Norm coherence for descent of level structures on formal deformations

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We give a formulation for descent of level structures on deformations of formal groups and study the compatibility between descent and a norm construction. Under this framework, we generalize Ando's construction of H_∞ complex orientations for Morava E-theories associated to the Honda formal groups over $\mathbb{F}_p$. We show the existence and uniqueness of such an orientation for any Morava E-theory associated to a formal group over an algebraic extension of $\mathbb{F}_p$ and, in particular, orientations for a family of elliptic cohomology theories. These orientations correspond to coordinates on deformations of formal groups that are compatible with norm maps along descent.

Keywords: deformation of a formal group, Morava E-theory, complex orientation, norm coherence

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Contents

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1 Introduction

1.1 Algebraic motivations and statement of results Let R be a commutative ring with 1 and let A be an algebra over R. Suppose that, as an R-module, A is finitely generated and free. The norm of A is a map $A \to R$ which sends a to $\det(a\cdot)$, the determinant of multiplication by a as an R-linear transformation on A. It is multiplicative but not additive in general. Such norms appear as an important ingredient in various contexts: arithmetic moduli of elliptic curves [Katz–Mazur 1985, § 1.8 and § 7.7], actions of finite group schemes on abelian varieties [Mumford 2008, § 12], isogenies of one-parameter formal Lie groups over p-adic integer rings [Lubin 1967, § 1]. These norm maps are closely related to construction of *quotient objects*.

The purpose of this paper is to examine an interaction between norms and the corresponding *subobjects*, more precisely, a functorial interaction with chains of subobjects, in the context of Lubin and Tate's formal deformations [Lubin–Tate 1966]. The functoriality amounts to descent of "level structures" on deformations (see § 6 and § 8). In this paper, a level structure on a formal group is a choice of finite subgroup scheme, from which we obtain a quotient morphism of formal groups. A norm map between their rings of functions then gets involved in making this quotient morphism into a homomorphism of formal group *laws* (2.3). This norm construction is compatible with successive quotient along a chain of subgroups.

On the other hand, given a deformation over a p-adic integer ring, there is a canonical (i.e., coordinate-free) descent of level structures via Lubin and Tate's universal deformations. Strickland studied the representability of this moduli problem [Strickland 1997] so that the descent can be realized as canonical lifts of Frobenius morphisms (5.13).

Our main result shows the existence and uniqueness of deformations of formal group laws (equivalently, formal groups with a choice of coordinate) on which the canonical lifts of Frobenius coincide with quotient homomorphisms from the norm construction. We say that these deformations are *norm coherent*, and call their corresponding coordinates *norm-coherent* ones (see § 6, specifically Definition 6.21).

Let k be an algebraic extension of $\mathbb{F}_p$, R a complete local ring with residue field containing k, G a formal group law over k of finite height, and F a deformation of G to R.

Theorem 1.2 *There exists a unique formal group law F' over R, $\star$-isomorphic to F, which is norm coherent. Moreover, when F is a Lubin–Tate universal deformation, F' is functorial under base change of G/k, under k-isogeny out of G, and under k-Galois descent.*

Cf. Theorem 7.22 and see Proposition 7.1 for a statement in terms of coordinates.

Remark 1.3 In the context of local class field theory via Lubin and Tate's theory of complex multiplication [Lubin–Tate 1965], Coleman's norm operator [Coleman 1979, Theorem 11] is used to compute norm groups. Walker observed its similarity to the norm construction above [Walker 2008, Chapter 5]. Specifically, he reformulated the norm-coherence condition (for a special case) in terms of a particular way in which Coleman's norm operator acts [Walker 2008, 5.0.10]. It would be interesting to have a conceptual understanding of this connection in the generality of Theorem 1.2.

Another instance where norms interact with descent of level structures appears in the Lubin–Tate tower for a formal group of height 1 (with full level structures) as in [Weinstein 2016, § 2.3]. A natural question would be the relevance of our results with their situation at a general height, as the two settings are closely related.

1.4 Topological motivations and statement of results The relevance to topology (and, further, to geometry and mathematical physics) of this functorial interaction between norms and finite formal subgroup schemes lies, for instance, in having highly coherent multiplications for *genera*. These are cobordism invariants of manifolds. Such multiplications refine the invariants by reflecting symmetries of the geometry (some known, some conjectural).

A prominent example is the Witten genus for string manifolds, which takes values in the ring of integral modular forms of level 1. Motivated by this, Hopkins and his collaborators developed highly structured multiplicative *orientations* (i.e., genera for *families* of manifolds) for elliptic cohomology theories and for a universal theory of topological modular forms [Hopkins 1995, Hopkins 2002]. In particular, in [Ando–Hopkins–Strickland 2004], they showed that their sigma orientation $MU\langle 6\rangle \to E$ for any elliptic cohomology E is H_∞, a commutativity condition on its multiplicative structure (2.8).

Their analysis of this H_∞ structure was based on [Ando 1995, Ando 1992], where the algebraic condition of norm coherence had made a first appearance. See specifically [Ando–Hopkins–Strickland 2004, 1.5 and Remark 4.16]. Specializing to the case of interest, our algebraic formulation of norm coherence from above is equivalent to the condition they required (see Proposition 8.17 below). As a topological application, Theorem 1.2 then produces $MU\langle 0\rangle$-orientations that are H_∞ for a family of generalized cohomology theories called Morava E-theories (2.7), including those treated by Ando and by Ando, Hopkins, Strickland.

Theorem 1.5 *Let k and G be as in Theorem 1.2. For the form of Morava E-theory associated to G/k, there exists a unique $MU\langle 0\rangle$-orientation that is an H_∞ map.*

Cf. Corollary 8.20 for a precise statement about uniqueness.

Remark 1.6 Rezk reminded us that the sigma orientations do not factor through these H_∞ $MU\langle 0\rangle$-orientations (8.1).

On the other hand, the coefficient ring of an E-theory (of height 2) is a certain completion of a ring of modular forms. In [Zhu 2020], as a first step, we related its elements to certain quasimodular forms (and to mock modular forms) via Rezk's logarithmic operations. See also [Rezk 2018, the second remark following Theorem 1.29]. Given Theorem 1.5, it would be interesting to acquire and analyze more exotic manifold invariants. In particular, we may investigate an analogue of the modular invariance of a sigma orientation [Ando–Hopkins–Strickland 2001, 1.3] in view of the uniqueness in Theorem 1.5.

Remark 1.7 A natural question is whether there exist E_∞ complex orientations for Morava E-theories and, more specifically, whether the orientation in Theorem 1.5 rigidifies to be an E_∞ map. See [Hopkins–Lawson 2018] for recent progress on E_∞ complex orientations, where the norm-coherence condition appears.

Finally, the expositions in [Rezk 2015] and [Rezk 2018, esp. § 4] provide some other perspectives. See also [Strickland, esp. § 29].

1.8 Outline of the paper In § 2, we recall some basic concepts from the theory of formal groups and homotopy theory, particularly quotient of formal groups (2.3), and set their notation.

In § 3, following a suggestion of Rezk, we introduce an enlarged category of formal groups (cf. [Katz–Mazur 1985, § 4.1]). This viewpoint will be helpful in clarifying deformations of Frobenius (5.2), descent of level structures (6.8, 6.11), the norm-coherence condition (6.19), and functoriality of norm coherence (7.21).

In § 4 and § 5, we give an account for the theorems of Lubin, Tate (4.12) and of Strickland (5.12) on deformations of formal groups. Our formulation follows Rezk's (e.g., in [Rezk 2014, § 4]) with an emphasis on formal group *laws* as well, for we are concerned with special coordinates. The purpose of these two sections is to provide a detailed exposition together with a precise setup which is crucial for the notion of norm coherence to follow in desired generality.

In § 6, we introduce the central notion of this paper, norm coherence (6.18–6.29), building on Ando's framework [Ando 1995, § 2]. We then generalize his theorem and prove Theorem 1.2 in § 7. Our main results are Proposition 7.1 and Theorem 7.22, the latter stated in a form suggested by Rezk.

In § 8, we discuss corresponding topological results for complex orientations, with (8.1) an introduction of further background on work of Ando, of Ando, Hopkins, Strickland, and of Ando, Strickland. In (8.3–8.15), we compare the setup for our results above with Ando, Hopkins, and Strickland's descent data and norm maps [Ando–Hopkins–Strickland 2004, Parts 1 and 3]. The purpose is to continue the exposition from § 5 while proving Theorem 1.5.

1.9 Acknowledgments The author learned most of what he knew about norm coherence and related questions from Charles Rezk. A good deal of the theory presented here was developed in discussions with him, including "norm coherent." The term is the author's choice over the synonym "Ando" and it is Matthew Ando who originally discovered this condition in algebra and applied it to topology.

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1.10 General conventions Unless otherwise indicated, a prime p is fixed throughout.

We often omit the symbol Spf and simply write R for $\mathrm{Spf}(R)$ when it appears as a base scheme. In particular, β^* means base change from R to S along $\beta\colon R \to S$, understood as $\beta\colon \mathrm{Spf}(S) \to \mathrm{Spf}(R)$.

We also write ψ^* for the pullback of functions along a morphism ψ of schemes.

Depending on the context, the symbol / stands for "over" (indicating the structure morphism of a scheme) or "modulo" (indicating a quotient).

More specific conventions are contained in (2.1) (twice) and (4.1) below, which apply to sections thereafter.

2 General notions

2.1 Formal groups, coordinates, and formal group laws Let R be a complete local ring with residue characteristic $p > 0$. A *formal group $\mathscr{G}$ over R* is a group object in the category of formal R-schemes, i.e., a pointed formal scheme

$$\mathscr{G} \to \mathrm{Spf}(R) \xrightarrow{0} \mathscr{G}$$

satisfying a set of group-like axioms, where 0 denotes the identity section with respect to the group law. It can be viewed as a covariant functor from the category of complete local R-algebras (and local homomorphisms) to the category of abelian groups.

Conventions In this paper, all formal groups will be commutative, one-dimensional, and affine. Let $\mathcal{O}_{\mathscr{G}}$ be the structure sheaf of $\mathscr{G}$. We will simply write $\mathcal{O}_{\mathscr{G}}$ for the ring $\Gamma(\mathscr{G}, \mathcal{O}_{\mathscr{G}})$ of global sections of $\mathcal{O}_{\mathscr{G}}$, and similarly for other sheaves.

A *coordinate X on $\mathscr{G}$* is a natural isomorphism $\mathscr{G} \xrightarrow{\sim} \hat{\mathbb{A}}^1 = \hat{\mathbb{A}}^1_R$ of functors to pointed sets. It gives an isomorphism $\Gamma(\mathscr{G}, \mathcal{O}_{\mathscr{G}}) \cong R[\![X]\!]$ of augmented R-algebras, as well as a trivialization of the ideal sheaf $\mathcal{I}_{\mathscr{G}}(0) = \mathcal{O}_{\mathscr{G}}(-0)$ of functions on $\mathscr{G}$ which vanish at the identity section 0.

A (*one-dimensional commutative*) *formal group law F over R* is a formal power series in two variables T_1 and T_2 with coefficients in R, often written $T_1 +_F T_2$, which satisfies a set of abelian-group-like axioms. In particular, the above data of $\mathscr{G}$ and X determines a formal group law G such that

$$X(P_1) \underset{G}{+} X(P_2) = X(P_1 \underset{\mathscr{G}}{+} P_2)$$

for any R-points P_1 and P_2 on $\mathscr{G}$ (where we identify an R-point on $\hat{\mathbb{A}}^1$ with an element in the maximal ideal of R). To be more explicit, the group map μ and the coordinate X give a map

$$\hat{\mathbb{A}}^1 \times \hat{\mathbb{A}}^1 \xleftarrow{\sim} \mathscr{G} \times \mathscr{G} \xrightarrow{\mu} \mathscr{G} \xrightarrow{\sim} \hat{\mathbb{A}}^1$$

of formal schemes. If $\hat{\mathbb{A}}^1 = \mathrm{Spf}(R[\![T]\!])$ and $\hat{\mathbb{A}}^1 \times \hat{\mathbb{A}}^1 = \mathrm{Spf}(R[\![T_1, T_2]\!])$, then these data determine the formal group law $G(T_1, T_2) = \mu^*(T)$. Conversely, given a formal group law F, it determines a formal group $\mathscr{F} = \mathrm{Spf}(R[\![X_F]\!])$ in a similar way.

Conventions Given the above relationship between a formal group $\mathscr{G}$, a coordinate X, and a formal group law G, we will sometimes write a pair $(\mathscr{G}, X)$ for a corresponding formal group law. Here and throughout the paper, as a visual reminder for the reader, we use calligraphic letters to denote formal groups and plain letters for formal group laws.

2.2 Subgroups and isogenies By (*finite*) *subgroups* of a formal group over R, we mean finite flat closed subgroup schemes. Their points are often defined over an extension $\widetilde{R}$ of R.

An *isogeny* $\psi\colon \mathscr{G} \to \mathscr{G}'$ over R is a finite flat morphism of formal groups. Along ψ^*, $\mathcal{O}_{\mathscr{G}}$ becomes a free $\mathcal{O}_{\mathscr{G}'}$-module of finite rank d, called the *degree of* ψ. Since the residue characteristic of R is p, d must be a power of p.

Suppose X and X' are coordinates on $\mathscr{G}$ and $\mathscr{G}'$. Then ψ induces a homomorphism $(\mathscr{G}, X) \to (\mathscr{G}', X')$ of formal group laws, i.e., $h(T) \in T \cdot R[\![T]\!]$ such that

$$h(T_1 \underset{G}{+} T_2) = h(T_1) \underset{G'}{+} h(T_2)$$

More explicitly, the composite

$$\hat{\mathbb{A}}^1 \xleftarrow{\sim} \mathscr{G} \xrightarrow{\psi} \mathscr{G}' \xrightarrow{\sim} \hat{\mathbb{A}}^1$$

together with the identities $\hat{\mathbb{A}}^1 = \mathrm{Spf}(R[\![T]\!])$ on the source and $\hat{\mathbb{A}}^1 = \mathrm{Spf}(R[\![T']\!])$ on the target determine the homomorphism $h(T) = \psi^*(T')$. Thus we will sometimes abuse notation by writing ψ for h, denote this homomorphism of formal group laws by $\psi\colon G \to G'$, and say it is an isogeny of degree d (cf. [Lubin 1967, 1.6]). By Weierstrass preparation, $h = mn$ with $m \in R[T]$ monic of degree d and $n \in R[\![T]\!]$ invertible.

2.3 Kernels and quotients The notions of subgroups and of isogenies are connected as follows.

Given $\psi\colon \mathscr{G} \to \mathscr{G}'$ as above, its *kernel* $\mathscr{K}$ is defined by $\mathcal{O}_{\mathscr{K}} = \mathcal{O}_{\mathscr{G}} \otimes_{\mathcal{O}_{\mathscr{G}'}} R$, where the tensor product is taken along ψ^* and the augmentation map of $\mathcal{O}_{\mathscr{G}'}$. It is naturally a subgroup of $\mathscr{G}$ and has degree d as an effective Cartier divisor in $\mathscr{G}$.

Conversely, given a subgroup $\mathscr{D} \subset \mathscr{G}$ over $\widetilde{R}$ of degree p^r, there is a corresponding isogeny $f_{\mathscr{D}}\colon \mathscr{G} \to \mathscr{G}/\mathscr{D}$ defined by an equalizer diagram

$$\mathcal{O}_{\mathscr{G}/\mathscr{D}} \xrightarrow{f_{\mathscr{D}}^*} \mathcal{O}_{\mathscr{G}} \underset{\pi^*}{\overset{\mu^*}{\rightrightarrows}} \mathcal{O}_{\mathscr{G}\times\mathscr{D}}$$

where $\mu, \pi\colon \mathscr{G} \times \mathscr{D} \to \mathscr{G}$ are the group, projection maps, and $\mathscr{G}/\mathscr{D}$ is naturally a formal group over $\widetilde{R}$. Moreover, given a coordinate X on $\mathscr{G}$,

$$X_{\mathscr{D}} \coloneqq \mathrm{Norm}_{f_{\mathscr{D}}^*}(X)$$

is a coordinate on $\mathscr{G}/\mathscr{D}$, where $\mathrm{Norm}_{f_{\mathscr{D}}^*}(X)$ equals the determinant of multiplication by X on $\mathcal{O}_{\mathscr{G}}$ as a finite free $\mathcal{O}_{\mathscr{G}/\mathscr{D}}$-module via $f_{\mathscr{D}}^*$. Explicitly,

$$f_{\mathscr{D}}^*(X_{\mathscr{D}}) = \prod_{Q\in\mathscr{D}(\widetilde{R})} \big(X \underset{G}{+} X(Q)\big) \tag{2.4}$$

By writing

$$f_D\colon G \to G/D$$

as an isogeny of formal group laws, we will always intend the above compatibility between corresponding coordinates. Sometimes we write more specifically

$$(f_{\mathscr{D}}, X)\colon (\mathscr{G}, X) \to (\mathscr{G}, X)/D \coloneqq (\mathscr{G}/\mathscr{D}, X_{\mathscr{D}})$$

Note that over the residue field of R, (2.4) becomes

$$f_{\mathscr{D}}^*(X_{\mathscr{D}}) = X^{p^r} \tag{2.5}$$

as a formal group over a field of characteristic p has exactly one subgroup of degree p^r. Thus $f_{\mathscr{D}}$ is a lift of the relative p^r-power Frobenius isogeny.

For more details, see [Lubin 1967, § 1, esp. Theorems 1.4 and 1.5], [Strickland 1997, § 5, esp. Theorem 19] (cf. Remark 8.15 below), and [Ando 1995, §§ 2.1–2.2].

2.6 Complex cobordism and orientations Let $MU\langle 0\rangle$ be the Thom spectrum of the tautological (virtual) complex vector bundle over $\mathbb{Z}\times BU$. We have

$$\pi_*(MU\langle 0\rangle)\cong \pi_*(MU)[\beta^{\pm 1}]$$

with $|\beta| = 2$. More generally, let $MU\langle 2k\rangle$ be the Thom spectrum associated to the $(2k-1)$-connected cover $BU\langle 2k\rangle \to \mathbb{Z}\times BU$.

The spectrum $MU\langle 0\rangle$ is often written MUP or MP for "periodic" (as can be seen from its homotopy groups). In fact, $MU\langle 0\rangle$ is the Thom spectrum associated to virtual bundles of any rank, while MU is the Thom spectrum associated to virtual bundles of rank 0, so that

$$MU\langle 0\rangle = \bigvee_{m\in\mathbb{Z}} \Sigma^{2m} MU$$

Thus $\pi_0(MU\langle 0\rangle)$ is the ring of cobordism classes of even dimensional stably almost complex manifolds. The spectrum $MU\langle 2\rangle = MU$. The homology of $MU\langle 2k\rangle$ is concentrated in even degrees if $0\le k\le 3$.

Let E be an even periodic ring spectrum. The formal scheme

$$\mathscr{G}_E := \operatorname{Spf}\bigl(E^0(\mathbb{C}P^\infty)\bigr)$$

is naturally a formal group over $E^0(\text{point}) = \pi_0(E)$.

An $MU\langle 0\rangle$*-orientation* for E is a map $g\colon\ MU\langle 0\rangle \to E$ of homotopy commutative ring spectra. Consider the natural map

$$\mathbb{C}P^\infty_+ \to (\mathbb{C}P^\infty)^{\mathscr{L}} \to \Sigma^2 MU \to MU\langle 0\rangle$$

where $\mathscr{L}$ is the tautological line bundle over $\mathbb{C}P^\infty$. Composing with this map, each $MU\langle 0\rangle$-orientation g gives an element $X_g\in E^0(\mathbb{C}P^\infty)$ whose restriction to the bottom cell is a generator (because $(\mathbb{C}P^\infty)^{\mathscr{L}} \to MU\langle 0\rangle \to E$ is a Thom class), and so induces an isomorphism $E^0(\mathbb{C}P^\infty)\cong \pi_0(E)[\![X_g]\!]$. Thus, from an $MU\langle 0\rangle$-orientation $g\colon\ MU\langle 0\rangle\to E$, this procedure produces a coordinate X_g on $\mathscr{G}_E$, and hence a formal group law $(\mathscr{G}_E, X_g)$ over $\pi_0(E)$. In particular, taking $E = MU\langle 0\rangle$ with its $MU\langle 0\rangle$-orientation the identity map, we obtain over $\pi_0(MU\langle 0\rangle)$ the universal formal group law of Lazard (see [Quillen 1969, Theorem 2] and [Adams 1974, Part II]).

An MU*-orientation* (or *complex orientation*) for E is a ring map $g\colon MU \to E$. Composing this with $(\mathbb{C}P^\infty)^{\mathscr{L}-1}\to MU$ gives an element

$$\xi_g\in E^0\bigl((\mathbb{C}P^\infty)^{\mathscr{L}-1}\bigr)\cong E^2\bigl((\mathbb{C}P^\infty)^{\mathscr{L}}\bigr)\cong \widetilde{E}^2(\mathbb{C}P^\infty)$$

whose restriction to $\widetilde{E}^2(S^2) \cong \widetilde{E}^0(S^0)$ is 1, since the following diagram commutes.

$$\begin{tikzcd}
(\mathbb{C}P^\infty)^{\mathscr{L}-1} \arrow[rr] & & MU \\
S^0 \arrow[u] \arrow[urr, "\eta"'] & &
\end{tikzcd}$$

Thus this procedure produces an "orientation" in the sense of [Adams 1974, Part II, 2.1] from an MU-orientation $g\colon MU \to E$.

In fact, the procedures from the previous two paragraphs induce bijections between orientations and corresponding objects [Ando–Hopkins–Strickland 2001, Corollary 2.50, Examples 2.51 and 2.52] (cf. [Ando 2000, Proposition 1.10 (ii)] and [Adams 1974, Part II, Lemma 4.6]).

2.7 Morava E-theories We now specialize the setup from the previous section to a family of cohomology theories. Let k be a perfect field of characteristic p, and $\mathscr{G}$ be a formal group over k of finite height n. Associated to this data, there is a generalized cohomology theory, called a *Morava E-theory (of height n at the prime p)*. It is represented by an even periodic ring spectrum $E = E^{\mathscr{G}}$. The above association has the property that the formal group $\mathscr{G}_E = \operatorname{Spf}\big(E^0(\mathbb{C}P^\infty)\big)$ is a universal deformation of $\mathscr{G}$ in the sense of Lubin and Tate (see § 4 below). We have

$$\pi_*(E) \cong W(k)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$$

where $|u_i| = 0$ and $|u| = 2$.[1]

Thus a Morava E-theory spectrum is a topological realization of a Lubin–Tate ring. Strickland showed that $E^0(B\Sigma_{p^r})/I_{\mathrm{tr}}$ is a finite free module over $\pi_0(E)$, where I_{tr} is the ideal generated by the images of transfers from proper subgroups of the symmetric group Σ_{p^r} on p^r letters. Moreover, this ring classifies degree-p^r subgroups of $\mathscr{G}_E$ [Strickland 1998, Theorem 1.1] (see § 5). Ando, Hopkins, and Strickland then assembled these into a topological realization of descent data for level structures on $\mathscr{G}_E$ in [Ando–Hopkins–Strickland 2004, § 3.2] (see § 8).

When $\mathscr{G}$ is the formal group of a supersingular elliptic curve, its corresponding E-theory (of height 2) is an *elliptic cohomology theory* [Ando–Hopkins–Strickland 2001, Definition 1.2] via the Serre–Tate theorem.

[1] For some purposes, it is convenient to instead have $W(\bar{k})$ or $|u| = -2$ in $\pi_*(E)$.

2.8 E_∞ and H_∞ structures Let Sp be a complete and cocomplete category of spectra, indexed over some universe, with an associative and commutative smash product $\wedge$ (e.g., the category of $\mathbb{L}$-spectra in [Elmendorf–Kriz–Mandell–May 1997, Chapter I]).

An *E_∞-ring spectrum* is a commutative monoid in Sp. Equivalently, it is an algebra for the monad $\mathbb{D}$ on Sp defined by

$$\mathbb{D}(-) \coloneqq \bigvee_{m \geq 0} \mathbb{D}_m(-) \coloneqq \bigvee_{m \geq 0} (-)^{\wedge m}/\Sigma_m$$

where Σ_m is the symmetric group on m letters acting on the m-fold smash product.

Weaker than being E_∞, an *H_∞-ring spectrum* is a commutative monoid in the homotopy category of Sp. It also has a description as an algebra for the monad which descends from $\mathbb{D}$ to the homotopy category. There are power operations D_m on the homotopy groups of such a spectrum (see [Bruner–May–McClure–Steinberger 1986, Chapter I]).

Complex cobordism MU and its variants above are E_∞-ring spectra [May 1977, § IV.2]. Morava E-theories E are also E_∞-ring spectra [Goerss–Hopkins 2004, Corollary 7.6]. A morphism of E_∞-ring (or H_∞-ring) spectra is called an E_∞ (or H_∞) *map*.

3 Categories of formal groups

3.1 The category FG and its subcategories Consider FG whose objects are formal groups $\mathscr{G} \xrightarrow{f} \mathrm{Spf}(k)$ of finite height over variable base fields of characteristic p, and whose morphisms are commutative squares

(3.2)
$$\begin{tikzcd}
\mathscr{G} \arrow[r, "\Psi"] \arrow[d, "f"'] & \mathscr{G}' \arrow[d, "f'"] \\
\mathrm{Spf}(k) \arrow[r, "\beta"] & \mathrm{Spf}(k')
\end{tikzcd}$$

of formal schemes such that the induced morphism of k-schemes

(3.3)
$$\mathscr{G} \xrightarrow{(\Psi, f)} \mathscr{G}' \underset{k'}{\times} k$$

is a homomorphism of formal groups over k.

We also have subcategories $\mathrm{FG}_{\mathrm{isog}}$ and $\mathrm{FG}_{\mathrm{iso}}$ when (3.3) is restricted to be an isogeny or isomorphism. Write $\mathrm{FG}(k)$, $\mathrm{FG}_{\mathrm{isog}}(k)$, and $\mathrm{FG}_{\mathrm{iso}}(k)$ for the subcategories where the base field is fixed and $\beta = \mathrm{id}$ in (3.2). In contrast to these subcategories, we think of FG, $\mathrm{FG}_{\mathrm{isog}}$, and $\mathrm{FG}_{\mathrm{iso}}$ as "wide" categories because of the factorization

(3.4)
$$\begin{tikzcd}
\mathscr{G} \arrow[r] \arrow[d, "f"'] & \mathscr{G}' \times_{k'} k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{G}' \arrow[d, "f'"] \\
\mathrm{Spf}(k) \arrow[r, no head] & \mathrm{Spf}(k) \arrow[r, "\beta"] & \mathrm{Spf}(k')
\end{tikzcd}$$

Example 3.5 (Frobenius endomorphisms in $\mathrm{FG}_{\mathrm{isog}}$) For our purpose, a key example of morphisms in FG is the following, where σ is the absolute p-power Frobenius and Frob is the relative one.

(3.6)
$$\begin{tikzcd}
\mathscr{G} \arrow[r, "\mathrm{Frob}"] \arrow[rr, bend left, "\sigma"] \arrow[d, "f"'] & \mathscr{G}^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{G} \arrow[d, "f"] \\
\mathrm{Spf}(k) \arrow[r, no head] & \mathrm{Spf}(k) \arrow[r, "\sigma"] & \mathrm{Spf}(k)
\end{tikzcd}$$

This is an endomorphism in $\mathrm{FG}_{\mathrm{isog}}$ on the object $\mathscr{G}/k$. Denote it by Φ. It is not a morphism in $\mathrm{FG}_{\mathrm{isog}}(k)$. The composite Φ^r corresponds to the p^r-power Frobenius.

3.7 Canonical factorization of Φ^r along an isogeny We begin with the following observation.

Lemma 3.8 *Any $\psi\colon \mathscr{G} \to \mathscr{G}'$ in $\mathrm{FG}_{\mathrm{isog}}(k)$, necessarily of degree p^r for some $r \geq 0$, has the same kernel as the relative p^r-power Frobenius $\mathrm{Frob}^r\colon \mathscr{G} \to \mathscr{G}^{(p^r)}$.*

Proof Since its base k is a field of characteristic p, the formal group $\mathscr{G}$ has a unique subgroup of degree p^r, namely, the divisor $p^r[0] = \mathrm{Spf}(\mathcal{O}_{\mathscr{G}}/X^{p^r})$, where X is a coordinate on $\mathscr{G}$ (see (2.3)). $\square$

Thus, given such an isogeny ψ as in the lemma, there is a unique factorization in $\mathrm{FG}_{\mathrm{isog}}$ of Φ^r along ψ as follows, where $\Phi^r = \Lambda_\psi \circ \psi$ with Λ_ψ in $\mathrm{FG}_{\mathrm{iso}}$.

(3.9)
$$\begin{tikzcd}
\mathscr{G} \arrow[r, "\psi"] \arrow[d] & \mathscr{G}' \arrow[r, "\sim"] \arrow[d] \arrow[rr, bend left, "\Lambda_\psi"] & \mathscr{G}^{(p^r)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{G} \arrow[d] \\
\mathrm{Spf}(k) \arrow[r, equal] & \mathrm{Spf}(k) \arrow[r, equal] & \mathrm{Spf}(k) \arrow[r, "\sigma^r"] & \mathrm{Spf}(k)
\end{tikzcd}$$

The canonical isomorphisms from $\mathscr{G}/(p^r[0])$ to $\mathscr{G}'$ and $\mathscr{G}^{(p^r)}$ give the unique isomorphism $\mathscr{G}' \xrightarrow{\sim} \mathscr{G}^{(p^r)}$ in this diagram.

Correspondingly, between rings of functions on the formal groups, we have

(3.10)
$$\begin{tikzcd}
\mathcal{O}_{\mathscr{G}} & \mathcal{O}_{\mathscr{G}'} \arrow[l, "\psi^*"'] & \mathcal{O}_{\mathscr{G}} \arrow[l, "\Lambda_\psi^*"'] \arrow[ll, bend left, "\sigma^r"]
\end{tikzcd}$$

Next, recall from linear algebra that the *norm* of a matrix is its determinant. It is multiplicative and, together with the trace, appears as a coefficient of the characteristic polynomial of the matrix and its corresponding linear transformation.

In our context, let R be a commutative ring and S be an R-algebra along the structure map $f\colon R \to S$. Suppose this makes S free of finite rank as an R-module. Define the *norm* (*map*)
$$\mathrm{Norm}_f\colon\ S \to R$$
by sending $s \in S$ to the determinant of multiplication by s as an R-linear transformation on S. Observe that it is functorial with respect to restriction of scalars, i.e., given $g\colon R' \to R$ with $f' := f \circ g$, we have $\mathrm{Norm}_f = g \circ \mathrm{Norm}_{f'}$.

Proposition 3.11 *Given* (3.9)*, the map* Λ_ψ^* *coincides with the norm* $\mathrm{Norm}_{\psi^*}\colon \mathcal{O}_{\mathscr{G}} \to \mathcal{O}_{\mathscr{G}'}$ *of* $\mathcal{O}_{\mathscr{G}}$ *as a finite free module over* $\mathcal{O}_{\mathscr{G}'}$ *along* ψ^*.

Proof First, consider the case of $\psi = \mathrm{Frob}^r$. Let $\mathcal{O}_{\mathscr{G}} = k[\![X]\!]$ and $\mathcal{O}_{\mathscr{G}'} = k[\![X']\!]$ with

$\psi^*(X') = X^{p^r}$. Let X_i be the roots of the minimal polynomial of X over $\mathcal{O}_{\mathscr{G}'}$. We have

$$\mathrm{Norm}_{\psi^*}(X) = \prod_{i=1}^{p^r} X_i = (-1)^{p^r+1} X' = X' \tag{3.12}$$

and $\mathrm{Norm}_{\psi^*}(c) = c^{p^r}$ for $c \in k$. Note that in characteristic p, the norm map is additive and hence a local homomorphism. Thus composing with the k-linear map ψ^*, it becomes the absolute p^r-power Frobenius σ^r as follows, where $h^{(p^r)}$ is the series obtained by twisting the coefficients of h with the p^r-power Frobenius (cf. (3.10) and also [Stacks 2020, Tag 0BCX]).

$$\begin{tikzcd} k[\![X]\!] & k[\![X']\!] \arrow[l, "\psi^*"'] & k[\![X]\!] \arrow[l, "\mathrm{Norm}_{\psi^*}"'] \\ h^{(p^r)}(X^{p^r}) & h^{(p^r)}(X') \arrow[l, mapsto] & h(X) \arrow[l, mapsto] \end{tikzcd} \tag{3.13}$$

The claim then follows by the uniqueness of the factorization (3.9).

In general, consider two such isogenies ψ_1 and ψ_2 out of $\mathscr{G}$, with $\psi_1 = \iota \circ \psi_2$ for a unique k-isomorphism ι. We need only observe that

$$\psi_1^* \circ \mathrm{Norm}_{\psi_1^*} = \psi_2^* \circ \iota^* \circ \mathrm{Norm}_{\psi_1^*} = \psi_2^* \circ \mathrm{Norm}_{\psi_2^*} \tag{3.14}$$

where the second equality is by restriction of scalars along ι^*. With $\psi_1 = \mathrm{Frob}^r$ from above, the claim for ψ_2 follows. □

4 Deformations of formal groups

4.1 Setup Let k be a field of characteristic $p > 0$, and $\mathscr{G}$ be a formal group over k of height $n < \infty$. Let R be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} \supset k$, and let $\pi\colon R \to R/\mathfrak{m}$ be the natural projection.

Conventions To ease notation, for complete local rings and for objects and morphisms over them, we will often use subscript 0 to indicate restriction to the special fiber. For example, $R_0 := R/\mathfrak{m}$, and $\mathscr{F}_0 := \pi^*\mathscr{F}$ if $\mathscr{F}$ is a formal group over R.

Recall the category $\mathrm{FG}_{\mathrm{iso}}$ from (3.1). Let us consider a similar category $\widetilde{\mathrm{FG}}_{\mathrm{iso}}$ where the formal groups are over complete local rings instead of fields only. In this category, an object is a formal group over a complete local ring of the form $\mathscr{F} \to \mathrm{Spf}(R)$ and a morphism is a pullback diagram

(4.2)
$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\Psi"] \arrow[d] & \mathscr{F}' \arrow[d] \\
\mathrm{Spf}(R) \arrow[r, "\alpha"] & \mathrm{Spf}(R')
\end{tikzcd}$$

i.e., the induced map $\mathscr{F} \to \alpha^* \mathscr{F}'$ is an isomorphism of formal groups over R.

4.3 Deformations and deformation structures Fix $\mathscr{G}/k$ from (4.1). A *deformation* $(\mathscr{F}, i, \eta)$ *of* $\mathscr{G}$ is a diagram in $\widetilde{\mathrm{FG}}_{\mathrm{iso}}$ of the form

(4.4)
$$\begin{tikzcd}
\mathscr{F} \arrow[d] & \mathscr{F}_0 \arrow[l] \arrow[r, "\eta"] \arrow[d] & \mathscr{G} \arrow[d] \\
\mathrm{Spf}(R) & \mathrm{Spf}(R_0) \arrow[l, "\pi"'] \arrow[r, "i"] & \mathrm{Spf}(k)
\end{tikzcd}$$

where $\mathscr{F}_0 = \pi^* \mathscr{F}$ in the morphism on the left. A morphism of deformations $(\mathscr{F}, i, \eta) \to (\mathscr{F}, i', \eta')$ is a commutative diagram (in $\widetilde{\mathrm{FG}}_{\mathrm{iso}}$) of the form

(4.5)
$$\begin{tikzcd}
\mathscr{F} \arrow[d, "\Psi"'] & \mathscr{F}_0 \arrow[l] \arrow[r, "\eta"] \arrow[d, "\Psi_0"'] & \mathscr{G} \\
\mathscr{F}' & \mathscr{F}_0' \arrow[l] \arrow[ur, "\eta'"'] &
\end{tikzcd}$$

over one of the form

$$\begin{tikzcd} \mathrm{Spf}(R) \arrow[d, "\alpha"'] & \mathrm{Spf}(R_0) \arrow[l] \arrow[r, "i"] \arrow[d, "\alpha_0"'] & \mathrm{Spf}(k) \\ \mathrm{Spf}(R') & \mathrm{Spf}(R_0') \arrow[l] \arrow[ur, "i'"'] & \end{tikzcd}$$

It is convenient to display the two pieces of data Ψ and α simply as (4.2). Write $\mathrm{Def}_{\mathrm{iso}}^{\mathscr{G}}$ for this category of deformations of $\mathscr{G}$.

Given a deformation $(\mathscr{F}, i, \eta)$ in $\mathrm{Def}_{\mathrm{iso}}^{\mathscr{G}}$, we call the pair (i, η) a *deformation structure attached to* $\mathscr{F}$ *with respect to* $\mathscr{G}/k$, and may simply call $\mathscr{F}$ a deformation of $\mathscr{G}$ if its deformation structure is understood.

Example 4.6 (Change of bases) Let $\beta\colon R \to S$ be a local homomorphism. Given a deformation $(\mathscr{F}, i, \eta)$ of $\mathscr{G}$ to R, writing $i' := \beta_0 \circ i$ (as ring homomorphisms), we obtain an induced deformation $(\beta^*\mathscr{F}, i', \eta)$ to S by base change along β,[2] together with an induced morphism in $\mathrm{Def}_{\mathrm{iso}}^{\mathscr{G}}$. In terms of deformation structures, we write $\beta^*(i, \eta) := (i', \eta)$.

4.7 $\star$-isomorphisms Upon choosing coordinates (see (2.1)), the definitions in (4.3) translate directly for formal group laws. Let (F, i, η) and (F', i', η') be deformations of G/k to R. A *$\star$-isomorphism* $(F, i, \eta) \to (F', i', \eta')$ consists of an equality $i = i'$ and an isomorphism $\psi\colon F \xrightarrow{\sim} F'$ of formal group laws over R such that $\eta' \circ \psi_0 = \eta$, as in the following commutative diagram over $\mathrm{Spf}(R) \leftarrow \mathrm{Spf}(R_0) \xrightarrow{i} \mathrm{Spf}(k)$.

(4.8)
$$\begin{tikzcd} F \arrow[d, "\psi"'] & F_0 \arrow[l] \arrow[r, "\eta"] \arrow[d, "\psi_0"'] & G \\ F' & F_0' \arrow[l] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

[2]Here, as an abuse of notation, we also denote by η the pullback of η along β_0.

We simply call $\psi\colon F \to F'$ a $\star$-*isomorphism* if in addition $\eta = \eta'$ so that $\psi_0 = \mathrm{id}$. We use the symbol $\overset{\star}{=}$ for this equivalence relation. Clearly it is preserved under base change.

Example 4.9 (Change of coordinates) Let $\mathscr{F}/R$ be a formal group with a deformation structure (i, η) with respect to $\mathscr{G}/k$. A change of coordinates $X \mapsto X'$ on $\mathscr{F}$ over R results in a $\star$-isomorphism $\big((\mathscr{F}, X), i, \eta\big) \to \big((\mathscr{F}, X'), i, \eta'\big)$. In particular, if X and X' are lifts of the same coordinate on $\mathscr{F}_0$ and $X \mapsto X'$ restricts to the identity map, then $\eta = \eta'$ and so this $\star$-isomorphism gives one between formal group laws.

4.10 Classification of deformations The Lubin–Tate theorem classifies deformations of formal group laws up to $\star$-isomorphisms [Lubin–Tate 1966, Theorem 3.1].

Let $\mathrm{Def}^{\mathscr{G}}_{\mathrm{iso}}$ be the category of deformations of $\mathscr{G}/k$ from (4.3). Strickland's coordinate-free reformulation of the Lubin–Tate theorem states that $\mathrm{Def}^{\mathscr{G}}_{\mathrm{iso}}$ has a terminal object [Strickland 1997, Proposition 20] (cf. [Strickland 1997a, Proposition 6.1]). Specifically, there is a formal group $\mathscr{F}_{\mathrm{univ}} \to \mathrm{Spf}(E)$ with $E = E^{\mathscr{G}} := W(k)[\![u_1, \ldots, u_{n-1}]\!]$ which participates in a deformation

$$\begin{tikzcd}
\mathscr{F}_{\mathrm{univ}} \arrow[d] & (\mathscr{F}_{\mathrm{univ}})_0 \arrow[l] \arrow[r, dash] \arrow[d] & \mathscr{G} \arrow[d] \\
\mathrm{Spf}(E) & \mathrm{Spf}(E_0) \arrow[l] \arrow[r, dash] & \mathrm{Spf}(k)
\end{tikzcd}$$

such that, for any deformation $(\mathscr{F}, i, \eta)$ as in (4.4), there is a unique morphism of deformations

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\Psi"] \arrow[d] & \mathscr{F}_{\mathrm{univ}} \arrow[d] \\
\mathrm{Spf}(R) \arrow[r, "\alpha"] & \mathrm{Spf}(E)
\end{tikzcd}$$

Remark 4.11 As a universal object, $(\mathscr{F}_{\mathrm{univ}}, \mathrm{id}, \mathrm{id})$ is unique up to unique isomorphism. Moreover, $\mathscr{F}_{\mathrm{univ}} = \mathscr{F}^{\mathscr{G}}_{\mathrm{univ}}$ is functorial with respect to $\mathscr{G}/k$ as an object in $\mathrm{FG}_{\mathrm{isog}}$.

The following is a generalized and reformulated version of the Lubin–Tate theorem (cf. [Ando 1995, Theorem 2.3.1] and [Rezk 2014, Proposition 4.2]). When η in the proposition is allowed to be the identity only, this is the Lubin–Tate theorem. The general statement, necessary for our construction of special coordinates in later sections, follows directly (see (2.1)) from Strickland's coordinate-free formulation above.

Corollary 4.12 *Let k and R be as in (4.1) and fix a formal group law G/k of height $n < \infty$. Then the functor*

$$R \mapsto \{\star\text{-isomorphism classes of deformations } (F, i, \eta) \text{ of } G \text{ to } R\}$$

from the category of complete local rings with residue field containing k to the category of sets is co-represented by the ring $E = E^G := W(k)[\![u_1, \ldots, u_{n-1}]\!]$.

Explicitly, there is a deformation $(F_{\mathrm{univ}}, \mathrm{id}, \mathrm{id})$ to E satisfying the following universal property. Given any deformation (F, i, η) of G to R, there is a unique local homomorphism

$$\alpha\colon\ E \to R$$

such that it reduces to $i\colon\ k = E_0 \to R_0$ and that there is a unique $\star$-isomorphism

$$(F, i, \eta) \to (\alpha^* F_{\mathrm{univ}}, i, \mathrm{id}) \tag{4.13}$$

5 Deformations of Frobenius

The flexibility of having an isomorphism η in a deformation of a formal group (law) buys us a notion of pushforward of deformation structures along *any* isogeny, compatible with Frobenius in a precise way.

We continue the setup and conventions in (4.1).

5.1 Pushforward of deformation structures along an isogeny Let $(\mathscr{F}, i, \eta)$ be a deformation of $\mathscr{G}$ to R. Let $\psi\colon\ \mathscr{F} \to \mathscr{F}'$ be an isogeny of formal group over R of degree p^r.

Applying Lemma 3.8 to the restriction ψ_0 over the special fiber, we see that $\mathscr{F}'$ can be endowed with a deformation structure (i', η') such that the following diagrams commute, where σ is the absolute p-power Frobenius and Frob is the relative one (cf. [Strickland 1997, § 13]).

(5.2)
$$\begin{tikzcd}
\mathscr{F} \arrow[d, "\psi"'] & \mathscr{F}_0 \arrow[l] \arrow[r, "\eta"] \arrow[d, "\psi_0"'] & i^*\mathscr{G} \arrow[r] \arrow[d, "i^*\mathrm{Frob}^r"'] & \mathscr{G} \arrow[d, "\mathrm{Frob}^r"'] \\
 & & i^*\mathscr{G}^{(p^r)} \arrow[r] \arrow[d, equal] & \mathscr{G}^{(p^r)} \arrow[d] \\
\mathscr{F}' & \mathscr{F}'_0 \arrow[l] \arrow[r, "\eta'"] & i'^*\mathscr{G} \arrow[r] & \mathscr{G}
\end{tikzcd}$$

over

$$\begin{tikzcd}
\mathrm{Spf}(R) \arrow[dd, equal] & \mathrm{Spf}(R_0) \arrow[l] \arrow[r, equal] \arrow[dd, equal] & \mathrm{Spf}(R_0) \arrow[r, "i"] \arrow[d, equal] & \mathrm{Spf}(k) \arrow[d, equal] \\
 & & \mathrm{Spf}(R_0) \arrow[r, "i"] \arrow[d, equal] & \mathrm{Spf}(k) \arrow[d, "\sigma^r"] \\
\mathrm{Spf}(R) & \mathrm{Spf}(R_0) \arrow[l] \arrow[r, equal] & \mathrm{Spf}(R_0) \arrow[r, "i'"] & \mathrm{Spf}(k)
\end{tikzcd}$$

We write $\psi_!(i,\eta) := (i',\eta')$ and call it the *pushforward of* (i,η) *along* ψ. Explicitly, the pair is determined by the equalities

$$i' = i \circ \sigma^r \qquad \text{and} \qquad \eta' \circ \psi_0 = i^*\mathrm{Frob}^r \circ \eta$$

Example 5.3 Let $\mathscr{D} \subset \mathscr{F}$ be a subgroup of degree p^r and $f_{\mathscr{D}}\colon \mathscr{F} \to \mathscr{F}/\mathscr{D}$ be the quotient map as in (2.3). Then $f_{\mathscr{D}\,!}(i,\eta) = (i \circ \sigma^r, \eta)$ by the proof of Lemma 3.8.

5.4 More categories of deformations Fix $\mathscr{G}/k$. Recall from (3.1) the categories $\mathrm{FG}_{\mathrm{iso}}$ and $\mathrm{FG}_{\mathrm{isog}}$ of formal groups over fields, and their subcategories $\mathrm{FG}_{\mathrm{iso}}(k)$ and $\mathrm{FG}_{\mathrm{isog}}(k)$ for a fixed base field k. We defined in (4.3) the category $\mathrm{Def}^{\mathscr{G}}_{\mathrm{iso}}$ of deformations (and isomorphisms). In view of (5.1), let us extend it to a category that allows isogenies, with the restriction of a fixed base ring.

Let $\mathrm{Def}^{\mathscr{G}}_{\mathrm{isog}}(R)$ be the category with objects deformations $(\mathscr{F}, i, \eta)$ of $\mathscr{G}$ to R and with morphisms $(\mathscr{F}, i, \eta) \to (\mathscr{F}', i', \eta')$, each consisting of an isogeny $\psi\colon \mathscr{F} \to \mathscr{F}'$ of formal groups over R and an equality $(i',\eta') = \psi_!(i,\eta)$. The degree of ψ must be p^r for some $r \geq 0$.

There is a corresponding category $\mathrm{Def}^{G}_{\mathrm{isog}}(R)$ with formal group laws instead of formal groups. Note that the isomorphisms in this category are precisely the $\star$-isomorphisms (cf. (4.8), when $r = 0$) and that the only automorphism of an object is the identity by the uniqueness in Corollary 4.12.

5.5 Deformations of Frobenius Given the diagram (5.2), we view a morphism $(\mathscr{F}, i, \eta) \to (\mathscr{F}', i', \eta')$ in $\mathrm{Def}^{\mathscr{G}}_{\mathrm{isog}}(R)$ as a deformation to R of Φ^r in the category $\mathrm{FG}_{\mathrm{isog}}$ (3.5). Thus, we call it a *deformation of Frobenius*, and simply call $\psi\colon \mathscr{F} \to \mathscr{F}'$ such if $\eta = \eta'$ (in the same sense of the footnote in (4.6), with $i' = \sigma^r \circ i$, σ being the absolute Frobenius on R_0) so that ψ_0 is a relative Frobenius (cf. [Rezk 2009, 11.3]).

In terms of formal group laws, we say that two deformations of Frobenius $(F_1, i_1, \eta_1) \to (F'_1, i'_1, \eta'_1)$ and $(F_2, i_2, \eta_2) \to (F'_2, i'_2, \eta'_2)$ are *isomorphic*, if (F_1, i_1, η_1) and (F_2, i_2, η_2) are $\star$-isomorphic and if (F'_1, i'_1, η'_1) and (F'_2, i'_2, η'_2) are $\star$-isomorphic.

5.6 Classification of deformations of Frobenius Extending (4.10), we now cast [Strickland 1997, Theorem 42] as a generalization of the Lubin–Tate theorem formulated in Proposition 20 there (cf. its § 13 as well).

For each $r \geq 0$, let $\mathrm{DefFrob}^{r\,\mathscr{G}}_{\mathrm{iso}}$ denote the following category of deformations of the p^r-power Frobenius Φ^r on $\mathscr{G}/k$. An object in $\mathrm{DefFrob}^{r\,\mathscr{G}}_{\mathrm{iso}}$ is a deformation of Frobenius to a complete local ring of the form (5.2), abbreviated $(\mathscr{F} \to \mathscr{F}')/R$. A morphism from $(\mathscr{F}_1 \to \mathscr{F}'_1)/R_1$ to $(\mathscr{F}_2 \to \mathscr{F}'_2)/R_2$ is a pair of pullback diagrams of the form (4.2)

$$\begin{tikzcd} \mathscr{F}_1 \arrow[r, "\Psi"] \arrow[d] & \mathscr{F}_2 \arrow[d] \\ \mathrm{Spf}(R_1) \arrow[r, "\alpha"] & \mathrm{Spf}(R_2) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F}'_1 \arrow[r, "\Psi'"] \arrow[d] & \mathscr{F}'_2 \arrow[d] \\ \mathrm{Spf}(R_1) \arrow[r, "\alpha"] & \mathrm{Spf}(R_2) \end{tikzcd}$$

compatible with deformation structures (cf. (4.5)), such that the following diagram of formal groups over R_1 commutes.

$$\begin{tikzcd} \mathscr{F}_1 \arrow[r, "\sim"] \arrow[d] & \alpha^*\mathscr{F}_2 \arrow[d] \\ \mathscr{F}'_1 \arrow[r, "\sim"] & \alpha^*\mathscr{F}'_2 \end{tikzcd}$$

Strickland's theorem gives a terminal object for each $\mathrm{DefFrob}^{r}{}^{\mathscr{G}}_{\mathrm{iso}}$ from the terminal object $(\mathscr{F}_{\mathrm{univ}}, \mathrm{id}, \mathrm{id})$ over E of $\mathrm{Def}^{\mathscr{G}}_{\mathrm{iso}}$ in (4.10) as follows.

Let $\mathrm{Sub}_r(\mathscr{F}_{\mathrm{univ}})$ be the affine formal scheme over E from [Strickland 1997, Theorem 42] that classifies degree-p^r subgroups of $\mathscr{F}_{\mathrm{univ}}$, and let A^r be its ring of functions. Observe that $A^0 = E$. The structure morphism $A^0 \to A^r$ of $\mathrm{Sub}_r(\mathscr{F}_{\mathrm{univ}})/A^0$ reduces to the identity between residue fields (see [Strickland 1997, § 13]). Thus $\mathscr{F}_{\mathrm{univ}} \times_{A^0} A^r$ inherits the deformation structure $(\mathrm{id}, \mathrm{id})$ from $\mathscr{F}_{\mathrm{univ}}$ along the base change. Let $\mathscr{D}^{(p^r)}_{\mathrm{univ}} \subset \mathscr{F}_{\mathrm{univ}} \times_{A^0} A^r$ be the subgroup of degree p^r classified by $\mathrm{id}\colon A^r \to A^r$, and let $\mathscr{F}^{(p^r)}_{\mathrm{univ}} := (\mathscr{F}_{\mathrm{univ}} \times_{A^0} A^r)/\mathscr{D}^{(p^r)}_{\mathrm{univ}}$ be the quotient group as in (2.3). Then the quotient map

$$\psi^{(p^r)}_{\mathrm{univ}}\colon\ \mathscr{F}_{\mathrm{univ}} \times_{A^0} A^r \to \mathscr{F}^{(p^r)}_{\mathrm{univ}}$$

of formal groups induces a deformation of Frobenius

$$(\mathscr{F}_{\mathrm{univ}} \times_{A^0} A^r, \mathrm{id}, \mathrm{id}) \to (\mathscr{F}^{(p^r)}_{\mathrm{univ}}, \sigma^r, \mathrm{id})$$

over A^r, where id appears in the deformation structure attached to $\mathscr{F}^{(p^r)}_{\mathrm{univ}}$ in view of the proof of Lemma 3.8.

Proposition 5.7 *For each $r \geq 0$, $(\mathscr{F}_{\mathrm{univ}} \times_{A^0} A^r \to \mathscr{F}^{(p^r)}_{\mathrm{univ}})/A^r$ is a terminal object of* $\mathrm{DefFrob}^{r}{}^{\mathscr{G}}_{\mathrm{iso}}$.

Proof Given any deformation $(\mathscr{F}, i, \eta) \to (\mathscr{F}', i', \eta')$ of Φ^r to R, we need to show that there exists a unique local homomorphism $\alpha^r\colon A^r \to R$ together with unique morphisms

$$(\mathscr{F}, i, \eta) \to (\mathscr{F}_{\mathrm{univ}} \times_{A^0} A^r, \mathrm{id}, \mathrm{id}) \quad \text{and} \quad (\mathscr{F}', i', \eta') \to (\mathscr{F}^{(p^r)}_{\mathrm{univ}}, \sigma^r, \mathrm{id}) \tag{5.8}$$

in $\mathrm{Def}^{\mathscr{G}}_{\mathrm{iso}}$ along α^r such that the following diagram of formal groups over R commutes.

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\sim"] \arrow[d] & \alpha^{r*}(\mathscr{F}_{\mathrm{univ}} \times_{A^0} A^r) \arrow[d] \\ \mathscr{F}' \arrow[r, "\sim"] & \alpha^{r*}\mathscr{F}^{(p^r)}_{\mathrm{univ}} \end{tikzcd} \tag{5.9}$$

By [Strickland 1997, Proposition 20], in $\mathrm{Def}^{\mathscr{G}}_{\mathrm{iso}}$ there are unique morphisms

$$\begin{tikzcd} \mathscr{F}_{\mathrm{univ}} \times_{A^0} A^r \arrow[r] \arrow[d] & \mathscr{F}_{\mathrm{univ}} \arrow[d] \\ \mathrm{Spf}(A^r) \arrow[r, "s^r"] & \mathrm{Spf}(A^0) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F}^{(p^r)}_{\mathrm{univ}} \arrow[r] \arrow[d] & \mathscr{F}_{\mathrm{univ}} \arrow[d] \\ \mathrm{Spf}(A^r) \arrow[r, "t^r"] & \mathrm{Spf}(A^0) \end{tikzcd} \tag{5.10}$$

Indeed, by uniqueness, s^r is the structure morphism of $\mathrm{Sub}_r(\mathscr{F}_{\mathrm{univ}})/A^0$. Similarly, there are unique morphisms

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}_{\mathrm{univ}} \arrow[d] \\ \mathrm{Spf}(R) \arrow[r, "\alpha"] & \mathrm{Spf}(A^0) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F}_{\mathrm{univ}} \arrow[d] \\ \mathrm{Spf}(R) \arrow[r, "\alpha'"] & \mathrm{Spf}(A^0) \end{tikzcd} \tag{5.11}$$

Let $\mathscr{D} \subset \alpha^* \mathscr{F}_{\mathrm{univ}}$ be the image of $\ker(\mathscr{F} \to \mathscr{F}')$ under the isomorphism $\mathscr{F} \xrightarrow{\sim} \alpha^* \mathscr{F}_{\mathrm{univ}}$ in (5.11). It is a subgroup of degree p^r. Then by [Strickland 1997, Theorem 42], there is a unique local homomorphism $\alpha^r \colon A^r \to R$ that classifies $\mathscr{D}$, with $\alpha^r \circ s^r = \alpha$. Moreover, $\alpha^r \circ t^r = \alpha'$ by the universal property of $(\mathscr{F}_{\mathrm{univ}}, \mathrm{id}, \mathrm{id})$, as we compute in $\mathrm{Def}^{\mathscr{G}}_{\mathrm{iso}}$ that

$$\begin{aligned} (\mathscr{F}', i', \eta') &\cong \big(\mathscr{F}/\ker(\mathscr{F} \to \mathscr{F}'), i', \eta\big) && \text{by (5.2) and the proof of Lemma 3.8} \\ &\cong (\alpha^* \mathscr{F}_{\mathrm{univ}}/\mathscr{D}, i', \mathrm{id}) && \text{by (5.11)} \\ &\cong \big(\alpha^{r*}(\mathscr{F}_{\mathrm{univ}} \underset{A^0}{\times} A^r)/\alpha^{r*}\mathscr{D}^{(p^r)}_{\mathrm{univ}}, i', \mathrm{id}\big) && \text{by [Strickland 1997, Theorem 42]} \\ &\cong (\alpha^{r*} \mathscr{F}^{(p^r)}_{\mathrm{univ}}, i', \mathrm{id}) && \text{by [Strickland 1997, Theorem 19 (v)]} \\ &\cong (\alpha^{r*} t^{r*} \mathscr{F}_{\mathrm{univ}}, i', \mathrm{id}) && \text{by (5.10)} \end{aligned}$$

Thus we obtain the desired morphisms (5.8) from (5.11), with readily checked compatibility as required in (5.9). □

For applications in later sections, we next deduce a corollary from Proposition 5.7 and its proof, in terms of formal group laws (cf. [Rezk 2014, Theorem 4.4]).

Corollary 5.12 *Let k, R, G, E be as in Corollary 4.12 and again fix G/k. Then for each $r \geq 0$ the functor*

$$R \mapsto \{\text{isomorphism classes of deformations } (F, i, \eta) \to (F', i', \eta') \text{ of } \Phi^r \text{ to } R\}$$

from the category of complete local rings with residue field containing k to the category of sets is co-represented by a ring A^r, which is a bimodule over $A^0 = E$ with structure maps local homomorphisms $s^r, t^r\colon\ A^0 \to A^r$.

Explicitly, there is a deformation $(F_{\mathrm{univ}}, \mathrm{id}, \mathrm{id})$ of G to A^0 satisfying the following universal property. Given any deformation $(F, i, \eta) \to (F', i', \eta')$ of Φ^r to R, there is a unique local homomorphism

$$\alpha^r\colon\ A^r \to R$$

such that $\alpha^r s^r, \alpha^r t^r\colon\ A^0 \to R$ reduce to $i, i'\colon\ k = E_0 \to R_0$ respectively and that there are unique $\star$-isomorphisms

$$(F, i, \eta) \to (\alpha^{r*} s^{r*} F_{\mathrm{univ}}, i, \mathrm{id}) \qquad \textit{and} \qquad (F', i', \eta') \to (\alpha^{r*} t^{r*} F_{\mathrm{univ}}, i', \mathrm{id})$$

5.13 Canonical lifts of Frobenius morphisms In view of Proposition 5.7 and Corollary 5.12, the ring A^r carries a universal example

$$s^{r*} F_{\mathrm{univ}} = F_{\mathrm{univ}} \times_{A^0} A^r \xrightarrow{\psi^{(p^r)}_{\mathrm{univ}}} F^{(p^r)}_{\mathrm{univ}} \overset{\star}{=} t^{r*} F_{\mathrm{univ}} \tag{5.14}$$

of deformations of Φ^r to R.[3] The central notion of norm coherence in this paper, to be introduced in the next section, concerns the question of when the $\star$-isomorphism in (5.14) is the identity.

6 Norm-coherent deformations

6.1 Setup Let k be an algebraic extension of $\mathbb{F}_p$ (in particular, k is perfect) and G be a formal group law over k of finite height n. Let R be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field $R_0 := R/\mathfrak{m} \supset k$. Let F/R be a deformation of G/k with deformation structure (i, id) as in (4.3).

Remark 6.2 Observe that, given any deformation (F, i, η), there exists a unique deformation $(\widetilde{F}, i, \mathrm{id})$ such that the two are in the same $\star$-isomorphism class, as shown

[3] See [Strickland 1997, § 10 and § 13] for more about the rings A^r. For an explicit example with $r = 1$ and G of height 2 over $k = \overline{\mathbb{F}}_p$, see [Zhu 2019, Theorems A and B (ii)], where $u_1 = h \in A^0$ and $t^1(u_1) = \psi^p(h) \in A^1$.

in the following diagram.[4]

$$\begin{tikzcd}
F & F_0 \arrow[l] \arrow[r, "\eta"] \arrow[d, "\eta"'] & G \arrow[dl, dash] \\
\widetilde{F} \arrow[from=u, "\Psi"'] & G \arrow[l] &
\end{tikzcd}$$

Without loss of generality, here we focus on the case of $\eta = \mathrm{id}$.

6.3 Outline of the section Having discussed the general theory of deformations in Sections 4 and 5, in (6.4–6.16) below we shall give a closer look at morphisms of deformations of formal group laws and set up notations (intended to keep in line with [Ando 1995]), before introducing the central notion of norm coherence.

6.4 Morphisms of deformations: quotient by the p-torsion subgroup As in (2.1) write $\mathscr{F}$ for the formal group over R whose group law is F (upon choosing a coordinate) and write $\mathscr{F}[p]$ for its subgroup scheme of p-torsions. This is defined over an extension $\widetilde{R}$ of R obtained by adjoining the roots of the p-series of F. Let $\mathscr{F}/\mathscr{F}[p] := (\mathscr{F} \times_R \widetilde{R})/\mathscr{F}[p]$ be the quotient group as in (2.3) with a particular group law $F/F[p]$ so that the isogeny

$$f_p\colon\ F \to F/F[p]$$

induced by the quotient morphism of formal groups is a deformation of Frobenius (5.5). Note that $\mathscr{F}[p](\widetilde{R})$ is stable under the action of $\mathrm{Aut}(\widetilde{R}/R)$. Thus f_p can be defined over R (cf. [Lubin 1967, Theorem 1.4]).

Remark 6.5 The restriction of f_p on the special fiber is the relative p^n-power Frobenius (see Example 5.3). It is not an endomorphism unless $k \subset \mathbb{F}_{p^n}$ (cf. [Ando 1995, proof of Proposition 2.5.1]).

6.6 Morphisms of deformations: the isogeny l_p By Corollary 5.12, there exists a unique local homomorphism $\alpha^n\colon A^n \to R$ together with a unique $\star$-isomorphism

[4] Here Ψ is any isomorphism lifting η. Such lifts always exist because the ring corepresenting (strict) isomorphisms between formal group laws over commutative rings is free polynomial. They are in fact unique by the uniqueness in [Lubin–Tate 1966, Theorem 3.1].

$(F/F[p], i\circ\sigma^n, \mathrm{id}) \to (\alpha^{n*}t^{n*}F_{\mathrm{univ}}, i\circ\sigma^n, \mathrm{id})$. According to the convention in (4.7), we simply write this as

$$g_p\colon\ F/F[p] \to \alpha_n^* t_n^* F_{\mathrm{univ}}$$

Define

$$l_p\colon\ F \to \alpha_n^* t_n^* F_{\mathrm{univ}}$$

to be the composite $g_p \circ f_p$.

Remark 6.7 The isogeny l_p of formal group laws over R is uniquely characterized by the following properties (cf. [Ando 1995, Proposition 2.5.4], the proof here being completely analogous).

(i) It has source F and target of the form $\alpha^* t^{n*} F_{\mathrm{univ}}$ for some local homomorphism $\alpha\colon A^n \to R$.

(ii) The kernel of l_p applied to $\mathscr{F}$ is $\mathscr{F}[p]$.

(iii) Over the residue field, l_p reduces to the relative p^n-power Frobenius.

Explicitly, with notation as in (5.2), f_p and l_p fit into the following commutative diagram. Their restrictions on the special fiber are highlighted with corresponding colors, which are in fact identical in this case (cf. Example 5.3).

(6.8)
$$\begin{tikzcd}
F \arrow[d, "f_p"'] \arrow[dd, bend right=60, "l_p"'] & F_0 \arrow[l] \arrow[r, equal] \arrow[d, "(f_p)_0"'] & i^*G \arrow[r] \arrow[d, "i^*\mathrm{Frob}^n"'] & G \arrow[d, "\mathrm{Frob}^n"'] \\
F/F[p] \arrow[d, "g_p"'] & (F/F[p])_0 \arrow[l] \arrow[r, equal] \arrow[d, equal, "(g_p)_0"'] & i^*G^{(p^n)} \arrow[r] \arrow[d, equal] & G^{(p^n)} \arrow[d] \\
\alpha^{n*}t^{n*}F_{\mathrm{univ}} & (\alpha^{n*}t^{n*}F_{\mathrm{univ}})_0 \arrow[l] \arrow[r, equal] & i'^*G \arrow[r] & G
\end{tikzcd}$$

Example 6.9 Let $k = \mathbb{F}_p$ and G be the Honda formal group law given by [Ando 1995, 2.5.5] with $[p]_G(t) = t^{p^n}$. Then the relative Frobenius Frob^n coincides with the absolute Frobenius automorphism on G and so $l_p = [p]_F$ (cf. [Ando 1995, Proposition 2.6.1]).

6.10 Morphisms of deformations: the isogenies l_D More generally, let $\mathscr{D} \subset \mathscr{F}$ be a subgroup of degree p^r, $\psi\colon F \to F'$ be any isogeny with kernel $\mathscr{D}$, and

$\psi \times \psi_!\colon (F, i, \mathrm{id}) \to (F', i', \eta')$ be the corresponding deformation of Frobenius. The diagram (6.8) generalizes as follows.

(6.11)
$$\begin{tikzcd}
F \arrow[d, "\psi"'] \arrow[dd, bend right=60, "l_D"'] & F_0 \arrow[l] \arrow[r, equal, "\eta = \mathrm{id}"] \arrow[d, "\psi_0"'] & i^*G \arrow[r] \arrow[d, "i^*\mathrm{Frob}^r"'] & G \arrow[d, "\mathrm{Frob}^r"'] \\
F' \arrow[d] & F'_0 \arrow[l] \arrow[r, "\eta'"] \arrow[d] & i^*G^{(p^r)} \arrow[r] \arrow[d, equal] & G^{(p^r)} \arrow[d] \\
\alpha^{r*}t^{r*}F_{\mathrm{univ}} & (\alpha^{r*}t^{r*}F_{\mathrm{univ}})_0 \arrow[l] \arrow[r, equal] & i'^*G \arrow[r] & G
\end{tikzcd}$$

In particular, when $\psi = f_D\colon F \to F/D$, we have the following commutative diagram (cf. (2.3) for f_D and (6.6) for g_D).

(6.12)
$$\begin{tikzcd}
F \arrow[r, "f_D"] \arrow[dr, "l_D"'] & F/D \arrow[d, "g_D"] \\
& \alpha^{r*}t^{r*}F_{\mathrm{univ}}
\end{tikzcd}$$

Remark 6.13 This construction of l_D is functorial under base change and under quotient, due to the functoriality of f_D and g_D (see [Strickland 1997, Theorem 19 (v)], [Ando 1995, Proposition 2.2.6], Proposition 5.7, and Remark 4.11). To be precise, given any local homomorphism $\beta\colon R \to R'$ and any finite subgroups $\mathscr{D}_1 \subset \mathscr{D}_2$ of $\mathscr{F}$, we have

$$l_{\beta^* D} = \beta^* l_D \qquad \text{and} \qquad l_{D_2/D_1} \circ l_{D_1} = l_{D_2}$$

where the composition is taken up to a $\star$-isomorphism, as shown in the following commutative diagrams.

(6.14)
$$\begin{tikzcd}
F & & \beta^* F \arrow[ll] \arrow[d, "\beta^* f_D"'] \arrow[dr, "f_{\beta^* D}"] & \\
F/D \arrow[from=1-1, d, "f_D"'] \arrow[d, "g_D"'] & & \beta^*(F/D) \arrow[ll] \arrow[d, "\beta^* g_D"'] & = \beta^* F/\beta^* D \arrow[dl, "g_{\beta^* D}"] \\
\alpha^{r*} t^{r*} F_{\mathrm{univ}} & & \beta^* \alpha^{r*} t^{r*} F_{\mathrm{univ}} \arrow[ll] &
\end{tikzcd}$$

(6.15)
$$\begin{tikzcd}
F \arrow[r, "f_{D_2}"] \arrow[d, "f_{D_1}"'] & F/D_2 \arrow[r, "g_{D_2}"] \arrow[d, equal] & \alpha^*_{r_2} t^*_{r_2} F_{\mathrm{univ}} \\
F/D_1 \arrow[r, "f_{D_2/D_1}"] \arrow[d, "g_{D_1}"'] & F/D_1\big/D_2/D_1 \arrow[d, "\tilde{g}_{D_1}"'] & \\
\alpha^*_{r_1} t^*_{r_1} F_{\mathrm{univ}} \arrow[r, "\tilde{f}_{D_2/D_1}"] & \alpha^*_{r_1} t^*_{r_1} F_{\mathrm{univ}}\big/D_{\mathrm{univ}}^{(p^{r_2-r_1})} \arrow[uur, "\tilde{g}_{D_2/D_1}"'] &
\end{tikzcd}$$

6.16 Morphisms of deformations: pushing forward coordinates As a followup to (5.1), given (6.11) and (5.13), let $F = (\mathscr{F}, X)$ and $\alpha^{r*} t^{r*} F_{\mathrm{univ}} = (\alpha^{r*} t^{r*} \mathscr{F}_{\mathrm{univ}}, X_{\mathrm{can}})$ (see (2.1)). Define the *pushforward of X along ψ* to be

$$\psi_!(X) := \iota^*(X_{\mathrm{can}}) \in \mathcal{O}_{\mathscr{F}'} \tag{6.17}$$

where ι is the $\star$-isomorphism from F' to $\alpha^{r*} t^{r*} F_{\mathrm{univ}}$ (or an isomorphism between corresponding formal groups as in Proposition 5.7). This is a coordinate on $\mathscr{F}'$ (cf. Example 4.9). Its pullback to $\mathscr{F}$ along ψ equals $l_D^*(X_{\mathrm{can}})$, where l_D carries the canonical descent of the level structure of a degree-p^r subgroup $\ker \psi$.

6.18 Norm coherence: idea Let $\mathscr{G}/k$ and $\mathscr{F}/R$ be the formal group and its deformation, equipped with suitable coordinates, that correspond to the formal group laws in (6.1) (see (2.1)).

Recall from the proof of Proposition 3.11 that when $\psi = \mathrm{Frob}^r \colon \mathscr{G} \to \mathscr{G}^{(p^r)}$, the norm map $\mathrm{Norm}_{\psi^*} \colon \mathcal{O}_{\mathscr{G}} \to \mathcal{O}_{\mathscr{G}'}$ sends a coordinate $X_{\mathscr{G}}$ on $\mathscr{G}$ to the coordinate on

$\mathscr{G}' = \mathscr{G}^{(p^r)}$ which pulls back along ψ to $X_{\mathscr{G}}^{p^r}$. Thus the norm map agrees with pushing forward a coordinate along the Frobenius isogeny in the sense of (6.16) ($\mathscr{G}$ is a trivial deformation of itself).

This agreement on $X_{\mathscr{G}}$ over k may not extend to R for an arbitrary coordinate X on $\mathscr{F}$ lifting $X_{\mathscr{G}}$.

On one hand, given a subgroup $\mathscr{D} \subset \mathscr{F}$ of degree p^r, the isogeny $f_{\mathscr{D}} \colon \mathscr{F} \to \mathscr{F}/\mathscr{D}$ lifts the norm map in the sense that

$$
\begin{aligned}
X_{\mathscr{D}}\big(f_{\mathscr{D}}(P)\big) &= \big(f_{\mathscr{D}}^*(X_{\mathscr{D}})\big)(P) \\
&= \prod_{Q \in \mathscr{D}(\widetilde{R})} \big(X(P) \underset{F}{+} X(Q)\big) && \text{by (2.4)} \\
&= \Big(\prod_{\sigma \in \mathrm{Aut}(\mathcal{O}_{\mathscr{F}}/\mathcal{O}_{\mathscr{F}/\mathscr{D}})} \sigma \cdot X\Big)(P) \\
&= \big(f_{\mathscr{D}}^* \mathrm{Norm}_{f_{\mathscr{D}}^*}(X)\big)(P) && \text{cf. (3.12)} \\
&= \mathrm{Norm}_{f_{\mathscr{D}}^*}(X)\big(f_{\mathscr{D}}(P)\big)
\end{aligned}
\tag{6.19}
$$

where $X_{\mathscr{D}}$ is the coordinate corresponding to the group law $(\mathscr{F}, X)/D$, P is any R-point on $\mathscr{F}$, and $\widetilde{R}$ is an extension of R to define the p^r points of $\mathscr{D}$.[5]

On the other hand, the isogeny $l_D = g_D \circ f_D$ of formal group laws lifts Frob^r canonically with respect to $\mathscr{D}$. Namely, if f_D' is another lift with kernel $\mathscr{D}$ and classifying $\star$-isomorphism g_D', then $g_D' \circ f_D' = l_D$ (see Remark 6.7 and (5.13)).

6.20 Norm coherence: definition Let (F, i, id) be a deformation of G to R as in (6.1). Suppose that $F = (\mathscr{F}, X)$ as in (2.1). Given any finite subgroup $\mathscr{D}$ of $\mathscr{F}$, let

[5]There is an analogue of this in the context of Galois theory, where the finite free extension $f_{\mathscr{D}}^* \colon \mathcal{O}_{\mathscr{F}/\mathscr{D}} \to \mathcal{O}_{\mathscr{F}}$ of rings is replaced by a finite Galois extension (see, e.g., [Rotman 2010, pp. 916–920, esp. Corollary 10.87]). Moreover, consider a coordinate on $\mathscr{F}$ as a map $\mathscr{F} \to \hat{\mathbb{A}}^1$ (2.1). We then have

$$\mathscr{F} \xrightarrow{f_{\mathscr{D}}} \mathscr{F}/\mathscr{D} \to \hat{\mathbb{A}}^1$$

and $\mathrm{Norm}_{f_{\mathscr{D}}^*}$ gives

$$\mathscr{F} \xrightarrow{X} \hat{\mathbb{A}}^1 \quad \mapsto \quad \mathscr{F}/\mathscr{D} \xrightarrow{X_{\mathscr{D}}} \hat{\mathbb{A}}^1$$

which is analogous to a *norm map* as a piece of structure in a Tambara functor [Tambara 1993, 3.1]. This last notion of a norm map has been packaged into equivariant stable homotopy theory and turned out as a key ingredient in recent advances in the field [Brun 2007, Hill–Hopkins 2016].

$f_D\colon (\mathscr{F},X)\to(\mathscr{F}/\mathscr{D},X_{\mathscr{D}})$ be the associated quotient map of formal group laws as in (2.3). Write $f_{D\,!}(X)$ for the pushforward of X along f_D as in (5.2).

Definition 6.21 We say that the coordinate X on the deformation $\mathscr{F}$ is *norm coherent* if the identity

$$f_{D\,!}(X)=X_{\mathscr{D}} \tag{6.22}$$

holds in $\mathcal{O}_{\mathscr{F}/\mathscr{D}}$ for all $\mathscr{D}$. In this case, we also say that (F,i,id), or simply F, is norm coherent.

More generally, given any deformation (F,i,η) of G to R, let $(\widetilde{F},i,\mathrm{id})$ be the unique deformation associated to it (Remark 6.2). We say that (F,i,η) is norm coherent if $(\widetilde{F},i,\mathrm{id})$ is.

6.23 Norm coherence: an alternative definition Roughly speaking, the condition (6.22) says that the canonical descent along l_D has the same effect as the descent along the norm map in terms of f_D.

To be precise, in view of (6.12), the equality (6.22) forces the $\star$-isomorphism g_D to be given by the identity series $h(T)=T$ (see (2.2)), which generalizes Ando's condition

$$l_D=f_D$$

in [Ando 1995, Theorem 2.5.7] and can be taken as an alternative definition for norm coherence.[6]

6.24 Norm coherence: an explicit criterion Recall from (4.10) the universal deformation $\mathscr{F}_{\mathrm{univ}}$ of $\mathscr{G}/k$ over $E=E^{\mathscr{G}}=W(k)[\![u_1,\ldots,u_{n-1}]\!]$.

Proposition 6.25 *A coordinate X on $\mathscr{F}_{\mathrm{univ}}$ is norm coherent if and only if, given any finite subgroup $\mathscr{D}$ of $\mathscr{F}_{\mathrm{univ}}$, we have*

$$h^{(p^r)}\big(l_D(X)\big)=\prod_{Q\in\mathscr{D}(A^r)}h\big(X\underset{F_{\mathrm{univ}}}{+}X(Q)\big) \tag{6.26}$$

for all $h(T)\in T\cdot E[\![T]\!]$, where $h^{(p^r)}$ denotes the series obtained by twisting the coefficients c_j of $h(T)=\sum c_jT^j$ with the automorphism of E given by the lift of the absolute p^r-power Frobenius on k to $W(k)$ (and leaving the generators u_i fixed).

[6] The reader may prefer to distinguish the identity series from the identity homomorphism id with the same source and target, and hence not to phrase the condition in terms of an *equality* $l_D=f_D$.

Proof Suppose that (6.26) holds for all h. When h is the identity series $h(T) = T$, we obtain $g_D \circ f_D(X) = f_{\mathscr{D}}^*(X_{\mathscr{D}})$ (cf. (6.12, 2.4)). This is the pullback of (6.22) to $\mathcal{O}_{\mathscr{F}_{\text{univ}}}$ along $f_{\mathscr{D}}$. Thus (6.22) holds in $\mathcal{O}_{\mathscr{F}_{\text{univ}}/\mathscr{D}}$, viewed as a subring of $\mathcal{O}_{\mathscr{F}_{\text{univ}}}$ via $f_{\mathscr{D}}^*$.

Conversely, a norm-coherent coordinate X forces $l_D = f_D$ (by an abuse of notation) as in (6.23). We view $h(T) \in T \cdot E[\![T]\!]$ as a homomorphism $\psi\colon F_{\text{univ}} \to F$ of formal group laws over E for some deformation $F = (\mathscr{F}, X_{\mathscr{F}})$ as in (2.2). Then

$$\begin{aligned}
h^{(p^r)}\big(l_D(X)\big) &= h^{(p^r)}\big(f_D(X)\big) \\
&= h^{(p^r)}\big(f_{\mathscr{D}}^*(X_{\mathscr{D}})\big) \\
&= h^{(p^r)}\big(f_{\mathscr{D}}^* \mathrm{Norm}_{f_{\mathscr{D}}^*}(X)\big) && \text{by (6.19)} \\
&= f_{\mathscr{D}}^* \mathrm{Norm}_{f_{\mathscr{D}}^*}\big(h(X)\big) && \text{cf. (3.13)} \\
&= f_{\mathscr{D}}^* \mathrm{Norm}_{f_{\mathscr{D}}^*}\big(\psi^*(X_{\mathscr{F}})\big) \\
&= \psi^* f_{\psi(\mathscr{D})}^* \mathrm{Norm}_{f_{\psi(\mathscr{D})}^*}(X_{\mathscr{F}}) \\
&= \psi^* f_{\psi(\mathscr{D})}^* (X_{\psi(\mathscr{D})}) && \text{by (6.19)} \\
&= \psi^*\Big(\prod_{Q' \in \psi(\mathscr{D})(A^r)} \big(X_{\mathscr{F}} \underset{F}{+} X_{\mathscr{F}}(Q')\big)\Big) \\
&= \prod_{Q \in \mathscr{D}(A^r)} \Big(h(X) \underset{F}{+} h\big(X(Q)\big)\Big) \\
&= \prod_{Q \in \mathscr{D}(A^r)} h\big(X \underset{F_{\text{univ}}}{+} X(Q)\big)
\end{aligned}$$

□

6.27 Norm coherence: a conceptual formulation Let $\psi \times \psi_!\colon (F, i, \eta) \to (F', i', \eta')$ be any deformation of Φ^r to R (5.5) and write $\mathscr{D} := \ker \psi$. Let $X[F, i, \eta] := X_{\mathscr{F}}$ be the coordinate on $\mathscr{F}$ corresponding to F, the latter equipped with deformation structure (i, η). Consider the identity

$$X[F', \psi_!(i, \eta)] = \mathrm{Norm}_{\psi^*}(X[F, i, \eta]) \tag{6.28}$$

in $\mathcal{O}_{\mathscr{F}'}$. The norm coherence of (F, i, η) is equivalent to the condition that (6.28) hold for *any* ψ.

Indeed, let us reduce to the universal case. The pushforward $\psi_!(i, \eta)$ of deformation structure (5.2) indicates a change of coordinates on $\mathscr{F}'$ under which the left-hand side of (6.28) corresponds to the formal group law $\alpha^{r*} t^{r*} F_{\text{univ}}$, i.e., the target of l_D (cf. Remark 6.2 and (6.11)) . Meanwhile, by functoriality of norm maps (i.e., restriction of scalars for determinants), the right-hand side changes to $\mathrm{Norm}_{l_D^*}(X[F,i,\eta])$ (cf. (3.14)). Thus, in the universal case, (6.28) becomes

$$X[\alpha^{r*}t^{r*}F_{\mathrm{univ}}, i\circ\sigma^r, \eta] = \mathrm{Norm}_{l_D^*}(X[F,i,\eta])$$

(The left-hand side with $\eta = \mathrm{id}$ was written as X_{can} in (6.16).) Pulling this back along g_D to $\mathcal{O}_{\mathscr{F}/\mathscr{D}}$, we see that it holds if and only if (F,i,η) is norm coherent.

We shall return to this formulation of norm coherence towards the end of Section 8 (in the proof of Proposition 8.17).

6.29 Norm coherence: functoriality Recall from Example 4.6 and (5.1) the operations of base change and pushforward of deformation structures. The notion of norm coherence in Definition 6.21 is preserved under both as follows.

Proposition 6.30 *Let (F,i,η) be a norm-coherent deformation of G to R.*

(i) Given any local homomorphism $\beta\colon R\to R'$, the deformation $\big(\beta^*F, \beta^*(i,\eta)\big)$ is norm coherent.

(ii) Given any isogeny $\psi\colon F\to F'$ over R, the deformation $\big(F', \psi_!(i,\eta)\big)$ is norm coherent. In particular, given any finite subgroup $\mathscr{D}\subset\mathscr{F}$ of degree p^r, the deformation $(F/D, i\circ\sigma^r, \eta)$ is norm coherent.

Proof We resort to the alternative definition of norm coherence in (6.23).

For (i), in view of Remark 6.2, first note that

$$\begin{aligned}
&(\widetilde{F}, i, \mathrm{id}) \overset{\star}{=} (F,i,\eta)\\
\Longrightarrow\ &\big(\beta^*\widetilde{F}, \beta^*(i,\mathrm{id})\big) \overset{\star}{=} \big(\beta^*F, \beta^*(i,\eta)\big)\\
\Longrightarrow\ &\beta^*\widetilde{F} = \widetilde{\beta^*F}
\end{aligned}$$

To see that $\beta^*\widetilde{F}$ is norm coherent, we have from (6.14)

$$l_{\beta^*D} = \beta^* l_D = \beta^* f_D = f_{\beta^*D}$$

For (ii), suppose that ψ is of degree p^r and let $i' = i\circ\sigma^r$. In view of

$$\big(F', \psi_!(i,\eta)\big) \overset{\star}{=} (F/D, i', \eta) \overset{\star}{=} (\widetilde{F/D}, i', \mathrm{id}) = (\widetilde{F}/D, i', \mathrm{id})$$

we are reduced to the special case of

$$(F,i,\mathrm{id}) \xrightarrow{f_D\times f_{D!}} (F/D, i\circ\sigma^r, \mathrm{id})$$

(see Example 5.3). Since the source is norm coherent, we have from (6.15)

$$l_D = f_D \qquad \text{and} \qquad l_{D'/D} \circ l_D = l_{D'} = f_{D'} = f_{D'/D} \circ f_D$$

where $\mathscr{D}'$ is any finite subgroup of $\mathscr{F}$ containing $\mathscr{D}$, and the first composition is on the nose because of the first identity in the display. Given that $g_{D'/D}$ is an isomorphism, we then deduce from these

$$l_{D'/D} = f_{D'/D}$$

which shows the norm coherence of $(F/D, i \circ \sigma^r, \mathrm{id})$. □

7 Existence and uniqueness of norm-coherent deformations

The following generalizes a result of Ando's (cf. [Ando 1995, Theorem 2.5.7]).

Proposition 7.1 *Let k, G, R, F be as in (6.1) and fix G/k. There exists a unique formal group law F' over R, $\star$-isomorphic to F, that is norm-coherent. In other words, given any coordinate $X_{\mathscr{G}}$ on the formal group $\mathscr{G}$ and a coordinate $X_{\mathscr{F}}$ on $\mathscr{F}$ that lifts $X_{\mathscr{G}}$, there exists a unique norm-coherent coordinate $X'_{\mathscr{F}}$ on $\mathscr{F}$ such that the formal group law $(\mathscr{F}, X'_{\mathscr{F}})$ is $\star$-isomorphic to $(\mathscr{F}, X_F)$.*

To show this, we will follow Ando's proof of his theorem, making alterations for greater generality whenever necessary (most significantly in (7.6)). The argument breaks into two parts, the first focusing on norm coherence for the p-torsion subgroup $\mathscr{F}[p]$ and the second showing functoriality for all finite subgroups. We begin with the following key lemma (cf. [Ando 1995, Theorem 2.6.4]).

Lemma 7.2 *Given any coordinate X_F on $\mathscr{F}$ that lifts $X_{\mathscr{G}}$, there exists a unique coordinate on $\mathscr{F}$ whose corresponding formal group law is $\star$-isomorphic to that of X_F and satisfies*

$$l_p = f_p \tag{7.3}$$

Proof **Existence** First we reduce the proof to the universal case. Let F_{univ} be a universal deformation of G/k to E as in Corollary 4.12, so that there is a unique local homomorphism

$$\alpha\colon\ E \to R$$

together with a unique $\star$-isomorphism

$$g\colon\ F \to \alpha^* F_{\mathrm{univ}}$$

Suppose that we can construct a coordinate X on $\mathscr{F}_{\text{univ}}$ such that $F'_{\text{univ}} = (\mathscr{F}_{\text{univ}}, X)$ satisfies (7.3) and is $\star$-isomorphic to F_{univ}. Taking $\mathscr{D} = \mathscr{F}_{\text{univ}}[p]$ in the proof of Proposition 6.30 (i), we then see that $\alpha^* F'_{\text{univ}}$ satisfies (7.3) and is $\star$-isomorphic to F.

We turn to the universal case. The proof is inductive, on powers of the maximal ideal I of E. Let Y be the coordinate corresponding to F_{univ} from above, so we may write $F_Y := F_{\text{univ}}$. We will also drop the subscript $_{\text{univ}}$ for the rest of this proof. With respect to Y, given that $g_p^Y \colon F_Y/F[p] \to \alpha^{n*} t^{n*} F_Y$ is defined over E as in (6.4), let $a(T) \in E[\![T]\!]$ be such that

$$g_p^Y(T) = T + a(T) \tag{7.4}$$

We shall construct a desired coordinate X on the universal formal group $\mathscr{F}$ by inductively modifying the coordinate Y so that $a(T) \equiv 0 \mod I^r$ for increasing r.

Let the inductive hypothesis be

$$a(T) = \sum_{j \geq 1} a_j T^j \qquad \text{with} \quad a_j \in I^{r-1} \tag{7.5}$$

Since g_p^Y is a $\star$-isomorphism, we get the case of $r = 2$. Let $\delta(T)$ be the power series

$$\delta(T) = T - a^{(-p^n)}(T) \tag{7.6}$$

where $a^{(-p^n)}(T)$ is the series obtained by twisting the coefficients a_j with the inverse of the local automorphism $\alpha^n t^n$ on $A^0 = E$,[7] and has its coefficients in I^{r-1} as well. The coordinate

$$Z := \delta(Y) \tag{7.7}$$

on $\mathscr{F}$ then yields a formal group law F_Z over E such that $\delta \colon F_Y \to F_Z$ is a $\star$-isomorphism. With respect to Z, let $b(T) \in E[\![T]\!]$ be such that

$$g_p^Z(T) = T + b(T) \tag{7.8}$$

We will show that this choice of coordinate Z gives

$$b(T) = \sum_{j \geq 1} b_j T^j \qquad \text{with} \quad b_j \in I^r \tag{7.9}$$

[7] This automorphism lifts the p^n-power Frobenius σ^n on k to $W(k)$ and fixes the generators u_i. Indeed, by [Lubin 1967, Theorem 1.5], the isogeny f_p differs by an E-isomorphism from the *endomorphism* $[p]_F$. Moreover, their targets can be equipped with deformation structures so that they are $\star$-isomorphic as deformations (cf. (6.8)). Since the generators u_i parameterize $\star$-isomorphism classes of deformations as in Corollary 4.12, they remain unchanged under $\alpha^n t^n$. Cf. [Rezk 2013, 4.15–4.16] and Proposition 6.25.

and in particular produces the equation

$$g_p^Z(T) \equiv T \mod I^r$$

Note that the formal group laws F_Y and F_Z coincide modulo I^{r-1}. Thus, by induction and Krull's intersection theorem, we will then obtain in the limit a coordinate X such that $g_p^X(T) = T$, or $l_p^X(T) = f_p^X(T)$, as desired.

Consider the diagram

$$\begin{tikzcd} F_Y \arrow[r, "\delta"] \arrow[d, "l_p^Y"'] & F_Z \arrow[d, "l_p^Z"] \\ \alpha^{n*}t^{n*}F_Y \arrow[r, "\tilde{\delta}"] & \alpha^{n*}t^{n*}F_Z \end{tikzcd} \tag{7.10}$$

where $\tilde{\delta} := \alpha^{n*}t^{n*}\delta$ is a $\star$-isomorphism.[8] By the unique characterization of l_p in Remark 6.7, we have $\tilde{\delta} \circ l_p^Y \circ \delta^{-1} = l_p^Z$. Thus the diagram commutes and we get $\tilde{\delta} \circ l_p^Y(T) = l_p^Z \circ \delta(T)$, or

$$\tilde{\delta} \circ g_p^Y \circ f_p^Y(T) = g_p^Z \circ f_p^Z \circ \delta(T) \tag{7.11}$$

We shall compare the two sides of (7.11) modulo I^r to show (7.9) and thus complete the induction.

The left-hand side of (7.11) can be evaluated modulo I^r as follows.

$$\begin{aligned} \tilde{\delta} \circ g_p^Y \circ f_p^Y(T) &= \tilde{\delta}\big(f_p^Y(T) + a \circ f_p^Y(T)\big) && \text{by (7.4)} \\ &\equiv \tilde{\delta}\big(f_p^Y(T) + a(T^{p^n})\big) && \text{by (2.5, 7.5)} \\ &= f_p^Y(T) + a(T^{p^n}) - a\big(f_p^Y(T) + a(Y^{p^n})\big) && \text{by (7.6)} \\ &\equiv f_p^Y(T) + a(T^{p^n}) - a\big(f_p^Y(T)\big) && \text{by (7.5)} \\ &\equiv f_p^Y(T) + a(T^{p^n}) - a(T^{p^n}) && \text{by (2.5, 7.5)} \\ &= f_p^Y(T) \end{aligned} \tag{7.12}$$

[8] The classifying maps for $F_Y/F[p]$ and $F_Z/F[p]$ are both $\alpha^n t^n$ because F_Y and F_Z are $\star$-isomorphic.

For the right-hand side of (7.11), first note that modulo I^r we have

$$
\begin{aligned}
(7.13)\quad f_p^Z \circ \delta(T) &= \prod_{c \in \mathscr{F}[p](\widetilde{E})} \big(\delta(T) \underset{F_Z}{+} Z(c)\big) && \text{by (2.4)}\\
&= \prod_c \delta\big(T \underset{F_Y}{+} Y(c)\big) && \text{by (7.7)}\\
&= \prod_c \big[\big(T \underset{F_Y}{+} Y(c)\big) - a^{(-p^n)}\big(T \underset{F_Y}{+} Y(c)\big)\big] && \text{by (7.6)}\\
&\equiv \prod_c \big(T \underset{F_Y}{+} Y(c)\big)\\
&\quad - \sum_c a^{(-p^n)}\big(T \underset{F_Y}{+} Y(c)\big) \prod_{d \neq c} \big(T \underset{F_Y}{+} Y(d)\big) && \text{by (7.5)}\\
&\equiv f_p^Y(T) - \sum_c a^{(-p^n)}(T)\, T^{p^n-1} && \text{by (2.4, 7.5, 2.5)}\\
&= f_p^Y(T) - p^n a^{(-p^n)}(T)\, T^{p^n-1}\\
&\equiv f_p^Y(T) && \text{by (7.5) and as } p \in I
\end{aligned}
$$

In particular, by (2.5), this gives

$$
(7.14)\qquad f_p^Z \circ \delta(T) \equiv T^{p^n} \mod I
$$

Thus, given $k \geq 2$, if in (7.8) we have

$$
b(T) = \sum_{j \geq 1} b_j T^j \qquad \text{with} \quad b_j \in I^{k-1}
$$

then for $k \leq r$ on the right-hand side of (7.11) we have

$$
\begin{aligned}
g_p^Z \circ f_p^Z \circ \delta(T) &= f_p^Z \circ \delta(T) + b\big(f_p^Z \circ \delta(T)\big) && \text{by (7.8)}\\
&\equiv f_p^Y(T) + b(T^{p^n}) \mod I^k && \text{by (7.13, 7.14)}
\end{aligned}
$$

Comparing this to (7.12), we get

$$
b(T) \equiv 0 \mod I^k
$$

Since g_p^Z in (7.8) is a $\star$-isomorphism, we can proceed by induction on k and obtain

$$
b(T) \equiv 0 \mod I^r
$$

which implies (7.9).

Uniqueness Let $\mathscr{F}/R$ be a deformation of $\mathscr{G}/k$. Let X and Y be two coordinates on $\mathscr{F}$, both lifting $X_{\mathscr{G}}$ on $\mathscr{G}$ and both satisfying (7.3). Suppose F_X and F_Y are in the same $\star$-isomorphism class so that there is a $\star$-isomorphism $\delta\colon F_X \to F_Y$ fitting into a commutative diagram analogous to (7.10).

$$\begin{tikzcd} F_X \arrow[r, "\delta"] \arrow[d, "l_p^X = f_p^X"'] & F_Y \arrow[d, "l_p^Y = f_p^Y"] \\ F_X/F[p] \arrow[r, "\tilde{\delta}"] & F_Y/F[p] \end{tikzcd}$$

Let $\mathfrak{m}$ be the maximal ideal of R. Let $c(T) \in R[\![T]\!]$ be such that

$$\delta(T) = T + c(T) \tag{7.15}$$

where

$$c(T) = \sum_{j\geq 1} c_j T^j \qquad \text{with} \quad c_j \in \mathfrak{m}$$

Since X and Y are distinct, there exists $r_0 \geq 2$ such that it is the largest r satisfying

$$c_j \in \mathfrak{m}^{r-1} \qquad \text{for all } j \tag{7.16}$$

Modulo $\mathfrak{m}^{r_0}$ we then have

$$\begin{aligned} f_p^Y(T) &= \tilde{\delta} \circ f_p^X \circ \delta^{-1}(T) \\ &= f_p^X \circ \delta^{-1}(T) + c^{(p^n)} \circ f_p^X \circ \delta^{-1}(T) && \text{by (7.15)} \\ &\equiv f_p^X \circ \delta^{-1}(T) + c^{(p^n)}(T^{p^n}) && \text{by (2.5, 7.16)} \\ &\equiv f_p^Y(T) + c^{(p^n)}(T^{p^n}) && \text{analogous to (7.13)} \end{aligned}$$

which is a contradiction. □

Proof of Proposition 7.1 (cf. [Ando 1995, proof of Proposition 2.6.15]) We need only show that the coordinate X on $\mathscr{F}$ constructed in Lemma 7.2 satisfies the stronger condition $l_D = f_D$ for any finite subgroup $\mathscr{D} \subset \mathscr{F}$. As in the proof of existence there, we are reduced to the universal case with $F_{\text{univ}} =: F = F_X$ over E.

Unlike in the proof of the lemma, since there is a single coordinate X (and its descendants $X_{\mathscr{D}}$) involved here, we will drop the superscripts and simply write, e.g., $f_D := f_D^X\colon F \to F/D$ and $f_{D'/D} := f_{D'/D}^{X_{\mathscr{D}}}\colon F/D \to F/D'$. We will also save the superscripts in α^r and t^r when the subgroup under classification is understood.

Given any $\mathscr{D} \subset \mathscr{F}$ of degree p^r, we will show that the $\star$-isomorphism

$$g_D\colon F/D \to \alpha^* t^* F \tag{7.17}$$

is the identity by the uniqueness from Lemma 7.2. Namely, the source and target are in the same $\star$-isomorphism class, and we show that both of them satisfy (7.3). That the target does is clear from the proof of Proposition 6.30 (i). For the source of (7.17), let $p^{-1}\mathscr{D} := \{c \in \mathscr{F} \mid pc \in \mathscr{D}\}$.[9] It contains both $\mathscr{D}$ and $\mathscr{F}[p]$ as subgroups. We need to show

$$l_{p^{-1}D/D} = f_{p^{-1}D/D} \tag{7.18}$$

Consider the following commutative diagram. The upper rectangle commutes due to the functoriality of the isogeny f under quotient [Ando 1995, Proposition 2.2.6]. The lower rectangle commutes (with identical classifying maps at the bottom-left and bottom-right corners) as a result of the functoriality from Corollary 5.12 of the $\star$-isomorphisms g under quotient.

$$\begin{tikzcd}
F \arrow[rr, "f_D"] \arrow[d, "f_p"'] \arrow[drr, "f_{p^{-1}D}"] & & F/D \arrow[d, "f_{p^{-1}D/D}"] \\
F/F[p] \arrow[rr, "f_{p^{-1}D/F[p]}"] \arrow[d, equal, "g_p"'] & & F/p^{-1}D \arrow[d, "g_{p^{-1}D/D}"] \\
\alpha^*t^*F \arrow[r, "\alpha^*t^*f_D"] & \alpha^*t^*F/\alpha^*t^*D = \alpha^*t^*(F/D) &
\end{tikzcd}$$

Note that $p^{-1}\mathscr{D}/\mathscr{F}[p] \cong \mathscr{D}$. In the lower rectangle, $g_p = \mathrm{id}$ and hence $f_{p^{-1}D/F[p]} = \alpha^*t^*f_D$. This then forces the $\star$-isomorphism $g_{p^{-1}D/D}$ to be the identity, and (7.18) follows. $\square$

Remark 7.19 In [Zhu 2020, §3.1], for the purpose of studying Hecke operators in elliptic cohomology, we showed the existence of an analogue of Ando's coordinate. It is conceptually different from the norm-coherent coordinates here. Note that there the base change is not along a *local* homomorphism (see [Zhu 2014, §4, footnote] and cf. (7.21) below).

Example 7.20 Let $k = \mathbb{F}_{p^2}$ and G be the formal group law of a supersingular elliptic curve over k. We choose this curve so that its p^2-power Frobenius endomorphism

[9] The notation $c \in \mathscr{F}$ means $[c] \subset \mathscr{F}$, where $[c]$ is the effective Cartier divisor defined by a section. To be precise, this set-theoretic description defines the subgroup scheme $p^{-1}\mathscr{D}$ of $\mathscr{F}$ as a sum of effective Cartier divisors.

coincides with the map of multiplication by $(-1)^{p-1}p$, as in [Zhu 2020, 3.1]. We then have $l_p = [-p]$, if $p = 2$, in view of Remark 6.7 and Footnote 7.

Let E be the Morava E-theory associated to $\mathscr{G}/k$ as in (2.7) and choose a *preferred* $\mathscr{P}_N$*-model* for E in the sense of [Zhu 2020, Definition 3.8]. In particular, there is a chosen coordinate u on the universal deformation of $\mathscr{G}/k$. Given [Zhu 2020, 3.5], the cotangent map along f_p^u is multiplication by p. Thus, by the criterion (7.3), u cannot be norm coherent if $p = 2$.

More explicitly, let us consider the supersingular elliptic curve $C_0/\mathbb{F}_2\colon\ y^2 + y = x^3$. A direct calculation shows that $\mathrm{Frob}^2 = [-2]$ on C_0. In [Rezk 2008], Rezk chose the coordinate $u := x/y$ for the formal group $\widehat{C}_0$ and for its universal formal deformation $\widehat{C}$ over $\mathbb{Z}_2[\![a]\!]$, with $C\colon\ y^2 + axy + y = x^3$. Let Q be the universal example of a point of exact order 2 of $\widehat{C}$. Rezk then chose $d := u(Q)$ as a parameter for the modular curve $X_0(2)$ near the supersingular locus, and computed its equation as

$$d^3 - ad - 2 = 0$$

Taking $\mathscr{D} = \widehat{C}[2]$ in (2.4), we see that the cotangent map along f_2^u is multiplication by $d_1 d_2 d_3 = 2$, with each d_i a root of the modular equation, whereas $l_2^u = [-2]$. If we instead choose $\widetilde{u} := -x/y$ as a coordinate for $\widehat{C}$, it restricts to u over $\mathbb{F}_2$ while satisfying $f_2^{\widetilde{u}} = l_2^{\widetilde{u}}$ by rigidity (cf. [Rezk 2013, Remark 4.16]). Thus $\widetilde{u}$ is the unique norm-coherent lift of u to $\mathbb{Z}_2[\![a]\!]$.[10]

Comparing with the more comprehensive list of [Zhu 2019, Examples 2.16, 2.17, 2.18, and 2.20], only the coordinate in 2.16 is not norm coherent (but still good for the purpose as explained in the last footnote).

7.21 Norm coherence: more functoriality We continue the discussion in (6.29) with varying G/k.

[10]Note that Rezk derived in [Rezk 2008, §4] formulas for power operations in E without using a norm-coherent coordinate. In fact, let $\widetilde{d} := \widetilde{u}(Q)$ and suppose that $d = s\widetilde{d}$ for some unit $s \in \mathbb{Z}_2[\![a]\!]$. With his notation, we then have

$$d' = s'\widetilde{d}' = s \cdot \frac{-2}{\widetilde{d}} = s \cdot \frac{-2}{s^{-1}d} = s^2 \cdot \frac{-2}{d} = s^2(a - d^2)$$

where the second equality relies on norm coherence and the last one follows from the modular equation above. Here s happens to be -1 so incidentally $s^2 = 1$. In general, to apply [Rezk 2009, Theorem B], we need norm-coherent coordinates to compute power operations for E-theories as studied in [Rezk 2008, Zhu 2014, Zhu 2020, Zhu 2019] (cf. Remark 8.8 below).

Let $\mathfrak{X}\colon \mathrm{FG}_{\mathrm{isog}} \to \mathrm{Set}$ be the functor

$$\mathscr{G}/k \mapsto \{\text{coordinates on } \mathscr{G}\} \subset \mathcal{O}_{\mathscr{G}}$$

Note that this is a "wide" functor in the sense that, given diagram (3.4), $\mathfrak{X}$ is contravariant along the right square and covariant along the left square. More specifically, $\mathfrak{X}$ is contravariant with respect to base change $\mathrm{Spf}(k) \to \mathrm{Spf}(k')$ and pullback along an isomorphism over k, hence contravariant with respect to any morphism in the subcategory $\mathrm{FG}_{\mathrm{iso}}$. On the other hand, given an isogeny $\mathscr{G} \to \mathscr{G}'$ over k of degree p^r, any coordinate X on $\mathscr{G}$ determines a unique coordinate on $\mathscr{G}^{(p^r)}$ which pulls back along Frob^r to X^{p^r}. This coordinate on $\mathscr{G}^{(p^r)}$ then corresponds to one on $\mathscr{G}'$ via the isomorphism between the two formal groups (cf. (6.16)). Thus $\mathfrak{X}$ is also covariant with respect to any morphism in the subcategory $\mathrm{FG}_{\mathrm{isog}}(k)$.

Let $\mathfrak{X}_{\mathrm{nc}}\colon \mathrm{FG}_{\mathrm{isog}} \to \mathrm{Set}$ be the functor

$$\mathscr{G}/k \mapsto \{\text{norm-coherent coordinates on } \mathscr{F}^{\mathscr{G}}_{\mathrm{univ}}/E\}$$

where $\mathscr{F}^{\mathscr{G}}_{\mathrm{univ}}$ is a functorial choice of universal deformation of $\mathscr{G}$ as in Remark 4.11. The "wideness" of $\mathfrak{X}_{\mathrm{nc}}$, in the same sense as above, follows from Proposition 6.30 and the universal property in Corollary 5.12.

Theorem 7.22 *The natural transformation $\rho\colon \mathfrak{X}_{\mathrm{nc}} \to \mathfrak{X}$ of functors by restricting a coordinate on $\mathscr{F}^{\mathscr{G}}_{\mathrm{univ}}$ to $\mathscr{G}$ is an isomorphism. Moreover, it satisfies Galois descent: given $\mathscr{G}/k$ in $\mathrm{FG}_{\mathrm{isog}}$ and a Galois extension K/k, the following diagram commutes, where the vertical maps take fixed points under the Galois action.*

$$\begin{tikzcd}
\mathfrak{X}_{\mathrm{nc}}(\mathscr{G} \times_k K) \arrow[r, "\sim"] \arrow[d] & \mathfrak{X}(\mathscr{G} \times_k K) \arrow[d] \\
\mathfrak{X}_{\mathrm{nc}}(\mathscr{G}) \arrow[r, "\sim"] & \mathfrak{X}(\mathscr{G})
\end{tikzcd}$$

This diagram is natural in $\mathscr{G}/k$ and K/k.

Proof On each object in $\mathrm{FG}_{\mathrm{isog}}$, the natural transformation ρ is an isomorphism by Proposition 7.1, and the descent is clear since the condition (6.22) of norm coherence is stable under Galois actions. Each of the naturality properties is straightforward to check. $\square$

8 Norm coherence and H_∞ complex orientations

8.1 Introduction Given a Morava E-theory spectrum E, consider its $MU\langle 0\rangle$-orientations, i.e., homotopy multiplicative maps $MU\langle 0\rangle \to E$ (2.6). A necessary and sufficient condition for such an orientation to be H_∞ (2.8) is that its corresponding coordinate on the formal group of E is norm coherent.

Ando showed this for E-theories associated to the Honda formal groups over $\mathbb{F}_p$ [Ando 1995, Theorem 4.1.1]. There, the norm-coherence condition (6.22) boils down to the identity $[p] = f_p$ (cf. (7.3) and (6.9)). Moreover, he established the existence and uniqueness of coordinates, hence orientations, with the desired property [Ando 1995, Theorem 2.6.4].

In fact, to show that norm coherence is necessary and sufficient for H_∞ orientations, Ando's proof does not depend on the choice of the formal groups being the Honda formal groups (see [Ando 1995, Lemma 4.4.4]). However, his setup does require them be defined over $\mathbb{F}_p$ so that the relative p^r-power Frobenius is an endomorphism for every $r \geq 0$ (cf. [Ando–Hopkins–Strickland 2004, Proposition 2.5.1] and Remark 6.5).

With results in sufficient generality about level structures on formal groups from [Strickland 1997], Ando, Hopkins, and Strickland extended the applicability of the above condition for H_∞ orientations: $MU\langle 0\rangle$ generalizes to $MU\langle 2k\rangle$, $k \leq 3$, and E generalizes to any even periodic H_∞-ring spectrum whose zeroth homotopy is a p-regular admissible local ring with perfect residue field of characteristic p and whose formal group is of finite height [Ando–Hopkins–Strickland 2004, Proposition 6.1]. They did this by first reformulating Ando's condition so that in particular it applies to E-theories associated to formal groups over any perfect field of positive characteristic [Ando–Hopkins–Strickland 2004, Proposition 4.13].

Based on this general condition, Ando, Hopkins, and Strickland showed the existence and uniqueness of H_∞ $MU\langle 6\rangle$-orientations for H_∞ elliptic spectra, called the sigma orientations, from corresponding norm-coherent cubical structures of elliptic curves [Ando–Hopkins–Strickland 2004, Proposition 16.5].

However, when the elliptic spectrum represents an E-theory associated to the formal group of a supersingular elliptic curve, such an orientation does not factor through $MU\langle 4\rangle$ due to obstruction from Weil pairings (see [Ando–Strickland 2001, proof of Theorem 1.4]). Thus, in this case, we cannot deduce the existence and uniqueness of H_∞ $MU\langle 2k\rangle$-orientations for $0 \leq k \leq 2$ from the sigma orientation.

8.2 Setup Let E be a Morava E-theory spectrum, with $\mathscr{G}_E = \mathscr{F}^{\mathscr{G}}_{\text{univ}}$ for some $\mathscr{G}/k$ whose group law is as in (6.1).

We will show the existence and uniqueness of H_∞ $MU\langle 0\rangle$-orientations for E by combining Proposition 7.1 with Ando, Hopkins, and Strickland's condition for H_∞ orientations. Indeed, we need only check that their criterion [Ando–Hopkins–Strickland 2004, 4.14] and our definition (6.22) for norm coherence agree.

8.3 Descent for level structures on formal deformations We carry out the needed comparison by recalling the canonical descent data for level structures on $\mathscr{G}_E = \mathscr{F}^{\mathscr{G}}_{\text{univ}}$ from [Ando–Hopkins–Strickland 2004, Part 3]. Since $\mathscr{G}$ is over k of characteristic p, the finite subgroups $\mathscr{D}$ of $\mathscr{G}_E$ must be of degree p^r for some $r \geq 0$.

Let A be an "abstract" finite abelian group of order p^r. Let $S_E = \operatorname{Spf}\big(\pi_0(E)\big)$ and $T = \operatorname{Spf}(R)$ with R as in (6.1). Let $i\colon T \to S_E$ be a morphism of formal schemes, faithfully flat and locally of finite presentation, which classifies a deformation of $\mathscr{G}/k$ to R. Write $\mathscr{A}_T$ for the constant formal group scheme of A over T. We have the following (cf. [Ando–Hopkins–Strickland 2004, Definitions 3.1, 9.9, Proposition 10.10 (i), 12.5]).

Definition 8.4 A morphism

$$\ell\colon \mathscr{A}_T \to i^*\mathscr{G}_E \tag{8.5}$$

of formal groups over T, equivalent to a group homomorphism $\phi_\ell\colon A \to i^*\mathscr{G}_E(T)$, is a *level A-structure on $\mathscr{G}_E$*, if the effective Cartier divisor $\mathscr{D}_\ell := \sum_{a\in A}[\phi_\ell(a)]$ of degree p^r is a subgroup of $i^*\mathscr{G}_E$.

Remark 8.6 Note that a level A-structure ℓ on $\mathscr{G}_E$ uniquely corresponds to a finite subgroup $\mathscr{D} = \mathscr{D}_\ell$, which is different from the scheme-theoretic image of $\mathscr{A}_T$ under ℓ (the latter automatically a subgroup, but possibly of smaller degree). Automorphisms of A correspond to automorphisms of $\mathscr{D}$ (cf. [Ando–Hopkins–Strickland 2004, Definition 3.1 (3)]).

Given a level A-structure $\ell\colon \mathscr{A}_{\operatorname{Spf}(R)} \to i^*\mathscr{G}_E$ on $\mathscr{G}_E$ as above, we have the following (cf. [Ando–Hopkins–Strickland 2004, Definition 3.9, Remark 3.12]).

Definition 8.7 Define $\psi^E_\ell\colon \pi_0(E) \to R$ to be the composite

$$\pi_0(E) \xrightarrow{D_{p^r}} \pi_0\big(E^{(B\Sigma_{p^r})_+}\big) \to \pi_0\big(E^{(B\Sigma_{p^r})_+}\big)/I_{\text{tr}} \xrightarrow{\alpha^r} R$$

where the power operation D_{p^r} arises from the H_∞-ring structure of E (2.8), I_{tr} is the ideal generated by the images of transfers from proper subgroups of Σ_{p^r}, and α^r classifies the subgroup of $i^*\mathscr{G}_E$ corresponding to ℓ (8.6, 2.7).

Remark 8.8 In the presence of a level structure as in Definition 8.7, the structure morphism i of T over S_E in Definition 8.4 is given by the classifying map

$$\alpha\colon\ A^0 \xrightarrow{s^r} A^r \xrightarrow{\alpha^r} R$$

from Corollaries 4.12 and 5.12, while ψ_ℓ^E is precisely the classifying map

$$\alpha'\colon\ A^0 \xrightarrow{t^r} A^r \xrightarrow{\alpha^r} R$$

(cf. [Rezk 2009, Theorem B] for the identification with t^r).

Let $F := E^{X_+}$ and $f\colon\ E \to F$ be the natural map of H_∞-ring spectra. Given any level A-structure $\ell\colon\ \mathscr{A}_T \to i^*\mathscr{G}_E$ on $\mathscr{G}_E$, let ℓ' be the unique level A-structure on $\mathscr{G}_F$ induced by f,[11] so that the following pullback squares commute.

$$\begin{tikzcd}
\mathscr{A}_{T'} \arrow[r, "\ell'"] \arrow[d] & i'^*\mathscr{G}_F \arrow[r] \arrow[d] & \mathscr{G}_F \arrow[d, "\mathscr{G}_f"] \\
\mathscr{A}_T \arrow[r, "\ell"] & i^*\mathscr{G}_E \arrow[r] & \mathscr{G}_E
\end{tikzcd}$$

over

$$\begin{tikzcd}
T' \arrow[r, equal] \arrow[d] & i^*S_F \arrow[r, "i'"] \arrow[d] & S_F \arrow[d, "S_f"] \\
T \arrow[r, equal] & T \arrow[r, "i"] & S_E
\end{tikzcd}$$

[11] *Level A-structures on $\mathscr{G}_F$* are defined analogously to those on $\mathscr{G}_E$ as in Definition 8.4.

Let $\psi^F_{\ell'}\colon T' \to S_F$ be the morphism analogous to ψ^E_ℓ in Definition 8.7, obtained by the naturality of power operations on $E^0(X)$. We then have the following definition (cf. [Ando–Hopkins–Strickland 2004, 3.13–3.15]).

Definition 8.9 Define $\psi^{F/E}_\ell$ to be the unique T-morphism that fits into the following commutative diagram.

(8.10)
$$\begin{tikzcd}
S_F \arrow[d, "s_f"'] & i^*S_F \arrow[l] \arrow[r, "\psi^{F/E}_\ell"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[rrr, bend left, "\psi^F_{\ell'}", start anchor=north east, end anchor=north west, phantom] \arrow[rr, bend left=40, "\psi^F_{\ell'}"] & \psi^{E*}_\ell S_F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S_F \arrow[d, "s_f"] \\
S_E & T \arrow[l, "i"'] \arrow[r, equal] & T \arrow[r, "\psi^E_\ell"] & S_E
\end{tikzcd}$$

In particular, when $F = E^{(\mathbb{C}P^\infty)_+}$, write $\psi^{F/E}_\ell$ as

$$\psi^{\mathscr{G}/E}_\ell\colon i^*\mathscr{G}_E \to \psi^{E*}_\ell \mathscr{G}_E$$

Remark 8.11 Let $F = E^{(\mathbb{C}P^\infty)_+}$. When $A = \mathbb{Z}/p$, the diagram (8.10) lifts (3.6).

More generally, let $\mathscr{D} \subset i^*\mathscr{G}_E$ correspond to ℓ as in Remark 8.6. Comparing (8.10) to the universal example (5.14) and Remark 8.8, we see that $\psi^{\mathscr{G}/E}_\ell$ is precisely the isogeny l_D from (6.10) if we assume without loss of generality that the $\star$-isomorphism (4.13) is the identity.

8.12 Norm maps In view of [Ando–Hopkins–Strickland 2004, Theorem 3.25], we have compared above the ingredients that constitute descent data for level structures on $\mathscr{G}_E$ (level structures ℓ, classifying maps i and ψ^E_ℓ, isogenies $\psi^{\mathscr{G}/E}_\ell$) with corresponding terms from the earlier sections of this paper. There is one more and key ingredient which goes into the condition [Ando–Hopkins–Strickland 2004, 4.14] for H_∞ $MU\langle 0\rangle$-orientations.

Let $\psi\colon \mathscr{G} \to \mathscr{G}'$ be an isogeny of formal groups with kernel $\mathscr{K}$. Let $\mu, \pi\colon \mathscr{G} \times \mathscr{K} \to \mathscr{G}$ be the group, projection maps, and $q\colon \mathscr{G} \to \mathscr{G}/\mathscr{K}$ be the quotient map, as in (2.3). We have the following (cf. [Ando–Hopkins–Strickland 2004, Definitions 10.1, 10.9]).

Definition 8.13 Define $N_\psi\colon \mathcal{O}_{\mathscr{G}} \to \mathcal{O}_{\mathscr{G}'}$ to be the horizontal composite

$$\begin{tikzcd}
\mathcal{O}_{\mathscr{G}} \arrow[rr, dashed] \arrow[dr, "\mu^*"'] & & \mathcal{O}_{\mathscr{G}/\mathscr{K}} \arrow[r, "\sim"] \arrow[d, "q^*"] & \mathcal{O}_{\mathscr{G}'} \\
& \mathcal{O}_{\mathscr{G}\times\mathscr{K}} \arrow[r, "\mathrm{Norm}_{\pi^*}"'] & \mathcal{O}_{\mathscr{G}} \arrow[d, shift right, "\mu^*"'] \arrow[d, shift left, "\pi^*"] & \\
& & \mathcal{O}_{\mathscr{G}\times\mathscr{K}} &
\end{tikzcd} \tag{8.14}$$

where the vertical maps exhibit $\mathcal{O}_{\mathscr{G}/\mathscr{K}}$ as an equalizer, Norm_{π^*} sends a to the determinant of multiplication by a on $\mathcal{O}_{\mathscr{G}\times\mathscr{K}}$ as a finite free $\mathcal{O}_{\mathscr{G}}$-module via π^*, and the factorization through $\mathcal{O}_{\mathscr{G}/\mathscr{K}}$ was shown, e.g., in [Strickland 1997, Theorem 19].

Remark 8.15 Since $q \circ \mu = q \circ \pi$, we have $\mathrm{Norm}_{\pi^*} \circ \mu^* = q^* \circ \mathrm{Norm}_{q^*}$ (by an argument similar to the proof of the factorization mentioned above). Thus the dashed arrow in (8.14) is Norm_{q^*} by uniqueness from the universal property of an equalizer.

Suppose that the isogeny ψ is over a field k of characteristic p, and is hence of degree p^r for some $r \geq 0$. Comparing [Strickland 1997, Theorem 19 (i)] and Proposition 3.11, we see that N_ψ is precisely the map $\Lambda_\psi^* = \mathrm{Norm}_{\psi^*}$ in (3.10).

8.16 H_∞ **orientations from norm coherence** Having set up the notation for norm maps as well as descent of level structures, we observe that the condition of Ando, Hopkins, and Strickland for H_∞ $MU\langle 0\rangle$-orientations can be stated without reference to topological apparatus such as power operations.

Proposition 8.17 *Let $g\colon MU\langle 0\rangle \to E$ be a map of homotopy commutative ring spectra, and $X = X_g$ be its corresponding coordinate on $\mathscr{G}_E$ as in (2.6). Then the condition of [Ando–Hopkins–Strickland 2004, 4.14] that for any level structure (8.5) the section X satisfies the identity*

$$\psi_\ell^{E*}(X) = N_{\psi_\ell^{\mathscr{G}/E}}\, i^*(X) \tag{8.18}$$

is equivalent to the norm-coherence condition on X as in (6.22).

Proof Let ψ be the isogeny $\psi_\ell^{\mathscr{G}/E}\colon i^*\mathscr{G}_E \to \psi_\ell^{E*}\mathscr{G}_E$ over $R = \mathcal{O}_T$ from Definition 8.9, and Y be any coordinate on $i^*\mathscr{G}_E$. In view of Remark 8.6, we have from [Ando–Hopkins–Strickland 2004, 10.11] that

$$\psi^* N_\psi(Y) = q^* N_q(Y) = \prod_{a\in A} T_a^*(Y) = \prod_{Q\in\mathscr{D}(R)} \big(Y \underset{i^*G_E}{+} Y(Q)\big)$$

where $T_a\colon i^*\mathscr{G}_E \to i^*\mathscr{G}_E$ translates any R-point P on $i^*\mathscr{G}_E$ to $P+Q$, with $Q=\phi_\ell(a)$ (cf. (3.14) for the first equality). Comparing this to (6.19), with $\mathscr{F}=i^*\mathscr{G}_E$, we see that

$$\psi^* N_\psi(Y) = f_D^*(Y_{\mathscr{D}})$$

Now, given any coordinate X on $\mathscr{G}_E$, write $Y=Y_X := i^*(X)$ and $Y'=Y'_X := \psi_\ell^{E\,*}(X)$. Pulling (8.18) back along $\psi_\ell^{\mathscr{G}/E}$, we then obtain an equivalent identity

$$l_D^*(Y') = f_D^*(Y_{\mathscr{D}}) \tag{8.19}$$

where $l_D=\psi_\ell^{\mathscr{G}/E}$ from Remark 8.11, and $Y'=\alpha^{r*}t^{r*}(X)$ from Remark 8.8. In view of (6.11, 6.12), we see that (8.19) is equivalent to (6.22) (cf. (6.23)). It follows that (8.18) and our norm-coherence condition agree (cf. (6.28)). □

Corollary 8.20 *Let E, $\mathscr{G}_E$, and $\mathscr{G}$ be as in (8.2). Given any coordinate $X_{\mathscr{G}}$ on $\mathscr{G}$, there exists a unique coordinate X on $\mathscr{G}_E$ lifting $X_{\mathscr{G}}$ such that its corresponding $MU\langle 0\rangle$-orientation for E is H_∞.*

Proof Given Proposition 8.17, the corollary follows from Proposition 7.1. In particular, as p is not a zero-divisor in $\pi_0(E)$, we may apply [Ando–Hopkins–Strickland 2004, Proposition 6.1] for H_∞ $MU\langle 2k\rangle$-orientations with $k=0$ (cf. the discussion following 1.6 there). □

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