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LAMBDA-RINGS, BINOMIAL DOMAINS, AND VECTOR BUNDLES OVER $CP(\infty)$

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This note is one aspect of work in progress studying the complex K-theory of a topological space X for which the loop space is p-equivalent to a finite complex (for technical reasons the preferred coefficients are to be the p-adic integers, $\hat{\mathbb{Z}}_p$). The notable examples of such spaces are when X is the classifying space of a compact Lie group. If G is connected then $K(BG, \hat{\mathbb{Z}}_p)$ can be described as a λ-ring as the $W(G)$-invariants of $K(BT, \hat{\mathbb{Z}}_p)$, for $T \to G$ a maximal torus and $W(G)$ its Weyl group, [Atiyah-Hirzebruch]. The existence of such an embedding and "Weyl group" is conjectured for the general case, in analogy to the results of [Adams-Wilkerson]. The proof of such a conjecture appears remote at this point.

This note is directed toward characterizing $K(BT, \hat{\mathbb{Z}}_p)$ itself as a λ-ring. The λ-ring structure is completely determined by the fact that $K(BT, \hat{\mathbb{Z}}_p)$ is the power series algebra

$\hat{\mathbb{Z}}_p[[x_1,\ldots,x_n]]$, where $\xi_i = 1 + x_i$ is the pullback of the Hopf line bundle over $\mathbb{C}P^\infty$ via the i^{th} projection map: hence $\lambda^i(\xi_i) = 0$ for $j > 1$. In theory, explicit formulae for $\lambda^j(\Sigma a_\ell x^\ell)$ could be developed. However, more qualitative questions are often difficult to answer.

Hereinafter, we focus on the determination of all elements $\{\eta\}$ in $K(BT^n, \hat{\mathbb{Z}}_p)$ which have the property in common with vector bundles that $\lambda^i(\eta) \equiv 0$ for $i \gg 0$ (i sufficiently large). By abuse of terminology these are the elements referred to as vector bundles in the title. The theorem below partially justifies this abuse.

Recall that by limiting arguments, the Adams operations $\{\psi^k\}$ can be extended to $\{\psi^\alpha : \alpha \in \hat{\mathbb{Z}}_p\}$ acting on $K(X, \hat{\mathbb{Z}}_p)$.

Theorem 3.3. If $\eta \in K(BT^1, \hat{\mathbb{Z}}_p)$ has $\lambda^i\eta \equiv 0$ for $i \gg 0$, there exist p-adic integers $\{\alpha_1,\ldots,\alpha_r\}$ such that $\eta = \psi^{\alpha_1}(\xi)+\ldots+\psi^{\alpha_r}(\xi)$. Here ξ is the Hopf line bundle and the $\{\alpha_j\}$ are unique up to order.

The analogous theorem and proof for $\mathbb{Z}$ or $\mathbb{Z}[1/N]$ coefficients holds, but other proofs are available in these cases ([Berstein], [Adams-Mahmud] [Wilkerson] [Adams]).

J.F. Adams [Adams] has a different proof for the analogous result in $K(X, \hat{\mathbb{Z}}_p)$, for $X = BT^n$, the classifying space of the n-torus. The proof of 3.3 dates before preprints of [Adams] were available. At the present time there is no inductive argument reducing the n-dimensional case to the one dimensional case of 3.3.

Nonetheless, the proof presented has features relevant to the general program of generalizing to λ-rings the [Adams-Wilkerson] work on algebras over the Steenrod algebra. Proposition 1.6 characterizes the algebraic closure of certain binomial domains, within a category of binomial domains, and 1.2 is a description of torsion-free (special) λ-rings in terms of only the Adams operations. In the last section, an application of Theorem 3.3 and its higher dimensional analogue to the determination of the λ-ring homomorphisms from $K(BG, \hat{\mathbb{Z}}_p)$ to $K(BG', \hat{\mathbb{Z}}_p)$ is given.

I. Binomial Domains

General references for this section are [Knutson] and [Atiyah-Tall]. The integers $\mathbb{Z}$ with the operations $\lambda^i(n) = \binom{n}{i}$ for $i > 0$, $n \in \mathbb{Z}$ form the primal λ-ring. The corresponding Adams operations are just

$$\psi^k(n) = n .$$

In general a commutative ring B with unit which embeds in $B \otimes \mathbb{Q}$ is said to be binomial if the expressions $\{\alpha(\alpha-1)\ldots(\alpha-i+1) \otimes 1/i! = \binom{\alpha}{i} \in B \otimes \mathbb{Q}\}$ actually lie in B, for all $i > 0$ and $\alpha \in B$. Clearly any field of characteristic zero is binomial, so the existence of the binomial structure is arithmetic in nature rather then algebraic. The main concern here is with B an integral domain, so that $\binom{\alpha}{i}$ can be considered as elements of the field of fractions K of B. A binomial ring satisfies the axioms for (special) λ-rings as given in [Atiyah-Tall].

In this section some known results [Warfield, Chapter 10] on the ideal pB, for p a prime integer, are rederived from the viewpoint of the formula

$$\psi^p x = x^p \bmod (pB)$$

valid in any special λ-ring. This avoids some manipulations with binomial coefficients, but the primary motivation is to continue for λ-rings the spirit of the [Adams-Wilkerson] work on algebras with a Steenrod algebra action.

Proposition 1.1: If B is binomial, then $r^p \equiv r \bmod pB$ for each prime p in $\mathbb{Z}$.

Proof: The equation $\psi^p r = r^p + p\theta_p(r)$, where $\theta_p(r)$ is a universal polynomial in $\{\lambda^i(r),\ i \le p\}$, is valid in any special λ-ring. Since $\psi^p r = r$ in a binomial ring, and $\lambda^i(r) \in B$, the result follows.

The converse to 1.1 is valid but requires some preparation. The idea is that the Newton formulae relating the exterior powers to Adams operations can be used to solve for the exterior powers, at least in $B \otimes \mathbb{Q}$. The mod p condition can then be used to show that the exterior powers thus defined take B into B. While this can be proved directly for binomial rings a corresponding statement holds more generally for special λ-rings.

Newton Formulae:

$$\psi^k(x) - \lambda^1(x)\psi^{k-1}(x) + \ldots + (-1)^{k-1}\lambda^{k-1}(x)\psi^1(x) = (-1)^{k+1} k\,\lambda^k(x)$$

for x an element of a (special) λ-ring.

Proposition 1.2: Let R be a commutative ring with unit which is torsion-free as a $\mathbb{Z}$-module. If $\{\psi^k : k \in \mathbb{Z}^+\}$ is a set of ring endomorphisms of R satisfying

i) $\psi^1 = Id_R$

ii) $\psi^k\psi^r = \psi^{kr}$ for all $r,k \in \mathbb{Z}^+$ and

iii) $\psi^p x = x^p \bmod (pR)$ for each $x \in R$ and p prime in $\mathbb{Z}$,

then the λ-operations on $R \otimes Q$ given by the inversion of the Newton formulae map R into R, and R is a special λ-ring with respect to these λ-operations.

Lemma: For any special λ-ring R

a) $x^p - \psi^p x = p\theta_p(x) = p[(-1)^p\lambda^p(x) + \text{products}]$

b) $\lambda^k \circ \lambda^q(x) = (-1)^{(k+1)(q+1)}\lambda^{kq}(x) + \text{products}.$

"Products" means products of $\{\lambda^i(x),\ i < p\}$ and $\{\lambda^i(x),\ i < kq\}$ respectively.

Proof: This follows easily from the "verification principle" of [Atiyah-Tall].

Proof of 1.2: Since $R \to R \otimes \mathbb{Q}$ is monic, and $\lambda^i : R \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$, by [Knutson, pp. 46-52], we have only to show that $\lambda^i : R \to R$. In fact, taking $\mathbb{Z}_{(p)}$ to be the p-local integers, we need only show that λ^i maps $R \otimes \mathbb{Z}_{(p)}$ into itself, since $R = \bigcap_p (R \otimes \mathbb{Z}_{(p)}) \subset R \otimes \mathbb{Q}$. For $p > i$, this follows from the Newton formulae and induction. For λ^p, $\psi^p x = x^p + p\theta_p(x) = x^p + p((p-1)!\lambda^p(x) + \ldots)$. Since p divides $\psi^p x - x^p$, and the other terms of $\theta_p(x)$ are in $R \otimes \mathbb{Z}_{(p)}$, $\lambda^p(x) \in R \otimes \mathbb{Z}_{(p)}$. For i relatively prime to p and larger than p, the Newton formulae and the inductive hypothesis show that $\lambda^i x \in R \otimes \mathbb{Z}_{(p)}$. However, if $i = jp$, by the lemma

$$\lambda^{jp}(x) = (-1)^{(p+1)(j+1)}\lambda^p \circ \lambda^j(x) + \text{products}.$$

This last term involves only $\{\lambda^1(x), \ldots, \lambda^{jp-1}(x)\}$ so the inductive hypothesis shows that $\lambda^{jp}(x) \in R \otimes \mathbb{Z}_{(p)}$.

Corollary 1.3:

a) If B is a special λ-ring and S is any multiplicative subset such that $\psi^k S \subset S$ for all k, then $S^{-1}B$ is a special λ-ring.

b) If B is a special λ-ring and τ is any metric topology on B such that B is a topological ring with respect to τ and the $\{\psi^k\}$ map Cauchy sequences to Cauchy sequences, then the τ-completion $\hat{B}$ is a special λ-ring.

Comment: The immediate applications are to B binomial, and τ given by some I-adic topology.

Proof: a) $S^{-1}B = \{x/s : s \in S,\ x'/s' \sim x/s \Leftrightarrow (x's - xs')u = 0$ for some $u \in S\}$. Extend ψ^k to $S^{-1}B$ by $\psi^k\{x/s\} = \{\psi^k x/\psi^k s\}$. This is well-defined, and $\psi^p(x/s) - (x/s)^p = \psi^p x/\psi^p s - x^p/s^p = [s^p(\psi^p x - x^p) + (s^p - \psi^p s)x^p]/s^p\psi^p s \in pS^{-1}B$.

b) Let $\{x_i\}$ be a Cauchy sequence representing a point in $\hat{B}$. Then $\{\psi^k x_i\}$ is again a Cauchy sequence, $\psi^p\{x_i\} - \{x_i{}^p\} = \psi^p\{x_i\} - \{x_i^p\} = \{\psi^p x_i - x_i^p\} = p\{\theta_p x\} \in p\hat{B}$.

Corollary: Any localization or I-adic completion of $\mathbb{Z}$ is a binomial ring. In particular, $\mathbb{Z}_{(p)}$ and $\hat{\mathbb{Z}}_p$ are binomial (here $\hat{\mathbb{Z}}_p$ denotes the p-adic integers). Any integrally closed subring of $\hat{\mathbb{Z}}_p$ is binomial.

Thus far the scenario has been much like the study of extension of Steenrod operations in [Adams-Wilkerson]. Indeed, next we will show that the possible extensions of binomial rings are very restricted:

Proposition 1.4: If B is a binomial ring, then pB is radical for each $p \in \mathrm{Spec}(\mathbb{Z})$. That, is if $x^N \in pB$, then $x \in pB$.

Proof: Let $x^N \in pB$. Now $\psi^{p^N} x = x$, since B is binomial. But $\psi^{p^N} x = x = x^{p^N} + pB = x^N x^{p^N - N} + pB$, so $x \in pB$.

Proposition 1.5: If the binomial domain B is an integral extension of $\mathbb{Z}[1/N]$, then $B = \mathbb{Z}[1/N]$.

Proposition 1.6: If the binomial domain B is an integral extension of $\hat{\mathbb{Z}}_p$, then $B = \hat{\mathbb{Z}}_p$.

Caution: 1.6 is not valid for $\mathbb{Z}_{(p)}$, e.g. $B = \mathbb{Z}_{(5)}[i]$, $i^2 = -1$, is binomial since $B \subset \hat{\mathbb{Z}}_5$, and B is integrally closed.

Proof of Proposition 1.6: B has a field of fractions K. We can reduce to the case that B has its fraction field finite dimensional over the field of p-adics, $\hat{\mathbb{Q}}_p$, since given an element $b \in B$, the smallest binomial ring containing b is contained in $\hat{\mathbb{Q}}_p[b]$. In fact, now we can assume that B itself is f.g. over $\hat{\mathbb{Z}}_p$. Then the valuation ν_p on $\hat{\mathbb{Z}}_p$ has an unique extension to B and K, and the only nonzero prime ideal of B is $\{b \in B : \nu_p b > 0\}$. By 1.4, pB is radical. We will show that it is prime. Since B is Noetherian, and the prime ideals associated to pB reduce to one ideal, pB is primary. But primary and radical together imply prime, so pB is the nonzero prime ideal in B. Hence B/pB is a finite field. But $b^p = b \bmod pB$, so the finite field is just $\mathbb{F}_p$. By Nakayama's lemma, $B = \hat{\mathbb{Z}}_p$.

Proof of Proposition 1.5: There seems to be no elementary proof of 1.5 in the generality stated. For $\mathbb{Z}$ itself, a simple proof appears in [Adams].

The fraction field K of B is a finite extension of $\mathbb{Q}$. The corresponding ring of integers $\mathcal{O}_K$ need not be B, since B is not necessarily integrally closed in K. However, there exists an integer M such that $S^{-1}B = S^{-1}\mathcal{O}_K$ for S the multiplicative subset generated by M. By 1.4, $p\mathcal{O}_K$ is a radical ideal in $\mathcal{O}_K$ for all but a finite number of $\mathbb{Z}$-primes p. But the set of primes p which split completely in $\mathcal{O}_K$ has density $1/n$, where $n = [K : \mathbb{Q}]$, by the Cebotarev Density Theorem, [p. 299, Cassels-Frolich]. Hence $n = 1$ and $K = \mathbb{Q}$. Since B is integral over $\mathbb{Z}[1/N]$ (N possibly 1) and contained in $\mathbb{Q}$, $B = \mathbb{Z}[1/N]$.

II. An Integrality Condition for Binomial Domains.

In the applications, we will have to know that certain quantities which are a priori known only to be elements of the algebraic closure field of some domain are actually in the domain. The main integrality result is

Proposition 2.1: Let $(x_1, x_2, \ldots)$ and $(b_1, \ldots)$ be formal variables related by the formula $\sum_{i=1}^{\infty} \binom{x_i}{k} = b_k$, where

$$\binom{x}{k} = x(x-1) \ldots (x-k+1)/k! \quad \text{for each positive integer } k .$$

Then a) $\sigma_n(x_1, \ldots)$, the n-th elementary symmetric function in $\{x_1, \ldots\}$ is a $\mathbb{Z}$-polynomial in $\binom{b_i}{j}$

b) $\sum_{i=1}^{\infty} \binom{\binom{x_i}{n}}{\ell}$ is a $\mathbb{Z}$-polynomial in $\{\binom{b_i}{j}\}$

Example: $\sigma_1 = b_1$, $\sigma_2 = \binom{b_1}{2} - b_2$ and $\sigma_3 = \binom{b_1}{3} - b_2 b_1 + 2b_3 + 2b_2$

Corollary 2.2: If B is a binomial domain, and $\{x_1,\ldots,x_N\}$ elements of some extension field of B which satisfy the equations $\Sigma \binom{x_i}{k} = b_k \in B$ for all k, then the $\{x_i\}$ are integral over B and generate an integral binomial extension of B. In particular, if $B = \mathbb{Z}, \mathbb{Z}_{[1/N]}$ or $\hat{\mathbb{Z}}_p$, $x_i \in B$ for all i.

Proof of Corollary: From part a) of the Proposition, the elementary symmetric functions in the $\{x_i\}$ are in B, since B is binomial. Hence the $\{x_i\}$ are integral over B, since they are roots of

$$\Pi(X - x_i) = \Sigma(-1)^i X^{N-i}\sigma_i(x_1,\ldots,x_N) .$$

Moreover, by part b) of the proposition, the same holds for $\{ \binom{x_i}{j} \}$. Hence $B\{ \binom{x_i}{j} \}$ is an extension binomial domain integral over B. The final sentence follows then from 1.5 and 1.6.

The proof of 2.1 requires several supporting lemmas. Proofs appear at the end of this section.

Lemma 2.3: Define $\binom{x}{I} = \binom{x_1}{i_1} \cdots \binom{x_n}{i_n}$ for sequences $I = (i_1,\ldots,i_n,0,0,\ldots)$ containing only finitely many nonzero entries. If $P(x_1,\ldots) = \sum_{|I| \le N} a_I x^I$ (where $a_I \in \mathbb{Q}$) has the property that $P(n_1,\ldots) \in \mathbb{Z}$ for all sequences $(n_1,\ldots)$ with only finitely many nonzero terms, then $P(x_1,\ldots) = \Sigma c_I \binom{x}{I}$, where $c_I \in \mathbb{Z}$.

Lemma 2.4: If $P(x_1,\ldots)$ as in 2.3 is symmetric under all permutations of the x_i, $P(x_1,\ldots) = \Sigma\, d_I\, \Sigma\, \sigma_* \binom{x}{I}$ for $d_I \in \mathbb{Z}$, where the left hand sum is over unordered indices I, and the right hand sum is over distinct permutations of I. Here $\sigma_*\binom{x}{I} = (\sigma^x_I\ I)$.

Assign a partial order to unordered index sets $\{i_1,\ldots\}$ with only finitely many entries by the rule $I' < I'' \Leftrightarrow \Sigma\, i_j' = \Sigma\, i_j''$ and th number of distinct entries in I' is greater than the number of distinct entries in I'', and finally, by the size of the largest entry (e.g. $(1,1,1) < (2,1)$, and $(2,3) < (4,1)$). Then

<u>Lemma 2.5</u>: $$\sum_{\substack{\sigma \\ \text{distinct} \\ \text{perm. of } I}} \sigma_* \binom{x}{I} = \binom{b_{i_1}}{r_1} \cdots \binom{b_{i_k}}{r_k} + \text{higher}$$

order terms.

Here i_j has multiplicity r_j in $I = \{i_1,\ldots\}$.

<u>Lemma 2.6</u>: $\Sigma\binom{x}{I}$ is a $\mathbb{Z}$-polynomial in $\binom{b_i}{j}$.

<u>Proof of 2.3</u>: $\binom{x}{I}$ has the property of mapping $\bigoplus_{\infty} \mathbb{Z} \to \mathbb{Z}$, and the $\binom{x}{I}$ with $|I| \le N$ form a $\mathbb{Q}$-basis for all polynomials of total degree $\le N$. Hence $P(x_1,\ldots) = \Sigma\, c_I\binom{x}{I}$, where $c_J \in \mathbb{Q}$ a priori. Order the I's lexicographically. $P(J) = \Sigma\, c_I\binom{j_1}{i_1} \cdots \binom{j_1}{i_1} \in \mathbb{Z}$. Since $\binom{j}{i} = 0$ if $i > j$, by ordering lexicographically, we get a system of equations

$$c_{I_1} = P(I_1)$$
$$\vdots$$
$$c_{I_1}\binom{I_2}{I_1} + c_{I_2} = P(I_2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots \text{ etc.}$$

which is solvable over $\mathbb{Z}$ for c_{I_j}, since the diagonal entries are 1's.

<u>Proof of 2.4</u>: By 2.3, $P(x_1,\ldots) = \sum_{\text{ordered } I} c_I\binom{x}{I}$. But since

$P(x_1,\ldots)$ is symmetric, any permutation of I also appears with the same coefficient c_I .

Proof of 2.5:
$$\binom{b_{i_j}}{r_j} = \sum_s \binom{\binom{x_s}{i_j}}{r_j}$$
$$= \sum_{[k_1+k_2+\ldots+k_r = r_j]} \binom{\binom{x_{n_1}}{i_j}}{k_1} \cdots \binom{\binom{x_{n_r}}{i_j}}{k_r}$$
$$= \sum_{\substack{r_j \text{ distinct } x\text{'s} \\ (n_1,\ldots,n_r)}} \binom{x_{n_1}}{i_j} \cdots \binom{x_{n_{r_j}}}{i_j}$$

+ terms involving fewer than r_j distinct values of x's . Taking the product of $\{\binom{b_{i_j}}{r_j}\}$'s gives leading term $\sum_\sigma \sigma\binom{x}{I}$ modulo terms with fewer than the naximum number of distinct x's . The lemma follows.

Proof of 2.6: Let $\Sigma\, i_j = N$. We induct on the number of distinct x's . If the number is 1 , $\Sigma\binom{x}{i_1} = b_{i_1}$. Now if the lemma is true for $\leq n-1$ distinct entries in I , lemma 2.5 shows that
$$\Sigma\binom{x}{I} = \binom{b_{i_1}}{r_1} \cdots \binom{b_{i_n}}{r_n} + g(x_1,\ldots)\ ,$$
where g is expressible in terms of $\binom{x}{J}$ with J having fewer than n distinct entries. By the inductive hypothesis, $g(x_1,\ldots)$ is a $\mathbb{Z}$-polynomial in the $\{\binom{b_i}{j}\}$.

Proof of Proposition 2.1:

a) $\sigma_n(x_1,\ldots)$ is a symmetric function taking integer values on $\oplus \mathbb{Z}$. By the lemmas, $\sigma_n(x_1,\ldots) = \Sigma\, a_I\binom{b}{I}$, where $a_I \in \mathbb{Z}$, since by 2.3, and 2.4

$$\sigma_n(x_1,\ldots) = \Sigma\, d_I\, \Sigma\, \sigma_*\binom{x}{I}\ ,\ d_I \in \mathbb{Z}\ ,$$

and by 2.6, $\Sigma\, \sigma_*\binom{x}{I}$ is a $\mathbb{Z}$-polynomial in the $\binom{b}{I}$.

b) $\sum \begin{pmatrix} x_i \\ \binom{}{n} \\ \ell \end{pmatrix}$ is symmetric, takes $\mathbb{Z}$-values $\underset{\infty}{\oplus} \mathbb{Z}$, so by above,

$$= \Sigma\, c_I\binom{b}{I} \text{ on with } c_I \in \mathbb{Z}\ .$$

III. Chern Classes and Bundles over $\mathbb{C}P^\infty$

We have for $K(X,\mathbb{Z})$ the Chern classes $c_n : K(X,\mathbb{Z}) \to H^{2n}(X,\mathbb{Z})$, with property that $c_n(x+y) = \Sigma\, c_{n-i}(x)c_i(y)$.

Amalgamating these, we have the total Chern class $c : K(X;\mathbb{Z}) \to H^*(X,\mathbb{Z})\,[[T]]$, transforming sums into products. In particular, $c(\xi) = 1 + zT$ for ξ the Hopf line bundle on $\mathbb{C}P^\infty$. We require this Chern class with coefficient groups larger than $\mathbb{Z}$. The main desideratum for such an extended Chern class is

$$c : K(X:\overline{\hat{\mathbb{Q}}}_p) \to H^*(X,\overline{\hat{\mathbb{Q}}}_p)\,[[T]]$$

such that $c(x+y) = c(x)c(y)$ and $c(\psi^\alpha x) = \Sigma\, \alpha^n(c_n x)T^n$. For $X = \mathbb{C}P^\infty$, such an extension can be defined almost by fiat. However, a more intrinsic definition is also possible. Recall that any special λ-ring R with a split augmentation $\epsilon : R \to B$, B binomial, has a Grothendieck γ-filtration $\{F_nR\}$ [e.g. Atiyah-Tall] defined. Here F_nR is the sub-B-module generated by elements of the form $\gamma^{i_1}(y_1)\ldots\gamma^{i_r}(y_r)$, where $y_i \in \ker\ \epsilon$ and $\Sigma\, i_j \geq n$. Atiyah defined "Chern classes" $c_i : R \to F_iR/F_{i+1}R$ by

$$c_i x = [\gamma^i(x - \epsilon x)] \ , \quad \text{[Atiyah]}$$

The property that $\gamma^n(x+y) = \Sigma\, \gamma^{n-i}(x)\, \gamma^i(x)$ then gives the formula $c(x+y) = c(x)c(y)$, while the fact that $\psi^k x = k^n x \bmod F_{n+1}$ for $x \in F_n$ gives the property that $c_i(\psi^\alpha x) = \alpha^n c_n x$. If $R = K(X)$, these classes coincide with the topologically defined classes after the identification of $E^0 R \otimes \mathbb{Q}$ with $H^*(X,\mathbb{Q})$.

If R is a special λ-ring with augmentation $\epsilon : R \to B$, and L is some binomial extension of B , we can add L coefficients by forming $\varprojlim_n (R/F_n R \otimes L) = \hat{R}_L$. The Adams operations extend to $\{L \setminus \{0\}\}\ \hat{R}_L \to \hat{R}_L$, and the resulting Chern class $c : \hat{R}_L \to E^0 R \otimes L[[T]]$ has the usual properties. If $E^0 R$ is not torsion-free this may do violence, but in the case $R = K(BT^n, \mathbb{Z})$ and L a field or $\hat{\mathbb{Z}}_p$, R injects into $\hat{R}_L$.

The decomposition of η into line bundles now proceeds by brute force. There exists an $n > 0$ such that $\lambda^n \eta \neq 0$ and $\lambda^{n+i}\eta = 0$ for $i > 0$. We need a line bundle of the form $\xi^\alpha = \psi^\alpha \xi = (1+x)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} x^i$ so that $\lambda^n(\eta - \xi^\alpha) = 0$. Since $\lambda^n(x+n-1) = \gamma^n x$ in general, this is equivalent to the equation

$$\gamma^n((\eta - n) - (\xi^\alpha - 1)) = 0 \ .$$

The first approximation to this is just

$$c_n(\eta - \xi^\alpha) = 0 \ .$$

But this is the coefficient of T^n in

$$c_n(\eta - \xi^\alpha) \ , \text{ or just } \Sigma\, (-1)^i \alpha^i\, c_{n-i}(\eta) c_1(\xi)^i \ .$$

Putting $c_1(\xi) = z$ and dividing by z^n leaves the scalar equation for α ,

$$\Sigma\ (-1)^i \alpha^i (c_{n-i}(\eta)/z^{n-i}) = 0\ .$$

Hence $\alpha \in \overline{\hat{\mathbb{Q}}}_p = L$.

Lemma 3.1: If $c_n(\eta - \xi^{\alpha}) = 0$, then ξ^{α} also satisfies $\lambda^n(\eta - \xi^{\alpha}) = 0$.

Proof: $K(BT^1,L)$ is a complete local ring in the x-adic topology. αx is an approximate root to the equation.

$$\gamma^n(\eta\text{-}n\text{-}(X)) = 0\ .$$

By Hensel's lemma, the equation has a root w congruent to $\alpha x \bmod x^2$. By the splitting principle [Atiyah-Tall 6.5] $1+w$ is a line bundle, and hence by 3.2 below has the form ξ^{α} .

Lemma 3.2: If $1+w \in K(BT^1, L) = L[[x]]$ (for $1+x$ the Hopf line bundle) has $\lambda^i(1+w) \equiv 0$ for $i > 1$, then $1+w = \psi^{\alpha}\xi = (1+x)^{\alpha}$, where $w \equiv \alpha x \bmod x^2$.

Proof: We proceed by induction. If $1+w = (1+x)^{\alpha} \bmod x^N$, we must show that this is also true for $N + 1$. Put $1+w = (1+x^{\alpha}) + ax^N + \dots$. Then $\psi^2(1+w) = (1+w)^2 = (1+x)^{2\alpha} + 2a(1+x)^{\alpha}x^N + \dots$. But this is also equal to $(1+x)^{2\alpha} + 2^N ax^N + \dots$ by properties of ψ^2 . Hence, comparing coefficients of x^N , we have $(2^N - 2)a = 0$ and $a = 0$ if $N \neq 1$. The hypothesis is true for $N = 2$, so $1+w = (1+x)^{\alpha}$.

Proof of Theorem 3.3: On the one hand $\eta = \Sigma\, a_j x^j$ where $\xi = 1+x$, and on the other hand $\eta = \Sigma\, \xi^{\alpha_k}$ from the splitting equations. Hence the equations

$$\Sigma \binom{\alpha_k}{j} = a_j \in \hat{\mathbb{Z}}_p \quad \text{or} \quad \mathbb{Z}_{1/N}$$

hold. By 2.2 $\alpha_k \in \hat{\mathbb{Z}}_p$ or $\mathbb{Z}_{[1/N]}$ resp.

Essentially the same proof applies in the higher dimensional cases to show that if η factors as a sum of line bundles in $K(BT^n, L)$, then it does so in $K(BT^n, \hat{\mathbb{Z}}_p)$. However, since for dimensional reasons there are elements in $K(BT^n, L)$ with $\lambda^i \eta = 0$ for $i \gg 0$ but yet such that η is not a sum of line bundles, the fact that $\eta \in K(BT^n, \hat{\mathbb{Z}}_p)$ cannot ignored until the last step as for $n = 1$.

IV. Induced Homorphisms

Theorem 3.3 and the higher dimensional analogue of [Adams] are not only natural steps in the classification of possible $K(X, \hat{\mathbb{Z}}_p)$, but have an application to the computation of the λ-ring maps between the K-theories of classifying spaces of Lie groups. This is only implicit in [Adams] for p-adic coefficients, but is in the spirit of analogous results in [Adams, Wilkerson] or [Wilkerson] for Steenrod algebra actions.

Recall that the line bundles in $K(BT^n, \hat{\mathbb{Z}}_p)$ form a free $\hat{\mathbb{Z}}_p$-module of rank n, under the group operation of tensor product and with $\hat{\mathbb{Z}}_p$ acting via the Adams operations. If W is any subgroup of $GL(n, \hat{\mathbb{Z}}_p)$, then W acts via λ-ring automorphisms on $K(BT^n, \hat{\mathbb{Z}}_p)$, by extension of the action on the module of line bundles.

__Theorem 4.1__: If R is the ring of invariants in $K(BT^n, \hat{\mathbb{Z}}_p)$ for some finite group W of $Gl(n, \hat{\mathbb{Z}}_p)$, then any λ-ring homomorphism

$$\Phi : R \rightarrow K(BT^m, \hat{\mathbb{Z}}_p)$$

extends to a λ-ring homomorphism

$$\tilde{\Phi} : K(BT^n, \hat{\mathbb{Z}}_p) \rightarrow K(BT^m, \hat{\mathbb{Z}}_p)$$

Lemma: Let A, B, and C be integral domains such that A is contained in B and B is integral over A. If $\Phi : A \to C$ is a homomorphism such that

1) there exists a set of elements $\{y_i\}$ of B such that for each y_i there exists a monic polynomial $f_i(X)$ in $A[X]$ such that $f_i(y_i) = 0$,
2) B is generated as an A-module by polynomials in the elements $\{y_i\}$, and
3) $\Phi_* f_i(X)$ splits completely in $C[X]$ for all i (here the induced polynomial is obtained by applying Φ to the coefficients of $f_i(X)$),

then there exists a extension of Φ to $\tilde{\Phi} : B \to C$.

Proof of Lemma: This is basically [Prop. 16, p. 250, Lang], modified to account for C not being an algebraically closed field. Let K be the field of fractions of C and let $\bar{K}$ be an algebraic closure of K. Then Prop. 16 of [Lang], under the hypothesis that B is integral over A, produces an extension $\tilde{\Phi} : B \to \bar{K}$. It would suffice to show that $\tilde{\Phi}(y_i)$ is in C in order to conclude that $\Phi(B)$ is actually in C. But $\Phi(y_i)$ is a root of $\Phi_* f_i(X)$, which by hypothesis 3) has all its roots in C.

Proof of Theorem: Let the $\{y_i\}$ of the Lemma be $\{\xi_i = 1+x_i,$ $i=1$ to $n\}$, the set of Hopf line bundles. Define

$$f_i(X) = \prod_W (X - w\xi_i) = \pm\lambda^{|W|}(\mathrm{Trace}(\xi_i) - X)$$

where X is regarded as a formal line bundle variable for the purposes of the right hand term. Then

$$\Phi_* f_i(X) = \lambda^{|W|}(\Phi(\mathrm{Trace}(\xi_i)) - X)$$

is simply the splitting equation of [Atiyah-Tall] for the vector bundle $\Phi(\mathrm{Trace}(\xi_i))$ in $K(BT^m, \hat{\mathbb{Z}}_p)$. If i is larger than $|W|$, $\lambda^i(\Phi(\mathrm{Trace}(\xi_i))) = 0$ by naturality. Hence $\Phi(\mathrm{Trace}(\xi_i))$ is a sum of line bundles in $K(BT^m, \hat{\mathbb{Z}}_p)$, by [Adams, Prop. 5.1]. To finish the application of the lemma we need only to show that the smallest subring of $K(BT^n, \hat{\mathbb{Z}}_p)$ containing the ring of invariants R and $\{\xi_i\}$ is the entire ring. Since R is the ring of invariants, it is closed in the augmentation ideal topology, and hence R is complete. $R\{\xi_i\}$ is a finitely generated R-module, and thus is also complete, and hence closed in $K(BT^n, \mathbb{Z}_p)$ in the augmentation ideal topology. Then

$$R\{\xi_i\}/R\{\xi_i\} \cap I^N \longrightarrow K(BT^n, \hat{\mathbb{Z}}_p)/I^N$$

is onto for all N, since $\{\xi_i\}$ are algebra generators for the right hand side. Thus the Lemma applies, and there exists a $\tilde{\Phi}$ extending Φ. By its construction, $\tilde{\Phi}$ maps $\{\xi_i\}$ into the roots of the splitting equation, which are in turn line bundles. Hence $\tilde{\Phi}$ is a λ-map, since the Laurent series $\hat{\mathbb{Z}}_p\,[\xi_1, \ldots, \xi_1^{-1}, \ldots]$ is dense in the I-adic topology.

Theorem 4.1 reduces the study of the maps between $K(BG)$ and $K(BG')$ to those between $K(BT)$ and $K(BT')$ (indeed, between the modules of line bundles) which are compatible with the action of the two Weyl groups in the sense of [Adams-Mahmud].

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