THEOREM 0.1. *$(K_{Tate})_A(pt)/I^A_{tr}$ classifies $Level(A^*, \mathbb{G}\oplus_{\mathbb{Z}}\mathbb{Q})$ when A is any finite abelian group.*

1. The case when $A = \mathbb{Z}/p^N \times \mathbb{Z}/p^N$

In this section we prove Theorem 0.1 when A is of the form $\mathbb{Z}/p^N \times \mathbb{Z}/p^N$ for p a prime and N a positive integer.

We use quasi-elliptic cohomology here because it's more convenient to compute it using representation theory. And the rank of it as a $\mathbb{Z}[q^{\pm}]-$module is the same as the rank of Tate K-theory as $\mathbb{Z}((q))-$module.

To simplify the symbol, let G denote the group $\mathbb{Z}/p^N \times \mathbb{Z}/p^N$.

Recall the definition of quasi-elliptic cohomology

$$QEll^*_G(X) := \prod_{\sigma\in G^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) = \Big(\prod_{\sigma\in G^{tors}} K^*_{\Lambda_G(\sigma)}(X^\sigma)\Big)^G. \tag{1.1}$$

where G^{tors}_{conj} is a set of representatives of $G-$conjugacy classes in G^{tors}. Because H and G are both finite abelian groups, so the set G^{tors}_{conj} is G itself and H^{tors}_{conj} is H.

We need this conclusion from the representation theory.

LEMMA 1.1. *Let $\phi : H \longrightarrow G$ be an inclusion of abelian groups with finite index. If $\phi^* : RG \longrightarrow RH$ is surjective, then as an ideal $\phi!(RH) = RG(\phi!(1))$ where $\phi!$ is the induced map.*

PROPOSITION 1.2. The rank of

$$QEll_{\mathbb{Z}/p^N\times\mathbb{Z}/p^N}(\mathrm{pt})/I_{tr}$$

is equal to

$$|GL_2(\mathbb{Z}/p^N)|,$$

where I_{tr} is

$$\sum \mathrm{Image}[I_H^{\mathbb{Z}/p^N\times\mathbb{Z}/p^N} : QEll(\mathrm{pt}/\!\!/H) \longrightarrow QEll(\mathrm{pt}/\!\!/\mathbb{Z}/p^N\times\mathbb{Z}/p^N)]$$

and the sum goes over all the proper subgroups of $\mathbb{Z}/p^N \times \mathbb{Z}/p^N$.

PROOF. We need to consider the induced maps of the form

$$I^G_H : \prod_{h\in H} K_{\Lambda_H(h)}(\mathrm{pt}) \longrightarrow \prod_{g\in G} K_{\Lambda_G(g)}(\mathrm{pt})$$

which is in fact

$$\prod_{h\in H}\Big(K_{\Lambda_H(h)}(\mathrm{pt}) \xrightarrow{Ind^{\Lambda_G(h)}_{\Lambda_H(h)}} K_{\Lambda_G(h)}(\mathrm{pt})\Big)$$

with $Ind^{\Lambda_G(h)}_{\Lambda_H(h)}$ the induced map of equivariant K-theories.

We take the quotients by the image of each I^G_H one by one and get $QEll_G(\mathrm{pt})/I_{tr}$ finally.

$$QEll_G(\mathrm{pt})/\mathrm{Image}I^G_H = \prod_{g\notin H} K_{\Lambda_G(g)}(\mathrm{pt}) \times \prod_{h\in H} K_{\Lambda_G(h)}(\mathrm{pt})/\mathrm{Image}(Ind^{\Lambda_G(h)}_{\Lambda_H(h)}).$$

Quotient by $\mathrm{Image}I^G_H$ only has something to do with the factors corresponding to $h \in H$.

We only need to consider those proper subgroups H that are maximal subgroups of G. The order of each maximal subgroup should be $p^{N-1} \times p^N$. Each maximal subgroups should be the kernel of some surjective homomorphism $\phi_{(a,b)} : G \longrightarrow \mathbb{Z}/p$ sending $[(x, y)]$ to $ax + by$ with a b some fixed integers.

The kernel of $\phi_{(a,b)}$ is
either Case I: $\mathbb{Z}/p^{N-1} \times \mathbb{Z}/p^N$ when $(a, p) = 1$ and $(b, p) = p$;
or Case II: $\mathbb{Z}/p^N \times \mathbb{Z}/p^{N-1}$ when $(a, p) = p$ and $(b, p) = 1$;
or Case III: $\langle(-b, a)\rangle \oplus \langle(p, 0)\rangle$ when $(a, p) = 1$ and $(b, p) = 1$.

Now let's start the tedious computation part. Consider an element $[(g, h)] \in G$. Assume $0 \leq g, h < p^N$.

If $(g, p) = 1$ and $(h, p) = p$, this type of elements is contained in the subgroup Case II above but in no other maximal subgroups of G. Then $\mathrm{Image}I^G_H$ in this case is equal to $\left(Ind^{\Lambda_{\mathbb{Z}/p^N}(h)}_{\Lambda_{\mathbb{Z}/p^{N-1}}(h)} K_{\Lambda_{\mathbb{Z}/p^{N-1}}(h)}(\mathrm{pt})\right) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_{\mathbb{Z}/p^N}(g)}(\mathrm{pt})$. And $K_{\Lambda_G(g,h)}(\mathrm{pt})/\mathrm{Image}I^G_H$ is
$\left(K_{\Lambda_{\mathbb{Z}/p^N}(h)}(\mathrm{pt})/Ind^{\Lambda_{\mathbb{Z}/p^N}(h)}_{\Lambda_{\mathbb{Z}/p^{N-1}}(h)} K_{\Lambda_{\mathbb{Z}/p^{N-1}}(h)}(\mathrm{pt})\right) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_{\mathbb{Z}/p^N}(g)}(\mathrm{pt})$.

According to Lemma 1.1, we only need to compute $Ind^{\Lambda_{\mathbb{Z}/p^N}(h)}_{\Lambda_{\mathbb{Z}/p^{N-1}}(h)} 1$. It is

$$1 + (x^{p^{N-1}}_{h,p^N} q^{-\frac{h}{p}}) + \cdots + (x^{p^{N-1}}_{h,p^N} q^{-\frac{h}{p}})^{p-1}$$

where $x_{h,p^N}([\alpha, t]) = e^{2\pi i(\frac{\alpha+ht}{p^N})}$ is a $\Lambda_{\mathbb{Z}/p^N}(h)-$representation. It is a generator of

$$K_{\Lambda_{\mathbb{Z}/p^N}(h)}(\mathrm{pt}) \cong \mathbb{Z}[q^\pm][x_{h,p^N}]/(x^{p^N}_{h,p^N} - q^h).$$

The rank of $(K_{\Lambda_G(g,h)}(\mathrm{pt})/\mathrm{Image}I^G_H)$ is $p^N \times p^{N-1}(p - 1)$. The number of elements (g, h) with $(g, p) = 1$ and $(h, p) = p$ or $(g, p) = p$ and $(h, p) = 1$ is $(p^N - p^{N-1})p^{N-1} \times 2$.

Then, let's consider those (g, h) with $(g, p) = 1$ and $(h, p) = 1$. Then (g, h) is contained in only one maximal subgroup of G, i.e. $H = \langle(g, h)\rangle \oplus \langle(p, 0)\rangle$. $\langle(g, h)\rangle$ is isomorphic to $\mathbb{Z}/p^N$.

Note that $G = \langle(g, h)\rangle \oplus \langle(1, 0)\rangle$. $K_{\Lambda_G(g,h)}(\mathrm{pt})/\mathrm{Image}I^G_H$ is isomorphic to

$$K_{\Lambda_{\mathbb{Z}/p^N}(g,h)}(\mathrm{pt}) \otimes_{\mathbb{Z}[q^\pm]} \left(K_{\Lambda_{\mathbb{Z}/p^N}(p)}(\mathrm{pt})/Ind^{\Lambda_{\mathbb{Z}/p^N}(p)}_{\Lambda_{\mathbb{Z}/p^{N-1}}(1)} K_{\Lambda_{\mathbb{Z}/p^{N-1}}(1)}(\mathrm{pt})\right).$$

We compute $Ind^{\Lambda_{\mathbb{Z}/p^N}(p)}_{\Lambda_{\mathbb{Z}/p^{N-1}}(1)} 1 = 1+(x^{p^{N-1}}_{p,p^N} q^{-1})+\cdots+(x^{p^{N-1}}_{p,p^N} q^{-1})^{p-1}$ where $x_{p,p^N}[\alpha, t] = e^{2\pi i(\frac{\alpha+pt}{p^{N-1}})}$.

The rank of $K_{\Lambda_G(g,h)}(\mathrm{pt})/\mathrm{Image}I^G_H$ in this case is $p^N \times p^{N-1}(p - 1)$. And the number of elements (g, h) in G with $(g, p) = 1$ and $(h, p) = 1$ is $(p^N - p^{N-1}) \times (p^N - p^{N-1})$.

The last case is the most complicated one. We consider elements (g, h) with g and h both can be divided by p. Then the maximal subgroup H containing (g, h) can be

- $H := \mathbb{Z}/p^{N-1} \times \mathbb{Z}/p^N$;
- $H' := \mathbb{Z}/p^N \times \mathbb{Z}/p^{N-1}$;

- $H_m := \langle (1,m)\rangle \oplus \langle (p,0)\rangle$ with $m = 1, 2, \cdots p-1$.

Then the transfer ideal is generated by the polynomials:

- $A := 1 + x^{p^{N-1}} + \cdots + x^{p^{N-1}(p-1)}$ where $x([a,b,t]) = e^{2\pi i \frac{a}{p^N}}$;
- $A' := 1 + y^{p^{N-1}} + \cdots + y^{p^{N-1}(p-1)}$ where $y([a,b,t]) = e^{2\pi i \frac{b}{p^N}}$;
- $A_m := 1 + (xy^{-m})^{p^{N-1}} + \cdots + (xy^{-m})^{p^{N-1}(p-1)}$ for $m = 1, 2, \cdots p-1$.

$y^{-p^{N-1}}([a,b,t]) = e^{-2\pi i \frac{b}{p}}$. Thus, $\sum\limits_{m=0}^{p-1} y^{-mp^{N-1}}([a,b,t]) = \sum\limits_{m=0}^{p-1} e^{-2\pi i \frac{mb}{p}} = 0$.

So

$$A + \sum_{m=1}^{p-1} A_m = \sum_{m=0}^{p-1} 1 = p. \tag{1.2}$$

Thus, $K_{\Lambda_G(g,h)}(\mathrm{pt})$ modulo the transfer ideal is

$$\mathbb{Z}[q^{\pm}][x,y]/(A, A', A_m) \cong (\mathbb{Z}/p)[q^{\pm}][x,y]/(A', A_m).$$

It is a torsion part.

The rank of the first two cases $p^N \times p^{N-1}(p-1) \times (p^N - p^{N-1})p^{N-1} \times 2 + p^N \times p^{N-1}(p-1)(p^N - p^{N-1}) \times (p^N - p^{N-1})$ equals $p^{4N-3}(p-1)^2(p+1)$, which is $|GL_2(\mathbb{Z}/p^N)|$. □

2. The general case

In this section we prove Theorem 0.1 for any finite abelian group A.

Lemma 2.1. *Let p_1, p_2 be two distinct primes and N_1, N_2 two different positive integers. We have the decompositions*

$$(K_{Tate})_{\mathbb{Z}/p_1^{N_1}\times\mathbb{Z}/p_2^{N_2}}(pt)/I_{tr}^{\mathbb{Z}/p_1^{N_1}\times\mathbb{Z}/p_2^{N_2}} \cong (K_{Tate})_{\mathbb{Z}/p_1^{N_1}}(pt)/I_{tr}^{\mathbb{Z}/p_1^{N_1}} \otimes_{\mathbb{Z}((q))} (K_{Tate})_{\mathbb{Z}/p_2^{N_2}}(pt)/I_{tr}^{\mathbb{Z}/p_2^{N_2}}.$$

$$Level((\mathbb{Z}/p_1^{N_1}\times\mathbb{Z}/p_2^{N_2})^*, \mathbb{G}\oplus_{\mathbb{Z}}\mathbb{Q}) \cong Level((\mathbb{Z}/p_1^{N_1})^*, \mathbb{G}\oplus_{\mathbb{Z}}\mathbb{Q}) \times Level((\mathbb{Z}/p_2^{N_2})^*, \mathbb{G}\oplus_{\mathbb{Z}}\mathbb{Q}).$$

The proof of Lemma 2.1 is straightforward.

Then the only case we need to study is when $A = \mathbb{Z}/p^k \times \mathbb{Z}/p^j$ with p prime and $k > j > 0$. In this case $A[p] = \mathbb{Z}/p \times \mathbb{Z}/p$. Fix any surjective group homomorphism $s : A \longrightarrow A[p]$. Each maximal subgroup H is the pullback

$$\begin{tikzcd} H \arrow[r] \arrow[d, "s"] & A \arrow[d, "s"] \\ H' \arrow[r] & A[p] \end{tikzcd} \tag{2.1}$$

with H' a maximal subgroup of $A[p]$. In this way we have a one-to-one correspondence between the maximal subgroups of $\mathbb{Z}/p^k \times \mathbb{Z}/p^j$ and those of $A[p]$.

Consider the pull back diagram as you suggest

$$\begin{tikzcd} Hom(A[p], \mathbb{G}\oplus_{\mathbb{Z}}\mathbb{Q}) & Level(A[p], \mathbb{G}\oplus_{\mathbb{Z}}\mathbb{Q}) \arrow[l] \\ Hom(A, \mathbb{G}\oplus_{\mathbb{Z}}\mathbb{Q}) \arrow[u] & Level(A, \mathbb{G}\oplus_{\mathbb{Z}}\mathbb{Q}) \arrow[l] \arrow[u] \end{tikzcd} . \tag{2.2}$$

Apply the ring of functions to (2.2), we get

$$
\begin{tikzcd}
O(Hom(A[p], \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q})) \arrow[r] \arrow[d] & O(Level(A[p], \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q})) \arrow[d] \\
O(Hom(A, \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q})) \arrow[r] & O(Level(A, \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q}))
\end{tikzcd} , \tag{2.3}
$$

which is a pushout square.

Then we consider the diagram

$$
\begin{tikzcd}
(K_{Tate})_{A[p]}(\mathrm{pt}) \arrow[r] \arrow[d, "s^*"'] & (K_{Tate})_{A[p]}(\mathrm{pt})/I_{tr}^{A[p]} \arrow[d, "s^*"] \\
(K_{Tate})_A(\mathrm{pt}) \arrow[r] & (K_{Tate})_A(\mathrm{pt})/I_{tr}^{A}
\end{tikzcd} . \tag{2.4}
$$

Lemma 2.2. *The right map s^* in (2.4) is well-defined.*

Proof. It suffices to show s^* is well-defined between quasi-elliptic cohomology and we can use the language of representation theory.

$$
\begin{tikzcd}
QEll_{A[p]}(\mathrm{pt}) \arrow[r] \arrow[d, "s^*"'] & QEll_{A[p]}(\mathrm{pt})/I_{tr}^{A[p]} \arrow[d, "s^*"] \\
QEll_A(\mathrm{pt}) \arrow[r] & QEll_A(\mathrm{pt})/I_{tr}^{A}
\end{tikzcd} .
$$

For each $g \in A$, s^* maps the factor $K_{\Lambda_{A[p]}(s(g))}(pt) \overset{s^*}{K}_{\Lambda_A(g)}(pt)$ is the restriction. For any $g \in A$, if $s(g)$ is in some maximal subgroup H' of $A[p]$, then g is in the maximal subgroup $H = pb(H')$ of A corresponding to it. Since restriction $\circ$ transfer = transfer $\circ$ restriction, we have $s^*(Ind|_{\Lambda_{H'}(s(g))}^{\Lambda_{A[p]}(s(g))} x) = Ind|_{\Lambda_H(g)}^{\Lambda_A(g)} s^* x$.

So the map $s^* : QEll_{\mathbb{Z}/p\times\mathbb{Z}/p}(\mathrm{pt})/I_{tr}^{A[p]} \longrightarrow QEll_A(\mathrm{pt})/I_{tr}^{A}$ is well-defined.

□

Lemma 2.3. *The diagram (2.4) is a pushout square.*

Proof. Consider the obvious isomorphism

$$
(K_{Tate})_A(\mathrm{pt}) \otimes_{(K_{Tate})_{A[p]}(\mathrm{pt})} (K_{Tate})_{A[p]}(\mathrm{pt}) \longrightarrow (K_{Tate})_A(\mathrm{pt}).
$$

Let H' be any maximal subgroup of $A[p] = \mathbb{Z}/p \times \mathbb{Z}/p$ and $H = pb(H')$ be the maximal subgroup of A corresponding to H'. Let $g \in H$. Then $s(g) \in H'$. We have the pullback square

$$
\begin{tikzcd}
\Lambda_H(g) \arrow[r] \arrow[d] & \Lambda_A(g) \arrow[d] \\
\Lambda_{H'}(s(g)) \arrow[r] & \Lambda_{A[p]}(s(g))
\end{tikzcd}
$$

and the pushout square of the representation rings

$$\begin{array}{ccc} R\Lambda_H(g) & \longleftarrow & R\Lambda_A(g) \\ \uparrow & & \uparrow \\ R\Lambda_{H'}(s(g)) & \longleftarrow & R\Lambda_{A[p]}(s(g)) \end{array} .$$

We have

$$Ind|^{\Lambda_A(g)}_{\Lambda_H(g)} R\Lambda_H(g) = \left(Ind|^{\Lambda_{A[p]}(s(g))}_{\Lambda_{H'}(s(g))} R\Lambda_{H'}(s(g))\right) \otimes_{R\Lambda_{A[p]}(s(g))} R\Lambda_A(g).$$

With g fixed, we have

$$\sum_{\text{all } H\ni g} Ind|^{\Lambda_A(g)}_{\Lambda_H(g)} R\Lambda_H(g) = \sum_{\text{all } H'\ni s(g)} \left(Ind|^{\Lambda_{A[p]}(s(g))}_{\Lambda_{H'}(s(g))} R\Lambda_{H'}(s(g))\right) \otimes_{R\Lambda_{A[p]}(s(g))} R\Lambda_A(g).$$

Then we can get

$$I^A_{tr} = I^{A[p]}_{tr} \otimes_{(K_{Tate})_{A[p]}(\mathrm{pt})} (K_{Tate})_A(\mathrm{pt}).$$

So we get the isomorphism

$$(K_{Tate})_A(\mathrm{pt}) \otimes_{(K_{Tate})_{A[p]}(\mathrm{pt})} (K_{Tate})_{A[p]}(\mathrm{pt})/I^{A[p]}_{tr} \cong (K_{Tate})_A(\mathrm{pt})/I^A_{tr}.$$

So (2.4) is a pushout.

□

PROPOSITION 2.4. For any finite abelian group A, $(K_{Tate})_A(\mathrm{pt})$ is the ring of functions on $Hom(A^*, \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q})$.

PROOF. We first consider the case $A = \mathbb{Z}/N$. Then $Hom(A^*, \mathbb{Q}/\mathbb{Z})$ is isomorphic to $\mathbb{Z}/N$. Let F_k denote the fiber of the natural map

$$Hom(A^*, G \oplus_{\mathbb{Z}} \mathbb{Q}) \longrightarrow Hom(A^*, \mathbb{Q}/\mathbb{Z})$$

at $\frac{k}{N}$. For any $\mathbb{Z}[q^{\pm}]-$algebra R with connected spectrum,

$$F_k(R) = \{f = (f, \frac{k}{N}) | f(1)^N = q^k\}.$$

It is isomorphic to $\{x \in R | x^N = q^k\}$. So F_k is isomorphic to

$$T_k[N] = \mathrm{Spec}(\mathbb{Z}[q^{\pm}][x]/\langle x^N - q^k\rangle) = \mathrm{Spec}(R\Lambda_{\mathbb{Z}/N}(k)).$$

So $O(Hom(A^*, G \oplus_{\mathbb{Z}} \mathbb{Q})) = \prod_k O(F_k) = \prod_k R\Lambda_{\mathbb{Z}/N}(k) = QEll_{\mathbb{Z}/N}(\mathrm{pt})$.

When A is any finite abelian group , it is the product of some cyclic groups $\mathbb{Z}/N$. And $QEll_A(\mathrm{pt})$ is the $\mathbb{Z}[q^{\pm}]-$tensor product of those $QEll_{\mathbb{Z}/N}(\mathrm{pt})$;

$$O(Hom(A^*, \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q}))$$

is the product of $O(Hom(\mathbb{Z}/N, \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q}))$. So we still have the isomorphism. □

Then we consider the natural maps from the square (2.4) to (2.3).

We have the isomorphisms

$$(K_{Tate})_{A[p]}(\mathrm{pt}) \cong O(Hom(A[p], \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q}));$$
$$(K_{Tate})_A(\mathrm{pt}) \cong O(Hom(A, \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q})).$$

So the pushouts

$$(K_{Tate})_A(\mathrm{pt})/I^A_{tr} \cong O(Level(A, \mathbb{G} \oplus_{\mathbb{Z}} \mathbb{Q})). \tag{2.5}$$