Higher-periodic homotopy types through Lubin–Tate towers

In their recent work, Barthel, Schlank, Stapleton, and Weinstein determined the periodic stable homotopy groups of the sphere spectrum rationally. They made an essential use of the structure afforded by perfectoid spaces for computing relevant group cohomology in the framework of condensed mathematics. These spaces appear in an equivariant isomorphism between two towers: (1) the Lubin–Tate tower that parametrizes deformations of a formal group of fixed height with level structures and (2) the Drinfeld tower that parametrizes those for shtukas. I'm obliged to introduce this exciting mathematical landscape to the greater "perfection" community, and appeal for further insights and collaborations. This also includes: (a) my ongoing joint work with Guozhen Wang which computes unstable higher-periodic homotopy types integrally, (b) Xuecai Ma's spectral realization of finite levels of the Lubin–Tate tower as non-even commutative ring spectra, which generalize Morava, Hopkins, Miller, Goerss, and Lurie's spectra at the ground level, and (c) Hongxiang Zhao's work which connects Ando's norm through homotopical descent of level structures along the tower, to Coleman's norm in the context of Lubin and Tate's explicit local class field theory.

https://bicmr.pku.edu.cn/content/show/17-3377.html

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Yifei Zhu

Southern University of Science and Technology

2024.6.25

Context and motivations

$$p\textbf{-adic geometry} \rightleftarrows \textbf{topology}$$

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$$\begin{tikzcd} \textbf{\textit{p}-adic geometry} \arrow[r, bend left] \arrow[r, phantom, "\text{homotopy theory}"] & \textbf{topology} \arrow[l, bend left] \end{tikzcd}$$

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p-adic geometry $\underset{\text{homotopy theory}}{\rightleftarrows}$ **topology**

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e.g., Tate diagonal
cyclotomic spectra à la Nikolaus and Scholze
Liu–Wang, Jingbang Guo

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In this talk, we focus on some of the recent progresses in the forward direction, but the *p*-adic geometry involved, as well as the relevant tools, may be of independent interest beyond topology.

Analytic geometry and homotopy groups of the $K(n)$-local sphere

Let $\mathbb{S}$ be the sphere spectrum

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Thick subcategories of **Sp** (Hopkins–Smith '98)

$$
\begin{tikzcd}[column sep=small]
\mathcal{P}_{2,\infty} \arrow[d, dash] & & \mathcal{P}_{3,\infty} \arrow[d, dash] & \cdots & \mathcal{P}_{p,\infty} \arrow[d, dash] & \cdots \\
\vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
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\mathcal{P}_{2,1} \arrow[drr, dash] & & \mathcal{P}_{3,1} \arrow[d, dash] & \cdots & \mathcal{P}_{p,1} \arrow[dll, dash] & \cdots \\
& & \mathcal{P}_{0,0} & & &
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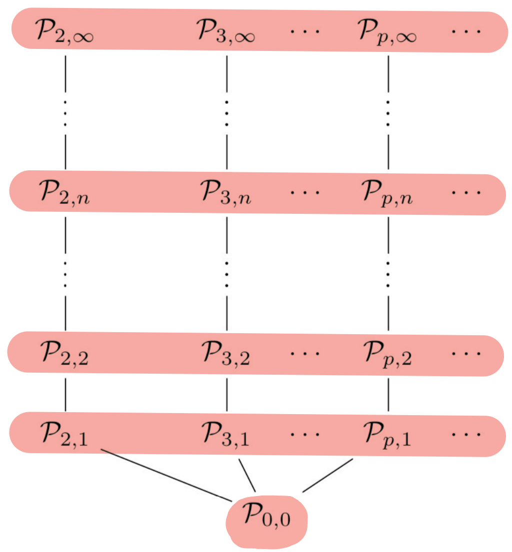

These correspond precisely to the stratification by heights and primes of the moduli stack of 1-dimensional commutative formal groups

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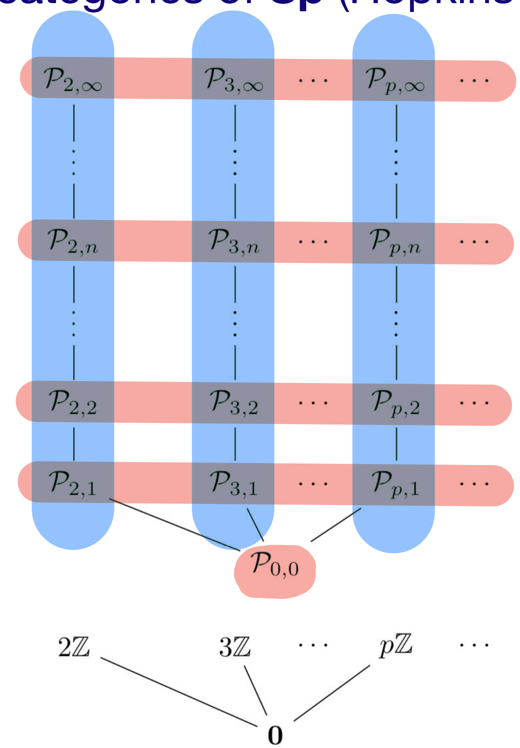

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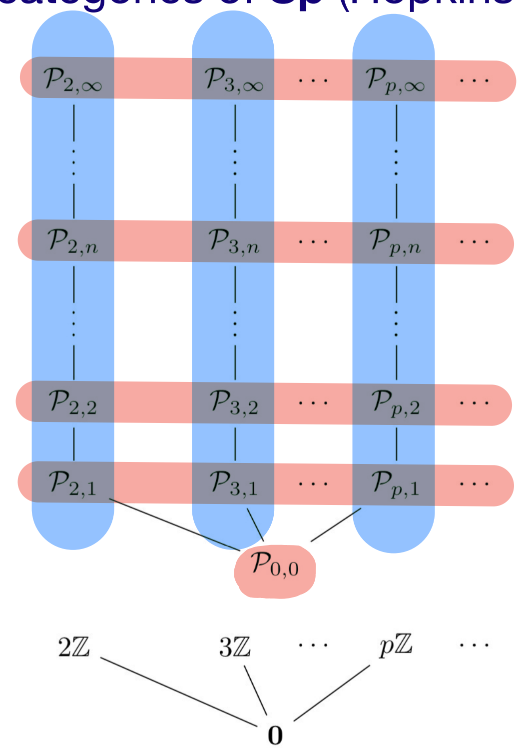

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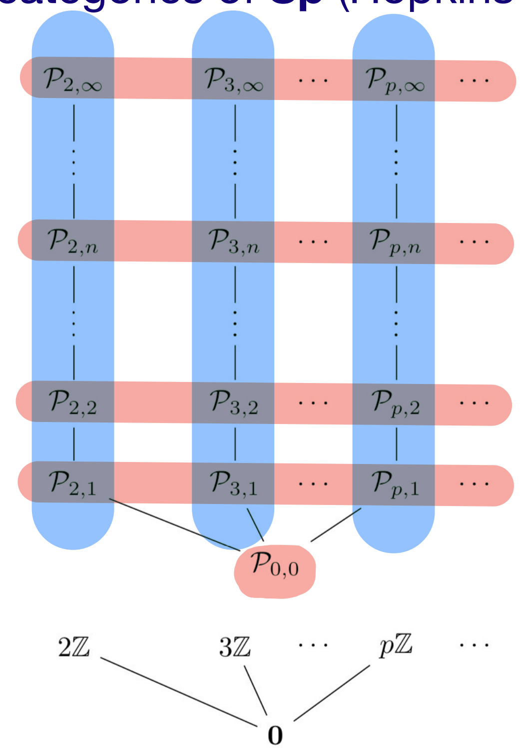

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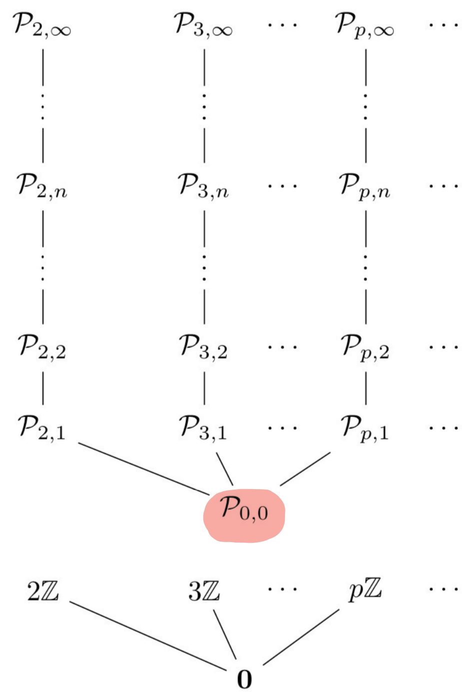

rational cohomology

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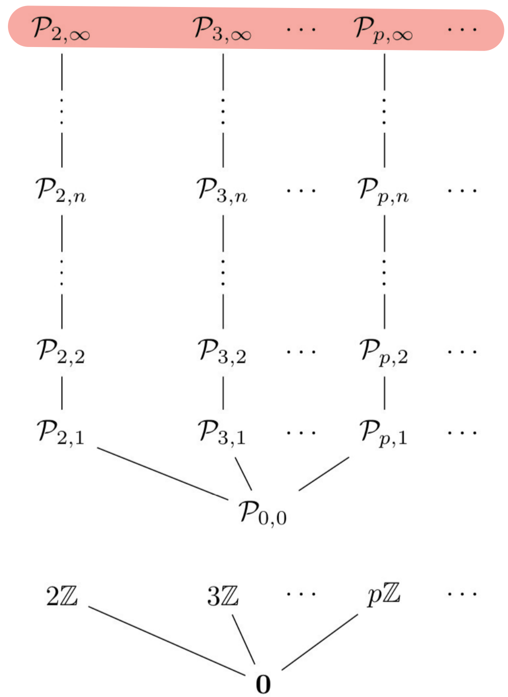

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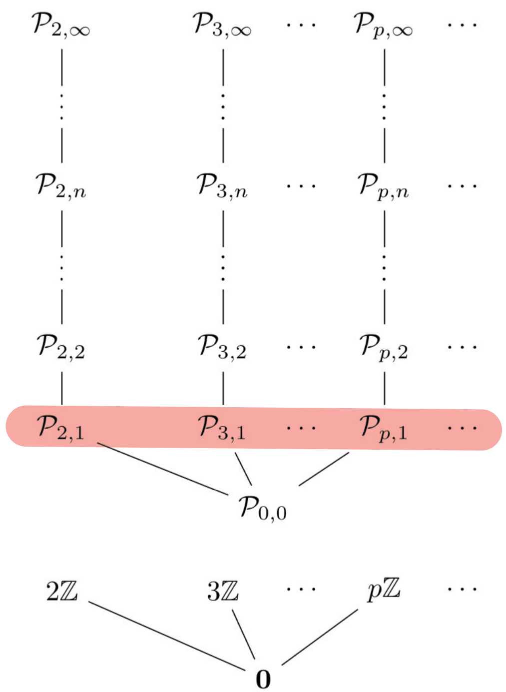

p-adic K-theory

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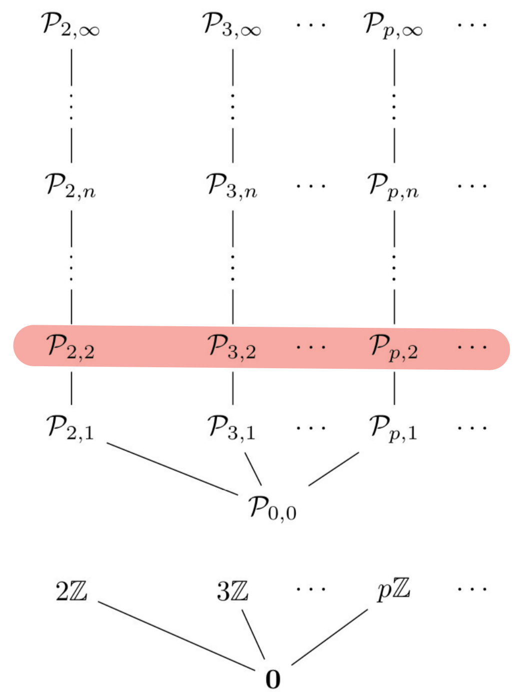

elliptic cohomology
(topological modular forms)

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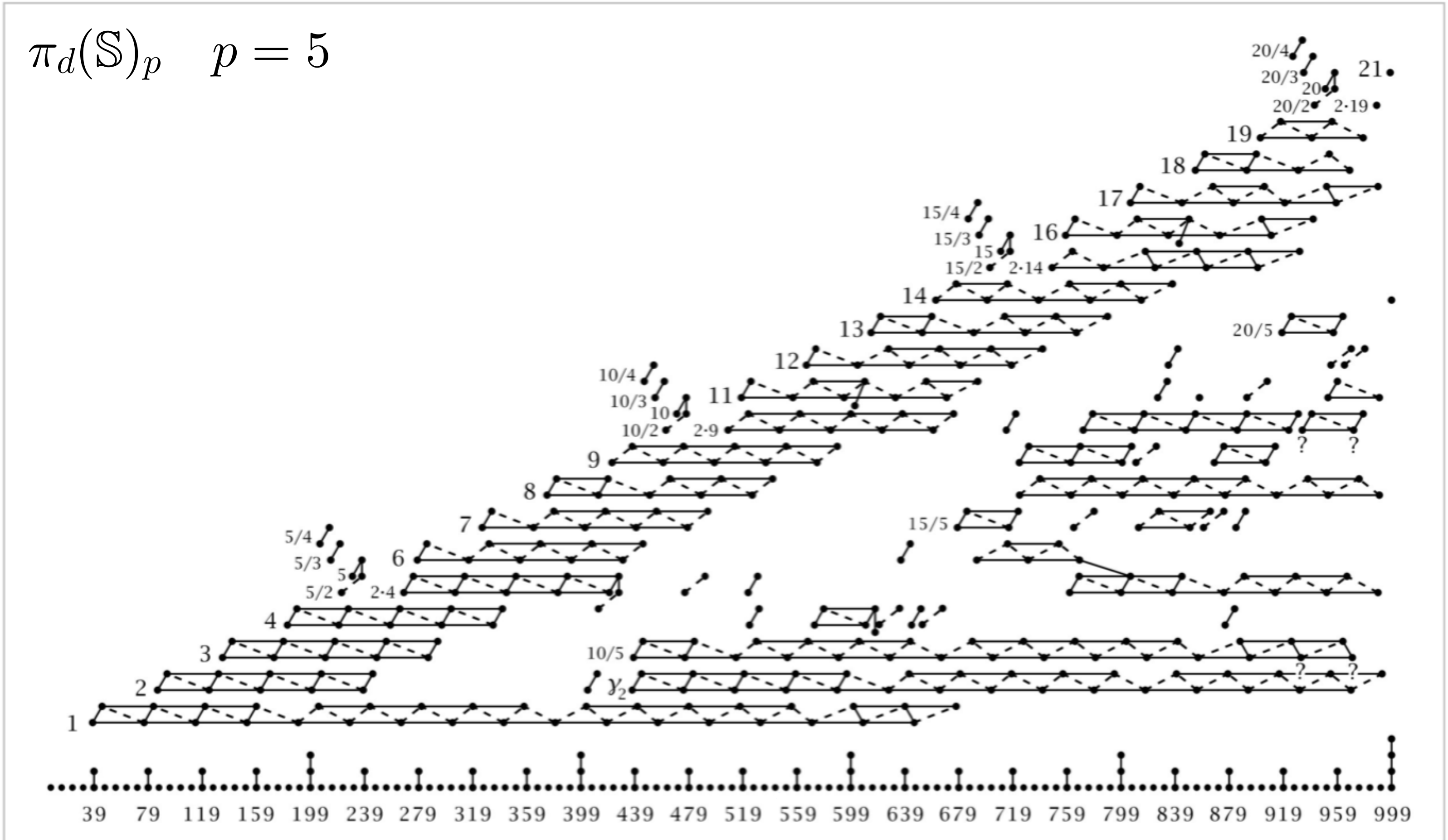
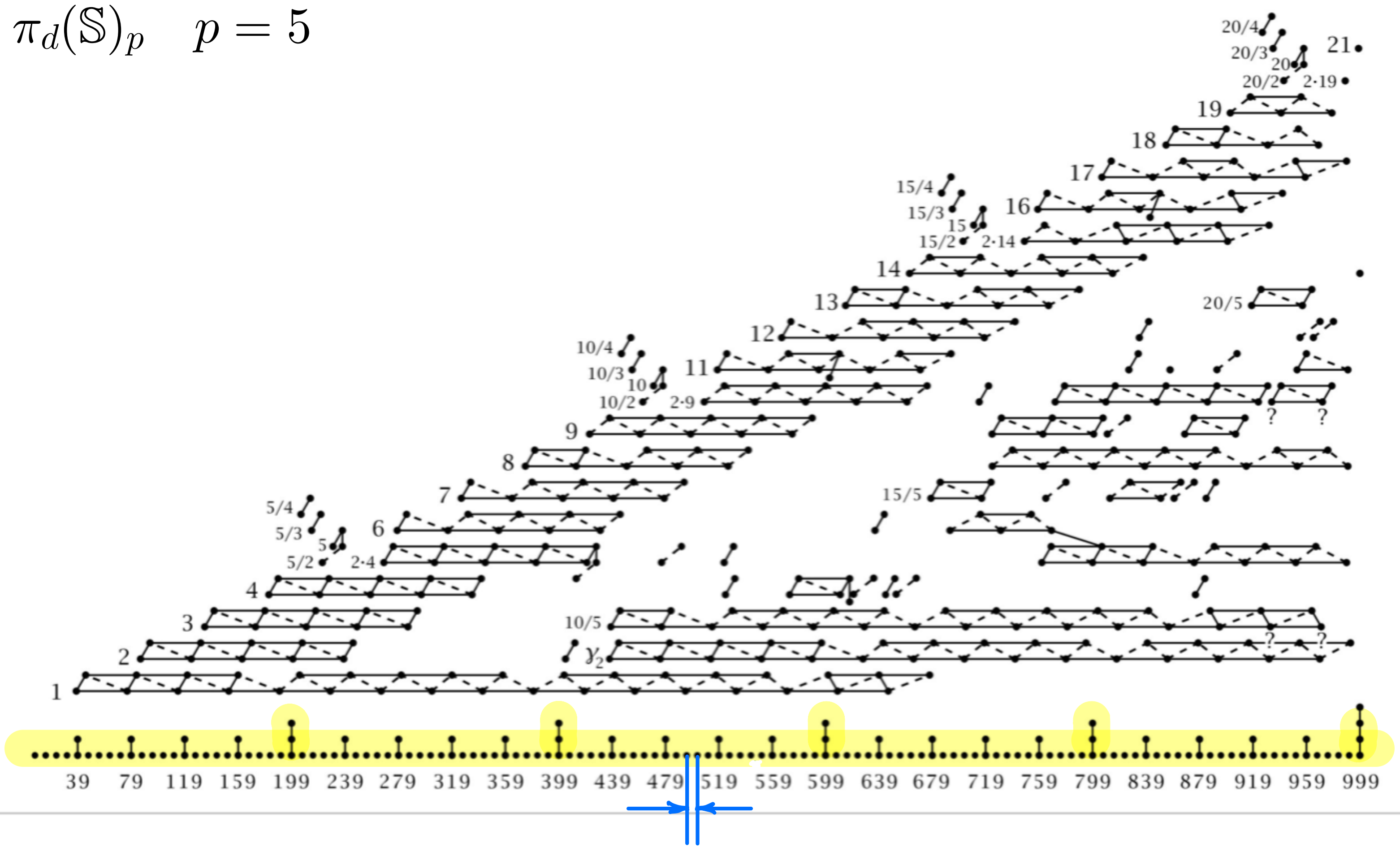


Picture: Hatcher
Computation: Ravenel

$\pi_d(\mathbb{S})_p \quad p = 5$



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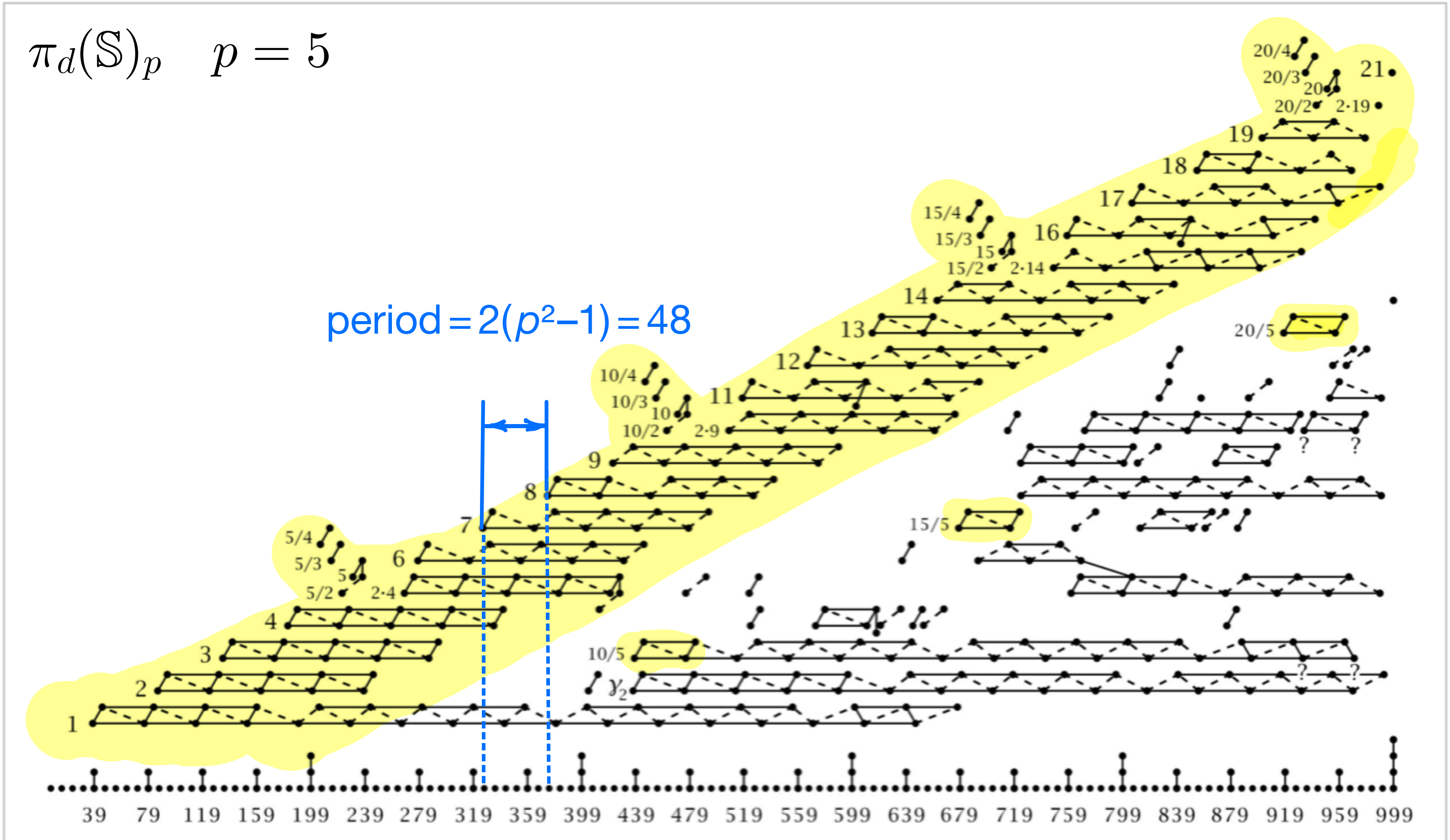
v_1-periodic elements



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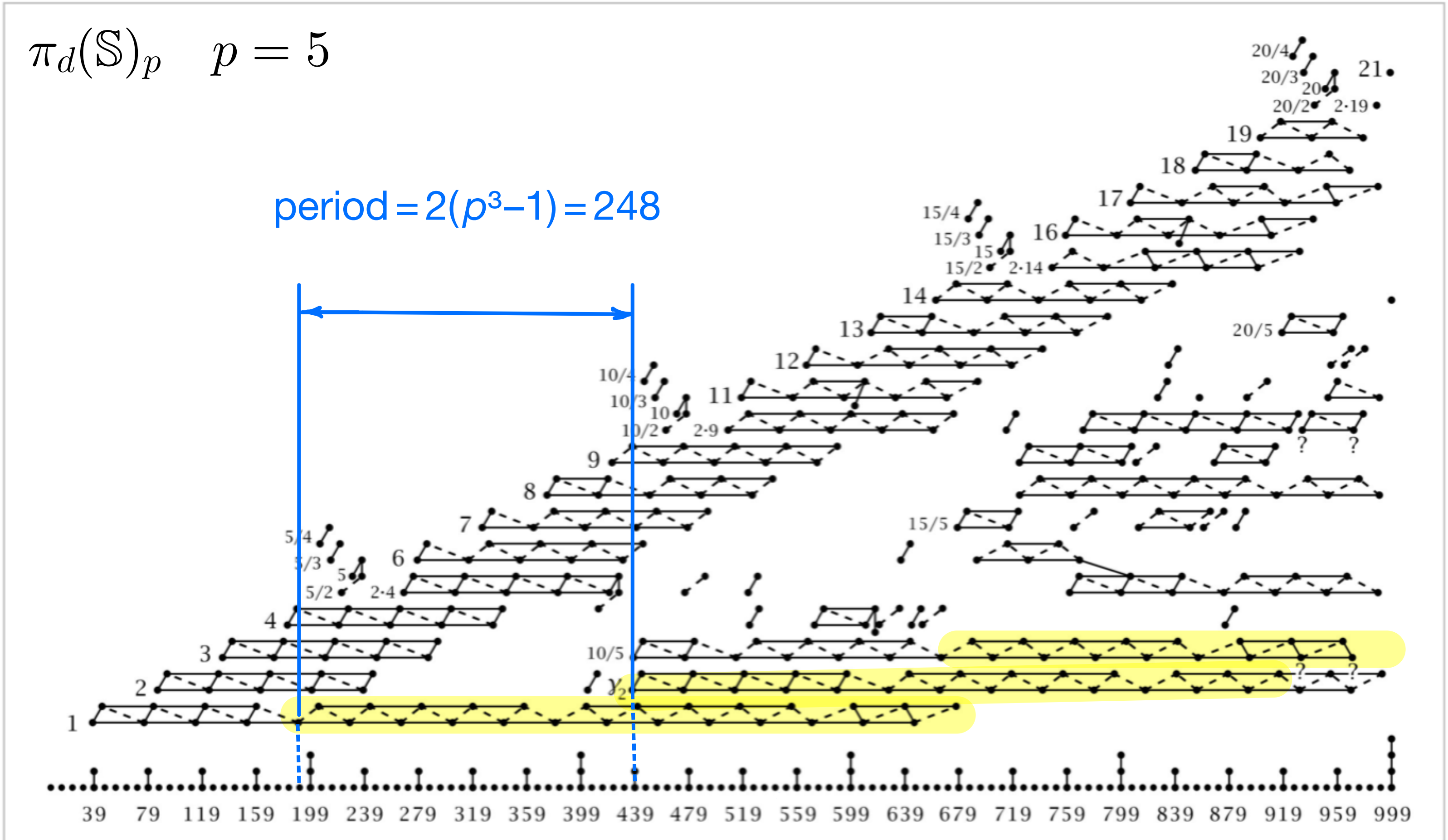
v_2-periodic elements



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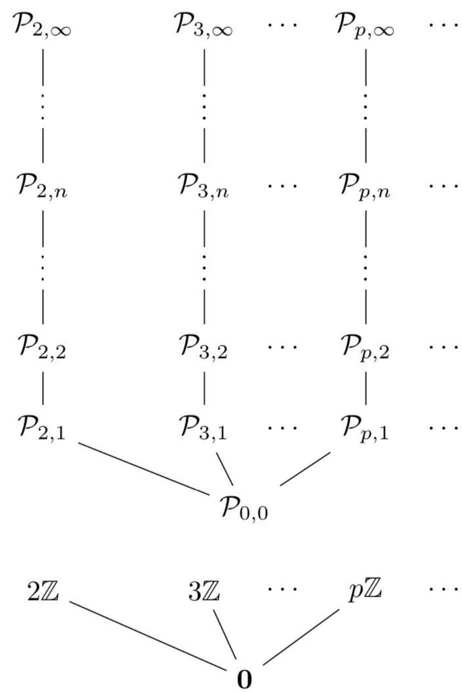

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$$\mathbb{Q} \otimes \pi_* L_{K(n)} \mathbb{S} \cong \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \dots, \zeta_n)$$

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[Goerss–Hopkins–Miller, Strickland '98] These moduli spaces are (partially) spectrally realized by the Morava E-theory spectrum $E = E_n(G/k)$, i.e.,

$$A_r \cong E^0(B\Sigma_{p^r})/I_{\mathrm{tr}}$$

[Fargues '08, Weinstein '15] As a family of spaces indexed by r, they stack into a tower with all levels literally:

$$\begin{tikzcd}
\mathcal{LT}_\infty \arrow[d] & \text{Perfectoid!} \\
\mathcal{LT}_r \arrow[d] & \\
\mathcal{LT}_0 = \coprod_{\mathbb{Z}} \mathring{\mathbb{B}}^{n-1} \arrow[d, "\substack{\text{Gross–Hopkins}\\ \text{period}}"'] & \\
\mathbb{P}^{n-1} &
\end{tikzcd}$$

Analytic geometry and homotopy groups of the $K(n)$-local sphere

Recall moduli spaces for deformations of a formal group with level structures.

Fix a formal group G/k. Its level-$\Gamma_0(p^r)$ deformations are classified:

- [Lubin–Tate '66] for $r = 0$, by an affine formal scheme Spf A_0 of dimension $n - 1$ over the Witt ring $\mathcal{O}_F$ of k (with F = max. unrf. ext. of p-adic comp. of k).
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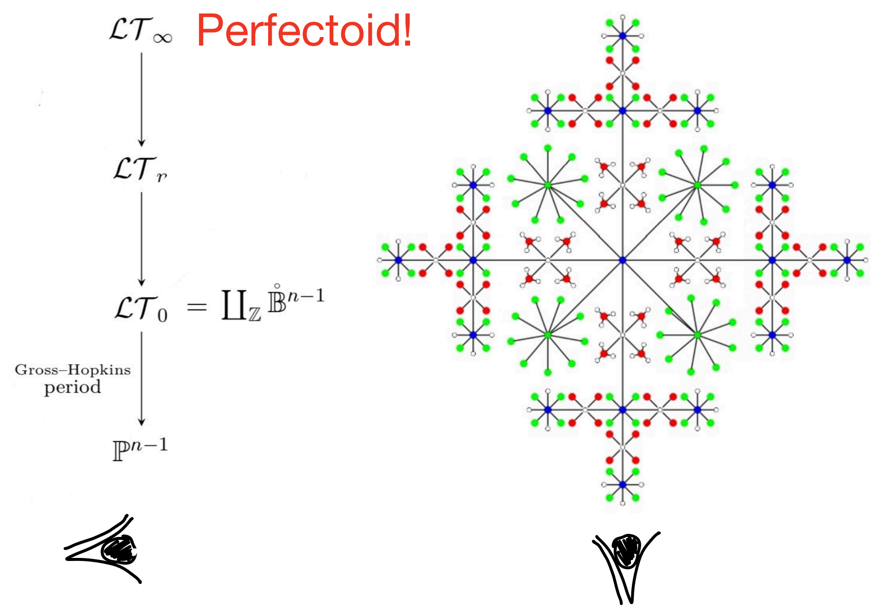

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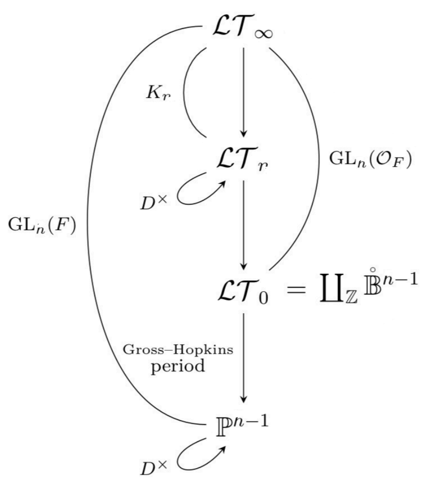

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- [Faltings, Fargues ’08, Scholze–Weinstein ’13] There is an equivariant isomorphism between the Lubin–Tate tower and another Drinfeld tower (parametrizing deformations of shtukas).

$$
\begin{tikzcd}
\mathcal{LT}_\infty \arrow[rrr, dashed, shift left] \arrow[d] \arrow[d, bend right=60, dash, "K_r"'] \arrow[dd, bend left=40, dash, "\mathrm{GL}_n(\mathcal{O}_F)"] \arrow[ddd, bend right=70, dash, "\mathrm{GL}_n(F)"'] \arrow[dddrrr, dashed] & & & \mathcal{D}r_\infty \arrow[lll, dashed, shift left] \arrow[d] \arrow[d, bend left=60, dash, "K_r"] \arrow[dd, bend right=40, dash, "\mathcal{O}_D^\times"'] \arrow[ddd, bend left=70, dash, "D^\times"] \arrow[dddlll, dashed] \\
\mathcal{LT}_r \arrow[d] \arrow[loop left, "D^\times"] & & & \mathcal{D}r_r \arrow[d] \arrow[loop right, "\mathrm{GL}_n(F)"] \\
\mathcal{LT}_0 \arrow[d, "\text{Gross–Hopkins period}"'] & = \coprod_{\mathbb{Z}} \mathring{\mathbb{B}}^{n-1} & \coprod_{\mathbb{Z}} \Omega = & \mathcal{D}r_0 \arrow[d, "\text{Period}"] \\
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Crucial to their work, Barthel, Schlank, Stapleton, and Weinstein leveraged these dual towers

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In view of a $\mathbb{G}_n$-equivariant splitting $\pi_0(E_n) \cong W \oplus A^c$, they computed

$$H^*\Big(R\Gamma(\mathcal{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h\mathcal{O}_D^\times}\Big) \otimes_W K \cong \Lambda_K(x_1, x_3, \ldots, x_{2n-1})[\epsilon] \oplus \big((A^c)^{h\mathcal{O}_D^\times} \otimes_W K\big)[\epsilon]$$

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to conclude that $H^*_{\text{cts}}(\mathbb{G}_n; A^c)$ is p-power torsion. This is a key step.

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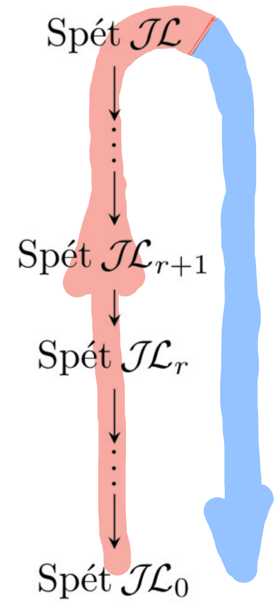

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$$E_n^{\mathrm{L}}(-) := \mathrm{Fun}(-, \mathscr{JL}^{h\mathbb{G}_n}) \quad \text{with} \quad \mathscr{JL} := \lim \mathscr{JL}_r$$

which may be used to construct alternative homotopy-fixed-point spectral sequences for computations, as well as leverage more pro-étale cohomology computations through condensed mathematics.

$$\begin{array}{c} \mathrm{Sp\acute{e}t}\,\mathscr{JL} \\ \downarrow \\ \vdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathscr{JL}_{r+1} \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathscr{JL}_r \\ \downarrow \\ \vdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathscr{JL}_0 \end{array}$$

Our related works

[Wang–Z., in progress] Aimed at computing unstable chromatic homotopy groups of spheres, we have constructed a filtered, equivariant, quasi-syntomic sheaf of Koszul complexes over the Lubin–Tate tower.
Moreover, in this case, there are homotopy-fixed-point spectral sequences too.

[Xuecai Ma '24] Ma (fully) spectrally realized finite levels of the Lubin–Tate tower as non-even commutative ring spectra, which generalize the Morava E-theory spectrum (due to Goerss–Hopkins–Miller, Lurie) at the ground level, in the framework of spectral algebraic geometry.
Moreover, he has defined a dual Morava E-theory

$$E_n^{\mathrm{L}}(-) := \mathrm{Fun}(-, \mathcal{JL}^{h\mathbb{G}_n}) \quad \text{with} \quad \mathcal{JL} := \lim \mathcal{JL}_r$$

which may be used to construct alternative homotopy-fixed-point spectral sequences for computations, as well as leverage more pro-étale cohomology computations through condensed mathematics.

$$\begin{array}{c} \mathrm{Sp\acute{e}t}\,\mathcal{JL} \\ \downarrow \\ \vdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathcal{JL}_{r+1} \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathcal{JL}_r \\ \downarrow \\ \vdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathcal{JL}_0 \end{array}$$

[Hongxiang Zhao '23] Zhao gave explicit evidence that, along Lubin–Tate towers, certain norm operators from local class field theory have spectral realizations.

Thank you.