Fundamental thermodynamic limit to the accuracy of quantum processors

Accurate processing of information is crucial both in nature and in technology. To achieve it, any information-processing machine needs an initial supply of systems away from thermal equilibrium. Here we establish a limit of principle on the maximum accuracy achievable with a given amount of nonequilibrium resources. The limit is valid for quantum systems, and is expressed in terms of an entropic quantity, which we call the thermodynamic complexity. When applied to the task of replicating quantum information, the limit reveals a thermodynamic advantage of coherent quantum information processing over all machines equipped with a purely classical memory. Our results shed light on the interplay between thermodynamic resources and information-theoretic tasks, and in the long term can inform the design of quantum devices approaching the ultimate efficiency limits.

INTRODUCTION

Many processes in nature depend on accurate processing of information. For example, the development of complex organisms relies on the replication of the information contained in their DNA. This process takes place with high accuracy, its error rate being estimated to be less than one basis per billion [1].

At the fundamental level, information is stored into patterns that stand out from the thermal fluctuations of the surrounding environment [2, 3]. In order to achieve deviations from thermal equilibrium, any information processing machine needs an initial supply of systems in a non-thermal state [4–6]. For example, an ideal copy machine for classical data requires at least a clean bit for every bit it copies [7–9]. These clean bits constitute an initial resource needed to carry out the replication process without errors.

The minimum amount of clean bits required by an information processing task is a lower bound to the amount of work required to carry out that task starting from reservoirs at thermal equilibrium. The reason is that, in order to generate clean bits from a thermal reservoir, a machine would need to erase thermal fluctuations, and Landauer's principle stipulates that the erasure of a bit has a work cost of at least $kT\ln 2$, where k is the Boltzmann constant and T is the temperature of the heat bath [8, 10]. For the task of replicating information, this implies that copying an n-bit string without errors requires at least $nkT\ln 2$ work. This lower bound determines the ultimate thermodynamic efficiency limit for every classical replication process, from the copy of a DVD to the replication of DNA [11, 12].

The above analysis, however, is inadequate at the quantum scale, where a perfect replication of non-orthogonal states is forbidden by the no-cloning theorem [13, 14]. As a consequence, quantum cloning machines cannot achieve perfect accuracy [15–17]. The question of thermodynamic efficiency then becomes a question about the ultimate tradeoff between accuracy and nonequilibrium: what is the minimum amount of nonequilibrium needed to copy quantum data with a target level of accuracy?

In recent years, there has been a growing interest in the interplay between quantum information processing and thermodynamics [18–20], motivated both by fundamental questions [6, 21–24] and by the experimental realisation of new quantum devices [25–27]. Research in this area led to the development of resource theoretic frameworks that can be used to study thermodynamics beyond from the macroscopic limit. These frameworks have been applied to the characterisation of the thermodynamically allowed state transitions [28–36], of the work cost of logical operations [37, 38], and to the study of erasure and work extraction [39–41].

Here, we use the resource theoretic approach to establish a limit on the accuracy of quantum information-processing machines. The limit is expressed in terms of an entropic quantity, which we call the thermodynamic complexity, and is attainable in a broad class of tasks, including the erasure and replication of classical and quantum information. For quantum replication, we find that achieving the ultimate limit requires machines with an ability to preserve quantum entanglement: every copy machine that breaks entanglement will necessarily be unable to achieve the maximum accuracy for a given amount of nonequilibrium resources. Our argument can be extended also to other tasks, providing a link between the thermodynamical efficiency of quantum machines and their ability to transfer quantum entanglement.

RESULTS

Thermodynamic complexity of a given information processing task. At the fundamental level, the goal of information processing is to induce a desired relation between an input and an output. For example, a deterministic classical computation amounts to transforming a bit string x into another bit string $f(x)$, where f is a given function. Information processing tasks are often associated to ideal state transformations $\rho_x \mapsto \rho'_x$, in which an input state ρ_x has to be converted into a target output state ρ'_x, x being a parameter in some given set X.

Since every realistic machine is subject to imperfections, the physical realisations of an ideal information processing task can have varying levels of accuracy. For an ideal transformation $\rho_x \mapsto \rho'_x$ with pure target

states ρ'_x, the accuracy can be quantified by the fidelity $F(\mathcal{M}) = \min_x \mathrm{Tr}[\rho'_x \mathcal{M}(\rho_x)]$, where $\mathcal{M}$ is the quantum channel (completely positive trace-preserving map) describing the action of the machine. Other types of accuracy measures and other types of state transformations are discussed in Methods.

We now introduce an entropic quantity that measures the thermodynamic hardness of a given information processing task. The motivation for this quantity comes from the structure of an optimisation problem discussed in Methods. For a state transformation with pure target states, the *thermodynamic complexity* κ is defined as

$$\kappa = \max_{\mathbf{p}} H_{\min}(A|B)_{\omega_{\mathbf{p}}}\,, \tag{1}$$

where the maximum ranges over all probability distributions $\mathbf{p} = (p_x)_{x\in\mathsf{X}}$, and $H_{\min}(A|B)_{\omega_{\mathbf{p}}} := -\log\min\{\mathrm{Tr}[\Lambda_B] \mid (I_A \otimes \Lambda_B) \geq \omega_{\mathbf{p}}\}$ is the conditional min-entropy [42–44] of the operator $\omega_{\mathbf{p}} := \sum_x p_x \Gamma_A^{-\frac{1}{2}} \rho_x^T \Gamma_A^{-1/2} \otimes \Gamma_B^{\frac{1}{2}} \rho'_x \Gamma_B^{\frac{1}{2}}$, where Γ_A and Γ_B are the Gibbs states of the input and output systems, labelled as A and B, respectively, and ρ_x^T is the transpose of the matrix ρ_x with respect to the energy basis (note that the operator $\omega_{\mathbf{p}}$ it is generally not a density matrix). Here and in the following, we take log to be the logarithm in base 2.

Crucially, κ depends only on the information processing task under consideration, and not on the specific quantum channel used to implement the task. In fact, the thermodynamic complexity is well defined even for tasks that cannot be perfectly achieved by any physical process, as in the case of ideal quantum cloning, and even for tasks that are not formulated in terms of state transitions. The extension of definition (1) general information processing tasks is discussed in to the Methods section.

To gain some intuition into the definition of the thermodynamic complexity, it is useful to apply it in some special cases. Consider first the case of a deterministic classical computation, corresponding to the evaluation of a function $y = f(x)$. In Supplementary Note 1, we show that the thermodynamic complexity (1) is given by the expression

$$\kappa_f = D_{\max}(p_f \| g_B) \tag{2}$$

where $D_{\max}(p\|q)$ is the max Rényi divergence between two probability distributions p and q [45], $g_B(y)$ is Gibbs distribution for the output system, and $p_f(y)$ is the probability distribution of the output of the function f when the input is sampled from the Gibbs distribution. Eq. (2) shows that the thermodynamical complexity of a classical computation is a measure of how much the computation transforms thermal fluctuations into an output that deviates from thermal equilibrium.

In general, however, the thermodynamic complexity is not just a measure of deviation from thermal equilibrium. An important feature of definition (1) is that it also captures genuinely quantum limitations to information processing. Consider for example the transposition task $\rho_x \mapsto \rho_x^T$ [46–51], where x parametrises all the possible pure states of a quantum system, and ρ_x^T is the transpose of the density matrix ρ_x with respect to the energy eigenbasis. This transformation does not generate any deviation from equilibrium, as it maps the Gibbs state into itself. On the other hand, a perfect transposition is forbidden by the laws of quantum mechanics. In Supplementary Note 1, we show that the thermodynamic complexity of the transposition is $\kappa_{\mathrm{trans}} = \log[(d+1)/2]$ for a d-dimensional system with fully degenerate energy levels. More generally, we show that the thermodynamic complexity is strictly positive for every impossible state transformation whose input states form an ensemble decomposition of the Gibbs state.

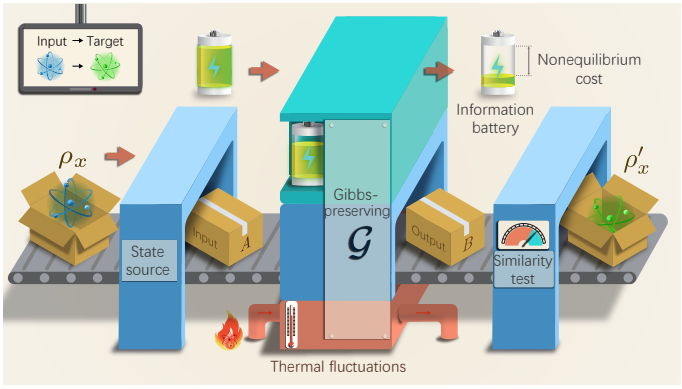

FIG. 1. **Performance test for an information processing machine.** A source generates the input states for the machine. The machine then uses thermal fluctuations and an information battery to transform the input states into an approximation of the ideal target states. Finally, the similarity between the output and the target states is assessed by a measurement. The number of clean qubits consumed by the machine is the nonequiibrium cost that needs to be payed in order to achieve the desired level of accuracy.

Limit on the accuracy of general quantum machines. Accurate information processing generally requires an initial supply of systems away from equilibrium. The amount of nonequilibrium can be rigorously quantified in a resource theoretic framework where equilibrium (Gibbs) states are regarded as freely available, and the only operations that can be performed free of cost are the Gibbs-preserving channels [34, 38], that is, the largest class of quantum channels that preserve the condition of thermal equilibrium. In this framework, the initial nonequilibrium resources can be put into a canonical form by introducing an information battery [37, 38]. The battery starts off with some qubits in a pure state, while all the remaining qubits are in the maximally mixed state. To implement the desired information processing task, the machine will operate jointly on the input system and on the information battery, as illustrated in

Figure 1. Once the machine has operated, the number of clean qubits in the battery will generally change.

The minimum number of clean qubits consumed in the execution of a task depends on the logical operation implemented by the machine, and on the input states on which the machine is required to operate [37, 38]. If the input states are contained in a proper subspace of the system's Hilbert space, it is sometimes possible to extract clean qubits from the input [31, 39]. When the input subspace is spanned by energy eigenstates, an exact expression for the minimum number of clean qubits consumed by the machine can be derived from Ref. [38], which yields

$$c(\mathcal{M}) = D_{\max}(\mathcal{M}(\Pi_A \Gamma_A \Pi_A) \,\|\, \Gamma_B)\,, \tag{3}$$

where $\mathcal{M}$ is the quantum channel describing the action of the machine, $c(\mathcal{M})$ is the minimum number of clean qubits consumed in the execution of $\mathcal{M}$, Π_A is the projector on the subspace containing the inputs of the channel, and $D_{\max}$ is the max-relative entropy [52] (see Supplementary Note 2).

The minimum number of clean qubits consumed by the machine, hereafter called the *nonequilibrium cost*, provides a fundamental lower bound to the amount of work that has to be invested in the realisation of the machine when using thermal operations [34]. Explicitly, the bound reads $W(\mathcal{M}) \geq kT \ln 2\, c(\mathcal{M})$ where $W(\mathcal{M})$ is the work cost of the machine.

The main question of this paper is: what is the minimum amount of nonequilibrium needed to achieve a desired level of accuracy? Or equivalently, what is the maximum accuracy accessible with a given amount of nonequilibrium? The answer to the both questions can be found by minimising the cost over all quantum machines that achieve accuracy above a given threshold F: matematically, the problem is to compute the minimum nonequilibrium cost $c = \min\{c(\mathcal{M}) \mid F(\mathcal{M}) \geq F\}$.

In the Methods section, we provide a semidefinite program for the exact value of c. For low dimensional systems, the semidefinite program can be numerically solved, thus providing the exact tradeoff between nonequilibrium and accuracy. Still, brute-force optimisation is intractable for high dimensional systems. For this reason, it is important to have computable bounds that can be applied in a broad spectrum of tasks. The central result of our paper is the following theorem:

Theorem 1. *For any information processing task where the inputs are constrained to a subspace spanned by energy eigenstates, the minimum nonequilibrium cost for achieving accuracy F satisfies the bound*

$$c \geq \kappa + \log F\,, \tag{4}$$

where κ is the thermodynamic complexity defined in Eq. (1).

The derivation is outlined in Methods, and the full details are provided in Supplementary Note 2.

The bound (4) implies that the amount of work needed to achieve accuracy F using thermal operations satisfies the inequality $W \geq kT \ln 2\,(\kappa + \log F)$. Eq. (4) can also be equivalently formulated as a limit on the accuracy attainable with a given budget of nonequilibrium resources: for a given number of clean bits/qubits, one has the bound

$$F \leq 2^{c-\kappa}\,. \tag{5}$$

This bound represents an in-principle thermodynamic constraint on the performance of every information processing machine.

Condition for achieving the limit. The appeal of the bounds (4) and (5) is that they are general and easy to use. But when are they attainable? Here we give a simple attainability criterion, which we show to be satisfied in a number of situations.

To discuss attainability of the bounds (4) and (5), it is important to first identify the parameter range in which these bounds are meaningful. First of all, the bound (4) is only meaningful when the desired accuracy does not exceed the maximum accuracy allowed by the laws of physics, hereafter denoted by $F_{\max}$.

Similarly, the bound (5) is only meaningful if the initial amount of nonequilibrium resources is no less than the minimum nonequilibrium cost over all possible physical processes, hereafter denoted by $c_{\min}$. By maximising the accuracy over all quantum channels with minimum cost $c_{\min}$, we then obtain a minimum value $F_{\min}$ below which reducing the accuracy does not result in any reduction of the nonequilibrium cost.

We now provide a criterion that guarantees the attainability of the bounds (4) and (5) in the full interval $[F_{\min}, F_{\max}]$. Since the two bounds are equivalent to one another, we will focus on bound (4).

Theorem 2. *If the bound (4) is attainable for a value of the accuracy F_0, then it is attainable for every value of the accuracy in the interval $[F_{\min}, F_0]$.*

In particular, the above theorem guarantees that the bound (4) is attainable for every value of the accuracy in the interval $[F_{\min}, F_{\max}]$ whenever it is attainable for the maximum value $F_{\max}$. In Supplementary Note 3 we prove the theorem by explicitly constructing a family of channels that achieve the bound (4).

Combining our attainability criterion with Equation (3), one can prove the attainability of the bound (4) for a variety of different tasks. For example, the bound (4) is attainable for every deterministic classical computation. Moreover, it is achievable for every quantum extension of a classical computation. A quantum extension of a classical function f is any quantum channel $\mathcal{M}$ satisfying the condition $\mathcal{M}(|x\rangle\langle x|) = |f(x)\rangle\langle f(x)|$, for

every x labelling the energy eigenstates. The corresponding information processing task is to transform a generic input state ρ_x into the target output state $\rho'_x = \mathcal{M}(\rho_x)$. In Supplementary Note 4 we show that the bound (4) is achievable for every such task: for every value of the accuracy, the nonequilibrium cost is the same for the classical computation and for its quantum extension, and therefore the achievability condition holds in both cases.

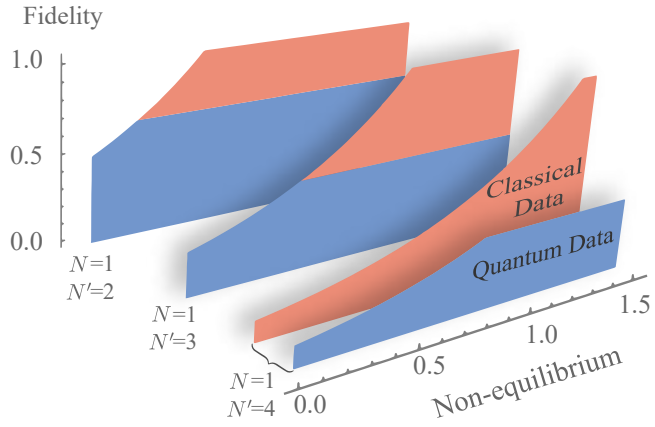

FIG. 2. **Maximum cloning fidelity for a given amount of nonequilibrium resources.** The optimal accuracy-nonequilibrium tradeoff is depicted for $1 \to 2$, $1 \to 3$, and $1 \to 4$ cloning machines. The fidelities for copying classical (red region) and quantum data (blue region) are limited by the same boundary curve, except that the fidelity of copying quantum data cannot reach to 1 due to the no-cloning theorem.

Nonequilibrium cost of classical replication. The task of information replication is to transform N identical copies of a generic pure state into $N' \geq N$ copies of the same state. The thermodynamic complexity of this task can be computed from Eq. (2), which gives

$$\kappa^{\mathrm{C}}_{\mathrm{clon}} = \frac{\Delta N\, \Delta A}{kT\, \ln 2}\,, \tag{6}$$

where $\Delta N := N' - N$ the number of extra copies, and ΔA is the maximum difference between the free energy of a single-copy pure state and the free energy of the single-copy Gibbs state. Physically, $\kappa^{\mathrm{C}}_{\mathrm{clon}}$ coincides with maximum amount of work needed to generate ΔN copies of a pure state from the thermal state [31].

Since replication is a special case of a deterministic classical computation, the results established in the previous section guarantee that the bound (4) is attainable for every value of the fidelity in the interval $[F_{\min}, F_{\max}]$, with $F_{\max} = 1$. In this interval, the minimum nonequilibrium cost of classical cloning is exactly

$$c^{\mathrm{clon}}_{\min} = \frac{\Delta N\, \Delta A}{kT\, \ln 2} + \log F\,. \tag{7}$$

This result generalizes previous results on the thermodynamics of classical replication [8, 10, 11], extending them to the approximate scenario and to the case of classical systems with non-degenerate energy levels.

Nonequilibrium cost of quantum replication. We now consider the task of replicating quantum information. By definition, the thermodynamic complexity of quantum cloning is at least as large as the thermodynamic complexity of classical cloning, namely $\kappa^{\mathrm{Q}}_{\mathrm{clon}} \geq \kappa^{\mathrm{C}}_{\mathrm{clon}}$. This bound follows immediately from Eq. (1), by restricting the optimization to probability distributions that are concentrated on the eigenstates of the energy.

In Supplementary Note 5, we show that *(i)* the bound (4) is attainable for quantum cloning, and *(ii)* the quantum and classical thermodynamic complexities coincide. These results imply that classical and quantum replication exhibit the same tradeoff between accuracy and nonequilibrium: for every value of the accuracy, the minimum nonequilibrium cost of information replication is given by Eq. (7) both in the classical and in the quantum case. In terms of accuracy/nonequilibrium tradeoff, the only difference between classical and quantum cloning is that the classical tradeoff curve goes all the way up to unit fidelity, while the quantum tradeoff curve stops at a maximum fidelity, which is strictly smaller than 1 due to the no-cloning theorem [13, 14].

Considering the differences between quantum and classical cloning, the fact that these two tasks share the same tradeoff curve is quite striking. Nevertheless, in the following we will show that quantum and classical cloning exhibit a fundamental difference in the way the ultimate thermodynamic limit is achieved: to be thermodynamically efficient, quantum machines cannot use classical copying strategies.

Limit on the accuracy of entanglement binding machines. Classical copy machines scan the input copies and produce replicas based on this information. Similarly, a classical machine for a general task can be modelled as a machine that measures the input and produces an output based on the measurement result. When this approach is used at the quantum scale, it leads to a special class of quantum machines, known as entanglement breaking [53]. Here we show that entanglement breaking copy machines cannot achieve the ultimate efficiency limit set by Eq. (7). To prove this result, we establish a stricter efficiency limit for a broader class of machines, called entanglement binding [54].

An entanglement binding channel is a quantum channel that transforms every entangled state into a bound entangled state [55, 56], from which no pure entangled states can be extracted. In Methods, we show that entanglement binding channels generally obey a more stringent bound than (4): the nonequilibrium cost of an entangle-

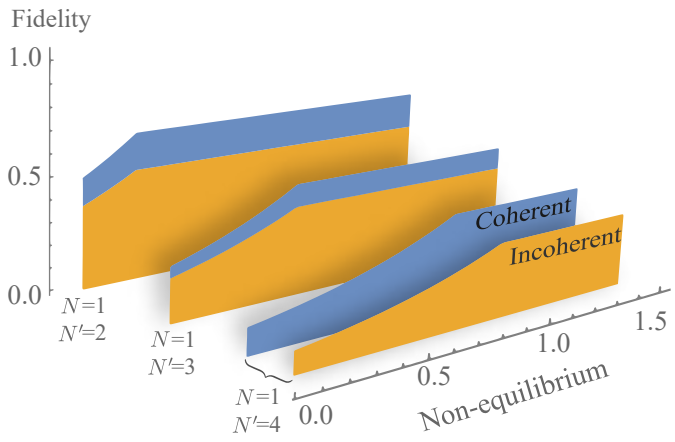

FIG. 3. **Comparison between entanglement binding machines and general quantum machines.** The figure illustrates the accessible regions for the cloning fidelity for various values of N and N' in the case of qubits with degenerate Hamiltonian. The values of the fidelity in the blue region are attainable by general quantum machines, while the values in the orange region are attainable by entanglement binding machines. The difference between the two regions indicates a thermodynamic advantage of quantum machines over all machines equipped with a purely classical memory.

ment binding machine is lower bounded as

$$c^{(\mathrm{eb})} \geq \max\{\kappa, \kappa^*\} + \log F\,, \tag{8}$$

where κ is the thermodynamic complexity for the state transformation $\rho_x \to \rho'_x$, and κ^* is the thermodynamic complexity for the transpose transformation $\rho_x \mapsto (\rho'_x)^T$.

This bound can be used to demonstrate quantum-enhanced accuracy under a constraint on the available nonequilibrium resources, as shown in the following section.

Quantum advantage in replication. For quantum replication, it turns out that no entanglement binding machine can achieve the ultimate thermodynamic limit. The reason for this is that the thermodynamic complexity of the transpose task is larger than the thermodynamic complexity of the direct task, namely $\kappa^*_{\mathrm{clon}} > \kappa_{\mathrm{clon}}$. In Supplementary Note 6 we prove the inequality

$$\kappa^*_{\mathrm{clon}} \geq \kappa_{\mathrm{clon}} + \log \frac{d_{N+N'}\, e^{-\frac{N'\Delta E}{kT}}}{d_{N'}}\,, \tag{9}$$

where ΔE is the difference between the maximum and minimum energy, and, $d_K = (K+d-1)!/[K!(d-1)!]$ for $K = N$ or $K = N+N'$. Inserting this inequality into Eq. (8), we conclude that every entanglement binding machine necessarily requires a larger number of clean qubits compared to the most general quantum machines.

When the energy levels are fully degenerate, we show that the bounds (8) and(9) are exact equalities. With this result at hand, we can compare the exact performance of entanglement binding machines and general quantum machines, showing that the latter achieve a higher accuracy for every given amount of nonequilibrium resources. The comparison is presented in Figure 3.

Our result shows that entanglement binding machines are thermodynamically inefficient for the task of information replication. Achieving the ultimate efficiency limit requires machines that are able to preserve free (*i. e.* non-bound) entanglement. This observation fits with the known fact that the preservation of entanglement is necessary to achieve the maximum accuracy allowed by quantum mechanics [16, 17, 57]. Here we have shown that not only entanglement binding machines are limited in their accuracy, but also that, to achieve such limited accuracy, they require a higher amount of nonequilibrium resources. Interestingly, the thermodynamic advantage of general quantum machines vanishes in the asymptotic limit $N' \to \infty$, in which the optimal quantum cloning can be reproduced by state estimation [58–60], which corresponds to a special case of entanglement binding machine.

DISCUSSION

An important feature of our bound (4) is that it applies also to state transformations that are forbiden by quantum mechanics, such as ideal quantum cloning or ideal quantum transposition. For state transformations that can be implemented perfectly, instead, it is interesting to compare our bound with related results in the literature. Existing bounds are typically on the work cost, evaluated in a resource-theoretic framework where the free operations are *thermal operations* [31], resulting from a joint energy-preserving evolution of the system together with auxiliary systems in the Gibbs state.

For example, consider the problem of generating a state ρ from the equilibrium state. By choosing a suitable figure of merit (see Methods for the details), we find that the nonequilibrium cost for the state transition $\Gamma \mapsto \rho$ is equal to $D_{\max}(\rho\|\Gamma)$, and coincides (up to the usual proportionality constant $kT\ln 2$) with the minimal amount of work needed to generate the state ρ without errors [31]. Similarly, one can consider the task of extracting work from the state ρ, in which case Ref. [31] showed that the maximum extractable work is $kT\ln 2\, D_{\min}(\rho\|\Gamma)$, where $D_{\min}$ is the min relative entropy introduced by Datta in Ref. [52]. This value can also be retrieved from our bound with a suitable choice of figure of merit (see Supplementary Note 7 for the details).

For a state general transformation $\rho_x \to \rho'_x$, a bound on the work cost can be deduced from Ref. [32], which adopted a different resource theoretic framework, restricting to thermal operations but allowing for catalysts. When the input and output states are diagonal in the en-

ergy basis, Ref. [32] implies the bound

$$W \geq kT \ln 2 \max_x \max_\alpha D_\alpha(\rho'_x \| \Gamma_B) - D_\alpha(\rho_x \| \Gamma_A)\,, \quad (10)$$

where $D_\alpha(\rho\|\sigma) := \log \mathrm{Tr}\big[\rho^\alpha \sigma^{1-\alpha}\big]/(\alpha-1)$ are α-Rényi relative entropies. The bound (10) can be compared with Eq. (4), which yields the bound $\kappa \geq \max_x D_{\max}(\rho'_x\|\Gamma_B) - D_{\max}(\rho_x\|\Gamma_A)$ for a suitable choice of figure of merit (see Methods). The resulting bound on the work cost is generally lower than Eq. (10) and the two bounds coincide when the input and output states are pure, or, more generally, when they have well-defined energy. Notably, they coincide when the Hamiltonians of systems A and B are both degenerate.

On the other hand, our bound (4) gives better estimates of the work cost for impossible quantum tasks like the universal transpose $\rho_x \to \rho_x^T$. In this case, the change in relative entropy from the input to the ideal output is zero, both for the α-Rényi relative entropies and for other quantum generalisations of the Rényi relative entropies [61–64]. In contrast, our bound implies the non-trivial lower bound $W \geq kT \ln 2\,(\log[(d+1)/2\, e^{-\frac{\Delta E}{kT}}] + \log F)$, which reduces to $W \geq \Delta E$ when setting $F = 2/(d+1)$, which is the maximum fidelity of transposition [46–48].

While the applications discussed in the paper focussed on one-shot tasks, our results also apply to the asymptotic scenario where the task is to implement the transformation $\rho_x^{\otimes n} \mapsto \rho_x'^{\otimes n}$ in the large n limit. In Methods we consider the amount of nonequilibrium per copy required by this transformation, allowing for small deviations in the input and output states. This setting leads to the definition of a smooth thermodynamic complexity, whose value per copy is denoted by κ_{iid} and is shown to satisfy the bound

$$\kappa_{\mathrm{iid}} \geq \max_x S(\rho'_x\|\Gamma_B) - S(\rho_x\|\Gamma_A)\,, \quad (11)$$

where $S(\rho\|\sigma) := \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$ is the quantum relative entropy.

In the special case where the state transformation $\rho_x \to \rho'_x$ can be implemented perfectly, and where $(\rho_x)_{x\in\mathsf{X}}$ is the set of all possible quantum states of the input system, the r.h.s. of Eq. (11) is coincides with the thermodynamic capacity introduced in by Faist, Berta, and Brandão in Ref. [65]. In this setting, the results of Ref. [65] imply that the thermodynamic capacity coincides with the amount of work per copy needed to implement the channel $\mathcal{M}$ that implements the transformation $\rho_x \to \rho'_x$. Since the amount of work cannot be smaller than the nonequilibrium requirement, this result implies that our bound is asymptotically achievable whenever the ideal task $\rho_x \to \rho'_x$ can be implemented without errors.

METHODS

General performance tests. The performance of a machine in a given information processing task can be operationally quantified by the probability to pass a test [66–68]. For example, a simple test is to check whether the output state has the same support of the target state. Operationally, the test is implemented by randomly choosing an input state ρ_x, letting the machine act, and then measuring the output using a projective measurement with two orthogonal projectors P_x and $P_x^\perp := I - P_x$, projecting onto the support of the target state ρ'_x and its orthogonal subspace, respectively. When the system is projected on the support of ρ'_x, the test is passed. Otherwise, the test is failed. The probability to pass such a test gives an indication of how close the actual output is to the target. When the target states are pure, the probability of passing this test is equal to the fidelity used in the main text.

Performance tests provide a more general way to define information processing tasks. Rather than specifying a desired state transformation $\rho_x \mapsto \rho'_x$, one can specify a test that assigns a score to the machine. Then, the information processing task is simply to maximise the expected score. In the one-shot scenario, a general test consists in preparing states of a composite system AR, consisting of the input of the machine and an additional reference system. The machine is requested to act on system A, while the reference system undergoes the identity process, or some other known process. Finally, the output systems undergo a joint measurement, described by a suitable observable whose outcomes give a score the the machine. The test is then described by the possible pairs $(\rho_x, O_x)_{x\in\mathsf{X}}$, consisting of an input state and an observable. In the worst case over all possible pairs, one gets the accuracy $F := \min_x \mathrm{Tr}[O_x\,(\mathcal{M}\otimes\mathcal{N})(\rho_x)]$, where $\mathcal{M}$ is the channel describing the machine's action, and $\mathcal{N}$ is the fixed channel acting on the reference system.

Performance tests can be expressed in a compact way via the Choi representation, which associates each channel $\mathcal{M}$ to its Choi operator $M := (\mathcal{I}_A \otimes \mathcal{M})(|I_A\rangle\langle I_A|)$, where $|I_A\rangle := \sum_i |i\rangle\otimes|i\rangle$ is the canonical (unnormalised) maximally entangled state [69]. In this representation, the performance test is described by a set of operators $(\Omega_x)_{x\in\mathsf{X}}$, which we call the *performance operators* [68], and the accuracy of the test has the simple expression $F = \min_x \mathrm{Tr}[M\,\Omega_x]$. Without loss of generality, we assume each operator Ω_x to be positive (if this is not the case, one can always consider the new performance operators $\Omega'_x = a\,I + \Omega_x$, where $a \geq 0$ is a sufficiently large positive constant that guarantees the condition $\Omega'_x \geq 0$ for every x).

Exact expression for the nonequilibrium cost. Consider the average accuracy $F_{\mathbf{p}} = \sum_x p_x\, \mathrm{Tr}[M\,\Omega_x]$ with respect to a given probability distribution $\mathbf{p} = (p_x)$. In Supplementary Note 2, we show that the minimisation of the nonequilibrium cost (3) over all machines achieving a desired level of accuracy can be reduced to a semidef-

inite program [70]. Explicitly, the minimum nonequilibrium cost has the exact expression

$$c = \log \max_{\substack{X \otimes I_B + z\,\Omega_{\mathbf{p}} \leq \Pi_A \Gamma_A \Pi_A \otimes Y \\ \mathrm{Tr}[\Gamma_B Y] \leq 1}} \mathrm{Tr}[X] + z\,F\,, \tag{12}$$

where F is the required value of the accuracy, and the maximisation runs over all Hermitian operators X (Y) acting on system A (B) and over the real number z.

A simpler optimisation problem arises by setting $X = 0$, which provides a lower bound to the nonequilibrium cost. The bound is $c \geq H(B|A)_{\omega_{\mathbf{p}}} + \log F$, with $\omega_{\mathbf{p}} := \sum_x p_x\,(\Gamma_A^{-1} \otimes \Gamma_B)^{1/2}\,\Omega_x\,(\Gamma_A^{-1} \otimes \Gamma_B)^{1/2}$. Taking the worst case over all probability distributions, one then gets the bound (4) (see Supplementary Note 2 for details).

When the test consists in the preparation of a set of states $(\rho_x)_{x\in\mathsf{X}}$ of system A, and in the measurement of a set of observables $(O_x)_{x\in\mathsf{X}}$ on system B, the thermodynamic complexity satisfies the lower bound

$$\kappa \geq \max_x\, -\log \mathrm{Tr}[\Gamma_B\, O_x] - D_{\max}(\rho_x \| \Gamma_A)\,, \tag{13}$$

with the equality when $|\mathsf{X}| = 1$. A natural choice of observable is $O_x = P_x$, where P_x is the projector on the support of the target state ρ'_x. In this case, the bound (13) becomes

$$\kappa \geq \max_x\, D_{\min}(\rho'_x \| \Gamma_B) - D_{\max}(\rho_x \| \Gamma_A)\,. \tag{14}$$

An alternative choice of observables is $O_x = \Gamma^{-1/2} |\psi_x\rangle\langle\psi_x| \Gamma^{-1/2} / \|\Gamma^{-1/2} \rho'_x \Gamma^{-1/2}\|$, where $|\psi_x\rangle$ is the normalised eigenvector corresponding to the maximum eigenvalue of $\Gamma_B^{-1/2} \rho'_x \Gamma_B^{-1/2}$. With this choice, the bound (13) becomes $\kappa \geq \max_x D_{\max}(\rho'_x|\Gamma_B) - D_{\max}(\rho_x|\Gamma_A)$, with the equality when $|\mathsf{X}| = 1$. Using this inequality, we obtain the following

Proposition 1. *If there exists a quantum channel $\mathcal{M}$ such that $\mathcal{M}(\rho_x) = \rho'_x$ for every $x \in \mathsf{X}$, then its nonequilibrium cost satisfies the bound $c(\mathcal{M}) \geq \max_x D_{\max}(\rho'_x \| \Gamma_B) - D_{\max}(\rho_x \| \Gamma_A)$.*

The proposition follows from Eq. (4) and from the observation that the channel $\mathcal{M}$ has unit accuracy in the test specified by the observables $(O_x)_{x\in\mathsf{X}}$.

Smooth thermodynamic complexity. For every information processing task, one can consider an approximate version obtained by replacing the performance operators $(\Omega_x)_{x\in\mathsf{X}}$ with a larger set $(\Omega_y)_{y\in\mathsf{Y}_\epsilon}$ where the parameter set Y_ϵ contains the original parameter set X, and reduces to it in the limit $\epsilon \to 0$. One can then define the smooth thermodynamic complexity κ_ϵ as the thermodynamic complexity associated to the new performance operators $(\Omega_y)_{y\in\mathsf{Y}_\epsilon}$. Note that, by definition, the smoothed thermodynamic complexity is at least as large as the thermodynamic complexity of the original task.

In particular, consider the test that consists in preparing an input state $\rho_x^{\otimes n}$ and measuring the observable $P_{x,n}$, where $P_{x,n}$ is the projector on the support of the target state $\rho_x'^{\otimes n}$. A natural approximation is to allow, for every $x \in \mathsf{X}$, all inputs $\rho_{y,n}$ that are ϵ-close to $\rho_x^{\otimes n}$, and all outputs $\rho_{y,n}$ that are ϵ-close to $\rho_x'^{\otimes n}$. With this choice, Eq. (14) gives the bound $\kappa_{\epsilon,n} \geq \max_x D^\epsilon_{\min}(\rho_x'^{\otimes n} \| \Gamma_B^{\otimes n}) - D^\epsilon_{\max}(\rho_x^{\otimes n} \| \Gamma_A^{\otimes n})$, where $D^\epsilon_{\min}$ and $D^\epsilon_{\max}$ are the smooth versions of $D_{\min}$ and $D_{\max}$ [52]. One can then define the regularised thermodynamic capacity $\kappa_{\mathrm{iid}} := \lim_{\epsilon\to 0} \sup_n \kappa_{\epsilon,n}/n$, where $\kappa_{\epsilon,n}$ is the smooth thermodynamic capacity. Using the relations $\lim_{\epsilon\to 0} \sup_n D^\epsilon_{\min}(\rho_x'^{\otimes n} \| \Gamma_B^{\otimes n})/n = S(\rho'_x \| \Gamma_B)$ and $\lim_{\epsilon\to 0} \inf_n D^\epsilon_{\max}(\rho_x^{\otimes n} \| \Gamma_A^{\otimes n})/n = S(\rho_x \| \Gamma_A)$ [52] we finally obtain the bound $\kappa_{\mathrm{iid}} \geq \max_x S(\rho'_x \| \Gamma_B) - S(\rho_x \| \Gamma_A)$.

Bound for entanglement binding channels. Entanglement binding channels generally satisfy a more stringent bound than (4). The derivation of this strengthened bound is as follows: first, the definition of an entanglement binding channel $\mathcal{P}$ implies that the map $\mathcal{P}^{\mathrm{PT}}$ defined by $\mathcal{P}^{\mathrm{PT}}(\rho) := [\mathcal{P}(\rho)]^T$ is a valid quantum channel. Then, note that the nonequilibrium cost of the channels $\mathcal{P}$ and $\mathcal{P}^{\mathrm{PT}}$ is the same. To see this, it is enough to use Eq. (3) and to observe that the max-relative entropy satisfies the relation $D_{\max}(\rho\|\sigma) = D_{\max}(\rho^T \| \sigma^T)$ for every pair of states ρ and σ. Since the Gibbs state is invariant under transposition with respect to the energy basis, we obtain the equality $c(\mathcal{P}) = c(\mathcal{P}^{\mathrm{PT}})$.

The second step is to note that the fidelity of the channel $\mathcal{P}$ for the task specified by the performance operators (Ω_x) is equal to the fidelity of the channel $\mathcal{P}^{\mathrm{PT}}$ for the task specified by the performance operators $(\Omega_x^{T_B})$, where T_B denotes the partial transpose over system B. Applying the bound (4) to channel $\mathcal{P}^{\mathrm{PT}}$, we then obtain the relation

$$\begin{aligned} c(\mathcal{P}) &= c(\mathcal{P}^{\mathrm{PT}}) \\ &\geq \kappa^* + \log F\,, \end{aligned} \tag{15}$$

where κ^* is the thermodynamic complexity of the task with performance operators $(\Omega_x^{T_B})$. Hence, an entanglement binding channel is subject to both bounds (4) and (15).

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