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We use the theory of barcodes as a new tool for studying dynamics of area-preserving homeomorphisms. We will show that the barcode of a Hamiltonian diffeomorphism of a surface depends continuously on the diffeomorphism, and furthermore define barcodes for Hamiltonian homeomorphisms.

Our main dynamical application concerns the notion of *weak conjugacy*, an equivalence relation which arises naturally in connection to C^0 continuous conjugacy invariants of Hamiltonian homeomorphisms. We show that for a large class of Hamiltonian homeomorphisms with a finite number of fixed points, the number of fixed points, counted with multiplicity, is a weak conjugacy invariant. The proof relies, in addition to the theory of barcodes, on techniques from surface dynamics such as Le Calvez's theory of transverse foliations.

In our exposition of barcodes and persistence modules, we present a proof of the isometry theorem which incorporates Barannikov's theory of simple Morse complexes.

37E30, 53D05, 53D40

1 Introduction and main results

Our goal in writing this paper is to use a new set of tools to study dynamics of area-preserving homeomorphisms. Floer homology has played an important role in studying

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dynamical features of Hamiltonian diffeomorphisms. However, it is not well defined for nonsmooth objects such as area-preserving homeomorphisms. As we will see in this article, *barcodes* provide us with a medium through which one can define and effectively apply Floer theory for studying dynamics of area-preserving homeomorphisms.

A barcode $\boldsymbol{B} = \{I_j\}_{j \in \mathbb{N}}$ is a countable collection of intervals (or bars) of the form $I_j = (a_j, b_j]$ with $a_j \in \mathbb{R}$ and $b_j \in \mathbb{R} \cup \{+\infty\}$ which satisfy certain finiteness assumptions. Using Hamiltonian Floer theory one can associate, in a canonical manner, a barcode $\boldsymbol{B}(\phi)$ to every Hamiltonian diffeomorphism ϕ which encodes a significant amount of information about the Floer homology of ϕ: it completely characterizes the filtered Floer complex associated to ϕ up to quasi-isomorphism, and hence it subsumes all filtered Floer-theoretic invariants of ϕ. Barcodes have recently found several interesting applications in PDEs, under the guise of the Barannikov complex (see Le Peutrec, Nier and Viterbo [40]), and also in symplectic topology, following the appearance of the article of Polterovich and Shelukhin [56]. We refer the reader to the recent book [55] of Polterovich, Rosen, Samvelyan and Zhang for a thorough introduction to the subject and applications to geometry and analysis.

We will show that the barcode $\boldsymbol{B}(\phi)$ depends continuously, with respect to the uniform topology, on ϕ, and, moreover, $\boldsymbol{B}(\phi)$ is well defined even when ϕ is a Hamiltonian homeomorphism. Of course, when ϕ is a diffeomorphism, the barcode $\boldsymbol{B}(\phi)$ may be interpreted as the (filtered) Floer homology of ϕ.

The barcode of a Hamiltonian homeomorphism ϕ is defined via a limiting process and so the nature of the information it carries about the dynamics of ϕ is by no means clear. To extract dynamical information from $\boldsymbol{B}(\phi)$, namely about the fixed points of ϕ, we will use Le Calvez's theory of transverse foliations for dynamical systems on surfaces.

1.1 Rokhlin property and Hamiltonian homeomorphisms

Before presenting our results on barcodes, we will describe our main dynamical applications.

Let (Σ, ω) be a closed surface equipped with an area form ω and denote by $\overline{\mathrm{Ham}}(\Sigma, \omega)$ the C^0–closure of Hamiltonian diffeomorphisms of (Σ, ω). This is often referred to as the group of *Hamiltonian homeomorphisms* of (Σ, ω). Although much is known about groups of volume-preserving homeomorphisms in dimensions greater than two (see Fathi [12]), the algebraic structure of this group is shrouded in mystery; for example, it is not known whether it is simple or not and, in the case where $\Sigma = S^2$, we do not know if it admits any homogeneous quasimorphisms; see Oh and Müller [49], Le Roux [42] and Entov, Polterovich and Py [11]. Investigating questions of this nature, F Béguin, S Crovisier and the first author of this paper were led to ask if there exists $\theta \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ whose conjugacy class is dense in $\overline{\mathrm{Ham}}(\Sigma, \omega)$?

The question of whether a given topological group has dense conjugacy classes has been of interest in ergodic theory; see Glasner and Weiss [20; 21]. Glasner and Weiss, inspired by influential results of Halmos and Rokhlin, refer to groups with dense conjugacy classes as *Rokhlin* groups. Examples of Rokhlin groups include the group of measure-preserving automorphisms of a standard measure space equipped with the weak topology (see page 5 of [21]), the group of unitary operators of a separable, infinite-dimensional Hilbert space equipped with the strong topology, and the group of homeomorphisms that are isotopic to the identity for any even-dimensional sphere equipped with the topology of uniform convergence; see [21] for further details on these examples.

Despite the last example of the previous paragraph, it turns out that the symplectic nature of $\overline{\mathrm{Ham}}(\Sigma, \omega)$ forces a certain form of rigidity on its conjugacy classes; indeed, $\overline{\mathrm{Ham}}(\Sigma, \omega)$ is not Rokhlin. In the case of surfaces with genus greater than one, this is a consequence of Gambaudo and Ghys [16], who proved existence of continuous homogeneous quasimorphisms on $\overline{\mathrm{Ham}}(\Sigma, \omega)$. The case of the surface of genus one follows from a combination of results from Entov, Polterovich and Py [11] and Gambaudo and Ghys [16] (the latter constructs homogeneous quasimorphism on the group, while the former proves their continuity). In the case of S^2, the negative answer was provided by the second author in [60]. In each case, the proof involves constructing continuous conjugacy invariants. One cannot easily associate a dynamical interpretation to these invariants, particularly in the case of S^2, where the invariant is the so-called spectral norm, which is derived from Hamiltonian Floer theory and whose construction is rather complicated.

In this article, we introduce, via the theory of barcodes, a very simple invariant, whose definition involves a *weighted count of fixed points*, to separate closures of conjugacy classes. In fact, we will see that our weighted sum of fixed points, and more generally barcodes, are invariants of a natural equivalence relation which we will refer to as *weak conjugacy*; this is the strongest Hausdorff equivalence relation on $\overline{\mathrm{Ham}}(M, \omega)$ which is weaker than the conjugacy relation.

The weak conjugacy relation is characterized by the following universal property: f is weakly conjugate to g if and only if $\theta(f) = \theta(g)$ for any continuous function

$\theta\colon \overline{\mathrm{Ham}}(M,\omega) \to Y$ such that θ is invariant under conjugation and Y is a Hausdorff topological space. See Definition 51 in Section 5.1 for further details.

It is evident from the above characterization of weak conjugacy that if $\overline{\mathrm{Ham}}(M,\omega)$ possessed a dense conjugacy class then the weak conjugacy relation would be trivial, ie any f and g would be weakly conjugate. We remark that f and g are weakly conjugate if they satisfy the following criterion: there exist $h_1,\dots,h_N \in \overline{\mathrm{Ham}}(M,\omega)$ such that $h_1 = f$, $h_N = g$ and $\overline{\mathrm{Conj}}(h_i) \cap \overline{\mathrm{Conj}}(h_{i+1}) \neq \varnothing$; here $\overline{\mathrm{Conj}}$ stands for closure of conjugacy class.

Observe that any continuous conjugacy invariant is a weak conjugacy invariant as well. As a consequence, the notion of weak conjugacy arises naturally in settings where one needs to consider continuous conjugacy invariants. This is for example the case in the study of mapping class group actions on the circle; see the article by K Mann and M Wolff [44].

1.1.1 Weak conjugacy and the Lefschetz index In order to formulate our results, we will need to introduce a few notions.

Let x be an isolated fixed point of a homeomorphism f of a manifold of dimension k. Recall that the Poincaré–Lefschetz index of the fixed point x, which we will denote by $L(f,x)$, is defined as follows: Let U be a chart centered at x and denote by S a small sphere (of codimension 1 in U) which is centered at x as well, oriented as the boundary of its interior in U. For S sufficiently small, the formula

$$x \mapsto \frac{f(x)-x}{\|f(x)-x\|}$$

yields a well-defined map from S to the unit sphere in $\mathbb{R}^k$ and $L(f,x)$ is the degree of this map. For further details on this index, we refer the interested reader to Katok and Hasselblatt [30] and Hatcher [23].

Let (Σ,ω) be a closed and connected symplectic surface. Given $f \in \overline{\mathrm{Ham}}(\Sigma,\omega)$, we will denote the set of contractible fixed points of f by $\mathrm{Fix}_c(f)$; recall that a fixed point x of f is said to be contractible if there exists an isotopy I_t, where $t \in [0,1]$, $I_0 = \mathrm{Id}$, $I_1 = f$ and $I_t \in \overline{\mathrm{Ham}}(\Sigma,\omega)$, such that the loop $I_t(x)$ is contractible. This notion is well defined, ie does not depend on the choice of the isotopy, because the proof of the Arnold conjecture implies that, for a loop in $\mathrm{Ham}(M,\omega)$ based at the identity, all trajectories are contractible (in our situation, if $\Sigma \neq \mathbb{S}^2$, then $\overline{\mathrm{Ham}}(\Sigma,\omega)$ is simply connected; see Polterovich [54] for a proof).

We are now ready to state our first result:

Theorem 1 *Suppose that $f, g \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ are smooth and have finitely many contractible fixed points. If f and g are weakly conjugate, then*

$$\sum_{x \in \mathrm{Fix}_c(f)} |L(f, x)| = \sum_{x \in \mathrm{Fix}_c(g)} |L(g, x)|.$$

We will refer to $\sum_{x \in \mathrm{Fix}_c(f)} |L(f, x)|$ as the *absolute Lefschetz number* of f. A few remarks are in order. First, observe that one can immediately conclude from the above that $\overline{\mathrm{Ham}}(\Sigma, \omega)$ is not Rokhlin: As mentioned earlier, we must produce f and g which are not weakly conjugate. It is very easy to produce f and g with different absolute Lefschetz numbers, and hence they cannot be weakly conjugate, by just creating a pair of fixed points of index ± 1.

Second, we should mention that if f is nondegenerate, then every fixed point has index ± 1 and so the absolute Lefschetz number is just the total number of fixed points of f. However, the above result would not be true if we were to replace the absolute Lefschetz number by the total number of fixed points. Indeed, it is possible to produce f and g which are weakly conjugate but do not have the same number of fixed points.

Third, observe that if f and g are weakly conjugate, then so are f^p and g^p for every $p \in \mathbb{Z}$. Hence, if f and g have finitely many periodic points of period p, then the absolute Lefschetz numbers of f^p and g^p coincide as well.

Finally, we remark that our proof of Theorem 1 yields the following refinement: We denote by $\mathrm{Spec}(f)$ the set of action values of fixed points of a given Hamiltonian diffeomorphism f. Recall that $\mathrm{Spec}(f)$ is well defined up to a shift. Now, suppose that $f \in \mathrm{Ham}(M, \omega)$ has finitely many fixed points. For every value $a \in \mathrm{Spec}(f)$ define $L(f, a) := \sum |L(f, x)|$, where the sum is taken over all $x \in \mathrm{Fix}_c(f)$ whose action is a. Define $\widetilde{\mathrm{Spec}}(f) := \{(a, L(f, a)) : a \in \mathrm{Spec}(f)\}$. If f and g are as in the statement of Theorem 1, then $\widetilde{\mathrm{Spec}}(f) = \widetilde{\mathrm{Spec}}(g)$. In fact, one can even refine this latter statement to take into account Conley–Zehnder indices of fixed points with a given action. These statements are immediate consequences of Theorem 54 and Proposition 58.

Our next result is a generalization of Theorem 1 to the setting where f and g are not assumed to be smooth. As will be explained in the next section, the removal of the smoothness assumption gives rise to significant complications. Our strategy for dealing with these complications passes through the following notion: Consider $f \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ with finitely many fixed points. We say that f is *smoothable* if there exists a sequence $f_i \in \mathrm{Ham}(\Sigma, \omega)$ which converges uniformly to f and such that $\mathrm{Fix}_c(f_i) = \mathrm{Fix}_c(f)$.

Theorem 2 *Suppose that $f, g \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ are smoothable and have finitely many contractible fixed points. If f and g are weakly conjugate, then*

$$\sum_{x \in \mathrm{Fix}_c(f)} |L(f, x)| = \sum_{x \in \mathrm{Fix}_c(g)} |L(g, x)|.$$

We conjecture that every $f \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ with finitely many fixed points is smoothable, and we verify this for a large class of homeomorphisms. It follows from our results that a nonsmoothable f (which conjecturally does not exist) would be dynamically very complicated near its (contractible) fixed-point set. In Section 7, we establish a precise criterion for smoothability which, in particular, implies the following statement:

Theorem 3 *Suppose that $f \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ has a finite number of contractible fixed points. Assume that there does not exist $x \in \mathrm{Fix}_c(f)$ which is accumulated by periodic orbits of every period. Then f is smoothable.*

The above is an immediate consequence of Theorem 77.

1.1.2 A homeomorphism which is not weakly conjugate to any diffeomorphism

It is well known that homeomorphisms could be dynamically more complicated than diffeomorphisms. The result below on the weak conjugacy relation tells us that barcodes are capable of detecting the wilder dynamics of homeomorphisms.

Theorem 4 *There exists $f \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ which is not weakly conjugate to any Hamiltonian diffeomorphism. In particular, the closure of the conjugacy class of f contains no Hamiltonian diffeomorphisms.*

1.2 Barcodes as invariants of Hamiltonian homeomorphisms

We will now explain how barcodes enter our story by presenting a brief outline of the proofs of the results mentioned in the previous section. Here, in order to avoid the complications which arise in the case of $\mathbb{S}^2$, we will focus on surfaces of positive genus.

As we will see in Section 3, using Hamiltonian Floer theory, one can associate a barcode $\boldsymbol{B}(H)$ to every Hamiltonian $H \in C^\infty(\mathbb{S}^1 \times \Sigma)$.[1] It turns out that if H and G are two Hamiltonians the time-1 maps of whose flows coincide, then there exists a constant c such that $\boldsymbol{B}(G) = \boldsymbol{B}(H) + c$, where $\boldsymbol{B}(H) + c$ is the barcode obtained from $\boldsymbol{B}(H)$ by shifting each of the bars in $\boldsymbol{B}(H)$ by c. This implies that we have a well-defined map $\boldsymbol{B} : \mathrm{Ham}(\Sigma, \omega) \to \hat{\mathcal{B}}$, where $\hat{\mathcal{B}}$ denotes the space of barcodes modulo the equivalence relation $\boldsymbol{B}_1 \sim \boldsymbol{B}_2$ if $\boldsymbol{B}_2 = \boldsymbol{B}_1 + c$ for some constant c.

Now, $\hat{\mathcal{B}}$ may be equipped with a natural distance d_{bot} which is called the bottleneck distance. Consider the mapping of metric spaces

$$\boldsymbol{B} : (\mathrm{Ham}(\Sigma, \omega), d_{C^0}) \to (\hat{\mathcal{B}}, d_{\mathrm{bot}}),$$

where d_{C^0} denotes the C^0–distance. We prove in Theorem 34 and Remark 35 that the above mapping is continuous and, moreover, extends continuously to $\overline{\mathrm{Ham}}(\Sigma, \omega)$. This is what allows us to associate barcodes to Hamiltonian homeomorphisms. This is generalized to aspherical manifolds in Buhovsky, Humilière and Seyfaddini [4].

It follows from the aforementioned continuity of barcodes, and standard properties of Hamiltonian Floer theory, that if $f, g \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ are weakly conjugate, then $\boldsymbol{B}(f) = \boldsymbol{B}(g)$; see Theorem 54. Clearly, to prove Theorems 1 and 2 it is sufficient to show that the absolute Lefschetz number of $f \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ is an invariant of its barcode $\boldsymbol{B}(f)$. This is achieved in Theorems 55 and 72: When f is smooth, we use local Floer homology to show that the absolute Lefschetz number of f is simply the total number of endpoints (counted with multiplicity) of the bars of $\boldsymbol{B}(f)$. We show that the same conclusion continues to hold when f is not smooth but is smoothable; we do so by proving that for f smoothable, one can find smooth functions f_n converging to f such that $\boldsymbol{B}(f_n) = \boldsymbol{B}(f)$. This will then allow us to prove Theorem 2 as a consequence of Theorem 1. Without the assumption of smoothability, we have little control over $\boldsymbol{B}(f_n)$ and so cannot say much about $\boldsymbol{B}(f)$.

We end this part of the introduction by giving a sketch of the proof of Theorem 4. As we will see, if h is a Hamiltonian diffeomorphism, then the endpoints of the bars in $\boldsymbol{B}(h)$ correspond to actions of fixed points of h. Now, the set of actions of fixed points of a Hamiltonian diffeomorphism (on a closed surface other than $\mathbb{S}^2$) is always compact. To prove the theorem, we simply produce a Hamiltonian homeomorphism f with the property that the set of endpoints of its barcode $\boldsymbol{B}(f)$ is not bounded.

[1] In fact, Hamiltonian Floer theory may be used to construct barcodes on symplectic manifolds far more general than surfaces; see Usher and Zhang [67].

1.3 Smoothability, transverse foliations and local dynamics

We will end the introduction with a few words on the proof of Theorem 3. This proof relies on techniques from surface dynamics, namely Le Calvez's theory of transverse foliations for dynamical systems of surfaces, as described in [37], and the notion of local rotation set for an isolated fixed point, which was introduced by the third author in [43].

The local rotation set is a topological conjugacy invariant associated to a germ of an orientation-preserving surface homeomorphism near a fixed point. It is a closed interval of $[-\infty, +\infty]$, defined modulo translation by integers, which captures the amount of asymptotic rotation of orbits around the fixed point. Furthermore, when the fixed point p is isolated and the homeomorphism f is area-preserving, the local rotation set is a subset of $[0,1]$. The precise content of Theorem 77, which implies Theorem 3, is the following: if the local rotation set is a *proper* subset of $[0,1]$, then there is an area-preserving homeomorphism g that is smooth near p, coincides with f outside some small neighborhood of p, and has the same fixed-point set as f.

The proof of the theorem is divided into two cases. It is known that the Lefschetz index $L(f,p)$ of an isolated fixed point for an area-preserving homeomorphism is less than or equal to 1. When $L(f,p) < 1$, we show that, after a first small perturbation, there exists a small disk D containing p which is in *canonical position* for f, which mainly means that $f(D) \cap D$ is connected. This first step makes a fundamental use of Le Calvez's transverse foliations. The canonical position is then used to approximate f by surgery, replacing the restriction of f to D by a smooth model. This approach is inspired by the technique designed by Schmitt and used by Slaminka to remove index 0 fixed points (see [59; 65], even though some arguments in these papers are rather hard to follow). When $L(f,p) = 1$, we use a different strategy. In this case, Le Calvez's transverse foliation provides a coordinates system for which the "θ" polar coordinate is essentially increasing along every trajectory of some isotopy from the identity to f. Assuming the local rotation set is included in $[0,1)$, we refine Le Calvez's result, finding a coordinates system in which, in addition, at least one ray $\theta =$ constant is mapped by f to another ray. This is the content of the "iterated leaf lemma" (Lemma 98 below), which is of independent interest. This property allows us to make a series of perturbations by pushing points in the direction positively transverse to the rays, adding no new fixed points and ending up with a map that admits a periodic ray of period 2. This map is finally (and easily) smoothed into a map that is a 2–periodic rotation near p.

Organization of the paper

In Section 2, we present a brief review of persistence modules, Barannikov complexes, barcodes and the metrics which are naturally associated to them. In Section 3, after presenting a brief review of Hamiltonian Floer theory, we explain how one can associate barcodes to Hamiltonian diffeomorphisms via Hamiltonian Floer homology. In Section 4 we prove our main results on C^0 continuity of barcodes. Section 5 contains the proofs of Theorems 1 and 2. Theorem 4 is proven in Section 6.

In Section 7, we state and prove a more general version of Theorem 3; see Theorem 77. This section contains a brief review of the necessary background from dynamical systems such as Le Calvez's theory of transverse foliations and the notion of local rotation set for isolated fixed points of area-preserving homeomorphisms.

Acknowledgments

Subsequent to the announcement of the results of this article, Theorem 34 on continuity of barcodes was generalized to higher-dimensional aspherical symplectic manifolds by Buhovsky, Humilière and the second author (see [4]). Their methods are very different than ours. Thus, Theorem 1 generalizes to the aspherical situation, but with $|L(f,x)|$ replaced by $r(f,x)$, the rank of the local Floer homology of x.

Continuity of barcodes could also be proven by combining the results of the articles by Shelukhin [62] and Seyfaddini [60]; see Kislev and Shelukhin [31, Corollary 6]. This fact was brought to the attention of the second author by Egor Shelukhin in the course of private communication. This is the strategy used in [4].

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2 Preliminaries on persistence modules, Barannikov complexes and barcodes

In this section, we introduce persistence modules, barcodes and Barannikov modules and the natural metrics associated to each of these objects. The main goal here is to prove that the above objects define categories with isomorphic objects and that the natural correspondences between them are isometries; see Proposition 23.

Persistence modules and barcodes have been studied extensively in the topological data analysis community; see [8; 5]. The isometric correspondence between barcodes and persistence modules is well known and is referred to as the isometry theorem. What seems to be less well known is the correspondence between barcodes and persistence homology, on the one hand, and Barannikov's simple Morse complex [1], on the other hand. This aspect of Proposition 23 is of rather folkloric nature and some features of it have already been exploited in the symplectic community; see for example [58; 40; 67].

We should add that our presentation of Barannikov's work is more closely aligned with the presentation in [40] (see also [33]) rather than Barannikov's original article [1].

For the rest of this section, we consider vector spaces over a fixed field $\mathbb{F}$. It is important to keep the same field all along as some Morse/Floer homological constructions of Section 3 depend on the choice of $\mathbb{F}$.

2.1 Persistence modules

We begin by defining persistence modules, their morphisms and the interleaving distance.

Definition 5 A *persistence module* V is a family $(V_s)_{s\in\mathbb{R}}$ of finite-dimensional vector spaces equipped with morphisms $i_{s,t}\colon V_s \to V_t$ for $s \leqslant t$ satisfying:

(1) For all $s\in\mathbb{R}$ we have $i_{s,s}=\mathrm{Id}$ and for every $s\leqslant t\leqslant u$ we have $i_{t,u}\circ i_{s,t}=i_{s,u}$.

(2) There exists a finite subset $F\subset\mathbb{R}$, often referred to as the spectrum of V, such that $i_{s,t}$ is an isomorphism whenever s and t belong to the same connected component of $\mathbb{R}\setminus F$.

(3) For all $t\in\mathbb{R}$, $\varinjlim_{s<t} V_s = V_t$; equivalently, for fixed t, $i_{s,t}$ is an isomorphism for $s<t$ sufficiently close to t,

(4) $V_s=\{0\}$ for $s\ll 0$.

The persistence module will be written as $(V_s, i_{s,t})$ or just V_s if the $i_{s,t}$ are implicit.

Here are two examples of persistence modules: First, consider an interval I of the form $(a,b]$ and define $Q_s(I)=\mathbb{F}$ if $s\in I$, and $Q_s(I)=\{0\}$ if $s\notin I$. Then $Q_s(I)$ is a persistence module, with $i_{s,t}$ equal to Id if $s,t\in I$ and 0 otherwise. Second, let $f\colon M\to\mathbb{R}$ be a continuous function on a closed manifold M. The family of vector spaces $V_s := H_*(M^s)$, the singular homology of $M^s := \{x \in M : f(x) < s\}$, is a persistence module, where the $i_{s,t}$ are induced by the inclusion of M^s into M^t.

Definition 6 We denote by $\mathcal{P}$ the category of persistence modules. Let $(V_t, i_{s,t})$ and $(W_t, j_{s,t})$ be objects in $\mathcal{P}$. An element in $\mathrm{Mor}(\boldsymbol{V}, \boldsymbol{W})$ is given by a family $u_s \colon V_s \to W_s$ commuting with $i_{s,t}$ and $j_{s,t}$:

$$\begin{tikzcd} V_s \arrow[r, "u_s"] \arrow[d, "i_{s,t}"] & W_s \arrow[d, "j_{s,t}"] \\ V_t \arrow[r, "u_t"] & W_t \end{tikzcd}$$

If $\boldsymbol{V}$ is a persistence module, we denote by $\tau_a \boldsymbol{V}$ the persistence module defined by $(\tau_a V)_s = V_{a+s}$. For $a > 0$, we have a natural morphism $i_a \colon \boldsymbol{V} \to \tau_a \boldsymbol{V}$ induced by $i_{s,s+a}$. We similarly have $j_a \colon \boldsymbol{W} \to \tau_a \boldsymbol{W}$ for $a > 0$.

The set of persistence modules can be equipped with the so-called interleaving pseudodistance.

Definition 7 Let $\boldsymbol{V} = (V_s)_{s\in\mathbb{R}}$ and $\boldsymbol{W} = (W_s)_{s\in\mathbb{R}}$ be two persistence modules. The pseudodistance $d_{\mathrm{int}}(\boldsymbol{V}, \boldsymbol{W})$, called the *interleaving* distance, is defined as the infimum of the set of positive ε such that there are morphisms $\varphi_s \colon V_s \to W_{s+\varepsilon}$ and $\psi_s \colon W_s \to V_{s+\varepsilon}$ compatible with the $i_{s,t}$ and $j_{s,t}$ as in

$$\begin{tikzcd} V_{s-\varepsilon} \arrow[r, "\varphi_{s-\varepsilon}"] \arrow[d, "i_{s-\varepsilon,t-\varepsilon}"] & W_s \arrow[r, "\psi_s"] \arrow[d, "j_{s,t}"] & V_{s+\varepsilon} \arrow[r, "\varphi_{s+\varepsilon}"] \arrow[d, "i_{s+\varepsilon,t+\varepsilon}"] & W_{s+2\varepsilon} \arrow[d, "j_{s+2\varepsilon,t+2\varepsilon}"] \\ V_{t-\varepsilon} \arrow[r, "\varphi_{t-\varepsilon}"] & W_t \arrow[r, "\psi_t"] & V_{t+\varepsilon} \arrow[r, "\varphi_{t+\varepsilon}"] & W_{t+2\varepsilon} \end{tikzcd}$$

such that $\psi_s \circ \varphi_{s-\varepsilon} = i_{s-\varepsilon,s+\varepsilon}$ and $\varphi_{s+\varepsilon} \circ \psi_s = j_{s,s+2\varepsilon}$. In other words, $\varphi \colon \boldsymbol{V} \to \tau_\varepsilon \boldsymbol{W}$ and $\psi \colon \boldsymbol{W} \to \tau_\varepsilon \boldsymbol{V}$ are such that $\varphi \circ \tau_\varepsilon \psi = j_{2\varepsilon}$ and $\psi \circ \tau_\varepsilon \varphi = i_{2\varepsilon}$.

Note that the pseudodistance takes value in $[0, +\infty]$.

Proposition 8 (see [5]) *The interleaving distance defines a pseudometric on the set of persistence modules. Furthermore, two persistence modules at distance zero are isomorphic.*

Proof We leave the proof of the fact that d_{int} is a pseudodistance as an exercise for the reader and will only sketch the proof of the second statement.

Let us assume that $d_{\mathrm{int}}(\boldsymbol{V}, \boldsymbol{W}) = 0$. Thus, we have morphisms $\varphi_\varepsilon \colon \boldsymbol{V} \to \tau_\varepsilon \boldsymbol{W}$ and $\psi_\varepsilon \colon \boldsymbol{W} \to \tau_\varepsilon \boldsymbol{V}$ such that $\varphi_\varepsilon \circ \tau_\varepsilon \psi_\varepsilon = j_{2\varepsilon}$ and $\psi_\varepsilon \circ \tau_\varepsilon \varphi_\varepsilon = i_{2\varepsilon}$, where $\varepsilon > 0$ may be picked to be as small as one wishes. We must show that $\boldsymbol{V} \simeq \boldsymbol{W}$. The idea is simple: as ε converges to zero, $i_{2\varepsilon}$ and $j_{2\varepsilon}$ "converge" to the identity morphism, and so we would like to show that φ_ε and ψ_ε "converge" to φ_0 and ψ_0 and get that $\varphi_0 \circ \psi_0 = \psi_0 \circ \varphi_0 = \mathrm{id}$.

Fix any $s \in \mathbb{R}$ which is neither in the spectrum of $\boldsymbol{V}$ nor in the spectrum of $\boldsymbol{W}$. For ε small enough, the vector spaces $W_{s+\varepsilon}$ and $V_{s+\varepsilon}$ are stationary in the sense that $W_s \simeq W_{s+\varepsilon}$ and $V_s \simeq V_{s+\varepsilon}$. Thus, we can identify $\varphi_\varepsilon\colon V_s \to W_{s+\varepsilon}$ and $\psi_\varepsilon\colon W_s \to V_{s+\varepsilon}$ to maps $\varphi_0\colon V_s \to W_s$ and $\psi_0\colon W_s \to V_s$, respectively. It is not difficult to see that $\varphi_0 \circ \psi_0 = \psi_0 \circ \varphi_0 = \mathrm{id}$. One can further check, since $V_s = V_{s-\varepsilon}$ for s in the spectrum and ε small enough, that for s in the spectrum of $\boldsymbol{V}$, or $\boldsymbol{W}$, one can still define $\varphi_0\colon V_s \to W_s$ and $\psi_0\colon W_s \to V_s$ and that the two maps continue to be inverse to one another. □

Note that for any $t \in \mathbb{R}$, there exists $\varepsilon > 0$ such that $i_{s,u}\colon V_s \to V_u$ is an isomorphism if $s, u \in (t-\varepsilon, t]$ or if $s, u \in (t, t+\varepsilon)$. Pick $t^- \in (t-\varepsilon, t]$ and $t^+ \in (t, t+\varepsilon)$ and let $j(t) = \dim(\mathrm{Ker}(i_{t^-,t^+})) + \mathrm{codim}(\mathrm{Im}(i_{t^-,t^+}))$. Observe that $j(t)$ is zero except for $t \in F$. We say that $\boldsymbol{V}$ is *generic* if $j(t) \leqslant 1$ for all $t \in \mathbb{R}$.

Proposition 9 *Generic persistence modules are dense for the interleaving distance.*

Proof Because singular values are isolated, we can assume $j(t_0) = d$ and t_0 is the only point in the spectrum of $\boldsymbol{V}$. For $s < t_0$ and close enough to t_0, we have an isomorphism $V_s \to V_{t_0}$ and, for $t > t_0$ close to t_0, $V_{t_0} \to V_t$ has kernel K_{t_0} and cokernel C_t. Let $u_1^t, \dots, u_d^t$ be a basis of C_t and W_{t_0} a complementary subspace to K_{t_0}. Set $W_t = i_{t_0,t}(W_{t_0})$ and note that W_t is a complement of C_t in V_t and these are all isomorphic. Since $V_t \to V_{t'}$ is an isomorphism for $t_0 < t < t'$, we can assume $i_{t,t'} u_j^t = u_j^{t'}$.

Similarly, let $v_1, \dots, v_r$ be a basis of K_{t_0}. Let $s_r < s_{r-1} < \dots < s_1 < s_0 = t_0 < t_1 < \dots < t_d$ be any real numbers close enough to t_0. Define:

(1) For $s \in (s_j, s_{j-1}]$,
$$V'_s = W_{t_0} \oplus k v_1 \oplus \dots \oplus k v_j.$$
In particular, for $s < s_r$, we have $V'_s = W_{t_0} \oplus C_{t_0} = V_{t_0}$.

(2) For $t \in (t_j, t_{j+1}]$,
$$V'_t = W_{t_0} \oplus k u_1^t \oplus \dots \oplus k v_j^t.$$
In particular, for $t > t_d$, we have $V'_t = W_{t_0} \oplus C_t = V_t$.

(3) For $s < s' \leqslant t_0$, $i'_{s,s'}$ is the obvious inclusion map.

(4) For $t_0 < t < t'$, the map $i'_{t,t'}$ is the obvious projection map.

(5) $i'_{s,t} = i'_{t_0,t} \circ i'_{s,t_0}$ for $s < t_0 < t$.

It is easy to check that $V' = (V'_t, i'_{s,t})$ defines a new persistence module with spectrum equal to the spectrum of V union the set of s_j and t_ℓ for $1 \leqslant j \leqslant r$ and $1 \leqslant \ell \leqslant d$. It is also easy to check that $j(s_j) = j(t_\ell) = 1$. Now we claim that if $t_0 - \varepsilon < s_r < t_d < t_0 + \varepsilon$, then we have $d_{\mathrm{int}}(V, V') \leqslant \varepsilon$. For this we just have to define maps $\varphi\colon V \to \tau_\varepsilon V'$ and $\varphi\colon V' \to \tau_\varepsilon V$. Now, for $s < t_0 - 2\varepsilon$ or $t > t_0 + 2\varepsilon$, we have $V'_s = V_s$ and $V_{s\pm\varepsilon} = V'_{s\pm\varepsilon}$, so we define φ and ψ as i_ε and then $\varphi \circ \psi = \psi \circ \varphi = i_{2\varepsilon}$. Otherwise:

(1) For $t_0 - 2\varepsilon < s < t_0$, we set

- $\varphi_s\colon V_s = W_s \oplus kv_1 \oplus \cdots \oplus kv_r \to V'_{s+\varepsilon}$ is the projection on
$$W_{s+\varepsilon} \oplus kv_1 \oplus \cdots \oplus kv_j$$
if $s_j < s + \varepsilon < s_{j-1} < t_0$ and the projection[2] on $W_{s+\varepsilon}$ if $s + \varepsilon \geqslant t_0$.

- $\psi_s\colon V'_s \to V_{s+\varepsilon}$ is the inclusion of $W_s \oplus kv_1 \oplus \cdots \oplus kv_j$ into
$$W_{s+\varepsilon} \oplus kv_1 \oplus \cdots \oplus kv_r = V_{s+\varepsilon}$$
if $s + \varepsilon < t_0$ and the composition of the projection on $W_{s+\varepsilon}$ and the inclusion of $W_s \simeq W_{s+\varepsilon}$ into $V_{s+\varepsilon}$ if $s + \varepsilon \geqslant t_0$.

(2) For $t_0 \leqslant t \leqslant t_0 + 2\varepsilon$, we set
$$\varphi_t\colon V_t = V_{t_0} = W_{t_0} \oplus C_{t_0} \to V'_{t+\varepsilon} = W_{t+\varepsilon} \oplus ku_1^{t+\varepsilon} \oplus \cdots \oplus kv_j^{t+\varepsilon}$$
to be the projection, that is, for $t_j < t + \varepsilon \leqslant t_{j+1}$, and
$$\begin{aligned}\psi_t\colon V'_t = W_t \oplus ku_1^t \oplus \cdots \oplus ku_j^t \to V_{t+\varepsilon} &= W_{t+\varepsilon} \oplus C_{t+\varepsilon}\\ &= W_{t+\varepsilon} \oplus ku_1^{t+\varepsilon} \oplus \cdots \oplus ku_d^{t+\varepsilon}\end{aligned}$$
to be the inclusion.

It is easy to check that $\varphi \circ \psi = i_{2\varepsilon}$ and $\psi \circ \varphi = i_{2\varepsilon}$. This proves that $d_{\mathrm{int}}(V, V') \leqslant \varepsilon$. $\square$

Remarks 10 (1) In the definition of persistence module, instead of assuming V_t is finite-dimensional, we could have assumed that for all $s < t$, both $\mathrm{Ker}(i_{s,t})$ and $\mathrm{Coker}(i_{s,t})$ are finite-dimensional (ie $j(t) < +\infty$). This would not change anything: the distance is still defined, and Proposition 9 still holds.

[2] Remember that we identify u_j^t and $u_j^{t'}$, and W_t and $W_{t'}$.

(2) The notion of persistence modules can be translated in terms of sheaves, as in [29]. We endow $\mathbb{R}$ with the (nonseparated) topology for which open sets are the sets $(-\infty, s)$, and denote by $\widetilde{\mathbb{R}}$ this topological space; then for a persistence module (V_s) the $V((-\infty, s)) = V_{-s}$ defines a presheaf on $\widetilde{\mathbb{R}}$. This will be a sheaf provided $V_t = V_{t+} = \lim_{s<t} V_s$, which we assumed. We moreover assume this sheaf is constructible, that is, there is a finite subset F of $\mathbb{R}$ such that $i_{s,t}$ is an isomorphism whenever s and t belong to the same connected component of $\mathbb{R} \setminus F$. The sheaf is flabby, and hence injective, if and only if $i_{s,t}$ is onto for all $s < t$. We shall not insist on the sheaf-theoretic point of view, for which we refer to [29].

(3) One can generalize the notion of persistence modules to that of persistence complexes, where V_s is graded in a manner compatible with the filtration, ie $V_s = \bigoplus_p V_s^p$, and the maps respect the grading and differentials. This corresponds to the abelian category of complexes of constructible sheaves on $\widetilde{\mathbb{R}}$.

2.2 Barannikov modules

This section is dedicated to the introduction of Barannikov modules. Before presenting the definition, let us recall that a filtration on a vector space C is a family C^s for $s \in \mathbb{R}$ of vector subspaces of C which is increasing (ie $C^s \subset C^t$ if $s < t$) and such that $C^s = \bigcup_{t<s} C^t$. We will furthermore require that $C^s = C$ for $s \gg 0$ and $C^{s_1} = C^{s_2}$ for $s_1, s_2 \ll 0$. A filtration on a chain complex (C, ∂) is a filtration of C such that $\partial(C^s) \subset C^s$.

Definition 11 A *Barannikov module*, also called a *simple module* or a *simple chain complex*, is a finite-dimensional filtered chain complex $\boldsymbol{C} = (C, \partial)$ which is endowed with a preferred basis B such that:

(1) C admits a decomposition of the form $C = C_+ \oplus C_- \oplus C_0$ which is compatible with the filtration, ie $C^s = C_+^s \oplus C_-^s \oplus C_0^s$, where C_+^s, C_-^s and C_0^s are the filtrations induced on C_+, C_- and C_0.

(2) The preferred basis B is compatible with the decomposition and the filtration in the sense that C_+^s, C_-^s and C_0^s are generated by $B \cap C_+^s$, $B \cap C_-^s$ and $B \cap C_0^s$, respectively. We will denote $B_+ = B \cap C_+$, $B_- = B \cap C_-$ and $B_0 = B \cap C_0$.

(3) The differential ∂ gives a bijection from B_+ to B_-, and $\partial(B_-) = \partial(B_0) = \{0\}$.

Let us emphasize that ∂ yields an isomorphism between C_+ and C_- sending basis elements to basis elements and that ∂ vanishes on $C_- \oplus C_0$.

We will denote by $\mathcal{C}$ the category of Barannikov modules. Given two Barannikov modules $\boldsymbol{C} = (C, \partial_C)$ and $\boldsymbol{D} = (D, \partial_D)$, an element in $\mathrm{Mor}(\boldsymbol{C}, \boldsymbol{D})$ is a chain map which respects the filtration and the preferred bases.

Given a Barannikov module $\boldsymbol{C} = (C, \partial)$ and any $a \in \mathbb{R}$, we define the Barannikov module $\tau_a \boldsymbol{C}$ by shifting the filtration by a. For $a > 0$, we have a natural map $i_a : (C_*, \partial_C) \to (\tau_a C_*, \partial_C)$.

As we will now see, the set of Barannikov modules may be equipped with a natural metric.

Definition 12 The distance between two simple modules, $\boldsymbol{C} = (C, \partial_C)$ and $\boldsymbol{D} = (D, \partial_D)$ is defined as the infimum of the set of ε such that there are filtration-preserving morphisms $\varphi : (C, \partial_C) \to (\tau_\varepsilon D, \partial_D)$ and $\psi : (D, \partial_D) \to (\tau_\varepsilon C, \partial_C)$ such that $\psi \circ \varphi = j_{2\varepsilon}$ and $\varphi \circ \psi = i_{2\varepsilon}$. We denote it by $d_s(\boldsymbol{C}, \boldsymbol{D})$.

We will now define maps $H : \mathcal{C} \to \mathcal{P}$ and $\gamma : \mathcal{P} \to \mathcal{C}$, which will be proven to be isometries, between the two categories of Barannikov and persistence modules.

Proposition 13 *Let $\boldsymbol{C} = (C, \partial_C)$ be a simple chain complex. Then the family of vector spaces $V_s := H(C^s, \partial_C)$, the homology of C^s, equipped with the maps $i_{s,t} : V_s \to V_t$ induced by the inclusion map $C^s \to C^t$, is a persistence module.*

Proof Property (1) is just functoriality of homology; (2) is obvious from the fact that the simple complex is finite-dimensional, therefore $\dim(C^s)$ can have only finitely many jumps. Property (3) follows from the property $C^s = \bigcup_{t<s} C^t$. Property (4) follows from the fact that C^s does not depend on s for $s \ll 0$. We leave the details of the proof to the reader. □

Note that for s large enough, we have $V_s = H(C^s, \partial_C) = C_0$. Note also that since a map between simple complexes (that is a chain map) induces a map in homology, H is indeed a functor.

The map $\gamma : \mathcal{P} \to \mathcal{C}$ is constructed/defined below. We will not prove in detail that γ is well defined, as the proof, albeit in a different setting, can be found in [40, Section 2].

Definition 14 Let $V = (V_s)_{s \in \mathbb{R}}$ be a generic persistence module. We define a simple module $C = \gamma(V)$ as follows:

The preferred basis B consists of the set of $t \in \mathbb{R}$ such that $j(t) = 1$; this is precisely the spectrum of V. We define C to be the span, over $\mathbb{F}$, of B. It is clear that C is a filtered vector space for the tautological filtration (ie C^t is generated by the part of the spectrum in $(-\infty, t]$).

We define B_+, the preferred basis for C_+, to be the set of $t \in B$ such that

$$\dim(\operatorname{Ker}(i_{t^-,t^+})) = 1$$

(and so $\operatorname{codim}(\operatorname{Im}(i_{t^-,t^+})) = 0$). Note that $B \setminus B_+$ consists of $t \in B$ such that $\operatorname{codim}(\operatorname{Im}(i_{t^-,t^+})) = 1$ (and so $\dim(\operatorname{Ker}(i_{t^-,t^+})) = 0$).

We will next define B_-, the preferred basis for C_-, and the differential $\partial\colon B_+ \to B_-$. Consider $t \in B_+$. One can show that there exists a unique $s \in B \setminus B_+$ satisfying the following property: Let $x \in V_{t^-}$ represent a nonzero element in $\operatorname{Ker}(i_{t^-,t^+})$. The element x is in the image of $i_{s^+,t^-}\colon V_{s^+} \to V_{t^-}$ but x is not in the image of $i_{s^-,t^-}\colon V_{s^-} \to V_{t^-}$. We set $\partial(t) = s$ and define B_- to be the set of all such s.

Lastly, we define B_0, the preferred basis for C_0, by $B_0 := B \setminus (B_+ \cup B_-)$ and we set $\partial = 0$ on $B_- \cup B_0$.

We shall not define γ at the level of morphisms. This can be done — see Section 2.4 — but is not "natural", and moreover we do not need it.

2.3 Barcodes

We introduce in this section barcodes and the bottleneck distance. Let us begin by introducing some preliminary notions. A family of intervals, $\mathfrak{B}$, is a list of intervals of the form $((a_j, b_j])_{j \in \{1,\dots,n\}}$, where $-\infty \leqslant a_j \leqslant b_j \leqslant +\infty$. It is convenient for us to allow trivial intervals of the form $(a, a]$ and so we do permit them in our families. We allow a segment to appear multiple times and we identify two families which may be obtained from each other by a permutation. We say two families are equivalent if removing all singletons $(a, a]$ from them yields the same family. For example, the families $\mathfrak{B}_1 = ((-2,-2], (3,3], (-1,0], (0,1])$, $\mathfrak{B}_2 = ((-1,0], (-5,-5], (0,1])$ and $\mathfrak{B}_3 = ((0,1], (-1,0])$ are equivalent.

Definition 15 A barcode $\boldsymbol{B}$ is the equivalence class of a family of intervals $\mathfrak{B}$.

Let us emphasize that in the above definition the same interval can appear multiple times and that the order in which we list the intervals is irrelevant. Note also that adding or removing empty intervals of the type $(c,c]$ to a barcode does not change it.

A barcode represented by a family of segments where no two intervals have a common finite endpoint is called a *generic* barcode.

Let $a \leqslant b$ and $c \leqslant d$ be four elements of $\mathbb{R} \cup \pm\infty$. We set $d((a,b],(c,d]) = \max\{|c-a|,|d-b|\}$. Note that if $c = d = \frac{1}{2}(a+b)$, then $d((a,b],(c,d]) = \frac{1}{2}(b-a)$.

Definition 16 Let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be barcodes and take representatives $\mathfrak{B}_1 = (I_j^1)_{j\in A_1}$ and $\mathfrak{B}_2 = (I_k^2)_{k\in A_2}$. The *bottleneck distance* between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, denoted by $d_{\mathrm{bot}}(\boldsymbol{B}_1,\boldsymbol{B}_2)$, is the infimum of the set of positive ε such that there is a bijection σ between two subsets A_1' and A_2' of A_1 and A_2 with the property that $d(I_j^1, I_{\sigma(j)}^2) \leqslant \varepsilon$ and all the remaining intervals I_j^1 and I_k^2 for $j \in A_1 \setminus A_1'$ and $k \in A_2 \setminus A_2'$ have length less than 2ε.

Note that the bottleneck distance is an *extended metric* as it takes value in $\mathbb{R}_+ \cup \{+\infty\}$.

Remark 17 An equivalent definition is that $d_{\mathrm{bot}}(\boldsymbol{B}_1,\boldsymbol{B}_2) \leqslant \varepsilon$ if and only if there are representatives $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ and a bijection σ between the segments of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ such that $d(I_j^1, I_{\sigma(j)}^2) \leqslant \varepsilon$.

Remark 18 If $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ consist of one interval each, say $I_1 = (x_1,y_1]$ and $I_2 = (x_2,y_2]$, respectively, then

$$d_{\mathrm{bot}}(\boldsymbol{B}_1,\boldsymbol{B}_2) = \min\bigl(\max\bigl(\tfrac{1}{2}(y_1-x_1), \tfrac{1}{2}(y_2-x_2)\bigr), \max(|x_2-x_1|, |y_2-y_1|)\bigr).$$

We denote by $\mathcal{B}$ the category of barcodes. We now give a description of $\mathrm{Mor}(\boldsymbol{B}_1,\boldsymbol{B}_2)$.

Definition 19 Let $I = (a,b]$, $J = (c,d]$ be two nontrivial intervals, ie $a < b$ and $c < d$; we write $I \leqslant J$ if $a \leqslant c < b \leqslant d$.

First, suppose that $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ (have representative which) consist of one interval each, say $I_1 = (x_1,y_1]$ and $I_2 = (x_2,y_2]$, respectively. Then, there exists a nontrivial morphism from $\boldsymbol{B}_1$ to $\boldsymbol{B}_2$ if and only if $I_2 \leqslant I_1$. In other words, $\mathrm{Mor}(\boldsymbol{B}_1,\boldsymbol{B}_2) = \mathbb{F}$ if $I_2 \leqslant I_1$, and $\mathrm{Mor}(\boldsymbol{B}_1,\boldsymbol{B}_2) = \{0\}$ otherwise. If either of I_1 or I_2 is a trivial interval, then we set $\mathrm{Mor}(\boldsymbol{B}_1,\boldsymbol{B}_2) = \{0\}$, even if $I_1 = I_2$.

A morphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ is a collection of morphisms between the intervals constituting them. More precisely, suppose that $\boldsymbol{B}_1 = (I_j^1)_{j\in A_1}$ and $\boldsymbol{B}_2 = (I_j^2)_{j\in A_2}$.

Note that some of the intervals may be trivial. A morphism from $\boldsymbol{B}_1$ to $\boldsymbol{B}_2$ is represented by a map $j \mapsto S_j$, from A_1 to the set of subsets of A_2, such that $I_j^1 \leqslant I_k^2$ for all $k \in S_j$. Equivalently, and more concisely, $\mathrm{Mor}(\boldsymbol{B}_1, \boldsymbol{B}_2) = \bigoplus \mathrm{Mor}(I_j^1, I_k^2)$ for $j \in A_1$ and $k \in A_2$.

Remark 20 For a real number a, we define $\tau_a(\boldsymbol{B})$ to be the barcode obtained by shifting all the intervals of $\boldsymbol{B}$ by $-a$, ie if $\boldsymbol{B} = ((x_j, y_j])_{j \in A}$, then

$$\tau_a \boldsymbol{B} = ((x_j - a, y_j - a])_{j \in A}.$$

There exists a canonical morphism $i_a \colon \boldsymbol{B} \to \tau_a \boldsymbol{B}$, which we will describe in the case when $\boldsymbol{B}$ consists of a single interval $(x, y]$, leaving the description in the case of more general barcodes to the reader. First, suppose that the interval is nontrivial. Then, for $0 \leqslant a < y - x$, we have a canonical morphism $i_a \colon \boldsymbol{B} \to \tau_a \boldsymbol{B}$, corresponding to the unique morphism $(x, y] \to (x - a, y - a]$. For other values of a, or when $\boldsymbol{B}$ consists of a trivial interval, the morphism i_a is zero as $\mathrm{Mor}(\boldsymbol{B}, \tau_a \boldsymbol{B}) = \{0\}$.

Note also that given a morphism $u \colon \boldsymbol{B}_1 \to \boldsymbol{B}_2$, there exists a corresponding morphism $\tau_a u \colon \tau_a \boldsymbol{B}_1 \to \tau_a \boldsymbol{B}_2$.

The bottleneck distance admits the following characterization:

Proposition 21 *The bottleneck distance $d_{\mathrm{bot}}(\boldsymbol{B}_1, \boldsymbol{B}_2)$ is given by the infimum of the set of $\delta > 0$ with the property that there are morphisms $u \colon \boldsymbol{B}_1 \to \tau_\delta \boldsymbol{B}_2$ and $v \colon \boldsymbol{B}_2 \to \tau_\delta \boldsymbol{B}_1$ such that $\tau_\delta u \circ v = i_{2\delta}^2$ and $\tau_\delta v \circ u = i_{2\delta}^1$.*

Proof We will only provide a sketch of the proof and leave the details to the reader.

We leave it to the reader to check that the proposition is true when $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ consist of single nontrivial intervals.

Suppose that u, v and δ are as in the statement of the proposition. We will show that this implies that $d_{\mathrm{bot}}(\boldsymbol{B}_1, \boldsymbol{B}_2) < \delta$. We may assume that $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ have no intervals of length less than 2δ. Indeed, one can check that we may remove from $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ intervals of length less than 2δ and modify u and v so that we still have $\tau_\delta u \circ v = i_{2\delta}^2$ and $\tau_\delta v \circ u = i_{2\delta}^1$; furthermore, this does not affect the inequality $d_{\mathrm{bot}}(\boldsymbol{B}_1, \boldsymbol{B}_2) < \delta$.

Now, if there are no intervals of length less than 2δ, then $i_{2\delta}^1$ and $i_{2\delta}^2$ yield bijections among the bars of $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, respectively. Using this, one could show that the maps

u and v correspond to a bijection σ between the intervals of $\boldsymbol{B}_1$ and the intervals of $\boldsymbol{B}_2$. Furthermore, we have $d(I_j^1, I_{\sigma(j)}^2) \leqslant \delta$. This implies that $d_{\mathrm{bot}}(\boldsymbol{B}_1, \boldsymbol{B}_2) < \delta$.

Next, suppose that $d_{\mathrm{bot}}(\boldsymbol{B}_1, \boldsymbol{B}_2) < \delta$. This means that we can find a bijection σ between the intervals of $\boldsymbol{B}_1$ and those of $\boldsymbol{B}_2$ which are of length greater than 2δ. Using σ, one can easily construct u and v satisfying the statement of the proposition. $\square$

We end this section by introducing the completion of the space of barcodes for the bottleneck distance. Note that in Definition 15 of a barcode we restricted ourselves to families of intervals consisting of a finite number of intervals. One could also consider families consisting of infinitely many intervals and define an infinite barcode to be the equivalence class of one such family. (The equivalence relation in consideration is, of course, the one introduced before Definition 15.)

Proposition 22 *The completion of $\mathcal{B}$ for the bottleneck distance consists of the set of infinite barcodes $\boldsymbol{B}$ which satisfy the following finiteness property: for any positive ε, only finitely many intervals of $\boldsymbol{B}$ have length greater than ε. We denote by $\overline{\mathcal{B}}$ the set of these barcodes.*

We leave the proof of the above proposition to the reader. We should add that barcodes satisfying similar finiteness conditions as above have appeared in [6, Chapter 5], where they are referred to as q–tame barcodes.

2.4 The functorial triangle

We now define the following commutative triangle of maps relating the categories of persistence modules, Barannikov modules and barcodes:

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, shift left, "\gamma"] \arrow[dr, bend left, "\beta"] & & \mathcal{C} \arrow[ll, shift left, "H"] \\
& \mathcal{B} \arrow[ul, bend left, "Q"] \arrow[ur, "\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(V_s)_{s\in\mathbb{R}} \arrow[rr, shift left, "\gamma"] \arrow[dr, bend left, "\beta"] & & (C^s)_{s\in\mathbb{R}} \arrow[ll, shift left, "H"] \\
& \boldsymbol{B} \arrow[ul, bend left, "Q"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

The maps appearing in the above diagrams are defined as follows:

(1) The functor H was defined in Section 2.2. Recall that it associates to a Barannikov module $\boldsymbol{C} = (C, \partial_C)$ the persistence module $V_s = H(C^s, \partial_C)$.

(2) The functor α associates to a barcode $\boldsymbol{B} = \{(a_j, b_j]\}_{j\in\{1,\dots,n\}}$ the simple module $C(\boldsymbol{B})$, whose generators (ie elements of the preferred basis) are as follows: If

$a_j < b_j < \infty$, then a_j and b_j are both generators. If $b_j = \infty$, then a_j is a generator, and if $a_j = b_j$, then neither one is a generator. We define the differential by $\partial_C b_j = a_j$ and $\partial_C a_j = 0$.

(3) We begin by defining the map β for a generic persistence module $V = (V_s)_{s\in\mathbb{R}}$.

Let B be the spectrum of V. Consider the set of $t \in B$ such that $\dim(\mathrm{Ker}(i_{t^-,t^+})) = 1$ (and so $\mathrm{codim}(\mathrm{Im}(i_{t^-,t^+})) = 0$). Label its elements $b_1, \dots, b_n$. For each b_j, there exists a unique $a_j \in \mathbb{R}$ with the following property: Let $x \in V_{b_j^-}$ represent a nonzero element in $\mathrm{Ker}(i_{b_j^-,b_j^+})$. The element x is in the image of $i_{a_j^+,t^-} : V_{a_j^+} \to V_{b_j^-}$ but x is not in the image of $i_{a_j^-,b_j^-} : V_{a_j^-} \to V_{b_j^-}$. One can easily check that a_j has the property that $\mathrm{codim}(\mathrm{Im}(i_{a_j^-,a_j^+})) = 1$ and thus each a_j belongs to the spectrum of V. We label the remaining elements of the spectrum of V by $\{c_1, \dots, c_m\}$. The barcode $\beta(V)$ consists of the list of intervals $\big((a_j, b_j], (c_k, \infty]\big)$ for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m$.

Let us explain how to extend β to the set of all persistence modules. One can check that β, defined as above, commutes with the shift morphisms τ_a, and as a consequence it is 1–Lipschitz. Since, according to Proposition 9, generic persistence modules are dense for the interleaving distance, we can extend β to $\mathcal{P}$.

(4) We define $Q := H \circ \alpha$. Let us point out that for a nontrivial interval $I = (a, b]$ we have $Q(I)_s = \mathbb{F}$ if $s \in I$, and $Q_s(I) = \{0\}$ if $s \notin I$.

(5) The map γ is defined as $\gamma := \alpha \circ \beta$. This definition of γ does coincide with Definition 14 in the generic case. Note that Q is an equivalence of categories, and in fact $\mathcal{B}$ is the skeleton of the category $\mathcal{P}$. While β is easily defined on objects, we have not worked out the definition of β on morphisms and leave this as an open problem for the reader. Notice that since Q is an equivalence of categories, because it is full, faithful and essentially surjective, then one should be able to define β on morphisms so that β is left adjoint to Q (that is, a quasi-inverse of Q) (see [28, Definition 1.1.7, page 25]). Usually defining the quasi-inverse explicitly on morphisms is more complicated. However, for our purpose of proving that β, and the other maps appearing in the above diagram, are isometries, we only need their action on objects, and the obvious fact that isomorphic objects are sent to isomorphic objects. Thus defining β on morphisms is not relevant for the sequel of the paper.

The definitions presented above only describe the action of the functors at the level of objects. We leave the description of how the functors should act on morphisms to the reader, as it is not needed for the results below.

Proposition 23 *The functors/maps defined above satisfy the following properties*:

(1) $\beta \circ H \circ \alpha = \mathrm{Id}_{\mathcal{B}}$.

(2) *Each of the maps* β, H *and* α *is an isometry.*

Proof To prove that $\beta \circ H \circ \alpha = \mathrm{Id}_{\mathcal{B}}$, it is sufficient to verify this for barcodes which consist of a single interval, which can be checked quite easily.

Now we present the proof of the second statement. We have a functor shifting the filtration by a on each of the three categories, which we also denote by τ_a. We also have natural maps from an object to the a–shift of the object, which are also denoted by i_a. In both $\mathcal{P}$ and $\mathcal{B}$, the distance between two objects X and Y is given as the smallest δ such that there are morphisms $u\colon X \to Y$ and $v\colon Y \to X$ such that $v \circ \tau_\delta u = i_{2\delta}$ and $u \circ \tau_\delta v = j_{2\delta}$; note that here we are relying on Proposition 21. So the fact that the functors β and $H \circ \alpha$ are isometries follows from the fact that they commute with τ_a and i_a combined with the first part of the proposition. One can easily verify that $\alpha\colon \mathcal{B} \to \mathcal{C}$ is an isometry, which in turn implies that H is also an isometry. □

Remarks 24 (1) There exists another, somewhat simpler, proof which avoids Proposition 21: One could directly check that β, H and α are contractions. Since the composition of these maps is the identity, they must all be isometries.

(2) Since we have isometries, the completion $\overline{\mathcal{B}}$ yields the completions of $\mathcal{P}$ and $\mathcal{C}$, which we denote by $\overline{\mathcal{P}}$ and $\overline{\mathcal{C}}$.

(3) It follows from the above proposition that Q and γ are also isometries as they are compositions of isometries. Furthermore, we also have $H \circ \gamma = \mathrm{Id}_{\mathcal{P}}$, $\gamma \circ H = \mathrm{Id}_{\mathcal{C}}$ and $Q \circ \beta = \mathrm{Id}_{\mathcal{P}}$. We will verify here that $H \circ \gamma = \mathrm{Id}_{\mathcal{P}}$ and leave the rest to the reader: We know that $\beta \circ H \circ \alpha = \mathrm{Id}_{\mathcal{B}}$, hence, composing with β on the right, $\beta \circ H \circ \gamma = \beta$; in other words, we have $H \circ \gamma = \mathrm{Id}_{\mathcal{C}}$ provided β is injective. But β is an isometry, and so is obviously injective.

Corollary 25 (isometry theorem [6, Chapter 5; 8]) *The map* β *is an isometry. Thus the interleaving distance between two persistence modules coincides with the bottleneck distance of their barcodes.*

Corollary 26 (structure theorem [70]) *Each persistence module can be written as* $V = \bigoplus_{I \in S} Q(I)$, *where* S *is a list of intervals.*

Proof Indeed, for a barcode $\boldsymbol{B}$ with representative $(I_j)_{j\in A}$ we have $Q(\boldsymbol{B}) = \bigoplus_{j\in A} Q(I_j)$ and Q is surjective, so we get the result. □

2.5 Filtered homology

We would like to state explicitly certain consequences of Proposition 23 which will be used in the following sections.

Proposition 27 *Let $\boldsymbol{V}$ be a persistence module and $\boldsymbol{C} = \gamma(\boldsymbol{V})$, where γ is as defined in the previous section. Then, for all $t \in \mathbb{R}$, we have an isomorphism $H(C^t, \partial_{\boldsymbol{B}}) \to V_t$.*

Proof This is simply stating that $H \circ \gamma = \mathrm{Id}_{\mathcal{P}}$, which we proved in Remarks 24. □

Corollary 28 *Suppose that f is a Morse function on M and let*

$$M^t := \{x \in M : f(x) < t\}.$$

Let $\boldsymbol{V}$ be the persistence module $(H(M^t))$ and let $\boldsymbol{C}$ be the associated Barannikov module. We have $H(C^t, \partial_{\boldsymbol{C}}) = H(M^t)$.

Remark 29 An analogous statement holds for Floer homology. This plays a crucial role in the following sections.

3 Barcodes via Hamiltonian Floer theory

The goal of this section is to explain how one may use Hamiltonian Floer theory to associate barcodes to Hamiltonian diffeomorphisms. In the case of surfaces other than the sphere, we will have maps

$$\boldsymbol{B}_j : \mathrm{Ham}(\Sigma, \omega) \to \widehat{\mathcal{B}}$$

for each $j \in \mathbb{Z}$. The index j corresponds to the Conley–Zehnder index and $\widehat{\mathcal{B}}$ denotes the space of barcodes considered up to shift. In the case of the sphere, we will have similar maps

$$\boldsymbol{B}_j : \mathrm{UHam}(\mathbb{S}^2, \omega) \to \widehat{\mathcal{B}},$$

which are only well defined on the universal cover of $\mathrm{Ham}(\mathbb{S}^2, \omega)$. Since this is a two-sheeted covering, for each $\varphi \in \mathrm{Ham}(\mathbb{S}^2, \omega)$ and each $j \in \mathbb{Z}$ we will have two barcodes $\boldsymbol{B}_j(\widetilde{\varphi}_1)$ and $\boldsymbol{B}_j(\widetilde{\varphi}_2)$. Lastly, we will also introduce the *total barcode* $\boldsymbol{B}$, which will combine all the $\boldsymbol{B}_j$ into a single barcode.

We should point out that one can associate barcodes to Hamiltonian diffeomorphisms on symplectic manifolds more general than what we consider here; see [67].

We will begin by introducing some of our conventions and recalling some basic symplectic geometry. Suppose that M is equipped with a symplectic form ω. A symplectic diffeomorphism is a diffeomorphism $\theta\colon M \to M$ such that $\theta^*\omega = \omega$. The set of all symplectic diffeomorphisms of M is denoted by $\mathrm{Symp}(M,\omega)$. Hamiltonian diffeomorphisms provide an important class of examples of symplectic diffeomorphisms. Recall that a smooth Hamiltonian $H : \mathbb{S}^1 \times M \to \mathbb{R}$ gives rise to a time-dependent vector field X_H which is defined via the equation $\omega(X_H(t),\cdot) = -dH_t$. The Hamiltonian flow of H, denoted by φ_H^t, is by definition the flow of X_H. A Hamiltonian diffeomorphism is a diffeomorphism which arises as the time-1 map of a Hamiltonian flow. The set of all Hamiltonian diffeomorphisms, which is denoted by $\mathrm{Ham}(M,\omega)$, forms a normal subgroup of $\mathrm{Symp}(M,\omega)$.

The inverse of a Hamiltonian flow $(\varphi_H^t)^{-1}$ is itself a Hamiltonian flow whose generating Hamiltonian is given by $\overline{H}(t,x) = -H(t,\varphi_H^t(x))$. Given two Hamiltonian flows φ_H^t and φ_K^t, the composition $\varphi_H^t\varphi_K^t$ is also a Hamiltonian flow, generated by $H\#K(t,x) := H(t,x) + K(t,(\varphi_H^t)^{-1}(x))$.

Given a Hamiltonian H, we define

$$\|H\| = \int_{\mathbb{S}^1} \big[\max_{x\in M} H(t,\cdot) - \min_{x\in M} H(t,\cdot)\big]\, dt.$$

Recall that the Hofer distance [26; 32] between $\phi,\psi \in \mathrm{Ham}(M,\omega)$ is defined to be $d_{\mathrm{Hofer}}(\phi,\psi) := \inf\{\|H-G\| : \phi = \varphi_H^1,\ \psi = \varphi_G^1\}$.

We will denote by $\mathrm{UHam}(M,\omega)$ the universal of cover of $\mathrm{Ham}(M,\omega)$. An element $\widetilde{\varphi}$ of $\mathrm{UHam}(M,\omega)$ consists of the homotopy class of a Hamiltonian path $\{\varphi_H^t\}_{0\leqslant t\leqslant 1}$, relative to its endpoints, which are Id and φ_H^1.

3.1 A brief review of Hamiltonian Floer theory

We will now briefly recall the aspects of Hamiltonian Floer theory which will be needed to construct barcodes for Hamiltonian diffeomorphisms. Throughout this section, unless otherwise stated, (M,ω) will denote a $2n$–dimensional, closed, connected and monotone symplectic manifold. The latter condition means that there exists $\lambda \geqslant 0$

such that $\omega|_{\pi_2} = \lambda c_1|_{\pi_2}$. All closed surfaces are examples of monotone symplectic manifolds.[3]

We denote by $\Omega_0(M)$ the space of contractible loops in M. The Novikov covering of $\Omega_0(M)$ is defined by

$$\widetilde{\Omega}_0(M) = \frac{\{[z,u] : z \in \Omega_0(M),\, u\colon D^2 \to M,\, u|_{\partial D^2} = z\}}{[z,u] = [z',u'] \text{ if } z = z' \text{ and } \omega(\bar{u} \# u') = 0},$$

where $\bar{u} \# u'$ is the sphere obtained by gluing u, with its orientation reversed, to u' along their common boundary. The disk u in $[z,u]$ is referred to as the capping disk of the loop z.

We should point out that in the case of surfaces of nonzero genus, since $\pi_2(M) = 0$, we have $\widetilde{\Omega}_0(M) = \Omega_0(M)$. More generally, in the case of the sphere and other monotone manifolds, $\widetilde{\Omega}_0(M)$ is a covering space of $\Omega_0(M)$ whose group of deck transformations is given by $\Gamma := \pi_2(M)/\ker(c_1) = \pi_2(M)/\ker([\omega]) \simeq \mathbb{Z}$. An element A of Γ acts on $\widetilde{\Omega}_0(M)$ by $A[z,u] = [z, u \# A]$.

The action functional and its spectrum Recall that the action functional

$$\mathcal{A}_H : \widetilde{\Omega}_0(M) \to \mathbb{R}$$

associated to a Hamiltonian H is defined by

$$\mathcal{A}_H([z,u]) = \int_0^1 H(t, z(t))\, dt - \int_{D^2} u^*\omega.$$

Note that $\mathcal{A}_H([z, u \# A]) = \mathcal{A}_H([z,u]) - \omega(A)$ for every $A \in \Gamma$.

The set of critical points of $\mathcal{A}_H$, denoted by $\mathrm{Crit}(\mathcal{A}_H)$, consists of equivalence classes, $[z,u] \in \widetilde{\Omega}_0(M)$, such that z is a 1–periodic orbit of the Hamiltonian flow φ_H^t. We will often refer to such $[z,u]$ as capped 1–periodic orbits of φ_H^t.

The action spectrum of H, denoted by $\mathrm{Spec}(H)$, is the set of critical values of $\mathcal{A}_H$; it has Lebesgue measure zero. It turns out that the action spectrum $\mathrm{Spec}(H)$ is independent of H in the following sense: If H' is another Hamiltonian such that $\varphi_H^1 = \varphi_{H'}^1$, then there exists a constant $C \in \mathbb{R}$ such that $\mathrm{Spec}(H) = \mathrm{Spec}(H') + C$, where $\mathrm{Spec}(H') + C$ is the set obtained from $\mathrm{Spec}(H')$ by adding the value C to every element of $\mathrm{Spec}(H')$. It then follows that we can define the action spectrum

[3] Symplectic manifolds on which $\omega|_{\pi_2} = 0 = c_1|_{\pi_2}$ are usually called symplectically aspherical. All closed surfaces other than the sphere fall into this category. Note that we are treating symplectically aspherical manifolds as monotone.

of a Hamiltonian diffeomorphism ϕ by setting $\mathrm{Spec}(\phi) = \mathrm{Spec}(H)$, where H is any Hamiltonian such that $\phi = \varphi^1_H$. Of course, $\mathrm{Spec}(\phi)$ is well defined up to a shift by a constant.

The Conley–Zehnder index We say a diffeomorphism φ is nondegenerate if its graph intersects the diagonal in $M \times M$ transversely. A Hamiltonian H is called nondegenerate if the time-1 map of its flow φ^1_H is nondegenerate. When H is nondegenerate, the set $\mathrm{Crit}(\mathcal{A}_H)$ can be indexed by the Conley–Zehnder index, $\mu_{\mathrm{CZ}}\colon \mathrm{Crit}(\mathcal{A}_H) \to \mathbb{Z}$, which satisfies the following properties:

(1) For every critical point p of a C^2–small Morse function f, we have $i_{\mathrm{Morse}}(p) = \mu_{\mathrm{CZ}}([p, u_p])$, where u_p is a trivial capping disk and i_{Morse} denotes the Morse index.

(2) For every $A \in \pi_2(M)$, we have

$$\mu_{\mathrm{CZ}}([z, u \# A]) = \mu_{\mathrm{CZ}}([z, u]) - 2c_1(A), \tag{1}$$

where $u \# A$ denotes the capping disk obtained by attaching the sphere A to the disk u.

Filtered Floer homology Fix a ground field $\mathbb{F}$ and let H be a nondegenerate Hamiltonian on M. Being nondegenerate means that at each fixed point x of the time-1 map φ^1_H, the derivative $D_x\varphi^1_H$ does not have 1 as an eigenvalue.

For $t \in (-\infty, \infty] \setminus \mathrm{Spec}(H)$, we define $CF^t_m(H)$ to be the $\mathbb{F}$–vector space of formal linear combinations of the form

$$\alpha = \sum_{[z,u] \in \mathrm{Crit}(\mathcal{A}_H)} \alpha_{[z,u]} [z, u],$$

where $\alpha_{[z,u]} \in \mathbb{F}$, $\mu_{\mathrm{CZ}}([z,u]) = m$ and $\mathcal{A}_H([z,u]) < t$. Furthermore, α is required to satisfy the following finiteness criterion: for any given $c \in \mathbb{R}$, the number of terms such that $\mathcal{A}_H([z,u]) > c$ and $\alpha_{[z,u]} \neq 0$ must be finite.[4]

The vector space $CF^t_*(H)$ is equipped with the Floer boundary map $\partial\colon CF^t_*(H) \to CF^t_{*-1}(H)$ which counts L^2 negative gradient trajectories of $\mathcal{A}_H$: these are maps $u(s,t)\colon \mathbb{R} \times S^1 \to M$ satisfying the PDE $\partial_s u + J(t,u)(\partial_t u + X_H) = 0$, where J denotes an almost complex structure on M which is compatible with ω. It is a theorem of Floer that $\partial^2 = 0$. The homology of the chain complex $(CF^t_*(H), \partial)$ is referred to as the *filtered Floer homology* of H and is denoted by $HF^t_*(H)$.

[4]On surfaces of positive genus this finiteness criterion is void as the set $\mathrm{Crit}(\mathcal{A}_H)$ is itself finite.

More generally, filtered Floer homology groups may be defined for any interval of the form (a,b), where $-\infty \leqslant a,b \leqslant \infty$ are not in $\mathrm{Spec}(H)$: $HF_*^{(a,b)}(H)$ is defined to be the homology of the quotient complex $CF_*^{(a,b)}(H) = CF_*^b(H)/CF_*^a(H)$. Let us remark that the filtered Floer homology groups do not depend on the choice of the almost complex structure J.

For our purposes, we will be needing the following property of filtered Floer homology groups: Suppose that two Hamiltonian paths $\{\varphi_{H_0}^t\}_{0\leqslant t\leqslant 1}$ and $\{\varphi_{H_1}^t\}_{0\leqslant t\leqslant 1}$ are homotopic relative to endpoints in $\mathrm{Ham}(M,\omega)$, ie they represent the same element of $\mathrm{UHam}(M,\omega)$. Then there exists a constant $c \in \mathbb{R}$ such that

$$HF_*^t(H_0) = HF_*^{t+c}(H_1) \quad \text{for all } t \in \mathbb{R}. \tag{2}$$

In other words, for $\tilde{\varphi} \in \mathrm{UHam}(M,\omega)$, the filtered homology groups $HF_*^t(\tilde{\varphi})$ are well defined up to a shift by a constant.

We should mention that the filtered Floer homology groups are well defined for degenerate H as well: given $a,b \notin \mathrm{Spec}(H)$, one simply defines $HF_*^{(a,b)}(H)$ to be $HF_*^{(a,b)}(\tilde{H})$ for $\tilde{H}$ which is nondegenerate and sufficiently C^2–close to H. The definition does not depend on the choice of $\tilde{H}$.

Note that the action of Γ on $\tilde{\Omega}_0(M)$, which we described above, induces natural isomorphisms on Floer homology: given $A \in \Gamma$, the action $[z,u] \mapsto [z, u \# A]$ induces an isomorphism of filtered chain complexes $CF_*^t(H)$ and $CF_{*-2c_1(A)}^{t-\omega(A)}(H)$.

3.2 Barcodes for Hamiltonians

Fix an integer j and let H be a strongly nondegenerate Hamiltonian. Being strongly nondegenerate means that H is nondegenerate and no two of its capped 1–periodic orbits have the same action. As explained in [57], the family of vector spaces $HF_j^s(H)$ forms a persistence module, where the morphisms $i_{s,t}\colon HF_j^s(H) \to HF_j^t(H)$ of Definition 5 are induced by the inclusion of $CF_j^s(H) \hookrightarrow CF_j^t(H)$.

Applying the map γ of Section 2.2 to this persistence module, we obtain a corresponding Barannikov complex, which we will denote by $(\mathrm{BC}_j^s(H), \partial_B)$. Let us emphasize that the generators of this complex are given by *actions*, $\mathcal{A}_H([z,u])$, of capped 1–periodic orbits $[z,u] \in \mathrm{Crit}(\mathcal{A}_H)$ whose Conley–Zehnder indices are either j or $j+1$. The boundary map, which is given by the construction carried out in Section 2.2, sends the action of an orbit of index $j+1$ to the action of an orbit of index j and it is zero on actions of orbits of index j.

Next, applying the map β of Section 2.4 to the persistence module $HF_j^s(H)$, we obtain a corresponding barcode, which we will denote by $\boldsymbol{B}_j(H)$. The lower ends of the bars in $\boldsymbol{B}_j(H)$ are actions of orbits of index j while the upper ends of the bars are actions of orbits of index $j+1$. It is easy to see that, since H is strongly nondegenerate, the action of an orbit appears as one endpoint of exactly one bar in exactly one of the barcodes $\boldsymbol{B}_j(H)$.

It can be shown, via standard Floer-theoretic arguments (see eg equation (4) in [56]), that barcodes are 1–Lipschitz with respect to Hofer distance, namely

$$d_{\mathrm{bot}}(\boldsymbol{B}_j(H), \boldsymbol{B}_j(G)) \leqslant \|H-G\|. \tag{3}$$

The above inequality allows us to define the barcode of any smooth, or even continuous, function $H:\mathbb{S}^1\times M\to\mathbb{R}$. Indeed, for an arbitrary smooth, or continuous, H, take H_i to be a sequence of nondegenerate Hamiltonians such that $\|H-H_i\|\to 0$, and define $\boldsymbol{B}_j(H) := \lim \boldsymbol{B}_j(H_i)$, where the limit is taken with respect to the bottleneck distance. Note that $\boldsymbol{B}_j(H)$ belongs to the completion of the space of barcodes $\overline{\mathcal{B}}$ which we described in Section 2.3; see Proposition 22. Hence, for each integer j, we obtain a map

$$\boldsymbol{B}_j : C^\infty(\mathbb{S}^1\times M)\to\overline{\mathcal{B}}$$

which continues to satisfy (3). The rest of this section is dedicated to describing some of the properties of the barcodes $\boldsymbol{B}_j(H)$.

Spectrality If H is nondegenerate, the set of endpoints of $\boldsymbol{B}_j(H)$ forms a subset of $\mathrm{Spec}(H)$. In fact, this statement continues to hold for any $H\in C^\infty(\mathbb{S}^1\times M)$. This can be proven by writing H as the limit, in the C^2 topology, of a sequence of nondegenerate Hamiltonians H_i and applying the continuity and spectrality properties to the H_i.

Symplectic-invariance For any $\psi\in\mathrm{Symp}(M,\omega)$ and any $H\in C^\infty(\mathbb{S}^1\times M)$, we have

$$\boldsymbol{B}_j(H\circ\psi) = \boldsymbol{B}_j(H). \tag{4}$$

The above follows from the fact that, for nondegenerate H, the filtered Floer complexes $CF_*^t(H)$ and $CF_*^t(H\circ\psi)$ are isomorphic. See for example [56] for further details.

Periodicity Let N and τ denote the minimal nonnegative generators of $c_1(\pi_2(M))$ and $\omega(\pi_2(M))$, respectively. Then

$$\boldsymbol{B}_j(H) = \boldsymbol{B}_{j-2N}(H) - \tau, \tag{5}$$

where $\boldsymbol{B}_{j-2N}(H)-\tau$ is the barcode obtained from $\boldsymbol{B}_{j-2N}(H)$ by shifting each of its bars by $-\tau$. The above follows immediately from the last paragraph of Section 3.1. Let us point out that in the case of surfaces this property is of interest only for S^2 as $N=\tau=0$ for other surfaces. In the case of the sphere, $N=2$ and the symplectic form may be normalized to impose $\tau=1$.

3.2.1 The total barcode of a Hamiltonian Let H denote a nondegenerate Hamiltonian and define the *total barcode* of H to be

$$\boldsymbol{B}(H) := \bigsqcup_j \boldsymbol{B}_j(H).$$

This is not a standard barcode in the sense of Definition 15 because it has infinitely many bars. In fact, we can see from the periodicity property that if I is a bar in $\boldsymbol{B}(H)$, then so are the shifted bars $I+n\tau$ for every integer n. Nevertheless, we can still work with these barcodes: The definition of the bottleneck distance easily extends to this class of barcodes. Note that $d_{\mathrm{bot}}(\boldsymbol{B}(H),\boldsymbol{B}(G)) \leqslant \sup_j\{d_{\mathrm{bot}}(\boldsymbol{B}_j(H),\boldsymbol{B}_j(G))\}$. One can easily check that the total barcode satisfies inequality (3) and so it can be defined for any H. Moreover, $\boldsymbol{B}(H)$ satisfies the spectrality and symplectic-invariance properties from the previous section. Lastly, observe that the periodicity property translates to the shift-invariance property

$$\boldsymbol{B}(H) = \boldsymbol{B}(H)+\tau, \tag{6}$$

where τ is the nonnegative generator of $\omega(\pi_2(M))$.

We should emphasize that in the case of aspherical manifolds, such as surfaces of positive genus, $\boldsymbol{B}_j(H)$ is nontrivial for only finitely many values of j. Note also that, on aspherical manifolds, the family of vector spaces $\bigoplus_j HF_j^t(H)$ is a persistence module and so, as in the previous section, we can associate a barcode to this persistence module. This barcode will be exactly the total barcode $\boldsymbol{B}(H)$.

The following property of the total barcode will be used in Section 5.

Proposition 30 *Let (a,b) denote an interval whose endpoints are not in the spectrum of H. The number of bars of $\boldsymbol{B}(H)$ which have exactly one endpoint in the interval (a,b) is given by the rank of $HF_*^{(a,b)}(H)$.*

In particular, if $c\in\operatorname{Spec}(H)$ is isolated, then the total number of bars with one endpoint at value c is given by the rank of $HF_*^{(c-\varepsilon,c+\varepsilon)}(H)$ for sufficiently small $\varepsilon>0$.

Proof Observe that a sufficiently C^2–small perturbation of H does not change the total number of bars which have exactly one endpoint in the interval (a,b); indeed, such a perturbation might create new bars but both endpoints of these bars will be either inside (a,b) or outside of it. On the other hand, $HF_*^{(a,b)}(H)$ is by definition $HF_*^{(a,b)}(\tilde{H})$, where $\tilde{H}$ is sufficiently C^2–close to H. We conclude that we may suppose that H is nondegenerate and that no two of its periodic orbits have the same action.

First, suppose that $\Sigma \neq \mathbb{S}^2$. In that case, the barcode $\boldsymbol{B}(H)$ is a standard (finite) barcode. We leave it to the reader to check that in this case the result follows from Proposition 27.

Now, suppose that $\Sigma = \mathbb{S}^2$. Since $\boldsymbol{B}(H)$ is not a standard barcode in the sense of Definition 15, we cannot immediately apply Proposition 23.[5] We will reduce to the case of finite barcodes by noting that the set of periodic orbits of H whose actions are in (a,b) is finite. Let N and M denote the minimal and maximal Conley–Zehnder indices, respectively, of periodic orbits whose actions are in the interval (a,b). Define the barcode $\boldsymbol{B}'(H) = \bigsqcup \boldsymbol{B}_j(H)$, where $N-1 \leqslant j \leqslant M$; recall that the action of a periodic orbit of index j is the endpoint of a bar in either $B_j(H)$ or $B_{j-1}(H)$. We leave it to the reader to check that any bar in $\boldsymbol{B}(H)$ which has an endpoint in (a,b) appears in $\boldsymbol{B}'(H)$.

The barcode $\boldsymbol{B}'(H)$ is the barcode of the persistence module $V_t = \bigoplus_j HF_j^t(H)$, where $N-1 \leqslant j \leqslant M$. It follows, once again from Proposition 23, that the number of bars in $\boldsymbol{B}'(H)$ with exactly one endpoint in (a,b) coincides with the rank of $\bigoplus_j HF_j^{(a,b)}(H)$, where $N-1 \leqslant j \leqslant M$. But the latter is exactly the rank of $HF_*^{(a,b)}(H)$. This completes the proof. $\square$

Let us point out a consequence of (6) which will play an important role in our story: Given a barcode $\boldsymbol{B}$ and an interval $I \subset \mathbb{R}$, denote by $\#\mathrm{Endpoints}(\boldsymbol{B}) \cap I$ the total number of endpoints of bars of $\boldsymbol{B}$ which are in the interval I. We should emphasize that, here, we count endpoints with their multiplicities, ie if the same value appears as an endpoint of k different bars, then it is counted k times.

Corollary 31 *Suppose that $\boldsymbol{B}$ is a barcode which is invariant under shift by τ, ie $\boldsymbol{B}+\tau = \boldsymbol{B}$. Then, for any $c \in \mathbb{R}$, we have*

$$\#\mathrm{Endpoints}(\boldsymbol{B}) \cap [0,\tau) = \#\mathrm{Endpoints}(\boldsymbol{B}+c) \cap [0,\tau).$$

[5] Although the proposition does not apply directly, one can check that its proof may be modified to encompass more general barcodes such as the one being considered here.

3.3 Barcodes for Hamiltonian diffeomorphisms

Recall that given a barcode $\boldsymbol{B} \in \overline{\mathcal{B}}$ and $c \in \mathbb{R}$, we have defined $\boldsymbol{B} + c$ to be the barcode obtained from $\boldsymbol{B}$ by shifting each of its intervals by c. Let $\sim$ denote the equivalence relation on $\overline{\mathcal{B}}$ given by $\boldsymbol{B}_1 \sim \boldsymbol{B}_2$ if $\boldsymbol{B}_2 = \boldsymbol{B}_1 + c$ for some $c \in \mathbb{R}$; we will denote the quotient space by $\widehat{\mathcal{B}}$. The bottleneck distance defines a distance on $\widehat{\mathcal{B}}$ by $d(\boldsymbol{B}_1, \boldsymbol{B}_2) = \inf_{c\in\mathbb{R}} d_{\mathrm{bot}}(\boldsymbol{B}_1, \boldsymbol{B}_2 + c)$, which we will continue to denote by d_{bot}.

It is an immediate consequence of (2) that if two Hamiltonian paths $\{\varphi^t_{H_0}\}_{0\leqslant t\leqslant 1}$ and $\{\varphi^t_{H_1}\}_{0\leqslant t\leqslant 1}$ are homotopic relative to endpoints in $\mathrm{Ham}(M,\omega)$, and so they represent the same element of $\mathrm{UHam}(M,\omega)$, then $\boldsymbol{B}_j(H_0) = \boldsymbol{B}_j(H_1)$ in $\widehat{\mathcal{B}}$. Hence, we obtain a map, which we will continue to denote by the same symbol,

$$\boldsymbol{B}_j : \mathrm{UHam}(M,\omega) \to \widehat{\mathcal{B}}.$$

The barcode $\boldsymbol{B}_j(\tilde{\varphi})$, where $\tilde{\varphi} \in \mathrm{UHam}(M,\omega)$, inherits appropriately restated versions of the properties listed in the previous section. Of course, we may also define the total barcode

$$\boldsymbol{B} : \mathrm{UHam}(M,\omega) \to \widehat{\mathcal{B}}.$$

Indeed, $\boldsymbol{B}(\tilde{\varphi})$ is simply $\boldsymbol{B}(H)$ considered as a barcode up to shift, where H is any Hamiltonian whose flow represents $\tilde{\varphi}$.

Remark 32 Alternatively to our approach in this article, one could define $\boldsymbol{B}_j(\tilde{\varphi})$ and $\boldsymbol{B}(\tilde{\varphi})$ to be $\boldsymbol{B}_j(H)$ and $\boldsymbol{B}(H)$, where H is a *mean-normalized* Hamiltonian whose flow is a representative of $\tilde{\varphi}$. Being mean-normalized means $\int_0^1\int_M H\omega^n = 0$. This defines $\boldsymbol{B}_j(\tilde{\varphi})$ and $\boldsymbol{B}(\tilde{\varphi})$ without any ambiguity as a barcode, as opposed to a barcode up to shift, and so one obtains maps $\boldsymbol{B}_j, \boldsymbol{B} : \mathrm{UHam}(M,\omega) \to \overline{\mathcal{B}}$. This is the manner in which barcodes are defined in [56], and in fact it is a more natural approach from the point of view of Hofer geometry. However, this approach is not suitable for our purposes as it yields barcodes which are not continuous in the uniform topology; see Remark 36.

As in the previous section, denote by τ the minimal positive generator of $\omega(\pi_2(M))$.

Definition 33 For any $\tilde{\varphi} \in \mathrm{UHam}(M,\omega)$, we define

$$\#\mathrm{Endpoints}(\boldsymbol{B}(\tilde{\varphi})) \cap [0,\tau) = \#\mathrm{Endpoints}(\boldsymbol{B}(H)) \cap [0,\tau),$$

where H is any Hamiltonian whose flow represents $\tilde{\varphi}$.

The above definition is well defined as a consequence of Corollary 31.[6]

3.3.1 Barcodes on surfaces of positive genus Let us first consider the case of a closed surface Σ which is of positive genus. It is well known that, in this case, the fundamental group of $\mathrm{Ham}(\Sigma, \omega)$ is trivial. Therefore, $\mathrm{UHam}(\Sigma, \omega) = \mathrm{Ham}(\Sigma, \omega)$ and hence, $\boldsymbol{B}_j : \mathrm{Ham}(\Sigma, \omega) \to \widehat{\mathcal{B}}$ is defined directly for Hamiltonian diffeomorphisms.

As an immediate consequence of (3) we see that

$$d_{\mathrm{bot}}(\boldsymbol{B}_j(\phi), \boldsymbol{B}_j(\psi)) \leqslant d_{\mathrm{Hofer}}(\phi, \psi). \tag{7}$$

The symplectic-invariance property described in Section 3.2 translates to the following conjugacy-invariance property: for any $\psi \in \mathrm{Symp}(\Sigma, \omega)$ and any $\phi \in \mathrm{Ham}(M, \omega)$,

$$\boldsymbol{B}_j(\psi^{-1}\phi\psi) = \boldsymbol{B}_j(\phi). \tag{8}$$

This follows from (4) and the fact that $\varphi^t_{H\circ\psi} = \psi^{-1}\varphi^t_H\psi$ for any Hamiltonian H.

Clearly, the total barcode is also well defined directly for Hamiltonian diffeomorphisms. Hence, we have a map $\boldsymbol{B} : \mathrm{Ham}(\Sigma, \omega) \to \widehat{\mathcal{B}}$ which also satisfies (7) and (8).

3.3.2 Barcodes on the sphere In the case of the sphere, the maps

$$\boldsymbol{B}_j : \mathrm{UHam}(\mathbb{S}^2, \omega) \to \widehat{\mathcal{B}}$$

do not descend to $\mathrm{Ham}(\mathbb{S}^2, \omega)$. This is because the fundamental group of $\mathrm{Ham}(\mathbb{S}^2, \omega)$ is nontrivial. In fact, it is $\mathbb{Z}/2\mathbb{Z}$ and is generated by a full rotation around the North–South axis of the sphere; see for example [54]. We will denote this rotation by Rot. Hence, we see that every element $\varphi \in \mathrm{Ham}(\mathbb{S}^2, \omega)$ has two lifts $\widetilde{\varphi}_1, \widetilde{\varphi}_2 \in \mathrm{UHam}(\mathbb{S}^2, \omega)$ with $\widetilde{\varphi}_2 = \mathrm{Rot} \circ \widetilde{\varphi}_1$. Therefore, for each $\varphi \in \mathrm{Ham}(\mathbb{S}^2, \omega)$, we have barcodes $\boldsymbol{B}_j(\widetilde{\varphi}_1)$ and $\boldsymbol{B}_j(\widetilde{\varphi}_2)$ as well as the total barcodes $\boldsymbol{B}(\widetilde{\varphi}_1)$ and $\boldsymbol{B}(\widetilde{\varphi}_2)$. These maps satisfy appropriately restated versions of (7) and (8).

Lastly, from (5) we obtain

$$\boldsymbol{B}_j(\widetilde{\varphi}) = \boldsymbol{B}_{j-4}(\widetilde{\varphi}). \tag{9}$$

Note that the shift by τ disappears as we're considering barcodes up to shift.

[6] The quantity $\#\mathrm{Endpoints}(\boldsymbol{B}(\widetilde{\varphi})) \cap [0, \tau)$ is well defined for $\varphi \in \mathrm{Ham}(M, \omega)$, ie $\#\mathrm{Endpoints}(\boldsymbol{B}(\widetilde{\varphi}_1)) \cap [0, \tau) = \#\mathrm{Endpoints}(\boldsymbol{B}(\widetilde{\varphi}_2)) \cap [0, \tau)$ for any two lifts $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ of φ to $\mathrm{UHam}(M, \omega)$. This fact, which will not be used in our paper, is a consequence of Proposition 57 and Remarks 69.

4 Continuity of barcodes on surfaces

In this section we prove our main results on continuity of barcodes and their extension to Hamiltonian homeomorphisms. The results of this section allow us to effectively define Hamiltonian Floer homology for Hamiltonian homeomorphisms of surfaces.

Before giving precise statements of these results we will introduce some of our conventions. Let d be a Riemannian distance on a closed manifold M. Given two maps $\phi, \psi : M \to M$, we write

$$d_{C^0}(\phi, \psi) = \sup_{x \in M} d(\phi(x), \psi(x)).$$

The C^0 topology on $\mathrm{Ham}(M, \omega)$ is the topology induced by d_{C^0}. We will denote by $\overline{\mathrm{Ham}}(M, \omega)$ the C^0–closure of $\mathrm{Ham}(M, \omega)$ taken inside the group of homeomorphisms of M. In the case where M is a surface, $\overline{\mathrm{Ham}}(M, \omega)$ coincides with the set of area-preserving homeomorphisms of M with vanishing flux or, equivalently, mean rotation vector; for a proof of this fact see [12]. We will refer to elements of $\overline{\mathrm{Ham}}(M, \omega)$ as Hamiltonian homeomorphisms.

Surfaces of positive genus

Here is our main result concerning continuity of barcodes on surfaces other than the sphere.

Theorem 34 *Let (Σ, ω) denote a closed symplectic surface other than the sphere. For each integer j, the mapping*

$$\boldsymbol{B}_j : (\mathrm{Ham}(\Sigma, \omega), d_{C^0}) \to (\widehat{\mathcal{B}}, d_{\mathrm{bot}})$$

is continuous. Furthermore, $\boldsymbol{B}_j$ extends continuously to $\overline{\mathrm{Ham}}(\Sigma, \omega)$.

Remark 35 The total barcode map $\boldsymbol{B} : (\mathrm{Ham}(\Sigma, \omega), d_{C^0}) \to (\widehat{\mathcal{B}}, d_{\mathrm{bot}})$ is also continuous and extends continuously to $\overline{\mathrm{Ham}}(\Sigma, \omega)$. This is not an immediate consequence of the previous theorem. The proof of this fact is identical to the proof of the above theorem and so we will omit it.

Remark 36 In our approach, the barcodes $\boldsymbol{B}_j(\varphi)$ and $\boldsymbol{B}(\varphi)$ are well defined up to a shift by a constant. As mentioned in Remark 32, one can remove this ambiguity by working with mean-normalized Hamiltonians. This would yield a map $\boldsymbol{B}_j : \mathrm{Ham}(\Sigma, \omega) \to \overline{\mathcal{B}}$.

However, for the above theorem, it is absolutely crucial to consider barcodes up to shift, ie the space $\widehat{\mathcal{B}}$. Indeed, the maps $\boldsymbol{B}_j, \boldsymbol{B} : (\mathrm{Ham}(\Sigma, \omega), d_{C^0}) \to (\overline{\mathcal{B}}, d_{\mathrm{bot}})$ are not continuous. This remark also applies to Theorem 37.

The sphere

In the case of the sphere, $\mathrm{Ham}(\mathbb{S}^2,\omega)$ and $\overline{\mathrm{Ham}}(\mathbb{S}^2,\omega)$ are the groups of area-preserving diffeomorphisms and homeomorphisms of $\mathbb{S}^2$, respectively. The fundamental group of each of these groups is $\mathbb{Z}/2\mathbb{Z}$,[7] and it is generated by the full rotation around the North–South axis of the sphere. We will denote by $\mathrm{UHam}(\mathbb{S}^2,\omega)$ and $\overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)$ the universal covers of these groups. Being covering spaces, they naturally inherit the C^0 topology from $\mathrm{Ham}(\mathbb{S}^2,\omega)$ and $\overline{\mathrm{Ham}}(\mathbb{S}^2,\omega)$. As we will explain now, the C^0 topology on $\mathrm{UHam}(\mathbb{S}^2,\omega)$ and $\overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)$ may equivalently be defined via a natural lift of d_{C^0}: take $\widetilde{\varphi}_1,\widetilde{\varphi}_2\in\mathrm{UHam}(\mathbb{S}^2,\omega)$ and define

$$\widetilde{d}_{C^0}(\widetilde{\varphi}_1,\widetilde{\varphi}_2):=\inf\Bigl\{\max_{0\leqslant t\leqslant 1} d_{C^0}(\varphi_1^t,\varphi_2^t)\Bigr\},$$

where the infimum is taken over all paths $\{\varphi_1^t\}_{0\leqslant t\leqslant 1}$ and $\{\varphi_2^t\}_{0\leqslant t\leqslant 1}$ which represent $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ in $\mathrm{UHam}(\mathbb{S}^2,\omega)$. Of course, this definition extends to $\overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)$. Here is our main result concerning continuity of barcodes on the sphere:

Theorem 37 *For each integer j, the mapping*

$$\boldsymbol{B}_j:(\mathrm{UHam}(\mathbb{S}^2,\omega),\widetilde{d}_{C^0})\to(\widehat{\mathcal{B}},d_{\mathrm{bot}})$$

is continuous. Furthermore, $\boldsymbol{B}_j$ extends continuously to $\overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)$.

Remark 38 Once again, the total barcode $\boldsymbol{B}$ is also continuous and extends continuously to $\overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)$.

The rest of this section is dedicated to the proofs of Theorems 34 and 37. Since the proofs of the two theorems are quite similar, we will give a detailed proof of Theorem 34 and will only indicate what must be modified to obtain the proof of Theorem 37.

4.1 Proofs of Theorems 34 and 37

We begin with the proof of Theorem 34. It is a consequence of the following two theorems:

Theorem 39 *For any $\varepsilon>0$, there exists $\delta>0$ with the following property: if $\psi\in\mathrm{Ham}(\Sigma,\omega)$ and $d_{C^0}(\psi,\mathrm{Id})<\delta$, then*

$$d_{\mathrm{bot}}(\boldsymbol{B}_j(\psi),\boldsymbol{B}_j(\mathrm{Id}))<\varepsilon,$$

for any $j\in\mathbb{Z}$.

[7] Here, we are working under the assumption that $\mathrm{Ham}(\mathbb{S}^2,\omega)$ is equipped with the C^0 topology. However, equipping it with the C^∞ topology, instead of C^0, would not affect its fundamental group.

Theorem 40 *Suppose that* $\mathrm{Id} \neq \eta \in \overline{\mathrm{Ham}}(\Sigma, \omega)$. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$, *depending on* η, *with the following property*: *if* $\phi, \psi \in \mathrm{Ham}(\Sigma, \omega)$, $d_{C^0}(\phi, \eta) < \delta$ *and* $d_{C^0}(\psi, \mathrm{Id}) < \delta$, *then*

$$d_{\mathrm{bot}}(\boldsymbol{B}_j(\phi\psi), \boldsymbol{B}_j(\phi)) < \varepsilon$$

for any $j \in \mathbb{Z}$.

The above two theorems are proven below in Sections 4.3 and 4.4. Let us explain why Theorem 34 follows from the above two results. Indeed, it is not difficult to see that Theorem 39 proves the continuity of $\boldsymbol{B}_j : (\mathrm{Ham}(\Sigma, \omega), d_{C^0}) \to (\widehat{\mathcal{B}}, d_{\mathrm{bot}})$ at the identity and Theorem 40 proves its continuity at every other Hamiltonian diffeomorphism. Furthermore, Theorem 40 implies that the map $\boldsymbol{B}_j$ extends continuously to $\overline{\mathrm{Ham}}(\Sigma, \omega)$. To see this, take $\eta \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ and let ϕ_i be a sequence in $\mathrm{Ham}(\Sigma, \omega)$ converging uniformly to η. For any $\varepsilon > 0$, there exists a positive integer N such that if $N \leqslant i, k$, then $d_{C^0}(\phi_i, \eta) < \delta$ and $d_{C^0}(\phi_i^{-1}\phi_k, \mathrm{Id}) < \delta$. It then follows from Theorem 40 that $d_{\mathrm{bot}}(\boldsymbol{B}_j(\phi_i), \boldsymbol{B}_j(\phi_k)) < \varepsilon$. This, of course, implies that the sequence of barcodes $\boldsymbol{B}_j(\phi_i)$ is convergent in $\widehat{\mathcal{B}}$, and hence we can define $\boldsymbol{B}_j(\eta)$ to be the limit of this sequence. We leave it to the reader to check that $\boldsymbol{B}_j(\eta)$ does not depend on the choice of the approximating sequence and that the extension of $\boldsymbol{B}_j$ to $\overline{\mathrm{Ham}}(\Sigma, \omega)$ is continuous.

The proof of Theorem 37 is very similar to that of Theorem 34. We first need to introduce a bit of notation. Recall that the identity in $\mathrm{Ham}(\mathbb{S}^2, \omega)$ has two lifts to $\mathrm{UHam}(\mathbb{S}^2, \omega)$: one represented by the homotopy class of the constant path based at the identity, which we will continue to denote by Id, and a second represented by the homotopy class of the nontrivial loop in $\pi_1(\mathrm{Ham}(\mathbb{S}^2, \omega))$, which we will denote by Rot. One must begin by proving the following two results, which are analogous to Theorems 39 and 40.

Theorem 41 *Let* $\tilde{\psi}$ *denote an element of* $\mathrm{UHam}(\mathbb{S}^2, \omega)$. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*:

(1) *If* $\tilde{d}_{C^0}(\tilde{\psi}, \mathrm{Id}) < \delta$, *then* $d_{\mathrm{bot}}(\boldsymbol{B}_j(\tilde{\psi}), \boldsymbol{B}_j(\mathrm{Id})) < \varepsilon$ *for any* $j \in \mathbb{Z}$.

(2) *If* $\tilde{d}_{C^0}(\tilde{\psi}, \mathrm{Rot}) < \delta$, *then* $d_{\mathrm{bot}}(\boldsymbol{B}_j(\tilde{\psi}), \boldsymbol{B}_j(\mathrm{Rot})) < \varepsilon$ *for any* $j \in \mathbb{Z}$.

Theorem 42 *Let* $\tilde{\eta} \in \overline{\mathrm{UHam}}(\mathbb{S}^2, \omega) \setminus \{\mathrm{Id}, \mathrm{Rot}\}$. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$, *depending on* η, *with the following property*: *if* $\tilde{\phi}, \tilde{\psi} \in \mathrm{UHam}(\mathbb{S}^2, \omega)$, $\tilde{d}_{C^0}(\tilde{\phi}, \tilde{\eta}) < \delta$

and $\widetilde{d}_{C^0}(\widetilde{\psi}, \mathrm{Id}) < \delta$, *then*

$$d_{\mathrm{bot}}(\boldsymbol{B}_j(\widetilde{\phi}\widetilde{\psi}), \boldsymbol{B}_j(\widetilde{\phi})) < \varepsilon$$

for any $j \in \mathbb{Z}$.

The reasoning as to why Theorem 37 follows from the above two theorems is almost identical to the one given above for why Theorem 34 follows from Theorems 39 and 40. Indeed, Theorem 41 proves continuity of $\boldsymbol{B}_j$ at Id and Rot while Theorem 42 proves continuity of $\boldsymbol{B}_j$ elsewhere and furthermore allows us to extend $\boldsymbol{B}_j$ continuously to $\overline{\mathrm{UHam}}(\mathbb{S}^2, \omega)$.

Proofs of the above theorems are, once again, very similar to the proofs of Theorems 39 and 40. The only difference here is that instead of working with diffeomorphisms (or homeomorphisms) we must work with paths of diffeomorphisms (or homeomorphisms) which are based at the identity. This is, of course, necessary as we have to deal with elements of $\mathrm{UHam}(S^2, \omega)$. We will not provide proofs for Theorems 41 and 42 as one could indeed prove them by making simple modifications to the proofs of Theorems 39 and 40.

4.2 Preliminary results on continuity of barcodes

The results of this subsection, which will be needed for the proof of the theorems of the previous section, hold on any symplectic manifold which is monotone, ie not necessarily surfaces. We begin with the following definition:

Definition 43 Let ϕ be a homeomorphism of M and δ a positive real number. We will say that a set U is δ*–shifted* by ϕ if $d(x, \phi(x)) > \delta$ for each $x \in U$.

Consider Hamiltonians K and H and recall that

$$K \# H(t,x) := K(t,x) + H(t, (\varphi_K^t)^{-1}(x))$$

is the Hamiltonian whose flow is the composition $\varphi_K^t \varphi_H^t$.

Proposition 44 *Let K and H be Hamiltonians satisfying the following criteria*:

(1) *The support of H is δ–shifted by φ_K^1 for some $\delta > 0$.*
(2) $d_{C^0}(\mathrm{Id}, \varphi_H^t) < \delta$ *for each* $t \in [0,1]$.

Then, for each integer j, $\boldsymbol{B}_j(K \# H) = \boldsymbol{B}_j(K)$.

Proof We will only give a sketch of the proof of the above statement as the proof is similar to the proof of the results contained in [60].

For each $s \in [0,1]$ consider the Hamiltonian $F_s(t,x) = K(t,x) + sH(st, (\varphi_K^t)^{-1}(x))$ and note that $\varphi_{F_s}^1 = \varphi_K^1 \circ \varphi_H^s$. We leave it to the reader to check that the assumptions that φ_K^1 δ–shifts the support of H and that $d_{C^0}(\mathrm{Id}, \varphi_H^s) < \delta$ imply that the Hamiltonian diffeomorphisms φ_K^1 and $\varphi_{F_s}^1 = \varphi_K^1 \circ \varphi_H^s$ have the exact same fixed points and, in fact, coincide on a neighborhood of their fixed-point sets; see for example the proof of Theorem 4 of [60]. Furthermore, the corresponding Hamiltonians K and F_s have the exact same action spectra, ie $\mathrm{Spec}(K) = \mathrm{Spec}(F_s)$. Although this last claim is not directly stated in [60], it is contained therein within the proof of Proposition 2.2 and hence we will not give a proof of this fact either.

We will now prove that $\boldsymbol{B}_j(F_s) = \boldsymbol{B}_j(F_0) = \boldsymbol{B}_j(K)$. Without loss of generality we may assume that the Hamiltonian K is nondegenerate. This implies that the Hamiltonians F_s are also nondegenerate because $\varphi_{F_s}^1$ and φ_K^1 coincide near their fixed-point sets. Denote by $\mathrm{Spec}_j(F_s)$ the subset of $\mathrm{Spec}(F_s)$ consisting of actions of those 1–periodic orbits of F_s which are of Conley–Zehnder index j; this set is well defined and finite by nondegeneracy of F_s. Moreover, by repeating the argument which proves that $\mathrm{Spec}(K) = \mathrm{Spec}(F_s)$, one can show that for all $j \in \mathbb{Z}$ we have

$$\mathrm{Spec}_j(K) = \mathrm{Spec}_j(F_s).$$

As we explained in Section 3.2, the endpoints of the bars of $\boldsymbol{B}_j(F_s)$ are contained in the set $\mathrm{Spec}_j(F_s) \cup \mathrm{Spec}_{j+1}(F_s) = \mathrm{Spec}_j(K) \cup \mathrm{Spec}_{j+1}(K)$; this is a finite subset of $\mathbb{R}$ because K is nondegenerate. Now, the barcode $\boldsymbol{B}_j(F_s)$ varies continuously with s by the continuity property of barcodes, and so the endpoints of $\boldsymbol{B}_j(F_s)$ vary continuously in the finite set $\mathrm{Spec}_j(K) \cup \mathrm{Spec}_{j+1}(K)$. Of course, this implies that $\boldsymbol{B}_j(F_s)$ is constant and, hence, $\boldsymbol{B}_j(F_s) = \boldsymbol{B}_j(F_0) = \boldsymbol{B}_j(K)$. Noting that $F_1 = K \# H$ completes the proof. □

We will be needing the following proposition as well. Results similar to the following proposition are ubiquitous within the literature on the theory of spectral invariants; see for example [68; 50; 48; 66].

Proposition 45 *Suppose that K and H are two Hamiltonians such that φ_K^1 displaces the support of H. Then $\boldsymbol{B}_j(K \# H) = \boldsymbol{B}_j(K)$ for any integer j.*

Proof The proof of this proposition is very similar to that of Proposition 44. For each $s \in [0,1]$ consider the Hamiltonian $F_s(t,x) = K(t,x) + sH(st, (\varphi_K^t)^{-1}(x))$.

As in Proposition 44, using the fact that $\varphi^1_{F_s} = \varphi^1_K \varphi^s_H$ for each $s \in [0,1]$, one can easily show that $\mathrm{Spec}(K) = \mathrm{Spec}(F_s)$ for each $s \in [0,1]$. Repeating the argument from the last paragraph of the proof of Proposition 44 would lead to the conclusion that $\boldsymbol{B}_j(F_s) = \boldsymbol{B}(F_0) = \boldsymbol{B}(K)$ for each $s \in [0,1]$. Since $F_1 = K \# H$, we conclude that $\boldsymbol{B}_j(K \# H) = \boldsymbol{B}_j(K)$, which completes the proof. □

4.3 Proof of Theorem 39

Our proof of Theorem 39 will rely on the following fragmentation result:

Claim 46 *There exists a covering of Σ by two proper open subsets V_1 and V_2 with the following property: for every $r > 0$, there exists a C^0–neighborhood ν of the identity in $\mathrm{Ham}(\Sigma,\omega)$ such that if $\psi \in \nu$, then ψ can be written as a composition $\psi = \psi_1\psi_2$, where ψ_1 is supported in V_1, ψ_2 is supported in V_2, and $d_{C^0}(\psi_k, \mathrm{Id}) < r$.*

Furthermore, one can find Hamiltonians H_1 and H_2 such that H_k is supported in V_k, $\psi_k = \varphi^1_{H_k}$ and $d_{C^0}(\varphi^t_{H_k}, \mathrm{Id}) < r$ for all $t \in [0,1]$.

We will postpone the proof of the above claim and present a proof of Theorem 39. Let $\varepsilon > 0$. For $k = 1, 2$, we pick a C^2–small Morse function f_k satisfying the following properties: f_k has no critical point which is in the closure of V_k, the critical points of f_k are the only fixed points of $\varphi^1_{f_k}$, and $\|f_k\| < \frac{1}{8}\varepsilon$. Observe that since $\varphi^1_{f_k}$ has no fixed points in the closure of V_k, there exists $R_k > 0$ such that $\varphi^1_{f_k}$ R_k–shifts V_k. We let $R = \min\{R_1, R_2\}$.

We pick $r > 0$ to be small in comparison to R. We will pick $\delta > 0$ small enough that if $d_{C^0}(\mathrm{Id}, \psi) < \delta$, then ψ belongs to the neighborhood ν of the identity given by Claim 46. Let V_1, V_2, ψ_1 and ψ_2, and H_1 and H_2, be as in the conclusion of Claim 46. Clearly, for $k = 1, 2$ the support of H_k is R–shifted by $\varphi^1_{f_k}$ and $d_{C^0}(\varphi^t_{H_k}, \mathrm{Id}) < r < R$. Thus, applying Proposition 44, we see that

$$\boldsymbol{B}_j(\varphi^1_{f_k}\psi_k) = \boldsymbol{B}_j(\varphi^1_{f_k}). \tag{10}$$

Furthermore, the map $\psi_2\varphi^1_{f_1}$ shifts the support of H_1 by more than $R - r$. Since r is small compared to R and $d_{C^0}(\varphi^t_{H_1}, \mathrm{Id}) < r$, we may again apply Proposition 44 and conclude

$$\boldsymbol{B}_j(\psi_2\varphi^1_{f_1}\psi_1) = \boldsymbol{B}_j(\psi_2\varphi^1_{f_1}). \tag{11}$$

Combining the above with the continuity property of barcodes (with respect to the Hofer distance), we obtain the chain of inequalities

$$\begin{aligned}
&d_{\mathrm{bot}}(\boldsymbol{B}_j(\psi_2\psi_1), \boldsymbol{B}_j(\mathrm{Id}))\\
&\qquad \leqslant d_{\mathrm{bot}}(\boldsymbol{B}_j(\psi_2\psi_1), \boldsymbol{B}_j(\psi_1)) + d_{\mathrm{bot}}(\boldsymbol{B}_j(\psi_1), \boldsymbol{B}_j(\mathrm{Id}))\\
&\qquad \leqslant d_{\mathrm{bot}}(\boldsymbol{B}_j(\psi_2\varphi^1_{f_1}\psi_1), \boldsymbol{B}_j(\varphi^1_{f_1}\psi_1)) + d_{\mathrm{bot}}(\boldsymbol{B}_j(\varphi^1_{f_1}\psi_1), \boldsymbol{B}_j(\mathrm{Id})) + 3\|f_1\|\\
&\qquad \leqslant d_{\mathrm{bot}}(\boldsymbol{B}_j(\psi_2\varphi^1_{f_1}), \boldsymbol{B}_j(\varphi^1_{f_1})) + d_{\mathrm{bot}}(\boldsymbol{B}_j(\varphi^1_{f_1}), \boldsymbol{B}_j(\mathrm{Id})) + 3\|f_1\|\\
&\qquad \leqslant d_{\mathrm{bot}}(\boldsymbol{B}_j(\psi_2), \boldsymbol{B}_j(\mathrm{Id})) + 6\|f_1\|\\
&\qquad \leqslant d_{\mathrm{bot}}(\boldsymbol{B}_j(\varphi^1_{f_2}\psi_2), \boldsymbol{B}_j(\mathrm{Id})) + \|f_2\| + 6\|f_1\|\\
&\qquad = d_{\mathrm{bot}}(\boldsymbol{B}_j(\varphi^1_{f_2}), \boldsymbol{B}_j(\mathrm{Id})) + \|f_2\| + 6\|f_1\|\\
&\qquad \leqslant 2\|f_2\| + 6\|f_1\| \leqslant \varepsilon.
\end{aligned}$$

In the above chain of inequalities, we have applied Hofer continuity in passing from the first line to the second, from the third to the fourth, from the fourth to the fifth and from the sixth to the seventh, and we have used (10) and (11) in passing from the second line to the third and from the fifth to the sixth.

It remains to explain why Claim 46 is true. The first half of Claim 46 follows immediately from the following statement by setting $V_1 = D_1, V_2 = D_2 \cup \cdots \cup D_{2g+2}$:

Claim 47 *Denote by (M,ω) a closed symplectic surface of genus g. There exists a cover of M by disks $D_1,\dots,D_{2g+2}$ with the following property: for every $r > 0$, there exists a neighborhood ν of the identity in $\mathrm{Ham}(M,\omega)$ such that, if $\psi \in \nu$, then ψ can be written as a composition $\psi = \psi_1\psi_2\cdots\psi_{2g+2}$, where each ψ_i is supported in one of the disks D_j and $d_{C^0}(\psi_i, \mathrm{Id}) < r$.*

The above is the statement of Theorem 3.1 from [60]. The proof given in [60] was obtained by modifying a similar result from [11]. The result in [11] was in turn inspired by earlier works of Fathi [12] and Le Roux [42].

It remains to prove the latter statement of Claim 46. We leave it to the reader to check that it follows from Claim 47 and Lemma 3.2 of [60], whose statement we will recall for the reader's convenience:

Let B denote the unit ball in $\mathbb{R}^{2n}$, ω_0 the standard symplectic form on $\mathbb{R}^{2n}$ and $\psi \in \mathrm{Ham}_c(B,\omega_0)$. There exists a Hamiltonian H supported in B such that $\varphi^1_H = \psi$ and $d_{C^0}(\mathrm{Id}, \varphi^t_H) < d_{C^0}(\mathrm{Id}, \psi)$ for all $t \in [0,1]$.

This completes the proof of Theorem 39.

4.4 Proof of Theorem 40

Let $\eta \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ as in the statement, and $\varepsilon > 0$. Let B be a disk in M such that $\eta(B) \cap B = \varnothing$. Such a disk exists because $\eta \neq \mathrm{Id}$. We may assume that ε is small in comparison to the area of B. We pick a small $\delta > 0$ and $\phi, \psi \in \mathrm{Ham}(\Sigma, \omega)$ such that $d_{C^0}(\phi, \eta) < \delta$ and $d_{C^0}(\psi, \mathrm{Id}) < \delta$.

The following claim proves the theorem in the special case where there exists a disk D which contains the support of ψ:

Claim 48 *Suppose that $\phi(B) \cap B = \varnothing$. Let $D \subset \Sigma$ be a disk. There exists a constant δ_D, depending only on D, such that if ψ is supported in D and $d_{C^0}(\psi, \mathrm{Id}) \leqslant \delta_D$, then for any $j \in \mathbb{Z}$ we have*

$$d_{\mathrm{bot}}(\boldsymbol{B}_j(\phi\psi), \boldsymbol{B}_j(\phi)) < \varepsilon.$$

Before proving the claim we will show how Theorem 40 follows from it. According to Claim 47, we can cover the surface M by disks $D_1, \ldots, D_{2g+2}$, where g denotes the genus of the surface, such that if ψ is sufficiently C^0–small, then it can be written as $\psi = \psi_1\psi_2\cdots\psi_{2g+2}$, where each ψ_k is still C^0–close to the identity and is supported in one of the disks D_j. We may assume that δ has been picked small enough that $d_{C^0}(\psi_k, \mathrm{Id}) \leqslant \delta_{D_j}$ and $\phi\psi_1\psi_2\cdots\psi_k(B) \cap B = \varnothing$ for each k. Then, for each k, we can apply Claim 48 to $\phi\psi_1\cdots\psi_k$ and ψ_{k+1} and conclude that $d_{\mathrm{bot}}(\boldsymbol{B}_j(\phi\psi_1\cdots\psi_{k+1}), \boldsymbol{B}_j(\phi\psi_1\cdots\psi_k)) < \varepsilon$. Finally, we apply the triangular inequality to get

$$d_{\mathrm{bot}}(\boldsymbol{B}_j(\phi\psi), \boldsymbol{B}_j(\phi)) < (2g+2)\varepsilon,$$

which implies the theorem.

The proof of Claim 48 relies on the following fragmentation-type lemma, whose proof will be postponed to the end of this section:

Lemma 49 *Consider the unit square $[0,1] \times [0,1]$ in $\mathbb{R}^2$ equipped with the standard symplectic structure. Let m be a positive integer and ρ a positive real number.*

Define a partition of $[0,1] \times [0,1]$ into rectangles $U_i = [0,1] \times [(i-1)/m, i/m]$ for $i = 1, \ldots, m$. There exists a constant $\delta_0 > 0$, depending on m, with the following property: if ψ is a Hamiltonian diffeomorphism whose support is compactly contained *in the interior of* $[0,1]\times[0,1]$ *and satisfies* $d_{C^0}(\psi, \mathrm{Id}) < \delta_0$*, then there exist Hamiltonian diffeomorphisms* $\theta, \psi_1, \dots, \psi_m$ *such that the support of* θ *is contained in a disjoint union of balls of total area less than* $\frac{1}{2}\rho$*, each* ψ_i *is supported in the interior of* U_i*, and* $\psi = \psi_1 \cdots \psi_m \theta$*.*

Proof of Claim 48 Let ϕ, ψ and D be as in the statement. We will make the assumption that $\mathrm{Area}(D) = 1$. Using an area-preserving identification[8] of the disk D with the unit square $[0,1]\times[0,1]$, we will assume that D is an embedding of the unit square $[0,1]\times[0,1]$ in M.

Let $U_1, \dots, U_m$ be a partition of D into rectangles as described in the statement of Lemma 49. Note that the rectangles U_i have the same area. We will pick m to be large enough that the area of each U_i is smaller than $\frac{1}{2}\varepsilon$ and will pick ρ to be small in comparison to ε.

Now, according to Lemma 49, there exists a constant δ_D, which depends on the choice of the map identifying D with $[0,1]\times[0,1]$, such that if $d_{C^0}(\mathrm{Id}, \psi) \leqslant \delta_D$, then we obtain Hamiltonian diffeomorphisms $\theta, \psi_1, \dots, \psi_m$ satisfying the properties listed in that lemma.

Let $N = \lceil 2\,\mathrm{Area}(D)/\mathrm{Area}(B) \rceil$. For each $k \in \{1, \dots, N\}$, let $W_k = \bigsqcup U_j$, where the (disjoint) union is taken over the set $\{j : j = k \bmod N\}$. Define Ψ_k to be the composition of those ψ_j such that $j = k \bmod N$. Clearly, Ψ_k is supported in W_k.

We leave it to the reader to check that the total area of each W_k is less than the area of B. Hence, since each W_K is a disjoint union of topological disks, we can find $f_k \in \mathrm{Ham}(M, \omega)$ such that $f_k(W_k) \subset B$. Furthermore, since each connected component of W_k is a topological disk of area less than $\frac{1}{2}\varepsilon$, the Hamiltonian diffeomorphism f_k can be picked such that $d_{\mathrm{Hofer}}(\mathrm{Id}, f_k) < \varepsilon$. Note also that since the support of θ is contained in a disjoint union of topological disks of total area less than $\frac{1}{2}\rho$, we can find $g \in \mathrm{Ham}(M, \omega)$ such that it maps the support of θ into B and $d_{\mathrm{Hofer}}(\mathrm{Id}, g) < \rho < \varepsilon$. Now, consider the Hamiltonian diffeomorphism

$$\psi' = f_1 \Psi_1 f_1^{-1} f_2 \Psi_2 f_2^{-1} \cdots f_N \Psi_N f_N^{-1} g \theta g^{-1}.$$

[8] Although the identification map will have to be nonsmooth (at four points) on the boundary of D, we can ensure that it is smooth in the interior of D and so this will allow us to identify ψ with a smooth area-preserving map of $[0,1]\times[0,1]$.

It is easy to see that ψ' is supported in B. Furthermore, because $\psi = \Psi_1 \cdots \Psi_N \theta$, using bi-invariance of the Hofer metric, we obtain the inequalities

$$\begin{aligned} d_{\mathrm{Hofer}}(\psi, \psi') &\leqslant \sum_{i=1}^{N} d_{\mathrm{Hofer}}(\Psi_i, f_i \Psi_i f_i^{-1}) + d_{\mathrm{Hofer}}(\theta, g\theta g^{-1}) \\ &\leqslant \sum_{i=1}^{N} 2d_{\mathrm{Hofer}}(\mathrm{Id}, f_i) + 2d_{\mathrm{Hofer}}(\mathrm{Id}, g) < 2(N+1)\varepsilon. \end{aligned}$$

Since $\phi(B) \cap B = \varnothing$, Proposition 45 tells us that $\boldsymbol{B}_j(\phi\psi') = \boldsymbol{B}_j(\phi)$. Furthermore, the fact that $d_{\mathrm{Hofer}}(\psi, \psi') < 2(N+1)\varepsilon$ implies that $d_{\mathrm{bot}}(\boldsymbol{B}_j(\phi\psi), \boldsymbol{B}_j(\phi\psi')) < 2(N+1)\varepsilon$. We conclude from the above that

$$d_{\mathrm{bot}}(\boldsymbol{B}_j(\phi\psi), \boldsymbol{B}_j(\phi)) < 2(N+1)\varepsilon.$$

Lastly, replacing ε by $\varepsilon/(2N+1)$ throughout the above argument completes the proof of the claim. □

We will finish this section by presenting a proof of Lemma 49.

Proof of Lemma 49 Our proof of this lemma relies on the following extension lemma, a proof of which can be found in [11, Lemma 6.3].

Lemma 50 *Let $V'' = [0, R] \times [-c'', c'']$ be a rectangle, equipped with some area form, and $V \subset V' \subset V''$ be two smaller rectangles of the form $V = [0, R] \times [-c, c]$ and $V' = [0, R] \times [-c', c']$ with $0 < c < c' < c''$. Let $\psi \colon V' \to V''$ be an area-preserving embedding such that*:

- *ψ is the identity near $\{0\} \times [-c', c']$ and $\{R\} \times [-c', c']$.*
- *The area in V'' bounded by $[0, R] \times \{y\}$ and its image is zero for some, and hence all, $y \in [-c', c']$.*

Then there exists a Hamiltonian diffeomorphism θ compactly supported in V'' such that $\theta|_V = \psi|_V$.

Pick $r > 0$ to be so small that $r < \rho/3(m-1)$. For each $1 \leqslant i \leqslant m-1$, consider the rectangles $V_i \subset V_i' \subset V_i''$ defined by $V_i = [0,1] \times [i/m - r, i/m + r]$, $V_i' = [0,1] \times [i/m - 2r, i/m + 2r]$ and $V_i'' = [0,1] \times [i/m - 3r, i/m + 3r]$. (We will assume that r is small enough that the $V_i'' \cap V_j'' = \varnothing$ if $j \neq i$.) For each $1 \leqslant i \leqslant m-1$, consider the restriction $\psi|_{V_i'}$. Pick δ_0 small enough that if $d_{C^0}(\mathrm{Id}, \psi) < \delta_0$, then $\psi(V_i') \subset V_i''$.

In other words, $\psi|_{V_i'}$ is an area-preserving embedding of V_i' into V_i''. We will leave it to the reader to check that the hypotheses of Lemma 50 are satisfied and hence, applying the lemma, we obtain Hamiltonian diffeomorphisms θ_i for $1 \leqslant i \leqslant m-1$ such that θ_i is supported in V_i'' and $\theta_i = \psi$ on V_i.

Let $\theta = \theta_1 \cdots \theta_{m-1}$ and note that, because $r < \rho/3(m-1)$, the total area of the support of θ is less than ρ. Next, observe that $\psi\theta^{-1}$ coincides with the identity on each of the rectangles V_i. This implies that the support of $\psi\theta^{-1}$ is contained in the disjoint union

$$[0,1] \times \left[0, \frac{1}{m} - r\right] \cup [0,1] \times \left[\frac{1}{m} + r, \frac{2}{m} - r\right] \cup \cdots \cup [0,1] \times \left[\frac{i}{m} + r, \frac{i+1}{m} - r\right] \cup \cdots \cup [0,1] \times \left[\frac{m-1}{m} + r, 1\right].$$

Define ψ_1 to be the restriction of $\psi\theta^{-1}$ to $[0,1]\times[0,1/m-r]$, ψ_i to be the restriction of $\psi\theta^{-1}$ to $[0,1] \times [i/m + r, (i+1)/m - r]$ for $2 \leqslant i \leqslant m-1$, and ψ_m to be the restriction of $\psi\theta^{-1}$ to $[0,1] \times [(m-1)/m + r, 1]$. Clearly, each ψ_i is compactly supported in the interior of U_i and $\psi = \psi_1 \cdots \psi_m \theta$. □

5 Barcodes as invariants of weak conjugacy classes and the proof of Theorem 1

In this section, we will prove Theorem 1 of the introduction, that is, in the case of diffeomorphisms, the absolute Lefschetz number is invariant under weak conjugacy.

5.1 The weak conjugacy relation

Let us begin by giving a precise definition of the weak conjugacy relation. Recall that the graph of an equivalence relation $\sim$ on a set Z is the set of pairs $(z, w) \in Z \times Z$ such that $z \sim w$. Given two equivalence relations $\sim_1$ and $\sim_2$, we say that $\sim_1$ is smaller than $\sim_2$ if the graph of $\sim_1$ is a subset of the graph of $\sim_2$.[9] An equivalence relation $\sim$ on a topological space Z is said to be Hausdorff if the quotient $Z/\sim$ is Hausdorff.

Definition 51 Let G be a topological group. The weak conjugacy relation is the smallest equivalence relation on G which is both Hausdorff and larger than the conjugacy relation. That is, its graph is the intersection of graphs of all Hausdorff equivalence relations which are larger than the conjugacy relation.

[9] If $\sim_1$ is smaller than $\sim_2$, then $z \sim_1 w$ implies $z \sim_2 w$, ie $\sim_1$ is stronger than $\sim_2$.

We leave it to the reader to check that the weak conjugacy relation may be characterized by the following universal property: z is weakly conjugate to w if and only if $\sigma(z) = \sigma(w)$ for any continuous function $\sigma\colon G \to Y$, where Y is a Hausdorff topological space and σ is invariant under conjugation.

For the rest of this section we will be primarily concerned with the weak conjugacy relation on the group $G = \overline{\mathrm{Ham}}(\Sigma, \omega)$. We will be needing the following lemma:

Lemma 52 *Suppose that $\Sigma = \mathbb{S}^2$ and that $f, g \in \overline{\mathrm{Ham}}(\mathbb{S}^2, \omega)$ are weakly conjugate. Then the lifts of f and g to $\overline{\mathrm{UHam}}(\mathbb{S}^2, \omega)$ are pairwise weakly conjugate, ie denoting the lifts of f by $\tilde{f}_1$ and $\tilde{f}_2$ and the lifts of g by $\tilde{g}_1$ and $\tilde{g}_2$, up to relabeling the lifts, we have that $\tilde{f}_i$ is weakly conjugate to $\tilde{g}_i$ for $i = 1, 2$.*

Proof Let $\tilde{f}$ be a lift of f. Recall that the other lift of f is given by $\mathrm{Rot} \circ \tilde{f}$, where Rot denotes the full rotation of the sphere around the North–South axis.

Denote the lifts of g by $\tilde{g}$ and $\mathrm{Rot} \circ \tilde{g}$. Observe that to prove the lemma it is sufficient to show that one of $\tilde{g}$ or $\mathrm{Rot} \circ \tilde{g}$ is weakly conjugate to $\tilde{f}$. In order to obtain a contradiction, we will suppose that this is not the case. This assumption has the following consequence:

Claim 53 *There exists a continuous mapping $\tau\colon \overline{\mathrm{UHam}}(\mathbb{S}^2, \omega) \to X$ such that X is a Hausdorff topological space, τ is invariant under conjugation, and $\tau(\tilde{f}) \neq \tau(\tilde{g})$ and $\tau(\tilde{f}) \neq \tau(\mathrm{Rot} \circ \tilde{g})$.*

Proof Since $\tilde{f}$ is not weakly conjugate to any of $\tilde{g}$ or $\mathrm{Rot} \circ \tilde{g}$, we can find continuous mappings $\tau_1\colon \overline{\mathrm{UHam}}(\mathbb{S}^2, \omega) \to X_1$ and $\tau_2\colon \overline{\mathrm{UHam}}(\mathbb{S}^2, \omega) \to X_2$ such that X_1 and X_2 are Hausdorff, τ_1 and τ_2 are invariant under conjugation, and $\tau_1(\tilde{f}) \neq \tau_1(\tilde{g})$ and $\tau_2(\tilde{f}) \neq \tau_2(\mathrm{Rot} \circ \tilde{g})$.

Let $X = X_1 \times X_2$ and define $\tau\colon \overline{\mathrm{UHam}}(\mathbb{S}^2, \omega) \to X$ by

$$\tilde{h} \mapsto (\tau_1(\tilde{h}), \tau_2(\tilde{h})).$$

The topological space X is clearly Hausdorff and the mapping τ is continuous and invariant under conjugation. Finally, one can easily check that $\tau(\tilde{f}) \neq \tau(\tilde{g})$ and $\tau(\tilde{f}) \neq \tau(\mathrm{Rot} \circ \tilde{g})$. □

Using the above claim, we will construct a continuous mapping $\sigma\colon \mathrm{Ham}(\mathbb{S}^2,\omega)\to Y$, where Y is a Hausdorff topological space and σ is invariant under conjugation, such that $\sigma(f)\neq\sigma(g)$. This is, of course, a contradiction and we conclude that the lemma must be true.

Let us describe the construction of $\sigma\colon \mathrm{Ham}(\mathbb{S}^2,\omega)\to Y$. Set Y to be the quotient of $X\times X$ by the equivalence relation $(z,w)\sim(w,z)$. We leave it to the reader to check that Y is a Hausdorff topological space. Define $\sigma\colon \overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)\to Y$ by

$$\sigma(\tilde f)=(\tau(\tilde f),\tau(\mathrm{Rot}\circ\tilde f)).$$

Note that σ is constructed such that $\sigma(\tilde f)=\sigma(\mathrm{Rot}\circ\tilde f)$. Thus, it yields a well-defined mapping $\sigma\colon \overline{\mathrm{Ham}}(\mathbb{S}^2,\omega)\to Y$. Furthermore, σ is invariant under conjugation because τ is invariant under conjugation. Finally, to complete the proof, we must check that $\sigma(f)\neq\sigma(g)$. Now, $\sigma(f)=(\tau(\tilde f),\tau(\mathrm{Rot}\circ\tilde f))$ and $\sigma(g)=(\tau(\tilde g),\tau(\mathrm{Rot}\circ\tilde g))$. By Claim 53, $\tau(\tilde f)\neq\tau(\tilde g)$ and $\tau(\tilde f)\neq\tau(\mathrm{Rot}\circ\tilde g)$. This immediately implies that $(\tau(\tilde f),\tau(\mathrm{Rot}\circ\tilde f))\neq(\tau(\tilde g),\tau(\mathrm{Rot}\circ\tilde g))$. □

5.2 Proof of Theorem 1

Theorem 1 is an immediate consequence of Theorems 54 and 55, stated below. Indeed, Theorem 54 tells us that the barcodes we constructed in Section 3 are invariants of weak conjugacy classes and Theorem 55 states that the absolute Lefschetz number of a Hamiltonian diffeomorphism is an invariant of its barcode.

Throughout this section, Σ will denote a closed surface which we assume is equipped with a symplectic form ω.

Theorem 54 *Suppose that $f,g\in\overline{\mathrm{Ham}}(\Sigma,\omega)$ are weakly conjugate.*

(1) *If $\Sigma\neq\mathbb{S}^2$, then $\boldsymbol{B}_j(f)=\boldsymbol{B}_j(g)$ for every index j. Furthermore, the same is true for the total barcodes, ie $\boldsymbol{B}(f)=\boldsymbol{B}(g)$.*

(2) *Suppose that $\Sigma=\mathbb{S}^2$. Let $\tilde f_1$ and $\tilde f_2$, and $\tilde g_1$ and $\tilde g_2$, denote the lifts of f and g to $\overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)$. Then, up to relabeling the lifts, we have that $\boldsymbol{B}_j(\tilde f_1)=\boldsymbol{B}_j(\tilde g_1)$ and $\boldsymbol{B}_j(\tilde f_2)=\boldsymbol{B}_j(\tilde g_2)$ for every index j. Furthermore, the same is true for the total barcodes, ie $\boldsymbol{B}(\tilde f_1)=\boldsymbol{B}(\tilde g_1)$ and $\boldsymbol{B}(\tilde f_2)=\boldsymbol{B}(\tilde g_2)$.*

Proof The result in the case where $\Sigma\neq\mathbb{S}^2$ follows immediately from conjugacy-invariance of barcodes (equation (8)) and their continuity (Theorem 34).

Now, suppose that $\Sigma = \mathbb{S}^2$. Let $\tilde{f}_1$ and $\tilde{f}_2$, and $\tilde{g}_1$ and $\tilde{g}_2$, denote the lifts of f and g to $\overline{\mathrm{UHam}}(\mathbb{S}^2, \omega)$. By Lemma 52, up to relabeling the lifts, we have that $\tilde{f}_i$ is weakly conjugate to $\tilde{g}_i$ for $i = 1, 2$. Once again, the result follows from conjugacy-invariance of barcodes and their continuity (Theorem 37). □

For the next theorem, it might be helpful for the reader to recall the definition of $\#\mathrm{Endpoints}(\boldsymbol{B}) \cap [0, 1)$ from Section 3.3. We suppose, for the statement of this theorem and the rest of this section, that $\int_\Sigma \omega = 1$ when $\Sigma = \mathbb{S}^2$.

Theorem 55 *Consider $f \in \mathrm{Ham}(\Sigma, \omega)$ with finitely many fixed points.*

(1) *Suppose that $\Sigma \neq \mathbb{S}^2$. Then*
$$\sum_{x \in \mathrm{Fix}_c(f)} |L(f, x)| = \#\mathrm{Endpoints}(\boldsymbol{B}(f)).$$

(2) *Suppose that $\Sigma = \mathbb{S}^2$. Let $\tilde{f}$ denote either of the two lifts of f to $\mathrm{UHam}(\mathbb{S}^2, \omega)$. Then*
$$\sum_{x \in \mathrm{Fix}_c(f)} |L(f, x)| = \#\mathrm{Endpoints}(\boldsymbol{B}(\tilde{f})) \cap [0, 1).$$

Remark 56 It follows from our proof of this theorem that the set of endpoints of bars of $\boldsymbol{B}(f)$ coincides with the set of actions of contractible periodic orbits for which the corresponding fixed points have nonzero Lefschetz index. Fixed points with zero Lefschetz index make no contribution to the barcodes of a Hamiltonian diffeomorphism. Another way to see this is to combine the continuity of barcodes with the fact that zero-index fixed points can be removed by a small perturbation.

5.3 Local Floer homology and Theorem 55

Theorem 55 is an immediate consequence of Propositions 57 and 58, which are stated below. Both of these propositions rely on the notion of local Floer homology groups $HF(\varphi, x)$ associated to an isolated fixed point x of a Hamiltonian diffeomorphism φ. The definition and certain properties of local Floer homology will be recalled in Section 5.3. It is important to notice that many statements in this section hold in any dimension. The only statement exclusively valid in dimension 2 is Proposition 58. We refer to Remarks 69 for a counterexample in the general case.

Denote by $r(\varphi, x)$ the rank of $HF(\varphi, x)$. The next two propositions relate local Floer homology to barcodes and the Lefschetz index.

Proposition 57 *Consider $f \in \mathrm{Ham}(\Sigma,\omega)$ with finitely many fixed points.*

(1) *Suppose that $\Sigma \neq \mathbb{S}^2$. Then*

$$\sum_{x \in \mathrm{Fix}_c(f)} r(f,x) = \#\mathrm{Endpoints}(\boldsymbol{B}(f)).$$

(2) *Suppose that $\Sigma = \mathbb{S}^2$. Let $\tilde{f}$ denote either of the two lifts of f to $\mathrm{UHam}(\mathbb{S}^2,\omega)$. Then*

$$\sum_{x \in \mathrm{Fix}_c(f)} r(f,x) = \#\mathrm{Endpoints}(\boldsymbol{B}(\tilde{f})) \cap [0,1).$$

Although the above result is stated for surfaces only, an appropriately restated version holds in higher dimensions. See Remarks 69.

Proposition 58 *Suppose that x is an isolated fixed point of $f \in \mathrm{Ham}(\Sigma,\omega)$. Then $|L(f,x)| = r(f,x)$.*

5.3.1 A review of local Floer homology We will provide a brief review of the definition and some properties of local Floer homology. For further details we refer the reader to [17; 18; 19]; see also the original works of Floer which gave rise to this notion [13; 14].

We begin by reviewing local Morse homology. Let $h\colon M \to \mathbb{R}$ be a smooth function with an isolated critical point x. Take U to be an open neighborhood of x containing no other critical points of h and let $\tilde{h}$ be a smooth function which, on the open set U, is both Morse and C^2–close to h. The local Morse complex $CM_*^{\mathrm{loc}}(\tilde{h},x)$ is generated, over the field $\mathbb{F}$, by those critical points of $\tilde{h}$ which are contained in U. It can be checked that every Morse trajectory of $\tilde{h}$ connecting two such critical points is contained entirely in U. The same is also true for broken Morse trajectories. It follows that the usual Morse boundary map induces a boundary map ∂ on $CM_*^{\mathrm{loc}}(\tilde{h},x)$ and that $\partial^2 = 0$. The homology of this complex, denoted by $HM_*^{\mathrm{loc}}(h,x)$, is called the local Morse homology of h at x. It does not depend on the choice of $\tilde{h}$. This can alternatively be defined using Conley theory (see [10]): any sufficiently small neighborhood U of z is an isolating block, and the homological Conley index is $H_*(U,\partial^- U)$, where $\partial^- U$ is the subset in the boundary of U where the trajectories of the gradient flow exit from U. Invariance by deformation implies that this quantity is unchanged by going from h to $\tilde{h}$, and the fact that for a Morse function the homological Conley index coincides with the homology of the Morse complex is a classical result of Floer [15, Theorem 1]).

Having introduced local Morse homology, we will move on to local Floer homology. Let $\overline{z} = [z, u]$ denote a capped 1–periodic orbit of a Hamiltonian $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$. Let x be the fixed point of φ_H^1 corresponding to z, assume x is isolated and take some neighborhood U of x containing no other fixed point.

Assume that z is isolated and take U to be an open neighborhood of z in $\mathbb{S}^1 \times M$ which meets no other 1–periodic orbit of H and let $\widetilde{H}$ be a Hamiltonian which coincides with H outside U and is C^2–close to H on U. The orbit z breaks into several orbits z_i of $\widetilde{H}$ which are all C^1–close to z; we pick $\widetilde{H}$ such that all these orbits are nondegenerate. Using the capping u of z we can produce cappings u_i of z_i such that $\mathcal{A}_H([z, u])$ is close to $\mathcal{A}_{\widetilde{H}}([z_i, u_i])$.

By definition, the local Floer complex $CF_*^{\mathrm{loc}}(H, \overline{z})$ is generated, over the field $\mathbb{F}$, by the capped 1–periodic orbits $[z_i, u_i]$ of $\widetilde{H}$. As explained in Section 3.2 of [17], every Floer trajectory connecting two such 1–periodic orbits is contained entirely in U. The same is also true for broken Floer trajectories. It follows that the usual Floer boundary map induces a boundary map ∂ on $CF_*^{\mathrm{loc}}(H, \overline{z})$ and that $\partial^2 = 0$. The homology of this complex, denoted by $HF_*^{\mathrm{loc}}(H, \overline{z})$, is called the local Floer homology of H at z. It does not depend on the choice of the C^2–nearby Hamiltonian $\widetilde{H}$.

Remark 59 If z is a nondegenerate 1–periodic orbit of H, then $HF_*^{\mathrm{loc}}(H, \overline{z})$ has rank one in degree $\mu_{\mathrm{CZ}}(\overline{z})$ and is zero in all other degrees.

Remark 60 Local Morse and Floer homologies are related in the following sense: Let z be an isolated critical point of a C^2–small autonomous Hamiltonian H. We will also denote by z the corresponding 1–periodic orbit with its trivial capping. Then $HF_*^{\mathrm{loc}}(H, z) = HM_*^{\mathrm{loc}}(H, z)$.

Local Floer homology groups serve as building blocks for filtered Floer homology in the following sense: Suppose that all capped 1–periodic orbits $\overline{z}_i = [z_i, u_i]$ of H with action c are isolated. Then, for sufficiently small ε, we have

$$HF_*^{(c-\varepsilon, c+\varepsilon)}(H) = \bigoplus_{\mathcal{A}_H(\overline{z}_i) = c} HF_*^{\mathrm{loc}}(H, \overline{z}_i). \tag{12}$$

It turns out that, up to a shift in degree, $HF_*^{\mathrm{loc}}(H, \overline{z})$ depends only on the Hamiltonian diffeomorphism φ_H^1 and the fixed point corresponding to z. Here is a more precise statement: Let H' be a Hamiltonian such that $\varphi_{H'}^1 = \varphi_H^1$. Let z' be a 1–periodic orbit of the flow of H' such that $z'(0) = z(0)$, and consider the capped orbits $\overline{z} = [z, u]$

and $\overline{z}' = [z', u']$ for any choice of cappings u and u'. Then there exists an integer[10] k such that

$$HF_*^{\mathrm{loc}}(H', \overline{z}') = HF_{*+k}^{\mathrm{loc}}(H, z). \tag{13}$$

Hence, given $\varphi \in \mathrm{Ham}(M, \omega)$ with an isolated fixed point x, we can define $HF^{\mathrm{loc}}(\varphi, x)$ the (ungraded) local Floer homology of φ at x, and denote by $r(\varphi, x)$ its rank. It only depends on the germ of φ near the point x.

Note that even if (M, ω) is aspherical, $HF^{\mathrm{loc}}(\varphi, x)$ is only determined by φ^1 and x up to a shift in index. To recover the precise $\mathbb{Z}$–grading we need to know the isotopy φ^t in a neighborhood of x. However, in all cases, the ungraded homology is well determined by φ^1 and x.

Remark 61 One can similarly define local Lagrangian Floer theory: if L_1 and L_2 are exact Lagrangians submanifolds in (M, ω) and p an isolated point of $L_1 \cap L_2$, we may consider $HF_*^{\mathrm{loc}}(L_1, L_2, p)$. In the next section, we will need the following two facts concerning local Lagrangian Floer homology.

If φ is a Hamiltonian diffeomorphism of M, and we set Γ_φ to be the graph of φ in $\overline{M} \times M$ (ie $(M \times M, -\omega \oplus \omega)$), then $HF_*^{\mathrm{loc}}(\Gamma_\varphi, \Delta_M; (p, p)) = HF_*^{\mathrm{loc}}(\varphi, p)$. This follows from the identification of the two Floer homologies (ie nonlocal version) as in [69] and the fact that, after a perturbation of H away from p, formula (12) reduces to a single term identifying HF_*^{loc} and $HF_*^{(c-\varepsilon, c+\varepsilon)}$.

Consider the case of a cotangent bundle T^*N equipped with its canonical symplectic structure. Let p be an isolated point of $L \cap O_N$, where L is an exact Lagrangian in T^*N and O_N is the zero section. Suppose that near the point p, the Lagrangian L can be written as the graph of the differential of a C^2 function $F\colon N \to \mathbb{R}$. Then, similarly to Remark 60, we have $HF_*^{\mathrm{loc}}(L, O_N, p) = HM_*^{\mathrm{loc}}(F, p)$. More generally, if $F(x, \xi)$ is defined on $U \times \mathbb{R}^k$, where U is a neighborhood of p in N and F coincides with a quadratic form in ξ at infinity of index i, then $HF_*^{\mathrm{loc}}(L, O_N, p) = HM_{*-i}^{\mathrm{loc}}(F, p)$.

5.3.2 Local Floer homology and local generating functions

We consider $\mathbb{R}^{2n}$ equipped with its standard symplectic structure ω_0. Let φ be a germ of a symplectic diffeomorphism at $0 \in \mathbb{R}^{2n}$ (with $\varphi(0) = 0$)). We write $\varphi(x, y) = (X, Y)$ for $x, y, X, Y \in \mathbb{R}^n$.

[10] If (M, ω) satisfies $c_1(TM)(\pi_2(M)) = 0$, like for surfaces of positive genus, then $k = 0$.

Consider $(\mathbb{R}^{2n}\times\mathbb{R}^{2n}, -\omega_0\oplus\omega_0)$ and let $\Delta\subset\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ denote the diagonal. Consider the symplectomorphism $\Psi\colon(\mathbb{R}^{2n}\times\mathbb{R}^{2n}, -\omega_0\oplus\omega_0)\to(T^*\Delta,\omega_{\mathrm{can}})$ given by

$$(x,y,X,Y)\mapsto(x,Y,y-Y,X-x),$$

which maps the diagonal in $\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ to the zero section in $T^*\Delta$. The symplectomorphism Ψ identifies the graph of φ with a Lagrangian, which we will denote by Γ_φ, in $T^*\Delta$.

Suppose that Γ_φ is a graph over Δ near the point $(0,0,0,0)\in T^*\Delta$. Then we can find a function $F(x,Y)$, defined near $(0,0)$, and such that $(X,Y)=\varphi(x,y)$ if and only if

$$y-Y=\frac{\partial F}{\partial x}(x,Y),\ X-x=\frac{\partial F}{\partial Y}(x,Y). \tag{14}$$

The function $F(x,Y)$ is called a local generating function for φ near the fixed point p. More generally, we can consider a local generating function $F(x,Y;\xi)$ such that

$$(x,Y;\xi)\mapsto\frac{\partial F}{\partial\xi}(x,Y;\xi)$$

has 0 as a regular value in a neighborhood of $(x,Y;\xi)=(0,0,0)$, and φ is given locally by

$$y-Y=\frac{\partial F}{\partial x}(x,Y;\xi),\ X-x=\frac{\partial F}{\partial Y}(x,Y;\xi),\ \frac{\partial F}{\partial\xi}(x,Y;\xi)=0 \tag{15}$$

It is well known that such a generating function always exists (see [27, Proposition 2.5.7, page 123]). In the following lemma φ denotes a Hamiltonian diffeomorphism of any closed monotone symplectic manifold with an isolated fixed point p. Recall that we say p is totally degenerate if all of the eigenvalues of the derivative of φ at p are one.

Lemma 62 *Suppose that p is a fixed point of φ and let $F(x,y;\xi)$ be a local generating function for φ near p having 0 as an isolated critical value corresponding to p. Then there exists some integer j such that we have*

$$HF_*^{\mathrm{loc}}(\varphi,z_0)=HM_{*-j}^{\mathrm{loc}}(F^\varepsilon,F^{-\varepsilon};z_0).$$

As a consequence, $r(\varphi,p)$ coincides with the rank of $HM_*^{\mathrm{loc}}(F,p)$, the local Morse homology of F at p.

Remark 63 This extends a rather classical result in the totally degenerate case, proved in [25; 17; 18].

The proof of the above lemma follows immediately from the Lagrangian case stated below.

Proposition 64 *Let L_1 and L_2 be exact Lagrangians in the monotone symplectic manifold (M,ω). Let z be an isolated intersection point of $L_1\cap L_2$, and F a local generating function near z of L_1 in a neighborhood of L_2, identified to DT^*L_2. Then there is an integer k such that*

$$HF_*^{\mathrm{loc}}(L_1,L_2;z)=HM_{*-k}^{\mathrm{loc}}(F,z).$$

The proof is based on three simple lemmas.

Lemma 65 *There exists a Hamiltonian flow φ^t supported in a neighborhood B of z such that*

$$\varphi^1(L_2)\cap B=L_1\cap B.$$

Lemma 66 *If L_1 and L_1' are such that $L_1\cap B=L_1'\cap B$, where B is a neighborhood of z, then there exists some integer k such that*

$$HF_*^{\mathrm{loc}}(L_1,L_2;z)=HF_*^{\mathrm{loc}}(L_1',L_2;z).$$

Lemma 67 *If S_2 and S_2' are local generating functions for $L_1\subset DT^*L_2$. Then there exists $k\in\mathbb{Z}$ such that $HM_*^{\mathrm{loc}}(S_1,z)=HM_{*-k}^{\mathrm{loc}}(S_1',z)$*

Proof of Lemma 65 Let us assume $L_2=\mathbb{R}^n\times\{0\}$ and consider a Lagrangian linear subspace transverse to both L_1 and L_2. Then we can, after a linear change of coordinates, assume $E=L_2\oplus L_2^*$ and L_1 is the graph of a germ of a smooth function $f_1\colon L_2\to\mathbb{R}$ having an isolated singularity at 0. Then, in these coordinates, set $H(q,p)=f_1(q)\cdot\chi(q,p)$, where $\chi=1$ for $q^2+p^2\leqslant\varepsilon$ and $\chi=0$ for $|q|^2+|p|^2\geqslant 2\varepsilon$. It is then easy to see that the flow φ^t of H satisfies that $\varphi^1(L_2)$ and L_1 have the same germ at 0. □

Proof of Lemma 66 Clearly if we apply the same perturbation of size ε to L_1 and L_1' near z, we get $\widetilde{L}_{1,\varepsilon}$ and $\widetilde{L}_{1,\varepsilon}'$, which coincide near z. As a result, $CF_*^{\mathrm{loc}}(\widetilde{L}_1,L_2;z)=HF_*^{\mathrm{loc}}(\widetilde{L}_1',L_2;z)$ with a possible index shift, and we just have to check that the boundary maps coincide. But a boundary map corresponds to a holomorphic strip, and we just have to show that if a holomorphic strip with boundary in $\widetilde{L}_1\cup L_2$ and energy less than ε goes through a neighborhood of z, then, for ε small enough, the strip remains in B. For this we essentially apply to the Lagrangian case the argument of Section 3.2 of [17] mentioned above. Indeed, a holomorphic strip connecting two intersection points $z_{1,\varepsilon}$ and $z_{2,\varepsilon}$ of $\widetilde{L}_{1,\varepsilon}\cap L_2$ will have area bounded by the difference in action

$A_{\tilde{L}_1,L_2}(z_{1,\varepsilon}, z_{2,\varepsilon}) = O(\varepsilon)$. But if this was not the case, as ε goes to 0, we would get a sequence of holomorphic strips exiting from B arbitrarily close to z, and with area going to 0. But this implies that the limit of the sequence is constant, by Gromov's compactness and contradicts the fact that it must touch both z and a point outside B. □

Proof of Lemma 67 According to [27, Corollary 3.1.8, page 63], any two local generating functions of the same Lagrangian germ are equivalent, that is, we can find $F_j''(x;\xi,\eta) = F_j'(x, \varphi_j(x,\xi,\eta))$, where $F_j'(x,\xi,\eta) = S_j(x,\xi) + Q_j(\eta)$, where Q_j is a nondegenerate quadratic form, such that $F_1'' = F_2''$. Equivalence obviously implies that the local homology is the same up to a constant shift in index. □

Proof of Proposition 64 Let φ^t be as in Lemma 65 and extend it as the identity in $M \setminus B$. Identify a neighborhood of L_2 to DT^*L_2, where DT^*L_2 is the unit cotangent bundle to L_2, and we may assume B is contained in DT^*L_2, so that $\varphi^t(L_2) \subset DT^*L_2$. According to [34; 63], $L_1' = \varphi^1(L_2)$ has a generating function quadratic at infinity, which we normalize so that the action of its critical points corresponds to the action of the corresponding intersection points. Then, according to [69], we have $HF_*^{\mathrm{loc}}(L_1', L_2; z) = HM_*^{\mathrm{loc}}(F', z)$. In fact [69] only proves this for the global homology (ie proves that $HF_*^{[a,b]}(L_1', L_2; z) = HM_*(F^b, F^a)$), but by a small perturbation of L_1' outside B, we may assume z is the unique intersection point with action 0, and then we have $HF_*^{[\varepsilon,-\varepsilon]}(L_1', L_2; z) = HF_*^{\mathrm{loc}}(L_1', L_2; z)$ and $HM_*^{\mathrm{loc}}(F, z) = HM_*(F^\varepsilon, F^{-\varepsilon})$. □

5.3.3 Proof of Proposition 57 Let H be a Hamiltonian such that $\varphi_H^1 = f$. In the case of $\Sigma = \mathbb{S}^2$, we pick H such that $\{\varphi_H^t\}_{0 \leqslant t \leqslant 1}$ is a representative of $\tilde{f}$. Clearly, it is sufficient to prove the proposition for the barcode $\boldsymbol{B}(H)$ instead of $\boldsymbol{B}(f)$.

Suppose that $c \in \mathrm{Spec}(H)$ and denote by $\#\mathrm{Endpoints}(\boldsymbol{B}(H)) \cap \{c\}$ the total number of bars of $\boldsymbol{B}(H)$ which have c as an endpoint. By Proposition 30, we have

$$\#\mathrm{Endpoints}(\boldsymbol{B}(H)) \cap \{c\} = \mathrm{rank}(HF_*^{(c-\varepsilon,c+\varepsilon)}(H))$$

for sufficiently small ε. Using (12), we obtain

$$\#\mathrm{Endpoints}(\boldsymbol{B}(H)) \cap \{c\} = \sum_{\mathcal{A}_H(\bar{z})=c} \mathrm{rank}(HF_*^{\mathrm{loc}}(H, \bar{z})).$$

First, suppose that $\Sigma \neq \mathbb{S}^2$. Then summing both sides of the above equation over all $c \in \mathrm{Spec}(H)$ yields the result. Next, suppose that $\Sigma = \mathbb{S}^2$. In this case, summing both sides of the above equation over all $c \in \mathrm{Spec}(H) \cap [0,1)$ yields

$$\#\mathrm{Endpoints}(\boldsymbol{B}(H)) \cap [0,1) = \sum_{\mathcal{A}_H(\bar{z}) \in [0,1)} \mathrm{rank}(HF_*^{\mathrm{loc}}(H,\bar{z})).$$

We must now show that

$$\sum_{\mathcal{A}_H(\bar{z}) \in [0,1)} \mathrm{rank}(HF_*^{\mathrm{loc}}(H,\bar{z})) = \sum_{x \in \mathrm{Fix}_c(f)} r(f,x).$$

We leave it to the reader to check that this is an immediate consequence of the following claim:

Claim 68 *For every periodic orbit z of H, there exists a unique capping v such that*

$$\mathcal{A}_H([z,v]) \in [0,1).$$

Proof Let A denote the generator of $\pi_2(\mathbb{S}^2)$ such that $\omega(A) = 1$. $\big($Recall our assumption that $\int_{\mathbb{S}^2} \omega = 1$.$\big)$ Now, fix a capping u of z. Any other capping of z is of the form $u \# kA$ for some $k \in \mathbb{Z}$. Recall that $\mathcal{A}_H([z, u\#kA]) = \mathcal{A}_H([z,u]) - k\omega(A) = \mathcal{A}_H([z,u]) - k$. Clearly, there exists a unique $k \in \mathbb{Z}$ such that $\mathcal{A}_H([z,u]) - k \in [0,1)$. The capping v is given by $u\#kA$. □

This completes the proof of Proposition 57.

Remarks 69 (1) The proof presented above may easily be generalized to prove the following statement in higher dimensions:

(a) Suppose that (M,ω) is aspherical. Then

$$\sum_{x \in \mathrm{Fix}_c(f)} r(f,x) = \#\mathrm{Endpoints}(\boldsymbol{B}(f)).$$

(b) Suppose that (M,ω) is monotone. Let $\tilde{f}$ denote any lift of f to $\mathrm{UHam}(M,\omega)$ and let τ be the positive generator of $\omega(\pi_2(M))$. Then

$$\sum_{x \in \mathrm{Fix}_c(f)} r(f,x) = \#\mathrm{Endpoints}(\boldsymbol{B}(\tilde{f})) \cap [0,\tau).$$

(2) Proposition 58 does not hold in higher dimensions. Indeed, let F be a local generating function for f. Notice first that if we have a Conley index pair[11] $(U, \partial^- U)$

[11] This means in our case that U is an open set with smooth boundary, and $\partial^- U$ is the exit set on ∂U.

for ∇F, then $HM_*^{\mathrm{loc}}(F,p) = H_*(U,\partial^- U)$. According to [45], if the ambient manifold has dimension n, we have

$$L(f,p) = (-1)^n \chi(H_*(U,\partial^- U)) = \sum_{j=1}^{n} (-1)^{n+j} \dim HM_j^{\mathrm{loc}}(F,p)$$

and of course $(-1)^n \chi(H_*(U,\partial^- U)) = (-1)^n \chi(HM_*^{\mathrm{loc}}(F,p))$. In the 2–dimensional case we have that $HM_j^{\mathrm{loc}}(F,p)$ is only nonzero in dimensions 0, 1 and 2, and, unless F has a local minimum or maximum at p, is only nonzero in dimension 1. In this last case $L(f,p) = -\dim HM_1^{\mathrm{loc}}(F,p)$. If p is a local maximum of F, we have that $HM_2^{\mathrm{loc}}(F,p)$ has rank 1 and $HM_1^{\mathrm{loc}}(F,p) = HM_0^{\mathrm{loc}}(F,p) = \{0\}$, so $L(f,p) = +1 = r(f,p)$. We argue similarly for p a local minimum.

But in higher dimensions, we can very well have homologies of consecutive degree which are nonzero. Then cancellations will occur in $\chi(HM_*^{\mathrm{loc}}(F,p))$ and $|L(f,p)| < r(f,p)$. For example, suppose F is C^1 small, and given near 0 by $F(t\cdot x) = t^P F_0(x)$, where $x \in S^n$, $t \in [0,1]$ and the function F_0 is defined on S^{n-1}. Then

$$HM_*(B \cap F^{\varepsilon}, B \cap F^{-\varepsilon}) = H_*(B, S \cap F^{<0}).$$

If we choose $F_0^{<0}$ to be a subset of the sphere with nonzero homology in ranks j and $j+1$ — for example for $n=3$ we take $F_0^{<0}$ to be an annulus, so that the local homology is $H_*(B^3, A)$, where A is an annulus in S^2 — then, by the long exact sequence of the pair (B,A), we have $H_2(B,A) = H_1(A) = \mathbb{Z}$ and $H_1(B,A) = H_0(A) = \mathbb{Z}$. Then $r(f,p) = 3$ while $L(f,p) = -1$.

5.3.4 Proof of Proposition 58 Let p be an isolated fixed point of f. We want to prove that $r(f,p) = |L(f,p)|$.

When p is nondegenerate, $L(f,p) = \pm 1$ and so the result follows from Remark 59. Thus, we will assume, for the rest of the proof, that p is a degenerate fixed point. Since we are working in dimension 2, this implies that p is totally degenerate.

To simplify notation, we will consider a small chart centered at p and work with a germ of f at p. Thus, we will think of f as a Hamiltonian diffeomorphism defined near $p = (0,0) \in \mathbb{R}^2$ with an isolated fixed point at p. By Lemma 62, f admits a local generating function of the form $F(x,Y)$ near p; see (14). Moreover, according to the same lemma, $r(f,p)$ coincides with the rank of the local Morse homology group $HM_*^{\mathrm{loc}}(F,p)$. Proposition 58 follows immediately from the following two lemmas.

Lemma 70 *$L(f, p) = L(\chi_F, p)$, where χ_F denotes the Hamiltonian vector field of F and $L(\chi_F, p)$ denotes its Lefschetz index at p.*

Proof Let $T = \{(x, y) : x^2 + y^2 = r^2\}$ for a very small value of r. Observe that $L(f, p)$ is the degree of the map $\Phi\colon T \to \mathbb{S}^1$ given by

$$(x, y) \mapsto \left(\frac{X - x}{\|X - x\|}, \frac{Y - y}{\|Y - y\|}\right).$$

The Lefschetz index $L(\chi_F, p)$ is, by definition, the degree of the map $\Psi\colon T' \to \mathbb{S}^1$ given by the formula $(x, Y) \mapsto \chi_F(x, Y)/\|\chi_F(x, Y)\|$, where T' can be taken to be the boundary of any disk whose interior contains p and no other zeros of χ_F. We will take T' to be the image of the circle T from the previous paragraph under the mapping $(x, y) \mapsto (x, Y)$. It follows from the definition of F (see (14)) that $\Psi\colon T' \to \mathbb{S}^1$ is given by the formula $(x, Y) \mapsto ((X - x)/\|X - x\|, (Y - y)/\|Y - y\|)$.

Therefore, to prove the lemma we must show that the map from T to T' given by $(x, y) \mapsto (x, Y)$ has degree one. It is easy to see that the degree of this map is given by the sign of $\partial Y/\partial y$. We claim that $\partial Y/\partial y$ is positive: Indeed, since p is totally degenerate, up to conjugation by a symplectomorphism, we may assume that f is C^1–close to the identity near p. Of course, this would imply that $\partial Y/\partial y$ is close to 1 near p. □

Lemma 71 *The rank of $HM_*^{\mathrm{loc}}(F, p)$ coincides with $|L(\chi_F, p)|$.*

Proof Using the fact that χ_F is an area-preserving vector field, one can show that it can be described as follows near the point p: either every orbit near p is periodic, or one can find a neighborhood of p which may be divided into a finite number, say h, of hyperbolic sectors of χ_F; see Figure 1. Hyperbolic sectors are defined in Section 7.2.2.

In the first case, where all the orbits of χ_F in a neighborhood of p are periodic, p is either a local maximum or minimum of F. In this case, it is easy to see that $L(\chi_F, p)$ and the rank of $HM_*^{\mathrm{loc}}(F, p)$ are both 1.

Now suppose that χ_F has h hyperbolic sectors. An easy computation would show that $L(\chi_F, p) = 1 - \frac{1}{2}h$. Suppose that $F(p) = 0$. The local Morse homology $HM_*^{\mathrm{loc}}(F, p)$ coincides with the singular homology of the pair $(F^{\varepsilon}, F^{-\varepsilon})$. Hence, it is sufficient to show that the singular homology $H_*(F^{\varepsilon}, F^{-\varepsilon})$ has rank $\frac{1}{2}h - 1$. Now, $H_*(F^{\varepsilon}, F^{-\varepsilon}) = H_*(F^{\varepsilon} \cap D, F^{-\varepsilon} \cap D)$, where D is a small disk centered at p. Recall

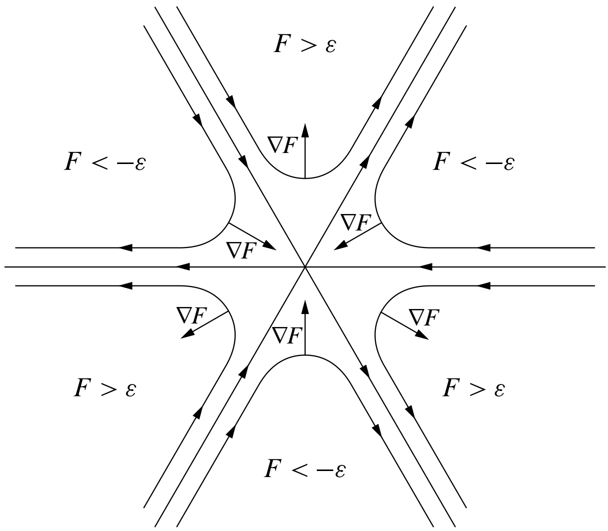

Figure 1: Phase portrait of χ_F near a singular point with six hyperbolic sectors. The Lefschetz index of the singularity is -2.

that the trajectories of χ_F are levels of F and F increases in the direction of ∇F (see Figure 1). Thus, it is clear that $(F^{\varepsilon} \cap D, F^{-\varepsilon} \cap D)$ has the homotopy type of the pair $(D, \partial D^-)$, where ∂D^- is the set of points on the unit circle where F is strictly negative. Note that trajectories of χ_F are contained in a level of F, and on a trajectory in a hyperbolic sector, F has positive value if ∇F points away from the origin (so that F increases as we move away from 0) and negative when ∇F points towards 0. Thus, ∂D^- is a union of intervals on the circle, one for each hyperbolic sector such that ∇F points towards the origin. We have $\frac{1}{2}h$ such sectors, so ∂D^- is a union of $\frac{1}{2}h$ disjoint intervals and $H_k(D, \partial D^-)$ has rank $\frac{1}{2}h - 1$ for $k = 1$ and 0 otherwise. This completes the proof. □

5.4 Absolute Lefschetz number for smoothable homeomorphisms

In this section we will prove Theorem 2 of the introduction: Suppose that f and g are Hamiltonian homeomorphisms which are weakly conjugate and have a finite number of fixed points. If f and g are smoothable, then they have the same absolute Lefschetz numbers.

The proof of Theorem 2 follows the same general outline as the proof of Theorem 1. Indeed, as in the case of Theorem 1, we use Theorem 54 to conclude that the barcodes we have constructed are invariants of weak conjugacy classes. We then use the theorem below to conclude that the absolute Lefschetz number of a smoothable Hamiltonian homeomorphism is an invariant of its barcode.

Theorem 72 *Suppose that $f \in \overline{\mathrm{Ham}}(\Sigma,\omega)$ has finitely many fixed points and that f is smoothable.*

(1) *Suppose that $\Sigma \neq \mathbb{S}^2$. Then*

$$\sum_{x \in \mathrm{Fix}_c(f)} |L(f,x)| = \#\mathrm{Endpoints}(\boldsymbol{B}(f)).$$

(2) *Suppose that $\Sigma = \mathbb{S}^2$. Let $\tilde{f}$ denote any of the two lifts of f to $\overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)$. Then*

$$\sum_{x \in \mathrm{Fix}_c(f)} |L(f,x)| = \#\mathrm{Endpoints}(\boldsymbol{B}(\tilde{f})) \cap [0,1).$$

Although the statement of the above theorem is similar to that of Theorem 55, the proof is different as local Floer theory is not well defined for homeomorphisms. One may interpret this theorem as a definition for local Floer homology of homeomorphisms.

Proof of Theorem 72 Since f is smoothable, there exists a sequence $f_n \in \mathrm{Ham}(M,\omega)$ such that f_n converges uniformly to f and $\mathrm{Fix}_c(f_n) = \mathrm{Fix}_c(f)$ for every n. Because the sets $\mathrm{Fix}_c(f_n)$ and $\mathrm{Fix}_c(f)$ all coincide, we will simply denote them by Fix_c.

In the case of $\Sigma = \mathbb{S}^2$, we pick lifts $\tilde{f}_n$ of f_n to $\mathrm{UHam}(\mathbb{S}^2,\omega)$ such that $\tilde{f}_n$ converges to $\tilde{f}$ with respect to the C^0 topology on $\overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)$. By Theorem 34, $\boldsymbol{B}_j(f_n) \to \boldsymbol{B}_j(f)$ and $\boldsymbol{B}(f_n) \to \boldsymbol{B}(f)$ when $\Sigma \neq \mathbb{S}^2$. By Theorem 37, $\boldsymbol{B}_j(\tilde{f}_n) \to \boldsymbol{B}_j(\tilde{f})$ when $\Sigma = \mathbb{S}^2$.

As the reader might anticipate, the strategy for the proof is to show that the number of endpoints for each barcode of f_n coincides with the number of endpoints for the corresponding barcode of f; for example, we will show that $\#\mathrm{Endpoints}(\boldsymbol{B}_j(f_n)) = \#\mathrm{Endpoints}(\boldsymbol{B}_j(f))$. What makes this nontrivial is that a priori $\boldsymbol{B}_j(f_n)$ might possess small bars which disappear as $n \to \infty$. This difficulty is tackled by the following lemma:

Lemma 73 *The sequences f_n and $\tilde{f}_n$ may be picked so that:*

(1) *When $\Sigma \neq \mathbb{S}^2$, we have $\boldsymbol{B}_j(f_n) = \boldsymbol{B}_j(f)$ and $\boldsymbol{B}(f_n) = \boldsymbol{B}(f)$.*

(2) *When $\Sigma = \mathbb{S}^2$, we have $\boldsymbol{B}_j(\tilde{f}_n) = \boldsymbol{B}_j(\tilde{f})$ and $\boldsymbol{B}(\tilde{f}_n) = \boldsymbol{B}(\tilde{f})$.*

Before proving this lemma, let us show that it implies Theorem 72. Indeed, by continuity of the Lefschetz index, for each $x \in \mathrm{Fix}_c$, we have $L(f,x) = L(f_n,x)$ for n sufficiently large. Hence, for n large enough,

$$\sum_{x \in \mathrm{Fix}_c} |L(f,x)| = \sum_{x \in \mathrm{Fix}_c} |L(f_n,x)|.$$

On the other hand, Theorem 55 tells us that if $\Sigma \neq \mathbb{S}^2$, then

$$\sum_{x \in \mathrm{Fix}_c} |L(f_n,x)| = \#\mathrm{Endpoints}(\boldsymbol{B}(f_n)),$$

and if $\Sigma = \mathbb{S}^2$, then

$$\sum_{x \in \mathrm{Fix}_c} |L(f_n,x)| = \#\mathrm{Endpoints}(\boldsymbol{B}(\tilde{f}_n)) \cap [0,1).$$

The theorem then follows from Lemma 73.

It remains to prove the lemma. Before getting into the proof, recall that the barcodes $\boldsymbol{B}_j(f_n)$ and $\boldsymbol{B}_j(f)$ belong to $\widehat{\mathcal{B}}$, which is the space of barcodes considered up to shift. It will be more convenient to work with barcodes as opposed to barcodes considered up to shift. To that end, we pick Hamiltonians H_n such that $\varphi^1_{H_n} = f_n$ and work with $\boldsymbol{B}_j(H_n)$ and $\boldsymbol{B}(H_n)$. In the case of the sphere, we pick the H_n such that $\{\varphi^t_{H_n}\}_{0 \leqslant t \leqslant 1}$ represents $\tilde{f}_n$ in $\mathrm{UHam}(\mathbb{S}^2,\omega)$. We start with any sequence H_n and we will show that for each fixed j we can modify the H_n so that the sequence of barcodes $\boldsymbol{B}_j(H_n)$ eventually stabilizes, ie we have $\boldsymbol{B}_j(H_n) = \boldsymbol{B}_j(H_{n+1})$ for sufficiently large n. Likewise, we will also get $\boldsymbol{B}(H_n) = \boldsymbol{B}(H_{n+1})$ for sufficiently large n. It is clear that Lemma 73 follows from the above.

We will begin by proving that, for each fixed j, the sequence $\boldsymbol{B}_j(H_n)$ eventually stabilizes. Denote by $E_j(H_n)$ the set of values in the spectrum of H_n which appear as endpoints of bars in $\boldsymbol{B}_j(H_n)$. Note that this is the set of endpoints of bars in $\boldsymbol{B}_j(H_n)$ counted without their multiplicities.[12]

We claim that $E_j(H_n)$ is finite for all n. Indeed, the cardinality of this set is bounded by the total number of fixed points; this is obvious in the case where $\Sigma \neq \mathbb{S}^2$, and we leave it to the reader to check that it continues to hold when $\Sigma = \mathbb{S}^2$.

Write $E_j(H_n) = \{\mathcal{A}^n_1, \mathcal{A}^n_2, \dots, \mathcal{A}^n_k\}$, where we suppose that $\mathcal{A}^n_1 < \mathcal{A}^n_2 < \cdots < \mathcal{A}^n_k$. Note that we're assuming here that the sets $E_j(H_n)$ have the same cardinality for

[12] Recall that, on the other hand, $\mathrm{Endpoints}(\boldsymbol{B}_j(H_n))$ denotes the set of endpoints of the bars of $\boldsymbol{B}_j(H_n)$ taken with their multiplicities.

different values of n. This is justified, up to passing to a subsequence of $\boldsymbol{B}_j(H_n)$, because the cardinality of $E_j(H_n)$ is bounded by the total number of fixed points.

Claim 74 *For every $\delta > 0$ there exists ε such that for every $a_1, a_2, \dots, a_k \in [-\varepsilon, \varepsilon]$ we can find a Hamiltonian K with C^2 norm less than δ such that, for every n:*

- *K is constant on a neighborhood of Fix_c and $\varphi_K^1 f_n$ has the same set of contractible fixed points as f_n.*
- *$E_j(\widetilde{H}_n) = \{\mathcal{A}_1^n + a_1, \mathcal{A}_2^n + a_2, \dots, \mathcal{A}_k^n + a_k\}$, where $\widetilde{H}_n = K \# H_n$ is a Hamiltonian with $\varphi^1_{\widetilde{H}_n} = \varphi^1_K f_n$.*

We leave the proof of the above claim as an exercise to the reader.

Next, shift each H_n by a constant, if necessary, to ensure that the minimum value in $E_j(H_n)$ is zero.

Claim 75 *The set $\bigcup_n E_j(H_n)$ is bounded.*

Proof Applying Claim 74, we may assume that, for each n, the difference between the largest value in $E_j(H_n)$ and the second largest value is at least ε. Therefore, each barcode $\boldsymbol{B}_j(H_n)$ has a bar $I_n = [c_n, d_n]$ of length at least ε. If $\bigcup_n E_j(H_n)$ was not bounded, we would conclude that $d_n \to \infty$. It is easy to see that these conditions force the sequence of barcodes $\boldsymbol{B}_j(H_n)$ not to have a limit in $\widehat{\mathcal{B}}$, the space of barcodes considered up to shift. (Of course, it does not have a limit in the space of barcodes either.) This contradicts the fact $\boldsymbol{B}_j(H_n) \to \boldsymbol{B}_j(f)$ when $\Sigma \neq \mathbb{S}^2$, and $\boldsymbol{B}_j(H_n) \to \boldsymbol{B}_j(\widetilde{f})$ when $\Sigma = \mathbb{S}^2$. □

We continue the proof of Lemma 73. Because the sequence of barcodes $\boldsymbol{B}_j(H_n)$ is convergent, to prove the lemma it is sufficient to pick the Hamiltonians H_n such that the sets $E_j(H_n)$ stabilize, ie $E_j(H_n) = E_j(H_{n+1})$ for n large.

Let us show that we can modify the H_n so that the sets $E_j(H_n)$ stabilize. By Claim 75, after passing to a subsequence, we may assume that $\mathcal{A}_i^n$ converges to a value which we will denote by $\mathcal{A}_i$. Since $\mathcal{A}_i^n - \mathcal{A}_i$ converges to 0, by Claim 74, we can perturb H_n to guarantee that $\mathcal{A}_i^n = \mathcal{A}_i$ for n sufficiently large. Thus, we have $E_j(H_n) = \{\mathcal{A}_1, \dots, \mathcal{A}_k\}$. This completes the proof of the fact that the sequence $\boldsymbol{B}_j(H_n)$ stabilizes for n large enough.

It remains to show that the sequence of total barcodes $\boldsymbol{B}(H_n)$ also stabilizes.[13] If $\Sigma \neq \mathbb{S}^2$, then we can repeat all of the above with $\boldsymbol{B}_j(H_n)$ replaced with $\boldsymbol{B}(H_n)$ and $E_j(H_n)$ replaced with $E(H_n)$, where $E(H_n)$ denotes the set of values in the spectrum of H_n which appear as endpoints of bars in $\boldsymbol{B}(H_n)$.

The above reasoning does not apply to the case of $\mathbb{S}^2$ because, on $\mathbb{S}^2$, the sets $E(H_n)$ are neither finite nor bounded. However, $\boldsymbol{B}(H_n)$ is the disjoint union of the sets $\boldsymbol{B}_j(H_n)$ and, by (9), we have $\boldsymbol{B}_j(H_n) = \boldsymbol{B}_{j-4}(H_n) - 1$. Therefore, the fact that for each j the sequence $\boldsymbol{B}_j(H_n)$ stabilizes implies that $\boldsymbol{B}(H_n)$ stabilizes as well. This completes the proof of Lemma 73. □

6 Proof of Theorem 4: homeomorphisms which are not weakly conjugate to diffeomorphisms

In this section we provide a proof for Theorem 4. We first treat the case of surfaces other than the sphere, leaving the case of the sphere to the end.

As mentioned in the introduction, it is sufficient to give an example of a Hamiltonian homeomorphism f such that the set of endpoints of $\boldsymbol{B}(f)$ is unbounded. Pick an area-preserving chart V with coordinates (x, y). We may identify V with a Euclidean disk in $\mathbb{R}^2$, whose radius we will denote by $R > 0$. We denote the origin in these coordinates by O and define $r(x, y)$ to be the usual Euclidean distance between (x, y) and O. Let $G\colon \Sigma \setminus \{O\} \to \mathbb{R}$ be a function whose support is compactly contained in V and which is of the form $G(x, y) = h\big(\frac{1}{2}r^2\big)$, where $h\colon \big(0, \frac{1}{2}R^2\big) \to \mathbb{R}$ is smooth, $h(s) = 1/s$ when $s \leqslant \frac{1}{2}r_0^2$, where r_0 is sufficiently small, and $h(s) = 0$ for s near R. Define $f\colon \Sigma \to \Sigma$ by $f(O) = O$ and $f(p) = \varphi_G^1(p)$ for all $p \in \Sigma \setminus \{O\}$.

Now, f is a Hamiltonian homeomorphism of Σ, because it is the uniform limit of $\varphi_{G_i}^1$, where G_i is a smooth Hamiltonian such that $G_i(x, y) = h_i\big(\frac{1}{2}r^2\big)$, where $h_i\colon [0, \infty) \to \mathbb{R}$ is smooth and $h_i\big(\frac{1}{2}r^2\big) = h\big(\frac{1}{2}r^2\big)$ for $r \geqslant \rho_i$ and $\rho_i \to 0$.

It is convenient here to work with a fixed representative of $\boldsymbol{B}(f)$ which is only defined up to shift. To pick a representative we normalize $\boldsymbol{B}(f)$ so that the ends of bars corresponding to the fixed points outside its support have action zero. This can be achieved as follows: Let $\boldsymbol{B}(G_i)$ be the barcode of the Hamiltonian G_i from the last

[13] When f is nonsmooth, it is possible that $B_j(f)$ is nonempty for infinitely many j, even in the case $\Sigma \neq \mathbb{S}^2$.

paragraph; note that $\boldsymbol{B}(G_i)$ is well defined as a barcode as opposed to a barcode up to shift. It can be checked that the sequence $\boldsymbol{B}(G_i)$ has a limit in the bottleneck distance and so we take this limit to represent $\boldsymbol{B}(f)$.

We remark that for $r \leqslant r_0$, the 1–periodic orbits of G appear at values r_k such that $h'\big(\frac{1}{2}r_k^2\big) = -2\pi k$, where k is a positive integer. A simple computation shows that the action of a periodic orbit corresponding to r_k is given by

$$c_k := h\big(\tfrac{1}{2}r_k^2\big) - \tfrac{1}{2}r_k^2 h'\big(\tfrac{1}{2}r_k^2\big).$$

One can easily check that $c_k = 2\sqrt{2\pi k}$. Now, the following claim tells us that the values c_k appear as endpoints of some bars in $\boldsymbol{B}(f)$. This, of course, implies that the set of endpoints of $\boldsymbol{B}(f)$ is not bounded as $c_k \to \infty$.

Claim 76 *The number of bars in $\boldsymbol{B}(f)$ with an endpoint at c_k is given by the rank of $HF^{(c_k-\delta,c_k+\delta)}(G)$ for sufficiently small δ. Furthermore, $HF^{(c_k-\delta,c_k+\delta)}(G)$ has rank 2.*

Proof At first glance, the first statement appears to be the content of Proposition 30. But this proposition does not apply verbatim as G is not smooth. However, G is a smooth Hamiltonian on a neighborhood of the set $G^{-1}(c_k-\delta, c_k+\delta)$ and it is not difficult to see that the proposition does apply in this setting.

Let us now prove that $HF^{(c_k-\delta,c_k+\delta)}(G)$ has rank 2. Computation of this type of Floer homology groups is a classical example which goes back to [7]; see also [47; 61]. Therefore, we will only sketch an outline of the computation.

Recall that we are considering periodic orbits of G corresponding to $r = r_k$. These orbits form circles which we will denote by S_k. Let U_k denote a small open neighborhood of S_k. Performing a C^2–small perturbation of G near S_k, one obtains a Hamiltonian $\widetilde{G}$ which has exactly two nondegenerate 1–periodic orbits in U_k. The Conley–Zehnder indices of these orbits are $2k-1$ and $2k$. Hence, the Floer chain complex $CF_*^{(c_k-\delta,c_k+\delta)}(\widetilde{G})$ has rank two and is supported in degrees $2k-1$ and $2k$. It is shown in Proposition 2.2 of [7] that the boundary map of this complex is zero. The result follows immediately. □

It remains to explain why f is not weakly conjugate to any Hamiltonian diffeomorphism in the case where $\Sigma = \mathbb{S}^2$. By Lemma 52, if f were weakly conjugate to $h \in \mathrm{Ham}(\mathbb{S}^2,\omega)$, then the lifts of f to $\overline{\mathrm{UHam}}(\mathbb{S}^2,\omega)$ would be pairwise weakly conjugate to the lifts of h. Hence, it is sufficient to show that f has a lift $\tilde{f} \in \overline{\mathrm{UHam}}(\mathbb{S}^2, \omega)$ which is not weakly conjugate to any $\tilde{h} \in \mathrm{UHam}(\mathbb{S}^2, \omega)$.

Note that the nonsmooth function G, introduced above, has a well-defined Hamiltonian flow, which we will denote by ϕ_G^t. We let $\tilde{f}$ be the lift of f given by the path $\{\phi_G^t\}_{t\in[0,1]}$. For each index j, we pick a representative of $\boldsymbol{B}_j(\tilde{f})$ by taking the barcode $\lim_{i\to\infty} \boldsymbol{B}_j(G_i)$, where the G_i are the smooth Hamiltonians introduced above.

In the case of $\mathbb{S}^2$, the action spectrum of no smooth Hamiltonian is bounded because of the effect of cappings. However, one can easily show that for $\tilde{h} \in \mathrm{UHam}(\mathbb{S}^2, \omega)$ the set of endpoints of $\boldsymbol{B}_j(\tilde{h})$ is bounded for any fixed j. We will show below that the set of endpoints of (at least) one of the two barcodes $\boldsymbol{B}_0(\tilde{f})$ and $\boldsymbol{B}_1(\tilde{f})$ is unbounded. This would then finish the proof in the case of $\mathbb{S}^2$.

Recall from Section 3.2 that the lower ends of the bars in $\boldsymbol{B}_j$ are actions of orbits of index j, while the upper ends of the bars are actions of orbits of index $j+1$. Combining this fact with Claim 76, we conclude that the value c_k is the endpoint of a bar in at least one of $\boldsymbol{B}_{2k-1}(\tilde{f})$ or $\boldsymbol{B}_{2k}(\tilde{f})$. Let $E = \{k : c_k \text{ is an endpoint of } B_{2k}\}$ and $O = \{k : c_k \text{ is an endpoint of } B_{2k-1}\}$ and observe that at least one of these sets is infinite. We will suppose that E is infinite, leaving the case where O is infinite to the reader.

For each $k \in E$, we pick a capped 1–periodic orbit $[z_k, u_k]$ of G corresponding to c_k. Here, u_k is the unique (up to homotopy) capping of z_k which is contained in the set V. Let A denote the generator of $\pi_2(S^2)$ with $\omega(A) > 0$. Consider the capped orbit $[z_k, u_k \# kA]$: As a consequence of (1), this orbit appears as an endpoint of a bar in $\boldsymbol{B}_0(\tilde{f})$. Furthermore, the action of this orbit is given by $c_k' := c_k - k\omega(A) = c_k - k$. Recall that $c_k = 2\sqrt{2\pi k}$ and thus $c_k' \to -\infty$ as $k \to \infty$. Therefore, the set of endpoints of $B_0(\tilde{f})$ is not bounded. This completes the proof.

7 Hamiltonian homeomorphisms and smoothability

Recall from the introduction that we say a Hamiltonian homeomorphism f, with a finite number of fixed points, is *smoothable* if there exists a Hamiltonian diffeomorphism g which is arbitrarily C^0–close to f and such that $\mathrm{Fix}_c(g) = \mathrm{Fix}_c(f)$. We conjecture that every such f is smoothable. In this section we will prove that a very large class of homeomorphisms consists of smoothable ones. We will be relying on techniques from local dynamics of surface homeomorphisms.

Let M be a surface, $f: M \to M$ be an area-preserving homeomorphism, and p an isolated fixed point of f. We say that f is *smoothable at* p if for every neighborhood U of p such that $\mathrm{Fix}(f) \cap U = \{p\}$, there exists an area-preserving homeomorphism $f': M \to M$ which coincides with f outside U, which is smooth near p and such that $\mathrm{Fix}(g) \cap U = \{p\}$. According to Proposition 78, if f is smoothable at every fixed point then it is smoothable.

The *local rotation set of* f *around* p is a closed interval included in $[-\infty, +\infty]$ defined up to an integer translation, which is a conjugacy invariant (see [43], and Section 7.4.1 below for more details). Assume furthermore that f preserves the area. If p is an isolated fixed point then the local rotation set contains no integer in its interior, and so it is included in $[0, 1]$ (up to integer translation). We will say that p is a *maximally degenerate* fixed point if the local rotation set is equal to $[0, 1] \bmod \mathbb{Z}$. Maximally degenerate fixed points are accumulated by periodic orbits of every period, and have Lefschetz index 1 [43, Section 3.2]. Thus, the following theorem, combined with Proposition 78, immediately implies Theorem 3 of the introduction.

Theorem 77 *Let $f: M \to M$ be an area-preserving homeomorphism, and let p be an isolated fixed point of f which is not maximally degenerate. Then f is smoothable at p.*

We split the proof into two cases. Recall that, in the case of area-preserving maps, the Lefschetz index of an isolated fixed point is always less than or equal to one [46; 64; 53; 35]; see the discussion at the beginning of Section 7.2.2 for a proof of this fact. We will use different strategies in the nonpositive index case and in the index one case. The two cases will be treated in Sections 7.3 and 7.4, respectively.

7.1 Local vs global smoothability

The following result shows that a homeomorphism which is smoothable at each fixed point is globally smoothable:

Proposition 78 *Let f be an area-preserving homeomorphism of a compact surface M with a finite number of contractible fixed points. Suppose that f is smoothable at every contractible fixed point. Then there exists an area-preserving diffeomorphism g, arbitrarily C^0–close to f, such that $\mathrm{Fix}_c(g) = \mathrm{Fix}_c(f)$. If furthermore $f \in \overline{\mathrm{Ham}}(M, \omega)$, then g can be chosen in $\mathrm{Ham}(M, \omega)$.*

The proof is a variation of the proof of the fact that, on surfaces, area-preserving diffeomorphisms are dense in the group of area-preserving homeomorphisms. Hence, we will only provide a sketch of the proof and leave much of the detail to the reader.

Proof By definition of smoothability at a point, there exists an area-preserving homeomorphism f' such that f' is C^0–close to f, $\mathrm{Fix}_c(f') = \mathrm{Fix}_c(f)$, and f' is smooth near every contractible fixed point. Let M' be a subsurface of M, whose complement in M is a neighborhood of $\mathrm{Fix}_c(f)$ on which f' is smooth. The first statement of the proposition is a consequence of the following claim:

Claim 79 *There exists an area-preserving diffeomorphism $g\colon M \to M$ which is arbitrarily C^0–close to f' and coincides with f' outside M'.*

Note that g has no new contractible fixed points since it remains C^0–close to f' on M' and f' has no contractible fixed points in M'.

We will finish the proof of the proposition before giving a proof of the above claim. If M is the sphere, there is nothing to do, since every area-preserving diffeomorphism is Hamiltonian. In the opposite case, assume that $f \in \overline{\mathrm{Ham}}(M, \omega)$. It then follows that the flux of g must be small, and so we can perform a C^0–small modification of g, far from the fixed point set, turning it into a Hamiltonian diffeomorphism: Indeed, consider some simple closed curve γ. We will explain how to modify g so that its flux through γ becomes zero and will leave the rest to the reader. Let γ' be a simple closed curve which intersects γ exactly once and which is far from the fixed point set. Denote by δ the flux of g through γ. It is easy to construct a symplectic diffeomorphism φ_δ, supported on a small tubular neighborhood of γ', whose flux through γ is equal to $-\delta$. Furthermore, since δ is small, φ_δ can be chosen to be C^0–close to the identity. Replacing g with $g\phi_\delta$, we obtain a map with zero flux through γ. □

Proof of Claim 79 There exists a diffeomorphism g' which is arbitrarily C^0–close to f' and coincides with f' outside M'. This follows from applying the handle-smoothing theorem in [24] successively to the vertices, edges and faces of a sufficiently fine triangulation of M.

Let $\Omega = g'^*\omega$. We will show that one can find a diffeomorphism $\Psi\colon M \to M$, arbitrarily C^0–close to the identity, such that $\Psi = \mathrm{Id}$ outside M' and $\Psi^*\Omega = \omega$. Of course, we will then set $g = g'\Psi$.

Let $\mathcal{T}$ be a triangulation of M'. Using Moser's method, independently on a neighborhood of each vertex of $\mathcal{T}$, we can find a diffeomorphism Ψ_1 such that $\Psi_1^*\Omega = \omega$ near the vertices of $\mathcal{T}$. Likewise, we can find a diffeomorphism Ψ_2 such that $\Psi_2^*\Psi_1^*\Omega = \omega$ near the 1–skeleton of $\mathcal{T}$. Furthermore, Ψ_1 and Ψ_2 can be picked to coincide with the identity outside M' and to be as C^0–small as one wishes (independently of $\mathcal{T}$); we will leave it to the reader to check this latter statement.

Claim 80 *Let $\Omega' = \Psi_2^*\Psi_1^*\Omega$. There exists a function $\eta\colon M \to \mathbb{R}$ such that*:

(1) *η is uniformly close to the constant function 1,*

(2) *$\eta = 1$ near the 1–skeleton of $\mathcal{T}$ and outside M',*

(3) *$\eta\Omega'(T) = \omega(T)$ for every triangle T of the triangulation $\mathcal{T}$.*

Proof Observe that for every triangle T of the triangulation, the ratio $\omega(T)/\Omega'(T)$ is close to 1 because Ω' is the pullback of ω by $g'\Psi_1\Psi_2$ and $g'\Psi_1\Psi_2$ is C^0–close to the area-preserving homeomorphism f'. It follows that, for every triangle T, we can pick a function $\eta_T\colon M \to \mathbb{R}$ such that η_T is uniformly close to the constant 1, $\eta_T - 1$ is supported in the interior of T, and $\int_T \eta_T\Omega' = \int_T \omega$.

Define the function η by setting $\eta(x) = \eta_T(x)$ if x is in the triangle T. We leave it to the reader to check that η satisfies all of the stated properties. □

Observe that $\int_M \eta\Omega' = \int_M \Omega'$. Therefore, by applying Moser's method, we find Ψ_3 such that $\Psi_3^*\Omega' = \eta\Omega'$. Because η is uniformly close to 1, the diffeomorphism Ψ_3 can be picked to be C^0–close to the identity; see Proposition 5 of [11]. Furthermore, it coincides with the identity outside M'.

Next, applying the Moser method again, we find a diffeomorphism Ψ_4 such that $\Psi_4^*\eta\Omega' = \omega$. Note that $\eta\Omega' = \omega$ near the 1–skeleton of the triangulation and outside M'. Thus, Ψ_4 may be picked such that it is supported in the union of the interiors of the triangles of $\mathcal{T}$. Thus, $d_{C^0}(\Psi_4, \mathrm{Id})$ is bounded by the maximum of the diameters of the triangles of $\mathcal{T}$ and so, by picking a sufficiently fine triangulation, Ψ_4 can be chosen as C^0–small as we wish. Note that since $\eta\Omega' = \omega$ outside M' we can ensure that $\Psi_4 = \mathrm{Id}$ outside M'. Finally, we set $\Psi = \Psi_1\Psi_2\Psi_3\Psi_4$. □

7.2 Transverse foliations for local area-preserving homeomorphisms

We begin this section by introducing some notions, as well as notation, from the theory of transverse foliations. For further details, we refer the reader to [38]. Let Σ be a surface, and fix p_0 to be a point of Σ. We will call (f, U) a *local homeomorphism* if $f: U \to f(U)$ is a homeomorphism between some open subsets U and $f(U)$ of Σ that fixes p_0. We will always assume tacitly that U and $f(U)$ are interiors of some closed topological disks in Σ. We will often assume that p_0 is an isolated fixed point, in which case, by diminishing U if necessary, we get a local homeomorphism for which p_0 is the only fixed point.

A *local isotopy* (I, V) for (f, U) is a continuous family $I = (f_t)_{t\in[0,1]}$ of local homeomorphisms (f_t, U_t) such that f_t fixes p_0 for each t, $V \subset U_t \subset U$, f_0 is the identity and $f_1 = f$ on U_1. We will say that (I, V) is *compactly supported in* U if $U_t = U$ for every t and f_t is the identity near the boundary of U. If (J, V') is another local isotopy with $J = (g_t)_{t\in[0,1]}$, with $g_1(V') \subset V$, then we may define the local isotopy (IJ, V') with the concatenation $IJ = (h_t)_{t\in[0,1]}$ defined by $h_t = g_{2t}$ when $t \in \left[0, \frac{1}{2}\right]$ and $h_t = f_{2t-1} g_1$ when $t \in \left[\frac{1}{2}, 1\right]$.

A curve is a continuous map $\gamma: [0,1] \to \Sigma$. The curve is closed if $\gamma(0) = \gamma(1)$. Given a local isotopy (I, V) and a point p in V, the trajectory of p is the curve $I.p: t \mapsto f_t(p)$, where $I = (f_t)_{t\in[0,1]}$.

7.2.1 Transverse foliations A *local foliation* $(\mathcal{F}, W)$ is a smooth (C^∞) oriented foliation $\mathcal{F}$ defined on $W \setminus \{p_0\}$, where W is some disk neighborhood of p_0; we will say that p_0 is the *singularity* of $\mathcal{F}$; note that no regularity is required at p_0. A curve included in $W \setminus \{p_0\}$ is *positively transverse to the foliation* if it crosses every leaf it meets from left to right: for every $t_0 \in [0,1]$ there is a chart Ψ for the foliation, defined on some neighborhood of $\gamma(t_0)$ and with values in the plane, that sends the foliation to the foliation by vertical lines oriented from bottom to top, and is such that the first coordinate of $\Psi \circ \gamma$ is an increasing function on some neighborhood of t_0. Following Le Calvez, we say that a local isotopy (I, V) and a local foliation $(\mathcal{F}, W)$ are *dynamically transverse* if there exists a neighborhood $V' \subset V \cap W$ of p_0 such that for every point $p \in V' \setminus \{p_0\}$, the trajectory $I.p$ is homotopic in $W \setminus \{p_0\}$ to a curve γ which is positively transverse to $\mathcal{F}$ (see [38, Section 3], where this property is called "localement dynamiquement transverse"; here we drop the word "locally" since everything is local). We should add that occasionally we will use topological (nonsmooth) foliations; for precise definitions we refer to [37]. Most of the time, in our context, there is not much difference between using topological or smooth foliations, but there will be one point for which the smoothness will be crucial (see the normal form Lemma 83 below).

Recall the definition of the Poincaré–Lefschetz index $L(f, p_0)$ from the introduction. We denote by $L(\mathcal{F}, p_0)$ the Lefschetz index of the foliation $\mathcal{F}$ at p_0: this is simply the Lefschetz index of a vector field which is tangent to $\mathcal{F}$ and vanishes at p_0. The index $L(\mathcal{F}, p_0)$ is often referred to as the Poincaré–Hopf index.

Theorem 81 *Let (f, U) be a local homeomorphism with an isolated fixed point p_0. Let (I, V) be a local isotopy for (f, U). Then there exists a local foliation $(\mathcal{F}, W)$ which is dynamically transverse to (I, V). Furthermore*:

- *If $L(f, p_0) = 1$ then $L(\mathcal{F}, p_0) = 1$.*
- *If $L(f, p_0) \neq 1$ then we can choose the isotopy (I, V) so that $L(\mathcal{F}, p_0) = L(f, p_0)$.*

Proof We will first prove the statement concerning the indices $L(f, p_0)$ and $L(\mathcal{F}, p_0)$, assuming the existence of the dynamically transverse local foliation. The proof requires passing through the notion of the index of an isotopy $L(I, p_0)$. This index is defined in [41]. Proposition 3.1 of [38] proves that the index of the transverse foliation is the same as the index of the isotopy, ie $L(\mathcal{F}, p_0) = L(I, p_0)$. Furthermore, according to Proposition 4.7 of [43], if $L(f, p_0) = 1$, then $L(I, p_0) = 1$, and if $L(f, p_0) \neq 1$, then we can choose the isotopy I so that $L(I, p_0) = L(f, p_0)$.[14]

It remains to prove the existence of the location foliation $(\mathcal{F}, W)$ which is dynamically transverse to (I, V). Without loss of generality, we may suppose that $U \subset \mathbb{R}^2$. By Appendix A of [43], we may also assume that the homeomorphism f is defined on the entire plane $\mathbb{R}^2$ and that p_0 is the only fixed point of f in $\mathbb{R}^2$. We pick an isotopy $I' = (f_t)_{t \in [0,1]}$ of $\mathbb{R}^2$ such that $f_0 = \mathrm{Id}$, $f_1 = f$ and $f_t(p_0) = p_0$ for all $t \in [0, 1]$. Furthermore, the isotopy I' may be picked so that for every point $p \in V$ the trajectories $I'.p$ and $I.p$ are homotopic relative to endpoints in $V \setminus \{p_0\}$: indeed, this can be achieved by replacing I' with $I'R^q$, where q is an integer and R is a full rotation of the plane around the point p_0.

According to Le Calvez [38], there exists a *topological*, ie not necessarily smooth, foliation $\mathcal{F}$ of $\mathbb{R}^2 \setminus \{p_0\}$ which is dynamically transverse to the isotopy I'. A priori, this foliation is only globally transverse to the isotopy I': for every point $p \in \mathbb{R}^2 \setminus \{p_0\}$ the trajectory $I'.p$ is homotopic to a curve γ which is transverse to the foliation $\mathcal{F}$. Now, pick W to be any sufficiently small neighborhood of p_0 and consider the local

[14] Here we follow the convention of [43]; note that [38] takes the value $L(I, p_0) - 1$ as the definition of the index of p_0 for the isotopy I.

foliation $(\mathcal{F}, W)$. By Proposition 3.4 of [36], global transversality of I' and $\mathcal{F}$ implies that (I', V) and $(\mathcal{F}, W)$ are locally transverse. Since we picked I' such that for every point $p \in V$ the trajectories $I'.p$ and $I.p$ are homotopic relative to endpoints in $V \setminus \{p_0\}$, we see that (I, V) and $(\mathcal{F}, W)$ are dynamically transverse as well.

We are not completely done yet because, as mentioned in the previous paragraph, Le Calvez [37] provides us with a foliation $\mathcal{F}$ of $\mathbb{R}^2 \setminus \{p_0\}$ which is a priori nonsmooth. We will now outline an argument which will allow us to perturb $\mathcal{F}$ to a smooth foliation. According to Proposition 3.3 of [43], on any surface, the set of foliations which are transverse to a given isotopy forms an open subset of the set of all foliations, where the set of all foliations is equipped with the Whitney topology. For the precise definition of the Whitney topology, please see Section 3 of [43]. Now, the natural argument would be to prove that the set of smooth foliations forms a dense subset of the set of all foliations and then it would follow that we may pick our foliation to be smooth. However, we have not been able to find a proof of the density of smooth foliations in the literature and so we will prove that in the very specific settings of our article the foliation $\mathcal{F}$ may be perturbed to a smooth foliation.

In this article we will only rely on Theorem 81 when (f, U) is area-preserving. As a consequence, and as explained at the beginning of Section 7.2.2, the transverse foliation $(\mathcal{F}, W)$ must be gradient-like. This implies, according to Appendix B of [43], that $(\mathcal{F}, W)$ is locally homeomorphic to a smooth foliation: up to possibly shrinking the set W, we can a find smooth local foliation, say $(\mathcal{F}', W')$, and a homeomorphism $\phi\colon W' \setminus \{p_0\} \to W \setminus \{p_0\}$ which maps $\mathcal{F}'$ to $\mathcal{F}$. As we will explain in the next paragraph, diffeomorphisms form a dense subset of homeomorphism, with respect to the Whitney topology, and so we may find a diffeomorphism $\psi\colon W' \setminus \{p_0\} \to W \setminus \{p_0\}$ which is arbitrarily close to ϕ in the Whitney topology. As a consequence, the smooth foliation $\psi(\mathcal{F}')$ will be Whitney-close to $\mathcal{F}$ and so (up to shrinking W) we may replace $(\mathcal{F}, W)$ with $\psi(\mathcal{F}', W')$.

It remains to explain why diffeomorphisms form a dense subset of homeomorphisms with respect to the Whitney topology. We will only provide a brief sketch of the argument as it is very similar to the usual argument for proving that, in the case of surfaces, diffeomorphisms are dense in homeomorphisms with respect to the uniform topology.

Let Σ be any surface. Let us recall the definition of a Whitney neighborhood of a homeomorphism ϕ: Consider an open cover (U_i) of Σ which is locally finite, ie no point is contained in infinitely many of the U_i. To each U_i we associate $\varepsilon_i > 0$. A basic open neighborhood of ϕ is given by the set of homeomorphisms $\theta\colon \Sigma \to \Sigma$ such that for each i we have $d(\phi(p), \theta(p)) < \varepsilon_i$ for all $p \in U_i$. Now, we will show that every basic open neighborhood of ϕ contains a diffeomorphism. Given a basic open neighborhood ϕ as above, we pick a smooth triangulation of W satisfying the following two criteria: first, each triangle T is contained in at least one of the sets U_i and second, if T is contained in U_i then the diameter of $\phi(T)$ is very small compared to ε_i. One can then apply the handle-smoothing theorem of [24] to obtain a diffeomorphism ψ by smoothing out ϕ, successively, near the vertices, edges and finally faces of the triangulation. This can be done so that for each point p the distance between $\phi(p)$ and $\psi(p)$ is controlled by the diameter of the triangle(s) of the triangulation which contain the point p. □

7.2.2 $(\mathcal{F}, W)$ is gradient-like when (f, U) is area-preserving We equip the surface Σ with a symplectic form ω and denote by μ_0 the measure induced on Σ. For the rest of the article, unless otherwise stated, it will be our standing assumption that (f, U) is area-preserving, meaning that the push-forward measure $f_*\mu_0$ coincides with μ_0. As we will explain below, an important consequence of this assumption is that the foliation given by Theorem 81 will be *gradient-like*.

Let (I, V) a local isotopy for (f, U), and $(\mathcal{F}, W)$ a local (smooth or topological) foliation dynamically transverse to (I, V). By Poincaré–Bendixson theory, a half-leaf of $\mathcal{F}$ which does not hit the boundary of W either converges to the singularity p_0, or spirals around a closed leaf (that is, a leaf homeomorphic to a topological circle) or a union of "petals", ie leaves whose α– and ω–limit sets are $\{p_0\}$. Let ℓ be a closed leaf of $\mathcal{F}$ and D be the topological open disk bounded by ℓ. A curve topologically transverse to $\mathcal{F}$ meets ℓ at most once, thus by transversality we have either $f(\overline{D}) \subset D$ or $\overline{D} \subset f(D)$. This contradicts the fact that f preserves the area. Thus, $\mathcal{F}$ has no closed leaf. Transversality also implies that $\mathcal{F}$ has no "petal", because this would (as in the case of a closed leaf) lead to the existence of a disk D such that $f(\overline{D}) \subset D$ or $\overline{D} \subset f(D)$. Thus in our situation, for every oriented leaf ℓ, the ω–limit set of ℓ is either empty or equal to the singularity $\{p_0\}$; that is, ℓ either hits the boundary of U or the singularity. The α–limit set shares the same properties. Hence we have three kinds of oriented leaves, namely those which go from the boundary of W to the boundary, from the boundary to the singularity, or from the singularity to the boundary. A local foliation with only these kinds of leaf is called *gradient-like* (see [37]).

Classification

Local gradient-like foliations may be classified up to homeomorphisms: $(\mathcal{F}, W)$ consists of *hyperbolic sectors* and *parabolic sectors* (see the appendix of [43] for more details). A *sector* S is a subset of W which is homeomorphic to the closed unit disk and whose boundary contains p_0. If α and β are two curves of which p_0 is an endpoint, we will say that S *is between* α *and* β, and we write $S = S(\alpha, \beta)$, if there exists a curve γ such that the concatenation of α, γ and β in that order is a simple closed curve that parametrizes the boundary of S, in such a way that S is on the left-hand side of this curve. Consider a sector between two pieces of leaves of the foliation $\mathcal{F}$. The sector is called *parabolic* if the restriction of the foliation is homeomorphic to the foliation of an angular sector by radial lines. In a parabolic sector either every leaf has its ω–limit set equal to $\{p_0\}$, or every leaf has its α–limit set equal to $\{p_0\}$; we call the parabolic sector *positive* in the first case and *negative* in the latter case. A sector is called *hyperbolic* if it is homeomorphic to the foliation of $\{x \geqslant 0,\ y \geqslant 0\}$ in the plane by the hyperbolæ $xy = \text{constant}$. The topological classification of local foliations states that any gradient-like local foliation is homeomorphic to a foliation obtained by gluing together a finite number of hyperbolic and parabolic sectors. Furthermore, the number of hyperbolic sectors is given by $N = 2(1 - L(\mathcal{F}, p_0))$. Thus, taking into account the orientation, there are exactly two topological types of gradient-like local foliation of index one (called sinks and sources). For indices different than one, there are 2^N foliations of a given index $L(\mathcal{F}, p_0)$; this is because between any two adjacent hyperbolic sectors one can choose to add a parabolic sector or not. Observe that the index of a gradient-like foliation is at most one. This, combined with Theorem 81, implies that the index of a fixed point of an area-preserving homeomorphism is at most one.

Note that the existence of a gradient-like foliation transverse to a local isotopy has interesting dynamical consequences on the local homeomorphism (f, U). For instance:

(1) Each hyperbolic sector is either locally attractive ($f(S \cap W') \subset S$ for some smaller neighborhood W' of the fixed point) or repulsive.

(2) If the foliation is a sink, then the local rotation set is included in $[0, \infty]$.

Both remarks will be crucial in what follows (see the proof of Lemma 87 and the beginning of Section 7.4).

We will be needing the following lemmas. The first of these is not difficult and so we leave the proof to the reader.

Lemma 82 *Let $(\mathcal{F}, W)$ be a local foliation and assume that there exists a closed curve γ included in $W \setminus \{p_0\}$ which is positively transverse to $\mathcal{F}$. Then γ is not contractible in $W \setminus \{p_0\}$. Furthermore, if $\mathcal{F}$ is gradient-like, then it must be a sink or a source.*

The next lemma tells us that we can find well-adapted area-preserving charts on a neighborhood of any parabolic sector. We should point out that the smoothness of the foliation is crucial here.

Lemma 83 (normal form for parabolic sectors) *Let $(\mathcal{F}, W)$ be a local foliation with a parabolic sector S, and O an open subset of W containing $S \setminus \{p_0\}$. Then there exists some open set O' with $S \setminus \{p_0\} \subset O' \subset O$ and a diffeomorphism $\Phi\colon O' \to \Phi(O') \subset \mathbb{R}^2 \setminus \{(0,0)\}$ with the following properties*:

(1) *$\Phi(O') \cup \{(0,0)\}$ contains $\{0\} \times [-\varepsilon, \varepsilon]$ for some positive ε.*

(2) *Φ sends the edges of S to Euclidean rays respectively included in*
$$\{(x,y) \in \mathbb{R}^2 : y = 0,\ x > 0\} \quad \textit{and} \quad \{(x,y) : y = x,\ x > 0\}.$$

(3) *$\Phi_* \mu_0 = \mathrm{Leb}$, where μ_0 is the measure induced by the symplectic form and* Leb *is the Lebesgue measure on $\mathbb{R}^2$.*

(4) *On a neighborhood of the origin in the half-plane $\{(x,y) \in \mathbb{R}^2 : x \geqslant 0\}$, the foliation $\Phi(\mathcal{F})$ is transverse to the foliation of the half-plane by vertical lines.*

Proof Our proof consists of two steps: In the first step, we will construct Φ on the parabolic sector S so that it satisfies all the requirements of the lemma on $\Phi(S)$. In the second step, we will extend Φ beyond the edges of S.

Step 1: constructing Φ on S Let T_1 be the triangle
$$\{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x\}.$$
We begin with the following claim:

Claim 84 *There exists a homeomorphism $\Phi'\colon S \to T_1$ such that $\Phi'(\mathcal{F})$ is the radial foliation of T_1 whose leaves are the lines $y = kx$ for $k \in [0,1]$. Furthermore, $\Phi'\colon S \setminus \{p_0\} \to T_1 \setminus \{(0,0)\}$ is a diffeomorphism.*

Proof Denote the edges of the parabolic sector by $\partial^- S$ and $\partial^+ S$. Let $\gamma\colon [0,1] \to W$ be a smooth curve which is transverse to the parabolic sector S such that $\gamma(0) \in \partial^- S$ and $\gamma(1) \in \partial^+ S$. Such γ exists for the following reason: The fact that S is topologically equivalent to a standard sector with its radial foliation implies that one may pick such γ which is continuous. One can then smooth out γ by an isotopy which preserves the leaves of the foliation (recall that the foliation is smooth on $W \setminus \{p_0\}$).

In the following we consider some Riemannian metric on our surface Σ. Consider the segment of the leaf of the foliation which starts at $\gamma(s)$ and ends at p_0; denote its length (which could be $+\infty$) by $\ell(s)$. Let us show that $S \setminus \{p_0\}$ may be parametrized by the set $U = \{(t,s) \in \mathbb{R}^2 : 0 \leqslant t < \ell(s), 0 \leqslant s \leqslant 1\}$: Pick X to be the unit-length vector field defined on $S \setminus \{p_0\}$ which is tangent to the leaves of $\mathcal{F}$ and such that X points into the sector S at every point on γ. Denote the flow of X by φ_X^t and define $\Psi_1 \colon U \to S \setminus \{p_0\}$ by $\Psi_1(t,s) = \varphi_X^t(\gamma(s))$. This is a diffeomorphism from U to $S \setminus \{p_0\}$ which sends the foliation of U by horizontal lines to the restriction of $\mathcal{F}$ to the sector S.

We leave it to the reader to check that one can find a diffeomorphism $\Psi_2 \colon T_1 \setminus \{(0,0)\} \to U$ which sends the radial foliation of T_1 to the horizontal foliation of U. Define $\Phi' = (\Psi_1 \Psi_2)^{-1} \colon S \setminus \{p_0\} \to T_1 \setminus \{(0,0)\}$ and set $\Phi'(p_0) = (0,0)$. It is easy to check this is our desired map. □

Observe that, because Φ' is smooth away from p_0, we can apply the change-of-variables formula to conclude that the measure $\mu = \Phi'_* \mu_0$ is of the form $\eta\,\mathrm{Leb}$, where η is a smooth function on $T_1 \setminus \{(0,0)\}$. We will adjust the coordinates *radially* so that the measure $\eta\,\mathrm{Leb}$ is sent to a new measure $\rho\,\mathrm{Leb}$ with the property that the vertical projections of $\rho\,\mathrm{Leb}$ and Leb onto the x–axis coincide. To achieve this, for each $X \in (0,1]$ consider the triangles $T_X = \{0 \leqslant x \leqslant X, 0 \leqslant y \leqslant x\}$ and define $H(X) = (2\mu(T_X))^{1/2}$. This is a homeomorphism between $(0,1]$ and some interval $(0,a]$. Now, note that x and the angular coordinate θ together define a smooth system of coordinates on $x > 0$. Consider the map

$$\Phi''(x,\theta) = (H(x),\theta).$$

This is a homeomorphism between the triangles T_1 and T_a which preserves the radial foliation. Moreover, it is smooth away from the origin and so it sends the measure $\eta\,\mathrm{Leb}$ to a measure of the form $\rho\,\mathrm{Leb}$, where ρ is a function on T_a which is defined everywhere except possibly at the origin. Furthermore,

$$\Phi''(T_X) = T_{H(X)} \quad \text{and} \quad \mathrm{Leb}(T_{H(X)}) = \mu(T_X)$$

for every $X \in (0,1]$, and thus the projections of $\rho\,\mathrm{Leb}$ and Leb onto the first coordinate are the measure $x\,dx$. In other words, $\int_0^x \rho(x,y)\,dy = x$ for every $x \in (0,a]$. Now, define Φ''' by the formula

$$\Phi'''(x, y) = \left(x, \int_0^y \rho(x,t)\, dt\right).$$

This map leaves the triangle T_a invariant and sends the measure $\rho\,\mathrm{Leb}$ to the Lebesgue measure. Furthermore, it preserves the vertical foliation and thus it maps the radial foliation to a foliation which is transverse to the vertical one.

The homeomorphism $\Phi = \Phi'''\Phi''\Phi' \colon S \mapsto T_a$ is an area-preserving chart which sends $\mathcal{F}$ to a foliation transverse to the vertical foliation. This completes the first step of our construction.

Step 2: extension of Φ beyond the edges of S Let $\partial^+ S$ and $\partial^- S$ denote the two leaves of the foliation which are at the boundary of the parabolic sector S. The map Φ constructed above sends $\partial^+ S$ and $\partial^- S$, diffeomorphically, to line segments which have an endpoint at the origin. By the following lemma, Φ may be extended to a neighborhood O' of S in $W \setminus \{p_0\}$ so that all the requirements of Lemma 83 are satisfied.

Lemma 85 *Suppose that $(\mathcal{F}, W)$ has a leaf F whose ω–limit, or α–limit, set is p_0. Let R be a line segment in $\mathbb{R}^2$ of the form $R = \{(x, kx) : x \in (0,1)\}$ for some $k \in \mathbb{R}$. Let $\Phi\colon F \to R$ be a smooth diffeomorphism. There exists an open set U containing F and included in $W \setminus \{p_0\}$, and an extension of Φ to an area-preserving diffeomorphism $\Phi\colon U \to \Phi(U) \subset \mathbb{R}^2$, such that:*

1. *$\Phi(U) \cup \{(0,0)\}$ contains the segment $\{0\} \times [-\varepsilon, \varepsilon]$ for some positive ε.*
2. *On the half-plane $\{(x,y) \in \mathbb{R}^2 : x \geqslant 0\}$, the leaves of $\Phi(\mathcal{F})$ are transverse to the foliation given by vertical lines.*

Proof The proof is very similar to the proof of Step 1; we will just sketch it and leave the details to the reader. We make a preliminary note that the linear map $(x, y) \mapsto (x, y - kx)$ preserves the Lebesgue measure and the vertical foliation and sends the ray R into the x–axis; using this map we can assume that $k = 0$. Using some curve which passes through the leaf F and which is transverse to the foliation, we first extend Φ into a diffeomorphism Φ' that sends $\mathcal{F}$ to the foliation by horizontal lines, and whose image contains a segment $\{0\} \times [-\varepsilon, \varepsilon]$ for some positive ε; the construction is similar to the proof of Claim 84. Then we postcompose Φ' by a diffeomorphism Φ'' which is the identity on $\Phi'(F)$ and sends the measure $\Phi'_* \mu_0$ to the Lebesgue measure; this construction is similar to the end of Step 1. □

This completes the second step of our construction. □

7.3 Smoothability of fixed points of nonpositive index

In this section, we will prove that a fixed point with nonpositive index is smoothable. Before going into the proof, let us reduce the problem to a purely local one. Let (f, U) be an area-preserving local homeomorphism for which p_0 is the only fixed point. We will say (f, U) is *smoothable* if there exists an area-preserving local homeomorphism (g, U) for which p_0 is the only fixed point, which is smooth near p_0 and which coincides with f near the boundary of U. Paradoxically enough, this notion is invariant under conjugacy by local *homeomorphisms*. Indeed, assume (f, U) is smoothable and let (g, U) be as above. Suppose that $\Psi\colon U \to U'$ is a homeomorphism, where U' is an open subset of some surface, and consider $(\Psi f \Psi^{-1}, U')$. By the handle-smoothing lemma there is a homeomorphism $\hat{\Psi}$ that coincides with Ψ near the boundary of U and that is smooth near p_0 [24]. Then, $(\hat{\Psi} g \hat{\Psi}^{-1}, U)$ is smooth near p_0 and coincides with $\Psi f \Psi^{-1}$ near the boundary of U', showing that $(\Psi f \Psi^{-1}, U')$ is also smoothable. We conclude that Theorem 77 is equivalent to the following purely local statement: *every area-preserving local homeomorphism (f, U) with a single fixed point p_0 which is not maximally degenerate is smoothable.* We will now prove that statement in the case when the index $L(f, p_0)$ is nonpositive. We will first give an outline of the proof, relying on several technical lemmas which will be proved later.

Proof of Theorem 77 in the nonpositive index case Let (f, U) be an area-preserving local homeomorphism with a single fixed point p_0, with $L(f, p_0) \leqslant 0$. According to Theorem 81 there is a local isotopy (I, V) for (f, U) and a local foliation $(\mathcal{F}, W)$ dynamically transverse to (I, V) such that $L(\mathcal{F}, p_0) = L(f, p_0)$. By applying Proposition 86 below, we only need to consider the case when the foliation $\mathcal{F}$ has no parabolic sectors.

Proposition 86 *Let (f, U) be an area-preserving local homeomorphism with a single fixed point p_0 of index $L(f, p_0) \leqslant 0$. There exists an area-preserving local homeomorphism (f', U) with a single fixed point p_0 that coincides with f near the boundary of U, and an isotopy (I', V) for (f', U) admitting a dynamically transverse local foliation that has the same index $L(f, p_0)$ and has no parabolic sectors.*

To fix ideas, from now on we assume $L(f, p_0) = -1$ (see the comments at the end of the proof for the other cases). We define the *model dynamics* to be the linear map $m_{-1}\colon (x, y) \mapsto \left(2x, \frac{1}{2}y\right)$ on the plane $\mathbb{R}^2$. Let D be a closed topological disk included in U and containing the fixed point p_0 in its interior. Assume $f(D)$ is included in U. We say that D is *in canonical position for* f if there is a continuous injective mapping $\Phi\colon U \to \mathbb{R}^2$ taking D to the unit disk and such that $\Phi f \Phi^{-1} = m_{-1}$ on the boundary of the unit disk (see [53; 3]). Note that this is a purely topological definition (no measure involved). The next lemma is purely topological, while the following one will take care of the area.

Lemma 87 *Let (I, V) be a local isotopy for a local homeomorphism (f, U), which is transverse to a local foliation $(\mathcal{F}, W)$ with $L(\mathcal{F}, p_0) = -1$ and no parabolic sector. There exists a disk D in canonical position for f.*

We consider the standard area on the plane. It is preserved by m_{-1}.

Lemma 88 *Let D be a disk in canonical position for f. Then there exists a continuous injective mapping $\Psi\colon U \to \mathbb{R}^2$ such that*

- *$f = \Psi^{-1} m_{-1} \Psi$ holds on ∂D,*
- *Ψ sends the area on U to the standard area,*
- *Ψ is smooth near p_0 and sends p_0 to 0.*

Let D be the disk given by the first lemma, and Ψ be given by the second lemma. Define g as the homeomorphism that coincides with f on $U \setminus D$ and with $\Psi^{-1} m_{-1} \Psi$ on D. Then g has only one fixed point and is smooth near p_0, as wanted. This concludes the proof in the case when $L(f, p_0) = -1$.

The proof when $L(f, p_0)$ is any nonpositive number p is entirely similar. Instead of m_{-1}, one uses a model map m_p of index p which is smooth and is made of $2(1-p)$ hyperbolic sectors (obtained, for instance, by integrating an autonomous Hamiltonian function with a degenerate critical point). Likewise, the proofs of Lemmas 87 and 88 are entirely similar for every nonpositive value of the index, and so, for expository reasons, they will be described only in the -1 case. □

Proof of Lemma 87 Remember that the isotopy (I, V) is dynamically transverse to the foliation $(\mathcal{F}, W)$; there exists a neighborhood $U' \subset V \cap W$ such that the trajectory of every point in $U' \setminus \{p_0\}$ is homotopic in $W \setminus \{p_0\}$ to a curve which is positively transverse to $\mathcal{F}$. Let us call a leaf of $\mathcal{F}$ whose α– or ω–limit set is $\{p_0\}$ a *separatrix*. Two consecutive separatrices in the cyclic order around p_0 are of opposite types, one is a stable separatrix oriented towards p_0 and the other one an unstable separatrix oriented from p_0. Let S be a sector in U bounded by two consecutive separatrices ℓ and ℓ', and which is on the right-hand side of both ℓ and ℓ'. By dynamical transversality we

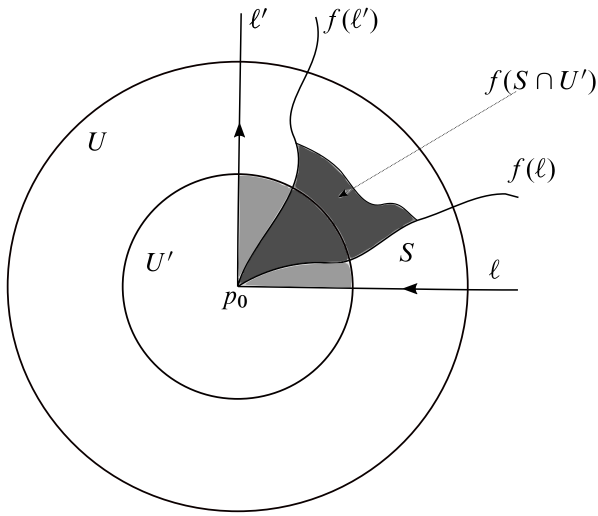

Figure 2: $f(\ell)$ and $f(\ell')$ are on the right-hand sides of ℓ and ℓ', respectively. As a consequence, $f(S\cap U')\subset S$.

have $f(S\cap U')\subset S$; see Figure 2. This implies that for every $n>0$, there is some neighborhood of p_0 in which the n first iterates of ℓ and ℓ' are pairwise disjoint, and their cyclic order around p_0 is either given by $\ell, f(\ell), \dots, f^n(\ell), f^n(\ell'), \dots, f(\ell'), \ell'$ or by the reverse order; see Figure 3.

Since $L(\mathcal{F}, p_0) = -1$ and $\mathcal{F}$ has no parabolic sector, $\mathcal{F}$ consists of four hyperbolic sectors. In particular, $\mathcal{F}$ has exactly four separatrices, which we denote by ℓ_0, ℓ_1, ℓ_2 and ℓ_3, choosing the numbering so that the sector between ℓ_0 and ℓ_1 is attracting, as

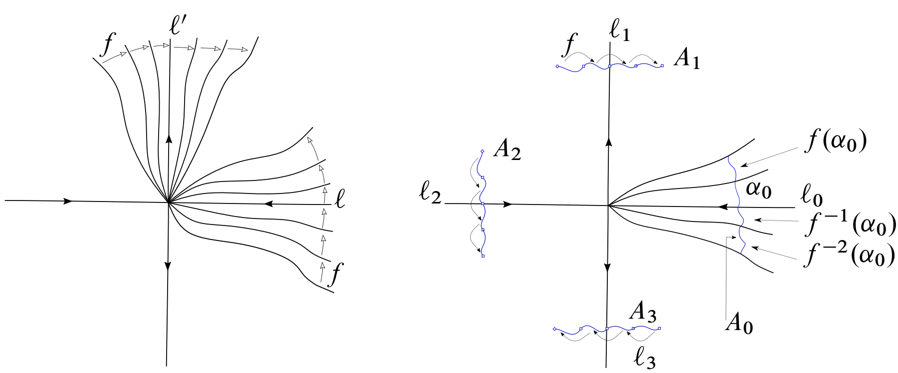

Figure 3: Left: the cyclic order of the images of ℓ and ℓ' under the iterates of f. Right: the construction of the curves A_0, A_1, A_2 and A_3.

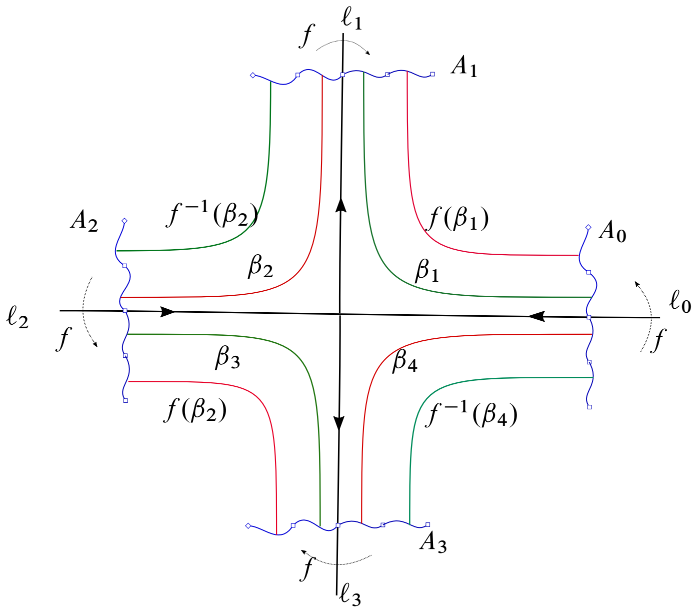

Figure 4: $\beta_1, \dots, \beta_4$ are leaves of the transverse foliation. The simple closed curve C is the union of β_1, a segment along A_1, $f^{-1}(\beta_2)$, a segment along A_2, β_3, a segment along A_3, $f^{-1}(\beta_4)$ and a segment along A_4. Here C consists of the green curves and segments along the blue curves. The image $f(C)$ consists of the red curves and segments along the blue curves.

in Figure 3, right. Let S_i denote a small open sector between ℓ_i and its image. Choose for each i a simple curve α_i close enough to the fixed point, joining one point on ℓ_i to its image, and whose interior is included in S_i. By local transversality the sector S_i is disjoint from $f(S_i)$, $f^2(S_i)$ and $f^3(S_i)$, so that $A_i = f^{-2}(\alpha_i) \cup f^{-1}(\alpha_i) \cup \alpha_i \cup f(\alpha_i)$ is a simple curve. Note that again by local transversality the A_i are pairwise disjoint; see Figure 3.

Choose for each $i = 1, \dots, 4$ a segment β_i of a leaf of the foliation included in the hyperbolic sector between ℓ_{i-1} and ℓ_i (where the indices are taken modulo 4), very close to the boundary $\ell_{i-1} \cup \ell_i$ of the sector, with

- one endpoint on α_{i-1} and the other on α_i for $i = 1, 3$,
- one endpoint on $f^{-1}(\alpha_{i-1})$ and the other on $f^{-1}(\alpha_i)$ for $i = 2, 4$,

and whose interior is disjoint from A_i and A_{i-1}; see Figure 4. Let C be the (only) simple closed curve included in the union of the curves

$$\beta_1, \quad A_1, \quad f^{-1}(\beta_2), \quad A_2, \quad \beta_3, \quad A_3, \quad f^{-1}(\beta_4), \quad A_4.$$

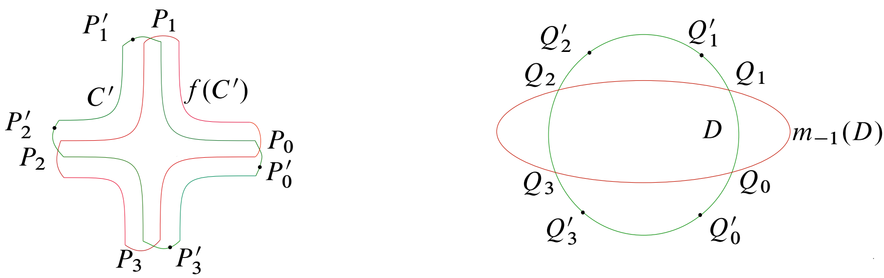

Figure 5: Here P_i' and Q_i' denote the preimages of P_i and Q_i under f and m_{-1}, respectively. The curve C' is in canonical position for f. Note that the cyclic order of the points P_0', P_0, P_1, P_1', P_2', P_2, P_3 and P_3' on C' matches that of the points Q_0', Q_0, Q_1, Q_1', Q_2', Q_2, Q_3 and Q_3'.

The reader may check that $C \cap f(C)$ is the union of four segments included respectively in A_1, A_2, A_3 and A_4; see Figure 4. We modify C slightly near these segments to get a Jordan curve C' such that $C' \cap f(C')$ consists in four points P_0, P_1, P_2 and P_3 respectively close to A_0, A_1, A_2 and A_3; see Figure 5.

We claim that the disk bounded by C' is in canonical position for f. To see this we define the homeomorphism Φ as follows. Let D be the unit disk and Q_0, Q_1, Q_2 and Q_3 be the points in $D \cap m_{-1}(D)$ as in Figure 5. We orient ∂D and C' in the counterclockwise direction. Note that the cyclic order of the points P_i and $f^{-1}(P_i)$ along C' coincides with the cyclic order of the points Q_i and $m_{-1}^{-1}(Q_i)$ on ∂D, so there exists an orientation-preserving homeomorphism from C' to ∂D that sends P_i to Q_i and $f^{-1}(P_i)$ to $m_{-1}^{-1}(Q_i)$ for every i. Let Φ coincide with this homeomorphism on C'. Then we extend Φ on $f(C')$ by the formula

$$\Phi = m_{-1}\Phi_{|C'} f^{-1}.$$

The complement of $C' \cup f(C')$ in U has six connected components. For each connected component, say Δ, it can easily be checked that $\Phi(\partial\Delta)$ is the boundary of some connected component Δ' of $\partial D \cup m_{-1}(\partial D)$ in the plane. Furthermore, the map $\Delta \mapsto \Delta'$ is a bijection. Thus, we can use Schoenflies' theorem independently on each Δ to extend Φ to a homeomorphism between U and $\mathbb{R}^2$, as required by the definition of canonical position. □

Proof of Lemma 88 If D is a disk in canonical position for f, then $U \setminus (\partial D \cup f(\partial D))$ has five bounded connected components (we call unbounded the connected component which contains a neighborhood of the boundary of U). We denote their respective

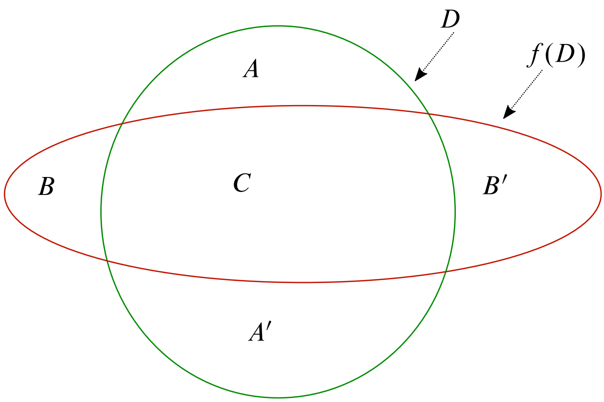

Figure 6: The areas of the five bounded connected components of $D \cup f(D)$ are denoted by A, A', B and B'. Note that $A + A' = B + B'$.

areas by A, A', B, B' and C so that

(1) C is the area of $D \cap f(D)$,

(2) $D \setminus f(D)$ has two connected components with areas A and A',

(3) $f(D) \setminus D$ has two connected components with areas B and B'.

Note that since f preserves the area we have $A + A' = B + B'$. See Figure 6.

We will make use of the following claim, whose proof is given below.

Claim 89 *There exists a disk D_0 in canonical position for the model map m_{-1} such that the above properties* (1), (2) *and* (3) *hold with the same values of A, A', B, B' and C when D and f are replaced by D_0 and m_{-1}.*

Postponing the proof of the above claim, we continue with the proof of Lemma 88. Using the definition of canonical position both for f and D and for m_{-1} and D_0, we get a local homeomorphism (Φ, U) taking D to D_0 such that $\Phi f \Phi^{-1} = m_{-1}$ on the boundary of D_0. We first define Ψ on $\partial D \cup f(\partial D)$ by setting $\Psi = \Phi$ there. Then we choose a small smooth disk δ around p_0, a smooth disk δ_0 around 0, included in D_0, having the same area as δ, and define Ψ on δ to be any area-preserving diffeomorphism between δ and δ_0 that sends p_0 to 0. Let M be one of the five bounded connected components of the complement of $(\partial D \cup f(\partial D)) \cup \delta$. Let M' be the corresponding connected component of the complement of $\partial D_0 \cup m_{-1}(\partial D_0) \cup \delta_0$. Then M' has the same area as M, so we can apply the Oxtoby–Ulam theorem [51, Part II, Corollary 1]

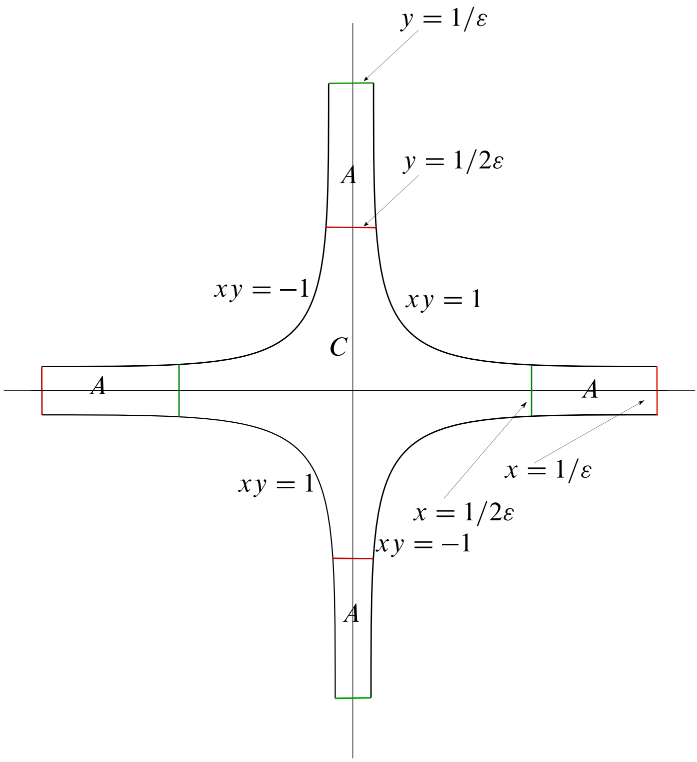

Figure 7: Obtaining small values of $\rho := A/C$ in the case where $A = A' = B = B'$: D_0 is the disk bounded by the curves $x = \pm 1/2\varepsilon$, $y = \pm 1/\varepsilon$ (in green) and segments along $xy \pm 1$. Note that $xy = \pm 1$ is invariant under m_{-1} and so the image of D_0 under m_{-1} is bounded by $x = \pm 1/\varepsilon$, $y = \pm 1/2\varepsilon$ (in red) and segments along $xy = \pm 1$. By picking ε to be small, we can attain arbitrarily large values of C. The disk D_0 is not in canonical position but one can make a small perturbation to place it in canonical position.

(see also Theorem 3.1 in [12]) and extend Ψ to an area-preserving homeomorphism from M to M'.

It remains to prove the claim. We first treat the case when $A = A'$ and $B = B'$. In this case we have $A = B$. Let $\rho = A/C$. The map m_{-1} commutes with the homothety $z \to \lambda z$, thus the image under $z \to \lambda z$ of a disk in canonical position for m_{-1} is still a disk in canonical position. Consequently we just have to check that we may obtain every value of ρ.

Figure 7 explains how to construct a disk in canonical position with an arbitrarily small value of ρ. On the other hand, consider given a disk in canonical position with some

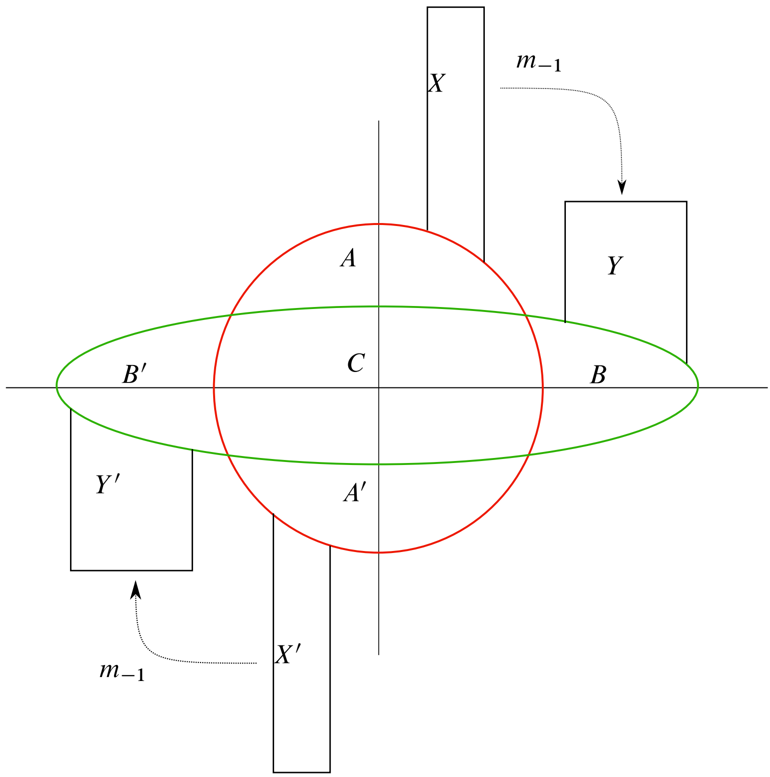

Figure 8: Obtaining values of $\rho > \rho_0$ in the case where $A = A' = B = B'$: The disk bounding the regions A, A' and C is in canonical position and $\rho_0 = A/C$. We attach two strips of equal area labeled X and X' as depicted above. The strips Y and Y' are the images of the previous strips under m_{-1}. The disk bounding the regions X, A, C, A' and X' is also in canonical position. For this disk we have $\rho = (A+X)/C$. By picking X suitably we can attain any value $\rho > \rho_0$.

value of $\rho = \rho_0$. Figure 8 explains how to modify the given disk to obtain a disk in canonical position for any value of $\rho > \rho_0$.

Now we turn to the general case. First note that the horizontal and vertical line symmetries commutes with m_{-1}, so we may apply a horizontal or a vertical symmetry to the target picture, thus exchanging the values of A and A', and/or those of B and B'. Thus we may assume that $A \leqslant A'$ and $B \leqslant B'$. Furthermore, since $A + A' = B + B$, we have $A \leqslant B \leqslant B' \leqslant A'$ or $B \leqslant A \leqslant A' \leqslant B'$. But the symmetry with respect to the line $x + y = 0$ conjugates m_{-1} and its inverse, and using this conjugacy exchanges

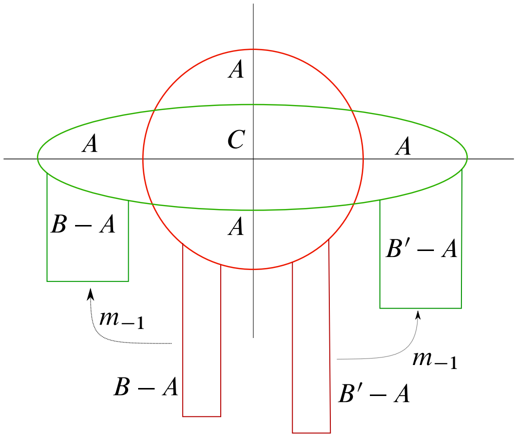

Figure 9: We construct the regions B and B' by attaching strips of area $B-A$ and $B'-A$ (in green) as above. The region A' is obtained by attaching the preimages, under m_{-1}, of the previous strips (in red).

the values of A and B on the one hand, and those of A' and B' on the other hand. Thus we may assume $A \leqslant B \leqslant B' \leqslant A'$.

We begin by constructing a disk, say D_0, in canonical position for which A and C have the required values and $B = B' = A' = A$. Such a disk exists by what was explained above. Now, we modify the disk D_0 by attaching two strips of areas $B-A$ and $B'-A$, respectively, to the regions which are supposed to eventually have areas B and B'; this is depicted in Figure 9. To adjust the area of the region which is supposed to have area A', we attach to it the preimages, under m_{-1}, of the previous two strips. It is easy to see that this region will have area $A+(B-A)+(B'-A)=A'$. □

Proof of Proposition 86 Let (f,U) be as in the statement; we apply Theorem 81 to get a local isotopy (I,V) for (f,U) and a local foliation $(\mathcal{F},W)$ which is dynamically transverse to I and has the right index. Since $L(\mathcal{F},p_0)=L(f,p_0)\leqslant 0$, the foliation $\mathcal{F}$ consists of at least two hyperbolic sectors, and maybe some parabolic sectors. If there is no parabolic sector then we are done. Assume that $\mathcal{F}$ has a parabolic sector. Proposition 86 is an immediate consequence of the following claim by a descending induction on the number of parabolic sectors around p_0. (Note that the modified local foliation may have a smaller domain, but it is important that the domain of the local homeomorphism is kept unchanged.)

Claim 90 *In this situation, there exists an area-preserving local homeomorphism (f',U) with a single fixed point p_0 that coincides with f near the boundary of U,*

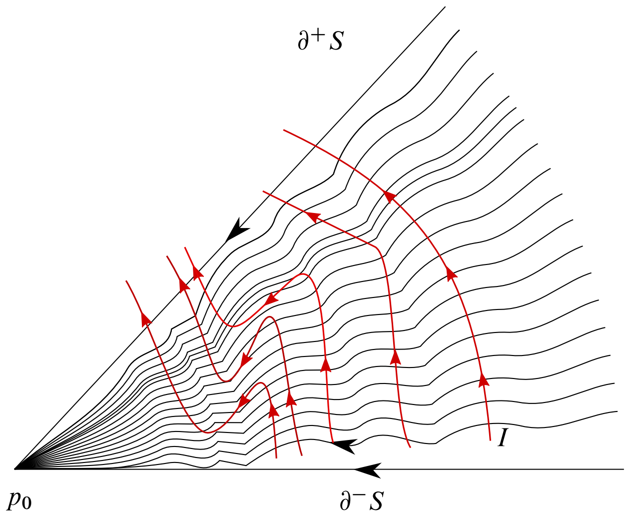

Figure 10: The isotopy I, whose trajectories are in red, sweeps out the parabolic sector S.

and a local isotopy (I', V) *for* f' *with a dynamically transverse foliation* $(\mathcal{F}', W')$ *that has one less parabolic sector than* $(\mathcal{F}, W)$.

It remains to prove the claim. We consider a parabolic sector S which is *maximal* in the following sense: for every parabolic sector S' that contains S, there exists a neighborhood U' of p_0 such that $S' \cap U' = S \cap U'$. Proposition 92 will apply and provide an isotopy (I', V) that "sweeps out the parabolic sector S" in the following sense:

Definition 91 Let (I, V) be a local isotopy, dynamically transverse to a local foliation $(\mathcal{F}, W)$, and S be a parabolic sector of $\mathcal{F}$. We denote by $\partial^- S$ and $\partial^+ S$ the two leaves of $\mathcal{F}$ which bound S near p_0, so that S is locally on the right-hand side of the oriented leaf $\partial^- S$ and on the left-hand side of $\partial^+ S$. We say that I *sweeps out the parabolic sector* S if the trajectory of every point x in $S \setminus \{p_0\}$ close enough to the fixed point p_0 has positive intersection with the leaf $\partial^+ S$; see Figure 10.

Indeed, point (3) of the proposition ensures that $I' = IJ$ sweeps out the sector S. In our situation $\mathcal{F}$ has nonpositive index, thus the last part of the proposition applies and says that p_0 is the only fixed point of $f' = fg$.

Proposition 92 (parabolic pushing) *Let* (I, V) *be a local isotopy for an area-preserving local homeomorphism* (f, U) *dynamically transverse to a local foliation* $(\mathcal{F}, W)$.

Assume p_0 is the only fixed point of f in U. Let $S \subset U$ be a parabolic sector of $\mathcal{F}$ and $O \subset U$ be some open set containing $S \setminus \{p_0\}$ which is the interior of a sector.

Then there exists an isotopy J for some area-preserving homeomorphism g, with the following properties:

(1) *J is supported in $\overline{O}$.*

(2) *The local isotopy (IJ, U) is dynamically transverse to $(\mathcal{F}, W)$.*

(3) *There exists some neighborhood U' of p_0 such that $g(U' \cap \partial^- S) \subset \partial^+ S$.*

Furthermore, under either of the following two hypotheses, we may require that p_0 be the only fixed point of fg in U:

(i) *$\mathcal{F}$ has nonpositive index.*

(ii) *For every p in O such that $f(p)$ is in O, the trajectory $I.p$ is homotopic in $U \setminus \{p_0\}$ to a curve included in O.*

Next, we apply Lemma 93 below to modify the foliation into a new transverse foliation $(\mathcal{F}', W)$ with the parabolic sector S replaced by a single leaf. To state the lemma we need the following notions. Let ℓ be a leaf of $\mathcal{F}$ whose α– or ω–limit set is $\{p_0\}$, and let p be a point on ℓ. We will refer to the connected component of $\ell \setminus \{p\}$ whose closure contains p_0 as a *half-leaf of $\mathcal{F}$*.

Lemma 93 (modification of the foliation) *Let (I, V) be a local isotopy, dynamically transverse to a gradient-like local topological foliation $(\mathcal{F}, W)$. Let S be a maximal parabolic sector of $\mathcal{F}$, and O be some open set containing $S \setminus \{p_0\}$, such that every half-leaf of $\mathcal{F}$ which is included in O is also included in S.*

Assume I sweeps out the sector S. Then there exists a local topological foliation $(\mathcal{F}', W')$ which is dynamically transverse to (I, V), equal to $\mathcal{F}$ on $W' \setminus \overline{O}$, and that has a single half-leaf included in O.

The new foliation has one less parabolic sector than the original one. This completes the proof of Claim 90, and thus also the proof of Proposition 86. □

Proof of Proposition 92 Up to conjugation by an orientation-reversing map, we may assume that the leaves of $\mathcal{F}$ through any point of S have their ω–limit set equal to $\{0\}$, that is, S is a positive parabolic sector. By the "normal form" lemma (Lemma 83),

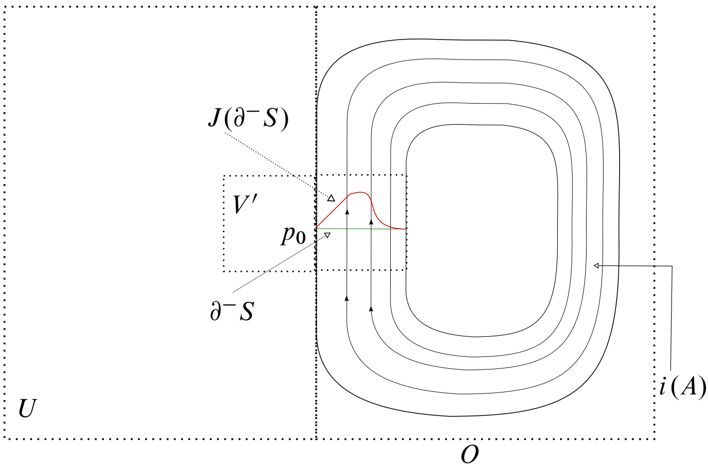

Figure 11: The isotopy J is supported in the annulus $i(A)$ and sends $\partial^- S$ to $\partial^+ S$ near p_0.

we may assume the following: $U, V, W \subset \mathbb{R}^2$, $p_0 = 0$, S locally coincides with the sector given by $0 < x$ and $0 \leqslant y \leqslant x$, the leaf $\partial^- S$ is included in $y = 0$ and oriented towards 0, O contains $(0, \varepsilon) \times (-\varepsilon, \varepsilon)$, and on this set the topological foliation $\mathcal{F}$ is transverse to the vertical foliation. Up to shrinking U and O, we may assume that $O = U \cap \{x > 0\} = (0, \varepsilon) \times (-\varepsilon, \varepsilon)$.

Let $V' = (-v, v) \times (-v, v) \subset U$ be an open neighborhood of 0 on which I is transverse to $\mathcal{F}$ (in the sense of the definition of local transversality, Section 7.2.1). Fix some small $a > 0$ and consider the annulus $A = [0, a] \times \mathbb{S}^1$ equipped with coordinates (r, θ) and the standard area form $dr \wedge d\theta$, and let $i \colon A \to \overline{O}$ be a symplectic embedding such that for every (r, θ) near $(0, 0)$, $i(r, \theta) = (r, \theta)$, where we use the cartesian coordinate system on the target $\mathbb{R}^2$. In particular, the curves $r =$ constant coincide with the vertical lines (see Figure 11). Up to shrinking V', we assume $i(r, \theta) = (r, \theta)$ on $V' \cap \{x > 0\}$.

Let $W' = (-w, w) \times (-w, w)$ with $w < v$. Since f has no fixed point in U, there is some $\delta > 0$ such that $d(p, f^{-1}(p)) \geqslant \delta$ for every $p \in U \setminus W'$. Let $J = (g_t)$ be an isotopy supported on the annulus $i(A)$ defined by $g_t(r, \theta) = (r, \theta + t\Delta(r))$, where $\Delta(r) = r$ near $r = 0$ and Δ is small enough that $g_1(W') \subset V'$ and $d(g_1(p), p) < \delta$ for every p.

We leave it to the reader to check that J meets the first and the third properties. As for the second property, note that for every p in W', the trajectory of p under J is a vertical segment, and thus it is transverse to the foliation. Note that if $p \in W'$, then $g(p) \in V'$ and thus the trajectory of p under IJ is homotopic to a curve positively transverse to $\mathcal{F}$; in particular, the local isotopy (IJ, V) is dynamically transverse to $(\mathcal{F}, W)$.

Now assume some point $p \in U \setminus \{p_0\}$ is fixed under fg. We will prove that none of the two hypotheses (i)–(ii) of the last part of the proposition hold. Since g is supported in $\overline{O}$ and f has no fixed point in $U \setminus \{p_0\}$, p must belong to O. By definition of δ, the map fg has no fixed point in $U \setminus W'$, thus p must belong to $W' \cap O$. By the previous paragraph, the trajectory $IJ.p$ is homotopic in U to a closed curve positively transverse to $\mathcal{F}$. According to Lemma 82, $IJ.p$ is not contractible in $U \setminus \{0\}$ and $\mathcal{F}$ must be a sink or a source. In particular, $L(\mathcal{F}, 0) = 1$, and hypothesis (i) does not hold. Since $J.p$ is included in O, the point $p' = g(p)$ is in O and its image $f(p') = p$ is also in O, and the trajectory $I.p'$ is not homotopic in $\mathbb{R}^2 \setminus \{0\}$ to a curve included in O (otherwise $IJ.p$ would be contractible in $O \subset \mathbb{R}^2 \setminus \{0\}$). Thus hypothesis (ii) does not hold. □

Proof of Lemma 93 As in the proof of the previous lemma, we may assume the parabolic sector S is positive in the sense that $\{0\}$ is the ω–limit set of any leaf of $\mathcal{F}$ which is contained in S. By dynamical transversality and the fact that the sector is swept out by the isotopy, up to decreasing V and W we may assume that

(1) the trajectory of every point in $V \setminus \{0\}$ is homotopic in $W \setminus \{0\}$ to a curve which is positively transverse to $\mathcal{F}$,

(2) the trajectory of every point in $S \cap V \setminus \{0\}$ has positive intersection number with $\partial^+ S$,

(3) $V \cap S \subset O \cup \{0\}$.

Since $\mathcal{F}$ is gradient-like and the parabolic sector is maximal, it is adjacent to two hyperbolic sectors. We consider a (non-area-preserving) chart in which $V = [-1, 1]^2$, $W = [-2, 2]^2$, the parabolic sector S is $W \cap \{x \geqslant 0,\ y \geqslant 0\}$ and $W \cap \{x \leqslant 0,\ y \geqslant 0\}$ is a hyperbolic sector H; see Figure 12. The open ball with center P and radius ε will be denoted by $B_\varepsilon(P)$.

Denote by f the time 1 of the isotopy I. Note that by transversality, $f((S \cup H) \cap V)$ is included in H, and for any leaf F in the interior of H, $f(F \cap V)$ is on the right-hand side of F in H; see Figure 13. The foliation $\mathcal{F}'$ will be obtained from $\mathcal{F}$ by modifying it on $V \cap \{y > 0\}$. Let $y \in (0, 1)$, let $P = (0, y)$ and define $F'_y = [0, 1] \times \{y\}$.

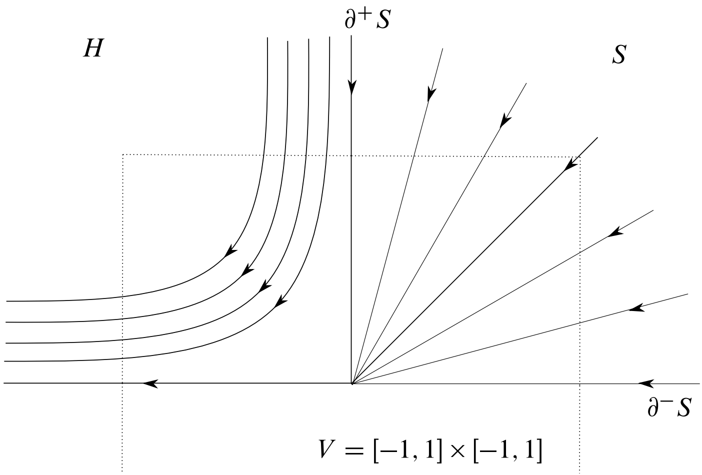

Figure 12: The maximal parabolic sector S and an adjacent hyperbolic sector H.

By dynamical transversality there exists $\varepsilon > 0$ such that the ball $B_\varepsilon(P)$ is included in $V \cap O$ and there exists a leaf F in the interior of H that separates in W the set $F'_y \cup B_{\varepsilon(y)}(P)$ from its image under f; we denote by $\varepsilon(y)$ the maximum of such numbers. Let us call F' a good curve if it is obtained the following way (see Figure 14). We choose a point $P_1 = (-2, y_1)$ with $y_1 > 0$ small enough that the leaf F through P_1 meets the ball $B_{\varepsilon(y)}(P)$. Let γ be a curve in that ball that joins a point P_2 in F to the point P and whose interior is on the left-hand side of F and on the right-hand side of $\partial^+ S$ in W. Finally, define F' to be the union of F'_y, γ and the piece of F from P_2 to P_1, oriented in this order. Note that the part of the good curve outside O

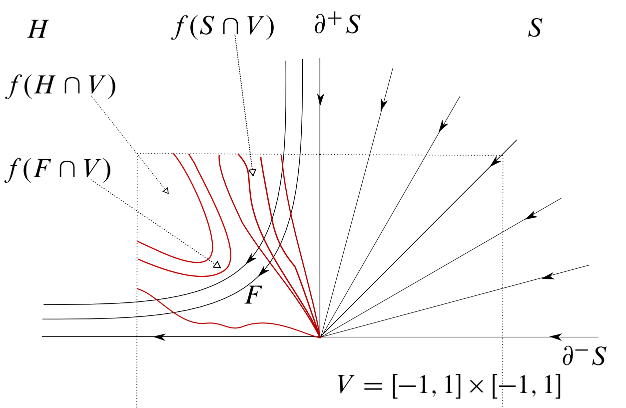

Figure 13: As a consequence of transversality $f((S \cup H) \cap V)$ is included in H and $f(F \cap V)$ is on the right-hand side of F in H.

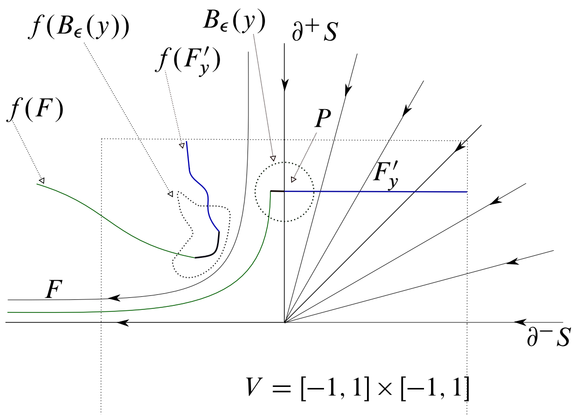

Figure 14: There exists a leaf of the foliation separating $F'_y \cup B_\varepsilon(y)$ from its image. The good curve F' consists of F'_y (in blue), γ (in black) and a part of the leaf F (in green).

coincides with a leaf of $\mathcal{F}$. Furthermore, it is easy to check that the image of $F' \cap V$ is on the right-hand side of F' in W. We leave it to the reader to check that there is a topological foliation $\mathcal{F}'$, defined on some neighborhood W' of 0, whose leaves in the upper half-plane $\{y > 0\}$ are good curves. Any such foliation is dynamically transverse to the isotopy I. Furthermore, it coincides with $\mathcal{F}$ outside $\overline{O}$ and the x–axis is the only half-leaf of $\mathcal{F}'$ included in O. $\square$

7.4 Smoothability of fixed points of index one

7.4.1 Local rotation set and transverse foliations Let (f, U) be an area-preserving local homeomorphism with fixed point p_0. Choose a local isotopy (I, V) for (f, U). Let us outline the basic properties of the local rotation set $\rho(I)$, and in particular the relationship with transverse foliations (we refer to [43] for more details). It is by definition a closed subset of $\mathbb{R} \cup \{\pm\infty\}$. Let (R, V) be a local isotopy from the identity to the identity that makes a full turn around p_0 in the positive direction. The formulas $\rho(I^p) = p\rho(I)$ and $\rho(R^p I) = \rho(I) + p$ hold.

Let $\alpha\colon [0,1] \to U$ be a simple curve with $\alpha(1) = p_0$, and orient α from $\alpha(0)$ to $\alpha(1)$. Let $\gamma\colon [0,1] \to U$ be a curve which is disjoint from $\alpha(0)$ and $\alpha(1)$ and such that $\gamma(t) \notin \alpha$ except for finitely many values of t. We define the intersection number $\sharp\gamma \wedge \alpha$ as follows. We count 1 each time γ crosses α from the left-hand side of α to the right-hand side, -1 when it crosses in the opposite direction, $\pm\frac{1}{2}$ if $\gamma(0) \in \alpha$ according to whether γ starts on the right-hand side or left-hand side of α, and similarly if $\gamma(1) \in \alpha$. We remark that we have $\sharp\gamma' \wedge \alpha = \sharp\gamma \wedge \alpha$ if γ' is another curve which is homotopic to γ in $U \setminus \{\alpha(0), \alpha(1)\}$ with respect to its endpoints. This remark allows us to extend the definition to the case of a curve γ that intersects α infinitely many times.

We say that α is a *positive arc* for the isotopy (I, V) if the intersection number $\sharp I.x \wedge \alpha$ is positive for every $x \in \alpha$ close enough to p_0. We say that α is a *direct positive arc* if $\sharp I.x \wedge \alpha = \frac{1}{2}$ for every $x \in \alpha$ close enough to p_0. Note that in this last case α is locally disjoint from its image, that is, there exists some neighborhood O of p_0 such that $f(\alpha) \cap \alpha \cap O = \{p_0\}$. The following simple claim will be used repeatedly (this is a special and simple case of the more general fact that the local rotation set is included in the *rotation interval*; see again [43]).

Claim 94 *If there is a positive arc for the local isotopy I, then the local rotation set $\rho(I)$ is included in $[0, +\infty]$. If there is a direct positive arc then $\rho(I)$ is included in $[0, 1]$.*

For the next claim, please note that the sector S is not between α and $f(\alpha)$ but between $f(\alpha)$ and α.

Claim 95 *Assume that α is a direct positive arc for I. Let $S = S(f(\alpha), \alpha)$ be a sector between $f(\alpha)$ and α. Then for every point x in the interior of S, close enough to p_0 and such that $f(x)$ also belongs to S, the trajectory $I.x$ is homotopic in $U \setminus p_0$ to a curve included in S.*

Proof If x is close enough to p_0 then its trajectory does not meet the endpoints of α and $f(\alpha)$. Since α is a direct positive arc, the intersection number $n(x) = \sharp I.x \wedge f(\alpha)$ vanishes for x in the interior of S and close to α. Furthermore, the function $x \mapsto n(x)$ is constant on the interior of S; indeed, the interior of S is connected and this function is locally constant, since, when x is in the interior of S, the endpoints of $I.x$ do not belong to $f(\alpha)$. Thus it vanishes identically on the interior of S. The claim follows. □

Assume now that there is a local foliation $(\mathcal{F}, W)$ dynamically transverse to (I, V). A leaf whose ω–limit set is $\{p_0\}$ is a positive arc for I, thus Claim 94 applies if there is such a leaf, and $\rho(I) \subset [0, +\infty]$. Symmetrically, if $\mathcal{F}$ admits a leaf whose α–limit set is $\{p_0\}$ then $\rho(I) \subset [-\infty, 0]$. Since f preserves the area, the foliation $\mathcal{F}$ is gradient-like, and every gradient-like foliation admits a leaf whose α– or ω–limit set is $\{p_0\}$; thus we see that $\rho(I)$ is either included in $[-\infty, 0]$ or in $[0, +\infty]$.

If p_0 is an isolated fixed point of f then, by Theorem 81, for every p the isotopy $R^p I$ admits a dynamically transverse local foliation, thus we see that $\rho(I)$ is included in an interval $[q, q+1]$ for some integer q. Up to replacing I by $R^{-q} I$, we may assume $\rho(I) \subset [0, 1]$. The same kind of arguments, applied to the powers of f, shows that the closed set $\rho(I)$ is actually an interval: every rational number in the interior of the convex hull of $\rho(I)$ is the rotation number of a sequence of periodic orbits accumulating p_0, and thus actually belongs to $\rho(I)$.

7.4.2 Strategy Let us start the proof of Theorem 77, namely that f is smoothable at an isolated fixed point p_0 provided that the local rotation set is not equal to $[0, 1]$ modulo 1. If the index $L(f, p_0)$ is nonpositive then the result follows from the previous section. Thus we may and we will assume that $L(f, p_0) = 1$. As explained at the beginning of Section 7.3, we only need to consider an area-preserving local homeomorphism (f, U) with a single fixed point p_0, and we want to find an area-preserving local homeomorphism (g, U) that fixes only p_0, coincides with f near the boundary of U, and in addition is smooth near p_0. By the above considerations we may choose a local isotopy (I, V) from the identity to f whose rotation set is included in $[0, 1]$. Theorem 81 provides a local foliation which is dynamically transverse to (I, V), and whose index is one. Since it is gradient-like, it is a sink or a source. The source case may happen only when $\rho(I) = \{0\}$, and we can avoid it by changing f into f^{-1}, I into I^{-1}, and $\mathcal{F}$ into the same foliation with reverse orientation.

By hypothesis, the fixed point is not totally degenerate, ie the local rotation set is a proper subinterval of $[0, 1]$. We first note that up to changing f into f^{-1} and changing I into RI^{-1}, we may assume that the local rotation set in included in $[0, 1)$. Let us describe the strategy of the proof. The starting point is a situation in which it is easy to smooth f at p_0; in this situation we assume that the fixed point belongs to the interior of a curve α such that f permutes the two connected components of $\alpha \setminus \{p_0\}$ (Lemma 96 below). This situation is certainly very special; in particular, it implies that the local rotation set equals $\{\frac{1}{2}\}$. In the general case the rotation set is positive, and the local isotopy is dynamically transverse to the sink foliation $\mathcal{F}$. The general idea of the proof is to push f transversally to the foliation $\mathcal{F}$, in the positive direction, in order to reduce the size of the rotation set until it becomes equal to $\{\frac{1}{2}\}$ and we can hope to have this special situation. The transverse foliation guarantees that the pushing creates no new fixed point with rotation number zero (points are already turning in the positive direction around p_0 and we push in the same direction). One difficulty will be to avoid the creation of fixed points with positive rotation number. This control will be made possible by using a transverse foliation with the following extra dynamical property: there is a leaf of $\mathcal{F}$ whose first iterates are also leaves of $\mathcal{F}$; this is the content of the "iterated leaf lemma", Lemma 98 below.

More concretely, in Proposition 97 we consider the case when the local rotation set is not too large, namely it is included in $\left[0, \frac{1}{2}\right)$, and we show how to apply the pushing strategy to get to the special situation that we have described at the beginning (Lemma 99). Finally, we will consider the general case, when the rotation set is included in $[0, 1)$. Roughly speaking we will apply the pushing strategy to reduce the rotation set to a subset of $\left[\frac{2}{3}, 1\right]$. Then, by considering f^{-1} and a well-chosen isotopy, this amounts to a rotation set included in $\left[0, \frac{1}{3}\right]$, and we conclude by applying Proposition 97.

We end this subsubsection by proving Proposition 97 $\left(\text{the } \left[0, \frac{1}{2}\right) \text{ case}\right)$. The general case is treated in the next subsubsection, and the last subsubsection contains the proof of the iterated leaf lemma.

Lemma 96 *Let (f, U) be an area-preserving local homeomorphism with a single fixed point p_0. Assume there is a simple curve γ in U such that p_0 is in the interior of γ, f exchanges both connected components of $\gamma \setminus \{p_0\}$, and f^2 is the identity on γ. Then (f, U) is smoothable.*

Proof Remember that the property of being smoothable is invariant under *topological* conjugacies (see the beginning of Section 7.3). Thus, by a (nonsmooth) conjugation we may assume U is in the plane, γ is a horizontal segment, the fixed point 0 is the midpoint of γ, and f coincides on γ with the Euclidean rotation R of angle π. Let D be a small disk centered at 0, and denote by D^+ and D^- the upper and lower half-disks. By continuity, if D is small enough, then $f(D^+)$ is included in the half-plane $\{y \leqslant 0\}$, and in particular the interior of D^+ is disjoint from its image, and likewise for D^-. Choose a Euclidean disk D' centered at 0 such that $fR(D')$ is included in the interior of D. Applying independently the Oxtoby–Ulam theorem on D^+ and D^-, we may find a homeomorphism Φ of D which preserves the area, is the identity on the circle ∂D and on the diameter γ, and coincides with fR on D'. We extend Φ by the identity outside D. Let $f' = \Phi^{-1} f$. Then f' preserves the area, coincides with f outside D, and with R on D'. Furthermore, since the interiors of D^+ and D^- are disjoint from their images, f' has no fixed point but 0. Of course, f' is smooth near 0. □

Proposition 97 *Let (f, U) be an area-preserving local homeomorphism with a single fixed point p_0. Let (I, V) be a local isotopy for (f, U), and assume that the local rotation set of I is included in $\left[0, \frac{1}{2}\right)$. Then (f, U) is smoothable.*

The proof of the proposition relies on the following fundamental lemma, which will be proved in the next subsubsection.

Lemma 98 (iterated leaf lemma) *Let (f, U) be an area-preserving local homeomorphism with a single fixed point p_0 such that $L(f, p_0) = 1$. Let (I, V) be a local isotopy for (f, U), and assume that the local rotation set of I is included in $[0, 1/q)$ for some positive integer q.*

Then there exists a local foliation $(\mathcal{F}, W)$ with singularity p_0 which is a sink or a source, which is dynamically transverse to the isotopy (I, V), and an open arc α of which p_0 is an endpoint, such that $\alpha, f(\alpha), \dots, f^q(\alpha)$ are included in some leaves of $\mathcal{F}$.

Note that under the hypothesis of this lemma, if furthermore the local rotation set is not $\{0\}$, then $\mathcal{F}$ is a sink, α is a direct positive arc for I, and $\alpha \setminus \{0\}, \dots, f^q(\alpha) \setminus \{0\}$ are pairwise disjoint and in this cyclic order around p_0. On top of the existence of such an arc α, the lemma provides the foliation $\mathcal{F}$ dynamically transverse to (I, V); this adds some information *only in the sector between $f^q(\alpha)$ and α* (in the other sectors, any topologically radial foliation is transverse to the isotopy), but this little piece of information will be crucial in what follows.

Proof of Proposition 97 Let f and I be as in the statement of the proposition. We may assume that $L(f, p) = 1$. Apply Lemma 98 with $q = 2$ to get a foliation $\mathcal{F}$ dynamically transverse to I with a half-leaf α such that $f(\alpha)$ and $f^2(\alpha)$ are also half-leaves of $\mathcal{F}$. As said before, the source case may happen only when the rotation set is $\{0\}$, thus we may always assume that $\mathcal{F}$ is a sink up to changing f into f^{-1} and I into I^{-1} and reversing the orientation of $\mathcal{F}$. The proposition is now an immediate consequence of Lemmas 96 and 99 below. □

Lemma 99 *In the situation of the previous proof, there is a local homeomorphism (f', U) that satisfies the hypotheses of Lemma 96 and which equals f near the boundary of U.*

Proof Since $\mathcal{F}$ is a sink the arc α is positive, and since α and $f(\alpha)$ are leaves of $\mathcal{F}$ and $\rho(I) \subset \left[0, \frac{1}{2}\right)$, α is a direct positive arc. Let β be a simple curve with $\beta(1) = p_0$ and otherwise included in the interior of a small sector between α and $f(\alpha)$. Then it is easy to see that β is also a direct positive arc.

Let S be a small parabolic sector of $\mathcal{F}$ between $f^2(\alpha)$ and α, and O be the interior of a small sector between $f(\beta)$ and β (see Figure 15). Claim 95 applies: for every point x

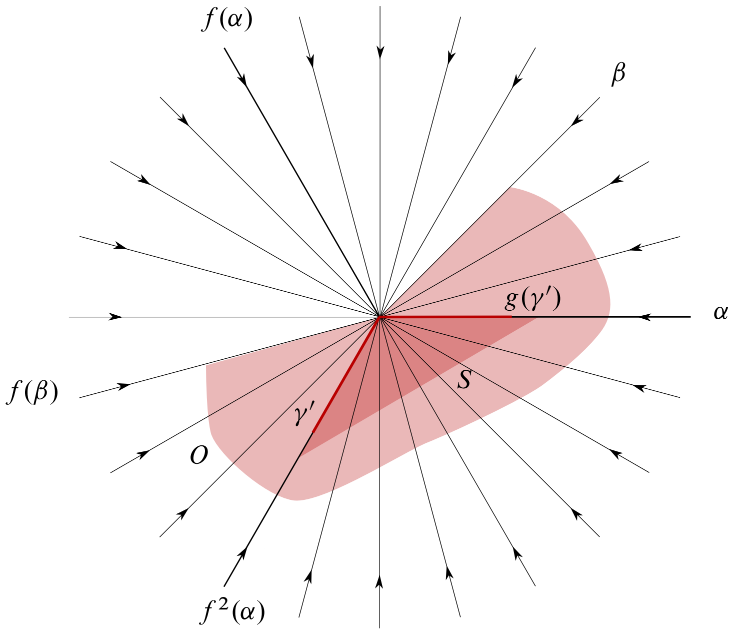

Figure 15: The foliation $\mathcal{F}$ and the parabolic sector S.

close enough to p_0 in O and such that $f(x)$ also belongs to O, the trajectory $I.x$ is homotopic in $U \setminus p_0$ to a curve included in O. In other words, the hypothesis (ii) at the end of Proposition 92 holds. We apply Proposition 92 to the parabolic sector S. We get a local isotopy (J, U) for some area-preserving homeomorphism (g, U) supported in O, dynamically transverse to $\mathcal{F}$, with some small subarc γ' of $f^2(\alpha)$ such that $g(\gamma') \subset \alpha$. Furthermore, p_0 is the only fixed point of fg in U.

We have $(fg)^2(\gamma') \subset fgf(\alpha) \subset f^2(\alpha)$ since g is the identity on $f(\alpha)$. So both γ' and $(fg)^2(\gamma')$ are included in $f^2(\alpha)$, which is disjoint from its image except at p_0. Let Z be an open set containing $\gamma' \cup (fg)^2(\gamma') \setminus \{p_0\}$, and sufficiently small that $fg(Z) \cap Z = \varnothing$. Using the Oxtoby–Ulam theorem, one can construct an area-preserving homeomorphism h, supported on Z and such that $h = (fg)^2$ on γ'. Let $f' = h^{-1} fg$. Since h is the identity on $fg(\gamma')$, we get that f'^2 is the identity on γ', and thus also on $\gamma = \gamma' \cup f'(\gamma')$. Furthermore, f' coincides with f near the boundary of U, as required by the lemma, and the proof is complete. □

7.4.3 Proof of the general case We now proceed to the proof of Theorem 77 in the case when the local rotation set is included in $[0, 1)$. We consider an area-preserving

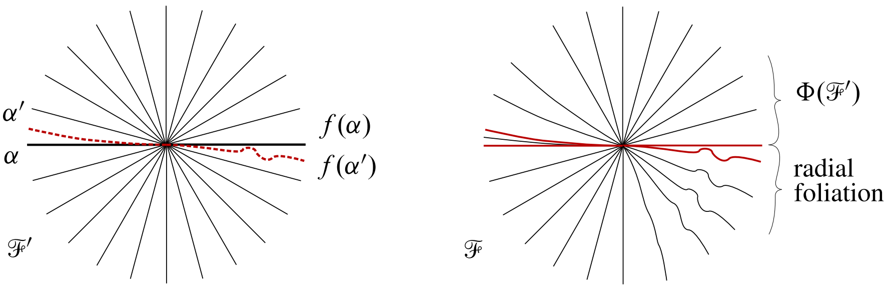

Figure 16: Proof of Step 1.

local homeomorphism (f, U) with a single fixed point p_0 and a local isotopy (I, V) for (f, U), and we assume that $\rho(I)$ is included in $[0, 1)$. By Proposition 97 we may also assume that $\rho(I)$ is not included in $\left[0, \frac{1}{2}\right)$, and in particular it is not $\{0\}$ and we have $L(f, p_0) = 1$. We want to show that (f, U) is smoothable. The strategy will be to make a small modification of f and I to get f'' and an isotopy I'' whose local rotation set is included in $\left[-\frac{1}{3}, 0\right]$. Then the inverse of I'' is a local isotopy whose time-1 map is f''^{-1}, and whose rotation set is included in $\left[0, \frac{1}{3}\right]$. Then, as explained in the strategy, Proposition 97 entails that (f''^{-1}, U) is smoothable, thus $(f'', 0)$ is also smoothable, and so is (f, U). It remains to describe the construction of f'', which will take three steps.

Lemma 100 (Step 1) *There exists a foliation $\mathcal{F}$ transverse to I with two leaves α and α' of $\mathcal{F}$ whose images are also leaves of $\mathcal{F}$, with α', α, $f(\alpha')$ and $f(\alpha)$ pairwise disjoint and in this cyclic order around p_0.*

Proof We apply Lemma 98 with $q = 1$ and get a first foliation $\mathcal{F}'$ with a leaf α whose image is also a leaf of $\mathcal{F}'$. We have assumed that $\rho(I)$ is included in $[0, 1)$ and is not $\{0\}$, thus $\mathcal{F}'$ is a sink, and α is a direct positive arc for I. We will modify $\mathcal{F}'$ into a foliation $\mathcal{F}$ having the desired property; this modification will be a small perturbation in the Whitney topology, and thus $\mathcal{F}$ will still be dynamically transverse to I by the stability property [43, Proposition 3.3]. Let α' be a simple curve with $\alpha'(1) = p_0$ and the remainder of α' included in the open sector between $f(\alpha)$ and α, and which is Whitney-close to α. Then $f(\alpha')$ is disjoint from α and included in the open sector between α and $f(\alpha)$. We choose a Whitney small local homeomorphism Φ that fixes $f(\alpha)$ and sends α to α'. By Whitney stability, the foliation $\Phi(\mathcal{F}')$ is still locally transverse to I, and $f(\alpha)$ and α' are leaves of this foliation. Let $\mathcal{F}$ be

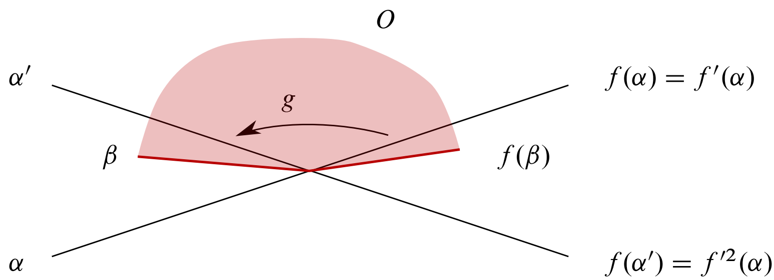

Figure 17: Proof of Step 2.

the foliation that coincides with $\Phi(\mathcal{F}')$ on $S(f(\alpha), \alpha')$, and which is a topologically radial foliation on $S(\alpha', f(\alpha))$ that includes α and $f(\alpha')$ as leaves; to construct $\mathcal{F}$, we can use coordinates in which all four curves are rays and take the foliation to be radial on that sector. Note that, near p_0, f sends the parabolic sector $S(\alpha', \alpha)$ to the disjoint parabolic sector $S(f(\alpha'), f(\alpha))$. We leave it to the reader to check, using this property, that the foliation $\mathcal{F}$ is again locally transverse to I. □

In the next lemmas we will make a slight language abuse by saying that the curves α and α' are disjoint if they meet only at p_0.

Lemma 101 (Step 2) *Consider f, U, α and α' as in Step 1. Then there exists a local homeomorphism (f', U) whose only fixed point is p_0 which coincides with f near the boundary of U and is such that the three curves α, $f'^2(\alpha)$ and $f'(\alpha)$ are pairwise disjoint and in this cyclic order around p_0.*

Proof Consider the foliation $\mathcal{F}$ provided by Step 1. Choose a curve β such that $\beta(1) = p_0$ between α' and α in the cyclic order around p_0 (see Figure 17). Note that $f(\beta)$ is between $f(\alpha')$ and $f(\alpha)$. Let $S = S(f(\alpha), \alpha')$ be a small parabolic sector of $\mathcal{F}$ included in U, and O be the interior of a small sector $S(f(\beta), \beta)$ containing $S \setminus \{p_0\}$. We apply the "parabolic pushing" Proposition 92 to the parabolic sector S to get an isotopy J compactly supported in U that pushes $f(\alpha)$ to α' near 0 and is locally transverse to $\mathcal{F}$. Let g be the time 1 of J. Claim 95 applies to O, and according to the end of the proposition the map $f' = fg$ has no fixed point in U but p_0. Finally, note that since α does not belong to the support of J, $f'(\alpha) = f(\alpha)$ and thus $f'^2(\alpha) = f(\alpha')$ and we get α, $f'^2(\alpha)$ and $f'(\alpha)$ as wanted. □

Lemma 102 (Step 3) *Let f', U and α be as in the conclusion of Step 2. Then there exists a local homeomorphism (f'', U) whose only fixed point is p_0, which coincides*

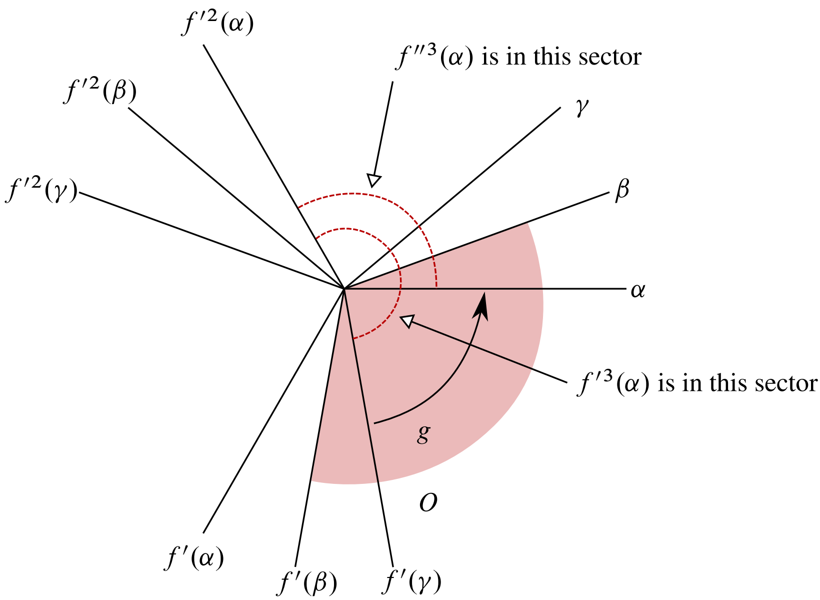

Figure 18: Proof of Step 3.

with f near the boundary of U and is such that the curves

$$\alpha, \quad f''^3(\alpha), \quad f''^2(\alpha), \quad f''(\alpha)$$

are pairwise disjoint and in that cyclic order around p_0. In particular, there exists a local isotopy I'' from the identity to f'' whose local rotation set is included in $\left[-\frac{1}{3}, 0\right]$.

Proof We can find two curves β and γ very close to α such that we get nine pairwise disjoint curves with the following cyclic order around p_0:

$$\alpha, \quad \beta, \quad \gamma, \quad f'^2\alpha, \quad f'^2\beta, \quad f'^2\gamma, \quad f'\alpha, \quad f'\beta, \quad f'\gamma$$

(see Figure 18). Furthermore, since $f'^2(\alpha)$ is included in the sector $S(\gamma, f'(\alpha))$, we get $f'^3\alpha \subset S(f'\gamma, f'^2\alpha)$. We consider coordinates in which all the above nine curves are straight rays. Note that in these coordinates I is dynamically transverse to the radial foliation. We consider some small parabolic sector $S = S(f'(\gamma), \alpha)$, and some small open set $O = S(f'(\beta), \beta)$ containing $S \setminus \{0\}$, such that the hypotheses of Proposition 92 are satisfied. Now we push as before by some local isotopy (J, V) for a local homeomorphism (g, U) supported in O that locally sends $f'(\gamma)$ to α. Near p_0 we have $g(f'^3(\alpha)) \subset g(S(f'\gamma, f'^2\alpha)) = S(\alpha, f'^2\alpha)$. By Proposition 92, $f'g$ has no fixed point but p_0 in U, thus so does $f'' = gf'$ since they are conjugate by g, which is supported on U. Since g is the identity on $f'(\alpha)$ and $f'^2(\alpha)$, we get $f''(\alpha) = f'(\alpha)$ and $f''^2(\alpha) = f'^2(\alpha)$, so the curves

$$\alpha, \quad f''^3(\alpha), \quad f''^2(\alpha) = f'^2(\alpha), \quad f''(\alpha) = f'(\alpha)$$

are pairwise disjoint and in that cyclic order around p_0, as wanted.

Let (I'', U) be a local isotopy from the identity to f'' that pushes α to $f''(\alpha)$ of less than one full turn in the negative direction; then it also pushes similarly $f''(\alpha)$ to $f''^2(\alpha)$ and $f''^2(\alpha)$ to $f''^3(\alpha)$. Since α is a positive arc for I^{-1}, the local rotation set $\rho(I'')$ is included in $[-\infty, 0]$ by Claim 94. On the other hand, α is also a positive arc for RI^3, and thus we also get $\rho(I'') \subset \left[-\frac{1}{3}, +\infty\right]$, and finally the local rotation set is included in $\left[-\frac{1}{3}, 0\right]$, as wanted. □

7.4.4 Proof of the "iterated leaf lemma" In order to prove the iterated leaf lemma, our strategy is first to extend our local dynamics to the whole plane. The "extension lemma" below shows that this extension can be done in such a way that the global rotation set around the fixed point is not too large, barely larger than the local rotation set of the initial local homeomorphism. In this global setting, we work in the infinite annulus $A = \mathbb{R}^2 \setminus \{0\}$. The dynamics lifts to a map $\widetilde{f}$ of the universal cover $\widetilde{A} \simeq \mathbb{R}^2$. We denote by T the deck transformation. The second step in the proof of the iterated leaf lemma consists in playing with maps of the form $T^p \widetilde{f}^{-q}$: we prove that if such a map T' has positive (global) rotation set then it is conjugate to a plane translation; this is the content of the "quotient lemma". Then $\widetilde{f}$ induces a dynamics in the annulus $\widetilde{A}/T'$; the key point in the construction of the foliation with an iterated leaf will consist in applying Le Calvez's transverse foliation in this quotient. Note that in this section all foliations are just topological (nonsmooth) foliations.

We begin by recalling some definitions in this global setting. We work in the plane $\mathbb{R}^2$, which we compactify by adding the point ∞. The set $\mathbb{R}^2 \setminus \{0\}$ is identified with the open annulus $A = \mathbb{S}^1 \times \mathbb{R}$, whose ends N and S correspond respectively to 0 and ∞ in this identification. We let $\widetilde{A}$ be the universal cover of A, with deck transformation T. In the course of the proof we will encounter another transformation T' of the plane $\widetilde{A}$, which will be conjugate to T. Then the quotient $A' = \widetilde{A}/T'$ is again an open annulus. We choose an invariant topological line for T' and orient it in the sense in which T' pushes points. In the quotient A', the line becomes an oriented circle. We let N' and S' be the two ends of the annulus A', so that N' is on the left-hand side of this circle and S' is on the right-hand side, and call them the North and South ends.

Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ that fixes 0, and I an isotopy that fixes 0, from the identity to f. This isotopy induces a lift $\tilde{f}:\tilde{A} \to \tilde{A}$. It is a classical theorem of Brouwer (of which Le Calvez's transverse foliation theorem is a generalization) that if $\tilde{f}$ has no fixed point, then it has no periodic point. Applying this to $T^{-p}\tilde{f}^q$, we get the following lemma, which will be useful below.

Lemma 103 *Assume f has some periodic point whose rotation number for I equals p/q in irreducible form. Then we can find such a point with period q.*

The (global) rotation set of I in A may be defined in this context. It is a closed subset of $\mathbb{R} \cup \{\pm\infty\}$ which, roughly speaking, contains the average speeds of rotation of long pieces of orbits that start and end not too close to one end of the annulus A. We refer to [43] or [9] for the precise definition. The definitions make it clear that this rotation set contains the local rotation set of I around the fixed point 0.

In this global setting we will apply Le Calvez's theorem, already mentioned in the local setting. Assume for simplicity that $\tilde{f}$ has no fixed point; this will always be the case in what follows. Then Le Calvez's theorem provides a topological oriented foliation $\mathcal{F}$ of the annulus A which is dynamically transverse to I: every trajectory of I is homotopic to a curve which crosses each leaf of $\mathcal{F}$ from left to right (this is made precise by using the charts of the foliation). As in the local setting, the property of being transverse is stable: any topological oriented foliation $\mathcal{F}'$ of A which is Whitney close to $\mathcal{F}$ is again dynamically transverse to the isotopy I (the proof is almost identical as the proof of the local stability; see [43, Section 4.4b, after Lemma 4.4.2]).

In what follows, we call *finite or infinite area* any Borel measure μ on $\mathbb{R}^2$ which is positive on open sets and has no atom. By the Oxtoby–Ulam theorem, such a measure is homeomorphic either to the standard area on the plane (in the infinite case) or to the standard area on the euclidean disk of area $\mu(\mathbb{R}^2)$.

Now assume in addition that f preserves a finite or infinite area on $\mathbb{R}^2$. Then the transverse foliation $\mathcal{F}$ is gradient-like near N (see Section 7.2): N is a sink, a source or a saddle singularity for $\mathcal{F}$. Moreover, $\mathcal{F}$ has no closed leaf in A. However, in the case when the area is infinite, there may be some leaf going from S to S ("petals"), and some leaf ϕ whose α–limit set is $\{N\}$ and whose ω–limit set contains a petal leaf (the leaf spirals around and accumulates on a family of petals). This phenomenon is unstable; indeed, in this situation, it is easy to prove that there exists an arbitrarily small perturbation of $\mathcal{F}$ that produces a foliation $\mathcal{F}'$ for which some leaf has α–limit set $\{N\}$ and ω–limit set $\{S\}$. Then, by Poincaré–Bendixson theory, every leaf of $\mathcal{F}'$

whose α–limit set is $\{N\}$ must have $\{S\}$ as its ω–limit set. By the above-mentioned stability property, $\mathcal{F}'$ is still dynamically transverse to the isotopy I. To summarize, up to perturbing $\mathcal{F}$, we may always assume that *every leaf that comes from N goes to S*, and likewise that *every leaf that goes to N comes from S*.

In particular, since N is a sink, a source or a saddle, there is at least one leaf going from N to S or from S to N. By transversality, the first case implies that the rotation set of I in A in included in $[0, +\infty]$, and the second case that it is included in $[-\infty, 0]$. A straightforward generalization of this argument shows that for every rational number p/q in irreducible form, if there exists no periodic orbit in A with rotation number p/q, then the rotation set is included in $[-\infty, p/q]$ or in $[p/q, +\infty]$. In particular, the density of rational numbers implies that the rotation set is an interval.

The following lemma allows us to extend a local homeomorphism into a homeomorphism of the plane, with a global rotation set equal to or slightly larger than the local rotation set of the local homeomorphism. For related (unpublished) results, see [52]; the techniques are already used in [43, Appendix A].

Lemma 104 (extension lemma) *Let (f, U) be an area-preserving local homeomorphism fixing 0, where U is a neighborhood of 0 in the plane. Let (I, V) be a local isotopy associated to f. Let $[\alpha, \beta]$ be the local rotation set of I.*

Then there exists a homeomorphism $\bar{f}: \mathbb{R}^2 \to \mathbb{R}^2$ which preserves a finite or infinite area, and an isotopy $\bar{I}$ on $\mathbb{R}^2$ for $\bar{f}$, such that $\bar{f}$ and $\bar{I}$ coincide respectively with f and I on a neighborhood of 0, and the rotation set of $\bar{I}$ in the annulus $A = \mathbb{R}^2 \setminus \{0\}$ is included in $[\alpha', \beta']$, where:

(i) *If $\alpha = p/q$ is a rational number (in irreducible form) different from β, and 0 is not accumulated by periodic orbits of period q and rotation number α, then $\alpha' = \alpha$; furthermore, in this case, $\bar{f}$ has no periodic point of rotation number α.*

(ii) *If $\alpha = -\infty$ then $\alpha' = \alpha$; in the other cases, α' can be chosen to be any real number less than α.*

And β' is defined by a symmetric set of conditions.

Note that since $\bar{I}$ extends I, the rotation set of $\bar{I}$ will automatically contain the local rotation set of I. The following remark is added for completeness, though we will not need it here. In the setting of the lemma, in the case when the local rotation set is a single rational number α and 0 is not accumulated by periodic orbits of rotation number α, a variation on the proof below produces an extension with rotation set included in an arbitrarily small interval $[\alpha-\varepsilon,\alpha+\varepsilon]$, one of whose endpoints is α.

Proof By a standard extension argument, we may assume that f is an area-preserving homeomorphism of the plane (with no assumption on the global rotation set).

In the case when $\alpha' = \alpha$ (case (i) of the statement and when $\alpha = -\infty$) we set $\alpha'' = \alpha' = \alpha$; otherwise choose any rational number α'' such that $\alpha' < \alpha'' < \alpha$. Define β'' symmetrically. If $\alpha'' = -\infty$ then we set $F_{\alpha''} = \varnothing$. In the opposite case we write $\alpha'' = p/q$ in irreducible form, and define $F_{\alpha''}$ to be the set of periodic points of f with period q and whose rotation number for I equals α'' in the annulus $A = \mathbb{R}^2 \setminus \{0\}$. Clearly $F_{\alpha''}$ is a (maybe empty) closed subset of the annulus A. By the assumption on periodic orbits in case (i), $F_{\alpha''}$ is also a closed subset of the plane, ie it does not accumulate 0. We define $F_{\beta''}$ in a similar fashion, and set $F = F_{\alpha''} \cup F_{\beta''}$.

Let M be the connected component of $\mathbb{R}^2 \setminus (F \cup \{0\})$ whose closure contains 0, which is an open set invariant under f. Let $\widetilde{M}$ be the universal cover of M, which is homeomorphic to the plane, and choose some covering map T' corresponding to a simple loop in M in the positive direction around 0. The annulus $A' = \widetilde{M}/T'$ is an intermediate cover of M, denote by $p\colon A' \to M$ the covering map. We remark that p restricts to a homeomorphism between some neighborhood $V_{N'}$ of the end N' of A' and some neighborhood V_0 of 0 in the plane. The isotopy I and the homeomorphism f lifts under p to an isotopy $\widetilde{I}$ and a homeomorphism $\widetilde{f}$ of the plane $A' \cup \{N'\}$. First note that, in the case when $\alpha'' = p/q$, by construction $\widetilde{f}$ has no periodic point of period q whose rotation number for $\widetilde{I}$ equals α''. Thus, by Lemma 103, $\widetilde{f}$ has no periodic point of rotation number α'', and the same holds for β''. Since p is a homeomorphism near N', we may find a homeomorphism Φ between $A' \cup \{N'\}$ and $\mathbb{R}^2$ which coincides with p near N'. The map $\bar{f}' = \Phi\widetilde{f}\Phi^{-1}$ is a homeomorphism of $\mathbb{R}^2$ that coincides with f near 0. Furthermore, there is a unique measure μ' such that $\mu'(D) = \mathrm{area}(p(D))$ for every topological disk D that projects one-to-one to the plane. This measure is positive on open sets and has no atoms. Then $\bar{f}'$ preserves the measure $\Phi^*\mu'$, which shares the same properties. Let $\bar{I}$ be an isotopy associated to $\bar{f}$ that extends (some restriction of) I. By conjugacy with $\widetilde{f}$, $\bar{f}$ has no periodic point of rotation number α'' or β''. Since the rotation set of $\bar{I}$ relative to the annulus $\mathbb{R}^2 \setminus \{0\}$ is an interval, and because every rational number in the interior of this interval is the rotation number of some periodic orbit, we deduce that this interval does not contain α'' or β'' in its interior. Furthermore, it contains the local rotation set

$[\alpha, \beta]$ of I at 0. Thus the rotation set of $\overline{I}$ is included in $[\alpha'', \beta'']$. In particular, it is included in $[\alpha', \beta']$, as required. □

Let $A = \widetilde{A}/T$ be an annulus as before, and denote the quotient map by $\pi\colon \widetilde{A} \to A$. Let μ be a measure on $\widetilde{A}$ which is invariant under T. This measure induces a unique measure $\pi_\star\mu$ on A with the property that for every topological disk D in $\widetilde{A}$ which projects injectively into A, $\pi_\star\mu(\pi(D)) = \mu(D)$. Conversely, given a measure μ_A on A, there is a unique measure $\pi^\star\mu_A$ on $\widetilde{A}$ which is invariant under T and such that μ_A is the measure induced by $\pi^\star\mu_A$; again we will say that $\pi^\star\mu_A$ is induced by μ_A. Given a measure μ_A on A, we say that *the end N has finite measure* if some neighborhood of N in A has finite measure.

A compact subset of the plane is said to be *full* if its complement is connected. Compact connected full subsets of the plane are classically characterized as those subsets which can be written as a decreasing intersection of closed topological disks (each disk is obtained by first approximating the full set with a connected surface with boundary, ie a disk with holes, and then filling in the holes).

Lemma 105 (quotient lemma) (1) *Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be homeomorphism of the plane that preserves a finite or infinite area μ. Assume g fixes 0, and let J be an associated isotopy fixing 0. Assume that the rotation set of J in the annulus $A = \mathbb{R}^2 \setminus \{0\}$ is included in $(0, +\infty]$. Then the associated lift $T' = \widetilde{g}\colon \widetilde{A} \to \widetilde{A}$ is conjugate to a translation.*

(2) *The measure μ induces a measure $\widetilde{\mu}$ on $\widetilde{A}$, and this measure in turn induces a measure μ' on $A' = \widetilde{A}/T'$. Assume furthermore that the local rotation set of J around 0 is bounded. Then the north end N' of A' has finite measure.*

(3) *Let $\pi\colon \widetilde{A} \to A$ and $\pi'\colon \widetilde{A}' = \widetilde{A} \to A'$ denote the quotient maps. Let $(K_k)_{k \geqslant 0}$ be a decreasing sequence of compact connected full subsets of the plane $A \cup \{0\}$ whose intersection is $\{0\}$, each of which is invariant under g. Define $K'_k = \pi'(\pi^{-1}(K_k)) \cup \{N'\}$ for each k. Then $(K'_k)_{k \geqslant 0}$ is a decreasing sequence of compact connected full subsets of the plane $A' \cup \{N'\}$ whose intersection is N'.*

Proof To prove the first point, we try to follow the argument from Section 3 of [2]. The hypothesis there was that g preserves a finite measure, so we have to adapt the proof in the case when the area is infinite. Note that here we may have some wandering points, so that even [22] does not apply directly. (Also note that the definition of the rotation numbers in the annulus used in [2] involves only recurrent points and thus is not adapted to the present setting.)

Let R be the isotopy that makes one turn around 0 in the positive direction, whose associated lift is the deck transformation T. Let g and J be as in the statement. Consider $q > 0$ such that the rotation set of J is included in $(1/q, +\infty]$. Then the rotation set of the isotopy $J^q R^{-1}$ is included in $(0, +\infty]$. Apply Le Calvez's theorem to get a singular foliation of the plane, with singularity 0, transverse to the isotopy $J^q R^{-1}$. Since g is area-preserving, so is the time-1 map $h = g^q$ of this isotopy. As explained at the beginning of Section 7.4.4, up to perturbing $\mathcal{F}$ we may assume that there is a leaf ϕ from ∞ to 0.

Now the argument of [2] applies, let us recall it. Given two oriented proper lines β and β', we will denote $\beta < \beta'$ if the two lines are disjoint, β is on the left-hand side of β' and β' is on the right-hand side of β. Note that this is a transitive relation which is preserved when we apply an orientation-preserving homeomorphism. Let $\widetilde{h} = \widetilde{g}^q T^{-1}$, which is the lift of h associated to the isotopy $J^q R^{-1}$. Let $\widetilde{\phi}$ be a lift of ϕ; we have

$$\widetilde{h}^{-1}(\widetilde{\phi}) < \widetilde{\phi} < \widetilde{h}(\widetilde{\phi})$$

($\widetilde{\phi}$ is said to be an *oriented Brouwer line* for $\widetilde{h}$). From this we first get

$$\widetilde{g}^{-q}(\widetilde{\phi}) < T^{-1}(\widetilde{\phi}) \quad \text{and} \quad T(\widetilde{\phi}) < \widetilde{g}^{q}(\widetilde{\phi}).$$

Let B' denote the closed strip bounded by $\widetilde{\phi}$ and $\widetilde{g}^q(\widetilde{\phi})$. We claim that the union of the iterates of B' under $\widetilde{g}^q$ cover the plane: indeed, from the previous inequalities we get, for every $n > 0$,

$$\widetilde{g}^{-nq}(\widetilde{\phi}) < T^{-n}(\widetilde{\phi}) \quad \text{and} \quad T^{n}(\widetilde{\phi}) < \widetilde{g}^{nq}(\widetilde{\phi})$$

and moreover the strip between $\widetilde{\phi}$ and $T(\widetilde{\phi})$ is a fundamental domain for T. We deduce that $\widetilde{g}^q$ is conjugate to a translation. By a classical lemma, $\widetilde{g}$ is also conjugate to a translation (indeed the quotient $\widetilde{A}/g$ is a quotient of the annulus $\widetilde{A}/g^q$ by a covering map of degree q, thus it is also an annulus).

Note that some orbits of g may accumulate on both ends of A, which is why it is not so easy to compare properties in A and in A' in what follows.

Let us prove the second point. We assume that the local rotation set of J around N is bounded. The annulus $\widetilde{A}/\widetilde{g}^q$ is a q–fold cover of the annulus A', thus the end N' of A' has finite measure if and only if the corresponding end of $\widetilde{A}/\widetilde{g}^q$ has finite measure for the measure induced by $\widetilde{\mu}$. Thus, up to changing g into g^q, we may assume that the rotation set of J in A is included in $(1, +\infty]$. Let ϕ be a topological line in A as in the first point, whose lift $\widetilde{\phi}$ satisfies

$$\widetilde{\phi} < T(\widetilde{\phi}) < \widetilde{g}(\widetilde{\phi}).$$

The band B' between $\widetilde{\phi}$ and $\widetilde{g}(\widetilde{\phi})$ is a fundamental domain for A'. Choose some curve $\widetilde{\alpha}$ joining a point on $\widetilde{\phi}$ to its image under $\widetilde{g}$, and whose interior is included in the interior of the band B'. Then $\alpha' = \pi'(\widetilde{\alpha})$ is a Jordan curve surrounding N'; we aim to prove that the disk bounded by α' in A' has finite μ'–measure. Denote by $\widetilde{\phi}_+$ the half-leaf of ϕ after the point $\widetilde{\alpha}(0)$ on $\widetilde{\phi}$. Let B'_+ be the connected component of $B' \setminus \widetilde{\alpha}$ bounded by $\widetilde{\phi}_+$, $\widetilde{\alpha}$ and $\widetilde{g}(\widetilde{\phi}_+)$, which projects into this disk. By perturbing ϕ if necessary, we may assume ϕ has zero measure, and then the measure of this disk is also the $\widetilde{\mu}$–measure of B'_+.

By modifying g far from 0, we may find a homeomorphism g_c of the plane $A \cup \{N\}$ which preserves μ, is compactly supported and has a lift $\widetilde{g}_c$ which is the identity on a half-leaf of $\widetilde{\phi}$ and coincides with $\widetilde{g}$ on $\widetilde{\phi}^+$. We denote by J_c the corresponding isotopy on A from the identity to g_c. To simplify the picture, up to modifying again g_c on a compact band of A, we may assume that $\widetilde{g}_c(\widetilde{\phi})$ is included in the closed half-plane on the right-hand side of $\widetilde{\phi}$, and in particular the rotation set of J_c in A is still nonnegative. Note that g_c is the identity except on a set of finite measure, thus we may apply the Poincaré recurrence theorem: μ–almost every point is recurrent under g_c. Let Z be the set of points which are on the right-hand side of $\widetilde{\phi}$ and on the left-hand side of $\widetilde{g}_c(\widetilde{\phi})$ (note that we do not care about measure zero sets, thus from now it is enough to define sets up to measure zero). Note that Z equals B'_+ up to a finite $\widetilde{\mu}$–measure set, and thus it is sufficient to prove that Z has finite measure. Moreover the local rotation set of J_c around N equals that of J, and thus is bounded by hypothesis. Since g_c has compact support in $A \cup \{N\}$, we deduce that the (global) rotation set of J_c in the annulus A is also bounded; say it is included in $[0, M]$. The idea of what follows is that the measure of Z is the mean rotation number of J_c in A, which is finite since the rotation set of J_c is bounded.

Here are the details. Let Q denote the band bounded by $\widetilde{\phi}$ and $T(\widetilde{\phi})$, $Q' = \widetilde{g}_c(Q)$ and $P_{i,j} = T^i(Q) \cap T^j(Q')$. The half-plane on the right-hand side of $\widetilde{\phi}$ is $\bigcup_{i \geqslant 0} T^i(Q)$, the half-plane on the left-hand side of $\widetilde{g}_c(\widetilde{\phi})$ is $\bigcup_{j<0} T^j(Q')$, and thus

$$Z = \bigcup_{i \geqslant 0, j<0} P_{i,j}.$$

For each i and j we have $T(P_{i,j}) = P_{i+1,j+1}$, and since $\widetilde{\mu}$ is invariant under T we get

$$\widetilde{\mu}(Z) = \sum_{i \geqslant 0} i\,\widetilde{\mu}(P_{i,0}).$$

Now we establish the relation with the mean rotation number. For almost every point x in A, we may define the algebraic intersection number $J_c.x \wedge \phi$ of the trajectory of x under J_c with the leaf ϕ. Furthermore, since ϕ is a leaf of a foliation transverse to J, the function $x \mapsto J.x \wedge \phi$ is nonnegative, and thus the function $x \mapsto J_c.x \wedge \phi$ is bounded below. In addition, it vanishes except on a subset of A which has finite area. In particular, its integral is well defined (maybe infinite). Let $x \in A$, let $\tilde{x}$ be a lift of x in the fundamental domain Q of A, and let $i = J_c.x \wedge \phi$. Then $\tilde{g}_c(\tilde{x})$ belongs to $T^i(Q) \cap Q' = P_{i,0}$. Thus, for every integer i, the points $\tilde{x}$ in Q for which $J_c.x \wedge \phi = i$ are exactly the elements of $\tilde{g}_c^{-1}(P_{i,0})$. Since $\tilde{\mu}$ is preserved by $\tilde{g}_c$, we get

$$\int_A (J_c.x \wedge \phi)\, d\mu = \sum_{i \geqslant 0} i\,\tilde{\mu}(\tilde{g}_c^{-1}(P_{i,0})) = \sum_{i \geqslant 0} i\,\tilde{\mu}(P_{i,0}) = \tilde{\mu}(Z).$$

It remains to show that this integral is finite. Since $x \mapsto J_c.x \wedge \phi$ is bounded below, we may apply Birkhoff's ergodic theorem for positive functions: the function

$$\rho(x) = \lim_{n \to +\infty} \frac{1}{n} J_c^n.x \wedge \phi$$

is defined μ–almost everywhere on A (allowing the value $+\infty$), and its integral equals that of $x \mapsto J_c.x \wedge \phi$. Let x be a recurrent point for which this limit exists; then it is easy to see that the number $\rho(x)$ belongs to the rotation set of J_c. Thus $\rho(x) \leqslant M$ almost everywhere, and since ρ vanishes outside a finite-measure subset, we get $\int_A \rho(x)\, d\mu < +\infty$. This completes the proof of the second point.

Let us prove the last point of the lemma. Let K be a compact connected full subset of $A \cup \{N\}$ which strictly contains N and is invariant under g. Write $U_K = A \setminus K$, which is an essential invariant subannulus of A. Let $\tilde{K} = \pi^{-1}(K)$ and $\tilde{U}_K = \pi^{-1}(U_K)$; then $\tilde{U}_K$ is a topological plane which is invariant by both T and $T' = \tilde{g}$. Since T' is conjugate to a translation, so is its restriction to $\tilde{U}_K$, and thus $\pi'(\tilde{U}_K)$ is an essential open subannulus of A' whose complement in $A' \cup \{N'\}$ is $K' := \pi'(\tilde{K}) \cup \{N'\}$, and we want to show that this is a compact connected full subset of the plane $A' \cup \{N'\}$. We can write

$$A' \cup \{S'\} = K'_{S'} \sqcup \pi'(\tilde{U}_K) \sqcup K'_{N'},$$

where $K'_{S'}$ and $K'_{N'}$ are the connected components of the complement of $\pi'(\tilde{U}_K)$ in the sphere $A' \cup \{S', N'\}$ containing S' and N', respectively. By considering a nested family of topological disks bounded by simple closed curves included in $\pi'(\tilde{U}_K)$, we see that $K'_{N'}$ is a compact connected full subset in the plane $A' \cup \{N'\}$, and likewise for $K'_{S'}$ in the plane $A' \cup \{S'\}$.

We will prove that $K'_{S'} = \{S'\}$. For this we consider an oriented topological line $\tilde{\phi}$ as in the proof of the first point, with $\tilde{\phi} < T(\tilde{\phi})$ and $T(\tilde{\phi}) < T'^q(\tilde{\phi})$. We denote by B the closed band between $\tilde{\phi}$ and $T(\tilde{\phi})$. Likewise let B' be the closed band between $\tilde{\phi}$ and $T'^q(\tilde{\phi})$, which we compactify by adding two ends $\tilde{N}$ and $\tilde{S}$, with $\tilde{N}$ on the left-hand side of any simple curve that goes from $\tilde{\phi}$ to $T'^q(\tilde{\phi})$. Note that B' contains B, and that $B \cup \{\tilde{S}, \tilde{N}\}$ is a compactification of the band B. The natural maps $B \cup \{\tilde{S}, \tilde{N}\} \to A \cup \{S, N\}$ and $B' \cup \{\tilde{S}, \tilde{N}\} \to A' \cup \{S', N'\}$ induced by the projections π and π' are continuous: indeed, B is a fundamental domain for A and B' is a fundamental domain for the annulus $\tilde{A}/T'^q$, and π' is the composition of the projection from $\tilde{A}$ to $\tilde{A}/T'^q$ and from $\tilde{A}/T'^q$ to $A' = \tilde{A}/T'$. We claim that the map from $B' \cup \{\tilde{S}, \tilde{N}\}$ to $A \cup \{S, N\}$ induced by π, which sends $\tilde{S}'$ to S and $\tilde{N}'$ to N, is also continuous. To see this, let us identify A with $\mathbb{S}^1 \times \mathbb{R}$; since ϕ is a topological line from S to N in A, we may assume that in these coordinates ϕ is parametrized by $\phi(t) = (0, t)$. Since g is continuous and fixes N and S, writing $g^q(\phi(t)) = (x(t), y(t))$ we get that the function y is proper, namely $y(t)$ tends to $\pm\infty$ when t tends to $\pm\infty$. In these coordinates the projection π becomes $\pi(\tilde{x}, \tilde{y}) = (\tilde{x} \bmod 1, \tilde{y})$, $\tilde{\phi}(t) = (0, t)$ and $\tilde{g}^k(\tilde{\phi}(t)) = (\tilde{x}(t), y(t))$. Let $M > 0$; since the function y is proper, we get that the intersection of the band B' with the horizontal band $\mathbb{R} \times [-M, M]$ is compact. Let $(x_k \bmod 1, y_k)$ be a sequence in B' that tends to $\tilde{S}$ (respectively $\tilde{N}$); then it has finitely many terms in each given compact subset, and (y_k) must tend to $-\infty$ (respectively $+\infty$). Thus the projection of the sequence in A tends to S (respectively N), which proves the claim.

Now assume by contradiction that $K'_{S'} \neq \{S'\}$. Then, since $K'_{S'}$ is connected, there exists a sequence (x'_k) in $K'_{S'} \neq \{S'\}$ that converges to S'. Let $(\tilde{x}_k)$ be a preimage under π' of this sequence in B', and note that it is included in $\tilde{K}$. Since the natural map $B' \cup \{\tilde{S}, \tilde{N}\} \to A' \cup \{S', N'\}$ is continuous, the sequence $(\tilde{x}_k)$ converges to $\tilde{S}$. Let $x_k = \pi(\tilde{x}_k)$; since the map $B' \cup \{\tilde{S}, \tilde{N}\} \to A \cup \{S, N\}$ induced by π is continuous, we get that the sequence (x_k) converges to S. But this last sequence is included in K, and this contradicts the fact that K is a compact subset of $A \cup \{N\}$.

Finally, if (K_k) and (K'_k) are as in the hypothesis of the lemma, then clearly (K'_k) is a decreasing sequence of compact connected full subsets of $A' \cup \{N'\}$, and it is easy to see that its intersection is $\{N'\}$. The proof of the lemma is complete. □

Proof of the "iterated leaf lemma" We assume the hypotheses of the lemma. Let $[\alpha, \beta]$ denote the local rotation set of (I, V); by hypothesis we have $[\alpha, \beta] \subset [0, 1/q)$, and p_0 is not accumulated by fixed points of rotation number 0. Thus we may apply case (i) of the "extension lemma", Lemma 104. In other words, we may assume that f is a homeomorphism of the plane that preserves a finite or infinite area, whose (global) rotation set is included in $[0, 1/q)$, and which has no fixed point of rotation number 0 (and thus no fixed point at all).

Let $\mathcal{F}$ be a transverse oriented foliation on $\mathbb{R}^2 \setminus \{0\}$ for I. Recall the following properties from Theorem 81 and Section 7.2.2. Since f preserves the area, $\mathcal{F}$ has no circle leaf, and no petal at 0. Since $L(f, 0) = 1$, the index of the singularity 0 for $\mathcal{F}$ must also be equal to 1. Thus 0 is a sink or a source for $\mathcal{F}$. If the local rotation set contains some positive number then 0 has to be a sink. If the local rotation set is $\{0\}$ and 0 is a source for $\mathcal{F}$, then, up to changing f into its inverse, we may assume again that 0 is a sink for $\mathcal{F}$, without changing the hypotheses of the lemma; note that if f^{-1} satisfies the conclusion of the lemma then so does f. Up to modifying $\mathcal{F}$ as in the proof of the quotient lemma, we may assume that there is a proper leaf that crosses the annulus, that is, its ω–limit set is 0 and its α–limit set is the other end S of the annulus. Note that as a consequence of the Poincaré–Bendixson theorem, this is the case for every leaf whose ω–limit set is 0.

Consider the isotopy $J = RI^{-q}$, and let A, $\widetilde{A}$ and T be as in the "quotient lemma", Lemma 105. Let $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ be the lift of f associated to I. The lift associated to J is $T' = T\widetilde{f}^{-q}$. Its rotation set is included in $(0, 1]$. According to the quotient lemma, T' is conjugate to a translation. We consider the quotient annulus $A' = \mathbb{R}^2/T'$, and the measure μ' induced by the finite or infinite area of the plane.

The map $\widetilde{f}$ commutes with T', thus it induces a homeomorphism f' of the annulus A' which fixes the end N' of A'. Let I' be an isotopy from the identity to f' on A', whose associated lift is $\widetilde{f}$. Note that the trajectories of any point $\widetilde{x}$ under $\widetilde{I}$ and $\widetilde{I}'$ are homotopic relative to their endpoints $\widetilde{x}$ and $\widetilde{f}(\widetilde{x})$, and thus an oriented foliation is transverse to one if and only if it is transverse to the other one.

The homeomorphism f' preserves the measure μ', and according to the quotient lemma the end N' of A' has finite measure. Let $\mathcal{F}'$ be a transverse oriented foliation for I' on A'. As a consequence, $\mathcal{F}'$ has no closed leaf and no petal at N', thus N' is either a source, a sink or a saddle singularity for $\mathcal{F}'$. Up to perturbing $\mathcal{F}'$, we can assume that it has a leaf that crosses A' either from N' to S' or from S' to N', and then this is the case for every leaf whose ω– or α–limit set is $\{N'\}$.

Claim 106 *N' is not a source for $\mathcal{F}'$.*

Let us first admit the claim and complete the proof of the lemma. According to the claim, N' is either a sink or a saddle singularity of $\mathcal{F}'$; in both cases there is a leaf α whose ω–limit set is $\{N'\}$. Let $\tilde{\alpha}$ be a lift of α in $\mathbb{R}^2$. The strip B' between $\tilde{\alpha}' := T'^{-1}(\tilde{\alpha}) = (T\tilde{f}^{-q})^{-1}(\tilde{\alpha})$ and $\tilde{\alpha}$ is a fundamental domain for A', thus the foliation $\mathcal{F}'$ lifts to a foliation $\tilde{\mathcal{F}}'$ of this strip. Furthermore, we have

$$\tilde{\alpha}' < \tilde{\alpha} = T'(\tilde{\alpha}') < T(\tilde{\alpha}') = \tilde{f}^q(\tilde{\alpha})$$

and from this we deduce that the strip B bounded by $\tilde{\alpha}'$ and $T(\tilde{\alpha}')$ is a fundamental domain for T (the argument is entirely analogous to the one used in the proof of the first point of the quotient lemma). In particular, the projection of $\tilde{\alpha}'$ in A is a proper line that goes from ∞ to 0.

Foliate the strip bounded by $\tilde{\alpha}$ and $\tilde{f}(\tilde{\alpha})$ with any foliation $\mathcal{F}_0$ homeomorphic to the foliation by parallel lines; then, for every $p \in \{1, \dots, q-1\}$, foliate the strip bounded by $\tilde{f}^p(\tilde{\alpha})$ and $\tilde{f}^{p+1}(\tilde{\alpha})$ by the foliation $\tilde{f}^p(\mathcal{F}_0)$. By gluing those foliations together with the foliation $\tilde{\mathcal{F}}'$ of B', we get a foliation $\tilde{\mathcal{F}}''$ of the fundamental domain B. Furthermore, this foliation is easily seen to be transverse to $\tilde{f}$. The projection of $\tilde{\mathcal{F}}''$ to our original plane yields a foliation $\mathcal{F}''$ of A which is transverse to I. Also note that since $\tilde{\alpha}' < T\tilde{\alpha}'$, the ω–limit set of the projection of $\tilde{\alpha}'$ in A is $\{0\}$. Thus 0 is either a saddle or a sink for $\mathcal{F}''$. But since $L(f, 0) = 1$, by Theorem 81, $L(\mathcal{F}'', 0) = 1$ and 0 cannot be a saddle singularity of $\mathcal{F}''$. Thus it is a sink and $\mathcal{F}''$ satisfies the conclusion of the lemma (by the way we also see now that N' was a sink of $\mathcal{F}'$ and not a saddle).

It remains to prove Claim 106. For this we first relate the rotation set $\mathcal{R}$ of I in the annulus A and the rotation set $\mathcal{R}'$ of I' in the annulus A' (note that $\mathcal{R}$ and $\mathcal{R}'$ are global rotation sets, not local ones). First note that I' has no contractible fixed point in A', because this would correspond to a fixed point of $\tilde{f}$. Next, we claim that $\mathcal{R}'$ is the image of $\mathcal{R}$ under the map

$$\rho \mapsto \frac{\rho}{1-q\rho}.$$

Indeed, this can be checked directly by using the definitions of the rotation sets in A and A'. (Note that in the case when $\mathcal{R}$ is a nontrivial interval, every rational number in the interior of $\mathcal{R}$ is realized by a periodic orbit, and then it suffices to check that if a point $\tilde{x}$ projects to a periodic point in A with rotation number ρ, then it projects to a periodic orbit in A' with rotation number $\rho/(1-q\rho)$.)

First assume that the rotation set $\mathcal{R}$ is not $\{0\}$. Since it is included in $[0, 1/q)$, it must contain some positive number. Then, by the above formula relating $\mathcal{R}$ and $\mathcal{R}'$, the rotation set $\mathcal{R}'$ also contains some positive number. Thus no leaf of $\mathcal{F}'$ can cross A' from N' to the other end. This shows that N' is not a source of $\mathcal{F}'$ (nor a saddle).

It remains to address the case when the rotation set satisfies $\mathcal{R} = \{0\}$. This case is conjectured not to happen, but in the absence of a proof we have to cope with it. In particular, there is no periodic orbit: indeed, since I is transverse to the foliation $\mathcal{F}$, a periodic orbit must have nonzero rotation number, contrarily to the assumption on $\mathcal{R}$.

Since I is transverse to a sink foliation $\mathcal{F}$ on A, the isotopy I has "some positive rotation". The idea of what follows is to try to see this positive rotation on the dynamics of some compact invariant set near the fixed point 0, and then to track this positive rotation in the universal cover $\widetilde{A} = \widetilde{A}'$, where it will in turn imply some positive rotation for I' in A'; this will prevent I' from being transverse to a source foliation.

To implement this idea we will heavily use the material in [36]. Following Le Calvez, we say that the point 0 is *indifferent* for f if for every small enough Jordan domain U containing 0, the connected component K of the set $\bigcap_{k\in\mathbb{Z}} f^{-k}(\overline{U})$ that contains 0 meets the boundary of U. Let us prove that in our situation, the fixed point is indifferent for f. Assume by contradiction that 0 is not indifferent. Since the area is preserved, it is neither a sink nor a source, and the Le Calvez–Yoccoz theorem [39] applies: there is some $q > 0$ such that $L(f^q, 0) < 1$. Then the local rotation set of $g = f^q$ is 0 modulo 1 (see for instance [43]). There is a local isotopy J from the identity to g whose local rotation set is $\{0\}$; this local isotopy is also characterized, up to homotopy, by having nonpositive index [41]. On the other hand, the isotopy I^q has index one since it is transverse to the sink foliation $\mathcal{F}$. Thus I^q is not homotopic to J, and its local rotation set is a nonzero integer. On the other hand, its local rotation set should be q times the local rotation set of I, which is $\{0\}$, a contradiction.

Thus the fixed point is indifferent. Let U be a small Jordan domain containing 0 in $\mathbb{R}^2$ such that K meets the boundary of U. Let U_K be the unbounded connected component of $\mathbb{R}^2 \setminus K$. This open set is an annulus, and the end corresponding to K admits the *prime-ends compactification*, which is a topology on the disjoint union $U_K \sqcup \mathbb{S}^1$ that is homeomorphic to the half-infinite annulus $\mathbb{S}^1 \times [0, +\infty)$ (see [36, Section 4]). Since $f(K) = K$, the restriction of f to U_K extends continuously to a circle homeomorphism $\hat{f}: \mathbb{S}^1 \to \mathbb{S}^1$.

The universal cover of $U_K \sqcup \mathbb{S}^1 \simeq \mathbb{S}^1 \times [0, +\infty)$ identifies with $\widetilde{U}_K \sqcup \Delta \simeq \mathbb{R} \times [0, +\infty)$, where $\widetilde{U}_K$ is included in $\widetilde{A}$ and Δ is the universal cover of the circle compactification $\mathbb{S}^1$. We denote by $\overline{T}$ the deck transformation of this universal cover which extends the restriction of T to $\widetilde{U}_K$. We orient Δ according to $\overline{T}$: in other words, we choose the identification of Δ with $\mathbb{R}$ in such a way that $\overline{p} < \overline{T}(\overline{p})$ for every $\overline{p} \in \Delta$. The lift $\tilde{f}$ of f to $\widetilde{U}_K$ induces on $\mathbb{R}$ a homeomorphism which is the natural lift $\overline{f}$ of $\hat{f}$. Since the rotation set of I in A is $\{0\}$, the circle rotation number of $\hat{f}$ is 0. In particular, we note for further use that there is a point $\overline{p}_0$ on Δ such that $\overline{f}(\overline{p}_0) = \overline{p}_0$.

A curve γ is called an *access arc* of K if it is included in U_K except for one endpoint $\gamma(0) = p$ which belongs to K. A classical result from prime-ends theory is that $\gamma \setminus \{p\}$ has a limit in the prime-ends compactification, ie there exists a point $\hat{p} \in \mathbb{S}^1$ such that $\gamma \setminus \{p\} \cup \{\hat{p}\}$ is a continuous curve in $U_K \sqcup \mathbb{S}^1$. The following claim uses access arcs to detect the sign of the rotation on $\mathbb{S}^1$:

Claim 107 *Let $\Gamma\colon (-\infty, 0] \to \widetilde{A}$ be an injective curve whose projection under the quotient map $\pi\colon \widetilde{A} \to A$ is proper, with $\pi \circ \Gamma(t)$ converging to the end opposite to 0 when t tends to $-\infty$, such that $\pi \circ \Gamma((-\infty, 0)) \subset U_K$ and $\pi \circ \Gamma(0) \in K \setminus \{0\}$. Assume that $\tilde{f}(\Gamma)$ and $T(\Gamma)$ are disjoint from Γ. Finally, let $\overline{p}$ be the point in Δ to which $\Gamma(t)$ converges when t tends to 0. Then $\overline{f}(\overline{p}) > \overline{p}$ if and only if $\tilde{f}(\Gamma)$ and $T(\Gamma)$ are in the same connected component of $\widetilde{U}_K \setminus \Gamma$.*

This claim follows from the considerations in [36, Section 5.1].

Let α be any leaf of $\mathcal{F}$ that goes from the other end of A to 0 and that meets $K \setminus \{0\}$; let $\tilde{\alpha}$ be some lift of α in $\widetilde{A}$. Then let $\Gamma \subset \tilde{\alpha}$ be obtained from $\tilde{\alpha}$ by running along α from $-\infty$ till the first point on K. Since $\mathcal{F}$ is positively transverse to I, $\tilde{f}(\Gamma)$ and $T(\Gamma)$ are both on the right-hand side of $\tilde{\alpha}$. Thus they are included in the same connected component of $\widetilde{U}_K \setminus \Gamma$. The criterion applies and provides a point $\overline{p}$ in Δ such that $\overline{f}(\overline{p}) > \overline{p}$.

Let $\widetilde{K}$ be the inverse image of K in $\widetilde{A}$. Let K' be the projection in A' of $\widetilde{K}$, to which we add the fixed point N'. According to point (3) of the quotient lemma, K' is a compact connected full set which is invariant under f' and included in some small neighborhood of N'. Note that $\widetilde{U}_K \sqcup \Delta$ also identifies with the universal cover of the prime-ends compactification of the complement $U_{K'}$ of K' in A'. Also note that the rotation number defined by Proposition 4.1 in [36] coincides with the element of our local rotation set by [43, Corollaire 3.14], and thus is equal to 0.

Remember that $T' = T\tilde{f}^{-q}$. Let $\overline{T}'$ be induced by T' on Δ; we get $\overline{T}' = \overline{T}\,\overline{f}^{-q}$. The point $\overline{p}_0$ of Δ which is fixed by $\overline{f}$ satisfies $\overline{T}'(\overline{p}_0) = \overline{T}(\overline{p}_0)$. Thus the orientations of Δ according to $\overline{T}$ and $\overline{T}'$ coincide. Now, to finish the proof of Claim 106, let us assume by contradiction that N' is a source for $\mathcal{F}'$. The above argument applies symmetrically to any leaf of $\mathcal{F}'$ that crosses the annulus A', and provides a point $\overline{p}'$ in Δ such that $\overline{f}(\overline{p}') < \overline{p}'$.

Now we will get a contradiction following an argument of [36]. The point 0 is indifferent and nonaccumulated. Theorem 9.4 of [36] applies and, since the rotation number is zero and $L(f,0) = 1$ in our case, says that the sequence of indices $L(f^k, 0)$ is constantly equal to 1. Then Proposition 9.5 of [36] applies and says that, given that the rotation number is zero, either every point p satisfies $\overline{f}(p) > p$ or every point p satisfies $\overline{f}(p) < p$. This contradicts the existence of the points $\overline{p}$ and $\overline{p}'$ and completes the proof of Claim 106.

Alternatively, we may argue as in point (c) of the proof of Proposition 9.5 in [36] (this argument is more precise, but it makes use of many definitions of [36]). Let O_+ be the projection in $\mathbb{S}^1$ of the connected component containing $\overline{p}$ of the complement in Δ of the set of fixed points of $\overline{f}$. Likewise, $\overline{p}'$ defines a component O_-. By identifying the connected components of the complement of $O_+ \cup O_-$ into two points, we get a quotient of $\mathbb{S}^1$ on which $\widehat{f}$ induces a North–South map. According to Lemma 7.1 and Proposition 7.2 of [36], there is a finer factor F of $\widehat{f}$ which is finite and not indifferent, and its restriction to $\mathbb{S}^1$ has at least one sink and one source with respect to the dynamics on $\mathbb{S}^1$. The sinks and sources are saddle points of F, and the formula of Proposition 8.1 says that $L(f,0) < 1$, which is a contradiction. □

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