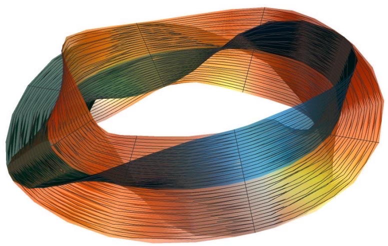
Topology and geometry of non-Hermitian Möbius bands and their deformations

Explicit examples of rank-2 and rank-3 Higgs bundles in the contexts of quantum materials and geometric Langlands program



Yifei Zhu

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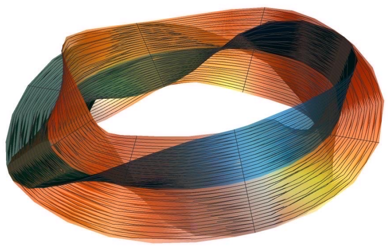

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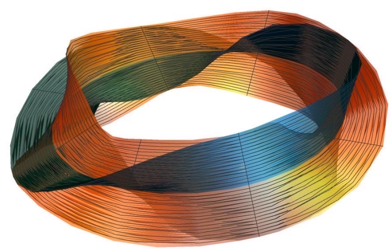

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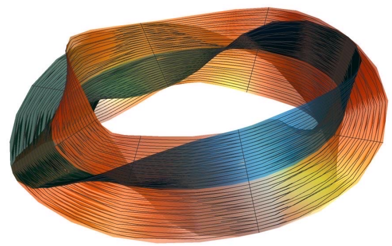

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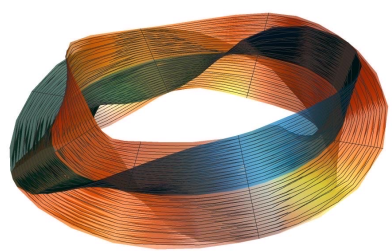

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Motivations: Quantum materials and their math modeling

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Holography, optical devices,
absorption devices, …

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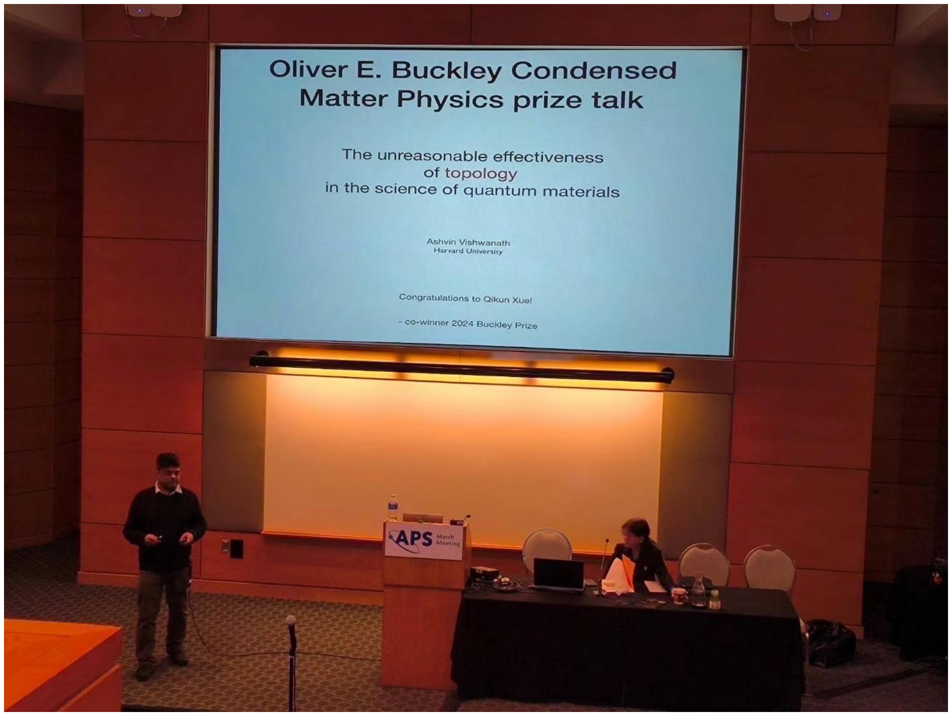

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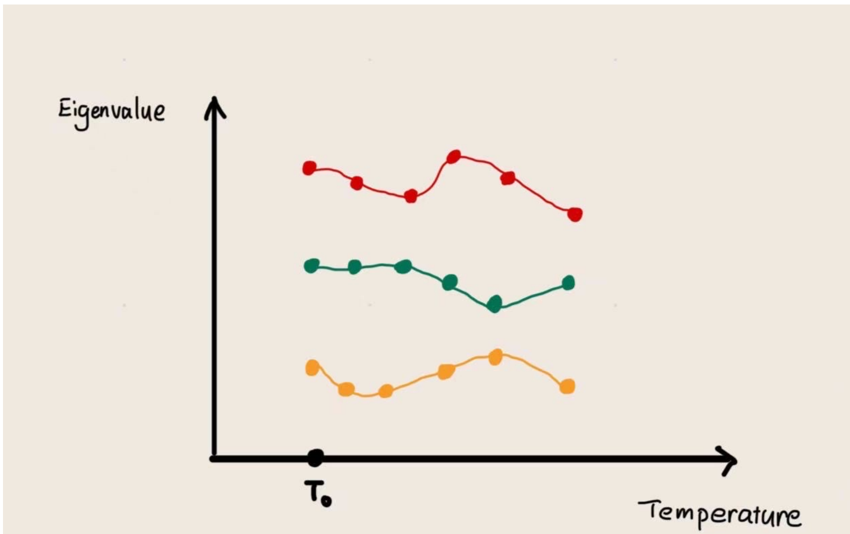
Mathematical modeling of electronic energy *band structures* therein



energy $\longleftrightarrow$ *eigenvalue*

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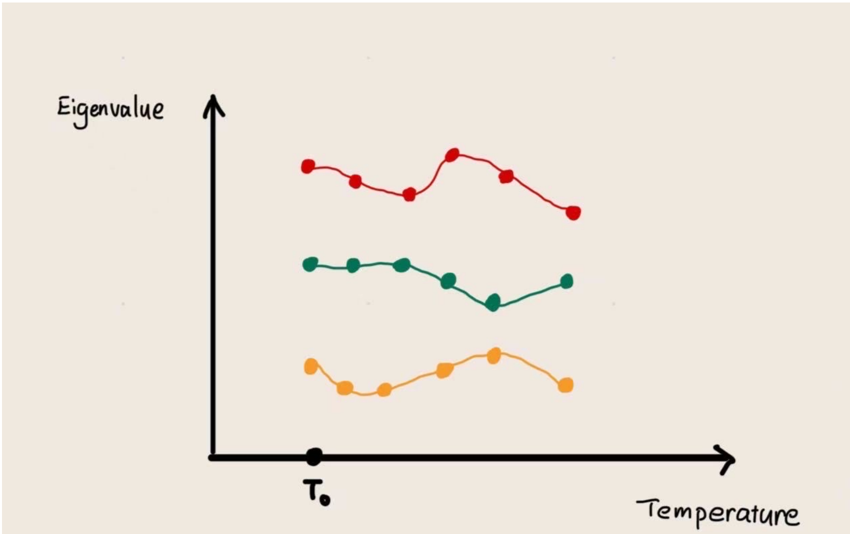

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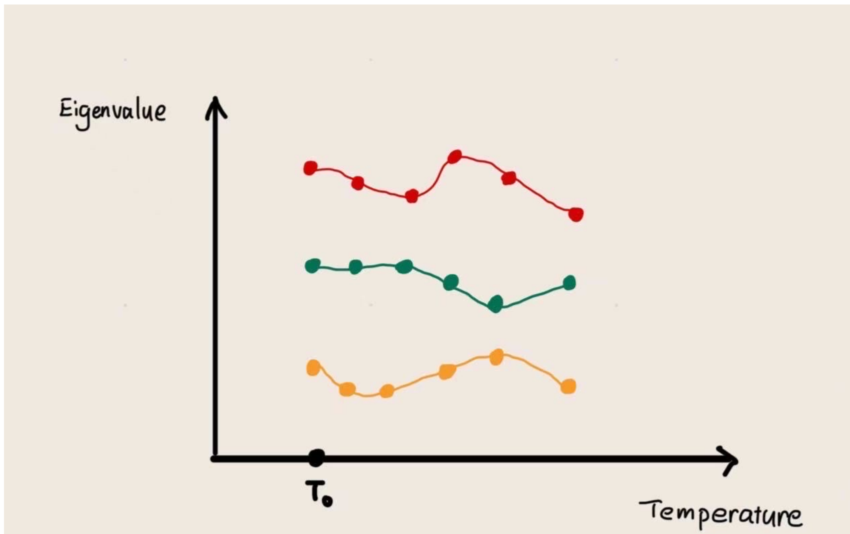

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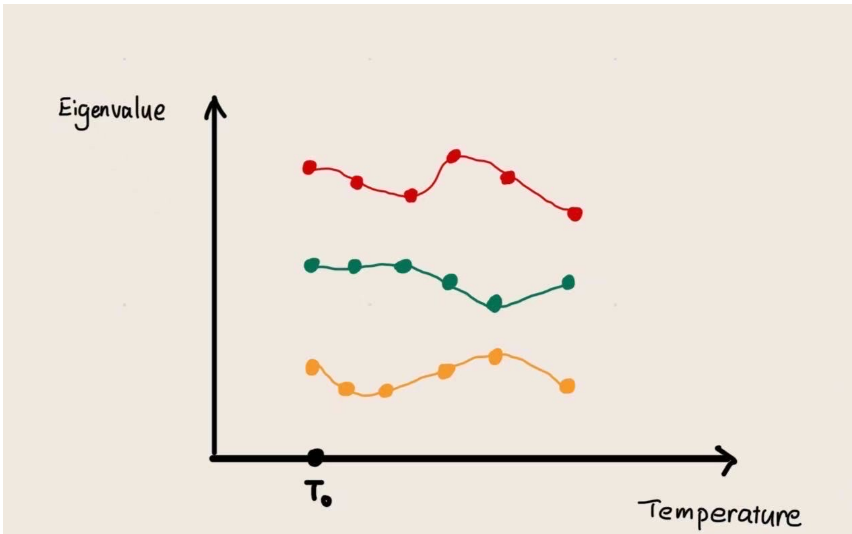

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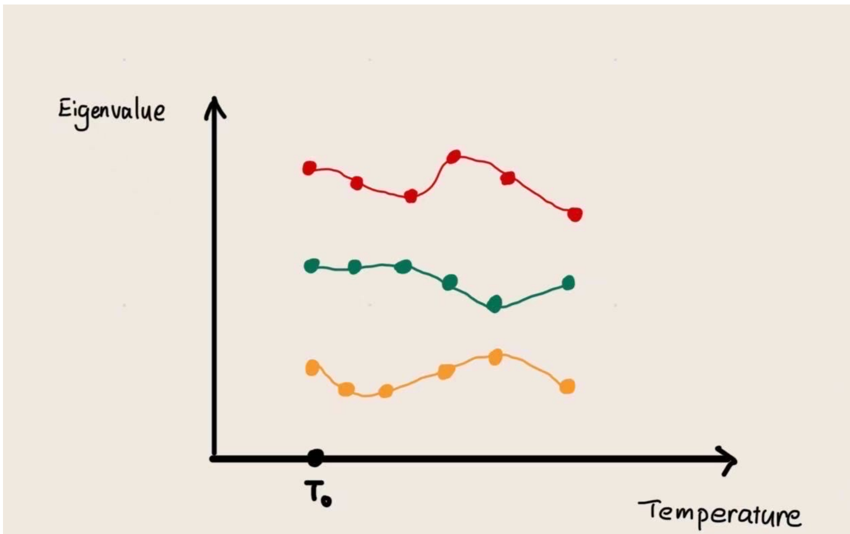

Hermitian vs. non-Hermitian

real eigenvalues (observable energies) vs. eigenvalues with imaginary part (counts for energy exchange with surrounding environment or other systems)

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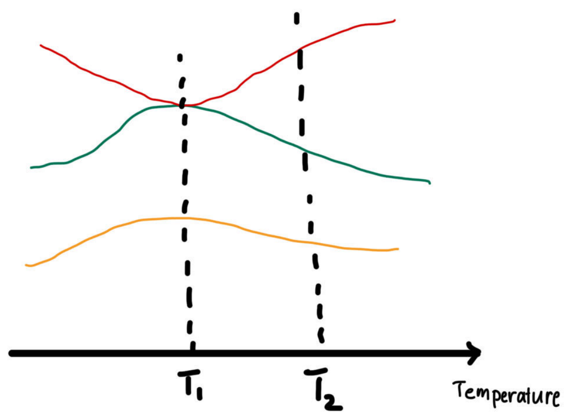

phases of matter
H_2O: *solid, liquid, gas*

T_1 : singular points (points where eigenvalues degenerate)
$H(T_1)$: gapless Hamiltonian
$H(T_2)$: gapped Hamiltonian

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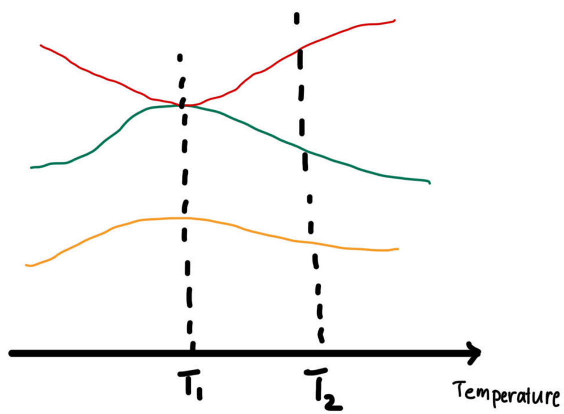

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Guzman et al., Model-free characterization of topological edge and corner states in mechanical networks, PNAS 2024.

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Thanks to Hopf bundles and Higgs bundles as *eigenbundles*, we now have a conceptually more systematic, visibly more intuitive understanding of the topic.

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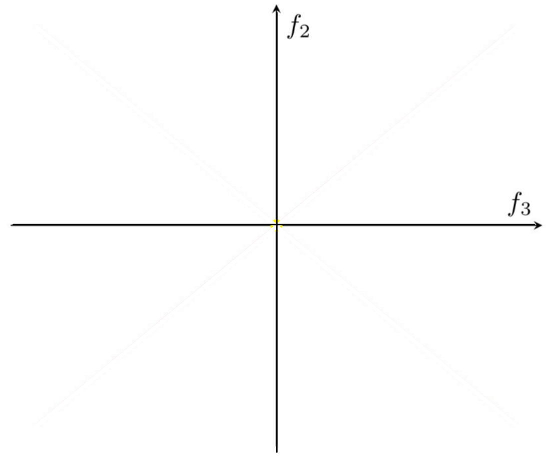

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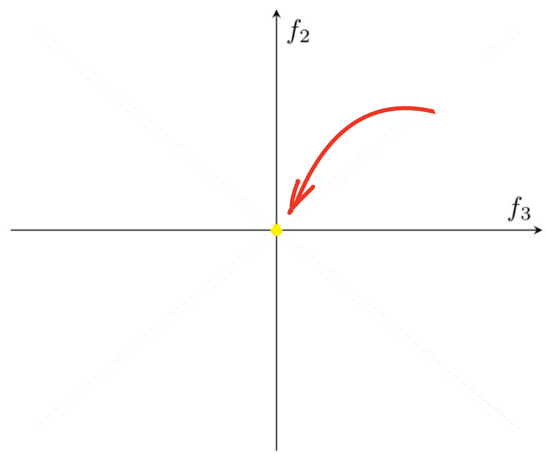

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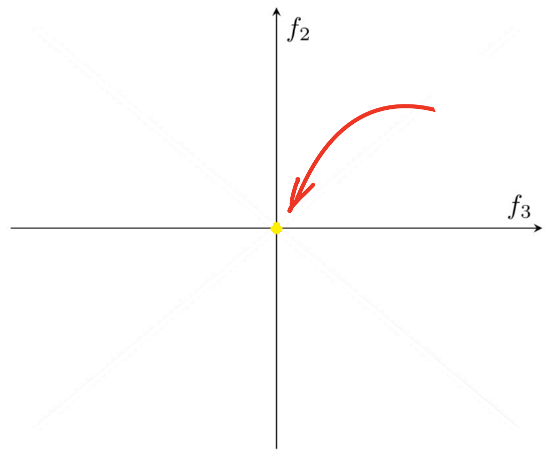

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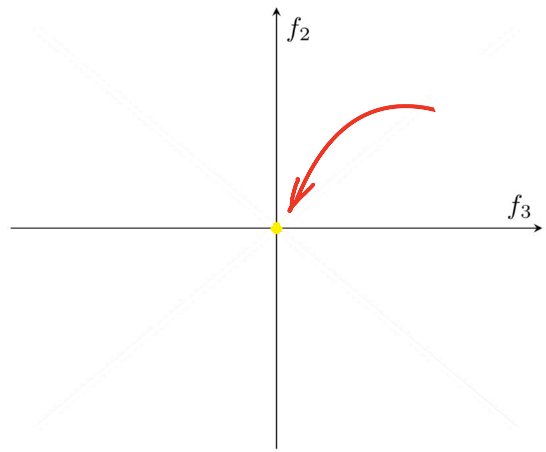

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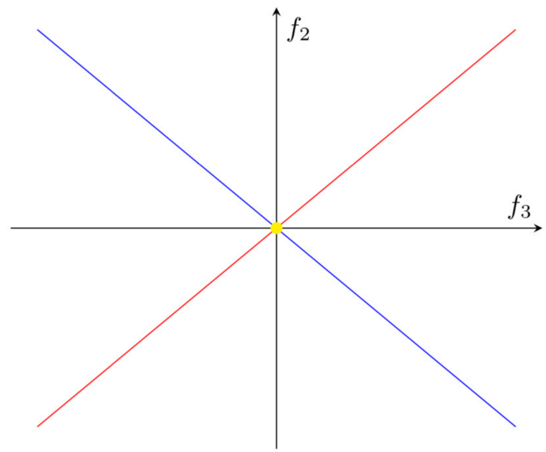

Its characteristic polynomial

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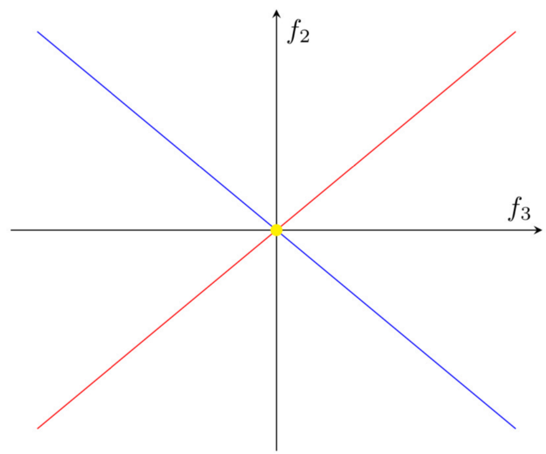

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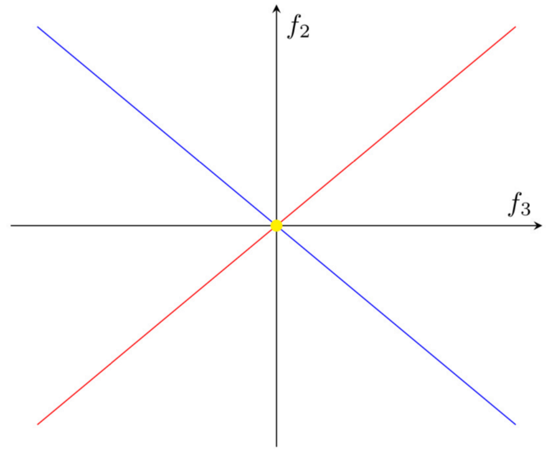

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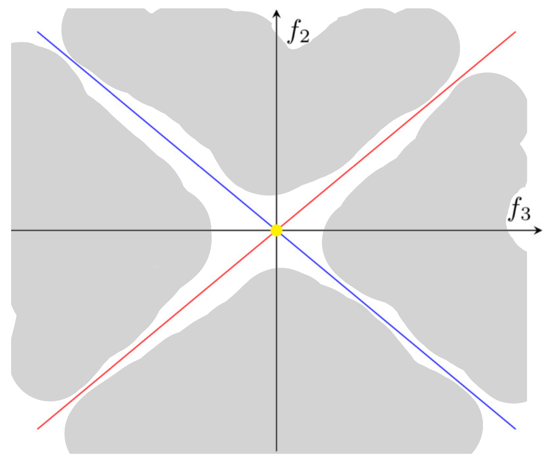

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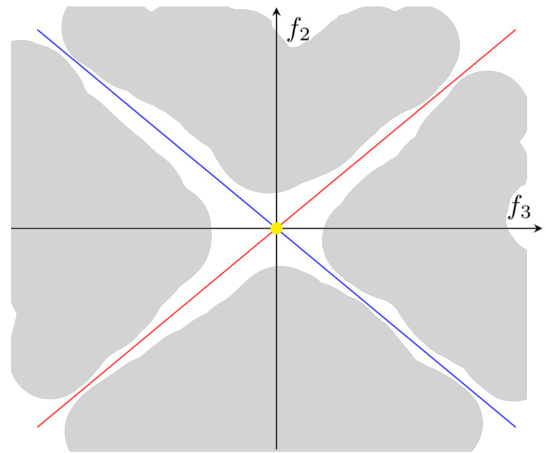

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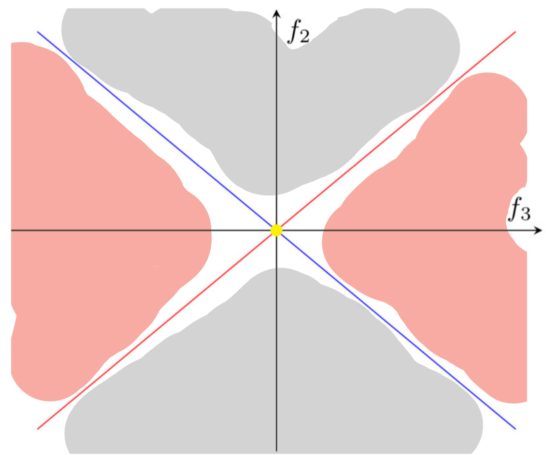

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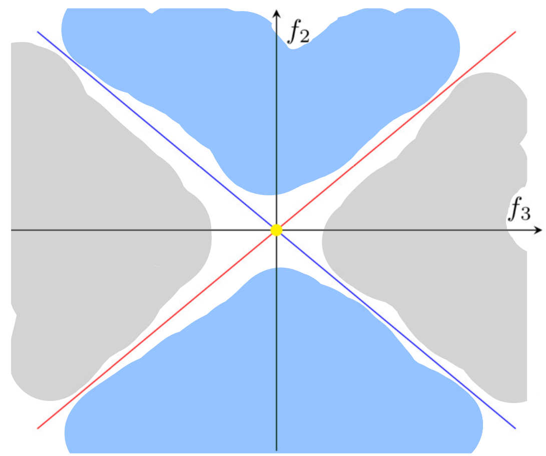

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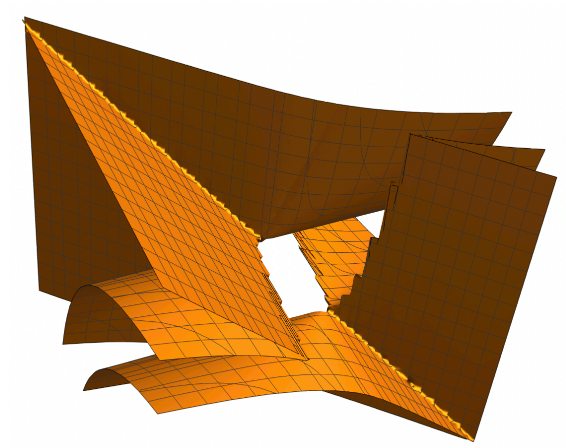
Governing eigenvalues with multiplicity, the discriminant surface of its characteristic polynomial is a pair of *swallowtails* in the $f_1 f_2 f_3$-space:

The equation for this surface is a non-homogeneous real polynomial in f_1, f_2, f_3 of degree 6.



Swallowtail couple $\mathrm{sw2}$

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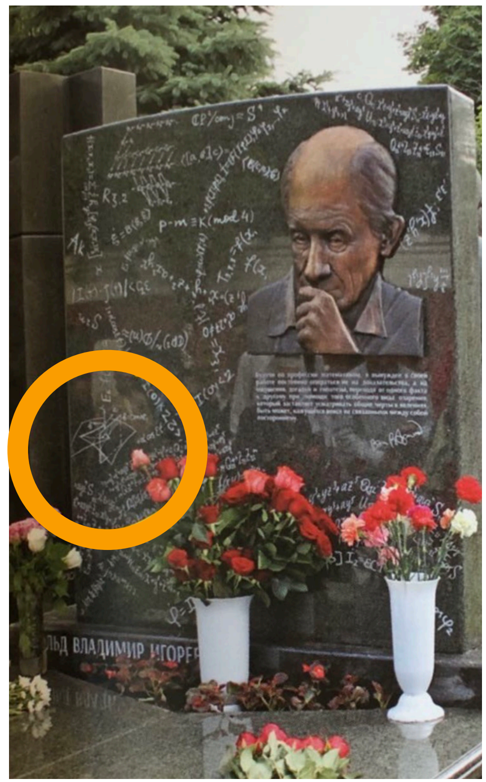

V. I. Arnold's tombstone at the Novodevichy Cemetery in Moscow

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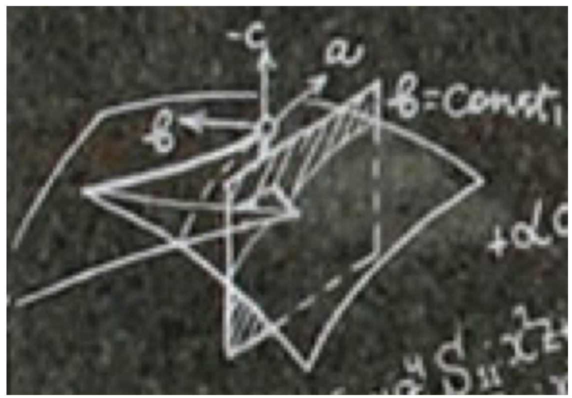

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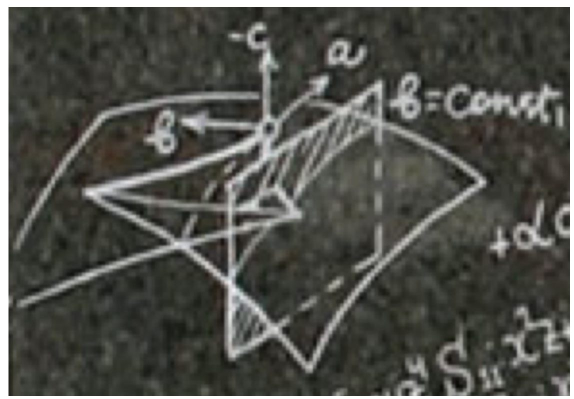

A local model for moduli spaces of 3-band Hamiltonians

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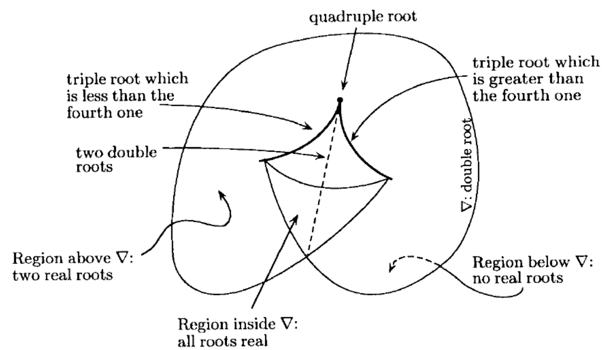

The space of polynomials $x^4 + ax^2 + bx + c$

Arnold, Braids of algebraic functions and the cohomology of swallowtails, 1968.

Homological stability of braid groups

Portrait from Gelfand, Kapranov, Zelevinsky, Discriminants, resultants, and multidimensional determinants.

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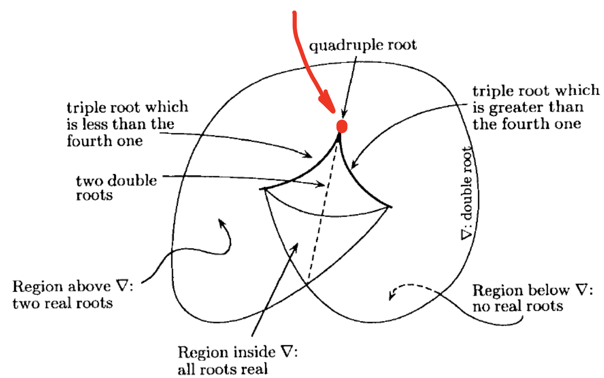

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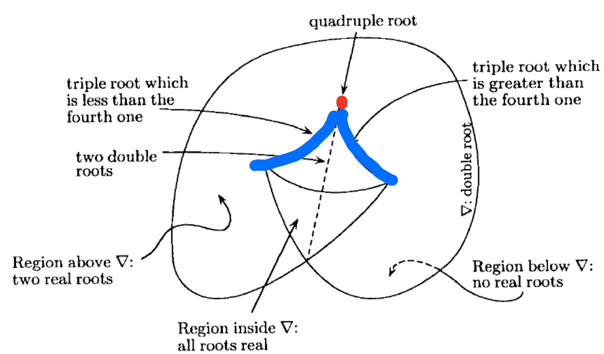

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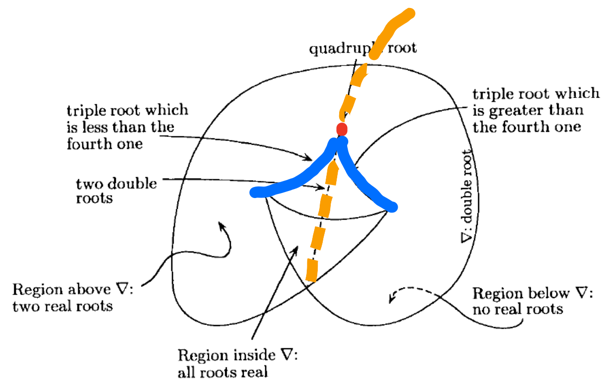

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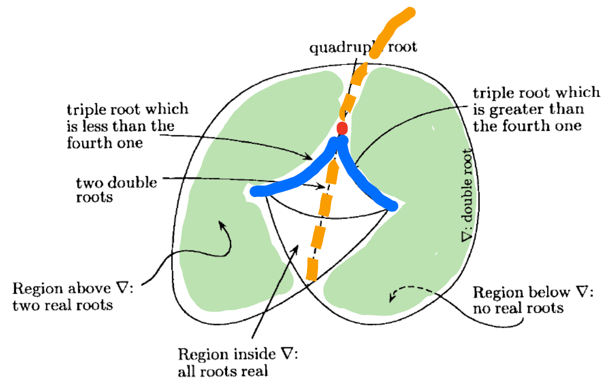

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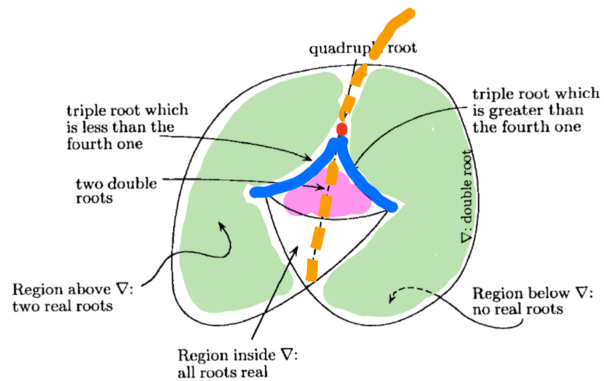

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Again, we aim to find computable algebraic invariants that systematically classify the evolutions of eigenvectors along loops in such stratified parameter spaces, including when they cross the discriminant surface resulting in degeneracies of various sorts.

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

The Hermitian case is simple, as the singularity is isolated, yet has profound physical implications already known to Arnold.

Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connections and quantum Hall effect, 1995.

Also: Polymathematics, 2000.

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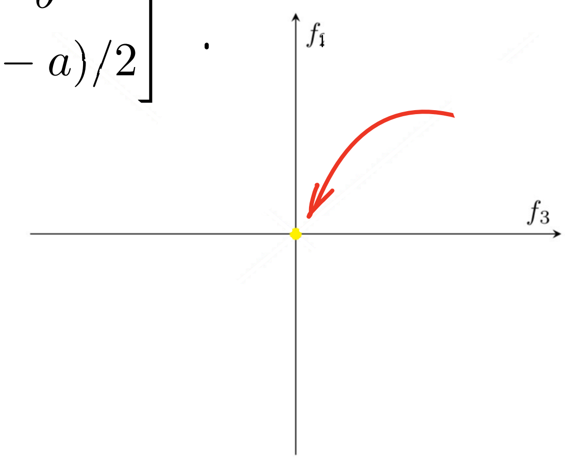

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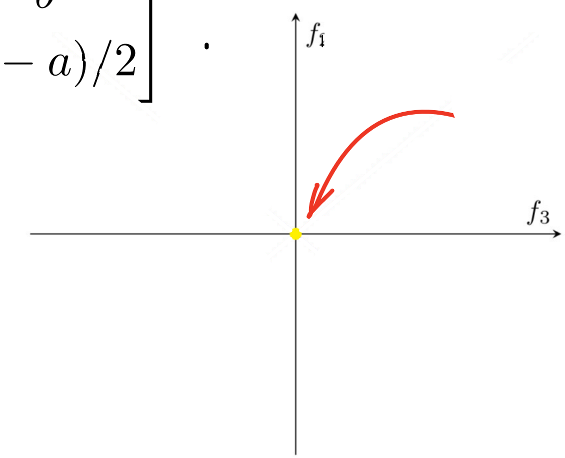

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Eigenframe rotation as vector bundles: Revisiting the Hermitian case

How does the eigenframe rotate over this stratified parameter plane?

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$$\begin{vmatrix} f_3 - \omega & f_1 \\ f_1 & -f_3 - \omega \end{vmatrix} = \omega^2 - f_1^2 - f_3^2 = 0 \implies \omega_\pm = \pm\sqrt{f_1^2 + f_3^2}$$

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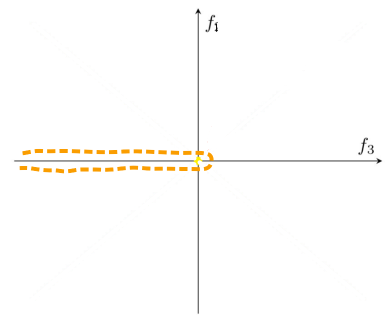

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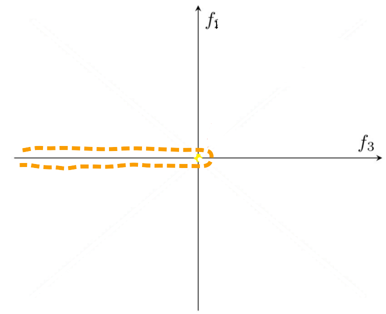

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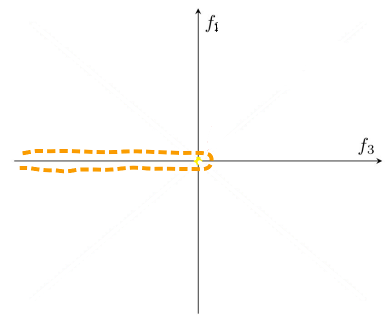

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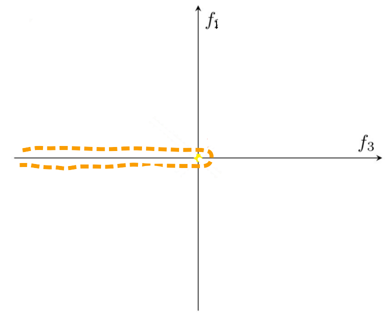

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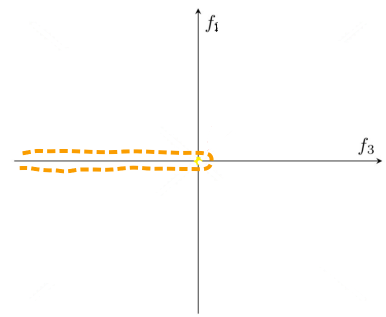

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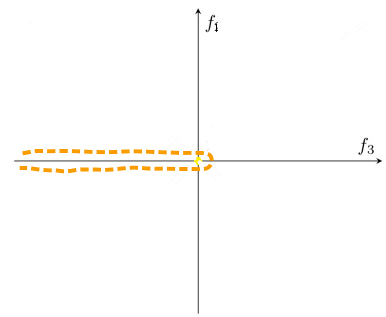

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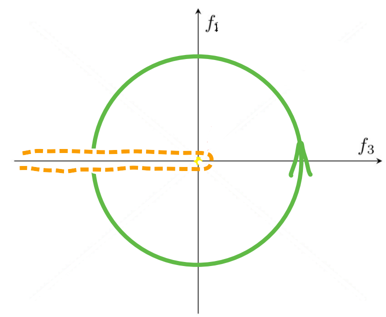

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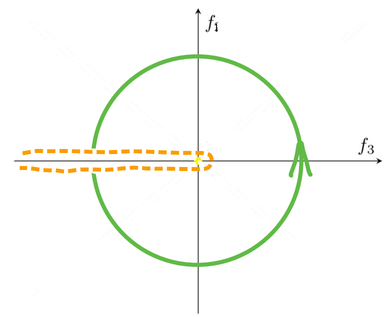

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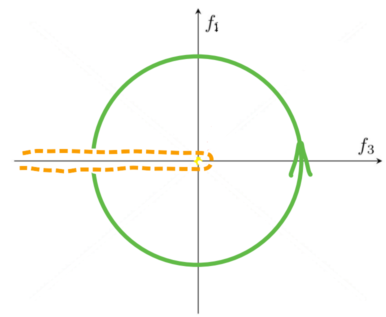

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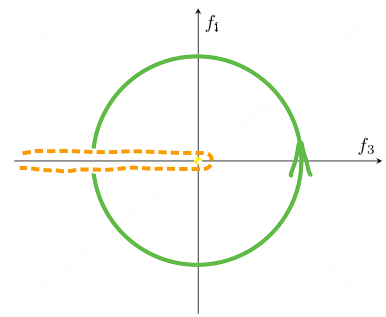

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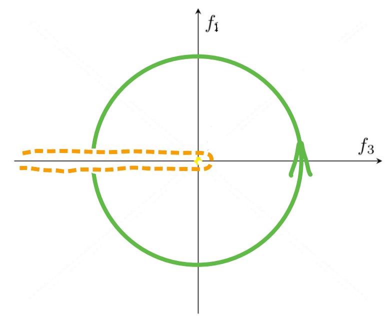

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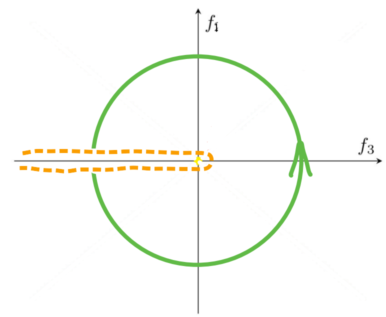

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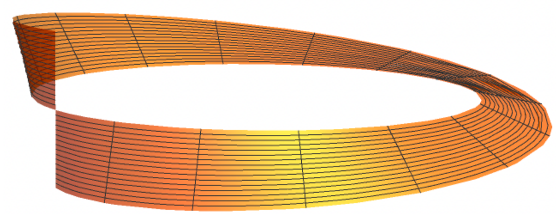

Eigenframe rotation as vector bundles: Revisiting the Hermitian case

Lemma. The universal rank-1 eigenbundle for Hermitian 2-band systems is given by the Hopf bundle

$$\begin{array}{lcl} S^0 \hookrightarrow S^1 \rightarrow S^1 & & \mathbb{R} \\ S^1 \hookrightarrow S^3 \rightarrow S^2 & \text{if the Hamiltonian is over} & \mathbb{C} \\ S^3 \hookrightarrow S^7 \rightarrow S^4 & & \mathbb{H} \end{array}$$

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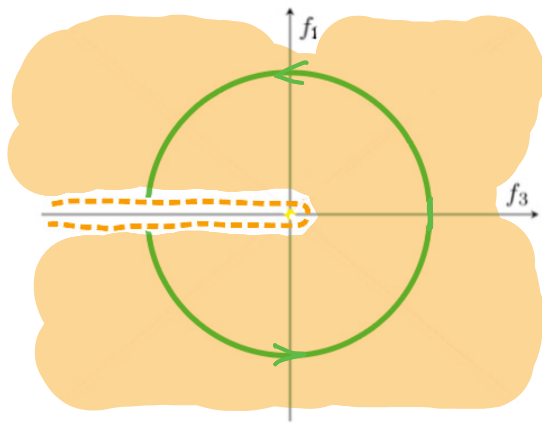

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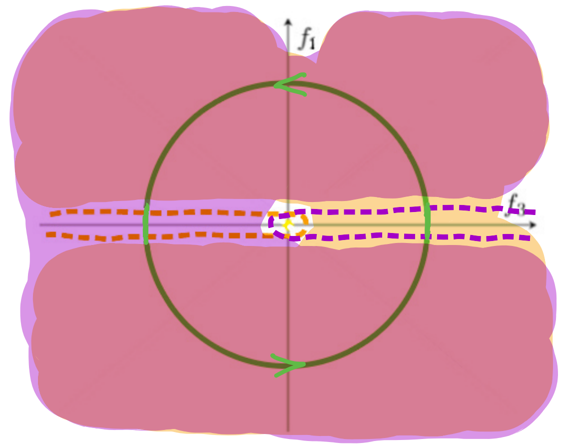

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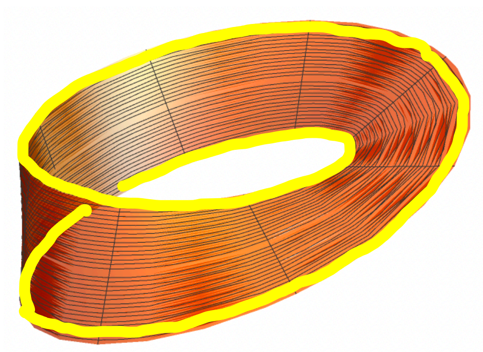

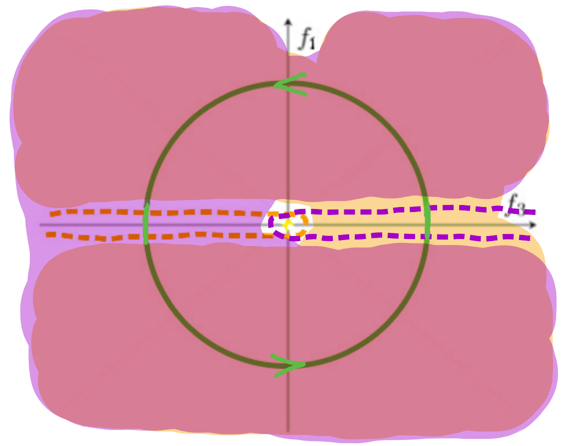

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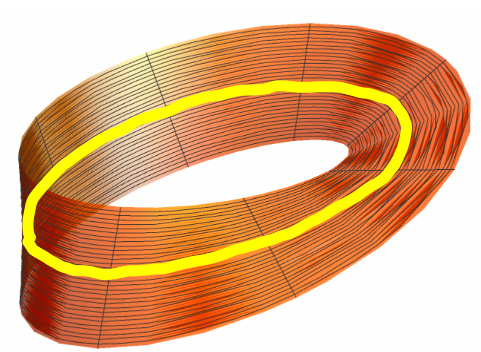

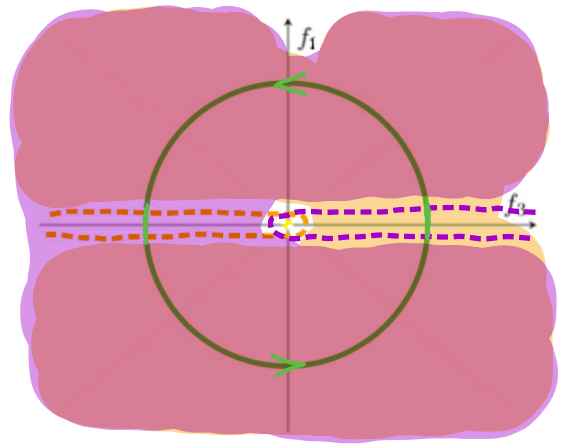

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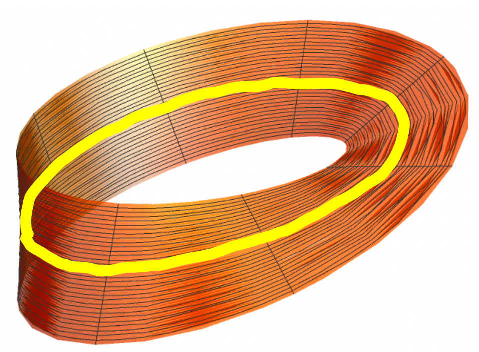

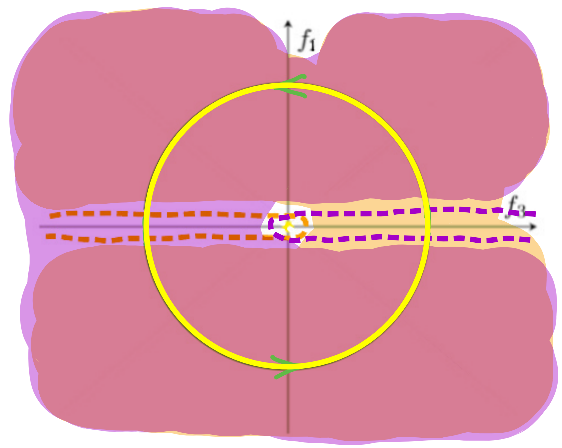

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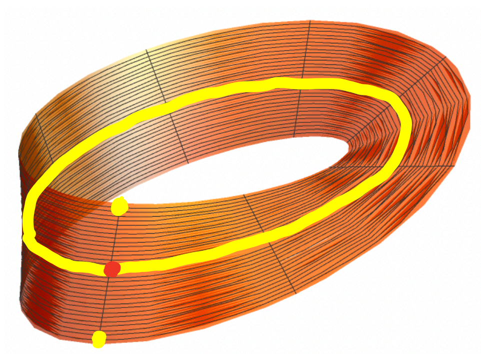

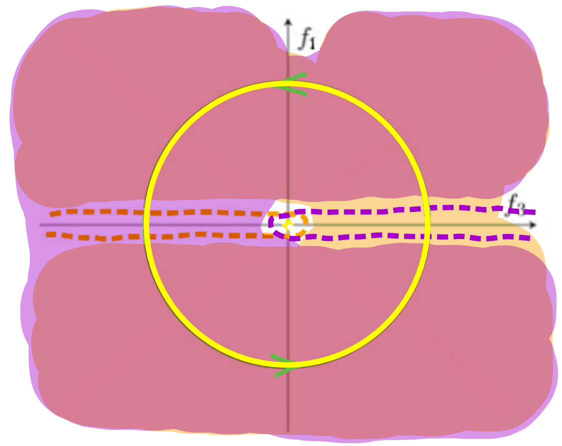

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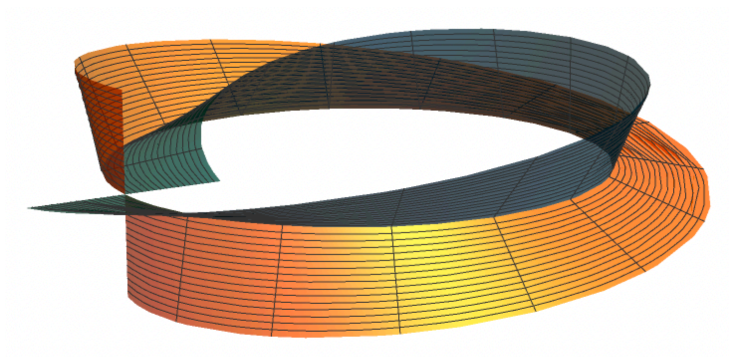

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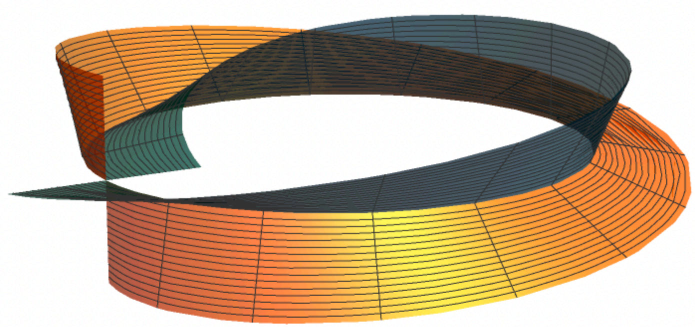

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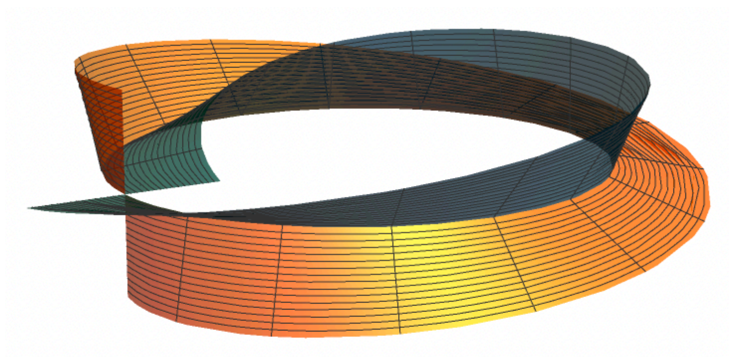

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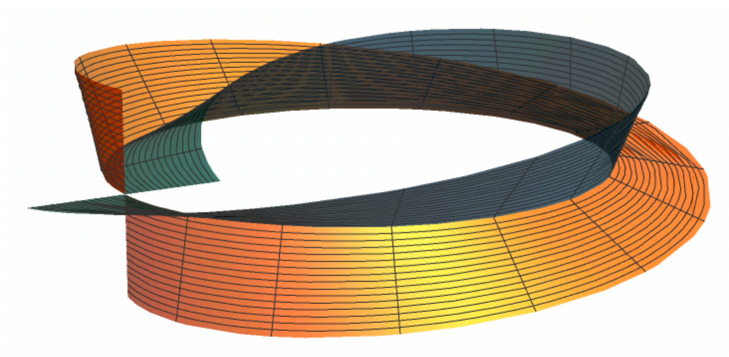

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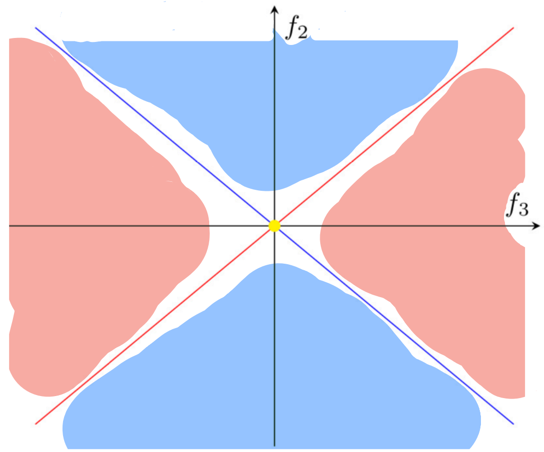

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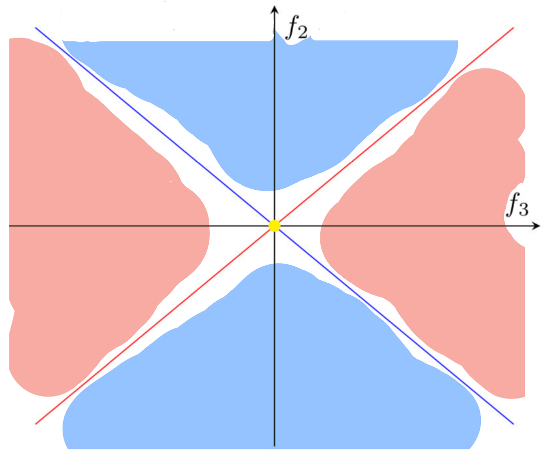

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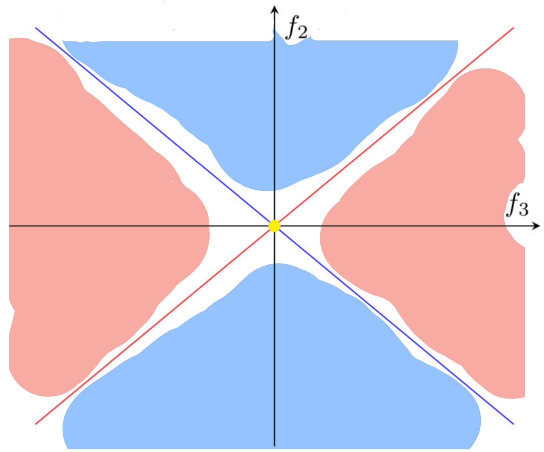

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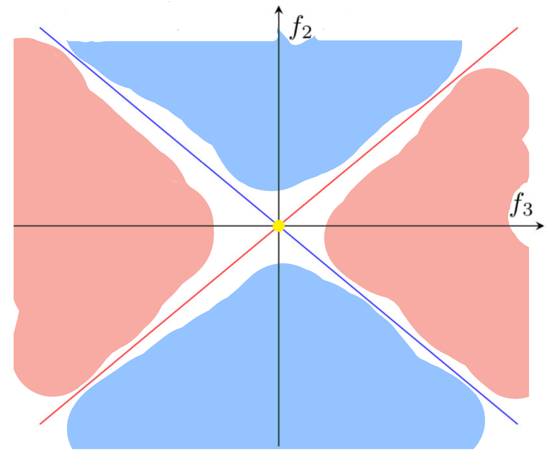

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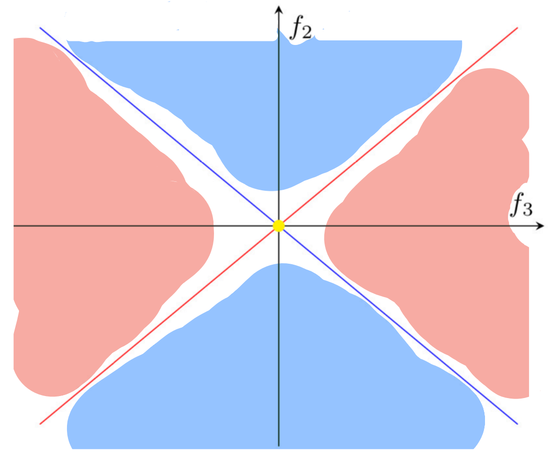

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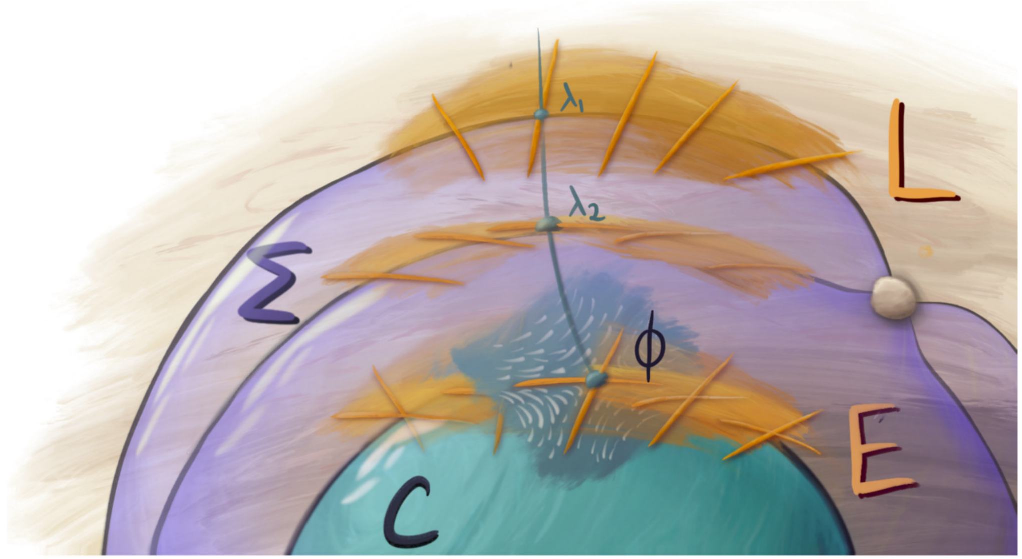
A *Higgs bundle* $(E, \phi) \to C$ is essentially a family of matrices

Peter Higgs (bosons)

Nigel Hitchin 1987
Carlos Simpson

C compact Riemann surface (or more generally Kähler manifold)
E holomorphic vector bundle
ϕ Higgs field: a holomorphic 1-form taking values in the bundle of endomorphisms of E such that $\phi \wedge \phi = 0$



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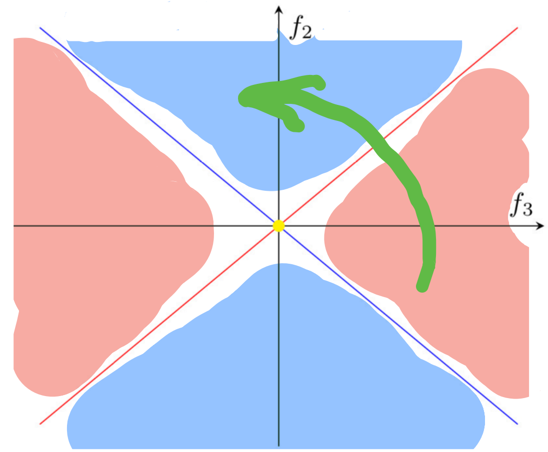

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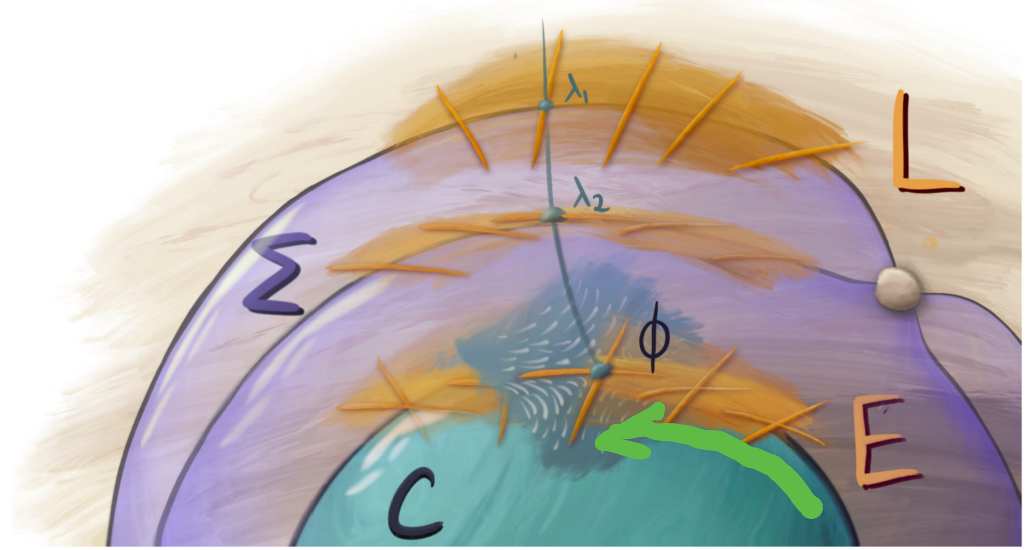
$$\phi_x \in \mathrm{End}(E_x), x \in C$$



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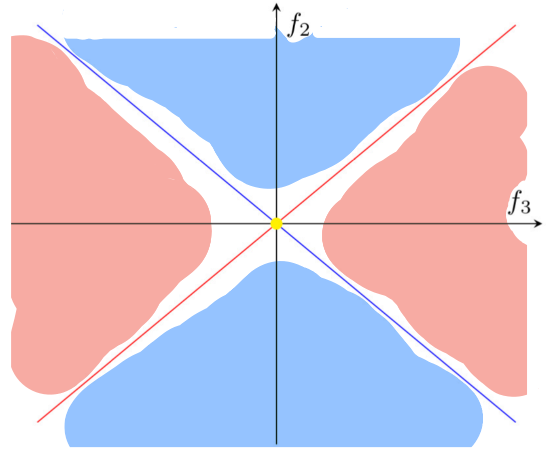

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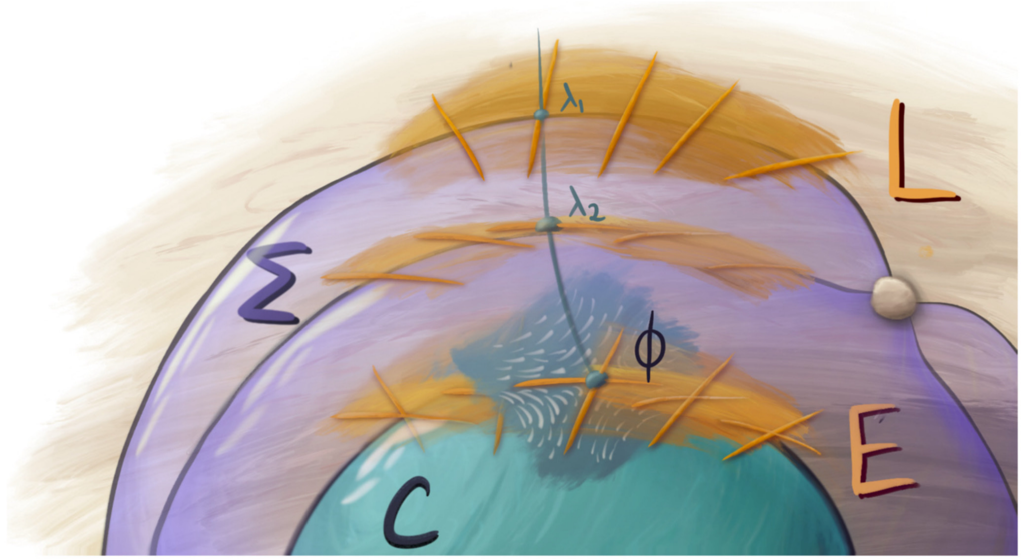

Portrait from Kienzle and Rayan, *Hyperbolic band theory through Higgs bundles*, ***Adv. Math.***, 2022.

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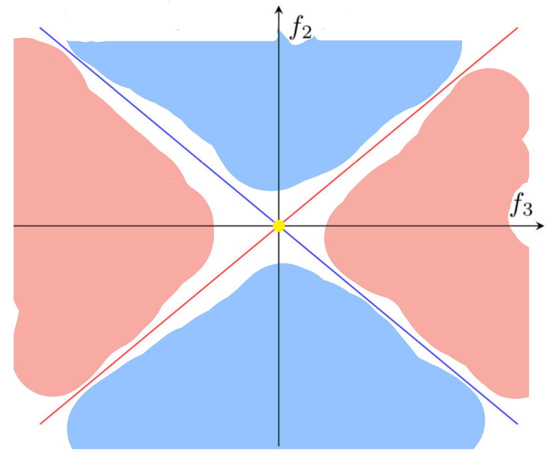

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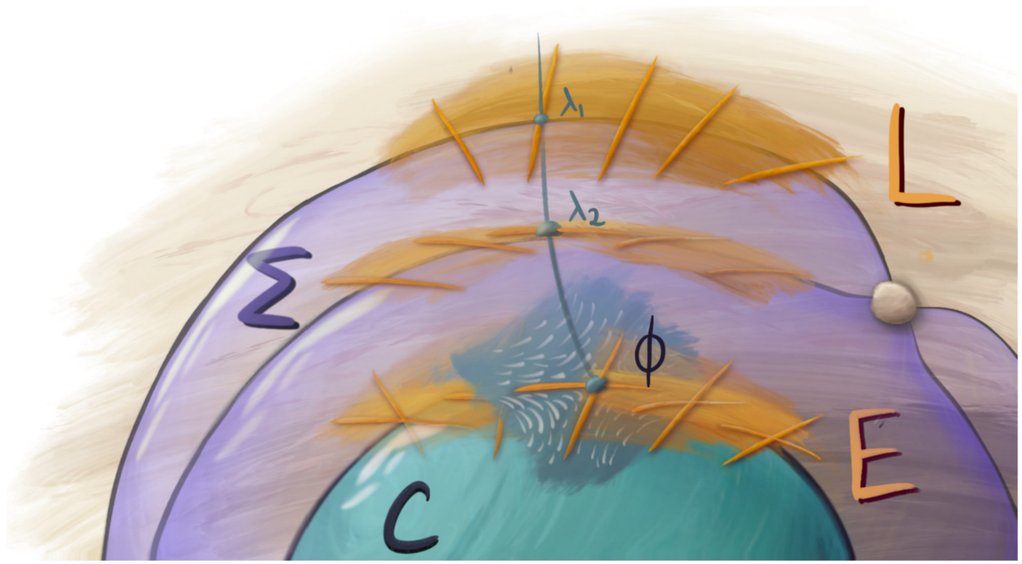

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Hyperbolic metric on the base C. Kollár et al., Nature, 2019. (Alicia Kollár, János)

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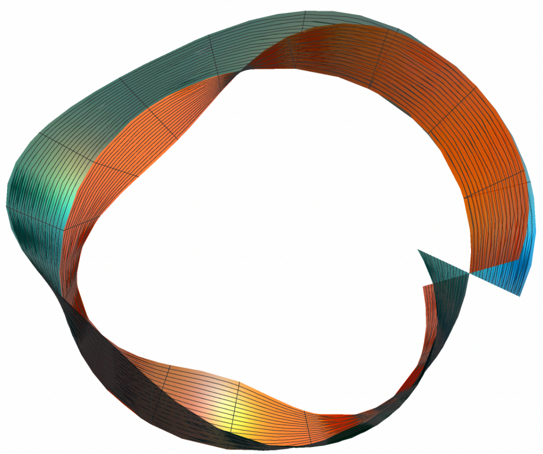

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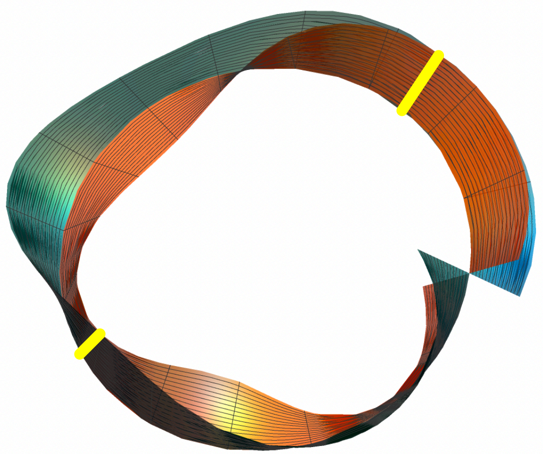

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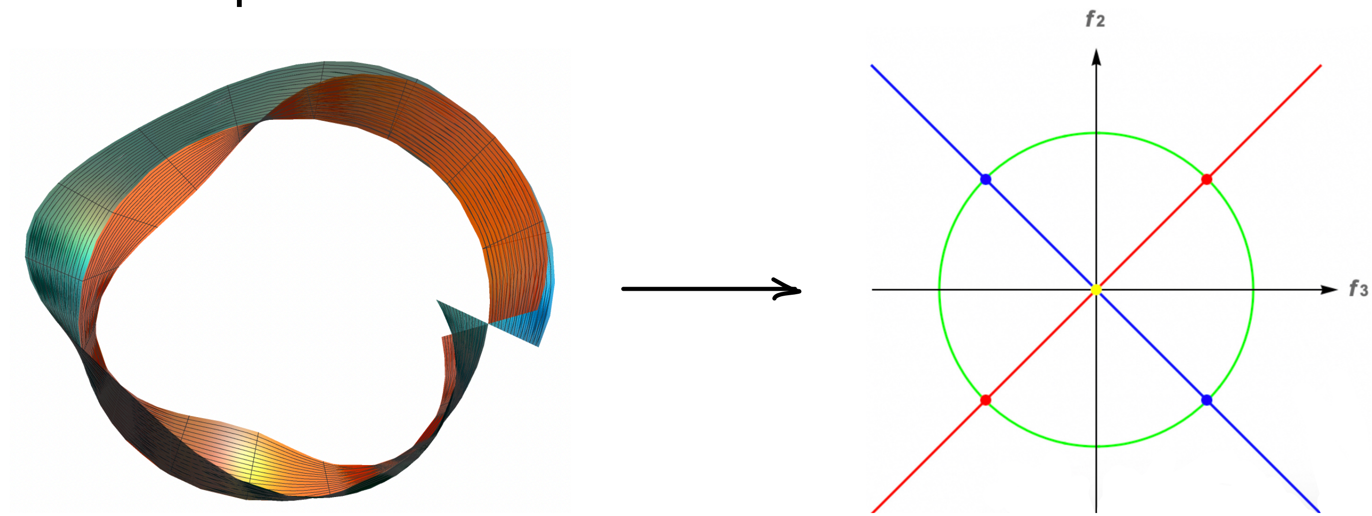

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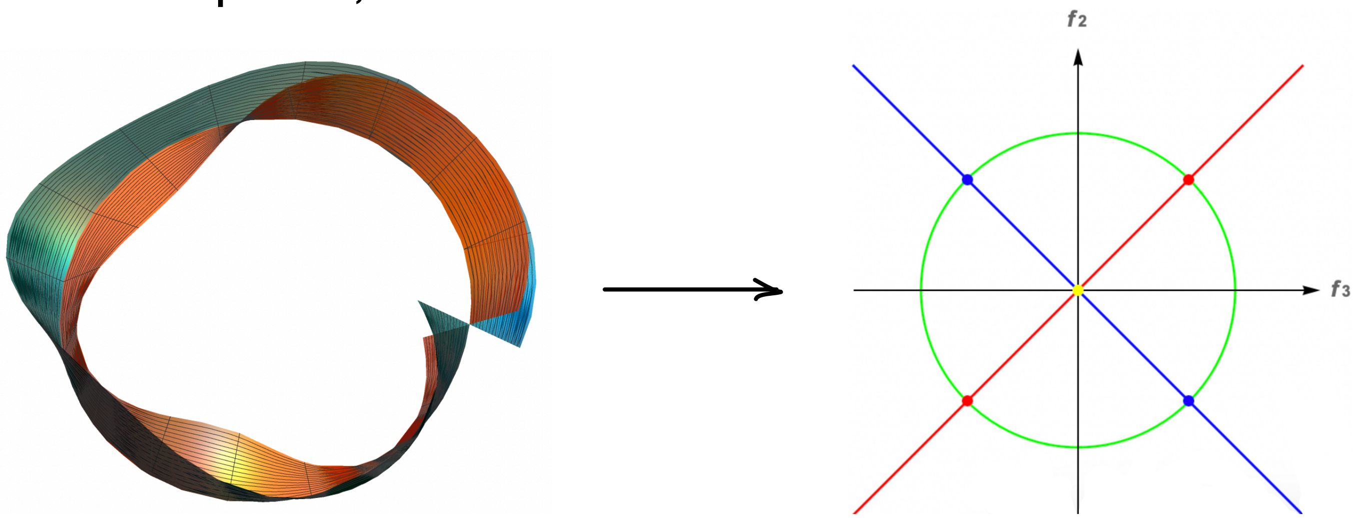

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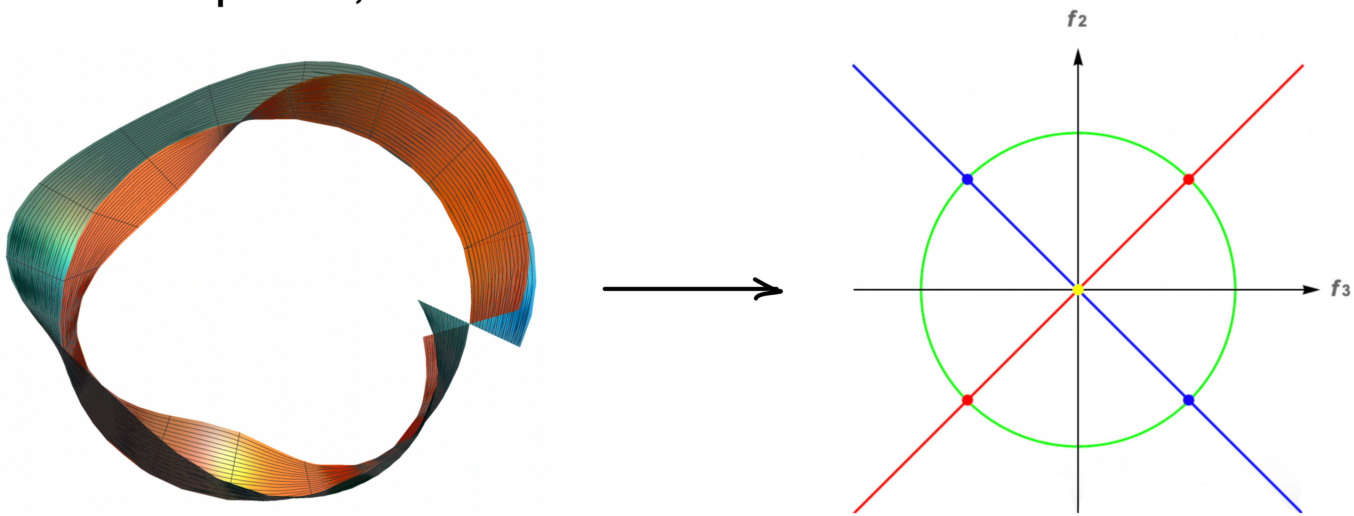

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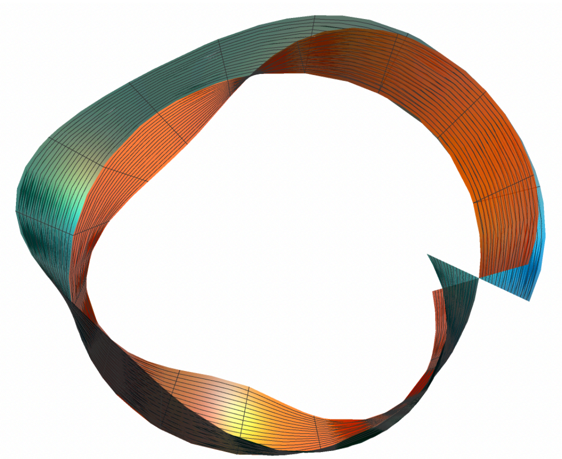

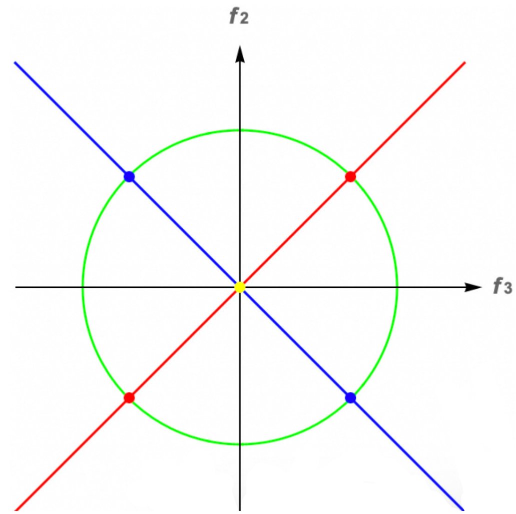

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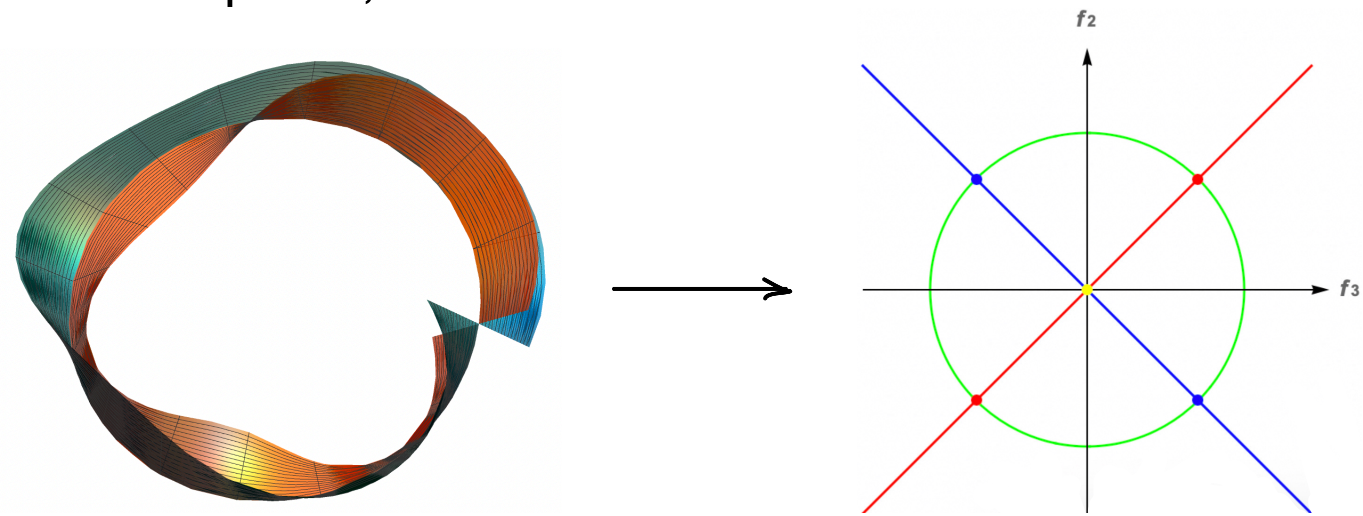

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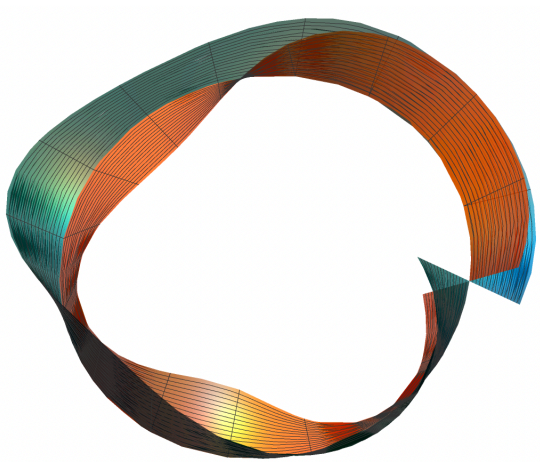

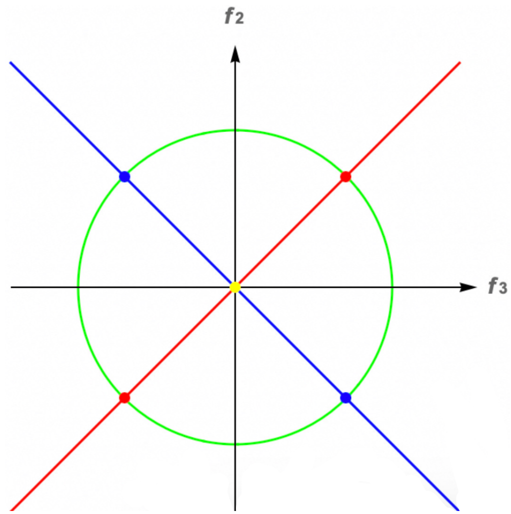

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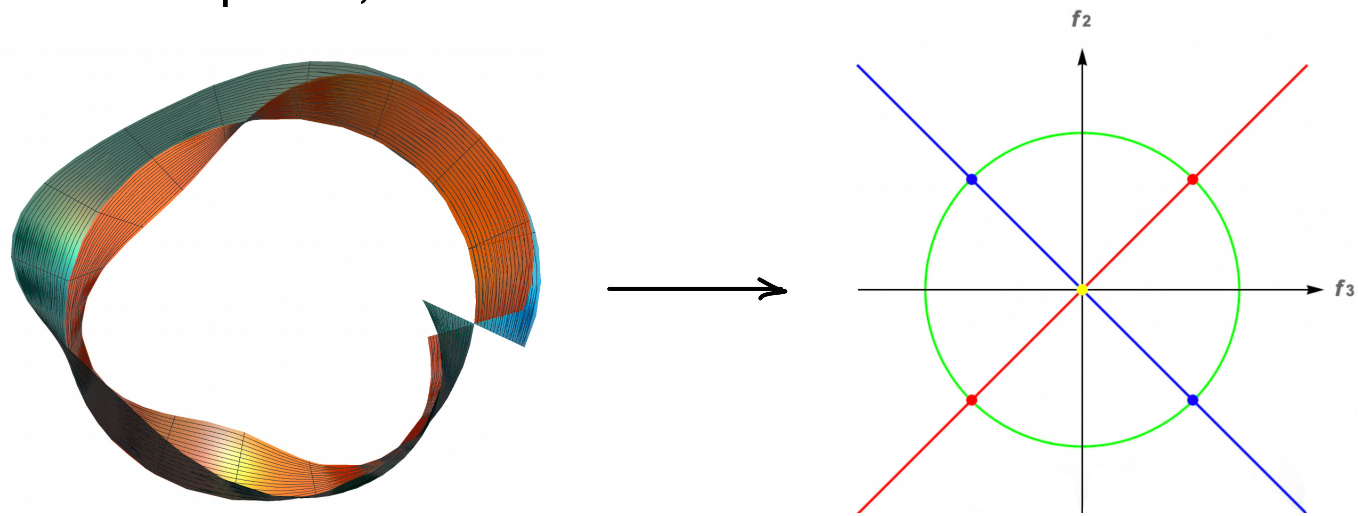

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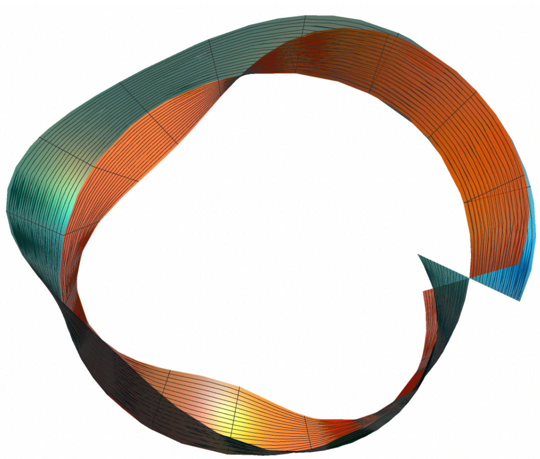
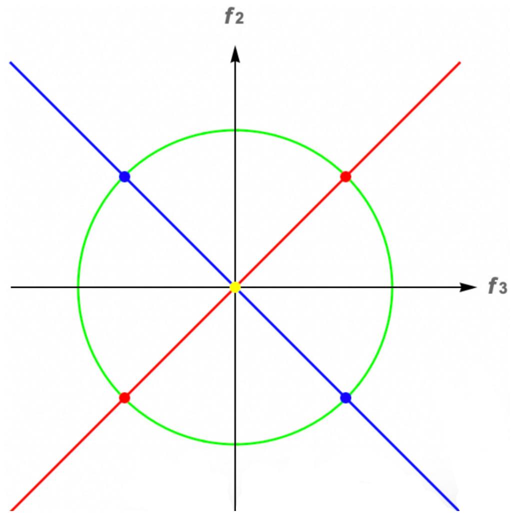
Eigenframe rotation as Higgs bundles: The non-Hermitian case

Proposition. The universal eigenbundle for non-Hermitian 2-band systems is given by a pair of kissing half Möbius bands over the stratified unit circle in the punctured parameter plane, whose 0-dimensional stratum consists of 4 points.



$\longrightarrow$



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Gajer, The intersection Dold–Thom theorem, Topology, 1996. (Ph.D. student of Blaine Lawson, 1993)

Goresky and MacPherson, 1974.

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$$\begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$$

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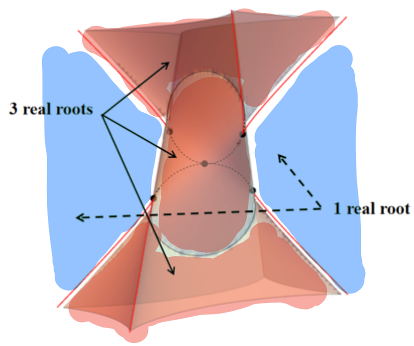

(a)

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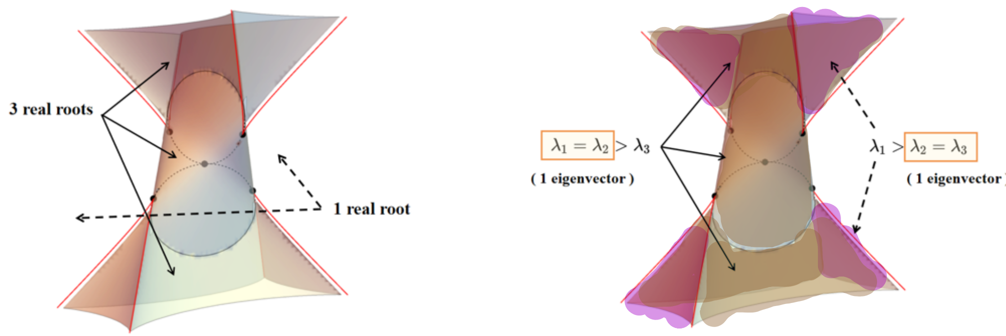

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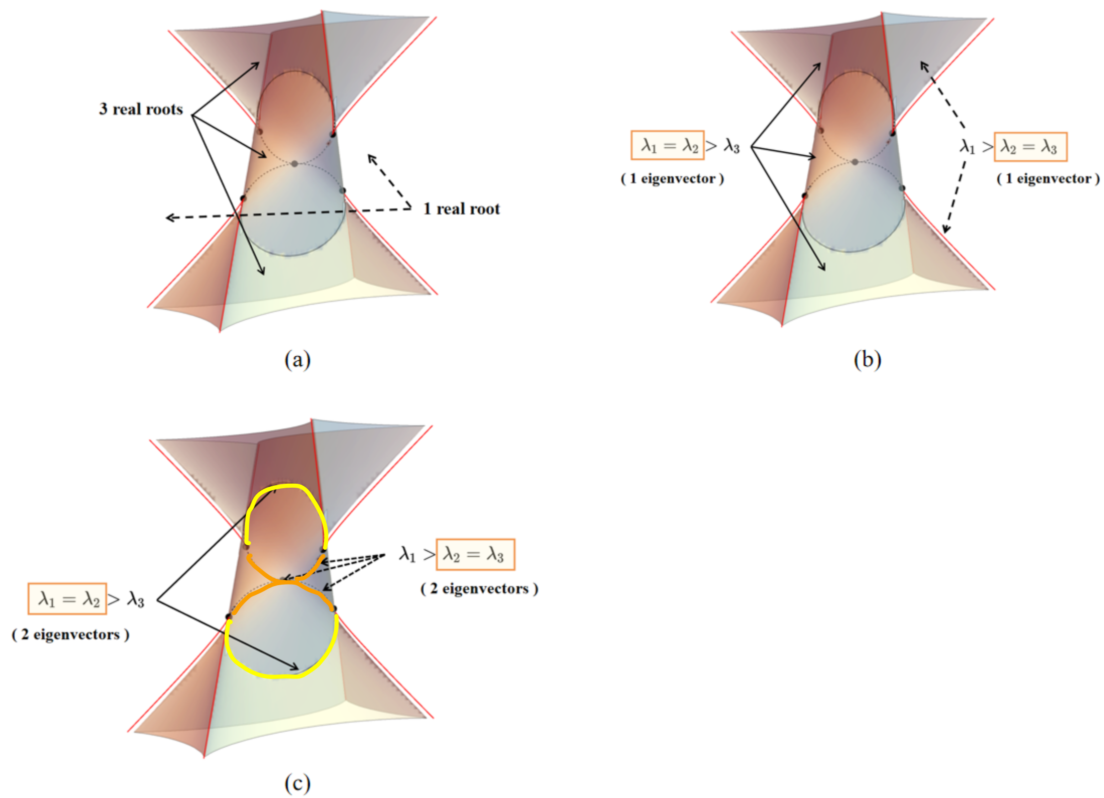

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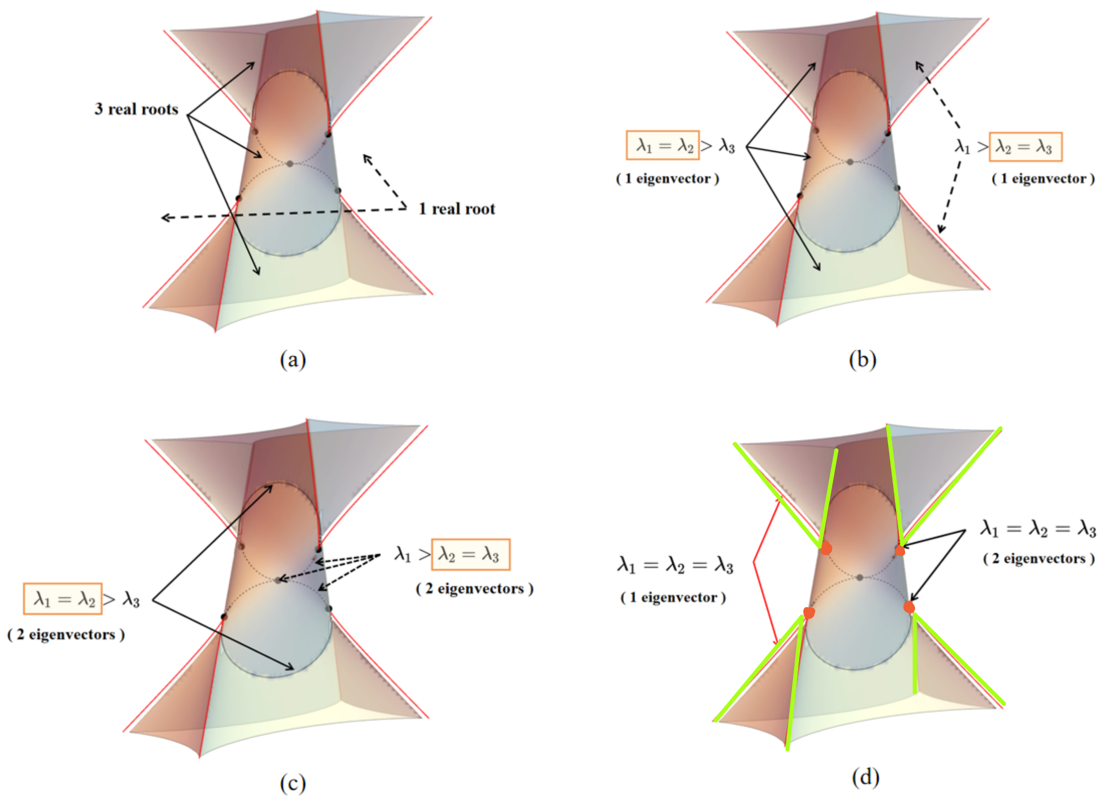

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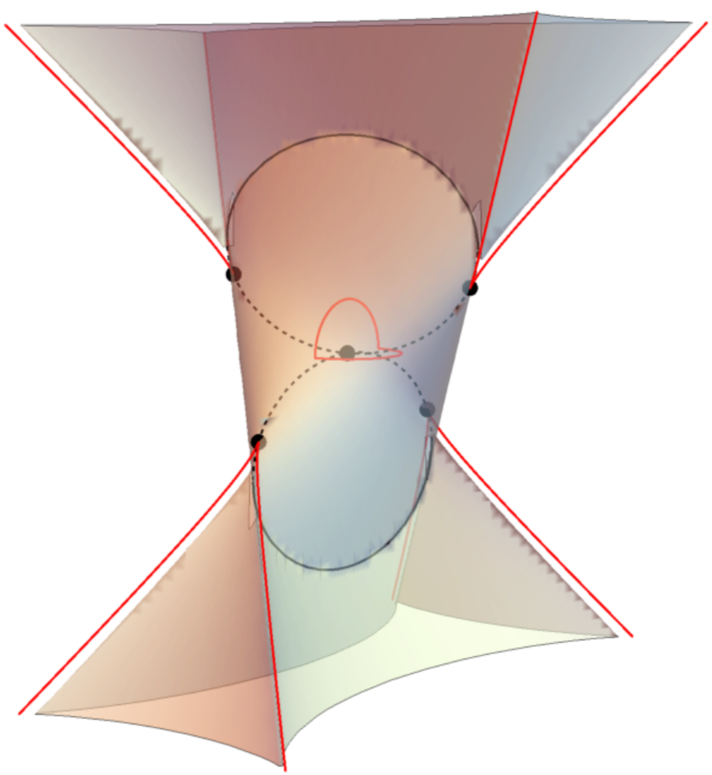

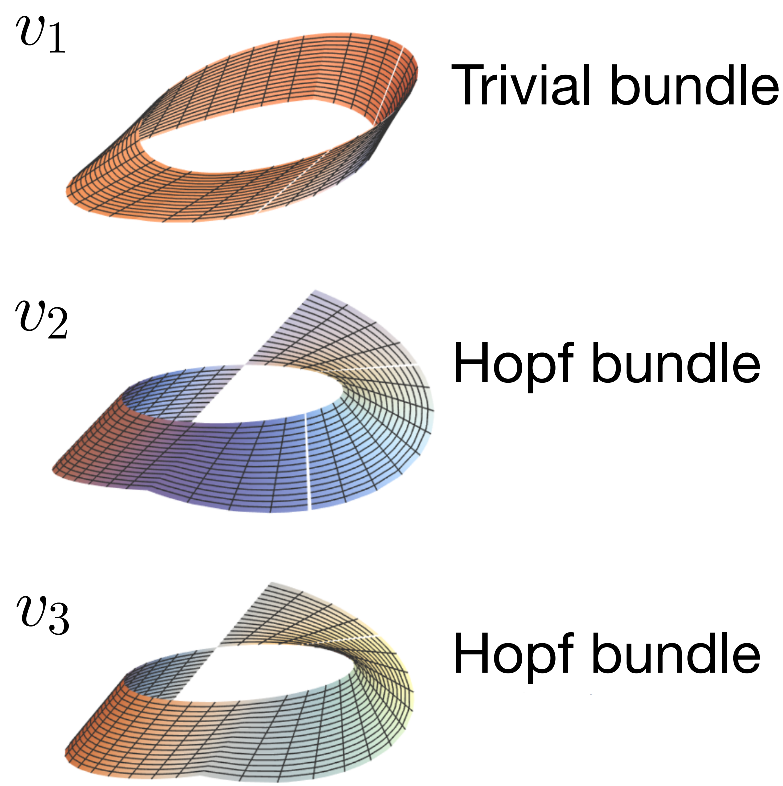

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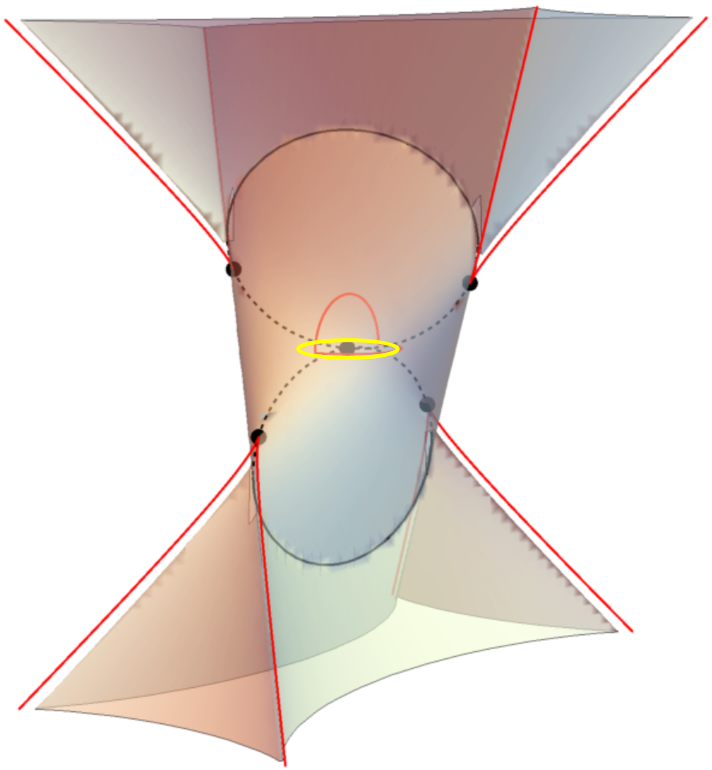
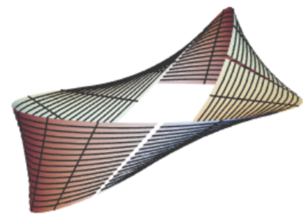
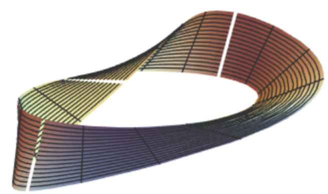


v_1



Trivial bundle

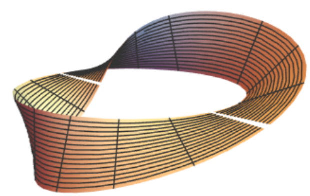
v_2



~~Hopf bundle~~

Trivial bundle

v_3



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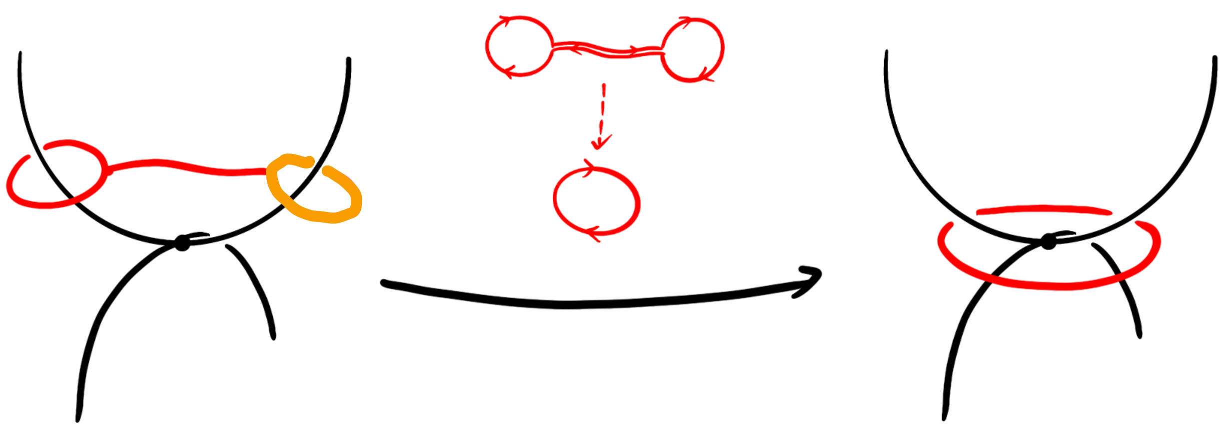

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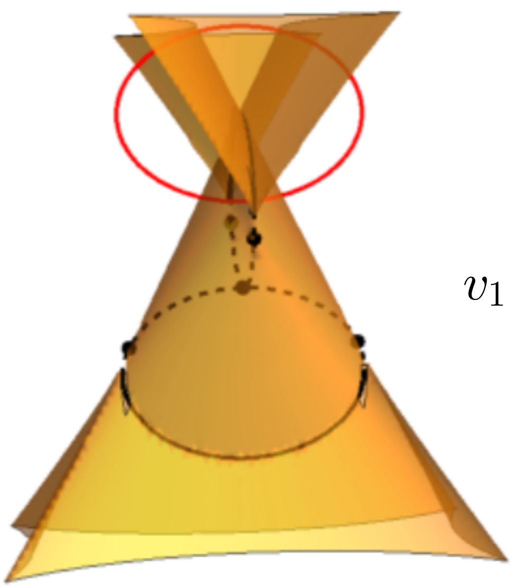

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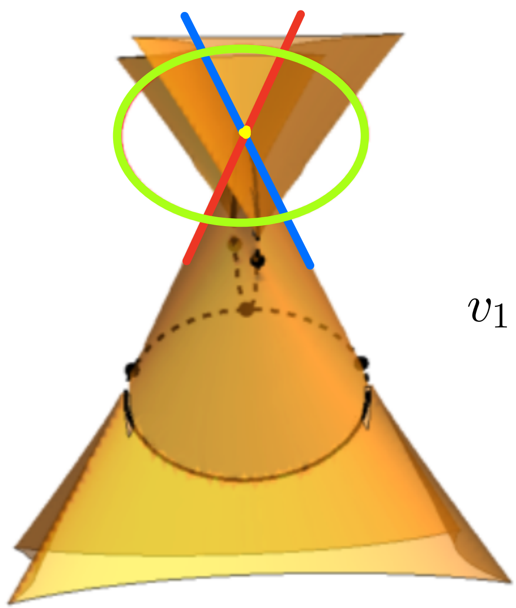

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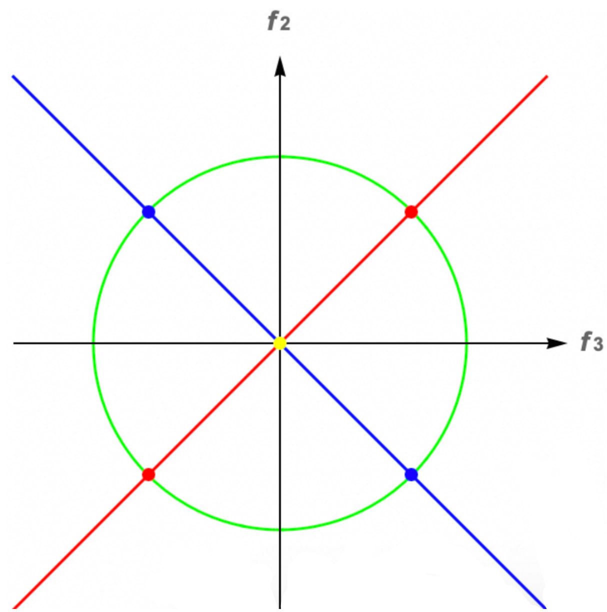

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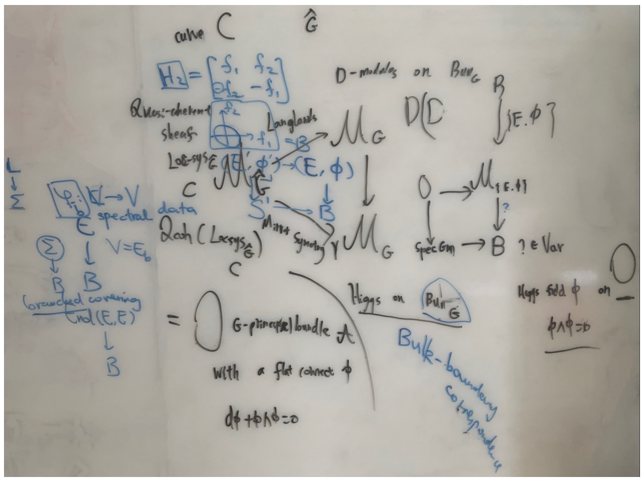

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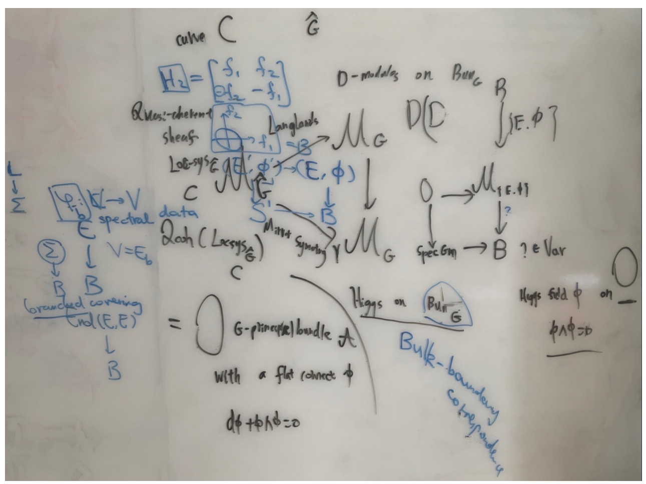
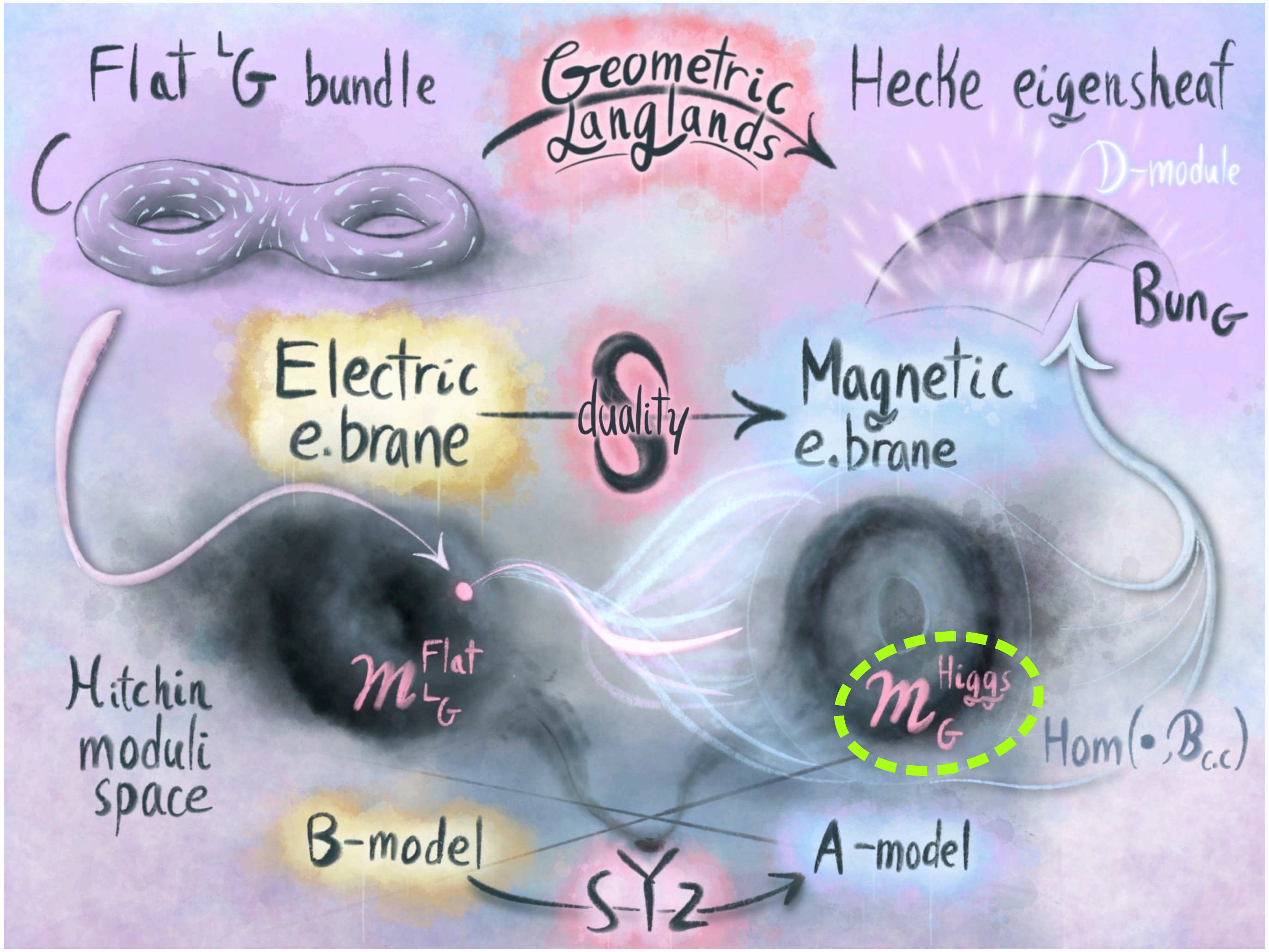
Indeed, Higgs bundles sit on one side of the geometric Langlands duality! We've at least found some testing ground.



Thank you.



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- pp. 58–59, Boris Khesin and Sergei Tabachnikov. Vladimir Igorevich Arnold, 12 June 1937 — 3 June 2010. Biogr. Mems Fell. R. Soc. 64, 7–26, 2018.
- p. 61, I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, Discriminants, resultants and multidimensional determinants. Birkhäuser, 1994.
- p. 114, Elliot Kienzle and Steven Rayan. Hyperbolic band theory through Higgs bundles. Adv. Math., 409:Paper No. 108664, 53, 2022.
- pp. 141–148, Chenlu Huang
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