$

commutes where $X \to \Omega P_n X$ is the inclusion and f_n is a loop map. Stasheff [83] shows $X \to \Omega P_n X$ is an A_n map, so f is also. □

If $f : X \to K(\mathbb{Z}_p, \ell)$ represents an A_{n-2} map, then the A_{n-1}-deviation is a cohomology class

$$A_{n-1}(f) \in H^{\ell-n+3}(X^{\wedge^{n-1}}; \mathbb{Z}_p)$$

The classical examples are the "transpotence elements".

A_n-deviations are often described by homotopies. If $f : X \to Y$ is an H-map between homotopy associative H-spaces. $A_3(f) : X \wedge X \wedge X \to \Omega Y$ can be described by the loop

$$\begin{tikzcd} f(x(yz)) \arrow[r, dash] \arrow[d, dash] & f(x)f(yz) \arrow[r, dash] & f(x)(f(y)f(z)) \arrow[d, dash] \\ f((xy)z) \arrow[r, dash] & f(xy)f(z) \arrow[r, dash] & (f(x)f(y))f(z) \end{tikzcd}$$

Here the paths are given by the H-structure of f and the homotopy associativity of X and Y. For $n > 3$, it is harder to see the A_n-obstructions, so the cofibration viewpoint is more useful. Note that there is quite a bit of indeterminancy in these obstructions, depending on your choice of A_{n-1}-structures for f, and choice of A_n-structures on X and Y. For additional details, look at [97].

Much of the literature deals with these homotopies, which can get quite complicated because of the indeterminacy. The cofibration method described in this chapter tends to be a cleaner approach.

6. Maps of H-spaces into fibrations

Our goal is to develop a general obstruction theory of maps between H-spaces. In this chapter we will show how the group $Cotor_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$ measures obstructions to lifting A_n-maps to A_{n-1} maps in a 2-stage Postnikov system. If the target space is the fibre of a map into an Eilenberg–MacLane space K_1 at $\mathbb{Z}_p$, then the obstruction takes values in the mod p cohomology. We illustrate this by the following example. Suppose

$$\begin{tikzcd} \Omega K_1 \arrow[d, "j"] \\ E \arrow[d, "p"] \\ K \arrow[r, "w"'] & K_1 \end{tikzcd}$$

is a stable 2-stage Postnikov system with K, K_1, $\mathbb{Z}_p$-Eilenberg–MacLane spaces. If X is an H-space and $f : X \to K$ is an H-map with a lifting $\bar{f} : X \to E$, when can $\bar{f}$ be chosen to be an H-map?

$$\begin{tikzcd} & \Omega K_1 \arrow[d, "j"] \\ & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dotted, "\bar{f}"] & K \arrow[r, "w"'] & K_1 \end{tikzcd}$$

From Section 5, we know $\tilde{f} : X \to E$ is an H-map if $\Sigma X \xrightarrow{\Sigma\tilde{f}} \Sigma E$ extends to $P_2X \to BE$.

In general, we have the following commutative diagram

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma\tilde{f}"'] \arrow[d] & \Sigma E \arrow[r] \arrow[d] & BE \arrow[d, "Bp"] \\ P_2X \arrow[r, "P_2f"] \arrow[d] & P_2K \arrow[r, "i_2"] & BK \arrow[d, "Bw"] \\ (\Sigma X)^{\wedge^2} \arrow[rr, dashed, "D"] \arrow[d, "\Sigma\theta_X"'] & & BK_1 \\ \Sigma^2 X \end{tikzcd}$$

By the cofibration and fibration properties there exists a $D : (\Sigma X)^{\wedge^2} \to BK_1$ that makes the diagram commute. Note that D is not unique; it can be altered by any map $(\Sigma X)^{\wedge^2} \xrightarrow{\Sigma\theta_X} \Sigma^2 X \to BK_1$. Williams [95] shows D is doubly adjoint to the H-deviation of $\tilde{f}$. (We will generalize this fact here.) Clearly if D is null homotopic there exists a map $\tilde{f}'$

$$\begin{tikzcd} & & BE \arrow[d] \\ \Sigma X \arrow[r, "i"] & P_2X \arrow[ur, "\tilde{f}'"] \arrow[r, "i_2(P_2 f)"'] & BK \end{tikzcd}$$

Hence $\tilde{f}$ can be chosen to be adjoint to $\tilde{f}'i$, and $\tilde{f}$ will be an H-map.

These ideas can be extended to A_n-maps. Suppose as above f is an a_n map and $\tilde{f}$ is an a_{n-1} map, what is the a_n obstruction? We have a diagram

$$\begin{tikzcd} P_{n-1}X \arrow[r] \arrow[d] & BE \arrow[d, "Bp"] \\ P_nX \arrow[r] \arrow[d] & BK \arrow[d, "Bw"] \\ (\Sigma X)^{\wedge^n} \arrow[r, dashed, "a_n(\tilde{f})'"] \arrow[d] & BK_1 \\ \Sigma P_{n-1}X & \end{tikzcd} \tag{6.1}$$

$a_n(\tilde{f})'$ exists and if it is trivial there exists a commutative diagram

$$\begin{tikzcd} & & BE \arrow[d, "Bp"] \\ \Sigma X \arrow[r] & P_nX \arrow[ur] \arrow[r] & BK \end{tikzcd}$$

adjointing this diagram, we get an A_n lift of $f : X \to K$. So we may consider $a_n(\tilde{f})'$ as an "a_n obstruction to lifting f to an a_n map" if it already lifts to an a_{n-1} map.

We describe the relation between $a_n(\tilde{f})'$ and $a_n(\tilde{f}) : (\Sigma X)^{\wedge^n} \to \Sigma P_n E$. We have a (noncommutative) diagram

$$\begin{array}{ccccc} (\Sigma X)^{\wedge^n} & \xrightarrow{(\Sigma\tilde{f})^{\wedge^n}} & (\Sigma E)^{\wedge^n} & & \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_E} & & \\ \Sigma P_{n-1}X & \xrightarrow{\Sigma P_{n-1}\tilde{f}} & \Sigma P_{n-1}E & \xrightarrow{i_E} & B^2E \end{array}$$

Note that $i_E\alpha_E \simeq *$ since we may express i_E

$$\begin{tikzcd} \Sigma P_{n-1}E \arrow[rr, "i_E"] \arrow[dr, "i_{n-1}"'] & & B^2E \\ & \Sigma P_n E \arrow[ur] & \end{tikzcd}$$

and $i_{n-1}\alpha_E \simeq *$. Therefore,

$$\begin{aligned} i_E a_n(\tilde{f}) &= i_E\big((\Sigma P_{n-1}\tilde{f})\alpha_X - \alpha_E(\Sigma\tilde{f})^{\wedge^n}\big) \\ &\simeq i_E(\Sigma P_{n-1}\tilde{f})\alpha_X. \end{aligned} \tag{6.2}$$

So we have a homotopy commutative diagram

$$\begin{tikzcd} (\Sigma X)^{\wedge^n} \arrow[rr, dashed] \arrow[d, "\alpha_X"'] & & BK_1 \arrow[d, "B^2 j"] \\ \Sigma P_{n-1}X \arrow[r, "\Sigma P_{n-1}\tilde{f}"'] \arrow[d, "i_{n-1}"'] & \Sigma P_{n-1}E \arrow[r, "i_E"'] \arrow[d] & B^2E \arrow[d, "B^2 p"] \\ \Sigma P_n X \arrow[r, "\Sigma P_n\tilde{f}"'] & \Sigma P_n K \arrow[r, "i_K"'] & B^2K \end{tikzcd}.$$

One checks that adjointing $i_K(\Sigma P_n\tilde{f})$ we get the map

$$P_nX \xrightarrow[P_n\tilde{f}]{} P_nK \xrightarrow[i_{K'}]{} BK.$$

This map may be placed above the dotted line to obtain

$$\begin{tikzcd} P_{n-1}X \arrow[r] \arrow[d] & BE \arrow[d] \\ P_nX \arrow[r, "i_{K'}P_n\tilde{f}"] \arrow[d] & BK \arrow[d] \\ (\Sigma X)^{\wedge^n} \arrow[r, "a_n(\tilde{f})'"] & BK_1 \end{tikzcd}.$$

Hence $a_n(\tilde{f})'$ may be chosen to be the dotted line. It follows that

THEOREM 6.1. $i_E a_n(\tilde{f}) \simeq B^2 j(a_n(\tilde{f})')$.

This construction is actually part of a general scheme of mapping cofibrations into fibrations. I state a basic fact pointed out to me by John Harper [26].

Suppose there is a homotopy commutative diagram

$$\begin{tikzcd} A \arrow[d, "\alpha"'] & \\ B \arrow[r, "r"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\ C \arrow[r, "s"] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \tag{6.3}$$

with $\beta\alpha$ and gf are null homotopic. So we have homotopies

$$\ell_1 : \ * \simeq \beta\alpha,$$
$$H : \ s\beta \simeq fr,$$
$$\ell_2 : \ gf \simeq *.$$

Then there are maps $\Sigma A \to C_\beta \to Z$; the first map is the coextension to C_β using ℓ_1, the second map extends gs using H and ℓ_2.

There is also a map

$$A \longrightarrow F_f \longrightarrow \Omega Z$$

where the first map is a lifting of r to the homotopy fibre of f using ℓ_1 and H while the second map is determined by ℓ_2. One checks these two maps are adjoint, up to reparameterization.

Now suppose f is actually an A_{n+1} map. Then the following diagram commutes

$$\begin{tikzcd} P_nX \arrow[r, "P_nf"] \arrow[d] & P_nK \arrow[d] & \\ P_{n+1}X \arrow[r, "P_{n+1}f"] \arrow[d] & P_{n+1}K \arrow[d] & \\ (\Sigma X)^{\wedge n+1} \arrow[r] \arrow[d, "\alpha_n"'] & (\Sigma K)^{\wedge n+1} \arrow[d] & \\ \Sigma P_nX \arrow[r, "\Sigma P_nf"] \arrow[d] & \Sigma P_nK \arrow[r, "i_n(K)"] \arrow[d] & B^2K \\ \Sigma P_{n+1}X \arrow[r] & \Sigma P_{n+1}K \arrow[ur] & \end{tikzcd} \tag{6.4}$$

Suspending (6.1) we get

$$\begin{tikzcd}
(\Sigma X)^{\wedge^{n+1}} \arrow[r, "\alpha_n"] \arrow[dr, "d"'] & \Sigma P_n X \arrow[r, "\Sigma P_n f"] \arrow[d] & \Sigma P_n K \arrow[r, "i_n(K)"] \arrow[d] & B^2 K \arrow[d] \\
 & \Sigma(\Sigma X)^{\wedge^n} \arrow[r, "\Sigma a_n(\bar{f})'"] & \Sigma B K_1 \arrow[r, "i_n(K_1)"] & B^2 K_1
\end{tikzcd}$$

Since $i_n(K)\Sigma P_n f \alpha_n = A_{n+1}(f)$ is trivial via the cofibration sequence, we have

$$d^*\big(i_n(K_1)\Sigma a_n(\bar{f})'\big) = 0 \quad \text{where } i_n(K_1)\Sigma a_n(\bar{f})' \in H^*\big(X^{\wedge^n}\big) = \bar{H}^*(X)^{\otimes^n}.$$

(Here, we are suppressing dimension shifts due to suspension.) Since d^* is the cobar differential, we may think of $i_n(K_1)\Sigma a_n(\bar{f})'$ as a cycle with respect to d^*.

Now by 6.1 $a_n(\bar{f})'$ may be altered by any map that factors as

$$(\Sigma X)^{\wedge^n} \to \Sigma P_{n-1} X \to BK_1.$$

If $i_n(K_1)a_n(\bar{f})'$ is altered by $im\, d$, this is realizable as a map

$$\begin{tikzcd}
(\Sigma X)^{\wedge^n} \arrow[r] \arrow[dr, "d"'] & \Sigma P_{n-1} X \arrow[r] \arrow[d] & BK_1 \\
 & \Sigma(\Sigma X)^{\wedge^{n-1}} \arrow[ur] &
\end{tikzcd} \tag{6.5}$$

So $\{i_n(K_1)a_n(\bar{f})'\}$ is a well defined class of $Cotor_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p) = \ker d/\operatorname{im} d$.

If $H^*(X)$ is a bicommutative Hopf algebra, $Cotor_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$ is very easy to compute. In fact, it is bigraded and generated as an algebra by elements that have first bidegree 1 or 2. If

$$H_*(X) = \wedge(x_1, \dots, x_\ell) \otimes \mathbb{Z}_p \frac{[y_k]}{y_k^{p^{f_k}}} \otimes \mathbb{Z}_p[z_m]$$

is a Borel decomposition of the homology, then

$$Cotor_{\wedge(\bar{x})}(\mathbb{Z}_p, \mathbb{Z}_p) = \Gamma(s\bar{x}, 1, \deg x),$$

$$Cotor_{\Gamma_{p^{f_k}}(\bar{y})}(\mathbb{Z}_p, \mathbb{Z}_p) = \wedge\big(s\bar{y}, 1, \deg \bar{y}\big) \otimes \mathbb{Z}_p\big[t\bar{y}, 2, 2p^{f_k} \deg \bar{y}\big],$$

$$Cotor_{\Gamma(\bar{z})}(\mathbb{Z}_p, \mathbb{Z}_p) = \wedge(s\bar{z}, 1, \deg \bar{z})$$

and

$$Cotor_{A\otimes B}(\mathbb{Z}_p, \mathbb{Z}_p) \cong Cotor_A(\mathbb{Z}_p, \mathbb{Z}_p) \otimes Cotor_B(\mathbb{Z}_p, \mathbb{Z}_p).$$

Thus $Cotor^{n,*}_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$ can be computed from the above formulae. If it vanishes in the appropriate degrees, a lifting $\tilde{f}$ can be chosen that is an A_n map.

As an example, to illustrate these techniques, suppose X is an A_{k+1}-space and $H^*(X;\mathbb{Z}_p) = \Lambda(x_1, \dots, x_\ell)$ where degree x_i are odd and let $w : K(\mathbb{Z}_p, 2n+1) \to K_1$ be a stable map where K_1 is a generalized Eilenberg–MacLane space in even degrees. Suppose $x \in PH^{2n+1}(X;\mathbb{Z}_p)$ is represented by a map $f : X \to K(\mathbb{Z}_p, 2n+1)$. Then if f is an A_{k+1} map, then f can always be lifted to an A_k-map $\tilde{f} : X \to E$ where E is the fibre of w

$$\begin{tikzcd}
 & \Omega K_1 \arrow[d] & \\
 & E \arrow[d, "p"] & \\
X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & K \arrow[r, "w"'] & K_1
\end{tikzcd}$$

The proof is by induction. If $\tilde{f}$ is an A_i map, for $i < k$ we have a diagram

$$\begin{tikzcd}
P_i X \arrow[r] \arrow[d] & BE \arrow[d] \\
P_{i+1} X \arrow[r] \arrow[d] & BK \arrow[d] \\
(\Sigma X)^{\wedge^i} \arrow[r, "a_i(\tilde{f})'"'] & BK_1
\end{tikzcd}$$

and $\{a_i(\tilde{f})'\} \in Cotor^{i,k}_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$ where $i+k$ is odd. Since $H^*(X)$ is exterior on odd degree generators, $Cotor_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a divided power algebra on elements sx_j of $\deg(1, \deg x_j)$. But $Cotor^{i,k}_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$ for $i+k$ odd, since $\deg x_j$ is odd.

Hence all the obstructions are trivial.

7. Universal examples

Recall in Section 3 (Property 2), that the vanishing of certain Steenrod operations in the projective plane implied the existence of nontrivial coproducts in the H-space. In this chapter, we develop a universal example for this phenomenon.

For convenience of exposition assume p is an odd prime. Let

$$Bw : K(\mathbb{Z}_p, 2n+1) \to K(\mathbb{Z}_p, 2np+2)$$

be defined by

$$(Bw)^*(i_{2np+2}) = \beta_1 \mathcal{P}^n i_{2n+1} \in H^{2np+2}\big((K(\mathbb{Z}_p, 2n+1); \mathbb{Z}_p\big).$$

(If $p = 2$, replace $\beta_1 \mathcal{P}^n$ with Sq^{2n+1}.) Let BE be the 2-stage Postnikov system with Bw as k-invariant.

$$\begin{array}{l} K(\mathbb{Z}_p, 2np+1) \\ \quad\quad \downarrow Bj \\ \quad\quad BE \\ \quad\quad \downarrow Bq \\ K(\mathbb{Z}_p, 2n+1) \xrightarrow[Bw]{} K(\mathbb{Z}_p, 2np+2) \end{array}$$

Note that the suspension map

$$\sigma^* : H^{2np+2}\big(K(\mathbb{Z}_p, 2n+1); \mathbb{Z}_p\big) \to H^{2np+1}\big(K(\mathbb{Z}_p, 2n); \mathbb{Z}_p\big)$$

has $\beta_1 \mathcal{P}^n i_{2n+1} \in \ker \sigma^*$. It follows that the loop map is null homotopic. So looping the 2-stage Postnikov system

$$\begin{array}{l} K(\mathbb{Z}_p, 2np) \\ \quad\quad \downarrow j \\ \quad\quad E \\ \quad\quad \downarrow q \\ K(\mathbb{Z}_p, 2n) \xrightarrow[w]{} K(\mathbb{Z}_p, 2np+1) \end{array}$$

E is the fibre of a null homotopic map w. Hence E splits as a product,

$$E \simeq K(\mathbb{Z}_p, 2n) \times K(\mathbb{Z}_p, 2np).$$

Note that $H^*(E; \mathbb{Z}_p)$ is a bicommutative Hopf algebra since E is the loops on an H-space. I claim that the element $v \in H^{2np}(E; \mathbb{Z}_p)$ corresponding to the fundamental class $i_{2np} \in H^{2np}(K(\mathbb{Z}_p, 2np); \mathbb{Z}_p)$ is not primitive. In fact

$$\bar{\Delta} v = \sum_{i=1}^{p-1} \binom{p}{i} \frac{1}{p} u^i \otimes u^{p-i} \quad \text{where } u = q^*(i_{2n}).$$

There are a number of ways to see this, [101], [49]. We will use $P_2 E$ to show this fact. We have

$$\Sigma E \xrightarrow{i} P_2E \overset{k}{\subseteq} BE$$

and if $u' \in H^{2n+1}(BE)$ has $\sigma^*(u') = u$ then $k^*(\beta_1 \mathcal{P}^n u') = \beta_1 \mathcal{P}^n k^*(u') = 0$.
But

$$u = i^*\big(k^*(u')\big) \neq 0$$

and in the cofibration sequence

$$\begin{tikzcd} H^*(P_2E) \arrow[rr, "i^*"] & & \overline{H}^*(E) \arrow[dl, "\bar{\Delta}"] \\ & \overline{H}^*(E) \otimes \overline{H}^*(E) \arrow[ul, "\lambda"] & \end{tikzcd}$$

we have by (3.1)

$$\lambda\left(\sum_{i=1}^{p-1} \binom{p}{i} \frac{1}{p} u^i \otimes u^{p-i}\right) = \beta_1 \mathcal{P}^n k^*(u') = 0$$

up to a constant.
By exactness, there exists a v with

$$\bar{\Delta} v = \sum_{i=1}^{p-1} \binom{p}{i} \frac{1}{p} u^i \otimes u^{p-i}.$$

Now v cannot be a Steenrod operation applied to u since all such elements are primitive. Nor can v lie in the subalgebra $H^*(K(\mathbb{Z}, 2n); \mathbb{Z}_p)$. (Note that v is dual to a pth power in homology.) The only other class in $H^{2np}(E; \mathbb{Z}_p)$ corresponds to the fundamental class of $K(\mathbb{Z}_p, 2np)$ so $j^*(v) \neq 0$.

It follows that E has a "twisted H-structure" even though E splits topologically as a product of two spaces. (The word "twisted" is used because the projection of E onto $K(\mathbb{Z}_p, 2np)$ is not an H-map.)

COROLLARY 7.1. *If there is a nontrivial H-map $X \xrightarrow{f} E$, with $f^*(i_{2n}) \neq 0$, then $H_*(X; \mathbb{Z}_p)$ contains a nontrivial p-th power.*

PROOF. We have

$$\begin{aligned} \bar{\Delta} f^*(i_{2np}) &= \big(f^* \otimes f^*\big) \bar{\Delta} i_{2np} \quad \text{since } f \text{ is an } H\text{-map} \\ &= \sum_{i=1}^{p-1} \binom{p}{i} \frac{1}{p} f^*(i_{2n})^i \otimes f^*(i_{2n})^{p-i}. \end{aligned}$$

If $\langle t, f^*(i_{2n})\rangle \neq 0$ then

$$\begin{aligned}
\langle t^p, f^*(i_{2np})\rangle &= \langle t \otimes t^{p-1}, \bar{\Delta} f^*(i_{2np})\rangle \\
&= \langle t \otimes t \otimes t^{p-2}, (1 \otimes \bar{\Delta})\bar{\Delta} f^*(i_{2np})\rangle \\
&= \cdots \\
&= \langle t \otimes \ldots \otimes t, (1 \otimes \cdots \otimes \bar{\Delta})(1 \otimes \cdots \bar{\Delta}) \cdots \bar{\Delta} f^*(i_{2np})\rangle \\
&= \langle t \otimes \cdots \otimes t, f^*(i_{2n}) \otimes \cdots \otimes f^*(i_{2n})\rangle \\
&= \langle t, f^*(i_{2n})\rangle^p \neq 0.
\end{aligned}$$

Hence $t^p \neq 0$. □

From this example, many others can be constructed. If $\beta_1 \mathcal{P}^n$ factors in $\mathcal{A}(p)$, say

$$\beta_1 \mathcal{P}^n = \sum a_i b_i, \quad a_i, b_i \in \mathcal{A}(p),$$

then let $Bw_1 = K(\mathbb{Z}_p, 2n+1) \to \Pi K(\mathbb{Z}_p, 2n+1+\deg b_i)$ be defined by

$$(Bw_1)^*(i_{2n+1+\deg b_i}) = b_i i_{2n+1}.$$

There is a commutative diagram

$$\begin{tikzcd}
K(\mathbb{Z}_p, 2n+1) \arrow[d, "Bw_1"'] \arrow[dr, "Bw"] & \\
\prod_i K(\mathbb{Z}_p, 2n+1+\deg b_i) \arrow[r, "B\theta"] & K(\mathbb{Z}_p, 2np+2)
\end{tikzcd}$$

$(B\theta)^*(i_{2np+2}) = \sum a_i i_{2n+1+\deg b_i}$. It follows that if BE_1 is the fibre of Bw_1, there is a commutative diagram

$$\begin{tikzcd}
\Pi K(\mathbb{Z}_p, 2n+\deg b_i) \arrow[r, "\theta"] \arrow[d, "Bj_1"'] & K(\mathbb{Z}_p, 2np+1) \arrow[d] \\
BE_1 \arrow[r, "Bh"] \arrow[d, "Bq_1"'] & BE \arrow[d] \\
K(\mathbb{Z}_p, 2n+1) \arrow[r, equal] & K(\mathbb{Z}_p, 2n+1)
\end{tikzcd}$$

Looping, we get an H-map $h : E_1 \to E$ with $h^*(i_{2n}) \neq 0$. By the Corollary 7.1, $H_*(E_1; \mathbb{Z}_p)$ has a nonzero p-th power, in fact there exists an element $h^*(i_{2np}) = v_1 \in H^{2np}(E_1)$ with $j_1^*(v_1) = \sum a_i(i_{2n-1+\deg b_i})$ and

$$\bar{\Delta} v_1 = \sum \binom{p}{i} \frac{1}{p} u_1^i \otimes u_1^{p-i}.$$

These examples can be used to analyze $H^*(X;\mathbb{Z}_p)$ for X an H-space. We give an example here.

THEOREM 7.2. *Suppose $H^*(X;\mathbb{Z}_p)$ is primitively generated and let $x \in H^{2n}(X;\mathbb{Z}_p)$ be a primitive generator. Then if $x^p = 0$, x lies in the image of β_1 plus decomposables.*

PROOF. We can factor $\beta_1\mathcal{P}^n$ as a_1b_1 where $a_1 = \beta_1$, $b_1 = \mathcal{P}^n$. Then we have a commutative diagram

$$\begin{tikzcd}
 & K(\mathbb{Z}_p, 2np-1) \arrow[d, "j_1"] & \\
 & E_1 \arrow[d, "q_1"] & \\
X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & K(\mathbb{Z}_p, 2n) \arrow[r, "w_1"'] & K(\mathbb{Z}_p, 2np)
\end{tikzcd}$$

where $f^*(i_{2n}) = x$, $w_1^*(i_{2np}) = \mathcal{P}^n i_{2n} = i_{2n}^p$. $\tilde{f}$ exists since $(w_1 f)^*(i_{2np}) = x^p = 0$. Further f is an H-map since x is primitive. We could look at the above diagram in a different way. Consider the following diagram

$$\begin{tikzcd}
\Sigma X \arrow[rr, "\tilde{f}'"] \arrow[d] & & BE_1 \arrow[d] \\
P_2X \arrow[r, "P_2f"] \arrow[d] & P_2K(\mathbb{Z}_p, 2n) \arrow[r, "i"] & K(\mathbb{Z}_p, 2n+1) \arrow[d, "Bw_1"] \\
(\Sigma X \wedge \Sigma X) \arrow[rr, dashed, "D'"] & & K(\mathbb{Z}_p, 2np+1)
\end{tikzcd} \; .$$

Here $\tilde{f}'$ is adjoint to $\tilde{f}$.

The upper square commutes since f is an H-map. There exists a D' using the cofibration property.

If D' is null homotopic, there exists a lifting of iP_2f.

$$\begin{tikzcd}
 & & & BE_1 \arrow[d] \\
\Sigma X \arrow[r] & P_2X \arrow[urr] \arrow[r] & P_2K(\mathbb{Z}_p, 2n) \arrow[r] & K(\mathbb{Z}_p, 2n+1)
\end{tikzcd}$$

If we adjoint the above diagram and use Lemma 5.3 of Section 5, we get that $\tilde{f}$ can be chosen to be an H-map. So D' can be considered the obstruction to lifting f' to an H-map. By the arguments of Section 6, D' is doubly adjoint to the H-deviation.

If we continue the cofibration, fibration ladder one more step, we obtain

$$\begin{tikzcd}
P_2X \arrow[r, "i(P_2f)"] \arrow[d, "\lambda"'] & K(\mathbb{Z}_p, 2n+1) \arrow[d, "Bw_1"] \\
(\Sigma X \wedge \Sigma X) \arrow[r, dashed, "D'"] \arrow[d, "\Sigma\theta_X"'] & K(\mathbb{Z}_p, 2np+1) \arrow[d, "\beta_1"] \\
\Sigma^2 X & K(\mathbb{Z}_p, 2np+2)
\end{tikzcd}.$$

We notice two things

(1) D' is not unique; it can be altered by any map

$$(\Sigma X) \wedge (\Sigma X) \xrightarrow{\Sigma\theta_X} \Sigma^2 X \xrightarrow{w} K(\mathbb{Z}_p, 2np+1).$$

This will alter D' by $\bar{\Delta}[w]$ where $[w] \in H^{2np-1}(X)$.

(2) $\beta_1(Bw_1)(iP_2f) = \beta_1\mathcal{P}^n u'$ where $u' \in H^{2n+1}(P_2X)$ restricts to $x \in H^{2n}(X;\mathbb{Z}_p)$. Now by Property 2 of Section 3

$$\lambda^*\left(\sum \binom{p}{i}\frac{1}{p}x^i \otimes x^{p-i}\right) = \beta_1\mathcal{P}^n u'.$$

So

$$\beta_1[D'] - \sum \binom{p}{i}\frac{1}{p}x^i \otimes x^{p-i} \in \ker\lambda^* = \text{image } \bar{\Delta}.$$

This places a restriction on $[D'] \in \overline{H}^*(X) \otimes \overline{H}^*(X)$. Suppose $x \notin \operatorname{im}\beta_1 +$ decomposables. Then there exists a primitive $t \in PH_{2n}(X;\mathbb{Z}_p)$ with $\langle t, x\rangle \neq 0$ and $\langle t, \operatorname{im}\beta_1\rangle = 0$. By the above equation

$$\beta_1[D'] = \sum \binom{p}{i}\frac{1}{p}x^i \otimes x^{p-i} + \bar{\Delta}w \quad \text{for some } w \in H^*(X;\mathbb{Z}_p).$$

Define $\bar{\Delta}^\ell$ inductively by $(1 \otimes \cdots \otimes \bar{\Delta})\bar{\Delta}^{\ell-1} = \bar{\Delta}^\ell$, $\bar{\Delta}^1 = \bar{\Delta}$. Applying $\bar{\Delta}^{p-2}$ we get

$$\beta_1\bar{\Delta}^{p-2}[D'] = x \otimes \cdots \otimes x + \bar{\Delta}^{p-1}w.$$

So

$$\begin{aligned}
0 = \left\langle t \otimes \cdots \otimes t, \beta_1\bar{\Delta}^{p-2}[D']\right\rangle &= \langle t \otimes \cdots \otimes t, x \otimes \cdots \otimes x\rangle + \\
&\quad + \left\langle t \otimes \cdots \otimes t, \bar{\Delta}^{p-1}w\right\rangle \\
&= \langle t, x\rangle^p = \langle t^p, w\rangle.
\end{aligned}$$

The left side is zero since $t \in \ker \beta_1$. $t^p = 0$ since $H_*(X;\mathbb{Z}_p)$ has no pth powers. (This is a Hopf algebra result since $H^*(X;\mathbb{Z}_p)$ is primitively generated.) We conclude

$$(t, x)^p = 0 \quad \text{which is a contradiction.}$$

So our original assumption is false and $x \in im\,\beta_1 +$ decomposables.

Variations of this argument are used repeatedly in the study of the cohomology of H-spaces. See [101], [49] for the details. As one can see, often we obtain information about the Steenrod algebra action on $H^*(X;\mathbb{Z}_p)$.

Perhaps it would be useful to give other examples of how one might use this technique to study other problems in H-space theory. Recall that in the 80s, topologists were studying the concept of "atomic spaces".

A space X is mod p *atomic* if

1. $X_{(p)}$ is $(n-1)$ connected and $H_n(X;\mathbb{Z}_p) = \mathbb{Z}_p$.
2. Given a map $f : X \to X$ such that f induces an isomorphism on $H_n(X;\mathbb{Z}_p)$, then $f_{(p)}$ is a homotopy equivalence.

Cohen, Mahowald and Peterson [12] show Ω^2S^5 and Ω^2S^9 are mod 2 atomic. They use a special argument of Peterson's to prove if

$$f : \Omega^2S^5 \to \Omega^2S^5$$

is nontrivial in $H_3(\Omega^2S^5;\mathbb{Z}_2)$, it is an isomorphism in degree 6. We will prove this in a different way using the techniques described above.

Note that $H^*(\Omega S^5;\mathbb{Z}_2) = \Gamma(x_4)$ that is, a divided power Hopf algebra on a 4 dimensional generator. Hence, using the Eilenberg–Moore spectral sequence, $H^*(\Omega^2S^5;\mathbb{Z}_2)$ is generated by $\gamma_2(y_3)$ in degree 6 where $y_3 = \sigma^*(x_4)$. In degree 7 we have $\sigma^*\gamma_2(x_4)$ with $Sq^1\gamma_2(y_3) = \sigma^*\gamma_2(x_4)$.

It suffices to prove

$$f^*\big(\sigma^*\gamma_2(x_4)\big) = \sigma^*\gamma_2(x_4).$$

We build the universal example. Note there is an Adem relation

$$Sq^5 = Sq^2Sq^1Sq^2 + Sq^4Sq^1.$$

Let $Bw : K(\mathbb{Z},4) \to K(\mathbb{Z}_2,6)$ be defined by $(Bw)^*(i_6) = Sq^2i_4$. Let BE be the fiber of Bw.

$$\begin{array}{lll}
K(\mathbb{Z}_2,5) & & \\
\big\downarrow{\scriptstyle Bj} & & \\
BE & & \\
\big\downarrow{\scriptstyle Bq} & & \\
K(\mathbb{Z},4) & \xrightarrow{Bw} & K(\mathbb{Z}_2,6)
\end{array}$$

By arguments similar to the odd prime case, there exists a $Bv \in H^8(BE)$ $(Bj)^*(Bv) = Sq^2Sq^1i_5$ and $\bar{\Delta}(Bv) = Bu \otimes Bu$, $Bu = (Bq)^*(i_4)$.

Now consider the commutative diagram

$$\begin{tikzcd} & & K(\mathbb{Z}_2,4) \arrow[d, "j"] & \\ & & E \arrow[d, "q"] & \\ \Omega^2S^5 \arrow[r, "f"] & \Omega^2S^5 \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & K(\mathbb{Z},3) \arrow[r, "w"'] & K(\mathbb{Z}_2,5) \end{tikzcd}$$

with $g^*(i_3) = \hat{y}_3$, an integral lift of y_3. $\tilde{g}$ exists since $Sq^2y_3 = 0$. Further $\tilde{g}$ can be chosen to be a loop map since g is a loop map and $BwBg$ is null homotopic.

Similarly, $\tilde{g}f$ and $\tilde{g}$ project to gf and g respectively and $gf \simeq g$. So $\tilde{g}f$ and $\tilde{g}$ differ by an element in $H^4(\Omega^2S^5;\mathbb{Z}_2)$ which is trivial, so

$$\tilde{g}f \simeq \tilde{g}.$$

Hence, we have a diagram

$$\begin{tikzcd} & K(\mathbb{Z}_2,5) \arrow[d] \\ & BE \arrow[d, "Bq"] \\ \Omega S^5 \arrow[ur, "B\tilde{g}"] \arrow[r, "Bg"'] & K(\mathbb{Z},4) \end{tikzcd}$$

Now $B\tilde{g}$ is an H-map since $D_{B\tilde{g}} \in H^5(\Omega S^5 \wedge \Omega S^5;\mathbb{Z}_2) = 0$. Hence

$$\bar{\Delta}(B\tilde{g})^*(Bv) = (B\tilde{g})^* \otimes (B\tilde{g})^*(Bu \otimes Bu) = x_4 \otimes x_4 \neq 0.$$

So $(B\tilde{g})^*(Bv) = \gamma_2(x_4)$ since that is the only nonzero element of $H^8(\Omega S^5;\mathbb{Z}_2)$. Hence $\tilde{g}^*(\sigma^*(Bv)) = \sigma^*\gamma_2(x_4) = f^*\tilde{g}^*(\sigma^*Bv))$. This implies $\sigma^*\gamma_2(x_4) = f^*(\sigma^*\gamma_2(x_4))$ which completes the proof.

The proof for Ω^2S^9 is exactly analogous. We can use the factorization $Sq^9 = Sq^8Sq^1 + Sq^2Sq^1Sq^6$.

Clearly one can introduce other notions of atomicity. We list a few here. Let X be an H-space with

$$H_\ell(X;\mathbb{Z}_p) = 0 \quad \text{for } 0 < \ell < n, \qquad H_n(X;\mathbb{Z}_p) = \mathbb{Z}_p.$$

1. X is mod p *H-atomic* if every self map $f : X \to X$ that is an H-map and an isomorphism on $H_n(X;\mathbb{Z}_p)$ induces an isomorphism on $H_*(X;\mathbb{Z}_p)$.

2. If X is an A_k-space, then X *is* mod p A_k*-atomic* if every self map $f : X \to X$ that is an A_k-map and an isomorphism on $H_n(X; \mathbb{Z}_p)$ induces an isomorphism on $H_*(X; \mathbb{Z}_p)$.

QUESTION 7.1. Let p be an prime, $S^{2p+1}\{p\}$ the fibre of a self map of degree p from S^{2p+1} to itself. Is $\Omega S^{2p+1}\{p\}$ mod p atomic?

This question was posed by Selick and is related to the mod p Arf invariant problem. See [78]. One can show that $\Omega S^{2p+1}\{p\}$ is mod p A_p-atomic without too much trouble using results of Selick, so a related question would be

QUESTION 7.2. Do there exist self maps $f : \Omega S^{2p+1}\{p\} \to \Omega S^{2\nu+1}\{p\}$ that are isomorphisms on $H_{2p}(\Omega S^{2p+1}\{p\}; \mathbb{Z}_p)$ but are not A_p-maps?

QUESTION 7.3. Which simple Lie groups are mod p atomic?

For large primes, a Lie group splits into a product of odd spheres. This is called "p-regular." Often H-spaces do also, or they split into other factors. Harper [24] showed F_4 splits at the prime 3, so he is able to produce a non-Lie finite H-space with p torsion in this manner. For other Lie groups the action of the Steenrod algebra often forces atomicity. This leads one to

QUESTION 7.4. Let X be a simply connected finite H-space with $H_*(X; \mathbb{Z}_2)$ associative. If $f : X \to X$ induces an isomorphism on $H_3(X; \mathbb{Z}_2)$ and $H_7(X; \mathbb{Z}_2)$, does it induce an isomorphism on $H_*(X; \mathbb{Z}_2)$?

Here we know all known examples of mod 2 finite H-spaces begin in degrees 1, 3 or 7 from [61]. One might throw in other hypotheses, for example f could be an A_k-map for appropriate k.

8. Homotopy commutativity

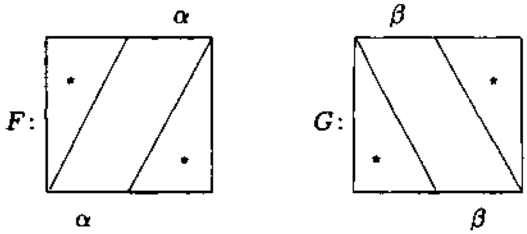
We begin by discussing some ideas of Stasheff and Sugawara. Homotopy commutativity has experienced a recent flurry of interest. Work of Hemmi, Kuhn, Slack and Williams, Zabrodsky, McGibbon, Foskey [85], [88], [32], [54], [97], [67], [68], [44], [45], [20], [21] and others has made it more accessible. Because of their work, there are many interesting open questions.

Recall a homotopy commutative H-space X has the property that the diagram

$$\begin{tikzcd}
X \times X \arrow[dr, "m"] \arrow[dd, "T"'] & \\
& X \\
X \times X \arrow[ur, "m"'] &
\end{tikzcd}$$

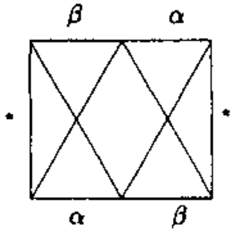
commutes up to homotopy.

Given an H-space X with basepoint acting as strict identity, ΩX is homotopy commutative. To see this, let α and β be loops in X, $\alpha, \beta : I, \{0,1\} \to X, *$. Then we can construct maps $F, G : I \times I \to X$ that look like



Here at the bottom of $I \times I$, F traverses α at double speed and then sits at the basepoint. G traverses the basepoint and then travels along β at double speed.

Consider $\mu_X(F,G) : I \times I \to X$. Its picture looks like



Hence this shows $\alpha\beta \simeq \beta\alpha$, relative endpoints.

Hubbuck [37] proves a finite 2 local H-space has the homotopy type of a torus. This generalizes the Lie group theorem that a commutative Lie group is a product of a torus with Euclidean space.

For homotopy commutative H-spaces, Stasheff [85] realized there is an "axial map"

$$\Sigma X \times \Sigma X \xrightarrow{c} P_2 X.$$

If we restrict to each factor, the induced maps

$$\Sigma X \xrightarrow{i_1} \Sigma X \times \Sigma X \xrightarrow{c} P_2 X,$$

$$i_1(x) = (x, *),$$

$$\Sigma X \xrightarrow{i_2} \Sigma X \times \Sigma X \xrightarrow{c} P_2 X,$$

$$i_2(x) = (x, *)$$

then ci_1 and ci_2 are homotopic to the inclusion map

$$\Sigma X \xrightarrow{j} P_2X. \tag{8.1}$$

We say c is "axial with respect to the maps j, j". We give the formal definition due to Hemmi [32].

DEFINITION. An H-pairing is a family of maps (μ, μ_1, μ_2)

$$\begin{aligned}
&\mu : X_1 \times X_2 \longrightarrow X, \\
&\mu_1 : X_1 \longrightarrow X \quad \text{with } \mu_1(x_1) = \mu(x_1, *), \\
&\mu_2 : X_2 \longrightarrow X \quad \text{with } \mu_2(x_2) = \mu(*, x_2).
\end{aligned}$$

We say μ is "axial with respect to μ_1, μ_2". If one performs the Hopf construction on the map c, we obtain a cofibration sequence

$$\Sigma(\Sigma X \wedge \Sigma X) \xrightarrow{\bar{c}(X)} \Sigma P_2X \xrightarrow{i} P_2^2X \xrightarrow{\lambda} \Sigma^2(\Sigma X \wedge \Sigma X).$$

Cup products in $H^*(P_2^2X)$ are related to elements of nonzero "c invariant in $H^*(X)$". We describe this phenomena here. Suppose

$$j^*i^*(a) = x, \quad j^*i^*(b) = y$$

where $j : \Sigma X \to P_2X$ is the inclusion. Then $\lambda^*(x \otimes y) = ab$. See [94], [32].

If $x \in PH^*(X)$, it corresponds to an H-map

$$X \xrightarrow{f} K(\mathbb{Z}_p, n), \quad [f] = x.$$

Since $X, K(\mathbb{Z}_p, n)$ are homotopy commutative, we can define

$$c(f) : X \wedge X \to \Omega K(\mathbb{Z}_p, n), \quad [c(f)] \in H^{n-1}(X \wedge X; \mathbb{Z}_p),$$

$c(f)$ is the loop

$$\begin{tikzcd}
f(x_1x_2) \arrow[r, dash] \arrow[d, dash] & f(x_2x_1) \arrow[d, dash] \\
f(x_1)f(x_2) \arrow[r, dash] & f(x_2)f(x_1)
\end{tikzcd} \tag{8.2}$$

Here, the lines correspond to a choice of H-structure for f and a choice of homotopy commutative structures for X and $K(\mathbb{Z}_p, n)$. Changing the H-structure of f changes

$c(f)$ by $im(1+T^*)$. Usually we assume the homotopy commutative structures are fixed. So $c(f)$ can be thought of as a coset in $H^*(X \wedge X)/im\, 1 + T^*$. See [97], [32].

Alternatively, since f is an H-map, there is an element $z \in H^*(P_2X)$ with $j^*(z) = sx$ where $[f] = x$. $c(f) = \bar{c}(X)^*(z)$. See [32]. Here $\bar{c}(X)$ will be defined explicitly in the following paragraph.

Hemmi proves $x \in PH^*(X)$ lies in the image of $(ij)^*$ if and only if $c(f)$ is trivial. The c obstruction can be rephrased as follows. Suppose $f : X \to Y$ is an H-map and X, Y are homotopy commutative. Then there is the following diagram (not necessarily commutative)

$$\begin{tikzcd}
\Sigma(\Sigma X \wedge \Sigma X) \arrow[r, "\Sigma(\Sigma f \wedge \Sigma f)"] \arrow[d, "\bar{c}(X)"'] & \Sigma(\Sigma Y \wedge \Sigma Y) \arrow[d, "\bar{c}(Y)"] \\
\Sigma P_2X \arrow[r, "\Sigma P_2 f"'] & \Sigma P_2Y
\end{tikzcd}$$

Here $\bar{c}(X)$ is defined to be the map induced by $-\Sigma j\pi_1 + \Sigma c - \Sigma j\pi_2$ where

$$\pi_1, \pi_2 : \Sigma X \times \Sigma X \to \Sigma X \quad \text{are the 2 projections.}$$

Define $\tilde{c}(f) : \Sigma(\Sigma X \wedge \Sigma X) \to \Sigma P_2Y$ by

$$\tilde{c}(f) = (\Sigma P_2 f)(\bar{c}(X)) - (\bar{c}(Y))\Sigma(\Sigma f \wedge \Sigma f).$$

Then if $Y = K(\mathbb{Z}_p, n)$ and $\theta : \Sigma P_2Y \to B^2K(\mathbb{Z}_p, n) = K(\mathbb{Z}_p, n+2)$, we have $[\theta\tilde{c}(f)] = [c(f)] \in H^{n-1}(X \wedge X; \mathbb{Z}_p)$. One easily checks if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of H-maps between homotopy commutative spaces, then

$$\tilde{c}(gf) \simeq \tilde{c}(g)\Sigma(\Sigma f \wedge \Sigma f) + (\Sigma P_2 g)\tilde{c}(f). \tag{8.3}$$

Hemmi [32] also proves the composition

$$\Sigma(\Sigma X \wedge \Sigma X) \xrightarrow{\bar{c}(X)} \Sigma P_2X \longrightarrow \Sigma(\Sigma X \wedge \Sigma X) \tag{8.4}$$

induces $1 - T^*$ on cohomology.

These properties can be used to restrict the H-deviation of certain liftings. For example, let E be a stable 2-stage Postnikov system, and suppose we have a commutative diagram

$$\begin{tikzcd}
 & \Omega K_1 \arrow[d, "j"] & \\
 & E \arrow[d, "q"] & \\
X \arrow[ur, "\bar{f}"] \arrow[r, "f"'] & K \arrow[r, "w"'] & K_1
\end{tikzcd}$$

where X is a homotopy commutative H-space and f is an H-map. We can transform this to a diagram involving projective planes.

$$\begin{tikzcd}
\Sigma X \arrow[r] \arrow[d] & P_2E \arrow[r] \arrow[d, "P_2q"] & BE \arrow[d, "Bq"] \\
P_2X \arrow[r, "P_2f"] \arrow[d] & P_2K \arrow[r, "i"] & BK \arrow[d, "Bw"] \\
\Sigma X \wedge \Sigma X \arrow[rr, "D'"] & & BK_1
\end{tikzcd}$$

Now if $(Bw)i(P_2f)$ is trivial, we could lift $i(P_2f)$ to BE. Taking adjoints, we would get an H-map for a lifting. Suspending the above diagram, we get

$$\begin{tikzcd}
\Sigma(\Sigma X \wedge \Sigma X) \arrow[r] \arrow[dr, "1-T"'] & \Sigma P_2X \arrow[r, "\Sigma P_2 f"] \arrow[d] & \Sigma P_2K \arrow[r] & \Sigma BK \arrow[r] \arrow[d] & B^2K \arrow[d, "B^2w"] \\
& \Sigma(\Sigma X \wedge \Sigma X) \arrow[rr, "\Sigma D'"] & & \Sigma BK_1 \arrow[r] & B^2K_1
\end{tikzcd} \quad . \tag{8.5}$$

It follows that

$$(1 - T^*)[D_{\tilde{f}}] = [c(f)] \tag{8.6}$$

So if f is a c-map, the H-deviation can be chosen to lie in kernel $(1 - T^*)$.

Zabrodsky gives a geometric proof of this fact in [97]. However, the proof here is potentially more easily generalizable since it becomes harder and harder to draw the homotopies, in the presence of higher homotopy commutativity.

Recall in Section 6, we proved if f is an A_3 map then $[D_{\tilde{f}}] \in Cotor^{2,*}_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$. Combining with (8.6) in the case $[c(f)]$ is trivial implies

$$[D_{\tilde{f}}] \in Cotor^{2}_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p) \cap \ker(1 - T^*).$$

This consists of $P\,Cotor^{2,*}_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p)$.

If $H^*(X)$ is a bicommutative Hopf algebra, such elements only occur in bidegrees $(2, 2p^j n)$. Here $j \geqslant 0$ for $p = 2$, $j > 0$ for p odd. So fairly often, one can choose $\tilde{f}$ to be an H-map.

For example, let X be a homotopy commutative, homotopy associative H-space and let $x \in PH^n(X; \mathbb{Z}_p)$ have the property that $c(x) = 0 = A_3(x)$.

Let $w : K(\mathbb{Z}_p, n) \to \Pi K(\mathbb{Z}_p, \ell_i) = K_1$ be an infinite loop map and let E be the fibre of w. Then if $w(x)$ is trivial, we get a commutative diagram

$$\begin{tikzcd} & \Omega K_1 \arrow[d, "j"] & \\ & E \arrow[d, "p"] & \\ X \arrow[ur, "\bar{f}"] \arrow[r, "x"'] & K(\mathbb{Z}_p, n) \arrow[r, "w"'] & K_1 \end{tikzcd}$$

If p is odd and $\ell_i - 1$ is not divisible by p for all i, then $\bar{f}$ can be chosen to be an H-map. This follows since $c(wx) \simeq * \simeq A_3(wx)$. So $[D_{\bar{f}}] \in P\,Cotor^{2,\ell_i-1}_{H^*(X)}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$.

QUESTION 8.1. Recall the transpotence element $\phi_{p^k}(x) \in PH^*(\Omega X)$ for X an H-space. What is the c obstruction of the H-map $\Omega X \to K(\mathbb{Z}_p, n)$ representing $\phi_{p^k}(x)$?

QUESTION 8.2. Zabrodsky [98] has shown that any p local H-space admits a multiplication that makes it homotopy commutative if p is an odd prime. See also [40]. If X originally has a homotopy associative multiplication, when is it possible to create a new multiplication that is both homotopy commutative and homotopy associative?

Slack has the following related theorem.

THEOREM 8.1 ([81]). *If X is a homotopy commutative homotopy associative H-space, and p is an odd prime $QH^{even}(X; \mathbb{Z}_p)$ lies in degrees $2p^j$ for $j \geqslant 0$.*

QUESTION 8.3 (Arkowitz, Lupton). Multiplication m on X is *quasicommutative* if $\theta : (X, m) \to (X, m^{op})$ is an H-equivalence. Clearly homotopy associative homotopy commutative multiplications are quasicommutative. It is well-known that *all* multiplications on spheres S^1, S^3, S^7 are quasicommutative. Williams [96] showed $\mathbb{R}P^3$ has a non-quasicommutative multiplication and stated that the same is true for $SU(3)$. Arkowitz–Lupton [5] gave an example of infinitely many multiplications on a product of three spheres $S^1 \times S^1 \times S^3$ which are not quasicommutative (Proposition 6.1).

PROBLEM. Is every multiplication on a product of two spheres $S^p \times S^q$, $p, q \in \{1, 3, 7\}$ quasicommutative?

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CHAPTER 23

Co-H-Spaces

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Contents

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0. Introduction

Co-H-spaces are important objects of study for at least two reasons. First of all, they are the duals, in the sense of Eckmann and Hilton, of H-spaces. The latter have played a significant and central role in topology for many years. Secondly, there is a large class of examples, namely the suspensions, which are co-H-spaces. It is the co-H-structure which enables one to add homotopy classes of maps defined on a suspension and which gives rise to the homotopy groups of a space. In this paper we survey the known results on co-H-spaces. While it has not been possible to cover everything that has been published, we have attempted to be comprehensive. In particular, we have presented sketches of the proofs of the major theorems and many illustrative examples. Although suspensions enter naturally into our exposition, our main focus is on co-H-spaces. We have therefore not included results which deal with suspensions per se.

We now fix our notation and state our conventions. All spaces are based and have the based homotopy type of a connected, CW-complex. All maps and homotopies preserve the base point. We do not distinguish notationally between a map and the homotopy class of the map. Thus equality of maps means equality of homotopy classes or homotopy of maps. The standard notation of homotopy theory will be used: $*$ for the base point or the constant map, I for the closed unit interval $[0,1]$, $[X,Y]$ for the set (of homotopy classes) of maps $X \to Y$, $\Delta : X \to X \times X$ for the diagonal map, $\nabla : X \vee X \to X$ for the folding map, Σ for (reduced) suspension, C for the (reduced) cone, Ω for the loop-space, 1 for the identity map, '$\approx$' for homeomorphism or isomorphism and '$\equiv$' for the relation of same homotopy type of spaces. Finally, for a map $f : X \to Y$, we let $f_* : [A,X] \to [A,Y]$ and $f^* : [Y,B] \to [X,B]$ be the induced functions, for any spaces A and B.

1. Definitions and basic properties

We begin with a number of definitions. Recall that equality of maps means equality of their homotopy classes.

DEFINITION 1.1. A pair (X,φ) consisting of a space X and a map $\varphi : X \to X \vee X$ is called a *co-H-space* if (i) $p_1\varphi = 1$ and (ii) $p_2\varphi = 1$, where $p_1, p_2 : X \vee X \to X$ are the two projections.

Co-H-spaces have been called H′-spaces or co-Hopf spaces. The map $\varphi : X \to X \vee X$ is called a *comultiplication*. Clearly (X,φ) is a co-H-space space if and only if $j\varphi = \Delta : X \to X \times X$, where $j : X \vee X \to X \times X$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. For a co-H-space (X,φ) it is common not to mention φ explicitly. The notion of co-H-space is homotopically invariant, in fact, if (X,φ) is a co-H-space and $i : A \to X$ and $r : X \to A$ are maps such that $ri = 1$, then $(r \vee r)\varphi i$ is a comultiplication of A. If X is a co-H-space and a finite CW-complex, we call X a *finite co-H-space*.

DEFINITION 1.2. Let (X,φ) and (X',φ') be co-H-spaces and $f : X \to X'$ a map. Then f is a *co-H-map* if $\varphi' f = (f \vee f)\varphi$.

A co-H-map is written $f : (X, \varphi) \to (X', \varphi')$ or just $f : X \to X'$. Co-H-maps have also been called primitive maps.

Clearly co-H-spaces and (homotopy classes of) co-H-maps form the objects and morphisms of a category.

DEFINITIONS 1.3. If $\varphi : X \to X \vee X$ is a map, then φ is called *associative* if $(1 \vee \varphi)\varphi = (\varphi \vee 1)\varphi : X \to X \vee X \vee X$. A *right inverse* for φ is a map $\rho : X \to X$ such that $\nabla(1 \vee \rho)\varphi = *$, where $\nabla : X \vee X \to X$ is the folding map and $* : X \to X$ is the constant map. A *left inverse* for φ is a map $\lambda : X \to X$ such that $\nabla(\lambda \vee 1)\varphi = *$.

The terms co-associative and co-inverse have been used for associative and inverse.

We can now define an important object of our study.

DEFINITION 1.4. A co-H-space (X, φ) is called a *cogroup* if φ is associative and has a right and left inverse.

The following is the primary example of a cogroup.

EXAMPLE 1.5. Let A be any space and ΣA the suspension of A. If $\sigma_A : \Sigma A \to \Sigma A \vee \Sigma A$ is the pinching map [56, p. 41], then it is well known that $(\Sigma A, \sigma_A)$ is a cogroup [56, pp. 47–48]. In particular, if S^n is the n-sphere, $\Sigma S^{n-1} \approx S^n$ for all $n \geqslant 1$ [37, p. 275] and so S^n is a cogroup for $n \geqslant 1$. If $f : A \to B$ is a map, then $\Sigma f : (\Sigma A, \sigma_A) \to (\Sigma B, \sigma_B)$ is clearly a co-H-map.

Many of the preceding definitions can be made for an abstract category. We briefly describe this. Let $\mathcal{C}$ be a category with zero morphisms and finite coproducts denoted by '$\sqcup$'. Then a pair (X, φ) consisting of an object X and a morphism $\varphi : X \to X \sqcup X$ is called a *co-H-object* in $\mathcal{C}$ if $p_1\varphi = 1 = p_2\varphi$ for $p_1, p_2 : X \sqcup X \to X$ the two projections. Similarly, one can define a cogroup object and a co-H-morphism. Thus we obtain new categories $\mathcal{C}_{CH}$ of co-H-objects and co-H-morphisms and $\mathcal{C}_{CG}$ of cogroup objects and co-H-morphisms. It is an interesting exercise to consider specific categories $\mathcal{C}$ and to determine $\mathcal{C}_{CH}$ and $\mathcal{C}_{CG}$. For example, the co-H-objects in the category of groups are precisely the free groups [42].

We return to spaces and let (X, φ) be a co-H-space and Y an arbitrary space. If $\alpha, \beta \in [X, Y]$, then define $\alpha + \beta \in [X, Y]$ to be the composition

$$X \xrightarrow{\varphi} X \vee X \xrightarrow{\alpha \vee \beta} Y \vee Y \xrightarrow{\nabla} Y.$$

This is a well-defined binary operation on the set $[X, Y]$ with the constant map $* \in [X, Y]$ a (two-sided) unit. We call this operation the *binary operation induced by* φ. It is clear that in homology, $(\alpha + \beta)_* = \alpha_* + \beta_*$, and a similar result holds for cohomology. If $f : Y \to Y'$, then the function $f_* : [X, Y] \to [X, Y']$ is a homomorphism (of sets with binary operation and unit). Furthermore, if (X, φ) is a cogroup, then the induced binary operation in $[X, Y]$ is a group such that f_* is a group homomorphism.

PROPOSITION 1.6. *Let X be a space. Then*

(1) *X is a co-H-space $\Leftrightarrow$ for every space Y, $[X,Y]$ has a binary operation with unit $*$ such that $f_* : [X,Y] \to [X,Y']$ is a homomorphism for all $f : Y \to Y'$.*

Next let (X,φ) be a co-H-space. Then

(2) *φ is associative $\Leftrightarrow$ for every space Y, the induced binary operation on $[X,Y]$ is associative;*

(3) *(X,φ) is a cogroup $\Leftrightarrow$ for every space Y, $[X,Y]$ with the induced binary operation is a group.*

PROOF. We only indicate '$\Leftarrow$' of (1) by showing how to define φ. Let $Y = X \vee X$ and $i_1, i_2 \in [X, X \vee X]$ be the inclusions. Then set $\varphi = i_1 + i_2 \in [X, X \vee X]$. It follows that (X,φ) is a co-H-space. The proof of the rest of the proposition is straightforward. □

It is clear that if (X,φ) is a co-H-space, then $\varphi = i_1 + i_2 \in [X, X \vee X]$, where '+' is the binary operation induced by φ.

COROLLARY 1.7. *Let $\varphi : X \to X \vee X$ be a map. Then (X,φ) is a cogroup $\Leftrightarrow$ φ is associative,* (i) *of Definition* 1.1 *holds and there is a right inverse for φ $\Leftrightarrow$ φ is associative,* (ii) *of Definition* 1.1 *holds and there is a left inverse for φ.*

COROLLARY 1.8. *If (X,φ) is a cogroup with right inverse ρ and left inverse λ, then $\rho = \lambda$.*

There are further analogies with group theory.

DEFINITION 1.9. A co-H-space (X,φ) is called *commutative* if $\tau\varphi = \varphi : X \to X \vee X$, where $\tau : X \vee X \to X \vee X$ is the map which interchanges coordinates.

The following result then complements Proposition 1.6.

PROPOSITION 1.10. *Let (X,φ) be a co-H-space. Then φ is commutative if and only if the induced binary operation in $[X,Y]$ is commutative for all spaces Y.*

We now give an example of a commutative co-H-space, in fact a commutative cogroup.

EXAMPLE 1.11. If A is a co-H-space, then $(\Sigma A, \sigma_A)$ is commutative. In particular, $(\Sigma^2 B, \sigma_{\Sigma B})$ is commutative. This follows from Proposition 1.10 since $[\Sigma A, Y] \approx [A, \Omega Y]$, and the latter group is commutative when A is a co-H-space [61, p. 124].

Other group theoretic concepts such as nilpotency are relevant to cogroups (see §8).

We next show that every 1-connected, associative co-H-space is a cogroup.

DEFINITION 1.12. Let L be a set with binary operation '+' and unit $e \in L$. Then L is called a *loop* if for every $a, b \in L$, the equations $a + x = b$ and $y + a = b$ have unique solutions $x, y \in L$.

The following result is the dual of a well known theorem of James for H-spaces.

PROPOSITION 1.13 ([36, Theorem 2.3]). *If X is a 1-connected co-H-space and Y is any space, then $[X,Y]$ with the induced binary operation is a loop.*

PROOF. A set L with binary operation '+' is a loop if and only if the functions $\theta, \mu : L \times L \to L \times L$ given by $\theta(a,b) = (a, a+b)$ and $\mu(a,b) = (a+b, b)$ are bijections. Let $t, m : X \vee X \to X \vee X$ be defined by $ti_1 = i_1$, $ti_2 = \varphi$, $mi_1 = \varphi$ and $mi_2 = i_2$, where $i_1, i_2 : X \to X \vee X$ are the inclusions. Then under the identification of $[X \vee X, Y]$ with $[X,Y] \oplus [X,Y]$, t^* and $m^* : [X \vee X, Y] \to [X \vee X, Y]$ correspond to θ and $\mu : [X,Y] \oplus [X,Y] \to [X,Y] \oplus [X,Y]$. Thus it suffices to prove that t and m are homotopy equivalences. But in homology, $\varphi_* = (i_1 + i_2)_* = i_{1*} + i_{2*}$, and so t_* and $m_* : H_*(X \vee X) \to H_*(X \vee X)$ correspond to μ and $\theta : H_*(X) \oplus H_*(X) \to H_*(X) \oplus H_*(X)$. The latter μ and θ are isomorphisms because $H_*(X)$ is a group. Since X is 1-connected, t and m are homotopy equivalences. □

REMARK 1.14. Proposition 1.13 remains true when X is not 1-connected provided that Y is a nilpotent space (see [36, Theorem 2.3]).

Since an associative loop is a group, we have

COROLLARY 1.15 ([1], [27]). *A 1-connected, associative co-H-space is a cogroup.*

This result does not hold without associativity. See §7 for an example of a co-H-space with different left and right inverses.

We conclude this section with a few remarks and questions. A co-H-map $f : X \to X'$ induces a homomorphism $f^* : [X', Y] \to [X, Y]$ for any space Y. In particular, if $g : A \to A'$, then $(\Sigma g)^* : [\Sigma A', Y] \to [\Sigma A, Y]$ is a homomorphism. A co-H-map $f : (X, \varphi) \to (X', \varphi')$ which is a homotopy equivalence is called a *co-H-equivalence*. This is an equivalence relation for co-H-spaces. Co-H-equivalent co-H-spaces are regarded as essentially the same.

The elementary considerations of this section raise a number of basic questions which are dealt with in subsequent sections. We mention some of them: When is a space a co-H-space? Are there nonassociative co-H-spaces (there are) and when is a co-H-space associative? Are there cogroups which are not suspensions (there are) and when is a cogroup co-H-equivalent to a suspension? When is a co-H-space commutative? A further complication which can be taken into account is that a given co-H-space may admit several homotopically distinct comultiplications. This raises several other questions, for example, is there a co-H-space such that no comultiplication on it is associative?

2. The topology of co-H-spaces

In this section we consider the classical algebraic topology of co-H-spaces. We will summarize the salient facts without proof or just outline the proof. We begin by noting, and leaving as an exercise, the fact that every co-H-space is path-connected. Thus $\pi_0(X) = 0$. We next consider the fundamental group.

PROPOSITION 2.1 ([26, p. 353]). *If (X, φ) is a co-H-space, then $\pi_1(X)$ is a free group.*

PROOF. We sketch the proof. We identify $\pi_1(X \vee X)$ with the free product $\pi_1(X) * \pi_1(X)$ and $\pi_1(X \times X)$ with the direct sum $\pi_1(X) \oplus \pi_1(X)$. Under these identifications $j_* :$

$\pi_1(X \vee X) \to \pi_1(X \times X)$ corresponds to the canonical homomorphism $\chi : \pi_1(X) * \pi_1(X) \to \pi_1(X) \oplus \pi_1(X)$ and we regard $\varphi_* : \pi_1(X) \to \pi_1(X) * \pi_1(X)$. Thus $\chi\varphi_* = \Delta : \pi_1(X) \to \pi_1(X) \oplus \pi_1(X)$ and so Image $\varphi_* \subseteq \chi^{-1}(\text{Image}\,\Delta)$. Furthermore, φ_* is a monomorphism since $p_1\varphi = 1$, and so it suffices to prove that $\chi^{-1}(\text{Image}\,\Delta)$ is a free group. This can be done directly by showing that the set of all $\alpha'\alpha'' \in \pi_1(X) * \pi_1(X)$, where α runs over all nontrivial elements of $\pi_1(X)$, is a free generating set for $\chi^{-1}(\text{Image}\,\Delta)$ (see [25, p. 211]). □

Although there seem to be no general statements that can be made for the higher homotopy groups of a co-H-space X (e.g., take $X = S^n$), much is known about the rational homotopy groups of X. Let $\mathbb{Q}$ be the group of rationals and let $s^{-1}V$ denote the desuspension of the graded vector space V (i.e. $(s^{-1}V)_n = V_{n+1}$). For any space Y, the total rational homotopy group $\pi_*(\Omega Y) \otimes \mathbb{Q}$ of ΩY is a graded Lie algebra over $\mathbb{Q}$ with Lie bracket given by the Samelson product. Clearly $\pi_*(\Omega Y) \otimes \mathbb{Q} = s^{-1}(\pi_*(Y) \otimes \mathbb{Q})$ with the Lie bracket in $\pi_*(\Omega Y) \otimes \mathbb{Q}$ corresponding to the Whitehead product in $\pi_*(Y) \otimes \mathbb{Q}$.

PROPOSITION 2.2. *Let X be a 1-connected, finite co-H-space. Then $\pi_*(\Omega X) \otimes \mathbb{Q}$ is the free Lie algebra generated by the graded vector space $s^{-1}(\widetilde{H}_*(X;\mathbb{Q}))$, where $\widetilde{H}_*$ denotes reduced singular homology.*

For a discussion of Proposition 2.2 and other rational homotopy results, see §7.

We next turn to cohomology.

PROPOSITION 2.3. *If (X, φ) is a co-H-space and $\alpha \in H^{p_1}(X; G_1)$, $\beta \in H^{p_2}(X; G_2)$ with $p_1, p_2 > 0$, then the cup product $\alpha\beta = 0$ in $H^{p_1+p_2}(X; G_1 \otimes G_2)$.*

PROOF. We regard cohomology as (homotopy classes of) maps into an Eilenberg–MacLane space. Let $K_i = K(G_i, p_i)$ and $K = K(G, p)$ be Eilenberg–MacLane spaces, where $G = G_1 \otimes G_2$ and $p = p_1 + p_2$, let $l : K_1 \vee K_2 \to K_1 \times K_2$ be the inclusion and let $q : K_1 \times K_2 \to K_1 \wedge K_2$ be the projection. Choose

$$\theta \in H^p(K_1 \wedge K_2; G) \approx \operatorname{Hom}\bigl(H_p(K_1 \wedge K_2), G\bigr) \approx \operatorname{Hom}(G, G)$$

corresponding to the identity homomorphism of G. Then $\alpha\beta$ is the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\alpha\times\beta} K_1 \times K_2 \xrightarrow{q} K_1 \wedge K_2 \xrightarrow{\theta} K.$$

If $\Delta = j\varphi$, then $\alpha\beta = \theta q l(\alpha \vee \beta)\varphi = 0$. □

A similar result holds for any multiplicative cohomology theory.

REMARK 2.4. If R is a ring, it follows from Proposition 2.3 that cup products in $H^*(X; R)$ of positive dimensional elements are trivial. More generally, Massey products are trivial. That is, if u_i are positive dimensional cohomology elements in $H^*(X; R)$, $i = 1, \dots, n$, with nonempty Massey product $\langle u_1, \dots, u_n\rangle \subseteq H^N(X; R)$, then $\langle u_1, \dots, u_n\rangle = \{0\}$. This is because $\langle u_1, \dots, u_n\rangle$ is in the kernel of the cohomology suspension $\sigma : H^N(X; R) \to H^{N-1}(\Omega X; R)$ [43, Theorem 5]. But σ is a monomorphism, since X is a co-H-space (Proposition 4.3). Similarly, matric Massey products are trivial [30, Corollary 5.14].

For an associative co-H-space X there is a *cohomology flat product*

$$H^{p+1}(X;G_1)\otimes H^{q+1}(X;G_2)\longrightarrow H^{p+q+1}(X;G_1\otimes G_2)$$

introduced in [1]. This is defined using the commutator (i_1,i_2) in the group $[X,X\vee X]$ and is dual to the Samelson product of an H-space. For a suspension $X=\Sigma A$, the flat product for ΣA is the cup product for A. For more details, see [1] and [8, pp. 160–163].

There are several homological results relating to co-H-spaces. The first is a generalization of the Bott–Samelson theorem on the homology of the loop-space of a suspension.

PROPOSITION 2.5 ([11]). *Let K be a field and (X,φ) a co-H-space. Then the Pontryagin algebra $H_*(\Omega X;K)$ is a free algebra. Furthermore, if φ is associative, there is a set $\{a_i\}$ of free generators of $H_*(\Omega X;K)$ such that*

$$(\Omega\varphi)_*(a_i)=a_i'+a_i''+\sum_{j,k}\alpha_{j,k}a_j'a_k'',$$

where $\alpha_{j,k}\in K$, $a'=(\Omega i_1)_*(a)$ and $a''=(\Omega i_2)_*(a)$, for any $a\in H_*(\Omega X;K)$.

A generalization of Proposition 2.5 has been given by Rutter [49, Theorem C].

The generators $\{a_i\}$ in Proposition 2.5 are called *semi-primitive* (with respect to $(\Omega\varphi)_*$).

These considerations lead to the Berstein–Scheerer coalgebra of a cogroup. Let $\mathcal{A}$ be the category of connected, graded, associative algebras over K and $\mathcal{C}$ the category of connected, graded, associative coalgebras over K. Then Berstein has defined a functor $S:\mathcal{A}_{CG}\to\mathcal{C}$, where $\mathcal{A}_{CG}$ is the category of cogroup objects of $\mathcal{A}$, and shown it to be an isomorphism of categories [11]. We briefly indicate the definition of S : Let (A,ψ) be a cogroup object in $\mathcal{A}$ and let $P\subseteq A$ be the vector space generated by a semi-primitive basis with respect to ψ. Then A is a free algebra generated by P and the homomorphism ψ gives A the structure of a Hopf algebra. It follows that $P\subseteq A$ is a subcoalgebra, and we set $S(A)=P$. If (X,φ) is a cogroup, then $(H_*(\Omega X;K),(\Omega\varphi)_*)$ is a cogroup object in $\mathcal{A}$, and we define the *Berstein–Scheerer coalgebra* of X by $B(X;K)=B(X,\varphi;K)=S(H_*(\Omega X;K))$. In the case $K=\mathbb{Q}$ and X is a rational space, B is a functor from the homotopy category of rational cogroups and co-H-maps to the category of connected, graded, associative, commutative coalgebras over $\mathbb{Q}$, and can be shown to be an equivalence of categories. In particular, two rational cogroups X and X' are co-H-equivalent $\Leftrightarrow B(X;\mathbb{Q})\approx B(X';\mathbb{Q})$ as coalgebras. Furthermore, the set of co-H-maps $[X,X']_{co\text{-}H}$ is in one-one correspondence with the coalgebra homomorphisms $\mathrm{Hom}(B(X;\mathbb{Q}),B(X';\mathbb{Q}))$. In addition, it can be shown that if X is a commutative cogroup, then $B(X;\mathbb{Q})$ has trivial diagonal. Finally, $B(\Sigma A;\mathbb{Q})\approx H_*(A;\mathbb{Q})$ as coalgebras. Therefore if A has finite type, the vector space dual $(B(\Sigma A;\mathbb{Q}))^*\approx H^*(A;\mathbb{Q})$ as algebras. Some of these results were first obtained by Baues [8, pp. 132–133]. For more details on the Berstein–Scheerer coalgebra, see [11] and [51]. For a generalization of the

Berstein–Scheerer coalgebra to 1-connected co-H-spaces and more general coefficient rings and for a discussion of similar functors defined on the category of co-H-spaces with values in other algebraic categories, see [52].

This concludes the summary of facts on the topology of co-H-spaces.

3. Examples of co-H-spaces

We begin with a simple, well-known result.

PROPOSITION 3.1. *If X is an $(n-1)$-connected complex of dim $\leqslant 2n-1$, where $n \geqslant 1$, then X admits a comultiplication. If, in addition, the dimension of $X \leqslant 2n-2$, then any two comultiplications of X are homotopic.*

PROOF. Assume that the $(n-1)$-skeleton of X is the base point. The diagonal map of X is homotopic to a cellular map Δ'. Since $\dim X \leqslant 2n-1$, Δ' factors through $X \vee X$. This proves the first assertion. The second is similar. □

For an $(n-1)$-connected complex X of dimension $\leqslant 2n$, the obstruction to the existence of a comultiplication on X can be identified with the cup square of the basic cohomology class. See [39, Proposition 5.3] for a more general result about spaces of category $\leqslant n$.

We now turn to some examples. By Proposition 3.1, the n-sphere S^n admits a comultiplication for $n \geqslant 1$ which is unique up to homotopy for $n \geqslant 2$. But $S^n = \Sigma S^{n-1}$, and so S^n is a cogroup for $n \geqslant 1$, commutative for $n \geqslant 2$. On S^1 there are many comultiplications, each corresponding to a certain element of $\pi_1(S^1 \vee S^1)$. More precisely, let x and y be the canonical generators of the free group $\pi_1(S^1 \vee S^1)$. The comultiplications on S^1 correspond to words w in x and y such that the sum of the exponents of the x's in w and of the y's in w are 1. Thus S^1 admits infinitely many distinct comultiplications.

More generally, we describe comultiplications on Moore spaces. For an abelian group G and integer $n \geqslant 2$, a *Moore space* of type (G, n) is a 1-connected space with a single nonvanishing reduced homology group G in dimension n. It is known that Moore spaces exist and that any two of type (G, n) have the same homotopy type. We denote the Moore space of type (G, n) by $M(G, n)$. Clearly, we can regard $M(G, n)$ as a CW-complex of dimension $\leqslant n+1$ ($\leqslant n$, if G is free-abelian). By Proposition 3.1, $M(G, n)$ has a comultiplication which is unique if $n \geqslant 3$. But $M(G, n) \equiv \Sigma M(G, n-1)$, and so $M(G, n)$ is a cogroup for $n \geqslant 3$. However, $M(G, 2)$ is also a suspension, and so all Moore spaces $M(G, n)$ are cogroups which are commutative for $n \geqslant 3$. The set of comultiplications of $M(G, 2)$ has been studied in [2] and shown to be in one-one correspondence with the group $\mathrm{Ext}(G, G \otimes G)$. In addition, if q is odd, all comultiplications of $M(\mathbb{Z}_q, 2)$ are commutative, and if q is even, no comultiplication of $M(\mathbb{Z}_q, 2)$ is commutative [2, Corollary 16].

Now Moore spaces $M(\mathbb{Z}_q, n)$ are complexes with two nontrivial cells, $M(\mathbb{Z}_q, n) = S^n \cup_q e^{n+1}$, where $q : S^n \to S^n$ is the map of degree q. Based on work of Berstein and

Hilton we next determine when a cell complex $X = S^n \cup_\alpha e^m$ with two nontrivial cells admits a comultiplication. We first give an important general result.

PROPOSITION 3.2. *If A and X are co-H-spaces and $f : A \to X$ is a co-H-map, then the mapping cone $X \cup_f CA$ is a co-H-space such that the inclusion $i : X \to X \cup_f CA$ is a co-H-map. If $A \equiv \Sigma A'$ and $X \equiv \Sigma X'$ are suspensions and $f = \Sigma g$ for $g : A' \to X'$, then $X \cup_f CA \equiv \Sigma(X' \cup_g CA')$.*

PROOF. Let λ and ψ be the comultiplications of A and X respectively and let $Y = X \cup_f CA$. Then $(i \vee i)\psi f = (i \vee i)(f \vee f)\lambda = (if \vee if)\lambda = 0$. Therefore there exists $\varphi : Y \to Y \vee Y$ such that $\varphi i = (i \vee i)\psi$. Now let $j : Y \vee Y \to Y \times Y$ be the inclusion and $p_1, p_2 : Y \vee Y \to Y$ and $\pi_1, \pi_2 : Y \times Y \to Y$ the projections. Then for $r = 1, 2$,

$$p_r \varphi i = p_r(i \vee i)\psi = i \quad \text{and so} \quad \pi_r j \varphi i = i = \pi_r \Delta i.$$

We consider the exact sequence

$$[\Sigma A, Y \times Y] \longrightarrow [Y, Y \times Y] \xrightarrow{i^*} [X, Y \times Y],$$

obtained from the cofibration $A \xrightarrow{f} X \xrightarrow{i} Y$ and note that $i^*(j\varphi) = i^*(\Delta)$. But for all spaces B, there is an operation of $[\Sigma A, B]$ on $[Y, B]$ whose orbits are the pre-images of $i^* : [Y, B] \to [X, B]$ [35, Chapter 15]. Therefore there is an $\alpha \in [\Sigma A, Y \times Y]$ such that $\alpha \cdot j\varphi = \Delta$. But $j_* : [\Sigma A, Y \vee Y] \to [\Sigma A, Y \times Y]$ is onto, and so $\alpha = j_*(\beta)$ for some $\beta \in [\Sigma A, Y \vee Y]$. Then $j_*(\beta \cdot \varphi) = \Delta$, and consequently $\beta \cdot \varphi \in [Y, Y \vee Y]$ is a comultiplication of Y such that $(\beta \cdot \varphi)i = (i \vee i)\psi$. This proves the first assertion of the proposition. The second assertion is easily proved and hence omitted. $\square$

Another approach to the result proved above has been given by Berstein and Harper [16]. They show that any homotopy between $(f \vee f)\lambda$ and ψf can be replaced by a primitive homotopy, i.e. one which is itself homotopic to a map built out of the homotopies $j_A \lambda \simeq \Delta_A$ and $j_X \psi \simeq \Delta_X$. The primitive homotopy can then be used to directly define a comultiplication on Y.

It follows from Proposition 3.2 that $S^n \cup_\alpha e^m$ is a co-H-space if $\alpha : S^{m-1} \to S^n$ is a co-H-map and that $S^n \cup_\alpha e^m$ is a suspension if $\alpha : S^{m-1} \to S^n$ is a suspension. We will see that the converses of these results hold.

We next define a Hopf invariant homomorphism for a co-H-space (X, φ). Let $\alpha \in \pi_p(X)$ and consider the element

$$\theta = -i_{2*}(\alpha) - i_{1*}(\alpha) + \varphi_*(\alpha)$$

in $\pi_p(X \vee X)$. From the exact homotopy sequence of the pair $(X \times X, X \vee X)$, we have the short exact sequence

$$0 \longrightarrow \pi_{p+1}(X \times X, X \vee X) \xrightarrow{d} \pi_p(X \vee X) \xrightarrow{j_*} \pi_p(X \times X) \longrightarrow 0.$$

Since $j_*(\theta) = 0$, there is a unique element $\mathcal{H}(\alpha) \in \pi_{p+1}(X \times X, X \vee X)$ such that $d\mathcal{H}(\alpha) = \theta$. This defines the *Hopf invariant homomorphism* $\mathcal{H} : \pi_p(X) \to \pi_{p+1}(X \times X, X \vee X)$. From the definition it follows that

$$\mathcal{H}(\alpha) = 0 \Leftrightarrow \theta = 0 \Leftrightarrow \alpha \text{ is a co-H-map.}$$

If $X = S^k$, then the homotopy groups of $X \vee X$ are isomorphic to the homotopy groups of an infinite product of spheres $S^k \times S^k \times S^{2k-1} \times S^{3k-2} \times S^{3k-2} \times \cdots$ [34, Theorem A]. Thus $\pi_{p+1}(X \times X, X \vee X) \approx \pi_{p+1}(S^{2k-1} \times S^{3k-2} \times S^{3k-2} \times \cdots)$ and so $\mathcal{H}$ determines homomorphisms $\mathcal{H}_0 : \pi_p(S^k) \to \pi_p(S^{2k-1})$, $\mathcal{H}_1 : \pi_p(S^k) \to \pi_p(S^{3k-2})$, $\mathcal{H}_2 : \pi_p(S^k) \to \pi_p(S^{3k-2})$, etc. which are the classical Hilton–Hopf invariants. Therefore for a sphere, $\mathcal{H}$ subsumes all of the Hilton–Hopf invariants. Now if $\alpha \in \pi_{m-1}(S^n)$, then $\mathcal{H}(\alpha) = 0$ implies that $S^n \cup_\alpha e^m$ is a co-H-space. In fact, the following result holds.

PROPOSITION 3.3 ([17, Theorem 3.20, Lemma 3.6]). *Let* $\alpha \in \pi_{m-1}(S^n)$ *with* $m - 1 \geqslant n \geqslant 2$, *then*

(1) $S^n \cup_\alpha e^m$ *is a co-H-space* $\Leftrightarrow \mathcal{H}(\alpha) = 0 \Leftrightarrow \alpha$ *is a co-H-map;*

(2) $S^n \cup_\alpha e^m$ *is a suspension* $\Leftrightarrow \alpha$ *is a suspension.*

Thus $S^n \cup_\alpha e^m$ is a co-H-space if and only if all Hilton–Hopf invariants are trivial (cf. [8, p. 43]).

We can now construct cell complexes $S^n \cup_\alpha e^m$ which are co-H-spaces but not suspensions.

EXAMPLE 3.4 ([17, p. 444]). Let p be an odd prime and $\alpha \in \pi_{2p}(S^3)$ an element of order p. Then for $i = 0, 1, 2, \ldots$, $\mathcal{H}_i(\alpha) \in \pi_{2p}(S^{2n_i+1})$ for $n_i > 1$. Since $\pi_{2p}(S^{2n_i+1})$ has no element of order p, $\mathcal{H}_i(\alpha) = 0$. Thus $S^3 \cup_\alpha e^{2p+1}$ is a co-H-space. However, α is not a suspension since $\pi_{2p-1}(S^2)$ has no element of order p. Thus $S^3 \cup_\alpha e^{2p+1}$ is not a suspension.

Using Proposition 2.5, Berstein has shown that $S^3 \cup_\alpha e^{2p+1}$ does not admit any associative comultiplication [10].

For further results on cell complexes with two nontrivial cells, see [17].

4. Characterizations of co-H-spaces and cogroups

In this section we give alternative formulations of the notion of co-H-space and cogroup which are duals to corresponding statements for H-spaces. The theorems for co-H-spaces, however, appear to be more difficult to prove because there are no known duals of certain fundamental H-space results and constructions (such as the James reduced product). The material of this section is based on [35, §17] and [27].

For a space X we denote by $\mathcal{C}(X) \subseteq [X, X \vee X]$ the (possibly empty) set of comultiplications of X. A map $\gamma = \gamma_X : X \to \Sigma\Omega X$ is called a *coretraction* of X if

$\nu_X\gamma_X = 1 : X \to X$, where $\nu = \nu_X : \Sigma\Omega X \to X$ is the canonical projection. Denote by $\mathcal{CR}(X) \subseteq [X, \Sigma\Omega X]$ the set of coretractions of X.

THEOREM 4.1 ([27, Theorem 1.1]). *The function* $((\nu_X \vee \nu_X)\sigma_{\Omega X})_* : [X, \Sigma\Omega X] \to [X, X \vee X]$ *induces a bijection from* $\mathcal{CR}(X)$ *to* $\mathcal{C}(X)$. *In particular,* X *admits a comultiplication if and only if* X *admits a coretraction.*

PROOF. If γ is a coretraction of X, clearly $(\nu \vee \nu)\sigma\gamma$ is a comultiplication of X. Thus $((\nu \vee \nu)\sigma)_*$ induces a function $F : \mathcal{CR}(X) \to \mathcal{C}(X)$. We will define an inverse function G after some preliminaries. Let W be the homotopy pull-back of j and Δ,

$$\begin{array}{ccc} W & \xrightarrow{q} & X \vee X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{\Delta} & X \times X \end{array},$$

with projections p and q. Thus

$$W = \{(x, \lambda, y) \mid x \in X,\ \lambda \in (X \times X)^I,\ y \in X \vee X,\ \lambda(0) = \Delta(x),\ \lambda(1) = j(y)\}.$$

Let E_0X be the paths in X starting at $*$ and E_1X the paths in X ending at $*$. Define $V \subseteq X^I$ by $V = E_0X \cup E_1X$. If ω is a path in X and $a, b \in I$, then $\omega_{a,b}$ denotes the path in X along ω from $\omega(a)$ to $\omega(b)$. (We allow $a > b$ in which case $\omega_{a,b}$ is traversed from $\omega(a)$ to $\omega(b)$ along the path opposite to ω.) Since E_0X and E_1X are contractible spaces whose intersection is ΩX, V has the homotopy type of $\Sigma\Omega X$. More precisely, define $\varepsilon : \Sigma\Omega X \to V$ by

$$\varepsilon(\omega, t) = \begin{cases} \omega_{0,2t} & \text{if } 0 \leqslant t \leqslant 1/2, \\ \omega_{2t-1,1} & \text{if } 1/2 \leqslant t \leqslant 1, \end{cases}$$

and show that ε is a homotopy equivalence [35, p. 211]. Moreover, $\theta : V \to W$ given by $\theta(\xi) = (\xi(\frac{1}{2}), (\xi_{\frac{1}{2},1}, \xi_{\frac{1}{2},0}), \xi(1), \xi(0))$ is a homeomorphism. If $\varphi \in \mathcal{C}(X)$, then φ and $1 : X \to X$ determine a map $\mu : X \to W$ such that $p\mu = 1$ and $q\mu = \varphi$. Now set $\gamma = \varepsilon^{-1}\theta^{-1}\mu : X \to \Sigma\Omega X$. Clearly γ is a coretraction, and so we define $G : \mathcal{C}(X) \to \mathcal{CR}(X)$ by $G(\varphi) = \gamma$. Since $(\nu \vee \nu)\sigma\gamma = \varphi$, $FG = 1$. To complete the proof we show $F : \mathcal{CR}(X) \to \mathcal{C}(X)$ is one-one. Consider the fibration $\pi' : X^I \to X \times X$ given by $\pi'(\omega) = (\omega(1), \omega(0))$ with fibre ΩX. By restricting the base to $X \vee X$, we obtain a fibre sequence $\Omega X \xrightarrow{i} V \xrightarrow{\pi} X \vee X$. Since i factors through a contractible space, $i = *$, and so $\pi_* : [\Sigma A, V] \to [\Sigma A, X \vee X]$ is a monomorphism for all spaces A. But $\pi\varepsilon = (\nu \vee \nu)\sigma : \Sigma\Omega X \to X \vee X$ with ε a homotopy equivalence. Thus

$$((\nu \vee \nu)\sigma)_* : [\Sigma A, \Sigma\Omega X] \to [\Sigma A, X \vee X]$$

is a monomorphism. Now let γ_1, γ_2 be two coretractions such that $F(\gamma_1) = F(\gamma_2)$. Hence $((\nu\vee\nu)\sigma)_*(\gamma_1\nu) = ((\nu\vee\nu)\sigma)_*(\gamma_2\nu)$, and so $\gamma_1\nu = \gamma_2\nu$. Thus $\gamma_1 = \gamma_1\nu\gamma_1 = \gamma_2\nu\gamma_1 = \gamma_2$. Therefore F is one-one. □

We see next that in Theorem 4.1 X is a cogroup if and only if the corresponding coretraction is a co-H-map. If $\gamma : X \to \Sigma\Omega X$ is a coretraction, then we say that (X, γ) is a *(homotopy) coalgebra* if the diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \Sigma\Omega X \arrow[d, "\Sigma e_{\Omega X}"] \\ \Sigma\Omega X \arrow[r, "\Sigma\Omega\gamma"] & \Sigma\Omega\Sigma\Omega X \end{tikzcd}$$

commutes, where $e_Y : Y \to \Omega\Sigma Y$ is the natural embedding.

THEOREM 4.2 ([27, Theorem 2.2]). *Let $\gamma : X \to \Omega\Sigma X$ be a coretraction with corresponding comultiplication $\varphi : X \to X \vee X$. Then the following are equivalent:*

(1) *(X, φ) is a cogroup;*

(2) *$\gamma : (X, \varphi) \to (\Sigma\Omega X, \sigma_{\Omega X})$ is a co-H-map;*

(3) *(X, γ) is a coalgebra.*

PROOF. We sketch the proof. If $\gamma : X \to \Sigma\Omega X$ is a co-H-map, then

$$\gamma^* : [\Sigma\Omega X, X \vee X \vee X] \to [X, X \vee X \vee X]$$

is an epimorphism from a group to a set with binary operation. Thus $[X, X \vee X \vee X]$ is associative, and so $i_1 + (i_2 + i_3) = (i_1 + i_2) + i_3$, where i_r is the inclusion of X into the rth summand of $X \vee X \vee X$. This shows that φ is associative. The proof of the existence of inverses is similar. Therefore (2) $\Rightarrow$ (1). The equivalence of (2) and (3) is based on the following general and easily verified facts: (a) Let (X, φ) and (X', φ') be co-H-spaces with corresponding coretractions γ and γ' and let $f : X \to X'$ be a map. Then f is a co-H-map $\Leftrightarrow$ the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma"'] & X' \arrow[d, "\gamma'"] \\ \Sigma\Omega X \arrow[r, "\Sigma\Omega f"] & \Sigma\Omega X' \end{tikzcd} .$$

(The proof of '$\Rightarrow$' uses the fact, established in the proof of Theorem 4.1, that $((\nu' \vee \nu')\sigma')_* : [X, \Sigma\Omega X'] \to [X, X' \vee X']$ is one-one.) (b) The coretraction corresponding to the suspension comultiplication $\sigma_A : \Sigma A \to \Sigma A \vee \Sigma A$ is $\Sigma e_A : \Sigma A \to \Sigma\Omega\Sigma A$. This just leaves the proof that (1) $\Rightarrow$ (2) which we omit because of its length. □

The alternative characterizations of co-H-spaces and cogroups of this section yield results about co-H-spaces and cogroups. As an illustration we prove

PROPOSITION 4.3. *If X is a co-H-space, then the homology suspension $\sigma_i : \widetilde{H}_i(\Omega X; G) \to H_{i+1}(X; G)$ is onto and the cohomology suspension $\sigma^i : H^{i+1}(X; G) \to \widetilde{H}^i(\Omega X; G)$ is one-one for all $i \geqslant 0$.*

PROOF. Let $\gamma : X \to \Sigma\Omega X$ be a coretraction corresponding to the given comultiplication of X. Since $\nu\gamma = 1$, $\nu_* : H_{i+1}(\Sigma\Omega X; G) \to H_{i+1}(X; G)$ is onto. But the homology suspension σ_i is just the composition of the isomorphism $\widetilde{H}_i(\Omega X; G) \approx H_{i+1}(\Sigma\Omega X; G)$ with ν_*, and so σ_i is onto. The result for the cohomology suspension is proved similarly. □

5. Connectivity/dimension results

In this section we consider when a co-H-space or a cogroup with connectivity and dimension restrictions is co-H-equivalent to a suspension. The main result is the following

THEOREM 5.1. *Let (X, φ) be a finite co-H-space which is $(n-1)$-connected.*

(1) *If $\dim X \leqslant 3n-3$, $n \geqslant 1$, then (X, φ) is co-H-equivalent to a suspension.*

(2) *If $\dim X \leqslant 4n-5$, $n \geqslant 2$, and φ is associative, then (X, φ) is co-H-equivalent to a suspension.*

Part (1) is due to Berstein and Hilton [18, Theorem A] and Part (2) is due to Ganea [27, Corollary 3.5]. Before discussing the proof of Theorem 5.1, we digress to describe a construction which is the dual of the Hopf construction of an H-space multiplication and which generalizes the Hopf invariant homomorphism defined in §3. Let $X \flat X$ be the space of paths in $X \times X$ beginning in $X \vee X$ and ending at the base point and let $i : X \flat X \to X \vee X$ assign to a path its initial point. Then we obtain a short exact sequence

$$0 \longrightarrow [\Omega X, \Omega(X \flat X)] \xrightarrow{(\Omega i)_*} [\Omega X, \Omega(X \vee X)] \xrightarrow{(\Omega j)_*} [\Omega X, \Omega(X \times X)] \longrightarrow 0.$$

The comultiplication $\varphi : X \to X \vee X$ determines an element

$$\mu = -\Omega i_2 - \Omega i_1 + \Omega\varphi \in [\Omega X, \Omega(X \vee X)]$$

such that $(\Omega j)_*(\mu) = 0$. Thus there exists a unique element $H(\varphi) \in [\Omega X, \Omega(X \flat X)]$ such that $(\Omega i)_*(H(\varphi)) = \mu$. We call $H(\varphi)$ the *dual Hopf construction (applied to φ)*. If A is any space and $\rho \in [\Sigma A, X]$, let $\overline{\rho} \in [A, \Omega X]$ denote the adjoint of ρ. Then the following two facts about the dual Hopf construction are easily verified: (i) Let $\alpha \in \pi_p(X)$ and let $\mathcal{H} : \pi_p(X) \to \pi_{p+1}(X \times X, X \vee X)$ be the Hopf invariant homomorphism of §3. We identify $\pi_{p+1}(X \times X, X \vee X)$ with $\pi_p(X \flat X)$ and regard $\mathcal{H}(\alpha) \in \pi_p(X \flat X)$. Then $H(\varphi)\overline{\alpha} = \overline{\mathcal{H}(\alpha)}$. (ii) $\beta : (\Sigma A, \sigma_A) \to (X, \varphi)$ is a co-H-map $\Leftrightarrow$ $H(\varphi)\overline{\beta} = 0$ in $[A, \Omega(X \flat X)]$. We note that (i) and (ii) yield the result in §3 that α is a co-H-map $\Leftrightarrow$ $\mathcal{H}(\alpha) = 0$.

We now sketch the proof of Theorem 5.1(1) following [18]. Let P be the fibre of $H(\varphi)$ and consider the fibre sequence

$$P \xrightarrow{\gamma} \Omega X \xrightarrow{H(\varphi)} \Omega(X \flat X).$$

Then if $\eta : \Sigma P \to X$ is the adjoint of γ (i.e. $\overline{\eta} = \gamma$), η is a co-H-map since $H(\varphi)\overline{\eta} = 0$. The main step in the proof is to show that $\eta_* : H_r(\Sigma P) \to H_r(X)$ is an isomorphism for

$r < 3n-3$ and an epimorphism for $r = 3n-3$. This is done by analyzing the homology homomorphism $(H(\varphi))_*$ (similar to the analysis of $(\Omega\varphi)_*$ in §2) and then applying the Serre spectral sequence of a fibration to obtain the desired result for η_*. The proof is completed by constructing a space P_0 of dimension $\leqslant 3n-4$ and a map $\nu : P_0 \to P$ using a modified homology decomposition of P such that $\nu_* : H_r(P_0) \to H_r(P)$ is an isomorphism for $r < 3n-4$ and

$$H_{3n-4}(P_0) \xrightarrow{\nu_*} H_{3n-4}(P) \approx H_{3n-3}(\Sigma P) \xrightarrow{\eta_*} H_{3n-3}(X)$$

is an isomorphism. The map $\eta\,\Sigma\nu : \Sigma P_0 \to X$ is then a co-H-equivalence.

Because of its length and complexity we omit the proof of Theorem 5.1(2). We note that Ganea's proof uses the characterization of a cogroup as a coalgebra (Theorem 4.2).

Theorem 5.1 suggest the possibility of a sequence of successively restrictive conditions $A_2, A_3, \ldots$ on a space X such that A_r and a connectivity/dimension condition (such as $\dim(X) \leqslant (r+1)n - (2r-1)$ for X $(n-1)$-connected) would imply that X is co-H-equivalent to a suspension. Furthermore, the condition A_2 should be that X is a co-H-space and the condition A_3 should be that X is an associative co-H-space. The prototype for this is Stasheff's A_n-theory for H-spaces. This theory for co-H-spaces has been developed in part by Saito [50], though the details are formidable. However, Saito has proved [50, Proposition 6.17] that an $(n-1)$-connected co-H-space of dimension $\leqslant 5n-7$ which satisfies an A_4-like condition is co-H-equivalent to a suspension.

For the remainder of this section we consider examples relating to Theorem 5.1. The general reference for homotopy groups of spheres is [58].

We begin by giving an example taken from [18] to show that the connectivity and dimension restriction in Theorem 5.1(1) is best possible. Let $\alpha \in \pi_6(S^3) = \mathbb{Z}_{12}$ be an element of order 3 and form $X = S^3 \cup_\alpha e^7$. By Example 3.4 with $p = 3$, X is a co-H-space which is not a suspension. Thus with $n = 3$ we see that the restriction in (1) of Theorem 5.1 is necessary.

We next discuss examples relating to Theorem 5.1(2). Let $\beta \in \pi_{15}(S^5) = \mathbb{Z}_{72} \oplus \mathbb{Z}_2$ be an element of order 9. Then $\mathcal{H}(\beta) = 0$ since $\pi_{15}(S^9) = \mathbb{Z}_2$ and $\pi_{15}(S^{13}) = \mathbb{Z}_2$. Thus β is a co-H-map and so $X = S^5 \cup_\beta e^{16}$ is a co-H-space. But β is not a suspension because $\pi_{14}(S^4) = \mathbb{Z}_{120} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2$. Therefore X is not a suspension. In [27, 4.2] Ganea made the following conjecture:

CONJECTURE 5.2. $X = S^5 \cup_\beta e^{16}$ admits an associative comultiplication.

If true, this conjecture would show that the dimensional restriction in Theorem 5.1(2) is best possible. We will see shortly that 5.2 is false. However, Ganea has given an example to show that some dimensional restriction is necessary. He considered $Y = S^3 \vee S^{15}$ and proved that Y has at least 72 associative comultiplications such that at most 56 of them are co-H-equivalent to a suspension [27, Proposition 4.3].

Conjecture 5.2 was taken up by Barratt and Chan in [7]. The spectral sequence of an inclusion with differentials d_r developed in [6] was applied to $S^{n-1} \subsetneq E^n$, n odd $\geqslant 3$. The following result was then proved.

PROPOSITION 5.3 ([7, Theorem 3.3]). *If $\beta \in \pi_{15}(S^5)$ has order* 9, *then $d_1(\beta) = 0$ and $d_2(\beta) \neq 0$.*

Proposition 5.3 disproves Conjecture 5.2 for the following reasons. For $\alpha \in \pi_q(S^n)$, $d_1(\alpha) = 0 \Leftrightarrow S^n \cup_\alpha e^{q+1}$ admits a comultiplication [6, §2]. For $\alpha \in \ker d_1 \subseteq \pi_q(S^n)$, $d_2(\alpha) = 0 \Leftrightarrow S^n \cup_\alpha e^{q+1}$ admits an associative comultiplication [20, Theorem 6.8.1]. If $\beta \in \pi_{15}(S^5)$ is an element of order 9, then $S^5 \cup_\beta e^{16}$ is a co-H-space such that no comultiplication is associative. As mentioned at the end of §3, the space $S^3 \cup_\alpha e^{2p+1}$, where $\alpha \in \pi_{2p}(S^3)$ has order p, also has this property. The following question was raised in [7, Remark 3.5].

QUESTION 5.4. Does there exist a finite complex X, more particularly, a cell complex with two nontrivial cells, which is not a suspension, but which admits an associative comultiplication?

This question was answered by Berstein and Harper in the affirmative. We state some of their results. First note that the p-primary components ${}_p\pi_{2p+2}(S^5) = \mathbb{Z}_p$, ${}_p\pi_{6p-3}(S^5) = \mathbb{Z}_p$ for $p \geqslant 5$ and ${}_3\pi_{15}(S^5) = \mathbb{Z}_9$. Let $\alpha \in {}_p\pi_{2p+2}(S^5)$ and $\gamma \in {}_p\pi_{6p-3}(S^5)$ be generators. Furthermore, ${}_3\pi_{34}(S^5) = \mathbb{Z}_3$ and we let η be a generator.

PROPOSITION 5.5 ([16, Theorems A, D]). *The space $X = S^5 \cup_\alpha e^{2p+3} \cup_\gamma e^{6p-2}$ is a cogroup which is not a suspension if p is a prime such that $p \equiv 1 (\mathrm{mod}\ 3)$. The space $Y = S^5 \cup_\eta e^{35}$ is a cogroup which is not a suspension.*

Berstein and Harper also show that $S^5 \cup_\gamma e^{6p-2}$ is a co-H-space which admits no associative comultiplication [16, Theorem B], thus providing more examples of this phenomena.

It is not known if the dimension restriction in Theorem 5.1(2) is best possible.

6. The Ganea conjecture

For a co-H-space (X, φ), Ganea conjectured that there is a homotopy equivalence

$$X \equiv Y \vee S,$$

where Y is 1-connected and S is a wedge of circles or a point [28, Problem 10]. We assume in this section that X is not 1-connected (so that S is not a point) and refer to this conjecture as the *Ganea conjecture*. In 1976 Berstein and Dror gave a condition on the comultiplication φ which implies the Ganea conjecture. As a consequence the Ganea conjecture holds for an associative comultiplication φ. We give the Berstein–Dror condition in Proposition 6.1. Our treatment is based on the paper of Hilton, Mislin and Roitberg [36]. We adopt the following notation: for a map f, $\pi_n(f)$ denotes the n-th induced homotopy homomorphism and $H_n(f)$ denotes the n-th induced homology homomorphism.

Since X is a co-H-space, $\pi_1(X)$ is free, and so there is a wedge of circles S with $\pi_1(X) \approx \pi_1(S)$. We choose a map $u : X \to S$ such that $\pi_1(u) : \pi_1(X) \to \pi_1(S)$

is an isomorphism. (Since S is an Eilenberg–MacLane space $K(\pi_1(X), 1)$, u could be the classifying map of the universal cover $\widetilde{X} \to X$.) We let $p_1 : X \vee S \to X$ and $p_2 : X \vee S \to S$ be the projections.

PROPOSITION 6.1. *The following are equivalent:*

(1) *For every space Z, $a \in [X, Z]$, and $b, c \in [S, Z]$,*

$$(a + bu) + cu = a + (bu + cu).$$

(2) *There exists a comultiplication μ on S such that $u : (X, \varphi) \to (S, \mu)$ is a co-H-map and there exists a $\psi : X \to X \vee S$ with $p_1\psi = 1$ and $p_2\psi = u$ which satisfies*

$$(\psi \vee 1)\psi = (1 \vee \mu)\psi.$$

PROOF. (1) $\Rightarrow$ (2): We first show the existence of μ such that u is a co-H-map without assuming (1). A map $\mu : S \to S \vee S$ is determined by $\pi_1(\mu)$, so we define μ by

$$\pi_1(\mu) = \pi_1(u \vee u)\pi_1(\varphi)\pi_1(u)^{-1}.$$

Clearly μ is a comultiplication and $u : (X, \varphi) \to (S, \mu)$ is a co-H-map. Now set $\psi = (1 \vee u)\varphi$. Then with $Z = X \vee S \vee S$, $a = i_1$, $b = i_2$ and $c = i_3$ (the three inclusions), the condition in (1) implies $(\psi \vee 1)\psi = (1 \vee \mu)\psi$.

(2) $\Rightarrow$ (1): If $\nabla : Z \vee Z \vee Z \to Z$ is the folding map then

$$(a + bu) + cu = \nabla(a \vee b \vee c)(\psi \vee 1)\psi \quad \text{and}$$
$$a + (bu + cu) = \nabla(a \vee b \vee c)(1 \vee \mu)\psi.$$

□

The map $\psi : X \to X \vee S$ with $(\psi \vee 1)\psi = (1 \vee \mu)\psi$ in (2) is called an *associative cooperation* of S on X. We will call either condition of Proposition 6.1 the *Berstein–Dror condition*. Then Berstein and Dror proved the following

THEOREM 6.2 ([12, Theorem 1.5]). *Let (X, φ) be a co-H-space and $u : X \to S$ a map which induces an isomorphism of fundamental groups. If (X, φ) satisfies the Berstein–Dror condition, then the Ganea conjecture holds for X, i.e. there is a 1-connected space Y such that $X \equiv Y \vee S$.*

PROOF. We sketch the proof in steps, omitting some details due to space limitations.

Step 1. A *retract* of a space Y is a triple (A, i, r) consisting of a space A and maps $i : A \to Y$ and $r : Y \to A$ such that $ri = 1$. An *idempotent* of Y is a map $e : Y \to Y$ such that $e^2 = e$. It is well known that there is a one-one correspondence between retracts of Y and idempotents of Y. This is seen as follows: If (A, i, r) is a retract of Y, then $e = ir$ is an idempotent of Y. Conversely, given an idempotent $e : Y \to Y$, we define a space Image e as the (homotopy) direct limit of $Y \xrightarrow{e} Y \xrightarrow{e} Y \xrightarrow{e} \cdots$ [29, p. 127]. Then the diagram

$$\begin{array}{ccccccc}
Y & \xrightarrow{1} & Y & \xrightarrow{1} & Y & \longrightarrow & \cdots \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \\
Y & \xrightarrow{e} & Y & \xrightarrow{e} & Y & \longrightarrow & \cdots \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \\
Y & \xrightarrow{1} & Y & \xrightarrow{1} & Y & \longrightarrow & \cdots
\end{array}$$

gives rise to maps $p_e : Y \to \text{Image}\, e$ and $i_e : \text{Image}\, e \to Y$ such that $i_e p_e = e$ and $p_e i_e = 1$. Thus $(\text{Image}\, e, i_e, p_e)$ is a retract of Y. For more details, see [36, §5] and [32].

Step 2. Now assume the hypothesis of Theorem 6.2. Then there is a map $f : S \to X$ such that $\pi_1(f) = \pi_1(u)^{-1}$. Therefore $uf = 1$ and so (S, f, u) is a retract of X. Then $e = fu$, the corresponding idempotent, is called the *canonical idempotent* of the co-H-space X. As in the proof of Proposition 6.1, there is a comultiplication $\mu : S \to S \vee S$ such that $u : (X, \varphi) \to (S, \mu)$ is a co-H-map. It now easily follows that $f : (S, \mu) \to (X, \varphi)$ is also a co-H-map. Consequently $e = fu : (X, \varphi) \to (X, \varphi)$ is a co-H-map.

Step 3. Let $e \in [X, X]$ be the canonical idempotent. If for every $a \in [X, X]$, there is a unique $x \in [X, X]$ such that $x + e = a$, then e is called *loop-like on the right.* (Note that we cannot use Proposition 1.13 to conclude that $[X, X]$ is a loop since X is not 1-connected.) The following is proved in [36, Theorem 6.1]: If e is loop-like on the right, then there is an idempotent $d \in [X, X]$ such that $d + e = 1$ and $X \equiv \text{Image}\, d \vee \text{Image}\, e$.

We make some remarks about the proof of this. The existence of $d : X \to X$ such that $d + e = 1$ is a consequence of e being loop-like on the right. We see that d is an idempotent as follows: $e = (d + e)e = de + e$, since e is a co-H-map. Therefore $de = 0$, and so $d = d(d + e) = d^2 + de = d^2$. Hence d is an idempotent. We omit the proof of the assertion that $X \equiv \text{Image}\, d \vee \text{Image}\, e$, but attempt to make it plausible by indicating an analogous, easily proved result in linear algebra. Let V be a vector space and $P : V \to V$ an idempotent linear transformation (i.e. $P^2 = P$). By setting $Q = 1 - P : V \to V$, we obtain an idempotent Q such that $Q + P = 1$. Then it is easy to show that $V \approx \text{Image}\, Q \oplus \text{Image}\, P$. The isomorphism is determined by the inclusions $\text{Image}\, Q \subseteq V$ and $\text{Image}\, P \subseteq V$. Returning to the co-H-space X with idempotents e and d, we have that $X \equiv \text{Image}\, d \vee \text{Image}\, e$, where the homotopy equivalence $\text{Image}\, d \vee \text{Image}\, e \to X$ is analogously defined by the inclusions $i_d : \text{Image}\, d \to X$ and $i_e : \text{Image}\, e \to X$. We show next that this decomposition of X yields the Ganea conjecture for X. For this it suffices to prove that Image e is a wedge of circles and that Image d is 1-connected. Now $\pi_i(\text{Image}\, e) = \text{Image}\, \pi_i(e)$ since π_i commutes with direct limits. But $e = fu$ factors through S, and hence Image e is a wedge of circles. Next note that $\pi_1(e) = 1$ by construction and so $H_1(e) = 1$. But we infer from $d + e = 1$ that $H_1(d) + H_1(e) = 1$. Therefore $H_1(d) = 0$ and thus $H_1(\text{Image}\, d) = \text{Image}\, H_1(d) = 0$. But Image d, being a retract of X, is a co-H-space. Hence $\pi_1(\text{Image}\, d)$ is a free group whose abelianization $H_1(\text{Image}\, d) = 0$. Thus $\pi_1(\text{Image}\, d) = 0$, and we conclude that Image d is 1-connected.

Step 4. To complete the proof we show that the hypothesis of the theorem implies that the canonical idempotent $e \in [X, X]$ is loop-like on the right. Let $\mu : S \to S \vee S$ be a comultiplication such that u and f are co-H-maps (Step 2). For any space Z and $\alpha, b, c \in [S, Z]$,

$$(\alpha + (b + c))u = ((\alpha + b) + c)u$$

by hypothesis. Therefore, since $u^* : [S, Z] \to [X, Z]$ is a monomorphism, $[S, Z]$ is associative. By Proposition 1.6, μ is associative. It follows that the group $[S, S]$ together with $\mu_* : [S, S] \to [S, S \vee S] = [S, S] * [S, S]$ (the free product, written additively) is an associative co-H-object in the category of groups. By a theorem of Kan, there is a set of free generators $\{\alpha_i\}$ of $[S, S]$ such that $\mu_*(\alpha_i) = \alpha_i' + \alpha_i''$ [42]. Define a left inverse $\lambda \in [S, S]$ for μ by $\lambda_*(\alpha_i) = -\alpha_i$. Then with this inverse, (S, μ) is a cogroup, and so $[S, Z]$ is a group for all spaces Z. Now $f \in [S, X]$ and $u^*(f) = e \in [X, X]$. Set $e' = u^*(-f)$. Then $e' + e = 0 = e + e'$. We show that e is loop-like on the right. Given $a \in [X, X]$, let $x = a + e'$. Then, using the Berstein–Dror condition, we have

$$x + e = (a + e') + e = a + (e' + e) = a.$$

For uniqueness, suppose that $x + e = y + e$. Then

$$x = x + (e + e') = (x + e) + e' = (y + e) + e' = y + (e + e') = y.$$

Thus e is loop-like on the right. This completes the sketch of the proof. □

Since every associative co-H-space satisfies the Berstein–Dror condition, the Ganea conjecture holds for every associative co-H-space.

To our knowledge, the Ganea conjecture is not known for arbitrary co-H-spaces.

7. Rational homotopy of co-H-spaces

In this section we use methods of rational homotopy theory to obtain information about rational co-H-spaces (and rational information about co-H-spaces). We assume some familiarity with rational homotopy, though we will summarize the Quillen theory below. Recall that a rational space is one whose homotopy groups are vector spaces over $\mathbb{Q}$. We denote the rationalization of a nilpotent space X of finite type by $X_{\mathbb{Q}}$.

We begin with a result of Scheerer which uses the Berstein–Scheerer coalgebra of §2.

PROPOSITION 7.1 ([51, Corollary 2, p. 68]). *If Y is a 2-connected, rational, associative co-H-space, then Y is co-H-equivalent to a suspension.*

PROOF. Let $B = B_*(Y)$ be the Berstein–Scheerer coalgebra of Y with $\mathbb{Q}$ coefficients. Then Quillen has shown that there is a 1-connected space Z such that $H_*(Z; \mathbb{Q})$ is isomorphic to B as a coalgebra [48]. Thus $B_*(\Sigma Z) \approx H_*(Z;\mathbb{Q}) \approx B_*(Y)$ as coalgebras. Consequently, Y is co-H-equivalent to ΣZ. □

This result is believed to be true for 1-connected Y, but no proof has yet appeared.

We next consider an early result on rational co-H-spaces. In 1961 Berstein proved that a 1-connected finite co-H-space X is equivalent modulo the class of finite groups to a wedge of spheres V, i.e. there is a map from X to V whose induced homotopy homomorphism has finite kernel and cokernel [9, Corollary 2.3]. With the later appearance of localization, the conclusion can be restated as follows: X is rationally equivalent to V (i.e. $X_{\mathbb{Q}} \equiv V_{\mathbb{Q}}$). This result has been generalized by Toomer [59, Corollary 13], Scheerer [51, p. 67] and Henn [33, p. 167] by weakening the finiteness and connectivity hypotheses. We state the result due to Henn which is the most general of these.

THEOREM 7.2 ([33, Theorem, p. 167]). *Let Y be a co-H-space such that $\pi_n(Y)$ is a rational vector space for all $n \geqslant 2$. Then Y is homotopically equivalent to a wedge of circles and rational spheres of dimension $\geqslant 2$.*

PROOF. The theorem does not assume that Y is 1-connected, but to shorten the exposition we will assume so. Thus Y is a 1-connected, rational co-H-space, and we show that

$$Y \equiv \bigvee_{i\geqslant 1} S_{\mathbb{Q}}^{m_i}, \quad \text{where } 2 \leqslant m_1 \leqslant m_2 \leqslant \cdots.$$

We proceed by a number of steps.

Step 1. $Y \equiv \bigvee_{i\geqslant 1} S_{\mathbb{Q}}^{m_i} \Leftrightarrow$ the Hurewicz homomorphism $h_Y : \pi_*(Y) \to H_*(Y)$ is an epimorphism. To prove '$\Leftarrow$', choose a collection of elements $\{\alpha_i\} \subseteq \pi_*(Y)$ such that $\{h_Y(\alpha_i)\}$ is a basis of $H_*(Y)$. Then the α_i determine a map

$$\bigvee_{i\geqslant 1} S_{\mathbb{Q}}^{m_i} \to Y$$

which is a homology isomorphism. The other implication is clear.

Step 2. For any 1-connected rational space Z,

$$\Sigma Z \equiv \bigvee_{i\geqslant 1} S_{\mathbb{Q}}^{p_i}, \quad 3 \leqslant p_1 \leqslant p_2 \leqslant \cdots.$$

We choose a homology decomposition of Z [35, Chapter 8] consisting of a nested sequence of 1-connected subcomplexes Z_n of Z and maps of Moore spaces

$$f_n : M(H_{n+1}, n) \to Z_n,$$

where $H_{n+1} = H_{n+1}(Z)$, such that

$$Z = \bigcup_{n\geqslant 2} Z_n,$$

Z_{n+1} is the mapping cone of f_n and $f_{n*} = 0$ on reduced homology. But the last statement implies $\Sigma f_n = 0$ [33, p. 165]. Thus by Proposition 3.2,

$$\Sigma Z \equiv \bigvee_{n\geqslant 2} M(H_{n+1}, n+1).$$

But each $M(H_n, n)$ is a wedge of $S^n_{\mathbb{Q}}$'s, and so the result follows.

Step 3. $\Sigma\Omega Y \equiv \bigvee_{i\geqslant 1} S^{m_i}_{\mathbb{Q}}$. Since ΩY is a connected, rational H-space, it is a product of Eilenberg–MacLane spaces,

$$\Omega Y \equiv \prod_{n\geqslant 1} K(\pi_n, n), \quad \text{where } \pi_n = \pi_n(\Omega Y).$$

Then, using the equivalence $\Sigma(A \times B) \equiv \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B)$, we have

$$\begin{aligned}\Sigma\Omega Y &\equiv \Sigma\Big(K(\pi_1, 1) \times \prod_{n\geqslant 2} K(\pi_n, n)\Big)\\ &\equiv \Sigma K(\pi_1, 1) \vee \Sigma\Big(\prod_{n\geqslant 2} K(\pi_n, n)\Big) \vee \Sigma\Big(K(\pi_1, 1) \wedge \prod_{n\geqslant 2} K(\pi_n, n)\Big).\end{aligned}$$

By Step 2, the second and third summands are equivalent to wedges of rational spheres. We show that the first summand $\Sigma K(\pi_1, 1)$ is equivalent to a wedge of rational spheres. If π_1 is finite-dimensional, we prove this by induction on the dimension of π_1. If $\dim \pi_1 = 1$, $K(\pi_1, 1) = S^1_{\mathbb{Q}}$ and so $\Sigma K(\pi_1, 1) = S^2_{\mathbb{Q}}$. Now assume the result is true for all vector spaces of dimension $\leqslant n$ and let π_1 be $(n+1)$-dimensional. Then $\pi_1 = V \oplus \mathbb{Q}$, where V is n-dimensional. Therefore $K(\pi_1, 1) \equiv K(V, 1) \times K(\mathbb{Q}, 1)$ and so

$$\Sigma K(\pi_1, 1) \equiv \Sigma K(V, 1) \vee \Sigma K(\mathbb{Q}, 1) \vee \Sigma\big(K(V, 1) \wedge K(\mathbb{Q}, 1)\big).$$

The first summand is a wedge of rational spheres by assumption, the second is $S^2_{\mathbb{Q}}$ and the third is a wedge of rational spheres by Step 2. This completes the induction. If π_1 is an arbitrary rational vector space, then a direct limit argument using Step 1 shows that $\Sigma K(\pi_1, 1)$ is equivalent to a wedge of rational spheres. This completes Step 3.

Step 4. We now prove the theorem. By Step 3, $\Sigma\Omega Y$ is equivalent to a wedge of rational spheres. By Step 1, $h_{\Sigma\Omega Y} : \pi_*(\Sigma\Omega Y) \to H_*(\Sigma\Omega Y)$ is an epimorphism. But Y is a coretraction of $\Sigma\Omega Y$ (Theorem 4.1). From this it easily follows that $h_Y : \pi_*(Y) \to H_*(Y)$ is an epimorphism. By Step 1, Y is equivalent to a wedge of rational spheres. $\square$

This proof follows [33] and can be extended to the non-1-connected case. Note that the 1-connected case together with the Ganea conjecture (§6) would imply Theorem 7.2. Note too that the equivalence between Y and the rational spheres is not asserted to be a co-H-map (and usually is not).

This theorem has many consequences. We discuss some of them next. For the remainder of this section all spaces will either be 1-connected, finite complexes, usually denoted X or $\overline{X}$, or 1-connected rational spaces with finite dimensional total homology, usually denoted by Y. All vector spaces and Lie algebras will be graded and over $\mathbb{Q}$.

We next recall some basic facts about the Quillen minimal Lie algebra L_Z of a space Z. It is a free Lie algebra $L_Z = \mathbb{L}(V)$ on a vector space V with differential d of degree -1 such that $H_*(L_Z) = H_*(\mathbb{L}(V)) = \pi_*(\Omega Z) \otimes \mathbb{Q}$, the rational homotopy Lie algebra. In addition, $V \approx s^{-1}\widetilde{H}_*(Z;\mathbb{Q})$, the desuspension of the reduced, rational homology of Z, and the quadratic part of the differential d of L_Z is determined by the coalgebra structure of $H_*(Z;\mathbb{Q})$. Furthermore, there is the notion of homotopy for two homomorphisms $L_Z \to L_{\overline{Z}}$ of Quillen minimal algebras. For more details, see [57].

Now let X be a finite co-H-space. By Theorem 7.2, $X_{\mathbb{Q}} \equiv S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_r+1}$, for unique integers $1 \leqslant n_1 \leqslant \cdots \leqslant n_r$. The integers $(n_1,\ldots,n_r)$ are called the *type* of the finite co-H-space X. (The type of a rational co-H-space is similarly defined.) Then $L_X = \mathbb{L}(V)$, where V has a basis $x_1,\ldots,x_r$ with degree $|x_i| = n_i$. We then write $L_X = \mathbb{L}(V) = \mathbb{L}(x_1,\ldots,x_r)$. Moreover, $d = 0$ since d is quadratic and the coalgebra structure of $H_*(X;\mathbb{Q})$ is trivial (Proposition 2.3). If X and $\overline{X}$ are finite co-H-spaces and $\alpha,\beta : L_X \to L_{\overline{X}}$ are homotopic homomorphisms, then $\alpha = \beta$ since α and β induce the same homology homomorphism from $H_*(L_X) = L_X$ to $H_*(L_{\overline{X}}) = L_{\overline{X}}$. Thus the following result holds (cf. [51, Proposition 2.7]).

PROPOSITION 7.3. *If X is a finite co-H-space of type $(n_1,\ldots,n_r)$, then $L_X = \mathbb{L}(x_1,\ldots,x_r)$ is the free Lie algebra on $x_1,\ldots,x_r$, where $|x_i| = n_i$, and the differential $d = 0$. For finite co-H-spaces X and $\overline{X}$ the set of homotopy classes $[L_X, L_{\overline{X}}]$ equals the set of Lie algebra homomorphisms Hom$(L_X, L_{\overline{X}})$.*

If X and $\overline{X}$ are co-H-spaces and $L_X = \mathbb{L}(V)$ and $L_{\overline{X}} = \mathbb{L}(\overline{V})$ then their coproduct (in the category of free Lie algebras) exists and is given by $L_X \sqcup L_{\overline{X}} = \mathbb{L}(V \oplus \overline{V}) = L_{X\vee\overline{X}}$.

Now let X be a finite co-H-space of type $(n_1,\ldots,n_r)$ with $L_X = \mathbb{L}(V) = \mathbb{L}(x_1,\ldots,x_r)$, $|x_i| = n_i$. Let $L'_X = \mathbb{L}(V')$ be another copy of L_X with V' isomorphic to V, and write $L'_X = \mathbb{L}(x'_1,\ldots,x'_r)$ with $|x'_i| = n_i$. Define $\pi,\pi' : L_X \sqcup L_{X'} \to L_X$ by $\pi(x_i) = x_i$, $\pi(x'_i) = 0$, $\pi'(x_i) = 0$ and $\pi'(x'_i) = x_i$. A homomorphism $\alpha : L_X \to L_X \sqcup L_{X'}$ is called a (*Lie algebra*) *comultiplication* if $\pi\alpha = 1$ and $\pi'\alpha = 1$. Note that (L_X, α) is a co-H-object in the category of free Lie algebras. For $i = 1,\ldots,r$, we have

$$\alpha(x_i) = x_i + x'_i + P(x_i),$$

where $P(x_i)$ is a linear combination of brackets of elements $x_1,\ldots,x_r,x'_1,\ldots,x'_r$ with no bracket containing only elements from $x_1,\ldots,x_r$ or only elements from $x'_1,\ldots,x'_r$. Equivalently, $P : V \to \mathbb{L}(V) \sqcup \mathbb{L}(V') = \mathbb{L}(V \oplus V')$ is a linear transformation such that $\pi P = 0$ and $\pi' P = 0$. We call P a *perturbation* of the comultiplication α.

PROPOSITION 7.4 ([4, §2]). *Let X be a finite co-H-space with Quillen minimal algebra $L_X = \mathbb{L}(V)$, where $V = s^{-1}\widetilde{H}_*(X;\mathbb{Q})$. Then there is a bijection between:*

(1) *The set of homotopy classes of comultiplications of* $X_{\mathbb{Q}}$;
(2) *The set of Lie algebra comultiplications on* $L_X = \mathbb{L}(V)$;
(3) *The set of perturbations* $V \to \mathbb{L}(V \oplus V')$.

The bijection between the sets in (1) and (2) follows from the basic result that for spaces X and $\overline{X}$, $[X_{\mathbb{Q}}, \overline{X}_{\mathbb{Q}}]$ and $[L_X, L_{\overline{X}}]$ are in one-one correspondence [57, Chapter 3].

Proposition 7.4 enables us to analyze comultiplications of the rationalization of a finite co-H-space or equivalently, of a rational co-H-space, by a purely algebraic investigation of perturbations. This was carried out by Arkowitz and Lupton and questions of associativity, commutativity and inverses for rational co-H-spaces were considered in [4]. We state one of these results for associativity. If φ and ψ are two comultiplications of Y, then φ is equivalent to ψ means that (Y, φ) and (Y, ψ) are co-H-equivalent.

THEOREM 7.5 ([4, Corollary 3.19]). *Let Y be a rational co-H-space of type $(n_1, \ldots n_r)$ with all n_i odd. Then any two associative comultiplications of Y are equivalent.*

We also give an example regarding inverses. The details can easily be worked out.

EXAMPLE 7.6 ([4, Example 5.7]). Let Y be a rational co-H-space of type (n_1, n_2, n_3) with $n_3 = 2n_1 + n_2$. Define a comultiplication α on $L_Y = \mathbb{L}(x_1, x_2, x_3)$, by setting $P(x_1) = 0 = P(x_2)$ and $P(x_3) = [x_1, [x_1, x_2']]$. Then the homomorphisms λ, $\rho : L_Y \to L_Y$ defined by $\lambda(x_i) = \rho(x_i) = -x_i$, $i = 1, 2$, $\lambda(x_3) = -x_3 - [x_1, [x_1, x_2]]$ and $\rho(x_3) = -x_3 + [x_1, [x_1, x_2]]$ are left and right inverses respectively of α with $\lambda \neq \rho$. Thus Y is a rational space with a comultiplication φ such that the left inverse and the right inverse for φ are different. For an example of this for a finite co-H-space, see [4, Example 6.13].

Finally we mention that the study of perturbations of comultiplications of rational spaces can be carried over to finite co-H-spaces which are wedges of spheres, though the presence of torsion in the homotopy groups of a wedge of spheres gives rise to additional difficulties. See [4, §6] for details.

8. Miscellaneous results

In this section we describe or state without proof a number of miscellaneous results on co-H-spaces which do not fit into earlier sections.

Co-H-Maps. (i) We begin with a result of Berstein and Hilton which is the counterpart of Theorem 5.1(1) for maps.

PROPOSITION 8.1 ([18, Theorem B]). *Let A be a finite complex of dimension $\leqslant 3n - 2$ and let B be an $(n-1)$-connected space of finite type, $n \geqslant 1$. Then every co-H-map $(\Sigma A, \sigma_A) \to (\Sigma B, \sigma_B)$ is a suspension.*

Now apply the dual Hopf construction of §5 to the comultiplication σ_B to obtain $H(\sigma_B) : \Omega\Sigma B \to \Omega(\Sigma B \flat \Sigma B)$. Then (as noted in [18, §5]) Proposition 8.1 can be interpreted as asserting the exactness of the following sequence under the hypothesis of Proposition 8.1:

$$[A,B] \xrightarrow{\Sigma} [\Sigma A, \Sigma B] \approx [A, \Omega\Sigma B]$$
$$\xrightarrow{H(\sigma_B)_*} [A, \Omega(\Sigma B \flat \Sigma B)] \approx [\Sigma A, \Sigma B \flat \Sigma B]$$

since $\alpha \in [\Sigma A, \Sigma B]$ is a co-H-map $\Leftrightarrow H(\sigma_B)\overline{\alpha} = 0$ (see §5). This exact sequence is a generalization of an extract of the well-known EHP sequence of G. Whitehead.

(ii) In [31] Harper considers a p-local space A and a map $f : \Sigma^2 A \to S^{2n+1}_{(p)}$, where p is an odd prime and $S^{2n+1}_{(p)}$ is the p-localization of S^{2n+1}. As in §3 let $h : \Omega S^{2n+1} \to \Omega S^{2np+1}$ denote any map which gives the Hilton–Hopf invariant. That is, h is the composition

$$\Omega S^{2n+1} \xrightarrow{\Omega\sigma} \Omega(S^{2n+1} \vee S^{2n+1}) \equiv \prod_i \Omega S^{2nr_i+1} \xrightarrow{q} \Omega S^{2np+1},$$

where σ is the comultiplication of S^{2n+1} and q is a projection. Let $h_{(p)} : \Omega S^{2n+1}_{(p)} \to \Omega S^{2np+1}_{(p)}$ be the p-localization of h and let $\overline{f} : \Sigma A \to \Omega S^{2n+1}_{(p)}$ be the adjoint of f.

PROPOSITION 8.2 ([31, Theorem 1]). *Let A be a p-local space and $f : \Sigma^2 A \to S^{2n+1}_{(p)}$. Then f is a co-H-map $\Leftrightarrow h_{(p)}\overline{f} = 0$.*

In addition, Harper has considered co-A-maps $f : \Sigma A \to Y$, where A is any space and Y is an associative co-H-space. These are co-H-maps with an added condition (namely, the vanishing of the co-A-deviation of f in $[\Sigma A, \Omega(Y \vee Y \vee Y)]$) which ensures that the induced comultiplication on the mapping cone of f is associative. The theory of co-A-deviations of a co-H-map is developed in [31, §2] in analogy to the A-deviations of an H-map. A necessary and sufficient condition is then given for $\Sigma^2 A \to S^{2n+1}_{(p)}$ to be a co-A-map, where p is an odd prime and A is p-local [31, Theorem 2].

(iii) Shi considers the set $[X, Y]_{\text{co-H}}$ of co-H-maps from a commutative cogroup X into a co-H-space Y and shows that it is a subgroup of $[X, Y]$ ([54] and [55]). An upper bound for the rank of this subgroup is given in terms of the Betti numbers of X and Y.

(iv) In [2, Theorem 14] the co-H-maps $M(\mathbb{Z}_m, 2) \to M(\mathbb{Z}_n, 2)$ of Moore spaces are determined for every comultiplication of $M(\mathbb{Z}_m, 2)$ and every comultiplication of $M(\mathbb{Z}_n, 2)$.

Sets of comultiplications. (i) If (X, φ) is a cogroup, let $\mathcal{C}(X) \subseteq [X, X \vee X]$ be the set of comultiplications of X. Then the sequence of groups

$$0 \to [X, X \flat X] \xrightarrow{i_*} [X, X \vee X] \xrightarrow{j_*} [X, X \times X] \longrightarrow 0$$

is exact. Since $\mathcal{C}(X) = j_*^{-1}(\Delta)$ and $j_*^{-1}(0) \approx [X, X \flat X]$, it follows that there is a bijection from $\mathcal{C}(X)$ to $[X, X \flat X]$. We see that this is also true if X is a 1-connected co-H-space by modifying the above argument with loops replacing groups. This was done by Navarro [46]. We now specialize to the case $X = \Sigma A$. By the Hilton–Milnor theorem [61, Chapter XI, §6], $\Omega(\Sigma A \vee \Sigma A)$ has the homotopy type of an infinite product

$$\prod_{i \geqslant 1} \Omega \Sigma P_i,$$

where $P_1 = A = P_2$, $P_3 = A \wedge A$, $P_4 = A \wedge A \wedge A = P_5$, etc., the factors P_i being determined by basic Lie brackets. Thus

$$\Omega(\Sigma A \flat \Sigma A) \equiv \prod_{i \geqslant 3} \Omega \Sigma P_i$$

and we obtain the following result due to Naylor.

PROPOSITION 8.3 ([47, Corollary 2]). *$\mathcal{C}(\Sigma A)$ is in one-one correspondence with*

$$\bigoplus_{i \geqslant 3} [\Sigma A, \Sigma P_i].$$

Naylor then gives necessary and sufficient conditions in terms of the Betti numbers of A for $\mathcal{C}(\Sigma A)$ to be finite [47, Theorem 3].

(ii) For a rational co-H-space Y, the finiteness of certain subsets of $\mathcal{C}(Y)$ is considered in [4]. In terms of the type of Y, necessary and sufficient conditions are given for (1) the existence of infinitely many associative comultiplications of Y [4, Theorem 3.14] (2) the existence of infinitely many nonassociative comultiplications of Y [4, Theorem 3.15] (3) the existence of infinitely many commutative comultiplications of Y [4, Proposition 4.2]. Similar results are obtained for wedges of spheres [4, §6].

(iii) The relation of co-H-equivalence is an equivalence relation on the set $\mathcal{C}(X)$. We let $\widetilde{\mathcal{C}}_a(X)$ denote the set of equivalence classes of associative comultiplications of X and $\widetilde{\mathcal{C}}_{ac}(X)$ denote the set of equivalence classes of associative and commutative comultiplications of X. Then Arkowitz and Lupton proved the following result.

THEOREM 8.4 ([5, Theorem 6.1]). *Let X be a finite, 1-connected complex which is a cogroup of type $(n_1, \dots, n_r)$.*

(1) *If, for every $i = 1, \dots, r$, we have that $n_i \neq n_j + n_k$ for $j \neq k$ and that $n_i \neq 2n_j$ for n_j even, then $\widetilde{\mathcal{C}}_a(X)$ is finite.*

(2) *$\widetilde{\mathcal{C}}_{ac}(X)$ is always finite.*

Operations with co-H-spaces. We consider some standard constructions on the homotopy category of topological spaces which carry co-H-spaces to co-H-spaces. One could determine (but we do not) which properties of co-H-spaces are preserved by these constructions. Clearly if X and Y are co-H-spaces, so is $X \vee Y$. If X is a co-H-space and Y is any space, then $X \wedge Y$ is a co-H-space. For, if Z is any space, then $[X \wedge Y, Z] \approx [X, Z^Y]$, where Z^Y is the space of all maps (not homotopy classes) $Y \to Z$ suitably topologized.

Since $[X, Z^Y]$ has a binary operation with unit $*$, $X \wedge Y$ is a co-H-space by Proposition 1.6(1). Note that if Y is a co-H-space, then Z^Y is an H-space [61, Chapter 3, Theorem 5.16].

Now let X be a 1-connected N dimensional complex and let $H_k = H_k(X)$. Recall that a homology decomposition of X consists of a sequence of mapping cone inclusions

$$X_2 \xrightarrow{i_2} X_3 \xrightarrow{i_3} \cdots \xrightarrow{i_{N-1}} X_N,$$

where X_{k+1} is the mapping cone of some $f_k : M(H_{k+1}, k) \to X_k$, and maps $g_k : X_k \to X$ compatible with the i_k such that g_{k*} is an isomorphism in dimensions $\leqslant k$ [35, Chapter 8]. Then Curjel proved the following result.

PROPOSITION 8.5 ([22, Lemma 2.3, Theorem I]). *If X is a 1-connected, finite dimensional co-H-space and $\{X_k; i_k, f_k, g_k\}$ is a homology decomposition of X, then each X_k admits a comultiplication such that all $i_k : X_k \to X_{k+1}$ and all $g_k : X_k \to X$ are co-H-maps. Furthermore, the elements f_k in the group $[M(H_{k+1}, k), X_k]$ have finite order.*

Finally, we consider localization and completion. If P is any set of primes, let X_P denote the P-localization of the nilpotent space X. If X is a 1-connected co-H-space, then it follows from standard functorial properties of P-localization that X_P is a co-H-space such that the canonical map $X \to X_P$ is a co-H-map. However, we cannot replace 1-connectedness with nilpotence in this result. For example, S^1_P is not a co-H-space. This follows since $S^1 = K(\mathbb{Z}, 1)$, and so $S^1_P = K(\mathbb{Z}_P, 1)$, where $\mathbb{Z}_P$ is the integers localized at P. But $\pi_1(S^1_P) = \mathbb{Z}_P$ is not free, and so by Proposition 2.1, S^1_P is not a co-H-space. We note that McGibbon has proved that the existence of a comultiplication on a space is a generic property with respect to the Mislin genus. That is, if X and Y are nilpotent spaces of finite type such that $X_{(p)} \equiv Y_{(p)}$ for all primes p and if X admits a comultiplication, then so does Y [44, Corollary 5.1].

The situation regarding completion is different from localization. It has been shown by McGibbon that $\widehat{S}^n_p$, the p-adic completion of S^n, is not a co-H-space. In fact, $\widehat{S}^n_p$ has nontrivial cup products in rational cohomology of arbitrary length [45].

Potpourri. (i) *Nilpotency of co-H-spaces.* If (X, φ) is a cogroup with inverse $\iota : X \to X$, then define the *(2-fold) commutator* $\theta_2 \in [X, X \vee X]$ to be the composition

$$X \xrightarrow{\varphi} X \vee X \xrightarrow{\varphi \vee \varphi} X \vee X \vee X \vee X \xrightarrow{1 \vee 1 \vee \iota \vee \iota} (X \vee X) \vee (X \vee X) \xrightarrow{\nabla} X \vee X,$$

where ∇ is the folding map for $X \vee X$. The *n-fold commutator* $\theta_n \in [X, {}^nX]$, where ${}^nX = X \vee \cdots \vee X$ (n times), is defined inductively as follows: Assume $\theta_{n-1} \in [X, {}^{n-1}X]$ already defined and let θ_n be the composition

$$X \xrightarrow{\theta_2} X \vee X \xrightarrow{1 \vee \theta_{n-1}} X \vee {}^{n-1}X = {}^nX.$$

Note that $\theta_n = (i_1, \ldots, (i_{n-1}, i_n) \ldots)$, the commutator in the group $[X, {}^nX]$ of the n inclusions $i_1, \ldots, i_{n-1}, i_n$ of X into ${}^nX$. We let $\theta_1 \in [X, X]$ be the identity map. Then

Berstein and Ganea defined the *conilpotency class* $\operatorname{conil}(X,\varphi)$ of the cogroup (X,φ) to be the least integer $n \leqslant \infty$ such that $\theta_{n+1} = 0$ [13, §1]. Clearly $\operatorname{conil}(X,\varphi) \leqslant n \Leftrightarrow$ for every space Y, the group $[X,Y]$ has nilpotency class $\leqslant n$. For a space A, let $\cup$-long A, be the least integer n, $0 \leqslant n \leqslant \infty$, such that for any coefficient ring, the cup product of any $n+1$ positive dimensional cohomology elements vanishes. Let $\operatorname{cat} A$ denote the Lusternik–Schnirelmann category of A, normalized so that $\operatorname{cat} S^n = 1$.

THEOREM 8.6 ([13, Theorem 5.8, Corollary 6.12]).

$$\cup\text{-long}\, A \leqslant \operatorname{conil}(\Sigma A, \sigma_A) \leqslant \operatorname{cat} A.$$

Examples of spaces have been given for which the inequalities are strict. For other results on the conilpotency class, see [13].

(ii) *Co-H-spaces which are H-spaces.* There are surprisingly few of these. If X is a noncontractible co-H-space and H-space of finite type, then West has shown that $X \equiv S^1$, S^3 or S^7 [60]. More generally, Holzsager has considered such spaces X which are not assumed to be of finite type. He has proved that either $X \equiv S^1$ or X is 1-connected [38, Theorem 1]. In the latter case a complete list of the possibilities for X is given – they are all wedges of Moore spaces [38, Theorem 2].

(iii) *Mod p decomposition of co-H-spaces.* Let X be a co-H-space of finite type and $f : X \to X$ a map. Let p be a prime or 0 and let $\mathbb{F}_p$ denote the field with p elements (with $\mathbb{F}_0 = \mathbb{Q}$). Then for each $i \geqslant 0$, $f_* : H_i(X;\mathbb{F}_p) \to H_i(X;\mathbb{F}_p)$ is a linear transformation which makes $H_i(X;\mathbb{F}_p)$ into an $\mathbb{F}_p[x]$-module. We form the primary decomposition

$$H_i(X;\mathbb{F}_p) = \bigoplus_{p(x)} H_i(X;\mathbf{F}_p)_{p(x)},$$

where $p(x)$ ranges over all monic, irreducible polynomials in $\mathbb{F}_p[x]$, and thence the decomposition

$$H_*(X;\mathbb{F}_p) = \bigoplus_{p(x)} H_*(X;\mathbf{F}_p)_{p(x)}.$$

Then Cooke and Smith show that there is a space $X_{p(x)}$ and a map $f_{p(x)} : X \to X_{p(x)}$ such that

$$f_{p(x)*}|_{H_*(X;\mathbb{F}_p)_{p(x)}} : H_*(X;\mathbb{F}_p)_{p(x)} \to H_*(X_{p(x)};\mathbb{F}_p)$$

is an isomorphism and

$$f_{p(x)*}|_{H_*(X;\mathbb{F}_p)_{q(x)}} = 0$$

for $q(x) \neq p(x)$ [21, Theorem 1.1]. Thus X is p-equivalent to $\bigvee_{p(x)} X_{p(x)}$. Applications and generalizations of this decomposition are given in [21].

(iv) *Extensions of co-H-spaces.* Castellet and Navarro consider the question of classifying the comultiplications on $X \vee Y$, where X and Y are co-H-spaces [19]. In analogy with group theory, they define equivalence classes CHE$(X,\varphi;Y,\psi)$ of co-H-extensions of the co-H-space (X,φ) by the commutative cogroup (Y,ψ). These turn out to be co-H-equivalence classes of comultiplications on $X \vee Y$. They establish a bijection from CHE$(X,\varphi;Y,\psi)$ to the cokernel of the homomorphism $[Y,X] \to [Y, X \flat X]$ which assigns to a map its deviation from being a co-H-map. They use this to investigate the case when X and Y are spheres.

(v) *Commutativity.* The following is a partial converse of Example 1.11.

PROPOSITION 8.7 ([14, Theorem 1]). *Let X be an $(n-1)$-connected complex of dimension $\leqslant 3n-2$, $n \geqslant 1$. If ΣX is commutative, then X admits a comultiplication.*

(vi) *k-fold suspensions.* From §4 we know that X is a co-H-space if and only if the projection $\nu : \Sigma\Omega X \to X$ has a right inverse. Thus Theorem 5.1 is essentially a special case of the following result of Berstein and Ganea.

PROPOSITION 8.8 ([15, Theorem 1.4]). *If X is an $(n-1)$-connected complex and the projection $\nu_k : \Sigma^k\Omega^k X \to X$ has a right inverse, $k \geqslant 1$ then X has the homotopy type of a k-fold suspension provided dimension $X \leqslant 3n-2k-1$ and $n-1 \geqslant k$.*

9. Generalizations

In this section we present a very brief discussion of a few of the generalizations of the theory of co-H-spaces. Some of these consist of working out the theory of co-H-objects, co-H-morphisms, etc. in specific topological categories. We are primarily interested in describing the results and in giving references for further study.

Lusternik–Schnirelmann category. Recall the homotopy theoretic characterization of spaces of normalized (Lusternik–Schnirelmann) category $\leqslant n$. Let $T^n(X) \subseteq X^n$ be the fat wedge and $j : T^n(X) \to X^n$ the inclusion. Then $\operatorname{cat} X \leqslant n-1$ if and only if there is $\varphi : X \to T^n(X)$ such that $j\varphi = \Delta : X \to X^n$, the n-fold diagonal. Clearly $\operatorname{cat} X \leqslant 1 \Leftrightarrow X$ admits a comultiplication. Some of the results of earlier sections for co-H-spaces have generalizations to spaces of cat $\leqslant n$. For a survey of category, see [39].

R-local homotopy. Let $r \geqslant 3$ and $R \subseteq \mathbb{Q}$ be a subring. Denote by $\bar{p}$ the smallest prime in R which is not invertible in R and set $m = r + 2\bar{p} - 4$. We next define a class of spaces CW_r^m. An R-local sphere is just a Moore space $M(R,n)$. In analogy to a CW-complex, an R-local CW-complex is a space built from a point by successively attaching cones by maps defined on R-local spheres. Then CW_r^m consists of based topological spaces X which are R-local CW-complexes of R-dimension $\leqslant m$ such that the $(r-1)$-skeleton $X^{r-1} = *$. Scheerer has proved analogues of some of the rational results of §7 for $X \in \mathrm{CW}_r^m$. In particular, the following are proved [53]: (i) If X is a co-H-space, then X has the homotopy type of a wedge of Moore spaces. (ii) If $H_*(X;R)$ is a free R-module, $r > 3$, and X is a cogroup, then X is co-H-equivalent to a suspension.

In addition, there are some results in [51, A3] on R-local co-H-spaces and the Berstein–Scheerer coalgebra $B_*(X;R)$.

Equivariant homotopy. Let G be a finite group and consider the category G-Top of pointed G-spaces and G-homotopy classes of pointed G-maps. Then G-co-H-spaces and G-cogroups are just co-H-objects and cogroup objects in the category G-Top. Of particular interest here are the appropriate Moore spaces. Kahn has given conditions for their existence and uniqueness up to homotopy [41]. From this it follows that certain Moore G-spaces are G-cogroups. However, Doman has given an example of infinitely many Moore G-spaces of the same type such that only one is a G-co-H-space [24]. Arkowitz and Golasinski describe the set of comultiplications of certain Moore G-spaces in [3, §3]. With regard to general G-co-H-spaces, Doman has proved the coretraction theorem (Theorem 4.1) for the category G-Top [23, Theorem 3.1]. He has also given conditions on a rational G-co-H-space for it to be G-homotopy equivalent to a wedge of Moore G-spaces [23, Theorems B, C, D].

Fibrewise pointed homotopy. For a fixed space B, a fibrewise pointed space consists of a space X and maps $p : X \to B$ and $s : B \to X$ such that $ps = 1$. Fibrewise pointed maps between fibrewise pointed spaces (X, p, s) and (X', p', s') are just continuous maps $f : X \to X'$ such that $p'f = p$ and $fs = s'$. There is a notion of homotopy for fibrewise pointed maps using an appropriate cylinder construction. This gives the fibrewise pointed homotopy category. James has studied co-H-objects – called fibrewise co-H-spaces – in this category. In the case where $p : X \to B$ is a fibration there are results about the relationship between an ordinary co-H-structure on the fibres and a fibrewise co-H-structure on X [40, §2]. Furthermore, James has shown that a sectioned $(q-1)$-sphere bundle over S^n is a fibrewise co-H-space if and only if the q-fold suspension of $\rho_*(\alpha)$ in $\pi_{n+q-1}(S^{2q-1})$ is zero, where $\alpha \in \pi_{n-1}(SO(q))$ is the characteristic element of the bundle and $\rho : SO(q) \to S^{q-1}$ is the evaluation map [40, Proposition 4.1].

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CHAPTER 24

Fibration and Product Decompositions in Nonstable Homotopy Theory

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Contents

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0. Introduction

The subject of this article is a survey of fibration and product decompositions which occur in parts of classical nonstable homotopy theory. A fundamental goal in this subject is an analysis of the structure of the set of pointed homotopy classes of maps $[X, Y]$ from a space X to a space Y.

Without some reasonable hypotheses, standard arguments applied to the set $[X, Y]$ are sometimes cumbersome. For example, the set $[\mathbb{R}P^2, \mathbb{R}P^2]$ is countably infinite although there is an exact sequence of sets $\mathbb{Z}/2\mathbb{Z} \to [\mathbb{R}P^2, \mathbb{R}P^2] \to \mathbb{Z}/2\mathbb{Z}$. This behavior traces back to the fact that maps in this exact sequence are not group homomorphisms. If either (i) X is a suspension given by ΣA or (ii) Y is a loop space given by ΩT, then the set $[X, Y]$ is naturally a group. Furthermore, the groups $[A, \Omega T]$ and $[\Sigma A, T]$ are canonically isomorphic by a standard adjoint functor argument. Thus much of this article will be restricted to spaces X which are suspensions or spaces Y which are loop spaces.

In the case that Y is a loop space ΩT, it is frequently the case that either (1) ΩT is homotopy equivalent to a nontrivial product or (2) ΩT admits a nontrivial fibration; the space T itself might not admit an "interesting" fibration while the space ΩT does. One might also consider more general pointed mapping spaces $\mathrm{map}_*(A, T)$ for which $\mathrm{map}_*(S^1, T)$ is the (pointed) loop space. The n-th homotopy group of $\mathrm{map}_*(A, T)$, $n \geqslant 1$, is isomorphic to the group $[\Sigma^n A, T]$. Thus fibrations for $\mathrm{map}_*(A, T)$ provide information about homotopy groups with coefficients.

Beautiful examples of this type of structure are given by (1) the EHP sequences due to James and Toda [37], [70] and (2) Selick's proof that the p-primary component of $\pi_q S^3$, p an odd prime, is a $\mathbb{Z}/p\mathbb{Z}$-vector space [61]. These types of fibrations and product decompositions occupy Sections 3–5 of this article. Further analogous results provide one of the main points of view in this article.

There are other related product decompositions of certain function spaces with targets given by spheres. Some of these are discussed in Section 5. In addition, stable analogues are given in Section 8. Applications to exponents of the homotopy groups of spheres and related problems are given in Section 6. Some of the main open questions at the prime 2 are also given.

Other related spaces admit certain product decompositions. For example, the loop spaces of double suspensions, at least after localization at p, are usually homotopy equivalent to products. The ur-example here is given by a mod-p^r Moore space which is described in Section 7 if p is odd; mod-2^r Moore spaces are considered in Sections 7, 11, and 12. General decompositions are given in Section 10.

Certain unresolved questions related to 2-primary homotopy theory are discussed in these sections. In addition, the question of the divisibility of the Whitehead square impinges on these product decompositions (or rather the lack of them). Thus a short discussion of this problem is given in Section 9.

Connections between compositions of Whitehead products and James–Hopf invariants have been of interest for over 40 years. The product decompositions alluded to above are, in part, obtained from general p-local product decompositions which follow from the relations between Whitehead products and Hopf invariants. However, the finer structure implicit here is more delicate (at least to the eyes of this author). In order to codify some of this finer information, combinatorial methods are applied. Thus this finer structure can be given in terms of generators and relations in "combinatorially defined" groups and is discussed in Section 13.

Unless otherwise stated spaces and maps are assumed to be in the category of compactly-generated weak Hausdorff spaces with nondegenerate base-points [68]. The general constructions given here fit well in this context. Several theorems in the literature admit proofs within this context although published proofs may not always be stated this way.

1. On some classical theorems

A fundamental feature of nonstable homotopy theory is reflected by one of Serre's theorems.

THEOREM 1.1 ([66]). (1) *If X is a simply-connected space of finite type, then $\pi_i X$ is a finitely generated abelian group for all i.*

(2) *If X is a simply-connected space having reduced homology which is entirely p-torsion, then $\pi_i X$ is a p-torsion abelian group for all i.*

(3) *If X is a simply-connected finite complex with nonvanishing reduced homology, then for each integer n, there is an integer $s_n > n$ such that $\pi_{s_n} X$ is nonzero.*

A beautiful improvement of part (3) is given by

THEOREM 1.2 ([46]). *If X is a simply-connected finite complex with nonvanishing reduced mod-p homology, then for each integer m there is an integer $s_m > m$ such that $\pi_{s_m} X$ has nonvanishing p-torsion subgroup.*

A classical example is

EXAMPLE 1.3 ([2], [71]). If p is an odd prime and $n \geqslant 3$, then $\pi_{kq+n-1} S^n$ contains a $\mathbb{Z}/p^r\mathbb{Z}$-summand for some r with $q = 2p - 2$. If $n = 3$, then $r = 1$.

A particularly nice way to see this result is to map $\Omega_0^3 S^3$ to the space J as given in [11]. Yet another attractive example is

THEOREM 1.4 ([26]). *If $i \geqslant 4$, then $\pi_i S^4$ is nonzero.*

As these theorems illustrate, the homotopy groups of a simply-connected finite complex are "large". Thus one point of view in the subject has been to focus on more specialized spaces such as spheres, Moore spaces, and bouquets of these spaces. Some of the basic tools which are used to study these spaces are given in the next section.

2. Formulae

Consider the bouquet $X_1 \vee X_2$ which is the subspace of the product $X_1 \times X_2$ given by $(X_1 \times e_2) \cup (e_1 \times X_2)$ where e_i is the base-point in X_i. The cofibre of the inclusion of

$X_1 \vee X_2$ in $X_1 \times X_2$ is the smash product $X_1 \wedge X_2$; the n-fold smash product is denoted by

$$X^{(n)} = \underbrace{X \wedge \cdots \wedge X}_{n}.$$

Let $F(X_1, X_2)$ denote the homotopy theoretic fibre of the inclusion of $X_1 \vee X_2$ in $X_1 \times X_2$.

THEOREM 2.1 ([28], [49]). *If X_1 and X_2 are path-connected, then $F(X_1, X_2)$ is homotopy equivalent to $\Sigma(\Omega X_1) \wedge (\Omega X_2)$.*

Thus if X_i are Eilenberg–MacLane spaces, then $X_1 \vee X_2$ frequently has interesting homotopy groups as illustrated next.

EXAMPLE 2.2. If $X_1 = X_2 = \mathbb{C}P^\infty$, then $\Sigma(\Omega X_1) \wedge (\Omega X_2)$ is homotopy equivalent to S^3. Thus there is an isomorphism

$$\pi_q(\mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \to \pi_q S^3, \quad q \geqslant 3.$$

A theorem of Ganea compares certain fibrations and cofibrations as follows: let $p : E \to B$ be a fibration with fibre F. Consider the cofibration $q : E \to E_\cup C(F)$ where the cone of F, $C(F)$, is attached by the inclusion of F in E. Thus $E_\cup C(F)$ is frequently written as E/F.

THEOREM 2.3 ([28]). *If B is path-connected, then the homotopy theoretic fibre of q is $\Sigma(F_\wedge \Omega B)$.*

The structure of $\Sigma\Omega X$ has similar useful properties some of which are used in evaluating secondary cohomology operations. Let $e : \Sigma\Omega X \to X$ be the evaluation map and $F\{e; X\}$ be the homotopy theoretic fibre of e.

THEOREM 2.4 ([4]). *If X is path-connected, then $F\{e; X\}$ is homotopy equivalent to $\Sigma(\Omega X) \wedge (\Omega X)$.*

The suspension of $\Omega\Sigma X$ has important applications.

THEOREM 2.5 ([37], [48]). *If X is path-connected, then there is a homotopy equivalence*

$$\bar\theta : \Sigma\Omega\Sigma X \to \Sigma\Big(\bigvee_{i \geqslant 1} X^{(i)}\Big).$$

Consider (1) the adjoint of $\bar\theta$

$$\theta : \Omega\Sigma X \to \Omega\Sigma\Big(\bigvee_{i \geqslant 1} X^{(i)}\Big),$$

and (2) the collapsing map

$$\pi_n : \bigvee_{i\geqslant 1} X^{(i)} \to X^{(n)}$$

which sends $X^{(i)}$ to the base-point if $i \neq n$ and which is the identity when restricted to $X^{(n)}$. The n-th James–Hopf invariant

$$H_n : \Omega\Sigma X \to \Omega\Sigma X^{(n)}$$

is the composite $\Omega\Sigma(\pi_n) \circ \theta$. This map has ubiquitous applications some of which are discussed in Sections 3, 4, 6, and 10.

The Hilton–Milnor theorem is basic in the subject. A precursor is as follows.

THEOREM 2.6 ([35], [30], [48], [74]). *If X and Y are path-connected, then there is a homotopy equivalence*

$$\Omega\Sigma X \times \Omega\Sigma\Big[Y \vee \bigvee_{i\geqslant 1} \left(X^{(i)} \wedge Y\right)\Big] \to \Omega\Sigma(X \vee Y).$$

Given spaces X_n let Y_n be the product $\prod_{i=1}^{n} X_i$. As X_n is pointed, there is a natural map $Y_n \to Y_{n+1}$. The weak infinite product $\prod Y_n$ is the colimit of the spaces Y_n.

A corollary of Theorem 2.6 is the Hilton–Milnor theorem which is given next.

THEOREM 2.7 ([35], [30], [48], [74]). *If X and Y are path-connected, then there is a homotopy equivalence*

$$\theta : \Omega\Sigma X \times \Omega\Sigma Y \times \prod_{(i,j)\in I} \Omega\Sigma(X^{(i)} \wedge Y^{(j)}) \to \Omega\Sigma(X \vee Y)$$

for a certain choice of index set I.

An elegant proof of this last theorem was given in [30] without specifying the choice of equivalence θ. Certain maps

$$S(i,j) : X_1^{(i)} \wedge X_2^{(j)} \to \Omega\Sigma(X_1 \vee X_2)$$

pervade nonstable homotopy theory and are also used to specify a choice of θ in Theorem 2.7. Namely, there are natural maps

$$E_k : X_k \to \Omega\Sigma(X_1 \vee X_2), \quad k = 1, 2,$$

given by the composite of the natural inclusion $X_k \to X_1 \vee X_2$ followed by the Freudenthal suspension $E : X_1 \vee X_2 \to \Omega\Sigma(X_1 \vee X_2)$. The maps $S(i,j)$ are gotten by choices of iterated Samelson products which are defined as follows. Regard $\Omega\Sigma(X_1 \vee X_2)$ as a group G. There are maps $S : G^{(2)} \to G$ induced by sending (g, h) to $ghg^{-1}h^{-1}$ [60]. The maps $S(i, j)$ are obtained by iterates of S composed with choices of maps E_k. After passage to adjoints, the resulting maps

$$\omega(i, j) : \Sigma X_1^{(i)} \wedge X_2^{(j)} \to \Sigma(X \vee Y).$$

are (iterated) Whitehead products. The equivalence θ is gotten by taking products of certain maps $\Omega\omega(i, j)$ which are indexed by a choice of basis for a free Lie algebra [35], [48], [74].

A generalization of the Hilton–Milnor theorem due to G. Porter is given below.

Let $Y_1, \ldots, Y_n$ be pointed CW-complexes with $T_i(Y_1, \ldots, Y_n)$ the subspace of the product $Y_1 \times \cdots \times Y_n$ where at least i coordinates are at the base-point.

THEOREM 2.8 ([58]). *If each Y_i is a suspension ΣX_i, then there is a homotopy equivalence*

$$\prod_{j=1}^{\infty} \Omega\Sigma X_j \to \Omega T_i(\Sigma X_1, \ldots, \Sigma X_n)$$

where X_j, $j > n$, is obtained by certain repeated smash products and suspensions of X_i, $1 \leqslant i \leqslant n$.

A general structure theorem for the localization at p of many loop spaces is given as follows.

THEOREM 2.9 ([47]). *Let X be a finite, 1-connected CW complex whose total rational homotopy rank is finite and nonzero. Then for almost all primes p, the loop space ΩX is p-equivalent to a product of spheres and loop spaces of spheres; that is there is a p-equivalence*

$$\Omega X \cong_p \prod_i S^{2m_i - 1} \times \prod_j \Omega S^{2n_j - 1}.$$

Very little is known at primes p for which the homology of ΩX has p-torsion. A basic example in this last case is discussed in Sections 6, 7, 10, 11, and 12. In addition one is led to wonder whether there are analogues of Theorem 2.9 and the results of Sections 7 and 10 when X is the localization at p of a simply-connected finite complex which has the rational homology of a sphere and has nontrivial p-torsion in integral homology.

3. The EHP sequence for $p = 2$

The q-th James–Hopf invariant $H_q : \Omega\Sigma X \to \Omega\Sigma X^{(q)}$ was described in Section 2 after Theorem 2.5. In this section X will be restricted to the n-sphere S^n. Furthermore all spaces are localized at the prime 2 unless otherwise stated. James' theorem given below yields the EHP sequence which has proven to be one of the important tools in the study of the homotopy groups of spheres.

THEOREM 3.1 ([37], [38]). *There is a 2-local fibration*

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H_2} \Omega S^{2n+1}$$

where E is the suspension given by the adjoint of the identity.

Toda obtained odd primary analogues of this fibration [70], [71]; some of this information is described in Section 4.

The EHP sequence is the result of applying the long exact sequence of homotopy groups to the fibration given in Theorem 3.1. Thus there is a long exact sequence after localization at 2:

$$\cdots \to \pi_i S^n_{(2)} \xrightarrow{E} \pi_{i+1} S^{n+1}_{(2)} \xrightarrow{H} \pi_{i+1} S^{2n+1}_{(2)} \xrightarrow{P} \pi_{i-1} S^n_{(2)} \to \cdots.$$

More information is obtained by looping the fibration in 3.1.

Namely the set $[X, Y]$ is naturally a group if Y is a loop space and the element H_2 in $[\Omega S^{n+1}, \Omega S^{2n+1}]$ always has infinite order. However the situation changes after looping. The next theorem anticipates some results in Sections 4 and 5.

THEOREM 3.2 ([37], [61], [14], [13]). *After localization at the prime p, the element ΩH_p has order p in the group $[\Omega^2 S^{2n+1}, \Omega^2 S^{2np+1}]$.*

James had already proven the very useful result that the James–Hopf invariant of twice an element in the homotopy groups of S^{2n+1} is zero [38]. The topological analogue given above is a modification of his methods. On the other hand, that maps, after looping, have finite order is a modification which also has very useful applications. This last approach is due to J.C. Moore.

If $p = 2$ the result in Theorem 3.2 was known to M.G. Barratt who never published it. Unwittingly the author of this article published a proof of this theorem. One consequence is the next result due to James.

THEOREM 3.3 ([38]). *The order of the 2-torsion in $\pi_q S^{2n+1}$ is bounded above by 2^{2n}.*

Other applications and improvements were given by Selick [61], Richter [59], and [15].

THEOREM 3.4 ([63], [15], [7]). *The order of the 2-torsion in $\pi_q S^{2n+1}$ is bounded above by $2^{2n-[\frac{n}{2}]}$.*

By extending the fibration in 2.1 in the natural way, there is a map

$$P : \Omega^2 S^{2n+1} \to S^n$$

such that the homotopy theoretic fibre of $E : S^n \to \Omega S^{n+1}$ is $\Omega^2 S^{2n+1}$ where P gives the map from the fibre to the total space. It was observed in [16] that one could factor the H-space squaring map on $\Omega^4 S^{4n+1}$ through $\Omega^2 S^{4n-1}$. The next result is an improvement of this remark where 2 denotes the H-space squaring map and E^2 denotes the double suspension.

THEOREM 3.5 ([59]). *There is a 2-local homotopy commutative diagram*

$$\begin{tikzcd}
\Omega^3 S^{4n+1} \arrow[r, "2"] \arrow[d, "\Omega P"'] & \Omega^3 S^{4n+1} \\
\Omega S^{4n-1} \arrow[r, "1"] & \Omega S^{4n-1} \arrow[u, "\Omega E^2"']
\end{tikzcd}$$

Let W_n denote the homotopy theoretic fibre of the double suspension $E^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$.

THEOREM 3.6 ([7]). *After localization at $p = 2$, the fourth power map of $\Omega^2 W_n$ is null-homotopic. Thus $\pi_q W_n$ has exponent bounded above by* 4.

The map P fits in with features of the tangent bundle of a sphere. Namely let $[q] : S^n \to S^n$ denote a map of degree q; write $S^n\{q\}$ for the homotopy theoretic fibre of $[q]$ and $P^{n+1}(q)$ for the cofibre of $[q]$. Let τS^n denote the unit sphere bundle in the tangent bundle of S^n; thus $\tau S^n = SO(n+1)/SO(n-1)$.

LEMMA 3.7 ([16]). *There is a 2-local fibration*

$$\bar{H}_2 : \Omega\tau(S^{2n}) \to \Omega S^{4n-1}$$

with homotopy theoretic fibre $S^{2n-1}\{2\}$. Furthermore there is a morphism of fibration sequences

$$\begin{tikzcd}
\Omega^2 S^{4n-1} \arrow[r, "d"] \arrow[d, "1"'] & S^{2n-1}\{2\} \arrow[r] \arrow[d] & \Omega\tau(S^{2n}) \arrow[r, "\bar{H}_2"] \arrow[d] & \Omega S^{4n-1} \arrow[d, "1"] \\
\Omega^2 S^{4n-1} \arrow[r, "P"] & S^{2n-1} \arrow[r] & \Omega S^{2n} \arrow[r, "H_2"] & \Omega S^{4n-1}
\end{tikzcd}$$

Thus P is the composite of two maps

$$\Omega^2 S^{4n-1} \xrightarrow{d} S^{2n-1}\{2\} \quad \textit{and} \quad S^{2n-1}\{2\} \to S^{2n-1}.$$

It is an elementary and open question as to whether the degree 2 map [2] induces multiplication by 2 on $\pi_q S^{2n-1}$ for all q. Part of this question is reflected in the features of $S^{2n-1}\{2\}$. This last space has some properties which are similar to those of H-spaces. In particular given a degree one map of a mod-2^r Moore space to $S^{2n-1}\{2\}$ say α :

$P^{2n-1}(2^r) \to S^{2n-1}\{2\}$, then this map extends to a map $\Omega\Sigma P^{2n-1}(2^r) \to S^{2n-1}\{2\}$ if and only if $r \geqslant 2$ [17]. In addition, there is a fibration

$$(\Omega^3 S^{2n-1})\{2\} \xrightarrow{\sigma} \Omega^2(S^{2n-1}\{2\}) \to (\Omega^2 S^{2n-1})\{4\}$$

where (1) $(\Omega X)\{q\}$ is the homotopy theoretic fibre of the q-th power map $q : \Omega X \to \Omega X$ and (2) $2\,\sigma$ is null [75]. Thus $8 \cdot \pi_*(S^{2n-1}\{2\}) = 0$.

A few of the many applications of James' EHP sequence are given in [5], [15], [38], [45], [63].

4. The EHP sequence for $p > 2$

The odd primary analogue of James' EHP sequence was obtained by Toda [70], [71]. Let p be an odd prime and let $J_q X$ denote the qth filtration of the James construction JX; if X is path-connected, then JX is homotopy equivalent to $\Omega\Sigma X$ [37]. In the case that X is S^n the q-th filtration of $J_q X$ is homotopy equivalent to the (nq)-skeleton of ΩS^{n+1}.

THEOREM 4.1 ([70]). *After localization at p, there are fibrations*

(i) $J_{p-1}S^{2n} \to \Omega S^{2n+1} \xrightarrow{H_p} \Omega S^{2np+1}$, *and*
(ii) $S^{2n-1} \to \Omega J_{p-1}S^{2n} \xrightarrow{T} \Omega S^{2np-1}$.

The map T has additional features which distinguish it from H_p.

THEOREM 4.2 ([31]). *After localization at p, $p > 2$, the map T may be chosen to be an H-map.*

Up to homotopy, there is only one T which is also an H-map. Furthermore, it is proven in [49] that this choice of T is homotopic to the one originally constructed by Toda.

Toda used the fibrations in Theorem 4.1 to obtain the odd primary analogues of James' exponent result (Theorem 3.3 here).

THEOREM 4.3 ([38], [70]). *If p is an odd prime, then p^{2n} annihilates the p-torsion in $\pi_q S^{2n+1}$.*

This last result has been improved to that which is best possible [61], [20], [52]. Some discussion of this is given in Sections 5 and 6.

Since ΩH_p has order p in the group $[\Omega^2 S^{2n+1}, \Omega^2 S^{2np+1}]$, there is a lift $\bar{H}_p$ which fits in a homotopy commutative diagram

$$\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\bar{H}_p} & (\Omega^2 S^{2np+1})\{p\} \\
{\scriptstyle 1}\big\downarrow & & \big\downarrow \\
\Omega^2 S^{2n+1} & \xrightarrow{\Omega H_p} & \Omega^2 S^{2np+1}
\end{array}$$

Selick showed that there is a choice of lift $\bar{H}_p$ which is an H-map if p is an odd prime [61]. The behavior of H_2 in mod-2 homology precludes any choice of $\bar{H}_2$ from being an H-map. Using these choices of $\bar{H}_p$, $p > 2$, which are H-maps Harper obtained the following factorization.

THEOREM 4.4 ([33]). *After localization at an odd prime p, there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^3 S^{2np+1} & \xrightarrow{p} & \Omega^3 S^{2np+1} \\ {\scriptstyle \Omega(\partial)}\downarrow & & \uparrow{\scriptstyle \Omega E} \\ \Omega S^{2np-1} & \xrightarrow[1]{} & \Omega S^{2np-1} \end{array}$$

Thus Theorem 3.5 is the 2-primary analogue of Harper's theorem. Additional related information is given in Section 6 where further factorizations of the pth power map are discussed.

There are further fibrations which fit with the ones given by Toda.

THEOREM 4.5 ([32], [49]). *If $n \geqslant 1$, there is a space BW_n together with a fibration*

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \longrightarrow BW_n$$

where E^2 is the double suspension.

Let $f : \Sigma^2 X \to S^{2n+1}$ be a map. Questions of whether f preserves the co-H-structure of the source and target have been studied in [34] by using Selick's choice of lift $\bar{H}_p$.

THEOREM 4.6 ([34]). *After localization at p for p an odd prime, f is a co-H map if and only if $H_p \circ f^*$ is null-homotopic where $f^* : \Sigma X \to \Omega S^{2n+1}$ is the adjoint of f.*

THEOREM 4.7 ([34]). *After localization at p for p an odd prime, f is a co-A map (co-associative) if and only if $\bar{H}_p \circ f^{**}$ is null-homotopic, where f^{**} is the double adjoint of f.*

As in the case $p = 2$, a few of the many applications of some of these constructions are given in ([32], [49], [61], [70], [71]).

5. Product decompositions related to spheres

Another basic theorem in the subject traces back to the classical Hopf fibrations

$\eta : S^3 \to S^2$ with fibre S^1,

$\nu : S^7 \to S^4$ with fibre S^3, and

$\sigma : S^{15} \to S^8$ with fibre S^7.

These fibrations immediately yield the following homotopy equivalences

$$S^{2n-1} \times \Omega S^{4n-1} \to \Omega S^{2n} \quad \text{if } n = 1, 2, 4.$$

In somewhat different language, Serre [66] exhibited p-local equivalences, $p > 2$,

$$S^{2n-1}_{(p)} \times \Omega S^{4n-1}_{(p)} \to \Omega S^{2n}_{(p)}.$$

These decompositions can be extended in at least 2 different ways. Either the sphere S^{2n} could be replaced by other spaces or the function space ΩS^{2n} could be replaced by other pointed mapping spaces $map_*(A, S^n)$. Product decompositions for either of these types of constructions yield useful information. The first analogue for loop spaces of double suspensions is studied in Sections 7, 10, 11, and 12 while $map_*(A, S^n)$ is considered here.

The notation in this section is as given earlier: W_n is the homotopy theoretic fibre of the double suspension $E^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$, $S^n\{q\}$ is the homotopy theoretic fibre of a degree q map $[q] : S^n \to S^n$, and $P^{n+1}(q)$ is the cofibre of $[q]$.

There is a p-local decomposition for the function space $map_*(P^3(p), S^{2p+1})$ which was first proven by P. Selick for $p > 2$ [61]. Since the p-th power map for $map_*(P^3(p), X)$ is null for p odd, it follows that $p\pi_* map_*(P^3(p), X) = 0$ if p is an odd prime. Let $X\langle n\rangle$ denote the n-connected cover of X.

THEOREM 5.1 ([61], [62]). *After localization at $p > 2$, there are homotopy equivalences*

$$\Omega^2 S^3\langle 3\rangle \times W_p \to map_*\big(P^3(p), S^{2p+1}\big).$$

As a corollary, Selick immediately obtains the next beautiful result.

COROLLARY 5.2 ([61]). *If p is an odd prime, then p annihilates the p-primary component of $\pi_q S^3$, $q \geqslant 4$.*

Analogous results for S^{2n+1} are described in Section 6.

The 2-primary analogue of Theorem 5.1 is the following.

THEOREM 5.3 ([16]). *There is a 2-local homotopy equivalence*

$$\Omega^2 S^3\langle 3\rangle \times W_2 \to map_*\big(P^3(2), S^5\big).$$

There are related splittings which reflect features of both the classical Hopf fibrations and the Whitehead square $w_n = [\iota_n, \iota_n]$ in $\pi_{2n-1}S^n$. In the following, a theorem of Gray stating that W_n is a loop space is used [32].

THEOREM 5.4 ([23]). *There are 2-local homotopy equivalences*

$$\begin{aligned} \Omega^2 S^3\langle 3\rangle \times W_2 &\to map_*\big(P^3(2), S^5\big), \\ BW_2 \times W_4 &\to map_*\big(P^3(2), S^9\big), \text{ and} \\ W_4 \times X_{17} &\to map_*\big(P^3(2), S^{17}\big). \end{aligned}$$

REMARK 5.5. It seems reasonable to guess that X_{17} is homotopy equivalent to ΩW_8 in this last theorem. In addition, it is not known whether these decompositions proliferate to $map_*(P^n(2), S^{2k+1})$ for n large. It is known that $map_*(P^3(2), S^n)$ cannot split nontrivially if $n \neq 2, 4, 8, 5, 9, 17$ [12].

One feature of the product decompositions given above is that they imply product decompositions of homotopy groups *with coefficients*. Thus, for example, Theorems 5.1 and 5.3 give the following isomorphisms where all spaces are localized at p:

$$\pi_q(S^{2p+1}; p) \xrightarrow{\simeq} \pi_{q-1} S^3 \oplus \pi_{q-3} W_p, \quad q \geqslant 2p+1,$$

and where $\pi_q(X;p) = [P^q(p), X]$. A thorough and careful study of mod-p homotopy groups $\pi_q(X;p)$, p odd, is given in [50].

Notice that these previous results give that the spaces $map_*(\mathbb{R}P^2, \Omega^i S^n)$, $i \geqslant 3$, decompose as nontrivial products if $n = 2, 4, 8, 5, 9$, or 17. One is led to wonder whether there are nontrivial decompositions of $map_*(\mathbb{R}P^b_a, \Omega^i S^n)$ for other values of n and i. An interesting case occurs when $\mathbb{R}P^b_a$ is the Spanier–Whitehead dual $S^{4n+2} - \mathbb{R}P^{2n}$. If either $n = 1$ or one restricts to the metastable range, then $\Omega_0^{2n+1} S^{2n+1}$ is a retract of the pointed mapping space $\mathrm{map}_*(S^{4n+2} - \mathbb{R}P^{2n}, S^{4n+1})$ [15]. Thus one might wonder how to interpolate these results for all n in order to obtain a nonstable analogue of the Kahn–Priddy theorem. A further question is whether this type of interpolation can settle exponent problems of the type addressed in the next section. A more precise discussion of this point is given in Section 8.

There are two related decompositions associated to $SO(3)$ and $SU(4)$. The splitting concerning $SU(4)$ due to D. Waggoner is given by

THEOREM 5.6 ([73]). *After localization at $p = 2$, there is a homotopy equivalence*

$$W_2 \times Y_4 \to map_*(P^3(2), SU(4))$$

for some space Y_4.

It is as yet unclear whether there are further analogous decompositions obtained by (1) replacing $SU(4)$ by $SU(n)$, $n > 4$, or other related Lie groups or (2) replacing $\Sigma\mathbb{R}P^2 = P^3(2)$ by $\mathbb{R}P^b_a = \mathbb{R}P^b/\mathbb{R}P^{a-1}$. However, there is the natural map

$$\gamma_n : \Sigma\mathbb{R}P^{n-1} \to BSO(n).$$

A theorem of Jie Wu which is stated in Section 12 here gives that γ_3 induces a split epimorphism on π_q after localization at 2 when $q \geqslant 5$.

6. On exponents for spheres

Throughout this section p denotes a prime and all spaces are assumed to be localized at p. A space Y is said to have exponent p^n (at p) provided p^n is the smallest power of p which annihilates the p-torsion subgroups of all of the homotopy groups of Y. An H-space X is said to have H-space exponent p^n if p^n is the least power of p such that the p^n-th power map $p^n : X \to X$ is null-homotopic. Since the self-map of X given by p^n induces multiplication by p^n on the level of homotopy groups, the existence of an H-space exponent implies that the space has an exponent which is bounded above by the H-space exponent. An example of a space with an exponent is the n-sphere; this follows from results of James and Toda [38], [70] which are listed here as Theorems 3.3 and 4.3. Improvements of these results were obtained by compressions of the pth power map for $\Omega^2 S^{2n+1}$ through the double suspension map $E^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$.

THEOREM 6.1 ([20], [52]). *After localization at an odd prime p, there is a map* $\pi : \Omega^2 S^{2n+1} \to S^{2n-1}$ *together with a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{p} & \Omega^2 S^{2n+1} \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle E^2} \\ S^{2n-1} & \xrightarrow{1} & S^{2n-1} \end{array}$$

A theorem of B. Gray gives that if p is an odd prime, then there exist elements of order exactly p^n in $\pi_* S^{2n+1}$ [29]. Combining this last fact with the fact that $\pi_i S^1$ is trivial if $i > 1$, the next corollary follows at once.

COROLLARY 6.2 ([20], [52]). *After localization at an odd prime p, the group* $p(\pi_q S^{2n+1})$ *is contained in the image of the double suspension* $E^2_*(\pi_{q-2} S^{2n-1})$. *Thus* S^{2n+1} *has exponent* p^n. *Furthermore,* $\Omega^{2n}(S^{2n+1}\langle 2n+1\rangle)$ *has H-space exponent* p^n.

Important refinements for "large" primes p were given in [3].

THEOREM 6.3 ([3, p. 7]). *If p is prime with* $p \geqslant 5$ *and* $n \geqslant 1$, *then there exist p-local spaces* T_∞ *together with p-local fibrations*

$$S^{2n-1} \xrightarrow{i} T_\infty \longrightarrow \Omega S^{2n+1}$$

and a homotopy commutative diagram

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{p} & \Omega^2 S^{2n+1} \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle E^2} \\ S^{2n-1} & \xrightarrow{1} & S^{2n-1} \end{array}$$

where

$$\Omega^2 S^{2n+1} \xrightarrow{\pi} S^{2n-1} \xrightarrow{i} T_\infty$$

is principal $\Omega^2 S^{2n+1}$*-fibration. Furthermore, the p-th power map for* $\Omega^2 S^{2n+1}$ *factors as* $\Omega^2 S^{2n+1} \xrightarrow{\pi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$.

The situation at the prime 2 is less clear. There is a factorization of the H-space squaring map for $\Omega^3 S^{4n+1}$ through ΩS^{4n-1}. However, the potential analogous factorization with $4n + 1$ replaced by $4n + 3$ fails. There are, however, upper and lower bounds for exponents at the prime 2.

Consider the canonical line bundle over $\mathbb{R}P^{2n}$ which has order $2^{\phi(2n)}$. Assume that all spaces are localized at 2 and consider the maps

$$i : \mathbb{R}P^{2n} \longrightarrow \Omega_0^{2n+1} S^{2n+1} \quad \text{and} \tag{6.1}$$

$$s : \Omega_0^{2n+1} S^{2n+1} \longrightarrow \Omega_0^{\infty} S^{\infty} \tag{6.2}$$

where i is Whitehead's map and s is the restriction to the base-point component of the $(2n + 1)$-fold looping of the stabilization map $S^{2n+1} \to QS^{2n+1}$. Since $s \circ i$ has order exactly $2^{\phi(2n)}$ in the group $[\mathbb{R}P^{2n}, \Omega_0^{\infty} S^{\infty}]$, the map s has order at least $2^{\phi(2n)}$. That s has order at most $2^{\phi(2n)}$ follows by naturality of the stable second James–Hopf invariant and the Kahn–Priddy theorem. That $\Omega_0^{2n+1} S^{2n+1}$ has an H-space exponent bounded above by $2^{2n-[\binom{n}{2}]}$ follows from ([64], [15], [7]).

Thus $\Omega_0^{2n+1} S^{2n+1}$ has an H-space exponent bounded below by $2^{\phi(2n)}$ and above by $2^{2n-[\binom{n}{2}]}$. However, it is not known whether there exist elements of order $2^{\phi(2n)}$ in $\pi_* S^{2n+1}$ for all n; it is known in many cases [45]. Barratt and Mahowald at one time conjectured that S^{2n+1} had exponent $2^{\phi(2n)}$. This conjecture is unsettled if $n \geqslant 3$, and also raises the possibility that the H-space exponent for $\Omega_0^{2n+1} S^{2n+1}$ may be different than the exponent (for homotopy groups).

It is also the case that H-space exponents for certain loop spaces of spheres are strictly larger than the exponent for spheres. Of course the power maps for ΩS^n are always essential.

THEOREM 6.4 ([53]). *If $q \leqslant 2n - 2$, and spaces are localized at the prime p, then the p^r-th power map on $\Omega^q(S^{2n+1}\langle 2n + 1\rangle)$ is essential for all r.*

A related and somewhat peculiar example is the exponent of $SU(3)$ at the prime 2. It was shown in [15], that $\Omega^4(SU(3)\langle 5\rangle)$ has an H-space exponent (localized at $p = 2$) which is bounded above by 16. However it was pointed out in [15] that $\pi_i SU(3)_{(2)}$ for $i \leqslant 10$ has exponent 4 and thus the v_1-periodic elements given by Oka [55] also have exponent bounded by 4. Thus the situation for $SU(3)$ is not very well understood.

These examples raise the possibility that H-space exponents may well be different than homotopy exponents. Again, the situation is not well understood. Namely, the proofs for the existence of exponents for the homotopy groups of certain finite complexes arise from the existence of an H-space exponent. Thus without "complete" knowledge of the homotopy groups for the complex, it is difficult to decide whether an exponent for homotopy groups is strictly less than a given H-space exponent.

One particular finite complex is a mod p^r-Moore space $P^n(p^r)$. The loop spaces of $P^n(p^r)$ were central in the proof of Theorem 6.1; they provide useful examples of spaces whose double loop spaces have exponents, and they are the subject of the next section.

7. On the homotopy theory of Moore spaces

The subject of this section is the homotopy theory of mod-q Moore spaces $P^{n+1}(q)$ the cofibre of a degree q self-map of S^n. The material in this section comes from ([19], [21], [51], [16], [54]).

Since the universal cover of $P^2(q)$ is homotopy equivalent to $\vee_{q-1} S^2$, the homotopy groups of $P^2(q)$ are given in terms of (1) the Hilton–Milnor theorem, and (2) the homotopy groups of spheres. Of course $P^2(q)$ does not have an exponent for $q > 2$. Thus, simply-connected mod-q Moore spaces are considered here.

Two useful features of mod-p Moore spaces are (1) they provide a tool for the analysis of p-th power maps on iterated loop spaces of spheres, and (2) the loop space of a simply-connected mod-p^r Moore always splits as a nontrivial product. An analysis of these splittings is the key to the results stated in Section 6 [20], [52]. The ur-example of these splittings is given in the theorem below.

THEOREM 7.1 ([19]). *If p is an odd prime and $n \geqslant 1$, then there is a homotopy equivalence*

$$S^{2n+1}\{p^r\} \times \Omega\Sigma\Big(\bigvee_{k\geqslant 0} P^{4n+2kn+2}(p^r)\Big) \to \Omega P^{2n+2}(p^r).$$

An analogous result is correct if $p = 2$ with either (i) $n = 1$ with $r \geqslant 3$ or (ii) $n = 3$ with $r \geqslant 4$. However these product decompositions fail if $2n + 2$ is not a power of 2 [15]. This failure traces directly to the failure of the Whitehead square on S^{2n+1} to be divisible by 2 in these cases, a problem which is discussed in the next section. In general, there are (nonsplit) fibrations

$$S^n\{2^r\} \to \Omega P^{n+1}(2^r) \to \Omega\Sigma\Big(\bigvee_{k\geqslant 0} P^{2n+k(n-1)}(2^r)\Big)$$

if $r \geqslant 2$ [16]. It is not, as yet, clear what the analogous fibrations should be if $r = 1$.

There is one other type of space which appears in the determination of the homotopy type of the loop space of a mod-p^r Moore space, $p^r > 2$. If p is an odd prime, and $n \geqslant 2$, then there is a choice of $(2n + 1)$-connected bouquets of Moore spaces

$$P(n, p^r) = \bigvee_{m_\alpha \in J} P^{m_\alpha}(p^r)$$

together with a map $\alpha : P(n, p^r) \to P^{2n+1}(p^r)$. The space $T^{2n+1}\{p^r\}$ is the homotopy theoretic fibre of α.

THEOREM 7.2 ([21], [52]). *If p is an odd prime, and $n \geqslant 2$, then the fibration*

$$\Omega P^{2n+1}(p^r) \to T^{2n+1}\{p^r\}$$

is split. Thus there is a homotopy equivalence

$$T^{2n+1}\{p^r\} \times \Omega P(n,p^r) \to \Omega P^{2n+1}(p^r).$$

Furthermore, there is a fibration

$$W_n \times \prod_{k\geqslant 1} S^{2np^k-1}\{p^{r+1}\} \to T^{2n+1}\{p^r\} \to \Omega S^{2n+1}\{p^r\}.$$

An application of (1) the Hilton–Milnor theorem, (2) the fact that $P^n(p^r) \wedge P^m(p^r)$ is homotopy equivalent to $P^{n+m}(p^r) \vee P^{n+m-1}(p^r)$ if $n+m \geqslant 5$ with $p^r > 2$, and (3) Theorems 7.1, 7.2 gives the next result.

THEOREM 7.3 ([21], [52]). *If p is an odd prime and $n \geqslant 3$, then $\Omega P^{n+1}(p^r)$ is homotopy equivalent to a weak infinite product of spaces*

$$S^{2k+1}\{p^r\} \quad \textit{and} \quad T^{2j+1}\{p^r\}$$

for certain choices of k and j.

The product decompositions given in Theorem 7.2 admit 2-primary analogues. If r, $n \geqslant 2$, then there exists an n-connected bouquet of Moore spaces

$$P(n,2^r) = \bigvee_{m_\alpha \in L} P^{m_\alpha}(2^r)$$

together with a map $\alpha : P(n,2^r) \to P^{n+1}(2^r)$ with homotopy theoretic fibre $T^{n+1}\{2^r\}$.

THEOREM 7.4 ([16]). *If r and n are at least 2, then the fibration $\Omega P^{n+1}(2^r) \to T^{n+1}\{2^r\}$ is split. Thus there is a homotopy equivalence*

$$T^{n+1}\{2^r\} \times \Omega P(n,2^r) \to \Omega P^{n+1}(2^r).$$

Furthermore, there are fibrations

$$\Omega\Big(S^{2n-1} \times \prod_{k\geqslant 2} S^{2^k n-1}\{2^r\}\Big) \to \Omega T^{2n+1}\{2^r\} \to \Omega^2 S^{2n+1} \times \Omega\Big(\prod_{k\geqslant 2} S^{2^k n-1}\{2\}\Big)$$

and

$$S^n\{2^r\} \to T^{n+1}\{2^r\} \to T^{2n+1}\{2^r\}.$$

The spaces $T^{n+1}\{2^r\}$ and $T^{2n+1}\{p^r\}$ do not usually split further. If p is an odd prime and $n > 1$, then $T^{2n+1}\{p^r\}$ is atomic [21]; a similar result applies to $T^{n+1}\{2^r\}$ if $n+1$

is not a power of 2 [16]. That the Moore spaces $P^{n+1}(p^r)$, $p^r > 2$, have exponents follows from the above theorems. However, the best possible exponents are known in case p is odd.

THEOREM 7.5 ([51]). *If p is an odd prime and $n \geqslant 3$, then $\Omega^2 P^n(p^r)$ has H-space exponent exactly p^{r+1}.*

This last result is best possible by the following theorem.

THEOREM 7.6 ([19], [14]). *If k and $n \geqslant 1$ with $p^r > 2$ for any prime p, then*

$$\pi_{2np^k-1} P^{2n+1}(p^r)$$

contains a $\mathbb{Z}/p^{r+1}\mathbb{Z}$-summand.

The analogue of this theorem in the case that $p^r = 2$ is discussed in Section 11 here.

The way in which mod-p^r Moore spaces fit with factorizations of power maps in Theorem 6.1 is described below where a short digression concerning maps of degree p^r $[p^r] : S^n \to S^n$ is given first. Consider the looping of $[p^r]$, $\Omega[p^r]$; this map is homotopic to the p^r-th power map on ΩS^n if either (i) S^n is an H-space (thus $n = 1, 3,$ or 7), (ii) all spaces are localized at p and $S^n_{(p)}$ is an H-space (thus if $p = 2, n = 1, 3,$ or 7 while if $p > 2$, then n is odd), or (iii) all spaces are localized at p with n odd and $p^r > 2$. The map $\Omega[2] : \Omega S^n \to \Omega S^n$ is homotopic to the H-space squaring map if and only if $n = 1, 3,$ or 7 [15]. Thus the notation $[p^r]$ is chosen in order to distinguish the self-maps p^r and $\Omega[p^r]$.

Next consider the homotopy commutative diagram

$$\begin{tikzcd} P^n(p^r) \arrow[r, "q"] \arrow[d] & S^n \arrow[d, "{[p^r]}"] \\ * \arrow[r] & S^n \end{tikzcd}$$

where q denotes the pinch map. Enlarging this diagram to one which gives morphisms of fibration sequences,

$$\begin{tikzcd} \Omega(S^n\{p^r\}) \arrow[r] \arrow[d] & E^n\{p^r\} \arrow[r] \arrow[d, "i"] & P^n(p^r) \arrow[r] \arrow[d] & S^n\{p^r\} \arrow[d] \\ \Omega S^n \arrow[r] \arrow[d, "{\Omega[p^r]}"'] & F^n\{p^r\} \arrow[r] \arrow[d, "j"] & P^n(p^r) \arrow[r, "q"] \arrow[d] & S^n \arrow[d, "{[p^r]}"] \\ \Omega S^n \arrow[r, "1"'] & \Omega S^n \arrow[r] & * \arrow[r] & S^n \end{tikzcd},$$

there is a factorization of $\Omega[p^r] : \Omega S^n \to \Omega S^n$ through $F^n\{p^r\}$, the homotopy theoretic fibre of q. Product decompositions of $\Omega E^n\{p^r\}$ and $\Omega F^n\{p^r\}$ which are given next imply Theorem 6.1.

THEOREM 7.7 ([20], [15]). *Let p be an odd prime. There are morphisms of p-local fibrations*

$$\begin{tikzcd}
W_n \times (\prod_{k\geqslant 1} S^{2np^k-1}\{p^{r+1}\}) \times \Omega P(n,p^r) \arrow[r] \arrow[d, "\partial\times 1\times 1"'] & \Omega E^{2n+1}\{p^r\} \arrow[d, "\Omega(i)"] \\
S^{2n-1} \times (\prod_{k\geqslant 1} S^{2np^k-1}\{p^{r+1}\}) \times \Omega P(n,p^r) \arrow[r] \arrow[d, "E^2\times *"'] & \Omega F^{2n+1}\{p^r\} \arrow[d, "\Omega(j)"] \\
\Omega^2 S^{2n+1} \arrow[r, "1"'] & \Omega^2 S^{2n+1}
\end{tikzcd}$$

where the horizontal arrows are homotopy equivalences.

In a similar direction, P. Selick has exhibited p-local decompositions of the loop spaces for $J_q S^{2n}$ [65]. His more complete results are in the cases q where $H^*(\Omega J_q S^{2n}; \mathbb{F}_p)$ have trivial Steenrod reduced power operations.

8. On product decompositions related to QX

This section represents a digression from the main points of the previous sections. There are product decompositions associated to $QX = \Omega^\infty \Sigma^\infty X$ of which the product decompositions in Section 5 are analogues. Thus the liberty of comparing product decompositions of QX with those given in previous sections is taken here.

Tørnehave exhibited a product decomposition of the space $SG = \Omega^\infty_{(1)} S^\infty$ [72]. This decomposition applies to other spaces in several different guises. In addition, a second decomposition due to Kahn and Priddy [40] provides a p-local homotopy product decomposition of $QB\Sigma_p$ where Σ_p denotes the symmetric group on p letters. Analogous product decompositions apply to spaces (1) $\Omega^n_0 QX$ for finite complexes X with $n \gg$ dimension of X, and (2) $(B\pi)^\wedge_p$ where π runs over various choices of groups like the stable automorphism group of free products of cyclic groups. These splittings are a reflection of the features of certain mapping spaces with targets given by spheres as given in Section 5.

First of all, the space J will be described. It is a path-connected H-space whose reduced integral homology is torsion. Thus J is homotopy equivalent to a wedge of its localizations $J_{(p)}$ for each prime p. The space $J_{(p)}$ can be chosen to be $BGL(\mathbb{F}_q)^\wedge_p$ if $p > 2$ where q is a prime such that $q^i - 1 \not\equiv 0 (\operatorname{mod} p)$ for $1 \leqslant i \leqslant p-2$ and $\nu_p(q^{p-1}-1) = 1$. If $p = 2$, $J_{(2)}$ can be taken to be $BSO(\mathbb{F}_3)^\wedge_2$ [56], [57], [27].

By work of Quillen [56] there is a map from $BGL(\mathbb{F}_q)$ to the homotopy theoretic fibre of $\psi^q - 1 : BU \to BU$ which is an equivalence after completing at an odd prime p and where ψ^q denotes the evident Adams operation. A similar result is correct at $p = 2$ with $\psi^3 - 1 : BSO \to BSO$ [56], [57], [27]. Thus $J_{(p)}$ may be taken to be the homotopy theoretic fibre of $\psi^{q-1} - 1 : BU \to BU$ at $p > 2$ and $\psi^3 - 1 : BSO \to BSO$ at $p = 2$.

Notice that the symmetric group on n letters Σ_n is isomorphic to the subgroup of permutation matrices in $GL(n, \mathbf{F}_q)$. Passage to limits and p-completions gives

$$\pi : B\Sigma_{\infty}{}^{\wedge}_{p} \to BGL(\mathbf{F}_q)^{\wedge}_{p}.$$

The homotopy theoretic fibre of π is the p-completion of Coker J. Among other results, Tørnehave showed that π admits a cross-section. An elementary proof is furnished in [22].

THEOREM 8.1 ([72]). *There is a homotopy equivalence*

$$J \times \operatorname{Coker} J \to \Omega_0^{\infty} S^{\infty}.$$

Assume that X is a connected finite complex which embeds in S^N and write $D(X, N)$ for the Spanier–Whitehead dual $S^N - X$. An application of Spanier–Whitehead duality gives the next result.

THEOREM 8.2 ([24]). *There is a homotopy equivalence*

$$QX \to map_*\big(D(X,N), QS^N\big).$$

Notice that $\Omega_0^N QX$ is homotopy equivalent to the component of the base-point in $\Omega^N map_*(D(X,N), QS^N)$, $map_*^0(D(X,N), \Omega_0^{\infty} S^{\infty})$. As $\Omega_0^{\infty} S^{\infty}$ splits by Theorem 8.1, the next corollary follows at once.

COROLLARY 8.3. *If X is a connected finite complex which embeds in S^N, then there is a homotopy equivalence*

$$\Omega_0^N QX \to map_*^0\big(D(X,N), J\big) \times map_*^0\big(D(X,N), \operatorname{Coker} J\big).$$

This last corollary of course provides an elementary description of the well-known decomposition of the stable homotopy groups of finite complexes where one summand is given by the homotopy groups of J with coefficients in the (suspensions) of the Spanier–Whitehead dual of X. Thus it is natural to wonder whether Coker J admits a nontrivial and useful fibering. It is worth remarking that similar decompositions apply to groups other than $GL(\mathbf{F}_q)$. Some examples are given by various "stable" automorphism groups such as the automorphism group of a free group or certain free products [22].

Let Σ_p denote the symmetric group on p-letters. Kahn and Priddy proved

THEOREM 8.4 ([40]). *There is a p-local homotopy equivalence*

$$QB\Sigma_p \to \Omega_0^{\infty} S^{\infty} \times Y(p)$$

for some choice of space $Y(p)$.

The product decompositions of $map_*(P^3(p), S^{2p+1})$ given in Theorems 5.3 and 5.4 fit with the above decomposition. Consider the natural map $s : \Omega_0^3 S^3 \to \Omega_0^\infty S^\infty$. The decompositions given in these theorems fit in a homotopy commutative diagram

$$\begin{tikzcd} map_*(P^4(p), S^{2p+1}) \arrow[r] \arrow[d] & \Omega_0^3 S^3 \times \Omega W_p \arrow[d, "s\times f"] \\ QB\Sigma_p \arrow[r] & \Omega_0^\infty S^\infty \times Y(p) \end{tikzcd}$$

where the horizontal arrows are p-local homotopy equivalences. Furthermore $QB\Sigma_p$ is naturally filtered with first filtration given by $map_*(P^4(p), S^{2p+1})$ [15]. The other filtrations are specified by certain mapping spaces which are analogues of

$$map_*\big(D\big(\mathbb{R}P^{2n}, 4n+2\big), S^{4n+1}\big) \quad \text{at } p=2.$$

For example, one might wonder when

$$map_*\big(D\big(\mathbb{R}P^{2n}, 4n+2\big), S^{4n+1}\big)$$

is homotopy equivalent to a product with one factor given by $\Omega_0^{2n+1} S^{2n+1}$ localized at 2. Very little is known about these mapping spaces other than the mod-p homology as a Hopf algebra and an exponent for their homotopy groups.

9. The strong form of the Kervaire invariant one problem

Consider the Whitehead square $w_n = [i_n, i_n]$ in $\pi_{2n-1}S^n$. That w_n is zero precisely when $n = 1, 3$, or 7 is equivalent to the classical problem of the existence of elements of Hopf invariant one which was solved completely in [1]. Restrict attention to integers n which are odd and not equal to 1, 3, or 7. Thus w_n is nonzero in these cases and is of order 2. There is a short exact sequence of abelian groups

$$0 \to \mathbb{Z}/2\mathbb{Z} \to {}_2\pi_{2n-1}S^n \to {}_2\pi_{n-1}^S \to 0$$

where ${}_2\pi_i^S$ is the 2-primary component of the i-th stable stem and w_n is the generator of $\mathbb{Z}/2\mathbb{Z}$. Hence w_n is divisible by 2 in $\pi_{2n-1}S^n$ if and only if this sequence fails to split. A small part of the interplay between stable homotopy theory and nonstable homotopy theory arises in asking for the structure of the *first* nonstable group $\pi_{2n-1}S^n$ in the homotopy groups of S^n; namely, is the extension above split or nonsplit?

It has been known since the 1950's that w_n is *not* divisible by 2 if n is *not* equal to $2^k - 1$ for some k. Of the remaining cases given by $n = 2^k - 1$, it is known that w_n is divisible by 2 if $n = 1, 3, 7, 15, 31$ or 63 [8], [41], [63]. The cases for which $n = 2^k - 1 > 63$ remain open.

The strong form of the Kervaire invariant one conjecture is that w_n is divisible by 2 when $n = 2^k - 1$. Several reformulations of this conjecture are listed below where n is odd and not equal to 1, 3, or 7. One form is given in the frontispiece of Ioan James' book [39] on the topology of Stiefel manifolds.

THEOREM 9.1 ([8], [63], [15]). *The following are equivalent where n is odd and not equal to* 1, 3, *or* 7.

(1) w_n *is divisible by* 2.

(2) *The exact sequence (on the level of* 2*-primary components)*

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \pi_{2n-1}S^n \to \pi_{2n}S^{n+1} \to 0$$

fails to split.

(3) *There is a map* $P^{2n}(2) \to \Omega S^{n+1}$ *which is nonzero in mod-2 homology.*

(4) *There exists a space* X *such that* (i) $H^i(X;\mathbb{F}_2)$ *is isomorphic to* $\mathbb{F}_2$ *in degrees* $i = 0, n+1, 2n+1$ *and* $2n+2$, (ii) $H^i(X;\mathbb{F}_2)$ *is zero in other degrees,* (iii) Sq^{n+1} *is nonzero on* $H^{n+1}(X;\mathbb{F}_2)$, *and* (iv) Sq^1 *is nonzero on* $H^{2n+1}(X;\mathbb{F}_2)$.

(5) *The self-map* $\Omega^2[-1]$ *of* $\Omega^2 S^{2n+1}$ *is homotopic to the inverse.*

Thus one might ask whether a degree -1 map $[-1] : S^n \to S^n$ induces multiplication by -1 on $\pi_* S^n$ for an odd integer n. A restatement of 9.1(5) is that the action of the map $[-1]$ on the abelian group $[\Sigma^2 X, S^n]$ by composition is given by multiplication by -1 for every double suspension $\Sigma^2 X$ if and only if w_n is divisible by 2. Thus in the cases that n is odd, $n \neq 2^k - 1$, there is a double suspension $\Sigma^2 X$ such that $[-1]$ does *not* act by multiplication by -1 on $[\Sigma^2 X, S^n]$; in this case, it suffices to use $X = \Omega^2 S^n$.

There have been a number of attempts to solve this question. Homotopy theoretic methods are used in [8]. One reformulation has been given in terms of a classical construction of L.E. Dickson which gives the quaternions, Cayley numbers and related algebras [17].

In addition, this conjecture is encountered when considering the structure of $\Omega P^{2n}(2^r)$. This last space is sometimes homotopy equivalent to

$$S^{2n-1}\{2^r\} \times \Omega\Sigma\Big(\bigvee_{k \geqslant 0} P^{4n-2+k(2n-1)}(2^r)\Big).$$

If there is such a product decomposition, then there is a map $P^{4n-2}(2^r) \to \Omega P^{2n}(2^r)$ which is nonzero in mod-2 homology. Thus there is an induced map $P^{4n-2}(2) \to \Omega S^{2n}$ which is nonzero in mod-2 homology gotten by precomposing with $P^{4n-2}(2) \to P^{4n-2}(2^r)$ and post-composing with the looping of the pinch map $P^{2n}(2^r) \to S^{2n}$. Furthermore, w_{2n-1} is then divisible by 2 by 9.1.

10. General splittings for loop spaces of double suspensions

Throughout this section X is assumed to be a suspension ΣA. Features of the k-fold Whitehead product map $\Sigma A^{(k)} \to \Sigma A$ are considered. Applications to natural p-local product decompositions of $\Omega\Sigma^2 A$ are given here. Applications to the homotopy theory of mod-2 Moore spaces are given in the next two sections. The material in this section is taken from [25].

Let $E : X \to \Omega\Sigma X$ be the Freudenthal suspension and write

$$ad(k-1) : X^{(k)} \to \Omega\Sigma X$$

for the k-fold Samelson product

$$[E[E[\underbrace{\cdots[E,E]}_{\leftarrow k \rightarrow}\cdots]$$

with the k-fold smash product of X denoted $X^{(k)}$. The symmetric group on k letters acts on $X^{(k)}$ by permuting coordinates. Define self-maps

$$\beta_k : X^{(k)} \to X^{(k)}$$

inductively by

$$\beta_2 = 1 - (1,2) \quad \text{and} \tag{10.1}$$

$$\beta_{k+1} = 1 \wedge \beta_k - (\beta_k \wedge 1) \circ (1, k+1, k, k-1, \ldots, 2). \tag{10.2}$$

The Dynkin–Specht–Wever relation is

$$\beta_k \beta_k = k\beta_k$$

in the group $[X^{(k)}, X^{(k)}]$ [36].

Let $k : \Omega\Sigma X \to \Omega\Sigma X$ denote the kth power map. Since X is itself a suspension, bilinearity of the Whitehead product $\Sigma X^{(k)} \to \Sigma X$ directly gives the next result.

LEMMA 10.1. *The equation $ad(k-1) \cdot \beta_k = k ad(k-1)$ holds in the group $[X^{(k)}, \Omega\Sigma X]$. Thus there is a homotopy commutative diagram*

$$\begin{array}{ccc} X^{(k)} & \xrightarrow{ad(k-1)} & \Omega\Sigma X \\ {\scriptstyle \beta_k}\downarrow & & \downarrow{\scriptstyle k} \\ X^{(k)} & \xrightarrow[ad(k-1)]{} & \Omega\Sigma X \end{array}$$

together with an induced map

$$\operatorname*{hocolim}_{\beta_k} X^{(k)} \to \operatorname*{hocolim}_{k} \Omega\Sigma X.$$

Let $L_k(X)$ denote $\mathrm{hocolim}_{\beta_k} X^{(k)}$. If X is a p-local space with $(p,k)=1$, then the natural map $\Omega\Sigma X \to \mathrm{hocolim}_k \Omega\Sigma X$ is a homotopy equivalence.

LEMMA 10.2. *If X is a p-local space with $(p,k)=1$, then there is a homotopy commutative diagram*

$$\begin{array}{ccc} X^{(k)} & \xrightarrow{ad(k-1)} & \Omega\Sigma X \\ \downarrow & & \downarrow k \\ L_k(X) & \xrightarrow[\phi(k)]{} & \Omega\Sigma X \end{array} .$$

Furthermore, there is a homotopy equivalence

$$X^{(k)} \to L_k(X) \vee M_k(X)$$

where $M_k(X) = \mathrm{hocolim}_{(k-\beta_k)} X^{(k)}$.

Combining the above results, one obtains general product decompositions of $\Omega\Sigma X$ as follows.

THEOREM 10.3. *If X is p-local with $(p,k)=1$, then any k-fold Samelson product $X^{(k)} \to \Omega\Sigma X$ factors through $L_k(X)$. Furthermore, the canonical multiplicative extension of $\phi(k)$,*

$$\Omega\Phi(k) : \Omega\Sigma L_k(X) \to \Omega\Sigma X$$

has a left inverse. Thus there is a homotopy equivalence

$$\Omega\Sigma X \to \Omega\Sigma L_k(X) \times A_k(X)$$

where $A_k(X)$ is the homotopy theoretic fibre of $\Phi(k) : \Sigma L_k(X) \to \Sigma X$.

REMARK 10.4. If ΣX is an odd sphere, then $L_k(X)$ is always contractible for $k \geqslant 2$. If ΣX is an even sphere S^{2n}, then $L_2(X)$ localized at an odd prime p is homotopy equivalent to $S^{4n-1}_{(p)}$. In this case Theorem 10.3 gives a p-local decomposition for ΩS^{2n}, $p>2$.

In the cases above $L_k(X)$ is homotopy equivalent to a $(k-1)$-fold suspension as β_k desuspends at least $(k-1)$ times. Thus the decompositions of $\Omega\Sigma X$ proliferate in at least 2 different ways. Two of these are illustrated below.

PROPOSITION 10.5. *Let X be a suspension which is homotopy equivalent to a 2-cell complex given by the cofibre of a map $\alpha : S^{n-1} \to S^m$, then $L_3(X)$ localized at $p \neq 3$ is homotopy equivalent to the localization at p of $\Sigma^{n+m}X$. Thus*

(1) *$L_3(P^n(2))$ is homotopy equivalent to $P^{3n-1}(2)$, and*

(2) $L_3(\Sigma^n\mathbb{C}P^2)_{(2)}$ *is homotopy equivalent to* $(\Sigma^{3n+2}\mathbb{C}P^2)_{(2)}$.

If X has more than 2 cells, then $L_3(X)_{(p)}$ grows in "size" quickly. For example, if

$$\sum_i \dim \tilde{H}_i(X;\mathbb{F}_2) = 3, \quad \text{then}$$

$$\sum_i \dim H_i(L_3(X);\mathbb{F}_2) = 8.$$

There are of course finer decompositions of the smash product which are useful for non-stable homotopy theory and which arise by comparing $L_k(X) \wedge X^{(n)}$ with $L_{k+n}(X)$. The proof of 10.2 gives that $L_{k+n}(X)$ is a p-local retract of $L_k(X) \wedge X^{(n)}$ if both k and $(n+k)$ are relatively prime to p. In the case that $X = P^n(2)$, $L_3(X) \wedge X^{(2)}$ is homotopy equivalent to $\Sigma^{2n-1}X^{(3)}$ by Proposition 10.5(i) and Lemma 10.2. It then follows that there are homotopy equivalences

$$L_3(P^n(2)) \wedge P^n(2)^{(2)} \to \Sigma^{2n-1}(P^n(2))^{(3)},$$
$$P^n(2)^{(3)} \to \left(\vee_2 P^{3n-1}(2)\right) \vee \Sigma^{3n-6}\left(\mathbb{C}P^2 \wedge P^2(2)\right).$$

Thus one has

LEMMA 10.6. *There is a homotopy equivalence*

$$L_5(P^n(2)) \to P^{5n-1}(2) \vee \left(\mathbb{C}P^2 \wedge P^{5n-5}(2)\right).$$

Statements 10.2, 10.3, and 10.6 have applications in Section 11 where infinitely many elements of order exactly 8 in $\pi_*(P^n(2)$, $n \geqslant 4$, are described.

Of course $X^{(n)}$ admits further decompositions after localization at p. Some of these have been used by Jeff Smith [67]. The decompositions here use "different" summands than those used by Smith. For example, $X^{(3)}$, after localization at $p = 2$, is homotopy equivalent to $L_3(X)_{(2)} \vee M_{3,1}(X)_{(2)} \vee M_{3,2}(X)_{(2)}$ where $M_{3,i}(X)$ is the telescope of θ_i, $i = 1, 2$, with $\theta_1 = 1 + (1,2,3) + (1,3,2)$ and $\theta_2 = 3 - \beta_3 - \theta_1$. If X is a 2-cell complex, then $M_{3,1}(X)$ is a 4-cell complex with the "top" and "bottom" cells of $X^{(3)}$. One is led to wonder about the structure of the indecomposable factors of $\Omega\Sigma X$ at least after localization at a prime p. In the special cases that X is a 1-connected mod-p^r Moore space with either $p > 2$ or $p = 2$ with $r > 1$, the factors of ΩX have been given in [21], [51], [16]. In all of these cases, the factors are indecomposable with the possible exception of the cases $X = P^n(2^r)$. In all cases, the factors or their loop spaces can be fibred iteratively in terms of spheres and their loop spaces. It is interesting to ask which other spaces X satisfy the property that ΩX is homotopy equivalent to a product of spaces which can be fibred (iteratively) in this way. One surprising example is given by some spaces which are the p-completions of BG for a finite group G. It is shown in [43] that $\Omega(BG_p^\wedge)$ sometimes has the loop space of a mod-p^r Moore space as a retract.

11. On the homotopy theory of $\Sigma^n\mathbb{R}P^2$, $n \geqslant 2$

Some product decompositions of $\Omega\Sigma^n\mathbb{R}P^2$, $n \geqslant 2$, are given in this section. Several applications follow. One such application is to the construction of an infinite family of elements of order exactly 8 in the homotopy groups of $\Sigma^n\mathbb{R}P^2$, $n \geqslant 2$.

A point worth mentioning concerns these elements of order 8 in $\pi_*\Sigma^n\mathbb{R}P^2$ when $n+2 \equiv 1 \pmod 4$. In these cases, the methods employed here give elements of order 8 in dimensions above roughly 15 times the stable range; the methods do not give elements of order 8 in lower degrees. One wonders whether this gap is an artifact of the methods or whether there is an anomaly worth studying. This section summarizes work in [25]. The space $\Sigma^n\mathbb{R}P^2$ is denoted $P^{n+2}(2)$.

THEOREM 11.1. *If $n \geqslant 3$, there exist spaces $X(n+1)$ and $Y(n+1)$ together with homotopy equivalences*

$$\Omega P^{n+1}(2) \to \Omega P^{3n}(2) \times X(n+1), \quad \textit{and}$$

$$\Omega P^{n+1}(2) \to \Omega P^{5n-1}(2) \times Y(n+1).$$

To find elements of order 8 in the homotopy groups of $P^n(2)$, $n \geqslant 3$, the previous theorem can be used in conjunction with the next result where the mod-2 homology of $\Omega P^{n+1}(2)$ is required. Here recall that $H_*(\Omega P^{n+1}(2); \mathbb{F}_2)$ is isomorphic to the tensor algebra $T[u,v]$ generated by a class u of degree $n-1$ and a class v of degree n [10].

THEOREM 11.2. *There exist elements λ_{2n} in $\pi_{4n-3}\Omega P^{2n}(2)$ with Hurewicz image in mod-2 homology given by the commutators $[u,v] = u\otimes v + v\otimes u$. Furthermore, the order of λ_{2n} is independent of the choice of an element with mod-2 Hurewicz image $[u,v]$ and is given by*

$$4 \quad \textit{if } n \equiv 0(2), \quad \textit{and} \quad 8 \quad \textit{if } n \equiv 1(2)$$

for all $n > 1$.

Fix natural numbers n and k and define an integer which is divisible by 4 with the equation

$$\mu(k,n) = 9^k(4n+1) - 1.$$

The next theorem follows at once from Theorems 11.1 and 11.2.

THEOREM 11.3. *Let n, k and $\mu(k,n)$ be as above.*

(1) *There exist spaces $B(k,n)$ and homotopy equivalences*

$$\Omega P^{4n+2}(2) \to \Omega P^{\mu(k,n)+2}(2) \times B(k,n).$$

Thus $\pi_{2+2\mu(k,n)}P^{4n+2}(2)$ contains a $\mathbb{Z}/8\mathbb{Z}$-summand for all $k \geqslant 1$.

(2) *There exist spaces $C(k,n)$ and homotopy equivalences*

$$\Omega P^{4n}(2) \to \Omega P^{\mu(k,5n-2)+2}(2) \times C(k,n).$$

Thus $\pi_{2+2\mu(k,5n-2)}P^{4n}(2)$ contains a $\mathbb{Z}/8\mathbb{Z}$-summand for all $k \geqslant 1$.

(3) *There exist spaces $D(k,n)$ and homotopy equivalences*

$$\Omega P^{4n+1}(2) \to \Omega P^{\mu(k,15n-2)+2}(2) \times D(k,n).$$

Thus $\pi_{2+2\mu(k,15n-2)}P^{4n+1}(2)$ contains a $\mathbb{Z}/8\mathbb{Z}$-summand for all $k \geqslant 1$.

(4) *There exist spaces $E(k,n)$ and homotopy equivalences*

$$\Omega P^{4n+3}(2) \to \Omega P^{\mu(k,3n+1)+2}(2) \times E(k,n).$$

Thus $\pi_{2+2\mu(k,3n+2)}P^{4n+3}(2)$ contains a $\mathbb{Z}/8\mathbb{Z}$-summand for all $k \geqslant 1$.

At this writing, the following questions arise in the study of $\Omega P^{n+1}(2)$.

(1) What are the "indecomposable" factors of $\Omega P^{n+1}(2)$, $n \geqslant 2$?

(2) Do the "indecomposable" factors of $\Omega P^{n+1}(2)$ admit (iterated) fibrations by spheres and their loop spaces if $n \geqslant 2$?

12. On the homotopy theory of $\Sigma\mathbb{R}P^2$, a theorem of Wu

J. Wu studies the natural map $\gamma_n : \Sigma\mathbb{R}P^{n-1} \to BSO(n)$ with homotopy theoretic fibre denoted $X(n)$ [75]. The results here are a partial synopsis of a portion of this study.

Let Z_n denote the homotopy theoretic fibre of γ_n. Let $\mathbb{R}P^b_a = \mathbb{R}P^b/\mathbb{R}P^{a-1}$.

THEOREM 12.1 ([75]). *If $n = 3$, Z_n is homotopy equivalent to $\Sigma^4\mathbb{R}P^2 \vee \Sigma\mathbb{R}P^4_2$.*

Of course γ_n does not admit a cross-section. However, Wu loops γ_3 beyond the connectivity of $\Sigma\mathbb{R}P^2$ to obtain

THEOREM 12.2 ([75]). *There is a homotopy equivalence*

$$\Omega^4_0\Sigma\mathbb{R}P^2 \to \Omega^4_0 BSO(3) \times \Omega^4_0 Z_3.$$

Thus $\Omega^4_0\Sigma RP^2$ is homotopy equivalent to

$$\Omega^3_0 S^3 \times \Omega^4_0(\Sigma^4\mathbb{R}P^2 \vee \Sigma\mathbb{R}P^4_2)$$

after localization at $p = 2$.

Notice that $BSO(3)$, and $\Sigma\mathbb{R}P^4_2$ have nontrivial rational homotopy groups. However, after looping enough to force the rational homotopy groups to vanish, Wu obtains a product decomposition. One corollary is of course that the natural multiplicative map

$$\Omega(\gamma_3) : \Omega\Sigma\mathbb{R}P^2 \to SO(3)$$

A table for $\pi_q(P^n(2))$ with $n \geqslant 3$ and $q - n \leqslant 7$ is given below:

$q-n$	$P^3(2)$	$P^4(2)$	$P^5(2)$	$P^6(2)$	$P^7(2)$	$P^8(2)$	$P^9(2)$	$P^{10}(2)$
-1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
0	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
1	$\mathbb{Z}/4$	$\mathbb{Z}/4$	$\mathbb{Z}/4$	$\mathbb{Z}/4$	$\mathbb{Z}/4$	$\mathbb{Z}/4$	$\mathbb{Z}/4$	$\mathbb{Z}/4$
2	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 2}$
3	$\mathbb{Z}/2^{\oplus 5}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
4	$\mathbb{Z}/2^{\oplus 2} \oplus \mathbb{Z}/4^{\oplus 2} \oplus \mathbb{Z}/8$	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/8$	$\mathbb{Z}/2$	0	0	0
5	$\mathbb{Z}/2^{\oplus 7} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
6	$\mathbb{Z}/2^{\oplus 10}$	$\mathbb{Z}/2^{\oplus 2} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2^{\oplus 2} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2^{\oplus 2}$
7	$\mathbb{Z}/2^{\oplus 5} \oplus \mathbb{Z}/4^{\oplus 3} \oplus \mathbb{Z}/8$	$\mathbb{Z}/2^{\oplus 4}$	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2^{\oplus 4}$	$\mathbb{Z}/2^{\oplus 4}$	$\mathbb{Z}/2^{\oplus 4}$

The notation $\mathbb{Z}/n^{\oplus r}$ is $\mathbb{Z}/n \oplus \cdot \oplus \mathbb{Z}/n$ means the direct sum of r-copies of $\mathbb{Z}/n$.

induces a split epimorphism on the 2-primary component of $\pi_q SO(3)$ for $q > 3$.

Other applications to spaces like S^5, $SU(3)$, and G_2 follow. For example, $\pi_5 \Sigma\mathbb{R}P^2$ is isomorphic to $\oplus_3 \mathbb{Z}/2\mathbb{Z}$; there is a choice of map $\bar{\eta} : S^5 \to \Sigma\mathbb{R}P^2$ such that $\Omega_0^3 SU(3)$ is a 2-local retract of $\Omega_0^4 F(\bar{\eta})$ where $F(\bar{\eta})$ is the homotopy theoretic fibre of $\bar{\eta}$. In addition, identification of infinitely many elements of order 8 in $\pi_* \Sigma\mathbb{R}P^2$ follows from Theorem 12.2 and the results in Section 11.

The product decomposition in Theorem 11.2 has immediate applications for the homotopy groups of $\Sigma\mathbb{R}P^2$. Combining the above with a further detailed analysis of the product decompositions given in Section 10 here, J. Wu [75] determines the following table for the homotopy groups of $P^n(2)$.

13. Hopf invariants and Whitehead products

A useful technique in nonstable homotopy theory is an analysis of the relations between Hopf invariants and Whitehead products. These relations are codified in the Hilton–Milnor theorem and have been studied since the 1950's [6], [4], [9], [69]. One application is a partial analysis of the effect of the qth power map in the group $[\Sigma X, \Sigma X]$ on the homotopy groups of ΣX [6].

Many features of these relations can be described by the structure of a group given by a certain combinatorial description. Using methods originating with work of W. Magnus [44] and M. Lazard [42], a description of this group is given in Theorem 13.1. Computations with the classical distributivity law then follow from relations in a group; this information is given in [18]. Several applications follow directly one of which is summarized here.

The purpose of this section is to present "universal examples" for certain groups which appear in classical nonstable homotopy theory. Consider the groups $[X^n, \Omega\Sigma X]$, $[J_n X, \Omega\Sigma X]$, and $[\Omega\Sigma X, \Omega\Sigma X]$ where X is assumed to be a suspension and $J_n X$ denotes the nth stage of the James construction JX [37]. The phrase "universal example" for $[J_n X, \Omega\Sigma X]$ is defined in the next paragraph.

Choose elements in the group $[\Omega\Sigma X, \Omega\Sigma X]$ as follows:

(1) the class of the kth power map $k : \Omega\Sigma X \to \Omega\Sigma X$ with k a natural number, and

(2) the class of the composite for each natural number k

$$\Omega\Sigma X \xrightarrow{h_k} \Omega\Sigma X^{(k)} \xrightarrow{\Omega\omega_k} \Omega\Sigma X$$

where h_k is the kth James–Hopf invariant, $X^{(k)}$ is the k-fold smash product, and $\omega_k : \Sigma X^{(k)} \to \Sigma X$ is a k-fold iterated Whitehead product.

Let $\mathcal{S}(X)$ denote the subgroup of $[\Omega\Sigma X, \Omega\Sigma X]$ generated by the elements (i)–(ii) above. Let $\mathcal{S}_n(X)$ denote the subgroup of $[J_n X, \Omega\Sigma X]$ given by the image of the restriction map $[\Omega\Sigma X, \Omega\Sigma X] \to [J_n X, \Omega\Sigma X]$ applied to $\mathcal{S}(X)$. Notice that $\mathcal{S}(X)$ and $\mathcal{S}_n(X)$ are quotients of the free group F generated by symbols k and $(\Omega\omega_k) \circ h_k$ one for each element listed in (i)–(ii) above. Define H_n to be the quotient of F modulo the normal subgroup given by the intersection of the kernels of composites $F \to \mathcal{S}_n(X) \to [J_n X, \Omega\Sigma X]$ for every space X which is a suspension. Thus H_n is the smallest group which admits a surjection to $S_n(X)$ for all suspensions X. Define the "universal example" of the groups $[J_nX, \Omega\Sigma X]$ to be the group H_n.

The first step in analyzing the structure of H_n is to analyze the analogous "universal example" for $[X^n, \Omega\Sigma X]$. Given a space X, there are n canonical choices of maps $p_i : X^n \to \Omega\Sigma X$ where p_i is the composite of ith coordinate projection $\pi_i : X^n \to X$ followed by the Freudenthal suspension map $E : X \to \Omega\Sigma X$. Let $F[p_1, \dots, p_n]$ denote the free group generated by the p_i and $\bar{\theta} : F[p_1, \dots, p_n] \to [X^n, \Omega\Sigma X]$ be the induced homomorphism. Let $K_n[x_1, \dots, x_n]$ be the smallest quotient group of $F[p_1, \dots, p_n]$ such that there is an induced homomorphism $\theta : K_n[x_1, \dots, x_n] \to [X^n, \Omega\Sigma X]$ for which $\theta(x_i) = \bar{\theta}(p_i)$ for every suspension $X = \Sigma(X')$.

The groups $K_n[x_1, \dots, x_n]$ are closely related to a noncommutative analogue of an exterior algebra. Here, let V be a free $\mathbb{Z}$-module of rank n with a fixed choice of basis $\{y_1, \dots, y_n\}$. Let J denote the two-sided ideal of the tensor algebra $T[V]$ generated by all monomials in this choice of basis elements given by $y_{i_1} \otimes y_{i_2} \otimes \dots \otimes y_{i_t}$ where $y_{i_j} = y_{i_k}$ for some $1 \leqslant j < k \leqslant n$. Define

$$A[V] = T[V]/J.$$

Let Γ_q denote the qth stage of the lower central series for the group $K[x_1, \dots, x_n]$.

THEOREM 13.1. *If $n \geqslant 1$, $K_n[x_1, \dots, x_n]$ is a torsion free nilpotent group of class n. The filtration quotients Γ_q/Γ_{q+1} are free abelian groups of rank $(q-1)!\binom{n}{q}$. Furthermore, $K_n[x_1, \dots, x_n]$ is isomorphic to the subgroup of the group of units of $A[V]$ generated by $1 + y_i$, $1 \leqslant i \leqslant n$.*

The centers of the groups $K_n[x_1, \dots x_n]$, denoted by Λ_n, are useful here and support an action of the symmetric group on n letters Σ_n. These centers are closely related to certain free Lie algebras over $\mathbb{Z}$. Namely, let $L[V]$ be the free Lie algebra generated by V (where V, of course, is ungraded). Define $\mathrm{Lie}(n)$ to be the linear span of the elements

$$[\cdots[y_{\sigma(1)}, y_{\sigma(2)}]y_{\sigma(3)}]\cdots\ y_{\sigma(n)}]$$

where σ runs over the elements in the symmetric group Σ_n. Thus $\mathrm{Lie}(n)$ is isomorphic to $\bigoplus_{(n-1)!}\mathbb{Z}$ as a $\mathbb{Z}[\Sigma_n]$-module. The module $\mathrm{Lie}(n)$ has appeared in several other contexts recently in connection with algebraic K-theory and conformal field theories. It was proven in the early '70's that $\mathrm{Lie}(n)$ is isomorphic to $H_{n-1}(P_n; \mathbb{Z})$ tensored with the sign representation as a $\mathbb{Z}[\Sigma_n]$-module where P_n is the pure braid group for n-stranded braids.

THEOREM 13.2. *Λ_n is isomorphic to $\mathrm{Lie}(n)$ as a $\mathbb{Z}[\Sigma_n]$-module.*

There is a quotient map $\pi : X^n \to J_nX$ defined in [37]. The natural homomorphism $\pi^* : [J_nX, \Omega Y] \to [X^n, \Omega Y]$ is a split monomorphism of sets. There exist subgroups

H_n of $K_n[x_1, \dots, x_n]$ which correspond to the image of π^* and which were defined above.

THEOREM 13.3. *There is a nonsplit central extension*

$$1 \to \Lambda_n \to H_n \xrightarrow{q_n} H_{n-1} \to 1.$$

Thus there is a tower of groups

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
\Lambda_n \arrow[r] & H_n \arrow[d, "q_n"] \\
\Lambda_{n-1} \arrow[r] & H_{n-1} \arrow[d] \\
 & H_{n-2} \arrow[d] \\
 & \vdots
\end{tikzcd}$$

where q_n is an epimorphism with kernel Λ_n which is isomorphic to Lie(n) as a $\mathbb{Z}[\Sigma_n]$-module. Define H_∞ to be the inverse limit of the groups H_n, $\varprojlim H_n$. Recall the groups $\mathcal{S}_n(X)$ defined in the third paragraph of this article. Restriction induces a surjection $\mathcal{S}_n(X) \to \mathcal{S}_{n-1}(X)$. Define $\mathcal{S}_\infty(X)$ to be $\varprojlim \mathcal{S}_n(X)$.

THEOREM 13.4. *There is a commutative diagram of groups*

$$\begin{tikzcd}
H_n \arrow[r, "\theta_n"] \arrow[d] & {[J_nX, \Omega\Sigma X]} \arrow[d] \\
H_{n-1} \arrow[r, "\theta_n"] & {[J_{n-1}X, \Omega\Sigma X]}
\end{tikzcd}$$

for every suspension X. Furthermore the natural homomorphism

$$\theta : H_\infty \to \varprojlim_n [J_nX, \Omega\Sigma X]$$

has image given by $\mathcal{S}_\infty(X)$.

Thus in the case that the natural homomorphism $[\Omega\Sigma X, \Omega\Sigma X] \to \varprojlim_n [J_nX, \Omega\Sigma X]$ is an isomorphism of groups H_∞ may be regarded as the universal example for $[\Omega\Sigma X, \Omega\Sigma X]$. There are finite p-groups which are p^r-torsion analogues of the groups

$K_n[x_1, \ldots, x_n]$ and H_n. These groups have analogous structure where Lie(n) is replaced by Lie$(n) \otimes_{\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z}$. Also, it should be pointed out that $1 : \Omega\Sigma X \to \Omega\Sigma X$ does not usually represent the identity element in the group $[\Omega\Sigma X, \Omega\Sigma X]$ as the constant map represents the identity. The results above codify the relations between power maps together with compositions of Hopf invariants and Whitehead products. These relations have been of interest since the 1950's; a sample in different form is given by [6], [9]. Also focus on specific compositions is given in [69].

The main point of the analysis of the groups H_n is that the information obtained gives a "global" picture of relations in a simple computable form. Namely the group structure "carries" the relations. This information in turn provides data about the kernel of the looping functor

$$\Omega_* : [J_nX, \Omega\Sigma X] \to [\Omega J_nX, \Omega^2\Sigma X]$$

which is the group homomorphism sending a map f to its looping. In particular if

$$\sum_{i>0} \dim \bar{H}_i(\Sigma X; \mathbb{F}_p) > 1,$$

then Ω_* always has a kernel. This structure implies information concerning essential maps in the kernel of Ω_* and thus about $\pi_*\Sigma X$. It should be remarked that these groups give no new information about the homotopy groups of spheres. However, they do give new information concerning double suspensions which are *not* homotopy equivalent to spheres. As a sample consider the pinch map $h_4 : J_4X \to X^{(4)}$ composed with the 4-fold Samelson product

$$\sigma = [[x_1, x_4], [x_2, x_3]] : X^{(4)} \to \Omega\Sigma X.$$

If X contains a bouquet of 4 mod-p Moore spaces, then $\sigma \circ h_4$ is nonzero mod-p homology. However the looping of $\sigma \circ h_4$ is null-homotopic. These relations after looping have consequences for the distributivity law implied by the Hilton–Milnor theorem; namely many terms vanish.

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CHAPTER 25

Phantom Maps

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Contents

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1. Definitions

Let X be a connected CW-complex and let X_n denote its n-skeleton. A map $f : X \to Y$ is called a *phantom map* if its restriction to each X_n is null homotopic. This paper is a study of such maps from a homotopy point of view.[1] Of course, if X is a finite dimensional space (and hence $X = X_n$ for some finite n) then every phantom map out of X is necessarily trivial up to homotopy. Hence essential phantom maps can occur only when the domain X is an infinite dimensional space. Similarly, if the range Y is a space with only finitely many nonzero homotopy groups, then it follows that the set of homotopy classes $[X,Y] \approx [X_n, Y]$ for some finite n and hence in this case there are no essential phantom maps into Y. So in the following we will deal with domains X which are infinite dimensional and ranges Y with infinitely many nonzero homotopy groups. Of course, there are lots of interesting spaces which satisfy these criteria.

The elusive wispy nature of phantom maps is readily apparent. Such maps appear to be null homotopic from a number of different points of view; indeed, they induce the trivial homomorphism on homotopy groups, in homology, and in cohomology. How then do we detect them? In what cases are they trivial? When a phantom map is essential, what does this imply about its domain and range? This survey will deal with questions such as these.

There is another slightly different notion of a phantom map in the literature. This second notion is more general than the one just given. In it a map $g : Z \to W$ is said to be a phantom map if for any finite complex K and any map $h : K \to Z$, the composition gh is null homotopic, e.g., see [39], [50], and [71]. Notice that if the domain X has a CW decomposition with only a finite number of cells in each dimension, then the two definitions agree. However for spaces not of finite type they do not agree. Indeed, in terms of the second definition, it is possible to have an essential phantom map coming out of a finite dimensional domain. The following example, due to Hilton, Mislin and Roitberg ([24, p. 84]), illustrates this.

EXAMPLE 1. Let $W = (S^n_A \vee S^n_B) \cup_\lambda e^{n+1}$ where $n \geqslant 2$, and A and B are complementary sets of primes, and $\lambda = (1,1) \in \pi_n(S^n_A \vee S^n_B) \approx \mathbb{Z}_{(A)} \oplus \mathbb{Z}_{(B)}$. Then the map $\kappa : W \to S^{n+1}$, which collapses $S_A \vee S_B$ to a point, is essential and yet $\kappa_{(p)} \simeq 0$ for all primes p.

Here X_L denotes the localization of X at a set of primes L, in the sense of Sullivan [66] or Bousfield and Kan [9]. If L contains just one prime p, the notation $X_{(p)}$ will be used. Recall from Sullivan's cellular construction of a localization that if X is a finite dimensional nilpotent complex, then so is X_L. Thus the domain W in Example 1 is finite dimensional. On the other hand, it is well known that if $f : K \to Y$ is a map from a finite complex into a nilpotent space such that the composite

$$K \xrightarrow{f} Y \longrightarrow Y_{(p)}$$

is null homotopic for all primes p, then f must be null homotopic ([24, p. 83]). It

[1] All spaces are assumed to have basepoints and all maps and homotopies are assumed to preserve them.

follows that the map κ in Example 1 is an essential phantom map according to the second definition but not according to the first.

Here is a second example involving the localized sphere. By the universal coefficient theorem,

$$H^{n+1}(S^n_{(p)};\mathbb{Z}) \approx \mathrm{Ext}(\mathbb{Z}_{(p)},\mathbb{Z})$$

and this Ext group is nonzero ([19, p. 226]). Hence there is an essential map from $S^n_{(p)}$ to the Eilenberg–MacLane space $K(\mathbb{Z}, n+1)$. Now $S^n_{(p)}$ can be constructed as an infinite mapping telescope using the self maps of S^n whose degrees are relatively prime to p. It follows by a compactness argument that every map of a finite complex into $S^n_{(p)}$ actually factors through S^n. Of course, there are no essential maps from S^n to $K(\mathbb{Z}, n+1)$, and so it follows that there exist essential phantom maps of the second kind from $S^n_{(p)}$ to a $K(\mathbb{Z}, n+1)^2$, or indeed into a sphere S^{n+1}.

It might seem unusual (maybe even unnatural) to consider maps from spaces which are p-local to spaces which are not. And yet, this is a typical situation where phantom maps of the second kind occur. For another unconventional example, consider first maps from finite complexes to Eilenberg–MacLane spaces. This, of course, is classical cohomology from a homotopy point of view. However, when one looks at maps going in the other direction; more precisely, from simply connected Eilenberg–MacLane spaces to finite complexes, then every such map is a phantom map of the first kind, and in general there are lots of them. This situation will be studied in Section 5.

In Section 3, we will study spaces out of which *all* phantom maps are trivial. Using the first definition given it will be shown that if a space X has this property then so too does any localization of it. This statement is false for phantom maps of the second kind, as the second example shows. This is one reason why I prefer the first notion; it behaves as expected with respect to localization. A second reason concerns the algebra involved; often questions involving phantoms maps of the first kind can be answered using *towers* (i.e. inverse sequences) of groups whereas those of the second kind involve inverse *systems* of groups. The latter are more general, of course, but they are also more difficult to work with. Hereafter all phantom maps in this paper will be of the first kind, unless specifically stated otherwise.

2. A bit of history

One of the first people to consider phantom maps was Alex Heller – it was he who named them[3]. The first published account of an essential phantom map – from $\Sigma\mathbb{C}P^\infty$ to an infinite bouquet of 4-spheres, was given by J.F. Adams and G. Walker in [1]. It was in response to a question from Paul Olum. One of the first detailed studies of phantom maps was done by Brayton Gray in his University of Chicago Ph.D. thesis, [20], written

[2] Thus phantom maps of the second kind do not necessarily induce the trivial homomorphism in cohomology, whereas those of the first kind do.

[3] I have been told that Heller felt the term phantom map was appropriate for something defined in terms of skeletons.

under the direction of Michael Barratt. Some of the results obtained there appeared in [22] and are covered here in Section 3.

An important algebraic tool in the computation of phantom maps is the derived functor lim^1. It will be described in detail in Section 4. To the best of my knowledge, a lim^1 construction was first given by Steenrod in 1940 in his paper, "Regular cycles on compact metric spaces" (see [64, p. 845]). Most topologists, however, would probably cite Milnor's 1962 paper "On axiomatic homology theory" as their first encounter with lim^1. Milnor introduced it there as the first derived functor of the inverse limit functor and used it to study the cohomology of infinite complexes. He also credited Steenrod as his source for lim^1. Others quickly saw the importance of this tool in algebra and in topology. Algebraists became interested in the derived functors lim^n in more general settings; e.g., the early work the work of Roos [56], Jensen [28], and Mitchell [46]. In 1966, Gray used a lim^1 computation in [21] to show that there are uncountably many phantom maps from $\mathbb{C}P^\infty$ to S^3. In [3], Anderson and Hodgkin proved the existence of essential phantom maps from various Eilenberg–MacLane spaces to BU, again using abelian lim^1 calculations.

In their book [9] Bousfield and Kan extended the definition of lim^1 to towers of nonabelian groups and showed that a number of its important properties carry over to this more general setting. One property in particular is the short exact sequence of pointed sets

$$* \longrightarrow \varprojlim{}^1[\Sigma X_n, Y] \longrightarrow [X,Y] \longrightarrow \varprojlim[X_n, Y] \longrightarrow *,$$

for any pointed complexes X and Y. This allows one to identify $lim^1[\Sigma X_n, Y]$ with $\mathrm{Ph}(X,Y)$, the set of all homotopy classes of phantom maps from X to Y. It thus enables one to determine $\mathrm{Ph}(X,Y)$ algebraically under very general conditions.

In the late 1970s Willi Meier made a number of interesting discoveries about phantom maps; let me mention just three of them. In [39], he noted the existence of essential phantoms maps which become trivial when localized at any prime. This is the subject of Section 6. In that same paper he gave a formula, which for certain spaces[4], almost reduces the computation of $\mathrm{Ph}(X,Y)$ to a rational calculation

$$\mathrm{Ph}(X,Y) \approx \left[\Sigma X, \left(\widehat{Y}\right)_o\right]/im\hat{\Delta} \approx \prod_k H^k\left(X; \pi_{k+1}(Y) \otimes \mathbb{R}\right)/im\hat{\Delta}.$$

In this formula $\widehat{Y}$ denotes the profinite completion of Y in the sense of Sullivan [66], and $\mathbb{R}$ denotes a rational vector space with the cardinality of the real numbers. Meier noted the relevance of the Sullivan conjecture to computations of this sort; it implies for certain X and Y, the $im\,\hat{\Delta}$ term is zero. This theme is taken up in Section 5.

A map $f : X \to X'$ induces the obvious function $f^* : \mathrm{Ph}(X',Y) \to \mathrm{Ph}(X,Y)$. What is not so obvious is that f^* is an epimorphism when f induces an isomorphism in rational

[4] Both domain and target have finite type and the target is a nilpotent space whose rationalization, Y_o, is an H-space.

homology. Meier noted some special cases of this in [40]. Roitberg and I subsequently pursued this connection between rational equivalences and phantom maps in [35] and [36]. The results we obtained are the subject of Section 7. Thus Sections 5, 6, and 7 of this survey deal with ideas first considered by Willi Meier.

In the early 1980s Haynes Miller proved the Sullivan conjecture; that the space of based maps $map_*(BG, Y)$ is contractible when G is locally finite and Y is finite dimensional [43]. Alex Zabrodsky was one of the first to recognize the implications of this important result in homotopy theory. Among other things he saw how to extend Miller's result to obtain $map_*(X, \widehat{Y}) \simeq *$ when the domain X has only a finite number of nonzero homotopy groups (subject to certain restrictions[5]) and Y is a finite complex. This implies that *every* map between from X to Y is a phantom map. Zabrodsky also saw that in this case the computation of $[X, Y]$ is essentially a rational calculation. He wrote up a preliminary version of [71] soon after the Sullivan conjecture was proved; it was revised and accepted for publication shortly before his untimely death in 1986. The paper, which is discussed more in Section 5, contains a number of interesting results and interesting errors; and both have led to new insights about phantom maps. Most of these offshoots of Zabrodsky's work are due to Joe Roitberg [50], [51], [52], and [23].

The first published mention of a universal phantom map appeared in a paper of J. Lannes [29] in 1987. Letting B denote $\mathbb{R}P^\infty$, Lannes notes in passing that the universal phantom map out of B is an example which shows that the restriction to finite type spaces in his theorem, $[B, Y] \approx \mathrm{Hom}_{\mathcal{K}}(H^*Y, H^*B)$, is necessary. It was here I first learned of it, thanks to Joe Neisendorfer. When I finally understood how the map worked I couldn't wait to tell the world about it. Soon I was drawing pictures of it for my wife, my kids, – anyone who would listen. At an AMS meeting, I had barely started the sketch for Brayton Gray, when he interrupted to say, "Oh yeah, that's the universal phantom map – I discovered it in my thesis 25 years ago." We decided then to combine our results in [22]. This paper is covered in the next section.

3. Universal phantom maps

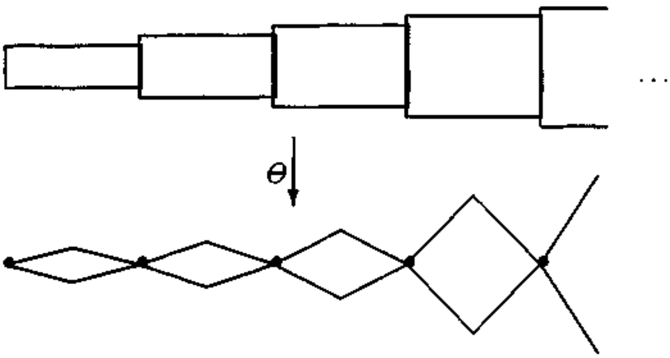
Given space X, how can you tell if it is the domain of an essential phantom map? We now deal with this question and its dual – when is X the target of an essential phantom map? The answer to the first question involves the universal phantom map out of X. This is a phantom map which factors every other phantom map out of X. It is also the *only* nontrivial phantom map that I know how to describe explicitly.

The universal phantom map out of X is a map $\Theta : X \to \overset{\infty}{\vee} \Sigma X_n$ that can be viewed as follows. Identify the space X with the direct limit of its CW-skeletons via the infinite telescope construction. Thus $X \simeq Tel(X)$ where

$$Tel(X) = \bigcup_{n \geqslant 1} X_n \times [n-1, n] / \sim .$$

[5] He also requires X to be simply connected of finite type.

Here each $X_n \times \{n\}$ is identified with its image in $X_{n+1} \times \{n\}$. Now lay the telescope on its side and collapse to a point, each joint at which the second coordinate is an integer.



Then collapse to a point the seam along the basepoint in the target. The resulting map is a surjection from $Tel(X)$ to the infinite wedge[6] of reduced suspensions

$$\overset{\infty}{\vee} \Sigma X_n = \Sigma X_1 \vee \Sigma X_2 \vee \Sigma X_3 \vee \cdots .$$

It is easy to see that the map just described is a phantom map. Indeed, restrict it to the first n stages of the telescope, and then deform that portion to the right into $X_n \times \{n\}$. This is a deformation retraction. Since $X_n \times \{n\}$ is sent to the base point in $\overset{\infty}{\vee} \Sigma X_n$, the assertion follows. Thus Θ is one phantom map which is easy to describe. The question of whether or not it is essential can, in many cases, be answered.

The results in this section appeared in [22] and [31]. A few proofs have been included to illustrate the ideas involved. Most of these results concern phantom maps out of a pointed path-connected space[7] with the homotopy type of a CW-complex. Of course, a space X could have many different CW-decompositions and so the universal phantom map out of X, as just defined, is not unique. It depends on which CW-decomposition is chosen. It will be assumed, in what follows, that a choice has been made.

THEOREM 3.1. *If X is a pointed connected CW-complex, then the map Θ is universal among phantom maps out of X. In other words, given another phantom map $f : X \to Y$, there exists a map $\bar{f}$ such that the following diagram commutes up to homotopy.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Theta"'] & Y \\ \overset{\infty}{\vee} \Sigma X_n \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

[6] This has the weak topology of a direct limit of wedges of finitely many spaces.

[7] The finite type restriction on a number of results in [22] have been removed here.

PROOF. Take the telescope $Tel(X)$, and collapse to a point its seam along the basepoint. Call the quotient space the reduced telescope $T(X)$. Now identify the n-skeleton of X with the image of $X_n \times \{n-1\}$ in $T(X)$. This defines an inclusion $i : \overset{\infty}{\vee} X_n \to T(X)$ which is a cofibration. To see this, let

$$R : I^2 \longrightarrow (0 \times I) \cup (I \times 0) \cup (1 \times I)$$

be the retraction given by stereographic projection from the point, say $(1/2, 3/2)$. Let $R_i(\ ,\ )$ denote the ith coordinate of the value. Now define another retraction

$$I \times Tel(X) \longrightarrow \bigl(0 \times Tel(X)\bigr) \cup \bigl(I \times \cup_n (X_n \times n-1)\bigr),$$

by sending

$$\bigl(s, \langle x, t\rangle\bigr) \mapsto \bigl(R_1(s, \bar{t}), \langle x, [t] + R_2(s, \bar{t})\rangle\bigr).$$

Here $t = \bar{t} + [t]$ where $[t]$ denotes the greatest integer less than or equal to t. This second retraction respects the identifications made in $Tel(X)$ in creating $T(X)$. The second retraction induces a third of $I \times T(X)$ onto $0 \times T(X) \cup I \times \overset{\infty}{\vee} X_n$. It follows by ([63, p. 57]) that $i : \overset{\infty}{\vee} X_n \to T(X)$ is a cofibration as claimed. There is also an obvious homotopy equivalence $\pi : T(X) \to X$, induced by projection on the first factor. Given a phantom map $f : X \to Y$, let $f' = f\pi$ in the following diagram

$$\begin{tikzcd} & Y & \\ \overset{\infty}{\vee} X_n \arrow[r, "i"] & T(X) \arrow[u, "f'"] \arrow[r, "\Theta"] & \overset{\infty}{\vee} \Sigma X_n \arrow[ul, "\bar{f}"'] \end{tikzcd}$$

Since the restriction of f' is null homotopic on each X_n, there is an extension, $\bar{f}$, to the cofiber of i, and so the result follows. □

It follows, of course, that every phantom map out of X is null homotopic if and only if Θ is. With regard to the dependence of Θ on the choice of CW-decomposition of X, this result suggests that one choice is as good as the next. The universal property in Theorem 1 leads to a very simple proof of the following.

COROLLARY 3.2. *If $f : X \to Y$ and $g : Y \to Z$ are two phantom maps, then the composition $gf : X \to Z$ is null homotopic.*

PROOF. The following commutative diagram, in which f and g are phantom maps, is an immediate consequence of Theorem 1.

$$
\begin{tikzcd}
 & & \overset{\infty}{\vee} \Sigma Y_n \arrow[d, "\bar{g}"] \\
X \arrow[r, "f"] \arrow[d, "\Theta"'] & Y \arrow[ur, "\Theta"] \arrow[r, "g"] & Z \\
\overset{\infty}{\vee} \Sigma Y_n \arrow[ur, "\bar{f}"'] & &
\end{tikzcd}
$$

The composition going up the diagonal is a phantom because the second map, Θ, is. The restriction of this composition to each ΣX_n is therefore null homotopic since each summand is finite dimensional. Since a map out of a bouquet is completely determined by such restrictions, we conclude the map along the diagonal is trivial. This, of course, implies that the horizontal composition gf must likewise be null homotopic. □

Recall that a space X is said to be *dominated* by a space Y if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq 1_X$. This, of course, is the homotopy analogue of saying that X is a retract of Y.

THEOREM 3.3. *If X is a pointed connected CW-complex, then all phantom maps out of X are trivial if and only if ΣX is dominated by $\overset{\infty}{\vee} \Sigma X_n$.*

The proof involves the extended cofiber sequence

$$\longrightarrow X \xrightarrow{\Theta} \overset{\infty}{\vee} \Sigma X_n \xrightarrow{\delta} \overset{\infty}{\vee} \Sigma X_n \longrightarrow \cdots$$

and the implications of $\Theta \simeq *$. For more details, see [22].

COROLLARY 3.4. *If ΣX is homotopy equivalent to a bouquet of finite dimensional complexes, then the universal phantom map out of X is trivial.*

The proof of this which was given in [22] is flawed; here is a different one. We can assume that there are cellular homotopy equivalences, say

$$\Sigma X \xrightarrow{\ f\ } \bigvee_{\alpha \in I} K_\alpha \xrightarrow{\ g\ } \Sigma X$$

where $gf \simeq 1$ and where each K_α is a finite dimensional complex. For each n, let $K[n]$ denote the subbouquet consisting of those K_α whose dimension is exactly n. Then $K[n]$ is a retract of the original bouquet and it is dominated by $(\Sigma X)_{n+1}$ since f and g are cellular. So for each n choose maps

$$K[n] \xrightarrow{g[n]} (\Sigma X)_{n+1} \xrightarrow{f[n]} K[n]$$

such that $f[n]g[n] \simeq 1$. It follows, using the maps $\overset{\infty}{\vee} f[n]$ and $\overset{\infty}{\vee} g[n]$ that $\overset{\infty}{\vee} K[n]$ is dominated by $\overset{\infty}{\vee} X_n$. But

$$\overset{\infty}{\vee} K[n] = \bigvee_{\alpha \in I} K_\alpha \simeq \Sigma X$$

and so the result follows.

A familiar example to which Corollary 3.4 applies is when $X = \Omega S^n$. Thus there are no essential phantom maps out of the first loop space of a sphere. More generally one has

EXAMPLE 3.5. If X has the homotopy type of $\Omega(\Sigma L_1 \times \cdots \times \Sigma L_n)$, where each L_i is a connected finite complex, then $\Sigma X \simeq \vee_\alpha K_\alpha$ where each K_α is a finite complex.

This example follows from the well known decomposition

$$\Sigma\Omega\Sigma L \simeq \Sigma\big(L \vee (L \wedge L) \vee (L \wedge L \wedge L) \vee \cdots\big).$$

Is the result in Corollary 3.4 best possible? More precisely we ask

QUESTION 3.6. If ΣX is dominated by $\overset{\infty}{\vee} \Sigma X_n$, does it follow that $\Sigma X \simeq \vee_\alpha K_\alpha$ where each summand K_α is finite dimensional?

The general answer to this question is no as the following example shows.

EXAMPLE 3.7. There exists a CW-complex X, with the property that ΣX is dominated by $\overset{\infty}{\vee} \Sigma X_n$ but ΣX has no nontrivial finite dimensional retracts. Moreover, this space X can be taken to have no odd dimensional cells, and at most one cell in each even dimension.

The construction of this example begins with two homotopy classes of essential maps, say α and β,

$$\begin{tikzcd} & & S^m \arrow[dll, "\alpha"'] \arrow[dl, "\beta"] \\ S^n & S^{n+t} = \Sigma^t S^n & \end{tikzcd}$$

whose orders are finite, stable, and relatively prime, and whose targets are different. In other words, if $\|\alpha\|$ denotes the order of α, assume that $\|\alpha\| = \|\Sigma^n \alpha\|$ for all n, that the same condition holds for β, that $(\|\alpha\|, \|\beta\|) = 1$, and that $t > 0$. For instance, one could take $\alpha = \alpha_1(7)$ in $\pi_{14} S^3$ and $\beta = \Sigma^8 \alpha_1(3)$ in $\pi_{14} S^{11}$ in this construction. The cell complex X can be built using the following suspensions of these maps

$$\begin{tikzcd}
 & S^m \arrow[dl, "\alpha"'] \arrow[d, "\beta"] & \Sigma^t S^m \arrow[dl, "\Sigma^t\alpha"'] \arrow[d, "\Sigma^t\beta"] & \Sigma^{2t} S^m \arrow[dl, "\Sigma^{2t}\alpha"'] \\
S^n & \Sigma^t S^n & \Sigma^{2t} S^n &
\end{tikzcd}$$

More precisely take the space X in this example to be the mapping cone,

$$X = \text{cofiber}\left\{\Psi : \bigvee_{k\geqslant 0} \Sigma^{kt} S^m \longrightarrow \bigvee_{k\geqslant 0} \Sigma^{kt} S^n\right\}$$

where the restriction of Ψ to $\Sigma^{kt}S^m$ is given by

$$\Sigma^{kt}\{S^m \xrightarrow{\ \nu\ } S^m \vee S^m \xrightarrow{\ \alpha\vee\beta\ } S^n \vee S^{n+t}\} \xrightarrow{\subseteq} \bigvee_{j\geqslant 0} \Sigma^{jt} S^n$$

The first map here is the standard comultiplication on the sphere – the one that pinches the equator to a point.

Notice that when ΣX is localized at $\|\alpha\|$, all the suspensions of β become trivial and the resulting space breaks apart into a wedge of mapping cones of the various suspensions of α and that this splitting will split no further. A similar splitting into irreducibles occurs when ΣX is localized at $\|\beta\|$. However if ΣX were to dominate some nontrivial finite dimensional complex, this would imply that at least one of these localized splittings was not of the form just described.

The proof that ΣX is dominated by $\overset{\infty}{\vee}\Sigma X_n$ is more complicated. It amounts to constructing a map

$$X \longrightarrow \bigvee_{k\geqslant 0} \Sigma^{kt} C(\alpha+\beta)$$

with a left inverse. The relative primeness of $\|\alpha\|$ and $\|\beta\|$ is crucial here. For details see [22, p. 381].

There are some special cases worth noting where the answer to Question 3.6 is yes. Here is one of them.

PROPOSITION 3.8. *If $H_n(X;\mathbb{Z})$ is finite for each n sufficiently large then the answer to Question* 3.6 *is yes. In other words, for such spaces X, the universal phantom map out of X is trivial if and only if ΣX decomposes into a bouquet of finite dimensional complexes.*

The proof involves splitting off finite dimensional complexes from ΣX through the use of idempotents. This same technique will be used in the proof of Theorem 3.9.

Recall that an H_o-space is one that, when rationalized, becomes an H-space. Odd dimensional spheres, connected compact Lie groups, and Stiefel manifolds provide familiar examples of H_o-spaces. Notice that if K is a 1-connected finite CW-complex and is also an H_o-space, then by Hopf's theorem it has the rational homotopy type of either a point or a finite product of odd dimensional spheres. The same is true of its double loop space, $\Omega^2 K$. In particular, this means that $\Omega^2 K$, which is almost always infinite dimensional, satisfies the hypothesis of Proposition 3.8. However, there are very few spaces K, that come to mind for which $\Omega^2 K$ splits apart into a bouquet of finite complexes after just one suspension. Consider, for example, the sphere S^n, with n odd. While a theorem of Snaith asserts that $\Omega^2 S^n$ stably splits into an infinite bouquet of finite spectra, this splitting is not achieved after one suspension. In fact, Snaith's splitting is never completely achieved, even 2-locally, after any finite number of suspensions according to Cohen and Mahowald [12]. This suggests the presence of essential 2-local phantom maps phantom maps coming out of $\Omega^2 S^n$. More will be said them in Section 8. But for now, localize at a prime p, and reconsider the domination of ΣX by $\overset{\infty}{\vee}\Sigma X_n$. The next result suggests that Question 3.6 is a problem that is best studied one prime at a time.

THEOREM 3.9. *Let X be a CW-complex with finite type and let p be a fixed prime. Then* $\mathrm{Ph}(X,Y) = *$ *for all p-local spaces Y if and only if $\Sigma X_{(p)}$ is equivalent to a bouquet of finite complexes.*

PROOF. The space ΣX will be assumed to be p-local in this proof, but the notation will not be burdened with this assumption. Reduced integral homology will be used throughout. The initial goal of the proof is to split ΣX into two pieces, say

$$\Sigma X \simeq K \vee L$$

where K is finite dimensional and the connectivity of L is strictly greater than that of ΣX. To this end, we can assume, without any real loss of generality, that the first nonzero homology group of ΣX occurs in degree 2. By hypothesis, there is an inclusion

$$\jmath : \Sigma X \longrightarrow \overset{\infty}{\vee}\Sigma X_n$$

that is a right inverse to the folding map $\mathcal{F} : \overset{\infty}{\vee}\Sigma X_n \to \Sigma X$. Since $H_2\Sigma X$ is a finitely generated $\mathbf{Z}_{(p)}$-module, its image under $\jmath_*$ is contained in a finitely generated summand, say

$$H_2\Big(\bigvee_{1\leqslant n\leqslant t}\Sigma X_n\Big) \subseteq H_2(\overset{\infty}{\vee}\Sigma X_n).$$

Let $\phi : \Sigma X \to \Sigma X$ be the following composition,

$$\Sigma X \xrightarrow{\ \jmath\ } \overset{\infty}{\vee}\Sigma X_n \xrightarrow{\ \ \pi_t\ \ } \bigvee_{1\leqslant n\leqslant t}\Sigma X_n \xrightarrow{\ \ \mathcal{F}_t\ \ } \Sigma X.$$

Note that in homology ϕ induces the identity in degree 2 and the zero map in degrees greater than t. If ϕ induces a pseudoprojection – that is, a homomorphism h such that $image(h) = image(h^2)$ – in all remaining degrees as well, then we will use it to form the telescope

$$Tel\{\Sigma X \xrightarrow{\phi} \Sigma X \xrightarrow{\phi} \Sigma X \xrightarrow{\phi} \cdots\}$$

whose homology realizes the image of ϕ_*. This telescope will have finite dimensional homology. It will be our K. The telescope corresponding to $1-\phi$ will be L. The initial splitting will have been achieved.

If ϕ does not induce a pseudoprojection in some degree d, where $2 < d \leqslant t$, then there is work to be done. We will follow Wilkerson [70] in obtaining the desired pseudoprojection. However, since his results pertain to finite dimensional spaces, a few changes are needed for our purposes.

Let $H = H_{\leqslant t}\Sigma X$, let T denote the torsion subgroup of H, and let $V = H/T$. Since V has finite rank, we can assume that if $r > 0$ is given, then some iterate of φ will induce an idempotent on the finite set, $V \otimes \mathbf{Z}/p^r$. Wilkerson proves the following algebraic fact in step 1 of his Theorem 3.3: if B is an endomorphism of V that induces an idempotent on $V \otimes \mathbf{Z}/p^r$, then there exists a pseudoprojection B' on V, such that

$$B' = B + p^r A.$$

We will combine this fact with the following result – the analogue of Wilkerson's Theorem 3.2. □

LEMMA 3.10. *There is an integer $r > 0$, such that that if A is any endomorphism of the graded $\mathbf{Z}_{(p)}$ module V, then $p^r A$ is realizable by a self-map of ΣX. Moreover this self-map can be taken to induce the zero map in degrees greater than t.*

Assume for the moment that this lemma is true. Take r large enough to satisfy the condition in the lemma, take $B = \phi_*/\text{torsion}$, and let $\alpha : \Sigma X \to \Sigma X$ be the map that realizes $p^r A$ on V. Use the suspension co-H-structure on ΣX to form the sum,

$$\psi = \phi + \alpha : \Sigma X \longrightarrow \Sigma X$$

Then on $H_*\Sigma X$, the self-map ψ induces the identity in degree 2, the zero map in degrees greater than t, and a pseudoprojection on V.

We claim that some iterate of ψ induces a pseudoprojection on H, and hence on all of $H_*\Sigma X$. To simplify the notation in the proof of this claim, let f denote the endomorphism of H induced by ψ. Since the torsion subgroup T is finite, it is clear that the nested sequence of subgroups $\{T \cap image(f^n)\}$ eventually stabilizes. Thus for n sufficiently large,

$$T \cap image(f^n) = T \cap image(f^{2n}).$$

Now if f induces a pseudoprojection on V, then so does f^n. This means that for any element $y \in H$,

$$f^n(y) = f^{2n}(x) + z$$

for some $x \in H$, and $z \in T$. This equation implies that z is in the image of f^n, and hence in $image(f^{2n})$, when n is sufficiently large. In this case, the claim

$$image(f^n) = image(f^{2n}) \quad \text{on } H$$

follows.

We have shown that ψ^n, for n sufficiently large, induces a pseudoprojection on $H_*\Sigma X$. As indicated earlier, we then use the telescope construction to obtain a splitting

$$\Sigma X \simeq K \vee L$$

where the homology of K realizes the image of ψ^n_*, and H_*L realizes the image of $(1-\psi^n)_*$. Notice that the space L is 2-connected. Thus the argument just used to split a finite dimensional retract off ΣX can be used again to do the same to L. The appropriate homology idempotent can be obtained as the composition

$$L \longrightarrow \Sigma X \longrightarrow \bigvee^{\infty} \Sigma X_n \longrightarrow \bigvee_{n=1}^{\tau} \Sigma X_n \longrightarrow \Sigma X \longrightarrow L$$

where $\tau > t$. Care should be taken here to choose the inclusion on the left and the retraction on the right to be compatible with the self map $1 - \phi$. In addition the integer τ should be taken large enough to ensure that in this composition of five maps , the middle three induce the identity on ΣX up through dimension t. Repeating this process over and over, and then taking limits, it follows that

$$\sigma X \simeq \bigvee_{\alpha} K_\alpha$$

where each summand k_α is a finite complex.

PROOF OF LEMMA 3.10. Choose an integer τ large enough that the composition

$$\Sigma X \xrightarrow{\jmath} \overset{\infty}{\vee} \Sigma X_n \xrightarrow{\pi_\tau} \bigvee_{1 \leqslant n \leqslant \tau} \Sigma X_n \xrightarrow{\mathcal{F}_\tau} \Sigma X_\tau$$

induces the identity on H. Here we have used the inclusion $\Sigma X_\tau \to \Sigma X$ to identify the homology of the two spaces in degrees $\leqslant t$. Now ΣX_τ has the rational homotopy type of a finite bouquet of spheres; let W denote the subbouquet consisting of those spheres of dimension $\leqslant t$. Since ΣX_τ is a finite co-H-space, Wilkerson's Theorem 3.2 shows that there is an integer r, and maps

$$\Sigma X_\tau \longrightarrow W \longrightarrow W \longrightarrow \Sigma X_\tau$$

whose composition realizes the endomorphism $p^r A$ on V. So take the composition $\Sigma X \to \Sigma X_\tau$ mentioned first, follow it by this one, and then compose that by the inclusion $\Sigma X_\tau \to \Sigma X$. This is the required map. □

For some spaces X it is easy to show that $\Sigma X_{(p)}$ is not homotopy equivalent to a bouquet of finite dimensional spaces. Consider the case where $H^*(X;\mathbb{Z}/p)$ is a polynomial algebra. The Steenrod algebra then acts in such a way that every nonzero orbit in positive degrees is an infinite set. As a result $\Sigma X_{(p)}$ can have no proper finite dimensional retracts.

It seems harder to verify the splitting of $\Sigma X_{(p)}$, when it occurs than to rule it out when it doesn't. A case in point is the following conjecture, which seems beyond the reach of current techniques.

CONJECTURE. If K is a 1-connected finite complex, then for all primes sufficiently large, $\Sigma\Omega K_{(p)} \simeq \vee_\alpha F_\alpha$ where each F_α is finite dimensional.

Recall that a space X is said to be atomic at a prime p, if any self map f, of its completion $\widehat{X_p}$, is either an equivalence or, under iteration, $f^n \to 0$ in the profinite topology on $[\widehat{X_p}, \widehat{X_p}]$. In particular, atomic spaces have no nontrivial idempotents and hence, they have no proper retracts.

COROLLARY 3.11. *Assume that X has finite type. If for some prime p, either $\Sigma X_{(p)}$ or $\widehat{\Sigma X_p}$ has an infinite dimensional atomic retract, then the universal phantom map out of X is nontrivial.*

Here is one simple but important application of this corollary.

EXAMPLE 3.12. Let G be a compact Lie group. Then the universal phantom map out of BG is trivial if and only if G is the trivial group.

This follows since for some prime $H^*(BG;\mathbb{Z}/p)$ must contain an element of infinite height.

One might be tempted to conclude that if the universal phantom map vanishes at every prime, then it must in fact be trivial. The next example shows that this is not the case.

EXAMPLE 3.13. Let

$$X = \text{cofiber}\left\{\alpha_1 : \bigvee_{p\geqslant 3} S^{2p} \longrightarrow S^3\right\}$$

where for each prime p, $\alpha_1 \mid S^{2p} = \alpha_1(p)$. There is a phantom map $X \to S^4$, that is stably essential and yet the universal phantom map out of X is trivial at each prime p.

Consider now phantom maps into targets of finite type. Since the outbound universal phantom map takes values in a space *not* of finite type, one might suspect that it is almost too sensitive; that, in some cases, it detects something never seen in a universe of finite types. It will be shown that such suspicion is justified.

QUESTION 3.14. For what spaces X, is $\mathrm{Ph}(X, Y) = *$ for every target Y of finite type?

Here are three rather different examples from [22].

EXAMPLE 3.15. Let X be a CW-complex whose homology groups, $H_n(X; \mathbf{Z})$, are torsion groups for all $n \geqslant 1$. Then $\mathrm{Ph}(X, Y) = *$ for every finite type target Y.

An important special case of this example is the classifying space, BG, for a finite group, G. Recall from Example 3.12 that the universal phantom map out of such a space is essential whenever $G \neq \{1\}$. So this is an instance where the sensitive nature of Θ is quite apparent. The next example represents the other extreme – with no torsion in its homology.

EXAMPLE 3.16. Fix an odd prime p, and take the cofiber of Toda's α-family on S^3. To be more precise, let

$$X = \operatorname{cofiber}\left\{\alpha : \bigvee_{t \geqslant 1} S^{2t(p-1)+2} \longrightarrow S^3\right\}$$

where for each t, $\alpha \mid S^{2t(p-1)+2} = \alpha_t$. Then the universal phantom map out of X is essential, but again $\mathrm{Ph}(X, Y) = *$ for all targets Y, of finite type.

The third example involves the loop space $\Omega^2 S^5$. It is the domain of an essential phantom map into a target of finite type, which vanishes when localized at any prime p.

EXAMPLE 3.17. There is an essential phantom map $\Omega^2 S^5 \to \mathbb{HP}^\infty$, and yet for every prime p and every $n \geqslant 1$, $\mathrm{Ph}(\Omega^2 S^{2n+1}, Y_{(p)}) = *$ for every nilpotent target Y of finite type.

In Section 7 we will return to this problem of targets of finite type. It will be shown in Theorem 7.1 that the question just raised has a rather simple answer. The remainder of this section deals with the Eckmann–Hilton dual problem, that of the universal phantom map *into* a pointed space Y. Let $Y^{(n)}$ denote the Postnikov approximation of Y up through dimension n. Thus $Y^{(n)}$ can be obtained from Y by attaching cells of dimension $n+2$ and higher to achieve a space all of whose homotopy groups are zero in dimensions greater than n. Consider the map

$$Y \xrightarrow{\quad \Delta \quad} \prod_{n \geqslant 1} Y^{(n)}$$

whose nth component is the inclusion $Y \to Y^{(n)}$. The infinite product here has the product topology.

THEOREM 3.18. *Up to a weak homotopy equivalence of the fiber, there is a fibration*

$$\prod_{n \geqslant 1} \Omega Y^{(n)} \xrightarrow{\quad \Gamma \quad} Y \xrightarrow{\quad \Delta \quad} \prod_{n \geqslant 1} Y^{(n)}$$

in which the map Γ is a phantom map. Moreover Γ is universal in the sense that for any CW-complex X and phantom map $f : X \to Y$, there is a lift $\bar{f} : X \to \prod \Omega Y^{(n)}$, so that $f = \Gamma\bar{f}$.

There is a technical point here that should be mentioned. Since the domain of Γ does not necessarily have the homotopy type of a CW-complex, the phrase "Γ is a phantom map" should be interpreted to mean that for any CW-approximation $h : W \to \prod \Omega Y^{(n)}$, the restriction of Γh to any skeleton of W is null homotopic. Similarly, in the next result the statement that the universal phantom map is trivial means that the composite Γh is null homotopic.

THEOREM 3.19. *The universal phantom map into Y is trivial if and only if ΩY is dominated by $\prod \Omega Y^{(n)}$.*

The previous two results appeared in [30]. They can be regarded as the Eckmann–Hilton duals of Theorem 3.1 and 3.3. The next result gives some simple conditions under which phantom maps vanish.

THEOREM 3.20. *The universal phantom map into a space Y is trivial (as is every other phantom map into Y) if any of the following four conditions hold:*

(i) *Y is the profinite completion of some space.*
(ii) *Y is the rationalization of some space.*
(iii) *$\pi_n Y$ is finite for each n.*
(iv) *$\Omega Y \simeq \prod_\alpha L_\alpha$ where each L_α has only finitely many nonzero homotopy groups.*

The following example shows that among familiar spaces, such as closed orientable manifolds, the universal inbound phantom map is almost always essential. It is an easy consequence of a result of Zabrodsky; see Theorems 5.2, 5.4, and 5.6.

EXAMPLE 3.21. If K is a 1-connected finite complex, then the universal phantom map into K is trivial if and only if K has the rational homotopy type of a point.

It was noted earlier that some of these results are the Eckmann–Hilton duals of certain theorems about the universal outbound phantom map. Let me close this section with an example that does not conform to this duality. Recall that Theorem 3.9 said that when localized at a prime p, ΣX is a retract of $\vee \Sigma X_n$ if and only if ΣX has the homotopy type of a bouquet of finite dimensional spaces. It is tempting to conjecture the dual result: when localized at a prime p, ΩY is a retract of $\prod \Omega Y^{(n)}$ if and only if ΩY has the homotopy type of a product $\prod L_\alpha$, where each L_α has only a finite number of nonzero homotopy groups. Here is a counterexample.

EXAMPLE 3.22. Take a Moore space, $Y = S^m \cup_p e^{m+1}$, with $m \geqslant 3$. Then ΩY is a retract of $\prod \Omega Y^{(n)}$, by Theorems 3.19 and 3.20, but ΩY does *not* have the homotopy type of a product $\prod_\alpha L_\alpha$ where each factor has only finitely many nonzero homotopy groups.

PROOF. By way of a contradiction, suppose that ΩY had the weak homotopy type of a product $\prod L_\alpha$ where each L_α has only a finite number of nonzero homotopy groups. Choose one of these factors, call it L, and assume that its last nonzero homotopy group occurs in dimension $n+1$. Since L is a retract of ΩY, $\Omega^n L$ is a retract of $\Omega^{n+1} Y$. However each component of $\Omega^n L$ is a $K(\pi, 1)$ where π is a finite but nontrivial abelian group. This implies there is an essential map of such a $K(\pi, 1)$ into $\Omega^{n+1} Y$. This, of course, contradicts the Sullivan conjecture. In view of Miller's confirmation of that conjecture [43] the example is verified. □

4. The tower approach

A tower of groups is an inverse sequence of groups and homomorphisms, say

$$G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} G_3 \xleftarrow{f_3} \cdots.$$

Such towers arise naturally in homotopy theory when, given spaces X and Y, one sets $G_n = [X, \Omega Y^{(n)}]$ and takes the homomorphism $G_{n-1} \leftarrow G_n$ to be the one induced by the Postnikov decomposition $\Omega(Y^{(n-1)} \leftarrow Y^{(n)})$. There is another slightly different tower whose nth term is $[\Sigma X_n, Y]$ and whose homomorphisms are induced by the inclusions of skeleta $\Sigma(X_{n-1} \to X_n)$. While the first tower has better naturality properties than the second, the two towers contain essentially the same information about phantom maps from X into Y. The fundamental fact is this: if X and Y have the homotopy type of pointed CW-complexes, then there are bijections of pointed sets

$$\varprojlim{}^1 [\Sigma X_n, Y] \approx \mathrm{Ph}(X, Y) \approx \varprojlim{}^1 [X, \Omega Y^{(n)}].$$

These bijections were given by Bousfield and Kan ([9, p. 254, 255]) although special cases of these equivalences go back to Milnor [44].

Before getting into a discussion about the functor lim^1 let us take a closer look at the sort of towers one encounters in the study of phantom maps between reasonably nice spaces. The following result shows that such towers are somewhat restricted.

PROPOSITION 4.1. *Let X and Y be connected nilpotent CW-complexes of finite type. If $G_1 \xleftarrow{\pi_1} G_2 \xleftarrow{\pi_2} \cdots$ denotes the tower $\{[X, \Omega Y^{(n)}]\}$, induced by a Postnikov decomposition of ΩY, then for each n,*

(i) *G_n is a finitely generated nilpotent group,*
(ii) *the kernel of π_n is central in G_{n+1}, and*
(iii) *the cokernel of π_n is a finite abelian group.*

A proof of this result is given in [37]. Notice that the third condition implies that such a tower rationalizes to a tower of epimorphisms. It seems like a difficult problem to characterize algebraically, even up to pro-isomorphism, those towers of the form $\{[X, \Omega Y^{(n)}]\}$. For more on this question see Section 9.

Let us begin with the oldest and most elementary description of $\lim^1$ in the case of abelian towers. Using this description it is easy to see that the $\lim^1$ term vanishes for a tower of epimorphisms or for a tower of trivial maps.

DEFINITION. Given a tower of abelian groups $A = \{A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} A_3 \xleftarrow{f_3} \cdots\}$, let $D : \prod A_n \to \prod A_n$ be defined by

$$(a_1, a_2, a_3, \cdots) \mapsto \big(a_1 - f_1(a_2), a_2 - f_2(a_3), a_3 - f_3(a_4), \ \cdots \ \big).$$

Then $\varprojlim A$ is the kernel of D and $\varprojlim^1 A$ is defined to be the cokernel of D.

The inverse limit can, of course, be viewed as the set of all coherent sequences. Although the lim^1 term is not as easy to describe in general, it does have an elementary interpretation in the following special case. Consider an abelian group A and a nested sequence of subgroups $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ such that $\cap A_n = 0$. These subgroups define a topology on A; they form a neighborhood system around zero. An element $\{a_n\} \in \prod A_n$ can then be regarded as a sequence which converges to zero in this topology on A. It then makes sense to ask, "If a sequence $\{a_n\}$ converges to zero in A, does the series $\sum a_n$ converge in A?" The lim^1 term, in this case, answers this basic question. Indeed, $lim^1 A_n = 0$ if and only if the answer is yes. To see this, remember that $lim^1 A_n = 0$ if and only if the map $D : \prod A_n \to \prod A_n$ is onto. Take a sequence $\{a_n\} \in \prod A_n$ and suppose that $D\{x_n\} = \{a_n\}$. This implies that

$$x_n - x_{n+1} = a_n \quad \text{for } n = 1, 2, 3, \ldots .$$

Then the partial sum $s_n = a_1 + \cdots + a_n = x_1 - x_{n+1}$ and since $x_n \to 0$, it follows that $\sum a_k = x_1$. Thus if the map D is onto, then every sequence converging to zero sums to a convergent series. Conversely, if the series $\sum a_n$ converges to x_1, then it is easy to check that $\{a_n\} = D\{x_n\}$ where $x_{n+1} = x_1 - s_n$.

The following result is an indispensable tool when dealing with towers and their limits.

THEOREM 4.2. *Given a short exact sequence of towers of abelian groups,*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there exists a six term exact sequence

$$0 \longrightarrow \varprojlim A \longrightarrow \varprojlim B \longrightarrow \varprojlim C \longrightarrow \varprojlim{}^1 A \longrightarrow \varprojlim{}^1 B \longrightarrow \varprojlim{}^1 C \longrightarrow 0.$$

Using this result it follows, for example, that an epimorphism between towers induces an epimorphism between lim^1 terms. For another example, consider a tower of proper inclusions, say $\{A_n\}$, of countable groups. Applying Theorem 4.2 to the short exact sequence

$$0 \longrightarrow \{A_n\} \longrightarrow \{A_1\} \longrightarrow \{A_1/A_n\} \longrightarrow 0$$

it follows easily that $lim^1 A_n$ must be uncountably large. The 6 term $lim - lim^1$ sequence is also useful in specific calculations, e.g., see [9, p. 253].

Consider the following three towers and their $\varprojlim^1$ terms.

$$A = \mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p},$$

$$B = \mathbb{Z} \xleftarrow{1} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{3},$$

$$C = \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{3} \mathbb{Z} \xleftarrow{5}.$$

The homomorphisms here are simply multiplication by the integers shown. In B, the nth homomorphism is multiplication by n while in C, the nth homomorphism is multiplication by the nth prime. It follows that

$$\begin{aligned}
\varprojlim{}^1 A &\approx \mathrm{Ext}(\mathbb{Z}_{(\frac{1}{p})}, \mathbb{Z}) \approx \mathbb{R} \oplus_{q \neq p} \mathbb{Z}/q^\infty, \\
\varprojlim{}^1 B &\approx \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \quad \approx \mathbb{R}, \\
\varprojlim{}^1 C &\approx \mathrm{Ext}(D, \mathbb{Z}) \quad \approx \mathbb{R} \oplus \mathbb{Q}/\mathbb{Z}.
\end{aligned}$$

To obtain these results, one can first use Jensen's formula (see Section 10.7). It asserts that $lim^1 A \approx \mathrm{Ext}(\mathrm{colim}(\mathrm{Hom}(A, \mathbb{Z}), \mathbb{Z}))$. In the last calculation, D denotes the additive group of rational fractions with square-free denominators. The symbol $\mathbb{R}$ denotes the real numbers, regarded only as a rational vector space. Using Theorem 4.2 it is not hard to show that $lim^1 A$ is also isomorphic to the p-adic integers mod $\mathbb{Z}$. The torsion in this case comes from p-local integers mod $\mathbb{Z}$.

It was Bousfield and Kan [9] who generalized the notion of lim^1 to towers of groups which are not necessarily abelian.

DEFINITION. Given a tower of groups, $G = \{G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} G_3 \xleftarrow{f_3} \cdots\}$, define $\varprojlim^1 G$ to be the quotient space of the action of the group $\prod G_n$ acting on the set $\prod G_n$ by

$$\{g_n\} \cdot \{x_n\} = \{g_n x_n (f_n(g_{n+1}))^{-1}\}.$$

It is a simple exercise to see that this definition agrees with the previous one when the tower is abelian. When the tower is nonabelian the lim^1 term has no obvious group structure; it is only a pointed set. Bousfield and Kan also showed that the 6 term $lim - lim^1$ sequence generalizes to the nonabelian setting; see [9, p. 252].

PROPOSITION 4.3. *Let $G = \{G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} G_3 \xleftarrow{f_3} \cdots\}$ be a tower where the G_i are compact Hausdorff*[8] *topological groups and the maps f_i are continuous. Then $\varprojlim^1 G = *$.*

[8] If one omits the Hausdorff condition here, as was done in [69, Lemma 2.2], the result is no longer true. For a counterexample take any discrete tower with a nontrivial lim^1 term and put the indiscrete topology on each G_i.

PROOF. Let $\{x_n\}$ denote a typical element in $\prod G_n$ and let $\{*_n\}$ denote the distinguished (identity) element in this product. It suffices to show that $\{x_n\}$ and $\{*_n\}$ lie in the same orbit under $\prod G_n$ action defined above. In other words I need to produce an element $\{g_n\}$ such that

$$g_n \cdot x_n \cdot (f_n(g_{n+1}))^{-1} = *_n \quad \text{for } n = 1, 2, 3, \dots .$$

For each n, define $f_n' : G_{n+1} \to G_n$ to be the map which sends $z \mapsto f_n(z) \cdot x_n^{-1}$. This is no longer a homomorphism, of course, but it is a continuous map. The existence of the element $\{g_n\}$ is then equivalent to the inverse limit of the sequence $\{G_i, f_i'\}$ being nonempty. The latter follows from [17, Theorem 3.6]. □

Using this latest definition of lim^1, let me indicate how a phantom map $f : X \to Y$ determines an element of $lim^1[\Sigma X_n, Y]$. For each n choose a null homotopy of the restriction of the phantom map f to X_n; regard this null homotopy as an extension of $f\,|X_n$ to the reduced cone over the n-skeleton, say $F_n : CX_n \to Y$. Regard the reduced suspension ΣX_n as the union of two cones on X_n. Define a map $\mathcal{F}_n : \Sigma X_n \to Y$ to be F_n on the bottom cone and to be the restriction of F_{n+1} to CX_n on the top cone. The sequence $\{\mathcal{F}_n\}$ then determines an element in $\prod[\Sigma X_n, Y]$ which is *not* well defined (different null homotopies can, of course, lead to different $\mathcal{F}_n$s). However, it is easy to see that in passing to the lim^1 term, all the different choices get sent to the same orbit.

Given a tower G, can one tell whether or not $lim^1 G = *$ without actually computing this term? Fortunately, the answer is often yes. The reason why is the *Mittag-Leffler property*[9] of towers which is often easy to verify and which will, in many important cases, settle this question.

DEFINITION. A tower $G = \{G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots\}$ is said to be *Mittag-Leffler* if for each n the nested sequences of images $G_n^{(m)} = image\{G_n \longleftarrow G_m\}$, $m \geqslant n$ satisfies a chain condition. In other words, these images do not grow ever smaller; instead they stabilize at some finite stage. More precisely , there exists an integer N, which depends on n, such that

$$G_n^{(N)} = G_n^{(m)} \quad \text{for all } m \geqslant N.$$

The abelian towers A, B, and C considered earlier clearly do not have this property. Which of the following examples are Mittag-Leffler?

(i) a tower of finite groups,
(ii) a tower of compact Hausdorff groups and continuous homomorphisms,
(iii) a tower of finite dimensional vector spaces over a field k with k-linear homomorphisms.

The descending chain conditions are evident in the first and third examples and thus these two are Mittag-Leffler. However, the second example, in general, is not. To see this consider Example 4.5, below, with each $H_i = \mathbb{Z}/2$.

[9] This property was so named by Dieudonné and Grothendieck [15] apparently in reference to Mittag-Leffler's theorem about inverse limits of complete Hausdorff uniform spaces (see [7, p. 187]).

Obviously a tower of epimorphisms has the Mittag-Leffler property. More generally, it is known that a tower is Mittag-Leffler if and only if it is pro-isomorphic to a tower of epimorphisms, e.g., see [57]. Since pro-isomorphic towers have isomorphic lim^1 terms it follows that if a tower is Mittag-Leffler, then its lim^1 term is trivial. The following result shows that for towers of countable groups the converse is also true.

THEOREM 4.4. *Let G denote a tower of countable groups. Then $lim^1 G = *$ if and only if the tower is Mittag-Leffler. Moreover if $\varprojlim^1 G \neq *$, then it is uncountably large.*

This theorem appeared in [33]. However the first part of its conclusion (that the Mittag-Leffler property is equivalent to the vanishing of lim^1 for countable towers) was already known in shape theory [16]. The second part of its conclusion was first noted in the abelian case by Gray in [21]. The following example shows that for towers which are not countable the situation is more complicated.

EXAMPLE 4.5. Let $H_1, H_2, H_3, \ldots$ be an infinite sequence of nontrivial groups. For each $n \geqslant 1$, let G_n be the kernel of the projection from $\prod H_i$ onto $H_1 \times \cdots \times H_n$. Then the tower of inclusions $G_1 \leftarrow G_2 \leftarrow \cdots$ is not Mittag-Leffler and yet $\varprojlim^1 G_n = *$.

There are a number of ways one can show that a tower is Mittag-Leffler. Some of them are described in the following three lemmas. The first one is a recent result due to J. Roitberg [55].

LEMMA 4.6. *Let G and H be towers of nilpotent groups. Assume these towers become isomorphic when localized at any prime p. Then G is Mittag-Leffler if and only if H is.*

Notice that this result together with Theorem 4.4 implies that if X and X' are two nilpotent finite type spaces which are locally homotopy equivalent[10] X at each prime p, then $\mathrm{Ph}(X, Y) = *$ if and only if $\mathrm{Ph}(X', Y) = *$ for any finite type target Y. A similar result holds when the domain is fixed and the target is allowed to vary (globally but not locally). Thus the cardinality of $\mathrm{Ph}(X, Y)$ can be regarded as a generic invariant for finite type spaces.

In the next result a homomorphism $f : G \to H$ is said to have a finite cokernel if the image of f has finite index in H. This result was used in [33] in the study of towers of the form $\{Aut\, X^{(n)}\}$.

LEMMA 4.7. *Let $G_1 \leftarrow G_2 \leftarrow \cdots$ be a tower of countable groups in which each map $G_n \leftarrow G_{n+1}$ has a finite cokernel. Then the tower $\{G_k\}$ is Mittag-Leffler if and only if the canonical map $\varprojlim G_k \to G_n$ has a finite cokernel for each n.*

It was observed earlier that a tower epimorphism induces an epimorphism on the lim^1 terms. The following result generalizes this observation.

LEMMA 4.8. *Assume that $f : G \to H$ is a homomorphism between towers of the type described in 4.1. Then the tower H is Mittag-Leffler if the tower G is Mittag-Leffler and the map f rationalizes to an epimorphism.*

[10] In this case the homotopy types of X and X' are said to be in the same *Mislin genus*. For a recent survey of this topic see [32].

In particular if $f : X \to X'$ is a rational homotopy equivalence between finite type domains and Y is a finite type target, then the induced map of towers $\Sigma f^* : \{[\Sigma X', Y^{(n)}]\} \to \{[\Sigma X, Y^{(n)}]\}$ rationalizes to a tower isomorphism. It follows then that $\mathrm{Ph}(X,Y) = *$ if $\mathrm{Ph}(X',Y) = *$.

It is possible to define lim, lim^1, and the Mittag-Leffler condition in settings which are much more general (for example, diagrams of groups indexed by a small category) than the towers considered here. Many of the basic results in this section generalize as well. The interested reader might consult Bousfield and Kan, Chapter 11 or [64].

5. The rationalization-completion approach

For nilpotent spaces X and Y, recall that $r : X \to X_o$ denotes the rationalization of the first and $\hat{e} : Y \to \widehat{Y}$ denotes the profinite completion of the second. Both constructions play central roles in the study of phantom maps. These roles will be described in the first three theorems of this section.

THEOREM 5.1. *Let X and Y be connected nilpotent complexes of finite type. A map $\varphi : X \to Y$ is a phantom map if and only if*

(i) *the composition $\hat{e}\varphi : X \to Y \to \widehat{Y}$ is null-homotopic, or*
(ii) *there is a diagram*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "r"'] & Y \\ X_o \arrow[ur] & \end{tikzcd}$$

which commutes up to homotopy.

Thus the first part characterizes $\mathrm{Ph}(X,Y)$ as the kernel of $\hat{e}_* : [X,Y] \to [X,\widehat{Y}]$ while the second describes it as the image of $r^* : [X_o, Y] \to [X,Y]$.

PROOF OF THEOREM 5.1. In part i) notice that if $\varphi : X \to Y$ is a phantom map, then so is the composite $\hat{e}\varphi : X \to \widehat{Y}$. But $\mathrm{Ph}(X,\widehat{Y}) = *$ because this is isomorphic to the lim^1 term of a tower of profinite groups and continuous homomorphisms which, in turn, is trivial by Proposition 4.3. Going the other way, if $\hat{e}\varphi : X \to \widehat{Y}$ is null homotopic, then so is its restriction to any finite skeleton of X. But this forces $\varphi \mid X_n$ to be null-homotopic by a basic result of Sullivan [66, Theorem 3.2].

In part ii), let X_τ denote the homotopy fiber of the rationalization map $r : X \to X_o$. This fiber is a space whose homotopy groups are torsion, as are its reduced integral homology groups. The fibration sequence

$$X_\tau \xrightarrow{i} X \xrightarrow{r} X_o$$

happens to be a cofibration sequence as well [42]. Now if $\varphi : X \to Y$ is a phantom map, then so is the composite $\varphi i : X_\tau \to Y$. But $\mathrm{Ph}(X_\tau, Y) = *$ by Example 3.15 and so φi must be null. Thus φ factors through the rationalization X_o as claimed. Going in the other direction, first notice that if A_o is the rationalization of a finitely generated abelian group and $\widehat{B}$ is the profinite completion of another finitely generated abelian group then

$$\mathrm{Hom}(A_o, \widehat{B}) = \mathrm{Ext}(A_o, \widehat{B}) = 0.$$

For a proof, see [19, Chapter IX]. Together with basic obstruction theory, this implies that every map from X_o to $\widehat{Y}$ is null homotopic. Therefore if φ factors through X_o it must be a phantom map by this observation and part. □

The theorem of Sullivan, mentioned in the proof, says that if K is a finite complex and Y is as in 5.1, then two maps $f, g : K \to Y$ are homotopic if and only if $\hat{e}f \simeq \hat{e}g$. This implies that for a domain Z, not necessarily of finite type, the kernel of $\hat{e}_* : [Z, Y] \to [Z, \widehat{Y}]$ is the set of all phantom maps of the second kind. In particular this means that in Theorem 5.1, the set $[X_o, Y]$ consists solely of phantom maps of the second kind. However, notice that the only phantom map (of the first kind) in $[X_o, Y]$ is the trivial one! This follows from Theorem 3.3 since the suspension of a rational space is a bouquet of rational spheres.

Both conclusions in Theorem 5.1 fail to hold if the finite type hypothesis on Y is dropped. To see this in part i), recall that if Y were a rational space, then its profinite completion would have the homotopy type of a point. If 5.1 still held this would mean that every map into a rational space is a phantom map, which is obviously nonsense. In part ii) consider the universal phantom map out of $\mathbb{R}P^\infty$. It is essential even though this domain has the rational homotopy type of a point. Thus the description of $\mathrm{Ph}(X, Y)$ as a quotient of $[X_o, Y]$ fails to hold in this generality.

It is apparent from Theorem 5.1 that, under the appropriate restrictions on the spaces involved, the set $[X_o, Y]$ serves as an upper bound on $\mathrm{Ph}(X, Y)$. The following theorem of Zabrodsky describes this upper bound in terms of ordinary rational invariants.

THEOREM 5.2. *If X and Y are 1-connected CW-complexes of finite type, then there is a bijection of pointed sets,*

$$[X_o, Y] \approx \prod_k H^k(X; \pi_{k+1}(Y) \otimes \mathbb{R}),$$

where $\mathbb{R}$ is a rational vector space whose cardinality equals that of the real numbers. Moreover if X is rationally a co-H-space or if Y is rationally an H-space, then there is a natural group structure on $[X_o, Y]$ so that the above bijection is an isomorphism of rational vector spaces.

PROOF. Let $j : Y \to \bar{Y}$ denote an integral approximation of Y. This means that the homotopy groups of Y are torsion-free and finitely generated, the loop space ΩY is homotopy equivalent to a product of Eilenberg–MacLane spaces, and the map j is a rational equivalence. Such approximations exist by [71, Lemma A]. Since the fiber of j

has finite homotopy groups, it follows that j induces a bijection between the sets $[X_o, Y]$ and $[X_o, \bar{Y}]$. Now apply $[\ , \bar{Y}]$ to the cofiber sequence

$$X_\tau \xrightarrow{i} X \xrightarrow{r} X_o$$

to obtain the sequence

$$[X, \bar{Y}] \xleftarrow{r^*} [X_o, \bar{Y}] \longleftarrow [\Sigma X_\tau, \bar{Y}] \xleftarrow{\Sigma i^*} [\Sigma X, \bar{Y}] \longleftarrow .$$

I first claim that the induced function r^* is trivial. To see this recall that every map from X_o to Y (and hence to $\bar{Y}$) is a phantom map of the second kind. When precomposed with r, these become phantom maps of the first kind. But according to Theorem 3.20, part 4, there are no essential phantom maps of the first kind into $\bar{Y}$. The claim follows.

The second claim is that the function Σi^* is injective. Since $\Omega\bar{Y}$ has the homotopy type of a product of $K(\mathbb{Z}, n)$s, this claim is easily seen to be equivalent to the assertion that $r^* : H^*(X_o; \mathbb{Z}) \to H^*(X; \mathbb{Z})$ is the trivial homomorphism. This is true because in each positive degree, the domain of r^* is a rational vector space while its target is a finitely generated abelian group.

Combining these two claims, it follows that there is a bijection between $[X_o, Y]$ and the cokernel of

$$[X_\tau, \Omega\bar{Y}] \xleftarrow{i^*} [X, \Omega\bar{Y}].$$

If one takes the product abelian multiplication on $\Omega\bar{Y}$, this cokernel then has the form

$$\prod H^n(X_o, F_n) \approx \prod Ext(H_{n-1}X_o, \pi_n Y),$$

where F_n is the maximal torsion-free quotient of $\pi_n Y$. The isomorphism here is a consequence of the universal coefficient theorem together with the observation that $\mathrm{Ext}(\mathbb{Q}, T) = 0$, when T is a finite abelian group. Since $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \approx \mathbb{R}$, the description of the set $[X_o, Y]$ given in the statement of this theorem follows.

As the set $[X_o, Y]$ consists solely of phantom maps of the second kind it can be identified as

$$[X_o, Y] \approx \underset{\longleftarrow K<X_o}{\lim{}^1} [K, \Omega Y]$$

where K runs through the finite subcomplexes of X_o, by [64, Theorem 3.3]. This bijection is clearly natural in the second variable. Naturality in the first variable is seen by noting that a map f between two CW-complexes induces a map between their systems of finite subcomplexes (send K to the minimal subcomplex containing the compact image $f(K)$). If X_o is a co-H-space, then it has the homotopy type of a bouquet of rational spheres and it has a CW-decomposition in which each finite subcomplex has the homotopy type of a bouquet of spheres. In this case each group $[K, \Omega Y]$ is abelian and hence so is the lim^1 term. If Y has the rational type of an H-space, then the integral approximation

$\bar{Y}$ can be taken to be an H-space. The map $j : Y \to \bar{Y}$ then induces a bijection $[X_o, Y] \to [X_o, \bar{Y}]$ and also one between the associated lim^1 sets. Since each $[K, \Omega\bar{Y}]$ is abelian the corresponding lim^1 term is again abelian.

The rational vector space structure on this lim^1 term is a consequence of the rational nature of the domain. To be more precise, let λ denote a power map on ΣX_o, with respect to the suspension co-H-structure. Then λ is a homotopy equivalence and so it must induce an isomorphism on the lim^1 term. But it also induces multiplication by λ on each group $[\Sigma K, Y]$ and hence it induces multiplication by λ on the abelian lim^1 term. The divisibility and the absence of torsion follow easily from these two observations. □

COROLLARY 5.3. *Let X and Y be simply connected spaces of finite type. If X is rationally a co-H-space, or if Y is rationally an H-space, then there is a natural group structure on* $\mathrm{Ph}(X, Y)$ *which is abelian and divisible. This group structure comes from identifying* $\mathrm{Ph}(X, Y)$ *with* $lim^1[X, \Omega Y^{(n)}]$.

PROOF. One way to see this is to combine the second part of 5.1 with 5.2. The map $r : X \to X_o$ induces an epimorphism from $\Theta(X_o, Y) = lim^1[K, \Omega Y]$ to $\mathrm{Ph}(X, Y)$. Here one notes that phantom maps of the second kind coincide with those of the first kind when the domain has finite type. As the quotient of a rational vector space, $\mathrm{Ph}(X, Y)$ is divisible. □

Another way to obtain this result is to work directly with the tower $\{[X, \Omega Y^{(n)}]\}$, as was done in [31]. See Theorem 6.3 for a slightly stronger result.

This group structure on $\mathrm{Ph}(X, Y)$ is natural in the following restricted sense. For the moment, let $\mathcal{C}$ denote the class of based spaces with the homotopy type of simply connected CW-complexes of finite type. If $f : X' \to X$ and $g : Y \to Y'$ are maps between members of $\mathcal{C}$, then the induced function

$$\mathrm{Ph}(f, g) : \mathrm{Ph}(X, Y) \to \mathrm{Ph}(X', Y')$$

is a homomorphism provided X, Y and X', Y' satisfy the hypothesis of this corollary.

The next result gives a condition under which the set $\mathrm{Ph}(X, Y)$ and the upper bound $[X_o, Y]$ coincide.

THEOREM 5.4. *Let X and Y be 1-connected CW-complexes of finite type. If the function space $map_*(X, \widehat{Y})$ is weakly contractible, then for every $k \geqslant 0$,*

$$[X, \Omega^k Y] = \mathrm{Ph}\left(X, \Omega^k Y\right) \approx [X_o, \Omega^k Y].$$

PROOF. The first equality follows immediately from the first part of Theorem 5.1. The proof of the other bijection begins with the fact $\mathrm{Ph}(X, Y) \approx r^*[X_o, Y]$ established in Theorem 5.1. In view of the long exact sequence

$$[X_\tau, Y] \longleftarrow [X, Y] \xleftarrow{\;r^*\;} [X_o, Y] \longleftarrow [\Sigma X_\tau, Y] \longleftarrow$$

it suffices to show that the set $[\Sigma X_\tau, Y]$ is trivial. This would follow, of course, from the weak contractibility of the space of based maps $map_*(X_\tau, Y)$. To prove this, use the following result from [43, Proposition 9.5] which is known nowadays as the Zabrodsky lemma.

LEMMA 5.5. *Let $E \to B$ be a principal bundle with structure group G. If $map_*(G, Z)$ is contractible, then $map_*(B, Z) \to map_*(E, Z)$ is a homotopy equivalence.*

Let $Z = \widehat{Y}$ and apply the Zabrodsky lemma to the principal fibration

$$\Omega X_o \longrightarrow X_\tau \longrightarrow X$$

Now $map_*(\Omega X_o, \widehat{Y})$ is weakly contractible, as noted in the proof of Theorem 5.1. Therefore one can take simplicial models which satisfy the requirements of the Zabrodsky lemma and deduce that $map_*(X_\tau, \widehat{Y})$ is weakly contractible. In other words, every map from X_τ to $\Omega^k Y$ is a phantom map of the second kind. However, since the reduced integral homology groups of X_τ are torsion groups, every phantom map of the second kind from this space to a finite type target is trivial. This is proved in the verification of Example 4.1 of [22]. The proof amounts to identifying this set of maps with the lim^1 term of a *system* of finite groups. Thus $map_*(X_\tau, Y)$ is weakly contractible and the theorem follows. □

THEOREM 5.6. *Let Y be a 1-connected finite complex. If*

(i) *X is a 1-connected Postnikov space (i.e. $\pi_n X = 0$ for n sufficiently large) of finite type, or*
(ii) *$X = BG$ where G is a connected Lie group, then $map_*(X, \widehat{Y})$ is weakly contractible.*

The first part of this theorem is due to Zabrodsky; its proof will be given shortly. The second part is a special case[11] of a result of Friedlander and Mislin; see [18]. Both results are consequences of H. Miller's celebrated theorem:

THEOREM 5.7. *If G is a locally finite group and Y is a finite dimensional complex, then $map_*(BG, Y)$ is weakly contractible.*

PROOF. Recall that a group is locally finite if every finite set of elements in it generates a finite subgroup. If π is such a group, then $map_*(K(\pi, n), Y)$ is weakly contractible for each $n \geqslant 1$. The proof of this is by induction, starting with Miller's theorem when $n = 1$. The induction step uses the principal fibration

$$K(G, n-1) \longrightarrow * \longrightarrow K(G, n).$$

Since $map_*(K(G, n-1), Y)$ and $map_*(*, Y)$ are weakly contractible, the Zabrodsky lemma implies the same is true of $map_*(K(G, n), Y)$. Notice that if X is a Postnikov

[11] They prove in Theorem 3.1 that $map_*(BG, Y)$ is weakly contractible when G is a Lie group with finitely many components and Y is any finite dimensional complex.

space, then so is X_τ. Moreover, the homotopy groups of X_τ are locally finite. A finite induction, using the fibrations

$$K(G,n) \longrightarrow X_\tau^{(n)} \longrightarrow X_\tau^{(n-1)}$$

and the Zabrodsky lemma, yields that $map_*(X_\tau^{(n)}, Y)$ is weakly contractible for each $n \geqslant 1$. Finally the same result holds for $map(X^{(n)}, \widehat{Y})$ using the fibrations

$$\Omega X_o^{(n)} \longrightarrow X_\tau^{(n)} \longrightarrow X^{(n)}$$

and the Zabrodsky lemma as was done in the proof of Theorem 5.3. □

Some of the results stated above are not as general as those stated in Zabrodsky's paper. For example, Zabrodsky states in [71, Theorem B(d)], that $\mathrm{Ph}(X,Y)$ is a divisible abelian group isomorphic to a quotient of $\prod H^{n-1}(X;\pi_n Y \otimes \mathbb{R})$ assuming *only* that X and Y are 1-connected with finite type. Even though he makes no claims of naturality, I still have some problems with this assertion and its proof. In particular, he contends (p. 137, last line) that $[X_\tau, \Omega\bar{Y}]$ is abelian and uses this fact to handle a double coset calculation of $\mathrm{Ph}(X,Y)$. Although $\Omega\bar{Y}$ does have the homotopy type of a product of Eilenberg–MacLane spaces, it does not follow that the group $[Z, \Omega\bar{Y}]$ is necessarily abelian. For example, take $\bar{Y}$ to be the two stage Postnikov tower with k-invariant $\iota^2 : K(\mathbb{Z}, 2n) \longrightarrow K(\mathbb{Z}, 4n)$ and let $Z = S^{2n-1} \times S^{2n-1}$. In this case $\Omega\bar{Y} \simeq K(\mathbb{Z}, 2n-1) \times K(\mathbb{Z}, 4n-1)$, but the group $[Z, \Omega\bar{Y}]$ is not even rationally abelian.

Joe Roitberg also wondered about the way in which Zabrodsky was adding phantom maps. In [50], Roitberg showed that if Y is group-like in $\mathcal{C}$, then the induced group structure on $\mathrm{Ph}(X,Y)$ was abelian, divisible and independent of the particular homotopy associative H-structure used on Y. In [51], he proved a dual result in terms of co-group structures on X. The naturality he obtained was thus limited to maps which preserve this set of homotopy associative structures.

Oda and Shitanda have worked out some equivariant versions of Zabrodsky's results on phantom maps in [48] and [49]. There are probably some other nuggets left in his paper worth mining, but the prospective prospector should be warned about some of the dangers. For example, when Zabrodsky claims (p. 131, midpage) that the integral approximation $\bar{Y}$ cannot be the target of an essential phantom map, he really means a phantom map of the first (and *not* of the second) kind. In his Theorem D, part 3, Zabrodsky seems to be claiming that every map from a Postnikov space into a finite *dimensional* complex is a phantom map. And since Miller's theorem holds for finite dimensional targets this might at first glance seem plausible. Nevertheless, there is an easy counterexample in this case; let $X = K(\mathbb{Z}, 2n-1)$ and let $Y = K(\mathbb{Q}, 2n-1)$. The latter is finite dimensional (it can be constructed as a telescope using self maps of the sphere) and there are clearly lots of nonphantom maps between these two spaces. Finally there is the application on p. 135 concerning S^n bundles over $K(\mathbb{Z}, m)$ which is wrong (for n odd) as explained in [33, p. 198].

6. Phantoms which vanish when localized

This section deals with the effects of localization on phantom maps. From an algebraic point of view it is also concerned with the curious failure of the functor lim^1 to commute with localization. Let us begin by considering two extreme examples. First, there are those phantom maps, such as the universal phantom map out of $\mathbb{C}P^\infty$, which remain essential when localized at any prime. At the other extreme is Example 3.17: Every essential phantom map

$$\Omega S^5 \longrightarrow \mathbb{H}P^\infty$$

(and there are uncountably many of them) becomes null homotopic when localized at any prime p. What is one to make of this?

The first example might seem closer to the norm when one recalls what happens in the case of finite complexes. If K is a finite complex and Y is a nilpotent space, then a map $f : K \to Y$ is null homotopic if and only if the composition

$$K \xrightarrow{f} Y \xrightarrow{e_p} Y_{(p)}$$

is null homotopic for each prime p. This is no longer true if one replaces "finite" by "finite dimensional", as Example 1 showed. Nor does it hold when K has finite type but is infinite dimensional, as the second example showed.

The second example might not seem so exotic when one recalls that every phantom map becomes null homotopic when rationalized (this is a consequence of Theorem 3.3). Loosely speaking, localizing at a single prime is not all that different from rationalizing. At any rate this example suggests that we take a closer look at the function

$$\delta_* : \mathrm{Ph}(X, Y) \longrightarrow \prod_p \mathrm{Ph}(X, Y_{(p)})$$

which is induced by the map $\delta : Y \to \prod Y_{(p)}$ whose pth projection is the map $e_p : Y \to Y_{(p)}$. According to a theorem of Steiner [65], the function δ_* is surjective when X and Y have finite type. The results which follow are concerned with the kernel of δ_*.

THEOREM 6.1. *The map δ_* is not injective when X and Y have finite type unless* $\mathrm{Ph}(X, Y) = *$.

This result, from [37], is maddening in how little it says; it claims only that there exist at least two phantom maps, say $f, g : X \to Y$, which are not globally homotopic but that are locally homotopic at every prime p. It would be nice to know if this phenomenon is uniformly distributed throughout $\mathrm{Ph}(X, Y)$ or not. The problem is, of course, that in this generality $\mathrm{Ph}(X, Y)$ is just a pointed set and the induced functions, such as δ_* are just pointed set maps with no other uniform properties known. Here are some special cases, again from [37] where we can say a bit more.

THEOREM 6.2. *Assume X and Y have finite type and* $\mathrm{Ph}(X,Y) \neq *$. *Then the map δ_* is an infinite-to-one covering if any of the following conditions hold:*

(i) *X has finite Lusternik–Schnirelmann category.*
(ii) *$H_n(X;\mathbb{Q}) = 0$ for n sufficiently large.*
(iii) *$\pi_n(Y) \otimes \mathbb{Q} = 0$ for n sufficiently large.*
(iv) $[X, \Omega Y] = \mathrm{Ph}(X, \Omega Y)$.

The case which is best understood is the next one where $\mathrm{Ph}(X,Y)$ has a natural abelian group structure and the map δ_* is a homomorphism. The following result was proved in [31].

THEOREM 6.3. *Assume that X is a CW-complex whose integral homology groups are finitely generated in each degree and that Y is a space whose homotopy groups are finitely generated in each degree. If X is a nilpotent space with the rational homotopy type of a co-H-space, or if the universal cover of Y has the rational homotopy type of an H-space, then*

(i) $\mathrm{Ph}(X,Y)$ *has a natural, divisible, abelian group structure, and*
(ii) *the kernel of δ_* is a divisible subgroup of* $\mathrm{Ph}(X,Y)$.

Remarks. The first part is a better result than was stated in Corollary 5.3. (The simply connected condition in 5.3 is unnecessary when working directly with towers.) The finiteness conditions on X and Y ensure that for each n, the group $[X, \Omega Y^{(n)}]$ is finitely generated and nilpotent. The rational conditions ensure that these groups are rationally abelian, or equivalently, that their commutator subgroups are finite. The lim^1 term of such a tower is seen to be isomorphic to that of its abelianization using the 6 term $lim - lim^1$ sequence and Lemma 3.2 of [69]. This fact enables one to identify $\mathrm{Ph}(X,Y)$ with $\mathrm{Ext}(A,\mathbb{Z})$, where $A = \mathrm{colim}(\mathrm{Hom}([X,\Omega Y^{(n)}],\mathbb{Z}))$. The second part is then a consequence of some homological algebra; namely that the obvious map

$$\mathrm{Ext}(A,\mathbb{Z}) \longrightarrow \prod_p \mathrm{Ext}(A,\mathbb{Z}_{(p)})$$

is always surjective and that it always has a nontrivial kernel (unless, of course, its domain, $\mathrm{Ext}(A,\mathbb{Z})$ is the trivial group). This particular result was one announced by Willi Meier in [39], but as far as I know, he never published a proof of it. Finally, since nonzero divisible groups are never finite, one concludes in part (ii) that the kernel of δ_* is infinitely large whenever $\mathrm{Ph}(X,Y)$ is nontrivial.

Here are some examples of abelian towers G in which the kernel of the induced map $\delta_* : lim^1 G \to \prod lim^1 G_{(p)}$ has been computed (see [37] for the details). The maps in each example are the obvious inclusions.

(i) *If $G_n = 2^n\mathbb{Z}$, then* $\mathrm{kernel}(\delta_*) \approx \mathbb{Z}_{(2)}/\mathbb{Z}$ *and hence is countably infinite.*

(ii) *If $G_n = m_n\mathbb{Z}$, where m_n is the product of the first n primes, then*

$$\text{kernel}(\delta_*) \approx \varprojlim{}^1 G \approx \mathbb{R} \oplus \mathbb{Q}/\mathbb{Z}.$$

(iii) *If $G_n = n!\mathbb{Z}$, then the kernel of δ_* is an uncountably large rational vector space.*

The failure of lim^1 to commute with localization was briefly mentioned earlier. These examples illustrate this phenomenon. The first tower, which is essentially tower A in Section 5, has a lim^1 term isomorphic to $\mathbb{R} \oplus \mathbb{Z}_{(2)}/\mathbb{Z}$. One can see directly that this tower becomes stationary when localized at any odd prime. On the other hand localizing the lim^1 term at an odd prime p leaves $\mathbb{R} \oplus \mathbb{Z}/p^\infty$.

The third example is isomorphic to the tower B in Section 5. In particular $lim^1 B \approx \mathbb{R}$. If the functor lim^1 commuted with localization the kernel of δ^* would necessarily be trivial in this case. Instead it turns out to be uncountably large!

7. Phantoms and rational homotopy equivalences

This section involves phantom maps between spaces of finite type. More precisely, a space X is called a *finite type domain* if each of its integral homology groups is finitely generated; a space Y is referred to as a *finite type target* if each of its homotopy groups is finitely generated. The results in this section are from joint work with Joe Roitberg [35], [36].

Those spaces X, out of which all phantom maps are trivial, were characterized in Section 3 by the property that ΣX is dominated by $\vee \Sigma X_n$. However, there are other spaces X, such as $\mathbb{R}P^\infty$ or the mapping cone in Example 3.16, for which the universal phantom map out of X is essential and yet every phantom map from X to a finite type target is trivial. The following theorem characterizes such domains.

THEOREM 7.1. *Let X be a finite type domain. Then the following statements are equivalent.*

(i) $\mathrm{Ph}(X, Y) = *$ *for every finite type target Y.*
(ii) $\mathrm{Ph}(X, S^n) = *$ *for every n.*
(iii) *There exists a map from ΣX to a bouquet of spheres $\vee S^{n_\alpha}$ that induces an isomorphism in rational homology.*

The example $X = \mathbb{R}P^\infty$ mentioned above, shows that the restriction to finite type targets is necessary in part (i). This same example also shows that sometimes there are no spheres in the bouquet in part (iii). In cases such as this, we define the empty bouquet to be a point. There is an Eckmann–Hilton dual result that goes as follows.

THEOREM 7.2. *Let Y be a finite type target. Then the following statements are equivalent.*

(i) $\mathrm{Ph}(X, Y) = *$ *for every finite type domain X.*
(ii) $\mathrm{Ph}(K(\mathbb{Z}, n), Y) = *$ *for every n.*

(iii) *There exists a weak rational equivalence from a product of Eilenberg–MacLane spaces* $\prod_\alpha K(\mathbb{Z}, n_\alpha)$ *to the basepoint component of* ΩY.

The direction of the rational equivalences in these theorems is important. In Theorem 7.1, for example, one can always construct a rational homology equivalence from an appropriate bouquet of spheres *into* ΣX, but one can not always find one going the other way. A case in point is $X = \mathbb{C}P^\infty$. Since there are essential phantoms from this space to other spaces of finite type, such as the 3-sphere, there is no rational equivalence from its suspension back to a bouquet of spheres. This lack of symmetry is truly an infinite dimensional phenomenon. With finite dimensional suspensions, or with loop spaces with only a finite number of nonzero homotopy groups, one can always get rational equivalences, in both directions, between these spaces and the models featured in Theorem 7.1.

One can, of course, regard $\mathrm{Ph}(X, Y)$ as a functor of two variables. The following results describe how it behaves when one variable is held constant and the other is allowed to vary, subject to certain rational conditions.

THEOREM 7.3. i) *If* $f : X \to X'$ *induces a monomorphism between the rational homology groups of two finite type domains, and* Y *is a finite type target, then*

$$\mathrm{Ph}(X, Y) \xleftarrow{f^*} \mathrm{Ph}(X', Y)$$

is an epimorphism of pointed sets.

ii) *If* $g : Y \to Y'$ *induces a rational epimorphism between the higher homotopy groups of two finite type targets, and* X *is a connected finite type domain, then*

$$\mathrm{Ph}(X, Y) \xrightarrow{g_*} \mathrm{Ph}(X, Y')$$

is an epimorphism of pointed sets.

More precisely, in (ii) it is only necessary that $g : \pi_n Y \otimes \mathbb{Q} \to \pi_n Y' \otimes \mathbb{Q}$ be an epimorphism for each $n \geqslant 2$. Thus Ωg, when rationalized, restricts to a retraction onto the basepoint component of $(\Omega Y')_o$. Dually, the rationalization of Σf, in part (i), has a left inverse.

Willi Meier was the first person to see a connection between phantom maps and rational equivalences. He announced in [40] a special case of 7.3, wherein he required his target to have the rational homotopy type of an H-space. He also assumed that $\mathrm{Ph}(X', Y) = *$ in the first part and that $\mathrm{Ph}(X, Y) = *$ in the second. His result was generalized in [22], where Gray and I removed the H_o hypothesis on Y. Clearly, that result is still a special case of Theorem 7.3.

The next result from [36] is quite different from the previous three in that it does not require any reasonable maps between the domains in part (i), or between the targets in part (ii). Instead, it only requires such maps between the suspensions of the domains, or between the loop spaces of the targets.

THEOREM 7.4. i) *If* X *and* X' *are finite type domains with rational homology equivalences between their suspensions,* ΣX *and* $\Sigma X'$, *in both directions, then* $\mathrm{Ph}(X, Y)$ *and*

Ph(X', Y), *are isomorphic as pointed sets, whenever Y is a finite type target.*

ii) *If Y and Y' are nilpotent finite type targets with rational homotopy equivalences between their loop spaces ΩY and $\Omega Y'$ in both directions, then* Ph(X, Y) *and* Ph(X, Y') *are isomorphic as pointed sets for any finite type domain X.*

The spaces S^3 and $K(\mathbb{Z}, 3)$ have the same rational homotopy type but, of course, there is a rational equivalence between their suspensions in only one direction. In view of Theorem 7.4, this is as it should be since $\text{Ph}(S^3, Y) = *$ for *all* spaces Y, whereas there are essential phantoms from $K(\mathbb{Z}, 3)$ into many finite type targets Y – the simplest example being $Y = S^4$, [41].

All four of the results just stated are proved using the tower approach. The proof of the first theorem will be given here to give some indication of the details involved.

PROOF OF THEOREM 7.1. The case when X has the rational homology of a point is exceptional and will be considered first. According to Example 3.15, statement (i) holds for all such X. This implies (ii) is true, as well. The proof of (iii) is a triviality for such X. Having finished this trivial case, let us henceforth assume that X does not have the rational homology of a point. Let $g : \vee S^{n_\alpha} \to \Sigma X$ be a rational homology equivalence, and let g_α denote its restriction to S^{n_α}. As remarked earlier, there is always a rational homology equivalence in the direction of g, but not always one going in the opposite direction. Now assume that $\text{Ph}(X, Y) = *$ for every finite type target Y. Then in particular

$$\text{Ph}\left(X, S^{n_\alpha}\right) = * \quad \text{for each } \; n_\alpha.$$

This is equivalent to saying that

$$\varprojlim{}^1 \left[(\Sigma X)_k, S^{n_\alpha}\right] = *.$$

Since each group in this tower is countable, the tower is Mittag-Leffler by Theorem 4.4. Moreover, by Lemma 4.7, the tower $G_k = [(\Sigma X)_k, S^{n_\alpha}]$, has the property that the image of the canonical map $\varprojlim G_k \to G_n$ has finite index. Since $[\Sigma X, S^{n_\alpha}]$ maps onto $\varprojlim G_k$, this implies that if

$$f : (\Sigma X)_k \longrightarrow S^{n_\alpha}$$

is a map some nonzero multiple of which extends to the t-skeleton of ΣX, for each $t \geqslant k$, then some nonzero multiple of f extends to all of ΣX. In particular take $k \geqslant n_\alpha$, and take f to a map such that the composite

$$S^{n_\alpha} \xrightarrow{\quad g_\alpha \quad} (\Sigma X)_k \xrightarrow{\quad f \quad} S^{n_\alpha}$$

is nontrivial. Since suspensions have the rational homotopy type of a bouquet of spheres and finite suspensions are universal in the sense of Mimura and Toda [45] it is clear that maps f with this property exist out of each skeleton of ΣX. Consequently one of them has an extension , call it f_α, to all of ΣX. Thus for each α there is a map f_α such that the composite

$$S^{n_\alpha} \xrightarrow{g_\alpha} \Sigma X \xrightarrow{f_\alpha} S^{n_\alpha}$$

is nontrivial. Now sum up these f_αs as follows: first let

$$\hat{f}_\alpha : X \longrightarrow \Omega\big(\vee S^{n_\alpha}\big)$$

denote the adjoint of f_α composed with the obvious inclusion. Then use the loop multiplication to form the finite product of maps into the nth Postnikov approximation,

$$\hat{f}_n = \prod_{n_\alpha \leqslant n} \hat{f}_\alpha^{(n)} : X \longrightarrow \Omega\big(\vee S^{n_\alpha}\big)^{(n)}.$$

The $\hat{f}_n$s form a coherent sequence and so there is a map, say

$$\hat{f} : X \longrightarrow \Omega \vee S^{n_\alpha}$$

that projects to $\hat{f}_n$ for each n. It is clear that the adjoint of $\hat{f}$ is the required rational equivalence.

The proof of (iii) $\Longrightarrow$ (i) amounts to showing that the tower $G_n = [(\Sigma X)_n, Y]$ has the Mittag-Leffler property. To this end, it is enough to show that the image of $[\Sigma X, Y]$ in G_n, induced by restriction to the n-skeleton, has finite index for each n. The sufficiency here follows since this particular image is a lower bound for all the other images of the G_{n+k} in G_n. If it has finite index, then there can be at most a finite number of distinct subgroups of G_n that contain it and so the Mittag-Leffler property follows as a consequence.

Assume now that the rational equivalence $f : \Sigma X \to \vee S^{n_\alpha}$ exists and consider the following diagram

$$\begin{tikzcd}
{[\Sigma X, Y]} \arrow[d] & {[\vee S^{n_\alpha}, Y]} \arrow[l, "f^*"'] \arrow[d] \\
{[\Sigma K, Y]} & {\Big[\bigvee_{n_\alpha \leqslant k} S^{n_\alpha}, Y\Big]} \arrow[l, "f_k^*"']
\end{tikzcd}$$

in which K is a certain $(k-1)$-dimensional subcomplex of X, chosen so that the restriction f_k is a rational equivalence from ΣK to the indicated subbouquet of spheres. This restriction would not necessarily be a rational equivalence if ΣK were the k-skeleton of ΣX; in that case f_k might have rational homology kernel in degree k. To get around this problem, let K be a complex of dimension $k-1$ with the following properties:

(i) $K_{k-2} = X_{k-2}$,

(ii) $H_{k-2}(K;\mathbf{Z}) \approx H_{k-2}(X;\mathbf{Z})$,
(iii) $H_{k-1}(K;\mathbf{Z}) \approx H_{k-1}(X;\mathbf{Z})/\text{torsion}$.

Both homology isomorphisms are to be induced by a map of K into X_{k-1}. Such a K, as well as the map, exists by Theorem 2.1 of [6]. For our homotopy theoretic purposes, we can (and will) regard K as a subcomplex of X by means of the mapping cylinder construction. It is then clear that f restricts, with the help of the cellular approximation theorem, to a rational homotopy equivalence from the subcomplex ΣK to the indicated subbouquet of spheres.

If the restriction, f_k, were a suspension, it would then follow by Lemma 7.1.2 below, that the image of f_k^* is a subgroup of finite index. This fact, together with a diagram chase, would imply that that the image of $[\Sigma X, Y]$ also has finite index in the lower left corner. However, neither f nor f_k was assumed to be a co-H map in this result and so we have to work a little harder.

Take $g \in [\Sigma K, Y]$. The group element g^n can be represented by the nth power map on ΣK followed by the map that represents g. The following lemma then shows that for some sufficiently large power, say λ, the element g^λ is in the image of f_k^*. □

LEMMA 7.1.1. *Let K be a connected complex whose suspension has the homotopy type of a finite complex and let*

$$f : \Sigma K \longrightarrow \bigvee_\beta S^{n_\beta}$$

be a rational equivalence. Then for some power map, λ, on ΣK, of sufficiently high power, there is a commutative diagram

$$\begin{tikzcd}
\Sigma K \arrow[rr, "\lambda"] \arrow[dr, "f"'] & & \Sigma K \\
& \bigvee_\beta S^{n_\beta} \arrow[ur] &
\end{tikzcd}$$

Assume for the moment that this lemma is true. The group $[\Sigma K, Y]$ is, of course, nilpotent and finitely generated. The image of $[\Sigma X, Y]$ in this group is easily seen to be a subgroup that contains the subset $image(f_k^*)$. Therefore the image of $[\Sigma X, Y]$ has finite index in $[\Sigma K, Y]$ by the previous lemma and the following bit of group theory.

LEMMA 7.1.2. *Let G be a finitely generated nilpotent group and assume that H is a subgroup of G such that for each g in G, $g^\lambda \in H$, where λ is some nonzero integer that may depend on g. Then H has finite index in G.*

The inclusions,

$$(\Sigma X)_{k-1} \subseteq \Sigma K \subseteq (\Sigma X)_k,$$

induce maps,

$$\big[(\Sigma X)_{k-1}, Y\big] \longleftarrow [\Sigma K, Y] \longleftarrow \big[(\Sigma X)_k, Y\big],$$

such that the image of the group on the right has finite index in the one on the left. It follows that the image of $[\Sigma X, Y]$ in the left group also has finite index. Since k was arbitrary here, the proof of Theorem 7.1 will be complete once we prove the two lemmas just used.

PROOF OF LEMMA 7.1.1. First choose a set of classes $\{g_\beta\} \subset \pi_* \Sigma K$ so that the map,

$$g : \bigvee_\beta S^{n_\beta} \longrightarrow \Sigma K, \quad \text{where } g\,|\,S^{n_\beta} = g_\beta,$$

is a rational homotopy equivalence and so that

$$f_*(g_\beta) = m_\beta \iota_\beta.$$

Here ι_β denotes the standard inclusion of S^{n_β} into the bouquet and each m_β is a nonzero integer. This is possible since $f : \Sigma K \to \vee S^{n_\beta}$ is a rational equivalence, and so for each β, some nonzero multiple $m_\beta \iota_\beta$ lies in the image of f_* and hence factors as a composition $f \cdot g_\beta$.

Next recall that the rationalization of ΣK can be constructed as an infinite telescope using the power maps given by the suspension co-H-structure. The composition

$$S^{n_\beta} \xrightarrow{\ g_\beta\ } \Sigma K \xrightarrow{\ r\ } (\Sigma K)_0$$

is certainly divisible by m_β, say

$$r_*(g_\beta) = m_\beta g'_\beta,$$

and it may be assumed that this equality holds at some finite stage of the telescope. In other words, for sufficiently large λ, the composition $\lambda \cdot g_\beta$ is divisible by m_β, say

$$\lambda_*(g_\beta) = m_\beta h_\beta.$$

Define $h : \vee S^{n_\beta} \longrightarrow \Sigma K$ by requiring

$$h_*(\iota_\beta) = h_\beta.$$

Since evidently,

$$r \cdot \lambda = r \cdot h \cdot f,$$

it follows from the infinite telescope description of $(\Sigma K)_o$ that for some nonzero power map, ℓ, on ΣK,

$$\ell \cdot \lambda = \ell \cdot h \cdot f.$$

Thus, by replacing λ by $\ell\lambda$ and h by ℓh, the lemma follows. □

PROOF OF LEMMA 7.1.2. If H is a normal subgroup of G, the quotient G/H is then a finitely generated, nilpotent, torsion group. It must, therefore, be a finite group. Thus, the lemma is true if H is normal in G. If H is not normal, consider the sequence of subgroups

$$H \leqslant H_1 \leqslant H_2 \leqslant \cdots$$

where each H_{i+1} is the normalizer in G of H_i. Since H_i contains the ith term of the upper central series of G, it follows that $G = H_n$ for some integer n. Any subgroup that contains H clearly satisfies the hypothesis of this lemma, and so as noted above, each quotient H_i/H_{i-1} is finite. The result follows. □

There is a p-local version of Theorem 7.1 whose proof is isomorphic to the one just given. For a finite type domain X, it says that $\mathrm{Ph}(X, Y) = *$ for any p-local space Y with finite type over $\mathbb{Z}_{(p)}$ if and only if there is a rational equivalence from $(\Sigma X)_{(p)}$ to a bouquet of p-local s spheres. Such a result would apply to the loop space $\Omega^2 S^5$ for example. Although there is no global rational equivalence $\Omega^2 S^5 \to S^3$, there does exist such a map at each prime. The existence of such maps is due to Cohen, Moore, and Neisendorfer [13] for $p \geqslant 5$, to Neisendorfer [47] for $p = 3$, and to Cohen [11] for $p = 2$.

8. Phantom maps out of loop spaces

Let X be a 1-connected finite CW-complex which is not contractible. It is known that for each $k \geqslant 1$, the iterated loop space $\Omega^k X$ has nonzero homology in infinitely many degrees. For $k = 1$ this is easy to see using the Serre spectral sequence. For $k \geqslant 2$ this follows from Hubbuck's theorem that the torus is the only connected finite complex with a homotopy commutative multiplication. It seems reasonable then to look for phantom maps coming out of such loop spaces. However, this setting seems to be a difficult one to search and aside from a few examples and some scattered results not much is known about it.

The first nontrivial example is, of course, the sphere. It was noted in Section 3 that there are no essential phantom maps out of ΩS^n. This follows from Corollary 3.4 and the decomposition

$$\Sigma\Omega S^{n+1} \simeq S^{n+1} \vee S^{2n+1} \vee S^{3n+1} \dots.$$

However, the situation for the higher loop spaces of S^n appears to be very different. When $k \geqslant 2$, the universal phantom map out of $\Omega^k S^n$ is often essential, as will be evident in Proposition 8.2, but I suspect that a much stronger result holds; namely,

CONJECTURE 8.1. The universal phantom map out of $\Omega^k S^n$ is essential at each prime p, for all $n \geqslant 2$, and for all $k \geqslant 2$.

Here is a proof for the case $k = p = 2$ when n is odd. In other words, I will show that $\Omega^2 S^{2m+1}$ does not become homotopy equivalent at the prime 2 to a bouquet of finite dimensional complexes after one suspension. To simplify notation, let $E = \Omega^2 S^{2m+1}$ localized at 2. If ΣE were dominated by $\overset{\infty}{\vee}\Sigma E_n$, it would follow, just as in the proof of Theorem 3.9, that for any finite degree d, there is an integer t and maps

$$E \longrightarrow \Omega\Sigma(E_t) \longrightarrow E$$

such that the composition induces isomorphisms on homotopy groups up to degree d. After taking d to be sufficiently large, this composition would have to be a homotopy equivalence by the results of [10]. But this would imply that the mod 2 homology of E, which is a polynomial algebra on infinitely many generators, can be embedded, as a module over the Steenrod algebra, into $H_*\Omega\Sigma E_t$, which is a finitely generated tensor algebra. I will show that no such embedding exists.

First recall that the mod 2 homology of $\Omega^2 S^{2m+1}$ is a polynomial algebra $\mathbb{Z}/2[x_1, x_2, x_3, \cdots]$ where the degree of x_i is $2^i m - 1$. The action of the Steenrod algebra[12] is the following: for each $i \geqslant 1$,

$$Sq^k x_{i+1} = \begin{cases} x_i^2 & \text{when } k = 1, \\ 0 & \text{when } k = 2, 4, 8, \dots. \end{cases}$$

The action of the Steenrod algebra on the tensor algebra $H_*\Omega\Sigma E_t$ is completely determined by its action on $H_* E_t$ and the Cartan formula.

Now suppose that a is the largest integer exponent such that $2^a m - 1 \leqslant t$. I will show that there is no place in the tensor algebra $H_*\Omega\Sigma E_t$ to send the next generator x_{a+1}. Notice back in $H_* E$ that

$$Sq^I x_{a+1} = (x_1)^{2^a} \neq 0,$$

[12] As usual, this action is the Hom-dual of the action of $\mathcal{A}_2$ on cohomology.

where $I = \{2^{a-1}, \ldots, 4, 2, 1\}$. Thus if the embedding takes x_{a+1} to an element y, it follows that $Sq^I y$ must be nonzero. Now a typical term in this tensor product is a linear combination of monomials, such as $\omega = [w_1][w_2]\cdots[w_n]$, where each w_i is a monomial in the generators, $x_1, x_2, \ldots, x_a$. Take the length of ω to be the obvious one obtained by forgetting the brackets and summing up the exponents; for example, the monomial $[x_1 x_2^2][x_1^3 x_3^4]$ has length 10. Notice that when Sq^1 is applied to a monomial, the result, if nonzero, is longer. Its length is increased by 1. This follows from the description of the action given earlier and the Cartan formula. More generally, if ω is a monomial of length ℓ, then $Sq^k\omega$, if nonzero, is a sum of monomials each of length $\ell + k$. It follows then if y is written as a sum of monomials, the shortest of which has length c, then $Sq^I y$ can be written as a sum of monomials, the shortest of which has length at least $c + 2^a - 1$. However, since the degree of y is $2^{a+1}m - 1$, the degree of $Sq^I y$ is $2^a(2m-1)$ and the longest monomial with this degree is $x_1^{2^a}$. Terms that seem to have greater length in this degree must in fact be zero. The assumption $Sq^I y \neq 0$ thus implies the existence of a monomial of length 1 with degree $2^{a+1}m - 1$ in the tensor algebra $H_*\Omega\Sigma E_t$. But this is absurd, and so the proof follows.

With regard to finite type targets, the following sums up what is known about phantom maps out of the iterated loop space of a sphere. As might be expected, the answer has a decidedly rational flavor.

PROPOSITION 8.2. *There exist essential phantom maps from $\Omega^k S^n$ into finite type targets if and only if $k \geqslant 2$ and*

$$\pi_q \Omega^k S^n \otimes \mathbb{Q} \neq 0 \quad \textit{for some } q \geqslant 2.$$

These targets can be taken to be spheres. However, for each prime p there are no essential phantom maps from $\Omega^k S^n$ to nilpotent p-local targets of finite type over $\mathbb{Z}_{(p)}$.

Remarks. The above conditions translate into $n - 2 \geqslant k \geqslant 2$ when n is odd and $2n - 3 \geqslant k \geqslant 2$ when n is even. The choice of essential targets among spheres is, of course, limited by Theorem 5.2. For example, when n is odd (and n and k satisfy the above inequalities), there is an essential phantom map $\Omega^k S^n \to S^m$ if and only if $m = n - k + 1$. When n is even there appear to be two choices: $m = 2n - k$ or $m = n - k + 1$. The first choice always works; the second works only if $k \geqslant 3$.

This proposition does not have much bearing on Conjecture 8.1. Of course, it implies that the universal phantom map out of $\Omega^k S^n$ is often essential (even, sometimes, when $k > n$) and so in this sense it supports 8.1. However, please note that it does not imply that the universal phantom map is trivial when localized at p; at most, it indicates the delicate nature of the problem.

PROOF OF PROPOSITION 8.2. The first step is to establish that

$$\mathrm{Ph}\left(\Omega^k S^{2n+1}, \Omega^{k-2} S^{2n}\right) \neq 0,$$

when $2n-1 \geqslant k \geqslant 2$. A special case of this (when $k=2$) was proved in [30] and so assume hereafter that $k \geqslant 3$. Using the tower approach, it suffices to show that the tower with jth term given by

$$[\Omega^k S^{2n+1}, (\Omega^{k-1} S^{2n})^{(j)}]$$

is not Mittag-Leffler. There is a map $\Omega S^{2n} \to S^{2n-1}$ which induces an epimorphism on rational homotopy groups. Therefore, by looping this map the appropriate number of times and using Theorem 7.3, it suffices to show that the tower $\{G_j\}$ is not Mittag-Leffler, where

$$G_j = [\Omega^k S^{2n+1}, (\Omega^{k-2} S^{2n-1})^{(j)}].$$

When $k \geqslant 4$, the group G_j is abelian and is easily seen to be isomorphic to $\mathbb{Z} \oplus F_j$ where F_j is some finite abelian group. When $k=3$, the group G_j may not be abelian but it has a finite commutator subgroup and its abelianization has the form just described. To see that the tower $\{G_j\}$ is not Mittag-Leffler, notice that at each odd prime p, there is no map, say

$$\varphi : \Omega^k S^{2n+1} \longrightarrow \Omega^{k-2} S^{2n-1},$$

with degree 1 on the bottom cell. Of course, there is such a map going in the other direction; it comes from the double suspension $S^{2n-1} \to \Omega^2 S^{2n+1}$. Since $\Omega^{k-2} S^{2n-1}$ is atomic at p, by [10], the existence of φ would imply that the image of the double suspension is a summand of $\pi_* S^{2n+1}$. But at each odd prime this is not the case, as can be seen in the unstable image of the J-homomorphism.

The observations just made imply that the tower $\{G_j\}$ is, modulo torsion, pro-isomorphic to a tower of the form

$$\mathbb{Z} \xleftarrow{3} \mathbb{Z} \xleftarrow{5} \mathbb{Z} \xleftarrow{7} \mathbb{Z} \xleftarrow{11} \cdots$$

where each odd prime occurs at least once. Such a tower is evidently not Mittag-Leffler. This completes the proof for the odd spheres.

When $k \geqslant 1$, there is a map $\Omega^k S^{4n-1} \to \Omega^k S^{2n}$ which induces a monomorphism in rational homology. Therefore the proof of 8.2 for the even spheres follows from that for the odd spheres and Theorem 7.3. That the targets can be taken to be spheres is a consequence of Theorem 7.1. The rest of Proposition 8.2 is proved in [30]. The proof of the last part is essentially a corollary of Theorem 8.7 (or a localized version of Theorem 7.1) and the existence of p-local maps $\Omega S^{2n+1} \to S^{2n-1}$ with degree p on the bottom cell. As mentioned at the end of Section 7, such maps were shown to exist by Cohen, Moore, and Neisendorfer. □

The rest of this section will concern only the *first* loop space of a finite complex. The obvious question is: Does there exist a finite complex K for which the universal phantom map out of ΩK is essential? The answer is yes.

EXAMPLE 8.3. Let G be the Lie group $Sp(2), Sp(3), G_2$ or F_4. In each case ΩG is stably atomic at the prime 2 and thus the universal phantom map out of it is essential. Indeed, it is stably essential. The first two of the four loop spaces just mentioned, were shown to be stably atomic by M. Hopkins in his Northwestern Ph.D. thesis [25]. J. Hubbuck then proved all four cases a different way in [27]. The claim in the example follows from this fact and Corollary 3.11.

The next example is an unstable one and in this respect it resembles $\Omega^2 S^{2n+1}$ at the prime 2.

EXAMPLE 8.4. Let $X = \Omega SU(3)$. Then X is stably equivalent to a bouquet of finite spectra, but the universal phantom map out of X is essential at $p = 2$.

The proof of this result is also similar to the one given above for the double loops on an odd sphere; it amounts to showing that one cannot embed the polynomial algebra $H_*\Omega SU(3)$, as a module over the mod 2 Steenrod algebra, into the corresponding finitely generated tensor algebra. The technical details are slightly different; see [22]. I suspect that the result holds for $\Omega SU(n)$, for all $n \geqslant 3$. Hubbuck has shown that $\Omega SU(n)$ is atomic at 2 for all n, [26]. A proof of the stable splitting of $\Omega SU(n)$ was given by M. Crabb and S. Mitchell in [14].

The examples just cited involved the smallest prime, $p = 2$. I do not know of a similar example involving a larger prime, although I suspect that they exist. Indeed, $\Omega Sp(2)$ at $p = 3$ looks like a good candidate. Of course, one would not expect $\Sigma\Omega K$ to be atomic at $p > 2$. It will break into $p - 1$ pieces on general principles. It seems likely that some of these pieces, at some small odd prime, do not decompose completely into bouquets of finite dimensional pieces.

But what about large primes? What happens to $\Sigma\Omega K$ when it is localized at a large prime? If K is one of the Lie groups mentioned in the previous two examples, it is easy to check that $\Sigma\Omega K$ is a bouquet of spheres at all primes $\geqslant 13$. The following theorem shows this phenomenon generalizes to a larger class of spaces. Recall that a space X is said to be *rationally elliptic* if it has finite type and its rational homology groups and its rational homotopy groups both vanish in all sufficiently large degrees. The prototypical example is G/H, where G is a Lie group and H is a closed subgroup. Wilkerson and I proved the following result in [38].

THEOREM 8.5. *If K is a 1-connected finite complex which is rationally elliptic, then for all sufficiently large primes p,*

$$\Omega K \simeq_p \prod_\alpha S^{2n_\alpha+1} \times \prod_\beta \Omega S^{2n_\beta+1}.$$

After one suspension, the product on the right decomposes into a bouquet of spheres. Thus the universal phantom map out of ΩK is trivial at almost all primes when K is a finite complex which is rationally elliptic. It is by no means clear that the same should hold for all finite complexes. For example, Anick has constructed a complex K consisting of eleven 4-cells attached to seven 2-cells, with the property that $H_*(\Omega K;\mathbb{Z})$ has torsion of order n in degree $n+1$ for every natural number n; see [4]. Nothing is known about p-local decompositions of $\Sigma\Omega K$ in this case.

The last topic in this section is that of phantom maps $\Omega K \to Y$, where K is a finite complex and Y has finite type. An essential phantom map under these circumstances has yet to be found! The best we can currently do is prove, in some cases, that none exist and examine its implications.

Given a 1-connected finite CW-complex K, it is a simple matter to construct an appropriate product of odd dimensional spheres and loop spaces of other odd dimensional spheres, along with a map,

$$\prod_{\alpha} S^{2n_\alpha+1} \times \prod_{\beta} \Omega S^{2n_\beta+1} \xrightarrow{\quad h \quad} \Omega K,$$

that induces an isomorphism in rational homology. Finding a map in the opposite direction that induces a rational homology isomorphism is, however, quite another matter. It seems to be a difficult unsolved problem in unstable homotopy theory. The following result deals with one of the few classes of compact spaces for which one knows that such a map exists.

PROPOSITION 8.6. *Let K be a 1-connected finite complex with the rational homotopy type of a product of spheres. Then* $\mathrm{Ph}(\Omega K, Y) = 0$ *for any finite type space Y.*

This result certainly applies to the Lie groups listed in Examples 8.3 and 8.4. Thus, for example, the universal phantom map out of ΩF_4 is essential but every phantom map from this loop space into a finite type target is trivial. This result would also apply to the complex or quaternionic Stiefel manifolds.

PROOF. By hypothesis, K has the rational homotopy type of a finite product of spheres, say P. It is not difficult to construct a map, $f : K \to P$, which induces an isomorphism in rational homology; see, e.g., [33, Proposition 5.1]. Then $\Omega f : \Omega K \to \Omega P$ is also a rational equivalence. Since the universal phantom map out of ΩP is trivial the result then follows from Theorem 7.3. □

Is it always possible to find a map

$$\prod_{\alpha} S^{2n_\alpha+1} \times \prod_{\beta} \Omega S^{2n_\beta+1} \xleftarrow{\quad g \quad} \Omega K,$$

which induces a rational homology isomorphism, when K is rationally elliptic? It is, at large primes, by Theorem 8.5, but I do not know if such maps exist at small primes too.

The last result in this section shows that the question of essential phantom maps $\Omega K \to Y$, where the target has finite type, is equivalent to the problem of finding a rational equivalence $\Omega K \to P$, where P is some product of odd spheres and loop spaces of odd spheres.

THEOREM 8.7. *Let X be a connected, nilpotent, CW-complex of finite type over some subring R of the rationals and assume that all other spaces in this theorem are R-local as well. Assume that X has the rational homotopy type of a product,*

$$P = \prod_{\alpha} S^{2n_\alpha - 1} \times \prod_{\beta} \Omega S^{2n_\beta + 1}.$$

Then there exists a map $X \to P$, which induces an isomorphism in rational homology if and only if $\mathrm{Ph}(X, Y) = *$ *for every finite type target Y.*

I proved this result in [30]. Notice the connection between it and Theorem 7.1. This result says a little bit more about a smaller class of spaces. Combining the two results one sees that to obtain a rational equivalence $\Omega K \to P$ it is enough to know that $\mathrm{Ph}(\Omega K, S^n) = *$ for every $n \geqslant 2$. Then by Theorem 5.2, one could further restrict the search to those n for which $H_{n-1}(\Omega K; \mathbb{Q}) \neq 0$.

One of the main reasons I became interested in finding rational equivalences $\Omega K \to P$ was John Moore's Exponent Conjecture. It says that in the homotopy groups of a finite complex, the exponent of the p-torsion is finite, for each prime p, if and only if the complex is rationally elliptic. This is the problem that motivated Theorem 8.5. Since spheres are known to satisfy this conjecture, so do all rationally elliptic spaces at large primes. But what about at small primes? This is still an open problem. To tackle it, one might start with a rational equivalence $\Omega K \to P$ of minimal degree and hope to derive information about the torsion in $\pi_* \Omega K$ in terms of a minimal map and knowledge of P – at least, that was my plan. However, to get started one needs a map...

There are other reasons for wanting a rational equivalence $\Omega K \to P$. When K is rationally elliptic, the existence of such a map implies that the pointed set $SNT(\Omega K)$, consisting of those homotopy types $[Y]$ with the same n-type as ΩK for all n, has just one element; see [33, Theorem 5]. This fact is crucial in establishing the finiteness of the Mislin genus $\mathcal{G}(\Omega K)$. At first glance, there is no apparent reason why ΩK, being infinite dimensional with respect to both homology and homotopy, should have a trivial SNT-set or a finite Mislin genus. For example, both $SNT(X)$ and $\mathcal{G}(X)$ are uncountably large when $X = BSP(2)$. Nevertheless, I know of no 1-connected finite complex K for which $SNT(\Omega K)$ is nontrivial or $\mathcal{G}(\Omega K)$ is an infinite set. For more details on these topics, see [32].

9. Open problems

This section contains most of the open problems previously mentioned and a few others that were not. The first five deal with problems considered in Section 8.

QUESTION 1. Is the universal phantom map out of $\Omega^k S^n$ essential at each prime p, for each $k \geqslant 2$, and for each $n \geqslant 2$?

QUESTION 2. If K is a finite 1-connected complex, does the universal phantom map out of ΩK become trivial when localized at sufficiently large primes?

By Theorem 3.9, this is equivalent to asking if $\Sigma\Omega K$ is homotopy equivalent to a bouquet of finite complexes at all sufficiently large primes. As mentioned in Section 8, the answer is known to be yes when K has only a finite number of nonzero rational homotopy groups. The answer is also yes when K has the rational homotopy type of a suspension. By a result of Anick [5] it is also true if K has Lusternik–Schnirelmann category two and $H_*(\Omega K;\mathbb{Z})$ has p-torsion for at most a finite number of primes.

The following problem was raised in [38] and is still open. An answer of yes would lend some credibility to the previous question.

QUESTION 3. If K is a 1-connected finite complex, do the Steenrod reduced powers $\mathcal{P}^t$ act trivially on $H^*(\Omega K;\mathbb{Z}/p)$ for all primes sufficiently large?

QUESTION 4. Does there exist a finite complex K and an essential phantom map from ΩK to a target of finite type?

QUESTION 5. Let G/H be a homogeneous space where G is a compact Lie group and H is a closed subgroup. Is there a map

$$\Omega(G/H) \longrightarrow \prod_\alpha S^{2n_\alpha-1} \times \prod_\beta \Omega S^{2n_\beta+1}$$

which induces an isomorphism in rational homology?

Question 5 is admittedly a special case of the one that preceded it, by Theorem 8.7. However, the homogeneous spaces provide good test cases. It is possible that techniques needed to handle the homogeneous spaces might also handle the rationally elliptic spaces at small primes. In particular, what is the answer when $X = U(2n)/Sp(n)$?

QUESTION 6. Suppose that X is an arbitrary CW-complex and that Y is group-like. Is the subgroup of phantom maps in $[X, Y]$ necessarily abelian?

This, of course, is true when X and Y have finite type, as was discussed in Section 5. But I can see no reason why the commutator of two phantom maps should be null homotopic in general.

QUESTION 7. Suppose that X and Y are simply connected finite complexes. Is it true that every map from $Q(X)$ to Y is a phantom map?

As usual, $Q(X) = \mathrm{colim}\,\Omega^n\Sigma^n X$. Thanks to the Sullivan conjecture we now know that $map_*(E, \widehat{Y}) \simeq *$ when E is an abelian Eilenberg–MacLane space. It seems reasonable to ask if the same result holds for other infinite loop spaces.

QUESTION 8. Given finite type spaces X and Y, define

$$A(X,Y) = colim_n\, \mathrm{Hom}\big([\Sigma X, \Omega Y^{(n)}], \mathbb{Z}\big).$$

Then $A(X,Y)$ is a countable torsion-free abelian group. Can every such group be obtained in this manner?

It seems like an impossible problem to characterize, even up to pro-isomorphism, those towers which occur as $\{[X, \Omega Y^{(n)}]\}$ for some spaces X and Y. Question 8 is a weaker version of this problem. I can show that every additive subgroup of $\mathbb{Q}$ can be obtained as $A(X,Y)$ for a suitable choice of X and Y.

QUESTION 9. Suppose that X and X' are two finite type domains that are homotopy equivalent after one suspension. Is it then true, for any target Y, that $\mathrm{Ph}(X,Y) = *$ if and only if $\mathrm{Ph}(X',Y) = *$?

The following problem is the algebraic version of Question 9. It deals with the extent to which $lim^1 G$ depends upon the actual group structures in the tower. To be more precise, suppose that $\phi : G \to G'$ is a bijection between two towers of groups. Thus for each n there is a commutative diagram

$$\begin{tikzcd} G_{n+1} \arrow[r, "\phi_{n+1}"] \arrow[d, "\pi_n"] & G'_{n+1} \arrow[d, "\pi'_n"] \\ G_n \arrow[r, "\phi_n"] & G_n \end{tikzcd}$$

where the horizontal maps are bijections, but not necessarily homomorphisms. Assume also that for each n

(i) ϕ_{n+1} restricts to an isomorphism between the abelian kernels, $\ker(\pi_n)$ and $\ker(\pi'_n)$, and
(ii) ϕ_n induces an isomorphism between the abelian cokernels, $\mathrm{coker}(\pi_n)$ and $\mathrm{coker}(\pi'_n)$.

Here is the problem.

QUESTION 10. Does it follow that $\varprojlim^1 G = *$ if and only if $\varprojlim^1 G' = *$?

When two spaces X and X' become homotopy equivalent after one suspension, the homotopy equivalence between the suspensions will induce a bijection between the towers $\{[\Sigma X, Y^{(n)}]\}$ and $\{[\Sigma X', Y^{(n)}]\}$, with the two properties just mentioned. This bijection is not necessarily a group isomorphism however. Thus Question 10 is the algebraic version of the problem which preceded it.

If the groups G_n are countable then the answer to Question 10 is yes. This can be shown using the Mittag-Leffler condition. Recall from Section 4 that this is a set theoretic condition which, for towers of countable groups, is equivalent to the vanishing of their lim^1 terms. Bijections between towers clearly preserve this property.

QUESTION 11. Do there exist towers of finitely generated nilpotent groups $\{G_n\}$ with the property that $\varprojlim_n{}^1 \varprojlim_m G_n^{(m)} \neq *$ or such that $\varprojlim_c{}^1 \varprojlim_n G_n^{[c]} \neq *$?

Here $G_n^{(m)}$ denotes the image of G_{m+n} in G_n while $H^{[c]}$ denotes the maximal nilpotent quotient of H of class $\leqslant c$. This is a problem that Steiner and I encountered in [37]. Simply put, we know very little about nonabelian towers and are unsure how complicated they can be.

Let G again denote a tower of finitely generated nilpotent groups and consider the function

$$\delta_* : \varprojlim{}^1 G \longrightarrow \prod_p \varprojlim{}^1 G_{(p)},$$

which was studied in Section 6. The main question left open is this:

QUESTION 12. If the kernel of δ_* is nontrivial, does it follow that $\delta_*^{-1}(y)$ is an infinite set for each y in $\varprojlim{}^1 \check{G}$?

This is known to be true in the abelian case, by Theorem 4 of [31] and it is also known to be true in certain special nonabelian cases considered in [37].

QUESTION 13. If $\mathrm{Ph}(X, Y)$ has a natural abelian group structure, does it follow that the torsion in $\mathrm{Ph}(X, Y)$ is contained in the kernel of δ_*?

The examples worked out in Section 6 make this look plausible but Steiner and I can show that this is not true in general on the algebraic level. That is we can construct towers G with torsion in $lim^1 G$ that is not in the kernel of δ_*. However we have yet to show that these towers can be realized as towers of the form $\{[X, \Omega Y^{(n)}]\}$. This is one reason for my interest in Question 8.

The last question deals with towers of large abelian groups and their lim^1 terms. First recall the finitely generated case. Jensen has shown that when G is a tower of finitely generated abelian groups, $lim^1 G \approx \mathrm{Ext}(A, \mathbf{Z})$ where A is a countable torsion-free abelian group. The connection between G and A is given by $A = \mathrm{colim}\ \mathrm{Hom}(G_n, \mathbf{Z})$. His result shows that the possible values of $lim^1 G$ are quite restricted. Indeed, $\mathrm{Ext}(A, \mathbf{Z})$ if nontrivial, is a divisible group with a torsion-free summand of cardinality $2^{\aleph_0}$ and, for each prime p, has p-torsion isomorphic to $\oplus \mathbf{Z}/p^\infty$ where the number of summands can either be finite or $2^{\aleph_0}$. Jensen also showed that for any countable torsion-free abelian group A, the group $\mathrm{Ext}(A, \mathbf{Z})$ can be realized as the lim^1 term of an appropriate tower of finitely generated abelian groups [28]. Roitberg has shown that each of these same Ext groups can be realized as groups of phantom maps $\mathrm{Ph}(X, Y)$ for appropriate choices of spaces X and Y [53].

Suppose that the finitely generated condition on the tower is weakened to where the groups in it are only assumed to be countable abelian. It is still true that the lim^1 term of such a tower is either 0 or uncountably large [33]. However, a nontrivial lim^1 term need no longer be divisible nor must it have a nontrivial torsion-free summand. For an example, consider an $\mathbb{F}_2$-vector space with a countable infinite basis. Take the tower of inclusions whose n-th term is the subspace of vectors whose first $n-1$ coordinates are zero. The lim^1 term in this case is $\prod^\infty \mathbb{F}_2/\oplus^\infty \mathbb{F}_2$. This prompts the following problem.

QUESTION 14. How does one characterize $lim^1 G$ when all the groups in the abelian tower are countable?

When all restrictions (except commutativity) are removed from the groups in the tower there is a very nice characterization of the possible lim^1 terms. Warfield and Huber have shown in this case that $lim^1 G$ is a *cotorsion* group and conversely that every cotorsion group can be realized as the lim^1 term of a tower of abelian groups [67]. Recall that an abelian group B is cotorsion if and only if $\mathrm{Ext}(\mathbb{Q}, B) = 0$. Thus every finite abelian group is cotorsion as is every countable bounded abelian group [19, Chapter 9]. Of course, the simplest example of a group which is *not* cotorsion is the integers $\mathbb{Z}$.

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CHAPTER 26

Wall's Finiteness Obstruction

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1. Historical background

1.1. *Borsuk's question*

A space X is called an *absolute neighborhood retract* or ANR, if every map $F \to X$ from a closed subspace $F \subset Y$ of a normal space Y admits an extension to some neighborhood of F in Y. A typical example is a compact topological n-manifold (compact Hausdorff space, locally homeomorphic to $\mathbb{R}^n$). It is easy to see that a compact ANR has the homotopy type of a CW-complex (see Section 2). Moreover, if X is a compact ANR and $\phi : X \to P$ a homotopy equivalence with P a CW-complex then, as $\phi(X)$ will be contained in a finite subcomplex Q of P, the restriction to Q of a homotopy inverse to ϕ shows that X is a retract up to homotopy of the *finite* complex Q.

Borsuk asked at the International Congress in Amsterdam in 1954 (cf. [7]) whether a compact *metric* ANR is actually homotopy equivalent to a finite CW-complex and he was able to settle this question positively for a special class of compact metric ANR's (admitting a "brick decomposition"). In the simply connected case one can use the concept of a homology decomposition (due to Eckmann and Hilton [18]) to see that such a space is of the homotopy type of a finite CW-complex. In subsequent years many more special cases were settled, in particular the case of a compact n-manifold (see Kirby and Siebenmann [37]). A positive solution to Borsuk's general question was given by West in [72]. Given a compact metric ANR X he constructed a *cell-like* mapping, thus a homotopy equivalence, from $Q \times I_\infty$ onto X, Q a finite polyhedron and I_∞ the Hilbert cube (a countably infinite product of intervals $[0, 1]$), showing that X is homotopy equivalent to Q. Later, Chapman found a different proof of the "Borsuk Conjecture" [13].

1.2. *Wall's answer to a more general question*

A natural generalization of Borsuk's question is the following. Consider an arbitrary topological space X which admits maps $\phi : X \to P$ and $\psi : P \to X$ with P a finite CW-complex, such that $\psi \circ \phi$ is homotopic to id_X; we call such an X *finitely dominated.* A finitely dominated space X is homotopy equivalent to a CW-complex (see Section 2), and the obvious question arises whether such a space is actually homotopy equivalent to a *finite* CW-complex. Indeed, that a finitely dominated space X is not far off from being of the homotopy type of a finite CW-complex can be seen from the following result due to Mather [43].

THEOREM 1.1. *Suppose X is a finitely dominated space. Then $X \times S^1$ is of the homotopy type of a finite CW-complex.*

Here is a sketch of the pretty argument. Let $\phi : X \to P$ and $\psi : P \to X$ be maps with $\psi \circ \phi \simeq id_X$, where P is a finite CW-complex. The *mapping torus* T of $\phi \circ \psi : P \to P$ is obtained from $P \times [0, 1]$ by identifying the points $(x, 0) \in P \times [0, 1]$ with those of the form $(\phi(\psi(x)), 1)$. The homotopy between id_X and $\psi \circ \phi$ gives rise to a homotopy equivalence between $X \times S^1$ and T. But T is obviously homotopy equivalent to a finite CW-complex and thus $X \times S^1$ is too. (We will later discuss a purely algebraic proof of this theorem, using the product formula for finiteness obstructions.) Note also, that Mather's result implies that a finitely dominated space X is homotopy equivalent to a *finite dimensional* CW-complex. Namely, if $X \times S^1 \simeq K$ with K a finite CW-complex of dimension say n, then X is homotopy equivalent to an infinite cyclic covering space of K, which is a locally finite CW-complex of dimension n.

First examples of finitely dominated spaces which are not homotopy equivalent to finite CW-complexes were discovered by de Lyra (see [40]). About at the same time Wall ([68], [69]) found the correct general setup to reduce the topological question to a purely algebraic one. For a finitely dominated connected space X Wall defined an invariant $\tilde{w}(X)$ in the reduced projective class group of $\mathbb{Z}\pi_1(X)$, such that $\tilde{w}(X) = 0$ if and only if X is homotopy equivalent to a finite CW-complex. Moreover, Wall showed that every element in the reduced projective class group of $\mathbb{Z}\pi$, π an arbitrary finitely presented group, may be realized as $\tilde{w}(X)$ for some connected finitely dominated space X with fundamental group π. (If X is a finitely dominated connected space, its fundamental group is necessarily finitely presented, because a retract of a finitely presented group is finitely presented [68]). To get examples of finitely dominated spaces which are not of the homotopy type of any finite CW-complex one therefore just needs to find a finitely presented group whose reduced projective class group is nontrivial. For a finite abelian group π the projective modules over $\mathbb{Z}\pi$ are closely related to ideals in rings of integers of number fields. In particular if $\pi = \mathbb{Z}/p\mathbb{Z}$ with p a prime, the reduced projective class group of π is isomorphic to the ideal class group of the cyclotomic field $\mathbb{Q}(\exp 2\pi\sqrt{-1}/p)$, a group which is known to be trivial for all primes $p \leqslant 19$ and nontrivial for all other primes.

In the following survey we will give an account of Wall's general theory together with some typical applications. A well written and comprehensive introduction to the topic is Varadarajan's book [67], in which the reader will find complete proofs of the many technical details which underlie the basic definitions, and which are omitted in these notes. It is interesting to investigate the question how topological properties of spaces are reflected in properties of their finiteness obstruction. There remain many open questions in that respect. For instance, it is not known whether a finitely dominated loop space (or H-space) is necessarily of the homotopy type of a finite CW-complex. It is also not known if every finitely dominated Eilenberg–MacLane space $K(G,1)$ is homotopy equivalent to a finite CW-complex. The fundamental group G of such a $K(G,1)$ is necessarily finitely presented and torsion-free; at present there are even no examples known of arbitrary torsion-free groups with nontrivial reduced projective class group! We will also discuss the behaviour of the finiteness obstruction in fibration and cofibration sequences, and we will indicate the close relationship with *Whitehead torsion*. For the class of finitely dominated nilpotent spaces the situation is relatively well understood, in the sense that one can give a precise description of the elements in the projective class group which can occur as finiteness obstructions of such spaces. For instance, a finitely dominated nilpotent space with *infinite* fundamental group is always of the homotopy type of a finite CW-complex, and the finiteness obstruction vanishes for finitely dominated nilpotent spaces with fundamental group cyclic of prime order. In the nilpotent situation, it is also natural to analyze the finiteness obstruction via p-local information, an approach which leads to several interesting examples and applications.

We also sketch briefly the relevance of the finiteness obstruction in the spherical space form problem.

There are many other applications, which we had to omit to keep this exposition relatively short and reasonably self-contained; this applies in particular to Siebenmann's exciting work on the finiteness obstruction to finding a boundary for an open manifold [57].

2. Finitely dominated spaces

Recall that X is said to be *dominated* by Y if there are maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq id_X$. A space X which is dominated by a CW-complex has the homotopy type of a CW-complex. Indeed, if $|S(X)|$ denotes the geometric realization of the singular complex of an arbitrary space X then the natural weak homotopy equivalence $\omega : |S(X)| \to X$ will induces for an arbitrary CW-complex P a bijection

$$\omega_* : [P, |S(X)|] \to [P, X]$$

of homotopy sets and, in case X is dominated by a CW-complex, one thus obtains an induced bijection

$$\omega_* : [X, |S(X)|] \to [X, X].$$

An inverse up to homotopy for ω is then given by a representative of $(\omega_*)^{-1}([id_X])$. If the connected topological space X is dominated by a CW-complex P then X is also dominated by a connected subcomplex Q of P, which is a path-connected space, and therefore X is path-connected too because $\pi_0(X)$ is a retract of $\pi_0(Q)$. We can therefore refer to the fundamental group of such a space X in an unambiguous way. Moreover, if a not necessarily connected space X is dominated by a finite CW-complex then X is the topological disjoint union of a finite number of connected (and path-connected) spaces each of which is dominated by a finite connected CW-complex. There is therefore no loss in generality to study finiteness conditions for the case of *connected* finitely dominated spaces and, if convenient, we may even assume our spaces to be actual CW-complexes.

REMARK. If X is a compact ANR, one can embed X in a linear space $\prod \mathbb{R}$ which is a normal space. Therefore, there exists an open neighborhood $N(X)$ of $X \subset \prod \mathbb{R}$ which retracts onto X. But such an $N(X)$ is easily seen to be homotopy equivalent to some open subset of a finite dimensional space $\mathbb{R}^n$, which is triangulable. It follows, that such an X is dominated by a (finite) CW-complex.

For an arbitrary connected CW-complex X and π-module M, π the fundamental group of X, we write

$$H_*(X; M) \quad \text{and} \quad H^*(X; M)$$

for the homology (respectively cohomology) of X with *local* coefficients in M. These groups are defined as the homology groups of the chain complex $M \otimes_\pi C_*^{cell}(\tilde{X})$ (respectively the cohomology of the cochain complex $\mathrm{Hom}_\pi(C_*^{cell}(\tilde{X}), M)$), where $\tilde{X}$ denotes the universal cover of X and $C_*^{cell}(\tilde{X})$ the cellular chain complex of $\tilde{X}$, considered as a π-complex. Note that if M is a trivial π-module, then $H_*(X;M)$ and $H^*(X;M)$ are just ordinary homology and cohomology of X with coefficients in M.

For an arbitrary connected CW-complex X we define

$$\mathrm{cd}(X) \leqslant n \Longleftrightarrow H^i(X;M) = 0, \quad \forall M,\ \forall i > n.$$

Here M stands as before for an arbitrary $\mathbb{Z}\pi_1(X)$-module. Clearly, if X happens to be homotopy equivalent to a connected m-dimensional CW-complex, then certainly $\mathrm{cd}(X) \leqslant m$. The converse is essentially true too.

THEOREM 2.1. *Suppose X is of the homotopy type of a connected CW-complex and $\mathrm{cd}(X) \leqslant n$. Then X is homotopy equivalent to a CW-complex of dimension* $\leqslant \max(3, n)$.

PROOF. Consider a homotopy equivalence $\phi : P \to X$. By restricting ϕ to the $(n-1)$-skeleton of P one obtains an $(n-1)$-connected map $\psi : P^{n-1} \to X$ (thus $\psi_* : \pi_i(P^{n-1}) \to \pi_i(X)$ is bijective for $i < n-1$ and surjective for $i = n-1$). We first consider the case $n > 2$ so that $\pi_1(P^{n-1}) \to \pi_1(X)$ is an isomorphism. One shows then, using the assumption $\mathrm{cd}(X) \leqslant n$, that the homotopy theoretic fiber $fib(\psi)$ has for $\pi_{n-1}(fib(\psi))$ a projective π-module, π the fundamental group of X. By replacing P^{n-1} by $P^{n-1} \vee (\bigvee S^{n-1})$, with $\bigvee S^{n-1}$ a suitable (possibly infinite) wedge of spheres, and extending ψ by a constant map over the wedge of spheres, we may assume that the new homotopy fiber has a *free* π-module as its $(n-1)$'st homotopy group. After attaching n-cells corresponding to a basis of that homotopy module, the map extends to a homotopy equivalence between $(P^{n-1} \vee (\bigvee S^{n-1})) \cup \{n\text{-cells}\}$ and X. If $n \leqslant 2$, the same argument shows that X is homotopy equivalent to a complex of dimension $\leqslant 3$, and we are done. □

REMARK. It is not known whether a CW-complex X with $\mathrm{cd}(X) \leqslant 2$ is actually homotopy equivalent to a 2-dimensional CW-complex. However, if $\mathrm{cd}(X) \leqslant 1$ it follows, using the fact that groups of cohomological dimension 1 are free, that X is homotopy equivalent to a 1-dimensional CW-complex, and therefore $X \simeq K(\pi_1(X), 1)$.

If X is dominated by a finite CW-complex then obviously all the finitely many connected components of X are of finite cohomological dimension and the following theorem is an immediate consequence of the previous one.

THEOREM 2.2. *If X is dominated by a finite complex, then X is homotopy equivalent to a finite dimensional CW-complex.*

Recall that a space of the homotopy type of a CW-complex is said to be of *finite type* if it is homotopy equivalent to a CW-complex with finite skeleta.

THEOREM 2.3. *Suppose X is finitely dominated. Then X is of finite type.*

PROOF. We will only sketch the proof. One may assume that X is connected. The fundamental group of X is then finitely presented, being a retract of a finitely presented group. Thus we can find a finite 2-dimensional CW-complex $W(2)$ and a 2-connected map $W(2) \to X$. One can show that π_2 of the homotopy fiber of that map is a *finitely generated* module over the fundamental group of X so that one can render the map 3-connected by attaching a finite number of 3-cells to $W(2)$, to obtain a new space $W(3)$. The map from $W(2)$ to X then extends to a 3-connected map from $W(3)$ to X. Continuing in this manner one obtains a coherent family of n-connected maps $W(n) \to X$, $n \geqslant 2$, giving rise to a homotopy equivalence $\bigcup W(n) \to X$. $\square$

The two preceding theorems constitute the basic homotopy theoretical properties of finitely dominated spaces. They have purely algebraic counter parts in the framework of chain complexes, which we will discuss in the next section.

3. Finitely dominated chain complexes

We denote by R an associative ring with 1. Modules over R are *unitary left modules*, and all chain complexes are supposed to be *non-negative*. A chain complex $C = \{C_i, d_i\}$ is said to be *projective* if the modules C_i are all projective. As usual, we write $H^*(C; M)$ for the cohomology of the cochain complex $\mathrm{Hom}_R(C, M)$ and we call C *acyclic* when C is exact. It is a basic fact that a morphism $f : C \to D$ of projective chain complexes is a chain homotopy equivalence if and only if the induced map of cohomology $H^*(\ ; R)$ is an isomorphism. We say that the chain complex $C = \{C_i, d_i\}$ is of type FP if each C_i is finitely generated and projective over R, and $C_i = 0$ for i large; C is said to be of type FF, if it is of type FP with all modules C_i free. For a chain complex $C = \{C_i, d_i\}$ of type FP the *Euler characteristic*

$$\chi(C) = \sum (-1)^i [C_i] \in K_0(R)$$

is defined, where $K_0(R)$ denotes the Grothendieck group of finitely generated projective R-modules, and $[C_i]$ stands for the class of the projective module C_i in $K_0(R)$. (Basic references for $K_0(R)$, the projective class group of R, are the books by Bass [2], Milnor [48] and Swan [61].) If

$$\cdots \to C_i \xrightarrow{d_i} C_{i-1} \to \cdots \to C_0 \to 0$$

is an acyclic complex of type FP then $\chi(C) = 0$, as one easily verifies by observing that all kernels and images of the homomorphisms d_i are projective in this case. More generally, if $\phi : C \to D$ is a chain homotopy equivalence between chain complexes of type FP then $\chi(C) = \chi(D)$. This follows by observing that the mapping cone M of ϕ is an acyclic chain complex of type FP with $M_i = C_i \oplus D_{i-1}$, thus M satisfies $\chi(M) = 0$, which implies $\chi(C) = \chi(D)$. The following definition is therefore meaningful.

DEFINITION 3.1. Let C be a chain complex which is chain homotopy equivalent to a chain complex D of type FP. Then the Euler characteristic

$$\chi(C) \in K_0(R)$$

is defined to be the Euler characteristic of D.

We will also use the following definitions.

DEFINITION 3.2. Let $C = \{C_i, d_i\}$ be a chain complex. Then

$$\mathrm{cd}(C) \leqslant n \iff H^i(C; M) = 0, \quad \forall M,\ \forall i > n.$$

Moreover, C is said to be of type FF_∞ if each module C_i is finitely generated and free over R.

In applications it is very useful to have a criterion for deciding when a chain complex is, up to chain homotopy, of type FF_∞. A homological criterion was first proved by Bieri and Eckmann for the case of resolutions of modules. Later Brown [9] dualized the criterion to a cohomological one and proved the following result for arbitrary chain complexes.

THEOREM 3.3. *A chain complex C is chain homotopy equivalent to a chain complex of type FF_∞ if and only if the functors $H^n(C; -)$ on the category of R-modules preserve direct limits for all n.*

A chain complex C is said to be of *finite cohomological dimension* if there is an N such that $\mathrm{cd}(C) \leqslant N$. Note that for X a connected CW-complex and C the $\mathbb{Z}\pi_1(X)$-complex $C_*^{cell}(\tilde{X})$, one has

$$\mathrm{cd}(X) \leqslant n \iff \mathrm{cd}(C_*^{cell}(\tilde{X})) \leqslant n,$$

and

$$X \text{ of finite type } \implies C_*^{cell}(\tilde{X}) \text{ is, up to chain homotopy, of type } FF_\infty.$$

The converse is true too if one assumes $\pi_1(X)$ to be finitely presented. In particular, the following cohomological characterization for finite domination holds.

THEOREM 3.4. *Let X be a connected CW-complex. Then*

$$X \textit{ is finitely dominated} \iff \begin{cases} \pi_1(X) \textit{ is finitely presented,} \\ \mathrm{cd}(X) < \infty \textit{ and} \\ \oplus_n H^n(X; -) \textit{ commutes with direct limits.} \end{cases}$$

REMARK. A chain complex $C = \{C_i, d_i\}$ is said to be of type FP_∞ if each C_i is finitely generated projective. It is easy to see that a chain complex is chain homotopy equivalent to a chain complex of type FP_∞ if and only if it is chain homotopy equivalent to one of type FF_∞.

The following simple theorem is basic in what follows. We denote by $\tilde{K}_0(R)$ the *reduced* projective class group of R; it is the factor group of $K_0(R)$ modulo the subgroup generated by $[R]$. Thus, the class $[P] \in K_0(R)$ of the finitely generated projective module P maps to zero in $\tilde{K}_0(R)$ if and only if P is *stably free*, meaning that there exists a finitely generated free module F such that $P \oplus F$ is free. We will denote the image of the Euler characteristic $\chi(C)$ in $\tilde{K}_0(R)$ by $\tilde{\chi}(C)$ and we call it the *reduced* Euler characteristic of C. We will use the notation $\{P\} \in \tilde{K}_0(R)$ for the class of the finitely generated projective module P. One easily checks that for finitely generated projective modules P and Q one has

$$\{P\} = \{Q\} \in \tilde{K}_0(P)$$

if and only there exist finitely generated *free* modules F and G such that $P \oplus F \cong Q \oplus G$; such modules P and Q are said to be *stably isomorphic*. Also, every element in $\tilde{K}_0(R)$ is of the form $\{P\}$ for some finitely generated projective P.

THEOREM 3.5. *Suppose that the chain complex C is chain homotopy equivalent to a chain complex D of type FF_∞. Furthermore, assume that C satisfies* $\mathrm{cd}(C) \leqslant n$. *Then C is chain homotopy equivalent to a chain complex of type FP and*

$$\tilde{\chi}(C) = (-1)^{N+1}\{B_N\} \in \tilde{K}_0(R)$$

where N is any integer $\geqslant n+1$ and $B_N = \mathrm{im}(d_{N+1} : D_{N+1} \to D_N) \subset D_N$, *the module of N-boundaries of D.*

PROOF. Since C is chain homotopy equivalent to D we have $\mathrm{cd}(D) \leqslant n$. The reader verifies easily that, in general, for a projective complex the condition $H^i(D; M) = 0, \forall M$ implies that $H_i(D; R) = 0$ and that $\mathrm{im}(d_i) \subset D_{i-1}$ is a direct summand. It follows that B_N is projective and that the inclusion of the truncated complex

$$0 \to B_N \to D_N \to D_{N-1} \to \cdots \to D_0 \to 0$$

into the complex D is a homology isomorphism, therefore a chain homotopy equivalence, because both complexes are projective. Clearly, as the D_i's are free, the reduced Euler characteristic of the truncated complex equals $(-1)^{N+1}\{B_N\}$ and the result follows. □

The following finiteness theorem is now plain and serves as a motivation for the corresponding topological result which we will discuss in the next section.

COROLLARY 3.6. *Let C be a chain complex of finite cohomological dimension, chain homotopy equivalent to a chain complex of type FF_∞. Then C is chain homotopy equivalent to a complex of type FF if and only if the reduced Euler characteristic $\tilde{\chi}(C)$ in $\tilde{K}_0(R)$ is zero.*

Namely, we have only to observe that once we have replaced C by the chain homotopy equivalent complex

$$0 \to B_N \to D_N \to \cdots \to D_0 \to 0$$

as above, with the D_i's free and $\tilde{\chi}(C) = (-1)^{N+1}\{B_N\} = 0$, the module B_N is stably free, and we can find a finitely generated free module F with $B_N \oplus F$ free. Therefore we can replaces the truncated complex above by a complex of the form

$$0 \to B_N \oplus F \to D_N \oplus F \to D_{N-1} \to \cdots \to D_0 \to 0$$

with obvious boundary maps, to obtain a complex of type FF, which is chain homotopy equivalent to the original complex C.

4. The finiteness obstruction

Let X denote a connected, finitely dominated CW-complex. Then, as X is also homotopy equivalent to a complex of finite type, the cellular chain complex $C_*^{cell}(\tilde{X})$ of the universal cover of X is chain homotopy equivalent to a chain complex of type FF_∞. Moreover, as X is homotopy equivalent to a finite dimensional CW-complex, $C_*^{cell}(\tilde{X})$ is of finite cohomological dimension. The Euler characteristic

$$\chi(C_*^{cell}(\tilde{X})) =: w(X) \in K_0(\mathbb{Z}\pi_1(X))$$

is therefore well defined, and we call it the *Wall* finiteness obstruction of X; the associated reduced Euler characteristic will be denoted by $\tilde{w}(X)$.

We have omitted the mention of *base points*. Indeed, if $x_0, x_1 \in X$ are two base points in the connected CW-complex X, then there exists a homotopy equivalence $f : X \to X$ taking x_0 to x_1 and thus inducing an isomorphism

$$f_* : K_0(\mathbb{Z}\pi_1(X, x_0)) \to K_0(\mathbb{Z}\pi_1(X, x_1)).$$

The point now is that f_* is independent of the particular choice of f. This can be seen by checking that an inner automorphism of a group π induces the identity on $K_0(\mathbb{Z}\pi)$, and using the well-known fact that the class of the homomorphism

$$f_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

modulo inner automorphisms of $\pi_1(X, x_1)$ is independent of the particular choice of f. As a consequence, we may neglect base points when dealing with Wall obstructions. In particular, if $f : X \to Y$ is any map of connected CW-complexes, one has a well-defined induced map

$$f_* : K_0(\mathbb{Z}\pi_1(X)) \to K_0(\mathbb{Z}\pi_1(Y)).$$

Note that if f is a homotopy equivalence and the spaces X and Y are finitely dominated, one has

$$f_* w(X) = w(Y),$$

as f (more precisely, a cellular approximation to f) induces a chain homotopy equivalence of cellular chain complexes of associated universal covers. The following corollary is useful to notice.

COROLLARY 4.1. *Let X be a connected, finitely dominated CW-complex and let f denote a self-homotopy equivalence of X. Then*

$$f_* w(X) = w(X).$$

5. Basic properties of $w(X)$

Let π denote an arbitrary group. The natural map $K_0(\mathbb{Z}\pi) \to \tilde{K}_0(\mathbb{Z}\pi)$ admits a splitting σ, given by $\sigma\{P\} = [P] - \mathrm{rk}(P) \cdot [\mathbb{Z}\pi]$, where the *rank* $\mathrm{rk}(P)$ of the finitely generated projective π-module P is defined to be the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_\pi P$. As a result, we have a canonical decomposition

$$K_0(\mathbb{Z}\pi) = \tilde{K}_0(\mathbb{Z}\pi) \oplus \mathbb{Z}, \qquad [P] = \big(\{P\}, \mathrm{rk}(P)\big),$$

where we have identified the cokernel of the splitting σ with $\mathbb{Z}$, which is the image of the *rank map*

$$\mathrm{rk} : K_0(\mathbb{Z}\pi) \to \mathbb{Z}.$$

LEMMA 5.1. *Let X be a connected, finitely dominated CW-complex and $w(X)$ its Wall obstruction. Then*

$$\mathrm{rk}\, w(X) = \chi(X),$$

where

$$\chi(X) = \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(X; \mathbb{Q})$$

denotes the ordinary Euler characteristic of X.

Namely, if we choose a chain complex P_* of type FP, chain homotopy equivalent to $C_*^{cell}(\tilde{X})$, then

$$w(X) = \left(\sum_i (-1)^i \{P_i\}, \mathrm{rk} \sum_i (-1)^i [P_i]\right)$$

and

$$\mathrm{rk} \sum_i (-1)^i [P_i] = \sum_i (-1)^i \dim_{\mathbb{Q}} (\mathbb{Q} \otimes_{\pi_1(X)} P_i) = \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(X; \mathbb{Q}).$$

The following basic theorem of C.T.C. Wall implies that the *reduced Wall obstruction* is indeed the *finiteness obstruction:*

THEOREM 5.2. *Let X be a connected, finitely dominated CW-complex. Then X is homotopy equivalent to a finite CW-complex if and only if*

$$\tilde{w}(X) = 0 \in \widetilde{K}_0(\mathbb{Z}\pi_1(X)).$$

Moreover, if π denotes a finitely presented group and y an arbitrary element of $K_0(\mathbb{Z}\pi)$, then there exists a connected, finitely dominated CW-complex Y with fundamental group π and Wall obstruction $w(Y) = y$.

For the proof see Wall's original papers [68], [69], or Varadarajan's book [67].

REMARK. It was proved by Ferry (cf. [25] and [26]) that if the space X is dominated by a finite CW-complex, then X is homotopy equivalent to a *compact metric* space Y; it is an open question whether Y can be chosen locally simply connected.

5.1. *A sum theorem for Wall's obstruction*

The following *Sum Theorem* is a very useful computational tool. If $f : G \to H$ denotes a group homomorphism, we write f_* for the induced homomorphism $K_0(\mathbb{Z}G) \to K_0(\mathbb{Z}H)$ (given by $[P] \mapsto [\mathbb{Z}H \otimes_G P]$); since $f_*[\mathbb{Z}G] = [\mathbb{Z}H]$, one also obtains an induced map of *reduced* projective class groups, which we will denote by the same symbol f_*. In case $Y \subset X$ denotes a connected, finitely dominated subcomplex of the connected and finitely dominated CW-complex X, and if ι_Y denotes the inclusion map $Y \to X$, then we obtain an induced map

$$\iota_{Y*} : K_0(\mathbb{Z}\pi_1(Y)) \to K_0(\mathbb{Z}\pi_1(X)).$$

One assumes here that a base point in X is chosen which lies in Y; the induced map of fundamental groups gives then rise to a map of projective class groups which is independent – in the obvious sense – of the particular choice of the basepoint (see also the corresponding remark earlier on), justifying our notation.

THEOREM 5.3. *Let $X = Y \cup Z$ be a connected, finitely dominated CW-complex, with Y, Z and $Y \cap Z$ connected and finitely dominated subcomplexes. Then*

$$w(X) = \iota_{Y*}(w(Y)) + \iota_{Z*}(w(Z)) - \iota_{Y\cap Z*}(w(Y \cap Z)) \in K_0(\mathbb{Z}\pi_1(X)),$$

where ι_Y, ι_Z and $\iota_{Y\cap Z}$ denote the inclusion maps.

We omit the proof and refer to Siebenmann [57], where also a version of this *Sum Theorem* in the nonconnected setting is discussed. Note also that under the hypothesis of the sum theorem one sees from the Mayer–Vietoris sequence for rational homology, that the Euler characteristics of the spaces involved satisfy

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

We could therefore have stated the *Sum Theorem* in exactly the same form with $w(-)$ replaced by the reduced Wall obstructions $\tilde{w}(-)$.

5.2. *The fibration theorem*

Let $F \to E \to B$ be a fibration sequence of connected CW-complexes. If F and B are finitely dominated then $\pi_1(E)$ is finitely presented (cf. [51]) and, by applying the cohomological characterization of finite domination to the Serre spectral sequence of the fibration $F \to E \to B$ one concludes that the total space E is finitely dominated too. We wish now to describe $w(E)$ in terms of the Wall obstructions of the base and fiber and possibly some other data present in the given fibration (the formula in Lal's paper [39] is incorrect; he neglected the action of the fundamental group of the base on the homology of the fiber). The simplest situation is that of a product fibration. For this, it is useful to consider the following pairing. Let G and H denote arbitrary groups. One defines

$$- \otimes - : K_0(\mathbb{Z}G) \times K_0(\mathbb{Z}H) \to K_0\big(\mathbb{Z}(G \times H)\big)$$

by putting

$$[P] \otimes [Q] := [P \otimes_{\mathbb{Z}} Q],$$

for P (respectively Q) a finitely generated projective $\mathbb{Z}G$- (respectively $\mathbb{Z}H$-) module. Note that the pairing induces also a pairing on the reduced projective class groups, which we will denote by the same symbol "$- \otimes -$".

The *Product Theorem* due to Gersten [32] and Siebenmann [57] can now be expressed by the following attractive formula.

THEOREM 5.4. *Let F and B be connected, finitely dominated CW-complexes. Then*

$$w(F \times B) = w(F) \otimes w(B) \in K_0\big(\mathbb{Z}\big(\pi_1(F) \times \pi_1(B)\big)\big).$$

For applications it is useful to decompose the Wall obstructions in this formula into their reduced and Euler characteristic parts, $w = (\tilde{w}, \chi)$, which yields the formula

$$\tilde{w}(F \times B) = \tilde{w}(F) \otimes \tilde{w}(B) + \chi(F) \cdot \iota_{B*}\big(\tilde{w}(B)\big) + \chi(B) \cdot \iota_{F*}\big(\tilde{w}(F)\big),$$

where ι_F and ι_B denote the obvious inclusion maps (as usual we assume that base-points have been chosen, where necessary).

COROLLARY 5.5. *Let X be a connected, finitely dominated CW-complex and Y a connected finite CW-complex with $\chi(Y) = 0$. Then $\tilde{w}(X \times Y) = 0$ and therefore $X \times Y$ is homotopy equivalent to a finite CW-complex.*

As a special case, we obtain Mather's result stated in the introduction: *If X is a finitely dominated space then $X \times S^1$ is homotopy equivalent to a finite CW-complex.* The formula stated in the *Product Theorem* is also correct for a certain wider class of fibrations with section $\iota_B : B \to E$ and retraction $E \to F$ (with associated decomposition $\pi_1(E) = \pi_1(F) \times \pi_1(B)$); for a precise statement, see Ehrlich [20]. To deal with the case of a general fibration, Ehrlich introduced in [19] a *geometric transfer homomorphism* (see also Pedersen [54]) denoted by

$$p^* : \tilde{K}_0(\mathbb{Z}\pi_1(B)) \to \tilde{K}_0(\mathbb{Z}\pi_1(E)),$$

associated with a fibration $p : E \to B$ of connected, finitely dominated CW-complexes, with finitely dominated fiber F. It is useful here to permit the case of a *disconnected* fiber F, e.g., a *finite covering space* situation (in which $\pi_1(E)$ is a subgroup of finite index of $\pi_1(B)$ and the geometric transfer map p^* is given by the usual "restriction map" of projective class groups). If $F = \cup F_i$, the disjoint union of its connected components, one puts $\tilde{w}(F)$ equal $\oplus_i \tilde{w}(F_i)$ in $\oplus_i \tilde{K}_0(F_i)$, and

$$\iota_{F*}(\tilde{w}(F)) = \sum_i \iota_{F_i *}(\tilde{w}(F_i)).$$

The general fibration theorem then reads as follows (for a proof see Ehrlich [19]).

THEOREM 5.6. *Given a fibration $p : E \to B$ of finitely dominated, connected CW-complexes with finitely dominated (not necessarily connected) fiber F. Then*

$$\tilde{w}(E) = p^*\tilde{w}(B) + \chi(B) \cdot \iota_{F*}(\tilde{w}(F)) \in \tilde{K}_0(\mathbb{Z}\pi_1(E)).$$

It seems to be difficult to compute p^* in general. But the composition p_*p^*, which is an endomorphism of $\tilde{K}_0(\mathbb{Z}\pi_1(B))$, can be described explicitly: *it is given by the $\bullet$-multiplication with the $\pi_1(B)$-module Euler characteristic* $\sum_i(-1)^i H_i(F;\mathbb{Z})$. This dot-multiplication is defined in the following way. For an arbitrary group π let $G(\mathbb{Z}\pi)$ denote the Grothendieck group of π-modules, which are finitely generated as abelian groups. One then defines a pairing

$$- \bullet - : G(\mathbb{Z}\pi) \times K_0(\mathbb{Z}\pi) \to K_0(\mathbb{Z}\pi)$$

by putting $[M] \bullet [P] = [M \otimes_{\mathbb{Z}} P]$, where M is a π-module which is finitely generated and *torsion-free* as an abelian group, and P an arbitrary finitely generated projective π-module. The definition extends to all of $G(\mathbb{Z}\pi)$ by observing that $G(\mathbb{Z}\pi)$ is generated by $[M]$'s which are torsion-free as abelian groups (one checks that the pairing is indeed well-defined, see the paper by Pedersen and Taylor [55]). The tensor product over $\mathbb{Z}$ turns $G(\mathbb{Z}\pi)$ into a commutative ring with unit (given by the class of the trivial π-module $\mathbb{Z}$), and $K_0(\mathbb{Z}\pi)$ becomes a module over $G(\mathbb{Z}\pi)$. One also obtains an induced action of $G(\mathbb{Z}\pi)$ on $\tilde{K}_0(\mathbb{Z}\pi)$, which we will denote by " $\bullet$ " too.

According to Pedersen and Taylor, the image $p_*p^*\tilde{w}(B)$, which by the fibration theorem is equal to $p_*\tilde{w}(E)$, is given by the following formula (their result is actually more general, compare [55]).

THEOREM 5.7. *Let $p : E \to B$ be a fibration of connected, finitely dominated CW-complexes, with (not necessarily connected) finitely dominated fiber. Then*

$$p_*\tilde{w}(E) = \left(\sum_i (-1)^i H_i(F;\mathbb{Z})\right) \bullet \tilde{w}(B) \in \tilde{K}_0\big(\mathbb{Z}\pi_1(B)\big).$$

Note that in case $\pi_1(B)$ acts trivially on the homology of the fiber F, the sum over the homology groups can be replaced by the ordinary Euler characteristic of F, and the formula reduces to

$$p_*\tilde{w}(E) = \chi(F) \cdot \tilde{w}(B).$$

5.3. *Relationship with Whitehead torsion*

For an associative ring R with unit one defines

$$K_1(R) = Gl(R)/\big[Gl(R), Gl(R)\big], \qquad Gl(R) = \bigcup_n Gl_n(R).$$

For the basic properties of the functor $K_1(-)$ see Milnor [47] and [48]. An isomorphism between two finitely generated free, based R-modules gives rise to a well defined "torsion element" in $K_1(\mathbb{Z}\pi)$, which is just the image of the matrix representing the isomorphism. More generally, if $C = \{C_i, d_i\}$ denotes an acyclic, based chain complex of type FF over R, its *torsion* $\operatorname{tor}(C) \in K_0(R)$ is well defined (see [47] or [14] for details). In case of $R = \mathbb{Z}\pi$, the group ring of the group π, we can consider

$$\pm\pi \subset Gl_1(\mathbb{Z}\pi) \to Gl(\mathbb{Z}\pi) \to K_1(\mathbb{Z}\pi)$$

and one defines the *Whitehead group* of π by

$$\operatorname{Wh}(\pi) := K_1(\mathbb{Z}\pi)/\operatorname{im}(\pm\pi).$$

One checks that an inner automorphism of π induces the identity on $\operatorname{Wh}(\pi)$ and thus $\operatorname{Wh}(\pi_1(X))$ is functorially defined for a path-connected space X. If $f : X \to Y$ is a homotopy equivalence of connected finite CW-complexes, then the *Whitehead torsion* $\tau(f) \in \operatorname{Wh}(\pi_1(Y))$ of f is defined (see Whitehead [73] in the simplicial setting).

Roughly speaking, the definition goes as follows. One replaces f up to homotopy by a cellular map g and considers the relative chain complex $C = C_*^{cell}(\tilde{M}_g, \tilde{X})$ with M_g the mapping cylinder of g, considered as a CW-complex in the obvious way. Since C is acyclic and has – because of the cell decomposition – a preferred $\mathbb{Z}\pi_1(M_f)$ basis (up to multiplication by elements $\pm x \in \pi_1(M_f)$), the torsion $\mathrm{tor}(C)$ is well defined modulo the image of $\pm\pi_1(M_f)$ in $K_1(\mathbb{Z}\pi_1(M_f))$, giving rise to a unique element denoted by $\tau(C) \in \mathrm{Wh}(\pi_1(M_f))$. The natural collapsing homotopy equivalence $p : M_f \to Y$ maps $\tau(C)$ to $\mathrm{Wh}(\pi_1(Y)$, leading to the definition

$$\tau(f) := p_*\tau(C) \in \mathrm{Wh}(\mathbb{Z}\pi_1(Y)).$$

REMARK. A homotopy equivalence f between connected, finite CW-complexes is called *simple*, if $\tau(f) = 0$; for a geometric interpretation see Cohen's book [14]. A basic result is the *topological invariance of Whitehead torsion* proved by Chapman in [11]: *If f is a homeomorphism between connected, finite CW-complexes, then $\tau(f) = 0$.* Indeed Chapman proves that a homotopy equivalence $f : X \to Y$ between connected finite CW-complexes is simple if and only if $f \times \mathrm{id} : X \times I_\infty \to Y \times I_\infty$ is homotopic to a homeomorphism (cf. [12]), where as earlier I_∞ stands for the Hilbert cube.

Let X be a connected, finitely dominated space so that one has maps $f : K \to X$ and $g : X \to K$ with K a connected finite CW-complex and $f \circ g$ homotopic to the identity of X. Consider a cellular map $\alpha : K \to K$ homotopic to $g \circ f$. Then the mapping torus $T(\alpha)$ is in a natural way a finite CW-complex. Choose any homotopy equivalence $\Phi : T(\alpha) \to X \times S^1$ and let Ψ be a homotopy inverse of Φ. Define the involution $\lambda : X \times S^1 \to X \times S^1$, $\lambda(x, z) = (x, \bar{z})$, and put

$$\sigma(X) := \Phi_*\big(\tau(\Psi \circ \lambda \circ \Phi)\big) \in \mathrm{Wh}\big(\pi_1(X \times S^1)\big).$$

This yields a well-defined homotopy invariant of X (cf. Ferry [27]).

Now, according to Bass, Heller and Swan [5], one has for any group π a natural decomposition

$$\mathrm{Wh}(\pi \times \mathbb{Z}) = \mathrm{Wh}(\pi) \oplus \mathrm{Nil}(\mathbb{Z}\pi) \oplus \mathrm{Nil}(\mathbb{Z}\pi) \oplus \tilde{K}_0(\mathbb{Z}\pi),$$

where $\mathrm{Nil}(-)$ denotes a certain functor on associative rings with 1. The relationship with Wall's obstruction is given by the following result due to Kwasik [38].

THEOREM 5.8. *Let X be a finitely dominated, connected CW-complex. Then one has $\sigma(X) = (0, 0, 0, \tilde{w}(X))$ if one identifies $\mathrm{Wh}(\pi_1(X \times S^1))$ with the sum*

$$\mathrm{Wh}\big(\pi_1(X)\big) \oplus \mathrm{Nil}\big(\mathbb{Z}\pi_1(X)\big) \oplus \mathrm{Nil}\big(\mathbb{Z}\pi_1(X)\big) \oplus \tilde{K}_0\big(\mathbb{Z}\pi_1(X)\big).$$

5.4. *Some results on projective class groups of group rings*

In order to be able to give examples of finitely dominated spaces which are not homotopy equivalent to finite CW-complexes, one needs to give examples of groups π with

$\tilde{K}_0(\mathbb{Z}\pi) \neq 0$. We list a few basic facts on projective class groups of group rings. The first one is due to Swan [60]:

$$\pi \text{ a finite group} \Longrightarrow \tilde{K}_0(\mathbb{Z}\pi) \text{ a finite group.}$$

Thus, for a finite group π the reduced projective class group $\tilde{K}_0(\mathbb{Z}\pi)$ is naturally isomorphic to the torsion subgroup of $K_0(\mathbb{Z}\pi)$. Note, however, that for an arbitrary finitely generated abelian group π the group $\tilde{K}_0(\mathbb{Z}\pi)$ is *not* finitely generated in general, see Bass and Murthy [4].

Because projective modules over free groups are free (cf. Bass [3], Gersten [31] and Stallings [58]), the following holds:

$$\pi \text{ a free group} \Longrightarrow \tilde{K}_0(\mathbb{Z}\pi) = 0.$$

Indeed, there is no torsion-free group known with nontrivial reduced projective class group (although there exist nonfree projective modules over certain torsion-free groups, see the examples of Dunwoody [17], Berridge and Dunwoody [6] and Artamonov [1]). On the other hand, there are many torsion-free groups known for which the reduced projective class group is trivial. The following are a few examples: finitely generated abelian groups, or more generally, Bieberbach groups (cf. Farrell and Hsiang [22]), poly-$\mathbb{Z}$ groups (cf. Farrell and Hsiang [23]) and – even more generally – fundamental groups of closed Riemannian manifolds all of whose sectional curvature values are nonpositive (cf. Farrell and Jones [24]). Actually, for all these groups π the Whitehead group $\mathrm{Wh}(\pi \times \mathbb{Z})$ vanishes. Note that if Γ is a direct limit of groups π_α all of which satisfy $\mathrm{Wh}(\pi_\alpha \times \mathbb{Z}) = 0$ then one has $\tilde{K}_0(\mathbb{Z}\Gamma) = 0$. In particular, because the Whitehead group vanishes for finitely generated free abelian groups, one concludes:

$$\pi \text{ a torsion-free abelian group} \Longrightarrow \tilde{K}_0(\mathbb{Z}\pi) = 0.$$

We finish this section by briefly describing examples of finite groups with *nontrivial* reduced projective class groups. If Λ denotes a Dedekind domain, we write $C(\Lambda)$ for the ideal class group of Λ, and $[I]_C$ for the class of a nonzero ideal $I \subset \Lambda$. Because ideals of Λ are finitely generated projective Λ-modules, and because for nonzero ideals I_1, I_2 of a Dedekind domain one has the well-known relation

$$I_1 \oplus I_2 \cong \Lambda \oplus (I_1 \otimes_\Lambda I_2),$$

there is a natural homomorphism

$$C(\Lambda) \to \tilde{K}_0(\Lambda), \qquad [I]_C \mapsto \{I\},$$

which is actually an isomorphism (see, e.g., Milnor's book [48]). Moreover, for p a prime number, the ideal class group $C(\mathbb{Z}[\exp(2\pi\sqrt{-1}/p)])$ is known to be trivial for $p < 23$, and nontrivial for all primes $p \geqslant 23$ (see Washington's book [71] for references on ideal class groups of number rings). The connection with projective class groups of group rings is given by the following basic theorem due to Rim [56].

THEOREM 5.9. *Let* $\pi = \langle x \rangle$ *be a cyclic group of prime order* p. *Consider the map* $\mathbb{Z}\pi \to \mathbb{Z}[\exp(2\pi\sqrt{-1}/p)]$ *given by mapping* x *to* $\exp(2\pi\sqrt{-1}/p)$. *Then the induced map of reduced projective class groups*

$$\tilde{K}_0(\mathbb{Z}\pi) \to \tilde{K}_0\big(\mathbb{Z}\big[\exp\big(2\pi\sqrt{-1}/p\big)\big]\big)$$

is an isomorphism.

COROLLARY 5.10. *For every prime* $p \geqslant 23$ *there exists a connected, finitely dominated CW-complex* X *with fundamental group* $\mathbb{Z}/p\mathbb{Z}$, *such that* X *is not homotopy equivalent to any finite CW-complex.*

There is a vast literature on projective class groups of finite groups. The following are a few examples. For cyclic groups of prime power order see Galovich [29], Kervaire and Murthy [36] and Ullom [66], for quaternion and dihedral 2-groups see Martinet [42], Keating [34], Fröhlich, Keating and Wilson [28], for metacyclic groups see Keating [35], Galovich, Reiner and Ullom [30], and for general p-groups see Taylor [62]. Examples of *infinite* groups (with torsion) and nonvanishing projective class group are discussed in Bass and Murthy [4] (abelian groups) and in Bürgisser [10] (arithmetic groups).

6. Nilpotent spaces

Let π be an arbitrary group, M a π-module and $I\mathbb{Z}\pi \subset \mathbb{Z}\pi$ the augmentation ideal. The group π is said to operate *nilpotently* on M, and M is termed a *nilpotent* π-module, if there is an $n > 0$ such that

$$(I\mathbb{Z}\pi)^n \cdot M = 0.$$

Recall that a connected CW-complex is called *simple* if $\pi_1(X)$ is abelian and operates trivially on all higher homotopy groups $\pi_i(X)$. The notion of a *nilpotent* space is a generalization as follows.

DEFINITION 6.1. A space is called nilpotent, if it is of the homotopy type of a connected CW-complex X such that $\pi_1(X)$ is a nilpotent group and all homotopy groups $\pi_i(X)$, $i > 1$, are nilpotent $\pi_1(X)$-modules.

A space X is called *quasi-finite* if $\oplus_n H_n(X;\mathbb{Z})$ is a finitely generated abelian group. For nilpotent spaces, the following criterion for finite domination holds (cf. [50]).

LEMMA 6.2. *A nilpotent space* X *is finitely dominated if and only if it is quasi-finite.*

The Wall obstruction of a finitely dominated nilpotent space is subjected to stringent restrictions. For the case of an infinite fundamental group, the following *vanishing theorem* holds (cf. [50]).

THEOREM 6.3. *Let X be a finitely dominated nilpotent space with infinite fundamental group. Then*

$$w(X) = 0 \in K_0(\mathbb{Z}\pi_1(X)).$$

Therefore, X is homotopy equivalent to a finite CW-complex and $\chi(X) = 0$.

PROOF. Since $\pi_1(X)$ is an infinite, finitely generated nilpotent group, it admits a surjective homomorphism $\pi_1(X) \to \mathbb{Z}$. The associated covering space $\overline{X}$ is classified by a map $p : X \to S^1$, which has as homotopy theoretic fiber the covering space $\overline{X}$. One checks that $\overline{X}$ is nilpotent and quasi-finite (cf.[50]), thus finitely dominated. Obviously,

$$\chi(X) = \chi(\overline{X}) \cdot \chi(S^1) = 0,$$

and, from the fibration theorem,

$$\tilde{w}(X) = p^*\tilde{w}(S^1) + \chi(S^1) \cdot \tilde{w}(\overline{X}) = 0,$$

which completes the proof. □

The next theorem is from [49]; we will only sketch its proof.

THEOREM 6.4. *Let X be a finitely dominated nilpotent space with finite cyclic fundamental group of prime order. Then X is homotopy equivalent to a finite CW-complex.*

PROOF. Let $\pi = \pi_1(X) \cong \mathbb{Z}/p\mathbb{Z}$ and $\Lambda = \mathbb{Z}[\exp(2\pi\sqrt{-1}/p)]$. Consider Λ as a π-module via a surjective homomorphism $\mathbb{Z}\pi \to \Lambda$. By Rim's theorem, the induced map

$$\theta : \tilde{K}_0(\mathbb{Z}\pi) \to \tilde{K}_0(\Lambda)$$

is an isomorphism. It therefore suffices to show that

$$\theta(\tilde{\chi}(C_*^{cell}(\tilde{X}))) = \tilde{\chi}(\Lambda \otimes_\pi C_*^{cell}(\tilde{X})) = 0.$$

Because Λ has finite cohomological dimension (actually $\operatorname{cd}\Lambda = 1$), one can define the *reduced Euler characteristic*

$$\tilde{\chi}(M) \in \tilde{K}_0(\Lambda)$$

for *any* finitely generated Λ-module M, by putting

$$\tilde{\chi}(M) = \sum_i (-1)^i \{P_i\},$$

where

$$0 \to P_n \to \cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to M$$

denotes a projective resolution of finite length of M, with all the modules P_i all finitely generated. One can now compute the Euler characteristic in question by passing to homology:

$$\tilde{\chi}\big(\Lambda \otimes_\pi C_*^{cell}(\tilde{X})\big) = \tilde{\chi}\big(H_i(X;\Lambda)\big) \in \tilde{K}_0(\Lambda).$$

Because of the nilpotency of X and using the homology sequence associated with the short exact sequence of π-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}\pi \to \Lambda \to 0,$$

one can show that, as abelian groups, all homology groups $H_i(X;\Lambda)$ are finite p-groups. On the other, for the trivial π-module $\mathbb{Z}/p\mathbb{Z}$, considered as a Λ-module by letting $\exp(2\pi\sqrt{-1}/p)$ operate via the identity map, one has

$$\tilde{\chi}(\mathbb{Z}/p\mathbb{Z}) = 0 \in \tilde{K}_0(\Lambda).$$

This is plain in view of the following short exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{1-\exp(2\pi\sqrt{-1}/p)} \Lambda \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

Using induction, it is now easy to prove that for *any* Λ-module M such that the underlying abelian group is a finite p-group, one has $\tilde{\chi}(M) = 0$. The claim of the theorem then follows readily. □

The previous theorem admits the following generalization. Let π denote an arbitrary finite group and write $\overline{\mathbb{Z}\pi}$ for a maximal $\mathbb{Z}$-order containing $\mathbb{Z}\pi$ in the rational group algebra $\mathbb{Q}\pi$. Define

$$D(\mathbb{Z}\pi) := \ker\big(j_* : K_0(\mathbb{Z}\pi) \to K_0(\overline{\mathbb{Z}\pi})\big),$$

where $j : \mathbb{Z}\pi \to \overline{\mathbb{Z}\pi}$ denotes the natural inclusion (one verifies that $D(\mathbb{Z}\pi)$ is independent of the particular choice of the maximal order). In case of $\pi = \mathbb{Z}/p\mathbb{Z}$ with p a prime, one has $\overline{\mathbb{Z}\pi} \subset \Lambda \times \mathbb{Z}$ with $\Lambda = \mathbb{Z}[\exp(2\pi\sqrt{-1}/p)]$, and Rim's theorem asserts that j_* is injective in that case, thus $D(\mathbb{Z}/p\mathbb{Z}) = 0$. The previous theorem then becomes a special case of the following result.

THEOREM 6.5. *Let X be a finitely dominated nilpotent space with nontrivial finite fundamental group $\pi_1(X) = \pi$. Then*

(i) $w(X) \in D(\mathbb{Z}\pi)$;

(ii) $|\pi| \cdot w(X) = 0$ *in case* π *is a* p*-group.*

For the proof, the reader is referred to [53]. Part (ii) of the theorem follows from results concerning the exponent of $D(\mathbb{Z}\pi)$ due to Ullom [64]. It is known that $D(\mathbb{Z}\pi) = 0$ for π a dihedral 2-group (cf. Fröhlich, Keating and Wilson [28]), and accordingly the finiteness obstruction vanishes for a finitely dominated nilpotent space with such fundamental groups. But in general, $D(\mathbb{Z}\pi)$ is rather large (see Taylor [62]). Note also that (ii) does not hold in general, if π is not a p-group; in [52] it is shown that there exists a finitely dominated nilpotent space with fundamental group cyclic of order 15 and Wall obstruction of order 2. It is also known that $D(\mathbb{Z}/22\mathbb{Z}) \neq 0$, although for a finitely dominated nilpotent space X with $\pi_1(X) \cong \mathbb{Z}/22\mathbb{Z}$ one has $w(X) = 0$ (cf. [52]). Thus, one is led to consider the subset

$$N(\mathbb{Z}\pi) \subset K_0(\mathbb{Z}\pi)$$

consisting of all those elements which arise as finiteness obstructions of finitely dominated nilpotent spaces with fundamental group π. Ewing, Löffler and Pedersen showed that for π a finite nilpotent group $N(\mathbb{Z}\pi)$ is actually a subgroup of $K_0(\mathbb{Z}\pi)$. For $\pi \neq \{e\}$ a finite nilpotent group, we have

$$N(\mathbb{Z}\pi) \subset D(\mathbb{Z}\pi)$$

by the previous theorem, and in general $N(\mathbb{Z}\pi) \neq D(\mathbb{Z}\pi)$, as the case of $\pi = \mathbb{Z}/22\mathbb{Z}$ illustrates. (Note that for the trivial group π one has $N(\mathbb{Z}\pi) = \mathbb{Z}$ but $D(\mathbb{Z}\pi) = 0$; that's why we have to assume $\pi \neq \{e\}$.) For p-groups, the following result holds (see Ewing, Löffler and Pedersen [21]).

THEOREM 6.6. *Let* π *be a nontrivial finite* p*-group. Then*

$$N(\mathbb{Z}\pi) = D(\mathbb{Z}\pi).$$

In particular it follows that the finiteness obstruction does not vanish in general for a finitely dominated nilpotent spaces whose fundamental group is a p-group. Indeed, for π a p-group of order p^n with $p > 2$ and $n \geqslant 5$ one has

$$|D(\mathbb{Z}\pi)| \geqslant p^{n-1},$$

see Taylor [62] (there is a similar result for the prime 2: if $\pi = \mathbb{Z}/2^n\mathbb{Z}$ with $n \geqslant 5$, one has $|D(\mathbb{Z}\pi)| \geqslant 2^{n-2}$, whereas for a generalized quaternion group Q_n of order $2^n \geqslant 8$ one has $|D(\mathbb{Z}Q_n)| = 2$). It should also be noted that one can even find *simple*, finitely dominated spaces not of the homotopy type of finite CW-complexes (cf. [52]). Of course, H-spaces are simple. However, no finitely dominated H-space with nonvanishing finiteness obstruction is known (some partial vanishing results on the Wall obstruction of H-spaces can be found in [50]).

7. Localization techniques

We will be concerned in this section with the study of the finiteness obstruction of a finitely dominated space X with finite fundamental group acting *trivially* on the rational homology of the universal cover of X (e.g., X a nilpotent space). In such a situation, after choosing a basis for the rational homology of X, the *p-local Reidemeister torsion* $\mathrm{RT}_p(X)$ is defined and is used to define the "p-part" $w_p(X)$ of the finiteness obstruction $w(X)$. It turns out that $w_p(X)$ may be nonzero, even if $w(X) = 0$; this is the reason why the "Zabrodsky mixing" of two finite CW-complexes can fail to be homotopy equivalent to a finite CW-complex. We will write as usual $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at the prime ideal (p), i.e. the subring of $\mathbb{Q}$ consisting of fractions with denominators prime to p, and we write $\mathbb{Z}[1/p]$ for the subring consisting of fractions with denominators involving only powers of p. For a nilpotent space X we will write $X_{(p)}$ for its p-localization, and $X_{(0)}$ for its rationalization (cf. [33]).

7.1. *p-local Reidemeister torsion*

Let X be an arbitrary CW-complex. We will call X *$\mathbb{Q}$-based*, if a fixed basis for $H_*(X;\mathbb{Q})$ is chosen. If X is a finite, $\mathbb{Q}$-based CW-complex, the chain complex $C_*^{cell}(X;\mathbb{Q})$ with its natural basis, together with the given homology basis defines a torsion invariant in $K_1(\mathbb{Q}) \cong \mathbb{Q}^\times$, see Milnor [47]. Now let X be a connected, finitely dominated $\mathbb{Q}$-based CW-complex with finite fundamental group π *operating trivially* on the rational homology of the universal cover of X. If we write $e \in \mathbb{Q}\pi$ for the idempotent

$$\frac{1}{|\pi|}\sum_{x\in\pi} x,$$

we obtain a decomposition

$$\mathbb{Q}\pi = e\cdot\mathbb{Q}\pi \times (1-e)\cdot\mathbb{Q}\pi = \mathbb{Q}\times \mathrm{A}, \qquad \mathrm{A} := (1-e)\cdot\mathbb{Q}\pi.$$

Let P be a chain complex of type FP, chain homotopy equivalent to the cellular chain complex of the universal cover of X. Since, according to Swan [60], for a finitely generated projective $\mathbb{Z}\pi$ module M the $\mathbb{Z}_{(p)}\pi$-module $M_{(p)} := M\otimes\mathbb{Z}_{(p)}$ is free, we can choose a basis for $P_{(p)} := P\otimes\mathbb{Z}_{(p)}$, and obtain from it an induced basis for $P_{(0)} := P\otimes\mathbb{Q}$. Using the natural splitting of $P_{(0)}$ into a $\mathbb{Q}$-complex and an A-complex,

$$P_{(0)} = e\cdot P_{(0)} \times (1-e)\cdot P_{(0)},$$

one observes that, because π operates trivially on the rational homology of X, the homology of $e\cdot P_{(0)}$ is naturally isomorphic via the projection $P_{(0)} \to P\otimes_\pi \mathbb{Q}$ to the homology of X with $\mathbb{Q}$-coefficients, and $H_*(e\cdot P_{(0)})$ has thus a natural basis because X is supposed to be $\mathbb{Q}$-based. On the other hand $(1-e)\cdot P_{(0)}$ is acyclic. From these two based complexes (with based homologies) one obtains thus a pair of torsion invariants in

$$K_1(\mathbb{Q}\pi) = K_1(\mathbb{Q}) \times K_1(A).$$

Since everything depends on the choice of the basis of $P_{(p)}$, the torsion invariant will only be well-defined if we calculate modulo the image of $K_1(\mathbb{Z}_{(p)}\pi)$ in $K_1(\mathbb{Q}\pi)$. The resulting invariant will be denoted by

$$\mathrm{RT}_p(X) \in K_1(\mathbb{Q}\pi)/\mathrm{im}\,\big(K_1(\mathbb{Z}_{(p)}\pi)\big)$$

and is called (by Ewing, Löffler and Pedersen [21]) the *p-local Reidemeister torsion* of the $\mathbb{Q}$-based space X.

For p any prime and π any group, the commutative diagram with obvious arrows

$$\begin{array}{ccc} \mathbb{Z}\pi & \longrightarrow & \mathbb{Z}_{(p)}\pi \\ \downarrow & & \downarrow \\ \mathbb{Z}[1/p]\pi & \longrightarrow & \mathbb{Q}\pi \end{array}$$

gives rise to a long exact sequence (see Bass [2]) of K-groups

$$\begin{aligned} &K_1(\mathbb{Z}\pi) \to K_1(\mathbb{Z}_{(p)}\pi) \times K_1(\mathbb{Z}[1/p]\pi) \to K_1(\mathbb{Q}\pi) \xrightarrow{\partial^p} \\ &\quad K_0(\mathbb{Z}\pi) \to K_0(\mathbb{Z}_{(p)}\pi) \times K_0(\mathbb{Z}[1/p]\pi) \to K_0(\mathbb{Q}\pi). \end{aligned}$$

We are interested in the *connecting homomorphism*

$$\partial^p : K_1(\mathbb{Q}\pi) \longrightarrow K_0(\mathbb{Z}\pi),$$

which is used in the following definition. Note that the image of $K_1(\mathbb{Z}_{(p)}\pi)$ in $K_1(\mathbb{Q}\pi)$ lies in the kernel of ∂^p.

DEFINITION 7.1. Let X be a connected, finitely dominated $\mathbb{Q}$-based CW-complex with finite fundamental group π operating trivially on the rational homology of the universal cover of X. Then for every prime p the *p-part* $w_p(X)$ of the Wall obstruction of X is

$$\partial^p\big(\mathrm{RT}_p(X)\big) =: w_p(X) \in K_0(\mathbb{Z}\pi).$$

In [21] Ewing, Löffler and Pedersen show that the p-parts $w_p(X)$ are zero for almost all primes p and it makes therefore sense to form their sum.

THEOREM 7.2. *Let X be a connected, finitely dominated $\mathbb{Q}$-based CW-complex with finite nontrivial fundamental group π operating trivially on the rational homology of the universal cover of X. Then*

$$\sum_p w_p(X) = w(X) \in K_0\big(\mathbb{Z}\pi_1(X)\big).$$

In case π is a finite nilpotent group, we write π_p for the Sylow p-subgroup of π, and one has a natural decomposition

$$\pi = \pi_p \times \pi',$$

with π' consisting of all elements of order prime to p. The idempotent

$$e_p := \frac{1}{|\pi'|} \sum_{x \in \pi'} x \in \mathbb{Q}\pi$$

provides a splitting of the group algebra

$$\mathbb{Q}\pi = e_p \cdot \mathbb{Q}\pi \times (1 - e_p) \cdot \mathbb{Q}\pi,$$

and yields a corresponding splitting

$$K_1(\mathbb{Q}\pi) = K_1(e_p \cdot \mathbb{Q}\pi) \times K_1\big((1 - e_p) \cdot \mathbb{Q}\pi\big).$$

Note that the projection $\pi \to \pi_p$ induces an isomorphism $e_p \cdot \mathbb{Q}\pi \to \mathbb{Q}\pi_p$ so that

$$K_1(e_p \cdot \mathbb{Q}\pi) \cong K_1(\mathbb{Q}\pi_p).$$

DEFINITION 7.3. Let π be a finite nilpotent group. Then

$$N_p(\mathbb{Z}\pi) := \partial^p\big(K_1(e_p \cdot \mathbb{Q}\pi) \times \{0\}\big) \subset K_0(\mathbb{Z}\pi).$$

Recall that for a finite nilpotent group π we defined $N(\mathbb{Z}\pi)$ to consists of those elements of the projective class group of π which are realizable as finiteness obstruction of finitely dominated nilpotent spaces with fundamental group π. The connection with the subgroups $N_p(\mathbb{Z}\pi)$ is given by the following theorem [21].

THEOREM 7.4. *Let π be a nontrivial finite nilpotent group. Then*

$$\sum_p N_p(\mathbb{Z}\pi) = N(\mathbb{Z}\pi) \subset K_0(\mathbb{Z}\pi).$$

If π is an arbitrary finite group, the *Swan subgroup*

$$\mathrm{T}(\mathbb{Z}\pi) \subset K_0(\mathbb{Z}\pi)$$

is defined as the image of the boundary map ∂^{T} in the Milnor exact sequence

$$\cdots \to K_1(\mathbb{Z}/|\pi|\mathbb{Z}) \xrightarrow{\partial^{\mathrm{T}}} K_0(\mathbb{Z}\pi) \to \cdots$$

associated with the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\pi & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}\pi/(\Sigma) & \longrightarrow & \mathbb{Z}/|\pi|\mathbb{Z} \end{array}$$

where (Σ) denotes the ideal generated by $\Sigma = \sum_{x\in\pi} x$ and the arrows are the obvious maps. The Swan subgroup is quite computable. Its elements are precisely those which can be written in the form $[(k,\Sigma)] - [\mathbb{Z}\pi]$ where $(k,\Sigma) \subset \mathbb{Z}\pi$ denotes the (projective) ideal generated by k and Σ, where $k \in \mathbb{Z}$ is prime to $|\pi|$. According to Ullom [65], the Swan subgroup is trivial for cyclic groups. In general, its exponent divides the Artin exponent of π, and if π is a p-group, $\mathrm{T}(\mathbb{Z}\pi)$ is cyclic (for π a p-group $\mathrm{T}(\mathbb{Z}\pi)$ is completely known, see Taylor [62]). It is not hard to see that in general

$$\mathrm{T}(\mathbb{Z}\pi) \subset D(\mathbb{Z}\pi),$$

and one can show that (cf. [52]) for a finite nilpotent group π one always has has

$$\mathrm{T}(\mathbb{Z}\pi) \subset \mathrm{N}(\mathbb{Z}\pi).$$

In the abelian case one even has the following result [52].

THEOREM 7.5. *Let π be a finite abelian group and $x \in \mathrm{T}(\mathbb{Z}\pi)$. Then there exist a finitely dominated connected,* simple *CW-complex X with fundamental group π and $w(X) = x$.*

REMARK. The definition of the p-part $w_p(X)$ of the Wall obstruction of X depended on the choice of a basis for the rational homology of X. It is shown in Ewing, Löffler and Pedersen [21] that a different choice results in a change of $w_p(X)$ by an element in $\mathrm{T}(\mathbb{Z}\pi)$. In particular, in case π is cyclic one has $\mathrm{T}(\mathbb{Z}\pi) = 0$ and thus $w_p(X)$ will be independent of that choice.

7.2. *Fiber-wise localization and genus*

For X a connected CW-complex with fundamental group π one has a natural fibration up to homotopy of the form

$$\tilde{X} \to X \to B\pi$$

which, for any prime p, admits a fiber-wise p-localization for which we will use the notation

$$\tilde{X}_{(p)} \to X_{(p-\pi)} \to B\pi,$$

where $\tilde{X}_{(p)}$ stands for $(\tilde{X})_{(p)}$. In case the group π is finite, of order prime to p and acting trivially on the p-local homology of $\tilde{X}$, this fibration is fiber-homotopy trivial, thus $X_{(p-\pi)} \simeq \tilde{X}_{(p)} \times B\pi$. Moreover, if X is nilpotent with finite fundamental group, one has

$$X_{(p-\pi)} \simeq X_{(p)} \times B\pi',$$

where π' denotes the subgroup of the fundamental group of X consisting of all elements of order prime to p. (For a definition of fiber-wise localizations on the level of spaces see Bousfield and Kan [8].)

As observed by Wojtkowiak in [75], the following holds.

THEOREM 7.6. *Let X be a connected CW-complex with finite fundamental group π and let Y be the universal cover of $X_{(p-\pi)}$ with its natural π action. Then the singular chain complexes $C_*^{sing}(\tilde{X}) \otimes \mathbb{Z}_{(p)}$ and $C_*^{sing}(Y) \otimes \mathbb{Z}_{(p)}$ are naturally chain homotopy equivalent over $\mathbb{Z}_{(p)}\pi$.*

Indeed, the natural π-map $\tilde{X} \to Y$ induces the equivalence. As a result, the p-local Reidemeister torsion $\mathrm{RT}_p(X)$ depends only on the homotopy type of $X_{(p-\pi)}$ and the choice of a basis for the rational homology of that space. In the nilpotent situation this implies that the finiteness obstruction depends, modulo the image of the Swan homomorphism, only on the homotopy types $X_{(p)}$. More precisely, the following holds. Recall that for a nilpotent space of finite type X the genus set $G(X)$ of X consists of all nilpotent homotopy types of finite type Y such that for every prime p one has $Y_{(p)} \simeq X_{(p)}$ (for basic properties of the genus see [33]).

THEOREM 7.7. *Let X be a finitely dominated nilpotent space with finite fundamental group and let $Y \in G(X)$. Then there is an isomorphism $\phi : \pi_1(Y) \to \pi_1(X)$ such that*

$$w(X) - \phi_* w(Y) \in \mathrm{T}(\mathbb{Z}\pi_1(X)) \subset K_0(\mathbb{Z}\pi_1(X)).$$

For a proof see [21] or [74]. Note that if Y is in the genus of a *finite*, nilpotent CW-complex with finite nontrivial fundamental group, then we can conclude that $w(Y)$ lies in the Swan subgroup (but it is not necessarily zero, for an example see [50]).

The same methods can be used to compute the Wall obstruction for a "Zabrodsky mixing".

THEOREM 7.8. *Suppose $P \cup Q$ is a partition of the set of primes and M, X, Y and Z are finitely dominated nilpotent spaces with finite fundamental groups, such that there is a pull-back diagram*

$$\begin{array}{ccc} M & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

with the horizontal arrows being P-equivalences, and the vertical ones Q-equivalences. Then there is a P-equivalence $\phi_X : \pi_1(X) \to \pi_1(M)$ and a Q-equivalence $\phi_Y : \pi_1(Y) \to \pi_1(M)$ such that

$$w(M) = \sum_{p\in P} \phi_{X*} w_p(X) + \sum_{q\in Q} \phi_{Y*} w_q(Y)$$

where the p-parts of the Wall obstructions are supposed to be computed with respect to bases of the rational homology of X and Y which correspond to each other via the induced isomorphisms

$$H_*(X;\mathbb{Q}) \xrightarrow{\cong} H_*(Z;\mathbb{Q}) \xleftarrow{\cong} H_*(Y;\mathbb{Q}).$$

An other interesting and closely related question is the following. Suppose X is a finitely dominated nilpotent space and p a prime. Is $X_{(p)}$ necessarily homotopy equivalent to $Y_{(p)}$ for some *finite* nilpotent CW-complex Y? The answer is "no" in general. Indeed, if we choose for X a finitely dominated nilpotent space with $\pi_1(X) \cong \mathbb{Z}/p^n\mathbb{Z}$ such that $w(X) \neq 0$, then one has obviously $w(X) = w_p(X) \neq 0$, but for Y a finite CW-complex with the same fundamental group, one has $w(Y) = w_p(Y) = 0$ (recall that $\mathrm{T}(\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}]) = 0$ so that there is no ambiguity in the definition of the local Wall obstructions). Thus, as $w_p(X)$ depends only on $X_{(p-\pi)} = X_{(p)}$, and similarly for $w_p(Y)$, we see that necessarily $X_{(p)} \not\simeq Y_{(p)}$. For a thorough discussion of these matters, see [21].

7.3. *The spherical space form problem*

A classical question asks to describe all topological manifolds M with universal cover homeomorphic to S^n. The fundamental group of such a manifold is necessarily a finite group with periodic cohomology and, in case M is orientable, the (minimal) period of $\pi_1(M)$ divides $(n+1)$. If M a nonorientable, then M is easily seen to be homotopy equivalent the projective space $\mathrm{P}^n(\mathbb{R})$ with n even; we shall concentrate on the orientable case in the sequel.

A finite group with periodic cohomology is called a $\mathcal{P}$-group. The $\mathcal{P}$-groups are characterized by the fact that all their abelian subgroups are cyclic. If the period of a $\mathcal{P}$-group π is k, then

$$H^k(\pi;\mathbb{Z}) = \mathbb{Z}/|\pi|\mathbb{Z}.$$

Conversely, if for a finite group π one has $H^i(\pi;\mathbb{Z}) = \mathbb{Z}/|\pi|\mathbb{Z}$, then π is a $\mathcal{P}$-group of period dividing i.

The following emerge as the natural and basic basic questions:

- *Which $\mathcal{P}$-groups admit free actions on spheres?*
- *If the $\mathcal{P}$-group π acts freely on S^n, what are the possible values of n? Is the minimal value of n equal to $k-1$, where k is the period of π?*

It has been known for a long time that not every $\mathcal{P}$-group admits a free action on some sphere. Namely, Milnor proved in [46] that if the finite group π acts freely on a sphere, then all elements of order two in π are central. Thus, for instance the symmetric group $\mathcal{S}_3$, which has period 4, does not act freely on any sphere. But according to Madsen, Thomas and Wall ([63], [41]), Milnor's condition is the only obstruction to finding an action, and the following converse holds.

THEOREM 7.9. *If the $\mathcal{P}$-group π has the property that all elements of order 2 are central, then π admits a free action on some sphere.*

This answers our first question completely. If the finite group π acts freely on S^n, the orbit space S^n/π is a compact topological manifold, which is an ANR and therefore homotopy equivalent to a finite CW-complex Y. Thus π acts freely and *cellularly* on the finite CW-complex $\tilde{Y}$ homotopy equivalent to S^n. One can therefore divide the second question up into two separate ones, a purely homotopy theoretical one, and a surgery problem:

- *Given a $\mathcal{P}$-group π, find all values of n such that π acts freely and cellularly on a finite CW-complex homotopy equivalent to a sphere S^n.*
- *Suppose the $\mathcal{P}$-group π acts freely and cellularly on a finite CW-complex homotopy equivalent to S^n. Does π admit a free action on S^n?*

We will only sketch how the homotopy problem can be reduced to an investigation concerning finiteness obstructions; a thorough analysis as well as a discussion of the surgery problem, which we won't address here, can be found in the excellent survey by Davis and Milgram [16].

If a finite group π acts freely and cellularly on the finite CW-complex X homotopy equivalent to S^n, preserving the orientation, then one obtains a complex

$$\cdots \to C_{n+1}^{cell}(X) \to C_n^{cell}(X) \to \cdots \to C_0^{cell}(X) \to \mathbb{Z} \to 0$$

with $B_n^{cell}(X) = \operatorname{im}(C_{n+1}^{cell}(X) \to C_n^{cell}(X))$ a direct summand. This direct summand is stably free because it represents the reduced finiteness obstruction for the finite CW-complex X/π. It is then easy to modify the complex to obtain a *periodic resolution*

$$0 \to \mathbb{Z} \to F_n \to F_{n-1} \to C_{n-2}^{cell}(X) \to \cdots \to C_0^{cell}(X) \to \mathbb{Z} \to 0,$$

with F_n, F_{n-1} and all the modules $C_i^{cell}(X)$, $0 \leqslant i \leqslant n-2$, finitely generated and free over $\mathbb{Z}\pi$. In particular, π is a $\mathcal{P}$-group of period dividing $n+1$. Conversely, according to Swan [59], if π is a $\mathcal{P}$-group of period k then π admits periodic resolutions

$$0 \to \mathbb{Z} \to P_{lk-1} \to P_{lk-2} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0, \quad l \geqslant 1,$$

with each P_i finitely generated projective. Swan proved in [59] that

$$\sum_{i=0}^{lk-1} (-1)^i [P_i] \in K_0(\mathbb{Z}\pi)$$

depends, modulo the Swan subgroup $\mathrm{T}(\mathbb{Z}\pi)$, only on the pair (π, l), and whence gives rise to elements

$$s_{lk}(\pi) \in K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi), \quad l \geqslant 1,$$

with k standing for the period of π. These elements are called *Swan obstructions*, and $s_k(\pi)$ is called *the* Swan obstruction of π. According to Swan the following holds (cf. [59]).

THEOREM 7.10. *Let π be a $\mathcal{P}$-group of period k. Then for every $l \geqslant 1$*

(i) $s_{lk}(\pi) = 0$ *if and only if π admits a free, cellular action on a finite CW-complex homotopy equivalent to S^{lk-1};*
(ii) $s_{lk}(\pi) = l \cdot s_k(\pi)$.

It is clear that the elements $s_{lk}(\pi)$ have finite order, that is, $\mathrm{rk}(s_{lk}(\pi)) = 0$. Indeed, because the period k of a $\mathcal{P}$-group is always *even* we obtain, by computing homology with $\mathbb{Q}$ coefficients,

$$\mathrm{rk}\left(\sum_{i=0}^{lk-1}(-1)^i[P_i]\right) = 1 - 1 = 0.$$

It follows that π acts on *some* finite CW-complex $X \simeq S^{lk-1}$, by taking for l the order of $s_k(\pi)$. Wall improved this considerably (cf. [70]) by showing that one can always take $l = 2$, that is

$$2 \cdot s_k(\pi) = 0 \in K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi).$$

For almost all families of $\mathcal{P}$-groups of period k one can show that the Swan obstruction $s_k(\pi)$ actually vanishes. Moreover, one can show that if $s_k(\pi) \neq 0$, the π must contain a subgroup of the form $Q(2^n a, b, c)$ with a, b, c coprime odd integers and $n \geqslant 3$ (the notation for these groups is due to Milnor [46]). These groups are defined as semi-direct products of the form

$$1 \to \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \times \mathbb{Z}/c\mathbb{Z} \to Q(2^n a, b, c) \to Q_n \to 1$$

with

$$Q_n = \mathrm{gp}\big(x, y \,|\, x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, yxy^{-1} = x^{-1}\big)$$

the quaternion group of order 2^n, acting so that x inverts the elements of $\mathbb{Z}/a\mathbb{Z}$ and $\mathbb{Z}/b\mathbb{Z}$ while y inverts those of $\mathbb{Z}/a\mathbb{Z}$ and $\mathbb{Z}/c\mathbb{Z}$. These groups all have period 4 and it turns out that the associated Swan obstruction does not vanish in general, but there is no simple minded pattern. For instance, according to Milgram [44] one has $s_4(Q(24, 5, 1)) \neq 0$ and $s_4(Q(24, 13, 1)) = 0$. The smallest group with nonvanishing Swan obstruction is $Q(16, 3, 1)$ a group of order 48 (cf. Davis [15]). Note that one can conclude that the groups $Q(24,5,1)$ and $Q(16,3,1)$ cannot be fundamental groups of 3-manifolds! The computation of the Swan obstruction relies on interpreting it as the image of some Reidemeister torsion element in the following way. Let $\mathbb{Z}_p$ denote the p-adic integers and $\mathbb{Q}_p$ the field of p-adic numbers. Consider the pull-back square

$$\begin{array}{ccc} \mathbb{Z}\pi & \longrightarrow & \mathbb{Z}[1/|\pi|]\pi \\ \downarrow & & \downarrow \\ \prod_{p||\pi|} \mathbb{Z}_p\pi & \longrightarrow & \prod_{p||\pi|} \mathbb{Q}_p\pi \end{array}$$

with associated Milnor sequence

$$\cdots \longrightarrow \prod_{p||\pi|} K_1(\mathbb{Q}_p\pi) \xrightarrow{\partial^\pi} K_0(\mathbb{Z}\pi) \longrightarrow \cdots.$$

Composing with the projection $K_0(\mathbb{Z}\pi) \to K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi)$ yields a homomorphism

$$\overline{\partial}^\pi : \prod_{p||\pi|} K_1(\mathbb{Q}_p\pi) \longrightarrow K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi).$$

It turns out that the Swan obstructions $s_{lk}(\pi)$ lie in the image of $\overline{\partial}^\pi$. Indeed, one can compute $s_{lk}(\pi)$ as follows. Given a $\mathcal{P}$-group π of period k and a periodic resolution of length lk,

$$\mathrm{P}(\pi, lk) : 0 \to \mathbb{Z} \to P_{lk-1} \to \cdots \to P_0 \to \mathbb{Z},$$

with each P_i finitely generated and projective over $\mathbb{Z}\pi$. One can think of $\mathrm{P}(\pi, lk)$ as being obtained as a pull-back of the form

$$\begin{array}{ccc} \mathrm{P}(\pi, lk) & \longrightarrow & \mathbb{Z}[1/|\pi|] \otimes \mathrm{P}(\pi, lk) \\ \downarrow & & \downarrow \\ \prod_{p||\pi|} \mathbb{Z}_p \otimes \mathrm{P}(\pi, lk) & \longrightarrow & \prod_{p||\pi|} \mathbb{Q}_p \otimes \mathrm{P}(\pi, lk) \end{array}.$$

This pull-back is completely determined by a family of "twisting isomorphisms"

$$\phi_{i,p} : \mathbb{Q}_p \otimes P_i \longrightarrow \mathbb{Q}_p \otimes P_i, \qquad p \big| |\pi|,$$

defining a Reidemeister torsion element

$$\tau\big(\mathrm{P}(\pi, lk)\big) := \sum_{i=0}^{lk-1} \sum_{p||\pi|} (-1)^i [\phi_{i,p}] \in \prod_{p||\pi|} K_1(\mathbb{Q}_p\pi),$$

which is uniquely determined modulo the image of

$$\left(\prod_{p||\pi|} K_1(\mathbb{Z}_p\pi)\right) \times K_1\big(\mathbb{Z}[1/|\pi|]\pi\big) \longrightarrow \prod_{p||\pi|} K_1(\mathbb{Q}_p\pi),$$

and it satisfies

$$\overline{\partial}^{\pi}\big(\tau\big(\mathrm{P}(\pi, lk)\big)\big) = s_{lk}(\pi) \in K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi).$$

It turns out that the computation of $s_{lk}(\pi)$ is closely related to the structure of the ring of integers (and strictly positive integers) in cyclotomic number fields. The interested reader is invited to consult [44], [45], [16] and [15] for more details.

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CHAPTER 27

Lusternik–Schnirelmann Category

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Category, in the sense of Lusternik and Schnirelmann, arose in the course of research into the theory of critical points. While the main applications of the classic theorem are in that area Smale [42] has found other applications in computer science, specifically in complexity theory. Both the original invariant and many variations are much studied by homotopy theorists. Some years ago I wrote a survey [28] of what was known at that time. Not surprisingly this had the effect of stimulating further research and much progress has been made since then. A comprehensive survey, up to the present time, would need to be quite lengthy. Rather than embark on this I have taken the opportunity presented by this article to describe some of the main ideas.

1. Introduction

Given a space X let us say that a subset V of X is *categorical* if V is contractible in X. It is not necessary for V to be contractible in itself, indeed V does not need to be connected. By a *categorical covering* of X we mean a finite numerable covering $\{V_1, \dots, V_k\}$ of X, for some k, by categorical subsets. We define the *category* $\mathrm{cat}(X)$ of X to be the least value of k for which such a covering exists. If no such covering exists we say that the category is infinite. However the class of spaces with finite category includes, for example, all compact manifolds and finite complexes.

In earlier work the covering is assumed to be either open or closed, rather than numerable. However when X is normal the definition given here is equivalent to the one in which the covering is required to be open, while when X is an ANR it is equivalent to the one in which the covering is required to be closed.

More generally we can define the category of a map $f : X \to Y$ in a similar fashion. We say that a subset V of X is *categorical with respect to* f, if the restriction $f \mid V$ is nulhomotopic. We define the *category* $\mathrm{cat}(f)$ of f to be the least value of k for which there exists a finite numerable covering $\{V_1, ..., V_k\}$ of X by subsets which are categorical with respect to f. If no such covering exists we say that the covering is infinite. Of course $\mathrm{cat}(f)$ reduces to $\mathrm{cat}(X)$ when $X = Y$ and f is the identity.

Of course $\mathrm{cat}(f) = 1$ if and only if f is nulhomotopic. Given a numerable covering $\{X_1, X_2\}$ of X we have

$$\mathrm{cat}(f) \leqslant \mathrm{cat}(f \mid X_1) + \mathrm{cat}(f \mid X_2).$$

For any two maps $f : X \to Y$ and $g : Y \to Z$ we have

$$\mathrm{cat}(g \circ f) \leqslant \min\left\{\mathrm{cat}(f), \mathrm{cat}(g)\right\}.$$

In particular

$$\mathrm{cat}(f) \leqslant \min\left\{\mathrm{cat}(X), \mathrm{cat}(Y)\right\}.$$

Clearly $\mathrm{cat}(f)$ depends only on the homotopy class of f, and hence $\mathrm{cat}\, X$ depends only on the homotopy type of X.

The Lusternik–Schnirelmann theorem has undergone various refinements over the years, at the hands of Palais, Schwartz, Smale and others. Here we follow the exposition of Clapp and Puppe [7].

Suppose that we have a paracompact C^1-Banach manifold M, possibly with boundary, and a C^1-function $f : M \to \mathbb{R}$. Consider the critical set K of f, i.e. the set of points of M where the derivative of f vanishes. Thus $f(K)$ is the set of critical values of f and $\mathbb{R} - f(K)$ is the set of regular values. For any $\alpha \in \mathbb{R}$ we write

$$M_\alpha = f^{-1}(-\infty, \alpha], \qquad K_\alpha = K \cap f^{-1}(\alpha).$$

The Lusternik–Schnirelmann theorem provides information about the topology of the sets K_α, under certain conditions.

If X and X' are subsets of M let us say that X is deformable into X' within M if there exists a homotopy $h_t : X \to M$ of the inclusion such that $h_1 X \subset X'$. The conditions we are going to assume are:

(D_1) For any α in the interior of the set of regular values of f there is an $\varepsilon > 0$ such that $M_{\alpha+\varepsilon}$ is deformable into $M_{\alpha-\varepsilon}$ within M.
(D_2) For any isolated critical value α of f and any neighborhood V of K in M there is an $\varepsilon > 0$ such that $M_{\alpha+\varepsilon} \backslash V$ is deformable into $M_{\alpha-\varepsilon}$ within M.
(D_3) If $\alpha > \sup f(K)$ then M is deformable into M_α within itself.

We refer to (D_1)—(D_3) as the *deformation conditions*. To ensure that they are satisfied some further assumptions on M and f are required. For example, suppose that M has no boundary. If M is a Hilbert manifold then M (being paracompact) admits Riemannian structure. If further $f : M \to \mathbb{R}$ is a C^2-function and a proper map (which is only possible when M is finite-dimensional) then standard methods of integrating the gradient field ∇f enable the deformation conditions to be established.

However, most of the important applications are to the infinite-dimensional case. Still assuming that M is a Hilbert manifold without boundary and that f is C^2 we impose

(C) For any $S \subset M$ such that f is bounded but $\|\nabla f\|$ is not bounded away from zero there exists a critical point in the closure $\bar{S} \subset M$.

This is often known as the *Palais–Smale condition*. When it is satisfied it can be shown, by more sophisticated arguments than in the finite-dimensional case, that the deformation conditions are satisfied.

So let $f : M \to \mathbb{R}$ be a C^1-function satisfying the deformation conditions. Write $\mathrm{cat}(M, X)$ for the category of the inclusion $X \subset M$, where X is any subset of M. Consider the function $m : \mathbb{R} \to \mathrm{N} \cup \{\infty\}$ given by $m(\alpha) = \mathrm{cat}(M, M_\alpha)$. We assert that

(i) the function m is (weakly) increasing.
(ii) In the interior of the set of regular values the function m is locally constant.
(iii) At any $\alpha \in \mathbb{R}$ which is an isolated critical value the function m jumps by $\mathrm{cat}(M, K_\alpha)$ at most.
(iv) When $\alpha > \sup f(K)$ then $m(\alpha) = \mathrm{cat}(M)$.

The proof of these assertions depends on just a few simple properties of the set function $n(X) = \mathrm{cat}(M, X)$, where X runs through the subsets of M, namely the following:

(1) *Monotonicity*: If $X' \subset X \subset M$ then $n(X') \leqslant n(X)$;

(2) *Subadditivity*: If $\{X_1, X_2\}$ is a numerable covering of $X \subset M$ then $n(X) \leqslant n(X_1) + n(X_2)$;

(3) *Deformation invariance*: If $X \subset M$ is deformable into X' within M then $n(X) \leqslant n(X')$;

(4) *Continuity*: If X is closed in M then $n(U) = n(X)$ for some neighborhood U of X in M.

The first three of these properties are trivial consequences of the definition while the fourth follows from the fact that M is an ANE. Returning to the four assertions made above we see that (i) follows directly from (1), that (ii) follows from (3) using (D_1), and that (iv) follows from (3) using (D_3). To prove (iii), the remaining assertion, take a neighborhood U of K_α such that $n(U) = n(K_\alpha)$. Let V be a closed neighborhood of K_α in the interior of U, and choose $\varepsilon > 0$ as in (D_2). Then

$$\begin{aligned} m(\alpha+\varepsilon) &= n(M_{\alpha+\varepsilon}) \\ &\leqslant n(M_{\alpha+\varepsilon}\backslash V) + n(U), \quad \text{by (2)}, \\ &\leqslant n(M_{\alpha-\varepsilon}) + n(K_\alpha), \quad \text{by } (\mathrm{D}_2) \text{ and (3)}, \\ &= m(\alpha-\varepsilon) + \mathrm{cat}(M, K_\alpha). \end{aligned}$$

This proves the assertions, which constitute the Lusternik–Schnirelmann theorem, from the modern standpoint. Note that if, in addition, f is bounded below then

$$\mathrm{cat}(M) \leqslant \sum_{\gamma \in \mathbf{R}} \mathrm{cat}(M, K_\gamma).$$

Palais [38] has shown how to extend the result to manifolds with boundary. Browder [6] has given an account of some of the applications. In the finite-dimensional case the Palais–Smale condition implies compactness. However it is possible, as shown in [29], to reformulate the theorem so that it applies to noncompact manifolds by modifying the notion of critical point, making use of ideas from the theory of ends.

2. Some variants

Basepoints play no role in the original definition of category in the Lusternik–Schnirelmann theorem. However although it is not always made explicit much of the literature is more concerned with the pointed version of category, as follows. Given a pointed space X let us say that a subset V of X (necessarily containing the basepoint) is *pointed categorical* if V is contractible in X in the pointed sense. By a *pointed categorical covering* of X we mean a finite numerable covering $\{V_1, \ldots, V_k\}$ of X, for some k, such that each member of the covering is pointed categorical. We define the *pointed category* $\mathrm{cat}^*(X)$ of X to be the least value of k for which such a covering exists. If no such covering exists we say that the pointed category is infinite. The pointed category of a pointed map is defined in a similar fashion.

Obviously $\operatorname{cat}(X) \leqslant \operatorname{cat}^*(X)$ in all cases. Equality holds provided (i) X is path-connected and (ii) the basepoint x_0 admits a numerically defined pointed categorical neighborhood in X. For then if V is a categorical subset of X containing x_0 then V is a pointed categorical subset, while if V is a categorical subset of X not containing x_0 we can form the union of V and a disjoint pointed categorical neighborhood of x_0 and thus obtain a pointed categorical superset of V. Thus a categorical covering can be converted into a pointed categorical covering and the conclusion follows.

G.W. Whitehead [45] gave a characterization of pointed category which the majority of homotopy theorists then adopted as their definition. As before let X be a pointed space with basepoint x_0. In the k-fold topological product $\Pi^k X$ $(k = 1, 2, \ldots)$ consider the "fat-wedge" subspace

$$T^k(X, x_0) = \pi_1^{-1}(x_0) \cup \cdots \cup \pi_k^{-1}(x_0),$$

where $\pi_i : \Pi^i X \to X \to X$ $(i = 1, \ldots, k)$ is the i-th projection. Whitehead showed that under fairly general conditions $\operatorname{cat}^*(X)$ is the least value of k for which the diagonal

$$\Delta : X \to \Pi^k X$$

can be deformed into $T^k(X, x_0)$, by a pointed homotopy. In fact it is sufficient that (i) X is normal and (ii) x_0 admits a pointed categorical neighborhood. Using the Whitehead definition we see that $\operatorname{cat}^*(X) \leqslant 2$ if and only if X admits co-Hopf structure.

Recall that the k-fold smash product $\Lambda^k(X)$ of X is obtained from $\Pi^k X$ by collapsing $T^k(X, x_0)$. Consider the projection

$$\Delta' : X \to \Lambda^k(X)$$

of the diagonal into the smash product. Obviously Δ' is nulhomotopic, in the pointed sense, if Δ can be deformed into $T^k(X, x_0)$, in the same sense. This observation led Berstein and Hilton [5] to define the *weak pointed category* $w\operatorname{cat}^*(X)$ of X to be the least value of k for which Δ' is pointed nulhomotopic. Clearly $w\operatorname{cat}^*(X) \leqslant \operatorname{cat}^*(X)$, in all cases, but examples are given in [5] and [22] where the two invariants are not the same.

Lower bounds for weak pointed category, and hence for pointed category, can be given using cohomology. Thus consider the reduced cohomology ring $\tilde{H}^*(X)$ of the pointed space X, with an arbitrary coefficient ring. If $w\operatorname{cat}^*(X)$ is defined then the cohomology ring is nilpotent and the index of nilpotency nil $\tilde{H}^*(X)$ cannot exceed it. Results of this type have a long history but this particular version can be found at the end of [5].

Although cohomological lower bounds yield important information it is not always easy to compute the index of nilpotency as, for example, in the case of the real Grassmannian where Stong [43], while improving earlier results of Hiller, has still not succeeded in completely solving what might appear to be a simple problem. Moreover, as we shall see in §7, it is not difficult to give examples where the category is infinite but the multiplicative structure of the cohomology ring is trivial.

3. More variants

The variants of the original concept we have described so far by no means exhaust the possibilities. We continue by discussing some examples of a different type. In fact each of these can, as we shall see, be regarded as a special case of category with respect to a map.

Given a fibrewise space X over B let us say that a subset W of B is *section-categorical* if X_W admits a section over W. By a *section-categorical covering* of B we mean a finite numerable covering $\{W_1, \dots, W_k\}$ of B, for some k, such that each member of the covering is section-categorical. We define the *sectional-category* $\mathrm{secat}(X)$ of X to be the least value of k for which such a covering exists. If no such covering exists we say that the sectional category is infinite. Note that $\mathrm{secat}(X) \leqslant \mathrm{cat}(B)$ when X is fibrant.

For paracompact B Schwartz has shown that $\mathrm{secat}(X) \leqslant k$ if and only if the k-fold fibrewise join

$$X^{(k)} = X *_B * \cdots *_B X \quad (k \text{ factors})$$

of X with itself admits a section. This result, for which §8 of [28] is a convenient reference, leads to an upper bound for sectional category as in (8.2) of [28]. Specifically, let B be a finite complex and let X be a fibre bundle over B with $(q-1)$-connected fibre, where $q \geqslant 1$. Then

$$\mathrm{secat}(X) < (q+1)^{-1} \dim B + 2.$$

For a lower bound we turn to cohomology again and consider the homomorphism

$$p^* : H^*(B) \to H^*(X)$$

induced by the projection. We find that

$$\mathrm{secat}(X) \geqslant \mathrm{nil}\ker p^*.$$

When X admits a section the sectional category itself is without interest, but then the polar category, which has somewhat similar properties, may be defined, as follows. Let us say that X is *polarized* if every section of $X - B$ is vertically homotopic in X to the standard section. For example, suppose that $X = \Sigma_B E$, the fibrewise suspension of a fibrewise space E. Then X comes equipped with a pair of "polar" sections, where the suspension parameter takes its extreme values. We choose one polar section, conventionally called the north, as standard, and refer to the other as the south polar section. Then X is polarized if and only if the polar sections are vertically homotopic. The latter condition is satisfied whenever the fibrewise space E admits a section. The converse holds when E is a $(q-1)$-sphere bundle over B and B is a finite complex such that $\dim B < 2q - 2$.

After these preliminaries we are ready to define the polar category. Given a fibrewise pointed space X over B let us say that a subset W of B is *polar categorical* if X_W

is polarized over W. By a *polar categorical covering* of B we mean a finite numerable covering $\{W_1, \ldots, W_k\}$ of B such that each member of the covering is polar categorical. We define the *polar category* polcat(X) of X to be the least value of k for which such a covering exists. If no such covering exists we say that the polar category is infinite.

In case $X = \Sigma_B E$, for some fibrewise space E, we have

$$\operatorname{polcat}(\Sigma_B E) \leqslant \operatorname{secat} E.$$

Conversely, suppose that E is a $(q-1)$-sphere-bundle over B and B is a finite complex such that $\dim B \leqslant q(k+1)-3$ for some k. Then $\operatorname{polcat}(\Sigma_B E) \leqslant k$ implies $\operatorname{secat}(E) \leqslant k$.

Another variant of the original concept arises in the theory of fibre bundles, as follows. Let X be a numerable G-bundle over B, where G is a topological group. By a *triviality covering* of B, with respect to X, we mean a finite numerable covering $\{W_1, \ldots, W_k\}$ of B, for some k, such that X is trivial over each member of the covering. We define the *triviality category* trivcat(X) of X to be the least value of k for which such a covering exists. If no such covering exists we say that the triviality covering is infinite. It is not difficult to show that trivcat(X) is equal to the category of the classifying map $B \to BG$ of X, where BG denotes the classifying space of G. An example where triviality category naturally arises is as follows.

Manifolds are usually described as the result of gluing together open subsets of a fixed Euclidean space. It is natural at the outset to ask how efficiently a given manifold can be constructed. In low dimensions it is often possible to study this problem by geometric methods, as Montejano and others have shown, but in general the geometry needs to be supplemented by the methods of homotopy theory.

In studying this problem Berstein [3] introduced the following invariants (all manifolds, embeddings and immersions are assumed to be C^∞-smooth). Let M be a closed n-manifold. The *embedding covering number* $N(M)$ is the least integer k such that M can be covered by k open sets, each of which embeds in $\mathbb{R}^n$. The *immersion covering number* $n(M)$ is the least integer k such that M can be covered by k open sets, each of which immerses in $\mathbb{R}^n$.

Although it is $N(M)$ we wish to determine, it cannot be less than $n(M)$, and thanks to the work of Hirsch and Smale $n(M)$ can be seen to be precisely the triviality category of the stable tangent bundle of M.

For various reasons it is of particular interest to determine the Berstein covering numbers in the case of P^n, the real projective n-space. Berstein himself gave upper and lower bounds for the covering numbers but in general they are wide apart. Much more recently Hopkins [27] succeeded in closing the gaps completely for the immersion problem, and has done so for the embedding problem except in two cases where there is an element of doubt. The specific results are as follows. Write $n+1 = 2^k m$, with m odd. Then

$$n(P^n) = \begin{cases} \max(2, m) \text{ if } k \leqslant 3, \\ \text{least integer } \geqslant \frac{n+1}{2(k+1)} \text{ if } k \geqslant 3. \end{cases}$$

Moreover $N(P^n) = n(P^n)$ with the possible exception of the values $n = 31$ and $n = 47$,

where

$$3 \leqslant N(P^{31}) \leqslant 4, \qquad 5 \leqslant N(P^{47}) \leqslant 6.$$

Unfortunately the arguments are too technical to be summarized here.

4. Equivariant category

Developing an equivariant version of the classical theory is not as easy as it might first appear. However it is clear enough how to begin. Let X be a G-space, where G is a topological group. We describe an invariant subspace V of X as *G-categorical* if there exists a G-homotopy $h_t : V \to X$ of the inclusion such that $h_1 V$ is the orbit Gx of some point x of X. By a *G-categorical covering* of X we mean a finite numerable covering $\{V_1, \ldots, V_k\}$ of X by G-categorical subsets. We define the *G-category* G-cat(X) of X to be the least value of k for which such a covering exists. If no such covering exists we say that the G-category is infinite. The G-category of a G-map is defined in a similar fashion.

After the initial stage one has a choice of several different treatments in the literature, which only partially overlap. Fadell [14], and Fadell and Husseni [15], [16], [17] have developed one approach. Mazantowicz [37] and Ramsay [39] have developed others, as have Barsch and Clapp [2], and Clapp and Puppe [7], [8], [9]. The differences originate from the different applications the authors have in mind and cannot be reconciled into a single theory.

Let us assume, for simplicity, that G is compact. Then each orbit $G.x$ is equivalent, as a G-space, to a factor space G/H, where H is the stabilizer of x. Relations exist between the equivariant category of the G-space X and the ordinary category of the orbit space X/G. We have

$$G\text{-cat}(X) \geqslant \operatorname{cat}(X/G)$$

in all cases, while equality holds when X has just one orbit type, in particular when the action is free (see 1.10 of [37], for example).

In this case, therefore, it is easy to convert results about ordinary category into results about equivariant category.

For some results it is convenient, perhaps necessary, to assume that X is a G-ANR. This class of G-spaces includes finite-dimensional G-CW complexes. It also includes finite-dimensional smooth G-manifolds, where G is a compact Lie group.

Mazantowicz [37] gives an upper bound for the equivariant category of the G-space X in terms of the dimension of X/G and another number depending on the orbit structure. In particular if X is a connected G-ANR and the fixed point set X^G is nonempty and connected then

$$G\text{-cat}(X) \leqslant \dim(X/G) + 1.$$

Here, as usual, dim means covering dimension.

Cohomological lower bounds for the equivariant category can be obtained in terms of Borel cohomology as follows. Let EG be a contractible space on which G acts freely. Consider, for each G-space X, the orbit space $EG \times_G X$ of the product $EG \times X$, with respect to the diagonal action. (Under certain conditions $EG \times_G X$ has the same homotopy type as X/G.) The *Borel cohomology* $H^*_G(X;R)$ of X, with coefficients in a ring R, is defined to be the Čech cohomology ring $H^*(EG \times_G X;R)$. The cup-product in $H^*_G(X;R)$ is defined in the usual way. We can regard $H^*_G(X;R)$ as a module over the coefficient ring $H^*_G(pt;R) = H^*(BG;R)$, where $BG = EG/G$ is the classifying space of G.

At this stage Mazantowicz and Ramsay introduce the strong assumption that all orbits satisfy the dimension axiom, i.e. that $H^i_G(G/H) = 0$ for all $i > 0$ and every closed subgroup H of G. Then

$$G\text{-cat}(X) \geqslant \operatorname{nil} \tilde{H}^*_G(X)$$

by the argument which is used in the ordinary theory. However the assumption is very restrictive and to avoid this Fadell, and Fadell and Husseini, prefer to disregard equivariant category and instead to seek cohomological lower bounds directly for the number of critical orbits of an invariant real-valued function on a given G-space.

Instead of pursuing these questions further let us turn now to the pointed version of the equivariant theory. Specifically let X be a pointed G-space (i.e. the basepoint x_0 is a fixed point). We describe an invariant subset V of G (necessarily containing the basepoint) as *pointed G-categorical* if there exists a pointed G-homotopy $h_t : V \to X$ of the inclusion such that $h_1 V = x_0$. By a *pointed G-categorical covering* of X we mean a finite numerable covering $\{V_1, \dots, V_k\}$ of X by pointed G-categorical subsets. We define the *pointed G-category* $G\text{-cat}^*(X)$ of X to be the least value of k for which such a covering exists. If no such covering exists we say that the pointed G-category is infinite. The pointed G-category of a pointed G-map is defined in a similar fashion.

Obviously $G\text{-cat}(X) \leqslant G\text{-cat}^*(X)$ in all cases. In fact equality holds if (i) X^H is path-connected for all closed subgroups H of G and (ii) there exists a numerically defined pointed G-categorical neighborhood of the basepoint.

Proceeding on the same lines as in §2 we can formulate a "Whitehead" form of the definition of pointed G-category and then a weak form. Without making any further assumptions a lower bound for the weak pointed G-category can be obtained in terms of Borel cohomology, using an equivariant version of the argument given at the end of [5], and of course this is a lower bound for pointed G-category itself.

5. Fibrewise category

Fibrewise category is a relatively new idea. The following outline is based on [30] and [31] where full details may be found. We describe a subset V of a fibrewise space X over a given base space B to be *fibrewise categorical* if the inclusion $V \to X$ is fibrewise nulhomotopic. By a *fibrewise categorical covering* of X we mean a finite numerable covering $\{V_1, \dots, V_k\}$ of X by fibrewise categorical subsets. We define the *fibrewise* *category* $\mathrm{cat}_B(X)$ of X to be the least value of k for which such a covering exists. If no such number exists the fibrewise category is said to be infinite. The fibrewise category of a fibrewise map is defined in a similar fashion.

Note that for any space A and map $\lambda : A \to B$ we have

$$\mathrm{cat}_A(\lambda^* X) \leqslant \mathrm{cat}_B(X),$$

where $\lambda^* X$ denotes the induced fibrewise space over A. In particular $\mathrm{cat}_B X$ is bounded below by the category of the fibres of X.

Of course $\mathrm{cat}_B(X) = 1$ if and only if X is fibrewise contractible. Also $\mathrm{cat}_B(X) \leqslant 2$ if X is a fibrewise suspension since then X is the union of two open fibrewise cones.

Let us turn now to the fibrewise pointed theory. We describe a subset V of a fibrewise pointed space X (necessarily containing the section) to be *fibrewise pointed categorical* if the inclusion $V \to X$ is fibrewise pointed nulhomotopic. By a *fibrewise pointed categorical covering* of X we mean a finite numerable covering $\{V_1, \ldots, V_k\}$ of X by fibrewise pointed categorical subsets. We define the *fibrewise pointed category* $\mathrm{cat}^B_B(X)$ of X to be the least value of k for which such a covering exists. If no such number exists the fibrewise pointed category is said to be infinite. The fibrewise pointed category of a fibrewise pointed map is defined in a similar fashion.

Note that for any space A and map $\lambda : A \to B$ we have

$$\mathrm{cat}^A_A(\lambda^* X) \leqslant \mathrm{cat}^B_B(X),$$

where $\lambda^* X$ denotes the induced fibrewise pointed space over A. In particular $\mathrm{cat}^B_B(X)$ is bounded below by the pointed category of the fibres of X.

Of course $\mathrm{cat}^B_B(X) = 1$ if and only if X is fibrewise pointed contractible. Also $\mathrm{cat}^B_B(X) \leqslant 2$ if X is the reduced fibrewise suspension of a fibrewise pointed space.

If we disregard the section then the fibrewise category $\mathrm{cat}_B X$ of X is defined and cannot exceed $\mathrm{cat}^B_B(X)$. The relation between these invariants will be considered below.

For any fibrewise pointed space X over B the k-fold fibrewise product $\Pi^k_B X$ is defined $(k = 1, 2, \ldots)$ and contains the union $T^k(X, B)$ of the preimages $\pi_i^{-1}(B)$ $(i = 1, \ldots, k)$ of the section. We may refer to $T^k(X, B)$ as the *fibrewise fat wedge*. Note that $\Pi^k_B X$ contains the diagonal ΔX of X while $T^k_B(X, B)$ contains the diagonal ΔB of B. In other words the pair

$$\Pi^k_B(X, B) = \left(\Pi^k_B X, T^k_B(X, B)\right)$$

contains the diagonal $\Delta(X, B) = (\Delta X, \Delta B)$ of the pair (X, B). By generalizing the argument used in the ordinary case we find (see [31]) that under fairly general conditions $\mathrm{cat}^B_B(X)$ is the least value of k for which the diagonal

$$\Delta : X \to \Pi^k_B X$$

can be deformed into $T^k_B(X, B)$ by a fibrewise pointed homotopy. If we adopt this criterion as our definition we see that $\mathrm{cat}^B_B(X) \leqslant 2$ if and only if X admits fibrewise coHopf structure.

There is an obvious connection between pointed category in the equivariant sense and pointed category in the fibrewise sense. Thus let P be a principal G-bundle over B, where G is a topological group. Let Y be a pointed G-space and let X be the associated bundle with fibre Y and section determined by the basepoint. Then pointed G-categorical subsets of Y correspond to fibrewise pointed categorical subsets of X, and so

$$\mathrm{cat}_B^B(X) \leqslant G\text{-}\mathrm{cat}^*(Y).$$

Recall that the k-fold fibrewise smash product $\Lambda_B^k X$ of X is obtained from $\Pi_B^k X$ by fibrewise collapsing $T_B^k(X, B)$. Consider the projection

$$\Delta' : X \to \Lambda_B^k X$$

of the diagonal into the fibrewise smash product. Obviously Δ' is fibrewise nulhomotopic, in the pointed sense, if Δ can be fibrewise deformed into $T_B^k(X, B)$, in the same sense. This suggests defining the *weak fibrewise pointed category* $w\mathrm{cat}_B^B(X)$ of X to be the least number k such that $\Delta' : X \to \Lambda_B^k X$ is fibrewise pointed nulhomotopic. Clearly $w\mathrm{cat}_B^B(X) \leqslant \mathrm{cat}_B^B(X)$, in all cases, but the two invariants do not always coincide, even when X is a sectioned sphere-bundle, as we shall see later.

Note that for any space A and map $\lambda : A \to B$ we have

$$w\mathrm{cat}_A^A(\lambda^* X) \leqslant w\mathrm{cat}_B^B(X).$$

In particular $w\mathrm{cat}_B^B(X)$ is bounded below by the weak pointed category of the fibres of X.

There is a useful functor which sends each fibrewise pointed space X into the mapping-cone C_s of the section s, and similarly for fibrewise pointed maps and fibrewise pointed homotopies. Then

$$\mathrm{cat}_B^B(X) \geqslant \mathrm{cat}^*(C_s) \leqslant \mathrm{cat}^*(B) + 1;$$

the first inequality resulting from the use of the functor, the second being due to Berstein and Ganea. Similarly

$$w\mathrm{cat}_B^B(X) \geqslant w\mathrm{cat}^*(C_s) \leqslant w\mathrm{cat}^*(B) + 1.$$

When the section s is a cofibration we may replace C_s by the pointed space X/B obtained from X by collapsing B. The index of nilpotency $H^*(X, B)$ of the cohomology ring of the pair (X, B), with arbitrary coefficients, is then a lower bound for $w\mathrm{cat}_B^B(X)$ and hence for $\mathrm{cat}_B^B(X)$.

It turns out that polar category, as defined in §4, appears in a relation between fibrewise category and fibrewise pointed category. Specifically, let X be a fibrewise pointed space over B such that the section admits a numerically defined fibrewise pointed categorical neighborhood in X. Then

$$\mathrm{cat}_B^B(X) \leqslant 1 + \mathrm{polcat}(X).\mathrm{cat}_B(X - B).$$

When X is a sectioned sphere-bundle this implies

$$\mathrm{cat}_B^B(X) \leqslant 1 + \mathrm{polcat}(X) \leqslant 1 + \mathrm{cat}(B).$$

These numerical invariants of fibrewise homotopy type can be evaluated in the case of sectioned sphere-bundles over spheres, or rather the problem of evaluation can be reduced to a computation in the homotopy groups of spheres. Specifically consider the sectioned oriented q-sphere bundle X_α over S^n formed by the clutching construction from an element $\alpha \in \pi_{n-1}SO(q)$. We find $\mathrm{cat}_B^B(X_\alpha) = 2$ if $\Sigma_*^n \rho_* \alpha = 0$, otherwise $\mathrm{cat}_B^B(X_\alpha) = 3$. We also find $w\mathrm{cat}_B^B(X_\alpha) = 2$ if $\Sigma_*^{n+1} \rho_* \alpha = 0$, otherwise $w\mathrm{cat}_B^B(X_\alpha) = 3$. Hence an example can be given of a sectioned 8-sphere bundle over S^{22} which has fibrewise pointed category 3 but weak fibrewise pointed category 2.

6. Strong category and homotopy colimits

Returning to the original definition of category it is natural to ask what difference it makes if we use coverings of the given space X by subsets which are contractible in themselves, rather than contractible in X. It was realized at an early stage that the number thus defined is not a homotopy invariant. However Ganea [21] considered the homotopy invariant which can be derived from it by running through all spaces of the same homotopy type as X. Specifically he defined the strong category $\mathrm{Cat}(X)$ of X to be the least number of contractible subsets required to numerably cover a space of the same homotopy type as X. It is easy to see that $\mathrm{Cat}(X) \leqslant k$ if and only if X is dominated by a space Z such that $\mathrm{cat}(Z) \leqslant k$. Moreover Takens [44] has shown, under fairly general conditions, that either $\mathrm{Cat}(X) = \mathrm{cat}(X)$ or $\mathrm{Cat}(X) = \mathrm{cat}(X) + 1$; both possibilities can occur.

Of course there is also a pointed version of the definition; the strong pointed category of the pointed space X will be denoted by $\mathrm{Cat}^*(X)$. If X is a CW complex one can use subcomplexes rather than subsets but this turns out to make no difference.

For path-connected CW-spaces X Ganea showed that $\mathrm{Cat}(X)$ coincides with another invariant, the "cone-length" $\mathrm{Cl}(X)$ of X. Specifically he defined $Cl(X)$ to be the least value of k such that there exists a sequence of cofibration sequences

$$Z_i \to X_i \to X_{i+1},$$

with $i = 1, ..., k-1$, with X_1 contractible, and with X_k of the same homotopy type as X. Recently Cornea [10], [11] has shown that the same is true for sequences in which each Z_i is required to be an i-fold suspension.

Another way of looking at strong category, and hence category, has been developed by Clapp and Puppe [8]. Essentially the same idea occurred to Hopkins [26] independently but Hopkins was more concerned with the dualization which led him to fresh insights into the right way to define cocategory. Following Clapp and Puppe I give an outline of this alternative method which yields comparatively straightforward proofs of some of the classical results.

Let K be a simplicial complex and let K also denote the poset of its simplices ordered by opposite inclusion. For any functor $\Phi : K \to \mathrm{Top}$ the homotopy colimit h-colim Φ of Φ is defined in the usual way. For example, if $K = \Delta^{k-1}$, the standard $k-1$ simplex, the k-fold mapping cylinder is the homotopy colimit of a functor $\Phi : \Delta^{k-1} \to \mathrm{Top}$.

Given a covering $\{V_0, ..., V_{k-1}\}$ of a space X we associate with it the functor $U : \Delta^{k-1} \to X$, where

$$U(\sigma) = \bigcap_{i \in \sigma} V_i \quad \text{and} \quad U(\tau \leqslant \sigma) : U(\tau) \subset U(\sigma).$$

The homotopy colimit of U is known as the classifying space BU of U and the canonical map $BU \to X$ is a homotopy equivalence when the covering is numerable.

Given a functor $\Phi : K \to \mathrm{Top}$ we can construct a canonical map

$$\phi : h\text{-colim}\,\Phi \to K.$$

The preimages under ϕ of the open stars $\mathrm{St}(v)$ of the vertices v of K form a numerable covering of h-colim Φ and the canonical contraction of $\mathrm{St}(v)$ to v lifts canonically to a fibrewise deformation retraction of $\phi^{-1}\mathrm{St}(v)$ to $\phi^{-1}(v) = \Phi(v)$. It follows that $\mathrm{Cat}^*(X) \leqslant k$ if and only if X has the pointed homotopy type of a k-fold mapping cylinder with vertices at the basepoint.

From this we can deduce the product inequality

$$\mathrm{Cat}^*(X \times Y) \leqslant \mathrm{Cat}^* X + \mathrm{Cat}^* Y - 1,$$

for any pointed spaces X, Y. For let $m = \mathrm{Cat}^*(X)$, $n = \mathrm{Cat}^*(Y)$. We may assume that X and Y are homotopy colimits of functors $U : \Delta^{m-1} \to \mathrm{Top}$ and $V : \Delta^{n-1} \to \mathrm{Top}$, vertices being mapped to the point-space. Consider the canonical simplicial subdivision K of $\Delta^{m-1} \times \Delta^{n-1}$. Then a functor $\Phi : K \to \mathrm{Top}$ is defined by

$$\Phi(\rho) = U(\sigma) \times V(\tau),$$

where σ and τ are the smallest simplices of Δ^{m-1} and Δ^{n-1}, respectively, such that $\rho \subset \sigma \times \tau$. Then $X \times Y$ is homeomorphic to h-colim Φ and the result follows.

Of course the product inequality for strong category implies the product inequality for ordinary category

$$\mathrm{cat}(X \times Y) \leqslant \mathrm{cat}(X) + \mathrm{cat}(Y) - 1.$$

Examples can be given where equality does not hold but these always possess torsion in both factors. Ganea conjectured that equality holds whenever one of the factors is a sphere. Only very limited progress had been made in proving the Ganea conjecture until recently when rational methods achieved a remarkable success. I will now describe some of these methods which have led to a much better understanding of the subject.

7. Rational methods

The rational homotopy theory, as developed by Quillen, now has an extensive literature, with contributions from Anick, Felix, Halperin, Hess, Jessup, Lemaire, Thomas and others. It was Felix and Halperin who first showed that rational methods were effective in dealing with problems about category, particularly the use of Sullivan's minimal models. This enabled Jessup and then Hess to obtain results, including the Ganea conjecture for simply-connected rational spaces, which seem beyond the reach of other methods.

A convenient introduction to the relevant rational homotopy theory has been given by Lemaire [36]. In this branch of the subject it is customary to denote spaces by the letters $S, T, \ldots$. From now on we assume that all such spaces are connected and simply-connected CW-spaces of finite type.

Recall that a rational space is one of which the homotopy groups are rational vector spaces of finite dimension. For any space S we denote by S_0 the rationalization of S, i.e. the localization with respect to all primes. Rational spaces can be modeled by (commutative) differential graded algebras, DG algebras for short. We recall that the Sullivan minimal model ΛX of S is a DG algebra, freely generated as a graded commutative algebra by the dual of $\pi_*(S)$.

So far, in this article, we have followed the traditional normalization of category, in which points have category one, spheres have category two, and so forth. From now on, however, we follow the practice in this branch of the subject and reduce the value of the invariant by unity, so that points have category zero, spheres have category one, and so forth.

Consider a fibration $p : E \to B$. Suppose first, that the fibre F is categorical in E. Then it follows from the homotopy lifting property that if V is a categorical subset of B then the preimage $p^{-1}V$ is categorical in E. Hence $\operatorname{cat}(E) \leqslant \operatorname{cat}(B)$. Does this conclusion hold if we simply assume that the induced homomorphism

$$p_* : \pi_*(E) \to \pi_*(B)$$

is injective, in all dimensions? Examples can be given to show that in general it does not. However, when E and B are rational spaces Felix and Halperin showed that the weak assumption is sufficient. For such spaces, therefore, we have some new information about the behaviour of category. This mapping theorem of Felix and Halperin opened up a line of investigation which has proved most fruitful.

In the Felix–Halperin mapping theorem it is asserted that if E and B are rational spaces then $\operatorname{cat}(E) \leqslant \operatorname{cat}(B)$ when p_* is injective. To establish this it is sufficient, as we have seen, to show that the fibre F of p is contractible in E. And this will be the case if we can show that the standard map $j : \Omega B \to F$ admits a right inverse up to homotopy. Now since

$$j_* : \pi_*(\Omega B) \to \pi_*(F)$$

is a surjection we may write

$$\pi_*(\Omega B) = U_* \oplus \ker j_*,$$

where the summand U_* is mapped isomorphically onto $\pi_*(F)$ under j_*. According to the well-known theorem of Milnor and Moore any loop-space has the rational homotopy type of a product of Eilenberg–MacLane spaces. So ΩB, here, has the homotopy type of the product

$$\prod_{n\geqslant 1} K(U_n, n) \times \prod_{n\geqslant 1} K\big(\ker(j_*), n\big).$$

Since the restriction of $j : \Omega B \to F$ to the factor

$$\prod_{n\geqslant 1} K(U_n, n)$$

is a homotopy equivalence, we deduce that j admits a right inverse, as required.

The rational category $\mathrm{cat}_0(S)$ of a space S is defined to be the ordinary category $\mathrm{cat}(S_0)$ of its rationalization S_0. So for spaces E and B, not necessarily rational spaces, the Felix–Halperin mapping theorem shows that $\mathrm{cat}_0(E) \leqslant \mathrm{cat}_0(B)$ when p_* is injective. The assumption that p is a fibration can be dropped since any map can be replaced by a fibration in the usual way.

The notion of category can be extended to DG algebras, as follows. If ΛX is the minimal model we denote by $\Lambda^{>k}X$ the algebra generated by products of length greater than k. The quotient $\Lambda X/\Lambda^{>k}X$ can be written in the form $\Lambda X \otimes \Lambda Y$, where ΛY is also a minimal model, and then the natural projection to the quotient takes the form of a morphism

$$p : \Lambda X \to \Lambda X \otimes \Lambda Y$$

of DG algebras. Felix and Halperin [18] define the rational category $\mathrm{cat}_0 \Lambda X$ of the DG algebra ΛX to be the least value of k for which ρ admits a right inverse, as a morphism of DG algebras. They show that if ΛX is the minimal model of a space S then $\mathrm{cat}_0(S)$, as previously defined, is equal to $\mathrm{cat}_0(\Lambda X)$.

Now suppose that we seek a right inverse of ρ not as a morphism of DG algebras but simply as a morphism of ΛX-modules. Halperin and Lemaire [24] define the module rational category $M\mathrm{cat}_0(\Lambda X)$ of ΛX to be the least integer k for which such a right inverse of ρ exists. Then they define $M\mathrm{cat}_0(S)$, for a space S, to be $M\mathrm{cat}_0(\Lambda X)$ where ΛX is the minimal model of S.

Berstein asked whether the Ganea conjecture might be true at least for rational spaces, in other words whether

$$\mathrm{cat}_0(T \times S^n) = \mathrm{cat}_0(T) + 1 \quad (n > 1)$$

for all rational T. Jessup [32] succeeded in establishing this with $M\mathrm{cat}_0$ in place of cat_0. Meanwhile Hess [25] had shown that cat_0 and $M\mathrm{cat}_0$ are equal. So Berstein's question is answered in the affirmative. The rational method tells us nothing when $n = 1$ nor does it help with the case when $\pi_1(T)$ is nontrivial, but nevertheless the Hess–Jessup theorem constitutes a major advance in this difficult area.

These and other algebraic invariants, related to category in the topological sense, seem destined to play an important role in future developments. The idea naturally suggests itself of formulating a notion of category (in the sense of Lusternik–Schnirelmann) for other categories (in the sense of Eilenberg–MacLane). An early exercise of this type is that of Eckmann and Hilton [13]. More recent exercises have usually been based on Quillen's closed model theory. In particular Doeraene [12] has developed a general framework which seems to include category in the ordinary sense and most, if not all, of these variants.

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Subject Index

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