SPECTRAL MODULI PROBLEMS FOR LEVEL STRUCTURES AND AN INTEGRAL JACQUET–LANGLANDS DUAL OF MORAVA E-THEORY

XUECAI MA AND YIFEI ZHU

ABSTRACT. Given an $\mathbb{E}_\infty$-ring spectrum R, with motivation from chromatic homotopy theory, we define relative effective Cartier divisors for a spectral Deligne–Mumford stack and prove that, as a functor from connective R-algebras to topological spaces, it is relatively representable. We then solve various moduli problems of level structures on spectral abelian varieties, overcoming difficulty at primes dividing the level. In particular, we obtain higher-homotopical refinement for finite levels of the Lubin–Tate tower as $\mathbb{E}_\infty$-rings, which generalize Morava, Hopkins, Miller, Goerss, and Lurie's spectral realization at the ground level. Moreover, passing to the infinite level and then descending along the equivariantly isomorphic Drinfeld tower, we obtain a Jacquet–Langlands dual to the Morava E-theory spectrum, along with homotopy fixed point spectral sequences dual to those studied by Devinatz and Hopkins. These serve as potential tools for computing higher-periodic homotopy types from pro-étale cohomology of p-adic general linear groups.

CONTENTS

Date: July 8, 2025. Some parts need polishing – please read with caution and understanding.

1. Introduction

The stable homotopy category is a central topic in algebraic topology. Structured ring spectra are the most common examples studied, such as H_∞ spectra and $\mathbb{E}_\infty$ spectra. In [Lur09a] and [Lur18b], Lurie uses spectral algebraic methods gives a proof of the Goerss-Hopkins-Miller theorem for topological modular forms. Except for the application of elliptic cohomology, Lurie also proved the $\mathbb{E}_\infty$ structures of Morava E-theories [Lur18b], which use the spectral version of deformation theory of certain p-divisible groups. The earliest proof of $\mathbb{E}_\infty$ structures of Morava E-theories is due to Goerss, Hopkins, and Miller [GH04]. They turned the problem into a moduli problem and developed an obstruction theory. One can finish the proof by computing the André–Quillen groups. Comparing with their method, Lurie's proof is more conceptual. There are more and more applications of spectral algebraic geometry in algebraic topology. Such as topological automorphic forms [BL10], Morava E-theories over any $\mathbb{F}_p$-algebra [Lur18b], not only just for a perfect field k. The construction of equivariant topological modular forms [GM23], elliptic Hochschild homology [ST23], and more.

On the other hand, moduli problems concerning deformations of formal groups with level structures are also representable, and moduli spaces of different levels form a Lubin–Tate tower [RZ96, FGL08, SW13]. We know that the universal objects of deformations of formal groups have higher algebraic analogs which are the Morava E-theories. A natural question is what are higher categorical analogues of moduli problems of deformations with level structures? And can we find higher categorical analogs of Lubin–Tate towers? Although the $\mathbb{E}_\infty$-structure of topological modular forms with level structures can be obtained from [HL16], we still hope that there exists a derived stack of spectral elliptic curves with level structures that provide us with a more moduli interpretation. Except this, in the computation of unstable homotopy groups of spheres, after applying the EHP spectral sequences and the Bousfield-Kuhn functor, we observe that some terms on the E_2-page also arise from the universal deformation of isogenies of formal groups. They are computed by the Morava E-theories on the classifying spaces of symmetric groups [Str97, Str98]. They can be viewed as sheaves on the Lubin–Tate tower. We hope to provide a more conceptual perspective on this fact within the higher categorical Lubin–Tate tower.

In this paper, we give an attempt to address this problem by studying specific moduli problems in spectral algebraic geometry. The main ingredient of our work is the derived version of Artin's representability theorem established in [Lur04, TV08]. We will use the spectral algebraic geometry version [Lur18c] in this paper. We study relative effective Cartier divisors in the context of spectral algebraic geometry. By imposing certain conditions, we define derived level structures of certain geometric objects in spectral algebraic geometry. Using this Artin representability theorem, we prove some representable results of moduli problems that arise from our derived level structures. We give some examples of applications involving derived level structures. We consider the moduli problem of spectral deformations with derived level structures of p-divisible groups. We prove that these moduli problems are representable by certain formal affine spectral Deligne–Mumford stacks and the corresponding spectra can provide us many interesting general cohomology theories.

We note here that the Goerss–Hopkins–Miller–Lurie sheaf does not directly apply to the moduli problems here due to the failure of étaleness (cf. [Dev23]). This is fixed by relative effective Cartier divisors analogous to Drinfeld's original approach to arithmetic moduli of (classical) elliptic curves [KM85, Introduction].

Outline. We work on spectral algebraic geometry in this paper. In Section 2, we define derived isogenies and prove that the kernel of a derived isogeny in some cases has the same phenomenon as in the classical case. This provides evidence that our derived versions of level structures must induce classical level structures. For representability reasons, we use moduli associated with sheaves to detect higher homotopy of derived versions of level structures. We define relative effective Cartier divisors in the context of spectral algebraic geometry. For a spectral Deligne–Mumford stack X over a spectral Deligne–Mumford stack S, a relative effective Cartier divisor is a morphism $D \to S$ of spectral Deligne–Mumford stacks such that $D \to X$ is a closed immersion, the ideal sheaf of D is a line bundle over X, and the morphism $D \to S$ is flat, proper and locally almost of finite presentation. We use Lurie's representability theorem to prove that the relative effective Cartier divisor is representable in certain cases. The main part of our proof involves computing of cotangent complex. Here is our first main result.

Theorem A (Theorem 2.17)**.** Suppose that E is a spectral algebraic space over a connective $\mathbb{E}_\infty$-ring R, such that $E \to R$ is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor

$$\begin{aligned} \mathrm{CDiv}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{cn} \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation over R.

In Section 3, we define derived level structures of spectral elliptic curves. Roughly speaking, for an abstract finite abelian group A, usually equals $\mathbf{Z}/N\mathbf{Z}$, $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$, a derived level-$A$ structure of a spectral elliptic curve E over an $\mathbb{E}_\infty$-ring R is just a relative effective Cartier divisor $D \to E$ satisfying its restriction to the heart comes from an ordinary level-A structure. We let $\mathrm{Level}(A, E/R)$ denote the space of derived level-A structures of a spectral elliptic curve E/R. We prove that moduli problems associated with derived level structures are representable.

Theorem B (Theorem 3.6)**.** Suppose that E is a spectral elliptic curve over a connective $\mathbb{E}_\infty$-ring R. Then the functor

$$\begin{aligned} \mathrm{Level}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

is representable by an affine spectral Deligne–Mumford stack which is locally almost of finite presentation over the $\mathbb{E}_\infty$-ring R.

In classical algebraic geometry, except one-dimensional group curves, we also care level structures of p-divisible groups, which come from the full sections of commutative finite flat group schemes. In Section 3.2, we consider derived level structures of spectral p-divisible groups. Let $\mathrm{Level}(k, G_R/R)$ denote the space of derived level-$(\mathbf{Z}/p^k\mathbf{Z})^n$ structures of a height n spectral p-divisible group G/R.

Theorem C (Theorem 3.19)**.** Suppose G is a spectral p-divisible group of height n over a connective $\mathbb{E}_\infty$-ring R. Then the functor

$$\mathrm{Level}^k_{G/R} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}; \quad R' \to \mathrm{Level}(k, G_{R'}/R')$$

is representable by an affine spectral Deligne–Mumford stack $S(k) = \operatorname{Spét} \mathcal{P}^k_{G/R}$.

In Section 4, we give some applications of derived level structures. We first prove that the moduli problem of spectral elliptic curves with derived level-A structures is representable by a spectral Deligne–Mumford stack.

Theorem D (Theorem 4.7)**.** Let $\mathrm{Ell}(\mathcal{A})(R)$ denote the space of spectral elliptic curves with derived level-A structures over the $\mathbb{E}_\infty$-ring R. Then the functor

$$\begin{aligned} \mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R) \end{aligned}$$

is representable by a spectral Deligne–Mumford stack and this stack is locally almost of finite presentation over the sphere spectrum $\mathbb{S}$.

In [Lur18b], Lurie considers the spectral deformations of classical p-divisible groups. As we have the concept of derived level structures, it is natural to consider the moduli of spectral deformations with derived level structures of certain p-divisible groups. Suppose G_0 is a p-divisible group of height n over a perfect $\mathbb{F}_p$-algebra R_0. We consider the following functor

$$\begin{aligned} \mathcal{M}^{\mathrm{or}}_k \quad : \quad & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{\mathrm{or}}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}^{\mathrm{or}}(G_0, R, k)$ is the ∞-category spanned by those quadruples (G, ρ, e, η)

(1) G is a spectral p-divisible group over R.
(2) ρ is a equivalence class of G_0-taggings of R.
(3) e is an orientation of the identity component of G.
(4) $\eta : D \to G$ is a derived $(\mathbf{Z}/p^k\mathbf{Z})^n$-level structure of G/R.

Our next main result is the following.

Theorem E (Theorem 4.9)**.** The functor $\mathcal{M}^{\mathrm{or}}_k$ is co-representable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_k$, where $\mathcal{JL}_k$ is an $R^{\mathrm{or}}_{G_0}$-algebra such that $\pi_0\mathcal{JL}_k$ is finite over $\pi_0 R^{\mathrm{or}}_{G_0}$, $R^{\mathrm{or}}_{G_0}$ is the orientation deformation ring of G_0 defined in [Lur18b].

We will give another example of spectra constructed by considering moduli of spectral deformations with p-power order subgroups level structures, which can be viewed as topological realizations of universal objects of Strickland's deformations of Frobenius.

Finally, in Section 5, for every classical p-divisible group, we construct an $\mathbb{E}_\infty$-spectrum $\mathcal{JL}$ called the Jacquet–Langlands spectrum. By taking homotopy fixed points, we get a Jacquet–Langlands dual of Morava E-theories. We have a diagram in algebraic geometry:

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "\mathrm{GL}_n(\mathbf{Z}_p)"'] \arrow[dr, "\mathbb{G}_n"] & \\ \mathrm{LT}_K & & \mathcal{H}, \end{tikzcd}$$

where LT_K is the moduli space of deformation of formal groups, $\mathcal{X}$ is the moduli space of defomration with level structures of formal groups, and $\mathcal{H}$ is the Drinfled upper half plane. It can be lifted to the following diagram in the level of $\mathbb{E}_\infty$-spectra.

$$\begin{tikzcd} & \mathcal{JL} \arrow[dl, "\mathrm{GL}_n(\mathbf{Z}_p)"'] \arrow[dr, "\mathbb{G}_n"] & \\ E_n & & {}^L E_n \end{tikzcd}$$

Question 1.1. Compute higher homotopy groups of the finite-level and infinite-level Jacquet–Langlands spectra. These should encode more refined arithmetic-geometric information. Cf. Remark 4.14 (and higher algebraic K-theory, higher stable motivic stems, classification of knots not just up to isotopy, and the Habiro ring of a number field (elementless vs. categorification of elements)). It is relevant to compute the (co)tangent complex of the corresponding moduli problem (cf., e.g., [MPR24]).

Notation and terminology.

- Let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings and $\mathrm{CAlg}^{\mathrm{cn}}$ denote the ∞-category of connective $\mathbb{E}_\infty$-rings.
- Let $\mathcal{S}$ denote the ∞-category of spaces (∞-groupoids).
- Given a spectral Deligne–Mumford stack $\mathsf{X} = (\mathcal{X}, \mathscr{O}_\mathcal{X})$, let $\tau_{\leq n}\mathsf{X}$ denote its n-truncation $(\mathcal{X}, \tau_{\leq n}\mathscr{O}_\mathcal{X})$ and $\mathsf{X}^\heartsuit$ denote its underlying ordinary stack $(\mathcal{X}^\heartsuit, \tau_{\leq 0}\mathscr{O}_\mathcal{X})$.
- By a spectral Deligne–Mumford stack X over an $\mathbb{E}_\infty$-ring R, we mean a morphism of spectral Deligne–Mumford stacks $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$. Given an R-algebra S, we sometimes write $\mathsf{X} \times_R S$ for the fiber product $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, S$.
- Let $\mathcal{M}_{\mathrm{ell}}$ denote the spectral Deligne–Mumford stack of spectral elliptic curves, as defined in [Lur18a], and $\mathcal{M}_{\mathrm{ell}}^{\mathrm{cl}}$ denote the (classical) Deligne–Mumford stack of (classical) elliptic curves.

2. Effective Cartier divisors of spectral Deligne–Mumford stacks

A main innovation of this paper concerns derived level structures. We begin with a derived version of isogenies and prove that, in certain cases, the kernel of a derived isogeny behaves similarly as in the classical setting. This gives evidence that our derived version of level structures must induce classical level structures. In Section 2.2, we define relative effective Cartier divisors in the setting of spectral algebraic geometry. We then use Lurie's representability theorem to prove that certain functors associated with relative effective Cartier divisors are representable by spectral Deligne–Mumford stacks. This paves the way for Section 3, where we establish specifically the representability of derived level structures for spectral elliptic curves and spectral p-divisible groups.

2.1. **Isogenies of spectral elliptic curves.** To define derived level structures, the first question we must address is what higher-categorical analogues of finite abelian groups are. Let us recall from [Lur17, Section 7.2.4] and [Lur18c, Section 2.7] some finiteness conditions in the context of $\mathbb{E}_\infty$-rings.

Let A be an $\mathbb{E}_\infty$-ring and M be an A-module. We say that M is

- *perfect*, if it is a compact object of the ∞-category LMod_A of left A-modules;
- *almost perfect*, if there exists an integer k such that $M \in (\mathrm{LMod}_A)_{\geq k}$ and M is an almost compact object of $(\mathrm{LMod}_A)_{\geq k}$, that is, $\tau_{\leq n}M$ is a compact object of $\tau_{\leq n}\big((\mathrm{LMod}_A)_{\geq k}\big)$ for all $n \geq 0$;

- *perfect to order n*, if given any filtered diagram $\{N_\alpha\}$ in $(\mathrm{LMod}_A)_{\leq 0}$, the canonical map $\varinjlim_\alpha \mathrm{Ext}^i_A(M, N_\alpha) \to \mathrm{Ext}^i_A(M, \varinjlim_\alpha N_\alpha)$ is injective for $i = n$ and bijective for $i < n$;
- *finitely n-presented*, if M is n-truncated and perfect to order $n+1$; and

Next we recall finiteness conditions on algebras. We say that a morphism $\phi\colon A \to B$ of connective $\mathbb{E}_\infty$-rings is

- *of finite presentation*, if B belongs to the smallest full subcategory of CAlg_A which contains $\mathrm{CAlg}_A^{\mathrm{free}}$ and is stable under finite colimits;
- *locally of finite presentation*, if B is a compact object of CAlg_A;
- *almost of finite presentation*, if B is an almost compact object of CAlg_A;
- *of finite generation to order n*, if the following condition holds;

  Let $\{C_\alpha\}$ be a filtered diagram of connective $\mathbb{E}_\infty$-rings over A having colimit C. Assume that each C_α is n-truncated and that each of the transition maps $\pi_n C_\alpha \to \pi_n C_\beta$ is a monomorphism. Then the canonical map
  $$\varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$$
  is a homotopy equivalence.
- *of finite type*, if it is of finite generation to order 0.

Proposition 2.1 ([Lur18c, Propositions 2.7.2.1 and 4.1.1.3])**.** *Let $\phi\colon A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. Then the following conditions are equivalent.*

- *The morphism ϕ is perfect to order 0 (resp. of finite type).*
- *The commutative ring $\pi_0 B$ is finite (resp. of finite type) over $\pi_0 A$.*

Definition 2.2 (cf. [Lur18c, Definition 4.2.0.1])**.** Let $f\colon \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne–Mumford Stacks. We say that f is *locally of finite type* (resp. *locally of finite generation to order n*, *locally almost of finite presentation*, *locally of finite presentation*) if the following condition holds. Given any commutative diagram

$$\begin{tikzcd}
\mathrm{Sp\acute{e}t}\, B \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "f"] \\
\mathrm{Sp\acute{e}t}\, A \arrow[r] & \mathsf{Y}
\end{tikzcd}$$

where the horizontal morphisms are étale, the $\mathbb{E}_\infty$-ring B is of finite type (resp. of finite generation to order n, almost of finite presentation, locally of finite presentation) over A.

Definition 2.3 ([Lur18c, Definition 5.2.0.1])**.** Let $f\colon (\mathcal{X}, \mathscr{O}_\mathcal{X}) \to (\mathcal{Y}, \mathscr{O}_\mathcal{Y})$ be a morphism of spectral Deligne–Mumford stacks. We say that f is *finite* if the following conditions hold.

- The morphism f is affine.
- The pushforward $f_* \mathscr{O}_\mathcal{X}$ is perfect to order 0 as a $\mathscr{O}_\mathcal{Y}$-module.

Remark 2.4. By [Lur18c, Example 4.2.0.2], a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne–Mumford stack is locally of finite type if and only if the underlying map of ordinary stacks is locally of finite type in the sense of classical algebraic geometry. Moreover, by [Lur18c, Remark 5.2.0.2], a morphism of $f\colon \mathsf{X} \to \mathsf{Y}$ is finite if and only if the underlying map $f^\heartsuit : \mathsf{X}^\heartsuit \to \mathsf{Y}^\heartsuit$ is finite. In particular, if X and Y are spectral algebraic spaces, then f is finite if and only if $f^{\heartsuit}$ is finite in the classical sense.

Recall that a morphism $f\colon \mathsf{X} \to \mathsf{Y}$ of spectral Deligne–Mumford stacks is surjective if for every field k and any map $\operatorname{Sp\acute{e}t} k \to \mathsf{Y}$, the fiber product $\operatorname{Sp\acute{e}t} k \times_{\mathsf{Y}} \mathsf{X}$ is nonempty [Lur18c, Definition 3.5.5.5].

Definition 2.5. Let R be a connective $\mathbb{E}_\infty$-ring and $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral abelian varieties over R. We call f an *isogeny* if it is finite, flat, and surjective.

Lemma 2.6. *Let $f : \mathsf{X} \to \mathsf{Y}$ be an isogeny of spectral abelian varieties. Then $f^{\heartsuit}\colon \mathsf{X}^{\heartsuit} \to \mathsf{Y}^{\heartsuit}$ is an isogeny in the classical sense.*

Proof. For ordinary abelian varieties, $f^{\heartsuit}$ being an isogeny means that it is surjective and its kernel is finite. This is equivalent to $f^{\heartsuit}$ being finite, flat, and surjective [Mil86, Proposition 7.1]. From Definition 2.5, it is clear that $f^{\heartsuit}$ is finite and flat. We need only show that $f^{\heartsuit}$ is surjective.

By the definition of surjectivity above for morphisms of spectral Deligne–Mumford stacks, we get a commutative diagram

$$\begin{tikzcd} \operatorname{Sp\acute{e}t} k' \arrow[r] \arrow[d] & \mathsf{X} \arrow[d] \\ \operatorname{Sp\acute{e}t} k \arrow[r] & \mathsf{Y} \end{tikzcd}$$

The upper horizontal morphism corresponds to a morphism $\operatorname{Sp\acute{e}t} k' \to \mathsf{X}^{\heartsuit}$ by the inclusion–truncation adjunction [Lur18c, Proposition 1.4.6.3]. On underlying topological spaces, this then corresponds to a point $|\operatorname{Sp\acute{e}t} k'| \to |\mathsf{X}^{\heartsuit}|$. It is clear that this point in $|\mathsf{X}^{\heartsuit}|$ is a preimage of $|\operatorname{Sp\acute{e}t} k|$ in $|\mathsf{Y}^{\heartsuit}|$. Therefore $f^{\heartsuit}$ is surjective. □

Lemma 2.7. *Let $f\colon \mathsf{X} \to \mathsf{Y}$ be an isogeny of spectral elliptic curves over a connective $\mathbb{E}_\infty$-ring R. Then $\operatorname{fib}(f)$ exists and is a finite and flat nonconnective spectral Deligne–Mumford stack over R.*

Proof. By [Lur18c, Proposition 1.4.11.1], finite limits of nonconnective spectral Deligne–Mumford stacks exist, so we can define $\operatorname{fib}(f)$. Let us consider the commutative diagram

$$\begin{tikzcd} \operatorname{fib}(f) \arrow[r] \arrow[d, "f'"] & \mathsf{X} \arrow[d, "f"] \arrow[ddr] & \\ * \arrow[r] \arrow[drr, "i"] & \mathsf{Y} \arrow[dr] & \\ & & \operatorname{Sp\acute{e}t} R \end{tikzcd}$$

where the square is a pullback diagram. We find that $\operatorname{fib}(f)$ is over $\operatorname{Sp\acute{e}t} R$. By [Lur18c, Remark 2.8.2.6], $f' : \operatorname{fib}(f) \to *$ is flat because it is a pullback of a flat morphism. Clearly $i\colon * \to \operatorname{Sp\acute{e}t} R$ is flat, so by [Lur18c, Example 2.8.3.12] (being a flat morphism is a property local on the source with respect to the flat topology), $i \circ f'\colon \operatorname{fib}(f) \to \operatorname{Sp\acute{e}t} R$ is flat.

Next we show that $\operatorname{fib}(f)$ is finite over R. Since $*$, X, and Y are all spectral algebraic spaces, so is $\operatorname{fib}(f)$. Moreover, $\operatorname{Sp\acute{e}t} R$ is a spectral algebraic space [Lur18c, Example 1.6.8.2]. By Remark 2.4, we need only prove that the underlying morphism is finite. Since the truncation functor is a right adjoint, it preserves limits. Thus we get a pullback diagram

$$\begin{tikzcd} \mathrm{fib}(f)^{\heartsuit} \arrow[r] \arrow[d] & \mathsf{X}^{\heartsuit} \arrow[d] \\ * \arrow[r] & \mathsf{Y}^{\heartsuit} \end{tikzcd}$$

So we are reduced to showing that given an isogeny $f^{\heartsuit}\colon \mathsf{X}^{\heartsuit} \to \mathsf{Y}^{\heartsuit}$ of ordinary abelian varieties over a commutative ring R, its kernel is finite over R. This is true in classical algebraic geometry [Mil86, Proposition 7.1]. □

Lemma 2.8. *Given an integer $N \geq 1$, let $f_N\colon \mathsf{E} \to \mathsf{E}$ be an isogeny of spectral elliptic curves over a connective $\mathbb{E}_\infty$-ring R such that the underlying morphism is the multiplication-by-N map $[N]\colon \mathsf{E}^{\heartsuit} \to \mathsf{E}^{\heartsuit}$. Then $\mathrm{fib}(f_N)$ is finite flat of degree N^2 in the sense of* [Lur18c, Definition 5.2.3.1]*. Moreover, if N is invertible in $\pi_0 R$, then $\mathrm{fib}(f_N)$ is an étale-locally constant sheaf.*

Proof. By [KM85, Theorem 2.3.1], we know that $[N]\colon \mathsf{E}^{\heartsuit} \to \mathsf{E}^{\heartsuit}$ is finite locally free of rank N^2 in the classical sense. When N is invertible in $\pi_0 R$, its kernel is an étale-locally constant sheaf. Now, from Lemma 2.7, $\mathrm{fib}(f_N)$ is a spectral algebraic space that is finite and flat, and its underlying space $\mathrm{fib}(f_N)^{\heartsuit} = \ker[N]$ is locally free of rank N^2. We need to prove that $\mathrm{fib}(f_N) \to \mathrm{Sp\acute{e}t}\, R$ is locally free of rank N^2 in spectral algebraic geometry. Observe that since $\mathrm{fib}(f_N)$ is finite and flat, it is affine. We are thus reduced to proving the above for affines, i.e., $f_N|_{\mathrm{Sp\acute{e}t}\, S}\colon \mathrm{Sp\acute{e}t}\, S \to \mathrm{Sp\acute{e}t}\, R$ is locally free of rank N^2 for any affine substack $\mathrm{Sp\acute{e}t}\, S$ of $\mathrm{fib}(f_N)$. This is equivalent to proving that $R \to S$ is locally free of rank N^2 in the sense of [Lur18c, Definition 2.9.2.1]. Therefore we need to prove the following:

(1) The ring S is locally free of finite rank over R (by [Lur17, Proposition 7.2.4.20], this is equivalent to saying that S is a flat and almost perfect R-module).
(2) For every $\mathbb{E}_\infty$-ring maps $R \to k$ with k a field, the vector space $\pi_0(k \otimes_R S)$ is an N^2-dimensional k-vector space.

For (1), we know that $\pi_0 S$ is a projective $\pi_0 R$-module and that S is a flat R-module, so by [Lur17, Proposition 7.2.2.18], S is a projective R-module. By [Lur17, Corollary 7.2.2.9], since $\pi_0 S$ is a finitely generated $\pi_0 R$-module, S is a retract of a finitely generated free R-module, and is therefore locally free of finite rank.

For (2), by [Lur17, Corollary 7.2.1.23], since R and S are connective, we have $\pi_0(k \otimes_R S) \simeq k \otimes_{\pi_0 R} \pi_0 S$, which is an N^2-dimensional k-vector space, as $\pi_0 S$ is a rank-N^2 free $\pi_0 R$-module from above.

We next show that if N is invertible in $\pi_0 R$, then $\mathrm{fib}(f_N)$ is a locally constant sheaf. Since $\mathrm{fib}(f_N)$ is a spectral Deligne–Mumford stack, its associated functor of points $\mathrm{fib}(f_N)\colon \mathrm{CAlg}_R \to \mathcal{S}$ is nilcomplete and locally almost of finite presentation. By [KM85, Theorem 2.3.1], $\mathrm{fib}(f_N)|_{\mathrm{CAlg}^{\heartsuit}_{\pi_0 R}}$ is a locally constant sheaf. The desired result then follows from the lemma below. □

Lemma 2.9. *Let R be a connective $\mathbb{E}_\infty$-ring. Let $\mathscr{F} \in \mathcal{S}\mathrm{hv}^{\text{ét}}(\mathrm{CAlg}^{\mathrm{cn}}_R)$ be nilcomplete and locally almost of finite presentation. Suppose that $\mathscr{F}|_{(\mathrm{CAlg}^{\mathrm{cn}}_R)^{\heartsuit}}$ is a locally constant presheaf. Then $\mathscr{F}$ is a (homotopy) locally constant sheaf (i.e., sheafification of a homotopy-locally constant presheaf).*

Proof. Let us choose an étale cover $\{U_i^0\}$ of $\pi_0 R$ such that $\mathscr{F}|_{U_i^0}$ is a constant sheaf for each i. By [Lur17, Theorem 7.5.1.11], this corresponds to an étale cover $\{U_i\}$ of R such that $\pi_0 U_i = U_i^0$. For each i and n, we consider the diagram

$$\begin{tikzcd} \tau_{\leq 0} R \arrow[r] \arrow[d] & \tau_{\leq 0} U_i \arrow[d] \\ \tau_{\leq n} R \arrow[r] & \tau_{\leq n} U_i \end{tikzcd}$$

which is a pushout diagram, since U_i is an étale R-algebra. This is a colimit diagram in $\tau_{\leq n}\mathrm{CAlg}_R$. Since $\mathscr{F}$ is a sheaf locally almost of finite presentation, we then get a pushout diagram

$$\begin{tikzcd} \mathscr{F}(\tau_{\leq 0} R) \arrow[r] \arrow[d] & \mathscr{F}(\tau_{\leq 0} U_i) \arrow[d] \\ \mathscr{F}(\tau_{\leq n} R) \arrow[r] & \mathscr{F}(\tau_{\leq n} U_i) \end{tikzcd}$$

Without loss of generality, we may assume that each U_i is connective. Thus the values $\mathscr{F}(\tau_{\leq 0} U_i)$ is independent of i. This implies that $\mathscr{F}(\tau_{\leq n} U_i)$ are all equivalent. Since $\mathscr{F}$ is nilcomplete, $\mathscr{F}(U_i) \simeq \varinjlim_n \mathscr{F}(\tau_{\leq n} U_i)$, and so all $\mathscr{F}(U_i)$ are equivalent. □

2.2. **Cartier divisors and an exercise of spectral Artin representability.** In this subsection, we define relative effective Cartier divisors in the context of spectral algebraic geometry. We then use Lurie's spectral Artin representability theorem to prove that relative effective Cartier divisors are representable in certain cases. Let us first recall this spectral analogue of Artin's representability criterion in classical algebraic geometry.

Theorem 2.10 ([Lur18c, Theorem 18.3.0.1])**.** *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Suppose that we have a natural transformation $f : X \to \operatorname{Spec} R$, where R is a Noetherian $\mathbb{E}_\infty$-ring with $\pi_0 R$ a Grothendieck ring. Given $n \geq 0$, X is representable by a spectral Deligne–Mumford n-stack which is locally almost of finite presentation over R if and only if the following conditions are satisfied:*

1. *For every discrete commutative ring A, the space $X(A)$ is n-truncated.*
2. *The functor X is a sheaf for the étale topology.*
3. *The functor X is nilcomplete, infinitesimally cohesive, and integrable.*
4. *The functor X admits a connective cotangent complex L_X.*
5. *The natural transformation f is locally almost of finite presentation.*

Given a locally spectrally ringed topos $\mathsf{X} = (\mathcal{X}, \mathscr{O}_\mathcal{X})$, we can consider its functor of points

$$h_\mathsf{X} \colon \infty\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}} \to \mathcal{S}, \quad \mathsf{Y} \mapsto \mathrm{Map}_{\infty\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X})$$

In particular, by [Lur18c, Remark 3.1.1.2], a closed immersion $f : (\mathcal{Y}, \mathscr{O}_\mathcal{Y}) \to (\mathcal{X}, \mathscr{O}_\mathcal{X})$ of locally spectrally ringed topoi corresponds to a morphism $\mathscr{O}_\mathcal{X} \to f_*\mathscr{O}_\mathcal{Y}$ of sheaves over $\mathcal{X}$ of connective $\mathbb{E}_\infty$-rings such that $\pi_0\mathscr{O}_\mathcal{X} \to \pi_0 f_*\mathscr{O}_\mathcal{Y}$ is an epimorphism. We denote this epimorphism by α. Given a closed immersion $f \colon \mathsf{D} \to \mathsf{X}$ of spectral Deligne–Mumford stacks, we let $\mathcal{I}(\mathsf{D})$ denote $\ker(\alpha)$, called the ideal sheaf of D.

To prove relative representability for effective Cartier divisors below, we need the representability of Picard functors. Given a map $f : \mathsf{X} \to \operatorname{Spét} R$ of spectral Deligne–Mumford stacks, we can define a functor

$$\mathscr{P}\mathrm{ic}_{\mathsf{X}/R} \colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathscr{P}\mathrm{ic}(\operatorname{Spét} R' \times_{\operatorname{Spét} R} \mathsf{X})$$

If f admits a section $x \colon \operatorname{Spét} R \to \mathsf{X}$, then pullback along x gives a natural transformation of functors $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R} \to \mathscr{P}\mathrm{ic}_{R/R}$. We let

$$\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R} \colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$$

denote the fiber of this map.

Theorem 2.11 ([Lur18c, Theorem 19.2.0.5])**.** *Let $f : \mathsf{X} \to \operatorname{Spét} R$ be a map of spectral algebraic spaces which is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected over an $\mathbb{E}_\infty$-ring R. Suppose that $x \colon \operatorname{Spét} R \to \mathsf{X}$ is a section of f. Then the functor $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ is representable by a spectral algebraic space which is locally of finite presentation over R.*

In the classical setting, schemes representing relative effective Cartier divisors are open subschemes of Hilbert schemes [Kol96, Theorem 1.13]. However, in the derived setting, the Hilbert functor is representable by a spectral algebraic space [Lur04, Theorem 8.3.3], and it is hard to establish an analogous relationship. We will directly study relative effective Cartier divisors and their spectral moduli as follows.

Definition 2.12 (Relative effective Cartier divisor)**.** Let X be a spectral Deligne–Mumford stack over a spectral Deligne–Mumford stack S. Define a *relative effective Cartier divisor of* X/S to be a closed immersion $\mathsf{D} \to \mathsf{X}$ such that it is flat, proper, locally almost of finite presentation and that the associated ideal sheaf of D over X is locally free of rank 1. We let $\mathrm{CDiv}(\mathsf{X}/\mathsf{S})$ denote the ∞-category of such closed immersions.

Remark 2.13. It is not hard to see that given any spectral Deligne–Mumford stack X over S, $\mathrm{CDiv}(\mathsf{X}/\mathsf{S})$ is a Kan complex, since all objects are closed immersions of X. Let $\mathsf{D} \to \mathsf{D}'$ be a morphism. Then we have a diagram

$$\begin{tikzcd} \mathsf{D} \arrow[rr, "f"] \arrow[dr] & & \mathsf{D}' \arrow[dl] \\ & \mathsf{X} & \end{tikzcd}$$

By the definition of closed immersions, they are all equivalent to the same substack of X, so f is an isomorphism (cf. [Lur18c, Remark 3.1.1.2]).

Lemma 2.14. *Let X/S be a spectral Deligne–Mumford stack as above, and $\mathsf{T} \to \mathsf{S}$ be a map of spectral Deligne–Mumford stacks. If we have a relative effective Cartier divisor $\mathsf{D} \to \mathsf{X}$, then D_T is a relative effective Cartier divisor of X_T.*

Proof. This is straightforward to check. We simply note that D_T is a closed immersion of X_T [Lur18c, Corollary 3.1.2.3]. After base change, D_T is flat, proper, and locally almost of finite presentation over T. It remains to show that $\mathcal{I}(\mathsf{D}_\mathsf{T})$ is a line bundle over X_T. Indeed, we have a fiber sequence

$$\mathcal{I}(\mathsf{D}) \to \mathscr{O}_\mathcal{X} \to \mathscr{O}_\mathcal{D}$$

By the flatness of D, pullback along the base change $f\colon \mathsf{T} \to \mathsf{S}$ gives another fiber sequence

$$f^*\big(\mathcal{I}(\mathsf{D})\big) \to \mathscr{O}_{\mathcal{X}_{\mathsf{T}}} \to \mathscr{O}_{\mathcal{D}_{\mathsf{T}}}$$

So we have that $\mathcal{I}(\mathsf{D}_{\mathsf{T}})$ is just $f^*\big(\mathcal{I}(\mathsf{D})\big)$, which is invertible. $\square$

Suppose that X is a spectral Deligne–Mumford stack over an affine spectral Deligne–Mumford stack $\mathsf{S} = \operatorname{Sp\acute{e}t} R$. From Definition 2.12, we then have a functor

$$\mathrm{CDiv}_{\mathsf{X}/R}\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{CDiv}(\mathsf{X}_{R'}/R')$$

Our main goal in this section is to prove that this functor is representable when X/R is a spectral algebraic space satisfying certain conditions. To achieve this, we need some preparations for computing the cotangent complex of a relative effective Cartier divisor functor. The main issue has to do with square-zero extensions, for which we need the following facts about pushouts of two closed immersions.

By [Lur18c, Theorem 16.2.0.1 and Proposition 16.2.3.1], given a pushout square of spectral Deligne–Mumford stacks

$$\begin{tikzcd} \mathsf{X}_{01} \arrow[r, "i"] \arrow[d, "j"] & \mathsf{X}_0 \arrow[d, "j'"] \\ \mathsf{X}_1 \arrow[r, "i'"] & \mathsf{X} \end{tikzcd}$$

such that i and j are closed immersions, the induced square of ∞-categories

$$\begin{tikzcd} \mathrm{QCoh}(\mathsf{X}_{01}) & \mathrm{QCoh}(\mathsf{X}_0) \arrow[l] \\ \mathrm{QCoh}(\mathsf{X}_1) \arrow[u] & \mathrm{QCoh}(\mathsf{X}) \arrow[l] \arrow[u] \end{tikzcd}$$

determines an embedding $\theta\colon \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X}_0) \times_{\mathrm{QCoh}(\mathsf{X}_{01})} \mathrm{QCoh}(\mathsf{X}_1)$, which restricts to an equivalence

$$\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{X}_0)^{\mathrm{cn}} \times_{\mathrm{QCoh}(\mathsf{X}_{01})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}_1)^{\mathrm{cn}}$$

between connective objects. Moreover, let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ and set

$$\mathscr{F}_0 = j'^* \in \mathrm{QCoh}(\mathsf{X}_0), \quad \mathscr{F}_1 = i'^*\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_1)$$

Then $\mathscr{F}$ is n-connective if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are n-connective, and this statement is also true for the conditions of almost connective, Tor-amplitude $\leq n$, flat, perfect to order n, almost perfect, perfect, and locally free of finite rank, respectively.

Also, by [Lur18c, Theorem 16.3.0.1], we have a pullback square of ∞-categories

$$\begin{tikzcd} \mathrm{SpDM}_{/\mathsf{X}} \arrow[r] \arrow[d] & \mathrm{SpDM}_{/\mathsf{X}_0} \arrow[d] \\ \mathrm{SpDM}_{/\mathsf{X}_1} \arrow[r] & \mathrm{SpDM}_{/\mathsf{X}_{01}} \end{tikzcd}$$

Let $f\colon \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne–Mumford stacks. Let $\mathsf{Y}_0 = \mathsf{X}_0 \times_{\mathsf{X}} \mathsf{Y}$, $\mathsf{Y}_1 = \mathsf{X}_1 \times_{\mathsf{X}} \mathsf{Y}$, and let $f_0 : \mathsf{Y}_0 \to \mathsf{X}_0$ and $f_1 : \mathsf{Y}_1 \to \mathsf{X}_1$ be the projection maps. Then we have that f is locally almost of finite presentation if and only if both f_0 and f_1 are locally almost of finite presentation. The statement remains true for the following individual conditions: locally of finite generation to order n, locally of finite presentation, étale, equivalence, open immersion, closed immersion, flat, affine, separated, and proper [Lur18c, Proposition 16.3.2.1].

Now, let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne–Mumford stack, $\mathscr{E} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ be a connective quasi-coherent sheaf, and $\eta \in \mathrm{Der}(\mathscr{O}_{\mathcal{X}}, \Sigma\mathscr{E})$ be a derivation, i.e., a morphism $\eta\colon \mathscr{O}_{\mathcal{X}} \to \mathscr{O}_{\mathcal{X}} \oplus \Sigma\mathscr{E}$. We let $\mathscr{O}_{\mathcal{X}}^{\eta}$ denote the square-zero extension of $\mathscr{O}_{\mathcal{X}}$ by $\mathscr{E}$ determined by η, so that we have a pullback diagram

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{X}}^{\eta} \arrow[r] \arrow[d] & \mathscr{O}_{\mathcal{X}} \arrow[d, "\eta"] \\
\mathscr{O}_{\mathcal{X}} \arrow[r, "0"] & \mathscr{O}_{\mathcal{X}} \oplus \Sigma\mathscr{E}
\end{tikzcd}$$

By [Lur18c, Proposition 17.1.3.4], $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{\eta})$ is a spectral Deligne–Mumford stack, which we will denote by X^{η}. In the case of $\eta = 0$, we denote it by $\mathsf{X}^{\mathscr{E}} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}} \oplus \mathscr{E})$. We then have a pushout square of spectral Deligne–Mumford stacks

$$\begin{tikzcd}
\mathsf{X}^{\mathscr{E}} & \mathsf{X} \arrow[l] \\
\mathsf{X} \arrow[u] & \mathsf{X}^{\Sigma\mathscr{E}} \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

such that f and g are closed immersions. In turn, by [Lur18c, Theorem 16.2.0.1], there is a pullback diagram

$$\begin{tikzcd}
\mathrm{QCoh}(\mathsf{X}^{\mathscr{E}})^{\mathrm{acn}} \arrow[r] \arrow[d] & \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}} \arrow[d] \\
\mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}} \arrow[r] & \mathrm{QCoh}(\mathsf{X}^{\Sigma\mathscr{E}})^{\mathrm{acn}}
\end{tikzcd}$$

of categories spanned by almost connective quasi-coherent sheaves. Passing to homotopy fibers over some $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}}$, we obtain an equivalence

$$\mathrm{QCoh}(\mathsf{X}^{\mathscr{E}})^{\mathrm{acn}} \times_{\mathrm{QCoh}(\mathsf{X})} \{\mathscr{F}\} \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}\big(\mathscr{F}, \Sigma(\mathscr{E} \otimes \mathscr{F})\big)$$

as in [Lur18c, Proposition 19.2.2.2]. Similarly, by passing to the homotopy fibers over some $\mathsf{Z} \in \mathrm{SpDM}_{/\mathsf{X}}$ with $f : \mathsf{Z} \to \mathsf{X}$, we obtain the classification of first-order deformations of X:

$$\mathrm{SpDM}_{/\mathsf{X}^{\mathscr{E}}} \times_{\mathrm{SpDM}_{/\mathsf{X}}} \{\mathsf{Z}\} \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Z})}(L_{\mathsf{Z}/\mathsf{X}}, \Sigma f^{*}\mathscr{E})$$

[Lur18c, Proposition 19.4.3.1].

Lemma 2.15. *Let $f : \mathsf{X} \to \mathrm{Spét}\, R$ be a morphism of spectral Deligne–Mumford stacks, and M be a connective R-module. Consider the ∞-category of Deligne–Mumford stacks X' equipped with a morphism $f'\colon \mathsf{X}' \to \mathrm{Spét}\,(R \oplus M)$ that fits into the pullback diagram*

$$\begin{tikzcd}
\mathsf{X} \arrow[r] \arrow[d, "f"'] & \mathsf{X}' \arrow[d, "f'"] \\
\mathrm{Spét}\, R \arrow[r] & \mathrm{Spét}\,(R \oplus M)
\end{tikzcd}$$

Then this ∞-category is a Kan complex, and it is canonically homotopy equivalent to the mapping space $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\,R}, \Sigma f^* M)$. Moreover, if f is flat, proper, and locally almost of finite presentation, then so is f'.

Proof. We have a pullback square of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} R \oplus M \arrow[r] \arrow[d] & R \arrow[d, "(\mathrm{Id},0)"] \\ R \arrow[r] & R \oplus \Sigma M \end{tikzcd}$$

which corresponds to a pushout square of spectral Deligne–Mumford stacks

$$\begin{tikzcd} \mathrm{Sp\acute{e}t}\,R \oplus M & \mathrm{Sp\acute{e}t}\,R \arrow[l] \\ \mathrm{Sp\acute{e}t}\,R \arrow[u] & \mathrm{Sp\acute{e}t}\,(R \oplus \Sigma M) \arrow[l] \arrow[u] \end{tikzcd}$$

such that the morphisms $\mathrm{Sp\acute{e}t}\,(R \oplus \Sigma M) \to \mathrm{Sp\acute{e}t}\,R$ are closed immersions. This exhibits $\mathrm{Sp\acute{e}t}\,(R \oplus M)$ as an "infinitesimal thickening" of $\mathrm{Sp\acute{e}t}\,R$ determined by $R \xrightarrow{(\mathrm{Id},0)} R \oplus \Sigma M$.

The first part of this lemma follows from the formula for first-order deformations of [Lur18c, Proposition 19.4.3.1]. The second part follows from properties of pushout of two closed immersions [Lur18c, Corollary 16.4.2.1]. $\square$

Lemma 2.16. *Suppose that we are given a pushout diagram of spectral Deligne–Mumford stacks*

$$\begin{tikzcd} \mathsf{X}_{01} \arrow[r, "i"] \arrow[d, "j"] & \mathsf{X}_0 \arrow[d] \\ \mathsf{X}_1 \arrow[r] & \mathsf{X} \end{tikzcd}$$

where i and j are closed immersions. Let $f : \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne–Mumford stacks. Let $\mathsf{Y}_0 = \mathsf{X}_0 \times_{\mathsf{X}} \mathsf{Y}$, $\mathsf{Y}_1 = \mathsf{X}_1 \times_{\mathsf{X}} \mathsf{Y}$, and let $f_0 : \mathsf{Y}_0 \to \mathsf{X}_0$ and $f_1 : \mathsf{Y}_1 \to \mathsf{X}_1$ be the projection maps. If f_0 and f_1 are both closed immersions and determine line bundles over Y_0 and Y_1 respectively, then f is a closed immersion and determines a line bundle over Y.

Proof. The statement concerning closed immersions follows from [Lur18c, Proposition 16.3.2.1]. For the line-bundle part, we note that by [Lur18c, Theorem 16.2.0.1 and Proposition 16.2.3.1], f determines a sheaf locally free of finite rank. To show that this sheaf is a line bundle, we proceed locally. By [Lur18c, Theorem 16.2.0.2], given a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

such that $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, there is an equivalence $F : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{Mod}_{A_0}^{\mathrm{cn}} \times_{\mathrm{Mod}_{A_{01}}^{\mathrm{cn}}} \mathrm{Mod}_{A_1}^{\mathrm{cn}}$. Moreover, this is a symmetric monoidal equivalence. Indeed, since $F(M) = (A_0 \otimes_A M, A_1 \otimes_A M, A_{01} \otimes_{A_0} A_0 \otimes_A M \simeq A_{01} \otimes_{A_1}$

$A_1 \otimes_A M$), we have $F(M \otimes_A N) \simeq F(M) \otimes F(N)$. By [Lur18c, Proposition 2.9.4.2], line bundles over A_1, A_{01}, and A_0 determine invertible objects of $\mathrm{Mod}^{\mathrm{cn}}_{A_1}$, $\mathrm{Mod}^{\mathrm{cn}}_{A_{01}}$, and $\mathrm{Mod}^{\mathrm{cn}}_{A_1}$ respectively, which in turn determine an invertible object of $\mathrm{Mod}^{\mathrm{cn}}_{A}$, hence a line bundle over A. □

Here is the main result of this section and the technical heart of the paper.

Theorem 2.17. *Given a connective $\mathbb{E}_\infty$-ring R, let E/R be a spectral algebraic space that is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor*

$$\begin{aligned} \mathrm{CDiv}_{\mathsf{E}/R} \colon \mathrm{CAlg}^{\mathrm{cn}}_R &\to \mathcal{S} \\ R' &\mapsto \mathrm{CDiv}(\mathsf{E}_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation over $\mathrm{Spét}\, R$.

Proof. We apply Lurie's spectral Artin representability theorem and verify the 5 criteria from Theorem 2.10 one by one, in the case of $n = 0$, as follows:

(1) Lemma 2.18;
(2) Lemma 2.19;
(3) Lemmas 2.20, 2.21, 2.22;
(4) Lemma 2.24; and
(5) Lemma 2.23.

These statements and their proofs occupy the rest of this section. □

Lemma 2.18. *For every discrete commutative R_0, the space $\mathrm{CDiv}_{\mathsf{E}/R}(R_0)$ is 0-truncated.*

Proof. Recall that $\mathrm{CDiv}_{\mathsf{E}/R}(R_0)$ consists of closed immersions $\mathsf{D} \to \mathsf{E} \times_R R_0$ such that D is flat and proper over R_0. Therefore, if R_0 is discrete, so are the objects D, and so $\mathrm{CDiv}_{\mathsf{E}/R}(R_0)$ is 0-truncated. □

Lemma 2.19. *The functor $\mathrm{CDiv}_{\mathsf{E}/R}$ is a sheaf for the étale topology.*

Proof. Let $\{R' \to U_i\}_{i \in I}$ be an étale cover of $\mathrm{Spét}\, R'$, and $U_\bullet$ be the associated Čech-simplicial object. We need to prove that the map

$$\mathrm{CDiv}_{\mathsf{E}/R}(R') \to \varprojlim_{\mathbf{\Delta}} \mathrm{CDiv}_{\mathsf{E}/R}(U_\bullet)$$

is an equivalence. Unwinding the definitions, we need only prove the following general result: Given a spectral Deligne–Mumford stack X/S and an étale cover $\mathsf{T}_i \to \mathsf{S}$, we have a homotopy equivalence

$$\mathrm{CDiv}(\mathsf{X}/\mathsf{S}) \to \varprojlim_{\mathbf{\Delta}} \mathrm{CDiv}(\mathsf{X} \times_{\mathsf{S}} \mathsf{T}_\bullet)$$

This follows from the fact that our conditions on relative effective Cartier divisors from Definition 2.12 are local with respect to the étale topology. □

Lemma 2.20. *The functor $\mathrm{CDiv}_{\mathsf{E}/R}$ is nilcomplete.*

Proof. By [Lur18c, Definition 17.3.2.1], we need to show that the canonical map

$$\mathrm{CDiv}_{\mathsf{E}/R}(R') \to \varprojlim_{n} \mathrm{CDiv}_{\mathsf{E}/R}(\tau_{\leq n} R')$$

is a homotopy equivalence for every $\mathbb{E}_\infty$-ring R'. This can be deduced from the following: Given a flat, proper, locally almost of finite presentation spectral algebraic space X over a connective $\mathbb{E}_\infty$-ring S, we have an equivalence

$$\mathrm{CDiv}(\mathsf{X}/S) \to \varprojlim_n \mathrm{CDiv}(\mathsf{X} \times_S \tau_{\leq n} S)$$

Let us now prove this equivalence. Given a relative effective Cartier divisor $\mathsf{D} \to \mathsf{X}$, we have the following commutative diagram

$$\begin{tikzcd}
\mathsf{D} \times_S \tau_{\leq n} S \arrow[r] \arrow[d, dashed] \arrow[dd, bend right=60] & \mathsf{D} \arrow[d] \\
\mathsf{X} \times_S \tau_{\leq n} S \arrow[r] \arrow[d] & \mathsf{X} \arrow[d] \\
\mathrm{Sp\acute{e}t}\, \tau_{\leq n} S \arrow[r] & \mathrm{Sp\acute{e}t}\, S
\end{tikzcd}$$

where we get an induced map $\mathsf{D} \times_S \tau_{\leq n} S \to \mathsf{X} \times_S \tau_{\leq n} S$. It is not hard to prove that this map is a closed immersion [Lur18c, Corollary 3.1.2.3]. Moreover, the map $\mathsf{D} \times_S \tau_{\leq n} S \to \mathrm{Sp\acute{e}t}\, \tau_{\leq n} S$ is flat, proper, and locally almost of finite presentation, since $\mathsf{D} \times_S \tau_{\leq n} S$ is the base change of D along $\mathrm{Sp\acute{e}t}\, \tau_{\leq n} S \to \mathrm{Sp\acute{e}t}\, S$. The associated ideal sheaf of $\mathsf{D} \times_S \tau_{\leq n} S$ remains a line bundle over $\mathsf{X} \times_S \tau_{\leq n} S$. Therefore $\mathsf{D} \times_S \tau_{\leq n} S$ is a relative effective Cartier divisor of $\mathsf{X} \times_S \tau_{\leq n} S$. Thus we define a functor

$$\theta\colon \mathrm{CDiv}(\mathsf{X}/S) \to \varprojlim_n \mathrm{CDiv}(\mathsf{X} \times_S \tau_{\leq n} S)$$
$$\mathsf{D} \mapsto \{\mathsf{D} \times_S \tau_{\leq n} S\}_n$$

This functor is fully faithful, since we have from [Lur18c, Proposition 19.4.1.2] an equivalence $\mathrm{SpDM}_{/S} \to \varprojlim_n \mathrm{SpDM}_{/\tau_{\leq n} S}$ defined by $\mathsf{X} \mapsto \mathsf{X} \times_S \tau_{\leq n} S$. For θ to be an equivalence, we need only show that it is essentially surjective.

Suppose $\{\mathsf{D}_n \to \mathsf{X} \times_S \tau_{\leq n} S\}_n$ is an object in $\varprojlim_n \mathrm{CDiv}(\mathsf{X} \times_S \tau_{\leq n} S)$. It is a morphism in $\varprojlim_n \mathrm{SpDM}_{/\tau_{\leq n} S}$. By [Lur18c, Proposition 19.4.1.2], there is a morphism $\mathsf{D} \to \mathsf{X}$ in $\mathrm{SpDM}_{/S}$ such that $\mathsf{D} \times_S \tau_{\leq n} S \to \mathsf{X} \times_S \tau_{\leq n} S$ are equivalent to $\mathsf{D}_n \to \mathsf{X} \times_S \tau_{\leq n} S$.

Next, we need to show that $\mathsf{D} \to \mathsf{X}$ from above is a relative effective Cartier divisor. The conditions that $\mathsf{D} \to \mathsf{X}$ is flat, proper, and locally almost of finite presentation follow immediately from [Lur18c, Proposition 19.4.2.1]. It remains to prove that $\mathsf{D} \to \mathsf{X}$ is a closed immersion and determines a line bundle over X.

Without loss of generality, we may assume that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, B$ is affine, so that we have closed immersions $\mathsf{D}_n \to (\mathrm{Sp\acute{e}t}\, B) \times_S \tau_{\leq n} S \simeq \mathrm{Sp\acute{e}t}\,(B \otimes_S \tau_{\leq n} S)$, the last equivalence from [Lur18c, Proposition 1.4.11.1(3)]. By [Lur18c, Theorem 3.1.2.1], each $\mathsf{D} \times_S \tau_{\leq n} S$ is equivalent to $\mathrm{Sp\acute{e}t}\, B'_n$ for some B'_n such that $\pi_0(B \otimes_S \tau_{\leq n} S) \to \pi_0 B'_n$ is surjective. Since $\tau_{\leq n+1} S \to B'_{n+1}$ is flat, we have

$$\begin{aligned}
\mathrm{Sp\acute{e}t}\, B'_n &= (\mathrm{Sp\acute{e}t}\, B'_{n+1}) \times_{\tau_{\leq n+1} S} \tau_{\leq n} S = \mathrm{Sp\acute{e}t}\,(B'_{n+1} \otimes_{\tau_{\leq n+1} S} \tau_{\leq n} S) \\
&\simeq \mathrm{Sp\acute{e}t}\, \tau_{\leq n} B'_{n+1}
\end{aligned}$$

Thus we obtain a spectrum B' such that $\mathrm{Sp\acute{e}t}\, \tau_{\leq n} B' \simeq \mathrm{Sp\acute{e}t}\, B'_n = \mathsf{D} \times_S \tau_{\leq n} S$. Consequently, $\mathsf{D} = \mathrm{Sp\acute{e}t}\, B'$ and $\pi_0 B \to \pi_0 B'$ is surjective, and so $\mathsf{D} = \mathrm{Sp\acute{e}t}\, B' \to \mathrm{Sp\acute{e}t}\, B = \mathsf{X}$ is a closed immersion.

Finally, to prove that the associated ideal sheaf of D is a line bundle, we note the pullback diagrams

$$\begin{tikzcd} I_n \arrow[r] \arrow[d] & B \otimes_S \tau_{\leq n} S \arrow[d] \\ * \arrow[r] & B' \otimes_S \tau_{\leq n} S \end{tikzcd}$$

where each I_n is an invertible module over $B \otimes_S \tau_{\leq n} S = \tau_{\leq n} B$. Passing to inverse limits, we obtain a pullback diagram

$$\begin{tikzcd} \varprojlim I_n \arrow[r] \arrow[d] & B \arrow[d] \\ * \arrow[r] & B' \end{tikzcd}$$

Consequently, we have $I(\mathsf{D}) \simeq \varprojlim I_n$. Now, by nilcompleteness of the Picard functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}/S}$ from [Lur18c, Proposition 19.2.4.7(1)], $I(\mathsf{D})$ is an invertible B-module. Therefore the associated ideal sheaf of D is a line bundle over X. □

Lemma 2.21. *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *is infinitesimally cohesive.*

Proof. This follows from Proposition 2.16 and [Lur18c, Proposition 16.3.2.1]. □

Lemma 2.22. *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *is integrable.*

Proof. Given a local Noetherian $\mathbb{E}_\infty$-ring R' which is complete with respect to its maximal ideal $\mathfrak{m} \subset \pi_0 R'$, we need to prove that the inclusion functor $\mathrm{Spf}\, R' \hookrightarrow \mathrm{Spec}\, R'$ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spec}\, R', \mathrm{CDiv}_{\mathsf{E}/R}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, R', \mathrm{CDiv}_{\mathsf{E}/R})$$

This can be deduced from the following result: Given a flat, proper, and separated spectral algebraic space X locally almost of finite presentation over a connective local Noetherian $\mathbb{E}_\infty$-ring S which is complete with respect to its maximal ideal, we have an equivalence

$$\mathrm{CDiv}(\mathsf{X}/S) \simeq \mathrm{CDiv}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S)$$

Indeed, let $\mathrm{Hilb}(\mathsf{X}/S)$ denote the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ consisting of those $\mathsf{D} \to \mathsf{X}$, such that each $\mathsf{D} \to \mathsf{X}$ is a closed immersion and is flat, proper, and locally almost of finite presentation. Then by the formal GAGA theorem [Lur18c, Corollary 8.5.3.4] and the base-change properties of being flat, proper, and locally almost of finite presentation, we have $\mathrm{Hilb}(\mathsf{X}/S) \simeq \mathrm{Hilb}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S)$.

To prove the above equivalence for relative effective Cartier divisors, we need to further check that $\mathsf{D} \to \mathsf{X}$ associates a line bundle over X if and only if $\mathsf{D} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S$ associates a line bundle over $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S$. Note that the morphism $f \colon \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S \to \mathsf{X}$ is flat by [Lur18c, Corollary 7.3.6.9], and so we have $\mathcal{I}(\mathsf{D} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S) = \mathcal{I}(f^*\mathsf{D}) \simeq f^*\mathcal{I}(\mathsf{D})$ over the pullback square

$$\begin{tikzcd} \mathsf{D} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S \arrow[r] \arrow[d] & \mathsf{D} \arrow[d] \\ \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S \arrow[r, "f"] & \mathsf{X} \end{tikzcd}$$

By [Lur18c, proof of Proposition 19.2.4.7], we have an equivalence

$$\mathrm{QCoh}(\mathsf{X}/S)^{\mathrm{aperf,cn}} \simeq \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S)^{\mathrm{aperf,cn}}$$

We need only restrict to the subcategories spanned by invertible objects via [Lur18c, Proposition 2.9.4.2] to complete the proof. □

Lemma 2.23. *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *is locally almost of finite presentation over* $\mathrm{Spec}\, R$*.*

Proof. By [Lur18c, Definition 17.4.1.1(b)], we need to prove that

$$\mathrm{CDiv}_{\mathsf{E}/R} \colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{CDiv}(\mathsf{E}_{R'}/R')$$

commutes with filtered colimits when restricted to each $\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}}$. We note that $\mathrm{CDiv}(\mathsf{E}_{R'}/R')$ is a full subcategory of $\mathrm{SpDM}_{/(\mathsf{E}_{R'} \to \mathrm{Sp\acute{e}t}\, R')}$ and first consider instead the functor

$$\mathrm{Var}^+ \colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty, \quad R' \mapsto \mathrm{Var}^+_{/(\mathsf{E}_{R'} \to \mathrm{Sp\acute{e}t}\, R')}$$

where $\mathrm{Var}^+_{/(\mathsf{E}_{R'} \to \mathrm{Sp\acute{e}t}\, R')}$ consists of diagrams

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[dr] & \mathsf{E}_{R'} \arrow[d] \\ & \mathrm{Sp\acute{e}t}\, R' \end{tikzcd}$$

such that $\mathsf{D} \to \mathrm{Sp\acute{e}t}\, R'$ is flat, proper, and locally almost of finite presentation. Then by [Lur18c, Proposition 19.4.2.1], this functor commutes with filtered colimits when restricted to $\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}}$. It remains to verify that when $\{\mathsf{D}_i \to \mathsf{E}_{i,R'}\}_{i \in I}$ are closed immersions and determine line bundles over $\{\mathsf{E}_{i,R'}\}$, $\varinjlim_{i\in I} \mathsf{D}_i \to \varinjlim_{i \in I} \mathsf{E}_{i,R'}$ are closed immersions and determine line bundles over $\varinjlim_{i\in I} \mathsf{E}_{i,R'}$. As we recalled earlier in this subsection, this follows from properties of closed immersions and the property of Picard functors that they are locally almost of finite presentation. □

Lemma 2.24. *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *admits a cotangent complex which is connective and almost perfect.*

Proof. Let S be a connective R-algebra, $\eta \in \mathrm{CDiv}_{\mathsf{E}/R}(S)$, and M be a connective S-module. We then have a pullback diagram

$$\begin{tikzcd} F_\eta(M) \arrow[r] \arrow[d] & \mathrm{CDiv}_{\mathsf{E}/R}(S \oplus M) \arrow[d] \\ \{\eta\} \arrow[r] & \mathrm{CDiv}_{\mathsf{E}/R}(S) \end{tikzcd}$$

From this we obtain a functor

$$F_\eta \colon \mathrm{Mod}_S \to \mathcal{S}, \quad M \mapsto F_\eta(M)$$

We first need to prove that the above functor is corepresentable. Here, η is to a morphism $\mathsf{D} \to \mathsf{E} \times_R S$, and $\mathsf{E} \times_R (S \oplus M)$ is a square-zero extension of $\mathsf{E} \times_R S$.

Thus by the classification of first-order deformations [Lur18c, Proposition 19.4.3.1], the space of spectral algebraic spaces D' which fit into the pullback diagram

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[d, "\eta"] & \mathsf{D}' \arrow[d] \\ \mathsf{E} \times_R S \arrow[r] \arrow[d, "p"] & \mathsf{E} \times_R (S \oplus M) \arrow[d] \\ \operatorname{Sp\acute{e}t} S \arrow[r] & \operatorname{Sp\acute{e}t}(S \oplus M) \end{tikzcd}$$

is equivalent to $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{D})}\big(L_{\mathsf{D}/(\mathsf{E}\times_R S)}, \Sigma\eta^*(p^*M)\big)$. Pushing forward along $p \circ \eta$, by [Lur18c, Proposition 6.4.5.3], we then have

$$\mathrm{Map}_{\mathrm{QCoh}(\mathsf{D})}\big(L_{\mathsf{D}/(\mathsf{E}\times_R S)}, \Sigma\eta^*(p^*M)\big) \simeq \mathrm{Map}_{\mathrm{QCoh}(\operatorname{Sp\acute{e}t} S)}\big(\Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)}), M\big)$$

By Lemma 2.16, any such $\mathsf{D}' \to \mathsf{E} \times_R (S \oplus M)$ is a closed immersion and determines a line bundle over $\mathsf{E} \times_R (S \oplus M)$. Since the diagram

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[d] & \mathsf{D}' \arrow[d] \\ \operatorname{Sp\acute{e}t} S \arrow[r] & \operatorname{Sp\acute{e}t}(S \oplus M) \end{tikzcd}$$

is a pullback square, D' is a square-zero extension of D. By [Lur18c, Proposition 16.3.2.1], $\mathsf{D}' \to \operatorname{Sp\acute{e}t}(S \oplus M)$ is flat, proper, and locally almost of finite presentation. Combining these facts, we find that

$$F_\eta(M) = \mathrm{Map}_{\mathrm{QCoh}(\operatorname{Sp\acute{e}t} S)}\big(\Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)}), M\big)$$

Consequently, the functor $\mathrm{CDiv}_{\mathsf{E}/R}$ satisfies condition (a) from [Lur18c, Example 17.2.4.4]. Condition (b) therein follows from the compatibility of $(p \circ \eta)_+$, as a left adjoint of the functor $(p \circ \eta)^*$, with base change (cf. [Lur18c, Construction 6.4.5.1 and Proposition 6.4.5.3]). Therefore the functor $\mathrm{CDiv}_{\mathsf{E}/R}$ admits a cotangent complex $L_{\mathrm{CDiv}_{\mathsf{E}/R}}$ satisfying $\eta^* L_{\mathrm{CDiv}_{\mathsf{E}/R}} = \Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)})$. Since the quasi-coherent sheaf $L_{\mathsf{D}/(\mathsf{E}\times_R S)}$ is connective and almost perfect [Lur18c, Proposition 17.1.5.1(3)], the S-module $\Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)})$ is (-1)-connective.

Next, we show that $L_{\mathrm{CDiv}_{\mathsf{E}/R}}$ is almost perfect. This follows from [Lur18c, 17.4.2.2] and Lemma 2.23.

Finally, we show that it is connective. As above, let S be a connective R-algebra and $\eta \in \mathrm{CDiv}_{\mathsf{E}/R}(S)$. We need to prove that $M_\eta \coloneqq \eta^* L_{\mathrm{CDiv}_{\mathsf{E}/R}} \in \mathrm{Mod}_S$ is connective. We already knew that M_η is (-1)-connective and almost perfect. In particular, the homotopy group $\pi_{-1}M_\eta$ is a finitely generated $\pi_0 S$-module. To prove that it in fact vanishes, by Nakayama's lemma, we note that this is equivalent to proving that

$$\pi_{-1}(\kappa \otimes_{\pi_0 S} M_\eta) \simeq \mathrm{Tor}_0^{\pi_0 S}(\kappa, \pi_{-1}M_\eta)$$

equals 0 for every residue field κ of $\pi_0 S$. Thus we may replace S by κ and assume κ is an algebraically closed field.

Let $A = \kappa[\epsilon]/(\epsilon^2)$. Unwinding the definitions, we find that the dual space $\mathrm{Hom}_\kappa(\pi_{-1}M_\eta, \kappa)$ can be identified with the set of automorphisms of the base change

η_A such that they restrict to be the identity of η. It remains to prove that this set is trivial. This boils down to the following assertion in classical algebraic geometry.

> Let X/κ be a scheme, L be a line bundle over X, and assume L_A is also a line bundle over X_A. If f is an automorphism of L_A such that $f|L$ is the identity on L, then f is the identity.

This can be proved, mutatis mutandis, as in the last part of [Lur18a, proof of Proposition 2.2.6]. $\square$

3. Level structures for spectral abelian varieties

For spectral Deligne–Mumford stacks, Theorem 2.17 gives the relative representability (with respect to a fixed E/R) of relative effective Cartier divisors (over Spét R). Their analogues in classical algebraic geometry are crucial to Drinfeld's approach to arithmetic moduli of elliptic curves with level structure over $\mathbf{Z}$, as developed in [KM85], which applies nicely at primes dividing the level. In this section, we define level structures on spectral abelian varieties and related objects from effective Cartier divisors. The applications we aim at are of a similar nature to those considered by the earlier authors, i.e., incorporating ramification or regardless of failure of étaleness, which we will discuss in the next two sections.

3.1. **Level structures on elliptic curves.** Let C be a one-dimensional smooth commutative group scheme over a base scheme S, and A be an abstract finite abelian group. Recall from [KM85, 1.5.1] that a homomorphism of abstract groups

$$\phi\colon A \to C(S)$$

is said to be an *A-structure on C/S* if the effective Cartier divisor $\sum_{a\in A}[\phi(a)]$ is a subgroup scheme of C/S.

The following result gives the relative representability of moduli problems of level structures.

Proposition 3.1 ([KM85, Proposition 1.6.2])**.** *Let C be a one-dimensional smooth commutative group scheme over S. Then the functor*

$$\begin{aligned} \mathrm{Level}^A_{C/S}\colon \mathrm{Sch}_S &\to \mathrm{Set} \\ T &\mapsto \text{the set of level-}A\text{ structures on } C_T/T \end{aligned}$$

is represented by a closed subscheme of $\mathrm{Hom}_{\mathrm{Grp}/S}(A, C)$*.*

Definition 3.2. Let R be an $\mathbb{E}_\infty$-ring and E/R be a spectral elliptic curve. A *(derived) level-A structure on* E is a pair (D, ϕ), where $\mathsf{D} \to \mathsf{E}$ is a relative effective Cartier divisor, and $\phi : A \to \mathsf{E}^\heartsuit(\pi_0 R)$ is an A-structure on $\mathsf{E}^\heartsuit/\pi_0 R$ as above, such that the underlying morphism $\mathsf{D}^\heartsuit \to \mathsf{E}^\heartsuit$, necessarily a closed immersion, is the inclusion of the associated relative effective Cartier divisor $\sum_{a\in A}[\phi(a)]$ into $\mathsf{E}^\heartsuit$. We denote by $\mathrm{Level}(A, \mathsf{E}/R)$ the ∞-category of level-A structures on E/R, whose objects can be viewed as relative effective Cartier divisors satisfying an extra property.

Given a spectral elliptic curve E/R, the ∞-category $\mathrm{Level}(A, \mathsf{E}/R)$ is an ∞-groupoid, since it is a full subcategory of $\mathrm{CDiv}(\mathsf{E}/R)$, which is an ∞-groupoid (see Remark 2.13).

We note that derived level structures are stable under base change, as follows.

Lemma 3.3. *Let E/R be a spectral elliptic curve, $\mathsf{S} = \operatorname{Spét} R$, and (D, ϕ) be a level structure. Suppose that $\mathsf{T} \to \mathsf{S}$ is a morphism of nonconnective spectral Deligne–Mumford stacks. Then the induced pair $(\mathsf{D}_\mathsf{T}, \phi_\mathsf{T})$ is a level structure on $\mathsf{E}_\mathsf{T}/\mathsf{T}$.*

Proof. First, the induced closed immersion $\mathsf{D}_\mathsf{T} \to \mathsf{E}_\mathsf{T}$ is a relative effective Cartier divisor by Lemma 2.14. It remains to check that $\phi_\mathsf{T} : A \to \mathsf{E}^\heartsuit(\mathsf{T}^\heartsuit) = \mathsf{E}_\mathsf{T}^\heartsuit(\mathsf{T}^\heartsuit)$ is a classical level structure, so that $\mathsf{D}_\mathsf{T}^\heartsuit$ is the associated classical relative effective Cartier divisor. This follows from the base-change property of classical level structures observed in [KM85, Section 1.5.1]. $\square$

We next recall a result on when a divisor becomes a (finite flat) subgroup.

Proposition 3.4 ([KM85, Corollary 1.3.7])**.** *Given a smooth curve C/S which is a group scheme over a scheme S along with a relative effective Cartier divisor D of C, there exists a closed subscheme Z of S with the property that, for any $T \to S$, D_T is a subgroup of C_T if and only if $T \to S$ factors through Z.*

Here we have an analogous incidence object for the relation "$\mathsf{D}^\heartsuit$ is a subgroup."

Lemma 3.5. *Let $\mathsf{E}/\operatorname{Spét} R$ be a spectral elliptic curve and $\mathsf{D} \to \mathsf{E}$ be a relative effective Cartier divisor. Then there exists a closed spectral Deligne–Mumford substack $\operatorname{Spét} B \subset \operatorname{Spét} R$ satisfying the following universal property:*

> *Given any $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, $\mathsf{D}_{R'}^\heartsuit$ is a subgroup of $\mathsf{E}_{R'}^\heartsuit$ if and only if $R \to R'$ factors through B.*

Proof. By the preceding proposition, if $\mathsf{D}_{R'}^\heartsuit/\pi_0 R'$ is a subgroup of $\mathsf{E}_{R'}^\heartsuit/\pi_0 R'$, the morphism $\operatorname{Spec} \pi_0 R' \to \operatorname{Spec} \pi_0 R$ must factor through a closed subscheme $Z = \operatorname{Spec} B_0$ of $\operatorname{Spec} \pi_0 R$. This corresponds to a closed spectral subscheme $\operatorname{Spét} B$ of $\operatorname{Spét} R$. In fact, since the map $R \to R'$ satisfies that $\pi_0 R \to \pi_0 R'$ factors through $\pi_0 R/I$ for some ideal I of $\pi_0 R$, we obtain a factorization of $R \to R'$ through $\Gamma_I R$ (see [Lur18c, Chapter 7, esp. Definition 7.1.2.1] for details about I-nilpotent R-modules). Conversely, suppose that $R \to R'$ factors through B. Then $\mathscr{O}_{\operatorname{Spét} R'}$ vanishes on some $I \subset \pi_0 R$. In other words, we have that $\pi_0 R \to \pi_0 R'$ factors through $\pi_0 R/\sqrt{I}$. This is equivalent to $\operatorname{Spec} \pi_0 R' \to \operatorname{Spec} \pi_0 R$ factoring through $\operatorname{Spec} \pi_0 R/I = \operatorname{Spec} B_0 = Z$, and so $\mathsf{D}_{R'}^\heartsuit$ is a subgroup of $\mathsf{E}_{R'}^\heartsuit$. $\square$

Below is our main result in this subsection on relative representability of level structures over the spectral moduli stack of spectral elliptic curves.

Theorem 3.6. *Let E/R be a spectral elliptic curve and A be an abstract finite abelian group. Then the functor*

$$\begin{aligned} \mathrm{Level}_{\mathsf{E}/R}^A \colon \mathrm{CAlg}_R^{\mathrm{cn}} &\to \mathcal{S} \\ R' &\mapsto \mathrm{Level}(A, \mathsf{E}_{R'}/R') \end{aligned}$$

is represented by a closed substack $\mathsf{S}(A)$ of $\mathrm{CDiv}_{\mathsf{E}/R}$. Moreover, $\mathsf{S}(A) = \operatorname{Spét} \mathcal{P}_{\mathsf{E}/R}$ for some $\mathbb{E}_\infty$-ring $\mathcal{P}_{\mathsf{E}/R}$, which is locally almost of finite presentation over R.

Proof. By definition, the functor $\mathrm{Level}_{\mathsf{E}/R}^A$ is a subfunctor of the representable functor $\mathrm{CDiv}_{\mathsf{E}/R}$ from Theorem 2.17. In view of Lemma 3.5, we consider a spectral Deligne–Mumford stack GrpCDiv defined by the pullback diagram of spectral

Deligne–Mumford stacks

$$\begin{tikzcd} \mathrm{GrpCDiv}_{\mathsf{E}/R} \arrow[r] \arrow[d] & \mathrm{CDiv}_{\mathsf{E}/R} \arrow[d] \\ \mathrm{Sp\acute{e}t}\, B \arrow[r] & \mathrm{Sp\acute{e}t}\, R \end{tikzcd}$$

where B is associated to the universal object $\mathsf{D}_{\mathrm{univ}} \to \mathsf{E} \times_R \mathrm{CDiv}_{\mathsf{E}/R}$. We verify that $\mathrm{GrpCDiv}_{\mathsf{E}/R}$ valued on an R-algebra R' is the space of relative effective Cartier divisors D of $\mathsf{E}_{R'}$ such that $\mathsf{D}^{\heartsuit}$ is a finite flat subgroup of $\mathsf{E}_{R'}^{\heartsuit}$.

Moreover, there is a clopen substack $\mathrm{CDiv}_{\mathsf{E}/R}^{A}$ of $\mathrm{GrpCDiv}_{\mathsf{E}/R}$ whose value on an R-algebra R' is the space of relative effective Cartier divisors D of $\mathsf{E}_{R'}$ such that $\mathsf{D}^{\heartsuit}$ is a subgroup of $\mathsf{E}_{R'}^{\heartsuit}$ finite locally free over $\pi_0 R'$ of rank equal to $\#A$. We then evoke [KM85, Proposition 1.6.5] and obtain $\mathsf{S}(A)$ representing $\mathrm{Level}_{\mathsf{E}/R}^{A}$ as a closed substack of $\mathrm{CDiv}_{\mathsf{E}/R}^{A}$ similarly as in Lemma 3.5.

To prove the remaining statement, we consider the morphism $\mathsf{S}(A) \to \mathrm{Sp\acute{e}t}\, R$, both being spectral algebraic spaces. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary algebraic spaces is finite in the sense of classical algebraic geometry. Thus we need only prove that $\mathsf{S}(A)^{\heartsuit}$ is finite over $\mathrm{Spec}\, \pi_0 R$. This is precisely the classical case: $\mathsf{S}(A)^{\heartsuit}$ is the representing object of the classical level-A structures, which is a finite $\pi_0 R$-scheme of finite presentation by [KM85, Corollary 1.6.3]. $\square$

3.2. **Level structures on p-divisible groups.** Before we move on and introduce derived level structures for spectral p-divisible groups, let us first recall some classical facts needed about level structures of commutative finite flat group schemes.

3.2.1. *Classical finite flat group schemes.* Let S be a scheme and X/S be a finite flat S-scheme of finite presentation and rank N. It can be proved that X/S is finite locally free of rank N. This means that for every affine scheme $\mathrm{Spec}\, R \to S$, the pullback scheme $X \times_S \mathrm{Spec}\, R$ over $\mathrm{Spec}\, R$ has the form $\mathrm{Spec}\, R'$, where R' is an R-algebra which is locally free of rank N. For an element $f \in R'$ acting on R' by multiplication, define an R-linear endomorphism of R'. Because R' is locally free of rank N, multiplication by f has a characteristic polynomial

$$\det(T - f) = T^N - \mathrm{trace}(f)\, T^{N-1} + \cdots + (-1)^N \mathrm{norm}(f)$$

Recall the following definition from [KM85, 1.8.2]. Let $\{P_1, \ldots, P_N\}$ be a set of N points not necessarily distinct in $X(S)$. We call it a *full set of sections of X/S* if one of the following two equivalent conditions is satisfied.

(1) For any $\mathrm{Spec}\, R \to S$ and $f \in R' = H^0(X_R, \mathscr{O})$, we have

$$\det(T - f) = \prod_{i=1}^{N} \big(T - f(P_i)\big)$$

(2) For any $\mathrm{Spec}\, R \to S$ and $f \in R' = H^0(X_R, \mathscr{O})$, we have

$$\mathrm{norm}(f) = \prod_{i=1}^{N} f(P_i)$$

Given N not necessarily distinct points $P_1, \dots, P_N$ in $X(S)$, we have a morphism

$$\mathscr{O}_X \to \bigotimes_i (P_i)_*(\mathscr{O}_S)$$

of sheaves over X. It is not hard to see that this morphism is surjective and defines a closed subscheme D of X which is flat and proper over S. Thus, given an abstract finite abelian group A and a map $\phi\colon A \to X(S)$ of sets, we can define a closed subscheme D of X by the sheaf $\bigotimes_{a \in A} \phi(a)_* \mathscr{O}_S$.

Lemma 3.7. *Given a finite flat S-scheme Z of finite presentation,* $\mathrm{Hom}(A, Z)$ *is an open subscheme of* $\mathrm{Hilb}_{Z/S}$.

Proof. Let $T \to S$ be an S-scheme. For any $D \to Y \coloneqq T \times_S Z$ in $\mathrm{Hilb}(Y)$, we need to prove that the set of points $t \in T$ over which $D_t \to Y_t$ comes from the closed subscheme associated to $\phi\colon A \to Z(T) = Y(T)$ is open in T. Since D is the closed subscheme defined by $\mathscr{O}_Y \to \mathscr{O}_D$, if D_t comes from $\mathscr{O}_{Y,t} \to \bigotimes_{a\in A} \phi(a)_* \mathscr{O}_{T,t}$, then by the definition of a stalk, there exists an open subset U of T such that $t \in U$ and D_U is defined by $\mathscr{O}_Y|_U \to \bigotimes_{a \in A} \phi(a)_* \mathscr{O}_T|_U$. ☐

Suppose that G/S is a finite flat commutative group scheme of finite presentation and A is an abstract finite abelian group of order N. Let K be a finite flat S-subgroup-scheme of G locally free of rank N, and $\phi\colon A \to G(S)$ be a homomorphism landing in $K(S)$. Recall from [KM85, Remark 1.10.10] that the pair (K, ϕ) is called an *A-structure on G/S* if the N points $\phi(a), a \in A$ form a full set of sections of K.

Lemma 3.8. *Suppose that G/S is a finite flat commutative group scheme of finite presentation and $K \subset G$ is a closed subscheme which is finite flat and finite presentation. Then there exists a closed subscheme $Z \subset S$ such that given any morphism of schemes $T \to S$, K_T is a subgroup scheme of G_T if and only if the morphism $T \to S$ factors through Z.*

Proof. This is an analogue of [KM85, Corollary 1.3.7] for finite flat group schemes. Following the proof strategy there, we need only prove: Given finite flat closed subschemes K_1, K_2 of G, there exists a closed subscheme $Z \subset S$ such that given any morphism of schemes $T \to S$, $(K_1)_T$ is a closed subscheme of $(K_2)_T$ if and only if the morphism $T \to S$ factors through Z (cf. [KM85, Lemma 1.3.4 (1)]).

Since we consider the case of finite flat group schemes of finite presentation and the question is local on S, we are reduced to proving:

> Let B be a finite free A-algebra, and $\mathrm{Spec}\, B/I_1, \mathrm{Spec}\, B/I_2$ be two closed subschemes of $\mathrm{Spec}\, B$ such that B/I_1 and B/I_2 are also free. Then there exits a closed subscheme $\mathrm{Spec}\, W$ of $\mathrm{Spec}\, A$ such that given any $A \to A'$, $\mathrm{Spec}\,(B/I_1 \otimes_A A')$ is a closed subscheme of $\mathrm{Spec}\,(B/I_2 \otimes_A A')$ if and only if $A \to A'$ factors through W.

Let $\bar{1} \in B/I_2$ be the identity. Since B/I_1 is a free A-module of finite rank, the image of $\bar{1}$ under the map

$$B/I_2 \to B/I_2 \otimes_A B/I_1$$

can be written as $\sum_{i=1}^d r_i e_i$, where d is the rank and $\{e_i\}_{1 \le i \le d}$ is an A-basis of $B/I_2 \otimes_A B/I_1$. It is not hard to see that $V(\{r_1, \cdots, r_d\})$ is the desired closed subscheme of $\mathrm{Spec}\, A$. ☐

Proposition 3.9. *Hypotheses and notations as above, the functor* $A\text{-}\mathrm{Str}(G/S)$ *on* S*-schemes defined by*

$$T \mapsto \big\{\big(K \subset G_T,\, \phi\colon A \to G(T)\big) \,\big|\, (K,\phi) \text{ is an } A\text{-structure on } G_T\big\}$$

is representable by a finite S*-scheme of finite presentation.*

Proof. This is a variant of [KM85, Lemma 1.10.11 and Proposition 1.10.13 (1)]. Let us proceed in 3 steps.

First, the functor

$$T \mapsto \{D_T \subset G_T \,|\, D_T \text{ is a closed subscheme finite flat over } T \text{ of rank } N\}$$

is representable by a finite S-scheme $\mathrm{Hilb}^N_{G/S}$ (a Grassmannian).

Second, applying the preceding lemma to the universal example over $\mathrm{Hilb}^N_{G/S}$, we obtain a finite S-scheme Z classifying finite flat subgroup schemes of G locally free of rank N.

Third, given such a subgroup scheme $K \subset G$, observe that the functor

$$T \mapsto \{\phi\colon A \to G(T) \,|\, (K_T,\phi) \text{ is an } A\text{-level structure on } G_T\}$$

is equivalent to the functor

$$T \mapsto \{\phi\colon A \to K(T) \,|\, \phi \text{ is an } A\text{-generator of } K_T\}$$

(cf. [KM85, Remark 1.10.10 and 1.10.5]). Since the latter is representable by a finite S-scheme of finite presentation by [KM85, Proposition 1.10.13 (1)], we further apply this representability to the universal example $K_{\mathrm{univ}} \subset G_Z$ to complete the proof. □

3.2.2. *Spectral finite flat group schemes.* Let R be a connective $\mathbb{E}_\infty$-ring and G be a commutative finite flat group scheme over R. By the definition of finite-flatness, we have $G = \mathrm{Spét}\, B$ for a finite flat R-algebra B [Lur18a, Definition 6.1.2]. We let $\mathrm{Hilb}(G/R)$ denote the full subcategory of $\mathrm{SpDM}_{/G}$ spanned by those $\mathsf{D} \to G$ such that $\mathsf{D} \to G$ is a closed immersion of spectral Deligne–Mumford stacks and that the composite $\mathsf{D} \to G \to \mathrm{Spét}\, R$ is flat, proper, and locally almost of finite presentation. Then $\mathrm{Hilb}(G/R)$ is equivalent to the ∞-category of diagrams of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & S & \end{tikzcd}$$

such that S is flat, proper, and locally almost of finite presentation over R subject to certain additional conditions. It is not hard to see that $\mathrm{Hilb}(G/R)$ is a Kan complex (cf. Remark 2.13), so that we can define a functor

$$\mathrm{Hilb}_{G/R}\colon \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}, \quad R' \mapsto \mathrm{Hilb}(G_{R'}/R')$$

The representability of this functor is a special case of [Lur04, Theorem 8.3.3], which we record below. Like that theorem and Theorem 2.17, it can be deduced from the spectral Artin representability theorem 2.10.

Theorem 3.10. *Suppose that* G *is a commutative finite flat group scheme over a connective* $\mathbb{E}_\infty$*-ring* R*. Then* $\mathrm{Hilb}_{G/R}$ *is representable by a spectral Deligne–Mumford stack which is locally almost of finite presentation over* R*.*

Corollary 3.11. *Hypotheses and notations as above, for each positive integer N, there exists a substack $\mathrm{Hilb}^N_{G/R}$ of $\mathrm{Hilb}_{G/R}$ such that given any R' in $\mathrm{CAlg}^{\mathrm{cn}}_R$, the space $\mathrm{Hilb}^N_{G/R}(R') =: \mathrm{Hilb}^N(G_{R'}/R') \subset \mathrm{Hilb}(G_{R'}/R')$ consists of those $\mathsf{D} \to G_{R'}$ locally free of rank N over R'.*

Definition 3.12. Let R be a connective $\mathbb{E}_\infty$-ring and G be a spectral commutative finite flat group scheme over R. Given an abstract finite abelian group A of order N, a *level-A structure on* G is a pair (D, ϕ), where $i\colon \mathsf{D} \to G$ is an object in $\mathrm{Hilb}^N(G/R)$ and $\phi\colon A \to G^\heartsuit(\pi_0 R)$ is a homomorphism, such that $(\mathsf{D}^\heartsuit, \phi)$ is an A-structure in the sense of [KM85, Remark 1.10.10], i.e., $\pi_0 i_* \mathscr{O}_\mathsf{D} = \bigotimes_{a\in A} \phi(a)_* \mathscr{O}_{\mathrm{Spec}\,\pi_0 R}$. We denote by $\mathrm{Level}(A, G/R)$ the ∞-category of level-A structures on G/R.

Remark 3.13. Given a level-A structure (D, ϕ) on G, D is locally free of rank N over R, since $\mathsf{D} \to G$ is a closed immersion, $\mathsf{D} \to \mathrm{Sp\acute{e}t}\, R$ is flat, and $\pi_0 i_* \mathscr{O}_\mathsf{D} = \bigotimes_{a\in A} \phi(a)_* \mathscr{O}_{\mathrm{Spec}\,\pi_0 R}$. The last identity also ensures the group structure on $\mathsf{D}^\heartsuit$.

Remark 3.14. Comparing Definition 3.12 with Definition 3.2, we see that [KM85, Proposition 1.10.6] establishes an equivalence between the two definitions in the classical case when $G^\heartsuit/\pi_0 R$ is embeddable as a closed subscheme of an elliptic curve $E^\heartsuit/\pi_0 R$. Thus these two definitions are compatible if the spectral group scheme G/R is embeddable as a closed substack of a spectral elliptic curve E/R.

To prove representability for the functor of level-A structures, we present a second result concerning the existence of incidence spectral Deligne–Mumford stacks (cf. Lemma 3.5).

Lemma 3.15. *Let G/R be a spectral commutative finite flat group scheme over a connective $\mathbb{E}_\infty$-ring R. Let A be an abstract finite abelian group of order N. Given an object $\mathsf{D} \to G$ in $\mathrm{Hilb}^N(G/R)$, there exists an $\mathbb{E}_\infty$-ring W satisfying the following universal property:*

> *For any $R \to R'$ in $\mathrm{CAlg}^{\mathrm{cn}}_R$, $\mathsf{D}_{R'}$ supports a level-A structure on $G_{R'}$ if and only if $R \to R'$ factors through W.*

Proof. Given R' in $\mathrm{CAlg}^{\mathrm{cn}}_R$, it is clear that $\mathsf{D}_{R'}$ is in $\mathrm{Hilb}(G_{R'}/R')$. For $\mathsf{D}_{R'}$ to support a level-A structure as in Definition 3.12, $\mathrm{Spec}\,\pi_0 R' \to \mathrm{Spec}\,\pi_0 R$ must factor through $\mathrm{Hom}(A, G^\heartsuit)$, which is open in $\mathrm{Hilb}_{G^\heartsuit/\pi_0 R}$ by Lemma 3.7. Thus $\pi_0 R \to \pi_0 R'$ factors through B_0 for some localization B_0 of $\pi_0 R$. This lifts to a factorization of $R \to R'$ through an $\mathbb{E}_\infty$-ring B, which is a localization of R with $\pi_0 B \simeq B_0$ (see [Lur18c, Remark 1.1.4.2]).

By now, along the map $\mathrm{Sp\acute{e}t}\, R' \to \mathrm{Sp\acute{e}t}\, B$, we already have $i : \mathsf{D}_{R'} \to G_{R'}$ in $\mathrm{Hilb}^N(G_{R'}/R')$ and a map $\phi : A \to G^\heartsuit(\pi_0 R')$ associated with $\pi_0 i_* \mathscr{O}_{\mathsf{D}_{R'}}$. For $(\mathsf{D}_{R'}, \phi)$ to be a level-A structure, $\bigotimes_{a\in A} \phi(a)_* \mathscr{O}_{\mathrm{Spec}\,\pi_0 R'} \to \pi_0 i_* \mathscr{O}_{\mathsf{D}_{R'}}$ needs to be an isomorphism, i.e., the N points $\phi(a), a \in A$ must form a full set of sections of $\mathsf{D}^\heartsuit_{R'}$. By [KM85, Proposition 1.9.1], $\mathrm{Spec}\,\pi_0 R' \to \mathrm{Spec}\,\pi_0 B$ must then factor through a closed subscheme of $\mathrm{Spec}\,\pi_0 B$. Thus $\pi_0 B \to \pi_0 R'$ factors through $W_0 = B_0/I$ for some ideal I. This lifts to a factorization of $B \to R'$ through the $\mathbb{E}_\infty$-ring $W = \Gamma_I(B)$, as desired.

We can show the converse by a similar argument to the one in the proof of Lemma 3.5. □

Proposition 3.16. *Suppose that G is a spectral commutative finite flat group scheme over a connective $\mathbb{E}_\infty$-ring R and A is an abstract finite abelian group.*

Then the functor

$$\begin{aligned}\mathrm{Level}^A_{G/R}\colon \mathrm{CAlg}^{\mathrm{cn}}_R &\to \mathcal{S}\\ R' &\mapsto \mathrm{Level}(A, G_{R'}/R')\end{aligned}$$

is representable by an affine spectral Deligne–Mumford stack $\mathsf{S}(A) = \mathrm{Sp\acute{e}t}\, \mathcal{P}_{G/R}$.

Proof. We first prove the representability. By definition, the functor $\mathrm{Level}^A_{G/R}$ is a subfunctor of the representable functor $\mathrm{Hilb}^N_{G/R}$, where $N = \#A$. In view of the previous lemma and its proof, let us consider from right to left the consecutive pullbacks of universal objects

$$\begin{tikzcd}
(G, \text{univ. level-}A \text{ str. on } G) \arrow[r] \arrow[d] & (G, \mathsf{D}_{\mathrm{univ}}, \phi_{\mathrm{univ}}) \arrow[r] \arrow[d] & (G, \mathsf{D}_{\mathrm{univ}}) \arrow[r] \arrow[d] & G \arrow[d] \\
W \arrow[r] & B \arrow[r] & \mathrm{Hilb}^N_{G/R} \arrow[r] & \mathrm{Sp\acute{e}t}\, R
\end{tikzcd}$$

It is straightforward to verify that $\mathsf{S}(A) \coloneqq W$ valued on an R-algebra R' is precisely the space of level-A structures on $G_{R'}$.

For the affineness property, we need to prove that $\mathsf{S}(A)$ is finite over R in the sense of spectral algebraic geometry. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary algebraic spaces is finite in the sense of classical algebraic geometry. Thus we need only prove that $\mathsf{S}(A)^{\heartsuit}$ is finite over $\pi_0 R$, which follows from Proposition 3.9. $\square$

3.2.3. *Spectral p-divisible groups.* Given an $\mathbb{E}_\infty$-ring R, let $\mathrm{FFG}(R)$ denote the ∞-category of spectral commutative finite flat group schemes over R. Let $X : (\mathrm{Ab}^p_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{FFG}(R)$ be a spectral p-divisible group of height h over an $\mathbb{E}_\infty$-ring R (see [Lur18a, Definition 6.5.1] and cf. [Lur18b, Definition 2.0.2]). For each non-negative integer r, we write $X[p^r]$ for the image of $\mathbf{Z}/p^r\mathbf{Z}$ under X, which is a degree-$(p^r)^h$ spectral commutative finite flat group scheme over R.

Definition 3.17. Let G be a spectral p-divisible group of height h over a connective $\mathbb{E}_\infty$-ring R. A *level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on* G is a level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on $G[p^r]$ as in Definition 3.12. We let $\mathrm{Level}(r, G/R)$ denote the ∞-groupoid of level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structures on G/R.

Remark 3.18. Recall that a level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on $G[p^r]$ is a pair (D, ϕ), where $\mathsf{D} \subset G[p^r]$ is a finite flat closed substack of rank $(\mathbf{Z}/p^r\mathbf{Z})^h$ over R, and $\phi : (\mathbf{Z}/p^r\mathbf{Z})^h \to G[p^r]^{\heartsuit}(\pi_0 R)$ is a homomorphism such that $(\mathsf{D}^{\heartsuit}, \phi)$ is a level-$(\mathbf{Z}/p^r\mathbf{Z})^h$ structure on $G[p^r]^{\heartsuit}/\pi_0 R$. Given such a structure (D, ϕ), since $G[p^r]$ is locally free of rank $(p^r)^h$ over R, the rank of $\mathsf{D}^{\heartsuit}$ over $\pi_0 R$ equals that of $G[p^r]^{\heartsuit}$. Since $\mathsf{D}^{\heartsuit}$ is a closed subscheme of $G[p^r]^{\heartsuit}$, they must then equal. Thus ϕ is a $(\mathbf{Z}/p^r\mathbf{Z})^h$*-generator of* $G[p^r]^{\heartsuit}(\pi_0 R)$ in the sense of [KM85, 1.10.5]. Note that as spectral Deligne–Mumford stacks, even though D has the same rank as $G[p^r]$, they are not equivalent, since closed immersions in spectral algebraic geometry are not categorical monomorphisms (see [Lur18a, Warning 6.2.3]). For this reason, we do not introduce the concept of A-generators when discussing derived level structures, and this is where the higher homotopical information of *derived* level structures resides.

Theorem 3.19. *Let G be a spectral p-divisible group of height h over a connective $\mathbb{E}_\infty$-ring R. Then the functor*

$$\begin{aligned}\mathrm{Level}^r_{G/R}\colon \mathrm{CAlg}^{\mathrm{cn}}_R &\to \mathcal{S}\\ R' &\mapsto \mathrm{Level}(r, G_{R'}/R')\end{aligned}$$

is representable by an affine spectral Deligne–Mumford stack $\mathsf{S}(r) = \mathrm{Spét}\,\mathcal{P}^r_{G/R}$.

Proof. We just notice that by the definition of a spectral p-divisible group, $G[p^r]$ is a spectral commutative finite flat group scheme. Thus the theorem follows from Proposition 3.16 above about general spectral commutative finite flat group schemes. □

Remark 3.20. Our derived level structure functor is defined over $\mathrm{CAlg}^{\mathrm{cn}}$. More generally, in view of [Lur18a, Remark 6.1.3], we can define such structures on G/R where R is not necessarily connective. We let

$$\mathrm{Level}(r, G/R) \coloneqq \mathrm{Level}(r, \tau_{\geq 0}G/\tau_{\geq 0}R)$$

The corresponding functor

$$\mathrm{Level}^r_{G/R}\colon \mathrm{CAlg}_R \to \mathcal{S}$$

is also representable. This will be useful in Section 4.2 when we consider *oriented* spectral p-divisible groups.

3.2.4. *Non-full level structures.* So far we have treated only full level structures on commutative finite flat group schemes. Here let us consider more general level structures, such as those relevant for power operations in Morava E-theories (see Section 4.3).

Definition 3.21. Suppose that G is a spectral commutative finite flat group scheme over a connective $\mathbb{E}_\infty$-ring R. We let $\mathrm{Level}_1(r, G/R)$ denote the ∞-groupoid of derived level-$(\mathbf{Z}/p^r\mathbf{Z})$ structures on G/R. We let $\mathrm{Level}_0(r, G/R)$ denote the ∞-groupoid of equivalence classes (D, ϕ) in $\mathrm{Level}_1(r, G/R)$ where two objects (D, ϕ) and (D', ϕ') are equivalent if the scheme-theoretic image of $\mathsf{D}^\heartsuit$ under ϕ and that of $(\mathsf{D}')^\heartsuit$ under ϕ' equal in $G^\heartsuit/\pi_0 R$.

Proposition 3.22. *Hypotheses and notations as above, the functor*

$$\mathrm{Level}^{1,r}_{G/R}\colon \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}, \quad R' \to \mathrm{Level}_1(r, G_{R'}/R')$$

is representable by an affine spectral Deligne–Mumford stack $\mathsf{S}_1(r) = \mathrm{Spét}\,\mathcal{P}^{1,r}_{G/R}$. The functor

$$\mathrm{Level}^{0,r}_{G/R}\colon \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}, \quad R' \to \mathrm{Level}_0(r, G_{R'}/R')$$

is representable by an affine spectral Deligne–Mumford stack $\mathsf{S}_0(r) = \mathrm{Spét}\,\mathcal{P}^{0,r}_{G/R}$.

Proof. The first statement is a direct consequence of the more general Proposition 3.16. For the second, we just notice that the classical level structure functor $\mathrm{Level}^{0,r}_{G^\heartsuit/\pi_0 R}$ is representable by a closed subscheme of the Grassmannian of all rank-p^r quotients of $G^\heartsuit[p^r]$ (cf. [KM85, Theorem 6.6.1 and proof of Proposition 6.5.1]). By an argument analogous to that for the case of full level structures, we obtain the desired result. □

Remark 3.23. From the above proposition, we obtain analogous representability results for spectral p-divisible groups as in Section 3.2.3.

4. Moduli problems of derived level structures

In this section, we apply the derived level structures and their representability results from Section 3 and discuss several related spectral moduli problems.

4.1. **Spectral elliptic curves with level structure.** In Section 3.1, given an abstract finite abelian group A, we defined level-A structures for spectral elliptic curves (Definition 3.2) and showed their representability relative to an object E/R (Theorem 3.6). Here, we consider their absolute representability (cf. [KM85, Sections 4.2–4.3]).

There exists a spectral Deligne–Mumford stack $\mathcal{M}_{\mathrm{ell}}$ whose functor of points is

$$\mathcal{M}_{\mathrm{ell}} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}, \quad R \mapsto \mathcal{M}_{\mathrm{ell}}(R)$$

where $\mathcal{M}_{\mathrm{ell}}(R) = \mathrm{Ell}(R)^{\simeq}$ is the underlying ∞-groupoid of the ∞-category of spectral elliptic curves over R [Lur18a, Theorem 2.4.1].

In classical algebraic geometry, we have the Deligne–Mumford stack of (ordinary) elliptic curves, which can be viewed as a spectral Deligne–Mumford stack

$$\mathcal{M}_{\mathrm{ell}}^{\mathrm{cl}} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}, \quad R \mapsto \mathcal{M}_{\mathrm{ell}}^{\mathrm{cl}}(\pi_0 R)$$

where $\mathcal{M}_{\mathrm{ell}}^{\mathrm{cl}}(\pi_0 R)$ is the groupoid of elliptic curves over the commutative ring $\pi_0 R$.

Moreover, if A equals $\mathbf{Z}/N\mathbf{Z}$ or $(\mathbf{Z}/N\mathbf{Z})^2$ with $N \geq 1$ an integer, we have the Deligne–Mumford stack of elliptic curves with level-A structures, which can also be viewed as a spectral Deligne–Mumford stack

$$\mathcal{M}_{\mathrm{ell}}^{\mathrm{cl},A} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}, \quad R \mapsto \mathcal{M}_{\mathrm{ell}}^{\mathrm{cl},A}(\pi_0 R)$$

where $\mathcal{M}_{\mathrm{ell}}^{\mathrm{cl},A}(\pi_0 R)$ is the groupoid of elliptic curves with level-A structure over the commutative ring $\pi_0 R$.

In Section 3.1, for derived level-A structures, the construction $\mathsf{X} \mapsto \mathrm{Level}(A, \mathsf{X}/R)$ determines a functor $\mathrm{Ell}(R) \to \mathcal{S}$ which classifies a left fibration $\mathrm{Ell}^A(R) \to \mathrm{Ell}(R)$ of ∞-categories by the unstraightening construction (see [Lur09b, Definition 3.3.2.2 and Section 2.2.1]). Objects of $\mathrm{Ell}^A(R)$ are triples $(\mathsf{E}, \mathsf{D}, \phi)$ where E is a spectral elliptic curve over R and (D, ϕ) is a derived level-A structure on E as in Definition 3.2.

For each $R \in \mathrm{CAlg}^{\mathrm{cn}}$, consider all spectral elliptic curves over R with level-A structure. This moduli problem can be thought of as a functor

$$\mathcal{M}_{\mathrm{ell}}^{A} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}, \quad R \mapsto \mathrm{Ell}^A(R)$$

where $\mathrm{Ell}^A(R)$ is the space of spectral elliptic curves E/R with a derived level-A structure (D, ϕ). To prove its representability, we proceed as follows.

Proposition 4.1. *The functor $\mathcal{M}_{\mathrm{ell}}^{A} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is an étale sheaf.*

Proof. Let $\{R \to U_i\}$ be an étale cover of R, and $U_\bullet$ be the associated Čech-simplicial object. Consider the diagram

$$\begin{tikzcd} \mathrm{Ell}^A(R)^{\simeq} \arrow[r, "f"] \arrow[d, "p"] & \varprojlim_{\Delta} \mathrm{Ell}^A(U_\bullet)^{\simeq} \arrow[d, "q"] \\ \mathrm{Ell}(R)^{\simeq} \arrow[r, "g"] & \varprojlim_{\Delta} \mathrm{Ell}(U_\bullet)^{\simeq} \end{tikzcd}$$

The map p is a left fibration between Kan complexes, and so is a Kan fibration by [Lur09b, Lemma 2.1.3.3]. The map q is a pointwise Kan fibration. By picking the projective model structure for the homotopy limit we may assume that q is a Kan fibration as well. The map g is an equivalence by [Lur18a, Theorem 2.4.1]. To show that f is an equivalence, we need only show that for every $\mathsf{E} \in \mathrm{Ell}(R)$, the map

$$p^{-1}(\mathsf{E}) \simeq \mathrm{Level}(A, \mathsf{E}/R) \to \varprojlim_{\Delta} \mathrm{Level}(A, \mathsf{E} \times_R U_\bullet/U_\bullet) \simeq q^{-1}g(\mathsf{E})$$

is an equivalence. Observe that $\mathrm{Level}(A, \mathsf{E}/R)$ is a full ∞-subcategory of $\mathrm{CDiv}(\mathsf{E}/R)$ and $\varprojlim_{\Delta} \mathrm{Level}(A, \mathsf{E} \times_R U_\bullet/U_\bullet)$ is a full ∞-subcategory of $\varprojlim_{\Delta} \mathrm{CDiv}(\mathsf{E} \times_R U_\bullet/U_\bullet)$. Since $\mathrm{CDiv}_{\mathsf{E}/R}$ is an étale sheaf by Lemma 2.19, the functor

$$\mathrm{Level}(A, \mathsf{E}/R) \to \varprojlim_{\Delta} \mathrm{Level}(A, \mathsf{E} \times_R U_\bullet/U_\bullet)$$

is fully faithful. To show that it is an equivalence, we need only show that it is essentially surjective.

Given any $\{(\mathsf{D}_{U_\bullet}, \phi_{U_\bullet})\}$ in $\varprojlim_{\Delta} \mathrm{Level}(A, \mathsf{E} \times_R U_\bullet/U_\bullet)$, clearly we can find a morphism $\mathsf{D} \to \mathsf{E}$ in $\mathrm{CDiv}(\mathsf{E}/R)$ whose image under the equivalence $\mathrm{CDiv}(\mathsf{E}/R) \simeq \varprojlim_{\Delta} \mathrm{CDiv}(\mathsf{E} \times_R U_\bullet/U_\bullet)$ is $\{\mathsf{D}_{U_\bullet} \to \mathsf{E} \times_R U_\bullet\}$, along with $\phi \colon A \to \mathsf{E}^{\heartsuit}(\pi_0 R)$ lifting $\{\phi_{U_\bullet}\}$. It remains to show that (D, ϕ) is a derived level-A structure. This is true since in the classical case, $\mathrm{Level}(A, \mathsf{E}^{\heartsuit}/\pi_0 R) \simeq \varprojlim_{\Delta} \mathrm{Level}(A, \mathsf{E}^{\heartsuit} \times_{\pi_0 R} \tau_{\leq 0} U_\bullet/\tau_{\leq 0} U_\bullet)$. □

Lemma 4.2. *$\mathcal{M}_{\mathrm{ell}}^A \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is a nilcomplete functor, i.e., $\mathcal{M}_{\mathrm{ell}}^A(R)$ is the limit of the diagram*

$$\cdots \to \mathcal{M}_{\mathrm{ell}}^A(\tau_{\leq m} R) \to \mathcal{M}_{\mathrm{ell}}^A(\tau_{\leq m-1} R) \to \cdots \to \mathcal{M}_{\mathrm{ell}}^A(\tau_{\leq 0} R)$$

Proof. For a spectral elliptic curve R, there is an obvious functor

$$\theta : \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R) \to \varprojlim_{n} \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(\tau_{\leq n} R)$$

define by $(E, \phi : D \to E) \mapsto \{(E \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R, \phi_n : D \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R \to E \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R)\}_n$. Here we notice that $(E \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R, \phi_n : D \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R \to E \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R)$ is in $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(\tau_{\leq n} R)$.

First, we prove that θ is essentially surjective. An object in $\varprojlim_{m} \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(\tau_{\leq m} R)$ can be written as a diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & D_{n+1} \arrow[r] \arrow[d] & D_n \arrow[r] \arrow[d] & D_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & D_0 \arrow[d] \\
\cdots \arrow[r] & E_{n+1} \arrow[r] & E_n \arrow[r] & E_{n-1} \arrow[r] & \cdots \arrow[r] & E_0
\end{tikzcd}$$

where each E_n is spectral elliptic curve over $\tau_{\leq n} R$ and $D_n \to E_n$ is a derived level structure, and satisfying $D_n = D_{n+1} \times_{\mathrm{Sp\acute{e}t}\, \tau_{\leq n+1} R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R, E_n = E_{n+1} \times_{\mathrm{Sp\acute{e}t}\, \tau_{\leq n+1} R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R$. By the nilcompletness of $\mathcal{M}_{\mathrm{ell}}$, we get a spectral elliptic curves E, such that $E \times_R \tau_{\leq n} R \simeq E_n$, and by the nilcompletness of Var_+ [Lur18c, Proposition 19.4.2.1], we get a spectral Deligne–Mumford stack D, such that $D_n = D \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R$. We need to prove the induced map $D \to E$ is a derived level structure, but this follows form nilcompletness of $\mathrm{Level}_{E/R}$.

Second, we need to prove that this functor is fully faithful. Unwinding the definitions, we need to prove that for every $(X, D_1 \to X), (Y, D_2 \to Y) \in \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R)$, the following map is a homotopy equivalence.

$$\mathrm{Map}_{\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R)}((X, D_X), (Y, D_Y)) \to \mathrm{Map}_{\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R)}(\varprojlim_n (X_n, D_{X,n}), \varprojlim_n (Y_m, D_{Y,m})).$$

where X_n is $\tau_{\leq n} X = X \times_R \tau_{\leq n} R$, and Y, $D_{X,n}, D_{Y,n}$ similarly.

But we notice that this is equivalent to the following equivalence

$$\mathrm{Map}_{\mathrm{SpDM}_{/R}}((X, D_X), (Y, D_Y)) \to \varprojlim_n \mathrm{Map}_{\mathrm{SpDM}_{\tau_{\leq n}}}((X_n, D_{X,n}), (Y_n, D_{Y,n})).$$

And this equivalence follows from [Lur18c, Proposition 19.4.1.2] $\square$

Lemma 4.3. *$\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is a cohesive functor.*

Proof. For every pullback diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d] & A \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ such that the underlying homomorphisms $\pi_0 A \to \pi_0 B \leftarrow \pi_0 C$ are surjective. We need to prove that

$$\begin{tikzcd} \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(D) \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(A) \arrow[d] \\ \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(C) \arrow[r] & \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(B) \end{tikzcd}$$

is a pullback diagram.

We have the following diagram in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$,

$$\begin{tikzcd} \mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \arrow[r, "g"] \arrow[dr, "f"'] & \mathcal{M}_{\mathrm{ell}} \arrow[d, "h"] \\ & * \end{tikzcd}$$

By [Lur18c, Remark 17.3.7.3], $\mathcal{M}_{\mathrm{ell}} * (\mathcal{A})$ is a cohesive fucntor if and only if f is cohesive. Since we have $\mathcal{M}_{\mathrm{ell}}$ is cohesive functor, h is a cohesive morphism in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$. And again by [Lur18c, Remark 17.3.7.3], f is cohesive if and only if g is cohesive. So we only need to prove that g is a cohesive morphism. But by [Lur18c, Proposition 17.3.8.4] g is cohesive if and only if each fiber of g is cohesive, i.e., for $R \in \mathrm{CAlg}^{cn}$ and a point $\eta_E \in \mathcal{M}_{\mathrm{ell}}(R)$ which represents a spectral elliptic curve E, the functor

$$f_E : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\}$$

is cohesive. But we have $R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\} \simeq \mathrm{Level}(\mathcal{A}, E \times_R R'/R') \simeq \mathrm{Level}_{E/R}(R')$. The cohesive of $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})$ then follows from the cohesive of $\mathrm{Level}_{E/R}$.

$\square$

Lemma 4.4. *The functor $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is integrable*

Proof. We need to prove that for R a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $m \subset \pi_0 R$, then there is an equivalence

$$\mathrm{Map}_{Fun(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Sp\acute{e}t}\, R', \mathcal{M}_{\mathrm{ell}}(\mathcal{A})) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Spf}\, R', \mathcal{M}_{\mathrm{ell}}(\mathcal{A})).$$

We have the following diagram in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$,

$$\begin{tikzcd} \mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \arrow[r, "g"] \arrow[dr, "f"'] & \mathcal{M}_{\mathrm{ell}} \arrow[d, "h"] \\ & * \end{tikzcd}$$

By [Lur18c, Remark 17.3.7.3], $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \to *$ is a integrable fucntor if and only if f is integrable. Since we have $\mathcal{M}_{\mathrm{ell}}$ is integrable functor, h is a integrable morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. And again by [Lur18c, Remark 17.3.7.3], f is integrable if and only if g is integrable. So we only need to prove that g is an integrable morphism. But by [Lur18c, Proposition 17.3.8.4] g is integrable if and only if each fiber of g is integrable, i.e., for $R \in \mathrm{CAlg}^{\mathrm{cn}}$ and a point $\eta_E \in \mathcal{M}_{\mathrm{ell}}(R)$ which represents a spectral elliptic curve E, the functor

$$f_E : \mathrm{CAlg}^{cn}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\}$$

is integrable. But we have $R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\} \simeq \mathrm{Level}(\mathcal{A}, E \times_R R'/R') \simeq \mathrm{Level}_{E/R}(R')$. The integrable of $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})$ then follows from the integrable of $\mathrm{Level}_{E/R}$.

□

Lemma 4.5. *The functor $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \mapsto \mathcal{S}$ admits a cotangent complex $L_{\mathcal{M}^{de}_{\mathrm{ell}}}$, and moreover $L_{\mathcal{M}^{de}_{\mathrm{ell}}}$ is connective and almost perfect.*

Proof. We have a commutative diagram in $\mathrm{CAlg}^{cn} \to \mathcal{S}$,

$$\begin{tikzcd} \mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \arrow[r, "g"] \arrow[dr, "f"'] & \mathcal{M}_{\mathrm{ell}} \arrow[d, "h"] \\ & * \end{tikzcd}$$

Since we have h is infinitesimally coheisve and admits a connective cotangent complex, and f,g is infinitesimally cohesive. By [Lur18c, Proposition 17.3.9.1], to prove that f admits a cotangent complex. We only need to prove g admits a relative cotangent complex. By [Lur18c, Proposition 17.2.5.7], a morphism $j : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$ admits a relative cotangent complex if and only if, for any corepresentbale $Y' = \mathrm{Map}(R, -) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ and any natural transformation $Y' \to U$, j' in the following pullback diagram admit a cotangent complex.

$$\begin{tikzcd} Y' \times_Y X \arrow[r] \arrow[d, "j'"] & X \arrow[d, "j"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

To prove that $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \to \mathcal{M}_{\mathrm{ell}}$ admits a cotangent complex, we just need to prove that for any $R \in \mathrm{CAlg}^{cn}$, and a spectral elliptic curve E which represents a natural transformation $\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{ell}}$. The functor

$$\mathrm{CAlg}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\}$$

admits a connective cotangent complex. But we have $\mathcal{M}_{\text{ell}}(\mathcal{A})(R')\times_{\mathcal{M}_{\text{ell}}(R')}\{\eta_E\} = \text{Level}(E\times_R R') = \text{Level}_{E/R}(R')$. So the results of $f:\mathcal{M}_{\text{ell}}(\mathcal{A})\to *$ admits a cotangent complex follows from $\text{Level}_{E/R}$ admits a cotangent complex. And the properties of connective and almost perfect also follow from the property of the cotangent complex of $\text{Level}_{E/R}$. ☐

Lemma 4.6. *The functor $\mathcal{M}_{\text{ell}}(\mathcal{A}) : \text{CAlg}^{cn} \mapsto \mathcal{S}$ is locally almost of finite presentation.*

Proof. Consider the functor $\mathcal{M}_{\text{ell}}(\mathcal{A})\to *$, it is infinitesimally cohesive and admits an almost perfect cotangent complex, so by [Lur18c, 17.4.2.2], it is locally almost of finite presentation. So $\mathcal{M}_{\text{ell}}(\mathcal{A})$ is locally almost of finite presentation, since $*$ is a final object of $\text{Fun}(\text{CAlg}^{cn},\mathcal{S})$.

☐

Theorem 4.7. *The functor*

$$\begin{aligned} \mathcal{M}_{\text{ell}}(A) \quad &: \quad \text{CAlg}\to\mathcal{S} \\ & \quad\; R\longmapsto \mathcal{M}_{\text{ell}}(\mathcal{A})(R)=\text{Ell}(\mathcal{A})(R)^{\simeq} \end{aligned}$$

is representable by a spectral Deligne–Mumford stack.

Proof. By the spectral Artin representability theorem, we need to prove that the functor $\mathcal{M}_{\text{ell}}(\mathcal{A})$ satisfies the following condition

(1) The space $\mathcal{M}_{\text{ell}}(\mathcal{A})(R_0)$ is n-truncated for every discrete commutative ring R_0.
(2) $\mathcal{M}_{\text{ell}}(\mathcal{A})$ is a sheaf for the étale topology.
(3) $\mathcal{M}_{\text{ell}}(\mathcal{A})$ is a nilcomplete, infinitesimally cohesive, and integrable functor.
(4) $\mathcal{M}_{\text{ell}}(\mathcal{A})$ admits a cotangent complex $L_{\mathcal{M}_{\text{ell}}(A)}$ which is connective.
(5) $\mathcal{M}_{\text{ell}}(\mathcal{A})$ is locally almost of finite presentation.

But these follow from the above series of lemmas. ☐

4.2. **Higher-homotopical Lubin–Tate towers.** We recall that for a height n p-divisible group G_0 over a commutative ring R_0 and suppose $A\in\text{CAlg}^{ad}_{cpl}$, a deformation of G_0 over R is a spectral p-divisible group over R together with an equivalence class of G_0-tagging of G. We let $\text{Level}(k,G/R)$ denote the space of derived $(\mathbf{Z}/p^k\mathbf{Z})^n$-level structure of a height n spectral p-divisible group. We consider the following functor

$$\begin{aligned} \mathcal{M}_k \quad &: \quad \text{CAlg}^{ad}_{cpl}\to\mathcal{S} \\ & \quad\; R\to \text{DefLevel}(G_0,R,k) \end{aligned}$$

where $\text{DefLevel}(G_0,R,k)$ is the ∞-category whose objects are triples (G,ρ,η)

(1) G is a spectral p-divisible group over R.
(2) ρ is an equivalence of G_0 taggings of R.
(3) $\eta : D\to G$ is a derived $(\mathbf{Z}/p^k\mathbf{Z})^n$-level structure of G.

Theorem 4.8. *The functor $\mathcal{M}_k$ is corepresentable by an $\mathbb{E}_\infty$-ring whose π_0 is finite over $\pi_0 R^{un}_{G_0}$.*

Proof. We let $E_{univ}/R^{\text{un}}_{G_0}$ denote the universal spectral deformation of G_0/R_0. Suppose that G is a spectral deformation G_0 to R, we get a map of $\mathbb{E}_\infty$-rings $R^{\text{un}}_{G_0}\to R$, and an equivalence $E_{univ} \times_{R^{un}_{G_0}} R \simeq G$ of spectral p-divisible groups. By the universal objects of level structures. We have the following equivalence

$$\mathrm{Level}(k, G/R) \simeq \mathrm{Level}(k, E_{univ} \times_{R^{un}_{G_0}} R) \simeq \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R^{un}_{G_0}}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R),$$

where $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is the universal object of derived level structure functor associated with the p-divisible group $E_{univ}/R^{un}_{G_0}$.

Then we consider the following moduli problem

$$\mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}, \quad R \mapsto \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R).$$

For $R \in \mathrm{CAlg}^{ad,cpl}_{R_0}$, $\mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R)$ can viewed the ∞-categories of pairs (α, f), where

$$\alpha : R^{un}_{G_0} \to R$$

is the classified map of a spectral p-divisible group G, which is a deformation of G_0, that is $\alpha = (G, \rho)$, and $f \in \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R^{un}_{G_0}}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R) = \mathrm{Level}(k, E_{univ} \times_{R^{un}_{G_0}} R)$ is a derived level structure of G/R. So we get $\mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R)$ is just the ∞-category of pairs (G, ρ, η). By lemma 3.19, $\pi_0\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is finite over $\pi_0 R^{un}_{G_0}$. So we have $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is the desired spectrum. □

Although we get spectra come from conceptually derived moduli problems, these spectra may be complicated, since we didn't know the homotopy groups. In algebraic topology, the orientation of $\mathbb{E}_\infty$-spectra makes E_2 page of Atiyah-Hirzebruch spectral sequences degenerating and give us the information of homotopy groups.

Let G_0 be a height n p-divisible group over R_{G_0}. We consider the following functor

$$\begin{aligned} \mathcal{M}^{\mathrm{or}}_k \quad : \quad & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{\mathrm{or}}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}^{\mathrm{or}}(G_0, R, k)$ is the space of four tuples (G, ρ, e, η), here

1. G is a spectral p-divisible over R.
2. ρ is an equivalence class of G_0 taggings of R.
3. $e : S^2 \to \Omega^\infty G^\circ(R)$ is an orientation of the G°, where G° is the identity component of G.
4. $\eta : D \to G$ is a derived $(\mathbf{Z}/p^r\mathbf{Z})^n$-level structure of G.

Theorem 4.9. *The functor $\mathcal{M}^{\mathrm{or}}_k : \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}$ is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_k$, an algebra over the orientated deformations ring $R^{or}_{G_0}$. Morover, $\pi_0\mathcal{JL}_k$ is finite over $\pi_0 R^{\mathrm{or}}_{G_0}$.*

Proof. Let $\mathrm{Def}^{or}(G_0, R)$ denote the ∞-groupoid of triples (G, ρ, e), where G is a p-divisible of over R, ρ is an equivalence class of G_0-taggings of R, and e is an orientation of the identity conpoment of G. By [Lur18b, Theorem 6.0.3 and Remark 6.0.7], the functor

$$\begin{aligned} \mathcal{M}^{or} \quad : \quad & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{Def}^{or}(G_0, R) \end{aligned}$$

is corepresnetable by the orientated deformation ring $R^{or}_{G_0}$, that is we have an equivalence of spaces

$$\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{or}_{G_0}, R) \simeq \mathrm{Def}^{or}(G_0, R).$$

Let E^{or}_{univ} be the associated universal orientation deformation of G_0 to $R^{or}_{G_0}$, then it is obvious that $\mathcal{JL}_k = \mathcal{P}_{E^{or}_{univ}/R^{or}_{G_0}}$, the universal object of derived level structures of $E^{or}_{univ}/R^{or}_{G_0}$, is the desired spectrum similar to th unorientated case. □

We call this spectrum $\mathcal{JL}_k$ the Jacquet-Langlands spectrum. It is easy to see that this $\mathcal{JL}_k$ admit an action of $GL_h(\mathbf{Z}/p^k\mathbf{Z}) \times \mathrm{Aut}(G_0)$. And when k varies, we have a tower

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathcal{JL}_k \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathcal{JL}_{k-1} \\ \downarrow \\ \vdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathcal{JL}_0. \end{array}$$

We call this tower a higher categorical Lubin–Tate tower.

In classical arithmetic geometry, the Lubin–Tate tower can be used to realize the Jacquet-Langlands correspondence [HT01]. Is there a topological realization of the Jacquet-Langlands correspondence? Actually, in a recent paper [SS23], they already realized a version of topological Jacquet-Langlands correspondence. But their method is based on the Goerss-Hopkins-Miller-Lurie sheaf. They consider the degenerate level structures such that representing objects is étale over representing objects of universal deformations.

We hope our higher categorical analogues of Lubin–Tate towers can also establish a topological version of the classical Langlands correspondence, which means that we construct representations on the category of spectra.

4.3. **Topological lifts of power operation rings.** We recall the deformation of formal groups. Let G_0 be a formal group over a perfect field k such that $\mathrm{char}\, k = p$, a deformation of G_0 to R is a triple (G, i, Φ) satisfying

- G is a formal group over R,
- There is a map $i : k \to R/m$
- There is an isomorphism $\Phi : \pi^*G \cong i^*G_0$ of formal groups over R/m.

Suppose that we have a complete local ring R whose residue filed has characteristic p. Let $\phi : R \to R, x \mapsto x^p$ be the Frobenius map. For each formal group G over R, the **Frobenius isogeny** $\mathrm{Frob} : G \to \phi^*G$ is the homomorphism of the formal group over R induced by the relative Frobenius map on rings. We write $\mathrm{Frob}^r : G \to (\phi^r)^*G$ which is the composition $\phi^*(\mathrm{Frob}^{r-1}) \circ \mathrm{Frob}$

Let G_0 be a formal group over k, (G, i, α) and (G', i', α') be two deformations of G_0 to R. A deformation of Frob^r is a homomorphism $f : G \to G'$ of formal groups over R which satisfying

(1) $i \circ \phi^r = i'$ and $i^*(\phi^r)^*G_0 = (i')^*G_0$.

$$\begin{tikzcd} k \arrow[r, "i'"] \arrow[d, "\phi^r"'] & R/m \\ k \arrow[ur, "i"'] & \end{tikzcd}$$

(2) the square

$$\begin{tikzcd} i^*G_0 \arrow[r, "i^*(\mathrm{Frob}^r)"] \arrow[d, "\alpha"'] & i^*(\phi^r)^*G_0 \arrow[d, "\alpha'"] \\ \pi^*G \arrow[r, "\pi^*(f)"] & \pi^*G' \end{tikzcd}$$

of homomorphisms of formal groups over R/m commutes.

We let Def_R denote the category whose objects are deformations fo G_0 to R, and whose morphisms are deformations of Frob^r for some $r \geq 0$. We will say that a morphism in Def_R has height r, if it is a deformation of Frob^r, and then we denote the corresponding subcategory as $\mathrm{Sub}^r R$. Let G be the deformation of G_0 to R, then it can be proved that the assignment $f \to \mathrm{Ker} f$ is a one-to-one correspondence between the morphisms in Sub^r_R with source G and the finite subgroup of G which have rank p^r.

Theorem 4.10. [Str97] *Let G_0/k be a height n formal group over a perfect field k. For each $r > 0$, there exists a complete local ring A_r which carries a universal height r morphism $f^r_{univ} : (G_s, i_s, \alpha_s) \mapsto (G_t, i_t, \alpha_t) \in \mathrm{Sub}^r(A_r)$. That is the operation $f^r_{univ} \to g^*(f^r_{univ})$ define a bijective relation from the set of local homomorphism $g : A_r \to R$ to the set Sub^r_R. Furthermore, we have:*

(1) *$A_0 \approx W(k)[[v_1, \cdots, v_{n-1}]]$ is the Lubin–Tate ring.*
(2) *There is a map $s : A_0 \to A_r$ which classifies the source of the universal height r map, i.e. $G_s = s^*G_E$, where $G_E = G_{univ}/A_0$ be the universal deformation of G_0, and A_r is finite and free as an A_0 module.*
(3) *There is a map $t : A_0 \to A_r$ which classifies the target of the universal height r map, i.e. $G_t = t^*G_E$.*
(4) *And there is a bijection $\{g : A_r \to R\} \to \mathrm{Sub}^r(R)$ given by $g \to g^*(f^r_{univ})(g^*G_s \to g^*G_t)$.*

We know that those rings $A_r, r \geq 0$ have topological meanings.

Theorem 4.11. [Str98] *The ring A_r in the universal deformation of Frobenuis is isomorphic to $E^0(B\Sigma_{p^r})/I$, i.e,*

$$A_r \cong E^0(B\Sigma_{p^r})/I$$

where I is the transfer ideal.

The collections $\{A_r\}$ have the structures of graded coalgerbas, for $s = s_k, t = t_k : A_0 \to A_k$, which is induced by E^0 cohomology on $B\Sigma \to *$, we have

$$\mu = mu_{k,l} : A_{k+l} : A_{k+l} \to A_k{}^s\otimes_{A_0}{}^t A_l$$

which classifies the source, target, and composite of morphisms. So for the power operation $R^k(X) \to R^k(X \times B\Sigma_m)$. For $x = *$, we have

$$\pi_0 R \to E^0(B\Sigma_{p^r})/I \otimes \pi_0 R = A[r] \otimes \pi_0 R$$

This make $\pi_0 R$ becomes a Γ-module, where Γ are duals of $A[r]$.

For more details about power operation in Morava E-theory, one can see [Rez24, Rez09] and [Rez13]. Direct computations are in [Rez08] for height 2 at the prime 2, [Zhu14] for height 2 at prime 3, [Zhu19] for height 2 at all primes. Cases of height > 2 are still lack of computations.

Because we have the assignment $f \to \mathrm{Ker} f$ is a one-to-one correspondence between the morphisms in Sub^r_R with source G and the finite subgroup of G which have rank p^r. So it is easy to see that A_r corepresents the following moduli problem

$$\begin{aligned} \mathcal{M}_{0,r} \quad : \quad & \mathrm{CAlg}_k^{\heartsuit} \to \mathcal{S} \\ & R \to \mathrm{Def}(G_0, R, p^r) \end{aligned}$$

where $\mathrm{Def}(G_0, R, p^r)$ consists of pairs (G, H) where G is an defomration G_0 to R, and H is a rank p^r subgroup of G.

Proposition 4.12. *For every integer $r \geq 1$, there exists an $\mathbb{E}_\infty$-ring $E_{n,r}$, such that $\pi_0 E_{n,r} = A_r$.*

Proof. For the formal group G_0 over a field k of characteristic p. We just consider the functor $\mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}$ by sending an $\mathbb{E}_\infty$-ring R to quadruples (G, ρ, e, η), where (G, ρ) is spectral deformation of G_0 to R. e is an orientation of G°, the identity component G, and $\eta \in \mathrm{Level}_0(r, G/R)$ is a derived level structure. Using the same argument in full-level structure and the fact $\mathrm{Level}^{0,r}_{G/R}$ is representable, see Remark 3.22. We get this proposition. $\square$

Remark 4.13. The $\mathbb{E}_\infty$-rings $E_{n,r}$ from above are not $K(n)$-local or complex oriented, in light of [Dev20, Theorems 1.3 and 3.5] and [Lur18b, Construction 5.1.1 and Remark 5.1.2]. They are not even a *finite* algebra over the Morava E-theory spectrum E, even though the morphism $\mathrm{Sp\acute{e}t}\, E_{n,r} \to \mathrm{Sp\acute{e}t}\, E$ of spectral Deligne–Mumford stacks *is* finite (cf. [Lur17, Definition 7.2.2.1], [Lur18c, Proposition 2.7.2.1], and the finiteness conditions we recalled at the beginning of Section 2.1).

Remark 4.14. Although we obtained spectra whose 0'th homotopy groups recover the power operation rings of Morava E-theories, we do not know yet the higher homotopy groups of these spectra concretely or explicitly, as these spectra are not 2-periodic in general unless $r = 0$ and they are not étale over E-theory spectra. This non-2-periodicity with $r > 0$ should be a manifestation of the structure of a *pile*, i.e., a presheaf of categories (rather than of groupoids), as indicated in [Rez14, Section 4.3].

5. More applications

5.1. **Jacquet–Langlands spectra.** The Langlands program is a project in mathematics which aims to relate many fileds in mathematics together, including number theory, representation theory, and harmonic analysis. The global Langlands correspondence is conjectural (bijection) between

(1) n-dimensional complex linear representations of the Galois group $\mathrm{Gal}(\bar{F}/F)$ of a given number field F.
(2) certain representations-called automorphic representations of the n dimensional general linear group $GL_n(\mathbb{A}_F)$ with coefficients in the ring of adeles of F, arising within the representations given by functions on the double coset space $GL_n(F) \setminus GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})$ (where $\mathcal{O} = \prod_v \mathcal{O}_p$ is the ring of integers of all formal completions of F).

which compatible with certain L-function conditions. Moreover, the group GL_n can be replaced by any reductive group. The Langlands correspondence has many specific examples in number theory. For the group GL_1, this correspondence is just global class field theory. The Langlands correspondence for GL_2 leads to the famous modularity theorem [Wil95], [TW95].

The Langlands correspondence has a local version. Let E be a local field, and G be a reductive group over E. The local Langlands correspondence predicts that for any irreducible smooth representation π of $G(E)$, we can naturally associate an L-parameter

$$\phi_E : W_E \to G(\mathbb{C}).$$

What we want to say in this paper is the Jacquet-Langlands correspondence. Let K be a p-adic filed, and D a division algebra with center K and dimension d^2 over K. We fix an integer $r \leq 1$, and Let $G = GL_n$, $G' = GL_r(D)$, where $n = rd$. The Jacquet Langlands correspondence aims to relate smooth irreducible representations of G to those of G', whereas the Langlands correspondence relates such representations to degree n-representations of the absolute Galois group of K.

We care about the case $r = 1$, i.e, D is a center algebra over K of dimension n^2. There is a bijection between

(1) square integrable irreducible representations of $D^\times$ and,
(2) square integrable irreducible representations of $GL_n(K)$.

In classical arithmetic geometry, the Lubin–Tate tower can be used to realize the Jacquet-Langlands correspondence [HT01]. Is there a topological realization of the Jacquet–Langlands correspondence? Actually, in a recent paper [SS23], they already realized a version of topological Jacquet-Langlands correspondence. But their method is based on the Goerss-Hopkins-Miller-Lurie sheaf. They actually consider the degenerate level structures such that representing objects are *étale* over representing objects of universal deformations. We hope our higher categorical analogues of Lubin–Tate towers can also establish a topological version of the classical Langlands correspondence, which means that we construct representations on the category of spectra. Our derived level structure give an attempt on this idea by considering certain function spectra.

On the other hand, we know the actions of certain Galois groups and automorphism groups on certain objects, like Morava E-theories, THH, TC. This means that these groups act on their homotopy groups. For example, we have the action of Morava stabilizer groups $\mathbb{G}_n$ on Morava E-theories E_n, it can be used to compute the stable homotopy group of spheres by the following spectral sequence

$$E_2^{s,t} \cong H^s_{cts}(\mathbb{G}_n, \pi_t E_n) \Longrightarrow \pi_{t-s} L_{K(n)} S^0.$$

But usually, it is complicated to compute the continuous cohomology of $\mathbb{G}_n$. This is common in Langlands correspondence that th Galois side is usually harder to understand than the automorphic side. One strategy for relevant problems is to transfer the problems in the Galois side to the automorphic side. Let's see an example first.

Theorem 5.1. *([BSSW24b]) There is an isomorphism of graded* $\mathbf{Q}$*-algebras*

$$\mathbf{Q} \otimes \pi_* L_{K(n)} S^0 \cong \Lambda_{\mathbf{Q}_p}(\zeta_1, \zeta_2, \cdots \zeta_n),$$

where the latter is the exterior $\mathbf{Q}_p$*-algebra with generators* ζ_i *in degree* $1 - 2i$*.*

The main of their proof of this theorem is they transfer the computation of cohomology of $\mathbb{G}_n$ to the cohomology of Drinfeld symmetric space $\mathcal{H}$.

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "\mathrm{GL}_n(\mathbf{Z}_p)"'] \arrow[dr, "\mathbb{G}_n"] & \\ \mathrm{LT}_K & & \mathcal{H}. \end{tikzcd}$$

In a continuous work [BSSW24a], they compute the Picard group of $K(n)$-local spectra by using some results of computation of Drinfeld symmetric space, which is due to Colmez–Dospinescu–Nizio [CDN20], [CDN21].

We know that LT has a higher categorical refinement, Morava E-theories. So it is a natural question how to lift this diagram to higher categorical setting and how to establish a more conceptual theory to transfer the computation of cohomology of $\mathbb{G}_n$ to the computation of cohomology of $\mathcal{H}$.

Let G_0 be a height n p-divisible group over R_{G_0}. We consider the following functor

$$\begin{aligned} \mathcal{M}_k^{\mathrm{or}} \quad : \quad & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{\mathrm{or}}(G_0, R, k) \end{aligned}$$

In subsection 3.3, we prove that this functor is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_k$. We defined the Jacquet-Langlands spectrum $\mathcal{JL}$ to be the limit of those $\mathcal{JL}_k$, i.e.,

$$\mathcal{JL} = \varprojlim_k \mathcal{JL}_k$$

Lemma 5.2. *$\mathcal{JL}$ is an $\mathbb{E}_\infty$-ring.*

Proof. This is because the ∞-category of $\mathbb{E}_\infty$-rings admits inverse limits, see[Lur17, Corollary 3.2.2.4] for details. ∎

The spectrum is the higher categorical realization of $\mathcal{X}$, the moduli of deformations with level structures. It was proved by Scholze and Weinstein [SW13] that $\mathcal{X}$ is a perfectoid space.

5.2. **Jacquet–Langlands duals of Morava E-theory spectra.** By the construction of Jacquet–Langlands spectra above, it is easy to see that this $\mathcal{JL}_k$ admits an action of $GL_n(Z/p^kZ) \times \mathbb{G}_n$. $\mathcal{JL}$ is the limit of $\mathcal{JL}_k$, so it admits an action of $\lim GL_n(Z/p^kZ) \times \mathbb{G}_n = GL_n(\mathbf{Z}_p) \times \mathbb{G}_n$.

Definition 5.3. We define the dual Morava E-theories ${}^L E_n$ to be $\mathcal{JL}^{hG_n}$.

The generic fibre of $\pi_0 {}^L E_n$ is just the Drinfled symmetric space. The Drinfled symmetric space was invented in [Dri76]. It is the rigid analytic space

$$\mathcal{H} = \mathbb{P}_K^{n-1} \setminus \bigcup_H H,$$

where $\mathbb{P}_K^{n-1}$ is a rigid analytic projective space, and H run over all K-rational hyperplanes in $\mathbb{P}_K^{n-1}$. It has a formal model $\mathfrak{h}$ which parametrizes the deformations of a special formal $\mathcal{O}_D$-module related to G_0. In future work, we will prove that ${}^L E_n$ can also come from some derived moduli problem.

In [Rog08], the author defines the Galois extension for commutative ring spectra. Suppose that E is an $\mathbb{E}_\infty$-ring spectra and F is an $\mathbb{E}_\infty$ E-algebra with an action of a finite group G. We say that F is a G-Galois extension of E if

(1) $F^{hG} \simeq E$, and
(2) $F \otimes_E F \to \prod_G F$ is an equivalence.

Proposition 5.4. *For every k, $\mathcal{JL}_k$ is a Galois extension of $R^{\mathrm{or}}_{G_0}$.*

Proof. Let $d = \sharp|\mathrm{GL_n}(\mathbf{Z}/p^k\mathbf{Z})|$. It is easy to see that $\mathcal{JL}_k^{h\mathrm{GL}_n(\mathbf{Z}/p^k\mathbf{Z})}$ is equivalent to $R^{\mathrm{or}}_{G_0}$, so we only need to prove that $\mathcal{JL}_k \otimes_{R^{\mathrm{or}}_{G_0}} \mathcal{JL}_k \simeq \prod_{\mathrm{GL}_n(\mathbf{Z}/p^r\mathbf{Z})} \mathcal{JL}_k$. By [Lur18c, Proposition 1.4.11.1], this is equivalent to say that $\mathrm{Spét}\, \mathcal{JL}_k \otimes_{\mathrm{Spét}\, R^{\mathrm{or}}_{G_0}} \mathrm{Spét}\, \mathcal{JL}_k \simeq \coprod_{\mathrm{GL}_n(\mathbf{Z}/p^k\mathbf{Z})} \mathrm{Spét}\, \mathcal{JL}_k$. We will prove this by using the moduli explanation. For a connective $\mathbb{E}_\infty$-ring R, we consider the diagram

$$\begin{tikzcd}
\coprod_{\mathrm{GL}_n(\mathbf{Z}/p^k\mathbf{Z})} \mathrm{Spét}\, \mathcal{JL}_k \arrow[dr] & & \\
& \mathrm{Spét}\, \mathcal{JL}_k \otimes_{R^{\mathrm{or}}_{G_0}} \mathrm{Spét}\, \mathcal{JL}_k \arrow[r] \arrow[d] & \mathrm{Spét}\, \mathcal{JL}_k \arrow[d] \\
& \mathrm{Spét}\, \mathcal{JL}_k \arrow[r] & \mathrm{Spét}\, R^{or}_{G_0}
\end{tikzcd}$$

(1) The moduli space $\mathrm{Spét}\, \mathcal{JL}_k \otimes_{R^{\mathrm{or}}_{G_0}} \mathrm{Spét}\, \mathcal{JL}_k$ parametrizes

$$\{(G_1, \rho_1, e_1, \eta_1), (G_2, \rho_2, e_2, \eta_2)\},$$

where (G_i, ρ_i, e_i) is a spectral orientated deformation of G_0 to R, e_i is an orientation of G_i^o, η_i is a derived level structure of G_i. Since the tensor product is over $\mathrm{Spét}\, R^{\mathrm{or}}_{G_0}$, so we have $G_1 = G_2, \rho_1 = \rho_2$ and $e_1 = e_2$.

(2) The moduli space $\coprod_{\mathrm{GL}_n(\mathbf{Z}/p^k\mathbf{Z})} \mathrm{Spét}\, \mathcal{JL}_k$ parametrizes

$$\{(G_1, \rho_1, e_1, \eta_1), \cdots, (G_d, \rho_d, e_d, \eta_d)\},$$

such that $(G_1, \rho_1, e_1) = \cdots, = (G_d, \rho_d, \eta_d)$ is an orientated deformation of G_0, and η_i is the derived level structure obtained from $\eta_a g_i \eta b$ for two derived level structures η_a, η_b.

It is clear that these two moduli spaces are equivalent, so we get

$$\mathcal{JL}_k \otimes_{R^{\mathrm{or}}_{G_0}} \mathcal{JL}_k \simeq \prod_{\mathrm{GL}_n(\mathbf{Z}/p^k\mathbf{Z})} \mathcal{JL}_k$$

□

Theorem 5.5. *E_n^L is an $\mathbb{E}_\infty$-ring spectrum.*

Proof. We have ${}^L E_n = \mathcal{JL}^{h\mathbb{G}_n} \simeq \varprojlim \mathcal{JL}_k^{h\mathbb{G}_n}$. It is sufficient to prove that $\mathcal{JL}_r^{h\mathbb{G}_n}$ is an $\mathbb{E}_\infty$-spectrum. We have the Galois extension

$$L_{K(n)}S^0 \to E(n) \to \mathcal{JL}_k.$$

We notice that $\mathcal{JL}_k$ is profinite $\mathbb{G}_n \times \mathrm{GL}_n(\mathbf{Z}/p^k\mathbf{Z})$-spectra in the sense of [QUI13] and [DQ16]. This means that $\mathcal{JL}_k$ is a $\mathbb{G}_n$-profinite spectra. By [QUI13, Proposition 3.23], we have

$$\mathcal{JL}_k^{h\mathbb{G}_n} \cong \mathrm{Tot}(\mathrm{Map}(G^\bullet), \mathcal{JL}_k).$$

The category of CAlg admits inverse limits, so $\mathcal{JL}_k^{h\mathbb{G}_n}$ is also an $\mathbb{E}_\infty$-ring. □

Proposition 5.6. *There are convergent spectral sequences*

$$E_2^{s,t} \cong H^s_{cts}(\mathbb{G}_n \times GL_n(\mathbf{Z}_p), \pi_t \mathcal{JL}) \Longrightarrow \pi_{t-s} L_{K(n)} S^0.$$

$$E_2^{s,t} \cong H^s_{cts}(GL_n(\mathbf{Z}_p), \pi_t {}^L E_n) \Longrightarrow \pi_{t-s} L_{K(n)} S^0.$$

Proof. This is just because for any profinite group G, and E is a G-equivariant spectrum, we always have

$$E_2^{s,t} \cong H^s_{cts}(G, \pi_t E) \Longrightarrow \pi_{t-s} E^{hG}.$$

see [May96] or [QUI13, Theorem 3.17] for more details. □

5.3. **Further problems.**

(1) Although we construct the higher categorical Lubin-Tate towers, but we still don't know the higher homotopy groups of this tower and their arithmetic meanings.

(2) In [GV18], Galatius and Venkatesh define and study derived Galois deformations. In [Ven19], Venkatesh defines the derived Hecke algebra to be derived enhancement of the classical Hecke algebra. It satisfies certain good properties like the classical Hecke algebra. These two constructions give us evidence about the homotopical version of Langland correspondence for general reductive group G, but the derived Hecke algebra doesn't come from the symmetry of derived objects.

In recent papers [CS24] and [Dav24], there are some constructions of Hecke operation on topological modular forms. We hope to establish a general theory of Hecke algebra in the derived algebra geometry context. In the geometric Langlands correspondence, the construction of the Hecke stack is an important ingredient. We want to find a reasonable construction of the derived Hecke stack that is compatible with Hecke algebra of topological modular forms.

Acknowledgements

This paper has its impetus from ongoing joint work of Guozhen Wang and the second author, and grew out of the first author's doctoral thesis. The authors would like to thank *... yet to be written.*

References

[BL10] Mark Behrens and Tyler Lawson. Topological automorphic forms. *Mem. Amer. Math. Soc.*, 204(958):xxiv+141, 2010.

[BSSW24a] Tobias Barthel, Tomer M. Schlank, Nathaniel Stapleton, and Jared Weinstein. On Hopkins' Picard group. 2024. arXiv:2407.20958.

[BSSW24b] Tobias Barthel, Tomer M. Schlank, Nathaniel Stapleton, and Jared Weinstein. On the rationalization of the $K(n)$-local sphere. 2024. arXiv:2402.00960.

[CDN20] Pierre Colmez, Gabriel Dospinescu, and Wiesława Nizioł. Cohomology of p-adic Stein spaces. *Invent. Math.*, 219(3):873–985, 2020.

[CDN21] Pierre Colmez, Gabriel Dospinescu, and Wiesława Nizioł. Integral p-adic étale cohomology of Drinfeld symmetric spaces. *Duke Math. J.*, 170(3):575–613, 2021.

[CS24] L. Candelori and A. Salch. Topological Hecke eigenforms. *Math. Z.*, 307(4):Paper No. 75, 40, 2024.

[Dav24] Jack Morgan Davies. Hecke operators on topological modular forms. *Adv. Math.*, 452:Paper No. 109828, 71, 2024.

[Dev20] Sanath Devalapurkar. Roots of unity in $K(n)$-local rings. *Proc. Amer. Math. Soc.*, 148(7):3187–3194, 2020.

[Dev23] Sanath K. Devalapurkar. Hodge theory for elliptic curves and the Hopf element ν. *Bull. Lond. Math. Soc.*, 55(2):826–842, 2023.

[DQ16] Daniel Davis and Gereon Quick. Profinite and discrete g–spectra and iterated homotopy fixed points. *Algebraic & Geometric Topology*, 16(4):2257–2303, 2016.

[Dri76] V.G. Drinfel'd. Coverings of p-adic symmetric regions. *Funct. Anal. Appl.*, 10(2):107–115, 1976.

[FGL08] Laurent Fargues, Alain Genestier, and Vincent Lafforgue. *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, volume 262 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2008.

[GH04] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.

[GM23] David Gepner and Lennart Meier. On equivariant topological modular forms. *Compos. Math.*, 159(12):2638–2693, 2023.

[GV18] S. Galatius and A. Venkatesh. Derived Galois deformation rings. *Adv. Math.*, 327:470–623, 2018.

[HL16] Michael Hill and Tyler Lawson. Topological modular forms with level structure. *Invent. Math.*, 203(2):359–416, 2016.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[Lur04] Jacob Lurie. *Derived algebraic geometry*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[Lur09a] J. Lurie. A survey of elliptic cohomology. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 219–277. Springer, Berlin, 2009.

[Lur09b] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. *Higher algebra*. 2017. `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[Lur18a] Jacob Lurie. Elliptic cohomology I: Spectal abelian varietes. 2018. `https://www.math.ias.edu/~lurie/papers/Elliptic-I.pdf`.

[Lur18b] Jacob Lurie. Elliptic cohomology II: Orientations. 2018. `https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf`.

[Lur18c] Jacob Lurie. *Spectral algebraic geometry*. 2018. `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`.

[May96] J. P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[Mil86] J. S. Milne. Abelian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 103–150. Springer, New York, 1986.

[MPR24] Sergej Monavari, Emanuele Pavia, and Andrea T. Ricolfi. Derived hyperquot schemes. 2024. arXiv:2409.16858.

[QUI13] GEREON QUICK. Continuous homotopy fixed points for lubin-tate spectra. *Homology, Homotopy and Applications*, 15(1):191–222, 2013.

[Rez08] Charles Rezk. Power operations for Morava E-theory of height 2 at the prime 2. 2008. arXiv:0812.1320.

[Rez09] Charles Rezk. The congruence criterion for power operations in Morava E-theory. *Homology Homotopy Appl.*, 11(2):327–379, 2009.

[Rez13] Charles Rezk. Power operations in Morava E-theory: structure and calculations (Draft). 2013. `https://rezk.web.illinois.edu/power-ops-ht-2.pdf`.

[Rez14] Charles Rezk. Isogenies, power operations, and homotopy theory. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1125–1145. Kyung Moon Sa, Seoul, 2014.

[Rez24] Charles Rezk. Lectures on power operations. 2024. `https://rezk.web.illinois.edu/power-operation-lectures.pdf`.

[Rog08] John Rognes. *Galois Extensions of Structured Ring Spectra/Stably Dualizable Groups: Stably Dualizable Groups*, volume 192. American Mathematical Soc., 2008.

[RZ96] M. Rapoport and Th. Zink. *Period spaces for p-divisible groups*, volume 141 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[SS23] Andrew Salch and Matthias Strauch. ℓ-adic topological Jacquet-Langlands duality. 2023. With an appendix by Andrew Salch and Matthias Strauch. arXiv:2311.10225.

[ST23] Nicolò Sibilla and Paolo Tomasini. Equivariant elliptic cohomology and mapping stacks I. 2023. arXiv:2303.10146.

[Str97] Neil P. Strickland. Finite subgroups of formal groups. *J. Pure Appl. Algebra*, 121(2):161–208, 1997.

[Str98] N. P. Strickland. Morava E-theory of symmetric groups. *Topology*, 37(4):757–779, 1998.

[SW13] Peter Scholze and Jared Weinstein. Moduli of p-divisible groups. *Camb. J. Math.*, 1(2):145–237, 2013.

[TV08] Bertrand Toën and Gabriele Vezzosi. *Homotopical algebraic geometry. II. Geometric stacks and applications*, volume 193. 2008.

[TW95] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.

[Ven19] Akshay Venkatesh. Derived Hecke algebra and cohomology of arithmetic groups. *Forum Math. Pi*, 7:e7, 119, 2019.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

[Zhu14] Yifei Zhu. The power operation structure on Morava E-theory of height 2 at the prime 3. *Algebr. Geom. Topol.*, 14(2):953–977, 2014.

[Zhu19] Yifei Zhu. Semistable models for modular curves and power operations for Morava E-theories of height 2. *Adv. Math.*, 354:Paper No. 106758, 29, 2019.

Institute for Theoretical Sciences, Westlake University, Hangzhou, Zhejiang 310030, P.R. China

Email address: maxuecai@westlake.edu.cn

Department of Mathematics, College of Science, Southern University of Science and Technology, Shenzhen, Guangdong 518055, P.R. China

Email address: zhuyf@sustech.edu.cn