MAT7064, Topics in Geometry and Topology

Assignment 3

Due in-class on Friday, October 11

Numbered exercises are from Hatcher's "Algebraic Topology."

1. Show that fibrations are closed under retracts: If there is a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

   of spaces such that both horizontal composites are identity maps, and $X \to Y$ is a (Serre) fibration, show that $A \to B$ is a (Serre) fibration.

2. Suppose $U \subset X$ is an open subset, and let j be the inclusion map. Show that the projection $p\colon M_j \to X$ from the mapping cylinder of j to X is a Serre fibration. Hint: Use one of the major theorems from point-set topology.

3. A map $f\colon X \to Y$ is called an *acyclic Serre fibration* if, whenever we have a commutative diagram

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X \arrow[d] \\ D^{n+1} \arrow[r] & Y \end{tikzcd}$$

   we can find a lift to a map $D^{n+1} \to X$ to make the diagram commute. Show that acyclic Serre fibrations are, in particular, Serre fibrations and that they give isomorphisms on homotopy groups.

4. In condensed matter physics, quantum mechanical systems take the mathematical form of families of matrices subject to prescribed symmetries, called *Hamiltonians*. The mathematical modeling of these systems concerns in part classifying spaces of certain *eigenvector bundles*, which capture the evolution of these matrices within a family. V.I. Arnold called

them *eigenvectors fibrations* and understood their significance in the quantum Hall effect in physics in communication with S.P. Novikov, who passed away this year.

As such a classifying space for eigenvector bundles, the *order-parameter space of* 3*-band Hamiltonians with parity–time symmetry* has been identified as $\mathrm{SO}(3)/D_2$. Here D_2 is the 3-dimensional dihedral *crystallographic point group*, which contains the identity and rotations by π around three perpendicular axes, i.e., $D_2 \cong \mathrm{O}(1) \times \mathrm{O}(1)$.

Compute $\pi_1\big(\mathrm{SO}(3)/D_2\big)$. Hint: Consider the fibration $\mathrm{SO}(3) \to \mathrm{SO}(3)/D_2$ and observe that $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{RP}^3$, whose fundamental group is computable. You may need to carefully analyze a group extension problem.

This algebraic invariant gives a nonabelian *topological charge* that characterizes intersections of exceptional surfaces in the momentum space for a class of solid materials, with potential applications to sensing and lasing devices. See QS. Wu *et al.*, **Science** 2019.