

MAT8021, Algebraic Topology

Assignment 3

Due in-class on Friday, March 21

Numbered exercises are from Hatcher's "Algebraic Topology."

1. In class, we defined subdivision maps $s_n^i : \Delta[n+1] \rightarrow \Delta[n] \times [0,1]$ for $0 \leq i \leq n$ by

$$s_n^i(t_1, \dots, t_{n+1}) = ((t_1, \dots, \widehat{t_{i+1}}, \dots, t_{n+1}), t_{i+1})$$

Show that these satisfy the relations

- $s_n^i d_{n+1}^j = \begin{cases} (d_n^{j-1}, \text{id}) \circ s_{n-1}^i & \text{if } i < j-1 \\ (d_n^j, \text{id}) \circ s_{n-1}^{i-1} & \text{if } i > j \end{cases}$
- $s_n^0 d_{n+1}^0 = i_0$
- $s_n^n d_{n+1}^{n+1} = i_1$
- $s_n^{i-1} d_{n+1}^i = s_n^i d_{n+1}^i$ for $1 \leq i < n+1$

Use this to show that the operator $h : C_n(X) \rightarrow C_{n+1}(X \times [0,1])$ given by

$$h\left(\sum a_\sigma \sigma\right) = \sum a_\sigma \sum_{i=0}^n (-1)^i (\sigma, \text{id}) \circ s_n^i$$

satisfies $\partial h(x) + h \partial(x) = \tilde{i}_0(x) - \tilde{i}_1(x)$, where $\tilde{i}_k = (\sigma, \text{id}) \circ i_k$.

2. Let C_* be the chain complex with

$$C_n = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let D_* be the chain complex with

$$D_n = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

such that the boundary map $\partial : D_1 \rightarrow D_0$ sends m to $2m$.

Show that the natural projection $\pi : D_* \rightarrow C_*$ is a map of chain complexes and it induces the zero map $H_*(D_*) \rightarrow H_*(C_*)$.

3. With notations as in the last question, show that there is no chain homotopy h such that $\partial h + h\partial = \pi$ (from π to zero).
4. For $Z \subset Y \subset X$ spaces, show that there is a short exact sequence of singular chain complexes

$$0 \rightarrow C_*(Y, Z) \rightarrow C_*(X, Z) \rightarrow C_*(X, Y) \rightarrow 0$$

What does the resulting long exact sequence of homology groups look like?

5. Hatcher, Exercise 12 on page 132.