MAT8021, Algebraic Topology

Assignment 6

Due in-class on Tuesday, March 30

1. Recall that a *topological group* G is a space with continuous maps

$$\begin{array}{ll} \mu\colon G \times G \to G & \text{multiplication} \\ \nu\colon G \to G & \text{inverse} \\ \iota\colon \{*\} \to G & \text{identity} \end{array}$$

   so that on the underlying set, we get a group with $g \cdot h = \mu(g,h)$, $g^{-1} = \nu(g)$, and $e = \iota(*)$.

   (a) Show that $H_*(G)$ is a ring by defining a multiplication on $C_*(G)$ (cf. Question 4 of Assignment 5). This is called a *Pontrjagin ring* structure on $H_*(G)$.

   (b) If G is abelian, show that $C_*(G)$ (and hence $H_*(G)$) is *graded commutative*, i.e., $x \cdot y = (-1)^{|x||y|} y \cdot x$ for any $x, y \in C_*(G)$, where $|x|$ and $|y|$ denote the degrees of x and y respectively.

2. (a) Let $G = \mathbb{R}$. What is the Pontrjagin ring structure on $H_*(\mathbb{R})$?

   (b) Show that $H_*(S^1)$ is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^2)$ with $|\alpha| = 1$.

   (c) More generally, it turns out that

$$H_*(S^1 \times S^1) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2, \alpha\beta + \beta\alpha) =: \Lambda[\alpha, \beta]$$

   is an *exterior algebra* on α, β with $|\alpha| = |\beta| = 1$. Similarly $H_*(S^1 \times S^1 \times S^1) \cong \Lambda[\alpha, \beta, \gamma]$, etc. In contrast, if $G = S^3$ regarded as the unit quaternions, what is the Pontrjagin ring structure on $H_*(S^3)$?

3. Let $G = \mathrm{SO}(3)$ be the 3×3 matrices over $\mathbb{R}$ with determinant 1.

   (a) Viewing it as the group of rotations in $\mathbb{R}^3$, describe a homeomorphism $\mathrm{SO}(3) \cong \mathbb{RP}^3$ by defining a map $D^3 \to \mathrm{SO}(3)$ that factors through $\mathbb{RP}^3$.

   (b) Give a presentation for $H_*\big(\mathrm{SO}(3)\big)$ as a ring.

   (c) What about $H_*\big(\mathrm{SO}(3); \mathbb{Z}/2\big)$? In particular, show that the square of the generator of $H_1\big(\mathrm{SO}(3); \mathbb{Z}/2\big)$ equals zero.

4. Consider the n-dimensional real projective space $\mathbb{RP}^n = S^n/(x \sim -x) = (\mathbb{R}^{n+1} \setminus \{O\})/\big((x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n), \lambda \in \mathbb{R}^\times\big)$ and write its points in homogeneous coordinates $[x_0 : \ldots : x_n]$. There are embeddings $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$ given by $[x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_n : 0]$.

   (a) Show that the complement $\mathbb{RP}^{n+1} \setminus \mathbb{RP}^n$ is homeomorphic to $\mathbb{R}^{n+1}$.

   (b) Deduce from part (a) that $\mathbb{RP}^{n+1}$ is formed by attaching an $(n+1)$-dimensional cell to $\mathbb{RP}^n$. In particular, check that the attaching map is injective on the interior $\mathring{D}^{n+1} \cong \{(x_0, \ldots, x_{n+1}) \in S^{n+1} | \, x_{n+1} > 0\}$.

   (c) Deduce from part (b) the groups in $C_*^{\mathrm{CW}}(\mathbb{RP}^n)$ as follows:

   $$\cdots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

   where the boundary maps turn out to alternate between 0 and multiplication by 2 (cf. page 119 of Jiang). Calculate all $H_k(\mathbb{RP}^n)$.

5. Consider the *complex* projective space $\mathbb{CP}^n$ defined analogously.

   (a) What is its (real) dimension as a CW complex? In particular, what is the space $\mathbb{CP}^1$?

   (b) Show that $\mathbb{CP}^{n+1}$ is formed by attaching a $(2n+2)$-dimensional cell to $\mathbb{CP}^n$.

   (c) Calculate all $H_k(\mathbb{CP}^n)$ from $C_*^{\mathrm{CW}}(\mathbb{CP}^n)$.

   (d) Generalize the above to projective spaces over the quaternions $\mathbb{H}$. (The failure of associativity in the octonions $\mathbb{O}$ creates a problem for further generalizations.)