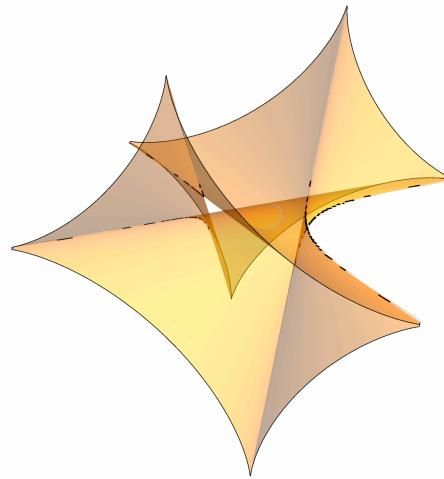


# **Explicit examples of Higgs bundles in the contexts of quantum materials and geometric Langlands correspondence**



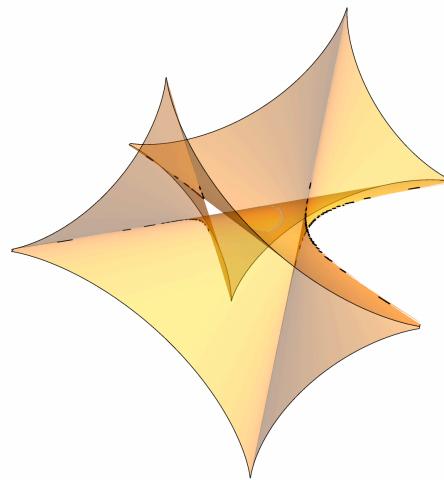
Yifei Zhu

Southern University of Science and Technology

2025.1.11

Physics + mathematical modeling

# Explicit examples of Higgs bundles in the contexts of quantum materials and geometric Langlands correspondence



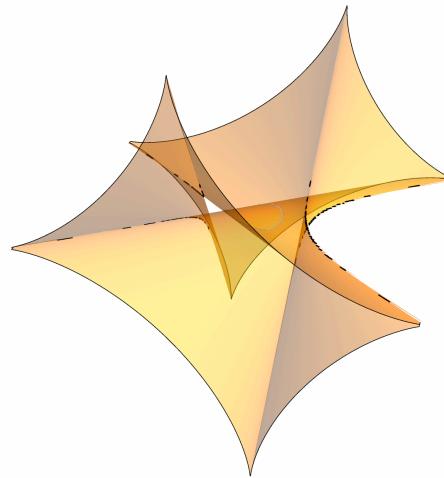
Yifei Zhu

Southern University of Science and Technology

2025.1.11

Physics + mathematical modeling (*not* Mathematical physics)

# Explicit examples of Higgs bundles in the contexts of quantum materials and geometric Langlands correspondence



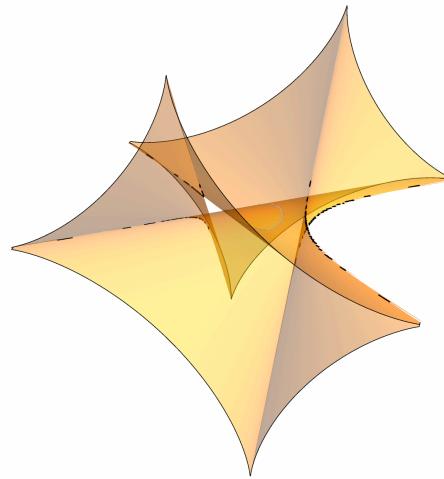
Yifei Zhu

Southern University of Science and Technology

2025.1.11

Physics + mathematical modeling (*not* Mathematical physics)

# Explicit examples of Higgs bundles in the contexts of quantum materials and geometric Langlands correspondence



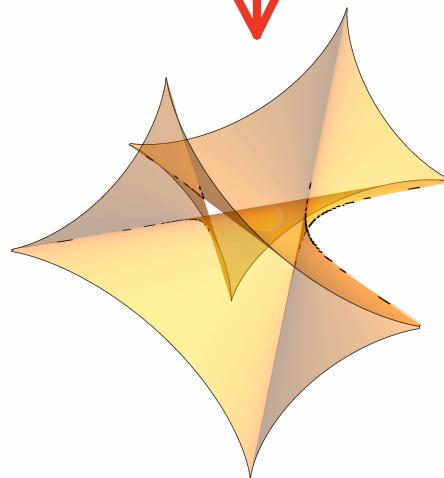
Yifei Zhu

Southern University of Science and Technology

2025.1.11

Physics + mathematical modeling (*not* Mathematical physics)

# Explicit examples of Higgs bundles in the contexts of quantum materials and geometric Langlands correspondence



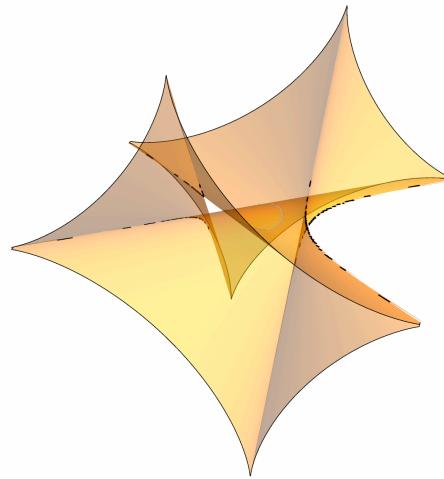
Yifei Zhu

Southern University of Science and Technology

2025.1.11

# Explicit examples of Higgs bundles in the contexts of quantum materials and geometric Langlands correspondence

Physics



Yifei Zhu

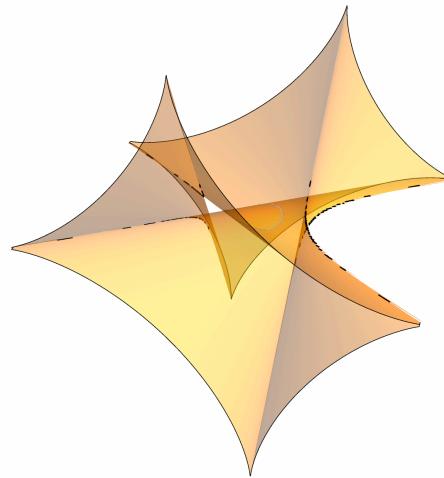
Southern University of Science and Technology

2025.1.11

# Explicit examples of Higgs bundles in the contexts of quantum materials and geometric Langlands correspondence

Physics

Mathematics



Yifei Zhu

Southern University of Science and Technology

2025.1.11

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems,
- Why “**hyperbolic** band theory” (after A.J. Kollár et al. ’19)

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems,
- Why “**hyperbolic** band theory” (after A.J. Kollár et al. ’19) and such “gapless” (**non-Hermitian**) systems

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems,
- Why “**hyperbolic** band theory” (after A.J. Kollar et al. ’19) and such “gapless” (**non-Hermitian**) systems are **natural** from the mathematical viewpoint of Higgs bundles

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems,
- Why “**hyperbolic** band theory” (after A.J. Kollar et al. ’19) and such “gapless” (**non-Hermitian**) systems are natural from the mathematical viewpoint of Higgs bundles, and
- Afforded by such systems, how the physical “**bulk–edge correspondence**”

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems,
- Why “**hyperbolic** band theory” (after A.J. Kollar et al. ’19) and such “gapless” (**non-Hermitian**) systems are natural from the mathematical viewpoint of Higgs bundles, and
- Afforded by such systems, how the physical “**bulk–edge correspondence**” may have a mathematical origin of the Langlands correspondence

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems,
- Why “**hyperbolic** band theory” (after A.J. Kollar et al. ’19) and such “gapless” (**non-Hermitian**) systems are natural from the mathematical viewpoint of Higgs bundles, and
- Afforded by such systems, how the physical “**bulk–edge correspondence**” may have a mathematical origin of the Langlands correspondence, as evidenced by the respective roles played by Higgs bundles.

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems,
- Why “**hyperbolic** band theory” (after A.J. Kollar et al. ’19) and such “gapless” (**non-Hermitian**) systems are natural from the mathematical viewpoint of Higgs bundles, and
- Afforded by such systems, how the physical “**bulk–edge correspondence**” may have a mathematical origin of the Langlands correspondence, as evidenced by the respective roles played by Higgs bundles.

This stems from ongoing joint work with  
H. Jia, J. Hu, C. T. Chan (physically)

## Overview

Our goal is to explain:

- With hindsight, how **explicit examples of rank-2 and rank-3 Higgs bundles** arise in topological classifications of (gapless) quantum mechanical systems,
- Why “**hyperbolic** band theory” (after A.J. Kollar et al. ’19) and such “gapless” (**non-Hermitian**) systems are natural from the mathematical viewpoint of Higgs bundles, and
- Afforded by such systems, how the physical “**bulk–edge correspondence**” may have a mathematical origin of the Langlands correspondence, as evidenced by the respective roles played by Higgs bundles.

This stems from ongoing joint work with

H. Jia, J. Hu, C. T. Chan (physically),

W. Yang, Z. Fang, C. Huang, Q. Qu, Z. Yu (mathematically), et al.

## **Motivations: Quantum materials and their math modeling**

## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials**

## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ... physical properties*

## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ... physical properties* at the *macroscopic* level

*Holography, optical devices,  
absorption devices, ...*

## Motivations: Quantum materials and their math modeling

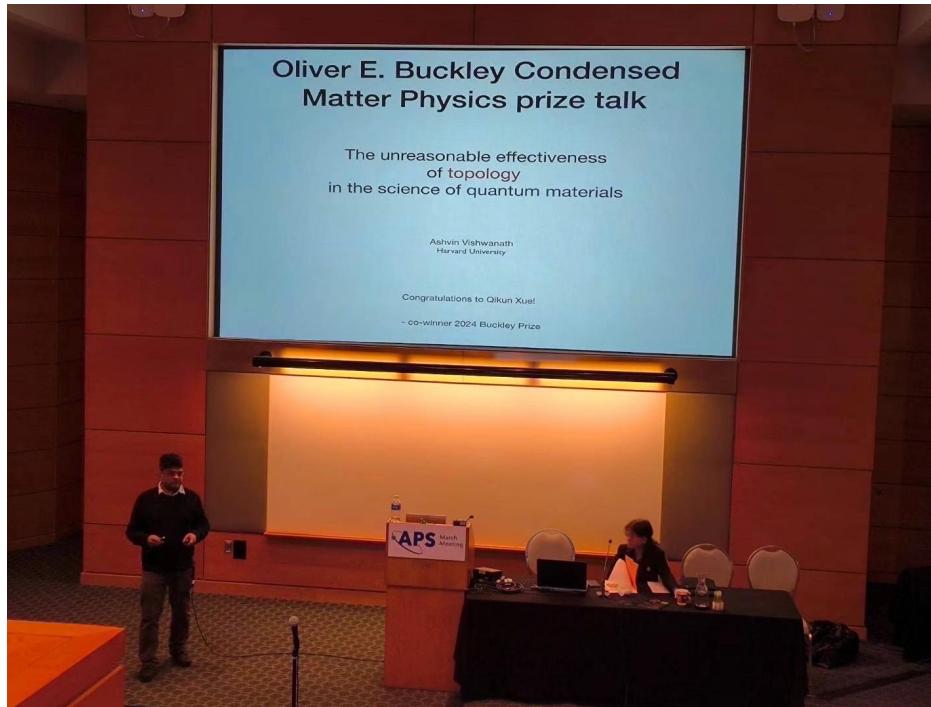
As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ...* physical properties at the **macroscopic** level that arise from the interactions of their electrons at the **microscopic** level

## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ...* physical properties at the *macroscopic* level that arise from the interactions of their electrons at the *microscopic* level, beginning at atomic and subatomic scales where the extraordinary effects of quantum mechanics cause **unique and unexpected behaviors**.

## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ...* physical properties at the *macroscopic* level that arise from the interactions of their electrons at the *microscopic* level, beginning at atomic and subatomic scales where the extraordinary effects of quantum mechanics cause unique and unexpected behaviors.



## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ...* physical properties at the *macroscopic* level that arise from the interactions of their electrons at the *microscopic* level, beginning at atomic and subatomic scales where the extraordinary effects of quantum mechanics cause unique and unexpected behaviors.

- *The unreasonable effectiveness of topology in the science of quantum materials*

## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ...* physical properties at the *macroscopic* level that arise from the interactions of their electrons at the *microscopic* level, beginning at atomic and subatomic scales where the extraordinary effects of quantum mechanics cause unique and unexpected behaviors.

- *The unreasonable effectiveness of topology in the science of quantum materials*, Ashvin Vishwanath of Harvard delivering the Buckley prize talk at last year's APS March Meeting in Minneapolis

## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ...* physical properties at the *macroscopic* level that arise from the interactions of their electrons at the *microscopic* level, beginning at atomic and subatomic scales where the extraordinary effects of quantum mechanics cause unique and unexpected behaviors.

- *The unreasonable effectiveness of topology in the science of quantum materials*, Ashvin Vishwanath of Harvard delivering the Buckley prize talk at last year's APS March Meeting in Minneapolis (薛其坤 of Tsinghua and SUSTech was the co-winner of the prize)

## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ...* physical properties at the *macroscopic* level that arise from the interactions of their electrons at the *microscopic* level, beginning at atomic and subatomic scales where the extraordinary effects of quantum mechanics cause unique and unexpected behaviors.

- *The unreasonable effectiveness of topology in the science of quantum materials*, Ashvin Vishwanath of Harvard delivering the Buckley prize talk at last year's APS March Meeting in Minneapolis (薛其坤 of Tsinghua and SUSTech was the co-winner of the prize)
- U.S. Department of Energy, Office of Science. *Basic research needs for quantum materials: Research to discover, harness, and exploit exotic electronic properties* (brochure), 2016.

## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ...* physical properties at the *macroscopic* level that arise from the interactions of their electrons at the *microscopic* level, beginning at atomic and subatomic scales where the extraordinary effects of quantum mechanics cause unique and unexpected behaviors.

- *The unreasonable effectiveness of topology in the science of quantum materials*, Ashvin Vishwanath of Harvard delivering the Buckley prize talk at last year's APS March Meeting in Minneapolis (薛其坤 of Tsinghua and SUSTech was the co-winner of the prize)
- U.S. Department of Energy, Office of Science. *Basic research needs for quantum materials: Research to discover, harness, and exploit exotic electronic properties* (brochure), 2016.
- 方忠 等, “拓扑电子材料计算预测”, 2023年度国家自然科学奖一等奖

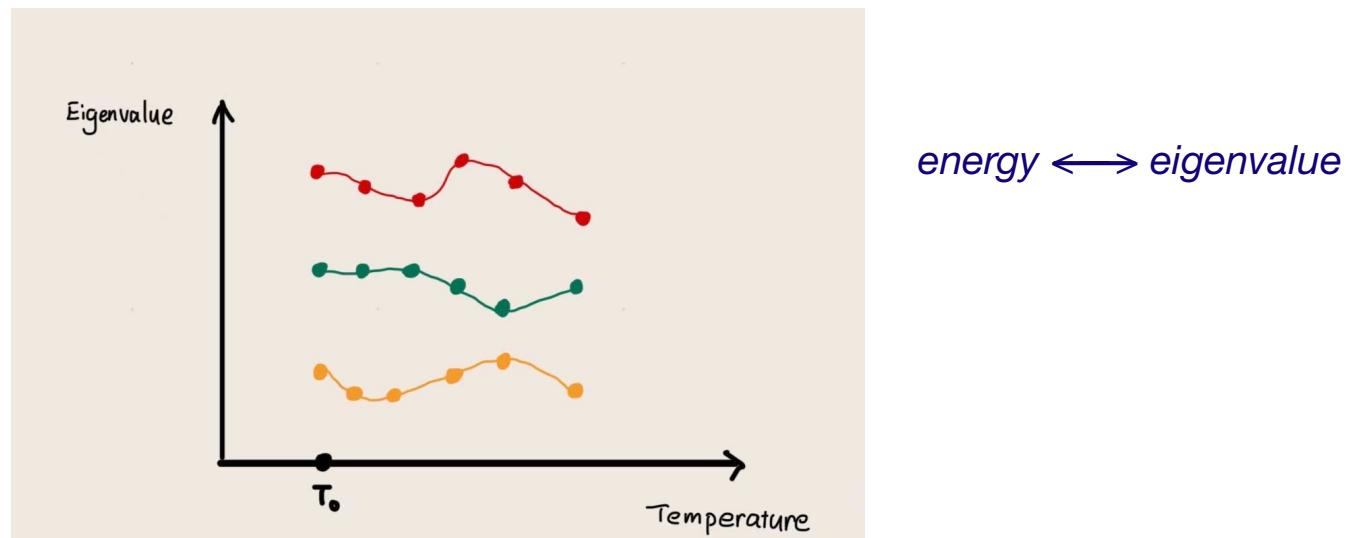
## Motivations: Quantum materials and their math modeling

As an umbrella term in condensed matter physics, **quantum materials** refer to solids with *exotic/exceptional/not found in nature ...* physical properties at the *macroscopic* level that arise from the interactions of their electrons at the *microscopic* level, beginning at atomic and subatomic scales where the extraordinary effects of quantum mechanics cause unique and unexpected behaviors.

- *The unreasonable effectiveness of topology in the science of quantum materials*, Ashvin Vishwanath of Harvard delivering the Buckley prize talk at last year's APS March Meeting in Minneapolis (薛其坤 of Tsinghua and SUSTech was the co-winner of the prize)
- U.S. Department of Energy, Office of Science. *Basic research needs for quantum materials: Research to discover, harness, and exploit exotic electronic properties* (brochure), 2016.
- 方忠 等, “**拓扑电子材料计算预测**”, 2023年度国家自然科学奖一等奖
- 第一届魅丽数学与交叉应用会议“**数学与生物医药、数学与先进材料**”, 2024年5月, 苏州

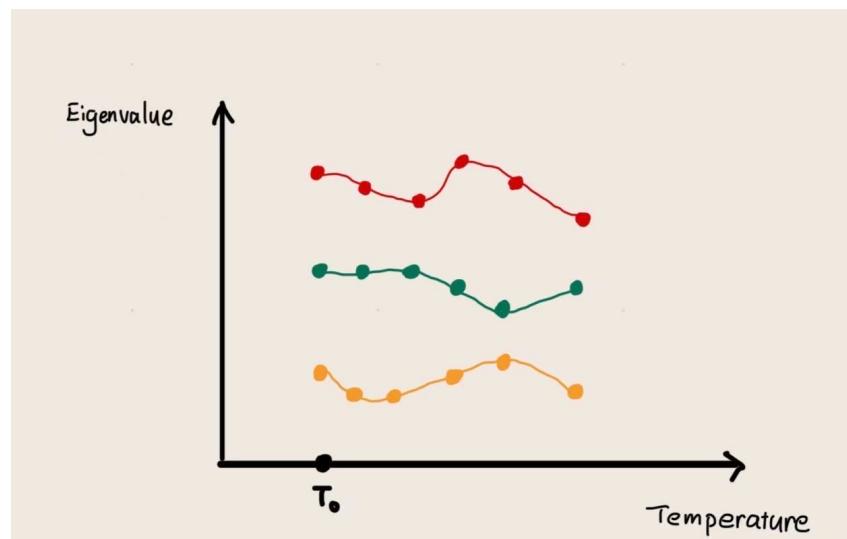
# Motivations: Quantum materials and their math modeling

Mathematical modeling of electronic energy *band structures* therein



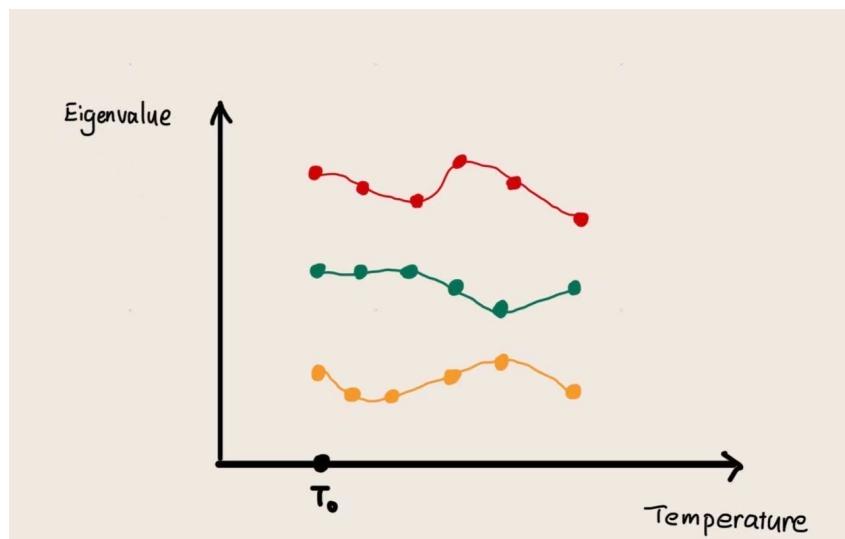
## Motivations: Quantum materials and their math modeling

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians*



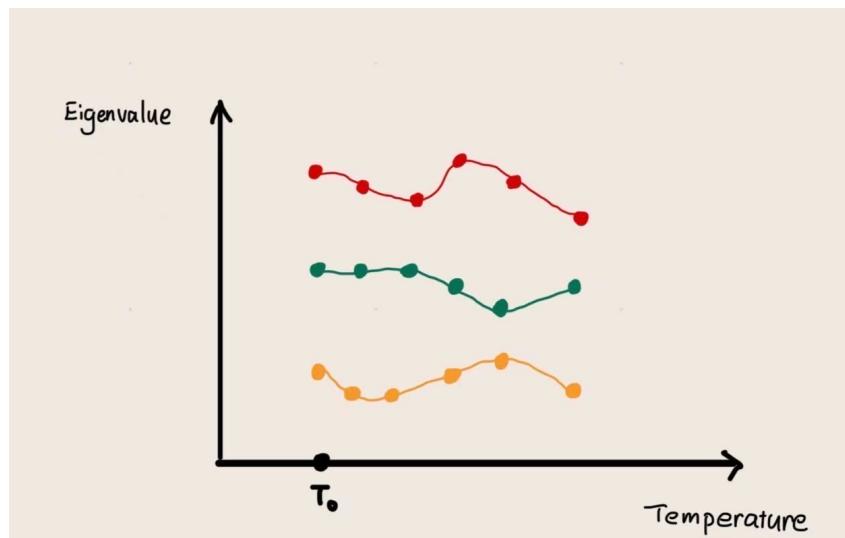
## Motivations: Quantum materials and their math modeling

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems



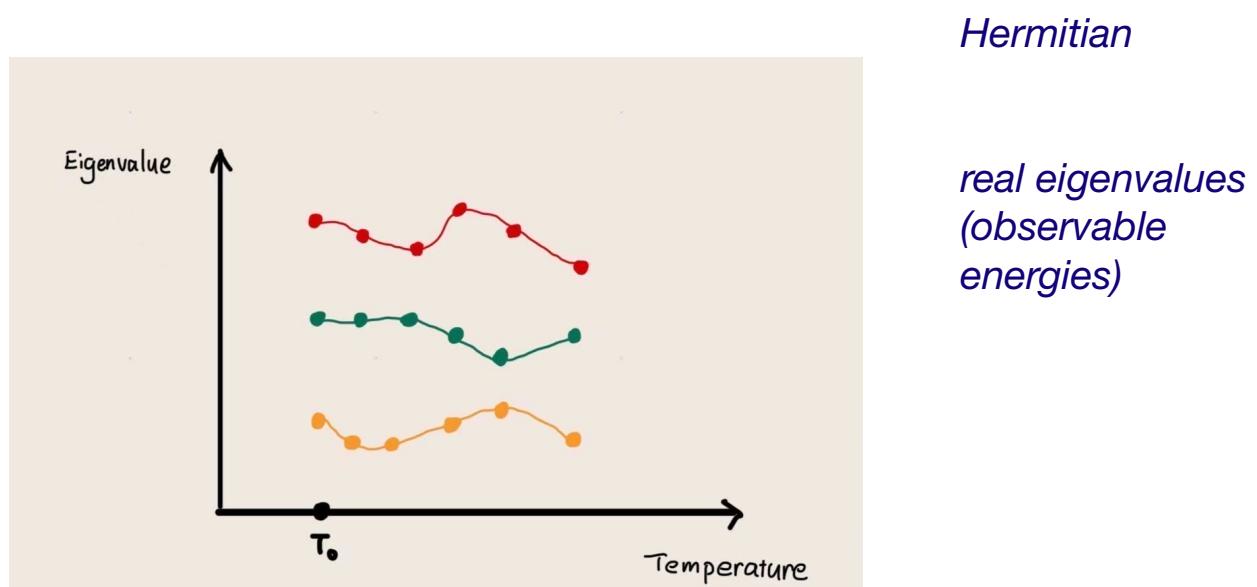
## Motivations: Quantum materials and their math modeling

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries]



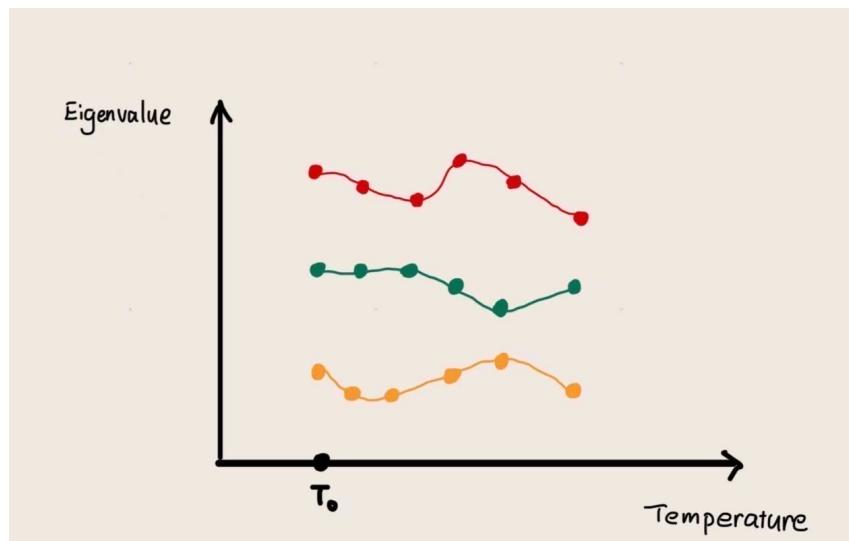
## Motivations: Quantum materials and their math modeling

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed **symmetries**]



# Motivations: Quantum materials and their math modeling

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed **symmetries**]

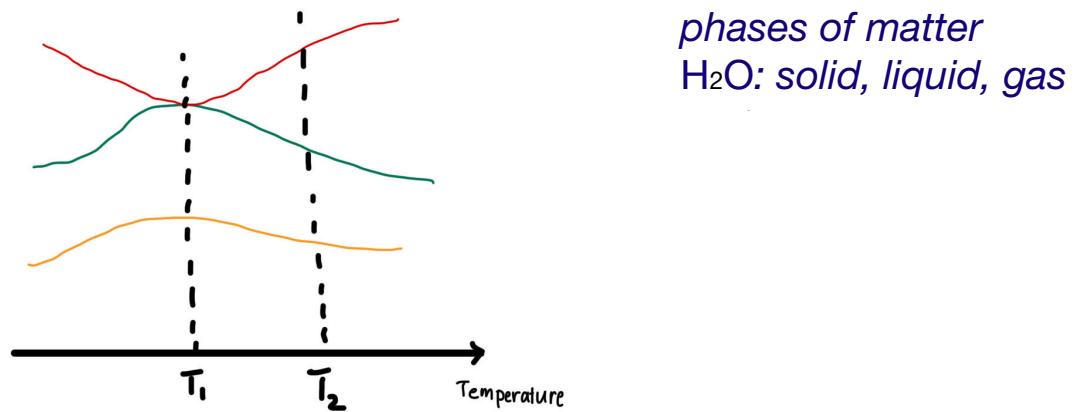


*Hermitian vs.  
non-Hermitian*

*real eigenvalues  
(observable  
energies) vs.  
eigenvalues with  
imaginary part  
(counts for  
energy exchange  
with surrounding  
environment or  
other systems)*

## Motivations: Quantum materials and their math modeling

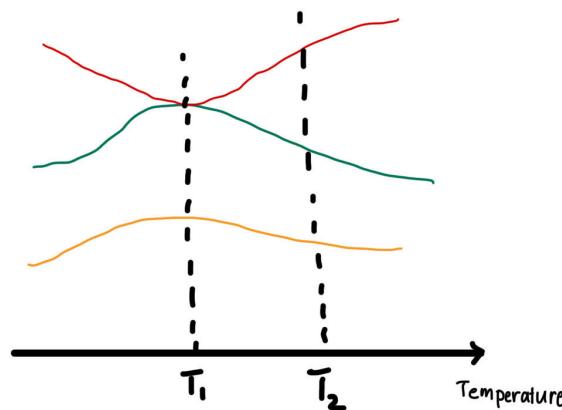
**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, **singularity/degeneracy** in the relevant moduli spaces



$T_1$  : singular points (points where eigenvalues degenerate)  
 $H(T_1)$  : gapless Hamiltonian  
 $H(T_2)$  : gapped Hamiltonian

## Motivations: Quantum materials and their math modeling

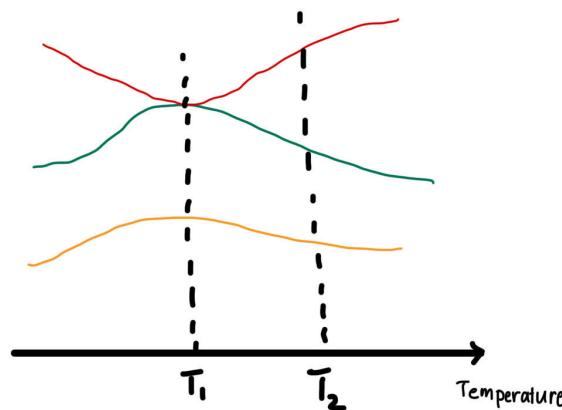
**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, **singularity/degeneracy** in the relevant moduli spaces, against which fine-tuning a system leads to **exceptional properties** of solid materials.



$T_1$  : singular points (points where eigenvalues degenerate)  
 $H(T_1)$  : gapless Hamiltonian  
 $H(T_2)$  : gapped Hamiltonian

## Motivations: Quantum materials and their math modeling

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, **singularity/degeneracy** in the relevant **moduli spaces**, against which fine-tuning a system leads to **exceptional properties** of solid materials.



$T_1$  : singular points (points where eigenvalues degenerate)  
 $H(T_1)$  : gapless Hamiltonian  
 $H(T_2)$  : gapped Hamiltonian

## **Mathematical digression: What is a moduli space?**

## **Mathematical digression: What is a moduli space?**

A moduli space is a space of parameters

## Mathematical digression: What is a moduli space?

A moduli space is a **space** of parameters, that is, a **set** of parameters **with extra structure**.

## Mathematical digression: What is a moduli space?

A moduli space is a **space** of parameters, that is, a **set** of parameters **with extra structure**.

*It is indeed “pointless” and is better understood as a **functor**!*

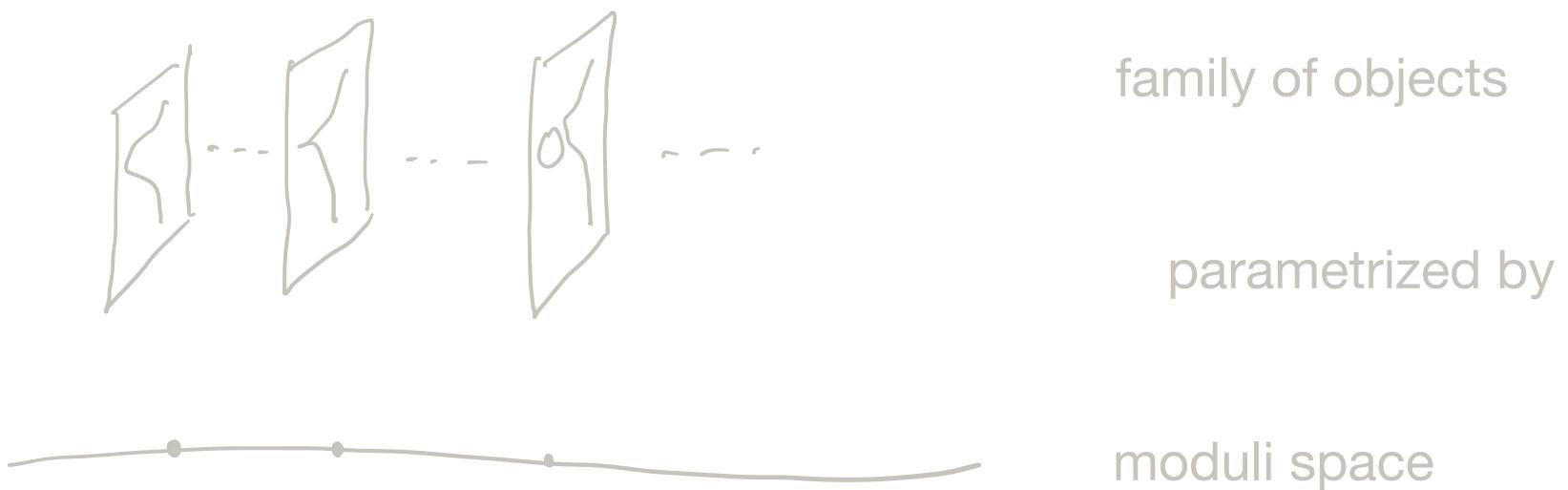
## Mathematical digression: What is a moduli space?

A moduli space is a space of parameters, that is, a set of parameters with extra structure. These parameters label objects we would like to study, often in a continuous fashion.

## Mathematical digression: What is a moduli space?

A **moduli space** is a **space** of parameters, that is, a **set** of parameters **with extra structure**. These **parameters** label objects we would like to study, often in a continuous fashion.

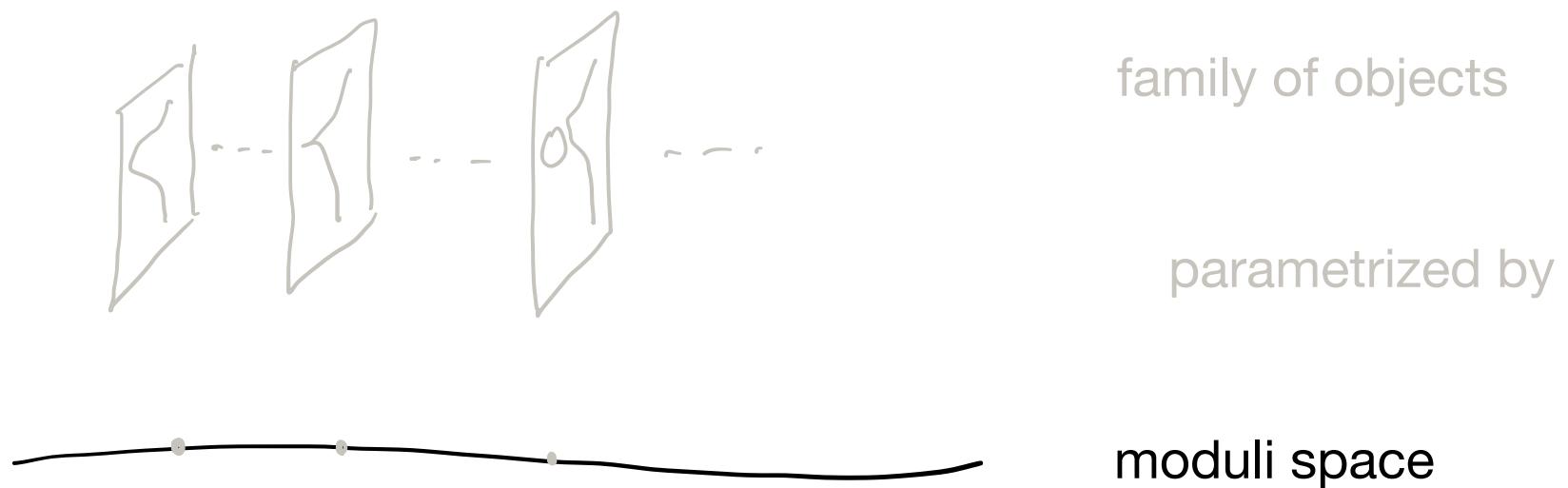
### Mental picture:



## Mathematical digression: What is a moduli space?

A **moduli space** is a **space** of parameters, that is, a **set** of parameters **with extra structure**. These **parameters** label objects we would like to study, often in a continuous fashion.

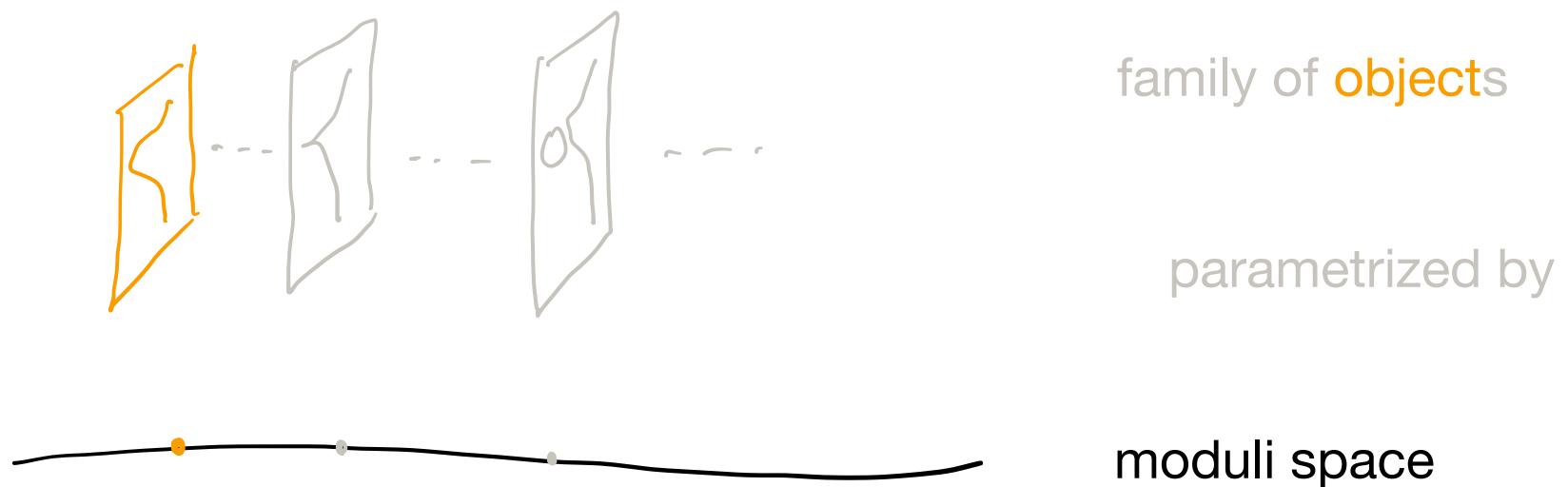
### Mental picture:



## Mathematical digression: What is a moduli space?

A **moduli space** is a **space** of parameters, that is, a **set** of parameters **with extra structure**. These **parameters** label objects we would like to study, often in a continuous fashion.

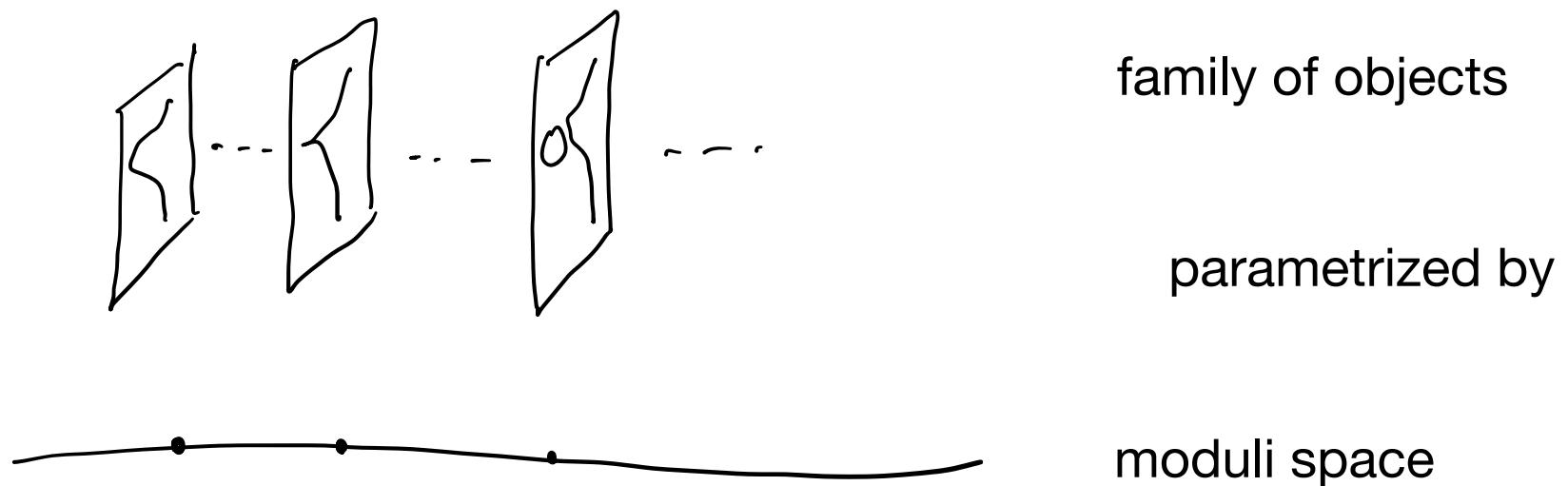
### Mental picture:



## Mathematical digression: What is a moduli space?

A **moduli space** is a **space** of parameters, that is, a **set** of parameters **with extra structure**. These **parameters** label objects we would like to study, often in a continuous fashion.

### Mental picture:



## **Why would you care about moduli spaces?**

## Why would you care about moduli spaces?

- Instead of dealing with an isolated or static object, we would really like to understand a continuous **family** of objects

## Why would you care about moduli spaces?

- Instead of dealing with an isolated or static object, we would really like to understand a continuous **family** of objects, or how an object **varies** as the parameters on which it depends change.

## Why would you care about moduli spaces?

- Instead of dealing with an isolated or static object, we would really like to understand a continuous **family** of objects, or how an object **varies** as the parameters on which it depends change.
- For these purposes, it is often fruitful to study this collection of parameters as a space on its own right.

## Why would you care about moduli spaces?

- Instead of dealing with an isolated or static object, we would really like to understand a continuous **family** of objects, or how an object **varies** as the parameters on which it depends change.
- For these purposes, it is often fruitful to study this collection of parameters as a space on its own right.
- While the objects in question have structure, the parameter space also has its own structure

## Why would you care about moduli spaces?

- Instead of dealing with an isolated or static object, we would really like to understand a continuous **family** of objects, or how an object **varies** as the parameters on which it depends change.
- For these purposes, it is often fruitful to study this collection of parameters as a space on its own right.
- While the objects in question have structure, the parameter space also has its own structure, often rich and distinct from that of the objects.

## Why would you care about moduli spaces?

- Instead of dealing with an isolated or static object, we would really like to understand a continuous **family** of objects, or how an object **varies** as the parameters on which it depends change.
- For these purposes, it is often fruitful to study this collection of parameters as a space on its own right.
- While the objects in question have structure, the parameter space also has its own structure, often rich and distinct from that of the objects.
- Understanding the moduli space offers in turn understanding of the objects individually

## Why would you care about moduli spaces?

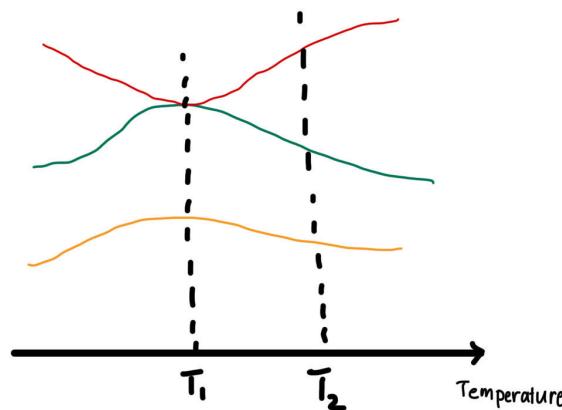
- Instead of dealing with an isolated or static object, we would really like to understand a continuous **family** of objects, or how an object **varies** as the parameters on which it depends change.
- For these purposes, it is often fruitful to study this collection of parameters as a space on its own right.
- While the objects in question have structure, the parameter space also has its own structure, often rich and distinct from that of the objects.
- Understanding the moduli space offers in turn understanding of the objects individually and as a whole.

## Why would you care about moduli spaces?

- Instead of dealing with an isolated or static object, we would really like to understand a continuous **family** of objects, or how an object **varies** as the parameters on which it depends change.
- For these purposes, it is often fruitful to study this collection of parameters as a space on its own right.
- While the objects in question have structure, the parameter space also has its own structure, often rich and distinct from that of the objects.
- Understanding the moduli space offers in turn understanding of the objects individually and as a whole.
- In this sense, studying moduli spaces is of the **second-order** nature.

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, **singularity/degeneracy** in the relevant **moduli spaces**, against which fine-tuning a system leads to **exceptional properties** of solid materials.



$T_1$  : singular points (points where eigenvalues degenerate)  
 $H(T_1)$  : gapless Hamiltonian  
 $H(T_2)$  : gapped Hamiltonian

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, singularity/degeneracy in the relevant moduli spaces, against which fine-tuning a system leads to exceptional properties of solid materials. This mathematical modeling is then followed by **experimental** realization, engineering, ...

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, singularity/degeneracy in the relevant moduli spaces, against which fine-tuning a system leads to exceptional properties of solid materials. This mathematical modeling is then followed by experimental realization, engineering, ... (though there is approach the other way around).

*Guzman et al., Model-free characterization of topological edge and corner states in mechanical networks, PNAS 2024.*

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed **symmetries**] and, in particular, singularity/degeneracy in the relevant moduli spaces, against which fine-tuning a system leads to exceptional properties of solid materials. This mathematical modeling is then followed by experimental realization, engineering, ... (though there is approach the other way around).

Our preliminary work explored the intriguing topological structures arising from certain novel ***non-Hermitian*** systems

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, singularity/degeneracy in the relevant moduli spaces, against which fine-tuning a system leads to exceptional properties of solid materials. This mathematical modeling is then followed by experimental realization, engineering, ... (though there is approach the other way around).

Our preliminary work explored the intriguing topological structures arising from certain novel *non-Hermitian* systems, whose moduli spaces have **stratified, non-isolated** singularities

- H. Jia, R.-Y. Zhang, J. Hu, Y. Xiao, S. Zhang, **Y. Zhu**, and C. T. Chan. *Topological classification for intersection singularities* of exceptional surfaces in pseudo-Hermitian systems. **Communication Physics**, 6:293, 2023.
- J. Hu, R.-Y. Zhang, Y. Wang, X. Ouyang, **Y. Zhu**, H. Jia, and C. T. Chan. *Non-Hermitian swallowtail catastrophe revealing transitions among diverse topological singularities*. **Nature Physics**, 19:1098–1103, 2023.

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, singularity/degeneracy in the relevant moduli spaces, against which fine-tuning a system leads to exceptional properties of solid materials. This mathematical modeling is then followed by **experimental realization, engineering, ...** (though there is approach the other way around).

Our preliminary work explored the intriguing topological structures arising from certain novel *non-Hermitian* systems, whose moduli spaces have stratified, non-isolated singularities, as well as their **circuit realizations**

- H. Jia, R.-Y. Zhang, J. Hu, Y. Xiao, S. Zhang, **Y. Zhu**, and C. T. Chan. *Topological classification for intersection singularities of exceptional surfaces in pseudo-Hermitian systems.* **Communication Physics**, 6:293, 2023.
- J. Hu, R.-Y. Zhang, Y. Wang, X. Ouyang, **Y. Zhu**, H. Jia, and C. T. Chan. *Non-Hermitian swallowtail catastrophe revealing transitions among diverse topological singularities.* **Nature Physics**, 19:1098–1103, 2023.

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, singularity/degeneracy in the relevant moduli spaces, against which fine-tuning a system leads to exceptional properties of solid materials. This mathematical modeling is then followed by experimental realization, engineering, ... (though there is approach the other way around).

Our preliminary work explored the intriguing topological structures arising from certain novel *non-Hermitian* systems, whose moduli spaces have stratified, non-isolated singularities, as well as their circuit realizations and **extraordinary physical consequences**.

### Preprints

- H. Jia, J. Hu, R.-Y. Zhang, Y. Xiao, D. Wang, M. Wang, S. Ma, X. Ouyang, **Y. Zhu**, and C. T. Chan. *Anomalous bulk-edge correspondence intrinsically beyond line-gap topology in non-Hermitian swallowtail gapless phase.*
- H. Jia, J. Hu, R.-Y. Zhang, Y. Xiao, D. Wang, M. Wang, S. Ma, X. Ouyang, **Y. Zhu**, and C. T. Chan. *Unconventional topological edge states beyond the paradigms of line-gap topology.*
- J. Hu, R.-Y. Zhang, M. Wang, D. Wang, S. Ma, X. Ouyang, **Y. Zhu**, H. Jia, and C. T. Chan. *Unconventional bulk-Fermi-arc linking paired exceptional points of order three and their splitting from a defective triple point.*

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, singularity/degeneracy in the relevant moduli spaces, against which fine-tuning a system leads to exceptional properties of solid materials. This mathematical modeling is then followed by experimental realization, engineering, ... (though there is approach the other way around).

Our preliminary work explored the intriguing topological structures arising from certain novel *non-Hermitian* systems, whose moduli spaces have stratified, non-isolated singularities, as well as their circuit realizations and extraordinary physical consequences. However, the mathematical modeling was rather **ad hoc** and the topological classifications remain **incomplete**.

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, singularity/degeneracy in the relevant moduli spaces, against which fine-tuning a system leads to exceptional properties of solid materials. This mathematical modeling is then followed by experimental realization, engineering, ... (though there is approach the other way around).

Our preliminary work explored the intriguing topological structures arising from certain novel *non-Hermitian* systems, whose moduli spaces have stratified, non-isolated singularities, as well as their circuit realizations and extraordinary physical consequences. However, the mathematical modeling was rather **ad hoc** and the topological classifications remain **incomplete**.

Thanks to Hopf bundles and Higgs bundles as **eigenbundles**, we now have a **conceptually more systematic**, visibly more intuitive understanding of the topic.

## Motivations: Quantum materials and their math modeling (cont'd)

**Mathematical modeling** of electronic energy band structures therein concerns topological/homotopical classification of *Hamiltonians* [= quantum mechanical systems = (families of) matrices with prescribed symmetries] and, in particular, singularity/degeneracy in the relevant moduli spaces, against which fine-tuning a system leads to exceptional properties of solid materials. This mathematical modeling is then followed by experimental realization, engineering, ... (though there is approach the other way around).

Our preliminary work explored the intriguing topological structures arising from certain novel *non-Hermitian* systems, whose moduli spaces have stratified, non-isolated singularities, as well as their circuit realizations and extraordinary physical consequences. However, the mathematical modeling was rather ad hoc and the topological classifications remain incomplete.

Thanks to Hopf bundles and Higgs bundles as *eigenbundles*, we now have a conceptually more systematic, visibly more intuitive understanding of the topic. The structure of **Higgs bundles** also hints at certain deeper aspects of mathematics as well as physics.

## **Mathematical set-up: Eigenframe evolution of non-Hermitian systems**

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the **real** matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a **variant of Hermitian symmetry** such that  $\eta H \eta^{-1} = \overline{H^t}$

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a **variant of Hermitian symmetry** such that  $\eta H \eta^{-1} = \overline{H^t}$  where

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is a Minkowski-like metric form.}$$

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is a Minkowski-like metric form.}$$

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$

has a double root if and only if  $f_2 = \pm f_3$ .

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

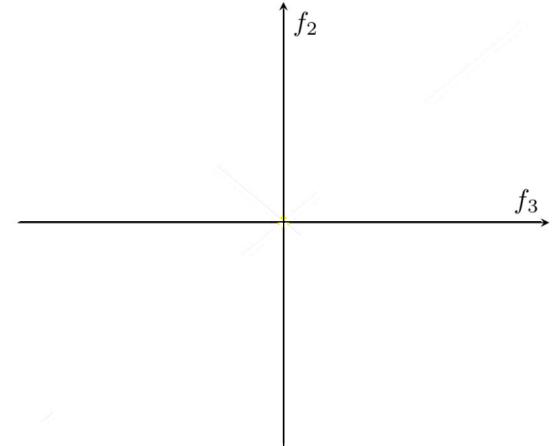
It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is a Minkowski-like metric form.}$$

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$

has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:



## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

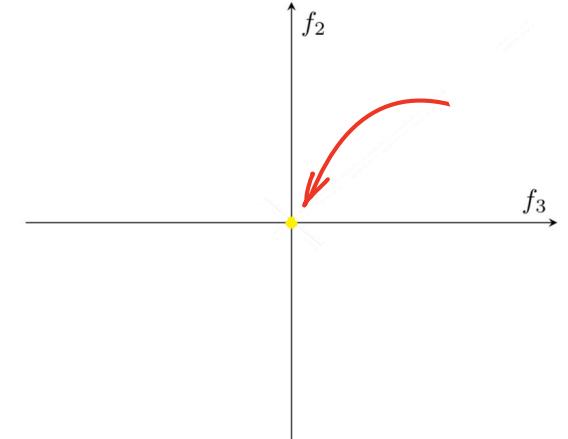
$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  is a Minkowski-like metric form.

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$

has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

0. Over  $\{(0, 0)\}$



## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

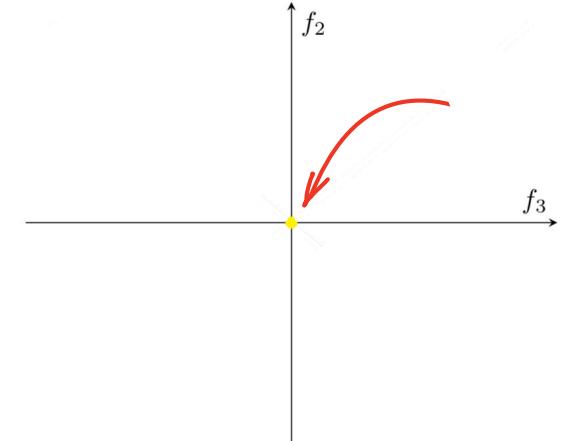
$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  is a Minkowski-like metric form.

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$

has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue



## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

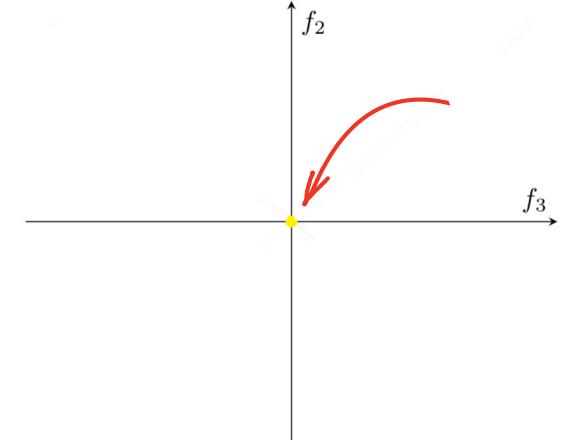
$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  is a Minkowski-like metric form.

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$

has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.



## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

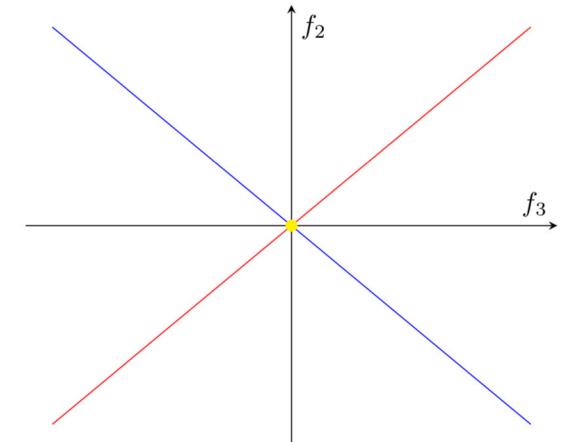
$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is a Minkowski-like metric form.}$$

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$



has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

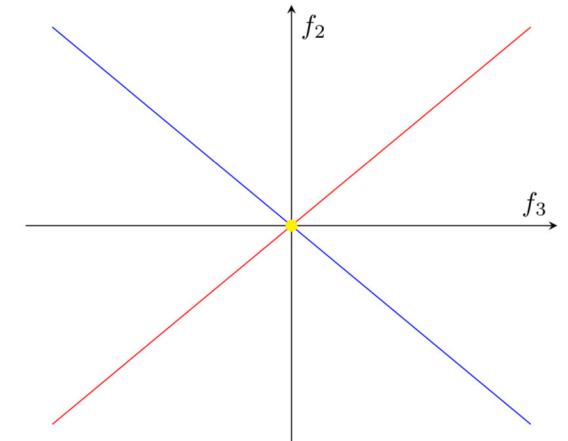
$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is a Minkowski-like metric form.}$$

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$



has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

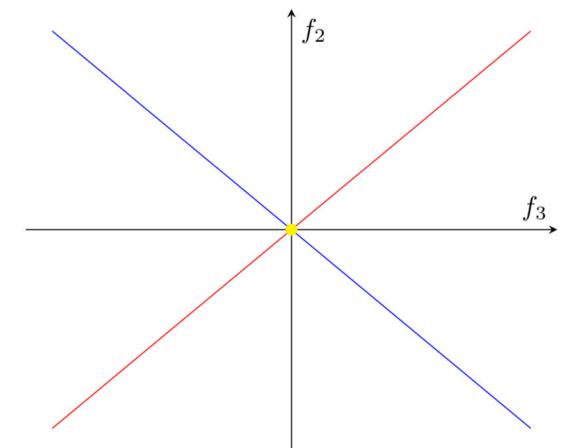
$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is a Minkowski-like metric form.}$$

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$



has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

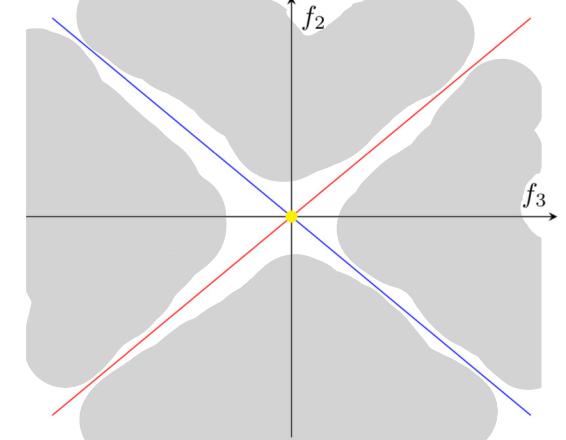
$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  is a Minkowski-like metric form.

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$

has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

2. Over  $\{f_2 \neq \pm f_3\}$



## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

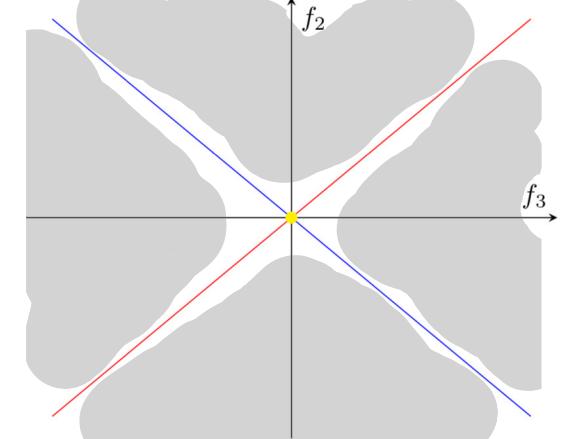
$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  is a Minkowski-like metric form.

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$

has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 **distinct** eigenvalues.



## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

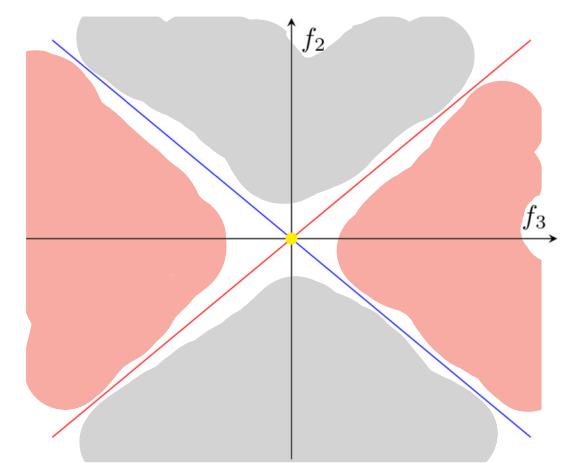
$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  is a Minkowski-like metric form.

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$



has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**.

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

With motivations from topological classifications of non-Hermitian gapless quantum mechanical systems and their applications to materials science, let us consider the real matrix (a Hamiltonian)

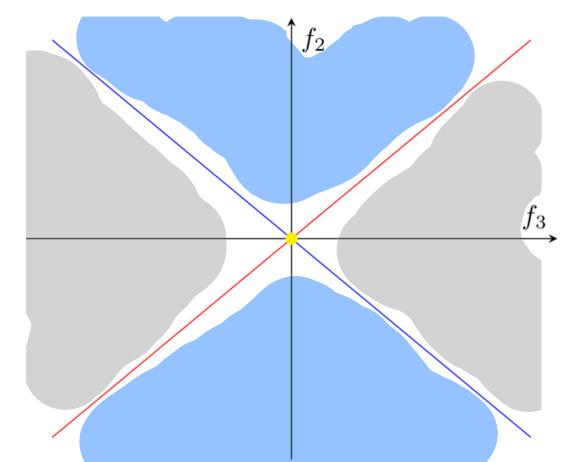
$$H = H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

It satisfies a variant of Hermitian symmetry such that  $\eta H \eta^{-1} = \overline{H^t}$  where

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is a Minkowski-like metric form.}$$

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_2 \\ -f_2 & -f_3 - \omega \end{vmatrix} = \omega^2 + f_2^2 - f_3^2$$



has a double root if and only if  $f_2 = \pm f_3$ . As a parameter space for this Hamiltonian  $H$ , the  $f_2 f_3$ -plane becomes a **stratified space**:

2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are real. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this **stratified space**

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy.

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us.

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

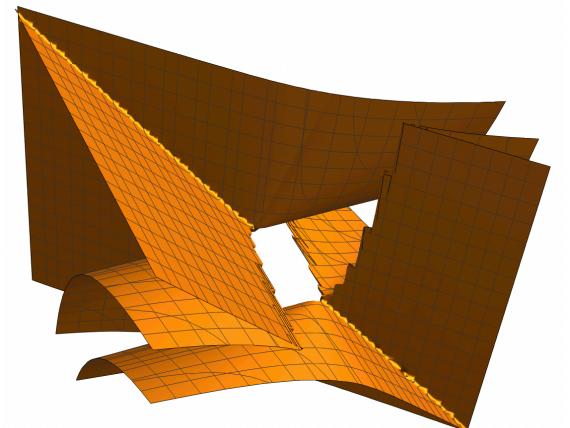
**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:

*The equation for this surface is a non-homogeneous real polynomial in  $f_1, f_2, f_3$  of degree 6.*



Swallowtail couple sw2

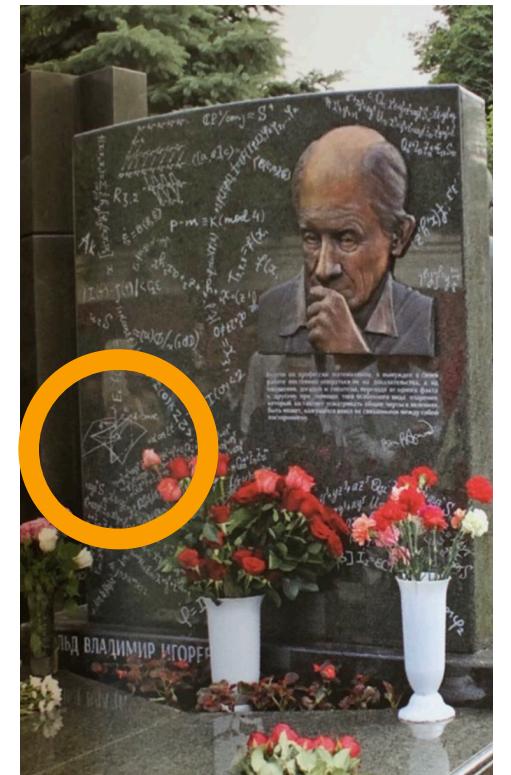
## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:



V. I. Arnold's tombstone at the Novodevichy Cemetery in Moscow

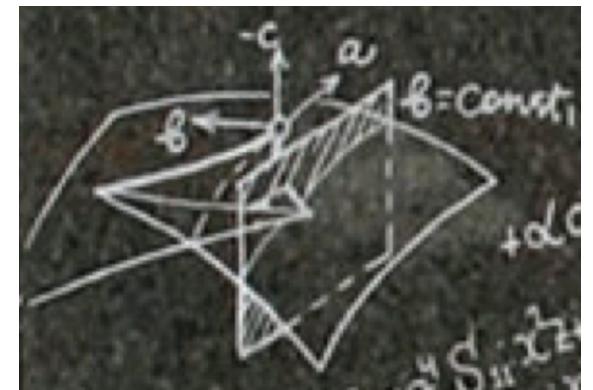
## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:



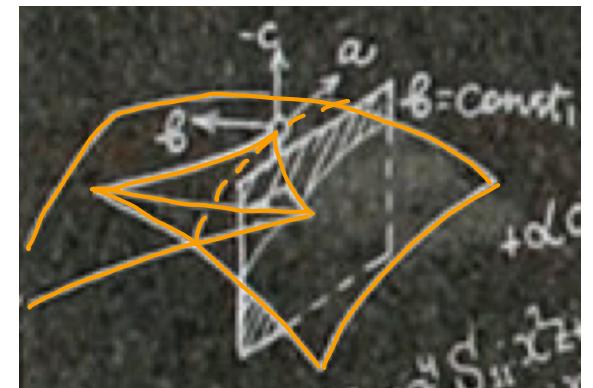
## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:



A **local** model for moduli spaces of 3-band Hamiltonians

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

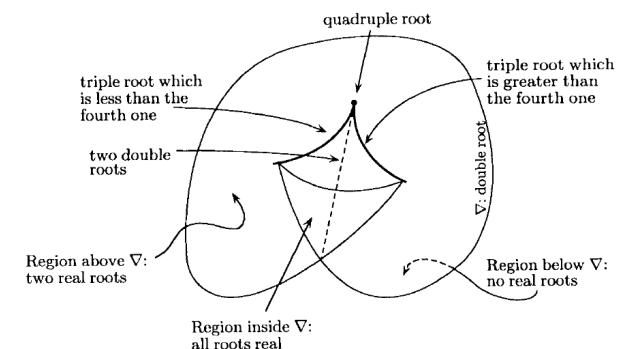
$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:

*Arnold, Braids of algebraic functions and the cohomology of swallowtails, 1968.*

*Homological stability of braid groups*

*Portrait from Gelfand, Kapranov, Zelevinsky,  
Discriminants, resultants, and multidimensional determinants.*



The space of polynomials  $x^4 + ax^2 + bx + c$

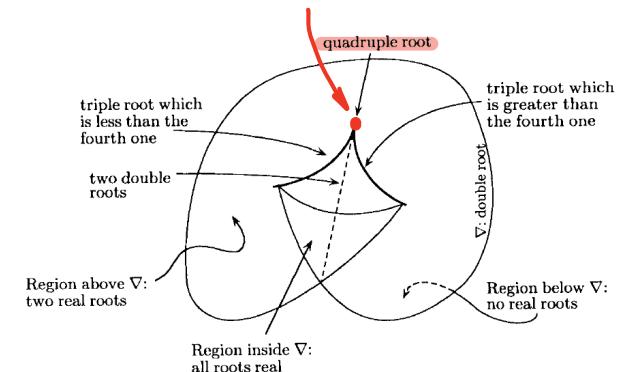
## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:



The space of polynomials  $x^4 + ax^2 + bx + c$

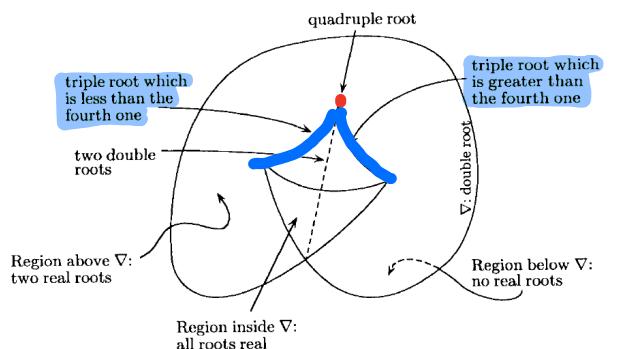
## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:



The space of polynomials  $x^4 + ax^2 + bx + c$

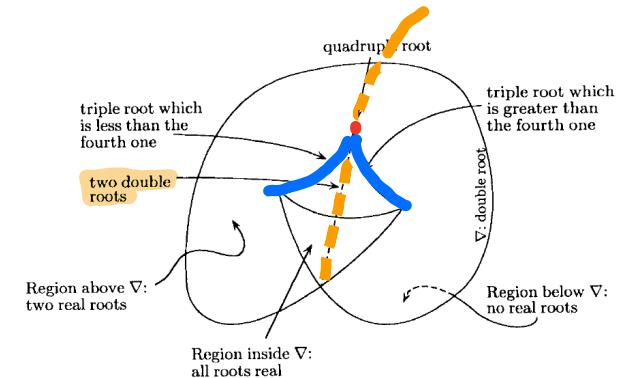
## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:



The space of polynomials  $x^4 + ax^2 + bx + c$

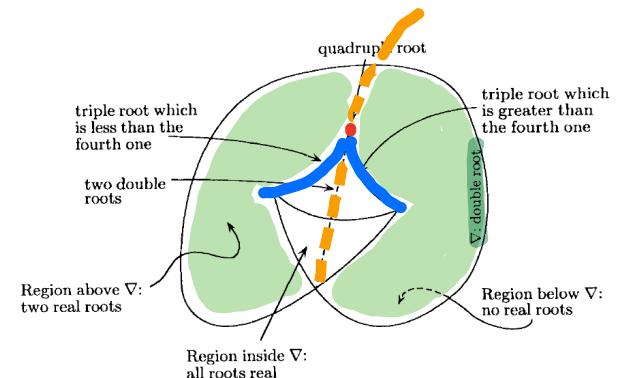
## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:



The space of polynomials  $x^4 + ax^2 + bx + c$

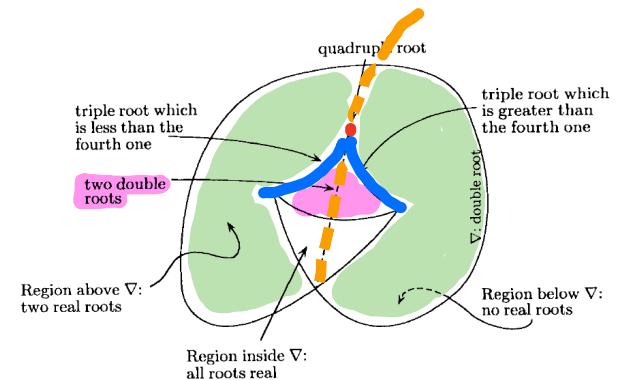
## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:



The space of polynomials  $x^4 + ax^2 + bx + c$

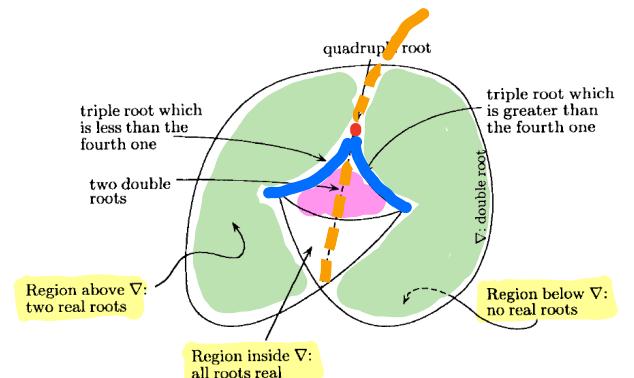
## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:



The space of polynomials  $x^4 + ax^2 + bx + c$

## Mathematical set-up: Eigenframe evolution of non-Hermitian systems

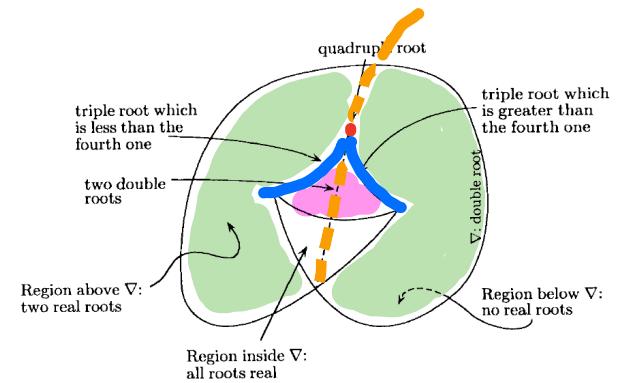
**Question.** We would like to classify, up to “intersection” homotopy, the loops in this stratified space, according to how the eigenvectors rotate and deform along each loop and the resulting monodromy. What would be computable algebraic invariants for such a “stratified vector bundle?”

Another example of non-Hermitian “3-band systems” is also of particular interest to us. One such Hamiltonian takes the form

$$H_3 = \begin{bmatrix} 1 - f_1 - f_2 & f_1 & f_2 \\ -f_1 & f_1 - f_3 & f_3 \\ -f_2 & f_3 & f_2 - f_3 \end{bmatrix}$$

Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the  $f_1 f_2 f_3$ -space:

Again, we aim to find computable algebraic invariants that systematically classify the evolutions of eigenvectors along loops in such stratified parameter spaces, including when they cross the discriminant surface resulting in degeneracies of various sorts.



The space of polynomials  $x^4 + ax^2 + bx + c$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is isolated

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is isolated, yet has profound physical implications already known to Arnold.

*Remarks on eigenvalues and eigenvectors of Hermitian matrices,  
Berry phase, adiabatic connections and quantum Hall effect, 1995.*

*Also: Polymathematics, 2000.*

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is isolated, yet has profound physical implications already known to Arnold. Let us consider the real Hamiltonian

$$H(f_1, f_3) = \begin{bmatrix} f_3 & f_1 \\ f_1 & -f_3 \end{bmatrix}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is isolated, yet has profound physical implications already known to Arnold. Let us consider the real Hamiltonian

$$H(f_1, f_3) = \begin{bmatrix} f_3 & f_1 \\ f_1 & -f_3 \end{bmatrix}$$

It represents all symmetric  $2 \times 2$  matrices **spectrally**, since any  $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$  has the same eigenvalues and eigenvectors as  $\begin{bmatrix} (a-c)/2 & b \\ b & (c-a)/2 \end{bmatrix}$ .

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is isolated, yet has profound physical implications already known to Arnold. Let us consider the real Hamiltonian

$$H(f_1, f_3) = \begin{bmatrix} f_3 & f_1 \\ f_1 & -f_3 \end{bmatrix}$$

It represents all symmetric  $2 \times 2$  matrices spectrally, since any  $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$  has the same eigenvalues and eigenvectors as  $\begin{bmatrix} (a-c)/2 & b \\ b & (c-a)/2 \end{bmatrix}$ .

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_1 \\ f_1 & -f_3 - \omega \end{vmatrix} = \omega^2 - f_1^2 - f_3^2$$

has a double root if and only if  $f_1 = f_3 = 0$ .

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is isolated, yet has profound physical implications already known to Arnold. Let us consider the real Hamiltonian

$$H(f_1, f_3) = \begin{bmatrix} f_3 & f_1 \\ f_1 & -f_3 \end{bmatrix}$$

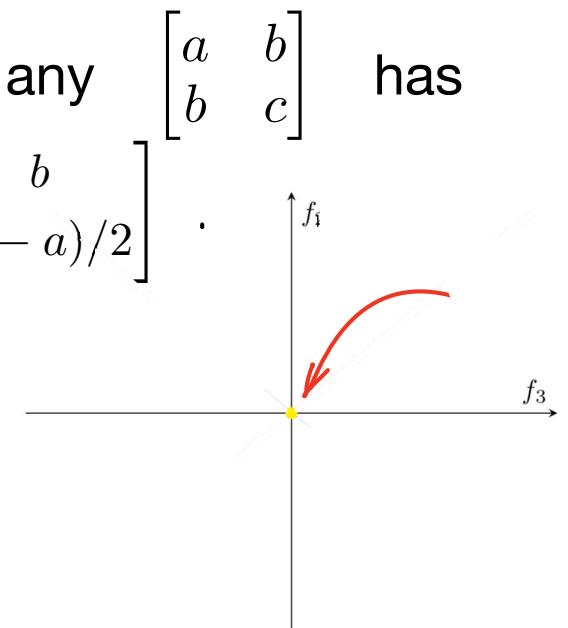
It represents all symmetric  $2 \times 2$  matrices spectrally, since any  $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$  has the same eigenvalues and eigenvectors as  $\begin{bmatrix} (a-c)/2 & b \\ b & (c-a)/2 \end{bmatrix}$ .

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_1 \\ f_1 & -f_3 - \omega \end{vmatrix} = \omega^2 - f_1^2 - f_3^2$$

has a double root if and only if  $f_1 = f_3 = 0$ .

The parameter  $f_1 f_3$ -plane thus has an **isolated singular point**  $(0, 0)$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is isolated, yet has profound physical implications already known to Arnold. Let us consider the real Hamiltonian

$$H(f_1, f_3) = \begin{bmatrix} f_3 & f_1 \\ f_1 & -f_3 \end{bmatrix}$$

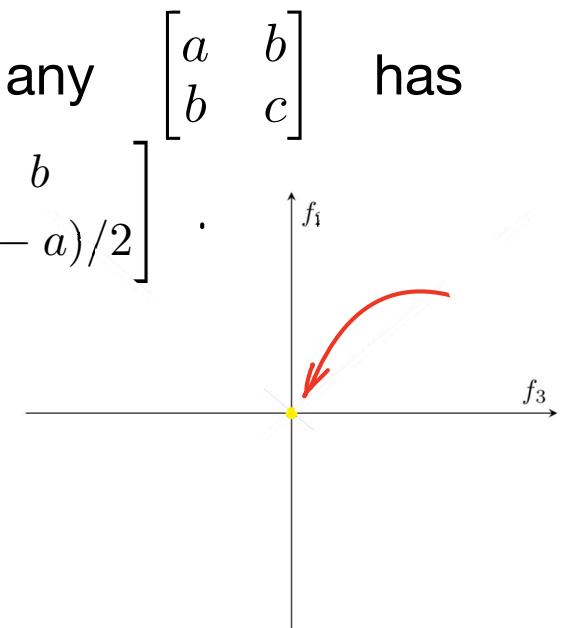
It represents all symmetric  $2 \times 2$  matrices spectrally, since any  $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$  has the same eigenvalues and eigenvectors as  $\begin{bmatrix} (a-c)/2 & b \\ b & (c-a)/2 \end{bmatrix}$ .

Its characteristic polynomial

$$\begin{vmatrix} f_3 - \omega & f_1 \\ f_1 & -f_3 - \omega \end{vmatrix} = \omega^2 - f_1^2 - f_3^2$$

has a double root if and only if  $f_1 = f_3 = 0$ .

The parameter  $f_1 f_3$ -plane thus has an **isolated singular point**  $(0, 0)$  and is a particularly simple **stratified space**.



## **Eigenframe rotation as vector bundles: Revisiting the Hermitian case**

How does the eigenframe rotate over this stratified parameter plane?

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

How does the eigenframe rotate over this stratified parameter plane?

As our starting point, previous work of Wu et al. [**Science**, 2019] classified the eigenframe rotation by

$$\pi_1(SO(2)/O(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

How does the eigenframe rotate over this stratified parameter plane?

As our starting point, previous work of Wu et al. [**Science**, 2019] classified the eigenframe rotation by

$$\pi_1(SO(2)/O(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Moreover, they obtained **non-Abelian** “topological charge” for  $n$ -band Hermitian systems when  $n > 2$ , such as

$$\pi_1\left(SO(3)/(O(1) \times O(1))\right) \cong \mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

How does the eigenframe rotate over this stratified parameter plane?

As our starting point, previous work of Wu et al. [**Science**, 2019] classified the eigenframe rotation by

$$\pi_1(SO(2)/O(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Moreover, they obtained non-Abelian “topological charge” for  $n$ -band Hermitian systems when  $n > 2$ , such as

$$\pi_1\left(SO(3)/(O(1) \times O(1))\right) \cong \mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$$

However, their explanation for the appearance of the  $O(1)^n$ -action in constructing the moduli spaces was rather ad hoc.

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

How does the eigenframe rotate over this stratified parameter plane?

As our starting point, previous work of Wu et al. [**Science**, 2019] classified the eigenframe rotation by

$$\pi_1(SO(2)/O(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Moreover, they obtained non-Abelian “topological charge” for  $n$ -band Hermitian systems when  $n > 2$ , such as

$$\pi_1\left(SO(3)/(O(1) \times O(1))\right) \cong \mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$$

However, their explanation for the appearance of the  $O(1)^n$ -action in constructing the moduli spaces was rather ad hoc.

One of our key steps is a more conceptual understanding of the above moduli spaces in the case of  $n=2$  through **bundle theory**.

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

How does the eigenframe rotate over this stratified parameter plane?

As our starting point, previous work of Wu et al. [**Science**, 2019] classified the eigenframe rotation by

$$\pi_1(SO(2)/O(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Moreover, they obtained non-Abelian “topological charge” for  $n$ -band Hermitian systems when  $n > 2$ , such as

$$\pi_1\left(SO(3)/(O(1) \times O(1))\right) \cong \mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$$

However, their explanation for the appearance of the  $O(1)^n$ -action in constructing the moduli spaces was rather ad hoc.

One of our key steps is a more conceptual understanding of the above moduli spaces in the case of  $n=2$  through **bundle theory**. To see how they rotate, let us compute the unit eigenvectors explicitly.

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

How does the eigenframe rotate over this stratified parameter plane?

As our starting point, previous work of Wu et al. [**Science**, 2019] classified the eigenframe rotation by

$$\pi_1(SO(2)/O(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Moreover, they obtained non-Abelian “topological charge” for  $n$ -band Hermitian systems when  $n > 2$ , such as

$$\pi_1\left(SO(3)/(O(1) \times O(1))\right) \cong \mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$$

However, their explanation for the appearance of the  $O(1)^n$ -action in constructing the moduli spaces was rather ad hoc.

One of our key steps is a more conceptual understanding of the above moduli spaces in the case of  $n=2$  through **bundle theory**. To see how they rotate, let us compute the unit eigenvectors explicitly.

$$\begin{vmatrix} f_3 - \omega & f_1 \\ f_1 & -f_3 - \omega \end{vmatrix} = \omega^2 - f_1^2 - f_3^2$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

How does the eigenframe rotate over this stratified parameter plane?

As our starting point, previous work of Wu et al. [**Science**, 2019] classified the eigenframe rotation by

$$\pi_1(SO(2)/O(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Moreover, they obtained non-Abelian “topological charge” for  $n$ -band Hermitian systems when  $n > 2$ , such as

$$\pi_1\left(SO(3)/(O(1) \times O(1))\right) \cong \mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$$

However, their explanation for the appearance of the  $O(1)^n$ -action in constructing the moduli spaces was rather ad hoc.

One of our key steps is a more conceptual understanding of the above moduli spaces in the case of  $n=2$  through **bundle theory**. To see how they rotate, let us compute the unit eigenvectors explicitly.

$$\begin{vmatrix} f_3 - \omega & f_1 \\ f_1 & -f_3 - \omega \end{vmatrix} = \omega^2 - f_1^2 - f_3^2 = 0 \implies \omega_{\pm} = \pm \sqrt{f_1^2 + f_3^2}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

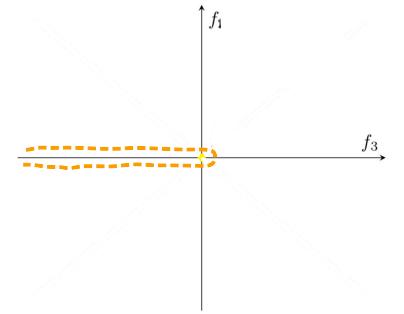
To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0}$$
$$\begin{bmatrix} \left(f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right)\left(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) & f_1\left(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{bmatrix}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

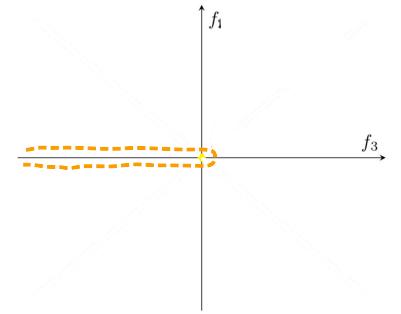
$$\left[ \begin{array}{cc} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{array} \right] \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0} \left[ \begin{array}{cc} \left( f_3 - \sqrt{\quad} \right) \left( -f_3 - \sqrt{\quad} \right) & f_1 \left( -f_3 - \sqrt{\quad} \right) \\ f_1 & -f_3 - \sqrt{\quad} \end{array} \right]$$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0}$$

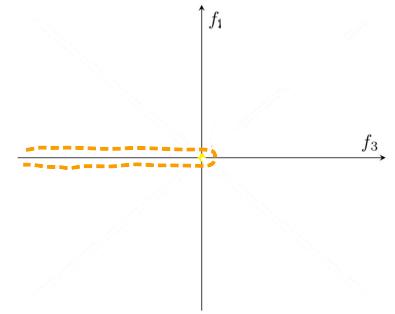


$$\begin{bmatrix} \left(f_3 - \sqrt{\quad}\right)\left(-f_3 - \sqrt{\quad}\right) & f_1\left(-f_3 - \sqrt{\quad}\right) \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \rightarrow \begin{bmatrix} f_1^2 & -f_1f_3 - f_1\sqrt{\quad} \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0}$$



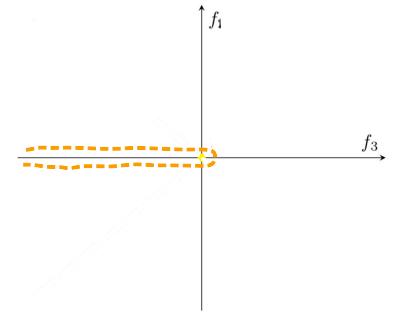
$$\begin{bmatrix} (f_3 - \sqrt{\quad})(-f_3 - \sqrt{\quad}) & f_1(-f_3 - \sqrt{\quad}) \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \rightarrow \begin{bmatrix} f_1^2 & -f_1 f_3 - f_1 \sqrt{\quad} \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0}$$



$$\begin{bmatrix} (f_3 - \sqrt{\quad})(-f_3 - \sqrt{\quad}) & f_1(-f_3 - \sqrt{\quad}) \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \rightarrow \begin{bmatrix} f_1^2 & -f_1 f_3 - f_1 \sqrt{\quad} \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \Rightarrow v_+ = \begin{bmatrix} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{bmatrix}$$

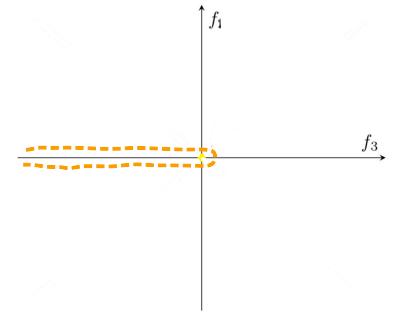
## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0}$$

$$\begin{bmatrix} (f_3 - \sqrt{\phantom{f_3^2}})(-f_3 - \sqrt{\phantom{f_3^2}}) & f_1(-f_3 - \sqrt{\phantom{f_3^2}}) \\ f_1 & -f_3 - \sqrt{\phantom{f_3^2}} \end{bmatrix} \rightarrow \begin{bmatrix} f_1^2 & -f_1 f_3 - f_1 \sqrt{\phantom{f_3^2}} \\ f_1 & -f_3 - \sqrt{\phantom{f_3^2}} \end{bmatrix}$$

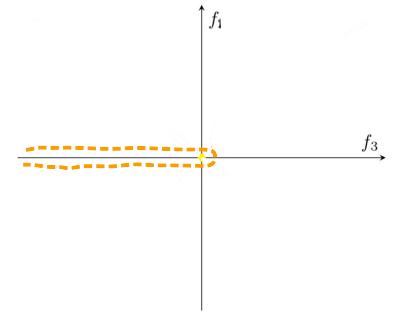
$$\rightarrow \begin{bmatrix} 0 & 0 \\ f_1 & -f_3 - \sqrt{\phantom{f_3^2}} \end{bmatrix} \Rightarrow v_+ = \begin{bmatrix} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{bmatrix} \underbrace{\quad}_{\begin{cases} f_3 = \cos \theta \\ f_1 = \sin \theta \end{cases} \quad -\pi < \theta < \pi}$$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0}$$



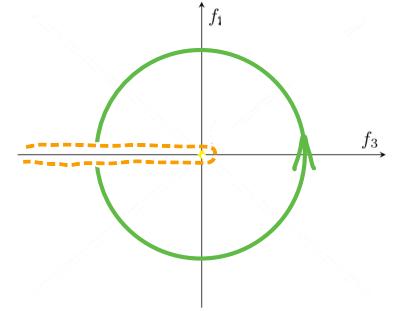
$$\begin{bmatrix} (f_3 - \sqrt{\quad})(-f_3 - \sqrt{\quad}) & f_1(-f_3 - \sqrt{\quad}) \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \rightarrow \begin{bmatrix} f_1^2 & -f_1 f_3 - f_1 \sqrt{\quad} \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \Rightarrow v_+ = \begin{bmatrix} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{bmatrix} \xrightarrow{\begin{cases} f_3 = \cos \theta \\ f_1 = \sin \theta \end{cases} \quad -\pi < \theta < \pi} \begin{bmatrix} \cos \theta + 1 \\ \sin \theta \end{bmatrix}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0}$$



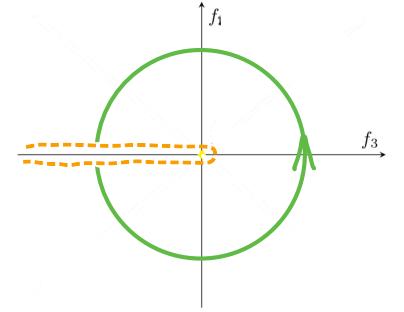
$$\begin{bmatrix} (f_3 - \sqrt{\quad})(-f_3 - \sqrt{\quad}) & f_1(-f_3 - \sqrt{\quad}) \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \rightarrow \begin{bmatrix} f_1^2 & -f_1 f_3 - f_1 \sqrt{\quad} \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \Rightarrow v_+ = \begin{bmatrix} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{bmatrix} \xrightarrow{\begin{cases} f_3 = \cos \theta \\ f_1 = \sin \theta \end{cases} \quad -\pi < \theta < \pi} \begin{bmatrix} \cos \theta + 1 \\ \sin \theta \end{bmatrix}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{bmatrix} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{bmatrix} \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0}$$



$$\begin{bmatrix} (f_3 - \sqrt{\quad})(-f_3 - \sqrt{\quad}) & f_1(-f_3 - \sqrt{\quad}) \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \rightarrow \begin{bmatrix} f_1^2 & -f_1 f_3 - f_1 \sqrt{\quad} \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix}$$

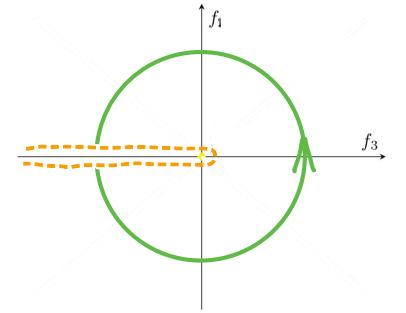
$$\rightarrow \begin{bmatrix} 0 & 0 \\ f_1 & -f_3 - \sqrt{\quad} \end{bmatrix} \Rightarrow v_+ = \begin{bmatrix} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{bmatrix} \xrightarrow{\begin{cases} f_3 = \cos \theta \\ f_1 = \sin \theta \end{cases} \quad -\pi < \theta < \pi} \begin{bmatrix} \cos \theta + 1 \\ \sin \theta \end{bmatrix}$$

Observe that when  $\theta \rightarrow (-\pi)_+$ , we have  $\cos \theta + 1 \rightarrow 0_+$  and  $\sin \theta \rightarrow 0_-$ ,

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{aligned}
 & \left[ \begin{array}{cc} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{array} \right] \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0} \\
 & \left[ \begin{array}{cc} \left(f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) \left(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) & f_1 \left(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \rightarrow \left[ \begin{array}{cc} f_1^2 & -f_1 f_3 - f_1 \sqrt{\phantom{f_1^2 + f_3^2}} \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \\
 & \rightarrow \left[ \begin{array}{cc} 0 & 0 \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \Rightarrow v_+ = \left[ \begin{array}{c} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{array} \right] \xrightarrow{\begin{cases} f_3 = \cos \theta \\ f_1 = \sin \theta \end{cases} \quad -\pi < \theta < \pi} \left[ \begin{array}{c} \cos \theta + 1 \\ \sin \theta \end{array} \right]
 \end{aligned}$$

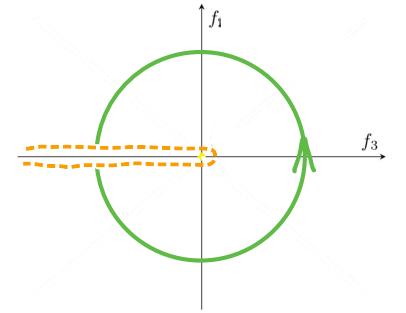


Observe that when  $\theta \rightarrow (-\pi)_+$ , we have  $\cos \theta + 1 \rightarrow 0_+$  and  $\sin \theta \rightarrow 0_-$ , whereas when  $\theta \rightarrow \pi_-$ , we have  $\cos \theta + 1 \rightarrow 0_+$  and  $\sin \theta \rightarrow 0_+$ .

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{aligned}
 & \left[ \begin{array}{cc} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{array} \right] \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0} \\
 & \left[ \begin{array}{cc} \left(f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) \left(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) & f_1 \left(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \rightarrow \left[ \begin{array}{cc} f_1^2 & -f_1 f_3 - f_1 \sqrt{\phantom{f_1^2 + f_3^2}} \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \\
 & \rightarrow \left[ \begin{array}{cc} 0 & 0 \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \Rightarrow v_+ = \left[ \begin{array}{c} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{array} \right] \xrightarrow{\begin{cases} f_3 = \cos \theta \\ f_1 = \sin \theta \end{cases} \atop -\pi < \theta < \pi} \left[ \begin{array}{c} \cos \theta + 1 \\ \sin \theta \end{array} \right]
 \end{aligned}$$



Observe that when  $\theta \rightarrow (-\pi)_+$ , we have  $\cos \theta + 1 \rightarrow 0_+$  and  $\sin \theta \rightarrow 0_-$ , whereas when  $\theta \rightarrow \pi_-$ , we have  $\cos \theta + 1 \rightarrow 0_+$  and  $\sin \theta \rightarrow 0_+$ .

We compute that

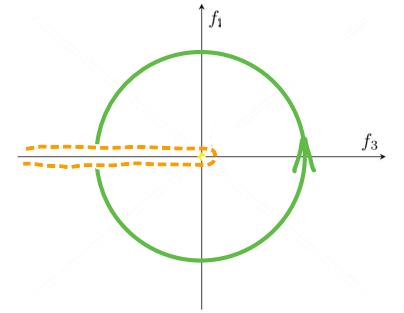
$$\lim_{\theta \rightarrow (-\pi)_+} \frac{v_+}{|v_+|} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\lim_{\theta \rightarrow \pi_-} \frac{v_+}{|v_+|} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

$$\begin{aligned}
 & \left[ \begin{array}{cc} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{array} \right] \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0} \\
 & \left[ \begin{array}{cc} \left(f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) \left(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) & f_1 \left(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}\right) \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \rightarrow \left[ \begin{array}{cc} f_1^2 & -f_1 f_3 - f_1 \sqrt{\phantom{f_1^2 + f_3^2}} \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \\
 & \rightarrow \left[ \begin{array}{cc} 0 & 0 \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \Rightarrow v_+ = \left[ \begin{array}{c} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{array} \right] \xrightarrow{\begin{cases} f_3 = \cos \theta \\ f_1 = \sin \theta \end{cases} \quad -\pi < \theta < \pi} \left[ \begin{array}{c} \cos \theta + 1 \\ \sin \theta \end{array} \right]
 \end{aligned}$$



Observe that when  $\theta \rightarrow (-\pi)_+$ , we have  $\cos \theta + 1 \rightarrow 0_+$  and  $\sin \theta \rightarrow 0_-$ , whereas when  $\theta \rightarrow \pi_-$ , we have  $\cos \theta + 1 \rightarrow 0_+$  and  $\sin \theta \rightarrow 0_+$ .

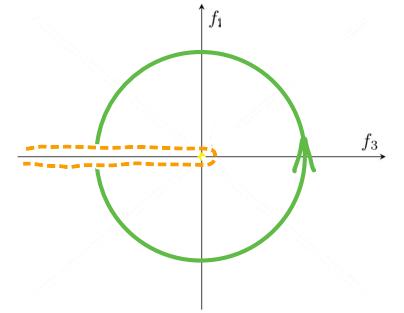
We compute that

$$\left. \begin{array}{l} \lim_{\theta \rightarrow (-\pi)_+} \frac{v_+}{|v_+|} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \lim_{\theta \rightarrow \pi_-} \frac{v_+}{|v_+|} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array} \right\} \text{Half Möbius band!}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

To solve for eigenvectors  $v_+$  corresponding to  $\omega_+$ , perform Gaussian elimination through elementary row operations:

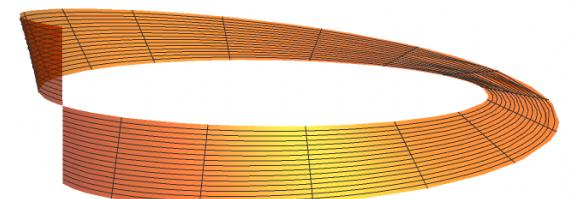
$$\begin{aligned}
 & \left[ \begin{array}{cc} f_3 - \sqrt{f_1^2 + f_3^2} & f_1 \\ f_1 & -f_3 - \sqrt{f_1^2 + f_3^2} \end{array} \right] \xrightarrow{-f_3 - \sqrt{f_1^2 + f_3^2} \neq 0 \iff f_1 = 0, f_3 \leq 0} \\
 & \left[ \begin{array}{cc} (f_3 - \sqrt{\phantom{f_1^2 + f_3^2}})(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}) & f_1(-f_3 - \sqrt{\phantom{f_1^2 + f_3^2}}) \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \rightarrow \left[ \begin{array}{cc} f_1^2 & -f_1 f_3 - f_1 \sqrt{\phantom{f_1^2 + f_3^2}} \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \\
 & \rightarrow \left[ \begin{array}{cc} 0 & 0 \\ f_1 & -f_3 - \sqrt{\phantom{f_1^2 + f_3^2}} \end{array} \right] \Rightarrow v_+ = \left[ \begin{array}{c} f_3 + \sqrt{f_1^2 + f_3^2} \\ f_1 \end{array} \right] \xrightarrow{\begin{cases} f_3 = \cos \theta \\ f_1 = \sin \theta \end{cases} \quad -\pi < \theta < \pi} \left[ \begin{array}{c} \cos \theta + 1 \\ \sin \theta \end{array} \right]
 \end{aligned}$$



Observe that when  $\theta \rightarrow (-\pi)_+$ , we have  $\cos \theta + 1 \rightarrow 0_+$  and  $\sin \theta \rightarrow 0_-$ , whereas when  $\theta \rightarrow \pi_-$ , we have  $\cos \theta + 1 \rightarrow 0_+$  and  $\sin \theta \rightarrow 0_+$ .

We compute that

$$\left. \begin{aligned} \lim_{\theta \rightarrow (-\pi)_+} \frac{v_+}{|v_+|} &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \lim_{\theta \rightarrow \pi_-} \frac{v_+}{|v_+|} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \right\} \text{Half Möbius band!}$$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{aligned} S^0 &\hookrightarrow S^1 \rightarrow S^1 & \mathbb{R} \\ S^1 &\hookrightarrow S^3 \rightarrow S^2 & \text{if the Hamiltonian is over } \mathbb{C} \\ S^3 &\hookrightarrow S^7 \rightarrow S^4 & \mathbb{H} \end{aligned}$$

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

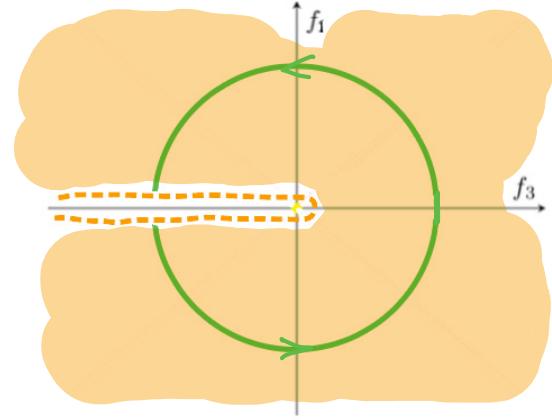
**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$S^0 \hookrightarrow S^1 \rightarrow S^1$$

$$S^1 \hookrightarrow S^3 \rightarrow S^2 \quad \text{if the Hamiltonian is over } \mathbb{R}$$

$$S^3 \hookrightarrow S^7 \rightarrow S^4 \quad \mathbb{C}$$

$$S^7 \hookrightarrow S^{15} \rightarrow S^8 \quad \mathbb{H}$$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

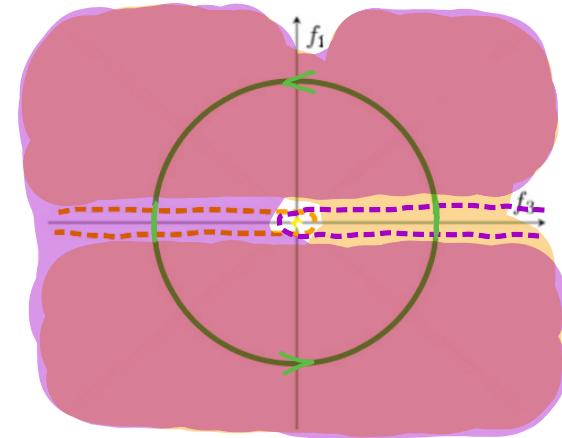
**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$S^0 \hookrightarrow S^1 \rightarrow S^1$$

$$S^1 \hookrightarrow S^3 \rightarrow S^2 \quad \text{if the Hamiltonian is over } \mathbb{R}$$

$$S^3 \hookrightarrow S^7 \rightarrow S^4 \quad \mathbb{C}$$

$$\quad \quad \quad \mathbb{H}$$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

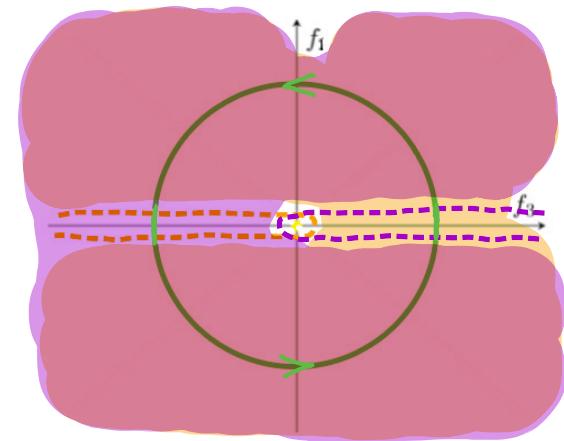
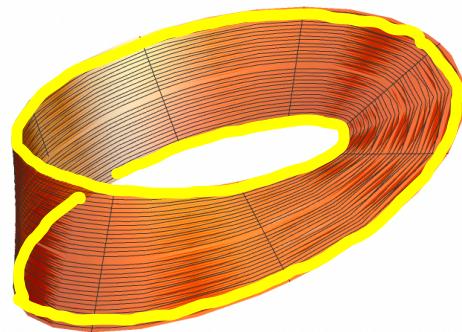
**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$S^0 \hookrightarrow S^1 \rightarrow S^1$$

$$S^1 \hookrightarrow S^3 \rightarrow S^2 \quad \text{if the Hamiltonian is over } \mathbb{R}$$

$$S^3 \hookrightarrow S^7 \rightarrow S^4 \quad \text{if the Hamiltonian is over } \mathbb{C}$$

$$S^7 \hookrightarrow S^{15} \rightarrow S^8 \quad \text{if the Hamiltonian is over } \mathbb{H}$$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

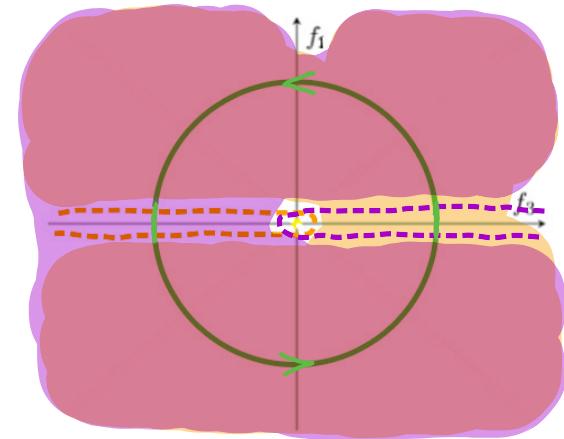
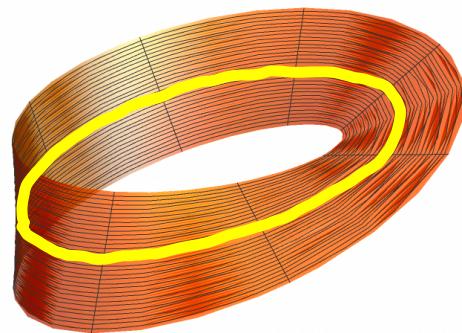
**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$S^0 \hookrightarrow S^1 \rightarrow \textcolor{red}{S}^1$$

$$S^1 \hookrightarrow S^3 \rightarrow S^2 \quad \text{if the Hamiltonian is over } \mathbb{R}$$

$$S^3 \hookrightarrow S^7 \rightarrow S^4 \quad \mathbb{C}$$

$$S^7 \hookrightarrow S^{15} \rightarrow S^8 \quad \mathbb{H}$$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

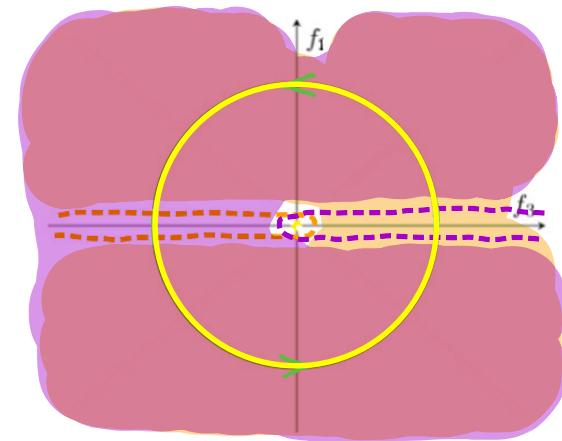
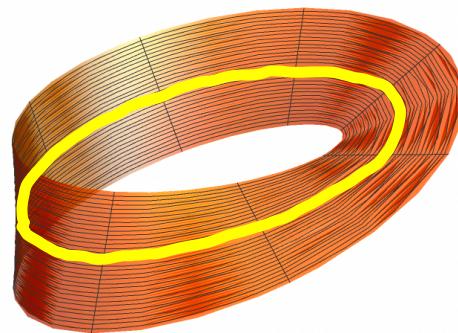
**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$S^0 \hookrightarrow S^1 \rightarrow \textcolor{red}{S}^1$$

$$S^1 \hookrightarrow S^3 \rightarrow S^2 \quad \text{if the Hamiltonian is over } \mathbb{R}$$

$$S^3 \hookrightarrow S^7 \rightarrow S^4 \quad \mathbb{C}$$

$$\mathbb{H}$$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

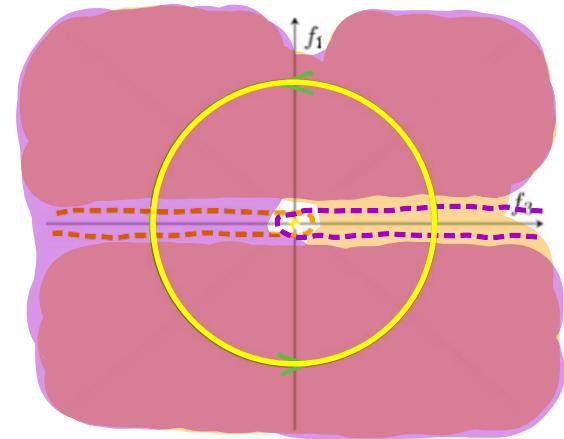
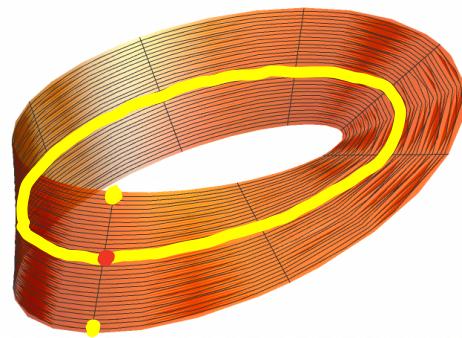
**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$S^0 \hookrightarrow S^1 \rightarrow S^1$$

$$S^1 \hookrightarrow S^3 \rightarrow S^2 \quad \text{if the Hamiltonian is over } \mathbb{R}$$

$$S^3 \hookrightarrow S^7 \rightarrow S^4 \quad \mathbb{C}$$

$$S^7 \hookrightarrow S^{15} \rightarrow S^8 \quad \mathbb{H}$$



## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{aligned} S^0 &\hookrightarrow S^1 \rightarrow S^1 & \mathbb{R} \\ S^1 &\hookrightarrow S^3 \rightarrow S^2 & \text{if the Hamiltonian is over } \mathbb{C} \\ S^3 &\hookrightarrow S^7 \rightarrow S^4 & \mathbb{H} \end{aligned}$$

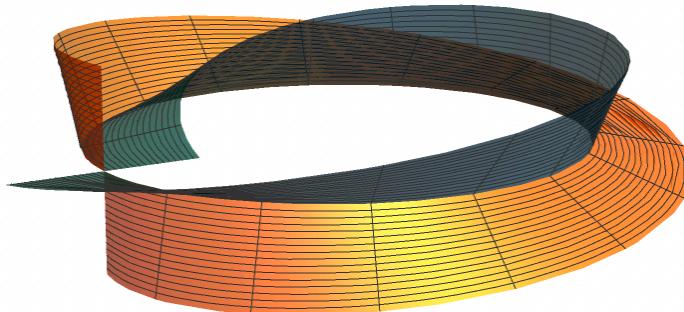
**Corollary.** The universal eigenbundle for real Hermitian 2-band systems is given by a pair of orthogonally intersecting half Möbius bands

## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{aligned} S^0 \hookrightarrow S^1 &\rightarrow S^1 & \mathbb{R} \\ S^1 \hookrightarrow S^3 &\rightarrow S^2 \quad \text{if the Hamiltonian is over} & \mathbb{C} \\ S^3 \hookrightarrow S^7 &\rightarrow S^4 & \mathbb{H} \end{aligned}$$

**Corollary.** The universal eigenbundle for real Hermitian 2-band systems is given by a pair of orthogonally intersecting half Möbius bands

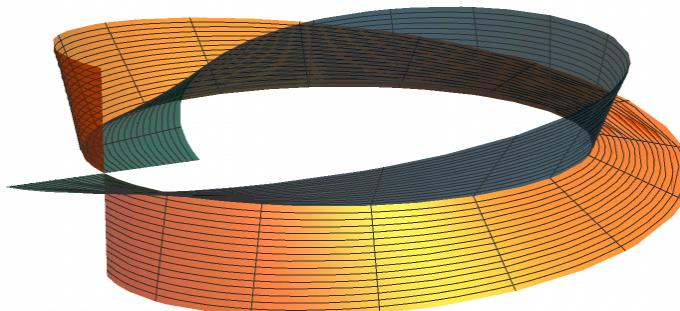


## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{aligned} S^0 &\hookrightarrow S^1 \rightarrow \textcolor{red}{S^1} & \mathbb{R} \\ S^1 &\hookrightarrow S^3 \rightarrow S^2 & \text{if the Hamiltonian is over } \mathbb{C} \\ S^3 &\hookrightarrow S^7 \rightarrow S^4 & \mathbb{H} \end{aligned}$$

**Corollary.** The universal eigenbundle for real Hermitian 2-band systems is given by a pair of orthogonally intersecting half Möbius bands over the **unit circle** in the punctured parameter plane.

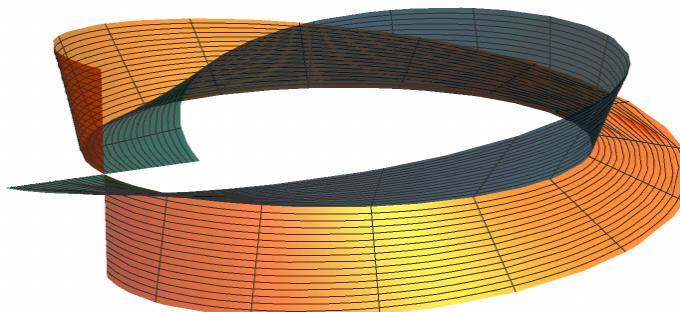


## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{aligned} S^0 &\hookrightarrow S^1 \rightarrow \textcolor{red}{S^1} & \mathbb{R} \\ S^1 &\hookrightarrow S^3 \rightarrow S^2 \quad \text{if the Hamiltonian is over} & \mathbb{C} \\ S^3 &\hookrightarrow S^7 \rightarrow S^4 & \mathbb{H} \end{aligned}$$

**Corollary.** The universal eigenbundle for real Hermitian 2-band systems is given by a pair of orthogonally intersecting half Möbius bands over the **unit circle** in the punctured parameter plane. In particular, eigenframe rotations along a generic loop in the moduli space are classified by  $\pi_1(S^1) \cong \mathbb{Z}$ .

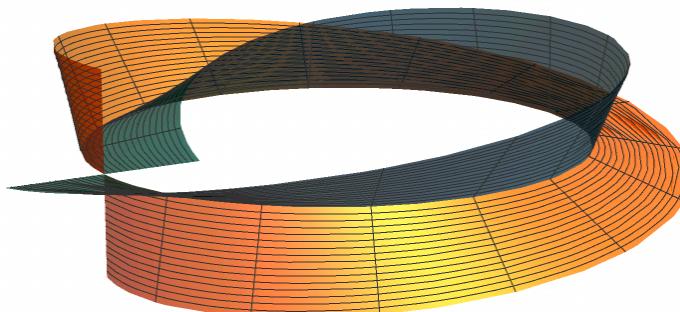


## Eigenframe rotation as vector bundles: Revisiting the Hermitian case

**Lemma.** The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{aligned} S^0 \hookrightarrow S^1 &\rightarrow \textcolor{red}{S^1} & \mathbb{R} \\ S^1 \hookrightarrow S^3 &\rightarrow S^2 \quad \text{if the Hamiltonian is over} & \mathbb{C} \\ S^3 \hookrightarrow S^7 &\rightarrow S^4 & \mathbb{H} \end{aligned}$$

**Corollary.** The universal eigenbundle for real Hermitian 2-band systems is given by a pair of orthogonally intersecting half Möbius bands over the unit circle in the **punctured** parameter plane. In particular, eigenframe rotations along a **generic** loop in the moduli space are classified by  $\pi_1(S^1) \cong \mathbb{Z}$ .



# Eigenframe evolution as Higgs bundles: The non-Hermitian case



## Eigenframe evolution as Higgs bundles: The non-Hermitian case

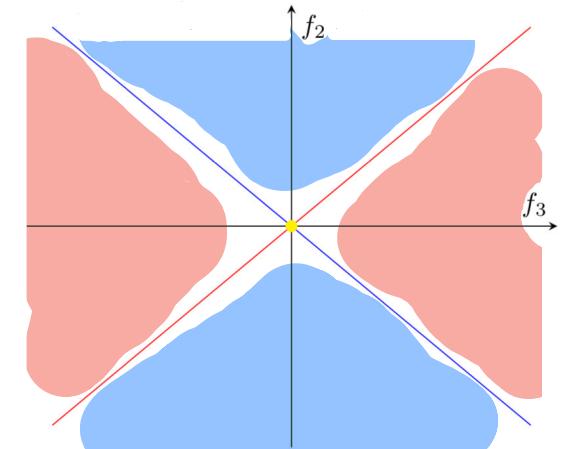
Recall that non-Hermitian 2-band systems

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

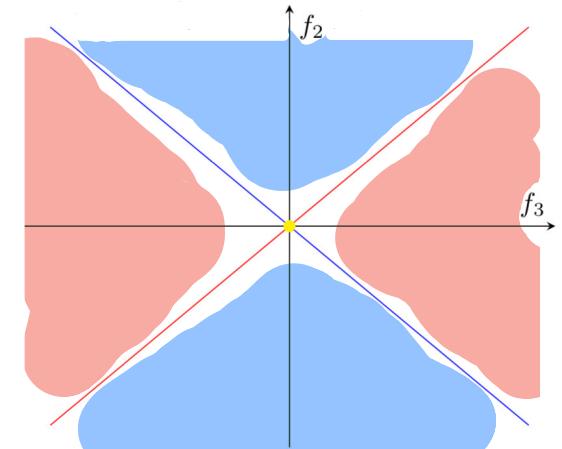


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.

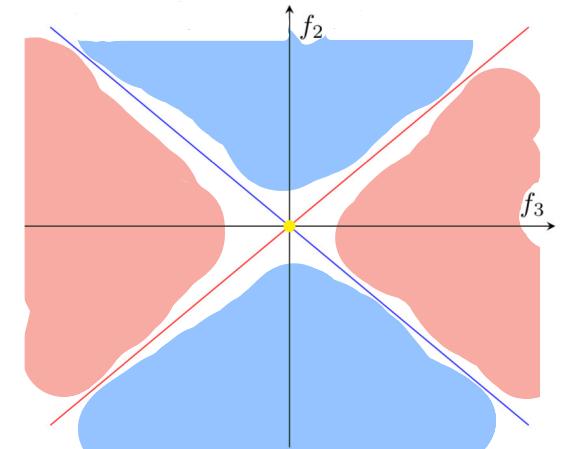


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.

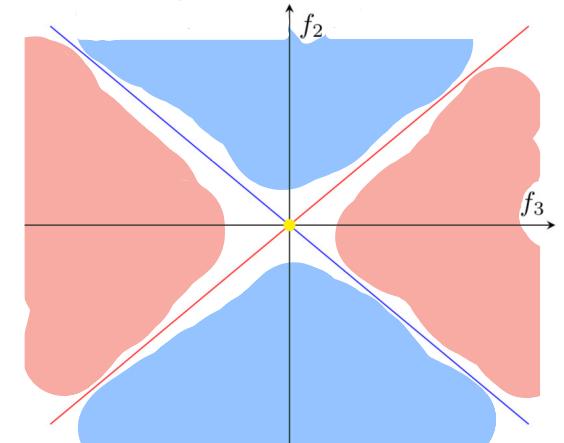


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.

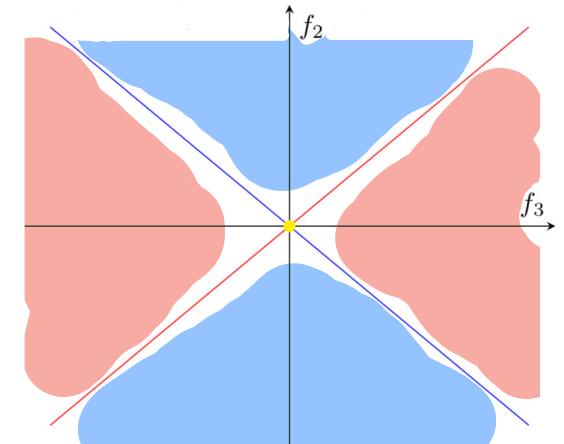


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

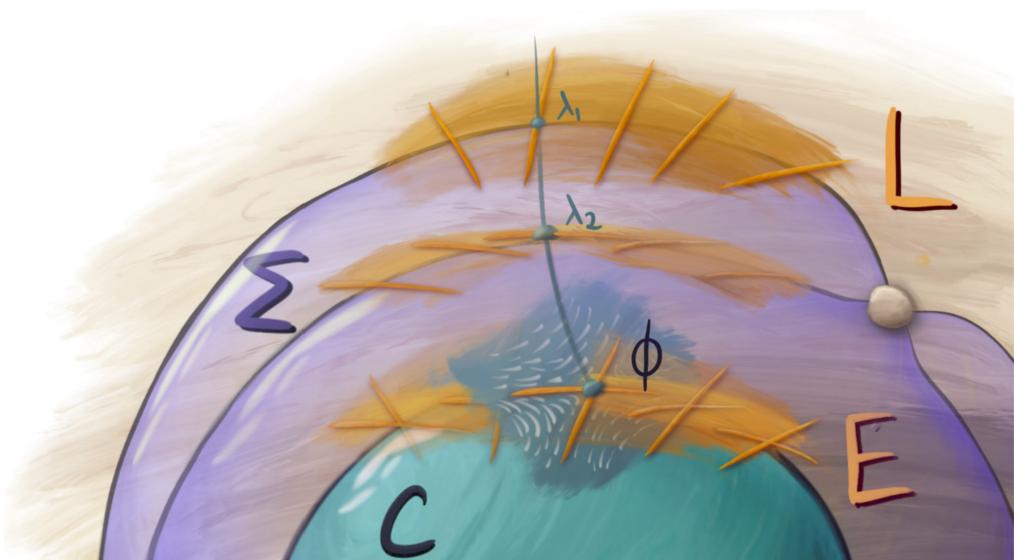
Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.



A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

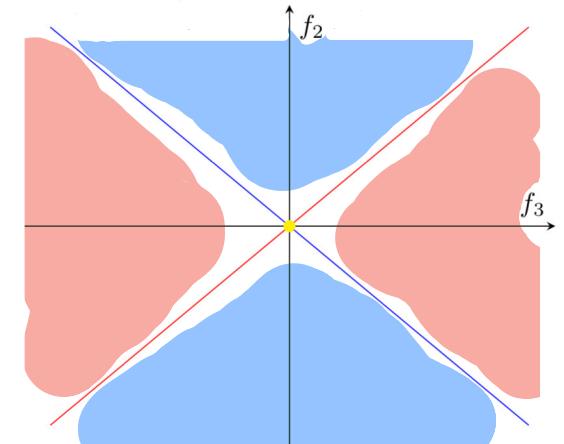


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

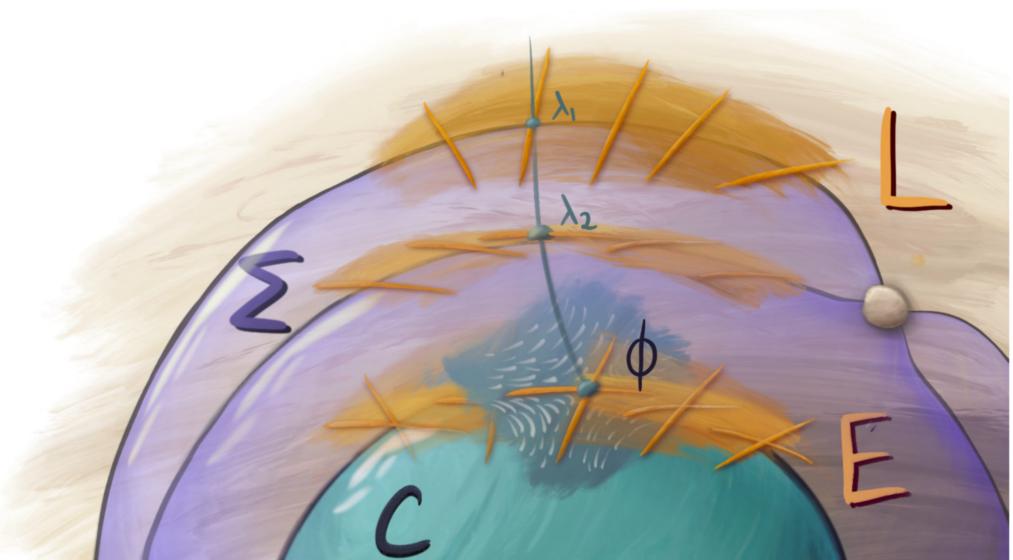
$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.



A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

Peter Higgs (bosons)

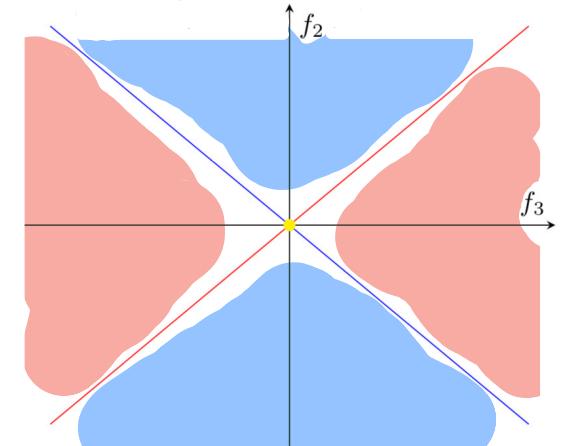


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.

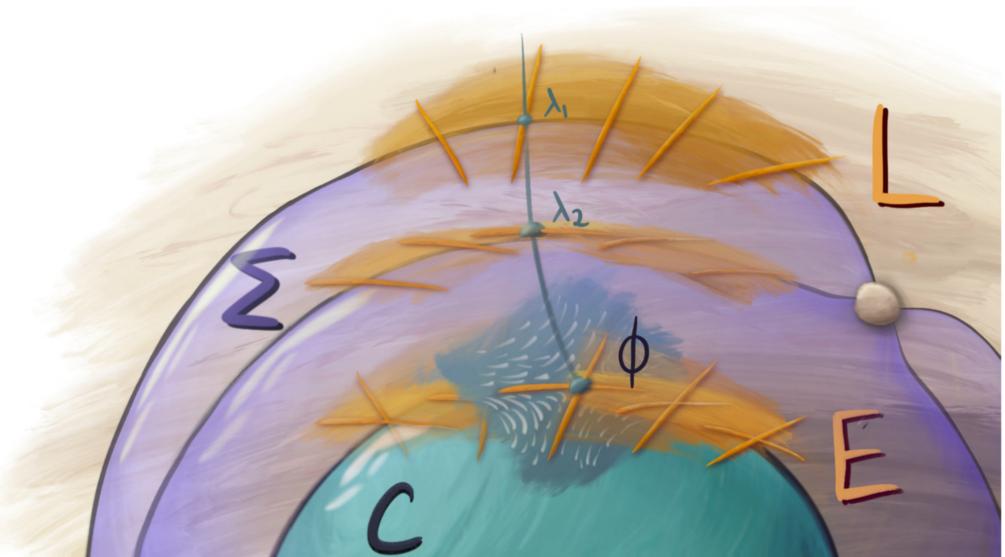


A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

Peter Higgs (bosons)



1929–2024

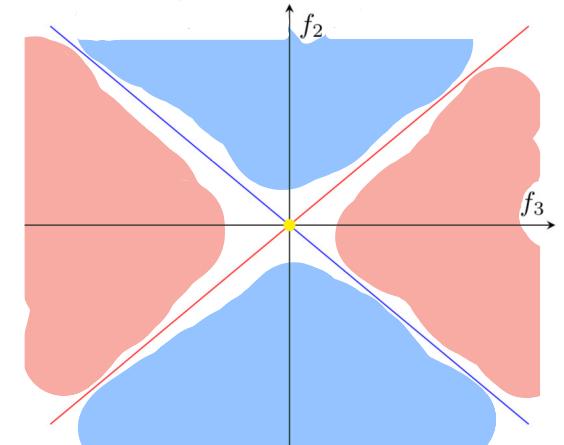


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

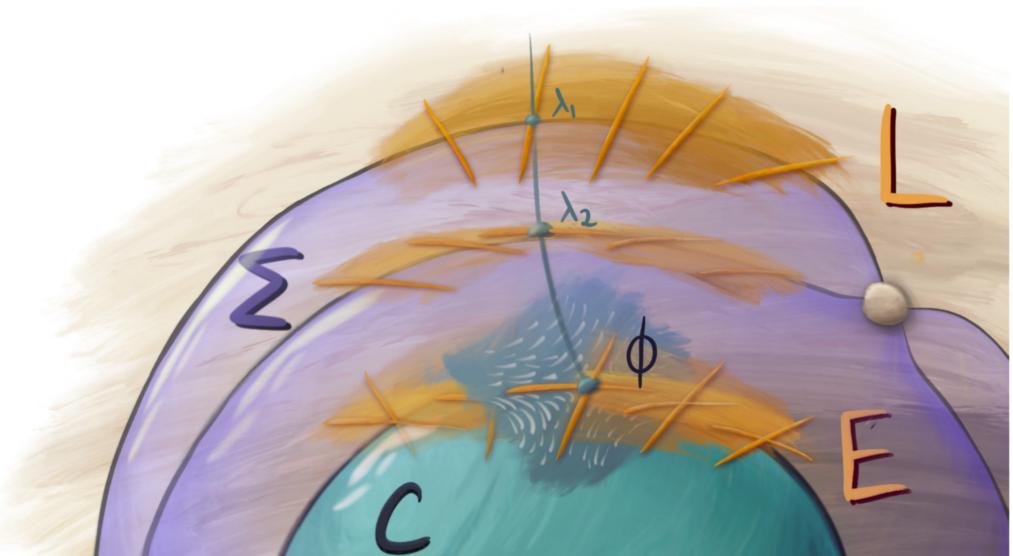
0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.



A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

Peter Higgs (bosons)

Nigel Hitchin 1987

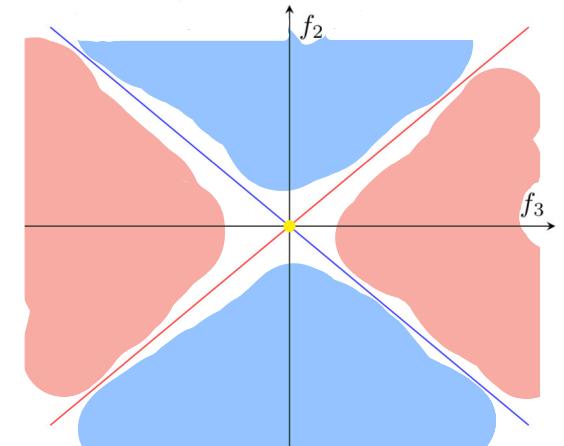


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.

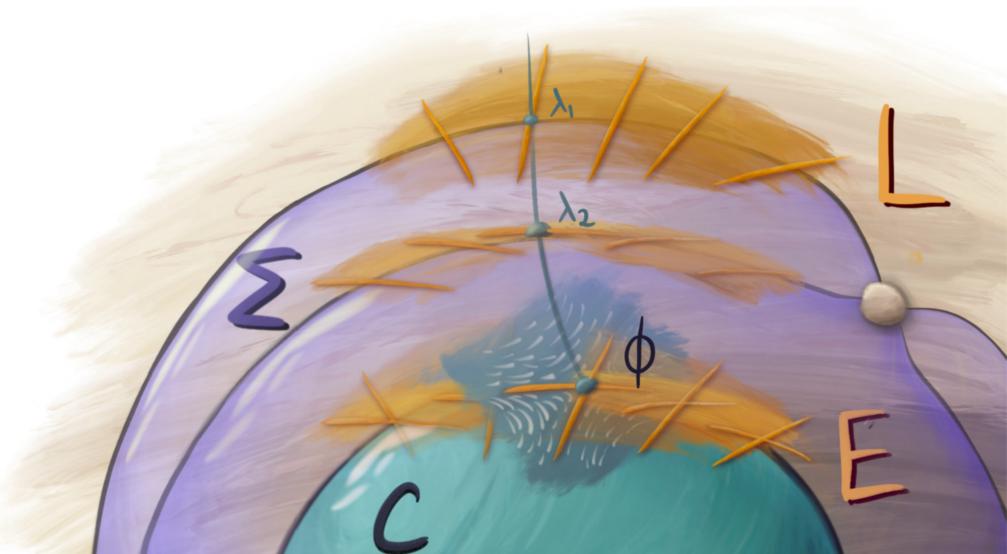


A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface

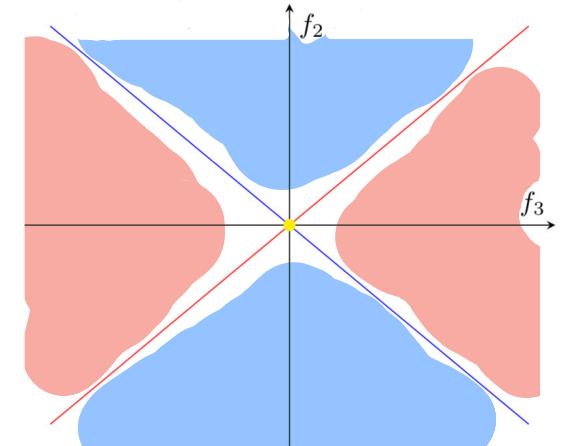


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.

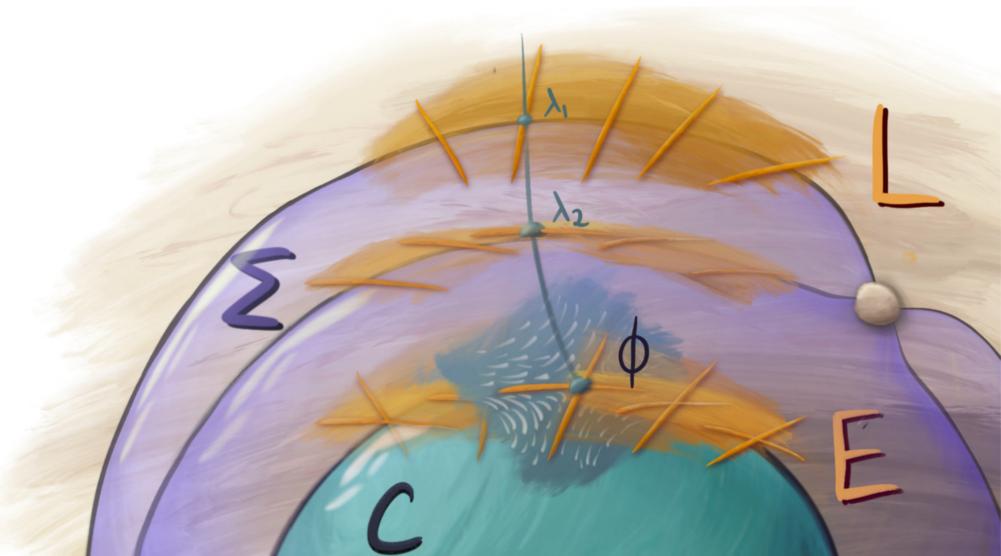


A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

Peter Higgs (bosons)

Nigel Hitchin 1987

C compact Riemann surface  
E holomorphic vector bundle

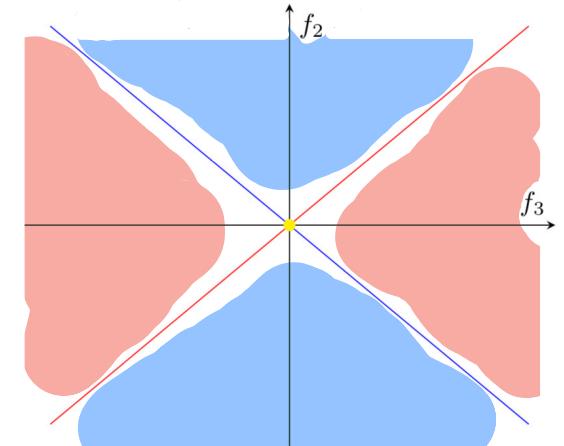


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.



A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

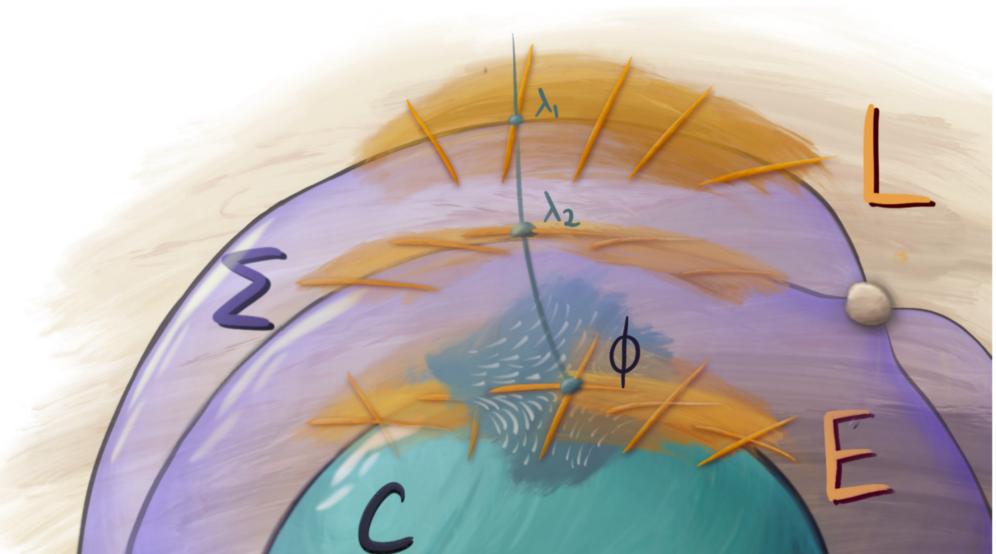
Peter Higgs (bosons)

Nigel Hitchin 1987

$C$  compact Riemann surface

$E$  holomorphic vector bundle

$\phi$  Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of  $E$

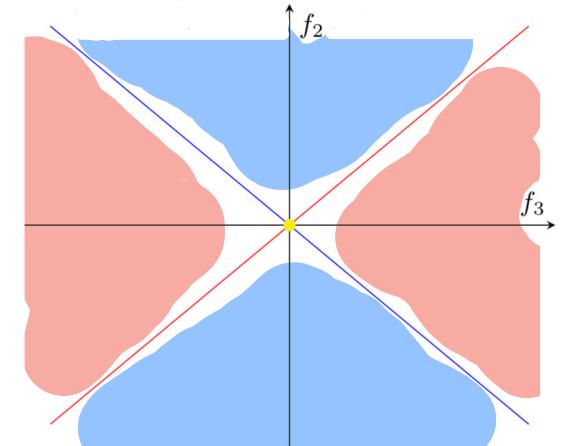


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.



A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of **matrices**

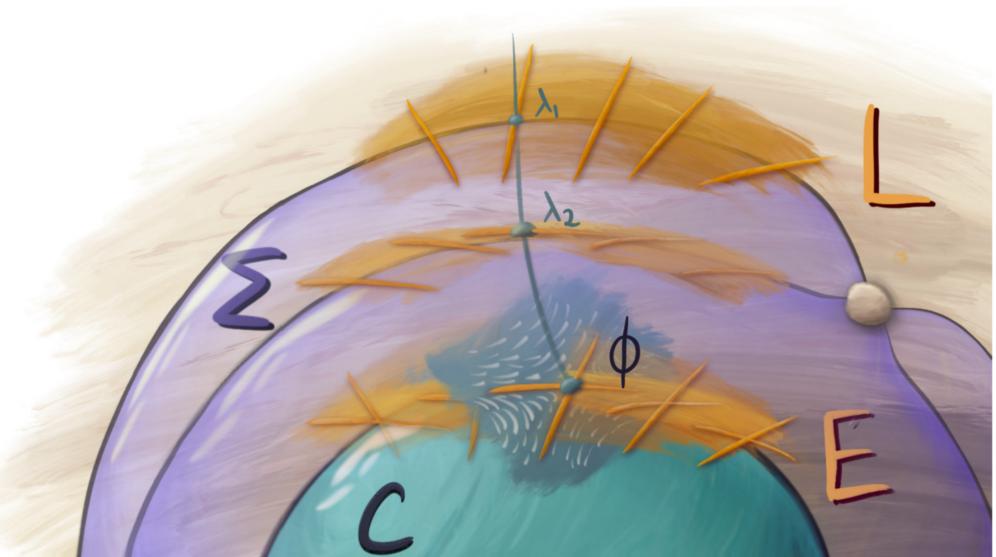
Peter Higgs (*bosons*)

Nigel Hitchin 1987

$C$  compact Riemann surface

$E$  holomorphic vector bundle

$\phi$  Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of  $E$

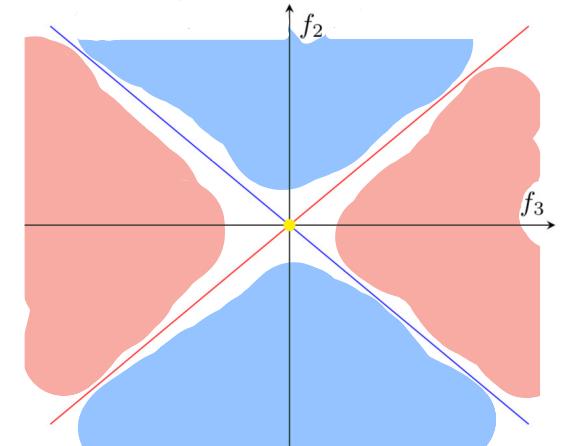


# Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.



A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices

Peter Higgs (bosons)

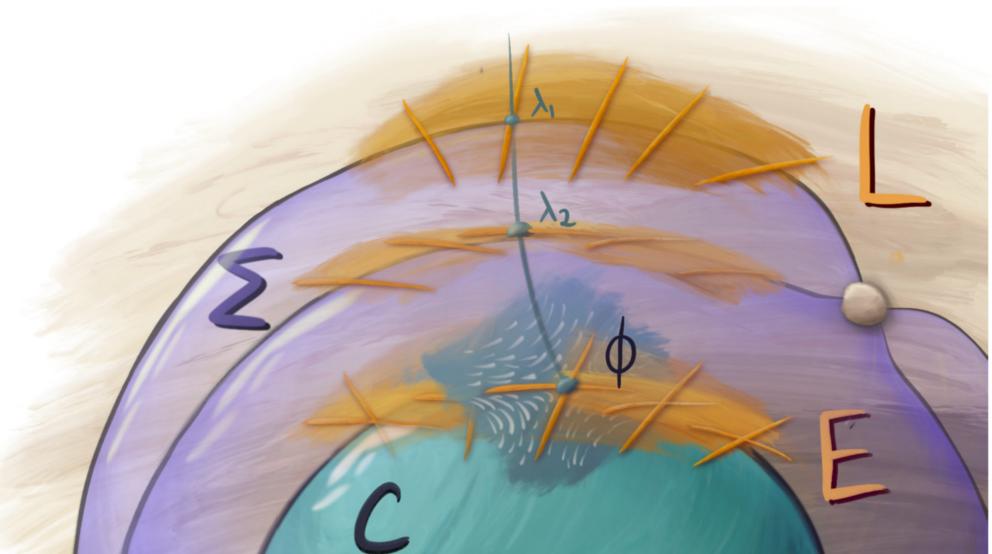
Nigel Hitchin 1987

Carlos Simpson

$C$  compact Riemann surface (or more generally Kähler manifold)

$E$  holomorphic vector bundle

$\phi$  Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of  $E$  such that  $\phi \wedge \phi = 0$

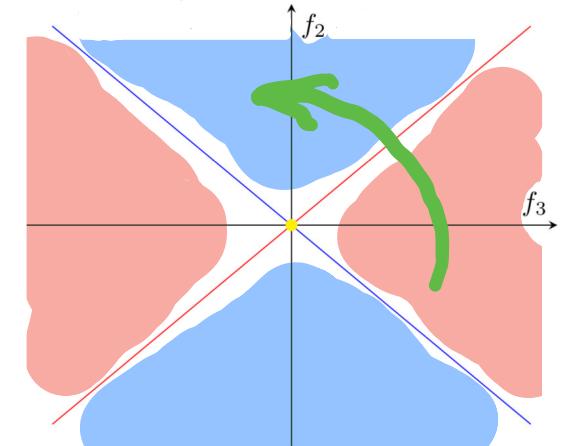


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

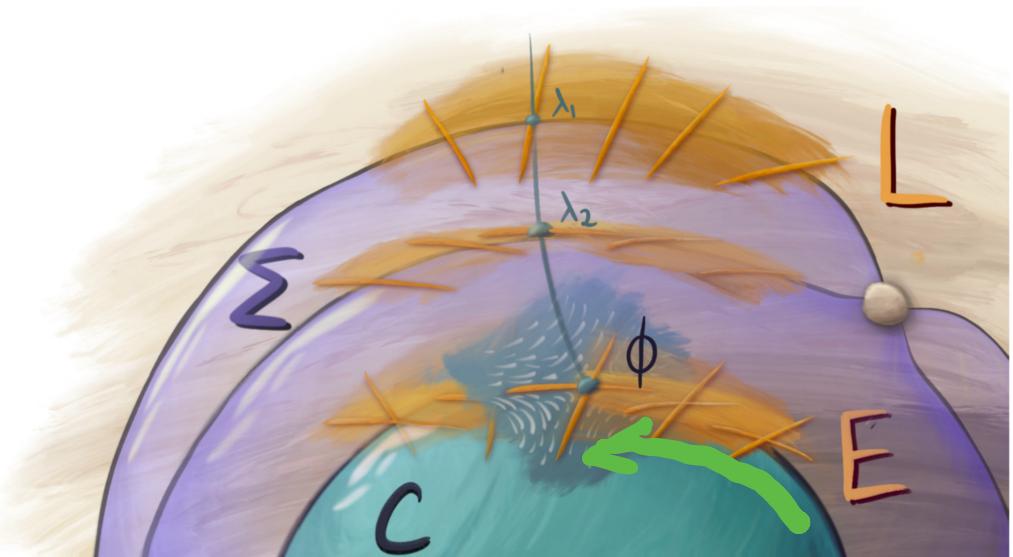
$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.



A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices, and if you try to diagonalize one you get a *spectral cover*.

$$\phi_x \in \text{End}(E_x), x \in C$$

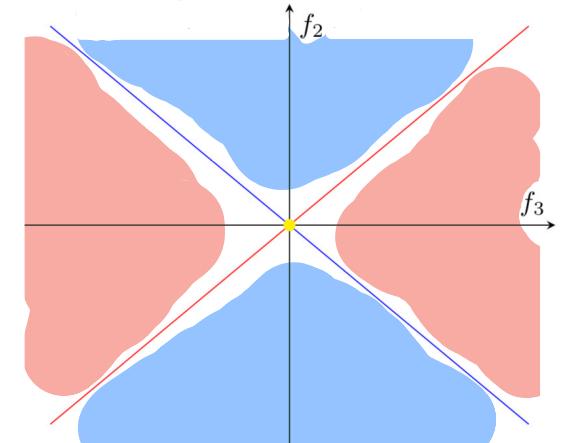


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

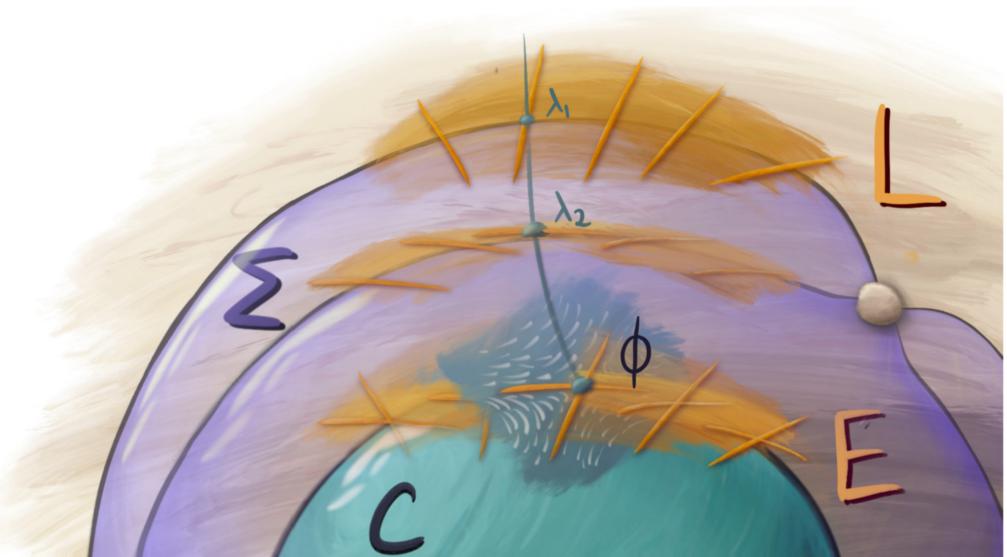
$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.



A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices, and if you try to diagonalize one you get a *spectral cover*.

Portrait from Kienzle and Rayan,  
*Hyperbolic band theory through Higgs bundles*, **Adv. Math.**, 2022.

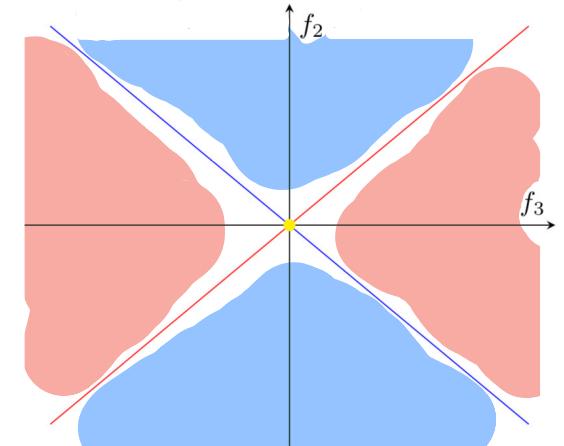


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

Recall that non-Hermitian 2-band systems have a stratified parameter plane:

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

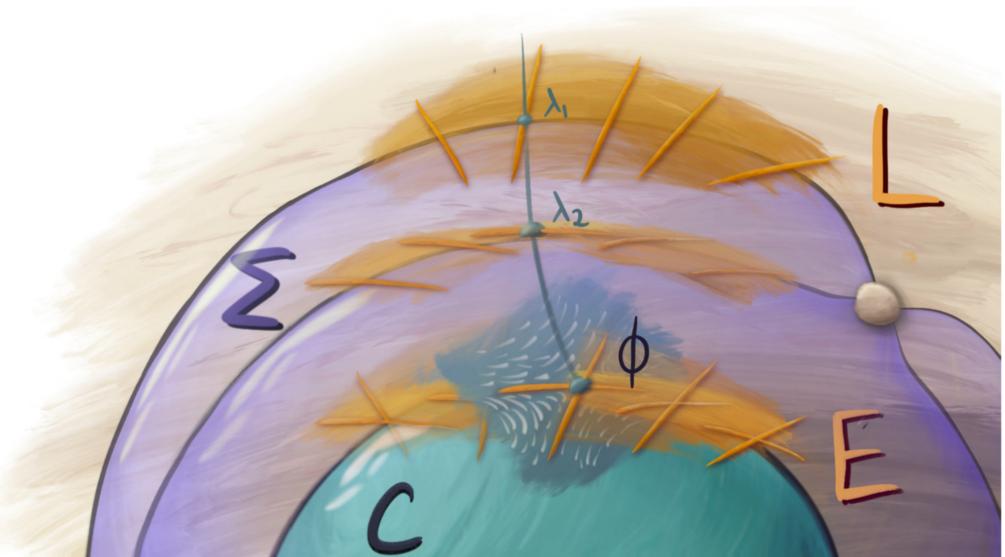
0. Over  $\{(0, 0)\}$ ,  $H$  has a **double** eigenvalue, whose eigenspace is 2-dimensional.
1. Over  $\{f_2 = \pm f_3\} - \{(0, 0)\}$ , again  $H$  has a double eigenvalue, but its eigenspace is of **dimension 1**.
2. Over  $\{f_2 \neq \pm f_3\}$ ,  $H$  has 2 distinct eigenvalues. When  $|f_2| < |f_3|$ , the eigenvectors are **real**. When  $|f_2| > |f_3|$ , the eigenvectors are **not real**.



A *Higgs bundle*  $(E, \phi) \rightarrow C$  is essentially a family of matrices, and if you try to diagonalize one you get a *spectral cover*.

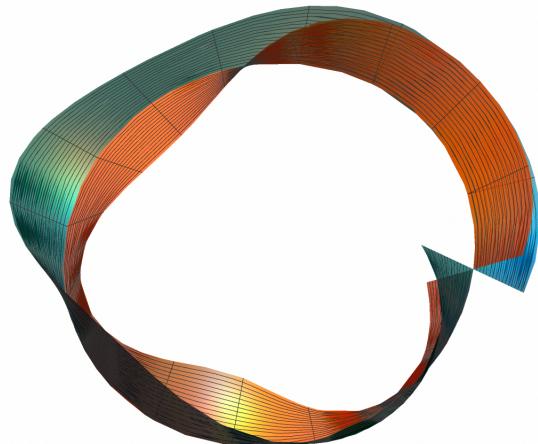
Portrait from Kienzle and Rayan,  
*Hyperbolic band theory through Higgs bundles*, *Adv. Math.*, 2022.

Hyperbolic metric on the base  $C$ . Kollár et al., *Nature*, 2019.



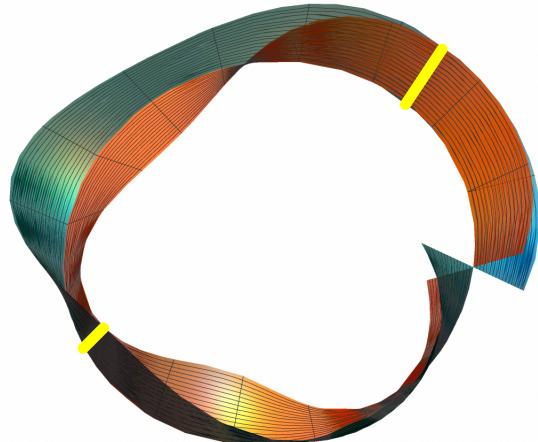
## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands



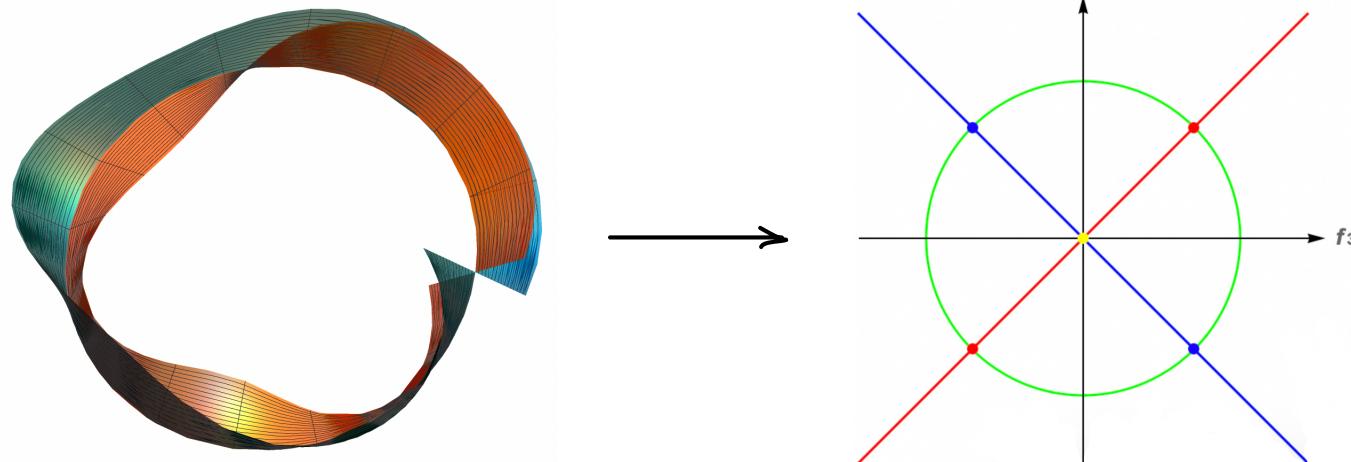
## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of **kissing** half Möbius bands



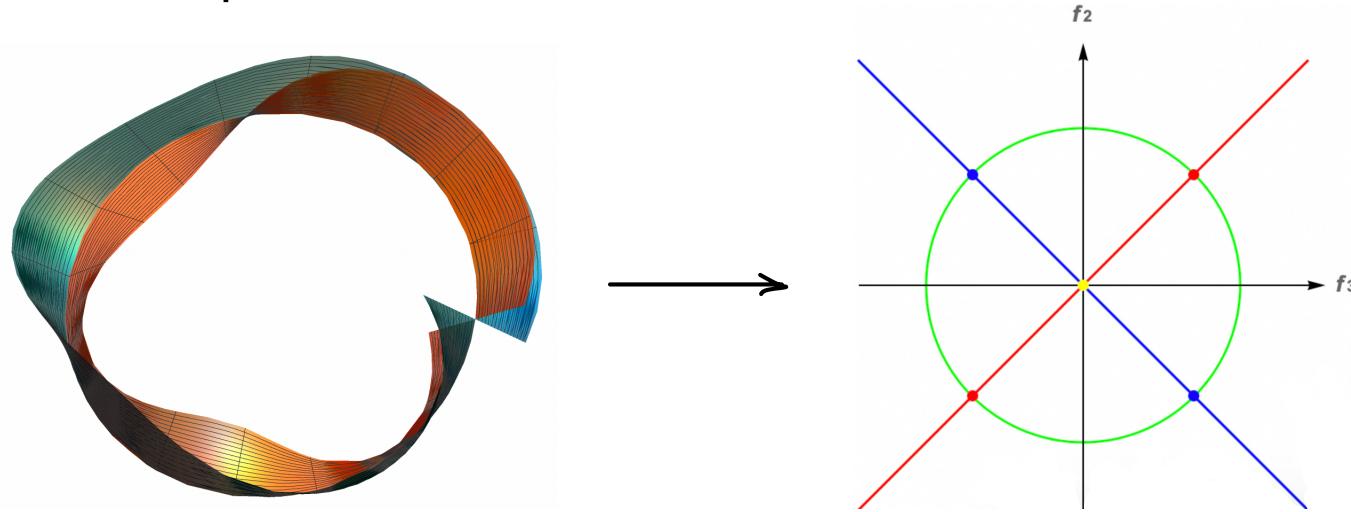
## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the **stratified** unit circle in the punctured parameter plane



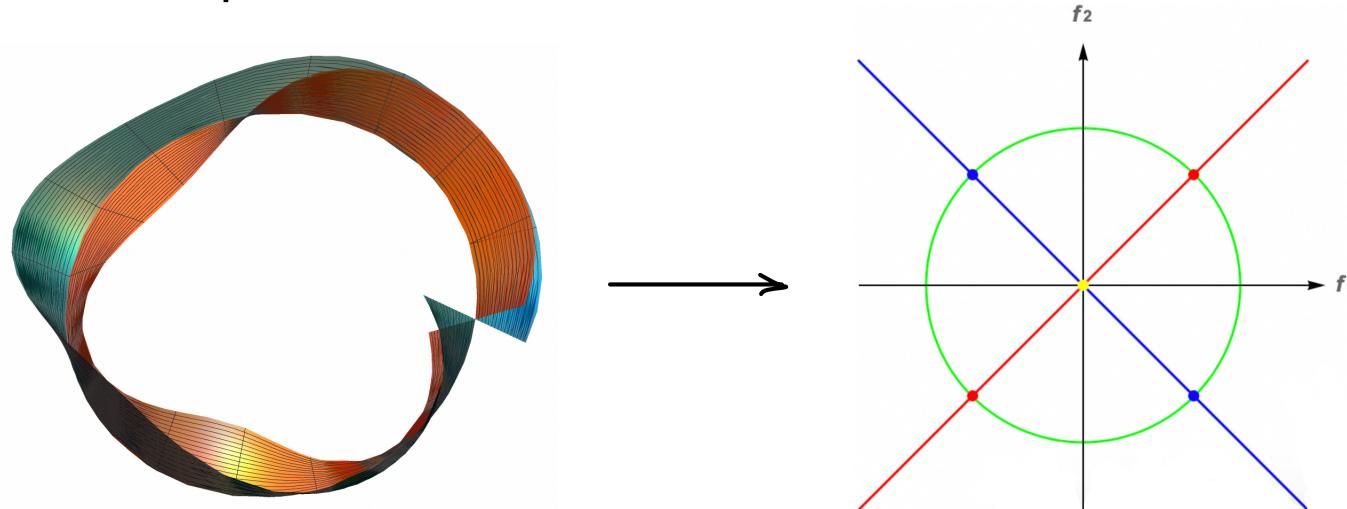
## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the **stratified** unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.



## Eigenframe evolution as Higgs bundles: The non-Hermitian case

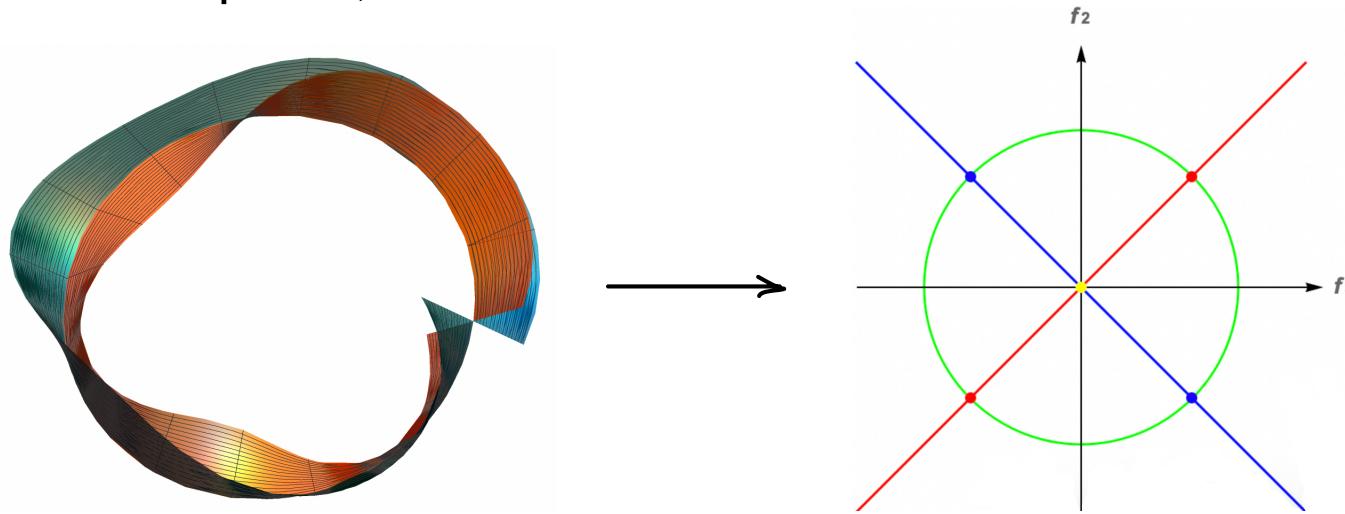
**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.



Here is a video showing the eigenframe evolution: <https://yifeizhu.github.io/swallowtail/rotate.mp4>

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.

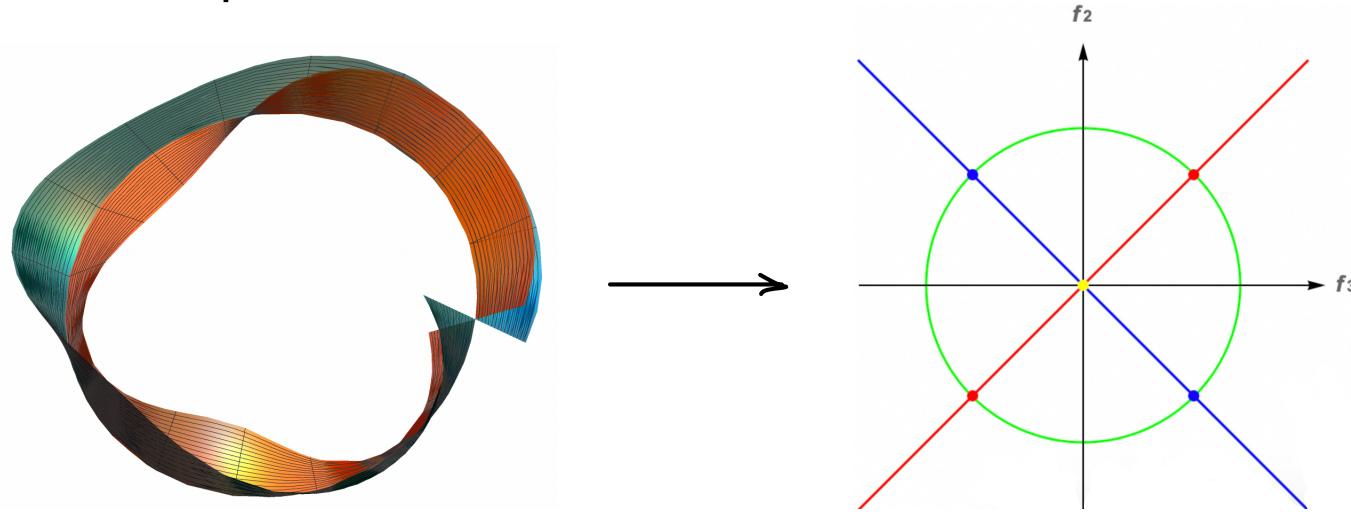


Here is a video showing the eigenframe evolution: <https://yifeizhu.github.io/swallowtail/rotate.mp4>

**Note.** In the non-Hermitian case, since the eigenvectors are in  $\mathbb{C}^2$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.

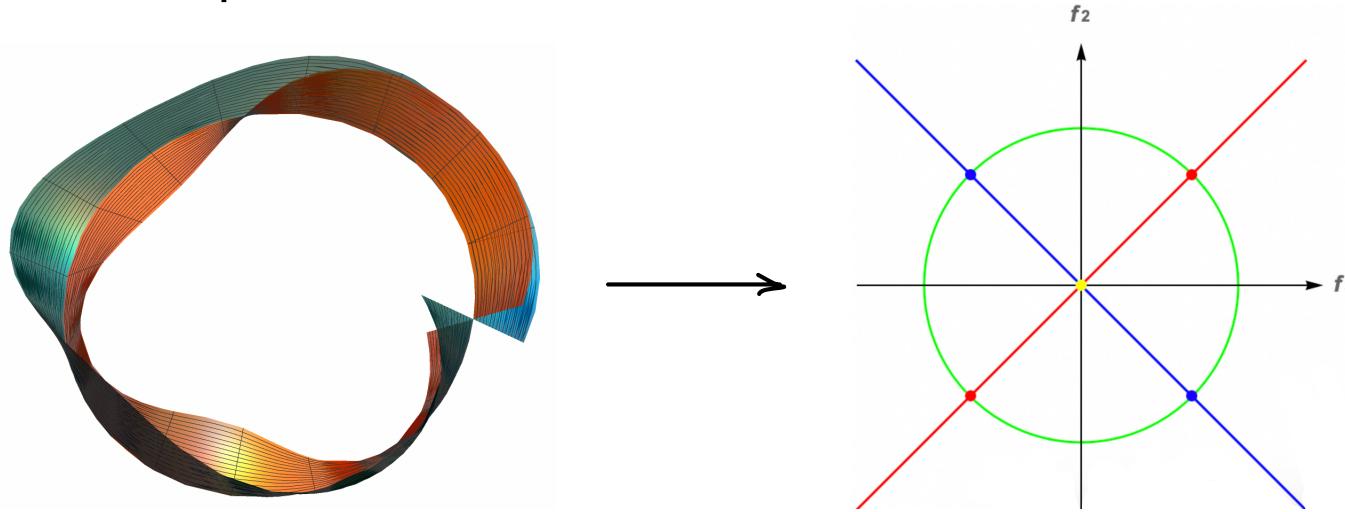


Here is a video showing the eigenframe evolution: <https://yifeizhu.github.io/swallowtail/rotate.mp4>

**Note.** In the non-Hermitian case, since the eigenvectors are in  $\mathbb{C}^2$ , we have adopted (a variant of) the *Hermitian angle* to properly characterize the eigenframe rotation and deformation:

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.



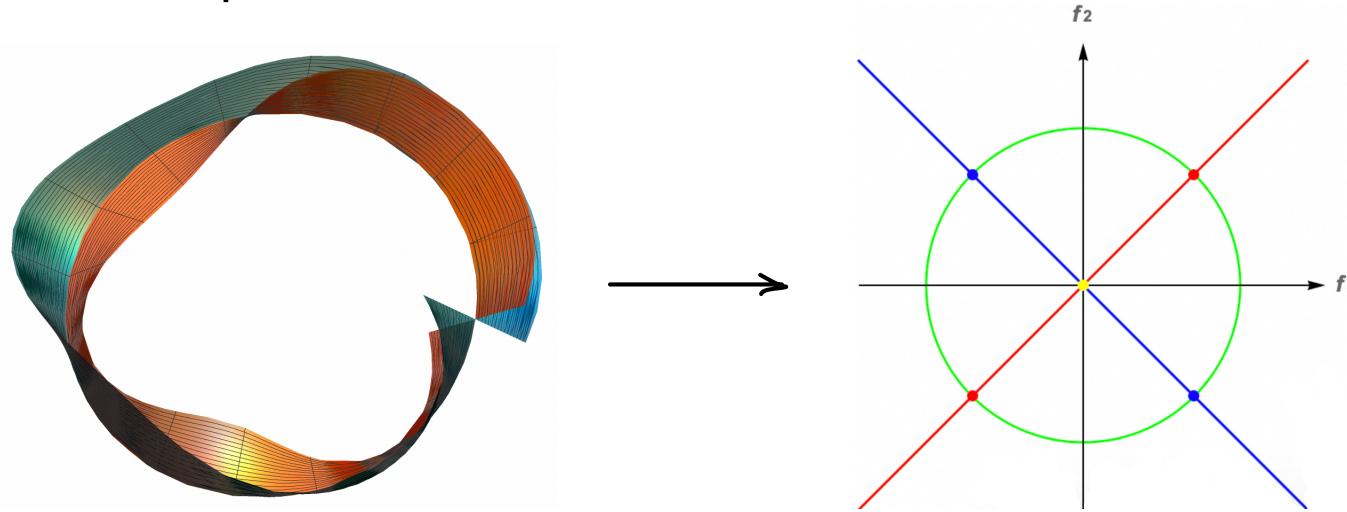
Here is a video showing the eigenframe evolution: <https://yifeizhu.github.io/swallowtail/rotate.mp4>

**Note.** In the non-Hermitian case, since the eigenvectors are in  $\mathbb{C}^2$ , we have adopted (a variant of) the *Hermitian angle* to properly characterize the eigenframe rotation and deformation:

$$\frac{\langle v_+, v_- \rangle_{\mathbb{C}}}{|v_+| |v_-|} = \rho e^{i\psi}$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.



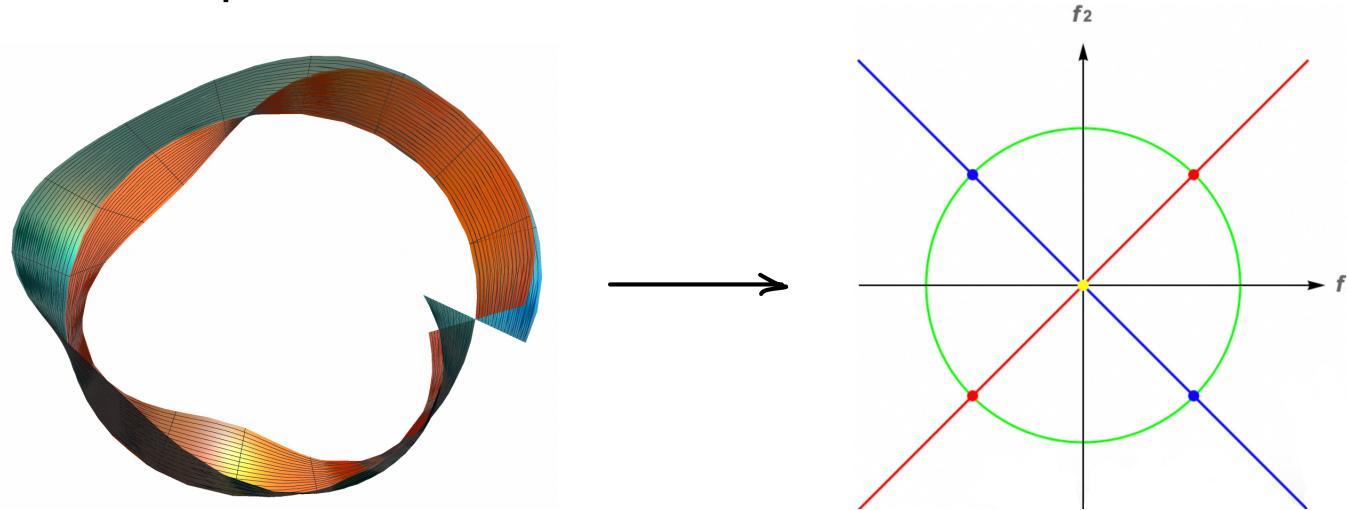
Here is a video showing the eigenframe evolution: <https://yifeizhu.github.io/swallowtail/rotate.mp4>

**Note.** In the non-Hermitian case, since the eigenvectors are in  $\mathbb{C}^2$ , we have adopted (a variant of) the *Hermitian angle* to properly characterize the eigenframe rotation and deformation:

$$\frac{\langle v_+, v_- \rangle_{\mathbb{C}}}{|v_+| |v_-|} = \rho e^{i\psi}, \quad \cos(v_+, v_-)_{\text{Herm}} := \rho$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.



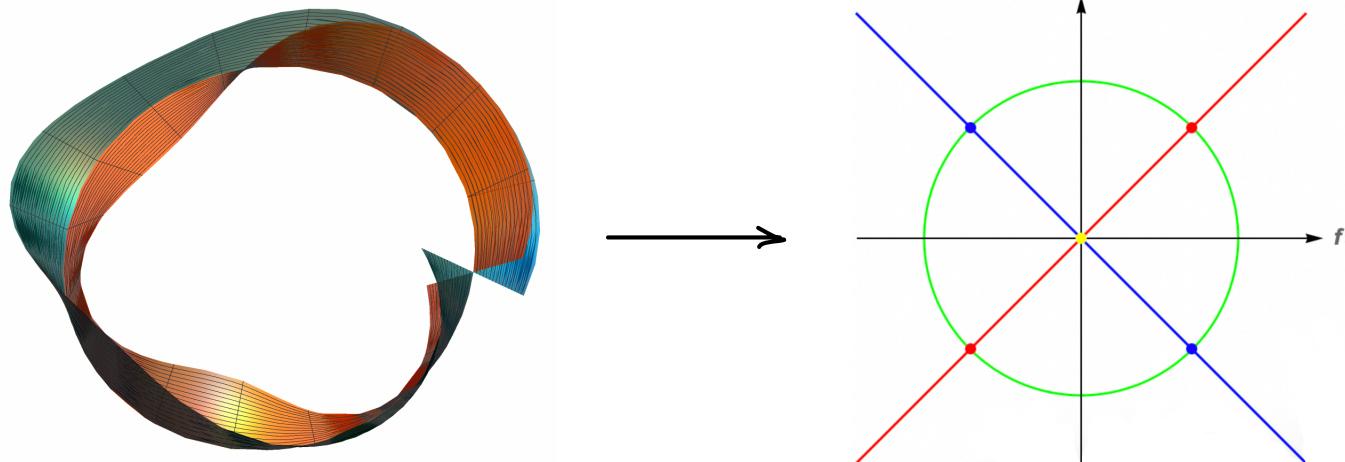
Here is a video showing the eigenframe evolution: <https://yifeizhu.github.io/swallowtail/rotate.mp4>

**Note.** In the non-Hermitian case, since the eigenvectors are in  $\mathbb{C}^2$ , we have adopted (a variant of) the *Hermitian angle* to properly characterize the eigenframe rotation and deformation:

$$\frac{\langle v_+, v_- \rangle_{\mathbb{C}}}{|v_+| |v_-|} = \rho e^{i\psi}, \quad \cos(v_+, v_-)_{\text{Herm}} := \rho$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Proposition.** The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.



Here is a video showing the eigenframe evolution: <https://yifeizhu.github.io/swallowtail/rotate.mp4>

**Note.** In the non-Hermitian case, since the eigenvectors are in  $\mathbb{C}^2$ , we have adopted (a variant of) the *Hermitian angle* to properly characterize the eigenframe rotation and deformation:

$$\frac{\langle v_+, v_- \rangle_{\mathbb{C}}}{|v_+| |v_-|} = \rho e^{i\psi}, \quad \cos(v_+, v_-)_{\text{Herm}} := \rho$$

*The intrinsic geometry should be independent of real/complex coordination, though.*

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**Mathematical interlude:** Classification of bundles

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**Mathematical interlude:** Classification of bundles

$$\begin{array}{ccc} V \\ \downarrow \\ X \end{array}$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

**Mathematical interlude:** Classification of bundles

$$\begin{array}{ccc} V & E & \textit{universal bundle} \\ \downarrow & \downarrow & \\ X & B & \textit{classifying space} \end{array}$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

**Mathematical interlude:** Classification of bundles

$$\begin{array}{ccc} V & E & \text{universal bundle} \\ \downarrow & \downarrow & \\ X & \xrightarrow{f} & B \\ & & \text{classifying space} \\ & & \text{classifying map} \end{array}$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

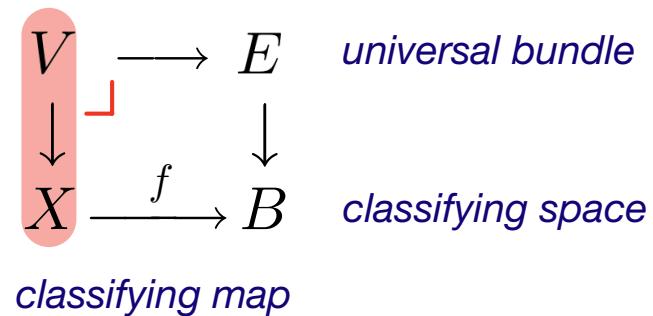
**Mathematical interlude:** Classification of bundles

$$\begin{array}{ccc} V & \longrightarrow & E & \text{universal bundle} \\ \downarrow & \lrcorner & \downarrow & \\ X & \xrightarrow{f} & B & \text{classifying space} \\ & & & \text{classifying map} \end{array}$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

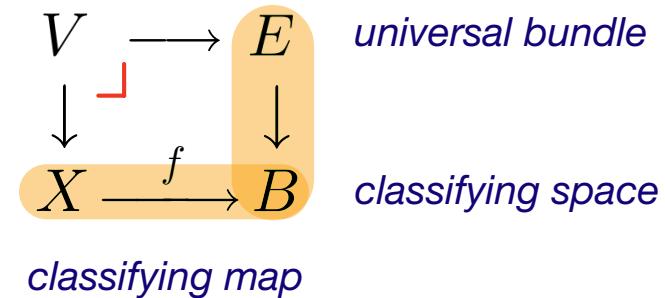
**Mathematical interlude:** Classification of bundles



# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

**Mathematical interlude:** Classification of bundles



## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

**Mathematical interlude:** Classification of bundles

$$\begin{array}{ccc} V & \longrightarrow & E & \text{universal bundle} \\ \downarrow & \lrcorner & \downarrow & \\ X & \xrightarrow{f} & B & \text{classifying space} \\ & & & \text{classifying map} \end{array}$$

$\{\text{isomorphism classes of bundles } V \rightarrow X\} \cong \{\text{homotopy classes of maps } X \rightarrow B\}$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

**Mathematical interlude:** Classification of bundles

$$\begin{array}{ccc} V & \longrightarrow & E & \text{universal bundle} \\ \downarrow & \lrcorner & \downarrow & \\ X & \xrightarrow{f} & B & \text{classifying space} \\ & & & \text{classifying map} \end{array}$$

$\{\text{isomorphism classes of bundles } V \rightarrow X\} \cong \{\text{homotopy classes of maps } X \rightarrow B\}$   
For eigenframe evolution, we take  $X = S^1$ , and the right side becomes  $\pi_1(B)$ .

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

**Mathematical interlude:** Classification of bundles

$$\begin{array}{ccc} V & \longrightarrow & E & \text{universal bundle} \\ \downarrow & \lrcorner & \downarrow & \\ X & \xrightarrow{f} & B & \text{classifying space} \\ & & & \text{classifying map} \end{array}$$

$\{\text{isomorphism classes of bundles } V \rightarrow X\} \cong \{\text{homotopy classes of maps } X \rightarrow B\}$

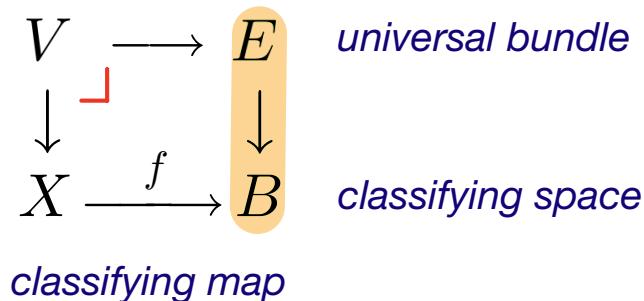
For eigenframe evolution, we take  $X = S^1$ , and the right side becomes  $\pi_1(B)$ .

This breaks the classification problem into two parts:

# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

**Mathematical interlude:** Classification of bundles



$\{\text{isomorphism classes of bundles } V \rightarrow X\} \cong \{\text{homotopy classes of maps } X \rightarrow B\}$

For eigenframe evolution, we take  $X = S^1$ , and the right side becomes  $\pi_1(B)$ .

This breaks the classification problem into two parts:

- Describe the **universal bundle**

# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the **topological charge**?

**Mathematical interlude:** Classification of bundles

$$\begin{array}{ccc} V & \longrightarrow & E & \text{universal bundle} \\ \downarrow & \lrcorner & \downarrow & \\ X & \xrightarrow{f} & B & \text{classifying space} \\ & & & \text{classifying map} \end{array}$$

$\{\text{isomorphism classes of bundles } V \rightarrow X\} \cong \{\text{homotopy classes of maps } X \rightarrow B\}$

For eigenframe evolution, we take  $X = S^1$ , and the right side becomes  $\pi_1(B)$ .

This breaks the classification problem into two parts:

- Describe the universal bundle
- Find **computable** and **effective algebraic invariants** (topological charge) for the classifying/moduli space

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the **stratified** moduli space.

*Gajer, The intersection Dold–Thom theorem,  
Topology, 1996. (Ph.D. student of Blaine Lawson, 1993)*

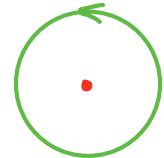
*Goresky and MacPherson, 1974.*

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st **intersection homology group** recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .



## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st **intersection homology group** recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .

$$\begin{array}{c|c|c|c} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \xrightarrow{\bar{p}}$$

Intersection homology of  $\mathbb{R}^2$  with one singular point: from top to bottom are  $I^{\bar{p}}H_0, I^{\bar{p}}H_1, I^{\bar{p}}H_2$ , where  $\bar{p}$  is the perversity function.

From blue to red regions, they detect the singular point.

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st intersection homology group recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .

$$\begin{array}{c|c|c|c} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \xrightarrow{\bar{p}}$$

Intersection homology of  $\mathbb{R}^2$  with one singular point: from top to bottom are  $I^{\bar{p}}H_0, I^{\bar{p}}H_1, I^{\bar{p}}H_2$ , where  $\bar{p}$  is the perversity function.  
*Tolerance of ill-behaved cycles*

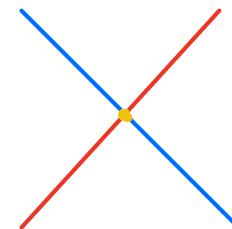
From blue to red regions, they detect the singular point.

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st intersection homology group recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .
- 0'th and 1st intersection homology groups reflect the stratification of the **non-Hermitian** 2-band parameter space.



# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st intersection homology group recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .
- 0'th and 1st intersection homology groups reflect the stratification of the **non-Hermitian** 2-band parameter space.

$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0
0	0	0	0
—	—	—	—
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0
0	0	0	0
—	—	—	—
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0
0	0	0	0
—	—	—	—
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0
0	0	0	0

Intersection homology of  $\mathbb{R}^2$  with a pair of intersecting singular lines:  
from top to bottom are  $I^{\bar{p}}H_*$  with  $* = 0, 1, 2$ .

From green to blue regions, they detect the singular lines.

From blue to red regions, they detect the intersection point.

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st intersection homology group recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .
- Still need compatibility with our earlier ad hoc non-Hermitian classification:

$$\pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

*May need to work at the chain level.*

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st intersection homology group recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .
- Still need compatibility with our earlier ad hoc non-Hermitian classification:

$$\pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

**Question.** How does the eigenframe evolution in the **non-Hermitian case** relate to that in the **Hermitian case**?

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st intersection homology group recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .
- Still need compatibility with our earlier ad hoc non-Hermitian classification:

$$\pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

**Question.** How does the eigenframe evolution in the non-Hermitian case relate to that in the Hermitian case?

**Conjecture.** It does so through a *deformation* (or homotopy) of Riemannian metrics

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st intersection homology group recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .
- Still need compatibility with our earlier ad hoc non-Hermitian classification:

$$\pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

**Question.** How does the eigenframe evolution in the non-Hermitian case relate to that in the Hermitian case?

**Conjecture.** It does so through a *deformation* (or homotopy) of Riemannian metrics, i.e., a 1-parameter continuous family  $\{\eta_t\}_{0 \leq t \leq 1}$  of metrics with

$$\eta_t = \begin{bmatrix} e^{i\pi t} & 0 \\ 0 & 1 \end{bmatrix}$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How to compute the topological charge?

**In progress:** Need to compute the *intersection fundamental group* of the stratified moduli space.

- 1st intersection homology group recovers the Hermitian 2-band charge of  $\mathbb{Z}$ .
- Still need compatibility with our earlier ad hoc non-Hermitian classification:

$$\pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

**Question.** How does the eigenframe evolution in the non-Hermitian case relate to that in the Hermitian case?

**Conjecture.** It does so through a *deformation* (or homotopy) of Riemannian metrics, i.e., a 1-parameter continuous family  $\{\eta_t\}_{0 \leq t \leq 1}$  of metrics with

$$\eta_t = \begin{bmatrix} e^{i\pi t} & 0 \\ 0 & 1 \end{bmatrix}$$

Here is a video of the eigenbundle deformation: <https://yifeizhu.github.io/swallowtail/deform.mp4>

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian **3-band** systems?

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered **cancellation of charges**

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered **cancellation of charges** as well as **reduction to the 2-band case**.

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered cancellation of charges as well as reduction to the 2-band case.

**Example** (Swallowtail quadruple sw4).

$$\begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$$

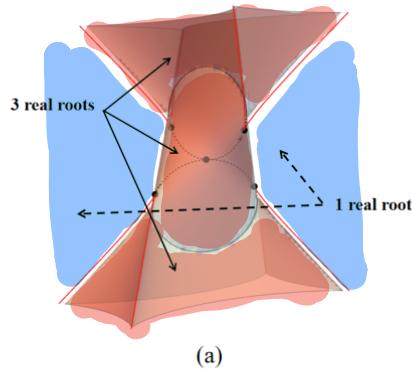
## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered cancellation of charges as well as reduction to the 2-band case.

**Example** (Swallowtail quadruple sw4).

$$\begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$$



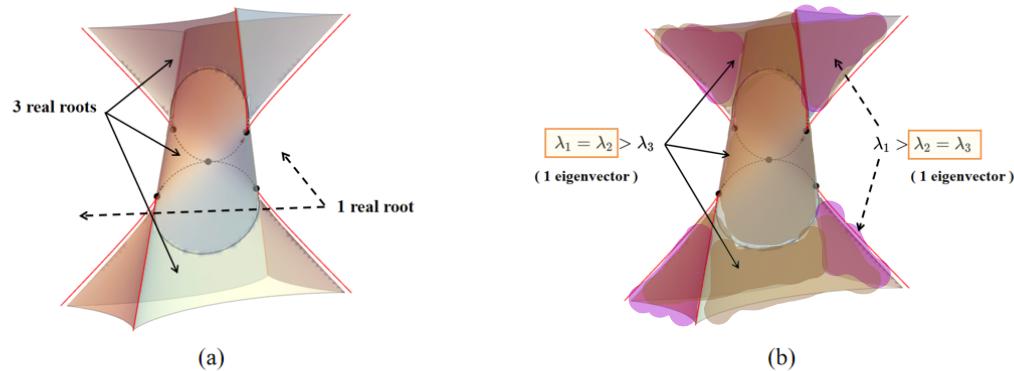
# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered cancellation of charges as well as reduction to the 2-band case.

**Example** (Swallowtail quadruple sw4).

$$\begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$$



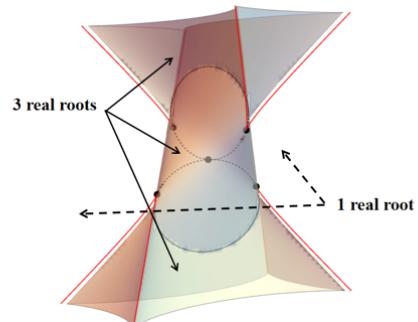
# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

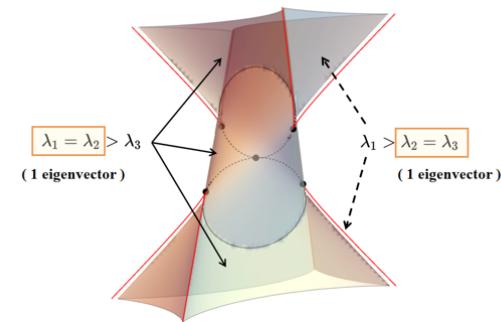
**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered cancellation of charges as well as reduction to the 2-band case.

**Example** (Swallowtail quadruple sw4).

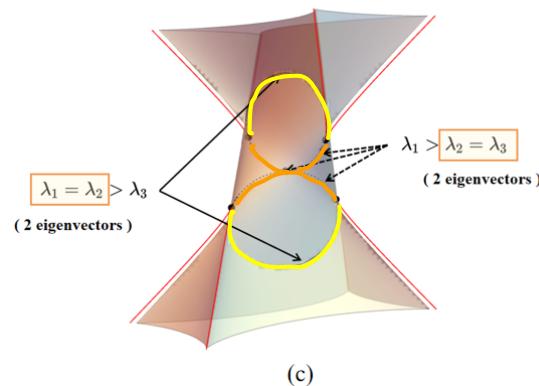
$$\begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$$



(a)



(b)



(c)

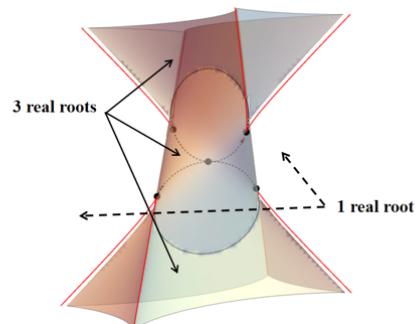
# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

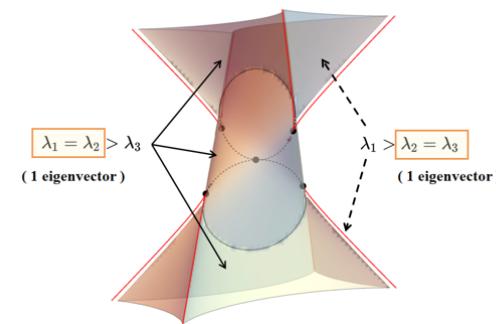
**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered cancellation of charges as well as reduction to the 2-band case.

**Example** (Swallowtail quadruple sw4).

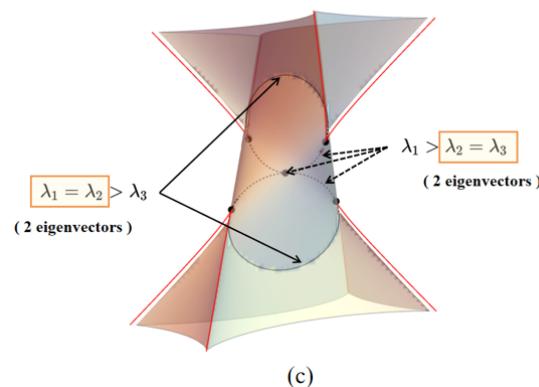
$$\begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$$



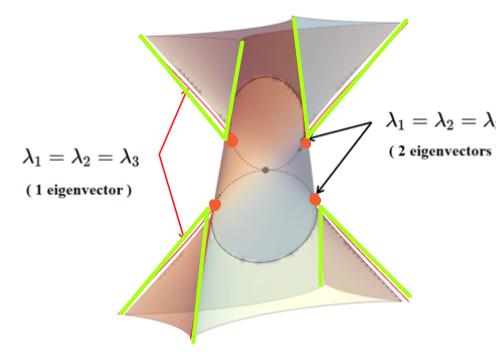
(a)



(b)



(c)



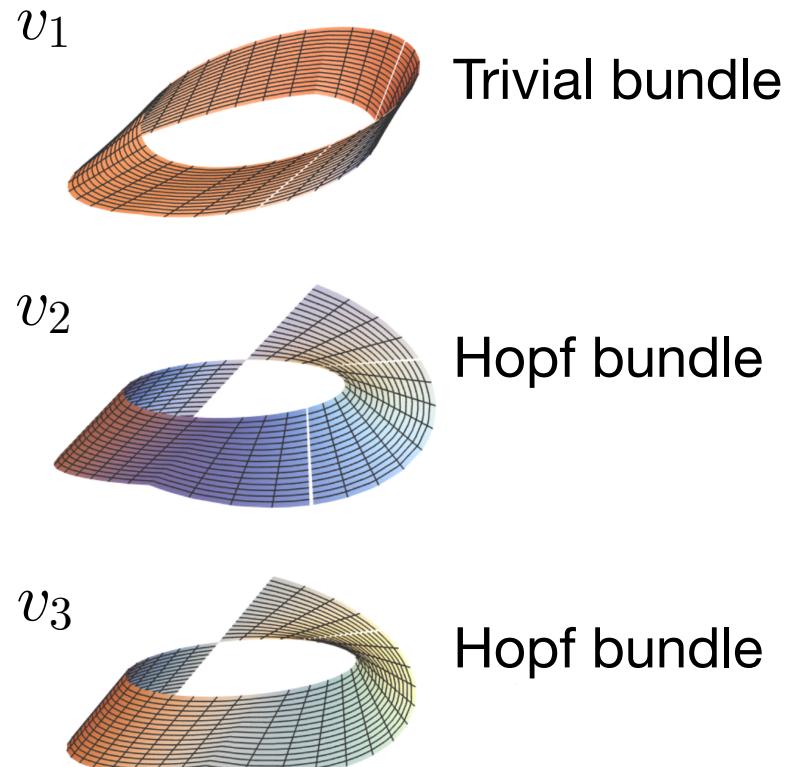
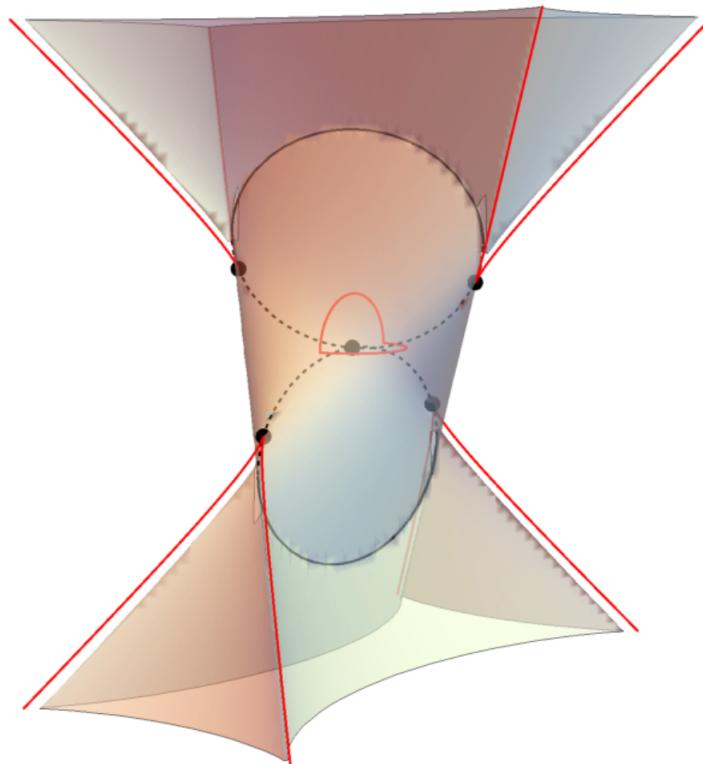
(d)

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered cancellation of charges as well as reduction to the 2-band case.

**Example** (Swallowtail quadruple sw4).

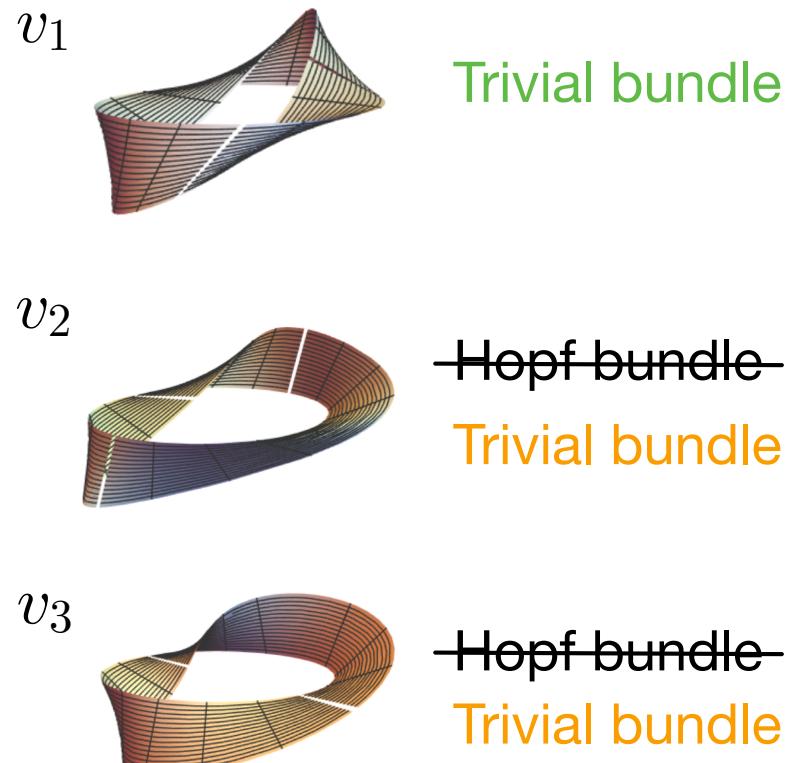
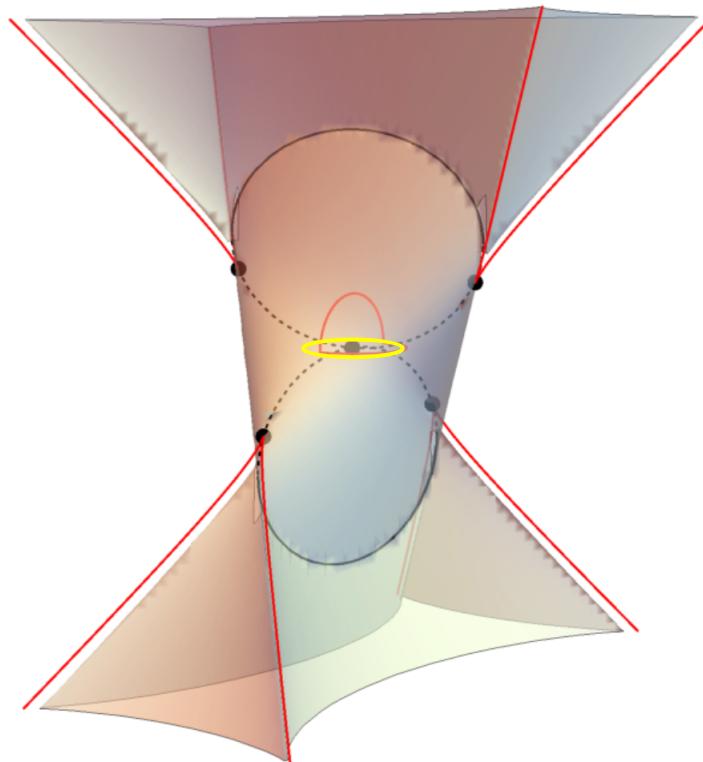


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered **cancellation of charges** as well as reduction to the 2-band case.

**Example** (Swallowtail quadruple sw4).

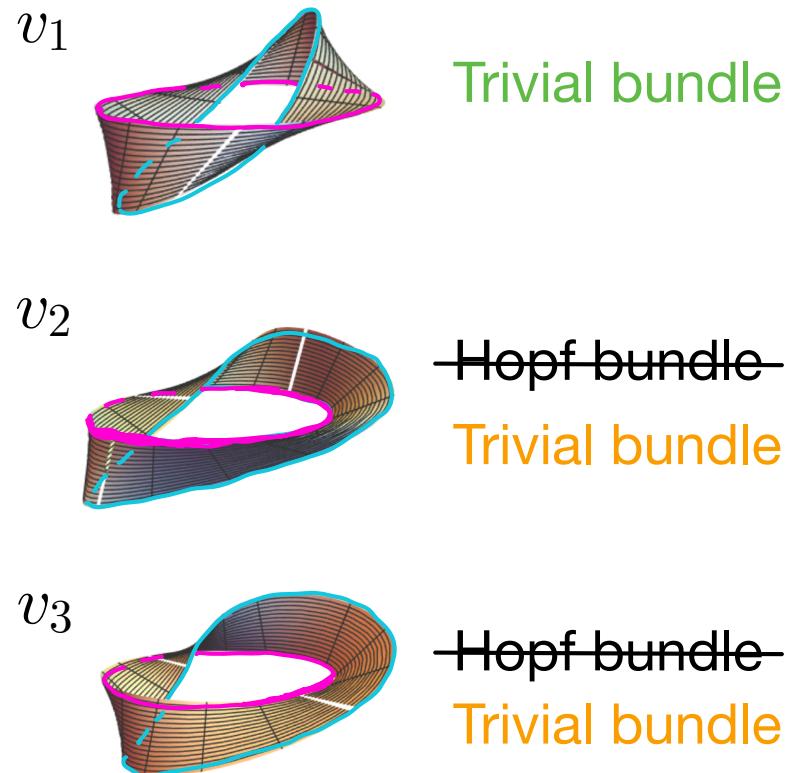
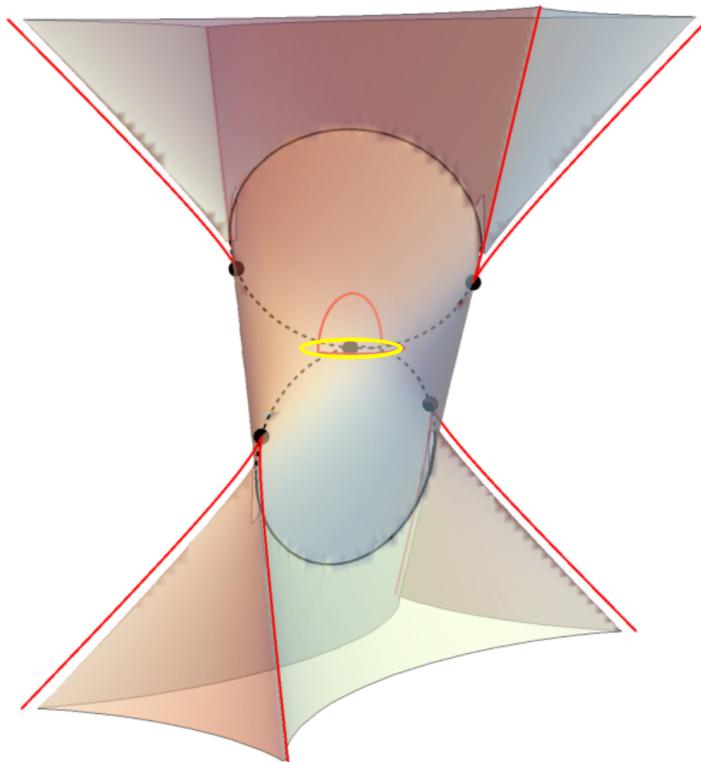


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered **cancellation of charges** as well as reduction to the 2-band case.

**Example** (Swallowtail quadruple sw4).



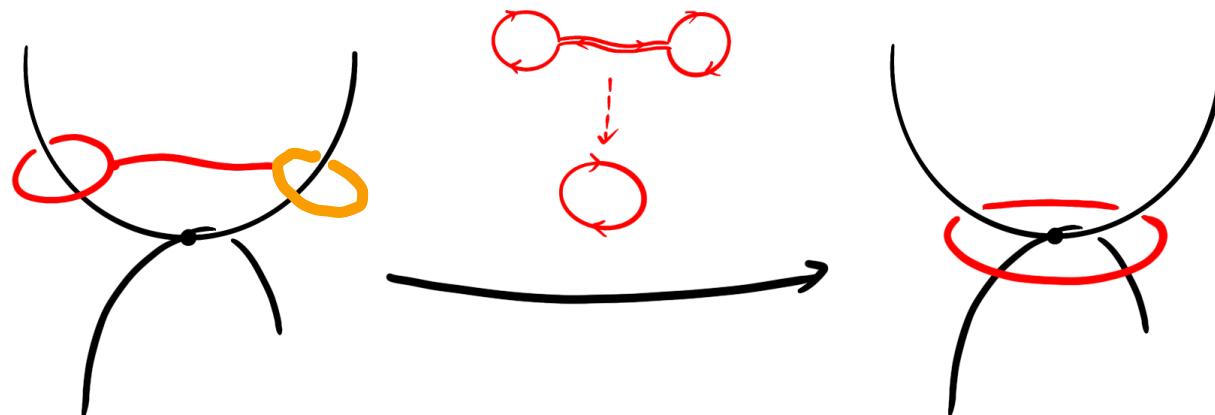
*Linking number is “over-sensitive” when it involves a loop in the **base space** and a loop in the **total space**.*

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered **cancellation of charges** as well as reduction to the 2-band case.

**Example** (Swallowtail quadruple sw4).

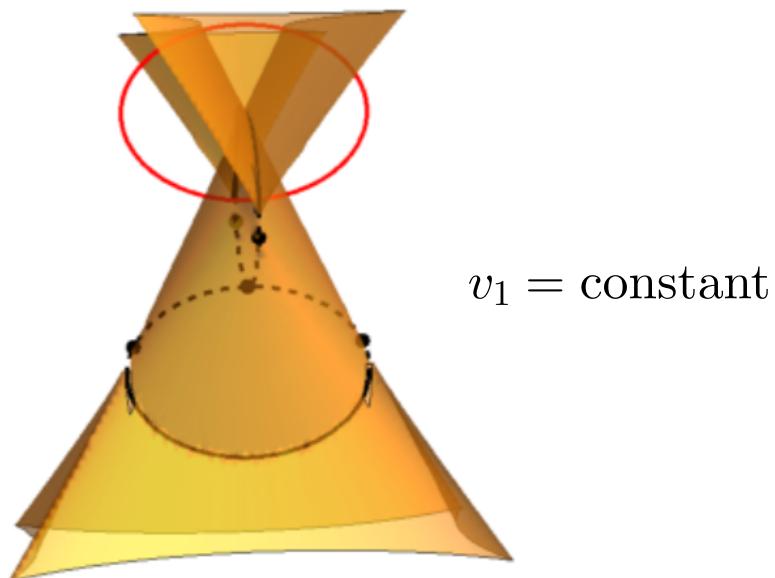


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered cancellation of charges as well as **reduction to the 2-band case**.

**Example** (Swallowtail quadruple sw4).

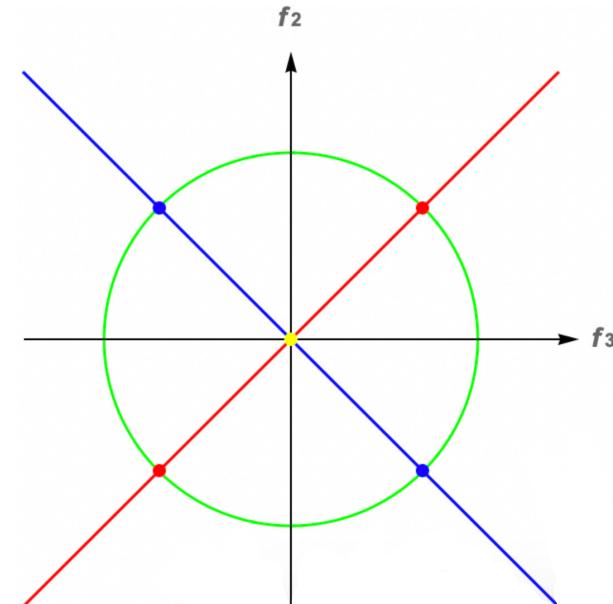
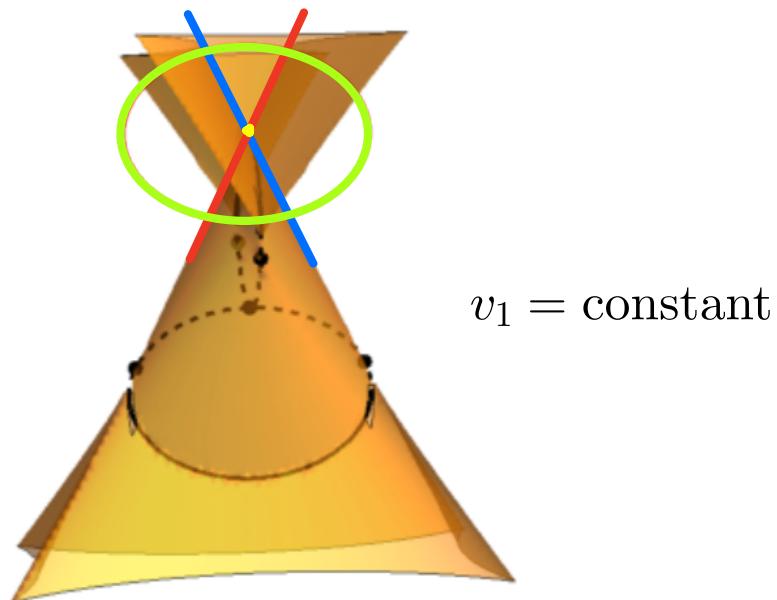


## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** We have investigated *slices* of the 3D moduli spaces containing swallowtails, and discovered cancellation of charges as well as **reduction to the 2-band case**.

**Example** (Swallowtail quadruple sw4).



## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces *as a family*

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D **moduli spaces as a *family***  
*a “**family of families**”*

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces *as a family*, and studied **interesting loops** therein

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces *as a family*, and studied **interesting loops** therein as well as proved **ruledness** as a geometric property of the discriminant surfaces

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces *as a family*, and studied **interesting loops** therein as well as proved **ruledness** as a geometric property of the discriminant surfaces, with physical implications.

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}, \text{ where each } g_i \text{ is a linear function of the parameters } f_j.$$

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Here is visualization from 3 angles of deforming sw4, with nodal lines degenerating:

<https://yifeizhu.github.io/swallowtail/sw4-defo-1.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-2.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-3.gif>

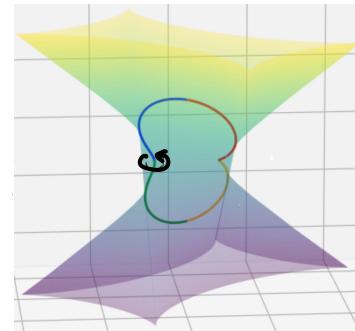
## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$



Here is visualization from 3 angles of deforming sw4, with nodal lines degenerating:

<https://yifeizhu.github.io/swallowtail/sw4-defo-1.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-2.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-3.gif>

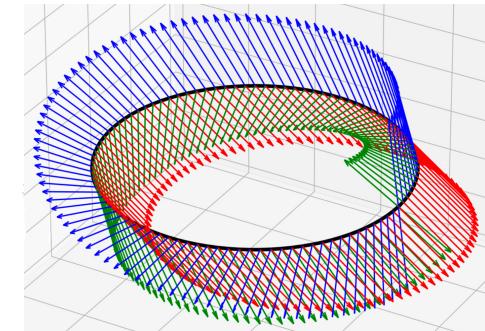
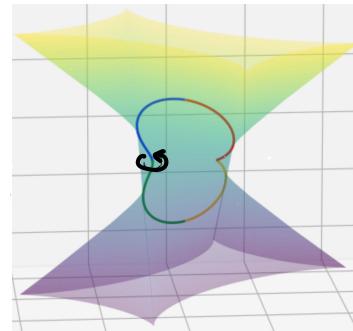
## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$



Here is visualization from 3 angles of deforming sw4, with nodal lines degenerating:

<https://yifeizhu.github.io/swallowtail/sw4-defo-1.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-2.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-3.gif>

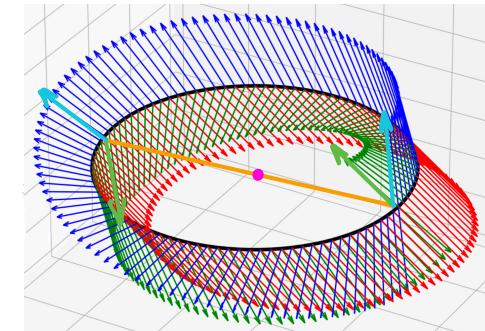
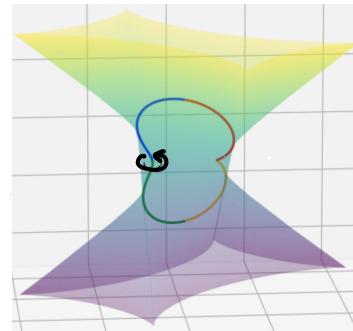
# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$



Here is visualization from 3 angles of deforming sw4, with nodal lines degenerating:

<https://yifeizhu.github.io/swallowtail/sw4-defo-1.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-2.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-3.gif>

Across the *center* (nodal line),  
the *blue* and *green* eigenstates  
swap

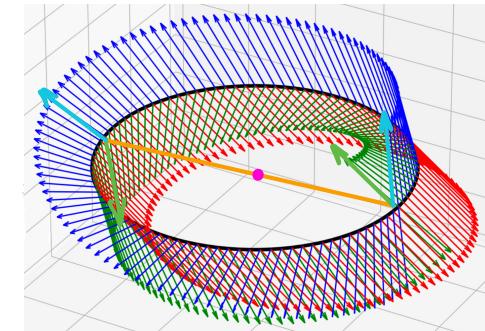
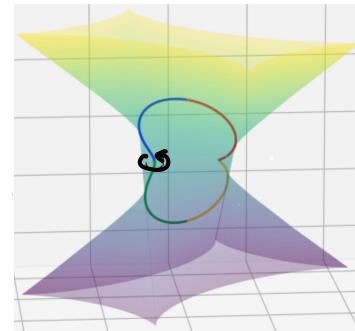
# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$



Here is visualization from 3 angles of deforming sw4, with nodal lines degenerating:

<https://yifeizhu.github.io/swallowtail/sw4-defo-1.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-2.gif>

<https://yifeizhu.github.io/swallowtail/sw4-defo-3.gif>

Across the *center* (nodal line),  
the *blue* and *green* eigenstates  
swap — “band inversion.”

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Visualizing at <https://www.wolframcloud.com/env/zhuwf0/Presentation.nb>

- Opening of 2 tunnels and a new “big” loop around, along which the rank-3 eigenbundle is trivial
- Merging of 8 cuspidal lines into 4
- Ruledness (specific parametrization)

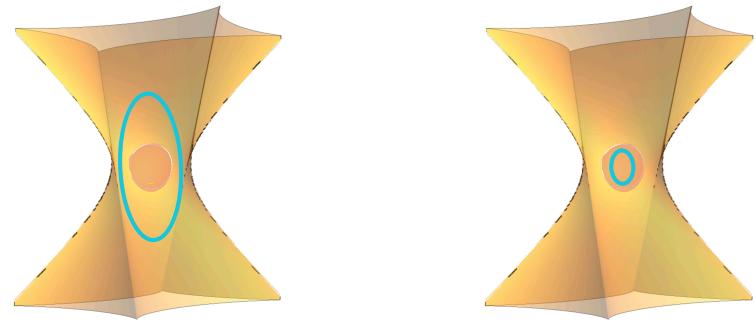
## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$



Visualizing at <https://www.wolframcloud.com/env/zhu0/Presentation.nb>

- Opening of 2 tunnels and a new “big” loop around, along which the rank-3 eigenbundle is trivial
- Merging of 8 cuspidal lines into 4
- Ruledness (specific parametrization)

*Shrinking this loop into the **enclosed region**, we find the eigenbundle along it remains trivial.*

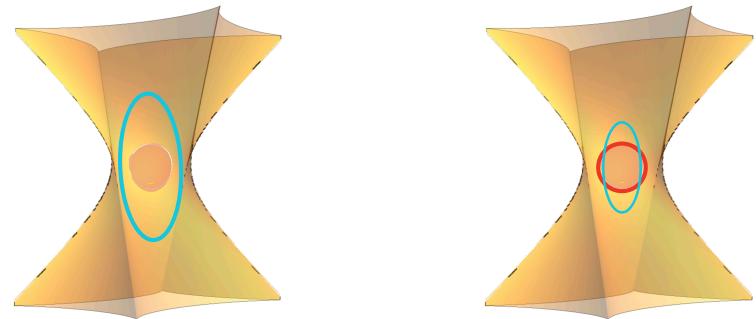
## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$



Visualizing at <https://www.wolframcloud.com/env/zhu0/Presentation.nb>

- Opening of 2 tunnels and a new “big” loop around, along which the rank-3 eigenbundle is trivial
- Merging of 8 cuspidal lines into 4
- Ruledness (specific parametrization)

*What about loops transversing the **nodal intersection lines**? Band inversion again?*

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Implications of ruledness:

- Improved precision with graphing and engineering

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Implications of ruledness:

- Improved precision with graphing and engineering
- In fact, **developable**

*tangent developable, along the **cuspidal lines***

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Implications of ruledness:

- Improved precision with graphing and engineering
- In fact, developable  $\Rightarrow$  Gaussian curvature = 0

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Implications of ruledness:

- Improved precision with graphing and engineering
- In fact, developable  $\Rightarrow$  Gaussian curvature = 0  
 $\Rightarrow$  boundary of a hyperbolic 3-manifold?

# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

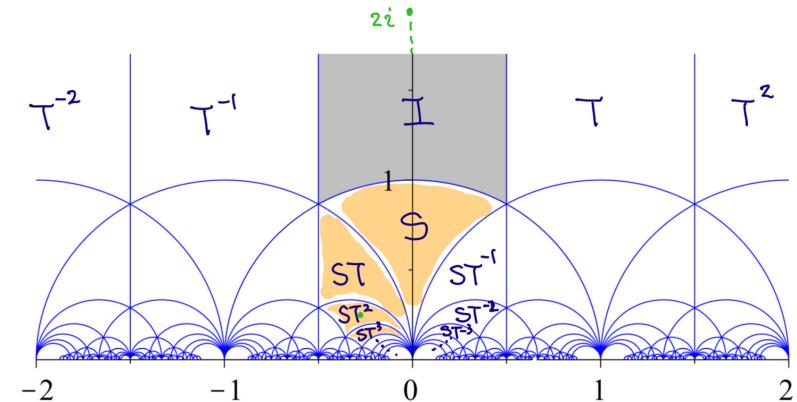
**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Implications of ruledness:

- Improved precision with graphing and engineering
- In fact, developable  $\Rightarrow$  Gaussian curvature = 0  
 $\Rightarrow$  boundary of a hyperbolic 3-manifold?



A prototypical 2D hyperbolic lattice  
with a *straight-line* boundary

# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

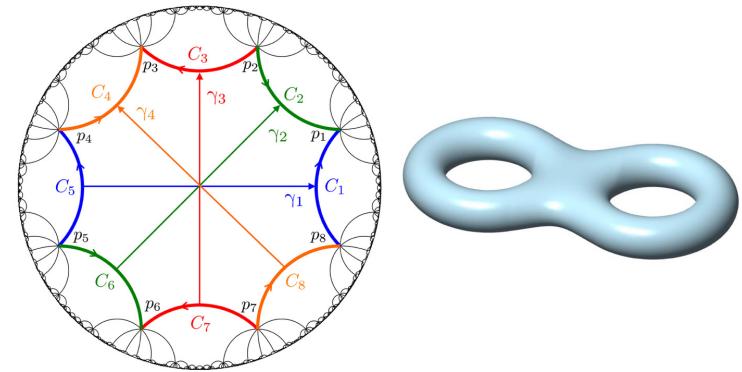
**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Implications of ruledness:

- Improved precision with graphing and engineering
- In fact, developable  $\Rightarrow$  Gaussian curvature = 0  
 $\Rightarrow$  boundary of a hyperbolic 3-manifold?



*Another basic example of a hyperbolic lattice associated to a genus-2 surface*  
(from Maciejko and Rayan, *Hyperbolic band theory*, **Sci. Adv.**, 2021)

# Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

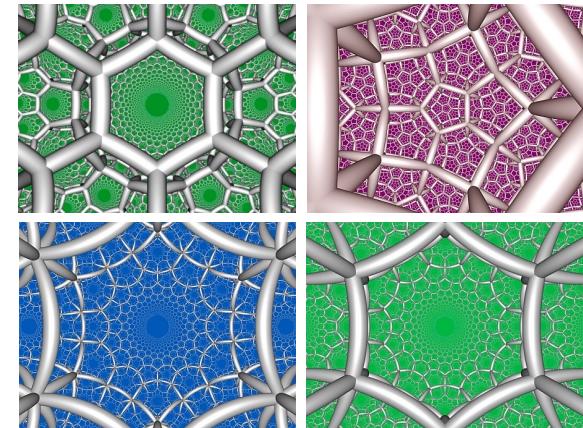
**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Implications of ruledness:

- Improved precision with graphing and engineering
- In fact, developable  $\Rightarrow$  Gaussian curvature = 0  
 $\Rightarrow$  boundary of a hyperbolic 3-manifold?



Four 3D hyperbolic lattices tiling up the hyperbolic 3-space  $\mathbb{H}^3$  (from John Baez's blog)

## Eigenframe evolution as Higgs bundles: The non-Hermitian case

**Question.** How does eigenframe evolve in non-Hermitian 3-band systems?

**In progress:** Moreover, we have investigated the 3D moduli spaces as a *family*, and studied interesting loops therein as well as proved ruledness as a geometric property of the discriminant surfaces, with physical implications.

**Example** (Family of swallowtail ensembles, including the couple sw2 and the quadruple sw4).

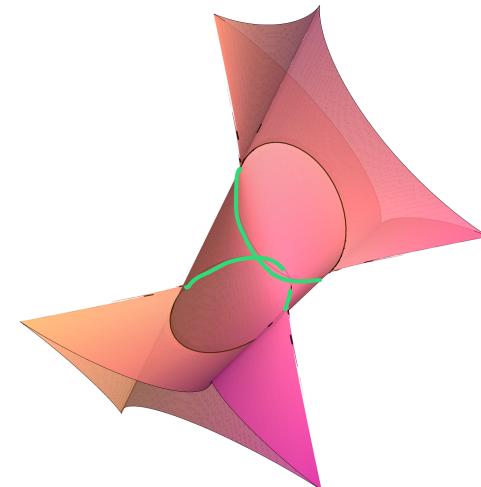
$$\begin{bmatrix} g_1(f_1, f_2, f_3) & f_1 & f_2 \\ -f_1 & g_2(f_1, f_2, f_3) & f_3 \\ -f_2 & f_3 & g_3(f_1, f_2, f_3) \end{bmatrix}$$

Implications of ruledness:

- Improved precision with graphing and engineering
- In fact, developable  $\Rightarrow$  Gaussian curvature = 0

$\Rightarrow$  boundary of a hyperbolic 3-manifold?

Existence of **nodal curves** inside also gives evidence, supporting nontrivial loops around (generating a free group on 3 letters) acting on a 3D hyperbolic lattice.

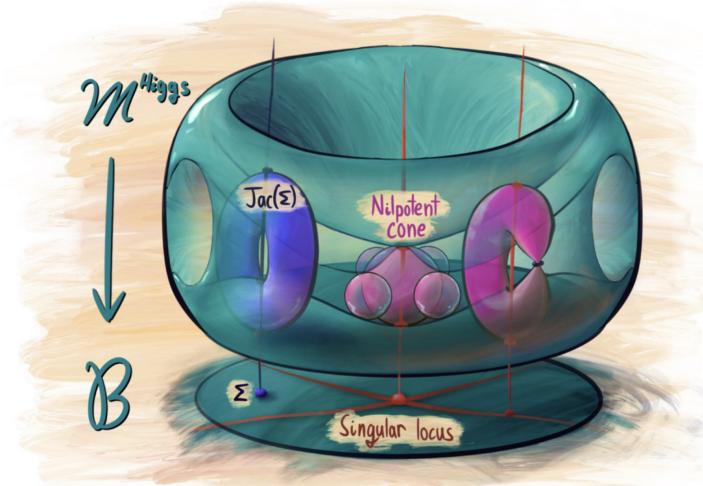


## **Moduli of Higgs bundles**

There are moduli spaces for Higgs bundles of various sorts.

## Moduli of Higgs bundles

There are moduli spaces for Higgs bundles of various sorts. An effective tool to study them is through **Hitchin's fibration**

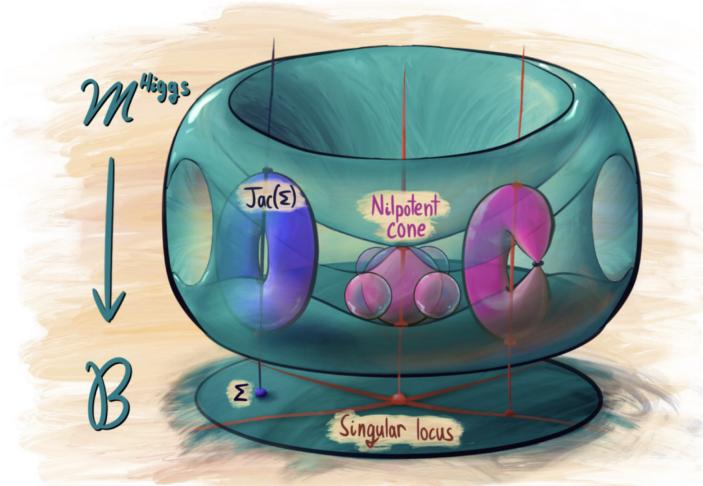


# Moduli of Higgs bundles

There are moduli spaces for Higgs bundles of various sorts. An effective tool to study them is through **Hitchin's fibration**

$$h: \mathcal{M}_{\text{Higgs}}^s(\text{SL}_n(\mathbb{C})) \longrightarrow \bigoplus_{i=2}^n H^0(\Sigma; K^i) =: \mathcal{B}$$
$$(E, \phi) \mapsto (p_2(\phi), \dots, p_n(\phi))$$

where each  $p_i$  is a homogeneous polynomial of degree  $i$ .

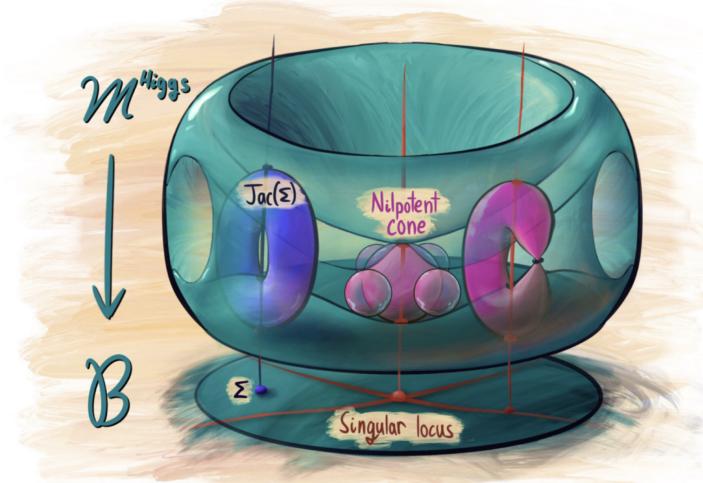


# Moduli of Higgs bundles

There are moduli spaces for Higgs bundles of various sorts. An effective tool to study them is through **Hitchin's fibration**

$$h: \mathcal{M}_{\text{Higgs}}^s(\text{SL}_n(\mathbb{C})) \longrightarrow \bigoplus_{i=2}^n H^0(\Sigma; K^i) =: \mathcal{B}$$
$$(E, \phi) \mapsto (p_2(\phi), \dots, p_n(\phi))$$

where each  $p_i$  is a homogeneous polynomial of degree  $i$ . A point  $b = (p_2(\phi), \dots, p_n(\phi)) \in \mathcal{B}$  determines a spectral curve  $\sigma_b: \omega^n + p_2(\phi)\omega^{n-2} + \dots + p_n(\phi) = \det(\omega I - \phi)$ .

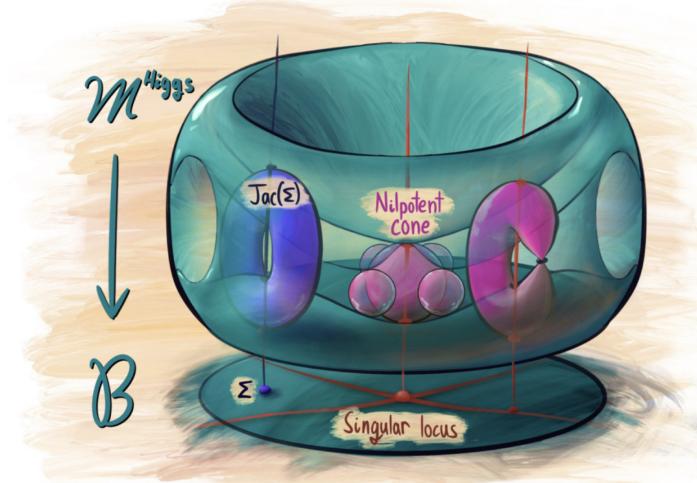


# Moduli of Higgs bundles

There are moduli spaces for Higgs bundles of various sorts. An effective tool to study them is through **Hitchin's fibration**

$$h: \mathcal{M}_{\text{Higgs}}^s(\text{SL}_n(\mathbb{C})) \longrightarrow \bigoplus_{i=2}^n H^0(\Sigma; K^i) =: \mathcal{B}$$
$$(E, \phi) \mapsto (p_2(\phi), \dots, p_n(\phi))$$

where each  $p_i$  is a homogeneous polynomial of degree  $i$ . A point  $b = (p_2(\phi), \dots, p_n(\phi)) \in \mathcal{B}$  determines a spectral curve  $\sigma_b: \omega^n + p_2(\phi)\omega^{n-2} + \dots + p_n(\phi) = \det(\omega I - \phi)$ .



- Regular locus: Consists of points  $b$  such that  $\sigma_b$  is regular.

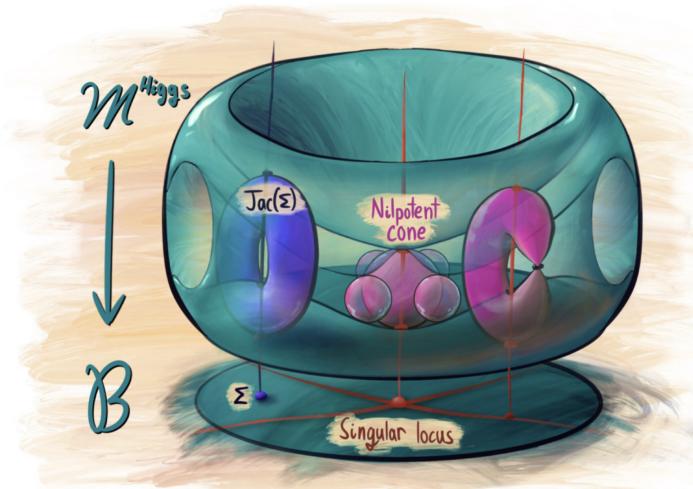
# Moduli of Higgs bundles

There are moduli spaces for Higgs bundles of various sorts. An effective tool to study them is through **Hitchin's fibration**

$$h: \mathcal{M}_{\text{Higgs}}^s(\text{SL}_n(\mathbb{C})) \longrightarrow \bigoplus_{i=2}^n H^0(\Sigma; K^i) =: \mathcal{B}$$

$$(E, \phi) \mapsto (p_2(\phi), \dots, p_n(\phi))$$

where each  $p_i$  is a homogeneous polynomial of degree  $i$ . A point  $b = (p_2(\phi), \dots, p_n(\phi)) \in \mathcal{B}$  determines a spectral curve  $\sigma_b: \omega^n + p_2(\phi)\omega^{n-2} + \dots + p_n(\phi) = \det(\omega I - \phi)$ .



- Regular locus: Consists of points  $b$  such that  $\sigma_b$  is regular.
- Singular locus: Consists of points  $b$  such that  $\sigma_b$  is singular. Over the singular locus, the spectral curve is singular, and the fiber can degenerate.

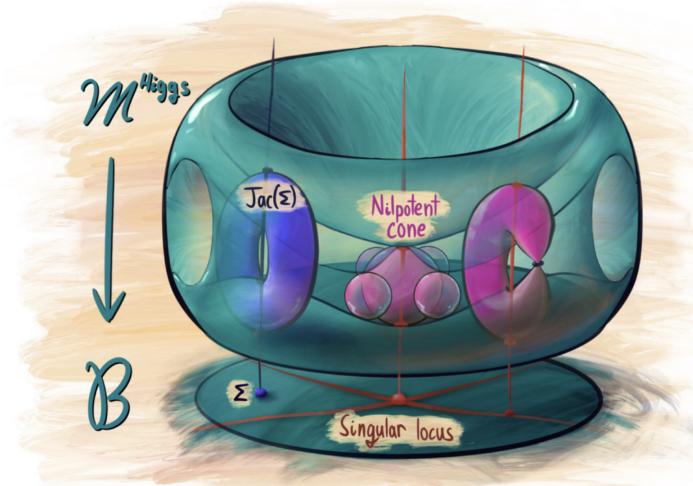
# Moduli of Higgs bundles

There are moduli spaces for Higgs bundles of various sorts. An effective tool to study them is through **Hitchin's fibration**

$$h: \mathcal{M}_{\text{Higgs}}^s(\text{SL}_n(\mathbb{C})) \longrightarrow \bigoplus_{i=2}^n H^0(\Sigma; K^i) =: \mathcal{B}$$

$$(E, \phi) \mapsto (p_2(\phi), \dots, p_n(\phi))$$

where each  $p_i$  is a homogeneous polynomial of degree  $i$ . A point  $b = (p_2(\phi), \dots, p_n(\phi)) \in \mathcal{B}$  determines a spectral curve  $\sigma_b: \omega^n + p_2(\phi)\omega^{n-2} + \dots + p_n(\phi) = \det(\omega I - \phi)$ .



- Regular locus: Consists of points  $b$  such that  $\sigma_b$  is regular.
- Singular locus: Consists of points  $b$  such that  $\sigma_b$  is singular. Over the singular locus, the spectral curve is singular, and the fiber can degenerate.
- Nilpotent cone: The most degeneration occurs over  $0 \in \mathcal{B}$ . The fiber  $h^{-1}(0)$  is called the **nilpotent cone**.

## Moduli of Higgs bundles

**Example** (2-band non-Hermitian system).

## Moduli of Higgs bundles

**Example** (2-band non-Hermitian system).

Recall

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

and compute that  $\text{tr } H = 0, \det H = f_2^2 - f_3^2$ .

## Moduli of Higgs bundles

**Example** (2-band non-Hermitian system).

Recall

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

and compute that  $\text{tr } H = 0$ ,  $\det H = f_2^2 - f_3^2$ .

Since there is no nontrivial  $H$ -invariant subbundle,  
this Higgs bundle is stable.

## Moduli of Higgs bundles

**Example** (2-band non-Hermitian system).

Recall

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

and compute that  $\text{tr } H = 0, \det H = f_2^2 - f_3^2$ .

Since there is no nontrivial  $H$ -invariant subbundle,  
this Higgs bundle is stable.

The spectral curve is given by

$$\det(\omega I - H) = \omega^2 + (f_2^2 - f_3^2)$$

so that

$$p_2(H) = f_2^2 - f_3^2$$

and the nilpotent cone is the fiber  $f_2^2 - f_3^2 = 0$ .

# Moduli of Higgs bundles

**Example** (2-band non-Hermitian system).

Recall

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

and compute that  $\text{tr } H = 0, \det H = f_2^2 - f_3^2$ .

Since there is no nontrivial  $H$ -invariant subbundle, this Higgs bundle is stable.

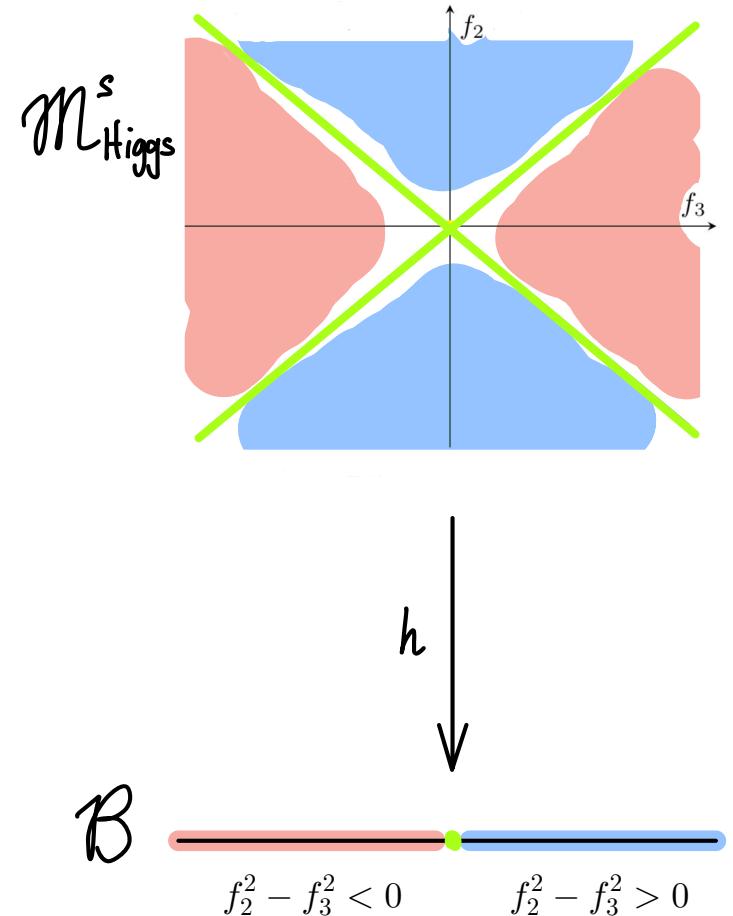
The spectral curve is given by

$$\det(\omega I - H) = \omega^2 + (f_2^2 - f_3^2)$$

so that

$$p_2(H) = f_2^2 - f_3^2$$

and the nilpotent cone is the fiber  $f_2^2 - f_3^2 = 0$ .



# Moduli of Higgs bundles

**Example** (2-band non-Hermitian system).

Recall

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

and compute that  $\text{tr } H = 0, \det H = f_2^2 - f_3^2$ .

Since there is no nontrivial  $H$ -invariant subbundle, this Higgs bundle is stable.

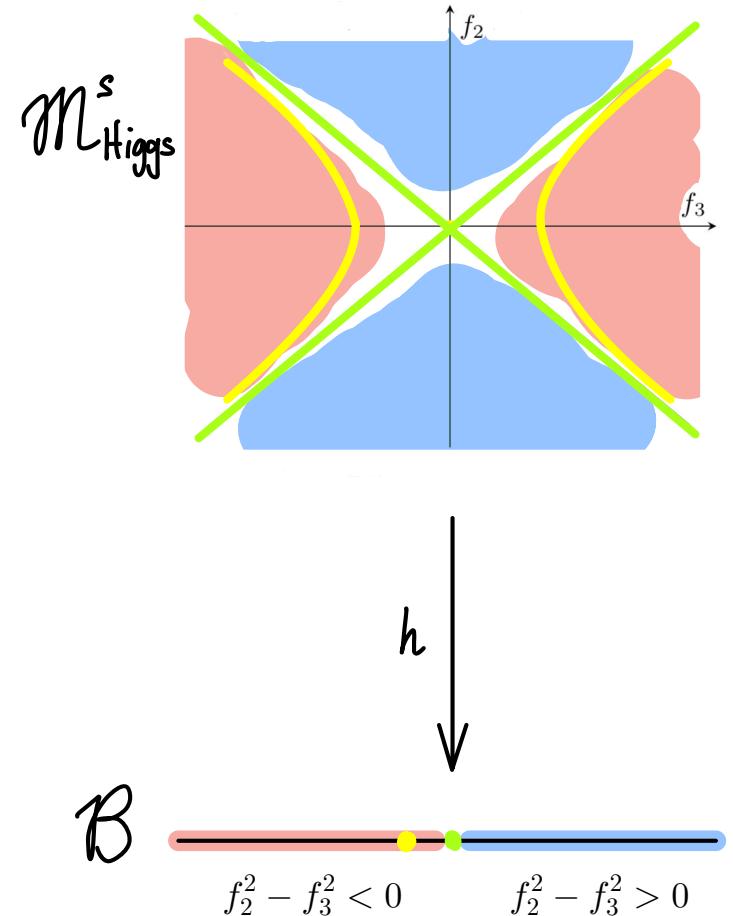
The spectral curve is given by

$$\det(\omega I - H) = \omega^2 + (f_2^2 - f_3^2)$$

so that

$$p_2(H) = f_2^2 - f_3^2$$

and the nilpotent cone is the fiber  $f_2^2 - f_3^2 = 0$ .



# Moduli of Higgs bundles

**Example** (2-band non-Hermitian system).

Recall

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

and compute that  $\text{tr } H = 0, \det H = f_2^2 - f_3^2$ .

Since there is no nontrivial  $H$ -invariant subbundle, this Higgs bundle is stable.

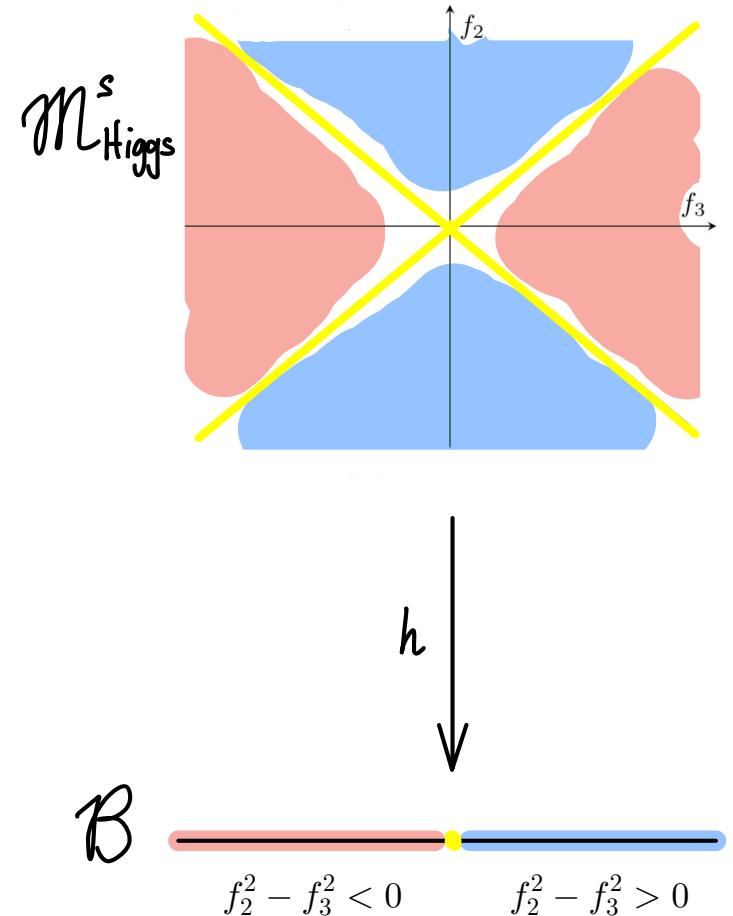
The spectral curve is given by

$$\det(\omega I - H) = \omega^2 + (f_2^2 - f_3^2)$$

so that

$$p_2(H) = f_2^2 - f_3^2$$

and the nilpotent cone is the fiber  $f_2^2 - f_3^2 = 0$ .



# Moduli of Higgs bundles

**Example** (2-band non-Hermitian system).

Recall

$$H(f_2, f_3) = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$$

and compute that  $\text{tr } H = 0, \det H = f_2^2 - f_3^2$ .

Since there is no nontrivial  $H$ -invariant subbundle, this Higgs bundle is stable.

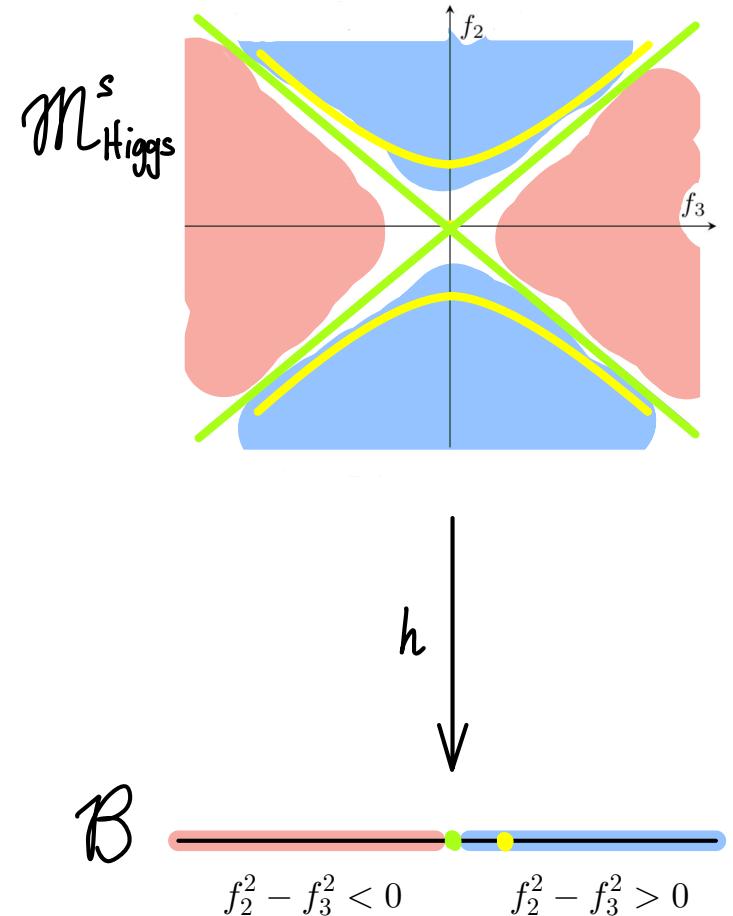
The spectral curve is given by

$$\det(\omega I - H) = \omega^2 + (f_2^2 - f_3^2)$$

so that

$$p_2(H) = f_2^2 - f_3^2$$

and the nilpotent cone is the fiber  $f_2^2 - f_3^2 = 0$ .



## Moduli of Higgs bundles

**Hyperbolic metric.** Higgs bundles naturally sit over hyperbolic base spaces.

## Moduli of Higgs bundles

**Hyperbolic metric.** Higgs bundles naturally sit over hyperbolic base spaces. The non-Euclidean metric form  $\eta$  in the definition of our **non-Hermitian symmetry** is **compatible** with this **hyperbolicity** through eigenframe **deformation**.

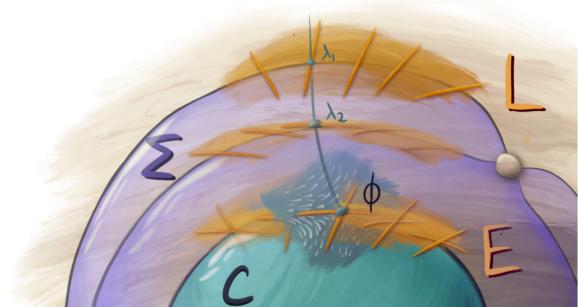
## Moduli of Higgs bundles

**Hyperbolic metric.** Higgs bundles naturally sit over hyperbolic base spaces. The non-Euclidean metric form  $\eta$  in the definition of our non-Hermitian symmetry is compatible with this hyperbolicity through eigenframe deformation.

In fact, the **non-Abelian Hodge correspondence** gives analytic isomorphisms

$$\mathcal{M}_{\text{Higgs}}^{\text{s}}(\text{SL}_n(\mathbb{C})) \cong \text{Rep}(\pi_1(\Sigma), \text{SL}_n(\mathbb{C})) \cong \mathcal{H}$$

where  $\mathcal{H}$  is the space of equivariant harmonic maps from the universal cover  $\tilde{\Sigma}$  to  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$ , modulo isometries.



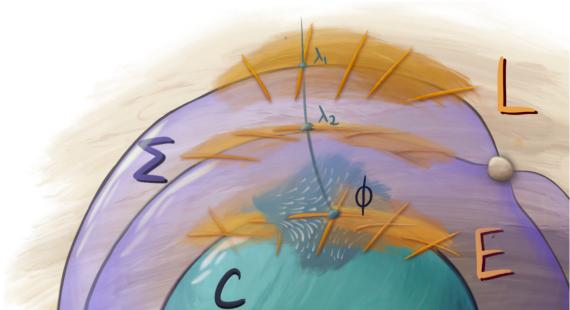
## Moduli of Higgs bundles

**Hyperbolic metric.** Higgs bundles naturally sit over hyperbolic base spaces. The non-Euclidean metric form  $\eta$  in the definition of our non-Hermitian symmetry is compatible with this hyperbolicity through eigenframe deformation.

In fact, the **non-Abelian Hodge correspondence** gives analytic isomorphisms

$$\mathcal{M}_{\text{Higgs}}^{\text{s}}(\text{SL}_n(\mathbb{C})) \cong \text{Rep}(\pi_1(\Sigma), \text{SL}_n(\mathbb{C})) \cong \mathcal{H}$$

where  $\mathcal{H}$  is the space of **equivariant** harmonic maps from the universal cover  $\tilde{\Sigma}$  to  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$ , modulo isometries. Here, the **equivariance** is with respect to a representation  $\rho: \pi_1(\Sigma) \rightarrow \text{SL}_n(\mathbb{C})$ .



## Moduli of Higgs bundles

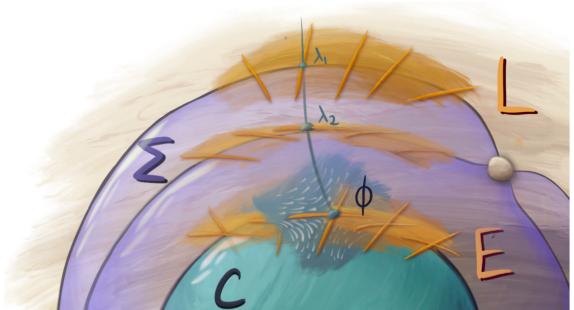
**Hyperbolic metric.** Higgs bundles naturally sit over hyperbolic base spaces. The non-Euclidean metric form  $\eta$  in the definition of our non-Hermitian symmetry is compatible with this hyperbolicity through eigenframe deformation.

In fact, the non-Abelian Hodge correspondence gives analytic isomorphisms

$$\mathcal{M}_{\text{Higgs}}^{\text{s}}(\text{SL}_n(\mathbb{C})) \cong \text{Rep}(\pi_1(\Sigma), \text{SL}_n(\mathbb{C})) \cong \mathcal{H}$$

where  $\mathcal{H}$  is the space of equivariant harmonic maps from the universal cover  $\tilde{\Sigma}$  to  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$ , modulo isometries. Here, the equivariance is with respect to a representation  $\rho: \pi_1(\Sigma) \rightarrow \text{SL}_n(\mathbb{C})$ .

Thus, given a Higgs bundle  $(E, \phi)$ , we get a harmonic map  $f: \tilde{\Sigma} \rightarrow \text{SL}_n(\mathbb{C})/\text{SU}(n)$ .



## Moduli of Higgs bundles

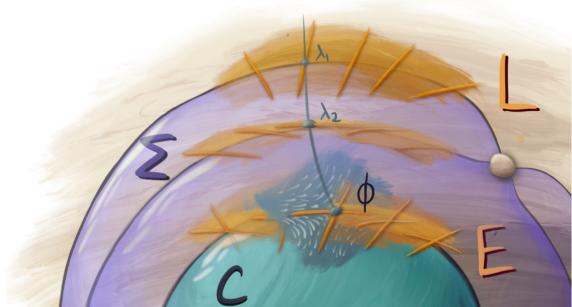
**Hyperbolic metric.** Higgs bundles naturally sit over hyperbolic base spaces. The non-Euclidean metric form  $\eta$  in the definition of our non-Hermitian symmetry is compatible with this hyperbolicity through eigenframe deformation.

In fact, the non-Abelian Hodge correspondence gives analytic isomorphisms

$$\mathcal{M}_{\text{Higgs}}^{\text{s}}(\text{SL}_n(\mathbb{C})) \cong \text{Rep}(\pi_1(\Sigma), \text{SL}_n(\mathbb{C})) \cong \mathcal{H}$$

where  $\mathcal{H}$  is the space of equivariant harmonic maps from the universal cover  $\tilde{\Sigma}$  to  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$ , modulo isometries. Here, the equivariance is with respect to a representation  $\rho: \pi_1(\Sigma) \rightarrow \text{SL}_n(\mathbb{C})$ .

Thus, given a Higgs bundle  $(E, \phi)$ , we get a harmonic map  $f: \tilde{\Sigma} \rightarrow \text{SL}_n(\mathbb{C})/\text{SU}(n)$ . The **negative semi-definite** Killing form on the Lie algebra  $\mathfrak{sl}_n(\mathbb{C})$  gives rise to a metric on  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$ , which then pulls back along  $f$  and descends to  $\Sigma$ .



# Moduli of Higgs bundles

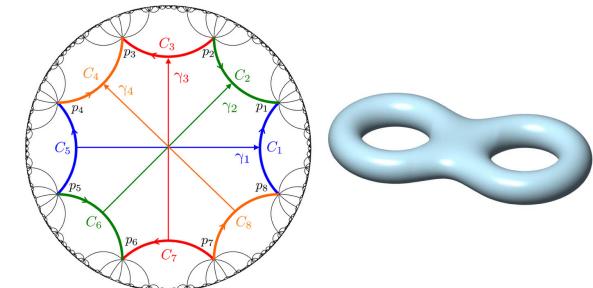
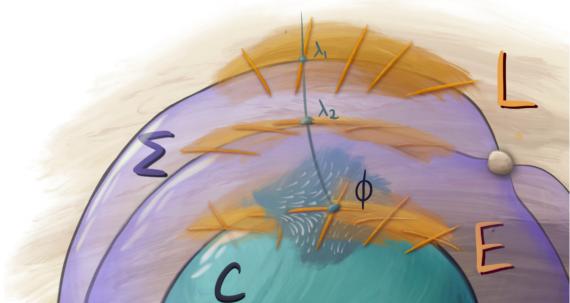
**Hyperbolic metric.** Higgs bundles naturally sit over hyperbolic base spaces. The non-Euclidean metric form  $\eta$  in the definition of our non-Hermitian symmetry is compatible with this hyperbolicity through eigenframe deformation.

In fact, the non-Abelian Hodge correspondence gives analytic isomorphisms

$$\mathcal{M}_{\text{Higgs}}^{\text{s}}(\text{SL}_n(\mathbb{C})) \cong \text{Rep}(\pi_1(\Sigma), \text{SL}_n(\mathbb{C})) \cong \mathcal{H}$$

where  $\mathcal{H}$  is the space of equivariant harmonic maps from the universal cover  $\tilde{\Sigma}$  to  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$ , modulo isometries. Here, the equivariance is with respect to a representation  $\rho: \pi_1(\Sigma) \rightarrow \text{SL}_n(\mathbb{C})$ .

Thus, given a Higgs bundle  $(E, \phi)$ , we get a harmonic map  $f: \tilde{\Sigma} \rightarrow \text{SL}_n(\mathbb{C})/\text{SU}(n)$ . The **negative semi-definite** Killing form on the Lie algebra  $\mathfrak{sl}_n(\mathbb{C})$  gives rise to a metric on  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$ , which then pulls back along  $f$  and descends to  $\Sigma$ . This metric has **non-positive** Gaussian curvature (**constant negative** in favorable cases).



## Moduli of Higgs bundles

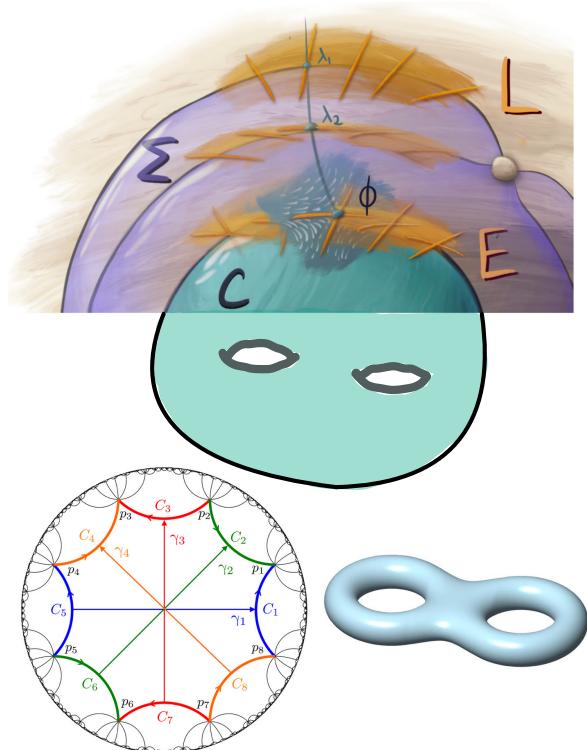
**Hyperbolic metric.** Higgs bundles naturally sit over hyperbolic base spaces. The non-Euclidean metric form  $\eta$  in the definition of our non-Hermitian symmetry is compatible with this hyperbolicity through eigenframe deformation.

In fact, the non-Abelian Hodge correspondence gives analytic isomorphisms

$$\mathcal{M}_{\text{Higgs}}^{\text{s}}(\text{SL}_n(\mathbb{C})) \cong \text{Rep}(\pi_1(\Sigma), \text{SL}_n(\mathbb{C})) \cong \mathcal{H}$$

where  $\mathcal{H}$  is the space of equivariant harmonic maps from the universal cover  $\tilde{\Sigma}$  to  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$ , modulo isometries. Here, the equivariance is with respect to a representation  $\rho: \pi_1(\Sigma) \rightarrow \text{SL}_n(\mathbb{C})$ .

Thus, given a Higgs bundle  $(E, \phi)$ , we get a harmonic map  $f: \tilde{\Sigma} \rightarrow \text{SL}_n(\mathbb{C})/\text{SU}(n)$ . The **negative semi-definite** Killing form on the Lie algebra  $\mathfrak{sl}_n(\mathbb{C})$  gives rise to a metric on  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$ , which then pulls back along  $f$  and descends to  $\Sigma$ . This metric has **non-positive** Gaussian curvature (**constant negative** in favorable cases).



## Bulk-edge correspondence

We have been **experimentally** investigating the *bulk–edge correspondence* for hypersurface singularities stemmed from our theoretical analysis with the swallowtail moduli spaces

## Bulk–edge correspondence

We have been **experimentally** investigating the **bulk–edge correspondence** for hypersurface singularities stemmed from our theoretical analysis with the swallowtail moduli spaces, i.e., the topology of **bulk states** (parametrized system) corresponds to the numerology of **edge states** (moduli space).

## Bulk-edge correspondence

We have been **experimentally** investigating the **bulk-edge correspondence** for hypersurface singularities stemmed from our theoretical analysis with the swallowtail moduli spaces, i.e., the topology of **bulk states** (parametrized system) corresponds to the numerology of **edge states** (moduli space).

*More precisely, e.g., the bulk-edge correspondence relates a topological invariant of the bulk insulator*

## Bulk-edge correspondence

We have been **experimentally** investigating the **bulk–edge correspondence** for hypersurface singularities stemmed from our theoretical analysis with the swallowtail moduli spaces, i.e., the topology of **bulk states** (parametrized system) corresponds to the numerology of **edge states** (moduli space).

*More precisely, e.g., the bulk–edge correspondence relates a topological invariant of the bulk insulator (the **first Chern number of the Bloch eigenbundle**, also called the Hall conductance)*

## Bulk-edge correspondence

We have been **experimentally** investigating the **bulk-edge correspondence** for hypersurface singularities stemmed from our theoretical analysis with the swallowtail moduli spaces, i.e., the topology of **bulk states** (parametrized system) corresponds to the numerology of **edge states** (moduli space).

*More precisely, e.g., the bulk-edge correspondence relates a topological invariant of the bulk insulator (the **first Chern number of the Bloch eigenbundle**, also called the Hall conductance) with an invariant of a surface state*

## Bulk-edge correspondence

We have been **experimentally** investigating the **bulk–edge correspondence** for hypersurface singularities stemmed from our theoretical analysis with the swallowtail moduli spaces, i.e., the topology of **bulk states** (parametrized system) corresponds to the numerology of **edge states** (moduli space).

*More precisely, e.g., the bulk–edge correspondence relates a topological invariant of the bulk insulator (the **first Chern number of the Bloch eigenbundle**, also called the Hall conductance) with an invariant of a surface state (the **winding number** about the Fermi energy in the complex **Bloch variety**).*

## Bulk-edge correspondence

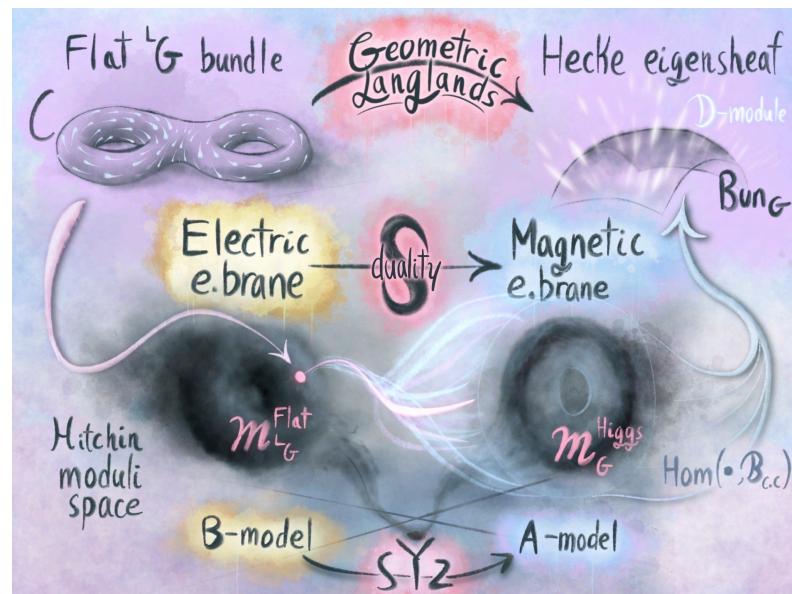
We have been **experimentally** investigating the **bulk–edge correspondence** for hypersurface singularities stemmed from our theoretical analysis with the swallowtail moduli spaces, i.e., the topology of **bulk states** (parametrized system) corresponds to the numerology of **edge states** (moduli space).

*More precisely, e.g., the bulk–edge correspondence relates a topological invariant of the bulk insulator (the **first Chern number of the Bloch eigenbundle**, also called the Hall conductance) with an invariant of a surface state (the **winding number** about the Fermi energy in the complex **Bloch variety**). Moreover, any topological invariant is determined from the band structure over the **nilpotent cone**.*

## Bulk-edge correspondence

We have been **experimentally** investigating the **bulk-edge correspondence** for hypersurface singularities stemmed from our theoretical analysis with the swallowtail moduli spaces, i.e., the topology of **bulk states** (parametrized system) corresponds to the numerology of **edge states** (moduli space).

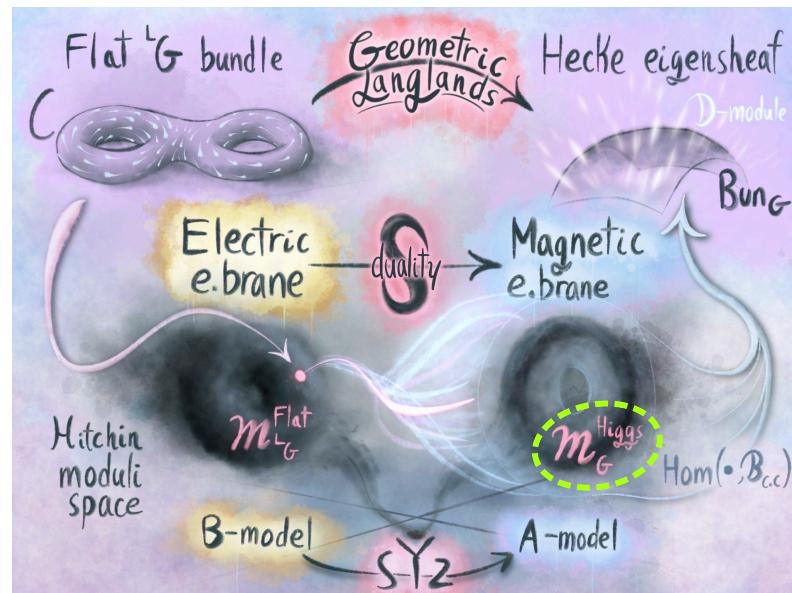
There has not been a rigorous mathematical explanation for such a correspondence in general, but it is reminiscent of the **Langlands duality**.

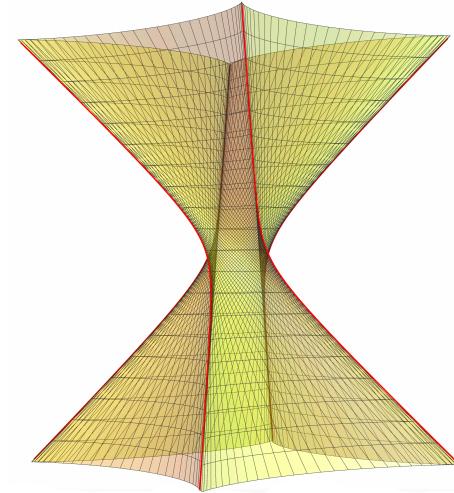
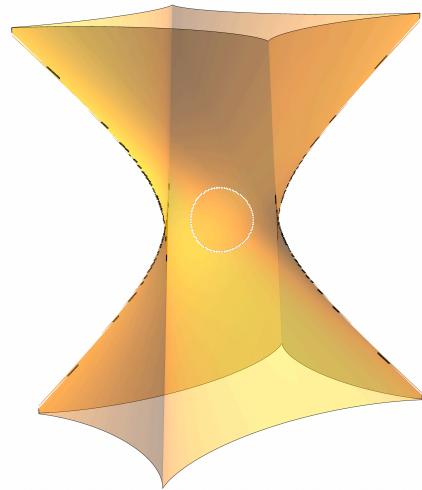
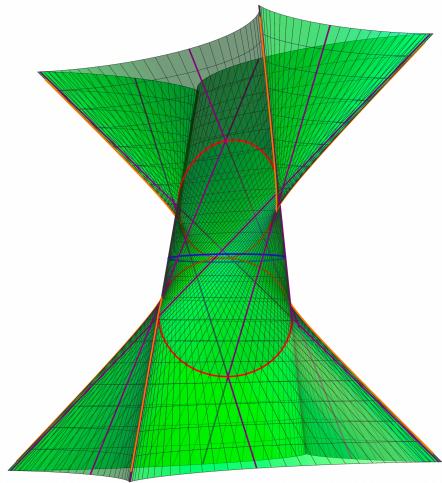


## Bulk-edge correspondence

We have been **experimentally** investigating the **bulk-edge correspondence** for hypersurface singularities stemmed from our theoretical analysis with the swallowtail moduli spaces, i.e., the topology of **bulk states** (parametrized system) corresponds to the numerology of **edge states** (moduli space).

There has not been a rigorous mathematical explanation for such a correspondence in general, but it is reminiscent of the **Langlands duality**. Indeed, **Higgs bundles** sit on one side of the geometric Langlands duality! We've at least found some **testing ground**.





*Thank you.*

