Quasi-Elliptic Cohomology and its Spectrum

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ABSTRACT. Ginzburg, Kapranov and Vasserot conjectured the existence of equivariant elliptic cohomology theories. In this paper, to give a description of equivariant spectra of the theories, we study an intermediate theory, quasi-elliptic cohomology. We formulate a new category of orthogonal $G-$spectra and construct explicitly an orthogonal $G-$spectrum of quasi-elliptic cohomology in it. The idea of the construction can be applied to a family of equivariant cohomology theories, including Tate K-theory and generalized Morava E-theories. Moreover, this construction provides a functor from the category of global spectra to the category of orthogonal $G-$spectra. In addition, from it we obtain some new idea what global homotopy theory is right for constructing global elliptic cohomology theory.

1. Introduction

An elliptic cohomology theory is an even periodic multiplicative generalized cohomology theory whose associated formal group is the formal completion of an elliptic curve. Elliptic cohomology theories serve as a family of algebraic variants reflecting the geometric nature of elliptic curves, which make themselves intriguing and significant subjects to study. One renowned conclusion on the representing spectra of the theories is Goerss-Hopkins-Miller theorem [**17**]. It constructs many examples of $E_\infty-$rings which represent elliptic cohomology theories, including Tate K-theory.

Moreover, as K-theory and many other cohomology theories, elliptic cohomology theories also have equivariant version. In [**10**], Ginzburg, Kapranov and Vasserot gave the axiomatic definition of $G-$equivariant elliptic cohomology theory. They have the conjecture that any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A. In his thesis [**8**], Gepner presented a construction of the equivariant elliptic cohomology that satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms. We have the question from another perspective whether we can construct an orthogonal $G-$spectrum representing each equivariant elliptic cohomology theory.

This question, however, is not easy to answer by studying elliptic cohomology theories themselves. They are intricate and mysterious theories. Instead, we turn to an intermediate theory, quasi-elliptic cohomology theory. The idea of quasi-elliptic cohomology is motivated by Ganter's construction of Tate K-theory. Rezk

2010 *Mathematics Subject Classification.* Primary 55.

The author was partially supported by NSF grant DMS-1406121.

established the theory in his unpublished manuscript [**22**]. The author gave a detailed description of the construction of the theory in Chapter 2, [**13**] and Section 2, 3, [**12**]. Currently the author is writing a survey on quasi-elliptic cohomology[**14**].

Quasi-elliptic cohomology theory is a variant of Tate K-theory, which is the generalized elliptic cohomology theory associated to the Tate curve. The Tate curve $Tate(q)$ is an elliptic curve over $\mathrm{Spec}\mathbb{Z}((q))$, which is classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity. A good reference for $Tate(q)$ is Section 2.6 of [**1**]. Tate K-theory itself is a distinctive subject to study. The relation between Tate K-theory and string theory is better understood than for most known elliptic cohomology theories. In addition, the definition of $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold, which is formulated explicitly in Section 2, [**7**].

Quasi-elliptic cohomology theory contains all the information of Tate K-theory and reflects the geometric nature of elliptic curves. Moreover, it has many advantages [**12**][**13**]. One large good feature that you can tell is it can be expressed explicitly by equivariant K-theories.

$$QEll_G^*(X) := \prod_{\sigma\in G^{tors}_{conj}} K^*_{\Lambda(\sigma)}(X^\sigma) = \Big(\prod_{\sigma\in G^{tors}} K^*_{\Lambda(\sigma)}(X^\sigma)\Big)^G. \tag{1.1}$$

Equivariant K-theory is a classical example of equivariant cohomology theories. It has been thoroughly studied and has many good features. Comparing with any elliptic cohomology theory, it is more practicable to construct the representing spectra of quasi-elliptic cohomology theory.

Then, how practicable is it? One immediate idea is that if we can construct a right adjoint functor r_σ of each fixed point functor $X \mapsto X^\sigma$, then a representing spectra $\{X_n, \psi_n\}_n$ of the theory $QEll^*_G(-)$ can be constructed by

$$X_n = \prod_{\sigma\in G^{tors}_{conj}} r_\sigma(KU_{\Lambda_G(\sigma),n}),\ \ \psi_n = \prod_{\sigma\in G^{tors}_{conj}} r_\sigma(\phi_n) \tag{1.2}$$

where $\{KU_{G,n}, \phi_n\}_n$ denotes a $G-$spectrum representing $K^*_G(-)$. However, the fixed point functor does not have right adjoint. Consequently, we introduce the concept of homotopical adjunction. Via homotopical right adjoints of fixed point functors, the representing spectrum that we obtain stays in a new category of orthogonal $G-$spectra GwS.

We are still studying whether this way of constructing the orthogonal $G-$spectrum of quasi-elliptic cohomology can be applied to the elliptic cohomology theories and whether the category GwS is the right category for equivariant elliptic spectra to reside at. But this idea can be applied to a family of theories, including generalized Morava E-theories and equivariant Tate K-theory. We can construct in the category GwS the orthogonal $G-$spectrum of any theory of the form

$$QE_G^*(X) := \prod_{\sigma\in G^{tors}_{conj}} E^*_{\Lambda(\sigma)}(X^\sigma) = \Big(\prod_{\sigma\in G^{tors}} E^*_{\Lambda(\sigma)}(X^\sigma)\Big)^G \tag{1.3}$$

with E any equivariant cohomology theory having the same key features as equivariant K-theory, as explained in detail at the beginning of Section 6.

As equivariant K-theories, quasi-elliptic cohomology also has the change-of-group isomorphism. In a conversation, Ganter indicated that it has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede [**25**].

However, this orthogonal $G-$spectrum of quasi-elliptic cohomology cannot arise from an orthogonal spectrum, i.e. this orthogonal $G-$spectrum is not the underlying orthogonal $G-$spectrum of any orthogonal spectrum. Instead, in a coming paper we construct a new global homotopy theory and show there is a global orthogonal spectrum in it that represents orthogonal quasi-elliptic cohomology. Some construction and idea of this new theory has already been presented in Chapter 6 and 7, [**13**].

It is worth mentioning that, other than Schwede's model for global homotopy theory, there is a presheaf model for the theory shown in [**9**]. In [**23**] Rezk briefly introduced this definition with differences in detail and he highlighted the role of "cohesion" in relating ordinary equivariant homotopy theory with global equivariant homotopy theory. This may be a better model to construct global elliptic cohomology theories though the author has not worked into it deeply.

1.1. Where should we construct the equivariant spectrum? As indicated above, the equivariant spectrum of quasi-elliptic cohomology cannot be constructed as in (1.2) because the right adjoint functor r_σ does not exist. We generalize the concept of right adjoints a little and introduce homotopical adjunction.

Definition 1.1 (homotopical adjunction). Let H and G be two compact Lie groups. Let

$$L : G\mathcal{T} \longrightarrow H\mathcal{T} \text{ and } R : H\mathcal{T} \longrightarrow G\mathcal{T} \tag{1.4}$$

be two functors. A *left-to-right homotopical adjunction* is a natural map

$$\mathrm{Map}_H(LX, Y) \longrightarrow \mathrm{Map}_G(X, RY), \tag{1.5}$$

which is a weak equivalence of spaces when X is a $G-$CW complex.

Analogously, a *right-to-left homotopical adjunction* is a natural map

$$\mathrm{Map}_G(X, RY) \longrightarrow \mathrm{Map}_H(LX, Y) \tag{1.6}$$

which is a weak equivalence of spaces when X is a $G-$CW complex.

L is called a *homotopical left adjoint* and R a *homotopical right adjoint*.

The homotopical right adjoint $\mathcal{R}_\sigma$ of the fixed point functor $X \mapsto X^\sigma$ exists. We give an explicit construction of it in Theorem 5.3. Via these $\mathcal{R}_\sigma$s, we construct a $G-$space $QE_{G,n}$. Its relation with the theory $QE^n_G(-)$ is

$$\pi_0(QE_{G,n}) = QE^n_G(S^0), \tag{1.7}$$

as shown in Theorem 5.4. This construction motivates us to construct the category GwT of G-spaces. It is defined to be the homotopy category of the category of $G-$spaces with the weak equivalence defined by

$$A \sim B \text{ if } \pi_0(A) = \pi_0(B). \tag{1.8}$$

Moreover, we can define the category GwS of orthogonal $G-$spectra, which is the homotopy category of the category of orthogonal $G-$spectra with the weak equivalence defined by

$$X \sim Y \text{ if } \pi_0(X(V)) = \pi_0(Y(V)), \tag{1.9}$$

for each faithful $G-$representation V. And an orthogonal $G-$spectrum X is said to represent a theory H^*_G in GwS if we have a natural map

$$\pi_0(X(V)) = H^V_G(S^0), \tag{1.10}$$

for each faithful $G-$representation V.

1.2. Orthogonal $G-$spectra. To construct an orthogonal $G-$spectrum strictly representing QE^*_G is far beyond our imagination. But the construction of an orthogonal $G-$spectrum representing it in GwS is down-to-earth.

The readers may have noticed that we cannot construct the structure maps in the same way as (1.2) to equip the spaces $\{QE_{G,n}\}_n$ the structure of a $G-$spectrum because we are using homotopical right adjoints. We have the same problem when trying to construct an orthogonal $G-$spectrum representing QE^*_G in the category GwS.

But when E_G is a $\mathcal{I}_G-$FSP, we can construct the structure maps for $QE(G,-)$ explicitly and obtain the main conclusion of this paper.

THEOREM 1.2. *If the equivariant cohomology theory E^*_G can be represented by a $\mathcal{I}_G-$FSP (E_G, η^E, μ^E), there is a $\mathcal{I}_G-$FSP*

$$(QE(G,-), \eta^{QE}, \mu^{QE})$$

representing QE^*_G in GwS. In the case that (E_G, η^E, μ^E) is commutative, $(QE(G,-), \eta^{QE}, \mu^{QE})$ is commutative.

The construction of $(QE(G,-), \eta^{QE}, \mu^{QE})$ gives us a functor.

THEOREM 1.3. *There is a well-defined functor $\mathcal{Q}$ from the full subcategory consisting of $\mathcal{I}_G-$FSP in GwS to the same category sending (E_G, η^E, μ^E) to $(QE(G,-), \eta^{QE}, \mu^{QE})$.*

The restriction of $\mathcal{Q}$ to the full subcategory consisting of commutative $\mathcal{I}_G-$FSP is a functor from that category to itself.

Moreover, we have the corollary for quasi-elliptic cohomology.

THEOREM 1.4. *There is a commutative $\mathcal{I}_G-$FSP $(QK(G,-), \eta^{QK}, \mu^{QK})$ representing quasi-elliptic cohomology in GwS.*

In addition, we construct the restriction maps $QE(G,V) \longrightarrow QE(H,V)$ for each group homomorphism $H \longrightarrow G$. This map is not a homeomorphism, but an $H-$weak equivalence.

As shown in Section 8, the orthogonal $G-$spectrum $(QE(G,-), \eta^{QE}, \mu^{QE})$ cannot arise from an orthogonal spectrum. This fact motivates us to construct a new global homotopy theory in a coming paper [**15**].

In Section 2 we recall the basics in equivariant homotopy theory. In Section 3 we recall the construction of quasi-elliptic cohomology. In Section 4 we introduce homotopical adjunction and construct the category GwS of orthogonal $G-$spaces. In Section 5, we construct a homotopical right adjoint of the fixed point functor and then show the construction of a space $E_{G,n}$ representing $E^n_G(-)$ in GwT. In Section 6 we construct an orthogonal $G-$spectrum for quasi-elliptic cohomology, which is a commutative $\mathcal{I}_G-$FSP in GwS. In Section 7, we define the restriction map of these equivariant orthogonal spectra. In Section 8 we give a brief introduciton of ideas related to global homotopy theory.

In Appendix A, we recall the definition and properties of join. In Appendix B we recall the basics of global homotopy theory and the construction of global K-theory. In Appendix C we construct some faithful representations of the group $\Lambda_G(g)$ which are essential in the construction of orthogonal $G-$spectrum. And we put the technical proofs of some conclusions in Appendix D.

Acknowledgement. I thank Charles Rezk for suggesting the idea of homotopical adjunction and weak spectra. He initiated the project and supported my work all the time. I also thank Matthew Ando, Stefan Schwede, Nathaniel Stapleton and Guozhen Wang for helpful remarks.

2. Notations in equivariant homotopy theory

In this section we give a sketch of the notations and conclusions in the equivariant homotopy theory that we need in further sections. The main references are [**3**], [**5**] and [**21**].

Let G be a compact Lie group. Let $\mathcal{T}$ denote the category of topological spaces and continuous maps. Let $G\mathcal{T}$ denote the category of $G-$spaces, namely, spaces X equipped with continuous $G-$action $G \times X \longrightarrow X$ and continuous $G-$maps.

Let H be a closed subgroup of G. Let X be a $G-$space and Y an $H-$space. Define

$$X^H := \{x|hx = x, \forall h \in H\}. \tag{2.1}$$

For $x \in X$, the isotropy group of x

$$G_x := \{h|hx = x\}. \tag{2.2}$$

Let $hG\mathcal{T}$ denote the homotopy category whose objects are $G-$spaces and morphisms are $G-$homotopy classes of continuous $G-$maps. Let $\overline{h}G\mathcal{T}$ denote the category constructed from $hG\mathcal{T}$ by adjoining formal inverses to the weak equivalences.

THEOREM 2.1 (Elemendorf's Theorem). *The category $\overline{h}\mathcal{T}^{\mathcal{O}_G^{op}}$ and $\overline{h}G\mathcal{T}$ are equivalent.*

Let $G\mathcal{C}$ denote the category of $G-$CW complexes and celluar maps. Proposition 2.2 is a conclusion needed for the construction later. It can be proved by induction over cells.

PROPOSITION 2.2. Let D be a complete category. Let $i : \mathcal{O}_G^{op} \longrightarrow G\mathcal{C}^{op}$ be the inclusion of subcategory. If $F_1, F_2 : G\mathcal{C}^{op} \longrightarrow D$ are two functors sending homotopy colimits to homotopy limits and if we have a natural transformation $p : F_1 \longrightarrow F_2$, which gives a weak equivalence at orbits, then it also gives a weak equivalence on $G\mathcal{C}$. Especially, if p gives a retract at each orbit, F_1 is a retract of F_2 at each $G-$CW complexes.

3. Quasi-elliptic cohomology

In this section we recall the definition of quasi-elliptic cohomology. The main reference is [**12**], [**13**] and [**22**]. Before that we discuss in Section 3.1 the representation ring of $\Lambda_G(g)$.

3.1. Preliminary: representation ring of $\Lambda_G(g)$. For any compact Lie group G and a torsion element $g \in G$, let $C_G(g)$ denote the centralizer of g in G, and let $\Lambda_G(g)$ denote the group

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle.$$

Let $\mathbb{T}$ denote the circle group $\mathbb{R}/\mathbb{Z}$. Let $q : \mathbb{T} \longrightarrow U(1)$ be the isomorphism $t \mapsto e^{2\pi i t}$. The representation ring $R\mathbb{T}$ is $\mathbb{Z}[q^{\pm}]$.

We have an exact sequence

$$1 \longrightarrow C_G(g) \longrightarrow \Lambda_G(g) \xrightarrow{\pi} \mathbb{T} \longrightarrow 0$$

where the first map is $g \mapsto [g, 0]$ and the second is

$$\pi([g, t]) = e^{2\pi i t}. \tag{3.1}$$

There is a relation between the representation ring of $C_G(g)$ and that of $\Lambda_G(g)$.

Lemma 3.1. *$\pi^* : R\mathbb{T} \longrightarrow R\Lambda_G(g)$ exhibits $R\Lambda_G(g)$ as a free $R\mathbb{T}-$module.*

In particular, there is an $R\mathbb{T}-$basis of $R\Lambda_G(g)$ given by irreducible representations $\{V_\lambda\}$, such that restriction $V_\lambda \mapsto V_\lambda|_{C_G(g)}$ to $C_G(g)$ defines a bijection between $\{V_\lambda\}$ and the set $\{\lambda\}$ of irreducible representations of $C_G(g)$.

The proof is in [**12**] and also [**13**].

Remark 3.2. We can make a canonical choice of $\mathbb{Z}[q^{\pm}]$-basis for $R\Lambda_G(g)$. For each irreducible G-representation $\rho : G \longrightarrow Aut(G)$, write $\rho(\sigma) = e^{2\pi i c} id$ for $c \in [0, 1)$, and set $\chi_\rho(t) = e^{2\pi i c t}$. Then the pair (ρ, χ_ρ) corresponds to a unique irreducible $\Lambda_G(g)$-representation

$$\rho \odot_{\mathbb{C}} \chi_\rho([h, t]) := \rho(h)\chi_\rho(t). \tag{3.2}$$

3.2. Quasi-elliptic cohomology. In this section we introduce the definition of quasi-elliptic cohomology $QEll^*_G$ in term of equivariant K-theory. This theory can also be constructed from Rezk's ghost loops defined in [**24**]. To see a full discussion about the relation between equivariant loop spaces and quasi-elliptic cohomology, please refer to Chapter 2 and 3 [**12**], Chapter 2 [**13**] and [**22**].

Let X be a $G-$space. Let $G^{tors} \subseteq G$ be the set of torsion elements of G. Let $\sigma \in G^{tors}$. The fixed point space X^σ is a $C_G(\sigma)-$space. We can define a $\Lambda_G(\sigma)-$action on X^σ by $[g, t] \cdot x := g \cdot x$.

Definition 3.3. The quasi-elliptic cohomology is defined by

$$QEll^*_G(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g) = \Big(\prod_{g \in G^{tors}} K^*_{\Lambda_G(g)}(X^g) \Big)^G, \tag{3.3}$$

where G^{tors}_{conj} is a set of representatives of $G-$conjugacy classes in G^{tors}.

We have the ring homomorphism $\mathbb{Z}[q^{\pm}] = K^0_{\mathbb{T}}(\mathrm{pt}) \xrightarrow{\pi^*} K^0_{\Lambda_G(g)}(\mathrm{pt}) \longrightarrow K^0_{\Lambda_G(g)}(X)$ where $\pi : \Lambda_G(g) \longrightarrow \mathbb{T}$ is the projection defined in (3.1) and the second is via the collapsing map $X \longrightarrow \mathrm{pt}$. So $QEll^*_G(X)$ is naturally a $\mathbb{Z}[q^{\pm}]-$algebra.

Similar to equivariant K-theories, we can construct the restriction map, the Künneth map on it, its tensor product and the change-of-group isomorphism of quasi-elliptic cohomology. We construct the restriction map and the change-of-group isomorphism in this section. For other constructions, please refer to [**12**].

Since each homomorphism $\phi : G \longrightarrow H$ induces a well-defined homomorphism $\phi_\Lambda : \Lambda_G(\tau) \longrightarrow \Lambda_H(\phi(\tau))$ for each τ in G, we can get the proposition below directly.

PROPOSITION 3.4. For each homomorphism $\phi : G \longrightarrow H$, it induces a ring map

$$\phi^* : QEll^*_H(X) \longrightarrow QEll^*_G(\phi^* X)$$

characterized by the commutative diagrams

$$\begin{array}{ccc} QEll^*_H(X) & \xrightarrow{\phi^*} & QEll^*_G(\phi^*X) \\ {\scriptstyle \pi_{\phi(\tau)}}\downarrow & & {\scriptstyle \pi_\tau}\downarrow \\ K^*_{\Lambda_H(\phi(\tau))}(X^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_G(\tau)}(X^{\phi(\tau)}) \end{array} \tag{3.4}$$

for any $\tau \in G$. So $QEll^*_G$ is functorial in G.

We also have the change-of-group isomorphism as in equivariant K-theory.

Let H be a subgroup of G and X an H-space. Let $\phi : H \longrightarrow G$ denote the inclusion homomorphism. The change-of-group map $\rho^G_H : QEll^*_G(G \times_H X) \longrightarrow QEll^*_H(X)$ is defined as the composite

$$\rho^G_H : QEll^*_G(G \times_H X) \xrightarrow{\phi^*} QEll^*_H(G \times_H X) \xrightarrow{i^*} QEll^*_H(X) \tag{3.5}$$

where ϕ^* is the restriction map and $i : X \longrightarrow G \times_H X$ is the $H-$equivariant map defined by $i(x) = [e, x]$.

PROPOSITION 3.5. The change-of-group map

$$\rho^G_H : QEll^*_G(G \times_H X) \longrightarrow QEll^*_H(X)$$

defined in (3.5) is an isomorphism.

PROOF. For any $\tau \in H_{conj}$, there exists a unique $\sigma_\tau \in G_{conj}$ such that $\tau = g_\tau \sigma_\tau g_\tau^{-1}$ for some $g_\tau \in G$. Consider the maps

$$\Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau \xrightarrow{[[a,t],x]\mapsto[a,x]} (G \times_H X)^\tau \xrightarrow{[u,x]\mapsto[g_\tau^{-1}u,x]} (G \times_H X)^\sigma. \tag{3.6}$$

The first map is $\Lambda_G(\tau)-$equivariant and the second is equivariant with respect to the homomorphism $c_{g_\tau} : \Lambda_G(\sigma) \longrightarrow \Lambda_G(\tau)$ sending $[u, t] \mapsto [g_\tau u g_\tau^{-1}, t]$. Taking a coproduct over all the elements $\tau \in H_{conj}$ that are conjugate to $\sigma \in G_{conj}$ in G, we get an isomorphism

$$\gamma_\sigma : \coprod_\tau \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau \longrightarrow (G \times_H X)^\sigma$$

which is $\Lambda_G(\sigma)-$equivariant with respect to c_{g_τ}. Then we have the map

$$\gamma := \prod_{\sigma\in G_{conj}} \gamma_\sigma : \prod_{\sigma\in G_{conj}} K^*_{\Lambda_G(\sigma)}(G\times_H X)^\sigma \longrightarrow \prod_{\sigma\in G_{conj}} K^*_{\Lambda_G(\sigma)}(\coprod_\tau \Lambda_G(\tau)\times_{\Lambda_H(\tau)} X^\tau) \tag{3.7}$$

It is straightforward to check the change-of-group map coincide with the composite

$$\begin{aligned} QEll^*_G(G \times_H X) \xrightarrow{\gamma} \prod_{\sigma\in G_{conj}} K^*_{\Lambda_G(\sigma)}(\coprod_\tau \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau) \longrightarrow &\prod_{\tau\in H_{conj}} K^*_{\Lambda_H(\tau)}(X^\tau) \\ &= QEll^*_H(X) \end{aligned}$$

with the second map the change-of-group isomorphism in equivariant $K-$theory. □

4. A new category of orthogonal $G-$spectra

To construct a concrete representing spectrum for elliptic cohomology is a difficult goal to achieve. We consider constructing a representing spectrum of quasi-elliptic cohomology first, which is not easy to realize, either.

In this section we first construct a new category of orthogonal $G-$spectra where quasi-elliptic cohomology resides.

The quasi-elliptic cohomology, as defined in (3.3), has the form

$$QEll^*_G(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g) = \Big(\prod_{g \in G^{tors}} K^*_{\Lambda_G(g)}(X^g) \Big)^G .$$

If we could construct the right adjoint of the fixed point functor $X \mapsto X^g$ from the category of $G-$spaces to that of $\Lambda_G(g)-$spaces, we can construct the representing spectrum of the theory afterwards. However, the fixed point functor does not preserve colimits, thus, does not have right adjoints. Instead, we consider a concept weaker than adjoints.

Definition 4.1 (homotopical adjunction). Let H and G be two compact Lie groups. Let

$$L : G\mathcal{T} \longrightarrow H\mathcal{T} \text{ and } R : H\mathcal{T} \longrightarrow G\mathcal{T} \tag{4.1}$$

be two functors. A *left-to-right homotopical adjunction* is a natural map

$$\mathrm{Map}_H(LX, Y) \longrightarrow \mathrm{Map}_G(X, RY), \tag{4.2}$$

which is a weak equivalence of spaces when X is a $G-$CW complex.

Analogously, a *right-to-left homotopical adjunction* is a natural map

$$\mathrm{Map}_G(X, RY) \longrightarrow \mathrm{Map}_H(LX, Y) \tag{4.3}$$

which is a weak equivalence of spaces when X is a $G-$CW complex.

L is called a *homotopical left adjoint* and R a *homotopical right adjoint*.

Homotopical adjunction is another way to describe the relation between $G-$equivariant homotopy theory and those equivariant homotopy theory for its closed subgroups. This definition can be generalized to functors between categories other than $H\mathcal{T}$ and $G\mathcal{T}$. Homotopical adjunction is a notion more ubiquitous in category theory than adjunctions.

Example 4.2. Let $G = \mathbb{Z}/2\mathbb{Z}$ and g be a generator of G. We want to find a homotopical right adjoint R of the functor $X \mapsto X^g$ from the category $G\mathcal{T}$ of G-spaces to the category $\mathcal{T}$ of topological spaces.

Let Y be a topological space. Suppose we have

$$\mathrm{Map}(X^g, Y) \simeq \mathrm{Map}_G(X, RY).$$

G has two subgroups, e and G.

$$RY^e = \mathrm{Map}_G(G/e, RY) \simeq \mathrm{Map}((G/e)^g, Y) \simeq \mathrm{pt};$$
$$RY^G = \mathrm{Map}_G(G/G, RY) \simeq \mathrm{Map}((G/G)^g, Y) = Y.$$

If Y is the empty set, $R\emptyset$ is EG. And generally for any Y, one choice of RY is the join $Y * EG$.

By Elmendorf's theorem 2.1, the space RY is unique up to $G-$homotopy. By definition, the functor R is a homotopical right adjoint to the fixed point functor $X \mapsto X^g$.

After we find a homotopical right adjoint R_g of the fixed point functor $X \mapsto X^g$, we can construct a space $QEll_{G,n}$ representing the $n-$th $G-$equivariant quasi-elliptic cohomology $QEll^n_G(-)$ up to the weak equivalence

$$\pi_0(QEll_{G,n}) = QEll^n_G(S^0), \tag{4.4}$$

In other words, $QEll_{G,n}$ represents $QEll^n_G(-)$ in the category GwT below. The explicit construction of $QEll_{G,n}$ can be found in Corollary 5.5.

Definition 4.3. The category GwT is the homotopy category of the category of $G-$spaces with the weak equivalence defined by

$$A \sim B \text{ if } \pi_0(A) = \pi_0(B). \tag{4.5}$$

A $G-$space A in GwT is said to represent H^n_G if we have a natural map

$$\pi_0(A) = H^n_G(S^0). \tag{4.6}$$

Moreover, we can consider the category below of orthogonal $G-$spectra.

Definition 4.4. The category GwS is the homotopy category of the category of orthogonal $G-$spectra with the weak equivalence defined by

$$X \sim Y \text{ if } \pi_0(X(V)) = \pi_0(Y(V)), \tag{4.7}$$

for each faithful $G-$representation V.

An orthogonal $G-$spectrum X in GwS is said to represent a theory H^*_G if we have a natural map

$$\pi_0(X(V)) = H^V_G(S^0), \tag{4.8}$$

for each faithful $G-$representation V.

The orthogonal $G-$spectrum representing quasi-elliptic cohomology in GwS is constructed in Section 6.

5. Equivariant spectra

Let $E^*_G(-)$ be a $G-$equivariant cohomology theory. Define

$$QE^*_G(X) := \prod_{\sigma\in G^{tors}_{conj}} E^*_{\Lambda(\sigma)}(X^\sigma) = \Big(\prod_{\sigma\in G^{tors}} E^*_{\Lambda(\sigma)}(X^\sigma)\Big)^G. \tag{5.1}$$

In this section, for each integer n, each compact Lie group G, we construct a space $QE_{G,n}$ representing the $n-$th $G-$equivariant QE^n_G up to weak equivalence.

The construction of the right homotopical adjoint in Theorem 5.2 needs the space $S_{G,g}$ below. For any compact Lie group G, let $\langle g\rangle$ denote the cyclic subgroup of G generated by $g \in G^{tors}$ and $*$ denote the join. Let

$$S_{G,g} := \mathrm{Map}_{\langle g\rangle}(G, *_K E(\langle g\rangle/K))$$

where K goes over all the maximal subgroups of $\langle g\rangle$ and $E(\langle g\rangle/K)$ is the universal space of the cyclic group $\langle g\rangle/K$. The action of $\langle g\rangle/K$ on $E(\langle g\rangle/K)$ is free. For this space $S_{G,g}$, it is classified up to $G-$homotopy, as shown in the following lemma.

Lemma 5.1. *For any closed subgroup $H \leqslant G$, $S_{G,g}$ satisfies*

$$S^H_{G,g} \simeq \begin{cases} pt, & \text{if for any } b\in G,\ b^{-1}\langle g\rangle b \nleqslant H; \\ \emptyset, & \text{if there exists a } b\in G \text{ such that } b^{-1}\langle g\rangle b \leqslant H. \end{cases} \tag{5.2}$$

Proof. For any closed subgroup H of G.

$$S^H_{G,g} = \mathrm{Map}_{\langle g\rangle}(G/H, *_K E(\langle g\rangle/K)) \tag{5.3}$$

where K goes over all the cyclic groups $\langle g^m\rangle$ with $\frac{|g|}{m}$ a prime.

If there exists an $b\in G$ such that $b^{-1}\langle g\rangle b \leqslant H$, it is equivalent to say that there exists points in G/H that can be fixed by g. But there are no points in $*_K E(\langle g\rangle/K)$ that can be fixed by g. So there is no $\langle g\rangle-$equivariant map from G/H to $*_K E(\langle g\rangle/K)$. In this case $S^H_{G,g}$ is empty.

If for any $b\in G$, $b^{-1}\langle g\rangle b \nleqslant H$, it is equivalent to say that there are no points in G/H that can be fixed by g. And for any subgroup $\langle g^m\rangle$ which is not $\langle g\rangle$ itself, $(*_K E(\langle g\rangle/K))^{\langle g^m\rangle}$ is the join of several contractible spaces $E(\langle g\rangle/K)^{\langle g^m\rangle}$, thus, contractible. So all the homotopy groups $\pi_n((*_K E(\langle g\rangle/K))^{\langle g^m\rangle})$ are trivial. For any $n\geq 1$ and any $\langle g\rangle-$equivariant map

$$f : (G/H)^n \longrightarrow *_K E(\langle g\rangle/K)$$

from the $n-$skeleton of G/H, the obstruction cocycle is zero.

Then by equivariant obstruction theory, f can be extended to the $(n+1)-$cells of G/H, and any two extensions f and f' are $\langle g\rangle-$homotopic.

So in this case $S^H_{G,g}$ is contractible. □

Theorem 5.2 is crucial to the construction of $QE_{G,n}$.

Theorem 5.2. *Let G be a compact Lie group and $g\in G^{tors}$. A homotopical right adjoint of the functor $L_g : G\mathcal{T} \longrightarrow C_G(g)\mathcal{T},\ X\mapsto X^g$ is*

$$R_g : C_G(g)\mathcal{T} \longrightarrow G\mathcal{T},\ Y\mapsto Map_{C_G(g)}(G, Y * S_{C_G(g),g}). \tag{5.4}$$

Proof. Let H be any closed subgroup of G.

First we show given a $C_G(g)-$equivariant map $f : (G/H)^g \longrightarrow Y$, it extends uniquely up to $C_G(g)-$homotopy to a $C_G(g)-$equivariant map $\widetilde{f} : G/H \longrightarrow Y * S_{C_G(g),g}$. f can be viewed as a map $(G/H)^g \longrightarrow Y * S_{C_G(g),g}$ by composing with the inclusion of one end of the join

$$Y \longrightarrow Y * S_{C_G(g),g},\ y\mapsto (1y, 0).$$

If $bH \in (G/H)^g$, define $\widetilde{f}(bH) := f(bH)$.

If bH is not in $(G/H)^g$, its stabilizer group does not contain g. By Lemma 5.1, for any subgroup K of it, $S^K_{C_G(g),g}$ is contractible. So $(Y * S_{C_G(g),g})^K = Y^K * S^K_{C_G(g),g}$ is contractible. In other words, if K occurs as the isotropy subgroup of a point outside $(G/H)^g$, $\pi_n((Y * S_{C_G(g),g})^K)$ is trivial. By equivariant obstruction theory, f can extend to a $C_G(g)-$equivariant map $\widetilde{f} : G/H \longrightarrow Y * S_{C_G(g),g}$, and any two extensions are $C_G(g)-$homotopy equivalent. In addition, $S^g_{C_G(g),g}$ is empty. So the image of the restriction of any map $G/H \longrightarrow Y * S_{C_G(g),g}$ to the subspace $(G/H)^g$ is contained in the end Y of the join.

Thus, $\mathrm{Map}_{C_G(g)}((G/H)^g, Y)$ is weak equivalent to $\mathrm{Map}_{C_G(g)}(G/H, Y*S_{C_G(g),g})$.

Moreover, we have the equivalence by adjunciton

$$\mathrm{Map}_G(G/H,\ \mathrm{Map}_{C_G(g)}(G, Y * S_{C_G(g),g})) \cong \mathrm{Map}_{C_G(g)}(G/H,\ Y * S_{C_G(g),g}) \tag{5.5}$$

So we get

$$R_gY^H = \mathrm{Map}_G(G/H, R_gY) \backsimeq \mathrm{Map}_{C_G(g)}((G/H)^g, Y) \tag{5.6}$$

Let X be of the homotopy type of a $G-$CW complex. Let X^k denote the $k-$skeleton of X. Consider the functors

$$\mathrm{Map}_G(-, R_gY) \text{ and } \mathrm{Map}_{C_G(g)}((-)^g, Y)$$

from $G\mathcal{T}$ to $\mathcal{T}$. Both of them sends homotopy colimit to homotopy limit. In addition, we have a natural map from $\mathrm{Map}_G(-, R_gY)$ to $\mathrm{Map}_{C_G(g)}((-)^g, Y)$ by sending a $G-$map $F : X \longrightarrow R_gY$ to the composition

$$X^g \xrightarrow{F^g} (R_gY)^g \longrightarrow Y^g \subseteq Y \tag{5.7}$$

with the second map $f \mapsto f(e)$. Note that for any $f \in (R_gY)^g$, $f(e) = (g \cdot f)(e) = f(eg) = f(g) = g \cdot f(e)$ so $f(e) \in (Y * S_{C_G(g),g})^g = Y^g$ and the second map is well-defined. It gives weak equivalence on orbits, as shown in (5.6). Thus, by Proposition 2.2, R_g is a homotopical right adjoint of L. □

THEOREM 5.3. *Let G be a compact Lie group, $g \in G^{tors}$, and Y a $\Lambda_G(g)-$space. The subgroup $\{[(1, t)] \in \Lambda_G(g) | t \in \mathbb{R}\}$ of $\Lambda_G(g)$ is isomorphic to $\mathbb{R}$. We use the same symbol $\mathbb{R}$ to denote it. Consider the functor $\mathcal{L}_g : G\mathcal{T} \longrightarrow \Lambda_G(g)\mathcal{T}, X \mapsto X^g$ where $\Lambda_G(g)$ acts on X^g by $[g, t] \cdot x = gx$. The functor $\mathcal{R}_g : \Lambda_G(g)\mathcal{T} \longrightarrow G\mathcal{T}$ with*

$$\mathcal{R}_gY = Map_{C_G(g)}(G, Y^{\mathbb{R}} * S_{C_G(g),g}) \tag{5.8}$$

is a homotopical right adjoint of $\mathcal{L}_g$.

PROOF. Let X be a $G-$space. Let H be any closed subgroup of G. Note for any $G-$space X, $\mathbb{R}$ acts trivially on X^g, thus, the image of any $\Lambda_G(g)-$equivariant map $X^g \longrightarrow Y$ is in $Y^{\mathbb{R}}$. So we have $\mathrm{Map}_{\Lambda_G(g)}(X^g, Y) = \mathrm{Map}_{C_G(g)}(X^g, Y^{\mathbb{R}})$.

First we show $f : (G/H)^g \longrightarrow Y^{\mathbb{R}}$ extends uniquely up to $C_G(g)-$homotopy to a $C_G(g)-$equivariant map $\widetilde{f} : G/H \longrightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$. f can be viewed as a map $(G/H)^g \longrightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$ by composing with the inclusion as the end of the join

$$Y^{\mathbb{R}} \longrightarrow Y^{\mathbb{R}} * S_{C_G(g),g},\ y \mapsto (1y, 0).$$

The rest of the proof is analogous to that of Theorem 5.2. □

Theorem 5.3 implies Theorem 5.4 directly.

THEOREM 5.4. *For any compact Lie group G and any integer n, let $E_{G,n}$ denote the space representing the $n-th$ $G-$equivariant $E-$theory. Then each $QE^n_G(-)$ is represented by the space*

$$QE_{G,n} := \prod_{g \in G^{tors}_{conj}} \mathcal{R}_g(KU_{\Lambda_G(g),n})$$

in the category GwT where $\mathcal{R}_g(E_{\Lambda_G(g),n})$ is the space

$$Map_{C_G(g)}(G, E^{\mathbb{R}}_{\Lambda_G(g),n} * S_{C_G(g),g}).$$

And we have the corresponding conclusion for quasi-elliptic cohomology.

COROLLARY 5.5. For any compact Lie group G and any integer n, let $KU_{G,n}$ denote the space representing the $n-$th $G-$equivariant $KU-$theory. The $n-$th quasi-elliptic cohomology $QEll^n_G(-)$ is represented by the space

$$QEll_{G,n} := \prod_{g\in G^{tors}_{conj}} \mathcal{R}_g(KU_{\Lambda_G(g),n})$$

in the category GwS where $\mathcal{R}_g(KU_{\Lambda_G(g),n})$ is the space

$$\mathrm{Map}_{C_G(g)}(G, KU^{\mathbb{R}}_{\Lambda_G(g),n} * S_{C_G(g),g}).$$

The construction of the orthogonal $G-$spectrum of $QE-$theory in Section 6.1 is based on that of $QE_{G,n}$.

6. Orthogonal $G-$spectrum of QE^*_G

In this section, we consider equivariant cohomology theories E that can be represented by $\mathcal{I}_G-$FSP (E_G, η^E, μ^E) and have the same key features as equivariant complex K-theories. More explicitly,

- The theories $\{E^*_G\}_G$ have the change-of-group isomorphism, i.e. for any closed subgroup H of G and $H-$space X, the change-of-group map $\rho^G_H : E^*_G(G\times_H X) \longrightarrow E^*_H(X)$ defined by $E^*_G(G\times_H X) \xrightarrow{\phi^*} E^*_H(G\times_H X) \xrightarrow{i^*} E^*_H(X)$ is an isomorphism where ϕ^* is the restriction map and $i : X \longrightarrow G\times_H X$ is the $H-$equivariant map defined by $i(x) = [e, x]$.
- There exists an orthogonal spectrum E such that for any compact Lie group G and "large" real $G-$representation V and a compact $G-$space B we have a bijection $E_G(B) \longrightarrow [B_+, E(V)]^G$. And (E_G, η^E, μ^E) is the underlying orthogonal $G-$spectrum of E.
- Let G be a compact Lie group and V an orthogonal $G-$representation. For every ample $G-$representation W, the adjoint structure map $\widetilde{\sigma}^E_{V,W} : E(V) \longrightarrow \mathrm{Map}(S^W, E(V\oplus W))$ is a $G-$weak equivalence.

In this section based on the spaces we construct in Section 5, we construct a $\mathcal{I}_G-$FSP representing the theory QE in the category GwS defined in Definition 4.4.

6.1. The construction of $QE(G,-)$. Let G be any compact Lie group. In this section we consider the case that the equivariant cohomology theory E can be represented by a global spectrum (E, η^E, μ^E) and show in Section 6.1.3 that there is a $\mathcal{I}_G-$FSP $(QE(G,-), \eta^{QE}, \mu^{QE})$ representing $QE^V_G(-)$ in the category GwS. Before that we construct each ingredient in the construction.

6.1.1. *The construction of $S(G,V)_g$.* First we construct an orthogonal version $S(G,V)_g := Sym(V)\setminus Sym(V)^g$ of the space $S_{G,g}$. It is the space classified by the condition (6.1) which is also the condition classifying $S_{G,g}$.

Let $g\in G^{tors}$ and V a real $G-$representation. Let $Sym^n(V)$ denote the $n-$th symmetric power $V^{\otimes n}$, which has an evident $G\wr\Sigma_n-$action on it. Let

$$Sym(V) := \bigoplus_{n\geq 0} Sym^n(V).$$

When V is an ample $G-$representation, $Sym(V)$ is a $G-$representation containing all the irreducible $G-$representations. Since in this case V is faithful $G-$representation, for any closed subgroup H of G, $Sym(V)$ is a faithful $H-$representation, thus, a complete $H-$universe.

We use $S(G,V)_g$ to denote the space $Sym(V)\setminus Sym(V)^g$. The complex conjugation on V induces an involution on it. Note that for any subgroup H of G containing g, $S(H,V)_g$ has the same underlying space as $S(G,V)_g$.

Proposition 6.1. Let V be an orthogonal $G-$representation. For any closed subgroup $H\leqslant C_G(g)$, $S(G,V)_g$ satisfies

$$S(G,V)_g^H \simeq \begin{cases}\mathrm{pt}, & \text{if } \langle g\rangle \nleqslant H;\\ \emptyset, & \text{if } \langle g\rangle\leqslant H.\end{cases} \tag{6.1}$$

Proof. If $\langle g\rangle\leqslant H$, $Sym(V)^H$ is a subspace of $Sym(V)^g$, so $(Sym(V)\setminus Sym(V)^g)^H$ is empty. If $\langle g\rangle \nleqslant H$, g is not in H. To simplify the symbol, let $Sym^{n,\perp}$ denote the orthogonal complement of $Sym^n(V)^g$ in $Sym^n(V)$.

$$\begin{aligned}(Sym(V)\setminus Sym(V)^g)^H &= colim_{n\longrightarrow\infty} Sym^n(V)^H\setminus (Sym^n(V)^g)^H\\ &= colim_{n\longrightarrow\infty}(Sym^n(V)^g)^H\times\big((Sym^{n,\perp})^H\setminus\{0\}\big)\end{aligned}$$

Let k_n denote the dimension of $(Sym^{n,\perp})^H$. Then $(Sym^{n,\perp})^H\setminus\{0\}\simeq S^{k_n-1}$. As n goes to infinity, k_n goes to infinity. When k_n is large enough, S^{k_n-1} is contractible. So $(Sym(V)\setminus Sym(V)^g)^H$ is contractible. □

6.1.2. *The construction of $F_g(G,V)$.* Next, we construct a space $F_g(G,V)$ representing the theory $E^{V^g}_{\Lambda_G(g)}(-)$.

The faithful real $\Lambda_G(g)-$representation constructed in Section C.2 serves as an essential component of the construction. If V is a faithful $G-$representation, by Proposition C.7, we have the faithful $\Lambda_G(g)-$representation $(V)^{\mathbb{R}}_g$. In addition, V^g can be considered as a $\Lambda_G(g)-$representation with trivial $\mathbb{R}-$action. The space $E((V)^{\mathbb{R}}_g\oplus V^g)$ represents $E^{(V)^{\mathbb{R}}_g\oplus V^g}_{\Lambda_G(g)}(-)$. So we have

$$\mathrm{Map}(S^{(V)^{\mathbb{R}}_g}, E((V)^{\mathbb{R}}_g\oplus V^g))$$

represents $E^{V^g}_{\Lambda_G(g)}(-)$ since

$$[X^g, \mathrm{Map}(S^{(V)^{\mathbb{R}}_g}, E((V)^{\mathbb{R}}_g\oplus V^g))]^{\Lambda_G(g)}$$

is isomorphic to

$$[X^g\wedge S^{(V)^{\mathbb{R}}_g}, E((V)^{\mathbb{R}}_g\oplus V^g)]^{\Lambda_G(g)} = E^{(V)^{\mathbb{R}}_g\oplus V^g}_{\Lambda_G(g)}(X^g\wedge S^{(V)^{\mathbb{R}}_g}) = E^{V^g}_{\Lambda_G(g)}(X^g).$$

To simplify the symbol, we use $F_g(G,V)$ to denote the space

$$\mathrm{Map}_{\mathbb{R}}(S^{(V)_g^{\mathbb{R}}}, E((V)_g^{\mathbb{R}} \oplus V^g)).$$

Its basepoint c_0 is the constant map to the basepoint of $E((V)_g^{\mathbb{R}} \oplus V^g)$.

$F_g : (G,V) \mapsto F_g(G,V)$ provides a functor from $\mathcal{I}_G$ to the category $C_G(g)\mathcal{T}$ of $C_G(g)-$spaces. It has the properties below.

Proposition 6.2. Let G and H be compact Lie groups. Let V be a real $G-$representation and W a real $H-$representation. Let $g \in G^{tors}$, $h \in H^{tors}$.
(i) We have the unit map $\eta_g(G,V) : S^{V^g} \longrightarrow F_g(G,V)$ and the multiplication

$$\mu^F_{(g,h)}((G,V),(H,W)) : F_g(G,V) \wedge F_h(H,W) \longrightarrow F_{(g,h)}(G \times H, V \oplus W)$$

making the unit, associativity and centrality of unit diagram commute. And $\eta_g(G,V)$ is $C_G(g)-$equivariant and $\mu^F_{(g,h)}((G,V),(H,W))$ is $C_{G\times H}(g,h)-$ equivariant.
(ii)Let Δ_G denote the diagonal map $G \longrightarrow G \times G,\ g \mapsto (g,g)$. Let $\widetilde{\sigma}_g(G,V,W) : F_g(G,V) \longrightarrow \mathrm{Map}(S^{W^g}, F_g(G, V \oplus W))$ denote the map

$$x \mapsto (w \mapsto \big(\Delta_G^* \circ \mu^F_{(g,h)}((G,V),(G,W))\big)\big(x, \eta_g(G,W)(w)\big)).$$

Then $\widetilde{\sigma}_g(G,V,W)$ is a $\Lambda_G(g)-$weak equivalence when V is an ample $G-$representation.
(iii) If (E, η^E, μ^E) is commutative, we have

$$\mu^F_{(g,h)}((G,V),(H,W))(x \wedge y) = \mu^F_{(h,g)}((H,W),(G,V))(y \wedge x) \tag{6.2}$$

for any $x \in F_g(G,V)$ and $y \in F_h(H,W)$.

Proof. (i) Let V_1 and V_2 be orthogonal $G-$representations and $f : V_1 \longrightarrow V_2$ be a linear isometric isomorphism. f gives the linear isometric isomorphisms $f_1 : (V_1)_g^{\mathbb{R}} \longrightarrow (V_2)_g^{\mathbb{R}}$, and $f_2 : (V_1)_g^{\mathbb{R}} \oplus V_1^g \longrightarrow (V_2)_g^{\mathbb{R}} \oplus V_2^g$. Then define $F_g(f) : F_g(V_1) \longrightarrow F_g(V_2)$ in this way: for any $\mathbb{R}-$equivariant map $\alpha : S^{(V_1)_g^{\mathbb{R}}} \longrightarrow E((V_1)_g^{\mathbb{R}} \oplus V_1^g)$, $F_g(f)(\alpha)$ is the composition

$$S^{(V_2)_g^{\mathbb{R}}} \xrightarrow{S(f_1^{-1})} S^{(V_1)_g^{\mathbb{R}}} \xrightarrow{\alpha} E((V_1)_g^{\mathbb{R}} \oplus V_1^g) \xrightarrow{E(f_2)} E((V_2)_g^{\mathbb{R}} \oplus V_2^g) \tag{6.3}$$

which is still $\mathbb{R}-$equivariant. It is straightforward to check $F_g(Id)$ is the identity map, and for morphisms $V_1 \xrightarrow{f} V_2 \xrightarrow{f'} V_3$ in $\mathcal{I}_G$, we have $F_g(f' \circ f) = F_g(f') \circ F_g(f)$.

(ii) Define the unit map $\eta_g(G,V) : S^{V^g} \longrightarrow F_g(G,V)$ by

$$v \mapsto (v' \mapsto \eta^E_{(V)_g^{\mathbb{R}} \oplus V^g}(v \wedge v')) \tag{6.4}$$

where $\eta^E_{(V)_g^{\mathbb{R}} \oplus V^g} : S^{(V)_g^{\mathbb{R}} \oplus V^g} \longrightarrow E((V)_g^{\mathbb{R}} \oplus V^g)$ is the unit map for global E-theory. Since $(V)_g^{\mathbb{R}} \oplus V^g$ is a $\Lambda_G(g)-$representation, $\eta^E_{(V)_g^{\mathbb{R}} \oplus V^g}$ is $\Lambda_G(g)-$equivariant. So $\eta_g(G,V)$ is well-defined and $\Lambda_G(g)-$equivariant.

Define the multiplication $\mu^F_{(g,h)}((G,V),(H,W)) : F_g(G,V) \wedge F_h(H,W) \longrightarrow F_{(g,h)}(G \times H, V \oplus W)$ by

$$\alpha \wedge \beta \mapsto (v \wedge w \mapsto \mu^E_{V,W}(\alpha(v) \wedge \beta(w))) \tag{6.5}$$

where $\mu^E_{V,W}$ is the multiplication for global $E-$theory. Since $\mu^E_{V,W}$ is $\Lambda_G(g) \times \Lambda_H(h)-$equivariant, $\mu^F_{(g,h)}((G,V),(H,W))$ is $C_{G\times H}(g,h)-$equivariant. It is straightforward to check the unit map and multiplication make the unit, associativity and centrality of unit diagram commute.

(iii) Since V is a faithful $G-$representation, by Proposition C.1, $(V)_g^{\mathbb{R}} \oplus V^g$ is a faithful $\Lambda_G(g)-$representation. By Theorem B.11, we have the $\Lambda_G(g)-$weak equivalence $E((V)_g^{\mathbb{R}} \oplus V^g) \xrightarrow{\widetilde{\sigma}^E} \mathrm{Map}(S^{(W)_g^{\mathbb{R}} \oplus W^g}, E((V \oplus W)_g^{\mathbb{R}} \oplus (V \oplus W)^g))$ where $\widetilde{\sigma}^E$ is the right adjoint of the structure map of E. Thus we have the $\Lambda_G(g)-$weak equivalence

$$\begin{aligned}\mathrm{Map}(S^{(V)_g^{\mathbb{R}}}, E((V)_g^{\mathbb{R}} \oplus V^g)) &\longrightarrow \mathrm{Map}(S^{(V)_g^{\mathbb{R}}}, \mathrm{Map}(S^{(W)_g^{\mathbb{R}} \oplus W^g}, E((V \oplus W)_g^{\mathbb{R}} \oplus (V \oplus W)^g))) \\ &= \mathrm{Map}(S^{W^g}, \mathrm{Map}(S^{(V \oplus W)_g^{\mathbb{R}}}, E((V \oplus W)_g^{\mathbb{R}} \oplus (V \oplus W)^g))),\end{aligned}$$

i.e. $F_g(G, V) \simeq_{C_G(g)} \mathrm{Map}(S^{W^g}, F_g(G, V \oplus W))$.

(iv) (6.2) comes directly from the commutativity of E. □

6.1.3. *The construction of $QE(G, V)$.* Recall in Theorem 5.4 we construct a $G-$space $QE_{G,n}$ representing $QE_G^n(-)$ in GwT. With $F_g(G, V)$ and $S(G, V)_g$ we can go further than that.

Apply Theorem 5.3, we get the conclusion below.

Proposition 6.3. Let V be a faithful orthogonal $G-$representation. Let $B'(G, V)$ denote the space

$$\prod_{g \in G_{conj}^{tors}} \mathrm{Map}_{C_G(g)}(G, F_g(G, V) * S(G, V)_g).$$

$QE_G^V(-)$ is represented by $B'(G, V)$ in GwT.

The proof of Proposition 6.3 is analogous to that of Theorem 5.5 step by step.

One disadvantage of $\{B'(G, V)\}_V$ is that it is not easy to see whether we can construct the structure maps to make it an orthogonal $G-$spectrum. Instead, we consider the $G-$weak equivalent spaces $\{QE(G, V)\}_V$ in Proposition 6.4.

Below is the main theorem in Section 6.1. We will use formal linear combination

$$t_1 a + t_2 b \text{ with } 0 \leqslant t_1, t_2 \leqslant 1, t_1 + t_2 = 1$$

to denote points in join, as talked in Appendix A.

Proposition 6.4. Let $QE_g(G, V)$ denote

$$\{t_1 a + t_2 b \in F_g(G, V) * S(G, V)_g \,|\, \|b\| \leqslant t_2\} / \{t_1 c_0 + t_2 b\}.$$

It is the quotient space of a closed subspace of the join $F_g(G, V) * S(G, V)_g$ with all the points of the form $t_1 c_0 + t_2 b$ collapsed to one point, which we pick as the basepoint of $QE_g(G, V)$, where c_0 is the basepoint of $F_g(G, V)$. $QE_g(G, V)$ has the evident $C_G(g)-$action. And it is $C_G(g)-$weak equivalent to $F_g(G, V) * S(G, V)_g$. As a result, $\prod\limits_{g \in G_{conj}^{tors}} \mathrm{Map}_{C_G(g)}(G, QE_g(G, V))$ is $G-$weak equivalent to $B'(G, V)$. So

$$QE(G, V) := \prod_{g \in G_{conj}^{tors}} \mathrm{Map}_{C_G(g)}(G, QE_g(G, V)) \tag{6.6}$$

represents $QE_G^V(-)$ in the category GwT.

PROOF. First we show $F_g(G,V) * S(G,V)_g$ is $C_G(g)-$homotopy equivalent to

$$QE'_g(G,V) := \{t_1 a + t_2 b \in F_g(G,V) * S(G,V)_g | \|b\| \leqslant t_2\}.$$

Note that $b \in S(G,V)_g$ is never zero. Let $j : QE'_g(G,V) \longrightarrow F_g(G,V) * S(G,V)_g$ be the inclusion. Let $p : F_g(G,V) * S(G,V)_g \longrightarrow QE'_g(G,V)$ be the $C_G(g)-$map sending $t_1 a + t_2 b$ to $t_1 a + t_2 \frac{min\{\|b\|, t_2\}}{\|b\|} b$. Both j and p are both continuous and $C_G(g)-$equivariant. $p \circ j$ is the identity map of $QE'_g(G,V)$. We can define a $C_G(g)-$homotopy

$$H : (F_g(G,V) * S(G,V)_g) \times I \longrightarrow F_g(G,V) * S(G,V)_g$$

from the identity map on $F_g(G,V) * S(G,V)_g$ to $j \circ p$ by shrinking. For any $t_1 a + t_2 b \in F_g(G,V) * S(G,V)_g$, Define

$$H(t_1 a + t_2 b, t) := t_1 a + t_2((1-t)b + t\frac{min\{\|b\|, t_2\}}{\|b\|} b). \tag{6.7}$$

Then we show $QE'_g(G,V)$ is $G-$weak equivalent to $QE_g(G,V)$. Let $q : QE'_g(G,V) \longrightarrow QE_g(G,V)$ be the quotient map. Let H be a closed subgroup of $C_G(g)$.

If g is in H, since $S(G,V)_g^H$ is empty, so $QE_g(G,V)^H$ is in the end $F_g(G,V)$ and can be identified with $F_g(G,V)^H$. In this case q^H is the identity map.

If g is not in H, $QE'_g(G,V)^H$ is contractible. The cone $\{c_0\} * S(G,V)_g^H$ is contractible, so $q\big((\{c_0\} * S(G,V)_g)^H\big) = q(\{c_0\} * S(G,V)_g^H)$ is contractible. Note that the subspace of all the points of the form $t_1 c_0 + t_2 b$ for any t_1 and b is $q\big((\{c_0\} * S(G,V)_g)^H\big)$. Therefore, $QE_g(G,V)^H = QE'_g(G,V)^H / q(\{c_0\} * S(G,V)_g)^H$ is contractible.

Therefore, $QE'_g(G,V)$ is $G-$weak equivalent to $F_g(G,V) * S(G,V)_g$. □

Moreover, generalizing the construction in Proposition 6.4, we have the conclusion below on homotopical right adjoints.

PROPOSITION 6.5. Let $g \in G^{tors}$. Let Y be a based $\Lambda_G(g)-$space. Let $\widetilde{Y}_g$ denote the $C_G(g)-$space

$$\{t_1 a + t_2 b \in Y^{\mathbb{R}} * S(G,V)_g | \|b\| \leqslant t_2\} / \{t_1 y_0 + t_2 b\}.$$

It is the quotient space of a closed subspace of $Y^{\mathbb{R}} * S(G,V)_g$ with all the points of the form $t_1 y_0 + t_2 b$ collapsed to one point, i.e the basepoint of $\widetilde{Y}_g$, where y_0 is the basepoint of Y. $\widetilde{Y}_g$ is $C_G(g)-$weak equivalent to $Y^{\mathbb{R}} * S(G,V)_g$. As a result, the functor $R_g : C_G(g)\mathcal{T} \longrightarrow G\mathcal{T}$ with $R_g \widetilde{Y} = \mathrm{Map}_{C_G(g)}(G, \widetilde{Y}_g)$ is a homotopical right adjoint of $L : G\mathcal{T} \longrightarrow C_G(g)\mathcal{T}$, $X \mapsto X^g$.

The proof is analogous to that of Theorem 5.3.

REMARK 6.6. We can consider $QE_g(G,V)$ as a quotient space of a subspace of $F_g(G,V) \times Sym(V) \times I$

$$\{(a,b,t) \in F_g(G,V) \times Sym(V) \times I | \|b\| \leqslant t; \text{ and } b \in S(G,V)_g \text{ if } t \neq 0\} \tag{6.8}$$

by identifying points $(a,b,1)$ with $(a',b,1)$, and collapsing all the points (c_0,b,t) for any b and t. In other words, the end $F_g(G,V)$ in the join $F_g(G,V) * S(G,V)_g$ is identified with the points of the form $(a,0,0)$ in (6.8).

Proposition 6.7. For each $g \in G^{tors}$,

$$QE_g : \mathcal{I}_G \longrightarrow C_G(g)\mathcal{T},\ (G, V) \mapsto QE_g(G, V)$$

is a well-defined functor. As a result,

$$QE : \mathcal{I}_G \longrightarrow G\mathcal{T},\ (G, V) \mapsto \prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, QE_g(G, V))$$

is a well-defined functor.

Proof. Let V and W be $G-$ representations and $f : V \longrightarrow W$ a linear isometric isomorphism. Then f induces a $C_G(g)-$homeomorphism $F_g(f)$ from $F_g(G, V)$ to $F_g(G, W)$ and a $C_G(g)-$homeomorphism $S_g(f)$ from $S(G, V)_g$ to $S(G, W)_g$. We have the well-defined map

$$QE_g(f) : QE_g(G, V) \longrightarrow QE_g(G, W)$$

sending a point represented by t_1a+t_2b in the join to that represented by $t_1F_g(f)(a)+t_2S_g(f)(b)$. And $QE(f) : QE(G, V) \longrightarrow QE(G, W)$ is defined by

$$\prod_{g \in G^{tors}_{conj}} \alpha_g \mapsto \prod_{g \in G^{tors}_{conj}} QE_g(f) \circ \alpha_g.$$

It is straightforward to check that all the axioms hold. □

6.2. Construction of η^{QE} and μ^{QE}. In this section we construct a unit map η^{QE} and a multiplication μ^{QE} so that we get a commutative $\mathcal{I}_G-$FSP representing the $QE-$theory in GwS.

Let G and H be compact Lie groups, V an orthogonal $G-$representation and W an orthogonal $H-$representation. We use x_g to denote the basepoint of $QE_g(G, V)$, which is defined in Proposition 6.4. Let $g \in G^{tors}$. For each $v \in S^V$, there are $v_1 \in S^{V^g}$ and $v_2 \in S^{(V^g)^\perp}$ such that $v = v_1 \wedge v_2$. Let $\eta^{QE}_g(G, V) : S^V \longrightarrow QE_g(G, V)$ be the map

$$\eta^{QE}_g(G, V)(v) := \begin{cases} (1 - \|v_2\|)\eta_g(G, V)(v_1) + \|v_2\|v_2, & \text{if } \|v_2\| \leqslant 1; \\ x_g, & \text{if } \|v_2\| \geqslant 1. \end{cases} \tag{6.9}$$

Lemma 6.8. *The map $\eta^{QE}_g(G, V)$ defined in (6.9) is well-defined, continuous and $C_G(g)-$equivariant.*

The proof of Lemma 6.8 is in Appendix D.1.

Remark 6.9. For any $g \in G^{tors}$, it's straightforward to check the diagram below commutes.

$$\begin{array}{ccc} S^{V^g} & \xrightarrow{\eta_g(G,V)} & F_g(G, V) \\ \downarrow & & \downarrow \\ S^V & \xrightarrow{\eta^{QE}_g(G,V)} & QE_g(G, V) \end{array}$$

where both vertical maps are inclusions. By Lemma 6.8, the map

$$\eta^{QE}(G, V) : S^V \longrightarrow \prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, QE_g(G, V)),\ v \mapsto \prod_{g \in G^{tors}_{conj}} (\alpha \mapsto \eta^{QE}_g(G, V)(\alpha \cdot v)), \tag{6.10}$$

is well-defined and continuous. Moreover, $\eta^{QE} : S \longrightarrow QE$ with $QE(G, V)$ defined in (6.6) is well-defined.

Next, we construct the multiplication map μ^{QE}. First we define a map

$$\mu^{QE}_{(g,h)}((G,V),(H,W)) : QE_g(G,V)\wedge QE_h(H,W) \longrightarrow QE_{(g,h)}(G\times H, V\oplus W)$$

by sending a point $[t_1a_1+t_2b_1]\wedge[u_1a_2+u_2b_2]$ to

(6.11)
$$\begin{cases} [(1-\sqrt{t_2^2+u_2^2})\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2) & \text{if } t_2^2+u_2^2\leq 1 \text{ and } t_2u_2\neq 0;\\ +\sqrt{t_2^2+u_2^2}(b_1+b_2)], & \\ [(1-t_2)\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2)+t_2b_1], & \text{if } u_2=0 \text{ and } 0<t_2<1;\\ [(1-u_2)\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2)+u_2b_2], & \text{if } t_2=0 \text{ and } 0<u_2<1;\\ [1\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2)+0], & \text{if } u_2=0 \text{ and } t_2=0;\\ x_{g,h}, & \text{Otherwise.}\end{cases}$$

where $\mu^{QE}_{(g,h)}((G,V),(H,W))$ is the one defined in (6.5) and $x_{g,h}$ is the basepoint of $QE_{(g,h)}(G\times H, V\oplus W)$.

Lemma 6.10. *The map $\mu^{QE}_{(g,h)}((G,V),(H,W))$ defined in (6.11) is well-defined and continuous.*

The proof of Lemma 6.10 is in Appendix D.2.

The basepoint of $QE(G,V)$ is the product of the basepoint of each factor $\mathrm{Map}_{C_G(g)}(G, QE_g(G,V))$, i.e. the product of the constant map to the base point of each $QE_g(G,V)$.

We can define the multiplication $\mu^{QE}((G,V),(H,W)) : QE(G,V)\wedge QE(H,W) \longrightarrow QE(G\times H, V\oplus W)$ by

$$(\prod_{g\in G^{tors}_{conj}}\alpha_g)\wedge(\prod_{h\in H^{tors}_{conj}}\beta_h)\mapsto \prod_{\substack{g\in G^{tors}_{conj}\\ h\in H^{tors}_{conj}}}\Big((g',h')\mapsto \mu^{QE}_{(g,h)}((G,V),(H,W))\big(\alpha_g(g')\wedge\beta_h(h')\big)\Big).$$

Lemma 6.11. *Let G, H, K be compact Lie groups. Let V be an orthogonal $G-$representation, W an orthogonal $H-$representation, and U an orthogonal $K-$representation. Let $g\in G^{tors}$, $h\in H^{tors}$, and $k\in K^{tors}$. Then we have the commutative diagrams below.*

(6.12)
$$\begin{array}{ccc} S^V\wedge S^W & \xrightarrow{\eta^{QE}_g(G,V)\wedge\eta^{QE}_h(H,W)} & QE_g(G,V)\wedge QE_h(H,W)\\ \downarrow\cong & & \downarrow \mu^{QE}_{(g,h)}((G,V),(H,W))\\ S^{V\oplus W} & \xrightarrow{\eta^{QE}_{(g,h)}(G\times H,V\oplus W)} & QE_{(g,h)}(G\times H, V\oplus W)\end{array}$$

(6.13)
$$\begin{array}{ccc} QE_g(G,V)\wedge QE_h(H,W)\wedge QE_k(K,U) & \xrightarrow{\mu^{QE}_g((G,V),(H,W))\wedge Id} & QE_{(g,h)}(G\times H,V\oplus W)\wedge QE_k(K,U)\\ \downarrow Id\wedge\mu^{QE}_{(h,k)}(H\times K,W\oplus U) & & \downarrow \mu^{QE}_{((g,h),k)}((G\times H,V\oplus W),(K,U))\\ QE_g(G,V)\wedge QE_{(h,k)}(H\times K,W\oplus U) & \xrightarrow{\mu^{QE}_{(g,(h,k))}((G,V),(H\times K,W\oplus U))} & QE_{(g,h,k)}(G\times H\times K, V\oplus W\oplus U)\end{array}$$

(6.14)

$$\begin{array}{ccccc} S^V \wedge QE_h(H,W) & \xrightarrow{\eta_g^{QE}(G,V)\wedge Id} & QE_g(G,V)\wedge QE_h(H,W) & \xrightarrow{\mu^{QE}_{(g,h)}((G,V),(H,W))} & QE_{(g,h)}(G\times H, V\oplus W) \\ \downarrow \tau & & & & \downarrow QE_{(g,h)}(\tau) \\ QE_h(H,W)\wedge S^V & \xrightarrow{Id\wedge \eta_g^{QE}(G,V)} & QE_h(H,W)\wedge QE_g(G,V) & \xrightarrow{\mu^{QE}_{(h,g)}((H,W),(G,V))} & QE_{(h,g)}(H\times G, W\oplus V) \end{array}$$

Moreover, we have

$$\mu^{QE}_{(g,h)}((G,V),(H,W))(x\wedge y) = \mu^{QE}_{(h,g)}((H,W),(G,V))(y\wedge x) \tag{6.15}$$

for any $x \in QE_g(G,V)$ *and* $y \in QE_h(H,W)$.

The proof of Lemma 6.11 is straightforward and is in Appendix D.3.

THEOREM 6.12. *Let* $\Delta_G : G \longrightarrow G\times G$ *be the diagonal map* $g \mapsto (g,g)$. *For* $G-$*representations* V *and* W, *let* $(\Delta_G)^*_{V\oplus W} : QE(G\times G, V\oplus W) \longrightarrow QE(G, V\oplus W)$ *denote the restriction map defined by the formula (7.7). Then* $QE : \mathcal{I}_G \longrightarrow G\mathcal{T}$ *together with the unit map* η^{QE} *defined in (6.10) and the multiplication* $\Delta_G^* \circ \mu^{QE}((G,-),(G,-))$ *gives a commutative* $\mathcal{I}_G-FSP$ *that weakly represents* $QE^*_G(-)$.

PROOF. Let G, H, K be compact Lie groups, V an orthogonal $G-$representation, W an orthogonal $H-$representation and U an orthogonal $K-$representation.

Let

$$X = \prod_{g\in G^{tors}_{conj}} \alpha_g \in QE(G,V);\ Y = \prod_{h\in H^{tors}_{conj}} \beta_h \in QE(H,W);\ Z = \prod_{k\in K^{tors}_{conj}} \gamma_k \in QE(K,U).$$

First we check the diagram of unity commutes. Let $v \in S^V$ and $w \in S^W$. $\mu^{QE}((G,V),(H,W)) \circ (\eta^{QE}(G,V)\wedge \eta^{QE}(H,W))(v\wedge w)$ is

(6.16)

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \Big((g',h') \mapsto \mu^{QE}_{(g,h)}((G,V),(H,W))\circ(\eta^{QE}_g(G,V)\wedge\eta^{QE}_h(H,W))(g'\cdot v\wedge h'\cdot w)\Big).$$

$$\eta^{QE}(G\times H, V\oplus W)(v\wedge w) = \prod_{\substack{g\in G^{tors}_{conj}\\ h\in H^{tors}_{conj}}} \Big((g',h') \mapsto \eta^{QE}_{(g,h)}(G\times H, V\oplus W)(g'\cdot v\wedge h'\cdot w)\Big),$$

is equal to (6.16) by Lemma 6.11.

Next we check the diagram of associativity commutes.

$\mu^{QE}((G\times H, V\oplus W),(K,U)) \circ (\mu^{QE}((G,V),(H,W))\wedge Id)(X\wedge Y\wedge Z)$ is

$$\prod_{g\in G^{tors}_{conj} h\in H^{tors}_{conj}, k\in G^{tors}_{conj}} \Big((g',h',k') \mapsto$$

$$\mu^{QE}_{((g,h),k)}((G\times H, V\oplus W),(K,U)) \circ (\mu^{QE}_{(g,h)}((G,V),(H,W))\wedge Id)(\alpha_g(g')\wedge\beta_h(h')\wedge\gamma_k(k'))\Big)$$

And $\mu^{QE}((G,V),(H\times K, W\oplus U)) \circ (Id \wedge \mu^{QE}(H\times K, W\oplus U))(X\wedge Y\wedge Z)$ is

$$\prod_{g\in G^{tors}_{conj} h\in H^{tors}_{conj}, k\in G^{tors}_{conj}} \Big((g',h',k') \mapsto$$

$$\mu^{QE}_{(g,(h,k))}((G,V),(H\times K, W\oplus U)) \circ (Id\wedge \mu^{QE}_{(h,k)}(H\times K, W\oplus U))(\alpha_g(g')\wedge\beta_h(h')\wedge\gamma_k(k'))\Big)$$

By Lemma 6.11, the two terms are equal.

In addition, $QE(\tau)\circ\mu^{QE}((G,V),(H,W))\circ(\eta^{QE}(G,V)\wedge Id)(v\wedge X)$ is

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}}\Big((h',g')\mapsto QE_{(g,h)}(\tau)\circ\mu^{QE}_{(g,h)}((G,V),(H,W))\circ(\eta^{QE}_g(G,V)\wedge Id)((g'\cdot v)\wedge\beta_h(h'))\Big)$$

And $\mu^{QE}((H,W),(G,V))\circ(Id\wedge\eta^{QE}(H,W))\circ\tau(v\wedge X)$ is

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}}\Big((h',g')\mapsto\mu^{QE}_{(h,g)}((H,W),(G,V))\circ(Id\wedge\eta^{QE}_h(H,W))\circ\tau((g'\cdot v)\wedge\beta_h(h'))\Big)$$

The two terms are equal. So the centrality of unit diagram commutes.

Moreover, by Lemma 6.11, $\mu^{QE}((G,V),(H,W))(X\wedge Y)=$

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}}\Big((g',h')\mapsto\mu^{QE}_{(g,h)}((G,V),(H,W))(\alpha_g(g')\wedge\beta_h(h'))\Big)$$
$$=\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}}\Big((h',g')\mapsto\mu^{QE}_{(h,g)}((H,W),(G,V))(\beta_h(h')\wedge\alpha_g(g'))\Big)$$

is $\mu^{QE}((H,W),(G,V))(Y\wedge X)$. Therefore we have the commutativity of QE. $\square$

Proposition 6.13. Let G be any compact Lie group. Let V be an ample orthogonal $G-$representation and W an orthogonal $G-$representation. Let $\sigma^{QE}_{G,V,W}:S^W\wedge QE(G,V)\longrightarrow QE(G,V\oplus W)$ denote the structure map of QE defined by the unit map $\eta^{QE}(G,V)$. Let $\widetilde{\sigma}^{QE}_{G,V,W}$ denote the right adjoint of $\sigma^{QE}_{G,V,W}$. Then $\widetilde{\sigma}^{QE}_{G,V,W}:QE(G,V)\longrightarrow \mathrm{Map}(S^W,QE(G,V\oplus W))$ is a $G-$weak equivalence.

Proof. From the formula of $\eta^{QE}(G,V)$, we can get an explicit formula for

$$\widetilde{\sigma}^{QE}_{G,V,W}:QE(G,V)\longrightarrow\mathrm{Map}(S^W,QE(G,V\oplus W)).$$

Let $\alpha:=\prod\limits_{g\in G^{tors}_{conj}}\alpha_g$ be any element in $QE(G,V)=\prod\limits_{g\in G^{tors}_{conj}}\mathrm{Map}_{C_G(g)}(G,QE_g(G,V))$, and w an element in S^W. For each $g\in G^{tors}_{conj}$, w has a unique decomposition $w=w^1_g\wedge w^2_g$ with $w^1_g\in S^{W^g}$ and $w^2_g\in S^{(W^g)^\perp}$. $\widetilde{\sigma}^{QE}_{G,V,W}$ sends α to

$$w\mapsto\Big(\prod_{g\in G^{tors}_{conj}}g'\mapsto\Delta^*_G\circ\mu^{QE}_{(g,g)}((G,V),(G,W))(\alpha_g(g'),\eta^{QE}_g(G,W)(g'\cdot w))\Big).$$

It suffices to show that for each $g\in G^{tors}_{conj}$, the map

$$\widetilde{\sigma}^{QE}_{G,g,V,W}:QE_g(G,V)\longrightarrow\mathrm{Map}_{C_G(g)}(S^W,QE_g(G,V\oplus W))$$
$$x\mapsto\Big(w\mapsto\Delta^*_G\circ\mu^{QE}_{(g,g)}((G,V),(G,W))(x,\eta^{QE}_g(G,W)(w))\Big)$$

is a $C_G(g)-$weak equivalence. We check for each closed subgroup H of $C_G(g)$, the map $(\widetilde{\sigma}^{QE}_{G,g,V,W})^H$ on the fixed point space is a homotopy equivalence.

Case I: $g\in H$.

$QE_g(G,V)^H$ is the space $F_g(G,V)^H$. By Proposition 6.2,

$$\widetilde{\sigma}_g(G,V,W)^H:F_g(G,V)^H\longrightarrow\mathrm{Map}_H(S^{W^g},F_g(G,V\oplus W))$$

is a weak equivalence.

By Theorem 5.3,

$$\mathrm{Map}_H(S^W, QE_g(G, V\oplus W)) \longrightarrow \mathrm{Map}_H(S^{W^g}, F_g(G, V\oplus W)),\ f\mapsto f|_{S^{W^g}}$$

is a homotopy equivalence. And we have the diagram below commutes.

$$\begin{tikzcd} F_g(G,V)^H \arrow[r, "\simeq"] \arrow[dr] & \mathrm{Map}_H(S^{W^g}, F_g(G, V\oplus W)) \\ & \mathrm{Map}_H(S^W, QE_g(G, V\oplus W)) \arrow[u, "\simeq"'] \end{tikzcd} \tag{6.17}$$

So $\widetilde{\sigma}^{QE}_{G,g,V,W}{:}F_g(G,V)^H{\longrightarrow}\mathrm{Map}_H(S^{W^g}, F_g(G,V\oplus W))$ is a homotopy equivalence.

Case II: g is not in H. In this case, $QE_g(G,V)^H$ is contractible. It suffices to show $\mathrm{Map}_H(S^W, QE_g(G, V\oplus W)$ is also contractible. Note that for any closed subgroup H' of H, $QE_g(G, V\oplus W)^{H'}$ is contractible. So for each $n-$cell $H/H'\times D^n$ of S^W, it's mapped to $QE_g(G,V\oplus W)^{H'}$ unique up to homotopy. So $\mathrm{Map}_H(S^W, QE_g(G, V\oplus W)$ is contractible.

Therefore $\widetilde{\sigma}^{QE}_{G,g,V,W}$ is a $C_G(g)-$weak equivalence. So $\widetilde{\sigma}^{QE}_{G,V,W}$ is a $G-$weak equivalence. □

By Proposition 6.4 and Proposition 6.13 we can get the conclusion below.

Corollary 6.14. For any compact Lie group G, $(QE(G,-), \eta^{QE}, \mu^{QE})$ represents $QE^{(-)}_G(-)$ in GwS.

Especially we have the conclusion for quasi-elliptic cohomology.

Corollary 6.15. For any compact Lie group G, $(QKU(G,-), \eta^{QKU}, \mu^{QKU})$ represents $QEll^{(-)}_G(-)$ in GwS.

At last, we get the main conclusion of Section 6.

Theorem 6.16. *There is a well-defined functor $\mathcal{Q}$ from the full subcategory consisting of $\mathcal{I}_G-FSP$ in GwS to the same category sending (E_G, η^E, μ^E) to $(QE(G,-), \eta^{QE}, \mu^{QE})$ that represents the cohomology theory QE in GwS.*

The restriction of $\mathcal{Q}$ to the full subcategory consisting of commutative $\mathcal{I}_G-FSP$ is a functor from that category to itself.

7. The Restriction map

In this section we construct the restriction maps $QE(G,V)\longrightarrow QE(H,V)$ for group homomorphisms $H\longrightarrow G$. The restriction maps for quasi-elliptic cohomology can be constructed in the same way.

Let $\phi: H\longrightarrow G$ be a group homomorphism and V a $G-$representation. For any homomorphism of compact Lie groups $\phi: H\longrightarrow G$ and $H-$space X, we have the change-of-group isomorphism $QE^*_G(G\times_H X)\cong QE^*_H(X)$. Thus, for any subgroup K of H, we have the isomorphism $QE^n_G(G/K) = QE^n_G(G\times_H H/K)\cong QE^n_H(H/K)$. So by Proposition 6.4 the space $QE(G,V)^K$ is homotopy equivalent to $QE(H,V)^K$ when V is a faithful $G-$representation. It implies when we consider $QE(G,V)$ as an $H-$space, it is $H-$weak equivalent to $QE(H,V)$.

As indicated in Remark **??**, the orthogonal $G-$spectrum $QE(G,-)$ cannot arise from an orthogonal spectrum. As a result, the restriction map $QE(G,V)\longrightarrow$

$QE(H,V)$ cannot be a homeomorphism. We construct in this section a restriction map ϕ_V^* that is $H-$weak equivalence such that the diagram below commutes.

$$\begin{array}{ccc} \pi_k(QE(G,V)) & \xrightarrow{\cong} & QE_G^V(S^k) \\ \Big\downarrow{\scriptstyle \pi^k(\phi_V^*)} & & \Big\downarrow{\scriptstyle \phi^*} \\ \pi_k(QE(H,V)) & \xrightarrow{\cong} & QE_H^V(S^k) \end{array} \tag{7.1}$$

where ϕ^* is the restriction map of quasi-elliptic cohomology.

Let X be a $G-$space. Let $g \in G^{tors}$ and $h \in H^{tors}$. The group homomorphism $\phi : H \longrightarrow G$ sends $C_H(h)$ to $C_G(g)$ and also gives

$$\phi_* : \Lambda_H(h) \longrightarrow \Lambda_G(\phi(h)),\ [h', t] \mapsto [\phi(h'), t].$$

ϕ induces an $H-$action on X. Especially, $X^g = X^h$ and ϕ_* induces a $\Lambda_H(h)-$action on it for each $h \in H^{tors}$. We consider the equivalent definition of the $QE-$theory

$$QE_G^*(X) = \prod_{g \in G^{tors}} E^*_{\Lambda_G(g)}(X^g).$$

With this definition, the restriction map can have a relatively simple form.

For each $g \in G^{tors}$, we first define a map

$$Res_{\phi,g} : \mathrm{Map}_{C_G(g)}(G, QE_g(G,V)) \longrightarrow \prod_{\tau} \mathrm{Map}_{C_H(\tau)}(H, QE_\tau(H,V))$$

in the form $\prod\limits_{\tau}\left(R_{\phi,\tau} : \mathrm{Map}_{C_G(g)}(G, QE_g(G,V)) \longrightarrow \mathrm{Map}_{C_H(\tau)}(H, QE_\tau(H,V))\right)$ where τ goes over all the elements τ in H^{tors} such that $\phi(\tau) = g$. Then we will combine all the $Res_{\phi,g}$s to define the restriction map ϕ_V^*.

The restriction map

$$\phi_V^* : QE(G,V) \longrightarrow QE(H,V)$$

to be defined should make the diagram (7.2) commute, which implies that (7.1) commutes.

$$\begin{array}{ccccccc} X^g & \longrightarrow & X & \xrightarrow{\widetilde{f}} & \mathrm{Map}_{C_G(g)}(G, QE_g(G,V)) & \xrightarrow{\alpha \mapsto \alpha(e)} & F_g(G,V) \\ {\scriptstyle =}\Big\downarrow & & {\scriptstyle =}\Big\downarrow & & \Big\downarrow{\scriptstyle R_{\phi,\tau}} & & \Big\downarrow{\scriptstyle res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}} \\ X^\tau & \longrightarrow & X & \xrightarrow{R_{\phi,\tau}\circ\widetilde{f}} & \mathrm{Map}_{C_H(\tau)}(H, QE_\tau(H,V)) & \xrightarrow{\beta \mapsto \beta(e)} & F_\tau(H,V) \end{array} \tag{7.2}$$

where $res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}$ is the restriction map defined in (7.3).

Let $\tau \in H^{tors}$ and $g = \phi(h)$. Then we have the isomorphism

$$a_\tau : (V)_g^{\mathbb{R}} \oplus V^g \longrightarrow (V)_\tau^{\mathbb{R}} \oplus V^\tau$$

sending v to v. For any $[b,t] \in \Lambda_H(h)$, $a_\tau([\phi(b), t]v) = [b,t]a_\tau(v)$.

In addition, we have the restriction map $res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} : F_g(G,V) \longrightarrow F_\tau(H,V)$ defined as below. Let $\beta : S^{(V)_g^{\mathbb{R}}} \longrightarrow E((V)_g^{\mathbb{R}} \oplus V^g)$ be an $\mathbb{R}-$equivariant map. Note that $S^{(V)_\tau^{\mathbb{R}}}$ and $S^{(V)_g^{\mathbb{R}}}$ have the same underlying space, and $(V)_g^{\mathbb{R}} \oplus V^g$ and

$(V)^{\mathbb{R}}_\tau \oplus V^\tau$ have the same underlying vector space. $res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}(\beta)$ is defined to be the composition

$$S^{(V)^{\mathbb{R}}_\tau} \xrightarrow{x\mapsto x} S^{(V)^{\mathbb{R}}_g} \xrightarrow{\beta} E((V)^{\mathbb{R}}_g \oplus V^g) \xrightarrow{E(a_\tau)} E((V)^{\mathbb{R}}_\tau \oplus V^\tau) \tag{7.3}$$

which is the identity map on the underlying spaces.

Let $\psi : K \longrightarrow H$ be another group homomorphism and $\psi(k) = h$ for some $k \in K$. Then we have

$$res|^{\Lambda_H(h)}_{\Lambda_K(k)} \circ res|^{\Lambda_G(g)}_{\Lambda_H(h)} = res|^{\Lambda_G(g)}_{\Lambda_K(k)} \tag{7.4}$$

Note $S(G,V)_g$ has the same underlying space as $S(H,V)_\tau$. Consider the join of maps

$$res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * Id : F_g(G,V) * S(G,V)_g \longrightarrow F_\tau(H,V) * S(H,V)_\tau \tag{7.5}$$

It is the identity map on the underlying space and has the equivariant property: for any $a \in C_H(\tau)$, $x \in H$,

$$res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * Id(\phi(a)\cdot x) = a \cdot res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * Id(x). \tag{7.6}$$

$res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * b_\tau$ gives a well-defined map on the quotient space $r_{\phi,\tau} : QE_g(G,V) \longrightarrow QE_\tau(H,V)$. It also has the equivariant property as (7.6). For any ρ in $\mathrm{Map}_{C_G(g)}(G, E_g(G,V))$, let $R_{\phi,\tau}(\rho)$ be the composition $H \xrightarrow{\phi} G \xrightarrow{\rho} QE_g(G,V) \xrightarrow{r_{\phi,\tau}} QE_\tau(H,V)$. $R_{\phi,\tau}(\rho)$ is $C_H(\tau)-$equivariant: $R_{\phi,\tau}(\rho)(ah) = r_{\phi,\tau}(\rho(\phi(ah))) = r_{\phi,\tau}(\rho(\phi(a)\phi(h))) = ar_{\phi,\tau}(\rho(\phi(h))) = a\cdot R_{\phi,\tau}(\rho)(h)$, for any $a \in C_H(\tau)$, $h \in H$.

For any $g \in Im\phi$, $Res_{\phi,g}$ is defined to be $\prod\limits_\tau R_{\phi,\tau}$ where τ goes over all the $\tau \in H^{tors}$ such that $\phi(\tau) = g$. The restriction map is defined to be

$$\phi^*_V := \prod_g Res_g : QE(G,V) \longrightarrow QE(H,V) \tag{7.7}$$

where g goes over all the elements in G^{tors} in the image of ϕ.

Lemma 7.1. *(i) $R_{\phi,\tau}$ is the restriction map making the diagram*

$$\begin{array}{ccc} Map_{C_G(g)}(G, QE_g(G,V)) & \xrightarrow{\alpha\mapsto\alpha(e)} & F_g(G,V) \\ \Big\downarrow{\scriptstyle R_{\phi,\tau}} & & \Big\downarrow{\scriptstyle res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}} \\ Map_{C_H(\tau)}(H, QE_\tau(H,V)) & \xrightarrow{\beta\mapsto\beta(e)} & F_\tau(H,V) \end{array} \tag{7.8}$$

commute. So the restriction map ϕ^*_V makes the diagram (7.2) commute.

(ii)Let $\phi : H \longrightarrow G$ and $\psi : K \longrightarrow H$ be two group homomorphism and V a $G-$representation. Then $\psi^*_V \circ \phi^*_V = (\phi\circ\psi)^*_V$. The composition is associative.

(iii) $Id^*_V : QE(G,V) \longrightarrow QE(G,V)$ is the identity map.

Proof. (i) $R_{\phi,\tau}(\alpha)(e) = r_{\phi,\tau}\circ\alpha(e) = res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}\alpha(e)$. So (7.8) commutes.

(ii) Let $\rho_g : G \longrightarrow QE_g(G,V)$ be a $C_G(g)-$equivariant map for each $g \in G^{tors}$. Note that if we have $\psi(\sigma) = \tau$ and $\phi(\tau) = g$, then $r_{\phi,\tau}\circ r_{\psi,\sigma} = r_{\phi\circ\psi,\sigma}$ since both sides are identity maps on the underlying spaces. Then we have for any $k \in K$,

$$\psi_V^* \circ \phi_V^*(\prod_{g\in G^{tors}} \rho_g) = \prod_g \prod_\tau \prod_\sigma r_{\psi,\sigma} \circ r_{\phi,\tau}\rho_g(\phi(\psi(k)))$$
$$= \prod_g \prod_\tau \prod_\sigma r_{\psi\circ\phi,\sigma}\rho_g(\phi\circ\psi(k)) = (\phi\circ\psi)_V^*(\prod_{g\in G^{tors}} \rho_g)$$

where τ goes over all the elements in H^{tors} with $\phi(\tau) = g$ and σ goes over all the elements in K^{tors} with $\psi(\sigma) = \tau$. So $\psi_V^* \circ \phi_V^* = (\phi\circ\psi)_V^*$.

(iii) For the identity map $Id : G \longrightarrow G$, by the formula of the restriction map, $Id_V^*(\prod\limits_{g\in G^{tors}} \rho_g) = \prod\limits_{g\in G^{tors}} \rho_g$, thus, is the identity. □

8. The Birth of a new global homotopy theory

At the early beginning of equivariant homotopy theory people noticed that certain theories naturally exist not only for one particular group but for all groups in a specific class. This observation motivated the birth of global homotopy theory. In [**25**] the concept of orthogonal spectra is introduced, which is defined from $\mathbb{L}-$functors with $\mathbb{L}$ the category of inner product real spaces. Each global spectra consists of compatible $G-$spectra with G across the entire category of groups and they reflect any symmetry. Globalness is a measure of the naturalness of a cohomology theory.

In Remark 4.1.2 [**25**], Schwede discussed the relation between orthogonal $G-$spectra and global spectra. We have the question whether the underlying orthogonal $G-$spectrum of the I_G-FSP $(QE(G,-), \eta^{QE}, \mu^{QE})$ in Theorem 6 can arise from an orthogonal spectrum. Ganter showed that $\{QEll_G^*\}_G$ have the change-of-group isomorphism, which is a good sign that quasi-elliptic cohomology may be globalized.

By the discussion in Remark 4.1.2 [**25**], however, the answer to this question is no. A $G-$spectrum Y is isomorphic to an orthogonal $G-$spectrum of the form $X\langle G\rangle$ for some orthogonal spectrum X if and only if for every trivial $G-$representation V the $G-$action on $Y(V)$ is trivial. $QE(V)$ is not trivial when V is trivial. So it cannot arise from an orthogonal spectrum.

Then it is even more difficult to see whether each elliptic cohomology theory, whose form is more intricate and mysterious than quasi-elliptic cohomology, can be globalized in the current setting.

Our solution is to establish a more flexible global homotopy theory where quasi-elliptic chomomology can fit into. We hope that it is easier to judge whether a cohomology theory, especially an elliptic cohomology theory, can be globalized in the new theory. In addition we want to show that the new global homotopy theory is equivalent to the current global homotopy theory.

We construct in [**13**] a category D_0 to replace $\mathbb{L}$ whose objects are (G, V, ρ) with V an inner product vector space, G a compact group and ρ a faithful group representations

$$\rho : G \longrightarrow O(V),$$

and whose morphism $\phi = (\phi_1, \phi_2) : (G, V, \rho) \longrightarrow (H, W, \tau)$ consists of a linear isometric embedding $\phi_2 : V \longrightarrow W$ and a group homomorphism $\phi_1 : \tau^{-1}(O(\phi_2(V))) \longrightarrow$

G, which makes the diagram (8.1) commute.

$$
\begin{tikzcd}
G \arrow[r, "\rho"] & O(V) \arrow[d, "\phi_{2*}"] \\
\tau^{-1}(O(\phi_2(V))) \arrow[u, "\phi_1"] \arrow[r, "\tau"] & O(W)
\end{tikzcd} \tag{8.1}
$$

In other words, the group action of H on $\phi_2(V)$ is induced from that of G. Intuitively, the category D_0 is obtained by adding the restriction maps between representations into the category $\mathbb{L}$.

Instead of the category of orthogonal spaces, we study the category of D_0-spaces. The category of orthogonal spaces is a full subcategory of the category D_0T of D_0-spaces. Apply the idea of diagram spectra in [**19**], we can also define D_0-spectra and D_0-FSP.

Combining the orthogonal $G-$spectra of quasi-elliptic cohomology together, we get a well-defined D_0-spectra and D_0-FSP. Thus, we can define global quasi-elliptic cohomology in the category of D_0-spectra.

Theorem 8.1. *(Theorem 7.2.3, [**13**])There is a D_0-FSP weakly representing quasi-elliptic cohomology.*

Equipping a homotopy theory with a model structure is like interpreting the world via philosophy. Model category theory is an essential basis and tool to judge whether two homotopy theories describe the same world. We build several model structures on D_0T. First by the theory in [**19**], there is a level model structure on D_0T.

Theorem 8.2. *(Theorem 6.3.4, [**13**])*
The category of D_0-spaces is a compactly generated topological model category with respect to the level equivalences, level fibrations and $q-$cofibrations. It is right proper and left proper.

D_0 is a generalized Reedy category in the sense of [**?**]. We can construct a Reedy model structure on D_0T.

Theorem 8.3. *(Theorem 6.4.5, [**13**])The Reedy cofibrations, Reedy weak equivalences and Reedy fibrations form a model structure, the Reedy model structure, on the category of D_0-spaces.*

We are constructing a global model structure on D_0T Quillen equivalent to the global model structure on the orthogonal spaces constructed by Schwede in [**25**]. Moreover, other than the new unstable global homotopy theory, we will also establish the new stable global homotopy theory.

Appendix A. Join

A.1. Definition.

Definition A.1. In topology, the join $A * B$ of two topological spaces A and B is defined to be the quotient space $(A \times B \times [0,1])/R$, where R is the equivalence relation generated by $(a, b_1, 0) \sim (a, b_2, 0)$ for all $a \in A$ and $b_1, b_2 \in B$ and $(a_1, b, 1) \sim (a_2, b, 1)$ for all $a_1, a_2 \in A$ and $b \in B$.

At the endpoints, this collapses $A \times B \times \{0\}$ to A and $A \times B \times \{1\}$ to B.

The join $A*B$ is the homotopy colimit of the diagram $A \longleftarrow A\times B \longrightarrow B$.

A nice way to write points of $A*B$ is as formal linear combination t_1a+t_2b with $0\leq t_1, t_2\leq 1$ and $t_1+t_2=1$, subject to the rules $0a+1b=b$ and $1a+0b=a$. The coordinates correspond exactly to the points in $A*B$.

PROPOSITION A.2. Join is associative and commutative. Explicitly, $A*(B*C)$ is homeomorphic to $(A*B)*C$, and $A*B$ is homeomrphic to $B*A$.

A.2. Group Action on the Join.

EXAMPLE A.3. Let G be a compact Lie group. Let A, B be $G-$spaces. Then $A*B$ has a $G-$structure on it by

$$g\cdot(t_1a+t_2b):=t_1(g\cdot a)+t_2(g\cdot b),\ \text{for any } g\in G, a\in A, b\in B,\ \text{and } t_1,t_2\geq 0, t_1+t_2=1. \tag{A.1}$$

It's straightforward to check (A.1) defines a continuous group action.

EXAMPLE A.4. Let G and H be compact Lie groups. Let A be a $G-$space and B a $H-$space. Then $A*B$ has a continuous $G\times H-$structure on it by

$$(g,h)\cdot(t_1a+t_2b):=t_1(g\cdot a)+t_2(h\cdot b),\ \text{for any } g\in G, a\in A, b\in B,\ \text{and } t_1,t_2\geq 0, t_1+t_2=1. \tag{A.2}$$

Appendix B. Equivariant Orthogonal spectra

In Section B.1, we recall the basics of equivariant orthogonal spectra. There are many references for this topic, such as [**2**], [**18**] [**25**], [**20**], etc. In Section B.3 we recall the global K-theory, which is a prominent example of global homotopy theory. Its properties will be applied in the construction of the orthogonal $G-$spectrum for quasi-elliptic cohomology.

B.1. Orthogonal G-spectra. Let G be a compact Lie group. Let $\mathcal{I}_G$ denote the category whose objects are pairs $(\mathbb{R}^n,\rho)$ with ρ a homomorphism from G to $O(n)$ giving $\mathbb{R}^n$ the structure of a $G-$representation. Morphisms $(\mathbb{R}^m,\mu)\longrightarrow(\mathbb{R}^n,\rho)$ are linear isometric isomorphisms $\mathbb{R}^m\longrightarrow\mathbb{R}^n$.

Let Top_G denote the category with objects based $G-$spaces and morphisms continuous based maps.

DEFINITION B.1. An $\mathcal{I}_G-$space is a $G-$continuous functor $X:\mathcal{I}_G\longrightarrow Top_G$. Morphisms between $\mathcal{I}_G-$spaces are natural $G-$transformations.

DEFINITION B.2. An orthogonal $G-$spectrum is an $\mathcal{I}_G-$space X together with a natural transformation of functors $\mathcal{I}_G\times\mathcal{I}_G\longrightarrow Top_G$

$$X(-)\wedge S^{-}\longrightarrow X(-\oplus -)$$

satisfying appropriate associativity and unitality diagrams. In other words, an orthogonal $G-$spectrum is an $\mathcal{I}_G-$space with an action of the sphere $\mathcal{I}_G-$space.

DEFINITION B.3. For $\mathcal{I}_G-$spaces X and Y, define the "external" smash product $X\overline{\wedge}Y$ by

$$X\overline{\wedge}Y=\wedge\circ(X\times Y):\mathcal{I}_G\times\mathcal{I}_G\longrightarrow Top_G; \tag{B.1}$$

thus $(X\overline{\wedge}Y)(V,W)=X(V)\wedge Y(W)$.

We have an equivariant notion of a functor with smash product (FSP).

DEFINITION B.4. An $\mathcal{I}_G-$FSP is an $\mathcal{I}_G-$space X with a unit $G-$map $\eta : S \longrightarrow X$ and a natural product $G-$map $\mu : X\overline{\wedge}X \longrightarrow X \circ \bigoplus$ of functors $\mathcal{I}_G \times \mathcal{I}_G \longrightarrow Top_G$ such that the evident unit, associativity and centrality of unit diagram also commutes.

LEMMA B.5. *An $\mathcal{I}_G-$FSP has an underlying $\mathcal{I}_G-$spectrum with structure $G-$map*

$$\sigma = \mu \circ (id\overline{\wedge}\eta) : X\overline{\wedge}S \longrightarrow X \circ \oplus.$$

B.2. Orthogonal spectra. The global homotopy theory is established to better describe certain theories naturally exists not only for a particular group, but for all groups of certain type in a compatible way. Some prominent examples of this are equivariant stable homotopy, equivariant K-theory, and equivariant bordism.

The idea of global orthogonal spectra was first inspired in the paper [**11**] by Greenlees and May where they introduce the concept of global $\mathcal{I}_*-$functors with smash product. The idea is developed by Mandell and May [**18**] and Bohmann [**2**]. Schwede develops another modern approach of global homotopy theory using a different categorical framework in [**25**], which is the main reference for Section B.2. For definition of orthogonal spectra in detail, please refer [**20**], [**19**], [**25**].

First we recall the definition of orthogonal spaces. Let $\mathbb{L}$ denote the category whose objects are inner product real spaces and whose morphism set between two objects V and W are the linear isometric embeddings $L(V, W)$.

DEFINITION B.6. An orthogonal space is a continuous functor $Y : \mathbb{L} \longrightarrow \mathcal{T}$ to the category of topological spaces. A morphism of orthogonal spaces is a natural transformation. We denote by spc the category of orthogonal spaces.

Orthogonal spectra is the stabilization of orthogonal spaces.

Let $\mathbb{O}$ denote the category whose objects are inner product real spaces and the morphisms $O(V, W)$ between two objects V and W is the Thom space of the total space

$$\xi(V, W) := \{(w, \phi) \in W \times L(V, W) | W \perp \phi(V)\}$$

of the orthogonal complement vector bundle, whose structure map $\xi(V, W) \longrightarrow L(V, W)$ is the projection to the second factor.

DEFINITION B.7. An orthogonal spectrum is a based continuous functor from $\mathbb{O}$ to the category of based compactly generated weak Hausdorff spaces. A morphism is a natural transformation of functors. Let Sp denote the category of orthogonal spectrum.

DEFINITION B.8. Given an orthogonal spectrum X and a compact Lie group G, the collection of $G-$spaces $X(V)$, for V a $G-$representation, and the equivariant structure maps $\sigma_{V,W}$ form an orthogonal $G-$spectrum. This orthogonal $G-$spectrum

$$X\langle G\rangle = \{X(V), \sigma_{V,W}\}$$

is called *the underlying orthogonal $G-$spectrum* of X.

B.3. Global K-theory and its variations. A classical example of orthogonal spectra is global K-theory. Quasi-elliptic cohomology can be expressed in terms of equivariant K-theory. And this example is especially important for our construction.

In [**16**] Joachim constructs G-equivariant K-theory as an orthogonal G-spectrum $\mathbb{K}_G$ for any compact Lie group G. In fact it is the only known $E_\infty-$version of equivariant complex K-theory when G is a compact Lie group.

For any real $G-$representation V, let $\mathbb{C}l_V$ be the Clifford algebra of V and $\mathcal{K}_V$ be the $G-C^*-$algebra of compact operators on $L^2(V)$. Let $s := C_0(\mathbb{R})$ be the graded $G-C^*-$algebra of continuous functions on $\mathbb{R}$ vanishing at infinity with trivial $G-$action. Then the orthogonal $G-$spectrum for equivariant K-theory defined by Joachim is the lax monoidal functor given by

$$\mathbb{K}_G(V) = Hom_{C^*}(s, \mathbb{C}l_V \otimes \mathcal{K}_V)$$

of $\mathbb{Z}/2-$graded $*-$homomorphisms from s to $\mathbb{C}l_V \otimes \mathcal{K}_V$.

Bohmann showed in her paper [**2**] that Joachim's model is "global", i.e. $\mathbb{K}$ is an orthogonal $\mathcal{G}-$spectrum. For more detail, please read [**2**] for reference.

Schwede's construction of global K-theory KR in [**25**] is a unitary analog of the construction by Joachim. It is an ultra-commutative ring spectrum whose $G-$homotopy type realizes Real $G-$equivariant periodic K-theory. He also shows that the spaces in the orthogonal spectrum KR represent Real equivariant K-theory.

For any complex inner product space W, let $\Lambda(W)$ be the exterior algebra W and $Sym(W)$ the symmetric algebra of it. The tensor product

$$\Lambda(W) \otimes Sym(W)$$

inherits a hermitian inner product from W and it's $\mathbb{Z}/2-$graded by even and odd exterior powers. Let $\mathcal{H}_W$ denote the Hilbert space completion of $\Lambda(W) \otimes Sym(W)$. Let $\mathcal{K}_W$ be the C^*-algebra of compact operators on $\mathcal{H}_W$. The orthogonal spectrum KR is defined to be the lax monoidal functor

$$KR(W) = Hom_{C^*}(s, \mathcal{K}_W).$$

Let uW denote the underlying euclidean vector space of W. There is an isomorphism of $\mathbb{Z}/2-$graded C^*-algebras

$$Cl(uW) \otimes_{\mathbb{R}} \mathcal{K}(L^2(W)) \cong \mathcal{K}_W.$$

So we get a homeomorphism

$$KR(W) \cong Hom_{C^*}(s, Cl(uW) \otimes_{\mathbb{R}} \mathcal{K}(L^2(W))) = \mathbb{K}(uW).$$

We have the relations below between the global Real K-theory KR, periodic unitary K-theory KU and periodic orthogonal real K-theory KO.

$$KU = u(KR);\ KO = KR^{\psi}.$$

In [**25**], Schwede shows that the spaces in the orthogonal spectrum KR represent real equivariant K-theory.

Theorem B.9. *For a compact Lie group G, a "sufficiently large" (i.e. faithful) real $G-$representation V and a compact $G-$space B, there is a bijection $\Psi_{G,B,V}$: $K_G(B) \longrightarrow [B_+, KU(V)]^G$ that is natural in B.*

We will use the orthogonal spectrum KU in the construction of orthogonal quasi-elliptic cohomology.

Definition B.10. An orthogonal $G-$representation is called ample if its complexified symmetric algebra is complete complex $G-$universe.

THEOREM B.11. *(i) Let G be a compact Lie group and V an orthogonal $G-$representation. For every ample $G-$representation W, the adjoint structure map*

$$\widetilde{\sigma}^K_{V,W} : KU(V) \longrightarrow Map(S^W, KU(V \oplus W))$$

is a $G-$weak equivalence.
(ii) Let G be an augmented Lie group and V a real $G-$representation such that $Sym(V)$ is a complete real $G-$universe. For every real $G-$representation W the adjoint structure map

$$\widetilde{\sigma}^K_{V,W} : KR(V) \longrightarrow Map(S^W, KR(V \oplus W))$$

is a $G-$weak equivalence.

Appendix C. Faithful representation of $\Lambda_G(g)$

We will apply the orthogonal spectrum KU of global K-theory to construct the orthogonal $G-$spectrum of QE^*_G. As indicated in Theorem B.9, we will need a faithful $\Lambda_G(g)-$representation. Thus, before construction in Section 6.1 and 6.2, we discuss complex and real $\Lambda_G(\sigma)-$representations in Section C.1 and C.2 respectively.

C.1. Preliminaries: faithful representations of $\Lambda_G(g)$. As shown in Theorem B.9, $KU(V)$ represents $G-$equivariant complex K-theory when V is a faithful $G-$representation. In this section, we construct a faithful $\Lambda_G(\sigma)-$representation from a faithful $G-$representation.

Let G be a compact Lie group and $\sigma \in G^{tors}$ with order l. Let ρ be a complex $G-$representation with underlying space V. Let $i : C_G(\sigma) \hookrightarrow G$ denote the inclusion. Let $\{\lambda\}$ denote all the irreducible complex representations of $C_G(\sigma)$. As said in [**6**], we have the decomposition of a representation into its isotypic components $i^*V \cong \bigoplus_\lambda V_\lambda$ where V_λ denotes the sum of all subspaces of V isomorphic to λ. Each $V_\lambda = Hom_{C_G(\sigma)}(\lambda, V) \otimes_{\mathbb{C}} \lambda$ is unique as a subspace. Note that σ acts on each V_λ as a diagonal matrix.

Each V_λ can be equipped with a $\Lambda_G(\sigma)-$action. Each $\lambda(\sigma)$ is of the form $e^{\frac{2\pi i m_\lambda}{l}} I$ with $0 < m_\lambda \leq l$ and I the identity matrix. As shown in Remark 3.2, we have the well-defined complex $\Lambda_G(\sigma)-$representations $V_\lambda)_\sigma := V_\lambda \odot_{\mathbb{C}} q^{\frac{m_\lambda}{l}}$ and

$$(V)_\sigma := \bigoplus_\lambda V_\lambda \odot_{\mathbb{C}} q^{\frac{m_\lambda}{l}} \tag{C.1}$$

Each $(V_\lambda)_\sigma$ is the isotypic component of $(V)_\sigma$ corresponding to the irreducible representation $\lambda \odot_{\mathbb{C}} q^{\frac{m}{l}}$.

PROPOSITION C.1. Let V be a faithful $G-$representation. And let $\sigma \in G^{tors}$.
(i) If V contains a trivial subrepresentation, $(V)_\sigma$ is a faithful $\Lambda_G(\sigma)-$representation.
(ii) $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$ is a faithful $\Lambda_G(\sigma)-$representation.
(iii) $(V)_\sigma \oplus V^\sigma$ is a faithful $\Lambda_G(\sigma)-$representation.

PROOF. (i) Let $[a, t] \in \Lambda_G(\sigma)$ be an element acting trivially on $(V)_\sigma$. Assume $t \in [0, 1)$. On $(V_1)_\sigma$, $[a, t]v_0 = e^{2\pi i t}v_0 = v_0$. So $t = 0$. Then on the whole space V_σ, since $C_G(\sigma)$ acts faithfully on it and for any $v \in V_\sigma$, $[a, 0] \cdot v = a \cdot v = v$, then $a = e$.

So $(V)_\sigma$ is a faithful $\Lambda_G(\sigma)-$representation.

(ii) Let $[a, t] \in \Lambda_G(\sigma)$ be an element acting trivially on V_σ. Consider the subrepresentations $(V_\lambda)_\sigma$ and $(V_\lambda)_\sigma \otimes_{\mathbb{C}} q^{-1}$ of $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$ respectively. Let

v be an element in the underlying vector space V_λ. On $(V_\lambda)_\sigma$, $[a,t]\cdot v = e^{\frac{2\pi i m_\lambda t}{l}} a\cdot v = v$; and on $(V_\lambda)_\sigma \otimes_{\mathbb{C}} q^{-1}$, $[a,t]\cdot v = e^{\frac{2\pi i m_\lambda t}{l} - 2\pi i t} a\cdot v = v$. So we get $e^{2\pi i t}\cdot v = v$. Thus, $t=0$. $C_G(\sigma)$ acts faithfully on V, so it acts faithfully on $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$. Since $[a,0]\cdot w = w$, for any $w\in (V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$, so $a=e$.

Thus, $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$ is a faithful $\Lambda_G(\sigma)-$representation.

(iii) Note that V^σ with the trivial $\mathbb{R}-$action is the representation $(V^\sigma)_\sigma \otimes_{\mathbb{C}} q^{-1}$. The representation $(V)_\sigma \oplus V^\sigma$ contains a subrepresentation $(V^\sigma)_\sigma \oplus (V^\sigma)_\sigma \otimes_{\mathbb{C}} q^{-1}$, which is a faithful $\Lambda_G(\sigma)-$representation by the second conclusion of Proposition C.1. So $(V)_\sigma \oplus V^\sigma$ is faithful. □

LEMMA C.2. *For any $\sigma \in G^{tors}$, $(-)_\sigma$ defined in (C.1) is a functor from the category of $G-$spaces to the category of $\Lambda_G(\sigma)-$spaces. Moreover, $(-)_\sigma \oplus (-)_\sigma \otimes_{\mathbb{C}} q^{-1}$ and $(-)_\sigma \oplus (-)^\sigma$ in Proposition C.1 are also well-defined functors from the category of $G-$spaces to the category of $\Lambda_G(\sigma)-$spaces.*

PROOF. Let $f: V \longrightarrow W$ be a $G-$equivariant map. Then f is $C_G(\sigma)-$equivairant for each $\sigma \in G^{tors}$. For each irreducible complex $C_G(\sigma)-$representation λ, $f: V_\lambda \longrightarrow W_\lambda$ is $C_G(\sigma)-$equivairant. And $f_\sigma: (V_\lambda)_\sigma \longrightarrow (W_\lambda)_\sigma,\ v\mapsto f(v)$ with the same underlying spaces is well-defined and is $\Lambda_G(\sigma)-$equivariant. It is straightforward to check if we have two $G-$equivariant maps $f: V\longrightarrow W$ and $g: U \longrightarrow V$, then $(f\circ g)_\sigma = f_\sigma \circ g_\sigma$. So $(-)_\sigma$ gives a well-defined functor from the category of $G-$representations to the category of $\Lambda_G(\sigma)-$representation.

The other conclusions can be proved in a similar way. □

PROPOSITION C.3. Let H and G be two compact Lie groups. Let $\sigma \in G$ and $\tau \in H$. Let V be a $G-$representation and W a $H-$representation.

(i) We have the isomorphisms of representations $(V\oplus W)_{(\sigma,\tau)} = (V_\sigma \oplus W_\tau)$ as $\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma)\times_{\mathbb{T}} \Lambda_H(\tau)-$representations;
$(V\oplus W)_{(\sigma,\tau)} \oplus (V\oplus W)_{(\sigma,\tau)} \otimes_{\mathbb{C}} q^{-1} = ((V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}) \oplus ((W)_\tau \oplus (W)_\tau \otimes_{\mathbb{C}} q^{-1})$ as $\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma)\times_{\mathbb{T}} \Lambda_H(\tau)-$representations;
and $(V\oplus W)_{(\sigma,\tau)} \oplus (V\oplus W)^{(\sigma,\tau)} = ((V)_\sigma \oplus V^\sigma) \oplus ((W)_\tau \oplus W^\tau)$ as $\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma)\times_{\mathbb{T}} \Lambda_H(\tau)-$representations.

(ii) Let $\phi: H\longrightarrow G$ be a group homomorphism. Let $\phi_\tau: \Lambda_H(\tau) \longrightarrow \Lambda_G(\phi(\tau))$ denote the group homomorphism obtained from ϕ. Then we have

$$\phi_\tau^*(V)_{\phi(\tau)} = (V)_\tau,$$
$$\phi_\tau^*((V)_{\phi(\tau)} \oplus (V)_{\phi(\tau)} \otimes_{\mathbb{C}} q^{-1}) = (V)_\tau \oplus (V)_\tau \otimes_{\mathbb{C}} q^{-1},$$
$$\phi_\tau^*((V)_{\phi(\tau)} \oplus V^{\phi(\tau)}) = (V)_\tau \oplus V^\tau$$

as $\Lambda_H(\tau)-$representations.

PROOF. (i) Let $\{\lambda_G\}$ and $\{\lambda_H\}$ denote the sets of all the irreducible $C_G(\sigma)-$representations and all the irreducible $C_H(\tau)-$representations. Then λ_G and λ_H are irreducible representations of $C_{G\times H}(\sigma,\tau)$ via the inclusion $C_G(\sigma)\longrightarrow C_{G\times H}(\sigma,\tau)$ and $C_H(\tau)\longrightarrow C_{G\times H}(\sigma,\tau)$.

The $\mathbb{R}-$representation assigned to each $C_{G\times H}(\sigma,\tau)-$irreducible representation in $V\oplus W$ is the same as that assigned to the irreducible representations of V and W. So we have

$$(V\oplus W)_{(\sigma,\tau)} = (V_\sigma \oplus W_\tau)$$

as $\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma)\times_{\mathbb{T}} \Lambda_H(\tau)-$representations.

Similarly we can prove the other two conclusions in (i).

(ii) Let $\sigma = \phi(\tau)$. If $(\phi_\tau^* V)_{\lambda_H}$ is a $C_H(\tau)-$subrepresentation of $\phi_\tau^* V_{\lambda_G}$, the $\mathbb{R}-$representation assigned to it is the same as that to V_{λ_G}. So we have $\phi_\tau^*(V)_{\phi(\tau)} = (V)_\tau$ as $\Lambda_H(\tau)-$representations.

Similarly we can prove the other two conclusions in (ii). $\square$

C.2. real $\Lambda_G(\sigma)-$representation. In this section we discuss real $\Lambda_G(\sigma)-$representation and its relation with the complex $\Lambda_G(\sigma)-$representations introduced in Lemma 3.1. The main reference is [**4**] and [**6**].

Let G be a compact Lie group, $\sigma \in G^{tors}$.

Definition C.4. A complex representation $\rho : G \longrightarrow Aut_{\mathbb{C}}(V)$ is said to be self dual if it is isomorphic to its complex dual $\rho^* : G \longrightarrow Aut_{\mathbb{C}}(V^*)$ where $V^* := Hom_{\mathbb{C}}(V, \mathbb{C})$ and $\rho^*(g) = \rho(g^{-1})^*$.

Example C.5. Let $\rho : C_G(g) \longrightarrow Aut_{\mathbb{R}}(V)$ be an irreducible complex $C_G(g)-$representation. Then as in Lemma 3.1, there exists a character $\eta : \mathbb{R} \longrightarrow \mathbb{C}$ such that $\rho(g) = \eta(1)I$. And $\rho \odot_{\mathbb{C}} \eta$ is an irreducible complex representation of $\Lambda_G(g)$. Since $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t)$, it is not self-dual if η is nontrivial. In this case it is of complex type. And $(V \odot_{\mathbb{C}} \eta) \oplus (V \odot_{\mathbb{C}} \eta)^*$ has irreducible real form.

If V is of real type, it is the complexification of a real $C_G(g)-$representation W. If $g = e$ and the character η we choose is trivial, $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t) = \rho(\alpha^{-1})^T = \rho(\alpha) = (\rho \odot_{\mathbb{C}} \eta)[\alpha, t]$ since V is self-dual. In this case W is a real $\Lambda_G(g)-$representation via $[\alpha, t] \cdot w = \alpha w$. And $V \odot_{\mathbb{C}} \eta$ is of real type since it is the complexification of W. For any nontrivial element g in G^{tors}, the $\Lambda_G(g)-$representation $V \odot_{\mathbb{C}} \eta$ is of complex type, then $(V \odot_{\mathbb{C}} \eta) \oplus (V \odot_{\mathbb{C}} \eta)^*$ is of the real type.

If V is of quaternion type, then $V = U_{\mathbb{C}}$ can be obtained from a quaternion $C_G(g)-$representation U by restricting the scalar to $\mathbb{C}$. If $g = e$ and η is trivial, $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t) = \rho(\alpha^{-1})^T = \rho(\alpha) = (\rho \odot_{\mathbb{C}} \eta)[\alpha, t]$ since V is self-dual. In this case W is a quaternion $\Lambda_G(g)-$representation with $[\alpha, t] \cdot w = \alpha w$. So $V \odot_{\mathbb{C}} \eta$ is of quaternion type.

Consider the case that V is of complex type. If $g = e$ and η is trivial, $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t) = \rho(\alpha^{-1})^T = \rho(\alpha) \neq (\rho \odot_{\mathbb{C}} \eta)[\alpha, t]$ since V is not self-dual. So $V \odot_{\mathbb{C}} \eta$ is of complex type.

For any compact Lie group, we use $RO(G)$ denote the real representation ring of G. In light of the analysis in Example C.5, we have the following conclusion.

Lemma C.6. *Let $\sigma \in G^{tors}$. Then the map $\pi^* : RO\mathbb{T} \longrightarrow RO\Lambda_G(\sigma)$ exhibits $RO\Lambda_G(\sigma)$ as a free $RO\mathbb{T}-$module.*

In particular there is an $RO\mathbb{T}-$basis of $RO\Lambda_G(\sigma)$ given by irreducible real representations $\{V_\Lambda\}$. There is a bijection between $\{V_\Lambda\}$ and the set $\{\lambda\}$ of irreducible real representations of $C_G(\sigma)$. When σ is trivial, V_Λ has the same underlying space V as λ. When σ is nontrivial, $V_\Lambda = ((\lambda \otimes_{\mathbb{R}} \mathbb{C}) \odot_{\mathbb{C}} \eta) \oplus ((\lambda \otimes_{\mathbb{R}} \mathbb{C}) \odot_{\mathbb{C}} \eta)^*$ where η is a complex $\mathbb{R}-$representation such that $(\lambda \otimes_{\mathbb{R}} \mathbb{C})(\sigma)$ acts on $V \otimes_{\mathbb{R}} \mathbb{C}$ via the scalar multiplication by $\eta(1)$. The dimension of V_Λ is twice as that of λ.

As in (C.1), we can construct a functor $(-)^{\mathbb{R}}_\sigma$ from the category of real $G-$representations to the category of real $\Lambda_G(\sigma)-$representations with

$$(V)^{\mathbb{R}}_\sigma = (V \otimes_{\mathbb{R}} \mathbb{C})_\sigma \oplus (V \otimes_{\mathbb{R}} \mathbb{C})^*_\sigma. \tag{C.2}$$

Proposition C.7. Let V be a faithful real $G-$representation. And let $\sigma \in G^{tors}$ and l denote its order. Then $(V)^{\mathbb{R}}_{\sigma}$ is a faithful real $\Lambda_G(\sigma)-$representation.

Proof. Let $[a,t] \in \Lambda_G(\sigma)$ be an element acting trivially on $(V)^{\mathbb{R}}_{\sigma}$. Assume $t \in [0,1)$. Let $v \in (V \otimes_{\mathbb{R}} \mathbb{C})_{\sigma}$ and let v^* denote its correspondence in $(V \otimes_{\mathbb{R}} \mathbb{C})^*_{\sigma}$. Then $[a,t] \cdot (v+v^*) = (ae^{2\pi imt} + ae^{-2\pi imt})(v+v^*) = v+v^*$ where $0 < m \leq l$ is determined by σ. Thus a is equal to both $e^{2\pi imt}I$, and $e^{-2\pi imt}I$. Thus $t=0$ and a is trivial.

So $(V)^{\mathbb{R}}_{\sigma}$ is a faithful real $\Lambda_G(\sigma)-$representation. □

Proposition C.8. Let H and G be two compact Lie groups. Let $\sigma \in G^{tors}$ and $\tau \in H^{tors}$. Let V be a real $G-$representation and W a real $H-$representation.

(i) We have the isomorphisms of representations $(V \oplus W)^{\mathbb{R}}_{(\sigma,\tau)} = (V^{\mathbb{R}}_{\sigma} \oplus W^{\mathbb{R}}_{\tau})$ as $\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)-$representations.

(ii) Let $\phi : H \longrightarrow G$ be a group homomorphism. Let $\phi_{\tau} : \Lambda_H(\tau) \longrightarrow \Lambda_G(\phi(\tau))$ denote the group homomorphism obtained from ϕ. Then $\phi^*_{\tau}(V)^{\mathbb{R}}_{\phi(\tau)} = (V)^{\mathbb{R}}_{\tau}$, as $\Lambda_H(\tau)-$representations.

The proof is left to the readers.

Appendix D. Tedious proofs

D.1. The proof of Lemma 6.8.

Proof. When v_1 is infinity, $\eta_g(G,V)(v_1)$ is the basepoint of $F_g(V)$. So by the construction of $QE_g(G,V)$ in Proposition 6.4, $v = v_1 \wedge v_2$ is mapped to the basepoint of $QE_g(G,V)$. When v_2 is infinity, $\eta^{QE}_g(G,V)(v)$ is the basepoint. So $\eta^{QE}_g(G,V)$ is well-defined. And since $\eta_g(G,V)$ is $C_G(g)-$equivariant, $\eta^{QE}_g(G,V)$ is $C_G(g)-$equivariant.

Next we prove $\eta^{QE}_g(G,V)$ is continuous by showing for each point in $QE_g(G,V)$, there is an open neighborhood of it whose preimage is open in S^V. Consider a point x in the image of $\eta^{QE}_g(G,V)$ represented by $t_1 a + t_2 b$.

Case I: $0 < t_2 < 1$ and a is not the basepoint of $F_g(G,V)$.

Let A be an open neighborhood of a in $F_g(G,V)$ not including the basepoint. We can find such an A since $F_g(G,V)$ is Hausdorff. Let $\delta > 0$ be a small enough value. Let $U_{x,\delta}$ be the open neighborhood of x

$$U_{x,\delta} := \{[s_1\alpha + s_2\beta] \in QE_g(G,V) | \alpha \in A, |s_2 - t_2| < \delta, \|\beta - b\| < \delta\}.$$

Then $\eta^{QE}_g(G,V)^{-1}(U_{x,\delta})$ is the smash product of $\eta_g(G,V)^{-1}(A)$, which is open in S^{V^g}, and an open subset of $S^{(V^g)^{\perp}}$

$$\{w \in S^{(V^g)^{\perp}} | t_2 - \delta < \|w\| < t_2 + \delta, \|w - b\| < \delta\}.$$

So it's open in S^V.

Case II: $t_2 = 0$ and a is not the basepoint of $F_g(G,V)$.

Let A be an open neighborhood of a in $F_g(G,V)$ not including the basepoint. Let $\delta > 0$ be a small enough value. Let $W_{x,\delta}$ be the open neighborhood of x

$$W_{x,\delta} := \{[s_1\alpha + s_2\beta] \in E_g(G,V) | \alpha \in A, |s_2| < \delta, \|\beta - b\| < \delta\}.$$

Then $\eta^{QE}_g(G,V)^{-1}(W_{x,\delta})$ is the smash product of $\eta_g(G,V)^{-1}(A)$, which is open in S^{V^g}, and an open subset of $S^{(V^g)^{\perp}}$

$$\{w \in S^{(V^g)^{\perp}} | \|w\| < \delta, \|w - b\| < \delta\}.$$

So it's open in S^V.

Case III: x is the basepoint x_g of $QE_g(G,V)$.

Let A_0 be an open neighborhood of the basepoint c_0. For any point w of the form $t_1c_0+t_2b$ in the the space $QE'_g(G,V)$ with $0<t_2<1$, let U_{w,δ_w} denote the open subset of $QE_g(G,V)$

$$\{[s_1\alpha+s_2\beta]\in QE_g(G,V)|\alpha\in A_0, |s_2-t_2|<\delta_w, \|\beta-b\|<\delta_w\}$$

with δ_w small enough. Let W_δ denote the open subset of $QE_g(G,V)$

$$\{[s_1\alpha+s_2\beta]\in QE_g(G,V)|\alpha\in A_0, |s_2|<\delta, \|\beta-b\|<\delta\}$$

with δ small enough.

For any $b\in S(G,V)_g$ with $\|b\|\leqslant 1$, let V_{b,δ_b} denote the open subset of $QE_g(G,V)$

$$\{[s_1\alpha+s_2\beta]\in QE_g(G,V)|s_2>1-\delta_b, \|\beta-b\|<\delta_b\}$$

with δ_b small enough.

We consider the open neighborhood U of x that is the union of the spaces defined above

$$U:=(\bigcup_w U_{w,\delta_w})\cup W_\delta\cup(\bigcup_b V_{b,\delta_b})$$

where w goes over all the points of the form $[t_1c_0+t_2b]$ in $QE_g(G,V)$ with $0<t_2<1$, and b goes over all the points in $S(G,V)_g$ with $\|b\|\leqslant 1$.

The preimage of each U_{w,δ_w} and W_δ is open, the proof of which is analogous to Case I and II. The preimage of V_{b,δ_b} is the smash product of S^{V^g} and the open set of $S^{(V^g)^\perp}$

$$\{w_2\in S^{(V^g)^\perp}|\|w_2\|>1-\delta_b, \|w_2-b\|<\delta_b\},$$

thus, is open.

The preimage of U is the union of open subsets in S^V, thus, open.

Therefore, The map $\eta_g^{QE}(G,V)$ defined in (6.9) is continuous. □

D.2. The proof of Lemma 6.10.

Proof. Note that when either a_1 is the basepoint of $F_g(G,V)$, or a_2 is the basepoint of $F_h(H,W)$, or $t_2=1$, or $u_2=1$, the point $[t_1a_1+t_2b_1]\wedge[u_1a_2+u_2b_2]$ is mapped to the basepoint $x_{g,h}$. The spaces $S(G,V)_g$ have the following properties:

(i) There is no zero vector in any $S(G,V)_g$ by its construction;

(ii) For any $b_1\in S(G,V)_g$, $b_2\in S(H,W)_h$, b_1, b_2 and b_1+b_2 are all in $S(G\times H, V\oplus W)_{(g,h)}$. b_1 and b_2 are orthogonal to each other, so $\|b_1+b_2\|^2=\|b_1\|^2+\|b_2\|^2$. Thus, if $t_2u_2\neq 0$, $\|b_1+b_2\|\leqslant\sqrt{t_1^2+t_2^2}$.

Therefore, $\mu^{QE}_{(g,h)}((G,V),(H,W))$ is well-defined.

Let $x=[s_1\alpha+s_2\beta]$ be a point in the image of $\mu^{QE}_{(g,h)}((G,V),(H,W))$. If s_2 is nonzero, there is unique $\beta_1\in S(G,V)_g\cup\{0\}$ and unique $\beta_2\in S(H,W)_h\cup\{0\}$ such that $\beta=\beta_1+\beta_2$.

For each point in the image, we pick an open neighborhood of it so that its preimage in $QE_g(G,V)\wedge QE_h(H,W)$ is open.

Case I: x is not the basepoint, $0<s_1,s_2<1$ and β_1 and β_2 are both nonzero.

Let $A(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G\times H, V\oplus W)$ not containing the basepoint. Let $\delta > 0$ be some small enough value. We consider the open neighborhood $U_{x,\delta}$ of x

$$U_{x,\delta} := \{[r_1 a + r_2 d] \mid \|d_1 - \beta_1\| < \delta, \|d_2 - \beta_2\| < \delta, a \in A(\alpha), |r_2^2 - s_2^2| < \delta\}$$

where $d = d_1 + d_2$ with $d_1 \in S(G,V)_g \cup \{0\}$ and $d_2 \in S(H,W)_h \cup \{0\}$.

The preimage of $U_{x,\delta}$ is

$$\{[t_1 a_1 + t_2 d_1] \wedge [u_1 a_2 + u_2 d_2] \mid a_1 \wedge a_2 \in \mu^F_{(g,h)}((G,V),(H,W))^{-1}(A(\alpha)),$$
$$\|d_1 - \beta_1\| < \delta, \|d_2 - \beta_2\| < \delta, |t_2^2 + u_2^2 - s_2^2| < \delta\},$$

where $\mu^F_{(g,h)}((G,V),(H,W))$ is the multiplication defined in (6.5).

Note that $QE_g(G,V) \wedge QE_h(H,W)$ is the quotient space of a subspace of the product of spaces

$$F_g(G,V) \times S(G,V)_g \times [0,1] \times F_h(H,W) \times S(H,W)_h \times [0,1]$$

and $U_{x,\delta}$ is the quotient of an open subset of this product. So it is open in $QE_g(G,V) \wedge QE_h(H,W)$.

Case II: x is not the basepoint, $0 < s_1, s_2 < 1$ and $\beta \in S(H,W)_h$.

Let $A(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G\times H, V\oplus W)$ not containing the basepoint. Let $\delta > 0$ be some small enough value. Consider the open neighborhood $W_{x,\delta}$ of x

$$W_{x,\delta} := \{[r_1 a + r_2 d] \mid \|d_1 - \beta_1\| < \delta, \|d_2\| < \delta, a \in A(\alpha), |r_2^2 - s_2^2| < \delta\}$$

where $d = d_1 + d_2$ with $d_1 \in S(G,V)_g \cup \{0\}$ and $d_2 \in S(H,W)_h \cup \{0\}$.

The preimage of $W_{x,\delta}$ is

$$\{[t_1 a_1 + t_2 d_1] \wedge [u_1 a_2 + u_2 d_2] \mid a_1 \wedge a_2 \in \mu^F_{(g,h)}((G,V),(H,W))^{-1}(A(\alpha)),$$
$$\|d_1 - \beta_1\| < \delta, \|d_2\| < \delta, |t_2^2 + u_2^2 - s_2^2| < \delta\}.$$

It is the quotient of an open subspace of the product

$$F_g(G,V) \times S(G,V)_g \times [0,1] \times F_h(H,W) \times S(H,W)_h \times [0,1].$$

So the preimage of $W_{x,\delta}$ is open in $QE_g(G,V) \wedge QE_h(H,W)$.

Case III: x is not the basepoint, $0 < s_1, s_2 < 1$ and $\beta \in S(G,V)_g$. We can show the map is continuous at such points in a way analogous to Case II.

Case IV x is not the basepoint and s_2 is zero.

Let $A(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G\times H, V\oplus W)$ not containing the basepoint. Let $\delta > 0$ be some small enough value.

Consider the open neighborhood of x

$$B_{x,\delta} := \{[r_1 a + r_2 d] \mid a \in A(\alpha), \|d_1\| < \delta, \|d_2\| < \delta, 0 \leqslant r_2^2 < \delta\}$$

where $d = d_1 + d_2$ with $d_1 \in S(G,V)_g \cup \{0\}$ and $d_2 \in S(H,W)_h \cup \{0\}$.

The preimage of $B_{x,\delta}$ is

$$\{[t_1 a_1 + t_2 d_1] \wedge [u_1 a_2 + u_2 d_2] \mid a_1 \wedge a_2 \in \mu^F_{(g,h)}((G,V),(H,W))^{-1}(A(\alpha)),$$
$$\|d_1\| < \delta, \|d_2\| < \delta, 0 \leqslant t_2^2 + u_2^2 < \delta\}.$$

It is the quotient of an open subspace of the product

$$F_g(G,V) \times S(G,V)_g \times [0,1] \times F_h(H,W) \times S(H,W)_h \times [0,1].$$

So the preimage of $B_{x,\delta}$ is open in $QE_g(G,V) \wedge QE_h(H,W)$.

Case V: $x=[s_1\alpha+s_2\beta]$ is the base point.

Let $A_0(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G\times H, V\oplus W)$. For any point w in $QE'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and b_1 b_2 both nonzero, let U_{w,δ_w} be the open subset of $QE_{(g,h)}(G\times H, V\oplus W)$

$$\{[r_1a+r_2d]\mid \|d_1-b_1\|<\delta_w, \|d_2-b_2\|<\delta_w, a\in A_0(\alpha), |r_2^2-t_2^2|<\delta_w\}$$

with δ_w small enough.

For each point y in $QE'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and $b\in S(H,W)_h$, let W_{y,δ_y} be the open subset

$$\{[r_1a+r_2d]\in QE_{(g,h)}(G\times H, V\oplus W)|\|d_1-b_1\|<\delta_y, \|d_2\|<\delta_y, a\in A_0(\alpha), |r_2^2-t_2^2|<\delta_y\}$$

with δ_y small enough. For each point z in $QE'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and $b\in S(G,V)_g$, let V_{z,δ_z} be the open subset

$$\{[r_1a+r_2d]\in QE_{(g,h)}(G\times H, V\oplus W)|\|d_2-b_2\|<\delta_z, \|d_1\|<\delta_z, a\in A_0(\alpha), |r_2^2-t_2^2|<\delta_z\}$$

with δ_z small enough. Let $B_{x_0,\delta}$ denote the open set

$$\{[r_1a+r_2d]\in QE_{(g,h)}(G\times H, V\oplus W)|\|d_2\|<\delta, \|d_1\|<\delta, a\in A_0(c_0), 0\leqslant r_2<\delta\}$$

with δ small enough. For each θ in $QE'_{(g,h)}(G\times H, V\oplus W)$ of the form $0+1b$, let D_{θ,δ_θ} be the open subset

$$\{[r_1a+r_2d]\in QE_{(g,h)}(G\times H, V\oplus W)|\|d-b\|<\delta_\theta, 1\geqslant r_2\geqslant 1-\delta_\theta\}$$

with δ_θ small enough.

Then we consider the open neighborhood of x in $QE_{(g,h)}(G\times H, V\oplus W)$ that is the union of the spaces above

$$U:=(\bigcup_w U_{w,\delta_w})\cup(\bigcup_y W_{y,\delta_y})\cup(\bigcup_z V_{z,\delta_z})\cup B_{x_0,\delta}\cup(\bigcup_\theta D_{\theta,\delta_\theta})$$

where w goes over all the points in $QE'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and b_1, b_2 both nonzero, y goes over all the points in $QE'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and $b\in S(H,W)_h$, z goes over all the points in $QE'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and $b\in S(G,V)_g$, and θ goes over all the points of the form $0+1b$ in $QE'_{(g,h)}(G\times H, V\oplus W)$.

The preimage of each U_{x,δ_x}, W_{y,δ_y}, V_{z,δ_z}, $B_{x_0,\delta}$ is open, the proof of which are analogous to that of Case I, II, III and IV. The preimage of D_{θ,δ_θ} is

$$\{[t_1a_1+t_2d_1]\wedge[u_1a_2+u_2d_2]\mid \|d_1+d_2-b\|<\delta_\theta, 1-\sqrt{t_2^2+u_2^2}<\delta_\theta\},$$

which is open. Therefore, the preimage of U is open.

Combining all the cases above, the multiplication $\mu^{QE}_{(g,h)}((G,V),(H,W))$ defined in (6.11) is continuous. □

D.3. The proof of Lemma 6.11.

Proof. In this proof, we identify the end $F_g(G,V)$ in the space $QE_g(G,V)$ with the points of the form $(a,0,0)$, i.e. $1a+00$, in the space (6.8) as indicated in Remark 6.6. If the coordinate t_2 in a point t_1a+t_2b is zero, then b is the zero vector.

(i) Unity.

Let $v \in S^V$ and $w \in S^W$. Let $v = v_1 \wedge v_2$, with $v_1 \in S^{V^g}$, and $v_2 \in S^{(V^g)^\perp}$, and $w = w_1 \wedge w_2$, with $w_1 \in S^{W^h}$, and $w_2 \in S^{(W^h)^\perp}$.

$$\mu^{QE}_{(g,h)}\big((G,V),(H,W)\big) \circ \big(\eta^{QE}_g(G,V) \wedge \eta^{QE}_h(H,W)\big)(v \wedge w)$$

is the basepoint if $\|v_2\|^2 + \|w_2\|^2 \geqslant 1$. If $\|v_2\|^2 + \|w_2\|^2 \leqslant 1$, it equals

(D.1)

$$\big[(1 - \sqrt{\|v_2\|^2 + \|w_2\|^2})\eta_g(G,V)(v_1) \wedge \eta_h(H,W)(w_1) + \sqrt{\|v_2\|^2 + \|w_2\|^2}(v_2 + w_2)\big].$$

On the other direction, $\eta^{QE}_{(g,h)}(G\times H, V\oplus W)(v\wedge w)$ is the basepoint if $\|v_2+w_2\| \geqslant 1$. Note that since v_2 and w_2 are orthogonal to each other, $\|v_2+w_2\|^2 = \|v_2\|^2+\|w_2\|^2$.

If $\|v_2 + w_2\| \leqslant 1$, it is

(D.2)

$$[(1 - \sqrt{\|v_2\|^2 + \|w_2\|^2})\eta_g(G,V)(v_1) \wedge \eta_h(H,W)(w_1) + \sqrt{\|v_2\|^2 + \|w_2\|^2}(v_2 + w_2)],$$

which is equal to the term in (D.1) by Proposition 6.2 (ii).

(ii) Associativity.

Let $x = [t_1a_1 + t_2b_1]$ be a point in $QE_g(G,V)$, $y = [s_1a_2 + s_2b_2]$ a point in $QE_h(H,W)$, and $z = [r_1a_3 + r_2b_3]$ a point in $QE_k(K,U)$.

$$\mu^{QE}_{((g,h),k)}((G\times H, V\oplus W),(K,U)) \circ (\mu^{QE}_{(g,h)}((G,V),(H,W)) \wedge Id)(x\wedge y\wedge z)$$

is the basepoint if $t_2^2 + s_2^2 + r_2^2 \geqslant 1$.

If $t_2^2 + s_2^2 + r_2^2 \leqslant 1$,

$$\begin{aligned}
&\mu^{QE}_{((g,h),k)}((G\times H, V\oplus W),(K,U)) \circ (\mu^{QE}_{(g,h)}((G,V),(H,W)) \wedge Id)(x\wedge y\wedge z)\\
=&\mu^{QE}_{((g,h),k)}((G\times H, V\oplus W),(K,U))\\
&([(1-\sqrt{t_2^2+s_2^2})\mu^F_{g,h}((G,V),(H,W))(a_1\wedge a_2) + \sqrt{t_2^2+s_2^2}(b_1+b_2)]\wedge z)\\
=&[(1-\sqrt{t_2^2+s_2^2+r_2^2})\mu^F_{((g,h),k)}((G\times H, V\oplus W),(K,U))(\mu^F_{g,h}((G,V),(H,W))(a_1\wedge a_2)\wedge a_3)\\
&+\sqrt{t_2^2+s_2^2+r_2^2}(b_1+b_2+b_3)]
\end{aligned}$$

Then consider the other direction.

$$\mu^{QE}_{(g,(h,k))}((G,V),(H\times K, W\oplus U)) \circ (Id \wedge \mu^{QE}_{(h,k)}(H\times K, W\oplus U))(x\wedge y\wedge z)$$

is the basepoint if $t_2^2 + s_2^2 + r_2^2 \geqslant 1$. If $t_2^2 + s_2^2 + r_2^2 \leqslant 1$,

$$\begin{aligned}
&\mu^{QE}_{(g,(h,k))}((G,V),(H\times K, W\oplus U)) \circ (Id \wedge \mu^{QE}_{(h,k)}(H\times K, W\oplus U))(x\wedge y\wedge z)\\
=&\mu^{QE}_{(g,(h,k))}((G,V),(H\times K, W\oplus U))\\
&(x\wedge[(1-\sqrt{r_2^2+s_2^2})\mu^F_{(h,k)}((H,W),(K,U))(a_2\wedge a_3) + \sqrt{r_2^2+s_2^2}(b_2+b_3)])\\
=&[(1-\sqrt{t_2^2+s_2^2+r_2^2})\mu^F_{(g,(h,k))}((G,V),(H\times K, W\oplus U))(a_1\wedge \mu^F_{(h,k)}((H,W),(K,U))(a_2\wedge a_3))\\
&+\sqrt{t_2^2+s_2^2+r_2^2}(b_1+b_2+b_3)], \text{ which by Proposition 6.2 (ii) is equal to}
\end{aligned}$$

$$\begin{aligned}
&[(1-\sqrt{t_2^2+s_2^2+r_2^2})\mu^F_{((g,h),k)}((G\times H, V\oplus W),(K,U))(\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2)\wedge a_3)\\
&+\sqrt{t_2^2+s_2^2+r_2^2}(b_1+b_2+b_3)].
\end{aligned}$$

(iii) Centrality of unit.

Let $v \in S^V$ and $x = [t_1 a + t_2 b]$ a point in $QE_h(H, W)$.

$$QE_{(g,h)}(\tau) \circ \mu^{QE}_{(g,h)}((G,V),(H,W)) \circ (\eta^{QE}_g(G,V) \wedge Id)(v \wedge x)$$

is the base point if $\|v_2\|^2 + t_2^2 \geqslant 1$. If $\|v_2\|^2 + t_2^2 \leqslant 1$, by Proposition 6.2 (ii) it is

$$\begin{aligned}&[(1-\sqrt{\|v_2\|^2+t_2^2})\mu^F_{(g,h)}((G,V),(H,W))(\eta_g(G,V)(v_1)\wedge a)+\sqrt{\|v_2\|^2+t_2^2}(v_2+b)]\\ =&[(1-\sqrt{\|v_2\|^2+t_2^2})\mu^F_{(h,g)}((H,W),(G,V))(a\wedge\eta_g(G,V)(v_1))+\sqrt{\|v_2\|^2+t_2^2}(v_2+b)].\end{aligned}$$

$$\mu^{QE}_{(h,g)}((H,W),(G,V)) \circ (Id \wedge \eta^{QE}_h(H,W)) \circ \tau(v \wedge x)$$

is the base point if $\|v_2\|^2 + t_2^2 \geqslant 1$. If $\|v_2\|^2 + t_2^2 \leqslant 1$, it is

$$[(1-\sqrt{\|v_2\|^2+t_2^2})\mu^F_{(h,g)}((H,W),(G,V))(a\wedge\eta_g(G,V)(v_1))+\sqrt{\|v_2\|^2+t_2^2}(v_2+b)].$$

(iv) $\mu^{QE}_{(g,h)}((G,V),(H,W))(x \wedge y) = [(1-\sqrt{t_2^2+s_2^2})\mu^F_{g,h}((G,V),(H,W))(a_1 \wedge a_2)+\sqrt{t_2^2+s_2^2}(b_1+b_2)] = [(1-\sqrt{t_2^2+s_2^2})\mu^F_{h,g}((H,W),(G,V))(a_2\wedge a_1)+\sqrt{t_2^2+s_2^2}(b_2+b_1)] = \mu^{QE}_{(h,g)}((H,W),(G,V))(y \wedge x)$. □

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