# SPECTRAL MODULI PROBLEMS FOR LEVEL STRUCTURES AND AN INTEGRAL JACQUET–LANGLANDS DUAL OF MORAVA E-THEORY

XUECAI MA AND YIFEI ZHU

Abstract. Given an $\mathbb{E}_\infty$-ring spectrum $R$, with motivation from chromatic homotopy theory, we define relative Cartier divisors for a spectral Deligne–Mumford stack and prove that, as a functor from connective $R$-algebras to topological spaces, it is relatively representable. We then solve various moduli problems of level structures on spectral abelian varieties, overcoming difficulty at primes dividing the level. In particular, we obtain higher-homotopical refinement for finite levels of the Lubin–Tate tower as $\mathbb{E}_\infty$-rings, which generalize Morava, Hopkins, Miller, Goerss, and Lurie's spectral realization at the ground level. Moreover, passing to the infinite level and then descending along the isomorphic Drinfeld tower, we obtain a Jacquet–Langlands dual to the Morava E-theory spectrum, along with homotopy fixed point spectral sequences dual to those studied by Devinatz and Hopkins. These serve as potential tools for computing higher-periodic homotopy types from pro-étale cohomology of $p$-adic general linear groups.

Contents



*Date*: February 9, 2025. Some parts need polishing – please read with caution and understanding.

## 1. Introduction

The stable homotopy category is a central topic in algebraic topology. Structured ring spectra are the most common examples studied, such as $H_\infty$ spectra and $\mathbb{E}_\infty$ spectra. In [Lur09a] and [Lur18b], Lurie uses spectral algebraic methods gives a proof of the Goerss-Hopkins-Miller theorem for topological modular forms. Except for the application of elliptic cohomology, Lurie also proved the $\mathbb{E}_\infty$ structures of Morava E-theories [Lur18b], which use the spectral version of deformation theory of certain $p$-divisible groups. The earliest proof of $\mathbb{E}_\infty$ structures of Morava E-theories is due to Goerss, Hopkins, and Miller [GH04]. They turned the problem into a moduli problem and developed an obstruction theory. One can finish the proof by computing the André–Quillen groups. Comparing with their method, Lurie's proof is more conceptual. There are more and more applications of spectral algebraic geometry in algebraic topology. Such as topological automorphic forms [BL10], Morava E-theories over any $\mathbb{F}_p$-algebra [Lur18b], not only just for a perfect field $k$. The construction of equivariant topological modular forms [GM23], elliptic Hochschild homology [ST23], and more.

On the other hand, moduli problems concerning deformations of formal groups with level structures are also representable, and moduli spaces of different levels form a Lubin–Tate tower [RZ96, FGL08, SW13]. We know that the universal objects of deformations of formal groups have higher algebraic analogs which are the Morava E-theories. A natural question is what are higher categorical analogues of moduli problems of deformations with level structures? And can we find higher categorical analogs of Lubin–Tate towers? Although the $\mathbb{E}_\infty$-structure of topological modular forms with level structures can be obtained from [HL16], we still hope that there exists a derived stack of spectral elliptic curves with level structures that provide us with a more moduli interpretation. Except this, in the computation of unstable homotopy groups of spheres, after applying the EHP spectral sequences and the Bousfield-Kuhn functor, we observe that some terms on the $E_2$-page also arise from the universal deformation of isogenies of formal groups. They are computed by the Morava E-theories on the classifying spaces of symmetric groups [Str97, Str98]. They can be viewed as sheaves on the Lubin–Tate tower. We hope to provide a more conceptual perspective on this fact within the higher categorical Lubin–Tate tower.

In this paper, we give an attempt to address this problem by studying specific moduli problems in spectral algebraic geometry. The main ingredient of our work is the derived version of Artin's representability theorem established in [Lur04, TV08]. We will use the spectral algebraic geometry version [Lur18c] in this paper. We study relative Cartier divisors in the context of spectral algebraic geometry. By imposing certain conditions, we define derived level structures of certain geometric objects in spectral algebraic geometry. Using this Artin representability theorem, we prove some representable results of moduli problems that arise from our derived level structures. We give some examples of applications involving derived level structures. We consider the moduli problem of spectral deformations with derived level structures of $p$-divisible groups. We prove that these moduli problems are representable by certain formal affine spectral Deligne–Mumford stacks and the corresponding spectra can provide us many interesting general cohomology theories.

We note here that the Goerss–Hopkins–Miller–Lurie sheaf does not directly apply to the moduli problems here due to the failure of étaleness (cf. [Dev23]). This is fixed by relative Cartier divisors analogous to Drinfeld's original approach to arithmetic moduli of (classical) elliptic curves [KM85, Introduction].

**Outline.** We work on spectral algebraic geometry in this paper. In Section 2, we define derived isogenies and prove that the kernel of a derived isogeny in some cases has the same phenomenon as in the classical case. This provides evidence that our derived versions of level structures must induce classical level structures. For representability reasons, we use moduli associated with sheaves to detect higher homotopy of derived versions of level structures. We define relative Cartier divisors in the context of spectral algebraic geometry. For a spectral Deligne–Mumford stack $X$ over a spectral Deligne–Mumford stack $S$, a relative Cartier divisor is a morphism $D \to S$ of spectral Deligne–Mumford stacks such that $D \to X$ is a closed immersion, the ideal sheaf of $D$ is a line bundle over $X$, and the morphism $D \to S$ is flat, proper and locally almost of finite presentation. We use Lurie's representability theorem to prove that the relative Cartier divisor is representable in certain cases. The main part of our proof involves computing of cotangent complex. Here is our first main result.

**Theorem A** (Theorem 2.17)**.** Suppose that $E$ is a spectral algebraic space over a connective $\mathbb{E}_\infty$-ring $R$, such that $E \to R$ is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor

$$\begin{aligned} \mathrm{CDiv}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{cn} \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation over $R$.

In Section 3, we define derived level structures of spectral elliptic curves. Roughly speaking, for a finite abstract abelian group $A$, usually equals $\mathbb{Z}/N\mathbb{Z}$, $\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}$, a derived level-$A$ structure of a spectral elliptic curve $E$ over an $\mathbb{E}_\infty$-ring $R$ is just a relative Cartier divisor $D \to E$ satisfying its restriction to the heart comes from an ordinary level-$A$ structure. We let $\mathrm{Level}(\mathcal{A}, E/R)$ denote the space of derived level-$A$ structures of a spectral elliptic curve $E/R$. We prove that moduli problems associated with derived level structures are representable.

**Theorem B** (Theorem 3.5)**.** Suppose that $E$ is a spectral elliptic curve over a connective $\mathbb{E}_\infty$-ring $R$. Then the functor

$$\begin{aligned} \mathrm{Level}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

is representable by an affine spectral Deligne–Mumford stack which is locally almost of finite presentation over the $\mathbb{E}_\infty$-ring $R$.

In classical algebraic geometry, except one-dimensional group curves, we also care level structures of $p$-divisible groups, which come from the full sections of commutative finite flat group schemes. In Section 3.2, we consider derived level structures of spectral $p$-divisible groups. Let $\mathrm{Level}(k, G_R/R)$ denote the space of derived level-$(\mathbb{Z}/p^k\mathbb{Z})^h$ structures of a height h spectral $p$-divisible group $G/R$.

**Theorem C** (Theorem 3.16)**.** Suppose $G$ is a spectral $p$-divisible group of height $h$ over a connective $\mathbb{E}_\infty$-ring R. Then the functor

$$\mathrm{Level}^k_{G/R} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}; \quad R' \to \mathrm{Level}(k, G_{R'}/R')$$

is representable by an affine spectral Deligne–Mumford stack $S(k) = \mathrm{Spét}\,\mathcal{P}^k_{G/R}$.

In Section 4, we give some applications of derived level structures. We first prove that the moduli problem of spectral elliptic curves with derived level-$A$ structures is representable by a spectral Deligne–Mumford stack.

**Theorem D** (Theorem 4.7)**.** Let $\mathrm{Ell}(\mathcal{A})(R)$ denote the space of spectral elliptic curves with derived level-$A$ structures over the $\mathbb{E}_\infty$-ring $R$. Then the functor

$$\begin{aligned} \mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R) \end{aligned}$$

is representable by a spectral Deligne–Mumford stack and this stack is locally almost of finite presentation over the sphere spectrum $\mathbb{S}$.

In [Lur18b], Lurie considers the spectral deformations of classical $p$-divisible groups. As we have the concept of derived level structures, it is natural to consider the moduli of spectral deformations with derived level structures of certain $p$-divisible groups. Suppose $G_0$ is a $p$-divisible group of height $h$ over a perfect $\mathbb{F}_p$-algebra $R_0$. We consider the following functor

$$\begin{aligned} \mathcal{M}^{or}_k \quad : \quad & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{or}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, k)$ is the $\infty$-category spanned by those quadruples $(G, \rho, e, \eta)$

1. G is a spectral $p$-divisible group over R.
2. $\rho$ is a equivalence class of $G_0$-taggings of $R$.
3. e is an orientation of the identity component of G.
4. $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of $G/R$.

Our next main result is the following.

**Theorem E** (Theorem 4.9)**.** The functor $\mathcal{M}^{or}_k$ is co-representable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_k$, where $\mathcal{JL}_k$ is a finite $R^{or}_{G_0}$-algebra, $R^{or}_{G_0}$ is the orientation deformation ring of $G_0$ defined in [Lur18b].

We will give another example of spectra constructed by considering moduli of spectral deformations with $p$-power order subgroups level structures, which can be viewed as topological realizations of universal objects of Strickland's deformations of Frobenius.

Finally, in Section 5, for every classical $p$-divisible group, we construct an $\mathbb{E}_\infty$-spectrum $\mathcal{JL}$ called the Jacquet–Langlands spectrum. By taking homotopy fixed points, we get a Jacquet–Langlands dual of Morava E-theories. We have a diagram in algebraic geometry:

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "\mathrm{GL}_n(\mathbb{Z}_p)"'] \arrow[dr, "\mathbb{G}_n"] & \\ \mathrm{LT}_K & & \mathcal{H}, \end{tikzcd}$$

where $LT_K$ is the moduli space of deformation of formal groups, $\mathcal{X}$ is the moduli space of defomration with level structures of formal groups, and $\mathcal{H}$ is the Drinfled upper half plane. It can be lift to the following diagram in the level of $\mathbb{E}_\infty$-spectra.

$$\begin{tikzcd} & \mathcal{JL} \arrow[dl, "\mathrm{GL}_n(\mathbb{Z}_p)"'] \arrow[dr, "\mathbb{G}_n"] & \\ E_n & & {}^L E_n \end{tikzcd}$$

**Question 1.1.** Compute higher homotopy groups of the finite-level and infinite-level Jacquet–Langlands spectra. These should encode more refined arithmetic-geometric information. Cf. higher algebraic K-theory, higher stable motivic stems, classification of knots not just up to isotopy, and the Habiro ring of a number field (elementless vs. categorification of elements).

**Notation and terminology.**

- Let CAlg denote the $\infty$-category of $\mathbb{E}_\infty$-rings and $\mathrm{CAlg}^{\mathrm{cn}}$ denote the $\infty$-category of connective $\mathbb{E}_\infty$-rings.
- Let $\mathcal{S}$ denote the $\infty$-category of spaces ($\infty$-groupoids).
- Given a spectral Deligne–Mumford stack $\mathsf{X} = (\mathcal{X}, \mathscr{O}_\mathcal{X})$, let $\tau_{\leq n}\mathsf{X}$ denote its $n$-truncation $(\mathcal{X}, \tau_{\leq n}\mathscr{O}_\mathcal{X})$ and $\mathsf{X}^\heartsuit$ denote its underlying ordinary stack $(\mathcal{X}^\heartsuit, \tau_{\leq 0}\mathscr{O}_\mathcal{X})$.
- By a spectral Deligne–Mumford stack $\mathsf{X}$ over an $\mathbb{E}_\infty$-ring $R$, we mean a morphism of spectral Deligne–Mumford stacks $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$. Given an $R$-algebra $S$, we sometimes write $\mathsf{X} \times_R S$ for the fiber product $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, S$.
- Let $\mathcal{M}_{\mathrm{ell}}$ denote the spectral Deligne–Mumford stack of spectral elliptic curves, as defined in [Lur18a], and $\mathcal{M}^{\mathrm{cl}}_{\mathrm{ell}}$ denote the (classical) Deligne–Mumford stack of (classical) elliptic curves.

## 2. Relative Cartier divisors of spectral Deligne–Mumford stacks

A main innovation of this paper concerns derived level structures. We begin with a derived version of isogenies and prove that, in certain cases, the kernel of a derived isogeny behaves similarly as in the classical setting. This gives evidence that our derived version of level structures must induce classical level structures. In Section 2.2, we define relative Cartier divisors in the setting of spectral algebraic geometry. We then use Lurie's representability theorem to prove that certain functors associated with relative Cartier divisors are representable by spectral Deligne–Mumford stacks. This paves the way for Section 3, where we establish specifically the representability of derived level structures for spectral elliptic curves and spectral $p$-divisible groups.

2.1. **Isogenies of spectral elliptic curves.** To define derived level structures, the first question we must address is what higher-categorical analogues of finite abelian groups are. Let us recall from [Lur17, Section 7.2.4] and [Lur18c, Section 2.7] some finiteness conditions in the context of $\mathbb{E}_\infty$-rings.

Let $A$ be an $\mathbb{E}_\infty$-ring and $M$ be an $A$-module. We say that $M$ is

- *perfect*, if it is a compact object of the $\infty$-category $\mathrm{LMod}_A$ of left $A$-modules;
- *almost perfect*, if there exists an integer $k$ such that $M \in (\mathrm{LMod}_A)_{\geq k}$ and $M$ is an almost compact object of $(\mathrm{LMod}_A)_{\geq k}$, that is, $\tau_{\leq n}M$ is a compact object of $\tau_{\leq n}\big((\mathrm{LMod}_A)_{\geq k}\big)$ for all $n \geq 0$;

- *perfect to order $n$*, if given any filtered diagram $\{N_\alpha\}$ in $(\mathrm{LMod}_A)_{\leq 0}$, the canonical map $\varinjlim_\alpha \mathrm{Ext}^i_A(M, N_\alpha) \to \mathrm{Ext}^i_A(M, \varinjlim_\alpha N_\alpha)$ is injective for $i = n$ and bijective for $i < n$;
- *finitely $n$-presented*, if $M$ is $n$-truncated and perfect to order $n+1$; and
- *finitely generated*, if it is perfect to order 0.

Next we recall finiteness conditions on algebras. We say that a morphism $\phi : A \to B$ of connective $\mathbb{E}_\infty$-rings is

- *of finite presentation*, if $B$ belongs to the smallest full subcategory of $\mathrm{CAlg}_A$ which contains $\mathrm{CAlg}_A^{\mathrm{free}}$ and is stable under finite colimits;
- *locally of finite presentation*, if $B$ is a compact object of $\mathrm{CAlg}_A$;
- *almost of finite presentation*, if $B$ is an almost compact object of $\mathrm{CAlg}_A$;
- *of finite generation to order $n$*, if the following condition holds;

  Let $\{C_\alpha\}$ be a filtered diagram of connective $\mathbb{E}_\infty$-rings over $A$ having colimit $C$. Assume that each $C_\alpha$ is $n$-truncated and that each of the transition maps $\pi_n C_\alpha \to \pi_n C_\beta$ is a monomorphism. Then the canonical map
  $$\varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$$
  is a homotopy equivalence.
- *of finite type*, if it is of finite generation to order 0; and
- *finite*, if $B$ is finitely generated as an $A$-module.

**Proposition 2.1** (cf. [Lur18c, Propositions 2.7.2.1 and 4.1.1.3])**.** *Let $\phi\colon A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. Then The following conditions are equivalent.*

- *The morphism $\phi$ is finite (resp. of finite type).*
- *The commutative ring $\pi_0 B$ is finite (resp. of finite type) over $\pi_0 A$.*

**Definition 2.2** (cf. [Lur18c, Definition 4.2.0.1])**.** Let $f\colon \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral Deligne–Mumford Stacks. We say that $f$ is *locally of finite type* (resp. *locally of finite generation to order $n$*, *locally almost of finite presentation*, *locally of finite presentation*) if the following condition holds. Given any commutative diagram

$$\begin{tikzcd} \mathrm{Sp\acute{e}t}\, B \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "f"] \\ \mathrm{Sp\acute{e}t}\, A \arrow[r] & \mathsf{Y} \end{tikzcd}$$

where the horizontal morphisms are étale, the $\mathbb{E}_\infty$-ring $B$ is of finite type (resp. of finite generation to order $n$, almost of finite presentation, locally of finite presentation) over $A$.

**Definition 2.3** ([Lur18c, Definition 5.2.0.1])**.** Let $f\colon (\mathcal{X}, \mathscr{O}_\mathcal{X}) \to (\mathcal{Y}, \mathscr{O}_\mathcal{Y})$ be a morphism of spectral Deligne–Mumford stacks. We say that $f$ is *finite* if the following conditions hold.

- The morphism $f$ is affine.
- The pushforward $f_*\mathscr{O}_\mathcal{X}$ is perfect to order 0 as a $\mathscr{O}_\mathcal{Y}$-module.

*Remark* 2.4. By [Lur18c, Example 4.2.0.2], a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne–Mumford stack is locally of finite type if and only if the underlying map of ordinary stacks is locally of finite type in the sense of classical algebraic geometry. Moreover, by [Lur18c, Remark 5.2.0.2], a morphism of $f : \mathsf{X} \to \mathsf{Y}$ is finite if and only if the underlying map $f^\heartsuit : \mathsf{X}^\heartsuit \to \mathsf{Y}^\heartsuit$ is finite. In particular, if $\mathsf{X}$ and $\mathsf{Y}$ are spectral algebraic spaces, then $f$ is finite if and only if $f^\heartsuit$ is finite in the classical sense.

Recall that a morphism $f : \mathsf{X} \to \mathsf{Y}$ of spectral Deligne–Mumford stacks is surjective if for every field $k$ and any map $\operatorname{Spét} k \to \mathsf{Y}$, the fiber product $\operatorname{Spét} k \times_{\mathsf{Y}} \mathsf{X}$ is nonempty [Lur18c, Definition 3.5.5.5].

**Definition 2.5.** Let $R$ be a connective $\mathbb{E}_\infty$-ring and $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of spectral abelian varieties over $R$. We call $f$ an *isogeny* if it is finite, flat, and surjective.

**Lemma 2.6.** *Let $f : \mathsf{X} \to \mathsf{Y}$ be an isogeny of spectral abelian varieties. Then $f^\heartsuit : \mathsf{X}^\heartsuit \to \mathsf{Y}^\heartsuit$ is an isogeny in the classical sense.*

*Proof.* For ordinary abelian varieties, $f^\heartsuit$ being an isogeny means that it is surjective and its kernel is finite. This is equivalent to $f^\heartsuit$ being finite, flat, and surjective [Mil86, Proposition 7.1]. From Definition 2.5, it is clear that $f^\heartsuit$ is finite and flat. We need only show that $f^\heartsuit$ is surjective.

By the definition of surjectivity above for morphisms of spectral Deligne–Mumford stacks, we get a commutative diagram

$$\begin{tikzcd} \operatorname{Spét} k' \arrow[r] \arrow[d] & \mathsf{X} \arrow[d] \\ \operatorname{Spét} k \arrow[r] & \mathsf{Y} \end{tikzcd}$$

The upper horizontal morphism corresponds to a morphism $\operatorname{Spét} k' \to \mathsf{X}^\heartsuit$ by the inclusion–truncation adjunction [Lur18c, Proposition 1.4.6.3]. On underlying topological spaces, this then corresponds to a point $|\operatorname{Spét} k'| \to |\mathsf{X}^\heartsuit|$. It is clear that this point in $|\mathsf{X}^\heartsuit|$ is a preimage of $|\operatorname{Spét} k|$ in $|\mathsf{Y}^\heartsuit|$. Therefore $f^\heartsuit$ is surjective. ☐

**Lemma 2.7.** *Let $f : \mathsf{X} \to \mathsf{Y}$ be an isogeny of spectral elliptic curves over a connective $\mathbb{E}_\infty$-ring $R$. Then $\operatorname{fib}(f)$ exists and is a finite and flat nonconnective spectral Deligne–Mumford stack over $R$.*

*Proof.* By [Lur18c, Proposition 1.4.11.1], finite limits of nonconnective spectral Deligne–Mumford stacks exist, so we can define $\operatorname{fib}(f)$. Let us consider the commutative diagram

$$\begin{tikzcd} \operatorname{fib}(f) \arrow[r] \arrow[d, "f'"] & \mathsf{X} \arrow[d, "f"] \arrow[ddr] & \\ * \arrow[r] \arrow[drr, "i"] & \mathsf{Y} \arrow[dr] & \\ & & \operatorname{Spét} R \end{tikzcd}$$

where the square is a pullback diagram. We find that $\operatorname{fib}(f)$ is over $\operatorname{Spét} R$. By [Lur18c, Remark 2.8.2.6], $f' : \operatorname{fib}(f) \to *$ is flat because it is a pullback of a flat morphism. Clearly $i : * \to \operatorname{Spét} R$ is flat, so by [Lur18c, Example 2.8.3.12] (being a flat morphism is a property local on the source with respect to the flat topology), $i \circ f' : \operatorname{fib}(f) \to \operatorname{Spét} R$ is flat.

Next we show that $\mathrm{fib}(f)$ is finite over $R$. Since $*$, $\mathsf{X}$, and $\mathsf{Y}$ are all spectral algebraic spaces, so is $\mathrm{fib}(f)$. Moreover, $\mathrm{Sp\acute{e}t}\, R$ is a spectral algebraic space [Lur18c, Example 1.6.8.2]. By Remark 2.4, we need only prove that the underlying morphism is finite. Since the truncation functor is a right adjoint, it preserves limits. Thus we get a pullback diagram

$$\begin{tikzcd} \mathrm{fib}(f)^{\heartsuit} \arrow[r] \arrow[d] & \mathsf{X}^{\heartsuit} \arrow[d] \\ * \arrow[r] & \mathsf{Y}^{\heartsuit} \end{tikzcd}$$

So we are reduced to showing that given an isogeny $f^{\heartsuit} : \mathsf{X}^{\heartsuit} \to \mathsf{Y}^{\heartsuit}$ of ordinary abelian varieties over a commutative ring $R$, its kernel is finite over $R$. This is true in classical algebraic geometry [Mil86, Proposition 7.1]. □

**Lemma 2.8.** *Given an integer $N \geq 1$, let $f_N : \mathsf{E} \to \mathsf{E}$ be an isogeny of spectral elliptic curves over a connective $\mathbb{E}_\infty$-ring $R$ such that the underlying morphism is the multiplication-by-$N$ map $[N] : \mathsf{E}^{\heartsuit} \to \mathsf{E}^{\heartsuit}$. Then $\mathrm{fib}(f_N)$ is finite flat of degree $N^2$ in the sense of* [Lur18c, Definition 5.2.3.1]*. Moreover, if $N$ is invertible in $\pi_0 R$, then $\mathrm{fib}(f_N)$ is an étale-locally constant sheaf.*

*Proof.* By [KM85, Theorem 2.3.1], we know that $[N] : \mathsf{E}^{\heartsuit} \to \mathsf{E}^{\heartsuit}$ is finite locally free of rank $N^2$ in the classical sense. When $N$ is invertible in $\pi_0 R$, its kernel is an étale-locally constant sheaf. Now, from Lemma 2.7, $\mathrm{fib}(f_N)$ is a spectral algebraic space that is finite and flat, and its underlying space $\mathrm{fib}(f_N)^{\heartsuit} = \ker[N]$ is locally free of rank $N^2$. We need to prove that $\mathrm{fib}(f_N) \to \mathrm{Sp\acute{e}t}\, R$ is locally free of rank $N^2$ in spectral algebraic geometry. Observe that since $\mathrm{fib}(f_N)$ is finite and flat, it is affine. We are thus reduced to proving the above for affines, i.e., $f_N|_{\mathrm{Sp\acute{e}t}\, S} : \mathrm{Sp\acute{e}t}\, S \to \mathrm{Sp\acute{e}t}\, R$ is locally free of rank $N^2$ for any affine substack $\mathrm{Sp\acute{e}t}\, S$ of $\mathrm{fib}(f_N)$. This is equivalent to proving that $R \to S$ is locally free of rank $N^2$ in the sense of [Lur18c, Definition 2.9.2.1]. Therefore we need to prove the following:

(1) The ring $S$ is locally free of finite rank over $R$ (by [Lur17, Proposition 7.2.4.20], this is equivalent to saying that $S$ is a flat and almost perfect $R$-module).
(2) For every $\mathbb{E}_\infty$-ring maps $R \to k$ with $k$ a field, the vector space $\pi_0(k \otimes_R S)$ is an $N^2$-dimensional $k$-vector space.

For (1), we know that $\pi_0 S$ is a projective $\pi_0 R$-module and that $S$ is a flat $R$-module, so by [Lur17, Proposition 7.2.2.18], $S$ is a projective $R$-module. By [Lur17, Corollary 7.2.2.9], since $\pi_0 S$ is a finitely generated $\pi_0 R$-module, $S$ is a retract of a finitely generated free $R$-module, and is therefore locally free of finite rank.

For (2), by [Lur17, Corollary 7.2.1.23], since $R$ and $S$ are connective, we have $\pi_0(k \otimes_R S) \simeq k \otimes_{\pi_0 R} \pi_0 S$, which is an $N^2$-dimensional $k$-vector space, as $\pi_0 S$ is a rank-$N^2$ free $\pi_0 R$-module from above.

We next show that if $N$ is invertible in $\pi_0 R$, then $\mathrm{fib}(f_N)$ is a locally constant sheaf. Since $\mathrm{fib}(f_N)$ is a spectral Deligne–Mumford stack, its associated functor of points $\mathrm{fib}(f_N) : \mathrm{CAlg}_R \to \mathcal{S}$ is nilcomplete and locally almost of finite presentation. By [KM85, Theorem 2.3.1], $\mathrm{fib}(f_N)|_{\mathrm{CAlg}^{\heartsuit}_{\pi_0 R}}$ is a locally constant sheaf. The desired result then follows from the lemma below. □

**Lemma 2.9.** *Let $R$ be a connective $\mathbb{E}_\infty$-ring. Let $\mathscr{F} \in \mathcal{S}\mathrm{hv}^{\acute{e}t}(\mathrm{CAlg}^{\mathrm{cn}}_R)$ be nilcomplete and locally almost of finite presentation. Suppose that $\mathscr{F}|_{(\mathrm{CAlg}^{\mathrm{cn}}_R)^{\heartsuit}}$ is a locally* *constant presheaf. Then $\mathscr{F}$ is a (homotopy) locally constant sheaf (i.e., sheafification of a homotopy-locally constant presheaf).*

*Proof.* Let us choose an étale cover $\{U_i^0\}$ of $\pi_0 R$ such that $\mathscr{F}|_{U_i^0}$ is a constant sheaf for each $i$. By [Lur17, Theorem 7.5.1.11], this corresponds to an étale cover $\{U_i\}$ of $R$ such that $\pi_0 U_i = U_i^0$. For each $i$ and $n$, we consider the diagram

$$\begin{tikzcd} \tau_{\leq 0} R \arrow[r] \arrow[d] & \tau_{\leq 0} U_i \arrow[d] \\ \tau_{\leq n} R \arrow[r] & \tau_{\leq n} U_i \end{tikzcd}$$

which is a pushout diagram, since $U_i$ is an étale $R$-algebra. This is a colimit diagram in $\tau_{\leq n}\mathrm{CAlg}_R$. Since $\mathscr{F}$ is a sheaf locally almost of finite presentation, we then get a pushout diagram

$$\begin{tikzcd} \mathscr{F}(\tau_{\leq 0} R) \arrow[r] \arrow[d] & \mathscr{F}(\tau_{\leq 0} U_i) \arrow[d] \\ \mathscr{F}(\tau_{\leq n} R) \arrow[r] & \mathscr{F}(\tau_{\leq n} U_i) \end{tikzcd}$$

Without loss of generality, we may assume that each $U_i$ is connective. Thus the values $\mathscr{F}(\tau_{\leq 0} U_i)$ is independent of $i$. This implies that $\mathscr{F}(\tau_{\leq n} U_i)$ are all equivalent. Since $\mathscr{F}$ is nilcomplete, $\mathscr{F}(U_i) \simeq \varinjlim_n \mathscr{F}(\tau_{\leq n} U_i)$, and so all $\mathscr{F}(U_i)$ are equivalent. $\square$

2.2. **Cartier divisors and an exercise of spectral Artin representability.** In this subsection, we define relative Cartier divisors in the context of spectral algebraic geometry. We then use Lurie's spectral Artin representability theorem to prove that relative Cartier divisors are representable in certain cases. Let us first recall this spectral analogue of Artin's representability criterion in classical algebraic geometry.

**Theorem 2.10** ([Lur18c, Theorem 18.3.0.1])**.** *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Suppose that we have a natural transformation $f : X \to \operatorname{Spec} R$, where $R$ is a Noetherian $\mathbb{E}_\infty$-ring with $\pi_0 R$ a Grothendieck ring. Given $n \geq 0$, $X$ is representable by a spectral Deligne–Mumford $n$-stack which is locally almost of finite presentation over $R$ if and only if the following conditions are satisfied:*

1. *For every discrete commutative ring $A$, the space $X(A)$ is $n$-truncated.*
2. *The functor $X$ is a sheaf for the étale topology.*
3. *The functor $X$ is nilcomplete, infinitesimally cohesive, and integrable.*
4. *The functor $X$ admits a connective cotangent complex $L_X$.*
5. *The natural transformation $f$ is locally almost of finite presentation.*

Given a locally spectrally ringed topos $\mathsf{X} = (\mathcal{X}, \mathscr{O}_\mathcal{X})$, we can consider its functor of points

$$h_\mathsf{X} \colon \infty\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}} \to \mathcal{S}, \quad \mathsf{Y} \mapsto \mathrm{Map}_{\infty\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathsf{Y}, \mathsf{X})$$

In particular, by [Lur18c, Remark 3.1.1.2], a closed immersion $f : (\mathcal{Y}, \mathscr{O}_\mathcal{Y}) \to (\mathcal{X}, \mathscr{O}_\mathcal{X})$ of locally spectrally ringed topoi corresponds to a morphism $\mathscr{O}_\mathcal{X} \to f_* \mathscr{O}_\mathcal{Y}$ of sheaves over $\mathcal{X}$ of connective $\mathbb{E}_\infty$-rings such that $\pi_0 \mathscr{O}_\mathcal{X} \to \pi_0 f_* \mathscr{O}_\mathcal{Y}$ is an epimorphism. We denote this epimorphism by $\alpha$. Given a closed immersion $f \colon \mathsf{D} \to \mathsf{X}$ of spectral Deligne–Mumford stacks, we let $\mathcal{I}(\mathsf{D})$ denote $\ker(\alpha)$, called the ideal sheaf of $\mathsf{D}$.

To prove relative representability for Cartier divisors below, we need the representability of Picard functors. Given a map $f : \mathsf{X} \to \operatorname{Spét} R$ of spectral Deligne–Mumford stacks, we can define a functor

$$\mathscr{P}\mathrm{ic}_{\mathsf{X}/R} \colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathscr{P}\mathrm{ic}(\operatorname{Spét} R' \times_{\operatorname{Spét} R} \mathsf{X})$$

If $f$ admits a section $x \colon \operatorname{Spét} R \to \mathsf{X}$, then pullback along $x$ gives a natural transformation of functors $\mathscr{P}\mathrm{ic}_{\mathsf{X}/R} \to \mathscr{P}\mathrm{ic}_{R/R}$. We let

$$\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R} \colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$$

denote the fiber of this map.

**Theorem 2.11** ([Lur18c, Theorem 19.2.0.5])**.** *Let $f : \mathsf{X} \to \operatorname{Spét} R$ be a map of spectral algebraic spaces which is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected over an $\mathbb{E}_\infty$-ring $R$. Suppose that $x \colon \operatorname{Spét} R \to \mathsf{X}$ is a section of $f$. Then the functor $\mathscr{P}\mathrm{ic}^x_{\mathsf{X}/R}$ is representable by a spectral algebraic space which is locally of finite presentation over $R$.*

In the classical setting, schemes representing relative Cartier divisors are open subschemes of Hilbert schemes [Kol96, Theorem 1.13]. However, in the derived setting, the Hilbert functor is representable by a spectral algebraic space [Lur04, Theorem 8.3.3], and it is hard to establish an analogous relationship. We will directly study relative Cartier divisors and their spectral moduli as follows.

**Definition 2.12** (Relative Cartier divisor)**.** Suppose that $\mathsf{X}$ is a spectral Deligne–Mumford stack over a spectral Deligne–Mumford stack $\mathsf{S}$. We let $\mathrm{CDiv}(\mathsf{X}/\mathsf{S})$ denote the $\infty$-category of closed immersions $\mathsf{D} \to \mathsf{X}$, such that $\mathsf{D}$ is flat, proper, locally almost of finite presentation over $\mathsf{S}$ and the associated ideal sheaf of $\mathsf{D}$ is locally free of rank one over $\mathsf{X}$.

*Remark* 2.13. It is easy to see that given any spectral Deligne–Mumford stack $\mathsf{X}$ over $\mathsf{S}$, $\mathrm{CDiv}(\mathsf{X}/\mathsf{S})$ is a Kan complex, since all objects are closed immersions of $\mathsf{X}$. Let $\mathsf{D} \to \mathsf{D}'$ be a morphism. Then we have a diagram

$$\begin{tikzcd} \mathsf{D} \arrow[rr, "f"] \arrow[dr] & & \mathsf{D}' \arrow[dl] \\ & \mathsf{X} & \end{tikzcd}$$

By the definition of closed immersions, they are all equivalent to the same substack of $\mathsf{X}$, so $f$ is an equivalence (cf. [Lur18c, Remark 3.1.1.2]).

**Lemma 2.14.** *Let $\mathsf{X}/\mathsf{S}$ be a spectral Deligne–Mumford stack as above, and $\mathsf{T} \to \mathsf{S}$ be a map of spectral Deligne–Mumford stacks. If we have a relative Cartier divisor $\mathsf{D} \to \mathsf{X}$, then $\mathsf{D}_\mathsf{T}$ is a relative Cartier divisor of $\mathsf{X}_\mathsf{T}$.*

*Proof.* This is straightforward to check. We simply note that $\mathsf{D}_\mathsf{T}$ is a closed immersion of $\mathsf{X}_\mathsf{T}$ [Lur18c, Corollary 3.1.2.3]. After base change, $\mathsf{D}_\mathsf{T}$ is flat, proper, and locally almost of finite presentation over $\mathsf{T}$. It remains to show that $\mathcal{I}(\mathsf{D}_\mathsf{T})$ is a line bundle over $\mathsf{X}_\mathsf{T}$. Indeed, we have a fiber sequence

$$\mathcal{I}(\mathsf{D}) \to \mathscr{O}_\mathcal{X} \to \mathscr{O}_\mathcal{D}$$

By the flatness of $\mathsf{D}$, pullback along the base change $f\colon \mathsf{T} \to \mathsf{S}$ gives another fiber sequence

$$f^*\big(\mathcal{I}(\mathsf{D})\big) \to \mathscr{O}_{\mathcal{X}_\mathsf{T}} \to \mathscr{O}_{\mathcal{D}_\mathsf{T}}$$

So we have that $\mathcal{I}(\mathsf{D}_\mathsf{T})$ is just $f^*\big(\mathcal{I}(\mathsf{D})\big)$, which is invertible. $\square$

Suppose that $\mathsf{X}$ is a spectral Deligne–Mumford stack over an affine spectral Deligne–Mumford stack $\mathsf{S} = \operatorname{Spét} R$. From Definition 2.12, we then have a functor

$$\mathrm{CDiv}_{\mathsf{X}/R}\colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{CDiv}(\mathsf{X}_{R'}/R')$$

Our main goal in this section is to prove that this functor is representable when $\mathsf{X}/R$ is a spectral algebraic space satisfying certain conditions. To achieve this, we need some preparations for computing the cotangent complex of a relative Cartier divisor functor. The main issue has to do with square-zero extensions, for which we need the following facts about pushouts of two closed immersions.

By [Lur18c, Theorem 16.2.0.1 and Proposition 16.2.3.1], given a pushout square of spectral Deligne–Mumford stacks

$$\begin{tikzcd} \mathsf{X}_{01} \arrow[r, "i"] \arrow[d, "j"] & \mathsf{X}_0 \arrow[d, "j'"] \\ \mathsf{X}_1 \arrow[r, "i'"] & \mathsf{X} \end{tikzcd}$$

such that $i$ and $j$ are closed immersions, the induced square of $\infty$-categories

$$\begin{tikzcd} \mathrm{QCoh}(\mathsf{X}_{01}) & \mathrm{QCoh}(\mathsf{X}_0) \arrow[l] \\ \mathrm{QCoh}(\mathsf{X}_1) \arrow[u] & \mathrm{QCoh}(\mathsf{X}) \arrow[l] \arrow[u] \end{tikzcd}$$

determines an embedding $\theta\colon \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X}_0) \times_{\mathrm{QCoh}(\mathsf{X}_{01})} \mathrm{QCoh}(\mathsf{X}_1)$, which restricts to an equivalence

$$\mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}} \to \mathrm{QCoh}(\mathsf{X}_0)^{\mathrm{cn}} \times_{\mathrm{QCoh}(\mathsf{X}_{01})^{\mathrm{cn}}} \mathrm{QCoh}(\mathsf{X}_1)^{\mathrm{cn}}$$

between connective objects. Moreover, let $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})$ and set

$$\mathscr{F}_0 = j'^* \in \mathrm{QCoh}(\mathsf{X}_0), \quad \mathscr{F}_1 = i'^*\mathscr{F} \in \mathrm{QCoh}(\mathsf{X}_1)$$

Then $\mathscr{F}$ is $n$-connective if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are $n$-connective, and this statement is also true for the conditions of almost connective, Tor-amplitude $\leq n$, flat, perfect to order $n$, almost perfect, perfect, and locally free of finite rank, respectively.

Also, by [Lur18c, Theorem 16.3.0.1], we have a pullback square of $\infty$-categories

$$\begin{tikzcd} \mathrm{SpDM}_{/\mathsf{X}} \arrow[r] \arrow[d] & \mathrm{SpDM}_{/\mathsf{X}_0} \arrow[d] \\ \mathrm{SpDM}_{/\mathsf{X}_1} \arrow[r] & \mathrm{SpDM}_{/\mathsf{X}_{01}} \end{tikzcd}$$

Let $f : \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne–Mumford stacks. Let $\mathsf{Y}_0 = \mathsf{X}_0 \times_\mathsf{X} \mathsf{Y}$, $\mathsf{Y}_1 = \mathsf{X}_1 \times_\mathsf{X} \mathsf{Y}$, and let $f_0 : \mathsf{Y}_0 \to \mathsf{X}_0$ and $f_1 : \mathsf{Y}_1 \to \mathsf{X}_1$ be the projection maps. Then we have that $f$ is locally almost of finite presentation if and only if both $f_0$ and $f_1$ are locally almost of finite presentation. The statement remains true for the following individual conditions: locally of finite generation to order $n$, locally of finite presentation, étale, equivalence, open immersion, closed immersion, flat, affine, separated, and proper [Lur18c, Proposition 16.3.2.1].

Now, let $\mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ be a spectral Deligne–Mumford stack, $\mathscr{E} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{cn}}$ be a connective quasi-coherent sheaf, and $\eta \in \mathrm{Der}(\mathscr{O}_{\mathcal{X}}, \Sigma\mathscr{E})$ be a derivation, i.e., a morphism $\eta\colon \mathscr{O}_{\mathcal{X}} \to \mathscr{O}_{\mathcal{X}} \oplus \Sigma\mathscr{E}$. We let $\mathscr{O}_{\mathcal{X}}^{\eta}$ denote the square-zero extension of $\mathscr{O}_{\mathcal{X}}$ by $\mathscr{E}$ determined by $\eta$, so that we have a pullback diagram

$$\begin{tikzcd} \mathscr{O}_{\mathcal{X}}^{\eta} \arrow[r] \arrow[d] & \mathscr{O}_{\mathcal{X}} \arrow[d, "\eta"] \\ \mathscr{O}_{\mathcal{X}} \arrow[r, "0"] & \mathscr{O}_{\mathcal{X}} \oplus \Sigma\mathscr{E} \end{tikzcd}$$

By [Lur18c, Proposition 17.1.3.4], $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{\eta})$ is a spectral Deligne–Mumford stack, which we will denote by $\mathsf{X}^{\eta}$. In the case of $\eta = 0$, we denote it by $\mathsf{X}^{\mathscr{E}} = (\mathcal{X}, \mathscr{O}_{\mathcal{X}} \oplus \mathscr{E})$. We then have a pushout square of spectral Deligne–Mumford stacks

$$\begin{tikzcd} \mathsf{X}^{\mathscr{E}} & \mathsf{X} \arrow[l] \\ \mathsf{X} \arrow[u] & \mathsf{X}^{\Sigma\mathscr{E}} \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

such that $f$ and $g$ are closed immersions. In turn, by [Lur18c, Theorem 16.2.0.1], there is a pullback diagram

$$\begin{tikzcd} \mathrm{QCoh}(\mathsf{X}^{\mathscr{E}})^{\mathrm{acn}} \arrow[r] \arrow[d] & \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}} \arrow[d] \\ \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}} \arrow[r] & \mathrm{QCoh}(\mathsf{X}^{\Sigma\mathscr{E}})^{\mathrm{acn}} \end{tikzcd}$$

of categories spanned by almost connective quasi-coherent sheaves. Passing to homotopy fibers over some $\mathscr{F} \in \mathrm{QCoh}(\mathsf{X})^{\mathrm{acn}}$, we obtain an equivalence

$$\mathrm{QCoh}(\mathsf{X}^{\mathscr{E}})^{\mathrm{acn}} \times_{\mathrm{QCoh}(\mathsf{X})} \{\mathscr{F}\} \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}\big(\mathscr{F}, \Sigma(\mathscr{E} \otimes \mathscr{F})\big)$$

as in [Lur18c, Proposition 19.2.2.2]. Similarly, by passing to the homotopy fibers over some $\mathsf{Z} \in \mathrm{SpDM}_{/\mathsf{X}}$ with $f : \mathsf{Z} \to \mathsf{X}$, we obtain the classification of first-order deformations of $\mathsf{X}$:

$$\mathrm{SpDM}_{/\mathsf{X}^{\mathscr{E}}} \times_{\mathrm{SpDM}_{/\mathsf{X}}} \{\mathsf{Z}\} \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{Z})}(L_{\mathsf{Z}/\mathsf{X}}, \Sigma f^{*}\mathscr{E})$$

[Lur18c, Proposition 19.4.3.1].

**Lemma 2.15.** *Let $f : \mathsf{X} \to \mathrm{Spét}\, R$ be a morphism of spectral Deligne–Mumford stacks, and $M$ a connective $R$-module. Consider the $\infty$-category of Deligne–Mumford stacks $\mathsf{X}'$ equipped with a morphism $f' : \mathsf{X}' \to \mathrm{Spét}\,(R \oplus M)$ that fits into the pullback diagram*

$$\begin{tikzcd} \mathsf{X} \arrow[r] \arrow[d, "f"'] & \mathsf{X}' \arrow[d, "f'"] \\ \mathrm{Spét}\, R \arrow[r] & \mathrm{Spét}\,(R \oplus M) \end{tikzcd}$$

*Then this $\infty$-category is a Kan complex, and it is canonically homotopy equivalent to the mapping space $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}, \Sigma f^* M)$. Moreover, if $f$ is flat, proper, and locally almost of finite presentation, then so is $f'$.*

*Proof.* We have a pullback square of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} R \oplus M \arrow[r] \arrow[d] & R \arrow[d, "(\mathrm{Id},0)"] \\ R \arrow[r] & R \oplus \Sigma M \end{tikzcd}$$

which corresponds to a pushout square of spectral Deligne–Mumford stacks

$$\begin{tikzcd} \mathrm{Sp\acute{e}t}\, R \oplus M & \mathrm{Sp\acute{e}t}\, R \arrow[l] \\ \mathrm{Sp\acute{e}t}\, R \arrow[u] & \mathrm{Sp\acute{e}t}\,(R \oplus \Sigma M) \arrow[l] \arrow[u] \end{tikzcd}$$

such that the morphisms $\mathrm{Sp\acute{e}t}\,(R \oplus \Sigma M) \to \mathrm{Sp\acute{e}t}\, R$ are closed immersions. This exhibits $\mathrm{Sp\acute{e}t}\,(R \oplus M)$ as an "infinitesimal thickening" of $\mathrm{Sp\acute{e}t}\, R$ determined by $R \xrightarrow{(\mathrm{Id},0)} R \oplus \Sigma M$.

The first part of this lemma follows from the formula for first-order deformations of [Lur18c, Proposition 19.4.3.1]. The second part follows from properties of pushout of two closed immersions [Lur18c, Corollary 16.4.2.1]. □

**Lemma 2.16.** *Suppose that we are given a pushout diagram of spectral Deligne–Mumford stacks*

$$\begin{tikzcd} \mathsf{X}_{01} \arrow[r, "i"] \arrow[d, "j"] & \mathsf{X}_0 \arrow[d] \\ \mathsf{X}_1 \arrow[r] & \mathsf{X} \end{tikzcd}$$

*where $i$ and $j$ are closed immersions. Let $f \colon \mathsf{Y} \to \mathsf{X}$ be a map of spectral Deligne–Mumford stacks. Let $\mathsf{Y}_0 = \mathsf{X}_0 \times_{\mathsf{X}} \mathsf{Y}$, $\mathsf{Y}_1 = \mathsf{X}_1 \times_{\mathsf{X}} \mathsf{Y}$, and let $f_0 : \mathsf{Y}_0 \to \mathsf{X}_0$ and $f_1 \colon \mathsf{Y}_1 \to \mathsf{X}_1$ be the projection maps. If $f_0$ and $f_1$ are both closed immersions and determine line bundles over $\mathsf{Y}_0$ and $\mathsf{Y}_1$ respectively, then $f$ is a closed immersion and determines a line bundle over $\mathsf{Y}$.*

*Proof.* The statement concerning closed immersions follows from [Lur18c, Proposition 16.3.2.1]. For the line-bundle part, we notice that by [Lur18c, Theorem 16.2.0.1 and Proposition 16.2.3.1], $f$ determines a sheaf locally free of finite rank. To show that this sheaf is a line bundle, we proceed locally. By [Lur18c, Theorem 16.2.0.2], given a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

such that $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, there is an equivalence $F$ : $\mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{Mod}_{A_0}^{\mathrm{cn}} \times_{\mathrm{Mod}_{A_{01}}^{\mathrm{cn}}} \mathrm{Mod}_{A_1}^{\mathrm{cn}}$. Moreover, this is a symmetric monoidal equivalence. Indeed, since $F(M) = (A_0 \otimes_A M, A_1 \otimes_A M, A_{01} \otimes_{A_0} A_0 \otimes_A M \simeq A_{01} \otimes_{A_1}$

$A_1 \otimes_A M)$, we have $F(M \otimes_A N) \simeq F(M) \otimes F(N)$. By [Lur18c, Proposition 2.9.4.2], line bundles over $A_1$, $A_{01}$, and $A_0$ determine invertible objects of $\mathrm{Mod}^{\mathrm{cn}}_{A_1}$, $\mathrm{Mod}^{\mathrm{cn}}_{A_{01}}$, and $\mathrm{Mod}^{\mathrm{cn}}_{A_1}$ respectively, which in turn determine an invertible object of $\mathrm{Mod}^{\mathrm{cn}}_{A}$, hence a line bundle over $A$. □

Here is the main result of this section and the technical heart of the paper.

**Theorem 2.17.** *Let $\mathsf{E}/R$ be a spectral algebraic space that is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor*

$$\begin{aligned} \mathrm{CDiv}_{\mathsf{E}/R} \colon \mathrm{CAlg}^{\mathrm{cn}}_R &\to \mathcal{S} \\ R' &\mapsto \mathrm{CDiv}(\mathsf{E}_{R'}/R') \end{aligned}$$

*is representable by a spectral algebraic space which is locally almost of finite presentation over $R$.*

*Proof.* We apply Lurie's spectral Artin representability theorem and verify the 5 criteria from Theorem 2.10 one by one, in the case of $n = 0$, as follows:

(1) Lemma 2.18;
(2) Lemma 2.19;
(3) Lemmas 2.20, 2.21, 2.22;
(4) Lemma 2.24; and
(5) Lemma 2.23.

These statements and their proofs occupy the rest of this section. □

**Lemma 2.18.** *For every discrete commutative $R_0$, the space $\mathrm{CDiv}_{\mathsf{E}/R}(R_0)$ is 0-truncated.*

*Proof.* Recall that $\mathrm{CDiv}_{\mathsf{E}/R}(R_0)$ consists of closed immersions $\mathsf{D} \to \mathsf{E} \times_R R_0$ such that $\mathsf{D}$ is flat and proper over $R_0$. Therefore, if $R_0$ is discrete, so are the objects $\mathsf{D}$, and so $\mathrm{CDiv}_{\mathsf{E}/R}(R_0)$ is 0-truncated. □

**Lemma 2.19.** *The functor $\mathrm{CDiv}_{\mathsf{E}/R}$ is a sheaf for the étale topology.*

*Proof.* Let $\{R' \to U_i\}_{i \in I}$ be an étale cover of $\mathrm{Sp\acute{e}t}\, R'$, and $U_\bullet$ be the associated Čech-simplicial object. We need to prove that the map

$$\mathrm{CDiv}_{\mathsf{E}/R}(R') \to \varprojlim_{\mathbf{\Delta}} \mathrm{CDiv}_{\mathsf{E}/R}(U_\bullet)$$

is an equivalence. Unwinding the definitions, we need only prove the following general result: Given a spectral Deligne–Mumford stack $\mathsf{X}/\mathsf{S}$ and an étale cover $\mathsf{T}_i \to \mathsf{S}$, we have a homotopy equivalence

$$\mathrm{CDiv}(\mathsf{X}/\mathsf{S}) \to \varprojlim_{\mathbf{\Delta}} \mathrm{CDiv}(\mathsf{X} \times_{\mathsf{S}} \mathsf{T}_\bullet)$$

This follows from the fact that our conditions on relative Cartier divisors from Definition 2.12 are local with respect to the étale topology. □

**Lemma 2.20.** *The functor $\mathrm{CDiv}_{\mathsf{E}/R}$ is nilcomplete.*

*Proof.* By [Lur18c, Definition 17.3.2.1], we need to show that the canonical map

$$\mathrm{CDiv}_{\mathsf{E}/R}(R') \to \varprojlim_{n} \mathrm{CDiv}_{\mathsf{E}/R}(\tau_{\leq n} R')$$

is a homotopy equivalence for every $\mathbb{E}_\infty$-ring $R'$. This can be deduced from the following: Given a flat, proper, locally almost of finite presentation spectral algebraic space $\mathsf{X}$ over a connective $\mathbb{E}_\infty$-ring $S$, we have an equivalence

$$\mathrm{CDiv}(\mathsf{X}/S) \to \varprojlim_n \mathrm{CDiv}(\mathsf{X} \times_S \tau_{\leq n} S)$$

Let us now prove this equivalence. Given a relative Cartier divisor $\mathsf{D} \to \mathsf{X}$, we have the following commutative diagram

$$\begin{tikzcd}
\mathsf{D} \times_S \tau_{\leq n} S \arrow[r] \arrow[d, dashed] \arrow[dd, bend right] & \mathsf{D} \arrow[d] \\
\mathsf{X} \times_S \tau_{\leq n} S \arrow[r] \arrow[d] & \mathsf{X} \arrow[d] \\
\mathrm{Sp\acute{e}t}\, \tau_{\leq n} S \arrow[r] & \mathrm{Sp\acute{e}t}\, S
\end{tikzcd}$$

where we get an induced map $\mathsf{D} \times_S \tau_{\leq n} S \to \mathsf{X} \times_S \tau_{\leq n} S$. It is not hard to prove that this map is a closed immersion [Lur18c, Corollary 3.1.2.3]. Moreover, the map $\mathsf{D} \times_S \tau_{\leq n} S \to \mathrm{Sp\acute{e}t}\, \tau_{\leq n} S$ is flat, proper, and locally almost of finite presentation, since $\mathsf{D} \times_S \tau_{\leq n} S$ is the base change of $\mathsf{D}$ along $\mathrm{Sp\acute{e}t}\, \tau_{\leq n} S \to \mathrm{Sp\acute{e}t}\, S$. The associated ideal sheaf of $\mathsf{D} \times_S \tau_{\leq n} S$ remains a line bundle over $\mathsf{X} \times_S \tau_{\leq n} S$. Therefore $\mathsf{D} \times_S \tau_{\leq n} S$ is a relative Cartier divisor of $\mathsf{X} \times_S \tau_{\leq n} S$. Thus we define a functor

$$\begin{aligned} \theta \colon \mathrm{CDiv}(\mathsf{X}/S) &\to \varprojlim_n \mathrm{CDiv}(\mathsf{X} \times_S \tau_{\leq n} S) \\ \mathsf{D} &\mapsto \{\mathsf{D} \times_S \tau_{\leq n} S\}_n \end{aligned}$$

This functor is fully faithful, since we have from [Lur18c, Proposition 19.4.1.2] an equivalence $\mathrm{SpDM}_{/S} \to \varprojlim_n \mathrm{SpDM}_{/\tau_{\leq n} S}$ defined by $\mathsf{X} \mapsto \mathsf{X} \times_S \tau_{\leq n} S$. For $\theta$ to be an equivalence, we need only show that it is essentially surjective.

Suppose $\{\mathsf{D}_n \to \mathsf{X} \times_S \tau_{\leq n} S\}_n$ is an object in $\varprojlim_n \mathrm{CDiv}(\mathsf{X} \times_S \tau_{\leq n} S)$. It is a morphism in $\varprojlim_n \mathrm{SpDM}_{/\tau_{\leq n} S}$. By [Lur18c, Proposition 19.4.1.2], there is a morphism $\mathsf{D} \to \mathsf{X}$ in $\mathrm{SpDM}_{/S}$ such that $\mathsf{D} \times_S \tau_{\leq n} S \to \mathsf{X} \times_S \tau_{\leq n} S$ are equivalent to $\mathsf{D}_n \to \mathsf{X} \times_S \tau_{\leq n} S$.

Next, we need to show that $\mathsf{D} \to \mathsf{X}$ from above is a relative Cartier divisor. The conditions that $\mathsf{D} \to \mathsf{X}$ is flat, proper, and locally almost of finite presentation follow immediately from [Lur18c, Proposition 19.4.2.1]. It remains to prove that $\mathsf{D} \to \mathsf{X}$ is a closed immersion and determines a line bundle over $\mathsf{X}$.

Without loss of generality, we may assume that $\mathsf{X} = \mathrm{Sp\acute{e}t}\, B$ is affine, so that we have closed immersions $\mathsf{D}_n \to (\mathrm{Sp\acute{e}t}\, B) \times_S \tau_{\leq n} S \simeq \mathrm{Sp\acute{e}t}\,(B \otimes_S \tau_{\leq n} S)$, the last equivalence from [Lur18c, Proposition 1.4.11.1(3)]. By [Lur18c, Theorem 3.1.2.1], each $\mathsf{D} \times_S \tau_{\leq n} S$ is equivalent to $\mathrm{Sp\acute{e}t}\, B'_n$ for some $B'_n$ such that $\pi_0(B \otimes_S \tau_{\leq n} S) \to \pi_0 B'_n$ is surjective. Since $\tau_{\leq n+1} S \to B'_{n+1}$ is flat, we have

$$\begin{aligned} \mathrm{Sp\acute{e}t}\, B'_n &= (\mathrm{Sp\acute{e}t}\, B'_{n+1}) \times_{\tau_{\leq n+1} S} \tau_{\leq n} S = \mathrm{Sp\acute{e}t}\,(B'_{n+1} \otimes_{\tau_{\leq n+1} S} \tau_{\leq n} S) \\ &\simeq \mathrm{Sp\acute{e}t}\, \tau_{\leq n} B'_{n+1} \end{aligned}$$

Thus we obtain a spectrum $B'$ such that $\mathrm{Sp\acute{e}t}\, \tau_{\leq n} B' \simeq \mathrm{Sp\acute{e}t}\, B'_n = \mathsf{D} \times_S \tau_{\leq n} S$. Consequently, $\mathsf{D} = \mathrm{Sp\acute{e}t}\, B'$ and $\pi_0 B \to \pi_0 B'$ is surjective, and so $\mathsf{D} = \mathrm{Sp\acute{e}t}\, B' \to \mathrm{Sp\acute{e}t}\, B = \mathsf{X}$ is a closed immersion.

Finally, to prove that the associated ideal sheaf of $\mathsf{D}$ is a line bundle, we notice the pullback diagrams

$$\begin{tikzcd} I_n \arrow[r] \arrow[d] & B \otimes_S \tau_{\leq n} S \arrow[d] \\ * \arrow[r] & B' \otimes_S \tau_{\leq n} S \end{tikzcd}$$

where each $I_n$ is an invertible module over $B \otimes_S \tau_{\leq n} S = \tau_{\leq n} B$. Passing to inverse limits, we obtain a pullback diagram

$$\begin{tikzcd} \varprojlim I_n \arrow[r] \arrow[d] & B \arrow[d] \\ * \arrow[r] & B' \end{tikzcd}$$

Consequently, we have $I(\mathsf{D}) \simeq \varprojlim I_n$. Now, by nilcompleteness of the Picard functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}/S}$ from [Lur18c, Proposition 19.2.4.7(1)], $I(\mathsf{D})$ is an invertible $B$-module. Therefore the associated ideal sheaf of $\mathsf{D}$ is a line bundle over $\mathsf{X}$. □

**Lemma 2.21.** *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *is infinitesimally cohesive.*

*Proof.* This follows from Proposition 2.16 and [Lur18c, Proposition 16.3.2.1]. □

**Lemma 2.22.** *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *is integrable.*

*Proof.* Given a local Noetherian $\mathbb{E}_\infty$-ring $R'$ which is complete with respect to its maximal ideal $\mathfrak{m} \subset \pi_0 R'$, we need to prove that the inclusion functor $\operatorname{Spf} R' \hookrightarrow \operatorname{Spec} R'$ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\operatorname{Spec} R', \mathrm{CDiv}_{\mathsf{E}/R}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\operatorname{Spf} R', \mathrm{CDiv}_{\mathsf{E}/R})$$

This can be deduced from the following result: Given a flat, proper, and separated spectral algebraic space $\mathsf{X}$ locally almost of finite presentation over a connective local Noetherian $\mathbb{E}_\infty$-ring $S$ which is complete with respect to its maximal ideal, we have an equivalence

$$\mathrm{CDiv}(\mathsf{X}/S) \simeq \mathrm{CDiv}(\mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S)$$

Indeed, let $\mathrm{Hilb}(\mathsf{X}/S)$ denote the full subcategory of $\mathrm{SpDM}_{/\mathsf{X}}$ consisting of those $\mathsf{D} \to \mathsf{X}$, such that each $\mathsf{D} \to \mathsf{X}$ is a closed immersion and is flat, proper, and locally almost of finite presentation. Then by the formal GAGA theorem [Lur18c, Corollary 8.5.3.4] and the base-change properties of being flat, proper, and locally almost of finite presentation, we have $\mathrm{Hilb}(\mathsf{X}/S) \simeq \mathrm{Hilb}(\mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S)$.

To prove the above equivalence for relative Cartier divisors, we need to further check that $\mathsf{D} \to \mathsf{X}$ associates a line bundle over $\mathsf{X}$ if and only if $\mathsf{D} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S$ associates a line bundle over $\mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S$. Note that the morphism $f \colon \mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S \to \mathsf{X}$ is flat by [Lur18c, Corollary 7.3.6.9], and so we have $\mathcal{I}(\mathsf{D} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S) = \mathcal{I}(f^* \mathsf{D}) \simeq f^* \mathcal{I}(\mathsf{D})$ over the pullback square

$$\begin{tikzcd} \mathsf{D} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S \arrow[r] \arrow[d] & \mathsf{D} \arrow[d] \\ \mathsf{X} \times_{\operatorname{Sp\acute{e}t} S} \operatorname{Spf} S \arrow[r, "f"] & \mathsf{X} \end{tikzcd}$$

By [Lur18c, proof of Proposition 19.2.4.7], we have an equivalence

$$\mathrm{QCoh}(\mathsf{X}/S)^{\mathrm{aperf,cn}} \simeq \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, S} \mathrm{Spf}\, S)^{\mathrm{aperf,cn}}$$

We need only restrict to the subcategories spanned by invertible objects via [Lur18c, Proposition 2.9.4.2] to complete the proof. □

**Lemma 2.23.** *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *is locally almost of finite presentation over* $\mathrm{Spec}\, R$*.*

*Proof.* By [Lur18c, Definition 17.4.1.1(b)], we need to prove that

$$\mathrm{CDiv}_{\mathsf{E}/R} \colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad R' \mapsto \mathrm{CDiv}(\mathsf{E}_{R'}/R')$$

commutes with filtered colimits when restricted to each $\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}}$. We notice that $\mathrm{CDiv}(\mathsf{E}_{R'}/R')$ is a full subcategory of $\mathrm{SpDM}_{/(\mathsf{E}_{R'} \to \mathrm{Sp\acute{e}t}\, R')}$ and first consider instead the functor

$$\mathrm{Var}^+ \colon \mathrm{CAlg}_R^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty, \quad R' \mapsto \mathrm{Var}^+_{/(\mathsf{E}_{R'} \to \mathrm{Sp\acute{e}t}\, R')}$$

where $\mathrm{Var}^+_{/(\mathsf{E}_{R'} \to \mathrm{Sp\acute{e}t}\, R')}$ consists of diagrams

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[dr] & \mathsf{E}_{R'} \arrow[d] \\ & \mathrm{Sp\acute{e}t}\, R' \end{tikzcd}$$

such that $\mathsf{D} \to \mathrm{Sp\acute{e}t}\, R'$ is flat, proper, and locally almost of finite presentation. Then by [Lur18c, Proposition 19.4.2.1], this functor commutes with filtered colimits when restricted to $\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}}$. It remains to verify that when $\{\mathsf{D}_i \to \mathsf{E}_{i,R'}\}_{i\in I}$ are closed immersions and determine line bundles over $\{\mathsf{E}_{i,R'}\}$, $\varinjlim_{i\in I} \mathsf{D}_i \to \varinjlim_{i\in I} \mathsf{E}_{i,R'}$ are closed immersions and determine line bundles over $\varinjlim_{i\in I} \mathsf{E}_{i,R'}$. As we recalled earlier in this subsection, this follows from properties of closed immersions and the property of Picard functors that they are locally almost of finite presentation. □

**Lemma 2.24.** *The functor* $\mathrm{CDiv}_{\mathsf{E}/R}$ *admits a cotangent complex* $L$ *which is connective and almost perfect.*

*Proof.* Let $S$ be a connective $R$-algebra, $\eta \in \mathrm{CDiv}_{\mathsf{E}/R}(S)$, and $M$ be a connective $S$-module. We then have a pullback diagram

$$\begin{tikzcd} F_\eta(M) \arrow[r] \arrow[d] & \mathrm{CDiv}_{\mathsf{E}/R}(S \oplus M) \arrow[d] \\ \{\eta\} \arrow[r] & \mathrm{CDiv}_{\mathsf{E}/R}(S) \end{tikzcd}$$

From this we obtain a functor

$$F_\eta \colon \mathrm{Mod}_S \to \mathcal{S}, \quad M \mapsto F_\eta(M)$$

We first need to prove that the above functor is corepresentable. Here, $\eta$ is to a morphism $\mathsf{D} \to \mathsf{E} \times_R S$, and $\mathsf{E} \times_R (S \oplus M)$ is a square-zero extension of $\mathsf{E} \times_R S$.

Thus by the classification of first-order deformations [Lur18c, Proposition 19.4.3.1], the space of spectral algebraic spaces $\mathsf{D}'$ which fit into the pullback diagram

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[d, "\eta"] & \mathsf{D}' \arrow[d] \\ \mathsf{E} \times_R S \arrow[r] \arrow[d, "p"] & \mathsf{E} \times_R (S \oplus M) \arrow[d] \\ \mathrm{Sp\acute{e}t}\, S \arrow[r] & \mathrm{Sp\acute{e}t}\,(S \oplus M) \end{tikzcd}$$

is equivalent to $\mathrm{Map}_{\mathrm{QCoh}(\mathsf{D})}\big(L_{\mathsf{D}/(\mathsf{E}\times_R S)}, \Sigma\eta^*(p^*M)\big)$. Pushing forward along $p \circ \eta$, by [Lur18c, Proposition 6.4.5.3], we then have

$$\mathrm{Map}_{\mathrm{QCoh}(\mathsf{D})}\big(L_{\mathsf{D}/(\mathsf{E}\times_R S)}, \Sigma\eta^*(p^*M)\big) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, S)}\big(\Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)}), M\big)$$

By Lemma 2.16, any such $\mathsf{D}' \to \mathsf{E} \times_R (S \oplus M)$ is a closed immersion and determines a line bundle over $\mathsf{E} \times_R (S \oplus M)$. Since the diagram

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[d] & \mathsf{D}' \arrow[d] \\ \mathrm{Sp\acute{e}t}\, S \arrow[r] & \mathrm{Sp\acute{e}t}\,(S \oplus M) \end{tikzcd}$$

is a pullback square, $\mathsf{D}'$ is a square-zero extension of $\mathsf{D}$. By [Lur18c, Proposition 16.3.2.1], $\mathsf{D}' \to \mathrm{Sp\acute{e}t}\,(S \oplus M)$ is flat, proper, and locally almost of finite presentation. Combining these facts, we find that

$$F_\eta(M) = \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, S)}\big(\Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)}), M\big)$$

Consequently, the functor $\mathrm{CDiv}_{\mathsf{E}/R}$ satisfies condition (a) from [Lur18c, Example 17.2.4.4]. Condition (b) therein follows from the compatibility of $(p \circ \eta)_+$, as a left adjoint of the functor $(p \circ \eta)^*$, with base change (cf. [Lur18c, Construction 6.4.5.1 and Proposition 6.4.5.3]). Therefore the functor $\mathrm{CDiv}_{\mathsf{E}/R}$ admits a cotangent complex $L_{\mathrm{CDiv}_{\mathsf{E}/R}}$ satisfying $\eta^* L_{\mathrm{CDiv}_{\mathsf{E}/R}} = \Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)})$. Since the quasi-coherent sheaf $L_{\mathsf{D}/(\mathsf{E}\times_R S)}$ is connective and almost perfect [Lur18c, Proposition 17.1.5.1(3)], the $S$-module $\Sigma^{-1}p_+(\eta_+ L_{\mathsf{D}/(\mathsf{E}\times_R S)})$ is $(-1)$-connective.

Next, we show that $L_{\mathrm{CDiv}_{\mathsf{E}/R}}$ is almost perfect. This follows from [Lur18c, 17.4.2.2] and Lemma 2.23.

Finally, we show that it is connective. As above, let $S$ be a connective $R$-algebra and $\eta \in \mathrm{CDiv}_{\mathsf{E}/R}(S)$. We need to prove that $M_\eta \coloneqq \eta^* L_{\mathrm{CDiv}_{\mathsf{E}/R}} \in \mathrm{Mod}_S$ is connective. We already knew that $M_\eta$ is $(-1)$-connective and almost perfect. In particular, the homotopy group $\pi_{-1}M_\eta$ is a finitely generated $\pi_0 S$-module. To prove that it in fact vanishes, by Nakayama's lemma, we note that this is equivalent to proving that

$$\pi_{-1}(\kappa \otimes_{\pi_0 S} M_\eta) \simeq \mathrm{Tor}_0^{\pi_0 S}(\kappa, \pi_{-1}M_\eta)$$

equals 0 for every residue field $\kappa$ of $\pi_0 S$. Thus we may replace $S$ by $\kappa$ and assume $\kappa$ is an algebraically closed field.

Let $A = \kappa[\epsilon]/(\epsilon^2)$. Unwinding the definitions, we find that the dual space $\mathrm{Hom}_\kappa(\pi_{-1}M_\eta, \kappa)$ can be identified with the set of automorphisms of the base change

$\eta_A$ such that they restrict to be the identity of $\eta$. It remains to prove that this set is trivial. This boils down to the following assertion in classical algebraic geometry.

> Let $X/\kappa$ be a scheme, $L$ be a line bundle over $X$, and assume $L_A$ is also a line bundle over $X_A$. If $f$ is an automorphism of $L_A$ such that $f|L$ is the identity on $L$, then $f$ is the identity.

This can be proved, mutatis mutandis, as in the last part of [Lur18a, proof of Proposition 2.2.6]. □

## 3. Level structures for spectral abelian varieties

3.1. **Level structures on elliptic curves.** Let $C$ be a one-dimensional smooth commutative group scheme over a base scheme $S$, and $A$ be an abstract finite abelian group. A homomorphism of abstract groups

$$\phi : A \to C(S)$$

is said to be an level-$A$ structure on $C/S$ if the effective Cartier divisor D in $C/S$ defined by

$$D = \Sigma_{a\in A}[\phi(a)]$$

is a subgroup of $C/S$.

The following result due to Katz-Mazur [KM85] gives the representability of level structures moduli problems.

**Proposition 3.1.** [KM85, Proposition 1.6.2] *Let $C/S$ be a one-dimensional smooth commutative group scheme over S. Then the functor*

$$\mathrm{Level}_{C/S} : \mathrm{Sch}_S \to \mathrm{Set}$$

$$T \mapsto \textit{the set of level-}A\textit{ structures on } C_T/T$$

*is representable by a closed subscheme of* $\mathrm{Hom}(A, C) \cong C[N_1] \times_S \cdots \times_S C[N_r]$.

**Definition 3.2.** Let $E/R$ be a spectral elliptic curve. In the level of objects, a derived level-$A$ structure is a relative Cartier divisor $\phi : D \to E$ of E, such that the underlying morphism $D^\heartsuit \to E^\heartsuit$ is the inclusion of the associated relative Cartier divisor $\Sigma_{a\in A}[\phi_0(a)]$ into $E^\heartsuit$, where $\phi_0 : A \to E^\heartsuit(R^\heartsuit)$ is any classical level structure. We let $\mathrm{Level}(\mathcal{A}, E/R)$ denote the $\infty$-category of derived level-$A$ structures of $E/R$, whose objects can be viewed as pairs $\phi = (D, \phi)$.

It is easy to see that for a spectral elliptic curve $E/R$, the $\infty$-category $\mathrm{Level}(\mathcal{A}, E/R)$ is a $\infty$-groupoid, since it is a full subcategory of $\mathrm{CDiv}(E/R)$, which is a $\infty$-groupoid.

**Lemma 3.3.** *Let $E/R$ be a spectral elliptic curve and $\phi_S : D \to E$ be a derived level structure. Suppose that $T \to S$ be a morphism of nonconnective spectral Deligne–Mumford stacks, then the induced morphism $\phi_S : D_T \to E_T$ is a derived level structure of $E_T/T$.*

*Proof.* We notice that the derived level structure is stable under base change. So $\phi_S^\heartsuit : A \to (E \times_S T)^\heartsuit(T_0) = E^\heartsuit(T_0)$ is a classical level structure, so $D_T^\heartsuit$ is the associated classical relative Cartier divisor of a classical level structure. And $D_T \to E_T$ is a relative Cartier divisor in spectral algebraic geometry, this is just the base change of the relative Cartier divisor (Lemma 2.14). □

We first recall a proposition in Katz and Mazur's book [KM85, Corollally 1.3.7]: Suppose that $C/S$ is a smooth group curve, and $D$ is a relative Cartier divisor of $C$, then exists a closed subscheme $Z$ of $S$, satisfying for any $T \to S$, $D_T$ is a subgroup of $C_T$ if and only if $T$ passing through $Z$.

**Lemma 3.4.** *Let $E/R$ be a spectral elliptic curve, and $D \to E$ be a relative Cartier divisor. There exists a closed spectral Deligne–Mumford substack $\operatorname{Spét} Z \subset \operatorname{Spét} R$, satisfying the following universal property:*

*For any $S \in \mathrm{CAlg}_R^{cn}$, such that the associated sheaf of $D_S$ is a relative Cartier divisor of $X_S$ and $(D_S)^\heartsuit$ is a subgroup of $(E_S)^\heartsuit$ if and only if $R \to S$ factor through $Z$.*

*Proof.* For a map $R \to S$, it is obvious that $D_S$ is a relative Cartier divisor of $X_S$. By [KM85, Corollally 1.3.7], we just notice that if $(D_S)^\heartsuit/\pi_0 S$ is a subgroup of $(E_S)^\heartsuit/\pi_0 S$, we have $\operatorname{Spec} \pi_0 S$ must passing through a closed subscheme $\operatorname{Spec} Z_0$ of $\operatorname{Spec} \pi_0 R$. This corresponds to a closed spectral subscheme $\operatorname{Spec} Z$ of $\operatorname{Spec} R$, sin

ce we have the map $R \to S$ such that $\pi_0 R \to \pi_0 S$ pass through $\pi_0 R/I$ for some ideal I of $\pi_0 R$, so we have $R \to S$ passing through $R^{Nil(I)}$, see [Lur18c, Chapter 7] for details about nilpotent R-module. Conversely, suppose that $R \to S$ passes through Z, then we have $\mathcal{O}_{\operatorname{Spét} S}$ vanishing on I. That is we have $\pi_0 R \to \pi_0 S$ passing through $\pi_0 R/\sqrt{I}$, but this is equivalent to say $\operatorname{Spec} \pi_0 S \to \operatorname{Spec} \pi_0 R$ passing through $\operatorname{Spec} \pi_0 R/I = \operatorname{Spec} Z_0$, and so $(D_S)^\heartsuit$ is a subgroup of $(E_S)^\heartsuit$. □

**Theorem 3.5.** *Let $E/R$ be a spectral elliptic curve, then the functor*

$$\begin{aligned} \mathrm{Level}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

*is representable by a closed substack $S(A)$ of $\mathrm{CDiv}_{X/R}$. Moreover, $S(A) = \operatorname{Spét} \mathcal{P}_{E/R}$ for an $\mathbb{E}_\infty$-ring $\mathcal{P}_{E/R}$, which is locally almost of finite presentation over R.*

*Proof.* By definition, the functor $\mathrm{Level}_{E/R}$ is a subfunctor of the representable functor $\mathrm{CDiv}_{X/R}$. We consider a spectral Deligne–Mumford stack GroupCDiv defined by the pullback diagram of spectral Deligne–Mumford stacks

$$\begin{tikzcd} \mathrm{GroupCDiv}_{E/R} \arrow[r] \arrow[d] & \mathrm{CDiv}_{E/R} \arrow[d] \\ \operatorname{Spét} Z \arrow[r] & \operatorname{Spét} R. \end{tikzcd}$$

It is easy to say that $\mathrm{GroupCDiv}_{E/R}$ valued on a R-algebra $R'$ is the space of relative Cartier divisors D of $E \times_{\operatorname{Spét} R} \operatorname{Spét} R'$, such that $D^\heartsuit$ is a subgroup of $(E \times_{\operatorname{Spét} R} \operatorname{Spét} R')^\heartsuit$. It is clear that

$$\mathrm{GroupCDiv}_{E/R} = \coprod_{A_0 \in \mathrm{FinAb}} A_0 - \mathrm{CDiv}_{E/R}$$

where $A_0 - \mathrm{CDiv}_{E/R}$ valued on a R-algebra $R'$ is the space of relative Cartier divisors D of $E \times_{\operatorname{Spét} R} \operatorname{Spét} R'$, such that $D^\heartsuit$ is an algebric subgroup of $(E \times_{\operatorname{Spét} R} \operatorname{Spét} R')^\heartsuit$ and $D^\heartsuit(R') = A_0$. It is cleared that $\mathrm{Level}_{E/R} = A - \mathrm{CDiv}_{E/R}$, so we have $\mathrm{Level}_{E/R}$ is representable by a open substack of $\mathrm{GroupCDiv}_{E/R}$.

To prove the second part, we consider the map $S(A) \to \operatorname{Spét} R$, they are all spectral algebraic spaces. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary spectral algebraic space is finite in ordinary algebraic geometry. So we only need to prove $S(A)^{\heartsuit}$ is finite over $\operatorname{Spec}\pi_0 R$, but this is just the classical case since $S(A)^{\heartsuit}$ is the representable object of the classical level structure, which is finite over $R_0$ by [KM85, Corollary 1.6.3]. $\square$

3.2. **Level structures on $p$-divisible groups.** Before we talk about derived level structures of spectral $p$-divisible groups, let us first review something about the classical level structures of commutative finite flat group schemes. Let $X/S$ be a finite flat $S$-scheme of finite presentation of rank $N$, it can be proved that $X/S$ is finite locally free of rank $N$. This means that for every affine scheme $\operatorname{Spec} R \to S$, the pullback scheme $X \times_S \operatorname{Spec} R$ over $\operatorname{Spec} R$ have the form $\operatorname{Spec} R'$, where $R'$ is an $R$-algebra which is locally free of rank $N$. For an element $f \in R'$ which can act on $R'$ by multiplication, define an R-linear endomorphism of $B'$. Because $R'$ is locally free of rank $N$. Then multiplication of $f$ can be representable by a $N \times N$ matrix $M_f$. Then we can define the characteristic polynomial of $f$ to be the characteristic polynomial of $M_f$, i.e.,

$$\det(T - f) = \det(T - M_f) = T^N - \operatorname{trace}(M_f) + \cdots + (-1)N\operatorname{Norm}(f).$$

Let $\{P_1, \cdots, P_N\}$ be a set of N points in $X(S)$, we say this set is a full set of sections of $X/S$ if one of the following two conditions are satisfied:

(1) For any $\operatorname{Spec} R \to S$, and $f \in B = H^0(X_R, \mathcal{O})$, we have the equality

$$\det(T - f) = \prod_{i=1}^{N}(T - f(p_i)).$$

(2) For every $\operatorname{Spec} R \to S$, and $f \in B = H^0(X_R, \mathcal{O})$, we have

$$\operatorname{Norm}(f) = \prod_{i=1}^{N} f(p_i).$$

Actually, these conditions are equivalent.

If we have $N$ not-necessarily-distinct points $\{P_1, \cdots, P_N\}$ in $X(S)$, then we have a morphism

$$\mathcal{O}_Z \to \bigotimes_i (P_i)_*(\mathcal{O}_S)$$

of sheave over $X$. It is easy to see that this map is surjective, and it defines a closed subscheme $D$ of $X$, which is flat, proper over $S$. So by the construction, for a $\phi : A \to X(S)$, we can define closed subscheme $D$ of $X$ which corresponds to the sheave $\otimes_{a\in A}\phi(a)_*\mathcal{O}_S$.

**Lemma 3.6.** *For a finite flat and finite presentation $S$-scheme $Z$,* $\operatorname{Hom}(A, Z)$ *is an open subscheme of* $\operatorname{Hilb}_{Z/S}$.

*Proof.* Let $T \to S$ be a S-scheme, for any $D \to Y = T \times_S Z$ in $\operatorname{Hilb}(Y) = \operatorname{Hilb}(T \times_S Z)$, we need to prove that the set of points $t \in T$ which satisfying $D_t \to Y_t$ is coming from the closed subscheme associated with a map $\phi : A \to Z(T) = Y(T)$ is an open subset of T. Since $D$ is the closed subscheme defined by $\mathcal{O}_Y \to \mathcal{O}_D$, if $D_t$ comes form $\mathcal{O}_Y|_t \to \bigotimes(P_i)_*(\mathcal{O}_T)|_t$. Then by the definition of stalks of sheaves, there exists an open subset U of D such that $t \in U$, and $D_U$ is defined by $\mathcal{O}_Y|_U \to \bigotimes(P_i)_*(\mathcal{O}_T)|_U$. $\square$

**Definition 3.7.** Suppose that $G/S$ is a rank $N$ commutative finite flat $S$-group scheme of finite presentation and $A$ is a finite abelian group of order $N$. A group homomorphism

$$\phi : A \to G(S)$$

is called an $A$-generator of $G/S$, if the $N$ points $\{\phi(a)\}_{a\in A}$ are a full subset of sections of $G(S)$. In these cases, we say $\phi$ is a Drinfeld level structure.

**Proposition 3.8.** [KM85, Proposition 1.10.13] *Suppose that $G$ is a rank $N$ finite flat commutative group scheme of finite presentation over $S$ and $A$ is a finite abelian group of order $N$. Then we have the following two propositions:*

(1) *The functor $A-\mathrm{Gen}(G/S)$ on $S$-schemes defined by*

$$T \mapsto \{\phi|\phi : A \to G(T) \text{ is a Drinfeld level structure}\}$$

*is representable by a finite $S$-scheme of finite presentation. Actually, it is the closed subscheme of $\mathrm{Hom}_{\mathrm{Sch}_S}(A, G)$ over which the image of sections $\{\phi_{univ}(a)\}_{a\in A}$ of the universal homomorphism $\phi_{univ} : A \to G$ form a full set of sections.*

(2) *If $G/S$ is finite étale over $S$ of rank $N$, we have*

$$A-\mathrm{Gen}(G/S) \simeq \mathrm{Isom}_{\mathrm{Sch}_S}(A, G),$$

*such that each connected component of $S$, $A-\mathrm{Gen}(S)$ is either empty or is a finite étale $\mathrm{Aut}(A)$-torsor.*

**Derived Level Structures of Spectral Finite Flat Group Schemes:** For a spectral commutative finite flat group scheme $G$ over $R$. By the definition of finite flat, we have $G = \mathrm{Spét}\, B$ for a finite flat R-algebra B. We let $\mathrm{Hilb}(G/R)$ denote the full subcategory of $\mathrm{SpDM}_{/G}$ spanned by those $D \to G$ such that $D \to G$ is a closed immersion of spectral Deligne–Mumford stacks, and the composition $D \to G \to R$ is flat, proper and locally almost of finite presentation. Then we find $\mathrm{Hilb}(G/R)$ is actually equivalent to the $\infty$-category of diagrams which have the form

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & R' & \end{tikzcd}$$

such that $R'$ is flat, proper and locally almost of finite presentation over $R$ and satisfies certain conditions. It is easy to see that $\mathrm{Hilb}(G/R)$ is a Kan complex. Then we can define a functor

$$\mathrm{Hilb}_{G/R} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$$
$$R' \to \mathrm{Hilb}(G_{R'})$$

**Theorem 3.9.** *Suppose that $G$ is a commutative finite flat group scheme over an $\mathbb{E}_\infty$-ring $R$, then $\mathrm{Hilb}_{G/R}$ is representable by a spectral Deligne–Mumford stack which is locally almost of finite presentation over $R$.*

*Proof.* This is just a special case of spectral algebraic geometry version of Lurie's theorem [Lur04, Theorem 8.3.3]. □

*Remark* 3.10. We can prove this theorem by the same argument of the proof of representability of relative Cartier divisors.

**Definition 3.11.** Let $G$ be a spectral commutative finite flat group scheme of rank $N$ over an $\mathbb{E}_\infty$-ring $R$, and $A$ be an abstract finite abelian group of order $N$, an level-$A$ structure of $G$ is an object $\phi : D \to G$ in $\mathrm{Hilb}(G/R)$, such that $\pi_0\phi_*\mathcal{O}_D \simeq \otimes\phi(a)_*\mathcal{O}_{\mathrm{Spec}\,\pi_0 R}$, where $\phi(a)_*\mathcal{O}_{\mathrm{Spec}\,\pi_0 R}$ comes from a map $\phi : A \to G^\heartsuit(\pi_0 R)$.

**Lemma 3.12.** *Let $G/R$ be a spectral commutative finite flat group scheme of rank $N$ over an $\mathbb{E}_\infty$-ring $R$ and let $D$ be a Hilbert closed subscheme of $G$. Then there exists a $\mathbb{E}_\infty$-ring $Z$, satisfying the following universal property:*

*For any $R \to R'$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, $(D_{R'})^\heartsuit$ is a derived level-$A$ structures of $(G_{R'})^\heartsuit$ if and only if $R \to R'$ factor through $Z$.*

*Proof.* For $R \to R'$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, it is obvious that $D_{R'}$ is in $\mathrm{Hilb}(G_{R'}/R')$. This means that $(D_{R'})^\heartsuit$ is a Hilbert closed subscheme of $(G_{R'})^\heartsuit$. For $D_{R'}$ to be a derived level structure, we have $D_{R'}^\heartsuit$ must lie in $\mathrm{Hom}(A, G^\heartsuit)(\pi_0 R')$, this means that $\mathrm{Spec}\,\pi_0 R' \to \mathrm{Spec}\,\pi_0 R$ must passing through an open of $\mathrm{Spec}\,\pi_0 R$, since $\mathrm{Hom}(A, G^\heartsuit)$ can be viewed as a open sub scheme of $\mathrm{Hilb}(G^\heartsuit/R^\heartsuit)$. Then we have $\pi_0 R \to \pi_0 R'$ passing through $W_0$, where $W_0$ is a localization of $\pi_0 R$, so we have $R \to R'$ must passing through $W$, where $W$ is an $\mathbb{E}_\infty$-ring, which is a localization of $R$. As for now, we already have a map $\mathrm{Spét}\,R' \to \mathrm{Spét}\,W$, such that $D_{R'}$ is a Hilbert closed subscheme of $G_{R'}$, and $\pi_0 i_*\mathcal{O}_{D_{R'}}$ comes from a map $\phi : A \to G^\heartsuit(\pi_0 R')$. For $D_{R'}$ want to be a derived level structure, $\mathcal{O}_{G^\heartsuit} \to \phi(a)_*(\mathcal{O}_{\mathrm{Spec}\,\pi_0 R'})$ needs to be an isomorphism, i.e., these $N$ points $\phi(a)_{a\in A}$ must be a full section of $G^\heartsuit(\pi_0 R')$. By [KM85, Proposition 1.9.1], for a set of $N$ points of $(G^\heartsuit(\pi_0 R'))$ to be a full section of $G^\heartsuit(\pi_0 R')$, $\mathrm{Spec}\,\pi_0 R' \to \mathrm{Spec}\,\pi_0 W$ must passing through a closed subscheme of $\mathrm{Spec}\,W_0$. Then $\pi_0 W \to \pi_0 R'$ must passing through $Z_0$, where $Z_0$ is equals $\pi_0 W/I$ for some ideal $I$ of $\pi_0 W$. This means that we have $W \to R'$ pass through $Z = W^{\mathrm{Nil}(I)}$. By the discussion above, we have $Z$ is the desired $\mathbb{E}_\infty$-ring. And the converse is also true by using the same discussion in the derived level structures of curves.

∎

**Proposition 3.13.** *Suppose that $G$ is a spectral commutative finite flat group scheme of rank $N$ over an $\mathbb{E}_\infty$-ring $R$ and $A$ is an abstract finite abelian group of order $N$. Then the following functor*

$$\mathrm{Level}^{\mathcal{A}}_{H/R} : \mathrm{CAlg}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}(\mathcal{A}, G_{R'}/R')$$

*is representable by an affine spectral Deligne–Mumford stack $S(A) = \mathrm{Spét}\,\mathcal{P}_{G/R}$.*

*Proof.* We first prove the representability. By definition, the functor $\mathrm{Level}^{\mathcal{A}}_{G/R}$ is a subfunctor of the representable functor $\mathrm{Hilb}_{G/R}$. We consider a spectral Deligne–Mumford stack $S(A)$ defined by the pullback diagram of spectral Deligne–Mumford stacks

$$\begin{tikzcd} S(A) \arrow[r] \arrow[d] & \mathrm{Hilb}_{G/R} \arrow[d] \\ \mathrm{Spét}\,Z \arrow[r] & \mathrm{Spét}\,R. \end{tikzcd}$$

It is easy to say that $S(A)$ valued on an R-algebra $R'$ is the Hilbert closed subscheme D of $E \times_{\mathrm{Spét}\,R} \mathrm{Spét}\,R'$, such that $D^\heartsuit$ is a derived level A-structure of $(E \times_{\mathrm{Spét}\,R} \mathrm{Spét}\,R')^\heartsuit$. Then $S(A)$ is the desired stack.

For the affine condition, we need to prove that $S(A)$ is finite in spectral algebraic geometry. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary spectral algebraic space is finite in ordinary algebraic geometry. We have $S(A)$ and $\operatorname{Spét} R$ are spectral spaces. So we only need to prove $S(A)^{\heartsuit}$ is finite over $R_0$, but this is just the classical case, which is finite by [KM85, Proposition 1.10.13]. $\square$

*Remark* 3.14. We let $\mathrm{FFG}(R)$ denote the $\infty$-category of spectral commutative finite flat group schemes over an $\mathbb{E}_\infty$-ring $R$. By [Lur18a, Proposition 6.5.8], there is another equivalent definition of spectral $p$-divisible group [Lur18b, Definition 6.0.2]. A spectral $p$-divisible group over a connective $\mathbb{E}_\infty$-ring $R$ is just a functor

$$G : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$$

which satisfies the following conditions:

(1) Suppose that $S \in \mathrm{CAlg}_R^{cn}$, the spectrum $G(S)$ is $p$-nilpotent, i.e., $G(S)[1/p] \simeq 0$.
(2) For $M$ be a finite abelian p-group, the functor

$$\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}, \quad S \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Z}}}(M, G(S))$$

is copresentable by a finite flat R-algebra.

Let $X$ be a spectral $p$-divisible group of height h over an $\mathbb{E}_\infty$-ring $R$, that is a functor

$$X : \mathrm{Ab}_{\mathrm{fin}}^p \to \mathrm{FFG(R)}.$$

For every $p^k \in \mathrm{Ab}_{\mathrm{fin}}^p$, we let $X[p^k]$ denote the image of $p^k$ of $X$. We find that $X[p^k]$ is a rank $(p^k)^h$ spectral commutative finite flat group schemes over $R$.

**Definition 3.15.** Let $G$ be a spectral $p$-divisible group of height h over a connective $\mathbb{E}_\infty$-ring $R$. For $A$ a finite abelian group, an derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of $G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure

$$\phi : D \to G[p^k]$$

of $G[p^k]$, which is a spectral commutative finite flat scheme over $R$. We let $\mathrm{Level}(k, G/R)$ denote the $\infty$-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structures of $G/R$.

**Theorem 3.16.** *Let $G$ be a spectral $p$-divisible group of height $h$ over an $\mathbb{E}_\infty$-ring $R$. Then the following functor*

$$\mathrm{Level}_{G/R}^k : \mathrm{CAlg}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}(k, G_{R'}/R')$$

*is representable by an affine spectral Deligne–Mumford stack $S(k) = \operatorname{Spét} \mathcal{P}_{G/R}^k$.*

*Proof.* We just notice that by the definition of spectral $p$-divisible group, $G[p^k]$ is a spectral commutative finite flat scheme. Then the theorem follows from the above result of the general spectral commutative finite flat group scheme. $\square$

**Non-Full Level Structures**

The above cases only care full-level structures of commutative finite flat schemes, actually we can define general-level structures of finite flat group schemes. Let $G$ be a spectral commutative finite flat group scheme of rank N over an $\mathbb{E}_\infty$-ring R, and $A$ be an abstract finite abelian group, an derived level-$A$ structure of $G$ is an object $\phi : D \to G$ in $\mathrm{Hilb}(G/R)$, such that $D^{\heartsuit}$ is a subgroup of $G$ and $G^{\heartsuit}(\pi_0 R)$ is isomorphic to $A$. We let $\mathrm{Level}_1(\mathcal{A}, G/R)$ denote the space of derived level-$A$

structure. And $\mathrm{Level}_0(\mathcal{A}, G/R)$ denote the space of equivalence class $D \to G$ in $\mathrm{Hilb}(G/R)$ such that $G^{\heartsuit}(\pi_0 R)$ is isomorphic to $A$, two object $D, D'$ are equivalent if the image of $D^{\heartsuit} \to G^{\heartsuit}$ and $D'^{\heartsuit} \to G^{\heartsuit}$ are same.

**Proposition 3.17.** *Suppose that $G$ is a spectral commutative finite flat group scheme of rank $N$ over an $\mathbb{E}_\infty$-ring $R$ and $A$ is an abstract finite abelian group of order not necessarily equal to $N$. Then the following functor*

$$\mathrm{Level}^{1,\mathcal{A}}_{G/R} : \mathrm{CAlg}^{cn}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}_1(\mathcal{A}, G_{R'}/R')$$

*is representable by an affine spectral Deligne–Mumford stack.*

*Proof.* We just noticed that the classical level structure functor $\mathrm{Level}(A, G^{\heartsuit}/\pi_0 R)$ is representable by a closed subscheme $\mathrm{Hom}(A, G)$, using the same discussion of full-level case, we get the desired result. $\square$

*Remark* 3.18. The above proposition is also true for $\mathrm{Level}^{0,\mathcal{A}}$. By the spectral commutative finite flat scheme cases, we can get the representability results of the spectral $p$-divisible group case.

We let $\mathrm{Level}_1(k, G/R)$ denote the $\infty$-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})$-level structures of $G/R$. Then the following functor

$$\mathrm{Level}^{1,k}_{G/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}_1(k, G_{R'}/R')$$

is representable by an affine spectral Deligne–Mumford stack $S_1(k) = \mathrm{Spét}\, \mathcal{P}^{1,k}_{G/R}$.

We let $\mathrm{Level}_0(k, G/R)$ denote the $\infty$-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})$-level generators of $G/R$. Then the following functor

$$\mathrm{Level}^{0,k}_{G/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}_0(k, G_{R'}/R')$$

is representable by an affine spectral Deligne–Mumford stack $S_0(k) = \mathrm{Spét}\, \mathcal{P}^{0,k}_{G/R}$.

## 4. Moduli problems associated with derived level structures

4.1. **Spectral elliptic curves with level structure.** There exists a spectral Deligne–Mumford stack $\mathcal{M}_{\mathrm{ell}}$ whose functor of points is

$$\begin{aligned} \mathcal{M}_{\mathrm{ell}} \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{\mathrm{ell}}(R), \end{aligned}$$

where $\mathcal{M}_{\mathrm{ell}}(R) = \mathrm{Ell}(R)^{\simeq}$ is the underline $\infty$-groupoid of the $\infty$-category of spectral elliptic curves over $R$.

And we have the classical Deligne–Mumford stack of classical elliptic curves, which can be viewed as a spectral Deligne–Mumford stack

$$\begin{aligned} \mathcal{M}^{cl}_{\mathrm{ell}} \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}^{cl}_{\mathrm{ell}}(\pi_0 R) \end{aligned}$$

where $\mathcal{M}^{cl}_{\mathrm{ell}}(\pi_0 R)$ is the groupoid of classical elliptic curves over the commutative ring $\pi_0 R$.

And for $A$ denote $\mathbb{Z}/N\mathbb{Z}$, or $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, we have the classical Deligne–Mumford stack of classical elliptic curves with level-$A$ structures, which can also be viewed as a spectral Deligne–Mumford stack.

$$\begin{aligned} \mathcal{M}^{cl}_{\mathrm{ell}}(A) \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}^{cl}_{\mathrm{ell}}(A)(\pi_0 R) \end{aligned}$$

where $\mathcal{M}^{cl}_{\text{ell}}(A)(\pi_0 R)$ is the groupoid of classical elliptic curves with level A-structures over the commutative ring $\pi_0 R$.

In last chapter, we define and study derived level structures. The construction $X \mapsto \text{Level}(\mathcal{A}, X/R)$ determines a functor $\text{Ell}(R) \to \mathcal{S}$ which is classified by a left fibration $\text{Ell}(\mathcal{A})(R) \to \text{Ell}(R)$. Objects of $\text{Ell}(\mathcal{A})(R)$ are pairs $(E, \phi)$, where $E$ is a spectral elliptic curve and $\phi$ is a derived level structure of $E$.

For every $R \in \text{CAlg}^{\text{cn}}$, we can consider all spectral elliptic curves over $R$ with derived level structures. This moduli problem can be thought of as a functor

$$\begin{aligned} \mathcal{M}_{\text{ell}}(\mathcal{A}) \quad : \quad & \text{CAlg}^{\text{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{\text{ell}}(\mathcal{A})(R) = \text{Ell}(\mathcal{A})(R) \end{aligned}$$

where $\text{Ell}(\mathcal{A})(R)$ is the space of spectral elliptic curves $E$ with a derived level structure $\phi : \mathcal{A} \to E$.

**Proposition 4.1.** *The functor $\mathcal{M}_{\text{ell}}(\mathcal{A}) : \text{CAlg}^{\text{cn}} \mapsto \mathcal{S}$ is an étale sheaf.*

*Proof.* Let $\{R \to U_i\}$ be an étale cover of $R$, and $U_\bullet$ be the associate check simplicial object. We consider the following diagram

$$\begin{tikzcd} \text{Ell}(\mathcal{A})(R)^{\simeq} \arrow[r, "f"] \arrow[d, "p"] & \lim_\Delta \text{Ell}(\mathcal{A})(U_\bullet)^{\simeq} \arrow[d, "q"] \\ \text{Ell}(R)^{\simeq} \arrow[r, "g"] & \lim_\Delta \text{Ell}(U_\bullet)^{\simeq}. \end{tikzcd}$$

The left map p is a left fibration between Kan complex, so is a Kan fibration [Lur09b, Lemma 2.1.3.3]. The right vertical map is pointwise Kan fibration. By picking a suit model for the homotopy limit we may assume that $q$ is a Kan fibration as well. We have $g$ is an equivalence by [Lur18a, Lemma 2.4.1]. To prove that $f$ is an equivalence. We only need to prove that for every $E \in \text{Ell}(R)$, the map

$$p^{-1}E \simeq \text{Level}(\mathcal{A}, E/R) \to \lim_{\Delta} \text{Level}(\mathcal{A}, E \times_R U_\bullet/U_\bullet) \simeq q^{-1}g(E)$$

is an equivalence. We have the $\text{Level}(\mathcal{A}, E)$ as full $\infty$-subcategory of $\text{CDiv}(E/R)$ and $\lim_\Delta \text{Level}(\mathcal{A}, E \times_R U_\bullet)$ as a full subcategory of

$$\lim_{\Delta} \text{CDiv}(E \times_R U_\bullet(U_\bullet))$$

But CDiv is an étale sheaf. So the functor

$$\text{Level}(\mathcal{A}, E/R) \to \lim_{\Delta} \text{Level}(\mathcal{A}, E \times_R U_\bullet/U_\bullet).$$

is fully faithful. To prove it is an equivalence, we only need to prove it is essentially surjective.

For any $\{\phi_{U_\bullet} : D \to E \times_R U_\bullet\}$ in $\lim_\Delta \text{Level}(\mathcal{A}, E \times_R U_\bullet/U_\bullet)$. Clearly, we can find a morphism $\phi_R : D \to E$ in $\text{CDiv}(E/R)$ whose image under the equivalence $\text{CDiv}(E/R) \simeq \lim_\Delta \text{CDiv}(E \times_R U_\bullet/U_\bullet)$ is $\{\phi_{U_\bullet} : D \to E \times_R U_\bullet\}$. We just need to prove: $\phi_R : D \to E$ is a derived level structure. This is true since in the classic case, $\text{Level}(A, E^\heartsuit(R_0)) \simeq \lim_\Delta \text{Level}(A, E^\heartsuit(\tau_{\leq 0} U_\bullet))$ and $\phi_R : D \to E$ is already a relative Cartier divisor. $\square$

**Lemma 4.2.** *$\mathcal{M}_{\text{ell}}(\mathcal{A}) : \text{CAlg}^{cn} \to \mathcal{S}$ is a nilcomplete functor, i.e., $\mathcal{M}_{\text{ell}}(\mathcal{A})(R)$ is the homotopy limit of the following diagram*

$$\cdots \to \mathcal{M}_{\text{ell}}(\mathcal{A})(\tau_{\leq m} R) \to \mathcal{M}_{\text{ell}}(\mathcal{A})(\tau_{\leq m-1} R) \to \cdots \to \mathcal{M}_{\text{ell}}(\mathcal{A})(\tau_{\leq 0} R)$$

*Proof.* For a spectral elliptic curve $R$, there is an obvious functor

$$\theta : \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R) \to \lim_{\leftarrow n} \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(\tau_{\leq n} R)$$

define by $(E, \phi : D \to E) \mapsto \{(E \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R, \phi_n : D \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R \to E \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R)\}_n$. Here we notice that $(E \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R, \phi_n : D \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R \to E \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R)$ is in $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(\tau_{\leq n} R)$.

First, we prove that $\theta$ is essentially surjective. An object in $\lim_{\leftarrow m} \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(\tau_{\leq m} R)$ can be written as a diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & D_{n+1} \arrow[r] \arrow[d] & D_n \arrow[r] \arrow[d] & D_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & D_0 \arrow[d] \\
\cdots \arrow[r] & E_{n+1} \arrow[r] & E_n \arrow[r] & E_{n-1} \arrow[r] & \cdots \arrow[r] & E_0
\end{tikzcd}$$

where each $E_n$ is spectral elliptic curve over $\tau_{\leq n} R$ and $D_n \to E_n$ is a derived level structure, and satisfying $D_n = D_{n+1} \times_{\mathrm{Sp\acute{e}t}\, \tau_{\leq n+1} R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R, E_n = E_{n+1} \times_{\mathrm{Sp\acute{e}t}\, \tau_{\leq n+1} R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R$. By the nilcompletness of $\mathcal{M}_{\mathrm{ell}}$, we get a spectral elliptic curves E, such that $E \times_R \tau_{\leq n} R \simeq E_n$, and by the nilcompletness of $\mathrm{Var}_+$ [Lur18c, Proposition 19.4.2.1], we get a spectral Deligne–Mumford stack $D$, such that $D_n = D \times_{\mathrm{Sp\acute{e}t}\, R} \mathrm{Sp\acute{e}t}\, \tau_{\leq n} R$. We need to prove the induced map $D \to E$ is a derived level structure, but this follows form nilcompletness of $\mathrm{Level}_{E/R}$.

Second, we need to prove that this functor is fully faithful. Unwinding the definitions, we need to prove that for every $(X, D_1 \to X), (Y, D_2 \to Y) \in \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R)$, the following map is a homotopy equivalence.

$$\mathrm{Map}_{\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R)}((X, D_X), (Y, D_Y)) \to \mathrm{Map}_{\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R)}(\lim_{\leftarrow n}(X_n, D_{X,n}), \lim_{\leftarrow n}(Y_m, D_{Y,m})).$$

where $X_n$ is $\tau_{\leq n} X = X \times_R \tau_{\leq n} R$, and $Y$, $D_{X,n}, D_{Y,n}$ similarly.

But we notice that this is equivalent to the following equivalence

$$\mathrm{Map}_{\mathrm{SpDM}_{/R}}((X, D_X), (Y, D_Y)) \to \lim_{\longleftarrow n} \mathrm{Map}_{\mathrm{SpDM}_{\tau_{\leq n}}}((X_n, D_{X,n}), (Y_n, D_{Y,n})).$$

And this equivalence follows from [Lur18c, Proposition 19.4.1.2] $\square$

**Lemma 4.3.** *$\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is a cohesive functor.*

*Proof.* For every pullback diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & A \arrow[d] \\
C \arrow[r] & B
\end{tikzcd}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ such that the underlying homomorphisms $\pi_0 A \to \pi_0 B \leftarrow \pi_0 C$ are surjective. We need to prove that

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(D) \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(A) \arrow[d] \\
\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(C) \arrow[r] & \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(B)
\end{tikzcd}$$

is a pullback diagram.

We have the following diagram in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$,

$$\begin{tikzcd} \mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \arrow[r, "g"] \arrow[dr, "f"'] & \mathcal{M}_{\mathrm{ell}} \arrow[d, "h"] \\ & * \end{tikzcd}$$

By [Lur18c, Remark 17.3.7.3], $\mathcal{M}_{\mathrm{ell}} * (\mathcal{A})$ is a cohesive fucntor if and only if $f$ is cohesive. Since we have $\mathcal{M}_{\mathrm{ell}}$ is cohesive functor, $h$ is a cohesive morphism in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$. And again by [Lur18c, Remark 17.3.7.3], $f$ is cohesive if and only if $g$ is cohesive. So we only need to prove that $g$ is a cohesive morphism. But by [Lur18c, Proposition 17.3.8.4] $g$ is cohesive if and only if each fiber of $g$ is cohesive, i.e., for $R \in \mathrm{CAlg}^{cn}$ and a point $\eta_E \in \mathcal{M}_{\mathrm{ell}}(R)$ which represents a spectral elliptic curve E, the functor

$$f_E : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\}$$

is cohesive. But we have $R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\} \simeq \mathrm{Level}(\mathcal{A}, E \times_R R'/R') \simeq \mathrm{Level}_{E/R}(R')$. The cohesive of $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})$ then follows from the cohesive of $\mathrm{Level}_{E/R}$.

□

**Lemma 4.4.** *The fucntor $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is integrable*

*Proof.* We need to prove that for $R$ a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $m \subset \pi_0 R$, then there is an equivalence

$$\mathrm{Map}_{Fun(\mathrm{CAlg}^{cn}, \mathcal{S})}(\mathrm{Sp\acute{e}t}\, R', \mathcal{M}_{\mathrm{ell}}(\mathcal{A})) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})}(\mathrm{Spf}\, R', \mathcal{M}_{\mathrm{ell}}(\mathcal{A})).$$

We have the following diagram in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$,

$$\begin{tikzcd} \mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \arrow[r, "g"] \arrow[dr, "f"'] & \mathcal{M}_{\mathrm{ell}} \arrow[d, "h"] \\ & * \end{tikzcd}$$

By [Lur18c, Remark 17.3.7.3], $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \to *$ is a integrable fucntor if and only if $f$ is integrable. Since we have $\mathcal{M}_{\mathrm{ell}}$ is integrable functor, $h$ is a integrable morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. And again by [Lur18c, Remark 17.3.7.3], $f$ is integrable if and only if $g$ is integrable. So we only need to prove that $g$ is an integrable morphism. But by [Lur18c, Proposition 17.3.8.4] $g$ is integrable if and only if each fiber of $g$ is integrable, i.e., for $R \in \mathrm{CAlg}^{\mathrm{cn}}$ and a point $\eta_E \in \mathcal{M}_{\mathrm{ell}}(R)$ which represents a spectral elliptic curve $E$, the functor

$$f_E : \mathrm{CAlg}^{cn}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\}$$

is integrable. But we have $R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\} \simeq \mathrm{Level}(\mathcal{A}, E \times_R R'/R') \simeq \mathrm{Level}_{E/R}(R')$. The integrable of $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})$ then follows from the integrable of $\mathrm{Level}_{E/R}$.

□

**Lemma 4.5.** *The functor $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) : \mathrm{CAlg}^{\mathrm{cn}} \mapsto \mathcal{S}$ admits a cotangent complex $L_{\mathcal{M}^{de}_{\mathrm{ell}}}$, and moreover $L_{\mathcal{M}^{de}_{\mathrm{ell}}}$ is connective and almost perfect.*

*Proof.* We have a commutative diagram in $\mathrm{CAlg}^{cn} \to \mathcal{S}$,

$$\begin{tikzcd} \mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \arrow[r, "g"] \arrow[dr, "f"'] & \mathcal{M}_{\mathrm{ell}} \arrow[d, "h"] \\ & * \end{tikzcd}$$

Since we have $h$ is infinitesimally coheisve and admits a connective cotangent complex, and f,g is infinitesimally cohesive. By [Lur18c, Proposition 17.3.9.1], to prove that $f$ admits a cotangent complex. We only need to prove $g$ admits a relative cotangent complex. By [Lur18c, Proposition 17.2.5.7], a morphism $j : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$ admits a relative cotangent complex if and only if, for any corepresentbale $Y' = \mathrm{Map}(R, -) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ and any natural transformation $Y' \to U$, $j'$ in the following pullback diagram admit a cotangent complex.

$$\begin{tikzcd} Y' \times_Y X \arrow[r] \arrow[d, "j'"] & X \arrow[d, "j"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

To prove that $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \to \mathcal{M}_{\mathrm{ell}}$ admits a cotangent complex, we just need to prove that for any $R \in \mathrm{CAlg}^{cn}$, and a spectral elliptic curve E which represents a natural transformation $\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{ell}}$. The functor

$$\mathrm{CAlg}_R \to \mathcal{S}, \quad R' \mapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\}$$

admits a connective cotangent complex. But we have $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R') \times_{\mathcal{M}_{\mathrm{ell}}(R')} \{\eta_E\} = \mathrm{Level}(E \times_R R') = \mathrm{Level}_{E/R}(R')$. So the results of $f : \mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \to *$ admits a cotangent complex follows from $\mathrm{Level}_{E/R}$ admits a cotangent complex. And the properties of connective and almost perfect also follow from the property of the cotangent complex of $\mathrm{Level}_{E/R}$. □

**Lemma 4.6.** *The functor $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) : \mathrm{CAlg}^{cn} \mapsto \mathcal{S}$ is locally almost of finite presentation.*

*Proof.* Consider the functor $\mathcal{M}_{\mathrm{ell}}(\mathcal{A}) \to *$, it is infinitesimally cohesive and admits an almost perfect cotangent complex, so by [Lur18c, 17.4.2.2], it is locally almost of finite presentation. So $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})$ is locally almost of finite presentation, since $*$ is a final object of $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$.

□

**Theorem 4.7.** *The functor*

$$\begin{aligned} \mathcal{M}_{\mathrm{ell}}(A) \quad : \quad & \mathrm{CAlg} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R)^{\simeq} \end{aligned}$$

*is representable by a spectral Deligne–Mumford stack.*

*Proof.* By the spectral Artin representability theorem, we need to prove that the functor $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})$ satisfies the following condition

(1) The space $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})(R_0)$ is n-truncated for every discrete commutative ring $R_0$.
(2) $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})$ is a sheaf for the étale topology.
(3) $\mathcal{M}_{\mathrm{ell}}(\mathcal{A})$ is a nilcomplete, infinitesimally cohesive, and integrable functor.

(4) $\mathcal{M}_{\text{ell}}(\mathcal{A})$ admits a cotangent complex $L_{\mathcal{M}_{\text{ell}}(A)}$ which is connective.
(5) $\mathcal{M}_{\text{ell}}(\mathcal{A})$ is locally almost of finite presentation.

But these follow from the above series of lemmas. $\square$

4.2. **Higher-homotopical Lubin–Tate towers.** We recall that for a height $h$ $p$-divisible group $G_0$ over a commutative ring $R_0$ and suppose $A \in \mathrm{CAlg}^{ad}_{cpl}$. We recall that a deformation of $G_0$ over $R$ is a spectral $p$-divisible group over $R$ together with an equivalence class of $G_0$-tagging of $G$. We let $\mathrm{Level}(k, G/R)$ denote the space of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of a height $h$ spectral $p$-divisible group. We consider the following functor

$$\begin{aligned} \mathcal{M}_k \quad : \quad & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}(G_0, R, k)$ is the $\infty$-category whose objects are triples $(G, \rho, \eta)$

(1) $G$ is a spectral $p$-divisible group over $R$.
(2) $\rho$ is an equivalence of $G_0$ taggings of $R$.
(3) $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of $G$.

**Theorem 4.8.** *The functor $\mathcal{M}_k$ is corepresentable by a $\mathbb{E}_\infty$-ring which is finite over the unoriented spectral deformation ring of $G_0$.*

*Proof.* We let $E_{univ}/R^{\text{un}}_{G_0}$ denote the universal spectral deformation of $G_0/R_0$. Suppose that $G$ is a spectral deformation $G_0$ to $R$, we get a map of $\mathbb{E}_\infty$-rings $R^{\text{un}}_{G_0} \to R$, and an equivalence $E_{univ} \times_{R^{un}_{G_0}} R \simeq G$ of spectral $p$-divisible groups. By the universal objects of level structures. We have the following equivalence

$$\mathrm{Level}(k, G/R) \simeq \mathrm{Level}(k, E_{univ} \times_{R^{un}_{G_0}} R) \simeq \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R^{un}_{G_0}}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R),$$

where $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is the universal object of derived level structure functor associated with the $p$-divisible group $E_{univ}/R^{un}_{G_0}$.

Then we consider the following moduli problem

$$\mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}, \quad R \mapsto \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R).$$

For $R \in \mathrm{CAlg}^{ad,cpl}_{R_0}$, $\mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R)$ can viewed the $\infty$-categories of pairs $(\alpha, f)$, where

$$\alpha : R^{un}_{G_0} \to R$$

is the classified map of a spectral $p$-divisible group $G$, which is a deformation of $G_0$, that is $\alpha = (G, \rho)$, and $f \in \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R^{un}_{G_0}}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R) = \mathrm{Level}(k, E_{univ} \times_{R^{un}_{G_0}} R)$ is a derived level structure of $G/R$. So we get $\mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R)$ is just the $\infty$-category of pairs $(G, \rho, \eta)$. By lemma 3.16, $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is finite over $R^{un}_{G_0}$. So we have $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is the desired spectrum. $\square$

Although we get spectra come from conceptually derived moduli problems, these spectra may be complicated, since we didn't know the homotopy groups. In algebraic topology, the orientation of $\mathbb{E}_\infty$-spectra makes $E_2$ page of Atiyah-Hirzebruch spectral sequences degenerating and give us the information of homotopy groups.

Let $G_0$ be a height $h$ $p$-divisible group over $R_{G_0}$. We consider the following functor

$$\begin{aligned}\mathcal{M}_k^{or} \quad : \quad & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{or}(G_0, R, k)\end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, k)$ is the space of four tuples $(G, \rho, e, \eta)$, where

(1) $G$ is a spectral $p$-divisible over $R$.
(2) $\rho$ is an equivalence class of $G_0$ taggings of $R$.
(3) $e : S^2 \to \Omega^\infty G^\circ(R)$ is an orientation of the $G^\circ$, where $G^\circ$ is the identity component of $G$.
(4) $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of $G$.

**Theorem 4.9.** *The functor $\mathcal{M}_k^{or} : \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S}$ is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JK}_k$, which is finite over the orientated deformations ring $R_{G_0}^{or}$.*

*Proof.* Let $\mathrm{Def}^{or}(G_0, R)$ denote the $\infty$-groupoid of triples $(G, \rho, e)$, where $G$ is a $p$-divisible of over $R$, $\rho$ is an equivalence class of $G_0$-taggings of $R$, and $e$ is an orientation of the identity conpoment of G. By [Lur18b, Theorem 6.0.3 and Remark 6.0.7], the functor

$$\begin{aligned}\mathcal{M}^{or} \quad : \quad & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{Def}^{or}(G_0, R)\end{aligned}$$

is corepresnetable by the orientated deformation ring $R_{G_0}^{or}$, that is we have an equivalence of spaces

$$\mathrm{Map}_{\mathrm{CAlg}_{cpl}^{ad}}(R_{G_0}^{or}, R) \simeq \mathrm{Def}^{or}(G_0, R).$$

Let $E_{univ}^{or}$ be the associated universal orientation deformation of $G_0$ to $R_{G_0}^{or}$, then it is obvious that $\mathcal{JL}_k = \mathcal{P}_{E_{univ}^{or}/R_{G_0}^{or}}$, the universal object of derived level structures of $E_{univ}^{or}/R_{G_0}^{or}$, is the desired spectrum similar to th unorientated case. $\square$

We call this spectrum $\mathcal{JL}_k$ the Jacquet-Langlands spectrum. It is easy to see that this $\mathcal{JL}_k$ admit an action of $GL_h(\mathbb{Z}/p^k\mathbb{Z}) \times \mathrm{Aut}(G_0)$. And when $k$ varies, we have a tower

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\, \mathcal{JL}_k \\ \downarrow \\ \mathrm{Sp\acute{e}t}\, \mathcal{JL}_{k-1} \\ \downarrow \\ \vdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\, \mathcal{JL}_0. \end{array}$$

We call this tower a higher categorical Lubin–Tate tower.

In classical arithmetic geometry, the Lubin–Tate tower can be used to realize the Jacquet-Langlands correspondence [HT01]. Is there a topological realization of the Jacquet-Langlands correspondence? Actually, in a recent paper [SS23], they already realized a version of topological Jacquet-Langlands correspondence. But their method is based on the Goerss-Hopkins-Miller-Lurie sheaf. They consider the degenerate level structures such that representing objects is étale over representing objects of universal deformations.

We hope our higher categorical analogues of Lubin–Tate towers can also establish a topological version of the classical Langlands correspondence, which means that we construct representations on the category of spectra.

4.3. **Topological lifts of power operation rings.** We recall the deformation of formal groups. Let $G_0$ be a formal group over a perfect field $k$ such that $\mathrm{char} k = p$ , a deformation of $G_0$ to $R$ is a triple $(G, i, \Phi)$ satisfying

- $G$ is a formal group over $R$,
- There is a map $i : k \to R/m$
- There is an isomorphism $\Phi : \pi^*G \cong i^*G_0$ of formal groups over $R/m$.

Suppose that we have a complete local ring $R$ whose residue filed has characteristic $p$. Let $\phi : R \to R, x \mapsto x^p$ be the Frobenius map. For each formal group $G$ over $R$, the **Frobenius isogeny** $\mathrm{Frob} : G \to \phi^*G$ is the homomorphism of the formal group over $R$ induced by the relative Frobenius map on rings. We write $\mathrm{Frob}^r : G \to (\phi^r)^*G$ which is the composition $\phi^*(\mathrm{Frob}^{r-1}) \circ \mathrm{Frob}$

Let $G_0$ be a formal group over $k$, $(G, i, \alpha)$ and $(G', i', \alpha')$ be two deformations of $G_0$ to $R$. A deformation of $\mathrm{Frob}^r$ is a homomorphism $f : G \to G'$ of formal groups over $R$ which satisfying

(1) $i \circ \phi^r = i'$ and $i^*(\phi^r)^*G_0 = (i')^*G_0$.

$$\begin{tikzcd} k \arrow[r, "i'"] \arrow[d, "\phi^r"'] & R/m \\ k \arrow[ur, "i"] & \end{tikzcd}$$

(2) the square

$$\begin{tikzcd} i^*G_0 \arrow[r, "i^*(\mathrm{Frob}^r)"] \arrow[d, "\alpha"'] & i^*(\phi^r)^*G_0 \arrow[d, "\alpha'"] \\ \pi^*G \arrow[r, "\pi^*(f)"] & \pi^*G' \end{tikzcd}$$

of homomorphisms of formal groups over $R/m$ commutes.

We let $\mathrm{Def}_R$ denote the category whose objects are deformations fo $G_0$ to $R$, and whose morphisms are deformations of $\mathrm{Frob}^r$ for some $r \geq 0$. We will say that a morphism in $\mathrm{Def}_R$ has height $r$, if it is a deformation of $\mathrm{Frob}^r$, and then we denote the corresponding subcategory as $\mathrm{Sub}^r R$. Let $G$ be the deformation of $G_0$ to $R$, then it can be proved that the assignment $f \to \mathrm{Ker} f$ is a one-to-one correspondence between the morphisms in $\mathrm{Sub}^r_R$ with source $G$ and the finite subgroup of $G$ which have rank $p^r$.

**Theorem 4.10.** [Str97] *Let $G_0/k$ be a height $n$ formal group over a perfect field $k$. For each $r > 0$, there exists a complete local ring $A_r$ which carries a universal height $r$ morphism $f^r_{univ} : (G_s, i_s, \alpha_s) \mapsto (G_t, i_t, \alpha_t) \in \mathrm{Sub}^r(A_r)$. That is the operation $f^r_{univ} \to g^*(f^r_{univ})$ define a bijective relation from the set of local homomorphism $g : A_r \to R$ to the set $\mathrm{Sub}^r_R$. Furthermore, we have:*

1. *$A_0 \approx W(k)[[v_1, \cdots, v_{n-1}]]$ is the Lubin–Tate ring.*
2. *There is a map $s : A_0 \to A_r$ which classifies the source of the universal height $r$ map, i.e. $G_s = s^*G_E$, where $G_E = G_{univ}/A_0$ be the universal deformation of $G_0$, and $A_r$ is finite and free as an $A_0$ module.*
3. *There is a map $t : A_0 \to A_r$ which classifies the target of the universal height $r$ map, i.e. $G_t = t^*G_E$.*
4. *And there is a bijection $\{g : A_r \to R\} \to \mathrm{Sub}^r(R)$ given by $g \to g^*(f^r_{univ})(g^*G_s \to g^*G_t)$.*

We know that those rings $A_r, r \geq 0$ have topological meanings.

**Theorem 4.11.** [Str98] *The ring $A_r$ in the universal deformation of Frobenuis is isomorphic to $E^0(B\Sigma_{p^r})/I$, i.e,*

$$A_r \cong E^0(B\Sigma_{p^r})/I$$

*where $I$ is the transfer ideal.*

The collections $\{A_r\}$ have the structures of graded coalgerbas, for $s = s_k, t = t_k : A_0 \to A_k$, which is induced by $E^0$ cohomology on $B\Sigma \to *$, we have

$$\mu = \ mu_{k,l} : A_{k+l} : A_{k+l} \to A_k{}^s\otimes_{A_0}{}^t A_l$$

which classifies the source, target, and composite of morphisms. So for the power operation $R^k(X) \to R^k(X \times B\Sigma_m)$. For $x = *$, we have

$$\pi_0 R \to E^0(B\Sigma_{p^r})/I \otimes \pi_0 R = A[r] \otimes \pi_0 R$$

This make $\pi_0 R$ becomes a $\Gamma$-module, where $\Gamma$ are duals of $A[r]$.

For more details about power operation in Morava E-theory, one can see [Rez24, Rez09] and [Rez13]. Direct computations are in [Rez08] for height 2 at the prime 2, [Zhu14] for height 2 at prime 3, [Zhu19] for height 2 at all primes. Cases of height $> 2$ are still lack of computations.

Because we have the assignment $f \to \mathrm{Ker} f$ is a one-to-one correspondence between the morphisms in $\mathrm{Sub}^r_R$ with source $G$ and the finite subgroup of $G$ which have rank $p^r$. So it is easy to see that $A_r$ corepresent the following moduli problem

$$\begin{aligned} \mathcal{M}_{0,r} \quad : \quad & \mathrm{CAlg}^{\heartsuit}_k \to \mathcal{S} \\ & R \to \mathrm{Def}(G_0, R, p^r) \end{aligned}$$

where $\mathrm{Def}(G_0, R, p^r)$ consists of pairs $(G, H)$ where $G$ is an defomration $G_0$ to $R$, and $H$ is a rank $p^r$ subgroup of G.

**Proposition 4.12.** *For every integer $r \geq 1$, there exists a $\mathbb{E}_\infty$-ring $E_{n,r}$, such that $\pi_0 E_{n,r} = A_r$.*

*Proof.* For the formal group $G_0$ over a field $k$ of characteristic $p$. We just consider the functor $\mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}$ by sending an $\mathbb{E}_\infty$-ring $R$ to quadruples $(G, \rho, e, \eta)$, where $(G, \rho)$ is spectral deformation of $G_0$ to R. e is an orientation of $G^\circ$, the identity component $G$, and $\eta \in \mathrm{Level}_0(k, G/R)$ is a derived level structure. Using the same argument in full-level structure and the fact $\mathrm{Level}^{0,k}_{G/R}$ is representable, see Remark 3.18. We get this proposition. □

*Remark* 4.13. Although we obtain spectra whose $\pi_0$ are the power operation rings of Morava E-theories. But we don't know higher homotopy groups of these spectra, since these spectra are not even periodic and they are not étale over Morava E-theories. We will continue to study such spectra in the future.

## 5. More applications

5.1. **Jacquet–Langlands spectra.** The Langlands program is a project in mathematics which aims to relate many fileds in mathematics together, including number theory, representation theory, and harmonic analysis. The global Langlands correspondence is conjectural (bijection) between

1. n-dimensional complex linear representations of the Galois group $\mathrm{Gal}(\bar{F}/F)$ of a given number field F.
2. certain representations-called automorphic representations of the n dimensional general linear group $GL_n(\mathbb{A}_F)$ with coefficients in the ring of adeles of F, arising within the representations given by functions on the double coset space $GL_n(F) \setminus GL_n(\mathbb{A}_F)/GL_n()$ (where $\mathcal{O} = \prod_v \mathcal{O}_p$ is the ring of integers of all formal completions of F).

which compatible with certain $L$-function conditions. Moreover, the group $GL_n$ can be replaced by any reductive group. The Langlands correspondence has many specific examples in number theory. For the group $GL_1$, this correspondence is just global class field theory. The Langlands correspondence for $GL_2$ leads to the famous modularity theorem [Wil95], [TW95].

The Langlands correspondence has a local version. Let $E$ be a local field, and $G$ be a reductive group over $E$. The local Langlands correspondence predicts that for any irreducible smooth representation $\pi$ of $G(E)$, we can naturally associate an $L$-parameter

$$\phi_E : W_E \to G(\mathbb{C}).$$

What we want to say in this paper is the Jacquet-Langlands correspondence. Let $K$ be a p-adic filed, and $D$ a division algebra with center $K$ and dimension $d^2$ over $K$. We fix an integer $r \leq 1$, and Let $G = GL_n$, $G' = GL_r(D)$, where $n = rd$. The Jacquet Langlands correspondence aims to relate smooth irreducible representations of $G$ to those of $G'$, whereas the Langlands correspondence relates such representations to degree n-representations of the absolute Galois group of $K$.

We care about the case $r = 1$, i.e, $D$ is a center algebra over $K$ of dimension $n^2$. There is a bijection between

1. square integrable irreducible representations of $D^\times$ and,
2. square integrable irreducible representations of $GL_n(K)$.

In classical arithmetic geometry, the Lubin–Tate tower can be used to realize the Jacquet-Langlands correspondence [HT01]. Is there a topological realization of the Jacquet–Langlands correspondence? Actually, in a recent paper [SS23], they already realized a version of topological Jacquet-Langlands correspondence. But their method is based on the Goerss-Hopkins-Miller-Lurie sheaf. They actually consider the degenerate level structures such that representing objects are *étale* over representing objects of universal deformations. We hope our higher categorical analogues of Lubin–Tate towers can also establish a topological version of the classical Langlands correspondence, which means that we construct representations on the category of spectra. Our derived level structure give an attempt on this idea by considering certain function spectra.

On the other hand, we know the actions of certain Galois groups and automorphism groups on certain objects, like Morava E-theories, THH, TC. This means that these groups act on their homotopy groups. For example, we have the action of Morava stabilizer groups $\mathbb{G}_n$ on Morava E-theories $E_n$, it can be used to compute the stable homotopy group of spheres by the following spectral sequence

$$E_2^{s,t} \cong H^s_{cts}(\mathbb{G}_n, \pi_t E_n) \Longrightarrow \pi_{t-s} L_{K(n)} S^0.$$

But usually, it is complicated to compute the continuous cohomology of $\mathbb{G}_n$. This is common in Langlands correspondence that th Galois side is usually harder to understand than the automorphic side. One strategy for relevant problems is to transfer the problems in the Galois side to the automorphic side. Let's see an example first.

**Theorem 5.1.** *([BSSW24b]) There is an isomorphism of graded* $\mathbf{Q}$*-algebras*

$$\mathbf{Q} \otimes \pi_* L_{K(n)} S^0 \cong \Lambda_{\mathbf{Q}_p}(\zeta_1, \zeta_2, \cdots \zeta_n),$$

*where the latter is the exterior* $\mathbf{Q}_p$*-algebra with generators* $\zeta_i$ *in degree* $1-2i$*.*

The main of their proof of this theorem is they transfer the computation of cohomoloy of $\mathbb{G}_n$ to the cohomoloyg of Drinfeld symmetric space $\mathcal{H}$.

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "\mathrm{GL}_n(\mathbb{Z}_p)"'] \arrow[dr, "\mathbb{G}_n"] & \\ \mathrm{LT}_K & & \mathcal{H}. \end{tikzcd}$$

In a continuous work [BSSW24a], they compute the Picard group of $K(n)$-local spectra by using some results of computation of Drinfeld symmetric space, which is due to Colmez–Dospinescu–Nizio [CDN20], [CDN21].

We know that $LT$ has a higher categorical refinement, Morava E-theories. So it is a natural question how to lift this diagram to higher categorical setting and how to estabilsh a more conceptual theory to transfer the computation of cohomogloy of $\mathbb{G}_n$ to the computation of cohomoloyg of $\mathcal{H}$.

Let $G_0$ be a height $h$ $p$-divisible group over $R_{G_0}$. We consider the following functor

$$\begin{aligned} \mathcal{M}_k^{or} \quad : \quad & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{or}(G_0, R, k) \end{aligned}$$

In subsection 3.3, we prove that this functor corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_k$. We defined the Jacquet-Langlands spectrum $\mathcal{JL}$ to be the limit of those $\mathcal{JL}_k$, i.e.,

$$\mathcal{JL} = \varprojlim_k \mathcal{JL}_k.$$

**Lemma 5.2.** $\mathcal{JL}$ *is an* $\mathbb{E}_\infty$*-ring.*

*Proof.* This is because the $\infty$-category of $\mathbb{E}_\infty$-rings admits inverse limits, see[Lur17, Corollary 3.2.2.4] for details. □

The spectrum is the higher categorical realization of $\mathcal{X}$, the moduli of deformations with level structures. It was proved by Scholze and Weinstein [SW13] that $\mathcal{X}$ is a perfectoid space.

5.2. **Jacquet–Langlands duals of Morava E-theory spectra.** By the construction of Jacquet–Langlands spectra above, it is easy to see that this $\mathcal{JL}_k$ admits an action of $GL_h(Z/p^kZ) \times \mathbb{G}_n$. $\mathcal{JL}$ is the limit of $\mathcal{JL}_k$, so it admits an action of $\lim GL_h(Z/p^kZ) \times \mathbb{G}_n = GL_h(\mathbb{Z}_p) \times \mathbb{G}_n$.

**Definition 5.3.** We define the dual Morava E-theories ${}^LE_n$ to be $\mathcal{JL}^{hG_n}$.

The generic fibre of $\pi_0{}^LE_n$ is just the Drinfled symmetric space. The Drinfled symmetric space was invented in [Dri76]. It is the rigid analytic space

$$\mathcal{H} = \mathbb{P}_K^{n-1} \setminus \bigcup_H H,$$

where $\mathbb{P}_K^{n-1}$ is a rigid analytic projective space, and $H$ run over all $K$-rational hyperplanes in $\mathbb{P}_K^{n-1}$. It has a formal model $\mathfrak{h}$ which parametrizes the deformations of a special formal $\mathcal{O}_D$-module related to $G_0$. In a future work, we will prove that ${}^LE_n$ can also come from some derived moduli problems.

**Theorem 5.4.** *$E_n^L$ is an $\mathbb{E}_\infty$-ring spectrum.*

*Proof.* □

**Proposition 5.5.** *There are convergent spectral sequences*

$$E_2^{s,t} \cong H^s_{cts}(\mathbb{G}_n \times GL_n(\mathbb{Z}_p), \pi_t\mathcal{JL}) \Longrightarrow \pi_{t-s}L_{K(n)}S^0.$$

$$E_2^{s,t} \cong H^s_{cts}(GL_n(\mathbb{Z}_p), \pi_t{}^LE_n) \Longrightarrow \pi_{t-s}L_{K(n)}S^0.$$

*Proof.* This is just because for any profinte group $G$, and $E$ is a $G$-equivarait spectrum, we always have

$$E_2^{s,t} \cong H^s_{cts}(G, \pi_tE) \Longrightarrow \pi_{t-s}E^{hG}.$$

see [May96] for more details. □

In [GV18], Galatius and Venkatesh define and study derived Galois deformations. Let $F$ be a global field, S is a finite set of places of F. Let k be a finite field, and $G$ be a split algebraic group over the Witt vectors $W(k)$. Let $\bar{\rho}$ be a representation of $\pi_1\mathrm{Spec}\,\mathcal{O}_F[1/S]$ in $G(k)$. Then we can define the Galois deformation functor $M^{\bar{\rho}}_{\mathcal{O}_F[1/s]}$ form the category of Artinian local $W(k)$-algebras augmented over k to the category of sets, by send A to the set of diagrams of the form

$$\begin{tikzcd} & G(A) \arrow[d] \\ \pi_1\mathrm{Spec}\,\mathcal{O}_F[1/s] \arrow[r, "\bar{\rho}"] \arrow[ur, dotted, "\rho"] & G(k) \end{tikzcd}$$

modulo conjugacy. We notice that the étale homotopy type of the scheme $\mathrm{Spec}\,\mathcal{O}_F[1/s]$ is equal to the classify space of $\pi_1\mathrm{Spec}\,\mathcal{O}_F[1/s]$. After applying the classifying space functor, and noticed that $G$ can be extended to simplicial rings. We then define the derived Galois deformation functor from the category of Artinian simplicial rings to the category of spaces by sending a simplicial ring $\mathcal{A}$ to diagrams

$$\begin{tikzcd} & BG(\mathcal{A}) \arrow[d] \\ \acute{E}t(\mathcal{O}_F[1/S]) \arrow[r, "\bar{\rho}"] \arrow[ur, dotted, "\rho"] & BG(k). \end{tikzcd}$$

It can be proved that this derived moduli problem is representable by a simplicial ring $\mathcal{R}^{\bar{\rho}}_{\mathcal{P}_F[1/S]}$, and its $\pi_0$ is the classical Galois deformation ring. There are variants of this construction, such as local derived deformation functor and crystalline deformation functor, see [GV18] for more details.

Let $G$ be a reductive group over a local field $K$, and $U \subset G$ be a compact open subgroup. Let $A$ be a commutative ring, we let $A[G(K)/U] = c - \mathrm{Ind}_U^{G(K)} A$ denote the induced representation of the trivial representation from $U$ to $G(K)$. The classical Hecke algebra for the pair $(G(K), U)$ is

$$H(G(K), U : A) := \mathrm{Hom}_{G(K)}(A[G(K)/U], A[G(K)/U]).$$

In [Ven19], Venkatesh defines the derived Hecke algebra to be

$$\mathcal{H}(G(K), U; A) := \mathrm{Ext}^*_{G(K)}(A[G(K)/U], A[G(K)/U]).$$

It satisfies certain good properties like the classical Hecke algebra.

These two constructions give us evidence about the homotopical version of Langland correspondence for general reductive group $G$, but the derived Hecke algebra doesn't come from the symmetry of derived objects.

In recent papers [CS24] and [Dav24], there is some constructions of Hecke operation on topological modular forms. We hope to establish a general theory of Hecke algebra in the derived algebra geometry context. In the geometric Langlands correspondence, the construction of the Hecke stack is an important ingredient. We want to find a reasonable construction of the derived Hecke stack that is compatible with Hecke algebra of topological modular forms.

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Institute for Theoretical Sciences, Westlake University, Hangzhou, Zhejiang 310030, P.R. China

*Email address*: `maxuecai@westlake.edu.cn`

Department of Mathematics, College of Science, Southern University of Science and Technology, Shenzhen, Guangdong 518055, P.R. China

*Email address*: `zhuyf@sustech.edu.cn`