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The Unstable Adams Spectral Sequence for S^3

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Abstract

This is a report on recent calculations which describe a portion of the unstable Adams spectral sequence (mod 2) for the three-sphere S^3. This report discusses $E_2^{*,*}(S^3)$ through stem 60, and calculates all differentials, and hence $E_\infty^{*,*}(S^3)$ through stem 52. This gives the 2-primary part of $\pi_{3+i}(S^3)$ for all $i \leq 52$.

1 Introduction

The homotopy groups of spheres $\pi_{n+q}(S^n)$ form long exact EHP sequences, which permits them to be calculated inductively. This is the way these groups were calculated by Toda ([25]) through stem 19 (that is, for $q \leq 19$), by Barratt through stem 22, and continued by Toda, Mimura, Mori, and Oda ([13], [14], [15], [16], [17], [18], [19]) through stem 30. In this approach, the main difficulty is computing the homomorphism P. The E_2 terms of the Unstable Adams Spectral Sequences for the spheres may be calculated by the use of similar EHP long exact sequences, with the advantage that the homomorphism P is given by a formula which is determined

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by the formula for the differential in the lambda algebra. The difficulty in computing the homomorphism P in homotopy becomes that of computing the higher Adams differentials. In this paper we use the computations of [5] for the groups $E_2^{*,*}(S^3)$. The higher Adams differentials are calculated by means of a map of (the 3-connected cover of) S^3 to the stable Moore space. We then use [8] for the differentials above the one-fifth for the stable Moore space to obtain the Adams differentials for the three-sphere S^3. These arguments are sometimes augmented by considerations of v_1-periodicity. To show that a class is a permanent cycle and hence persists to homotopy, we give an explicit construction of the class as a composition or a Toda bracket, or show that there is nothing in the Adams spectral sequence that the class can hit. After the calculations of the differentials, we present the first fifty-two stems of $E_\infty^{*,*}(S^3)$. We indicate group extensions where we can easily determine them. The proofs of these extensions are similar to the corresponding proofs in [12].

In this paper, we concentrate on the prime 2; all spaces and groups are assumed to be localized at 2. The sphere of dimension n is S^n, and the stable sphere is S^0. The mod-2 Moore space with cells in dimension $n-1$ and n will be denoted M^n. The Steenrod Algebra is A, and the Lambda algebra is Λ.

2 Presentation of $E_2^{*,*}(S^3)$.

Among the calculations of the $E_2^{*,*}(S^n)$ are Whitehead's Tables [26], Tangora's Memoir [24], and the computer calculations of [5]. This latter owes a lot to both [26] and [24]. The calculation of $E_2^{*,*}(S^n)$ has been done through the 60 stem for all spheres S^n. In this paper, we use the results of [5] for $E_2^{*,*}(S^3)$ through stem dimension 60.

We will present $E_2^{*,*}(S^3)$ in four forms, each with some information not readily available from the others. In the first presentation, each element is named according to its genealogy in the EHP sequence. For each n, the EHP sequence is a long exact sequence:

$$\cdots \longrightarrow E_2^{s,t}(S^n) \xrightarrow{E} E_2^{s,t}(S^{n+1}) \xrightarrow{H} E_2^{s-1,t-n-1}(S^{2n+1}) \xrightarrow{P} E_2^{s+1,t}(S^n) \longrightarrow \cdots$$

The name for a typical class in $E_2^{s,t}(S^n)$ is a sequence of integers $I = (i_1, i_2, \ldots, i_s)$. The sphere of origin is $i_1 + 1$, and the remaining sequence $(i_2, i_3, \ldots, i_s)$ is the name for the Hopf invariant of the class. To be consistent, the generator ("one") of $E_2^{0,0}(S^n)$ would be indicated by the empty sequence; the elements of Hopf invariant "one" are $2^i - 1$ for $i \geq 1$. The parentheses may be left off, and commas omitted. For example, the class 2 2 3 3 is born on S^3 and its Hopf invariant is 2 3 3. We also follow the conventions of [5]. That is, the sequence 2 4 1 1 is replaced by $*$ and certain other four and five digit sequences occur so commonly, that for convenience, they are written without spaces; e.g., 45333 for (4, 5, 3, 3, 3). The classes named by their genealogy are displayed in the Table in the Appendix, which is a standard Adams table. For each sequence, the filtration s is the length of the sequence, and the stem dimension $t - s$ is the sum of the indices.

This naming of a class in $E_2^{*,*}(S^n)$ by its EHP genealogy is related to the Λ algebra description, for which we introduce some notation (the same as may be found in [3], [5], [24] and elsewhere). Λ is the algebra (over $Z/2$) with a generator λ_i for each integer $i > 0$. There are relations: whenever $2i < j$,

$$\lambda_i \lambda_j = \sum_{k>0} \binom{j-2i-2-k}{k} \lambda_{j-i-k-1} \lambda_{2i+k+1}$$

Then Λ becomes a differential algebra, with

$$d(\lambda_i) = \sum_{k>1} \binom{i-k}{k} \lambda_{i-k}\lambda_{k-1}$$

For each sequence $I = (i_1, i_2, ..., i_s)$ of non-negative integers, λ_I denotes the product $\lambda_{i_1} \cdot \lambda_{i_2} \cdots \lambda_{i_s}$. A sequence I is called admissible if for each j, $2i_j \geq i_{j+1}$. The initial of such a sequence I is its first index i_1. It follows immediately from the relations that Λ has for basis (over $Z/2$) the set of all monomials λ_I, where I is admissible. For each positive integer n, $\Lambda(n)$ is defined to be the submodule of Λ spanned by the admissible λ_I with initial i_1 strictly less than n. One of the main results of [3] is the following.

Theorem 2.2 *There is an isomorphism*

$$E_2^{*,*}(S^n) \cong H^*(\Lambda(n))$$

The EHP sequences (2.1) for the $E_2^{s,t}(S^n)$ come about as follows. For each n, there is a short exact sequence:

$$0 \longrightarrow \Lambda(n) \xrightarrow{i} \Lambda(n+1) \xrightarrow{h} \Lambda(2n+1) \longrightarrow 0$$

Here i is the inclusion; h is defined by omitting λ_n when it occurs as the initial of a basis element λ_I and $h(\lambda_I)$ is zero if the initial is less than n. This short exact sequence of differential modules gives rise to a long exact sequence in homology which is the EHP sequence (2.1), and the map P comes from the differential in Λ. As in [4], [5], and [24], each class in $H^*(\Lambda)$ is represented by the leading term of a minimal representative. (The "leading term" and "minimal" representative are standard terms used in dealing with the Λ-algebra. A good reference is [24,Section 2.4].) When there is at most one element in a given bidegree, the genealogy name I refers to the same class that λ_I represents in the Λ algebra (after completion of λ_I to a cycle). The programs in [5] calculate $H^*(\Lambda)$ by an algorithm based on these EHP sequences. The multiplications (and Massey products) can also be found by Λ algebra calculations.

The second way we present $E_2^{*,*}(S^3)$ is in the form of a chart, as in Figure 1. Figure 1 is essentially the Table simplified, with some information added. A dot at a location indicates the presence of a non-zero class of that bidegree. A vertical line indicates multiplication (on the right) by h_0 (λ_0 in the Λ algebra); a line slanted to the right (at 45) indicates multiplication (on the right) by h_1 (λ_1 in the Λ algebra). When there is at most one class of a given bidegree, we may refer to location $(t-s,s)$ for that class, where the stem dimension $t-s$ is the sum of the subscripts, and the filtration s is the length of the sequence. Referring to the Table, we see that below stem dimension 40, each bidegree contains at most one class, and through stem dimension 60, each bidegree contains at most two classes. It is the purpose of Section 4 to establish the higher Adams differentials.

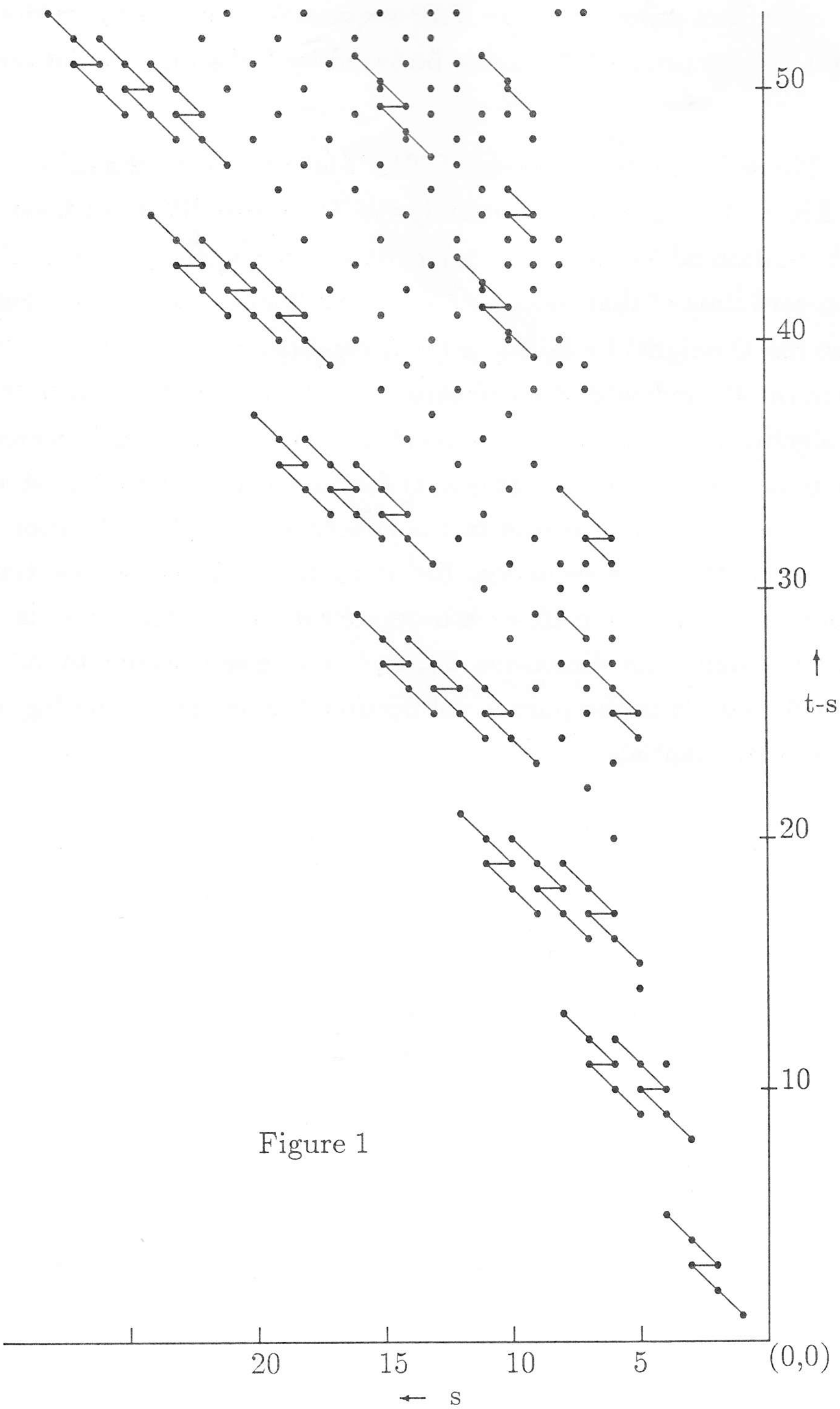

Figure 1

3 Patterns in $E_2^{*,*}(S^3)$.

A quick glance at Figure 1 gives the impression of a random collection of dots. However, they fall into patterns which can be understood through several theorems. For each non-negative integer k, let $A(k)$ be the subalgebra of the Steenrod algebra A generated by Sq^i, for $i \leq 2^k$. Thus, in our notation, $A(0) \cong \Sigma M^0$.

Theorem 3.1 [10] *There is a homomorphism*

$$E_2^{s,t}(S^3) \cong Ext_A^{s-1,t-2}(A(0), Z/2)$$

which is an isomorphism for $s > [(t-s)/5] + 2$.

Theorem 3.2 [11] *There is a geometric map*

$$f : \Omega^2 S^3 < 3 > \to Q\Sigma RP^2$$

which lifts to a map of resolutions, and induces the isomorphism of Theorem (3.1).

These theorems will be used to describe $E_2(S^3)$ in a more understandable form. These will be the third and fourth descriptions promised earlier. Let

$$B^{s,t} = Ext_{A(1)}^{s,t}(A(0), Z/2)$$

The groups $B^{s,t}$ form an infinite sequence of lightning flashes, periodically with period (4,12) in the standard Adams (s,t) indexing. Let

$$W^{s,t} = (F_2[v_1, h_{2,1}])^{s,t}$$

where $F_2[\ldots]$ stands for the polynomial algebra; v_1 has bidegree $(s,t) = (1,3)$ and $h_{2,1}$ has bidegree $(s,t) = (1,6)$. The groups $W^{s,t}$ form what is called a wedge. A wedge is usually the companion of a lightning flash, as follows. Let

$$C^{s,t} = B^{s,t} \oplus W^{s-1,t-6}$$

For easy reference, figure 2 is a chart of a wedge summand together with its companion lightning flash, "generated" by 24333.

Observation 3.3. For $t-s<55$

$$E_2^{s,t}(S^3) \cong B^{s-1,t-2} \oplus C^{s-5,t-25} \oplus C^{s-9,t-48} \oplus C^{s-9,t-62} \oplus F^{s,t}$$

where $F^{s,t}$ is given in figure 3. This representation does not contain all the structure. Compositions within a summand are valid but there are compositions connecting the summands which are not indicated.

The "generators " of B are 1 and 233 (i.e., λ_1 and $\lambda_2\lambda_3\lambda_3$. The "generators" of C are 24333 (with companion wedge generator 2 45333), 24733 6653 (with companion wedge generator 2 45553 6633), and 2 3 5 10 11 3577 (with companion wedge generator 2 4 5 9 3 59777). The B and C summands are v_1 periodic and the classes in F are all v_1 torsion in this representation. Some classes in F admitt v_1 multiplications into one of the C summands. All the classes in B and C inject into $Ext_A^{s,t}(A(0), Z/2)$ by the map of 3.1. We will use Observation 3.3 and Theorem 3.2 to calculate the higher Adams differentials.

Another observation (which predates the calculations) gives a different picture of these classes. This is the basis of our last description. Let K be the minimum stable A-module with

$$Sq^8Sq^4Sq^2Sq^1 \neq 0.$$

This is a complex with 5 cells. Let $N = K \otimes A(0)$.

Observation 3.4. There is a complex

$$\begin{aligned} Ext_{A(0)}^{s,t}(Z/2, Z/2) &\xrightarrow{d_0} Ext_{A(1)}^{s-1,t-2}(A(0), Z/2) \\ &\xrightarrow{d_1} Ext_{A(2)}^{s-3,t-11}(A(0), Z/2) \\ &\xrightarrow{d_2} Ext_{A(3)}^{s-5,t-29}(N, Z/2). \end{aligned}$$

The homology of this complex is $E_0(E_2^{s,t}(S^3))$ for $t-s<55$. The differentials d_0 and d_1 are zero, and d_2 is determined by $d_2 v_2^4 h_1 \neq 0$. There is no standard notation for labeling classes in

$$Ext_A(2)(Z/2, Z/2)$$

but this should be a self-explanatory name. Of course this differential implies a small number of other-differentials because of compositions. This description is very illuminating. If the reader is not familiar with Ext calculations over $A(2)$, he should consult [9]. In the range where we use it, the Ext calculation over $A(3)$ is essentially a stable calculation. It is also interesting to note that all of the differentials in the Adams spectral sequence which we give in Section 4 occur in the summand $Ext_{A(2)}^{s-3,t-11}(A(0), Z/2)$. Of course this will not continue. The differential described above suggests that the classes in $Ext_{A(2)}^{*,*}(N, Z/2)$ have a close connection with v_2 periodic phenomena. In particular, 23577 should be v_2^8 periodic, but the calculations to verify this have not been made.

Observation 3.4 also suggests that there are modules N_i and maps

$$\dots \xrightarrow{d_{i-1}} Ext_{A(i)}^{*,*}(N_i, Z/2) \xrightarrow{d_i} Ext_{A(i+1)}^{*,*}(N_{i+1}, Z/2) \xrightarrow{d_{i+1}} \dots$$

that would represent $E_2^{*,*}(S^3)$. The next one should be valid to about $t - s < 119$. The start of each pattern in the resolution corresponds to the first place that λ_{2^i-1} appears in $E_2^{*,*}(S^3)$.

A possible explanation for this pattern is the following. It is easy to see that the Λ algebra is filtered by differential submodules Λ_i, where Λ_i is generated by λ_I, I admissible, $I = (i_1, \dots, i_s)$ with $i_{s-j} \equiv -1 \ mod \ 2^{i-j}$ for $j = 0, 1, \dots, i-1$. Let $\Lambda_i(3)$ be the corresponding filtration of $\Lambda(3)$. This gives a decreasing filtration of cochain complexes

$$\dots \subseteq \Lambda_2(3) \subseteq \Lambda_1(3) \subseteq \Lambda_0(3) = \Lambda(3)$$

This gives rise to a spectral sequence with

$$E_1^n \cong H^*(\Lambda_n(3)/\Lambda_{n+1}(3)) \Rightarrow H^*\Lambda(3).$$

The above calculations give:

$$H^*(\Lambda_0(3)/\Lambda_1(3), d) = Ext^{*,*}_{A(0)}(Z/2, Z/2)$$

$$H^*(\Lambda_1(3)/\Lambda_2(3), d) = Ext^{*-1,*-2}_{A(1)}(A(0), Z/2)$$

We conjecture:

$$H^*(\Lambda_2(3)/\Lambda_3(3), d) = Ext^{*-3,*-11}_{A(2)}(A(0), Z/2)$$

$$H^*(\Lambda_3(3)/\Lambda_4(3), d) = Ext^{*-5,*-29}_{A(3)}(N, Z/2)$$

This is verified for $t - s < 60$ by our calculations. Indeed, this is essentially Observation 3.4 where it is further noted that $E_2 = E_\infty$ for $t - s \leq 60$ and that d_1 is mostly zero.

Further study of the calculations also suggest:

$$H^*(\Lambda_4(3)/\Lambda_5(3), d) = Ext^{s-6,t-60}_{A(4)}(A(0), Z/2) \oplus Ext^{s-7,t-65}_{A(4)}(\bar{N}_4, Z/2)$$

where

$$\bar{N}_4 = (A(4)/A(4)\{Sq^4, Sq^8, Sq^{16}\}) \otimes A(0).$$

If this is correct then there is a nontrivial $d_2 : E_2^2 \to E_2^4$ for $t - s$ about 80.

This spectral sequence probably has many differentials but certainly is among the simplest in homotopy theory. It is easy to see that E_1^5 will have its first non-zero value in (s,t) = (8,127).

These results suggest the following question. Is

$$E_1^n = \oplus_i \; Ext_{A(n)}(N_{n,i}, Z/2)$$

for some A(n) modules $N_{n,i}$? If this is true and the patterns of differentials holds up, then one might be able to compute $E_2^{*,*}(S^3)$ out to large stems.

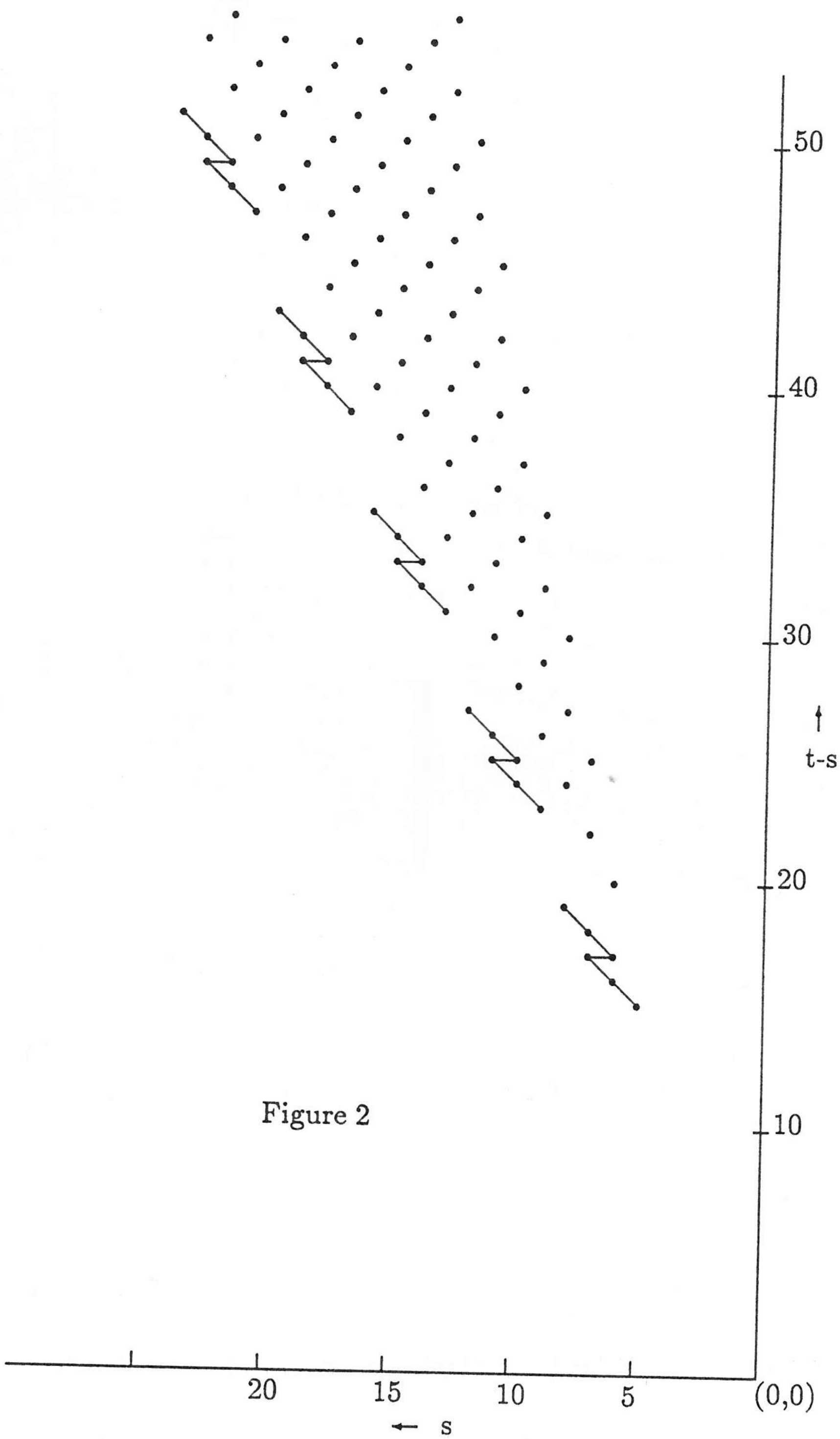

Figure 2

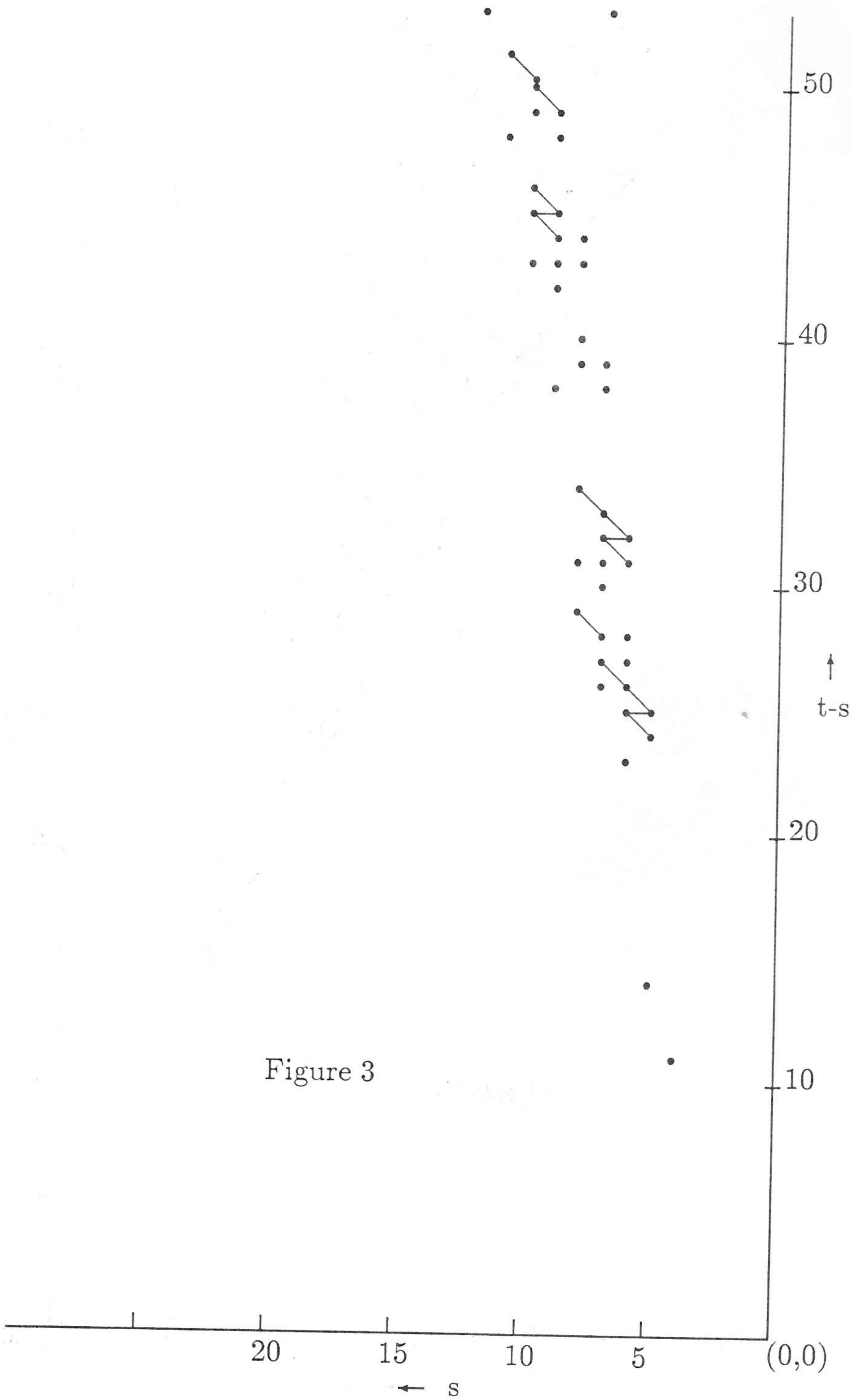

Figure 3

4 Adams differentials in $E_*^{*,*}(S^3)$

The main tools for finding differentials in $E_*^{*,*}(S^3)$ are Theorems 3.1 and 3.2, the differentials for the stable Moore space as calculated in [9], and the periodicity operator v_1^4, which will be denoted P. (In the rest of this paper we will not use the letter P for the P in the EHP sequence.) Each of the differentials is periodic under P . We label the classes in two ways. The first is by its genealogy name as in the Table. The second is by giving its $(t-s, s)$ location, which nearly always uniquely determines the class, as Figure 1 shows.

The first differential we encounter is at location (20,6), where we must show that

$$d_2(2\ 45333) = 2\ 1\ 1\ 24333$$

To show this, we apply the periodicity operator (three times), and consider the classes

$$P^3(2\ 45333) = 2^{13}\ 45333$$
$$P^3(2\ 1\ 1\ 24333) = 2\ 1\ 1\ (*)^3\ 24333$$

These classes are in the range covered by the isomorphism of Theorem 3.1. The geometric map of Theorem 3.2, and the differential in [9; Theorem 4.1] implies that

$$d_2(2^{13}\ 45333) = 2\ 1\ 1\ (*)^3\ 24333$$

Then because P commutes with d_2, we must also have

$$d_2(2\ 45333) = 2\ 1\ 1\ 24333$$

Then again by periodicity, we also have for all $k > 0$,

$$d_2(2^{4k+1}\ 45333) = 2\ 1\ 1\ (*)^k\ 24333$$

In a similar way, we have the differentials listed below as (a) to (v).

We give each of the differentials generically and for easy reference to the tables, we also give the lowest dimensional version both by its genealogy name and location. The proofs of the remaining (b) to (v) are similar, first using [9] to obtain the result in the range where Theorems 3.1 and 3.2 apply, and then extending by periodicity.

(a)	For each $k \geq 0$,	$d_2(2^{4k+1}\ 45333)$	$=$	$2\ 1\ 1\ (*)^k\ 24333$
	starting with	$d_2(2\ 45333)$	$=$	$2\ 1\ 1\ 24333$
		$d_2(20, 6)$	$=$	$(19, 8)$
(b)	For each $k \geq 1$,	$d_2(2^{4k}\ 45333)$	$=$	$1\ 1\ (*)^k\ 24333$
	starting with	$d_2(2^4\ 45333)$	$=$	$1\ 1\ *\ 24333$
		$d_2(26, 9)$	$=$	$(25, 11)$
(c)	For each $k \geq 0$,	$d_3(2^{4k+2}\ 35733)$	$=$	$(*)^{k+1}\ 24333$
	starting with	$d_3(2\ 2\ 35733)$	$=$	$1\ *\ 24333$
		$d_3(25, 7)$	$=$	$(24, 10)$
(d)	For each $k \geq 1$,	$d_3(2^{4k+3}\ 35733)$	$=$	$2\ 1\ (*)^{k+1}\ 24333$
	starting with	$d_3(2^3\ 35733)$	$=$	$2\ 1\ *\ 24333$
		$d_3(27, 8)$	$=$	$(26, 11)$
(e)	For each $k \geq 0$,	$d_4(2^{4k+3}\ 3\ 3\ 6653)$	$=$	$(*)^{k+2}\ 24333$
	starting with	$d_4(2^3\ 3\ 3\ 6653)$	$=$	$*\ *\ 24333$
		$d_4(32, 9)$	$=$	$(31, 13)$
(f)	For each $k \geq 0$,	$d_4(2^{4k+4}\ 3\ 3\ 6653)$	$=$	$2\ (*)^{k+2}\ 24333$
	starting with	$d_4(2^4\ 3\ 3\ 6653)$	$=$	$2\ *\ *\ 24333$
		$d_4(34, 10)$	$=$	$(33, 14)$
(g)	For each $k \geq 0$	$d_4(2^{4k+1}\ 3\ 5\ 5\ 3\ 6653)$	$=$	$2^{4k+8}\ 35733$
	starting with	$d_4(2\ 3\ 5\ 5\ 3\ 6653)$	$=$	$2^8\ 35733$
		$d_4(38, 9)$	$=$	$(37, 13)$
(h)	For each $k \geq 1$,	$d_4(2^{4k-1}\ 4\ 3\ 3\ 3\ 6653)$	$=$	$2^{4k+6}\ 45333$
	starting with	$d_4(2^3\ 4\ 3\ 3\ 3\ 6653)$	$=$	$2^{10}\ 45333$
		$d_4(39, 11)$	$=$	$(38, 15)$

(i) For each $k \geq 0$, $d_3(2^{4k+1}\,4733)\,6653 = 2^{4k+6}\,3\,3\,6653$
starting with $d_3(2\,4733\,6653) = 2^6\,3\,3\,6653$
$d_3(39,9) = (38,12)$

(j) For each $k \geq 0$, $d_4(2^{4k+2}\,3553\,6653) = 2^{4k+9}\,35733$
starting with $d_4(2\,2\,3553\,6653) = 2^9\,35733$
$d_4(40,10) = (39,14)$

(k) For each $k \geq 1$, $d_4(2^{4k}\,4\,3\,3\,6653) = 2^{4k+7}\,45333$
starting with $d_4(2^4\,4\,3\,3\,6653) = 2^{11}\,45333$
$d_4(41,12) = (40,16)$

(l) For each $k \geq 0$, $d_2(2^{4k+1}\,45553\,6653) = 2\,1\,1\,(*)^4\,24733\,6653$
starting with $d_2(2\,45553\,6653) = 2\,1\,1\,24733\,6653$
$d_2(44,10) = (43,12)$

(m) For each $k \geq 0$, $d_4(1\,2^{4k+1}\,45553\,6653) = 2^{4k+8}\,2\,3\,3\,6653$
starting with $d_4(1\,2\,45553\,6653) = 2^8\,2\,3\,3\,6653$
$d_4(45,11) = (44,15)$

(n) For each $k \geq 0$, $d_3(2^{4k+2}\,45553\,6653) = 2^{4k+5}\,24333\,6653$
starting with $d_3(2\,2\,45553\,6653) = 2^5\,24333\,6653$
$d_3(46,11) = (45,14)$

(o) For each $k \geq 0$, $d_3(2^{4k+1}\,43565\,23577) = 1\,(*)^{k+1}\,24733\,6653$
starting with $d_3(2\,43565\,23577) = 1\,*\,24733\,6653$
$d_3(49,11) = (48,14)$

(p) For each $k \geq 1$, $d_2(2^{4k}\,45553\,6653) = 1\,1\,(*)^k\,24733\,6653$
starting with $d_2(2^4\,45553\,6653) = 1\,1\,*\,24733\,6653$
$d_2(50,13) = (49,15)$

(q) For each $k \geq 0$, $d_3(2^{4k+2}\,43565\,23577) = 2\,1\,2^{4k+1}\,24733\,6653$
starting with $d_3(2\,2\,43565\,23577) = 2\,1\,2\,24733\,6653$
$d_3(51,12) = (50,15)$

(r) For each $k \geq 0$, $d_4(1\,2^{4k+2}\,43565\,23577) = 2^{4k+8}\,24333\,6653$
starting with $d_4(1\,2\,2\,43565\,23577) = 2^8\,24333\,6653$
$d_4(52,13) = (51,17)$

(s) For each $k \geq 1$, $d_4(2^{4k+1}\ 3\ 35565\ 23577) = 2\ 1\ 2\ (*)^{k+1}\ 24733\ 6653$
starting with $d_4(2\ 3\ 35565\ 23577) = 2\ 1\ 2\ *\ 24733\ 6653$
$d_4(53, 12) = (52, 16)$

(t) For each $k \geq 1$, $d_4(2^{4k+6}\ 45553\ 23577) = 2^{4k+5}\ 24333\ 6653$
starting with $d_4(2^6\ 45553\ 23577) = 2^9\ 24333\ 6653$
$d_3(54, 15) = (53, 18)$

(u) For each $k \geq 1$, $d_4(2\ 2\ 1\ 2\ (*)^k\ 24733\ 6653) = 2^{4k+11}\ 35733$
starting with $d_4(2\ 2\ 1\ 2\ *\ 24733\ 6653) = 2^{15}\ 35733$
$d_4(54, 17) = (53, 21)$

(v) For each $k \geq 1$, $d_4(1\ 2^{4k+2}\ 45553\ 6653) = 2^{4k+13}2\ 3\ 3\ 6653$
starting with $d_4(1\ 2^6\ 45553\ 6653) = 2^{17}\ 2\ 3\ 3\ 6653$
$d_4(55, 16) = (54, 20)$

The differentials in $E_*^{*,*}(S^3)$ are indicated in the figure 4.

Theorem 4.1 *The classes in stems through stem dimension 52 which are not included in the above as (a) to (v) are all permanent cycles.*

In particular, besides the elements in the two periodic lightning flashes starting with $1(=\lambda_1)$ at $(1,1)$ and $2\ 3\ 3\ at(8,3)$, and the partial flash starting at $(15,5)$, the nonzero classes through stem dimension 52 will be constructed in the following proof. The chart for $E_\infty^{*,*}(S^3)$ is given in Figure 5. We also indicate without proof "exotic" right multiplications by 2 and by η by dotted lines.

Proof. In order to show that each class not eliminated by the differentials (a) to (v) is a homotopy class, we construct it as a composite (sometimes a Toda bracket) or show that the class is a permanent cycle because there is nothing in the UASS for S^3 for it to hit. When we say that a class named I is a composite $J \circ K$, we mean that J and K in $E_\infty^{*,*}(S^n)$ lift to homotopy classes, say α and β, and that the composite $\alpha \circ \beta$ projects to I in $E_\infty^{*,*}(S^3)$.

Similarly for brackets, $I = \{J, K, L\}$, means that J, K and L lift to classes α, β, γ, and that the Toda bracket $\{\alpha, \beta, \gamma\}$ projects to I. For example, at location $(t-s, s) = (8, 3)$, the class 233 may be constructed as the bracket $\{1, 0, 33\}$. In homotopy, this is the standard construction of ϵ as the Toda bracket $\{\eta, 2\iota, \nu^2\}$. There is a possible source of confusion here since compositions in the Lambda algebra are written in the reverse of the usual order. The Toda bracket $\{\eta, 2\iota, \nu^2\}$ arises from the sequence of maps

$$S^{10} \xrightarrow{\nu^2} S^4 \xrightarrow{2\iota} S^4 \xrightarrow{\eta} S^3.$$

We will write compositions and Toda brackets in the order which makes the connection with the Lambda algebra clear.

At (11,4), 2333 is the composition of two permanent cycles 233 and 3. In homotopy, this class is the composition $\epsilon \circ \nu$.

At (14,5), 23333 is the composition of two permanent cycles 2333 and 3. In homotopy, this class is the composition $\epsilon \circ \nu^2$.

At (22,7), a calculation in the Lambda algebra shows that the class 2 2 45333 is the composition of two permanent cycles 233 and 6233. In homotopy, this class is the composition $\epsilon \circ \kappa$. The needed calculation is d applied to 2 4 2 3 5 7 + 2 4 4 7 3 3 + 2 4 4 5 5 3 + 2 3 6 4 5 3 + 2 4 8 3 3 3 + 2 3 4 8 3 3. This is a tedious calculation but most of the time the calculation in the Lambda algebra is much shorter.

At (23, 6), 2 35733 is a permanent cycle, because Table I shows that there are no classes in (22, s), with $s \geq 8$. The class 2 35733 may be constructed as the bracket $\{233, 0, 6233\}$. In homotopy, this class is the Toda bracket $\{\epsilon, 2\iota, \kappa\}$.

At (23,9), the class $*$ 24333 $= P(24333)$.

At (24,5), the class 23577 is a permanent cycle. The only other possibility is that $d_4(23577)$ might be $*$ 24333. But $*$ 24333 is a class which suspends to Ph_1d_0 in $E_2^{*,*}(S^0)$, and there it is not a boundary in any $E_r^{*,*}(S^0)$. Hence it is not a boundary in any $E_r^{*,*}(S^3)$. The class 23577 may be constructed as the bracket $\{233, 53, 7\}$. In homotopy, this class is the Toda bracket $\{\epsilon, \overline{\nu} + \epsilon, \sigma\}$.

At (24,8), 222 45333 is in the composite 2233 with 6233. In homotopy, this class is the composition $\epsilon' \circ \kappa$.

At (25,5), The class 24577 is in the bracket $\{1, 0, 4577\}$, and so appears on S^3 with Hopf Invariant 4577. In homotopy, this class is the Toda bracket $\{\eta, 2\iota, \phi\}$. and appears on S^3 with Hopf Invariant ϕ.

At (25,6), the class 1 23577 is the composite of 1 and 23577.

At (25,10), $2 * 24333 = P(2\ 24333)$. Also $2 * 24333$ may be constructed as $\{1, 0, *24333\}$.

At (26,6), a calculation in the Lambda algebra shows that the class $2\ 23577 = 24577\ 1$, which is the composition of two permanent cycles.

At (26,7), 2 3 57333 is the composite of 2 35733 and 3.

At (27,6), the class 2 3 3577 is the composition of two permanent cycles 233 and 577. In homotopy, this class is the composition $\epsilon \circ \overline{\sigma}$. Also, 2 3 3577 is the composition of 23577 and 3.

At (28,6), a calculation in the Lambda algebra shows that the class 2 4 3577 is the composition of two permanent cycles 24577 and 3. This class may also be constructed as the bracket $\{233, 0, 577\}$. In homotopy, this class is the Toda bracket $\{\epsilon, 2\iota, \overline{\sigma}\}$.

At (28,7), the class 2 3 3 6653 is the composition of two permanent cycles 233 and 6653. In homotopy, this class is the composition $\epsilon \circ \overline{\kappa}$.

At (29,8), 1 2 3 3 6653 is the composition of 1 with 233 6653.

At (29,9), $2222\ 35733 = P(35733\ )$ and so must also be a permanent cycle. (35733 first appears on S^4)

At (30,7), a calculation in the Lambda algebra shows that the class 2 2 4 3577 is the composition of two permanent cycles 233 and 3577. In particular, d(2 4 6 5 7 7) = 233 3577 + (2 2 4 3577 + 2 2 3 4 5 7 7 + 1 2 3 5 5 7 7). The first term is a cycle by itself and the the last three terms complete 2 2 4 3577 to a cycle. This is typical of the appropriate Lambda algebra calculations that are used in the proof. We will not give as much detail in what follows.

At (30,8), 2 2 3 3 6653 is the composition of two permanent cycles 2 2 3 6653 and 3.

At (31,6), The class 2 35777 is the composition of permanent cycles 23577 and 7.

At (31,7), a calculation in the Lambda algebra shows that the class 2 4 3 3577 is the composition of two cycles 2 4 3577 and 3.

At (31,8), 2333 6653 is the composition of two permanent cycles 2333 and 6653.

At (31,10), 2^5 35733 $= P(2\ 35733)$.

At (32,6), the class 245777 is the composition of permanent cycles 24577 and 7.

At (32,7), 1 2 35777 is the composition of two permanent cycles 1 and 2 35777.

At (32,12), 2^7 45333 $= P(2^3\ 45333)$.

At (33,7), a calculation in the Lambda algebra shows that the class 2 2 35777 is the composition of two permanent cycles 2 45777 and 1.

At(34,8), 2 1 2 35777 a calculation in the Lambda algebra shows that the class is the composition of two permanent cycles 2 2 35777 and 1.

At (35,9), 24333 6653 is the composition of two permanent cycles 24333 and 6653.

At (36,11), 2^5 3 3 6653 $= P(233\ 6653)$.

At(37,10), 2 24333 6653 is a permanent cycle because it is not periodic in $E_5^{*,*}(S^3)$ because P (37,10) = (45,14) which is hit by a d_4. The only possible targets for differentials land in periodic groups. Therefore (37,10) must be a permanent cycle.

At (38,7), 2 3 57777 is the composition of two permanent cycles 235777 and 7.

At (39,7), 2 4 57777 is the composition of two permanent cycles 2 45777 and 7.

At (39,8), 2 3 5 7 3577 is not periodic, and the only possible differentials land in periodic groups. Therefore (39,8) must be a permanent cycle.

At (40,8), 2457 3577 must be a permanent cycle because there is nothing left from $E_2^{*,*}(S^3)$ in stem 39 for it to hit.

At (40,10), 1 24733 6653 must be a permanent cycle because there is nothing left from $E_2^{*,*}(S^3)$ in stem 39 for it to hit.

At (41,10), 2 24733 6653 must be a permanent cycle because there is nothing left from $E_2(^{*,*}(S^3)$ in stem 40 for it to hit.

At (41,11), 1 1 24733 6653 is the composite 1 1 24733 with 6653.

At (42,9), 23357 3577 is the composition of two permanent cycles 2357 3577 and 3.

At (42,11), There are two classes. One of them is 1 2 24733 6653, which is the composition of two permanent cycles 1 and 2 24733 6653. The other is 2 1 24733 6653 which is the composition of two permanent cycles 2 24733 6653 and 1.

At (43,8), 233 59777 is the composition of two permanent cycles 233 and 59777.

At (43,9), 24357 3577 is the composition of two permanent cycle 2457 3577 and 3.

At (43,10), 23365 23577 must be a permanent cycle because there is nothing left in filtration 12 in the 42-stem except periodic groups, and 23365 23577 is not periodic.

At (43,13), 2^4 24333 6653 $= P(24333\ 6653)$.

At(44,8), 2 4 3 59777 We cannot have $d_5(44,8) = (43,13)$ by periodicity. The only remaining possibility is that $d_2(44,8)$ might hit (43,10). Taking Hopf Invariants, this would imply that $d_2(42,7)$ would be (41,9). To show this doesn't occur, we give a Toda bracket construction for 4 3 59777, as follows. There is a map $f : S^{48} \to \Sigma^{10}M$, because 3 59777 has order 2. Composing with the map $\Sigma^{10}M \to \Sigma^9 M$ is null, so there is a map $\alpha : S^{48} \to RP(6,9)$. Let $\beta : RP(6,9) \to S^5$ be the attaching map (which would give RP(5,10)). Then $\beta \circ \alpha$ is represented by 4 3 59777 in $E_2^{*,*}(S^5)$.

At (44,9), 233 5 9 3577 is the composite of two permanent cycles 24577 and 577.

At (44,12), 2 1 2 24733 6653 must be a permanent cycle because there is nothing left above filtration 13 in the 43-stem except groups that persist to $E_\infty^{*,*}(S^3)$

At (45,9), 2 4 3 5 9 3577 is the composition of 2 4 3 59777 and 1. This class is the Toda bracket $\{(40,8),1,3\}$

At (45,10), 1 2 3 3 5 9 3577 is the composite of two permanent cycles 1 23359 3577. Also, 1 2 3 3 5 9 3577 is the composite of two permanent cycles 233 5 7 3577 and 3.

At (46,10), 2 2 3 3 5 9 3577 must be a permanent cycle because there is nothing left above filtration 11 in the 45 stem except periodic groups, and (46,10) is not periodic. Also, (46,10) may be constructed as the composite of (45,9) and 1. This class at (46,10) is also the composite of (40,8) and 3 3. Also 2 2 3 3 5 9 3577 is the composite of (45,9) and 1, and this class is the composite of (40,8) and 3 3.

At (47,12), 1 2 2 45553 6653 must be a permanent cycle because there is nothing left above filtration 13 in the 46-stem.

At (48,9), under $f : \Omega^3 S^3 \to QRP^2$, the class 2 3 5 3 59777 maps to $h_1 B$ on the bottom cell in RP^2, which is non-zero. There it is a cycle, and all potential targets for $d_r(48, 9)$ map non-zero to QRP^2. Therefore all differentials on (48,9) are null.

At (48,11), this class at 2 3 3 5 6 5 23577 must be a permanent cycle because there is nothing left above filtration 12 in the 47-stem. We cannot have $d_1(48, 11) = (47, 12)$ because this would be a d_1 in the Lambda algebra. Under f_*, this class maps to $e_0 r$ on the bottom cell in QRP^2, where it is non-zero.

At (48,12), 222 45553 6653 must be a permanent cycle because there is nothing left above filtration 13 in the 47-stem. Under f_*, this class maps to gj on the top cell in QRP^2, where it is non-zero.

At (49,9), under $f : \Omega^3 S^3 \to QRP^2$, the class 2 4 5 3 59777 maps to $h_1 B$ on the top cell, which is non-zero There it is a cycle, and all potential targets for $d_r(48, 9)$ map non-zero to QRP^2. Therefore all differentials on (48,9) are null.

At (49,10), under $f : \Omega^3 S^3 \to QRP^2$, the class 2 3 5 3 5 9 3577 maps to $h_1 P c_0 h_5$) on the bottom cell, which is non-zero. There it is a cycle, and all potential targets for $d_r(48, 9)$ map non-zero to QRP^2. Therefore all differentials on (49,10) are null.

At (49,13), the class 1 222 45553 6653 is the composite of 1 with 222 45553 6653. Also, there is nothing left for it to hit above filtration 14 in the 48-stem.

At (49,14), the class 2 $*$ 24733 6653 = P(2 24733 6653). Also, there is nothing left for it to hit above filtration 15 in the 48-stem.

At (50,10), there are two classes. One of them is 2 2 3 5 3 59777 which is 2 4 5 3 59777 composed with 1. The other is 2 4 5 3 5

9 3577, which under $f : \Omega^3 S^3 \to QRP^2$, maps to $h_1 P c_0 h_5$) on the top cell. There it is a cycle, and all potential targets for $d_r(48,9)$ map non-zero to QRP^2. Therefore all differentials on (49,10) are null.

At (50,12), The only possibility would be $d_2(50,12) = (49,14)$, but this cannot happen because (50,12) is 1 composed with (49,11) and $d_2(49,11) = 0$. Also a non-zero d_2 on (50,12) would contradict the differential $d_2(50,13) = (49,15)$.

At (50,15), the class 2 1 $*$ 24733 6653 = P(2 1 24733 6653). Also, there is nothing left for it to hit above filtration 16 in the 49-stem.

At (51,11), the class 2 2 3 5 9 3577 is the composite of 2 4 5 3 5 9 3577 and 1.

At (51,14), the class the class 1 2222 45553 6653 must be a permanent cycle because there is nothing left for it to hit above filtration 16 in the 50-stem.

There are no classes remaining in the 52-stem except periodic ones.

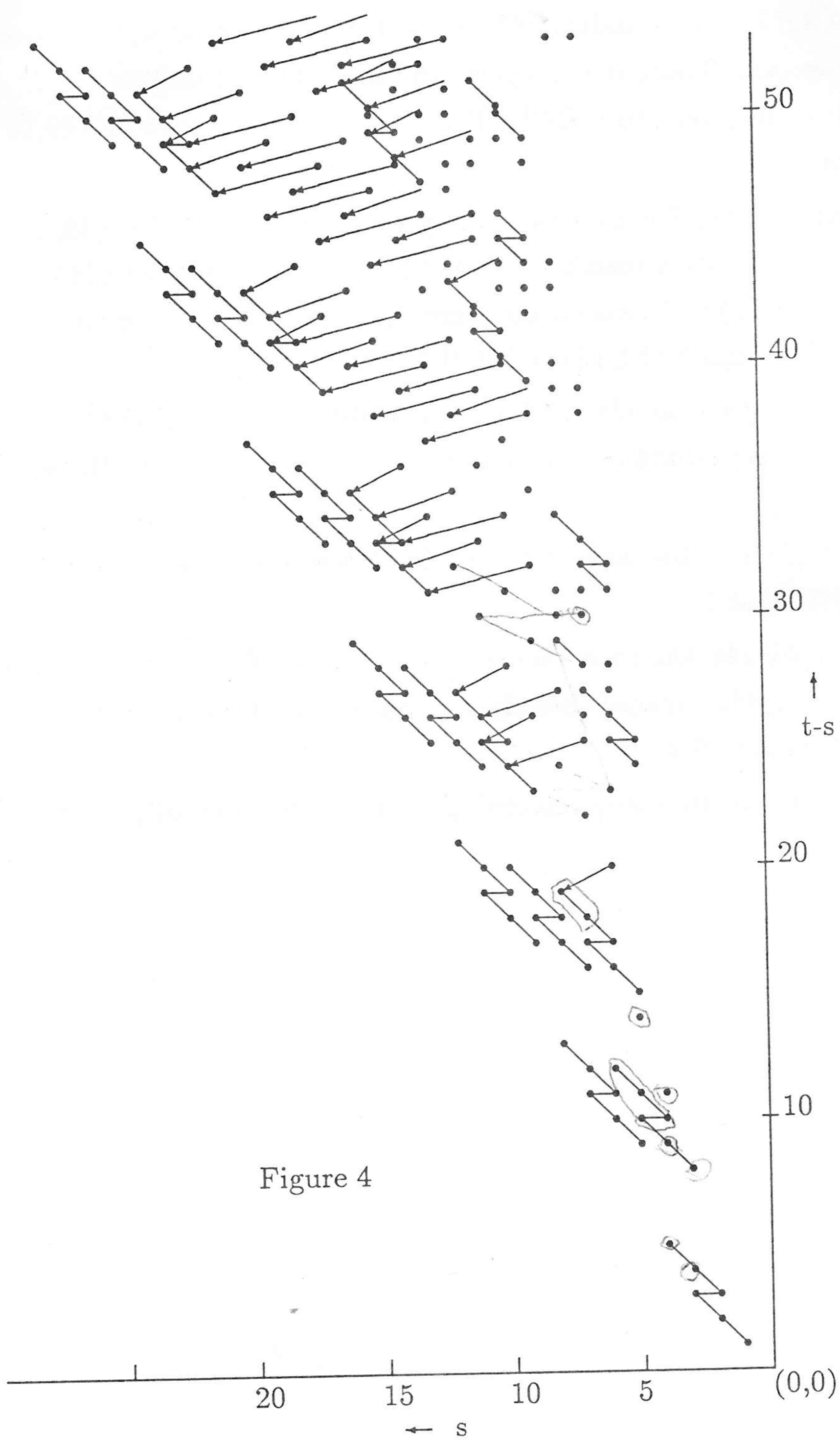

Figure 4

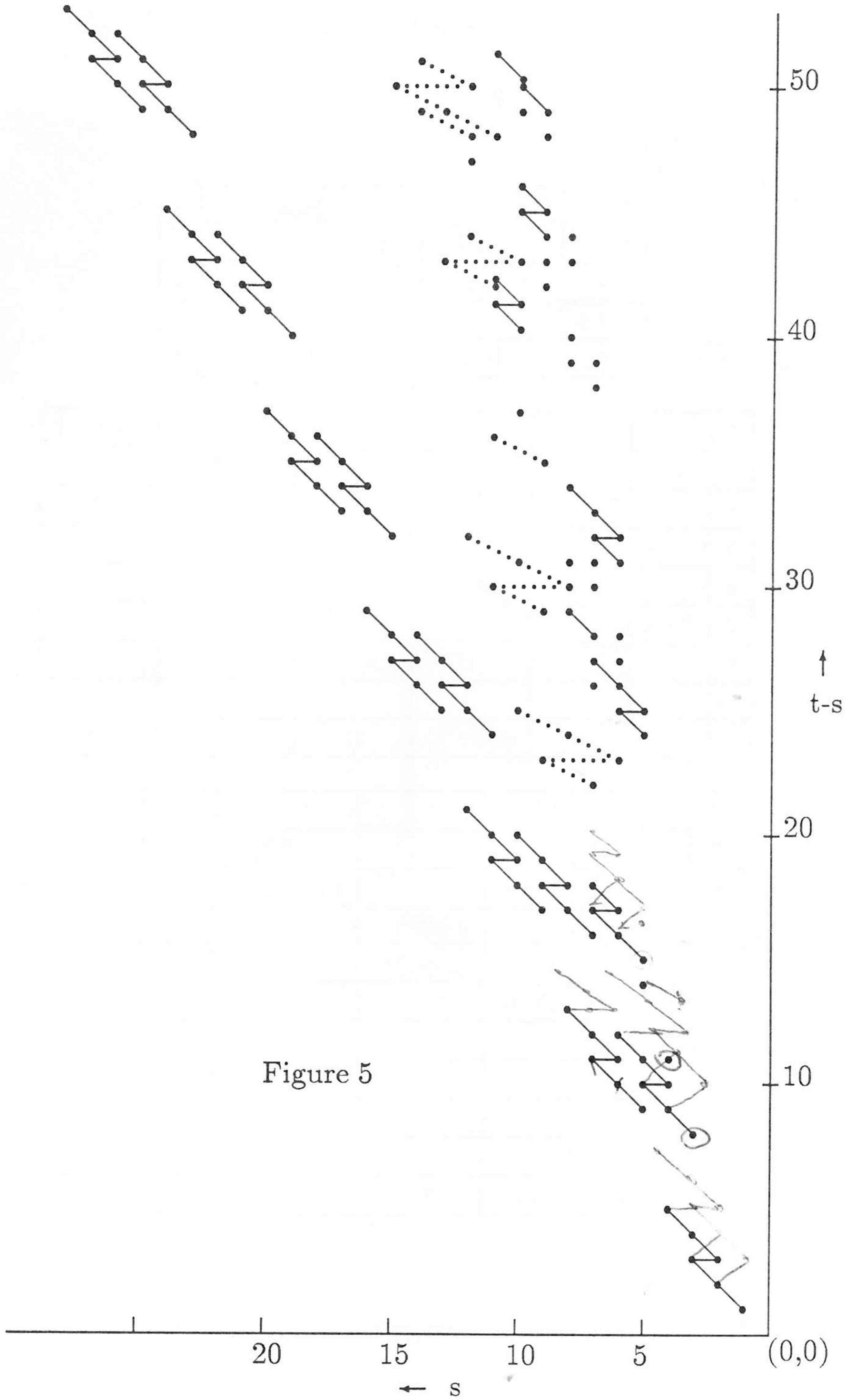

Figure 5

APPENDIX

Table of $E_2^{s,t}(S^3)$

4					2 1 1 1		
3			1 1 1	2 1 1			
2		1 1	2 1				
1	1						
	1	2	3	4	5	6	7

8						2 1 1 * 1
7				1 1 * 1	2 1 * 1	
6			1 * 1	2 * 1	2 1 1 2 3 3	
5		* 1	1 1 2 3 3	2 1 2 3 3		
4		1 2 3 3	2 2 3 3	2 3 3 3		
3	2 3 3					
2						
1						
	8	9	10	11	12	13

10					1 * * 1
9				* * 1	1 1 2 34411 1
8				1 2 34411 1	2 2 34411 1
7			2 34411 1	1 1 24333	2 1 24333
6			1 24333	2 24333	
5	2 3 3 3 3	24333			
4					
3					
2					
1					
	14	15	16	17	18

12			2 1 1 * * 1	
11	1 1 * * 1	2 1 * * 1		
10	2 * * 1	2 1 1 2 34411 1		
9	2 1 2 34411 1			
8	2 1 1 24333			
7				2 2 45333
6		2 45333		
5				
4				
3				
2				
1				
	19	20	21	22

14				1 * * * 1
13			* * * 1	1 1 2 34411 * 1
12			1 2 34411 * 1	2 2 34411 * 1
11		2 34411 * 1	1 1 * 24333	2 1 * 24333
10		1 * 24333	2 * 24333	
9	* 24333			2..2 45333
8		2.2 45333		
7			2 2 35733	2 3 35733
6	2 35733		1 23577	2 23577
5		23577	2 4 5 7 7	
4				
3				
	23	24	25	26

16			2 1 1 * * * 1	
15	1 1 * * * 1	2 1 * * * 1		
14	2 * * * 1	2 1 1 2 34411 * 1		
13	2 1 2 34411 * 1			
12	2 1 1 * 24333			
11				2....2 45333
10		2...2 45333		
9			2..2 35733	
8	2.2 35733		1 2 3 36653	2 2 3 36653
7	2 1 23577	2 3 36653		2 2 4 3577
6	2 3 3577	2 4 3577		
5				
4				
	27	28	29	30

18				1 * * * * 1
17			* * * * 1	1 1 2 34411 * * 1
16			1 2 34411 * * 1	2 2 34411 * * 1
15		2 34411 * * 1	1 1 * * 24333	2 1 * * 24333
14		1 * * 24333	2 * * 24333	
13	* * 24333			2......2 45333
12		2.....2 45333		
11			2....2 35733	
10	2...2 35733			2..2 3 36653
9		2 2 2 3 36653		
8	2 3 3 36653			2 1 2 35777
7	2 4 3 3577	1 2 35777	2 2 35777	
6	2 35777	2 45777		
5				
	31	32	33	34

20			2 1 1 * * * * 1	
19	1 1 * * * * 1	2 1 * * * * 1		
18	2 * * * * 1	2 1 1 2 34411 * * 1		
17	2 1 2 34411 * * 1			
16	2 1 1 * * 24333			
15				2........2 45333
14		2.......2 45333		
13			2......2 35733	
12	2.....2 35733			2....2 3 36653
11		2...2 3 36653		
10			2 24333 6653	
9	24333 6653			2 3 5 5 36653
8				
7				2 3 5 7 7 7 7
6				
5				
	35	36	37	38

22				1 * * * * * 1
21			* * * * * 1	1 1 2 34411 * * * 1
20			1 2 34411 * * * 1	2 2 34411 * * * 1
19		2 34411 * * * 1	1 1 * * * 24333	2 1 * * * 24333
18		1 * * * 24333	2 * * * 24333	
17	* * * 24333			2..........2 45333
16		2.........2 45333		
15			2........2 35733	
14	2.......2 35733			2......2 3 36653
13		2.....2 3 36653		
12			2 2 2 24333 6653	
11	2 2 24333 6653		1 1 24733 6653	2 1 24733 6653 1 2 24733 6653
10		2 2 3 5 5 36653 1 24733 6653	2 24733 6653	
9	24733 6653			2 3 3 5 7 3577
8	2 3 5 7 3577	2 4 5 7 3577		
7	2 4 5 7 7 7 7			
6				
5				
	39	40	41	42

24			2 1 1 * * * * * 1	
23	1 1 * * * * * 1	2 1 * * * * * 1		
22	2 * * * * * 1	2 1 1 2 34411 * * * 1		
21	2 1 2 34411 * * * 1			
20	2 1 1 * * * 24333			
19				2............2 45333
18		2...........2 45333		
17			2..........2 35733	
16	2.........2 35733			2........2 3 36653
15		2.......2 3 36653		
14			2....24333 6653	
13	2...24333 6653			2 2 1 2 24733 6653
12	2 1 1 24733 6653	2 1 2 24733 6653		
11			1 2 45553 6653	2 2 45553 6653
10	2 3 3 6 5 23577	2 45553 6653	1 2 3 3 5 93577	2 2 3 3 5 9 3577
9	2 4 3 5 7 3577	2 3 3 5 9 3577	2 4 3 5 9 3577	
8	2 3 3 59777	2 4 3 59777		
7				
6				
	43	44	45	46

26				1 * * * * * * 1
25			* * * * * * 1	1 1 2 34411 * * * * 1
24			1 2 34411 * * * * 1	2 2 34411 * * * * 1
23		2 34411 * * * * 1	1 1 * * * * 24333	2 1 * * * * 24333
22		1 * * * * 24333	2 * * * * 24333	
21	* * * * 24333			2..............2 45333
20		2............2 45333		
19			2...........2 35733	
18	2...........2 35733			2..........2 3 36653
17		2.........2 3 36653		
16			2......24333 6653	
15	2.....24333 6653		1 1 * 24733 6653	2 1 * 24733 6653 1 2 * 24733 6653
14		2.2 1 2 24733 6653 1 * 24733 6653	2 * 24733 6653	
13	* 24733 6653		1 2 2 2 45553 6653	2..2 45553 6653
12	1 2 2 45553 6653	2 2 2 45553 6653		1 2 43565 23577
11		2 3 3 5 6 5 23577	2 43565 23577	
10			2 3 5 3 5 9 3577	2 4 5 3 5 9 3577 2 2 3 5 3 59777
9		2 3 5 3 59777	2 4 5 3 59777	
8				
7				
	47	48	49	50

28			2 1 1 * * * * * * 1	
27	1 1 * * * * * * 1	2 1 * * * * * * 1		
26	2 * * * * * * 1	2 1 1 2 34411 * * * * 1		
25	2 1 2 34411 * * * * 1			
24	2 1 1 * * * * 24333			
23				2...............2 45333
22		2..............2 45333		
21			2.............2 35733	
20	2............2 35733			2...........2 3 36653
19		2..........2 3 36653		
18			2........24333 6653	
17	2.......24333 6653			2 2 1 2 * 24733 6653
16	2 1 1 * 24733 6653	2 1 2 * 24733 6653		
15			1 2...2 45553 6653	2....2 45553 6653
14	1 2..2 45553 6653	2...2 45553 6653		1 2.2 43565 23577
13		1 2 2 43565 23577	2.2 43565 23577	
12	2 2 43565 23577		2 3 35565 23577	2 4 35565 23577
11	2 2 3 5 3 5 9 3577			2 4 2 3 5 3 59777 2 3 4 6 3 5 9 3577
10				1 2 3 5 10 11 3577
9			2 3 5 10 11 3577	2 4 8 5 59777
8			2 4 6 9 11 7 7 7	
7				
6				2 4 7 11 15 15
5				
4				
3				
2				
1				
	51	52	53	54

30				1 * * * * * * * 1
29			* * * * * * * 1	1 1 2 34411 * * * * * 1
28			1 2 34411 * * * * * 1	2 2 34411 * * * * * 1
27		2 34411 * * * * * 1	1 1 * * * * * 24333	2 1 * * * * * 24333
26		1 * * * * * 24333	2 * * * * * 24333	
25	* * * * * 24333			2..................2 45333
24		2.................2 45333		
23			2................2 35733	
22	2...............2 35733			2.............2 3 36653
21		2.............2 3 36653		
20			2..........24333 6653	
19	2.........24333 6653		1 1 * * 24733 6653	2 1 * * 24733 6653 1 2 * * 24733 6653
18		2.2 1 2 * 24733 6653 1 * * 24733 6653	2 * * 24733 6653	
17	* * 24733 6653		1 2.....2 45553 6653	2......2 45553 6653
16	1 2....2 45553 6653	2.....2 45553 6653		1 2...2 43565 23577
15		1 2..2 43565 23577	2...2 43565 23577	
14	2..2 43565 23577		1 2 2 4 35565 23577	2.2 4 35565 23577 2 1 1 2 42353 59777
13	1 2 4 35565 23577	2 2 4 35565 23577 1 1 2 42353 59777		
12	1 2 42353 59777	2 2 42353 59777	2 1 1 2 3 5 10 11 3577	
11	2 4 5 5 3 5 9 3577 1 1 2 3 5 10 11 3577	2 1 2 3 5 10 11 3577 1 1 2 4 8 5 59777	2 1 2 4 8 5 59777	
10	2 2 3 5 10 11 3577 1 2 4 8 5 59777	2 3 3 5 10 11 3577 2 2 4 8 5 59777	2 3 4 8 5 59777 2.2 4 6 9 11 7 7 7	2 4 5 9 3 59777
9	2 2 4 6 9 11 7 7 7	1 2 3 5 9 15 7 7 7	2 2 3 5 9 15 7 7 7	2 3 3 5 9 15 7 7 7 2 1 1 2 4 7 11 15 15
8	2 3 5 9 15 7 7 7	2 4 5 9 15 7 7 7 1 1 2 4 7 11 15 15	2 3 5 7 11 15 7 7 2 1 2 4 7 11 15 15	2 4 5 7 11 15 7 7
7	1 2 4 7 11 15 15	2 2 4 7 11 15 15	2 3 4 7 11 15 15	2 3 5 7 11 15 15
6				
5				
4				
3				
2				
1				
	55	56	57	58

31	1 1 * * * * * * * 1	2 1 * * * * * * * 1
30	2 * * * * * * * 1	2 1 1 2 34411 * * * * * 1
	2 1 2 34411 * * * * * 1	
	2 1 1 * * * * * 24333	
27		
		2..................2 45333
25		
	2................2 35733	
		2..............2 3 3 6653
22		
	2..........2 24333 6653	
20	2 1 1 * * 24733 6653	2 1 2 * * 24733 6653
19		
18	1 2......2 4 5 5 5 3 6653	2.......2 4 5 5 5 3 6653
		1 2....2 4 3 5 6 5 23577
	2....2 4 3 5 6 5 23577	
15	1 2 2 2 4 3 5 5 6 5 23577	2..2 4 3 5 5 6 5 23577
		1 2 4 3 5 5 5 6 5 23577
	2 4 3 5 5 5 6 5 23577	
12	2 4 5 4 5 3 5 9 3577 2 3 3 5 9 3 5 7 3577	2 4 3 5 9 3 5 7 3577
11	2..2 4 6 9 11 7 7 7 1 2 4 5 9 3 59777	2 3 3 4 8 5 59777 2 2 4 5 9 3 59777
10		
9	2 4 3 5 9 15 7 7 7	2 2 4 5 7 11 15 7 7
		2 3 3 4 7 11 15 15
	2 4 5 7 11 15 15	
6		
5		
4		
3		
2		
1		
	59	60

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