QUASI-ELLIPTIC COHOMOLOGY

BY

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DISSERTATION

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Abstract

We introduce and study quasi-elliptic cohomology, a theory related to Tate K-theory but built over the ring $\mathbb{Z}[q^{\pm}]$. In Chapter 2 we build an orbifold version of the theory, inspired by Devoto's equivariant Tate K-theory [19]. In Chapter 3 we construct power operation in the orbifold theory, and prove a version of Strickland's theorem on symmetric equivariant cohomology modulo transfer ideals. In Chapter 4 we construct representing spectra but show that they cannot assemble into a global spectrum in the usual sense. In Chapter 6 we construct a new global homotopy theory containing the classical theory. In Chapter 7 we show quasi-elliptic cohomology is a global theory in the new category.

To Mother and Father.

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Chapter 1

Introduction

An elliptic cohomology theory is a generalized cohomology theory corresponding to an elliptic curve. It's an old idea of Witten, as shown in [37], that the elliptic cohomology of a space X is related to the $\mathbb{T}-$equivariant K-theory of the free loop space $LX = \mathbb{C}^{\infty}(S^1, X)$, with the circle $\mathbb{T}$ acting on LX by rotating loops.

It's surprisingly difficult to make this precise. One reason is that, in application, one needs to consider the case that a group G acts on X. In this case the loop space LX has rich structures as an orbifold. In this paper I will investigate the relation between Tate K-theory and the loop space, which leads me to a new theory, quasi-elliptic cohomology.

Before that, let's recall what Tate K-theory is. A complex elliptic curve is determined up to isomorphism by its $j-$invariant, which is a complex number. As a result, the moduli space of elliptic curves is isomorphic to $\mathbb{C}$, which can be compactified by adjoining an ∞ point. Correspondingly, the theory of elliptic cohomology can be extended to a formal neighborhood of $j = \infty$ by adding a nodal singularity, which gives a generalized elliptic curve, i.e. the Tate curve. It is a curve over $\mathbb{Z}[q]$ which models the complex situation near ∞, with $q = 0$ corresponding to ∞. It is an elliptic curve over $\mathbb{Z}((q))$. There is an elliptic cohomology theory associated to the Tate curve. Since the formal group of the Tate curve is the formal multiplicative group, it is a form of K-theory, called the Tate K-theory. Grojnowski constructed a complex equivariant theory, based the idea in [27] and [29].

The definition of $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold, which is formulated in [19]. Devoto's work also shows the relation between this theory and the level structure on the Tate curve.

The main subject I'm working on is the quasi-elliptic cohomology $QEll^*$,

$$QEll^*_G(X) := \prod_{\sigma \in G^{tors}_{conj}} K^*_{\Lambda(\sigma)}(X^{\sigma}) = \Big(\prod_{\sigma \in G^{tors}} K^*_{\Lambda(\sigma)}(X^{\sigma}) \Big)^G \tag{1.1}$$

which can be constructed from equivariant K-theories. The definition (1.1) is fully explained in Section 2.4.2.

The idea of quasi-elliptic cohomology is first motivated by Nora Ganter. Instead of a theory over $\mathbb{Z}((q))$, we consider a theory over $\mathbb{Z}[q^{\pm}]$. It is not an elliptic cohomology, but from it we can formulate the Tate K-theory. One advantage of it is we can consider $G-$equivariant cohomology theories not only for finite G but also for all the compact Lie groups. Moreover, this new theory can be interpreted in a neat form by equivariant K-theories, which makes many constructions and computations easier and more natural than those on the Tate K-theories. Some formulations can be generalized to other equivariant cohomology theories.

The elliptic cohomology of orbifolds involves a rich interaction between the orbifold structure and the elliptic curve. Ganter [25] explores this interaction in the case of the Tate curve, describing K^*_{Tate} for an orbifold X in term of the equivariant K-theory and the groupoid structure of X. In Section 2.4.2 we introduce the construction of $QEll^*$ for quotient orbifolds. In Section 3.33 we establish $QEll^*$ for general orbifolds in a way similar to Ganter's.

There are more relations between quasi-elliptic cohomology and Tate K-theory. Ganter define equivariant power operations in Tate K-theory in her thesis [23], which is elliptic in the sense of [2]. In Chapter 3, we study power operation in orbifold quasi-elliptic cohomology. We begin with the case of a global quotient orbifold, where we exhibit rich interaction between the power operations and loop spaces of symmetric power of orbifolds. On the loop spaces of symmetric powers of orbifolds we construct equivariant power operations for quasi-elliptic cohomology in Section 3.2 in the sense of [14] VIII. 1.1, as shown in Theorem 3.2.1. Moreover, Ganter spelled out the axioms for orbifold theories with power operations in [25], and constructed the power operation of orbifold Tate K-theory in the same paper. These power operation are closely related to the level structure and isogenies of the Tate curve, and they are consistent with the power operation on K_{Tate} in the sense of [2]. In Section 3.4 we construct a family of power operations for the orbifold quasi-elliptic cohomology satisfying Ganter's axioms. With the families of Ganter's power operations, in Section 3.5.2 we prove that the Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve, which is analogous to the principal result in Strickland [60]. With the equivariant power operations we formulate in Section 3.2, we prove the parallel conclusion for quasi-elliptic cohomology.

Goerss, Hopkins and Miller have proved that the moduli stack of elliptic curves can be covered by E_∞ elliptic spectra. It is not known whether this result can be extended to global elliptic cohomology theories and global ring spectra. In Chapter 4 we construct an orthogonal $G-$spectrum for each compact Lie group G which weakly represents quasi-elliptic cohomology. However, we show that it cannot arise from an orthogonal spectrum. Instead, in Chapter 6 we construct a new global homotopy theory and in Chapter 7 we show there is a global orthogonal spectrum in this new global homotopy theory that weakly represents orthogonal quasi-elliptic cohomology.

Let X be a $G-$space. Let $KU_{G,n}$ denote the space representing the $n-$th $G-$equivariant $K-$theory. Recall

$$QEll^*_G(X) = \prod_{\sigma \in G^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma).$$

For each compact Lie group G and each integer n, we construct a space $QEll_{G,n}$ representing $QEll^n_G(-)$ in the sense of (4.1).

$$\pi_0(QEll_{G,n}) = QEll^n_G(S^0), \text{ for each } k. \tag{1.2}$$

First we construct in Theorem 4.3.7 a homotopical right adjoint R_g for the functor $X \mapsto X^g$ from the category of G-spaces to the category of $\Lambda_G(g)$-spaces. Then we get

$$\prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{\Lambda_G(g)}(X^g, KU_{\Lambda_G(g),n})$$

is weakly equivalent to

$$\mathrm{Map}_G(X, \prod_{g \in G^{tors}_{conj}} R_g(KU_{\Lambda_G(g),n})),$$

as stated in Theorem 4.3.8.

So $QEll_{G,n} := \prod\limits_{g \in G^{tors}_{conj}} R_g(KU_{\Lambda_G(g),n})$ is one choice of the classifying space we want.

Based on the construction of $QEll_{G,n}$, we construct for each faithful $G-$representation V a space $E(G,V)$ that weakly represents $QEll^V_G(-)$ in the sense of (4.2),

$$\pi_k(E(G,V)) = QEll^V_G(S^k), \text{ for each } k. \tag{1.3}$$

We also construct the structure maps making E an orthogonal $G-$spectra and an $\mathcal{I}_G-$FSP.

Moreover, we construct the restriction maps $E(G,V) \longrightarrow E(H,V)$ for each group homomorphism $H \longrightarrow G$. This map is not a homeomorphism, but an $H-$weak equivalence.

The orthogonal $G-$spectra $E(G, -)$ cannot arise from an orthogonal spectrum. This fact motivates us to construct a new global homotopy theory.

We construct a new global homotopy theory in Chapter 6. An orthogonal space is a continuous functor from the category $\mathbb{L}$ of inner product real spaces to the category of topological spaces. We enlarge the category $\mathbb{L}$ by adding restriction maps to it, which are identity morphisms on the underlying vector spaces in $\mathbb{L}$, and form a category D. Instead of orthogonal spaces, we study the subcategory D_0 of it corresponding to finite groups and the category D_0T of D_0-spaces. There is a fully faithful functor from the category of $\Sigma-$spaces to the category of D_0-spaces where Σ is the category of finite sets and injective maps. In other words, D_0T contains all the information of $\mathbb{L}T$.

We establish several model structures on D_0T. As a category of diagram spaces, it's equipped the level model structure, as shown in [42]. Moreover, D_0 is a generalized Reedy category in the sense of [10]. Thus, there is a Reedy model structure on D_0T, as shown in Section 6.4.

It's conjectured that there is a model structure on D_0T Quillen equivalent to the global model structure on the category of orthogonal spaces constructed in [56]. We construct a global model structure on D_0T in Section 6.5. In Section 6.6 we show this model structure on a full subcategory D_0^wT of D_0T is Quillen equivalent to the global model structure on the orthogonal spaces.

In Chapter 7 we construct a global spectrum representing Quasi-elliptic cohomology in the sense developed in Chapter 6. Based on the orthogonal $G-$spectrum $(E(G, V), \eta^E(G, V), \mu^E(G, V))$ constructed in Section 4.5, we show the construction of the Real version ER and the real version EO in Section 7.1. They weakly represent the Real and real quasi-elliptic cohomology respectively. We show ER is a unitary $D^{\mathbb{C}}-$space and, in Section 7.2, a $D^{\mathbb{C}}-$FSP over S. Similarly one can show that E and EO are both D_0-spaces and D_0-FSP over S.

Chapter 2

Quasi-elliptic Cohomology

Quasi-elliptic cohomology is modelled on the orbifold K-theory of the constant loops in the free loop space. So to understand $QEll_G^*(X)$, it is essential to understand the orbifold loop space.

Let G be a compact Lie group and X be a $G-$space. Let $X//G$ denote the translation groupoid. In Example 2.1.5 I discuss the Lie groupoid $Loop_1(X//G)$ consisting of bibundles from $S^1//*$ to $X//G$, and introduce two other models $Loop_2(X//G)$ and $Loop_3(X//G)$.

Other than the $G-$action on it, we also consider the rotation by the circle group on the objects and form the Lie groupoids $Loop_1^{ext}(X//G)$ and $Loop_3^{ext}(X//G)$. The automorphism group on each object may not be a compact Lie group. To avoid these cases it is convenient to restrict to those objects $\gamma : \mathbb{R} \longrightarrow X$ with $\gamma(t+1) = \gamma(t)g$ when g is a *torsion* element of G and, instead of the whole automorphism group, we consider a subgroup of it. The resulting groupoid is

$$\mathcal{L}(X//G) := \coprod_{g \in G^{tors}_{conj}} \mathcal{L}_g X//\Lambda_G(g).$$

Each component $\mathcal{L}_g X = \mathrm{Map}_{\mathbb{Z}/l\mathbb{Z}}(\mathbb{R}/l\mathbb{Z}, X)$ is equipped with an evident $C_G(g)-$action. And there is a $G-$action on the whole space $\mathcal{L}(X//G)$. In addition, the circle group $\mathbb{T}$ acts on $\mathbb{R}/l\mathbb{Z}$ by rotation, and so in principle on the orbifold $\mathcal{L}_g X$.

We write $\Lambda(X//G)$ as the full subgroupoid of $\mathcal{L}(X//G)$ consisting of the constant loops. The quasi-elliptic cohomology $QEll_G^*(X)$ is an explicit form of

$$K^*_{orb}(\Lambda(X//G)).$$

In order to unravel the relevant notations in the construction of $QEll_G^*(X)$, we study the orbifold loop space with its $\mathbb{T}-$action in Section 2.1.2 and Section 2.1.3.

Quasi-elliptic cohomology is closely related to equivariant Tate K-theory, which is defined from the orbifold K-theory of the full subgroupoid of

$$\coprod_{g\in G^{tors}_{conj}} \mathcal{L}_g X//C_G(g)$$

consisting of the constant loops. The relation between the two cohomology theories is

$$QEll^*_G(X) \cong K^*_{Tate}(X//G) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)). \tag{2.1}$$

Moreover, there is a nice interpretation of the groupoid $\Lambda(X//G)$ in terms of the groupoid $GhLoop(X//G)$ of ghost loops. I introduce it in Section 2.1.3. The main reference for the ghost loops in this paper is [52].

In Section 2.1.1 I introduce basic notions on Lie groupoids and bibundle, and discuss an example $Loop_1(X//G) := Bibun(S^1//*, X//G)$ relevant to the construction of quasi-elliptic cohomology. The references are [47] and [54]. In Section 2.1.2 I interpret the Lie groupoids $Loop_1(X//G)$ and introduce the groupoid of main interest, $\mathcal{L}(X//G)$. In Section 2.1.3 I show quasi-elliptic cohomology can be interpreted by ghost loops constructed by Rezk in his unpublished manuscript [52]. The action of ghost loop groups is defined by the extension of the action of appropriate gauge groups. In Section 2.2 we recall the basics of equivariant K-theory and in Section 2.3 I briefly introduce the orbifold K-theory. In Section 2.4.1 I show the representation ring of $\Lambda_G(g)$ and the relevant groups. In Section 2.4.2 I introduce the construction of quasi-elliptic cohomology in term of orbifold K-theory first and then equivariant K-theory. I showed some properties of the theory in Section 2.4.3.

The readers may refer [3] and [47] for a reference on groupoids and orbifolds.

2.1 Loop space

2.1.1 Bibundles

Definition 2.1.1 (Lie groupoids). *A Lie groupoid $\mathbb{G}$ is a groupoid object, $\mathbb{G} = (\mathbb{G}_1 \rightrightarrows \mathbb{G}_0)$, in the category of smooth manifolds in which the source map and the target map $s, t : \mathbb{G}_1 \longrightarrow \mathbb{G}_0$ are surjective submersions (so that the domain $\mathbb{G}_1 \underset{s}{\times}\underset{t}{} \mathbb{G}_1$ of the composition m is a smooth manifold).*

Functors and natural transformations are defined as functors and natural transformations inside the category of smooth manifolds. Especially, a homormophism is a smooth *functor.*

Example 2.1.2 (Translation Groupoids). *Let $X//G$ denote the translation groupoid with as objects the points $x \in X$, and as arrows $\alpha : x \longrightarrow y$ those $\alpha \in G$ for which $\alpha \cdot x = y$. It is a topological groupoid. In the case when X is a manifold, G is a Lie group acting on it smoothly, $X//G$ is a Lie groupoid.*

When the group G is trivial, any smooth manifold X can be viewed as a Lie groupoid with only identity morphisms.

For any manifold X, let Man_X denote the category of manifolds over X, that is, the category whose objects are manifolds Y equipped with a smooth map $Y \longrightarrow X$, and whose morphisms are smooth maps $Y \longrightarrow Y'$ making the following triangle commute.

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y' \arrow[dl] \\ X & \end{tikzcd}$$

Definition 2.1.3. *Let $\mathbb{G}$ and $\mathbb{H}$ be Lie groupoids. A (left principal) bibundle from $\mathbb{H}$ to $\mathbb{G}$ is a smooth manifold P together with*

1. A map $\tau : P \longrightarrow \mathbb{G}_0$, and a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.

2. Action maps in $Man_{G_0 \times H_0}$

$$\mathbb{G}_1 \,{}_s\!\times_\tau P \longrightarrow P$$

$$P \,{}_\sigma\!\times_t \mathbb{H}_1 \longrightarrow P$$

which we denote on elements as $(g,p) \mapsto g \cdot p$ and $(p,h) \mapsto p \cdot h$, such that

1. $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $(g_1, g_2, p) \in \mathbb{G}_1 \,{}_s\!\times_t \mathbb{G}_1 \,{}_s\!\times_\tau P$;

2. $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$ for all $(p, h_1, h_2) \in P \,{}_\sigma\!\times_t \mathbb{H}_1 \,{}_s\!\times_t \mathbb{H}_1$;

3. $p \cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p)) \cdot p = p$ for all $p \in P$.

4. $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for all $(g, p, h) \in \mathbb{G}_1 \,{}_s\!\times_\tau P \,{}_\sigma\!\times_t \mathbb{H}_1$.

5. The map

$$\mathbb{G}_1 \,{}_s\!\times_\tau P \longrightarrow P \,{}_\sigma\!\times_\sigma P$$

$$(g,p) \mapsto (g \cdot p, p)$$

is an isomorphism.

Definition 2.1.4. *A bibundle map is a map $P \longrightarrow P'$ over $\mathbb{H}_0 \times \mathbb{G}_0$ which commutes with the $\mathbb{G}-$ and $\mathbb{H}-$actions, i.e. the following diagrams commute.*

$$
\begin{tikzcd}
\mathbb{G}_1 \underset{s\ \ \tau}{\times} P \arrow[r] \arrow[d] & P \arrow[d] & P \underset{\sigma\ \ t}{\times} \mathbb{H}_1 \arrow[r] \arrow[d] & P \arrow[d] \\
\mathbb{G}_1 \underset{s\ \ \tau}{\times} P' \arrow[r] & P' & P' \underset{\sigma\ \ t}{\times} \mathbb{H}_1 \arrow[r] & P'
\end{tikzcd}
$$

For each pair of Lie groupoids $\mathbb{H}$ and $\mathbb{G}$, we have a category $Bibun(\mathbb{H}, \mathbb{G})$ with as objects bibundles from $\mathbb{H}$ to $\mathbb{G}$ and as morphisms the bundle maps. The category of smooth functors from $\mathbb{H}$ to $\mathbb{G}$ is a subcategory of $Bibun(\mathbb{H}, \mathbb{G})$.

Example 2.1.5 **($Loop_1(X//G)$).** *Let G be a Lie group acting smoothly on a manifold X. Consider the translation groupoids $X//G$ and $S^1//*$ with trivial morphisms. Let's use $Loop_1(X//G)$ to denote the category $Bibun(S^1//*, X//G)$. By Definition 2.1.3, the objects are the diagrams*

$$P := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

where $P \xrightarrow{\pi} S^1$ is a principal $G-$bundle over S^1 and $f : P \longrightarrow X$ is a $G-$equivariant map. I use the same symbol P to denote both the object and the smooth manifold when there is no confusion. A morphism $P \longrightarrow P'$ is a $G-$bundle map $\alpha : P \longrightarrow P'$ making the diagram below commutes.

$$
\begin{tikzcd}
S^1 & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"] & X \\
 & P' \arrow[ul, "\pi'"] \arrow[ur, "f'"'] &
\end{tikzcd}
$$

Thus, the morphisms in $Loop_1(X//G)$ from P to P' are the bundle isomorphisms. $Loop_1(X//G)$ is a Lie groupoid.

In particular, $Loop_1(*//G)$ is the category with principal $G-$bundles over S^1 as objects and bundle isomorphisms as morphisms.

2.1.2 Orbifold Loop Space

Before introducing the ghost loops in Section 2.1.3, let's recall the orbifold loop group and loop group.

For any space X, we have the free loop space of X

$$LX := \mathbb{C}^\infty(S^1, X). \tag{2.2}$$

It comes with an evident action by the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by rotating the circle

$$t \cdot \gamma := (s \mapsto \gamma(s+t)),\ t \in S^1,\ \gamma \in LX. \tag{2.3}$$

Let G be a compact Lie group. Now let's suppose X is a right G-space. The free loop space LX is equipped with an action by the loop group LG

$$\delta \cdot \gamma := (s \mapsto \delta(s) \cdot \gamma(s)),\ \text{for any } s \in S^1,\ \delta \in LX,\quad \gamma \in LG. \tag{2.4}$$

Combining the action by group of automorphisms $Aut(S^1)$ on the circle and the action by LG, we get an action by the extended loop group ΛG on LX. $\Lambda G := LG \rtimes \mathbb{T}$ is a subgroup of

$$LG \rtimes Aut(S^1),\ (\gamma, \phi) \cdot (\gamma', \phi') := (s \mapsto \gamma(s)\gamma'(\phi^{-1}(s)), \phi \circ \phi') \tag{2.5}$$

with $\mathbb{T}$ identified with the group of rotations on S^1. ΛG acts on LX by

$$\delta \cdot (\gamma, \phi) := (t \mapsto \delta(\phi(t)) \cdot \gamma(\phi(t))),\ \text{for any } (\gamma, \phi) \in \Lambda G,\ \text{and } \delta \in LX. \tag{2.6}$$

It's straightforward to check (2.6) is a well-defined group action.

Let G^{tors} denote the set of torsion elements in G. Let $k \geq 0$ be an integer. Let $g \in G^{tors}$ and l denote the order of g. Let $L_g^k G$ be the twisted loop group

$$\{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+k) = g^{-1}\gamma(s)g\}. \tag{2.7}$$

The multiplication of it is defined by

$$(\delta \cdot \delta')(t) = \delta(t)\delta'(t),\ \text{for any } \delta, \delta' \in L_g^k G,\ \text{and } t \in \mathbb{R}. \tag{2.8}$$

The identity element e is the constant map sending all the real numbers to the identity element of G. Similar to ΛG, we can define $L_g^k G \rtimes \mathbb{T}$ whose multiplication is define by

$$(\gamma, t) \cdot (\gamma', t') := (s \mapsto \gamma(s)\gamma'(s+t), t+t'). \tag{2.9}$$

The set of constant maps $\mathbb{R} \longrightarrow G$ in $L_g^k G$ is a subgroup of it, i.e. the centralizer $C_G(g)$.

I will show in Lemma 2.1.11 each $L_g^1 G$ is isomorphic to the gauge group of a principal bundle.

Example 2.1.6 **(*$Loop_2(X//G)$*).** *Let G be a Lie group acting smoothly on a manifold X. Let $Loop_2(X//G)$ denote the groupoid whose objects are (σ, γ) with $\sigma \in G$ and $\gamma : [0,1] \longrightarrow X$ a continuous map such that $\gamma(1) = \gamma(0) \cdot \sigma$. A morphism $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ is a continuous map $\alpha : [0,1] \longrightarrow G$ satisfying $\gamma'(s) = \gamma(s)\alpha(s)$. Note that $\alpha(0)\sigma' = \sigma\alpha(1)$.*

$Loop_2(X//G)$ is equivalent to the groupoid $Loop_1(X//G)$ talked in Example 2.1.5. We can formulate the equivalence by constructing a functor $\psi : Loop_1(X//G) \longrightarrow Loop_2(X//G)$. It sends an object

$$S^1 \xleftarrow{\pi} P \xrightarrow{f} X$$

to the object (σ, γ) with $\gamma(t) := f([t, e])$ and $\sigma = \gamma(0)^{-1}\gamma(1)$. And it sends a morphism

$$\begin{tikzcd} S^1 & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "F"] & X \\ & P' \arrow[ul, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

to $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ with $\alpha(t) := F([t, e])^{-1}$.

ψ is an equivalence of groupoids.

Similarly, for each positive integer k, we can define the groupoid $Loop_2^k(X//G)$ whose objects are (σ, γ) with $\sigma \in G$ and $\gamma : [0,k] \longrightarrow X$ a continuous map such that $\gamma(k) = \gamma(0) \cdot \sigma$. A morphism $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ is a continuous map $\alpha : [0,k] \longrightarrow G$ satisfying $\gamma'(s) = \gamma(s)\alpha(s)$. Each $Loop_2^k(X//G)$ is equivalent to $Loop_1(X//G)$.

Example 2.1.7 **(*$Loop_3(X//G)$*).** *Let G be a Lie group acting smoothly on a manifold X. Let $Loop_3(X//G)$ denote the groupoid whose objects are (σ, γ) with $\sigma \in G$ and $\gamma : \mathbb{R} \longrightarrow X$ a continuous map such that $\gamma(s+1) = \gamma(s) \cdot \sigma$, for any $s \in \mathbb{R}$. A morphism $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ is a continuous map $\alpha : \mathbb{R} \longrightarrow G$ satisfying $\gamma'(s) = \gamma(s)\alpha(s)$. Note that $\alpha(s)\sigma' = \sigma\alpha(s+1)$, for any $s \in \mathbb{R}$.*

$Loop_3(X//G)$ is equivalent to the groupoid $Loop_1(X//G)$ in Example 2.1.5 and is isomorphic to the groupoid $Loop_2(X//G)$ in Example 2.1.6. The equivalence $Loop_1(X//G) \longrightarrow Loop_3(X//G)$ can be constructed analogous to ψ in Example 2.1.6. We can also construct the isomorphism $\phi : Loop_2(X//G) \longrightarrow Loop_3(X//G)$ in this way: an object (σ, γ) is sent to

$$\phi(\sigma, \gamma) = \gamma * \gamma\sigma * \gamma\sigma^2 * \cdots$$

where $*$ is the composition of paths. For any real number r between positive integers n and $n+1$, $\phi(\sigma, \gamma)(r) = \gamma(r-n)\sigma^n$. A morphism $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ is mapped to the morphism

$$\phi(\alpha) = \alpha * \sigma^{-1}\alpha\sigma * \sigma^{-2}\alpha\sigma^2 * \cdots .$$

For any r between positive integers n and $n+1$, $\phi(\alpha)(r) = \sigma^{-n}\alpha(r-n)\sigma^n$.

Similarly, for each positive integer k, we can define the groupoid $Loop_3^k(X//G)$ whose objects are (σ, γ) with $\sigma \in G$ and $\gamma : \mathbb{R} \longrightarrow X$ a continuous map such that $\gamma(s+k) = \gamma(s)\cdot\sigma$, for any $s \in \mathbb{R}$. A morphism $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ is a continuous map $\alpha : \mathbb{R} \longrightarrow G$ satisfying $\gamma'(s) = \gamma(s)\alpha(s)$, for any $s \in \mathbb{R}$. Each $Loop_3^k(X//G)$ is equivalent to $Loop_1(X//G)$.

For $g \in G$, let l denote its order. If g is not a torsion element, $l = 0$. The objects of $Loop_3^k(X//G)$ can be identified with the space

$$\coprod_{g\in G} {}_k\mathcal{L}_g X$$

where

$$ {}_k\mathcal{L}_g X := Map_{\mathbb{Z}/l\mathbb{Z}}(\mathbb{R}/kl\mathbb{Z}, X). \tag{2.10}$$

$\mathbb{Z}/l\mathbb{Z}$ is isomorphic to the subgroup $k\mathbb{Z}/kl\mathbb{Z}$ of $\mathbb{R}/kl\mathbb{Z}$. The isomorphism $\mathbb{Z}/l\mathbb{Z} \longrightarrow k\mathbb{Z}/kl\mathbb{Z}$ sends the generator $[1]$ corresponding to 1 to the generator $[k]$ of $k\mathbb{Z}/kl\mathbb{Z}$ corresponding to k. $k\mathbb{Z}/kl\mathbb{Z}$ acts on $\mathbb{R}/kl\mathbb{Z}$ by group multiplication. Thus, via the isomorphism, $\mathbb{Z}/l\mathbb{Z}$ acts on $\mathbb{R}/kl\mathbb{Z}$. $\mathbb{Z}/l\mathbb{Z}$ is also isomorphic to the cyclic group $\langle g\rangle$ by identifying the generater $[1]$ with g. So it acts on X via the $G-$action on it. And ${}_k\mathcal{L}_g X//L_g^k G$ is a full subgroupoid of $Loop_3^k(X//G)$.

Now let's consider the extended loop spaces with richer morphism spaces.

Example 2.1.8 **($Loop_1^{ext}(X//G)$ and $Loop_3^{ext}(X//G)$).** *Let $Loop_1^{ext}(X//G)$ denote the groupoid with the same objects as $Loop_1(X//G)$. A morphism*

$$\{S^1 \xleftarrow{\ \pi\ } P \xrightarrow{\ f\ } X\} \longrightarrow \{S^1 \xleftarrow{\ \pi'\ } P' \xrightarrow{\ f'\ } X\}$$

consists of the pair (t, α) where $t \in \mathbb{T}$ is a rotation of the circle and α is a morphism between the objects in $Loop_1(X//G)$ such that the diagrams below commute.

$$\begin{tikzcd} S^1 \arrow[d, "t"'] & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"'] & X \\ S^1 & P' \arrow[l, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

$Loop_1(X//G)$ is a subgroupoid of $Loop_1^{ext}(X//G)$.

For any $G-$equivariant map $f : P_\sigma \longrightarrow X$, it's of the form

$$f([s, g]) = \delta_f(s) \cdot g,$$

for any $[s, g] \in P_\sigma$, where $\delta_f : \mathbb{R} \longrightarrow X$ is a continuous map and $\delta_f(s) = f([s, e])$ for any $s \in \mathbb{R}$. Since $[s+1, g] = [s, \sigma g]$, we have $\delta_f(s+1)g = \delta_f(s)\sigma g$, thus, $\delta_f(s+1) = \delta_f(s)\sigma$. So $\delta_f(s)$ is in ${}_1\mathcal{L}_\sigma X$.

In addition, each vertical map $P \longrightarrow P'$ is of the form $[s, g]$ to $[t+s, \delta(s) \cdot g]$ for some $t \in \mathbb{R}$ and $\delta \in L^1_\sigma G$.

Let $Loop_3^{ext}(X//G)$ denote the groupoid with the same objects as $Loop_3(X//G)$. A morphism

$$(\sigma, \gamma) \longrightarrow (\sigma', \gamma')$$

consists of the pair (α, t) with $\alpha : \mathbb{R} \longrightarrow G$ a continuous map and $t \in \mathbb{R}$ a rotation on S^1 satisfying $\gamma'(s) = \gamma(s-t)\alpha(s-t)$.

$Loop_3(X//G)$ is a subgroupoid of $Loop_3^{ext}(X//G)$.

$Loop_3^{ext}(X//G)$ is equivalent to $Loop_1^{ext}(X//G)$. An equivalence

$$F : Loop_1^{ext}(X//G) \longrightarrow Loop_3^{ext}(X//G)$$

is constructed by sending an object

$$S^1 \xleftarrow{\pi} P \xrightarrow{f} X$$

to (σ, γ) *with* $\gamma(t) := f([t, e])$ *and* $\sigma = \gamma(0)^{-1}\gamma(1)$ *and sending a morphism*

$$\begin{tikzcd}
S^1 \arrow[d, "t"'] & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "F"] & X \\
S^1 & P' \arrow[l, "\pi'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

to $(\alpha, t) : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ *with* $\alpha(s) := F([s, e])^{-1}$.

Similarly, for each positive integer k*, we can define the groupoid* $Loop_3^{k,ext}(X//G)$ *with the same objects as* $Loop_3^k(X//G)$*. A morphism*

$$(\sigma, \gamma) \longrightarrow (\sigma', \gamma')$$

consists of the pair (α, t) *with* $\alpha : \mathbb{R} \longrightarrow G$ *a continuous map and* $t \in \mathbb{T}$ *a rotation on* S^1 *satisfying* $\gamma'(s) = \gamma(s - kt)\alpha(s - kt)$.

For each $g \in G$, ${}_k\mathcal{L}_g X // L_g^k G \rtimes \mathbb{T}$ *is a full subgroupoid of* $Loop_3^{k,ext}(X//G)$ *where* $L_g^k G \rtimes \mathbb{T}$ *acts on* ${}_k\mathcal{L}_g X$ *by*

$$\delta \cdot (\gamma, t) := (s \mapsto \delta(s + t) \cdot \gamma(s + t)), \text{ for any } (\gamma, t) \in L_g^k G \rtimes \mathbb{T}, \text{ and } \delta \in {}_k\mathcal{L}_g X. \tag{2.11}$$

The action by g *on* ${}_k\mathcal{L}_g X$ *coincides with that by* $k \in \mathbb{R}$*. So we have the isomorphism*

$$L_g^k G \rtimes \mathbb{T} = L_g^k G \rtimes \mathbb{R} / \langle (\overline{g}, -k) \rangle, \tag{2.12}$$

where $\overline{g}$ *represents the constant loop* $\mathbb{T} \longrightarrow \{g\} \subseteq G$*. Let's use* $\Lambda_g^k G$ *to denote the extended twisted loop group* $L_g^k G \rtimes \mathbb{T}$.

Example 2.1.8 implies the following result.

Proposition 2.1.9. *Let* G *be a compact Lie group. The groupoid*

$$\coprod_g {}_1\mathcal{L}_g X // \Lambda_g^1 G,$$

where the coproduct goes over conjugacy classes in $\pi_0 G$, is a skeleton of $Loop_3^{ext}(X//G)$. Thus, it is equivalent to $Loop_3^{ext}(X//G)$ and $Loop_1^{ext}(X//G)$.

Before we go on, let's see the meaning of the groupoid $Loop_1^{ext}(X//G)$ in physics. Recall that the gauge group of a principal bundle is defined to be the group of its vertical automorphisms. The readers may refer [46] for more details on gauge groups. For a $G-$bundle $P \longrightarrow S^1$, let $L_P G$ denote its gauge group.

We have the well-known facts below.

Lemma 2.1.10. *The principal $G-$bundles over S^1 are classified up to isomorphism by homotopy classes*

$$[S^1, BG] \cong \pi_0 G/conj.$$

Up to isomorphism every principal $G-$bundle over S^1 is isomorphic to one of the forms $P_\sigma \longrightarrow S^1$ with $\sigma \in G$ and

$$P_\sigma := \mathbb{R} \times G/(s+1, g) \sim (s, \sigma g).$$

A complete collection of isomorphism classes is given by a choice of representatives for each conjugacy class of $\pi_0 G$.

For the gauge group $L_{P_\sigma} G$ of the bundle P_σ, we have the conclusion.

Lemma 2.1.11. *For the bundle $P_\sigma \longrightarrow S^1$, $L_{P_\sigma} G$ is isomorphic to the twisted loop group $L^1_\sigma G$.*

Proof. Each automorphism f of an object $S^1 \overset{\pi}{\leftarrow} P_\sigma \overset{\widetilde{\delta}}{\rightarrow} X$ in $Loop_1^{ext}(X//G)$ has the form

$$\begin{tikzcd} P' \arrow[r, "{[s,g] \mapsto [s, \gamma_f(s) g]}"] \arrow[d] & P \arrow[d] \\ S^1 \arrow[r, "="] & S^1 \end{tikzcd} \tag{2.13}$$

for some $\gamma_f : \mathbb{R} \longrightarrow G$. The morphism is well-defined if and only if $\gamma_f(s+1) = \sigma^{-1}\gamma_f(s)\sigma$.

So we get a well-defined map

$$F : L_{P_\sigma} G \longrightarrow L^1_\sigma G,\ f \mapsto \gamma_f.$$

It's a bijection. Moreover, by the property of group action, F sends the identity map to the constant map $\mathbb{R} \longrightarrow G$, $s \mapsto e$, which is the trivial element in $L^1_\sigma G$, and for two automorphisms f_1 and f_2 at the object, $F(f_1 \circ f_2) = \gamma_{f_1} \cdot \gamma_{f_2}$. So $L_{P_\sigma} G$ is isomorphic to $L^1_\sigma G$.

$\square$

Example 2.1.12 ($Loop_3^{ext,tors}(X//G)$)**.** *Let $Loop_3^{k,ext,tors}(X//G)$ denote the full subgroupoid of $Loop_3^{k,ext}(X//G)$ whose objects are the pairs (σ, γ) with $\sigma \in G^{tors}$ and $\gamma : \mathbb{R} \longrightarrow X$ a continuous map such that $\gamma(s+k) = \gamma(s) \cdot \sigma$, for any $s \in \mathbb{R}$.*

Its objects is the space

$$\coprod_{g \in G^{tors}} {}_k\mathcal{L}_g X.$$

Especially, when $k = 1$, let's use $Loop_3^{ext,tors}(X//G)$ to denote $Loop_3^{1,ext,tors}(X//G)$.

One interesting case is when we consider the constant loops, namely, the full subgroupoid $\Lambda^k(X//G)$ of $Loop_3^{k,ext,tors}(X//G)$ consisting of the constant loops,

$$\coprod_{g \in G^{tors}} X^g.$$

When $k = 1$, let's use $\Lambda(X//G)$ to denote $\Lambda^1(X//G)$.

The groupoid $Loop_3^{k,ext,tors}(X//G)$ contains all the information we want. But is it convenient enough to study? When G is not finite, The isotropy group $\Lambda^k_g G$ of an object in ${}_k\mathcal{L}_g X$ is an infinite dimensional topological group. We need even smaller groups to define a good orbifold loop space. Let's consider those elements $[\gamma, t] \in \Lambda^k_g G$ with γ a constant loop. They form a subgroup of $\Lambda^k_g G$ which is the quotient group of $C_G(g) \times \mathbb{R}/kl\mathbb{Z}$ by the normal subgroup generated by $(g, -k)$. Let's denote it by $\Lambda^k_G(g)$. When $k = 1$, $\Lambda^1_G(g)$ is the group $\Lambda_G(g)$ defined in [50]. When G is a compact Lie group, $\Lambda^k_G(g)$ is also a compact Lie group.

Therefore, instead of $Loop_3^{k,ext,tors}(X//G)$, let's consider a subgroupoid ${}_k\mathcal{L}(X//G)$ of it, as shown in Example 2.1.14. Before that, we need the definition below.

Definition 2.1.13. *Let $C_G(g, g')$ denote the set*

$$\{x \in G | gx = xg'\}.$$

Let $C^k_G(g)$ denote the quotient group of $C_G(g) \times \mathbb{Z}/kl\mathbb{Z}$ by the normal subgroup generated by $(g, -k)$. And let $C^k_G(g, g')$ denote the quotient of $C_G(g, g') \times \mathbb{Z}/kl\mathbb{Z}$ under the equivalence

$$(x, t) \sim (gx, t-k) = (xg', t-k).$$

Let $\Lambda^k_G(g,g')$ denote the quotient of $C_G(g,g')\times\mathbb{R}/kl\mathbb{Z}$ under the equivalence

$$(x,t)\sim(gx,t-k)=(xg',t-k).$$

Example 2.1.14 (Orbifold Loop Space ${}_k\mathcal{L}(X//G)$). *Let ${}_k\mathcal{L}(X//G)$ the groupoid with the same objects as $Loop_3^{k,ext,tors}(X//G)$, i.e. the space*

$$\coprod_{g\in G^{tors}} {}_k\mathcal{L}_gX,$$

and with morphisms the space $\coprod\limits_{g,g'\in G^{tors}}\Lambda^k_G(g,g')\times X^g$.

For $\delta\in{}_k\mathcal{L}_gX$, $[a,t]\in\Lambda^k_G(g,g')$,

$$\delta\cdot([a,t],\delta):=(s\mapsto\delta(s+t)\cdot a)\in{}_k\mathcal{L}_{g'}X. \tag{2.14}$$

in the same way as (2.11).

When $k=1$, let's use $\mathcal{L}(X//G)$ to denote ${}_1\mathcal{L}(X//G)$.

$\Lambda^k(X//G)$ defined in Example 2.1.12 is the full subgroupoid of ${}_k\mathcal{L}(X//G)$ with constant loops as objects. Quasi-elliptic cohomology $QEll^*_G(X)$ is defined to be $K^*_{orb}(\Lambda(X//G))$, which is shown in detail in Section 2.4.2.

If g and g' are conjugate in G, the two groupoids ${}_k\mathcal{L}_gX//\Lambda^k_G(g)$ and ${}_k\mathcal{L}_{g'}X//\Lambda^k_G(g')$ are isomorphic, as shown below:

Let a be an element in G such that $g'=a^{-1}ga$. So we have $C_G(g')=a^{-1}C_G(g)a$. Let

$$\beta_a:{}_k\mathcal{L}_gX//\Lambda^k_G(g)\longrightarrow{}_k\mathcal{L}_{g'}X//\Lambda^k_G(g')$$

be the functor sending an object γ to $\gamma\cdot a$, a morphism $([h,t],\gamma)$ to $([a^{-1}ha,t],\gamma\cdot a)$. Similarly we can define the functor

$$\beta_{a^{-1}}:{}_k\mathcal{L}_{g'}X//\Lambda^k_G(g')\longrightarrow{}_k\mathcal{L}_gX//\Lambda^k_G(g)$$

sending an object γ' to $\gamma'\cdot a^{-1}$ and a morphism $([h,t],\gamma')$ to $([aha^{-1},t],\gamma\cdot a^{-1})$. Then the composition $\beta_a\cdot\beta_{a^{-1}}$ and $\beta_{a^{-1}}\cdot\beta_a$ are both identity maps. So β_a and $\beta_{a^{-1}}$ are both *isomorphisms of categories.*

Let G^{tors}_{conj} denote a set of representatives of $G-$conjugacy classes in G^{tors}. We have the well-defined groupoid

$$\coprod_{g\in G^{tors}_{conj}} {}_k\mathcal{L}_g X//\Lambda^k_G(g),$$

which is a skeleton of ${}_k\mathcal{L}(X//G)$ containing all the information of it. It does not depend on the choice of representatives of the $G-$conjugacy classes.

Let's also use the symbol ${}_k\mathcal{L}(X//G)$ to denote this skeleton when there is no confusion.

Remark 2.1.15. *In her paper [24], to study Devoto's equivariant Tate K-theory, Nora Ganter equipped each component ${}_k\mathcal{L}_g X$ with an action by $\mathbb{Z}/kl\mathbb{Z}$. A groupoid can be formulated with objects the points in*

$$\coprod_{g\in G^{tors}} {}_k\mathcal{L}_g X,$$

and whose morphisms are $\coprod\limits_{g,g'\in G^{tors}} C^k_G(g,g')\times {}_k\mathcal{L}_g X$. A point $([h,m],\gamma)\in C^k_G(g,g')\times {}_k\mathcal{L}_g X$ is viewed as a morphism $(g,\gamma(-))\mapsto (g',\gamma(-+m)\cdot h)$.

Now we want to emphasize there is an S^1-action on each component and on the whole space

$$\coprod_{g\in G^{tors}} {}_k\mathcal{L}_g X.$$

Let's consider the groupoid whose objects are the points in $\coprod\limits_{g\in G^{tors}} {}_k\mathcal{L}_g X$ as the previous examples, and whose morphisms are $\coprod\limits_{g,g'\in G^{tors}} \Lambda^k_G(g,g')\times {}_k\mathcal{L}_g X$. A point $([h,s],\gamma)\in \Lambda^k_G(g,g')\times {}_k\mathcal{L}_g X$ is viewed as a morphism $(g,\gamma(-))\mapsto (g',\gamma(-+s)h)$. And the composition of morphisms is defined by

$$[h_1,t_1]\cdot[h_2,t_2]=[h_1h_2,t_1+t_2].$$

2.1.3 Ghost Loops

In this section I introduce a subgroupoid $GhLoop(X//G)$ of $Loop^{ext}_1(X//G)$, which can be computed locally. $\Lambda(X//G)$ is a full subgroupoid of it. When G is finite, $GhLoop(X//G)$ is isomorphic to $\Lambda(X//G)$.

Definition 2.1.16 (Ghost Loops). *The ghost loops corresponds to the full subgroupoid $GhLoop(X//G)$ of $Loop^{ext}_1(X//G)$ consisting of objects $S^1\leftarrow P\xrightarrow{\widetilde{\delta}} X$ such that $\widetilde{\delta}(P)\subseteq X$ contained in a single $G-$orbit.*

For a given $\sigma \in G$, such maps correspond to the $\Lambda_\sigma G-$invariant subspace

$$GhLoop_\sigma(X//G) := \{\delta \in {}_1\mathcal{L}_\sigma X | \delta(\mathbb{R}) \subseteq G\delta(0)\}. \tag{2.15}$$

By Proposition 2.1.9, $GhLoop(X//G)$ is equivalent to

$$\coprod_{[\sigma]} GhLoop_\sigma(X//G)//\Lambda^1_\sigma G$$

where the coproduct goes over conjugacy classes in $\pi_0 G$.

Unlike true loops, ghost loops have the property that they can be computed locally. In other words we have the lemma below. The proof is left to the readers.

Lemma 2.1.17. *If $X = U \cup V$ where U and V are $G-$invariant open subsets, then*

$$GhLoop(X//G) \cong GhLoop(U//G) \cup_{GhLoop((U\cap V)//G)} GhLoop(V//G).$$

Thus, the ghost loop construction satisfies Mayer-Vietoris property.

If G is a finite group, it has the discrete topology. In this case, LG consists of constant loops and, thus, is isomorphic to G. And $GhLoop(X//G)$ can be identified with X. For $\sigma \in G$ and any integer k, $L^k_\sigma G$ can be identified with $C_G(\sigma)$ and $\Lambda^k_\sigma G \cong C_G(\sigma)\times\mathbb{R}/\langle(\sigma, -k)\rangle$; and $GhLoop_\sigma(X//G)$ can be identified with X^σ.

In this case we have the isomorphism of groupoids

$$GhLoop(X//G) \cong \Lambda(X//G), \tag{2.16}$$

where $\Lambda(X//G)$ is the full subgroupoid of ${}_1\mathcal{L}(X//G)$ with constant loops as objects. We have seen $\Lambda(X//G)$ in Example 2.1.12 and will study it in detail in Section 2.4.2 where I introduce quasi-elliptic cohomology.

Proposition 2.1.18. *Let G be a compact Lie group and H a closed subgroup of it. It acts on the space of left cosets G/H by left multiplication. Let pt denote the single point space with the trivial $H-$action. Then we have the equivalence of topological groupoids between $Loop^{ext}_1((G/H)//G)$ and $Loop^{ext}_1(pt//H)$. Especially, there is an equivalence of topological groupoids between $GhLoop((G/H)//G)$ and $GhLoop(pt//H)$.*

Proof. First we define a functor $F : Loop_1^{ext}((G/H)//G) \longrightarrow Loop_1^{ext}(\mathrm{pt}//H)$ sending an object $S^1 \leftarrow P \xrightarrow{\widetilde{\delta}} G/H$ to $S^1 \leftarrow Q \rightarrow \{eH\} = \mathrm{pt}$ where $Q \longrightarrow eH$ is the constant map, and $Q \longrightarrow S^1$ is the pull back bundle

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & \{eH\} \arrow[d, hook] \\ P \arrow[r] & G/H. \end{tikzcd}$$

It sends a morphism

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G/H \\ S^1 \arrow[r] & S^1 & \end{tikzcd}$$

to the morphism

$$\begin{tikzcd} Q' \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \{eH\} \arrow[d] \\ P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G/H \\ S^1 \arrow[r] & S^1 & \end{tikzcd}$$

where all the squares are pull-back.

In addition, we can define a functor $F' : Loop_1^{ext}(\mathrm{pt}//H) \longrightarrow Loop_1^{ext}((G/H)//G)$ sending an object $S^1 \leftarrow Q \rightarrow \mathrm{pt}$ to $S^1 \leftarrow G \times_H Q \rightarrow G \times_H \mathrm{pt} = G/H$ and sending a morphism

$$\begin{tikzcd} Q' \arrow[r] \arrow[d] & Q \arrow[d] \\ S^1 \arrow[r] & S^1 \end{tikzcd}$$

to

$$\begin{tikzcd} G \times_H Q' \arrow[r] \arrow[d] & G \times_H Q \arrow[r] \arrow[d] & G \times_H \mathrm{pt} = G/H \\ S^1 \arrow[r] & S^1 & \end{tikzcd}$$

$F \circ F'$ and $F' \circ F$ are both identity maps. So the topological groupoids $Loop_1^{ext}((G/H)//G)$ and $Loop_1^{ext}(\mathrm{pt}//H)$ are equivalent.

We can prove the equivalence between $GhLoop((G/H)//G)$ and $GhLoop(\mathrm{pt}//H)$ in the same way. $\square$

Remark 2.1.19. *In general, if H^* is an equivariant cohomology theory, Proposition 2.1.18 implies the functor*

$$X//G \mapsto H^*(GhLoop(X//G))$$

gives a new equivariant cohomology theory. When H^* has the change of group isomorphism, so does $H^*(GhLoop(-))$.

2.2 Equivariant K-theory

In Section 2.4.2 I show the definition of quasi-elliptic cohomology in terms of orbifold K-theory and that in terms of equivariant K-theory. I introduce some of its properties in Section 2.4.3, which resemble the properties of equivariant K-theory. So before the construction of quasi-elliptic cohomology let's recall the basic concepts and properties of equivariant K-theory.

2.2.1 Construction

Let G be a compact group and X be a compact G-space. By a G-vector bundle on X, we mean a vector bundle $V \longrightarrow X$ together with a G-action on V such that the projection $V \longrightarrow X$ is G-equivariant.

For any two G-vector bundles, there is a natural action of G on their direct sum, so the set of G-vector bundles forms a commutative monoid. The equivariant K-theory $K_G(X)$ is defined to be the group completion of this monoid. Tensor product of the vector bundles equips $K_G(X)$ a ring structure.

Compactness of X is essential. If we define $K_G(Y)$ as group completion for infinite $G-$cell complex Y, it doesn't satisfy Milnor axiom:

$$K_G(\bigsqcup_i Y_i) \neq \prod_i K_G(Y_i),$$

where Y_i is the $i-$th $G-$skeleton of Y. As in non-equivariant K-theory, K_G defines a functor from the homotopy category of G-spaces to the category of abelian groups. There is also an analogue of Mayer-Vietoris sequence in equivariant K-theory. We need the equivariant

Bott periodicity theorem to define the positive degree K_G groups. For more details, please see [7] and [8].

K_G is also functorial in G. If there is a homomorphism $H \longrightarrow G$, then it induces $K_G(X) \longrightarrow K_H(X)$.

As the non-equivariant $K-$theory, K_G also has an invariant using complexes of vector bundles. The corresponding element in K-theory of a complex

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

is the sum $\Sigma(-1)^j V_j$. This defines a element of $K_G(X, Y)$ if this complex is exact on Y.

2.2.2 Elementary properties

$K_G(\mathrm{pt})$ is represented by the formal difference of two G-vector spaces. So this is the representation ring RG of G. As in any cohomology theory, for any X, $K_G(X)$ is an RG-module.

As enlightened by Segal in his paper [57], equivariant K-theory has the change of group isomorphism.

Let H be a closed subgroup of G and X a H-space. Let $\phi : H \longrightarrow G$ denote the inclusion homomorphism. The change-of-group map $c_H^G : K_G^*(G \times_H X) \longrightarrow K_H^*(X)$ is defined as the composite

$$K_G^*(G \times_H X) \xrightarrow{\phi^*} K_H^*(G \times_H X) \xrightarrow{i^*} K_H^*(X) \tag{2.17}$$

where ϕ^* is the restriction map and $i : X \longrightarrow G \times_H X$ is the $H-$equivariant map defined by $i(x) = [e, x]$.

Proposition 2.2.1. *Let G be a compact Lie group and H a closed subgroup of G. Let X be a $G-$space. Then the change-of-group map*

$$c_H^G : K_G(G \times_H X) \longrightarrow K_H(X)$$

defined in (2.17) is an isomorphism.

Proof. Any $G-$vector bundle E over $G \times_H X$ is determined completely by its restriction on the subspace $X = H \times_H X$. The restriction c_H^G is an equivalence between $G-$vector bundles on $G \times_H X$ and $H-$vector bundles on X. It's the inverse to the extension $E \mapsto G \times_H E$. $\square$

Proposition 2.2.2. *If G acts trivially on X, then any G-bundle on X is a bundle $V \longrightarrow X$*

with the $G-$action identity on the base, i.e a family of G-representations parametrized by X and we have

$$K_G(X) \cong K(X) \otimes RG.$$

Then let's see the other extreme case.

Proposition 2.2.3. *If G acts freely on X, the map $X \longrightarrow X/G$ induces $K(X/G) \otimes RG \longrightarrow K_G(X)$. In particular, we have the map $K(X/G) \longrightarrow K_G(X)$, i.e. any G-vector bundle on X can descend to a vector bundle on X/G whose sections are the G-invariant sections over X. And the map $K(X/G) \longrightarrow K_G(X)$ is an isomorphism.*

More generally, we have:

Theorem 2.2.4. *Let N be a closed normal subgroup of G which acts freely on a compact G-space X. Then $K_G(X) \cong K_{G/N}(X/N)$.*

2.2.3 Transfer and Transfer Ideal

In Section 2.2.3 I introduce the construction of the transfer map and transfer ideal of equivariant K-theory. A reference for Section 2.2.3 is [38] and [48].

Moreover, given a finite covering $f : X \longrightarrow Y$ of compact $G-$spaces, the transfer map

$$f_! : K_G(X) \longrightarrow K_G(Y)$$

is the unique homomorphism such that any isomorphism class represented by a $G-$bundle $V \longrightarrow X$ is mapped to that represented by the $G-$bundle $f_!V \longrightarrow Y$ with

$$(f_!V)_y = \prod_{x \in f^{-1}(y)} V_x$$

and the $G-$action

$$g \cdot (v_x)_{x \in f^{-1}(y)} \mapsto (gv_{g^{-1}x'})_{x' \in f^{-1}(gy)}.$$

Consider any closed subgroup H of G. There is a covering $G \times_H X \longrightarrow X$. The induced transfer is defined to be the composition

$$Ind_H^G : K_H(X) \cong K_G(G \times_H X) \xrightarrow{f_!} K_G(X) \tag{2.18}$$

where the first isomorphism is the change of group isomorphism introduced in Proposition 2.2.1. The induced transfer is also called an induction map.

In particular, if $X = \text{pt}$, Ind_H^G is the induction map for the representation rings $RH \longrightarrow RG$.

The transfer map $f_!$ have some interesting properties.

Proposition 2.2.5. *(i) Transfers are natural, in the sense that given a sequence* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *of covers, the transfers satisfy* $(g \circ f)_! = g_! \circ f_!$.

(ii) Given a cover $f : X \longrightarrow Y$ *of* $G-$*spaces, and an* $H-$*space* Z*, we have*

$$(f \times 1_Z)_!(a \times c) = f_!(a) \times c,$$

where $a \in K_G(X)$ *and* $c \in K_H(Z)$.

(iii) Given a pullback square

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \\ Y' \arrow[r, "h"'] & Y \end{tikzcd}$$

where f *and* f' *are coverings, we have*

$$h^* f_! = (f')_! g^*.$$

(iv) Given a cover $f : X \longrightarrow Y$*, we have the formula*

$$f_!(af^*(b)) = f_!(a)b$$

for $a \in K_G(X)$ *and* $b \in K_G(Y)$.

Let X be a space. There is an evident Σ_m-action on $X^{\times m}$.

Let I_{tr} denote the subgroup of $K_{\Sigma_m}(X^{\times m})$

$$I_{tr} = \sum_{\substack{i+j=m,\\ m>j>0}} \text{Image}[Ind_{\Sigma_i \times \Sigma_j}^{\Sigma_m} : K_{\Sigma_i \times \Sigma_j}(X^{\times m}) \longrightarrow K_{\Sigma_m}(X^{\times m})]. \tag{2.19}$$

By Proposition 2.2.5 (iv), I_{tr} is in fact an ideal. It is called the proper transfer ideal.

2.2.4 $\lambda-$Rings

In Section 2.2.4 I introduce basics on $\lambda-$rings. As shown in Example 2.2.12, equivariant $K-$theory $K^0_G(X)$ can be equipped with a $\lambda-$ring structure. I show in Section 2.4.2 I discuss the $\lambda-$ring structure on quasi-elliptic cohomology. The main reference for Section 2.2.4 is [62].

Fix a ring R. Let $R[x_1, \cdots x_n]$ be the polynomial ring over R in n independent variables $x_1, \cdots x_n$.

Example 2.2.6 (Elementary symmetric functions). *For $1 \leqslant k \leqslant n$, let $s_k \in R[x_1, \cdots x_n]$ be the polynomial*

$$s_k = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Each s_k is a symmetric function since s_k is the coefficient of t^{n-k} in

$$\prod_{k=1}^{n} (t - x_k) = t^n - s_1 t^{n-1} + s_2 t^{n-2} - \cdots + (-1)^n s_n.$$

s_k is called the kth elementary symmetric function on $x_1, \cdots x_n$.

Example 2.2.7 (Universal polynomials P_n and $P_{n,m}$). *The coefficient of each t^n in the polynomial*

$$g(t) := \prod_{1 \leqslant i_1 < \cdots < i_m \leqslant nm} (1 + x_{i_1} \cdots x_{i_m} t)$$

is a symmetric polynomial. So it can be expressed uniquely as a polynomial with integer coefficients in the elementary symmetric functions $s_1, \cdots s_{nm}$.

In other words, there is a universal polynomial $P_{n,m}$ in nm variables with integer coefficients such that $P_{n,m}(s_1, \cdots s_{nm})$ is the coefficient of t^n in $g(t)$.

Let $y_1, \cdots y_n$ be another set of variables and let $\sigma_1, \cdots \sigma_n$ be their elementary symmetric functions.

The coefficient of each t^k in

$$h(t) := \prod_{i,j=1}^{n} (1 + x_i y_j t)$$

is a symmetric functions on the x_is and the y_js. So it can be expressed uniquely as a polynomial with integer coefficients in the elementary symmetric functions $s_1, \cdots s_n$ and $\sigma_1, \cdots \sigma_n$.

Thus, there exists a universal polynomial P_n in $2n$ variables with integer coefficients such that $P_n(s_1, \cdots s_n; \sigma_1, \cdots \sigma_n)$ is the coefficient of t^n in $h(t)$.

Definition 2.2.8 ($\lambda-$Ring). *A $\lambda-$ring is a ring R together with functions*

$$\lambda^n : R \longrightarrow R$$

for each $n \geqslant 0$ called a $\lambda-$operations, such that for all $x, y \in R$, the following axioms are satisfied:

(1) $\lambda^0(x) = 1$;

(2) $\lambda^1(x) = x$;

(3) $\lambda^n(1) = 0$ for $n \geqslant 2$;

(4) $\lambda^n(x+y) = \sum\limits_{i+j=n} \lambda^i(x)\lambda^j(y)$;

(5) $\lambda^n(xy) = P_n(\lambda^1(x), \cdots \lambda^n(x); \lambda^1(y), \cdots \lambda^n(y))$;

(6) $\lambda^n(\lambda^m(x)) = P_{n,m}(\lambda^1(x), \cdots \lambda^{nm}(x))$;

where P_n and $P_{n,m}$ are the universal polynomials with integer coefficients described in Example 2.2.7. If only axioms (1), (2) and (4) are satisfied, R is called a pre$-\lambda-$ring.

In a $\lambda-$ring R, we write

$$\lambda_t(x) = \sum_{n=0}^{\infty} \lambda^n(x)t^n, \text{ for any } x \in R. \tag{2.20}$$

Example 2.2.9. *The simplest $\lambda-$ring is the ring of integers $\mathbb{Z}$, with the $\lambda-$operations defined by*

$$\lambda^i(n) = \binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

$\lambda^i(n)$ is the coefficient of t^n in the polynomial $(1+t)^n$.

Example 2.2.10. *If R is a $\lambda-$ring with $\lambda-$operations $\{\lambda_R^i\}_i$, the power series ring $R((x))$ is a $\lambda-$ring, with $\lambda_t(x) = 1 + xt$. For a polynomial $\sum\limits_k r_k x^k$,*

$$\lambda_t(\sum_k r_k x^k) := \prod_k \lambda_t(r_k x^k)$$

is defined using the axioms for $\lambda^n(xy)$ and $\lambda^n(x+y)$.

Then by induction, the power series ring $R((x_1, \cdots x_n))$ is a $\lambda-$ring with

$$\lambda_t(x_i) = 1 + x_i t, \text{ for } 1 \leqslant i \leqslant n.$$

Example 2.2.11. *The topological $K-$theory $K(X)$ of a paracompact Hausdorff space X is a $\lambda-$ring, with the $\lambda-$operation λ^i induced by the ith exterior power of vector bundles over X.*

Example 2.2.12. *For a group G, the complex representation ring $R(G)$ is a $\lambda-$ring, with each λ^i induced by the ith exterior power on representations of G.*

Especially, if G is the trivial group $\{e\}$, $R(G) \cong \mathbb{Z}$ is the $\lambda-$ring described in Example 2.2.9.

Moreover, the equivariant $K-$theory $K_G(X)$ of a compact $G-$space X is a $\lambda-$ring, with the $\lambda-$operation λ^i induced by the ith exterior power of $G-$bundles over X.

Definition 2.2.13. *Let R and S be $\lambda-$rings. A $\lambda-$homomorphism $f : R \longrightarrow S$ is a ring homomorphism such that $f\lambda^i = \lambda^i f$ for $i \geqslant 0$.*

2.3 Orbifold K-theory

Via orbifold-K theory, quasi-elliptic cohomology can be constructed in an elegant way. Moreover, it's essential when I define quasi-elliptic cohomology for general orbifolds in Section 3.3.1. So before that in Section 2.3 I briefly introduce the relevant knowledge about orbifold K-theory. In this section I introduce the notion of orbifold vector bundles using the language of groupoids and define orbifold K-theory for compact orbifold groupoids. The main reference for Section 2.3 is [3] and [47].

Definition 2.3.1 (Orbifold Structure). *An orbifold structure on a paracompact Hausdorff space X consists of an orbifold groupoid $\mathbb{G}$, i.e. a proper Lie groupoid, and a homeomorphism $f : |\mathbb{G}| \longrightarrow X$. If $\phi : \mathbb{H} \longrightarrow \mathbb{G}$ is an equivalence, then $|\phi| : |\mathbb{H}| \longrightarrow |\mathbb{G}|$ is a homeomorphism, and the composition $f \circ |\phi| : |\mathbb{H}| \longrightarrow X$ defines an equivalent orbifold structure on X.*

Definition 2.3.2 (Orbifold). *An orbifold $\mathcal{X}$ is a space X equipped with an equivalence class of orbifold structures. A specific such structure, given by $\mathbb{G}$ and a homeomorphism $f : |\mathbb{G}| \longrightarrow X$, is called a presentation of the orbifold $\mathcal{X}$.*

If X is paracompact Haursdorff, then $\mathcal{X}$ is called an effective orbifold.

If X is compact, then $\mathcal{X}$ is called a compact orbifold.

If a compact Lie group G acts smoothly and effectively on a smooth manifold M, the associated orbifold $\mathcal{X} = M/G$ is called an effective global quotient.

By Theorem 1.23 in [3], every effective orbifold $\mathcal{X}$ is isomorphic to a global quotient orbifold.

Let $\mathbb{G}$ be an orbifold groupoid. Let $\mathbb{G}_0$ denote its orbjects and $\mathbb{G}_1$ denote its arrows. Let's recall the definition of $\mathbb{G}-$space.

Definition 2.3.3. *Let $\mathbb{G}$ be an orbifold groupoid. A left $\mathbb{G}-$space is a manifold E equipped with an action by $\mathbb{G}$. Such an action is given by two maps: an anchor $\pi : E \to \mathbb{G}_0$, and an action $\mu : \mathbb{G}_1 \times_{\mathbb{G}_0} E \longrightarrow E$. The latter map is defined on pairs (g, e) with $\pi(e) = s(g)$, and written $\mu(g, e) = g \cdot e$. It satisfies the usual identities for an action: $\pi(g \cdot e) = t(g)$, $1_x \cdot e = e$, and $g \cdot (h \cdot e) = (gh) \cdot e$, for $x \xrightarrow{h} y \xrightarrow{g} z$ in $\mathbb{G}_1$ and $e \in E$ with $\pi(e) = x$.*

Definition 2.3.4. *A $\mathbb{G}-$vector bundle over an orbifold groupoid $\mathbb{G}$ is a $\mathbb{G}-$space E for which $\pi : E \to \mathbb{G}_0$ is a vector bundle, such that the action of $\mathbb{G}$ on E is fiberwise linear. Namely, any arrow $g : x \to y$ induces a linear isomorphism $g : E_x \to E_y$. In particular, E_x is a linear representation of the isotropy group G_x for each $x \in \mathbb{G}_0$.*

Orbibundles over an orbifold X can be described as $\mathbb{G}-$vector bundles with $\mathbb{G}$ any orbifold groupoid presentation of X. They behave naturally under vector space constructions, sums, tensor products, exterior products, etc.

Definition 2.3.5. *Given a compact orbifold groupoid $\mathbb{G}$, let $K_{orb}(\mathbb{G})$ to be the Grothendieck ring of isomorphism classes of $\mathbb{G}-$vector bundles on $\mathbb{G}$. When X is an orbifold, we define $K_{orb}(X)$ to be $K_{orb}(\mathbb{G})$, where $\mathbb{G}$ is any groupoid presentation of X.*

Under an orbifold morphism $F : \mathbb{H} \longrightarrow \mathbb{G}$, orbifold bundles over $\mathbb{G}$ pull back to orbifold bundles over $\mathbb{H}$.

Proposition 2.3.6. *Each orbifold morphism $F : \mathbb{H} \longrightarrow \mathbb{G}$ induces a ring homomorphism $F^* : K_{orb}(\mathbb{G}) \longrightarrow K_{orb}(\mathbb{H})$.*

In particular, if two groupoids $\mathbb{G}$ and $\mathbb{H}$ are Morita equivalent, $K_{orb}(\mathbb{G}) \cong K_{orb}(\mathbb{H})$.

An important example of orbifold morphisms is the projection map $p : M \longrightarrow M/G$ where G is a compact Lie group acting almost freely on the manifold M.

Similar to the conclusion in equivariant K-theory, we have

Proposition 2.3.7. *Let $X = M/G$ be a quotient orbifold. Then the projection map $p : M \longrightarrow M/G$ induces an isomorphism $p^* : K_{orb}(X) \longrightarrow K_G(M)$.*

In particular, if X is an effective orbifold, we can identify its orbifold K-theory with the equivariant K-theory of its frame bundle.

2.4 Quasi-elliptic cohomology $QEll^*_G$

In Section 2.4.2 I show basic constructions of $QEll^*_G$, which is defined from orbifold loop space introduced in Example 2.1.14. The main reference for Section 2.4 is [50]. Before that I discuss in Section 2.4.1 the representation ring of $\Lambda_G(g)$, which is denoted by $\Lambda^1_G(g)$ in Section 2.1.2.

2.4.1 Preliminary: representation ring of $\Lambda_G(g)$

For any compact Lie group G and a torsion element $g \in G$, Recall $\Lambda_G(g)$ is the group

$$\Lambda^1_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$$

defined in Section 2.1.2 and $\mathbb{T}$ is the circle group $\mathbb{R}/\mathbb{Z}$. Let $q : \mathbb{T} \longrightarrow U(1)$ be the isomorphism $t \mapsto e^{2\pi i t}$. The representation ring $R\mathbb{T}$ of the circle group is $\mathbb{Z}[q^{\pm}]$.

We have an exact sequence

$$1 \longrightarrow C_G(g) \longrightarrow \Lambda_G(g) \xrightarrow{\pi} \mathbb{T} \longrightarrow 0$$

where the first map is $g \mapsto [g, 0]$ and the second map is

$$\pi([g, t]) = e^{2\pi i t}. \tag{2.21}$$

There is a relation between the representation ring of $C_G(g)$ and that of $\Lambda_G(g)$, which is shown as Lemma 1.2 in [50].

Lemma 2.4.1. *$\pi^* : R\mathbb{T} \longrightarrow R\Lambda_G(g)$ exhibits $R\Lambda_G(g)$ as a free $R\mathbb{T}-$module.*

In particular, there is an $R\mathbb{T}-$basis of $R\Lambda_G(g)$ given by irreducible representations $\{V_\lambda\}$, such that restriction $V_\lambda \mapsto V_\lambda|_{C_G(g)}$ to $C_G(g)$ defines a bijection between $\{V_\lambda\}$ and the set $\{\lambda\}$ of irreducible representations of $C_G(g)$.

Proof. Let l be the order of the torsion element g. Note that $\Lambda_G(g)$ is isomorphic to

$$C_G(g) \times \mathbb{R}/l\mathbb{Z}/\langle (g,-1) \rangle.$$

Thus, it is the quotient of the product of two compact Lie groups.

Let $\lambda : C_G(g) \longrightarrow GL(n, \mathbb{C})$ be an $n-$dimensional $C_G(g)-$representation with representation space V and $\eta : \mathbb{R} \longrightarrow GL(n, \mathbb{C})$ be a representation of $\mathbb{R}$ such that $\lambda(g)$ acts on V via scalar multiplication by $\eta(1)$. Define

$$\lambda \odot_{\mathbb{C}} \eta([h,t]) := \lambda(h)\eta(t). \tag{2.22}$$

It's straightforward to verify $\lambda \odot_{\mathbb{C}} \eta$ is a $n-$dimensional $\Lambda_G(g)-$representation with representation space V.

Any irreducible $n-$dimensional representation of the quotient group $\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g,-1) \rangle$ is an irreducible $n-$dimensional representation of the product $C_G(g) \times \mathbb{R}/\langle (g,-1) \rangle$. And any finite dimensional irreducible representation of the product of two compact Lie groups is the tensor product of an irreducible representation of each factor. So any irreducible representation of the quotient group $\Lambda_G(g)$ is the tensor product of an irreducible representation λ of $C_G(g)$ with representation space V and an irreducible representation η of $\mathbb{R}$. Any irreducible complex representation η of $\mathbb{R}$ is one dimensional. So the representation space of $\lambda \otimes \eta$ is still V. Let l be the order of g. $\eta(1)^l = I$. We need $\eta(1) = \lambda(g)$. So $\eta(1) = e^{\frac{2\pi i k}{l}}$ for some $k \in \mathbb{Z}$. So

$$\eta(t) = e^{\frac{2\pi i (k+lm)t}{l}}.$$

Any $m \in \mathbb{Z}$ gives a choice of η in this case. And η is a representation of $\mathbb{R}/l\mathbb{Z} \cong \mathbb{T}$.

Therefore, we have a bijective correspondence between

(1) isomorphism classes of irreducible $\Lambda_G(g)-$representation ρ, and

(2) isomorphism classes of pairs (λ, η) where λ is an irreducible $C_G(g)-$representation and $\eta : \mathbb{R} \longrightarrow \mathbb{C}^*$ is a character such that $\lambda(g) = \eta(1)I$. $\lambda = \rho|_{C_G(g)}$. □

Remark 2.4.2. *We can make a canonical choice of $\mathbb{Z}[q^{\pm}]$-basis for $R\Lambda_G(g)$. For each irreducible G-representation $\rho : G \longrightarrow Aut(G)$, write $\rho(\sigma) = e^{2\pi i c} id$ for $c \in [0,1)$, and set $\chi_\rho(t) = e^{2\pi i c t}$. Then the pair (ρ, χ_ρ) corresponds to a unique irreducible $\Lambda_G(g)$-representation.*

Example 2.4.3 **($G = \mathbb{Z}/N\mathbb{Z}$).** *Let $G = \mathbb{Z}/N\mathbb{Z}$ for $N \geq 1$, and let $\sigma \in G$. Given an integer $k \in \mathbb{Z}$ which projects to $\sigma \in \mathbb{Z}/N\mathbb{Z}$, let x_k denote the representation of $\Lambda_G(\sigma)$ defined by*

$$\Lambda_G(\sigma) = (\mathbb{Z} \times \mathbb{R})/(\mathbb{Z}(N,0) + \mathbb{Z}(k,1)) \xrightarrow{[a,t]\mapsto[(kt-a)/N]} \mathbb{R}/\mathbb{Z} = \mathbb{T} \xrightarrow{q} U(1). \tag{2.23}$$

$R\Lambda_G(\sigma)$ is isomorphic to the ring $\mathbb{Z}[q^{\pm}, x_k]/(x_k^N - q^k)$.

For any finite abelian group $G = \mathbb{Z}/N_1\mathbb{Z} \times \mathbb{Z}/N_2\mathbb{Z} \times \cdots \times \mathbb{Z}/N_m\mathbb{Z}$, let $\sigma = (k_1, k_2, \cdots k_n) \in G$. We have

$$\Lambda_G(\sigma) \cong \Lambda_{\mathbb{Z}/N_1\mathbb{Z}}(k_1) \times_{\mathbb{T}} \cdots \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/N_m\mathbb{Z}}(k_m).$$

Then

$$\begin{aligned} R\Lambda_G(\sigma) &\cong R\Lambda_{\mathbb{Z}/N_1\mathbb{Z}}(k_1) \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} R\Lambda_{\mathbb{Z}/N_m\mathbb{Z}}(k_m) \\ &\cong \mathbb{Z}[q^{\pm}, x_{k_1}, x_{k_2}, \cdots x_{k_m}]/(x_{k_1}^{N_1} - q^{k_1}, x_{k_2}^{N_2} - q^{k_2}, \cdots x_{k_m}^{N_m} - q^{k_m}) \end{aligned}$$

where all the x_{k_j}'s are defined as x_k in (2.23).

Example 2.4.4 **($G = \Sigma_3$).** *$G = \Sigma_3$ has three conjugacy classes represented by* 1, (12), (123) *respectively.*

$\Lambda_{\Sigma_3}(1) = \Sigma_3 \times \mathbb{T}$, thus, $R\Lambda_{\Sigma_3}(1) = R\Sigma_3 \otimes R\mathbb{T} = \mathbb{Z}[X,Y]/(XY - Y, X^2 - 1, Y^2 - X - Y - 1) \otimes \mathbb{Z}[q^{\pm}]$ where X is the sign representation on Σ_3 and Y is the standard representation.

$C_{\Sigma_3}((12)) = \langle (12) \rangle = \Sigma_2$, thus, $\Lambda_{\Sigma_3}((12)) \cong \Lambda_{\Sigma_2}((12))$. So we have $R\Lambda_{\Sigma_3}((12)) \cong R\Lambda_{\Sigma_2}((12)) = \mathbb{Z}[q^{\pm}, x_1]/(x_1^2 - q) \cong \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

$C_{\Sigma_3}(123) = \langle (123) \rangle = \mathbb{Z}/3\mathbb{Z}$, thus, $\Lambda_{\Sigma_3}((123)) \cong \Lambda_{\mathbb{Z}/3\mathbb{Z}}(1)$. So we have $R\Lambda_{\Sigma_3}((123)) \cong \mathbb{Z}[q^{\pm}, x_1]/(x_1^3 - q) \cong \mathbb{Z}[q^{\pm\frac{1}{3}}]$.

Example 2.4.5 **($G = \Sigma_4$).** *$G = \Sigma_4$ has conjugacy classes* 1, (12), (123), (1234), *and* (12)(34).

$\Lambda_{\Sigma_4}(1) = \Sigma_4 \times \mathbb{T}$, thus, $R\Lambda_{\Sigma_4}(1) = R\Sigma_4 \otimes \mathbb{Z}[q^{\pm}]$.

$C_{\Sigma_4}((12)) = \langle (12) \rangle \times \langle (34) \rangle$. $\Lambda_{\Sigma_4}((12)) \cong \Lambda_{\Sigma_2}((12)) \times_{\mathbb{T}} \Lambda_{\Sigma_2}(1)$. Thus, $R\Lambda_{\Sigma_4}((12)) \cong R\Lambda_{\Sigma_2}((12)) \otimes_{\mathbb{Z}[q^{\pm}]} R\Lambda_{\Sigma_2}(1) \cong \mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes \mathbb{Z}[s]/(s^2 - 1)$.

$C_{\Sigma_4}((123)) = \langle(123)\rangle$. $R\Lambda_{\Sigma_4}((123)) \cong R\Lambda_{\mathbb{Z}/3\mathbb{Z}}(1) = \mathbb{Z}[q^{\pm\frac{1}{3}}]$.

$C_{\Sigma_4}((1234)) = \langle(1234)\rangle$. $R\Lambda_{\Sigma_4}((1234)) \cong R\Lambda_{\mathbb{Z}/4\mathbb{Z}}(1) = \mathbb{Z}[q^{\pm\frac{1}{4}}]$.

$C_{\Sigma_4}((12)(34)) = \Sigma_2 \wr \Sigma_2$, *which is the Dihedral group. We can define a map*

$$\Psi : \Lambda_{\Sigma_2\wr\Sigma_2}((12)(34)) \longrightarrow (\Lambda_{\Sigma_2}((12)) \times_{\mathbb{T}} \Lambda_{\Sigma_2}((12))) \ltimes \Sigma_2$$
$$([(g_1, g_2, \sigma), t]) \mapsto ([g_1, t], [g_2, t], \sigma).$$

The semidirect product $(\Lambda_{\Sigma_2}((12)) \times_{\mathbb{T}} \Lambda_{\Sigma_2}((12))) \ltimes \Sigma_2$ *is a closed subgroup the wreath product* $\Lambda_{\Sigma_2}((12)) \wr \Sigma_2$. Ψ *is an isomorphism.*

To compute $R\Lambda_{\Sigma_4}((12)(34))$*, we need the representation theory for wreath product. By computing the inertia group of each irreducible representation of* $\Lambda_{\Sigma_2}((12)) \times_{\mathbb{T}} \Lambda_{\Sigma_2}((12))$ *and applying Theorem 4.3.34 in [34], we can get all the irreducible representations of* $(\Lambda_{\Sigma_2}((12)) \times_{\mathbb{T}} \Lambda_{\Sigma_2}((12))) \ltimes \Sigma_2$.

Let k be any integer. I'll show the relation between $\Lambda^k_G(g)$ and $\Lambda_G(g)$ and that between their representation rings.

Recall $\Lambda^k_G(g)$ is defined to be the quotient group $C_G(g) \times \mathbb{R}/\langle(g, -k)\rangle$. There is an exact sequence

$$1 \longrightarrow C_G(g) \xrightarrow{g \mapsto [g,0]} \Lambda^k_G(g) \xrightarrow{\pi_k} \mathbb{R}/k\mathbb{Z} \longrightarrow 0$$

where the second map $\pi_k : \Lambda^k_G(g) \longrightarrow \mathbb{R}/k\mathbb{Z}$ is $\pi_k([g, t]) = e^{2\pi i t}$.

Analogous to Lemma 2.4.1, we have the conclusion about $R\Lambda^k_G(g)$ below.

Lemma 2.4.6. *The map* $\pi^*_k : R\mathbb{R}/k\mathbb{Z} \longrightarrow R\Lambda^k_G(g)$ *exhibits it as a free* $\mathbb{Z}[q^{\pm\frac{1}{k}}]-$*module. There is a* $\mathbb{Z}[q^{\pm\frac{1}{k}}]-$*basis of* $R\Lambda^k_G(g)$ *given by irreducible representations* $\{\rho_k\}$ *such that the restrictions* $\rho_k|_{C_G(g)}$ *of them to* $C_G(g)$ *are precisely the* $\mathbb{Z}$*-basis of* $RC_G(g)$ *given by irreducible representations.*

In other words, any irreducible $\Lambda^k_G(g)-$*representation has the form* $\rho \odot_{\mathbb{C}} \chi$ *where* ρ *is an irreducible representation of* $C_G(g)$*,* $\chi : \mathbb{R}/k\mathbb{Z} \longrightarrow GL(n, \mathbb{C})$ *such that* $\chi(k) = \rho(g)$*, and*

$$\rho \odot_{\mathbb{C}} \chi([h, t]) := \rho(h)\chi(t), \text{ for any } [h, t] \in \Lambda^k_G(g). \tag{2.24}$$

$R\Lambda^k_G(g)$ is a $\mathbb{Z}[q^{\pm}]-$module via the inclusion $\mathbb{Z}[q^{\pm}] \longrightarrow \mathbb{Z}[q^{\pm\frac{1}{k}}]$.

There is a group isomorphism $\alpha_k : \Lambda^k_G(g) \longrightarrow \Lambda_G(g)$ sending $[g,t]$ to $[g, \frac{t}{k}]$.

Observe that there is a pullback square of groups

$$\begin{tikzcd} \Lambda^k_G(g) \arrow[r, "\alpha_k"] \arrow[d, "\pi_k"] & \Lambda_G(g) \arrow[d, "\pi"] \\ \mathbb{R}/k\mathbb{Z} \arrow[r, "t\mapsto \frac{t}{k}"] & \mathbb{R}/\mathbb{Z} \end{tikzcd} \tag{2.25}$$

By Lemma 2.4.6, we can make a $\mathbb{Z}[q^{\pm\frac{1}{k}}]$-basis $\{\rho \odot_{\mathbb{C}} \chi_{\rho,k}\}$ for $R\Lambda^k_G(g)$ with each $\rho : G \longrightarrow Aut(G)$ an irreducible G-representation and $\chi_{\rho,k}(t) = e^{2\pi i\frac{ct}{k}}$ with $c \in [0,1)$ such that $\rho(\sigma) = e^{2\pi ic} id$. This collection $\{\rho \odot_{\mathbb{C}} \chi_{\rho,k}\}$ gives a $\mathbb{Z}[q^{\pm\frac{1}{k}}]-$basis of $R\Lambda^k_G(g)$.

So we have the commutative square of a pushout square in the category of $\Lambda-$rings.

$$\begin{tikzcd} R\Lambda^k_G(g) & R\Lambda_G(g) \arrow[l] \\ R(\mathbb{R}/k\mathbb{Z}) \arrow[u] & R\mathbb{T} \arrow[l] \arrow[u] \end{tikzcd} \tag{2.26}$$

Let $q^{\frac{1}{k}} : \mathbb{R}/k\mathbb{Z} \longrightarrow U(1)$ denote the composition

$$\mathbb{R}/k\mathbb{Z} \xrightarrow{t\mapsto \frac{t}{k}} \mathbb{R}/\mathbb{Z} \xrightarrow{q} U(1).$$

The representation ring $R\mathbb{R}/k\mathbb{Z}$ of $\mathbb{R}/k\mathbb{Z}$ is $\mathbb{Z}[q^{\pm\frac{1}{k}}]$. And there is a canonical isomorphism of $\Lambda-$rings

$$R\Lambda_G(g) \longrightarrow R\Lambda^k_G(g)$$

sending q to $q^{\frac{1}{k}}$.

Moreover let's consider

$$\Lambda_n(\sigma) := \Lambda_{C_G(\sigma)}(\sigma^n). \tag{2.27}$$

It's a subgroup of $\Lambda_G(\sigma^n)$. Let $\beta : \Lambda_n(\sigma) \longrightarrow \Lambda_G(\sigma^n)$ denote the inclusion. And we can define

$$\alpha : \Lambda_n(\sigma) \longrightarrow \Lambda_G(\sigma), (g,t) \mapsto (g, nt). \tag{2.28}$$

We have the pullback square of groups

$$\begin{array}{ccc} \Lambda_n(\sigma) & \xrightarrow{\alpha} & \Lambda_G(\sigma) \\ \downarrow & & \downarrow \\ \mathbb{T} & \xrightarrow{e^{2\pi it}\mapsto e^{2\pi int}} & \mathbb{T} \end{array} \tag{2.29}$$

In addition, $R\Lambda_n(\sigma)$ is the Λ-ring pushout of $\mathbb{Z}[q^{\pm}] \xrightarrow{R\pi} R\Lambda_G(\sigma)$ along the inclusion $\mathbb{Z}[q^{\pm}] \longrightarrow \mathbb{Z}[q^{\pm\frac{1}{n}}]$.

$$\begin{array}{ccc} R\Lambda_n(\sigma) & \xleftarrow{\alpha^*} & R\Lambda_G(\sigma) \\ \uparrow & & \uparrow \\ \mathbb{T} & \xleftarrow{R[n]} & \mathbb{T} \end{array} \tag{2.30}$$

In particular, there is a canonical isomorphism of Λ-rings

$$R\Lambda_n(\sigma) \xrightarrow{\sim} R\Lambda_G(\sigma)[q^{\pm\frac{1}{n}}]. \tag{2.31}$$

Moreover, we have the conclusion below about the relation between the induced representations $Ind|_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(-)$ and $Ind|_{C_H(\sigma)}^{C_G(\sigma)}(-)$.

Lemma 2.4.7. *Let G be a compact Lie group and H a closed subgroup. Let σ be a torsion element in H. Let m denote $[C_G(\sigma) : C_H(\sigma)]$. Let V denote a $\Lambda_H(\sigma)-$representation $\lambda \odot_{\mathbb{C}} \chi$ with λ a $C_H(\sigma)-$representation, χ a $\mathbb{R}-$representation and $\odot_{\mathbb{C}}$ is defined in Lemma 2.4.1. Then the induced representation*

$$Ind|_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \chi)$$

is isomorphic to the $\Lambda_G(\sigma)-$representation

$$(Ind|_{C_H(\sigma)}^{C_G(\sigma)}\lambda) \odot_{\mathbb{C}} \chi.$$

Their underlying vector spaces are both $V^{\oplus m}$.

Proof. First I show $[\Lambda_G(\sigma) : \Lambda_H(\sigma)]$ is the same as $[C_G(\sigma) : C_H(\sigma)]$. We have the inclusion

$$i : C_G(\sigma)/C_H(\sigma) \longrightarrow \Lambda_G(\sigma)/\Lambda_H(\sigma),\ aC_H(\sigma) \mapsto a\Lambda_H(\sigma).$$

i is well-defined. For a, $b \in C_G(\sigma)$, if $a\Lambda_H(\sigma) = b\Lambda_H(\sigma)$, then $a^{-1}b \in \Lambda_H(\sigma)$. Since $a^{-1}b$ is also in $C_G(\sigma)$, $a^{-1}b \in C_H(\sigma)$. So $aC_H(\sigma) = bC_H(\sigma)$. In addition, for each $[g, t] \in$

$\Lambda_G(\sigma)$, $[g,t] = [g,0][e,t]$. Since $[e,t] \in \Lambda_H(\sigma)$, $[g,t]\Lambda_H(\sigma) = [g,0]\Lambda_H(\sigma)$. So $i(gC_H(\sigma)) = [g,t]\Lambda_H(\sigma)$.

Therefore, i is a bijection.

Let f be an element in

$$Ind|_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \chi) = \{\alpha : \Lambda_G(\sigma) \longrightarrow V | \alpha([h,s][x,t]) = \lambda(h)\chi(s)f([x,t]),$$
$$\forall [x,t] \in \Lambda_G(\sigma),\, [h,s] \in \Lambda_H(\sigma)\}.$$

Note $f([x,t]) = f([1,t][x,0]) = \chi(t)f([x,0])$.

Define a map

$$A : Ind|_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \chi) \longrightarrow (Ind|_{C_H(\sigma)}^{C_G(\sigma)}\lambda) \odot_{\mathbb{C}} \chi$$

by sending f to $A(f) := \widetilde{f} \odot_{\mathbb{C}} \chi$ with

$$\widetilde{f} : C_G(\sigma) \longrightarrow V,\, g \mapsto f(g).$$

We can check $\widetilde{f}$, which is restriction of f to $C_G(\sigma)$, is well-defined. For any $h \in C_H(\sigma)$, $\widetilde{f}(hg) = \lambda(h)f(g) = \lambda(h)\widetilde{f}(g)$. In addition, A is $\Lambda_G(\sigma)-$equivariant. For any $[g,s],[x,t] \in \Lambda_G(\sigma)$, $([g,s]\cdot f)([x,t]) = f([x,t][g,s]) = f([xg,t+s]) = \chi(t+s)f(xg)$. We have

$$A([g,s]\cdot f)([x,t]) = \chi(t)(\widetilde{[g,s]\cdot f})(x) = \chi(t)f(x[g,s]) = \chi(t)\chi(s)f(xg) = \chi(t+s)f(xg)$$

and $([g,s]\cdot A(f))([x,t]) = A(f)([x,t][g,s]) = A(f)([xg,t+s]) = \chi(t+s)\widetilde{f}(xg) = \chi(t+s)f(xg) = A([g,s]\cdot f)([x,t])$.

In addition, we have

$$P : (Ind|_{C_H(\sigma)}^{C_G(\sigma)}\lambda) \odot_{\mathbb{C}} \chi \longrightarrow Ind|_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \chi).$$

For any $f' \in (Ind|_{C_H(\sigma)}^{C_G(\sigma)}\lambda) \odot_{\mathbb{C}} \chi$, define

$$P(f')([x,t]) := \chi(t)f'(x).$$

For any $[h,s] \in \Lambda_H(\sigma)$,

$$P(f')([h,s][x,t]) = P(f')([h,s+t][x,0]) = \lambda(h)\chi(s+t)f'(x)$$

and

$$(\lambda \odot_{\mathbb{C}} \chi)([h,s])f'([x,t]) = \lambda(h)\chi(s)\chi(t)f'(x) = \lambda(h)\chi(s+t)f'(x) = P(f')([h,s][x,t]).$$

Thus, $P(f')$ is indeed in $Ind|^{\Lambda_G(\sigma)}_{\Lambda_H(\sigma)}(\lambda \odot_{\mathbb{C}} \chi)$.

$P \circ A$ and $A \circ P$ are both identity maps. So the conclusion is proved.

□

Remark 2.4.8. *By Lemma 2.4.7, the computation of the induced representation*

$$Ind|^{\Lambda_G(\sigma)}_{\Lambda_H(\sigma)}(\lambda \odot_{\mathbb{C}} \chi)$$

can be reduced to the computation of

$$(Ind|^{C_G(\sigma)}_{C_H(\sigma)}\lambda) \odot_{\mathbb{C}} \chi.$$

Thus, when $C_G(\sigma)$ and $C_H(\sigma)$ are both finite, the conclusions about induced representations for finite groups can apply to the computation of $Ind|^{\Lambda_G(\sigma)}_{\Lambda_H(\sigma)}(\lambda \odot_{\mathbb{C}} \chi)$.

Remark 2.4.9. *Let G be a compact Lie group and H a closed subgroup. Let σ be a torsion element in H. Let V denote a $\Lambda_H(\sigma)-$representation $\lambda \odot_{\mathbb{C}} \chi$ with λ a $C_H(\sigma)-$representation, χ a $\mathbb{R}-$representation and $\odot_{\mathbb{C}}$ is defined in Lemma 2.4.1.*

Then it's straightforward to check

$$res|^{\Lambda_G(\sigma)}_{\Lambda_H(\sigma)}(\lambda \odot_{\mathbb{C}} \eta) = (res|^{C_G(\sigma)}_{C_H(\sigma)}\lambda) \odot_{\mathbb{C}} \eta. \tag{2.32}$$

2.4.2 Quasi-elliptic cohomology

In this section I introduce the definition of quasi-elliptic cohomology $QEll^*_G$ in term of orbifold K-theory, and then express it via equivariant K-theory. The readers may read Chapter 3 in [3] and [47] for a reference of orbifold K-theory.

In Example 2.1.12 we see the full subgroupoid $\Lambda(X//G)$ of the orbifold loop space $\mathcal{L}(X//G)$ consisting of constant loops and is relevant to the definition of $QEll^*_G$. Before I describe $\Lambda(X//G)$ in detail, let's recall what Inertia groupoid is. A reference for that is [39].

Definition 2.4.10. *Let $\mathbb{G}$ be a groupoid. The Inertia groupoid $I(\mathbb{G})$ of $\mathbb{G}$ is defined as follows.*

An object a is an arrow in $\mathbb{G}$ such that its source and target are equal. A morphism v joining two objects a and b is an arrow v in $\mathbb{G}$ such that

$$v \circ a = b \circ v.$$

In other words, b is the conjugate of a by v, $b = v \circ a \circ v^{-1}$.

The torsion Inertia groupoid $I^{tors}(\mathbb{G})$ of $\mathbb{G}$ is a full subgroupoid of of $I(\mathbb{G})$ with only objects of finite order.

Example 2.4.11. *Let G be a compact Lie group and X a $G-$space. The torsion inertia groupoid $I^{tors}(X//G)$ of the translation groupoid $X//G$ is the groupoid with*

***objects**: the space $\coprod\limits_{g\in G^{tors}} X^g$*

***morphisms**: the space $\coprod\limits_{g,g'\in G^{tors}} C_G(g,g') \times X^g$ where $C_G(g,g') = \{\sigma \in G | g'\sigma = \sigma g\} \subseteq G$.*

For $x \in X^g$ and $(\sigma, g) \in C_G(g,g') \times X^g$, $(\sigma, g)(x) = \sigma x \in X^{g'}$.

The groupoid $\Lambda(X//G)$ has the same objects as $I^{tors}(X//G)$ but richer morphisms

$$\coprod_{g,g'\in G^{tors}} \Lambda^1_G(g,g') \times X^g$$

where $\Lambda^1_G(g,g')$ is the space defined in Section 2.1.2. For an object $x \in X^g$ and a morphism $([\sigma,t],g) \in \Lambda^1_G(g,g') \times X^g$, $([\sigma,t],g)(x) = \sigma x \in X^{g'}$. The composition of the morphisms is defined by

$$[\sigma_1, t_1][\sigma_2, t_2] = [\sigma_1\sigma_2, t_1 + t_2]. \tag{2.33}$$

We have a homomorphism of orbifolds

$$\pi : \Lambda(X//G) \longrightarrow \mathbb{T}$$

sending all the objects to the single object in $\mathbb{T}$, and a morphism $([\sigma,t],g)$ to $e^{2\pi it}$ in $\mathbb{T}$.

The quasi-elliptic cohomology $QEll^*_G(X)$ is defined to be $K^*_{orb}(\Lambda(X//G))$. We can unravel the definition and express it via equivariant K-theory.

Let X be a $G-$space. Let $G^{tors} \subseteq G$ be the set of torsion elements of G. Let $\sigma \in G^{tors}$. The fixed point space X^σ is a $C_G(\sigma)-$space. We can define a $\Lambda_G(\sigma)-$action on X^σ by

$$[g, t] \cdot x := g \cdot x.$$

Then quasi-elliptic cohomology is defined by

Definition 2.4.12.

$$QEll^*_G(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g) = \Big(\prod_{g \in G^{tors}} K^*_{\Lambda_G(g)}(X^g) \Big)^G. \qquad (2.34)$$

By computing the representation rings of $R\Lambda_G(g)$, we get $QEll^*_G(-)$ for contractible spaces. Then, using Mayer-Vietoris sequence, we can compute $QEll^*_G(-)$ for any G-CW complex by patching the G-cells together.

We have the ring homomorphism

$$\mathbb{Z}[q^{\pm}] = K^0_{\mathbb{T}}(\mathrm{pt}) \xrightarrow{\pi^*} K^0_{\Lambda_G(g)}(\mathrm{pt}) \longrightarrow K^0_{\Lambda_G(g)}(X)$$

where $\pi : \Lambda_G(g) \longrightarrow \mathbb{T}$ is the projection defined in (2.21) and the second is via the collapsing map $X \longrightarrow \mathrm{pt}$. So $QEll^*_G(X)$ is naturally a $\mathbb{Z}[q^{\pm}]-$algebra.

The λ-ring structure on $QEll^*_G(X)$ is the direct product of the exterior algebra on each equivariant K-group, with componentwise multiplication.

For each $QEll^*_G$ we can equip a special family of $\lambda-$ring homomorphisms

$$\mu^n : QEll^*_G(X) \cong \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g) \longrightarrow \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_n(g)}(X^g) \cong QEll^*_G(X)[q^{\pm\frac{1}{n}}]$$

defined by

$$QEll^*_G(X) \longrightarrow K^*_{\Lambda_G(g^n)}(X^{g^n}) \xrightarrow{\beta^*} K^*_{\Lambda_n(g)}(X^{g^n}) \longrightarrow K^*_{\Lambda_n(g)}(X^g), \qquad (2.35)$$

where the first map is projection, the second is the restriction via the inclusion $\beta : \Lambda_n(g) \longrightarrow \Lambda_G(g^n)$, and the third is restriction along $X^g \subseteq X^{g^n}$.

In addition, we can express the $\lambda-$ring isomorphism (2.31) and this family of $\lambda-$ring homomorphisms $\{\mu^n\}_n$ in terms of orbifold K-theory, which are fairly neat.

Let $\Lambda_n(g,g')$ denote the quotient of $C_G(g,g')\times\mathbb{R}$ under the action of $\mathbb{Z}$ where the action of the generator of $\mathbb{Z}$ is given by $(\sigma,t)\mapsto(\sigma g^n,t-1)=(g^n\sigma',t-1)$. Then we can define another groupoid $\Lambda_n(X//G)$ with the same objects as $\Lambda(X//G)$ and morphisms

$$\coprod_{g,g'\in G^{tors}}\Lambda_n(g,g')\times X^g$$

such that for each $x\in X^g$, $([\sigma,t],g)(x)=\sigma x\in X^{g'}$. We can also define $\pi:\Lambda_n(X//G)\longrightarrow\mathbb{T}$ sending all the objects to the single object in $\mathbb{T}$ and a morphism $([\sigma,t],g)$ to the morphism $e^{2\pi it}$ in $\mathbb{T}$.

Let $\alpha:\Lambda_n(X//G)\longrightarrow\Lambda(X//G)$ be the homomorphism of orbifolds sending an object $x\in X^g$ to $x\in X^g$ and a morphism $[\sigma,t]:x\longrightarrow x'$ to $[\sigma,nt]:x\longrightarrow x'$.Let $\beta:\Lambda_n(X//G)\longrightarrow\Lambda(X//G)$ be the functor sending an object $x\in X^g$ to $x\in X^{g^n}$ and a morphism $[\sigma,t]:x\longrightarrow x'$ to $[\sigma,t]:x\longrightarrow x'$.

Since we have the pullback square of groups (2.29) and the pushout square of groups (2.30), we have the pullback square of groupoids

$$\begin{array}{ccc}\Lambda_n(X//G) & \xrightarrow{\alpha} & \Lambda(X//G)\\ \downarrow & & \downarrow\\ \mathbb{T} & \xrightarrow{e^{2\pi it}\mapsto e^{2\pi nit}} & \mathbb{T},\end{array}\tag{2.36}$$

and the pushout square in the category of Λ-rings

$$\begin{array}{ccc}K^*_{orb}(\Lambda_n(X//G)) & \xleftarrow{\alpha^*} & K^*_{orb}(\Lambda(X//G))\\ \uparrow & & \uparrow\\ K^*_{orb}(*//\mathbb{T}) & \longleftarrow & K^*_{orb}(*//\mathbb{T}).\end{array}\tag{2.37}$$

It induces a natural isomorphism

$$K^*_{orb}(\Lambda(X//G))[q^{\pm\frac{1}{n}}]\xrightarrow{\sim}K^*_{orb}(\Lambda_n(X//G)).\tag{2.38}$$

The λ-ring homorphism μ^n can be constructed by

$$\mu^n:K^*_{orb}(\Lambda(X//G))\xrightarrow{\beta^*}K^*_{orb}(\Lambda_n(X//G))\cong K^*_{orb}(\Lambda(X//G))[q^{\pm\frac{1}{n}}].$$

2.4.3 Properties

$QEll_G^*$ inherits most properties of equivariant K-theory. In this section I discuss some properties of $QEll_G^*$, including the restriction map, the Künneth map on it, its tensor product and the change of group isomorphism.

Since each homomorphism $\phi : G \longrightarrow H$ induces a well-defined homomorphism $\phi_\Lambda : \Lambda_G(\tau) \longrightarrow \Lambda_H(\phi(\tau))$ for each τ in G^{tors}, we can get the proposition below directly.

Proposition 2.4.13. *For each homomorphism $\phi : G \longrightarrow H$, it induces a ring map*

$$\phi^* : QEll_H^*(X) \longrightarrow QEll_G^*(\phi^* X)$$

characterized by the commutative diagrams

$$\begin{array}{ccc} QEll_H^*(X) & \xrightarrow{\phi^*} & QEll_G^*(\phi^* X) \\ {\scriptstyle \pi_{\phi(\tau)}}\big\downarrow & & {\scriptstyle \pi_\tau}\big\downarrow \\ K^*_{\Lambda_H(\phi(\tau))}(X^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_G(\tau)}(X^{\phi(\tau)}) \end{array} \tag{2.39}$$

for any $\tau \in G^{tors}$. So $QEll_G^*$ is functorial in G.

More generally, we have

Proposition 2.4.14. *For any groupoid homomorphism $\phi : X//G \longrightarrow Y//H$, we have the groupoid homomorphism $\Lambda(\phi) : \Lambda(X//G) \longrightarrow \Lambda(Y//H)$ sending an object (x, g) to $(\phi(x), \phi(g))$, and a morphism $([\sigma, t], g)$ to $([\phi(\sigma), t], \phi(g))$. Thus, we get a ring map*

$$\phi^* : QEll^*(Y//H) \longrightarrow QEll^*(X//G)$$

characterized by the commutative diagrams

$$\begin{array}{ccc} QEll^*(Y//H) & \xrightarrow{\phi^*} & QEll^*(X//G) \\ {\scriptstyle \pi_{\phi(\tau)}}\big\downarrow & & {\scriptstyle \pi_\tau}\big\downarrow \\ K^*_{\Lambda_H(\phi(\tau))}(Y^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_G(\tau)}(X^{\tau}) \end{array} \tag{2.40}$$

for any $\tau \in G^{tors}$.

And, we can define Künneth map on $QEll_G^*$ induced from that on equivariant K-theory.

Let G and H be two compact Lie groups. X is a G-space and Y is a H-space. Let $\sigma \in G^{tors}$ and $\tau \in H^{tors}$. Let $\Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$ denote the fibered product of the morphisms

$$\Lambda_G(\sigma) \xrightarrow{\pi} \mathbb{T} \xleftarrow{\pi} \Lambda_H(\tau).$$

It's isomorphic to $\Lambda_{G\times H}(\sigma, \tau)$ under the correspondence

$$([\alpha, t], [\beta, t]) \mapsto [\alpha, \beta, t].$$

Consider the map below

$$T : K_{\Lambda_G(\sigma)}(X^\sigma) \otimes K_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_G(\sigma)\times\Lambda_H(\tau)}(X^\sigma \times Y^\tau) \xrightarrow{res} K_{\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)}(X^\sigma \times Y^\tau)$$
$$\xrightarrow{\cong} K_{\Lambda_{G\times H}(\sigma,\tau)}((X \times Y)^{(\sigma,\tau)}).$$

where the first map is the Künneth map of equivariant K-theory, the second is the restriction map and the third is the isomorphism induced by the group isomorphism $\Lambda_{G\times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$.

For any compact Lie group G and $g \in G^{tors}$, let 1 denote the trivial line bundle over X^g and let q denote the line bundle $1 \odot q$ over X^g. The map T above sends both $1 \otimes q$ and $q \otimes 1$ to q. So we get the well-defined map

$$K^*_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K^*_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_{G\times H}(\sigma,\tau)}((X \times Y)^{(\sigma,\tau)}). \tag{2.41}$$

(2.41) gives the Künneth map of quasi-elliptic cohomology.

Thus, we can define

$$QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) := \prod_{\sigma\in G^{tors}_{conj}, \tau\in H^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K^*_{\Lambda_H(\tau)}(Y^\tau). \tag{2.42}$$

and

$$QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) \longrightarrow QEll^*_{G\times H}(X \times Y),$$

which is the direct product of those maps defined in (2.41).

From this definition, we have

$$QEll_G^*(\mathrm{pt})\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}QEll_H^*(\mathrm{pt}) = QEll_{G\times H}^*(\mathrm{pt}).$$

Proposition 2.4.15. *Let G and H be two compact Lie groups. Let X be a $G\times H-$space with trivial $H-$action and let pt be the single point space with trivial $H-$action. Then we have*

$$QEll_{G\times H}(X) \cong QEll_G(X)\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}QEll_H(pt).$$

Especially, if G acts trivially on X, we have

$$QEll_G(X) \cong QEll(X)\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}QEll_G(pt).$$

Here $QEll^*(X)$ is $QEll^*_{\{e\}}(X) = K^*_{\mathbb{T}}(X)$.

Proof.

$$\begin{aligned} QEll_{G\times H}(X) &= \prod_{\substack{g\in G^{tors}_{conj}\\ h\in H^{tors}_{conj}}} K_{\Lambda_{G\times H}(g,h)}(X^{(g,h)}) \cong \prod_{\substack{g\in G^{tors}_{conj}\\ h\in H^{tors}_{conj}}} K_{\Lambda_G(g)\times_{\mathbb{T}}\Lambda_H(h)}(X^g) \\ &\cong \prod_{\substack{g\in G^{tors}_{conj}\\ h\in H^{tors}_{conj}}} K_{\Lambda_G(g)}(X^g)\otimes_{\mathbb{Z}[q^{\pm}]} K_{\Lambda_H(h)}(\mathrm{pt}) = QEll_G(X)\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}QEll_H(pt). \end{aligned}$$

□

Proposition 2.4.16. *If G acts freely on X,*

$$QEll_G^*(X) \cong QEll_e^*(X/G).$$

Proof. Since G acts freely on X,

$$X^\sigma = \begin{cases} \emptyset, & \text{if } \sigma \neq e; \\ X, & \text{if } \sigma = e. \end{cases}$$

Thus, $QEll_G^*(X) \cong \prod_{\sigma\in G^{tors}_{conj}} K^*_{\Lambda_G(\sigma)/C_G(\sigma)}(X^\sigma/C_G(\sigma)) \cong K^*_{\mathbb{T}}(X/G)$.

Since $\mathbb{T}$ acts trivially on X, we have $K^*_{\mathbb{T}}(X/G) = QEll_e^*(X/G)$ by definition. And it is isomorphic to $K^*(X/G)\otimes R\mathbb{T}$. □

We also have the change-of-group isomorphism as that in equivariant K-theory.

Let H be a closed subgroup of G and X a H-space. Let $\phi : H \longrightarrow G$ denote the inclusion homomorphism. The change-of-group map $\rho_H^G : QEll_G^*(G \times_H X) \longrightarrow QEll_H^*(X)$ is defined as the composite

$$QEll_G^*(G \times_H X) \xrightarrow{\phi^*} QEll_H^*(G \times_H X) \xrightarrow{i^*} QEll_H^*(X) \tag{2.43}$$

where ϕ^* is the restriction map and $i : X \longrightarrow G \times_H X$ is the $H-$equivariant map defined by $i(x) = [e, x]$.

Proposition 2.4.17. *The change-of-group map*

$$\rho_H^G : QEll_G^*(G \times_H X) \longrightarrow QEll_H^*(X)$$

defined in (2.43) is an isomorphism.

Proof. For any $\tau \in H_{conj}^{tors}$, there exists a unique $\sigma_\tau \in G_{conj}^{tors}$ such that $\tau = g_\tau \sigma_\tau g_\tau^{-1}$ for some $g_\tau \in G$. Consider the maps

$$\Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau \xrightarrow{[[a,t],x] \mapsto [a,x]} (G \times_H X)^\tau \xrightarrow{[u,x] \mapsto [g_\tau^{-1} u, x]} (G \times_H X)^\sigma. \tag{2.44}$$

The first map is $\Lambda_G(\tau)-$equivariant and the second is equivariant with respect to the homomorphism $c_{g_\tau} : \Lambda_G(\sigma) \longrightarrow \Lambda_G(\tau)$ sending $[u, t] \mapsto [g_\tau u g_\tau^{-1}, t]$. Taking a coproduct over all the elements $\tau \in H_{conj}^{tors}$ that are conjugate to $\sigma \in G_{conj}^{tors}$ in G, we get an isomorphism

$$\gamma_\sigma : \coprod_\tau \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau \longrightarrow (G \times_H X)^\sigma$$

which is $\Lambda_G(\sigma)-$equivariant with respect to c_{g_τ}. Then we have the map

$$\gamma := \prod_{\sigma \in G_{conj}^{tors}} \gamma_\sigma : \prod_{\sigma \in G_{conj}^{tors}} K_{\Lambda_G(\sigma)}^*(G \times_H X)^\sigma \longrightarrow \prod_{\sigma \in G_{conj}^{tors}} K_{\Lambda_G(\sigma)}^*(\coprod_\tau \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau) \tag{2.45}$$

It's straightforward to check the change-of-group map coincide with the composite

$$QEll^*_G(G\times_H X)\xrightarrow{\gamma}\prod_{\sigma\in G^{tors}_{conj}}K^*_{\Lambda_G(\sigma)}(\coprod_{\tau}\Lambda_G(\tau)\times_{\Lambda_H(\tau)}X^{\tau})\longrightarrow\prod_{\tau\in H^{tors}_{conj}}K^*_{\Lambda_H(\tau)}(X^{\tau})$$
$$=QEll^*_H(X)$$

with the second map the change-of-group isomorphism in equivariant $K-$theory, introduced in Proposition 2.2.1. □

2.4.4 Formulas for Induction

In this section 2.4.4 I introduce the induction formula for quasi-elliptic cohomology. Before that I'd like to show the induction formula for Tate K-theory that Nora Ganter constructed in [25].

Let $H\subseteq G$ be an inclusion of compact Lie groups and X be a $G-$space. Then we have the inclusion of the groupoids

$$j: X//H\longrightarrow X//G.$$

Let $a=\prod\limits_{\sigma\in H^{tors}_{conj}}a_{\sigma}$ be an element in

$$K_{Tate}(X//H)\subseteq\prod_{\sigma\in H^{tors}_{conj}}K_{C_G(\sigma)}(X^{\sigma})((q^{\frac{1}{|g|}})).$$

Let

$$Ind^G_H: K_H(X)\longrightarrow K_G(X)$$

be the induced transfer of equivariant K-theory defined in (2.18).

As equivariant K-theory, Tate K-theory also has the change of group isomorphism

$$K_{Tate}(X//H)\cong K_{Tate}((G\times_H X)//G). \tag{2.46}$$

Recall $I^{tors}(X//G)$ is the torsion Inertia groupoid defined in Example 2.4.11. We have the isomorphism of the groupoids

$$\phi: I^{tors}(G\times_H X//G))\longrightarrow G\times_H I^{tors}(X//H) \tag{2.47}$$

sending an object $(\sigma, [g, x])$ to $[g, (g^{-1}\sigma g, x)]$ and a morphism $(g', (\sigma, [g, x]))$ to $(g', [g, (g^{-1}\sigma g, x)])$. As indicated by Proposition 2.23 in [25], the finite covering map

$$f : I^{tors}(G \times_H X//G) \xrightarrow{\phi} G \times_H I^{tors}(X//H) \longrightarrow I^{tors}(X//G)$$

where the second functor sends an object $[g, (\sigma, x)]$ to $(g\sigma g^{-1}, gx)$ and a morphism $(g', [g, (\sigma, x)])$ to $(g', (g\sigma g^{-1}, gx))$. The transfer of Tate K-theory

$$I_H^G : K_{Tate}(X//H) \longrightarrow K_{Tate}(X//G)$$

is defined to be the composition

$$K_{Tate}(X//H) \xrightarrow{\cong} K_{Tate}((G \times_H X)//G) \longrightarrow K_{Tate}(X//G) \tag{2.48}$$

where the first map is the change of group isomorphism and the second is the finite covering. The formula for I_H^G is

$$I_H^G(a)_g = \sum_r r \cdot a_{r^{-1}gr}$$

where r goes over a set of representatives of $(G/H)^g$, in other words, $r^{-1}gr$ goes over a set of representatives of conjugacy classes in H conjugate to g in G.

Thus

$$I_H^G(a)_g = \begin{cases} Ind_{C_H}^{C_G}(a_g) & \text{if } g \text{ is conjuate to some element } h \text{ in } H; \\ 0 & \text{if there is no element conjugate to } g \text{ in } H. \end{cases} \tag{2.49}$$

The induced transfer $\mathcal{I}_H^G$ for $QEll$ is constructed in a similar way. Let $a' = \prod\limits_{\sigma \in H_{conj}^{tors}} a'_\sigma$ be an element in

$$QEll(X//H) = \prod_{\sigma \in H_{conj}^{tors}} K_{\Lambda_H(\sigma)}(X^\sigma)$$

where σ goes over all the conjugacy classes in H.

We can define the finite covering map

$$f' : \Lambda(G \times_H X//G) \longrightarrow \Lambda(X//G)$$

sending an object $(\sigma, [g, x])$ to (σ, gx) and a morphism $([g', t], (\sigma, [g, x]))$ to $([g', t], (gx, \sigma))$.

The transfer of quasi-elliptic cohomology

$$\mathcal{I}_H^G : QEll(X//H) \longrightarrow QEll(X//G)$$

is defined to be the composition

$$QEll(X//H) \xrightarrow{\cong} QEll((G \times_H X)//G) \longrightarrow QEll(X//G) \tag{2.50}$$

where the first map is the change of group isomorphism and the second is the finite covering.

Thus

$$\mathcal{I}_H^G(a')_g = \sum_r r \cdot a'_{r^{-1}gr}$$

where r goes over a set of representatives of $(G/H)^g$, in other words, $r^{-1}gr$ goes over a set of representatives of conjugacy classes in H conjugate to g in G.

$$\mathcal{I}_H^G(a')_g = \begin{cases} Ind_{\Lambda_H}^{\Lambda_G}(a'_g) & \text{if } g \text{ is conjuate to some element } h \text{ in } H; \\ 0 & \text{if there is no element conjugate to } g \text{ in } H. \end{cases} \tag{2.51}$$

There is another way to describe the transfer, which is shown in [50] for quasi-elliptic cohomology. The transfer of Tate K-theory can be described similarly.

Chapter 3

Power Operation

In Section 3.2 I define power operations for equivariant quasi-elliptic cohomology $QEll^*_G(-)$. And I show in Theorem 3.2.1 that they satisfies the axioms that Nora Ganter established in her paper [23] for equivariant power operations.

In Section 3.3 I define orbifold quasi-elliptic cohomology in Definition 3.33. In Section 3.4 I define power operation for orbifold quasi-elliptic cohomology, which satisfies the axioms that Nora Ganter formulated in [25] for power operations for orbifold theories.

Strickland showed in [60] that the quotient of the Morava E-theory of the symmetric group by a certain transfer ideal can be identified with the product of rings $\prod\limits_{k\geq 0} R_k$ where each R_k classifies subgroup-schemes of degree p^k in the formal group associated to $E^0\mathbb{C}P^\infty$. In Section 3.5 I prove similar conclusions for quasi-ellitpic cohomology and Tate K-theory, which gives a classification of the finite subgroups of the Tate curve.

3.1 Loop Space of Symmetric Power

Before constructing the power operation

$$\mathbb{P}_n : K_{orb}(\Lambda(X//G)) = QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) = K_{orb}(\Lambda(X^{\times n}//(G\wr\Sigma_n))),$$

to make it explicit I introduce the groupoid $D((X//G)\wr\Sigma_n)$ in Section 3.1.1, which is isomorphic to

$$\mathcal{L}((X//G)\wr\Sigma_n) \cong \coprod_{[\underline{g},\sigma]\in(G\wr\Sigma_n)^{tors}_{conj}} \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma),$$

as shown in Theorem 3.1.11. With it, it's convenient for constructing the explicit formula of the power operation.

3.1.1 The groupoid $D((X//G) \wr \Sigma_n)$

In this section I define the symmetric powers of orbifolds. In the case of a global quotient orbifold $X//G$, the $n-$th symmetric power is $X^{\times n}//(G \wr \Sigma_n)$. The main subject I introduce in Section 3.1.1 is the intermediate groupoid $D((X//G) \wr \Sigma_n)$, which will be shown in Section 3.1.2 isomorphic to the orbifold loop space $\mathcal{L}((X//G) \wr \Sigma_n)$ of the symmetric power.

Before introducing $D((X//G) \wr \Sigma_n)$, for each torsion element $(\underline{g}, \sigma)$ in $G \wr \Sigma_n$ I construct the groupoid $A_{(\underline{g},\sigma)}(X)$ and a subgroupoid $D_{(\underline{g},\sigma)}(X)$ of it. Then by combining all the groupoids $D_{(\underline{g},\sigma)}(X)$ together I construct $D((X//G) \wr \Sigma_n)$.

Let G be a compact Lie group and X a right $G-$space. Recall $X//G$ is the translation groupoid. Let Σ_n denote the $n-$th symmetric group.

Recall that the wreath product $G \wr \Sigma_n$ has elements

$$(\underline{g}, \sigma) \in G^n \times \Sigma_n$$

whose multiplication is defined by

$$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) = (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau).$$

$G \wr \Sigma_n$ acts on $X^{\times n}$ by

$$\underline{x} \cdot (\underline{g}, \sigma) = (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)}), \text{ for any } \underline{x} \in X^{\times n}, \ (\underline{g}, \sigma) \in G \wr \Sigma_n. \tag{3.1}$$

Remark 3.1.1. *Consider the space*

$$\underline{\underline{n}} := \{1, 2, \cdots n\}$$

with discrete topology. The symmetric group Σ_n acts on $\underline{\underline{n}}$ by permutation. For each $(\underline{g}, \sigma) \in G \wr \Sigma_n$, we have the commutative diagram

$$\begin{tikzcd} G \times \underline{\underline{n}} \arrow[r, "(\underline{g},\sigma)"] \arrow[d] & G \times \underline{\underline{n}} \arrow[d] \\ \underline{\underline{n}} \arrow[r, "\sigma"] & \underline{\underline{n}} \end{tikzcd} \tag{3.2}$$

where the vertical maps are the projection to the second factor and $(\underline{g}, \sigma)$ *acts on* $\underline{\underline{n}}$ *by*

$$(x, k) \cdot (\underline{g}, \sigma) := (xg_k, \sigma^{-1}(k)), \text{ for each } (x, k) \in G \wr \Sigma_n.$$

So $G \wr \Sigma_n$ *gives a family of bundle isomorphisms*

$$G \times \underline{\underline{n}} \longrightarrow G \times \underline{\underline{n}}$$

between the $G-$*bundles.*

Generally, for any orbifold groupoid

$$\mathbb{G} = (\mathbb{G}_0, \mathbb{G}_1, s, t, u, i)$$

we can define the $n-$th symmetric power $\mathbb{G} \wr \Sigma_n$ of $\mathbb{G}$.

Definition 3.1.2 (Symmetric Power of $\mathbb{G}$). *The objects of the* $n-$*th symmetric power* $\mathbb{G} \wr \Sigma_n$ *are the points*

$$\underline{x} := (x_1, \cdots x_n)$$

in $\mathbb{G}_0^{\times n}$ *and the arrows of it are* $(\underline{g}, \sigma)$ *with each* g_i *an arrow of* $\mathbb{G}$ *starting at* x_i *and* $\sigma \in \Sigma_n$*.*

$$\underline{x} \cdot (\underline{g}, \sigma) = (x_{\sigma(1)} g_{\sigma(1)}, x_{\sigma(2)} g_{\sigma(2)}, \cdots x_{\sigma(n)} g_{\sigma(n)}).$$

The unit map sends the object $\underline{x}$ *to* $(u(x_1), \cdots u(x_n), Id)$*. The composition of arrows* $(\underline{g}, \sigma)$ *and* $(\underline{h}, \tau)$ *where* $g_{\sigma(i)} = h_i$ *is given by*

$$(\underline{g}, \sigma) \circ (\underline{h}, \tau) = (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}).$$

The inverse of $(\underline{g}, \sigma)$ *is given by*

$$(i(g_{\sigma(1)}), \cdots i(g_{\sigma(n)}), \sigma^{-1}).$$

The symmetric power $(X//G) \wr \Sigma_n$ is isomorphic to the translation groupoid $X^{\times n}//(G \wr \Sigma_n)$. $\mathcal{L}((X//G) \wr \Sigma_n)$ is isomorphic to $\mathcal{L}(X^{\times n}//(G \wr \Sigma_n))$ and as defined in Example 2.1.14

is the coproduct

$$\coprod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma).$$

Let $\sigma \in \Sigma_n$ correspond to the partition $n = \sum\limits_k kN_k$, i.e. it has N_k $k-$cycles. Assume that for each cycle of σ, we have fixed a first element i_1, and thus a representation as $(i_1, \cdots i_k)$.

For $(\underline{g},\sigma) \in G \wr \Sigma_n$, let's consider the orbits of the bundle $G \times \underline{\underline{n}} \longrightarrow \underline{\underline{n}}$ under the action by $(\underline{g},\sigma)$. The orbits of $\underline{\underline{n}}$ under the action by σ corresponds to the cycles in the cycle decomposition of σ. The bundle $G \times \underline{\underline{n}} \longrightarrow \underline{\underline{n}}$ is the disjoint union of the $G-$bundles

$$\bigsqcup_{(i_1\cdots i_k)} (G \times \{i_1, \cdots i_k\} \longrightarrow \{i_1, \cdots i_k\})$$

where $(i_1, \cdots i_k)$ goes over all the cycles of σ. Each bundle $G \times \{i_1, \cdots i_k\} \longrightarrow \{i_1, \cdots i_k\}$ is an orbit of $G \times \underline{\underline{n}} \longrightarrow \underline{\underline{n}}$ under the action by $(\underline{g},\sigma)$.

Two $G-$subbundles

$$G \times \{i_1, \cdots i_k\} \longrightarrow \{i_1, \cdots i_k\} \text{ and } G \times \{j_1, \cdots j_m\} \longrightarrow \{j_1, \cdots j_m\}$$

are $(\underline{g},\sigma)-$equivariant equivalent if and only if $k = m$ and $C_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$ is nonempty. For each k-cycle $i = (i_1, \cdots i_k)$ of σ, let W^σ_i denote the set of all the $G-$subbundles $G\times\{j_1, \cdots j_m\} \longrightarrow \{j_1, \cdots j_m\}$ that are $(\underline{g},\sigma)-$isomorphic to $G\times\{i_1, \cdots i_k\} \longrightarrow \{i_1, \cdots i_k\}$. There is a bijection between W^σ_i and the set

$$\{j = (j_1, \cdots j_k) \mid (j_1, \cdots j_k) \text{ is a k-cycle of } \sigma \text{ and } C_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \text{ is nonempty.}\}.$$

Let M^σ_i denote the size of the set W^σ_i. Let $\alpha^i_1, \alpha^i_2, \cdots \alpha^i_{M^\sigma_i}$ denote all the elements of the set W^σ_i. Obviously, $i = (i_1, \cdots i_k)$ is in W^σ_i. So we can assume α^i_1 is i.

For any $k-$cycle i and $m-$cycle j of σ, if $k \neq m$, W^σ_i and W^σ_j are disjoint. In the case that $k = m$, if $C_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$ is nonempty, W^σ_i are W^σ_j are the same set; otherwise, W^σ_i and W^σ_j are disjoint. The set of all the $k-$cycles of σ can be divided into the disjoint union of several W^σ_is. We can pick a set of representatives θ_k of $k-$cycles of σ

such that the set of $k-$cycles of σ equals the disjoint union

$$\coprod_{i\in\theta_k} W_i^\sigma.$$

Example 3.1.3 (The groupoids $A_{(\underline{g},\sigma)}(X)$ and $D_{(\underline{g},\sigma)}(X)$). *Now let's introduce a groupoid $A_{(\underline{g},\sigma)}(X)$. The objects of it are the points of the space*

$$\prod_k \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X$$

where the second product goes over all the $k-$cycles of σ. The set of morphisms starting from an object x is

$$\{(\times_k \times_{i\in\theta_k} ([a_1^i, t_1^i], \cdots [a^i_{M_i^\sigma}, t^i_{M_i^\sigma}], \tau_i), x) \mid \tau_i \in \Sigma_{M_i^\sigma}, [a_j^i, t_j^i] \in \Lambda_G^k(g_{\tau_i(j)_k}\cdots g_{\tau_i(j)_1}, g_{j_k}\cdots g_{j_1})$$
$$\text{for each } j \in W_i^\sigma.\},$$

where τ_i permutes the cycles $\alpha_1^i, \alpha_2^i, \cdots \alpha^i_{M_i^\sigma}$ in W_i^σ. The composition is defined by

$$[\times_k \times_{i\in\theta_k} ([a_1^i, t_1^i], \cdots [a^i_{M_i^\sigma}, t^i_{M_i^\sigma}], \tau_i)] \circ [\times_k \times_{i\in\theta_k} ([b_1^i, p_1^i], \cdots [b^i_{M_i^\sigma}, p^i_{M_i^\sigma}], \varrho_i)] \qquad (3.3)$$
$$= \times_k \times_{i\in\theta_k}([a_1^i b^i_{\tau_i^{-1}(1)}, t_1^i + p^i_{\tau_i^{-1}(1)}], \cdots [a^i_{M_i^\sigma} b^i_{\tau_i^{-1}(M_i^\sigma)}, t^i_{M_i^\sigma} + p^i_{\tau_i^{-1}(M_i^\sigma)}], \tau_i\varrho_i) \qquad (3.4)$$

And for any $\times_k \times_{i\in\theta_k} (\gamma_{i,1}, \cdots \gamma_{i,M_i^\sigma}) \in \prod_k \prod_{(i_1,\cdots i_k)\in\theta_k} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X$,

$$[\times_k \times_{i\in\theta_k} (\gamma_{i,1}, \cdots \gamma_{i,M_i^\sigma})] \cdot [\times_k \times_{i\in\theta_k} ([a_1^i, t_1^i], \cdots [a^i_{M_i^\sigma}, t^i_{M_i^\sigma}], \tau_i)]$$
$$= \times_k \times_{i\in\theta_k}(\gamma_{i,\tau_i(1)}[a^i_{\tau_i(1)}, t^i_{\tau_i(1)}], \cdots \gamma_{i,\tau_i(M_i^\sigma)}[a^i_{\tau_i(M_i^\sigma)}, t^i_{\tau_i(M_i^\sigma)}]).$$

And as shown in (2.11), each

$$(\gamma_{i,\tau_i(j)} \cdot [a^i_{\tau_i(j)}, t^i_{\tau_i(j)}])(t) = (\gamma_{i,\tau_i(j)} \cdot a^i_{\tau_i(j)})(t + t^i_{\tau_i(j)}).$$

Let $D_{(\underline{g},\sigma)}(X)$ denote the subgroupoid of $A_{(\underline{g},\sigma)}(X)$ whose objects are the same as $A_{(\underline{g},\sigma)}(X)$, and whose morphisms are those of $A_{(\underline{g},\sigma)}(X)$ with all the t_j^is having the same image under the quotient map $\mathbb{R}/k\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$, for each k, each $k-$cycle i, and each $j = 1, \cdots M_i^\sigma$.

Combining all the groupoids $D_{(\underline{g},\sigma)}(X)$, we can define the groupoid

$$D((X//G)\wr\Sigma_n).$$

Before that we need to recall some properties of $C_{G\wr\Sigma_n}((\underline{g},\sigma),(\underline{g}',\sigma'))$.

$(\underline{h},\tau)$ is in $C_{G\wr\Sigma_n}((\underline{g},\sigma),(\underline{g}',\sigma'))$ if and only if $\tau\sigma'=\sigma\tau$ and $g_{\sigma(\tau(i))}h_{\tau(i)}=h_{\tau(\sigma'(i))}g'_{\sigma'(i)}$, $\forall i$.

We can reinterpret the two conditions above. Since $\tau\in C_{\Sigma_n}(\sigma,\sigma')$, τ maps a k-cycle $i=(i_1,\cdots i_k)$ of σ' to a k-cycle $j=(j_1,\cdots j_k)$ of σ. τ will still used to denote its map on the cycles, such as $\tau(r)=s$. For each $l\in\mathbb{Z}/k\mathbb{Z}$, let $\tau(i_l)=j_{l+m_i}$ where m_i depends only on τ and the cycle i. Then, the second conditions can be expressed as

$$\forall l\in\mathbb{Z}/k\mathbb{Z}, g_{j_l}h_{j_{l-1}}=h_{j_l}g'_{i_{l-m_i}}. \tag{3.5}$$

From this equivalence, we can induce that the element

$$h_{j_k}g'^{-1}_{i_{1-m_i}}\cdots g'^{-1}_{i_{k-1}}g'^{-1}_{i_k}=g^{-1}_{j_1}\cdots g^{-1}_{j_{m_i}}h_{j_{m_i}}$$

maps $g_{j_k}\cdots g_{j_1}$ to $g'_{i_k}\cdots g'_{i_1}$ by conjugation. In other words,

$$\beta^{\underline{h},\tau}_{j,i}:=h_{j_k}g'^{-1}_{i_{1-m_i}}\cdots g'^{-1}_{i_{k-1}}g'^{-1}_{i_k} \tag{3.6}$$

is an element in $C_G(g_{j_k}\cdots g_{j_1},g'_{i_k}\cdots g'_{i_1})$. Thus, $C_G(g_{j_k}\cdots g_{j_1},g'_{i_k}\cdots g'_{i_1})$ is nonempty.

Definition 3.1.4 ($D((X//G)\wr\Sigma_n)$). *Let $D((X//G)\wr\Sigma_n)$ denote the groupoid with objects the points of the space*

$$\coprod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}}\prod_k\prod_{(i_1,\cdots i_k)}{}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}}X$$

where $(i_1,\cdots i_k)$ goes over all the $k-$cycles of σ. The set of morphisms starting from an object

$$x\in\prod_k\prod_{(j_1,\cdots j_k)}{}_k\mathcal{L}_{g_{j_k}\cdots g_{j_1}}X$$

to an object in the component $\coprod_k \coprod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g'_{i_k}\cdots g'_{i_1}} X$ *corresponding to* $(\underline{g}', \sigma')$ *is*

$$\{(\times_k \times_{j\in\theta_k}([a^j_1, t^j_1], \cdots [a^j_{M^\sigma_j}, t^j_{M^\sigma_j}], \tau_j), x) \mid \tau_j \in \Sigma_{M^\sigma_j},$$
$$[a^j_\rho, t^j_\rho] \in \Lambda^k_G(g'_{\tau_j(\rho)_k} \cdots g'_{\tau_j(\rho)_1}, g_{\rho_k} \cdots g_{\rho_1}) \text{ for each } \rho \in W^\sigma_j;$$
$$\text{all the } t^j_\rho\text{s have the same image under the quotient map } \mathbb{R}/k\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}.\},$$

where τ_j *permutes the cycles* $\alpha^j_1, \alpha^j_2, \cdots \alpha^j_{M^\sigma_j}$ *in* W^σ_j.

The formula for the composition is

$$[\times_k \times_{i\in\theta_k} ([a^i_1, t^i_1], \cdots [a^i_{M^\sigma_i}, t^i_{M^\sigma_i}], \tau_i)] \circ [\times_k \times_{i\in\theta_k} ([b^i_1, p^i_1], \cdots [b^i_{M^\sigma_i}, p^i_{M^\sigma_i}], \varrho_i)] \tag{3.7}$$
$$= \times_k \times_{i\in\theta_k}([a^i_1 b^i_{\tau_i^{-1}(1)}, t^i_1 + p^i_{\tau_i^{-1}(1)}], \cdots [a^i_{M^\sigma_i} b^i_{\tau_i^{-1}(M^\sigma_i)}, t^i_{M^\sigma_i} + p^i_{\tau_i^{-1}(M^\sigma_i)}], \tau_i\varrho_i) \tag{3.8}$$

And for any $\times_k \times_{i\in\theta_k} (\gamma_{i,1}, \cdots \gamma_{i,M^\sigma_i}) \in \prod_k \prod_{(i_1,\cdots i_k)\in\theta_k} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X$,

$$[\times_k \times_{i\in\theta_k} (\gamma_{i,1}, \cdots \gamma_{i,M^\sigma_i})] \cdot [\times_k \times_{i\in\theta_k} ([a^i_1, t^i_1], \cdots [a^i_{M^\sigma_i}, t^i_{M^\sigma_i}], \tau_i)]$$
$$= \times_k \times_{i\in\theta_k}(\gamma_{i,\tau_i(1)}[a^i_{\tau_i(1)}, t^i_{\tau_i(1)}], \cdots \gamma_{i,\tau_i(M^\sigma_i)}[a^i_{\tau_i(M^\sigma_i)}, t^i_{\tau_i(M^\sigma_i)}]).$$

And as in (2.11), each

$$(\gamma_{i,\tau_i(j)} \cdot [a^i_{\tau_i(j)}, t^i_{\tau_i(j)}])(t) = (\gamma_{i,\tau_i(j)} \cdot a^i_{\tau_i(j)})(t + t^i_{\tau_i(j)}).$$

A skeleton for $D((X//G) \wr \Sigma_n)$ is a subgroupoid of the disjoint union

$$\coprod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} D_{(\underline{g},\sigma)}(X)$$

whose morphisms consists of those with all the rotation t^i_js having the same image under the quotient map $\mathbb{R}/k\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$.

Proposition 3.1.5 shows the groupoid $A_{(\underline{g},\sigma)}(X)$ is a translation groupoid. As a corollary, $D_{(\underline{g},\sigma)}(X)$ is also a translation groupoid. Thus, to study $K_{orb}(A_{(\underline{g},\sigma)}(X))$, $K_{orb}(D_{(\underline{g},\sigma)}(X))$ and $K_{orb}(D((X//G) \wr \Sigma_n))$, we can start by studying the representation ring of a group.

Proposition 3.1.5. *Each $A_{(\underline{g},\sigma)}(X)$ is isomorphic to the translation groupoid*

$$(\prod_k \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X)//(\prod_k \prod_{i\in\theta_k} \Lambda^k_G(\Gamma_{i,1}) \wr \Sigma_{M^\sigma_i})$$

where $\Gamma_{i,p}$ is defined in (3.9).

Proof. First let's see some facts about the centralizers.

To simplify the symbol, let $\Gamma_{i,p}$ denote the element

$$g_{\alpha^i_{p_k}} \cdots g_{\alpha^i_{p_1}}, \tag{3.9}$$

where $\alpha^i_p = ((\alpha^i_p)_1, \cdots (\alpha^i_p)_k)$ is an element in W^σ_i and $p = 1, 2 \cdots M^\sigma_i$. Assume the representative $i = (i_1, \cdots i_k)$ of W^σ_i is α^k_1.

We have a 1-1 correspondence between $C_G(u_1)$ and $C_G(u_1, u_2)$ for any $u_1, u_2 \in G$ if $C_G(u_1, u_2)$ is nonempty. With a representative $\delta \in C_G(u_1, u_2)$ fixed, the correspondence is

$$\begin{array}{ccc} C_G(u_1) & \longrightarrow & C_G(u_2, u_1) \\ h & \mapsto & \delta h. \end{array}$$

And we can define a map

$$\begin{array}{ccc} C_G(u_2, u_1) \times C_G(u_3, u_2) & \longrightarrow & C_G(u_3, u_1) \\ (\delta_1, \delta_2) & \mapsto & \delta_2\delta_1. \end{array}$$

Let's fix representatives for $C_G(\Gamma_{i,t}, \Gamma_{i,s})$ in a compatible way. First fix representatives $\sigma_{i,2,1} \in C_G(\Gamma_{i,2}, \Gamma_{i,1})$, $\sigma_{i,3,2} \in C_G(\Gamma_{i,3}, \Gamma_{i,2})$, $\cdots$, $\sigma_{i,M^\sigma_i,M^\sigma_i-1} \in C_G(\Gamma_{i,M^\sigma_i}, \Gamma_{i,M^\sigma_i-1})$. And get all the other representatives by inverting a representative or multiplying several of them. Then we are ready to get a map

$$\begin{array}{ccccc} C_G(\Gamma_{i,j}, \Gamma_{i,s}) & \longrightarrow & C_G(\Gamma_{i,s}) & \longrightarrow & C_G(\Gamma_{i,1}) \\ h & \mapsto & \sigma^{-1}_{i,j,s} h & \mapsto & \sigma^{-1}_{i,s,1}(\sigma^{-1}_{i,j,s} h)\sigma_{i,s,1}, \end{array}$$

where $\sigma^{-1}_{i,s,1}(\sigma^{-1}_{i,j,s} h)\sigma_{i,s,1} = \sigma_{i,1,s}\sigma_{i,s,j} h \sigma_{i,s,1} = \sigma_{i,1,j} h \sigma_{i,s,1}$.

Let's consider the group

$$\prod_k \prod_{i\in\theta_k} \Lambda^k_G(\Gamma_{i,1}) \wr \Sigma_{M^\sigma_i}.$$

For each $[b^i_j, t^i_j] \in \Lambda^k_G(\Gamma_{i,1})$, the product $[\sigma_{i,\tau(j),1}, 0][b^i_j, t^i_j][\sigma_{i,1,j}, 0] \in \Lambda^k_G(\Gamma_{i,\tau(j)}, \Gamma_{i,j})$. It acts on any $\times_k \times_{i\in\theta_k} (\gamma_{i,1}, \cdots \gamma_{i,M^\sigma_i}) \in \prod\limits_k \prod\limits_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X$ by

$$\begin{aligned}
&(\times_k \times_{i\in\theta_k} (\gamma_{i,1}, \cdots \gamma_{i,M^\sigma_i})) \cdot (\times_k \times_{i\in\theta_k} ([b^i_1, t^i_1], \cdots [b^i_{M^\sigma_i}, t^i_{M^\sigma_i}], \tau_i)) \\
:=&(\times_k \times_{i\in\theta_k} (\gamma_{i,1}, \cdots \gamma_{i,M^\sigma_i})) \cdot \\
&(\times_k \times_{i\in\theta_k} ([\sigma_{i,\tau(1),1}, 0][b^i_1, t^i_1][\sigma_{i,1,1}, 0], \cdots [\sigma_{i,\tau(M^\sigma_i),1}, 0][b^i_{M^\sigma_i}, t^i_{M^\sigma_i}][\sigma_{i,1,M^\sigma_i}, 0], \tau_i)) \\
=& \times_k \times_{i\in\theta_k} (\gamma_{i,\tau_i(1)}([\sigma_{i,\tau(1),1}, 0][b^i_1, t^i_1][\sigma_{i,1,1}, 0]), \cdots \gamma_{i,\tau_i(M^\sigma_i)}([\sigma_{i,\tau(M^\sigma_i),1}, 0][b^i_{M^\sigma_i}, t^i_{M^\sigma_i}][\sigma_{i,1,M^\sigma_i}, 0])).
\end{aligned}$$

It's quite straightforward to check that this is a well-defined group action and the translation groupoid associated to this action is isomorphic to $A_{(\underline{g},\sigma)}(X)$.

□

And we have the corollary below.

Corollary 3.1.6. *Let*

$$\prod_k {}_{\mathbb{T}} \prod_{i\in\theta_k} {}_{\mathbb{T}}\Lambda^k_G(\Gamma_{i,1}) \wr_{\mathbb{T}} \Sigma_{M^\sigma_i}$$

denote the subgroup of $\prod\limits_k \prod\limits_{i\in\theta_k} \Lambda^k_G(\Gamma_{i,1}) \wr \Sigma_{M^\sigma_i}$ *consisting of those elements*

$$(\times_k \times_{i\in\theta_k} ([b^i_1, t^i_1], \cdots [b^i_{M^\sigma_i}, t^i_{M^\sigma_i}], \tau_i))$$

with all the t^i_j*s having the same image under the quotient map* $\mathbb{R}/k\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$*, for each* k*, each* $k-$*cycle* i*, and each* $j = 1, \cdots M^\sigma_i$*.* $D_{(\underline{g},\sigma)}(X)$ *is the translation groupoid*

$$(\prod_k \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X)//(\prod_k {}_{\mathbb{T}} \prod_{i\in\theta_k} {}_{\mathbb{T}}\Lambda^k_G(\Gamma_{i,1}) \wr_{\mathbb{T}} \Sigma_{M^\sigma_i}).$$

To study $K_{orb}(A_{(\underline{g},\sigma)}(X))$ and $K_{orb}(D_{(\underline{g},\sigma)}(X))$, we need to study the representation ring of wreath product. Theorem 3.1.7 below gives all the irreducible representations of a wreath product. It's a conclusion from [34].

Theorem 3.1.7. *Let* $\{\rho_k\}^N_1$ *be a complete family of irreducible representations of* G *and let* V_k *be the corresponding representation space for* ρ_k*. Let* (n) *be a partition of* n*.* $(n) = (n_1, \cdots n_N)$*. Let* $D_{(n)}$ *be the representation*

$$\rho_1^{\otimes n_1} \otimes \cdots \otimes \rho_N^{\otimes n_N}$$

on the space $V_1^{\otimes n_1} \otimes \cdots \otimes V_N^{\otimes n_N}$. *Let* $\Sigma_{(n)} = \Sigma_{n_1} \times \cdots \times \Sigma_{n_N}$.

Let $(D_{(n)})^{\sim}$ *be the extension of* $D_{(n)}$ *from* $G^{\times n}$ *to* $G \wr \Sigma_{(n)}$ *defined by*

$$(D_{(n)})^{\sim}((g_{1,1}, \cdots g_{1,n_1}, \cdots g_{N,1}, \cdots g_{N,n_N}; \sigma))(v_{1,1} \otimes \cdots \otimes v_{1,n_1} \otimes \cdots \otimes v_{N,1} \otimes \cdots \otimes v_{N,n_N}) \tag{3.10}$$

$$= \bigotimes_{k=1}^{N} \rho_k(g_{k,1}) v_{k,\sigma_k^{-1}(1)} \otimes \cdots \otimes \rho_k(g_{k,n_k}) v_{k,\sigma_k^{-1}(n_k)}, \tag{3.11}$$

where $\sigma = \sigma_1 \times \cdots \times \sigma_N$ *with each* $\sigma_k \in \Sigma_{n_k}$.

Let D_τ *with* $\tau \in R\Sigma_{(n)}$ *be the representation of* $G \wr \Sigma_{(n)}$ *by*

$$D_\tau((g_{1,n_1}, \cdots g_{N,n_N}; \sigma)) := \tau(\sigma). \tag{3.12}$$

Then,

$$\{Ind|_{G\wr\Sigma_{(n)}}^{G\wr\Sigma_n} (D_{(n)})^{\sim} \otimes D_\tau | (n) = (n_1, \cdots n_N) \text{ goes over all the partition;}$$
$$\tau \text{ goes over all the irreducible representations of } \Sigma_{(n)}.\}$$

goes over all the irreducible representation nonrepeatedly of $G \wr \Sigma_n$.

And we can get all the irreducible representations of the fibred wreath product

$$\Lambda_G(\sigma) \wr_{\mathbb{T}} \Sigma_N$$

in a similar way.

Note that

$$\{\rho_1 \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} \rho_n \mid \text{Each } \rho_j \text{ is an irreducible representation of } \Lambda_G(\sigma).\}$$

goes over all the irreducible representation of

$$\Lambda_G(\sigma) \times_{\mathbb{T}} \cdots \times_{\mathbb{T}} \Lambda_G(\sigma).$$

The proof of Theorem 3.1.8 is analogous to that of Theorem 3.1.7 in [34], applying Clifford's theory in [15] and [16].

Theorem 3.1.8. *Let $\{\rho_k\}_1^N$ be a basis of the $\mathbb{Z}[q^{\pm}]-$module $R\Lambda_G(\sigma)$ and let V_k be the corresponding representation space for ρ_k. Let (n) be a partition of n. $(n) = (n_1, \cdots n_N)$. Let $D^{\mathbb{T}}_{(n)}$ be the $\Lambda_G(\sigma)^{\times_{\mathbb{T}} n}-$representation*

$$\rho_1^{\otimes_{\mathbb{Z}[q^{\pm}]} n_1} \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} \rho_N^{\otimes_{\mathbb{Z}[q^{\pm}]} n_N}$$

on the space $V_1^{\otimes n_1} \otimes \cdots \otimes V_N^{\otimes n_N}$. Let $\Sigma_{(n)} = \Sigma_{n_1} \times \cdots \times \Sigma_{n_N}$.

Let $(D^{\mathbb{T}}_{(n)})^{\sim}$ be the extension of $D_{(n)}$ from $\Lambda_G(\sigma)^{\times_{\mathbb{T}} n}$ to $\Lambda_G(\sigma) \wr_{\mathbb{T}} \Sigma_{(n)}$ defined by

$$(D^{\mathbb{T}}_{(n)})^{\sim}(([g_{1,1}, t], \cdots [g_{1,n_1}, t], \cdots [g_{N,1}, t], \cdots [g_{N,n_N}, t]; \sigma))(v_{1,1} \otimes \cdots \otimes v_{1,n_1} \otimes \cdots \otimes v_{N,1} \otimes \cdots \otimes v_{N,n_N}) \tag{3.13}$$

$$= \bigotimes_{\mathbb{Z}[q^{\pm}]} \rho_k([g_{k,1}, t]) v_{k,\sigma_k^{-1}(1)} \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} \rho_k([g_{k,n_k}, t]) v_{k,\sigma_k^{-1}(n_k)}, \tag{3.14}$$

where k is from 1 to N and $\sigma = \sigma_1 \times \cdots \times \sigma_N$ with each $\sigma_k \in \Sigma_{n_k}$.

Let $D^{\mathbb{T}}_{\tau}$ with $\tau \in R\Sigma_{(n)}$ be the representation of $\Lambda_G(\sigma) \wr_{\mathbb{T}} \Sigma_{(n)}$ by

$$D^{\mathbb{T}}_{\tau}(([g_{1,n_1}, t], \cdots [g_{N,n_N}, t]; \sigma)) := \tau(\sigma). \tag{3.15}$$

Then,

$$\{Ind|^{\Lambda_G(\sigma)\wr_{\mathbb{T}}\Sigma_n}_{\Lambda_G(\sigma)\wr_{\mathbb{T}}\Sigma_{(n)}} (D^{\mathbb{T}}_{(n)})^{\sim} \otimes D^{\mathbb{T}}_{\tau} \mid (n) = (n_1, \cdots n_N) \text{ goes over all the partition;}$$
$$\tau \text{ goes over all the irreducible representations of } \Sigma_{(n)}.\}$$

goes over all the irreducible representation nonrepeatedly of $\Lambda_G(\sigma) \wr_{\mathbb{T}} \Sigma_n$.

From Theorem 3.1.7, the representation ring of each $\Lambda^k_G(\Gamma_{i,1}) \wr \Sigma_{M_i^{\sigma}}$ is a $\mathbb{Z}[q^{\pm\frac{1}{k}}]-$module. Thus, the representation ring of

$$\prod_k \prod_{i \in \theta_k} \Lambda^k_G(\Gamma_{i,1}) \wr \Sigma_{M_i^{\sigma}},$$

which is the tensor product of the representation rings of each $\Lambda^k_G(\Gamma_{i,1}) \wr \Sigma_{M_i^{\sigma}}$, is a $\mathbb{Z}[q^{\pm}]-$module via the map

$$\mathbb{Z}[q^{\pm}] \longrightarrow \mathbb{Z}[q^{\pm\frac{1}{k}}],\ q \mapsto q^{\pm\frac{1}{k}}.$$

Thus, the representation ring

$$R(\prod_k \prod_{i\in\theta_k} \Lambda^k_G(\Gamma_{i,1}) \wr \Sigma_{M_i^\sigma}) \cong \bigotimes_k \bigotimes_{i\in\theta_k} R(\Lambda^k_G(\Gamma_{i,1}) \wr \Sigma_{M_i^\sigma})$$

is a $\mathbb{Z}[q^\pm]-$module.

Moreover, $K_{orb}(D_{(\underline{g},\sigma)}(X))$ is a $\mathbb{Z}[q^\pm]-$module via the map

$$R(\prod_k \prod_{i\in\theta_k} \Lambda^k_G(\Gamma_{i,1}) \wr \Sigma_{M_i^\sigma}) \cong K^0_{orb}(A_{(\underline{g},\sigma)}(\mathrm{pt})) \longrightarrow K^0_{orb}(A_{(\underline{g},\sigma)}(X)) \longrightarrow K^*_{orb}(D_{(\underline{g},\sigma)}(X)), \tag{3.16}$$

where the second map is induced by $X \longrightarrow \mathrm{pt}$ and the third one is the restriction.

3.1.2 Loop Space of Symmetric powers

In this section I show that the groupoid $\mathcal{L}((X//G) \wr \Sigma_n)$ is isomorphic to $D((X//G) \wr \Sigma_n)$ defined in Definition 3.1.4, as shown in Theorem 3.1.11.

First I show each component $\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$ is isomorphic to the groupoid $D_{(\underline{g},\sigma)}(X)$.

Theorem 3.1.9. *The two groupoids $\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$ and $D_{(\underline{g},\sigma)}(X)$ are isomorphic. Thus, this isomorphism induces a $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)-$action on the space*

$$\prod_k \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X.$$

Remark 3.1.10. *In [24] where Nora Ganter studied the stringy power operation of Devoto's equivariant Tate K-theory, she proved a conclusion analogous to Theorem 3.1.9 without considering the circle action on the space. I show in Theorem 3.1.9 the homeomorphism between the spaces $\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$ and*

$$\prod_k \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X$$

is not only $C_{G\wr\Sigma_n}(\underline{g},\sigma)-$equivariant but also $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)-$equivariant, with the equipped S^1-action.

Proof. I'm going to construct two functors

$$F_{(\underline{g},\sigma)} : \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G \wr \Sigma_n}(\underline{g},\sigma) \longrightarrow D_{(\underline{g},\sigma)}(X)$$

and

$$J_{(\underline{g},\sigma)} : D_{(\underline{g},\sigma)}(X) \longrightarrow \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G \wr \Sigma_n}(\underline{g},\sigma).$$

$F_{(\underline{g},\sigma)}$ sends a path

$$\gamma = (\gamma_1, \cdots \gamma_n) \in \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$$

to the product of $\sum\limits_k N_k$ paths,

$$\prod_k \prod_{(i_1,\cdots i_k)} \gamma_{i_k} * \gamma_{i_1} g_{i_1} * \cdots * \gamma_{i_{k-1}} g_{i_{k-1}} \cdots g_{i_1},$$

which is an object of $D_{(\underline{g},\sigma)}(X)$.

A morphism in $\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G \wr \Sigma_n}(\underline{g},\sigma)$ is of the form $[(\underline{h},\tau),t]$ with $(\underline{h},\tau) \in C_{G \wr \Sigma_n}(\underline{g},\sigma)$ and $t \in \mathbb{R}$.

Let τ send the $k-$cycle $i = (i_1, \cdots i_k)$ of σ to a $k-$cycle $j = (j_1, \cdots j_k)$ of σ and $\tau(i_1) = j_{1+m_i}$. According to the definition above, the path of $F_{(\underline{g},\sigma)}(\gamma \cdot [(\underline{h},\tau),t_0])$ in ${}_k\mathcal{L}_{g_{i_k} \cdots g_{i_1}} X$ is

$$(\gamma_{j_{m_i}} h_{j_{m_i}} * \gamma_{j_{m_i+1}} h_{j_{m_i+1}} g_{i_1} * \cdots * \gamma_{j_{m_i-1}} h_{j_{m_i-1}} g_{i_{k-1}} \cdots g_{i_1})(t+t_0),$$

which is

$$(\gamma_{j_k} * \gamma_{j_1} g_{j_1} * \cdots * \gamma_{j_{k-1}} g_{j_{k-1}} \cdots g_{j_1})(t + m_i + t_0) \cdot \beta^{\underline{h},\tau}_{j,i},$$

where $\beta^{\underline{h},\tau}_{j,i}$ is the symbol defined in (3.6).

$F_{(\underline{g},\sigma)}$ maps the morphism $[(\underline{h},\tau),t]$ to

$$\times_k \times_{i\in\theta_k} ([\beta^{\underline{h},\tau}_{\tau(1),1}, m_1 + t], \cdots [\beta^{\underline{h},\tau}_{\tau(M^\sigma_i),M^\sigma_i}, m_{M^\sigma_i} + t], \tau|_{W^\sigma_i})$$

where $\tau|_{W^\sigma_i}$ denotes the permutation induced by τ on the set $W^\sigma_i = \{\alpha^i_1, \alpha^i_2, \cdots \alpha^i_{M^\sigma_i}\}$, $\tau^{-1}(j)$ is short for $\tau^{-1}(\alpha^i_j)$ and $\tau(j_l) = \tau(j)_{l+m_j}$.

It sends the identity map $[(1, \cdots, 1, \mathrm{Id}), 0]$ to the identity map $\times_k \times_{i\in\theta_k}([1,0], \cdots [1,0], \mathrm{Id})$. And $F_{(\underline{g},\sigma)}$ preserves composition of morphisms. So it is a well-defined functor.

Now let's construct $J_{(\underline{g},\sigma)}$. For an object$\times_k \times_{i\in\theta_k} \nu_{i,k}$ in $D_{(\underline{g},\sigma)}(X)$, it sends $\nu_{i,k} \in$

${}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}}X$ to $\{\nu_m\}_1^n$ with $\{\nu_{i_s}\}_{s=1}^k$ with

$$\nu_{i_k} = \nu_{i,k}|_{[0,1]}$$

and

$$\nu_{i_s}(t) := \nu_{i,k}(s+t)g_{i_1}^{-1}\cdots g_{i_s}^{-1}.$$

For each morphism

$$\prod_k \prod_{i\in\theta_k}((u_1^i, m_1'^i), (u_2^i, m_2'^i), \cdots (u^i_{M_i^\sigma}, m'^i_{M_i^\sigma}), \varrho_i^k)$$

in $D_{(\underline{g},\sigma)}(X)$, we can get a unique $[(\underline{h},\tau),t] \in \Lambda_{G\wr\Sigma_n}((\underline{g},\sigma))$.

Let t be a representative of the image of $m_1'^i$ in $\mathbb{R}/\mathbb{Z}$. Then, let $m_k^i = m_k'^i - t$ for each $k = 1, 2\cdots n$. Each m_k^i is an integer.

When we know how $\tau \in C_{\Sigma_n}(\sigma)$ permutes the cycles of σ, whose information is contained in those $\varrho_i^k \in \Sigma_{M_i^\sigma}$, and the numbers $m_1^i, \cdots m^i_{M_i^\sigma}$, we can get a unique τ. Explicitly, for any number $j_r = 1, 2\cdots n$, if j_r is in a $k-$cycle $(j_1, \cdots j_k)$ of σ and is in the set W_i^σ, then τ maps j_r to $\varrho_i^k(j)_{r+m_j^i}$, i.e. the $r+m_j^i$-th element in the cycle $\varrho_i^k(j)$ of σ.

For any $a \in W_i^\sigma$, $\forall k$ and i, I want $u_a^i = \beta^{\underline{h},\tau}_{\tau(a),a}$ for some $\underline{h}$. Thus,

$$h_{\tau(a)_k} = u_a^i g_{a_k}\cdots g_{a_{1-m_a^i}}. \tag{3.17}$$

And by (3.5) we can get all the other $h_{\tau(a)_j}$.

It can be checked straightforward that $J_{(\underline{g},\sigma)}$ is a well-defined functor. And it does not depend on the choice of the representative t.

In addition, $J_{(\underline{g},\sigma)} \circ F_{(\underline{g},\sigma)} = \mathrm{Id}$; $F_{(\underline{g},\sigma)} \circ J_{(\underline{g},\sigma)} = \mathrm{Id}$. So the conclusion is proved. $\square$

Theorem 3.1.11. *The two groupoids*

$$\mathcal{L}((X//G)\wr\Sigma_n) = \coprod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}} \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$$

and

$$D((X//G)\wr\Sigma_n)$$

are isomorphic.

Proof. If $(\underline{g}, \sigma)$ and $(\underline{g}', \sigma')$ are conjugate in $G \wr \Sigma_n$, each $[(\underline{h}, \tau), t_0] \in \Lambda_{G\wr\Sigma_n}((\underline{g}, \sigma), (\underline{g}', \sigma'))$ gives an isomorphism from $\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g}, \sigma)$ to $\mathcal{L}_{(\underline{g}',\sigma')}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g}', \sigma')$ by

$$\gamma \mapsto \gamma \cdot [(\underline{h}, \tau), t_0].$$

There is a canonical isomorphism $F([(\underline{h}, \tau), t_0])$ from $D_{(\underline{g},\sigma)}(X)$ to $D_{(\underline{g}',\sigma')}(X)$ so that the diagram below commutes in the category of Lie groupoids.

$$\begin{tikzcd}
\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n}) \arrow[r, "F_{(\underline{g},\sigma)}"] \arrow[d, "{[(\underline{h},\tau),t_0]}"] & D_{(\underline{g},\sigma)}(X) \arrow[d, "{F([(\underline{h},\tau),t_0])}"] \\
\mathcal{L}_{(\underline{g}',\sigma')}(X^{\times n}) \arrow[r, "F_{(\underline{g}',\sigma')}"] & D_{(\underline{g}',\sigma')}(X)
\end{tikzcd}$$

Let $\gamma = (\gamma_1, \cdots \gamma_n)$ be a path in $\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$. Then, the path of $F_{(\underline{g}',\sigma')}(\gamma \cdot [(\underline{h}, \tau), t_0])(t)$ in ${}_k\mathcal{L}_{g'_{i_k}\cdots g'_{i_1}}X$ is

$$\gamma_{\tau(i_k)} h_{\tau(i_k)} * \gamma_{\tau(i_1)} h_{\tau(i_1)} g'_{i_1} * \cdots * \gamma_{\tau(i_{k-1})} h_{\tau(i_{k-1})} g'_{i_{k-1}} \cdots g'_{i_1}(t + t_0). \tag{3.18}$$

The path of $F_{(\underline{g},\sigma)}(\gamma)$ in ${}_k\mathcal{L}_{g_{j_k}\cdots g_{j_1}}X$ is

$$\gamma_{j_k} * \gamma_{j_1} g_{j_1} * \cdots * \gamma_{j_{k-1}} g_{j_{k-1}} \cdots g_{j_1}.$$

By the relation (3.5), each $\gamma_{\tau(i_r)} h_{\tau(i_r)} g'_{i_r} \cdots g'_{i_1}$ in (3.18) is equal to

$$\gamma_{j_{r+m_i}} g_{j_{r+m_i}} \cdots g_{j_1}(g_{j_1}^{-1} \cdots g_{j_{m_i}}^{-1} h_{j_{m_i}}).$$

The element $g_{j_1}^{-1} \cdots g_{j_{m_i}}^{-1} h_{j_{m_i}} = h_{j_k} g'^{-1}_{i_{1-m_i}} \cdots g'^{-1}_{i_k}$.

So the map of $F([(\underline{h}, \tau), t_0])$ on the path $\nu_j \in {}_k\mathcal{L}_{g_{j_k}\cdots g_{j_1}}X$ corresponding to $k-$cycle $(j_1, \cdots j_k)$ of σ is defined by

$$F([(\underline{h}, \tau), t_0])(\nu_j)(t) = \nu_j(t + m_i + t_0) h_{j_k} g'^{-1}_{i_{1-m_i}} \cdots g'^{-1}_{i_k} \in {}_k\mathcal{L}_{g'_{i_k}\cdots g'_{i_1}}X.$$

$F([(\underline{h}, \tau), t_0])$ is an isomorphism between the two groupoids.

$\square$

Remark 3.1.12. *Recall $\Lambda_k(\underline{g}, \sigma)$ is the quotient group of $C_{G\wr\Sigma_n}(\underline{g}, \sigma) \times \mathbb{R}$ by the normal* *subgroup generated by* $(g^k,-1)$. $\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$ *is a* $\Lambda(\underline{g},\sigma)$*-space. For each* k,

$$\prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X$$

is a $\Lambda_k(\underline{g},\sigma)$*-space since the action of* $(\underline{g},\sigma;0)$ *on*

$$\prod \gamma : [0,k] \longrightarrow X$$

is the same as $(\underline{e}, Id; 1)$ *on it, as indicated in Example 4.4 in [24].*

In addition, I state in Proposition 3.1.13 some properties about the functor $F_{(\underline{g},\sigma)}$ constructed in the proof of Theorem 3.1.9 and Theorem 3.1.11.

Proposition 3.1.13. *(i) If* $\sigma = (1) \in \Sigma_1$*, the morphism* $F_{(g,(1))}$ *is the identity map on* $\mathcal{L}_g(X)$.

(ii) Let $(\underline{g},\sigma) \in G \wr \Sigma_n$ *and* $(\underline{h},\tau) \in G \wr \Sigma_m$.

$$\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n}) \times \mathcal{L}_{(\underline{h},\tau)}(X^{\times m}) \text{ and } \mathcal{L}_{(\underline{g},\underline{h},\sigma\tau)}(X^{\times(m+n)})$$

are $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma) \times_{\mathbb{T}} \Lambda_{G\wr\Sigma_m}(\underline{h},\tau)$*-equivariant homeomorphic.*

(iii)$F_{(\underline{g},\sigma)}$ *preserves cartesian product of loops. The following diagram of* $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma) \times_{\mathbb{T}} \Lambda_{G\wr\Sigma_m}(\underline{h},\tau)-$*spaces commutes.*

$$\begin{tikzcd}
\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n}) \times \mathcal{L}_{(\underline{h},\tau)}(X^{\times m}) \arrow[r, "(\gamma_1,\gamma_2)\mapsto\gamma_1\times\gamma_2"] \arrow[dr, "F_{(\underline{g},\sigma)}\times F_{(\underline{h},\tau)}"'] & \mathcal{L}_{(\underline{g},\underline{h},\sigma\tau)}(X^{\times(n+m)}) \arrow[d, "\eta^*\circ F_{(\underline{g},\underline{h},\sigma\tau)}"'] \\
& \prod_k \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X \times \prod_j \prod_{(r_1,\cdots r_j)} {}_j\mathcal{L}_{h_{r_j}\cdots h_{r_1}} X
\end{tikzcd}$$

where $(i_1,\cdots i_k)$ *goes over all the* $k-$*cycles of* σ *and* $(r_1,\cdots r_j)$ *goes over all the* $j-$*cycles of* τ.

Proof. (i) has already been indicated in the proof of Theorem 3.1.9.

(ii) Let $\gamma_1 = (\gamma_{1,1},\cdots\gamma_{1,n}) \in \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$ and $\gamma_2 = (\gamma_{2,1},\cdots\gamma_{2,m}) \in \mathcal{L}_{(\underline{h},\tau)}(X^{\times m})$. Define

$$\Phi : \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n}) \times \mathcal{L}_{(\underline{h},\tau)}(X^{\times m}) \longrightarrow \mathcal{L}_{(\underline{g},\underline{h},\sigma\tau)}(X^{\times(m+n)})$$

by

$$\Phi(\gamma_1, \gamma_2) := (\gamma_{1,1}, \cdots \gamma_{1,n}, \gamma_{2,1}, \cdots \gamma_{2,m}) \in \mathcal{L}_{(\underline{g}, \underline{h}, \sigma\tau)}(X^{\times(m+n)}).$$

Φ is a homeomorphism.

Let $\eta : \Lambda_{G\wr\Sigma_n}(\underline{g}, \sigma) \times_{\mathbb{T}} \Lambda_{G\wr\Sigma_m}(\underline{h}, \tau) \longrightarrow \Lambda_{G\wr\Sigma_{m+n}}(\underline{g}, \underline{h}; \sigma\tau)$ be the inclusion

$$([\alpha, t], [\beta, t]) \mapsto [\alpha, \beta, t],$$

where $\alpha \in C_{G\wr\Sigma_n}(\underline{g}, \sigma)$, $\beta \in C_{G\wr\Sigma_m}(\underline{h}, \tau)$, and $t \in \mathbb{R}$. Then η induces a $\Lambda_{G\wr\Sigma_n}(\underline{g}, \sigma) \times_{\mathbb{T}} \Lambda_{G\wr\Sigma_m}(\underline{h}, \tau)$-action on the space $\mathcal{L}_{(\underline{g}, \underline{h}, \sigma\tau)}(X^{\times(m+n)})$. And Φ is a $\Lambda_{G\wr\Sigma_n}(\underline{g}, \sigma) \times_{\mathbb{T}} \Lambda_{G\wr\Sigma_m}(\underline{h}, \tau)$-equivariant homeomorphism between

$$\mathcal{L}_{(\underline{g}, \sigma)}(X^{\times n}) \times \mathcal{L}_{(\underline{h}, \tau)}(X^{\times m})$$

and

$$\eta^*(\mathcal{L}_{(\underline{g}, \underline{h}, \sigma\tau)}(X^{\times(m+n)})).$$

(iii) $F_{(\underline{g}, \sigma)}$ sends a path

$$\gamma = (\gamma_1, \cdots \gamma_n) \in \mathcal{L}_{(\underline{g}, \sigma)}(X^{\times n})$$

to the product of $\sum\limits_k N_k$ paths,

$$\prod_k \prod_{(i_1, \cdots i_k)} \gamma_{i_k} * \gamma_{i_1} g_{i_1} * \cdots * \gamma_{i_{k-1}} g_{i_{k-1}} \cdots g_{i_1},$$

which is an object of $D_{(\underline{g}, \sigma)}(X)$.

$F_{(\underline{g}, \underline{h}, \sigma\tau)}(\gamma_1 \times \gamma_2)$ is the product

$$\prod_k \prod_{(i_1, \cdots i_k)} \gamma_{i_k} * \gamma_{i_1} g_{i_1} * \cdots * \gamma_{i_{k-1}} g_{i_{k-1}} \cdots g_{i_1} \prod_m \prod_{(j_1, \cdots j_m)} \gamma_{j_m} * \gamma_{j_1} h_{j_1} * \cdots * \gamma_{j_{m-1}} h_{j_{m-1}} \cdots h_{j_1},$$

where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ and $(j_1, \cdots j_m)$ goes over all the $m-$cycles of τ. It is exactly $F_{(\underline{g}, \sigma)}(\gamma_1) \times F_{(\underline{h}, \tau)}(\gamma_2)$.

It's straightforward to check both maps are $\Lambda_{G\wr\Sigma_n}(\underline{g}, \sigma) \times_{\mathbb{T}} \Lambda_{G\wr\Sigma_m}(\underline{h}, \tau)-$equivariant.

$\square$

Proposition 3.1.14 (Naturality). *Let $f : X//G \longrightarrow Y//H$ be a homomorphism between groupoids. Let $(\underline{g}, \sigma) \in G\wr\Sigma_n$. Let $(\underline{f(g)}, \sigma)$ denote the element $(f(g_1), \cdots f(g_n), \sigma) \in H\wr\Sigma_n$. Then f induces homomorphisms*

$$D_{(\underline{g},\sigma)}(f) : D_{(\underline{g},\sigma)}(X) \longrightarrow D_{(\underline{f(g)},\sigma)}(Y)$$

and

$$\mathcal{L}_{(\underline{g},\sigma)}(f) : \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g}, \sigma) \longrightarrow \mathcal{L}_{(\underline{f(g)},\sigma)}(Y^{\times n})//\Lambda_{H\wr\Sigma_n}(\underline{f(g)}, \sigma).$$

Then the following diagram commutes in the category of Lie groupoids.

$$\begin{array}{ccc}
\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma}(\underline{g}, \sigma) & \xrightarrow{\mathcal{L}_{(\underline{g},\sigma)}(f)} & \mathcal{L}_{(\underline{f(g)},\sigma)}(Y^{\times n})//\Lambda_{H\wr\Sigma_n}(\underline{f(g)}, \sigma) \\
{\scriptstyle F_{(\underline{g},\sigma)}}\big\downarrow & & {\scriptstyle F_{(\underline{f(g)},\sigma)}}\big\downarrow \\
D_{(\underline{g},\sigma)}(X) & \xrightarrow{D_{(\underline{g},\sigma)}(f)} & D_{(\underline{f(g)},\sigma)}(Y).
\end{array} \tag{3.19}$$

Proof. For any $\gamma : [0, k] \longrightarrow X$ such that $\gamma(k) = \gamma(0) \cdot g$, let $f_*\gamma : [0, k] \longrightarrow Y$ denote the map $f_*\gamma(t) = f \circ \gamma(t)$. We have $f_*\gamma(k) = f_*\gamma(0) \cdot f(g)$. So we have a well-defined functor

$${}_k\mathcal{L}_g(f) : {}_k\mathcal{L}_g(X)//\Lambda_G(g) \longrightarrow {}_k\mathcal{L}_{f(g)}(Y)//\Lambda_H(f(g))$$

sending an object γ to $f_*(\gamma)$ and sending a morphism $[\alpha, t]$ to $[f(\alpha), t]$.

Similarly we can define

$$\mathcal{L}_{(\underline{g},\sigma)}(f) : \mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g}, \sigma) \longrightarrow \mathcal{L}_{(\underline{f(g)},\sigma)}(Y^{\times n})//\Lambda_{H\wr\Sigma_n}(\underline{f(g)}, \sigma)$$

by sending an object γ to $f_*(\gamma)$ and a morphism $[\underline{\alpha}, \tau, t]$ to $[\underline{f(\alpha)}, \tau, t]$. And define

$$D_{(\underline{g},\sigma)}(f) : D_{(\underline{g},\sigma)}(X) \longrightarrow D_{(\underline{f(g)},\sigma)}(Y)$$

to be the homomorphism sending an object

$$\prod_k \prod_{(i_1,\cdots i_k)} \gamma_{(i_1,\cdots i_k)} \in \prod_k \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X$$

to

$$\prod_k \prod_{(i_1,\cdots i_k)} f_*\gamma_{(i_1,\cdots i_k)} \in \prod_k \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{h_{i_k}\cdots h_{i_1}} X$$

where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ. It sends a morphism

$$\times_k \times_{i\in\theta_k} (([a^i_1, t^i_1], \cdots [a^i_{M^\sigma_i}, t^i_{M^\sigma_i}], \tau_i), x)$$

of $D_{(\underline{g},\sigma)}(X)$ to

$$\times_k \times_{i\in\theta_k} (([f(a^i_1), t^i_1], \cdots [f(a^i_{M^\sigma_i}), t^i_{M^\sigma_i}], \tau_i), f_*x)$$

where θ_k is a fixed set of representatives of $k-$cycles of σ.

Then it's straightforward to check the diagram (3.19) commutes. $\square$

Let's consider the full subgroupoid of $\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})//\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$ whose objects are the constant loops, i.e. the points in $(X^{\times n})^{(\underline{g},\sigma)}$. $\Lambda(\underline{g},\sigma)$ acts on $(X^{\times n})^{(\underline{g},\sigma)}$ by

$$\underline{x}\cdot[(\underline{h},\tau),t] := \underline{x}\cdot(\underline{h},\tau), \text{ for any } x\in(X^{\times n})^{(\underline{g},\sigma)},\ [(\underline{h},\tau),t]\in\Lambda(\underline{g},\sigma).$$

And let $d_{(\underline{g},\sigma)}(X)$ denote the full subgroupoid of $D_{(\underline{g},\sigma)}(X)$ whose objects are the constant loops, i.e. the points in $\prod\limits_k \prod\limits_{(i_1,\cdots i_k)} X^{g_{i_k}\cdots g_{i_1}}$.

Let $f_{(\underline{g},\sigma)}$ denote the restriction of the functor $F_{(\underline{g},\sigma)}$ to the full subgroupoid consisting of constant loops, which is an isomorphism from $(X^{\times n})^{(\underline{g},\sigma)}//\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$ to $d_{(\underline{g},\sigma)}(X)$. Thus, $f_{(\underline{g},\sigma)}$ induces a $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)-$action on the space

$$\prod_k \prod_{(i_1,\cdots i_k)} X^{g_{i_k}\cdots g_{i_1}},$$

where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ.

Moreover, let $d((X//G)\wr\Sigma_n)$ denote the full subgroupoid of $D((X//G)\wr\Sigma_n)$ consisting of constant loops. Then we have

Corollary 3.1.15. *The groupoids $\Lambda(X^{\times n}//G\wr\Sigma_n)$ and $d((X//G)\wr\Sigma_n)$ are isomorphic.*

Remark 3.1.16. *Let X be a G-space and $g\in G$. Recall the group $\Lambda_k(g) = \Lambda_{C_G(g)}(g^k)$.*

The map $\mu_k : K_{\Lambda(g^k)}(X^{g^k}) \longrightarrow K_{\Lambda_k(g)}(X^g)$ constructed in (2.35) can also be defined for loop spaces. We have seen $\mathcal{L}_gX$ is a $\Lambda_G(g)$-space. It can also be viewed as a $\Lambda_k(g)$-space *with the action*

$$[h,s]\cdot\gamma(t) := \gamma(t+ks)\cdot h.$$

Let

$$j : \mathcal{L}_g X//\Lambda_k(g) \longrightarrow \mathcal{L}_{g^k} X//\Lambda_k(g)$$

send $\gamma(t)$ *to*

$$\gamma * \gamma\cdot g * \cdots * \gamma\cdot g^{k-1}(kt).$$

j *is well-defined and* $\Lambda_k(g)-$*equivariant. When restricted to the space of constant loops, it is exactly the inclusion* $X^g \hookrightarrow X^{g^k}$.

μ_k *is defined to be the composition*

$$K^*_{orb}(\mathcal{L}_{g^k}X//\Lambda_G(g^k)) \xrightarrow{\beta^*} K^*_{orb}(\mathcal{L}_{g^k}X//\Lambda_k(g)) \xrightarrow{j^*} K^*_{orb}(\mathcal{L}_g X//\Lambda_k(g))$$

where the first map is induced by the inclusion $\beta : \Lambda_k(g) \longrightarrow \Lambda_G(g^k)$.

3.2 Total Power Operation of $QEll^*_G$

In this section I construct the total power operations for quasi-elliptic cohomology and give its explicit formula in (3.30). In Theorem 3.2.1 I check they satisfy the axioms that Ganter concluded in [23] for equivariant power operation.

The power operation of quasi-elliptic cohomology is of the form

$$\begin{aligned}\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)} :&\\ QEll^*_G(X) \longrightarrow QEll^*_{G\wr\Sigma_n}(X^{\times n}) = &\prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}),\end{aligned}$$

where $\mathbb{P}_n$ maps a bundle over the groupoid

$$\Lambda(X//G)$$

to a bundle over

$$\Lambda(X^{\times n}//(G\wr\Sigma_n)),$$

and each $\mathbb{P}_{(\underline{g},\sigma)}$ maps a bundle over

$$\Lambda(X//G)$$

to a $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)-$bundle over the space $(X^{\times n})^{(\underline{g},\sigma)}//\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$.

I construct each $\mathbb{P}_{(\underline{g},\sigma)}$ as the composition below.

$$\begin{aligned} QEll^*_G(X) &\xrightarrow{U^*} K^*_{orb}(\Lambda^1_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^\Lambda_k} K^*_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X)) \\ &\xrightarrow{\boxtimes} K^*_{orb}(d_{(\underline{g},\sigma)}(X)) \xrightarrow{f^*_{(\underline{g},\sigma)}} K^*_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}), \end{aligned}$$

where $k \in \mathbb{Z}$ and $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ. I will explain in detail what each map in this formula is: $U : \Lambda^1_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda(X//G)$ is the groupoid homomorphism defined in (3.20). The second map is the pullback $(\ )^\Lambda_k$ defined in (3.25). The third map $\boxtimes$ is the external product in (3.29). The fourth one is the pullback by $f_{(\underline{g},\sigma)}$, which is defined before Corollary 3.1.15.

The Functor U

For each torsion element $(\underline{g},\sigma) \in G \wr \Sigma_n$, $r \in \mathbb{Z}$, let $\Lambda^r_{(\underline{g},\sigma)}(X)$ denote the groupoid with objects

$$\coprod_k \coprod_{(i_1,\cdots i_k)} X^{g_{i_k}\cdots g_{i_1}}$$

where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ, and with morphisms

$$\coprod_k \coprod_{(i_1,\cdots i_k),(j_1,\cdots j_k)} \Lambda^r_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}},$$

where $(i_1, \cdots i_k)$ and $(j_1, \cdots j_k)$ go over all the $k-$cycles of σ respectively. It may not be a subgroupoid of $\Lambda^r(X//G)$ because there may be cycles $(i_1, \cdots i_k)$ and $(j_1, \cdots j_m)$ such that

$$g_{i_k}\cdots g_{i_1} = g_{j_m}\cdots g_{j_1}.$$

Let

$$U : \Lambda^1_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda(X//G) \tag{3.20}$$

denote the functor sending x in the component $X^{g_{i_k}\cdots g_{i_1}}$ to the x in the component $X^{g_{i_k}\cdots g_{i_1}}$ of $\Lambda(X//G)$, and send each morphism $([h,t],x)$ in $\Lambda^1_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}}$ to

$([h,t],x)$ in $\Lambda^1_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})\times X^{g_{i_k}\cdots g_{i_1}}$. In the case that $g_{i_k}\cdots g_{i_1}$ and $g_{j_k}\cdots g_{j_1}$ are equal, $([h,t],x)$ is an arrow inside a single connected component.

The Functors $(\ )_k$ and $(\ )^{\Lambda}_k$

For each integer k, there is a functor of groupoids $(\ )_k : {}_k\mathcal{L}(X//G) \longrightarrow \mathcal{L}(X//G)$ sending an object $s\mapsto\gamma(s)$ in a component ${}_k\mathcal{L}_gX$ to $t\mapsto\gamma(kt)$ in the component $\mathcal{L}_gX$; and $(\ )_k$ sends a morphism $([h,t_0],(s\mapsto\gamma(s)))$ to $([h,\frac{t_0}{k}],(t\mapsto\gamma(kt)))$. The composition has the equivalence $((\ )_k)_r = (\ )_{kr}$. Its restriction to the constant loops $\Lambda^k(X//G)\longrightarrow\Lambda(X//G)$ is also well-defined. I will use the same symbol $(\ )_k$ to denote the functor on the subgroupoid of constant loops.

In addition, the functor $(\ )_k$ gives a well-defined map

$$K_{orb}(\mathcal{L}(X//G))\longrightarrow K_{orb}({}_k\mathcal{L}(X//G))$$

by pull back of bundles. Let $\mathcal{V}$ denote a $\mathcal{L}(X//G)-$vector bundle over $\mathcal{L}(X//G)$. $\mathbb{R}/k\mathbb{Z}$ acts on the pull-back bundle $(\mathcal{V})_k$ by $(\ )_k$ via precomposing with the map

$$\mathbb{R}/k\mathbb{Z}\longrightarrow\mathbb{R}/\mathbb{Z},\ a\mapsto\frac{a}{k}$$

For example, if S^1 acts on $\mathcal{V}$ by the representation q, it acts on $(\mathcal{V})_k$ via

$$\begin{aligned} q^{\frac{1}{k}} : \mathbb{R}/k\mathbb{Z} &\longrightarrow U(1)\\ a &\mapsto e^{\frac{2\pi i a}{k}}. \end{aligned}$$

More generally, if $\mathcal{V}$ has the decomposition

$$\mathcal{V}=\bigoplus_{j\in\mathbb{Z}}V_jq^j, \tag{3.21}$$

then the pull-back bundle has the decomposition

$$(\mathcal{V})_k=\bigoplus_{j\in\mathbb{Z}}V_jq^{\frac{j}{k}}. \tag{3.22}$$

Restricting $(\ )_k$ to the subgroupoids $\Lambda(X//G)$ and $X^\sigma//\Lambda_G(\sigma)$ of $\mathcal{L}(X//G)$, we obtain the rescaling maps

$$K_{orb}(\Lambda(X//G)) \longrightarrow K_{orb}(\Lambda^k(X//G)) \tag{3.23}$$

and

$$K_{orb}(X^\sigma//\Lambda_G(\sigma)) \longrightarrow K_{orb}(X^\sigma//\Lambda^k_G(\sigma)). \tag{3.24}$$

Let's still use the symbol $(\ )_k$ to denote it when there is no confusion.

Let $\Lambda^{var}_{(\underline{g},\sigma)}(X)$ be the groupoid with the same objects as $\Lambda^1_{(\underline{g},\sigma)}(X)$ and morphisms

$$\coprod_k \coprod_{(i_1,\cdots i_k),(j_1,\cdots j_k)} \Lambda^k_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}},$$

where $(i_1,\cdots i_k)$ and $(j_1,\cdots j_k)$ go over all the $k-$cycles of σ respectively.

We can define a similar functor

$$(\ )^\Lambda_k : \Lambda^{var}_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda^1_{(\underline{g},\sigma)}(X) \tag{3.25}$$

that is identity on objects and sends each $[g,t] \in \Lambda^k_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$ to $[g,\frac{t}{k}] \in \Lambda^1_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$.

And we use the same symbol $(\ )^\Lambda_k$ to denote the pull back

$$K_{orb}(\Lambda^1_{(\underline{g},\sigma)}(X)) \longrightarrow K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X)). \tag{3.26}$$

The external product $\boxtimes$

Let G and H be compact Lie groups. Let X be a G-space and Y a H-space. Let $\sigma \in G^{tors}$ and $\tau \in H^{tors}$.

In Section 2.4.3 I introduce the Künneth map and the completed tensor product $\widehat{\otimes}_{\mathbb{Z}[q^\pm]}$ of quasi-elliptic cohomology. To simplify the symbol in this section, for $x \in K^*_{\Lambda_G(g)}(X^g)$ and $y \in K^*_{\Lambda_H(h)}(Y^h)$, let's use

$$x \boxtimes y$$

to denote their external product $x \otimes_{\mathbb{Z}[q^\pm]} y$ in $K^*_{\Lambda_{G\times H}(g,h)}(X\times Y)^{(g,h)}$; and for $\widetilde{x} \in QEll^*_G(X)$ and $\widetilde{y} \in QEll^*_H(Y)$, I use

$$\widetilde{x} \wedge \widetilde{y} := \prod_{(g,h)\in G^{tors}_{conj}\times H^{tors}_{conj}} (\widetilde{x} \wedge \widetilde{y})_{(g,h)}$$

to denote their external product as $\mathbb{Z}[q^{\pm}]-$algebras in $QEll^*_{G\times H}(X\times Y)$, where each $(\widetilde{x}\wedge\widetilde{y})_{(g,h)}$ is $\widetilde{x}_g\boxtimes\widetilde{y}_h$.

Similar to the Künneth map (2.41) of quasi-elliptic cohomology, we can also define the Künneth map

$$K^*_{orb}(d_{(\underline{g},\sigma)}(X))\otimes_{\mathbb{Z}[q^{\pm}]}K^*_{orb}(d_{(\underline{h},\tau)}(X))\longrightarrow K^*_{orb}(d_{(\underline{g},\sigma)}(X)\times_{\mathbb{T}}d_{(\underline{h},\tau)}(X)),$$

which is compatible with that of quasi-elliptic cohomology in the sense of (3.28).

First let's talk about the external product of $K^*_{orb}(D_{(\underline{g},\sigma)}(X))$ and $K^*_{orb}(D_{(\underline{h},\tau)}(X))$ with $(\underline{g},\sigma)\in G\wr\Sigma_n$ and $(\underline{h},\tau)\in G\wr\Sigma_m$. Each $K^*_{orb}(D_{(\underline{g},\sigma)}(X))$ is a $\mathbb{Z}[q^{\pm}]-$algebra, as shown at the end of Section 3.1.1. The external product in the theory $K^*_{orb}(D_{(\underline{g},\sigma)}(-))$ is defined to be the tensor product of $\mathbb{Z}[q^{\pm}]-$algebras.

Recall that a morphism in $D_{(\underline{g},\sigma)}(X)$ is of the form

$$(\times_k\times_{i\in\theta_k}([a^i_1,t^i_1],\cdots[a^i_{M^\sigma_i},t^i_{M^\sigma_i}],\tau_i),x)$$

in which $\tau_i\in\Sigma_{M^\sigma_i},[a^i_j,t^i_j]\in\Lambda^k_G(g_{\tau_i(j)_k}\cdots g_{\tau_i(j)_1},g_{j_k}\cdots g_{j_1})$, for each $j\in W^\sigma_i$. Each t^i_j in it has the same image under the quotient map $\mathbb{R}/k\mathbb{Z}\longrightarrow\mathbb{R}/\mathbb{Z}$. So we have $q(t^i_j)=q(t^{i'}_{j'})$.

Let $D_{(\underline{g},\sigma)}(X)\times_{\mathbb{T}}D_{(\underline{h},\tau)}(X)$ denote the groupoid with objects of the form

$$x\times y$$

where x is an object in $D_{(\underline{g},\sigma)}(X)$ and y is an object in $D_{(\underline{h},\tau)}(X)$, and with morphisms starting at $x\times y$ of the form

$$f\times g$$

where f is a morphism starting at x in $D_{(\underline{g},\sigma)}(X)$ and g is a morphism starting at y in $D_{(\underline{h},\tau)}(X)$, and all the t part has the same image under the quotient map.

$D_{(\underline{g},\sigma)}(X)\times_{\mathbb{T}}D_{(\underline{h},\tau)}(X)$ is a subgroupoid of $D_{(\underline{g},\underline{h},\sigma\tau)}(X)$ when we consider $(\underline{g},\underline{h},\sigma\tau)$ as an element in $G\wr\Sigma_{n+m}$ instead of $G\wr(\Sigma_n\times\Sigma_m)$. $D_{(\underline{g},\sigma)}(X)\times_{\mathbb{T}}D_{(\underline{h},\tau)}(X)$ and $D_{(\underline{g},\underline{h},\sigma\tau)}(X)$ have the same objects.

We have the Künneth map

$$K^*_{orb}(D_{(\underline{g},\sigma)}(X))\otimes_{\mathbb{Z}[q^{\pm}]}K^*_{orb}(D_{(\underline{h},\tau)}(X))\longrightarrow K^*_{orb}(D_{(\underline{g},\sigma)}(X)\times_{\mathbb{T}}D_{(\underline{h},\tau)}(X)).\qquad(3.27)$$

Recall $d_{(\underline{g},\sigma)}(X)$, which is defined before Corollary 3.1.15, is the full subgroupoid of $D_{(\underline{g},\sigma)}(X)$ consisting of the constant loops, i.e. those points in $\prod\limits_{k}\prod\limits_{(i_1,\cdots i_k)} X^{g_{i_k}\cdots g_{i_1}}$.

By Proposition 3.1.13 (iii), restricting the Künneth map (3.27) to the subgroupoid $d_{(\underline{g},\sigma)}(X)$

$$K^*_{orb}(d_{(\underline{g},\sigma)}(X)) \otimes_{\mathbb{Z}[q^{\pm}]} K^*_{orb}(d_{(\underline{h},\tau)}(X)) \longrightarrow K^*_{orb}(d_{(\underline{g},\sigma)}(X) \times_{\mathbb{T}} d_{(\underline{h},\tau)}(X)).$$

It is compatible with the Künneth map (2.41) of the quasi-elliptic cohomology in the sense that we have the commutative diagram

$$\begin{tikzcd}
K^*_{orb}(d_{(\underline{g},\sigma)}(X)) \otimes_{\mathbb{Z}[q^{\pm}]} K^*_{orb}(d_{(\underline{h},\sigma)}(X)) \arrow[r] \arrow[d, "f^*_{(\underline{g},\sigma)}\otimes_{\mathbb{Z}[q^{\pm}]} f^*_{(\underline{h},\sigma)}"'] & K^*_{orb}(d_{(\underline{g},\sigma)}(X) \times_{\mathbb{T}} d_{(\underline{h},\sigma)}(X)) \arrow[d, "f^*_{(\underline{(g,h)},\sigma)}"'] \\
K^*_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^n)^{(\underline{g},\sigma)}) \otimes_{\mathbb{Z}[q^{\pm}]} K^*_{\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}((Y^m)^{(\underline{h},\sigma)}) \arrow[r] & K^*_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{(g,h)},\sigma)}((X\times Y)^n)^{(\underline{(g,h)},\sigma)}
\end{tikzcd} \tag{3.28}$$

where the horizontal maps are Künneth maps.

In addition, if we have a vector bundle $E = \coprod\limits_{k}\coprod\limits_{(i_1,\cdots i_k)} E_{g_{i_k}\cdots g_{i_1}}$ over $\Lambda_{(\underline{g},\sigma)}(X)$, the external product

$$\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} E_{g_{i_k}\cdots g_{i_1}}$$

is a vector bundler over $d_{(\underline{g},\sigma)}(X)$. This defines a map

$$K_{orb}(\Lambda^1_{(\underline{g},\sigma)}(X)) \longrightarrow K_{orb}(d_{(\underline{g},\sigma)}(X)) \tag{3.29}$$

Composing all the relevant functors, we get the explicit form of $\mathbb{P}_{(\underline{g},\sigma)}$

$$\mathbb{P}_{(\underline{g},\sigma)}(\mathcal{V}) = f^*_{(\underline{g},\sigma)}(\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (\mathcal{V}_{g_{i_k}\cdots g_{i_1}})_k). \tag{3.30}$$

where $(\ )_k$ is the map cited in (3.24).

$\mathbb{P}_{(\underline{g},\sigma)}$ is natural. And by the proof of Theorem 3.1.11, if $(\underline{g},\sigma)$ and $(\underline{h},\tau)$ are conjugate in $G\wr\Sigma_n$, then $\mathbb{P}_{(\underline{g},\sigma)}(E)$ and $\mathbb{P}_{(\underline{h},\tau)}(E)$ are isomorphic.

Theorem 3.2.1. *The family of maps*

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)} : QEll^*_G(X) \longrightarrow QEll^*_{G\wr\Sigma_n}(X^{\times n}),$$

defined in (3.30) satisfies

(i) $\mathbb{P}_1 = Id$, $\mathbb{P}_0(x) = 1$.

(ii) Let $x \in QEll^*_G(X)$, $(\underline{g}, \sigma) \in G \wr \Sigma_n$ *and* $(\underline{h}, \tau) \in G \wr \Sigma_m$. *The external product of two power operations*

$$\mathbb{P}_{(\underline{g},\sigma)}(x) \boxtimes \mathbb{P}_{(\underline{h},\tau)}(x) = res|^{\Lambda_{G\wr\Sigma_{m+n}}(\underline{g},\underline{h};\sigma\tau)}_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{G\wr\Sigma_m}(\underline{h},\tau)} \mathbb{P}_{(\underline{g},\underline{h};\sigma\tau)}(x).$$

(iii) The composition of two power operations is

$$\mathbb{P}_{((\underline{\underline{h},\tau});\sigma)}(\mathbb{P}_m(x)) = res|^{\Lambda_{G\wr\Sigma_{mn}}(\underline{\underline{h}},(\underline{\tau},\sigma))}_{\Lambda_{(G\wr\Sigma_m)\wr\Sigma_n}((\underline{\underline{h},\tau});\sigma)} \mathbb{P}_{(\underline{\underline{h}},(\underline{\tau},\sigma))}(x)$$

where $(\underline{\underline{h},\tau}) \in (G \wr \Sigma_m)^{\times n}$, *and* $\sigma \in \Sigma_n$. $(\underline{\tau}, \sigma)$ *is in* $\Sigma_m \wr \Sigma_n$, *thus, can be viewed as an element in* Σ_{mn}.

(iv) $\mathbb{P}$ *preserves external product. For* $\underline{(g,h)} = ((g_1, h_1), \cdots (g_n, h_n)) \in (G \times H)^{\times n}$, $\sigma \in \Sigma_n$,

$$\mathbb{P}_{(\underline{(g,h)},\sigma)}(x \wedge y) = res|^{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{(g,h)},\sigma)} \mathbb{P}_{(\underline{g},\sigma)}(x) \boxtimes \mathbb{P}_{(\underline{h},\sigma)}(y).$$

Proof. Let's check each one respectively.

(i) When $n = 1$, all the cycles of a permutation is 1-cycle. $(\ )_1$ and the homeomorphism $f_{(g,(1))}$ are both identity maps. Directly from the formula (3.30), $\mathbb{P}_1(x) = x$.

(ii)

$$\begin{aligned}\mathbb{P}_{(\underline{g},\sigma)}(x) \boxtimes \mathbb{P}_{(\underline{h},\tau)}(x) &= f^*_{(\underline{g},\sigma)}(\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x_{g_{i_k}\cdots g_{i_1}})_k) \boxtimes f^*_{(\underline{h},\tau)}(\boxtimes_j \boxtimes_{(r_1,\cdots r_j)} (x_{h_{r_j}\cdots h_{r_1}})_j) \\ &= res|^{\Lambda_{G\wr\Sigma_{m+n}}(\underline{g},\underline{h};\sigma\tau)}_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{G\wr\Sigma_m}(\underline{h},\tau)} f^*_{(\underline{g},\underline{h};\sigma\tau)}((\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x_{g_{i_k}\cdots g_{i_1}})_k) \\ &\quad \boxtimes (\boxtimes_j \boxtimes_{(r_1,\cdots r_j)} (x_{h_{r_j}\cdots h_{r_1}})_j)).\end{aligned}$$

where $(i_1, \cdots i_k)$ goes over all the k-cycles of σ and $(r_1, \cdots r_j)$ goes over all the j-cycles of τ and $(\ )_k$ is the map cited in (3.24). The second step is from Proposition 3.1.13 (iii).

$$f^*_{(\underline{g},\underline{h};\sigma\tau)}((\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x_{g_{i_k}\cdots g_{i_1}})_k) \boxtimes (\boxtimes_j \boxtimes_{(r_1,\cdots r_j)} (x_{h_{r_j}\cdots h_{r_1}})_j))$$

is exactly

$$\mathbb{P}_{(\underline{g},\underline{h};\sigma\tau)}(x).$$

(iii) Recall that for an element $(\underline{\tau}, \sigma) = (\tau_1, \cdots \tau_n, \sigma) \in \Sigma_{mn}$, it acts on the set with mn elements

$$\{(i, j) | 1 \leq i \leq n, 1 \leq j \leq m\}$$

in this way:

$$(\underline{\tau}, \sigma) \cdot (i, j) = (\sigma(i), \tau_{\sigma(i)}(j)).$$

That also shows how to view it as an element in Σ_{mn}.

Then for any integer q,

$$(\underline{\tau}, \sigma)^q \cdot (i, j) = (\sigma^q(i), \tau_{\sigma^q(i)} \tau_{\sigma^{q-1}(i)} \cdots \tau_{\sigma(i)}(j)). \tag{3.31}$$

To find all the cycles of $(\underline{\tau}, \sigma)$ is exactly to find all the orbits of the action by $(\underline{\tau}, \sigma)$. If i belongs to an s-cycle of σ and j belongs to a r-cycle of $\tau_{\sigma^s(i)} \tau_{\sigma^{s-1}(i)} \cdots \tau_{\sigma(i)}$, then the orbit containing (i, j) has sr elements by (3.31). In other words, $(i_1, \cdots i_s)$ is an s-cycle of σ and $(j_1, \cdots j_r)$ is a r-cycle of $\tau := \tau_{i_s} \cdots \tau_{i_1}$ if and only if

$$\begin{aligned}
\Big(&(i_1, \tau_{i_1}(j_{r-1}))(i_2, \tau_{i_2}\tau_{i_1}(j_{r-1})) \cdots (i_s, j_r) \\
&(i_1, \tau_{i_1}(j_{r-2}))(i_2, \tau_{i_2}\tau_{i_1}(j_{r-2})) \cdots (i_s, j_{r-1}) \\
&\cdots \\
&(i_1, \tau_{i_1}(j_1))(i_2, \tau_{i_2}\tau_{i_1}(j_1)) \cdots (i_s, j_2) \\
&(i_1, \tau_{i_1}(j_r))(i_2, \tau_{i_2}\tau_{i_1}(j_r)) \cdots (i_s, j_1) \Big)
\end{aligned}$$

is an sr-cycle of $(\underline{\tau}, \sigma)$.

$$\begin{aligned}
\mathbb{P}_{((\underline{\underline{h}}, \tau); \sigma)}(\mathbb{P}_m(x)) &= f^*_{((\underline{\underline{h}}, \tau); \sigma)}[\boxtimes_k \boxtimes_{(i_1, \cdots i_k)} (\mathbb{P}_{((\underline{h}_{i_k}, \tau_{i_k}) \cdots (\underline{h}_{i_1}, \tau_{i_1}))}(x))_k] \\
&= f^*_{((\underline{\underline{h}}, \tau); \sigma)}[\boxtimes_k \boxtimes_{(i_1, \cdots i_k)} [f^*_{((\underline{h}_{i_k}, \tau_{i_k}) \cdots (\underline{h}_{i_1}, \tau_{i_1}))}(\boxtimes_r \boxtimes_{(j_1, \cdots j_r)} (x_{H_{\underline{i}\underline{j}}})_r)]_k] \\
&= (f^*_{((\underline{\underline{h}}, \tau); \sigma)} \circ \prod_{k, (i_1, \cdots i_k)} f^*_{((\underline{h}_{i_k}, \tau_{i_k}) \cdots (\underline{h}_{i_1}, \tau_{i_1}))})[\boxtimes_{k, (i_1, \cdots i_k)} \boxtimes_{r, (j_1, \cdots j_r)} (x_{H_{\underline{i}, \underline{j}}})_{kr}] \\
&= f^*_{(\underline{\underline{h}}, (\underline{\tau}, \sigma))}[\boxtimes_{k, (i_1, \cdots i_k)} \boxtimes_{r, (j_1, \cdots j_r)} (x_{H_{\underline{i}, \underline{j}}})_{kr}]
\end{aligned}$$

where

$$
\begin{aligned}
H_{\underline{i}\underline{j}} :=\ & h_{i_k,j_1} h_{i_{k-1},\tau_{i_k}^{-1}(j_1)} \cdots h_{i_1,(\tau_{i_k}\cdots\tau_{i_2})^{-1}(j_1)} \\
& h_{i_k,j_2} h_{i_{k-1},\tau_{i_k}^{-1}(j_2)} \cdots h_{i_1,(\tau_{i_k}\cdots\tau_{i_2})^{-1}(j_2)} \\
& \cdots \\
& h_{i_k,j_r} h_{i_{k-1},\tau_{i_k}^{-1}(j_r)} \cdots h_{i_1,(\tau_{i_k}\cdots\tau_{i_2})^{-1}(j_r)} \\
=\ & h_{i_k,j_1} h_{i_{k-1},\tau_{i_{k-1}}\cdots\tau_{i_2}\tau_{i_1}(j_1)} \cdots h_{i_1,\tau_{i_1}(j_r)} \\
& h_{i_k,j_2} h_{i_{k-1},\tau_{i_{k-1}}\cdots\tau_{i_2}\tau_{i_1}(j_2)} \cdots h_{i_1,\tau_{i_1}(j_1)} \\
& \cdots \\
& h_{i_k,j_r} h_{i_{k-1},\tau_{i_{k-1}}\cdots\tau_{i_2}\tau_{i_1}(j_{r-1})} \cdots h_{i_1,\tau_{i_1}(j_{r-1})}
\end{aligned}
$$

where $(i_1, \cdots i_k)$ goes over all the k-cycles of $\sigma \in \Sigma_m$ and $(j_1, \cdots j_r)$ goes over all the r-cycles of $\tau_{i_k} \cdots \tau_{i_1} \in \Sigma_n$. The last step is by Proposition 4.11 in [24].

$$f^*_{(\underline{\underline{h}},(\underline{\tau},\sigma))}[\boxtimes_{k,(i_1,\cdots i_k)} \boxtimes_{r,(j_1,\cdots j_r)} (x_{H_{\underline{i},\underline{j}}})_{kr}]$$

is the same space as $\mathbb{P}_{(\underline{\underline{h}},(\underline{\tau},\sigma))}(x)$, but the action is restricted by

$$res|^{\Lambda_{G\wr\Sigma_{mn}}(\underline{\underline{h}},(\underline{\tau},\sigma))}_{\Lambda_{(G\wr\Sigma_m)\wr\Sigma_n}((\underline{\underline{h},\tau});\sigma)}.$$

(iv)We have

$$
\begin{aligned}
&\mathbb{P}_{(\underline{(g,h)},\sigma)}(x \wedge y) = f^*_{(\underline{(g,h)},\sigma)}(\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} ((x\wedge y)_{(g_{i_k}\cdots g_{i_1}, h_{i_k}\cdots h_{i_1})})_k) \\
=&f^*_{(\underline{(g,h)},\sigma)}(\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x_{g_{i_k}\cdots g_{i_1}})_k \boxtimes (y_{h_{i_k}\cdots h_{i_1}})_k) \\
=&f^*_{(\underline{(g,h)},\sigma)}(\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x_{g_{i_k}\cdots g_{i_1}})_k) \boxtimes (\boxtimes_j \boxtimes_{(r_1,\cdots r_j)} (y_{h_{r_j}\cdots h_{r_1}})_j) \\
=&res|^{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{(g,h)},\sigma)} (f^*_{(\underline{g},\sigma)} \times f^*_{(\underline{h},\sigma)})(\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x_{g_{i_k}\cdots g_{i_1}})_k) \boxtimes (\boxtimes_j \boxtimes_{(r_1,\cdots r_j)} (y_{h_{r_j}\cdots h_{r_1}})_j) \\
=&res|^{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{(g,h)},\sigma)} f^*_{(\underline{g},\sigma)}[\boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x_{g_{i_k}\cdots g_{i_1}})_k] \boxtimes f^*_{(\underline{h},\sigma)}[\boxtimes_j \boxtimes_{(r_1,\cdots r_j)} (y_{h_{r_j}\cdots h_{r_1}})_j],
\end{aligned}
$$

where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ and $(r_1, \cdots r_j)$ goes over all the $j-$cycles of σ. It equals to

$$res|^{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{g},\sigma)} \mathbb{P}_{(\underline{g},\sigma)}(x) \boxtimes \mathbb{P}_{(\underline{h},\sigma)}(y).$$

$\square$

Example 3.2.2. *Let G be the trivial group and X a space. Let $\sigma \in \Sigma_n$. Then $QEll_G^*(X) = K_{\mathbb{T}}^*(X)$. $F_{(\underline{1},\sigma)}$ is the homeomorphism*

$$\mathcal{L}_{(\underline{1},\sigma)}(X^{\times n}) \cong \prod_k \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}X$$

in Theorem 3.1.9, where the second direct product goes over all the k-cycles of σ. By (3.30), the power operation is

$$\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k.$$

When $n = 2$, $\mathbb{P}_{(\underline{1},(1)(1))}(x) = x \boxtimes x$ and $\mathbb{P}_{(\underline{1},(12))}(x) = (x)_2$.

When $n = 3$, $\mathbb{P}_{(\underline{1},(1)(1)(1))}(x) = x \boxtimes x \boxtimes x$, $\mathbb{P}_{(\underline{1},(12)(1))}(x) = (x)_2 \boxtimes x$, and $\mathbb{P}_{(\underline{1},(123))}(x) = (x)_3$.

When $n = 4$, $\mathbb{P}_{(\underline{1},(1)(1)(1)(1))}(x) = x \boxtimes x \boxtimes x \boxtimes x$, $\mathbb{P}_{(\underline{1},(12))}(x) = (x)_2 \boxtimes x \boxtimes x$, $\mathbb{P}_{(\underline{1},(123))}(x) = (x)_3 \boxtimes x$, $\mathbb{P}_{(\underline{1},(1234))}(x) = (x)_4$, and $\mathbb{P}_{(\underline{1},(12)(34))}(x) = (x)_2 \boxtimes (x)_2$. Note that there is a Σ_2-action permuting the two $(x)_2$ in $\mathbb{P}_{(\underline{1},(12)(34))}(x)$.

When G is trivial, $\mu_k(x) = (x)_k$, though generally they are not equal.

Remark 3.2.3. *We have the relation between equivariant Tate K-theory and quasi-elliptic cohomology*

$$QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K_{Tate}(X//G). \tag{3.32}$$

It extends uniquely to a power operation for Tate K-theory

$$QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \longrightarrow QEll_{G \wr \Sigma_n}(X^{\times n}) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q))$$

which is the stringy power operation P_n^{string} constructed in [24]. It is elliptic in the sense of [2].

3.3 Orbifold quasi-elliptic cohomology and some basic constructions

The elliptic cohomology of orbifolds involves a rich interaction between the orbifold structure and the elliptic curve. [25] explores this interaction in the case of the Tate curve, describing

K_{Tate} for an orbifold X in term of the equivariant K-theory and the groupoid structure of X. I give a description of quasi-elliptic cohomology for an orbifold in Section 3.3.1.

In Section 3.3.2 I recall the symmetric power of groupoid, which is introduced in [25]. In addition, I discuss the inertia groupoid of symmetric power and the groupoids needed for the construction of the power operation in Section 3.4.

3.3.1 Definition

First let's recall some relevant constructions and notations. The main reference is [25].

Consider the category of groupoids $\mathcal{G}pd$ as a 2-category with small topological groupoids as the objects and with

$$1\mathrm{Hom}(X, Y) = Fun(X, Y),$$

the groupoid of continuous functors from X to Y.

The center of a groupoid X is defined to be the group

$$\mathrm{Center}(X) := 2\mathrm{Hom}(\mathrm{Id}_X, \mathrm{Id}_X) = \mathcal{N}at(\mathrm{Id}_X, \mathrm{Id}_X)$$

of natural transformations from Id_x to Id_x.

Let $\mathcal{G}pd^{cen}$ denote the 2-category whose objects are pairs (X, ξ) with ξ a center element of X, and the set of morphisms from (X, ξ) to (Y, ν) is

$$1\mathrm{Hom}((X, \xi), (Y, \nu)) \subset Fun(X, Y)$$

with

$$f\xi = \nu f$$

for each morphism f. I will assume all the center elements have finite order.

Example 3.3.1. *If G is a finite group, Center$(pt//G)$ is the center of the group G.*

Example 3.3.2. *The Inertia groupoid $I(X)$ of a groupoid X, which is defined in Definition 2.4.10, is isomorphic to*

$$Fun(pt//\mathbb{Z}, X).$$

Each object of $I(X)$ can be viewed as pairs (x, g) with $x \in ob(X)$ and $g \in aut(x)$, $gx = x$. A morphism from (x_1, g_1) to (x_2, g_2) is a morphism $h : x_1 \longrightarrow x_2$ in X satisfying $h \circ g_1 = g_2 \circ h$

in X. So in $I(X)$,

$$Hom((x_1, g_1), (x_2, g_2)) = \{h : x_1 \longrightarrow x_2 | h \circ g_1 = g_2 \circ h\}.$$

Recall $I^{tors}(X)$ is a full subgoupoid of $I(X)$ with elements (x, g) where g is of finite order. Let ξ^k denote the center element of $I^{tors}(X)$ sending (x, g) to (x, g^k). I will use ξ to denote ξ^1.

For any $k \in \mathbb{Z}$, we have the 2-functor

$$\mathcal{G}pd \longrightarrow \mathcal{G}pd^{cen}$$
$$X \mapsto (I^{tors}(X), \xi^k).$$

Example 3.3.3. *In the global quotient case, as indicated in Example 2.4.11, $I^{tors}(X//G)$ is isomorphic to $\coprod\limits_{g \in G^{tors}_{conj}} X^g//C_G(g)$. And the center element $\xi^k|_{X^g} = g^k$.*

Let $\mathrm{pt}//\mathbb{R} \times_{1\sim\xi} I^{tors}(X)$ denote the groupoid

$$(\mathrm{pt}//\mathbb{R}) \times I^{tors}(X)/\sim$$

with $\sim$ generated by $1 \sim \xi$.

Definition 3.3.4. *For any topological groupoid X, the quasi-elliptic cohomology $QEll^*(X)$ is the orbifold K-theory*

$$K^*_{orb}(pt//\mathbb{R} \times_{1\sim\xi} I^{tors}(X)). \tag{3.33}$$

In other words, for a topological groupoid X, $QEll(X)$ is defined to be a subring of $K_{orb}(X)[\![q^{\pm\frac{1}{|\xi|}}]\!]$ that is the Grothendieck group of finite sums

$$\sum_{a \in \mathbb{Q}} V_a q^a$$

satisfying:

for each $a \in \mathbb{Q}$, the coefficient V_a is an $e^{2\pi i a}-$ eigenbundle of ξ.

In the global quotient case,

$$QEll^*(X//G) = QEll^*_G(X).$$

Remark 3.3.5. *For any topological groupoid X, we can also consider the category*

$$Loop_1(X) := Bibundle(S^1//*, X)$$

and formulate $Loop_1^{ext}(X)$ by adding the rotation action by circle, as in Section 2.1.2. And then, we can formulate the subgroupoid $\Lambda(X)$ of $Loop_1^{ext}(X)$ consisting of the constant loops, which is isomorphic to $pt//\mathbb{R} \times_{1\sim\xi} I^{tors}(X)$.

3.3.2 Symmetric powers of orbifolds and its torsion Inertia groupoid

In this section I introduce the total symmetric power of groupoids, the torsion Inertia groupoid of it, the total symmetric power of the torsion Inertia groupoid, and other groupoids that are relevant in the construction of the power operation. In addition, in Lemma 3.3.10, Lemma 3.3.12 and Lemma 3.3.13 I show the relation between these groupoids.

$\mathcal{G}pd$ is a symmetric bimonoidal category with the monoidal structures given by $(\sqcup, \emptyset)$ and $(\times, \mathrm{pt})$. Each groupoid $X \in \mathrm{ob}(\mathcal{G}pd)$ is a monoid with respect to $(\sqcup, \emptyset)$, via the folding map

$$d : X \sqcup X \longrightarrow X.$$

X is a monoid with respect to $(\times, \mathrm{pt})$ if and only if X itself is a symmetric monoidal category.

Example 3.3.6 (Total Symmetric Power). *I introduce the $n-th$ symmetric power $X \wr \Sigma_n$ of a groupoid X in Definition 3.1.2. In addition, the total symmetric power of X is defined to be*

$$S(X) := \coprod_{n\geq 0} X \wr \Sigma_n. \tag{3.34}$$

S is a functor from $\mathcal{G}pd$ to $\mathcal{G}pd$.

The functor S is exponential. We have

$$S(X \sqcup Y) = \coprod_{n\geq 0} (X \sqcup Y) \wr \Sigma_n = (\coprod_{n\geq 0} X \wr \Sigma_n) \times (\coprod_{m\geq 0} Y \wr \Sigma_m) = S(X) \times S(Y).$$

In other words, the functor

$$S : (\mathcal{G}pd, \sqcup, \emptyset) \longrightarrow (\mathcal{G}pd, \times, pt)$$

is monoidal.

Moreover, S maps monoids to monoids. $S(X)$ is a monoid with respect to $(*, (\ ))$ where

$$* := S(d),$$

i.e. the concatenation, and the unit $(\ )$ is the unique object in $X \wr \Sigma_0$. The monoidal structure is symmetric.

The triple

$$(S(X), *, (\ ))$$

is "free" symmetric monoidal category on X in the sense that the functor

$$X \mapsto (S(X), *, (\ ))$$

is left adjoint to the forgetful functor from the category of monoids in $\mathcal{G}pd$ to $\mathcal{G}pd$.

Example 3.3.7 (The torsion Inertia groupoid of the total symmetric power). *Now let's consider the groupoid $I^{tors}(S(X))$. It's the disjoint union*

$$\coprod_{n \geq 0} I^{tors}(X \wr \Sigma_n).$$

Each object of it is of the form $(\underline{x}, \underline{g}, \sigma, n)$ where $n \geq 0$ is an integer, $\underline{x} \in ob(X \wr \Sigma_n)$, $(\underline{g}, \sigma) \in G \wr \Sigma_n$. Since $(\underline{g}, \sigma)$ is an automorphism of $\underline{x}$ in $X \wr \Sigma_n$, each arrow g_i is a morphism from x_i to $x_{\sigma^{-1}(i)}$. The information of each object in $I^{tors}(S(X))$ is contained in $(\underline{g}, \sigma)$.

$I^{tors}(S(X))$ inherits a monoidal structure from $S(X)$:

$$(\underline{g}, \sigma) * (\underline{h}, \tau) = (\underline{g}, \underline{h}, \sigma \sqcup \tau).$$

An object $(\underline{g}, \sigma)$ of $I^{tors}(S(X))$ is indecomposable with respect to $*$ if and only if $n > 0$ and $\sigma \in \Sigma_n$ has only one orbit.

Moreover, for groupoids like $\mathrm{pt}//\mathbb{R} \times_{k \sim \xi} X$, instead of $S(\mathrm{pt}//\mathbb{R} \times_{k \sim \xi} X)$, let's consider a subgroupoid

$$S^R(\mathrm{pt}//\mathbb{R} \times_{k\sim\xi} X)$$

of it.

Example 3.3.8 (The Groupoid $S^R(\mathbf{pt}//\mathbb{R} \times_{k\sim\xi} X)$). *Let*

$$\rho_k : pt//\mathbb{R} \times_{k\sim\xi} X \longrightarrow pt//(\mathbb{R}/\mathbb{Z})$$

be the functor sending all the objects to the single point, and an arrow

$$[g, t]$$

to

$$t \ mod \ \mathbb{Z}.$$

Let $\times_{\mathbb{R}}(pt//\mathbb{R} \times_{k\sim\xi} X)$ *denote the limit of the diagram*

$$pt//\mathbb{R} \times_{k\sim\xi} X \xrightarrow{\rho_k} pt//(\mathbb{R}/\mathbb{Z}) \xleftarrow{\rho_k} pt//\mathbb{R} \times_{k\sim\xi} X \ .$$

And let

$$\times^n_{\mathbb{R}}(pt//\mathbb{R} \times_{k\sim\xi} X)$$

denote the limit of n *morphisms* ρ_k*s.*

$\times^n_{\mathbb{R}}(pt//\mathbb{R} \times_{k\sim\xi} X)$ *inherits a* Σ_n-*action on it by permutation from that on the product* $(pt//\mathbb{R} \times_{k\sim\xi} X)^{\times n}$.

Let $S^R_n(pt//\mathbb{R} \times_{k\sim\xi} X)$ *denote the groupoid with the same objects as*

$$\times^n_{\mathbb{R}}(pt//\mathbb{R} \times_{k\sim\xi} X)$$

and morphisms of the form $([g_1, t_1], \cdots [g_n, t_n]; \sigma)$ *with* $([g_1, t_1], \cdots [g_n, t_n])$ *a morphism in* $\times^n_{\mathbb{R}}(pt//\mathbb{R} \times_{k\sim\xi} X)$ *and* $\sigma \in \Sigma_n$*. This new groupoid* $S^R_n(pt//\mathbb{R} \times_{k\sim\xi} X)$ *is a subgroupoid of*

$$(pt//\mathbb{R} \times_{k\sim\xi} X) \wr \Sigma_n.$$

Let

$$S^R(pt//\mathbb{R}\times_{k\sim\xi} X) := \coprod_{n\geq 0} S^R_n(pt//\mathbb{R}\times_{k\sim\xi} X).$$

The triple

$$(S^R(pt//\mathbb{R}\times_{k\sim\xi} X), *, (\ ))$$

is a symmetric monoid.

$S^R(pt//\mathbb{R}\times_{k\sim\xi} X)$ *is the symmetric product that we will use to formulate the power operation of quasi-elliptic cohomology.*

Example 3.3.9 (The groupoid $\Phi(X)$). *Let $\Phi_k(X)$ denote the full subgroupoid of $I^{tors}(S(X))$ whose objects are $(\underline{x}, \underline{g}, \varsigma_k)$ with ς_k a generater in the cyclic group $\langle(12\cdots k)\rangle$. And let $\phi_k \in$Center(Φ_k) denote the restriction of $S_k(\xi)$ to Φ_k.*

Let $\Phi(X) := \coprod_{k\geq 1} \Phi_k(X)$. Then

$$\phi := \coprod_{k\geq 1} \phi_k \in \mathit{Center}(\Phi)$$

is the restriction of $S(\xi)$ to Φ.

Thus, the essential image of $\Phi(X)$ in $I^{tors}(SX)$ is the subgroupoid consisting of indecomposable objects. And the functor Φ is additive: $\Phi(\emptyset) = \emptyset$ and it preserves $\sqcup$.

Lemma 3.3.10. *For each integer $k \geq 1$, there is an equivalence between*

$$pt//\mathbb{R}\times_{1\sim\phi_k} \Phi_k(X)$$

and the groupoid $pt//\mathbb{R}\times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X)[\xi^{\frac{1}{k}}]$ which identifies ϕ_k with $\xi^{\frac{1}{k}}$. Here $\xi^{\frac{1}{k}}$ is an added element such that the composition of k $\xi^{\frac{1}{k}}$s is ξ.

Proof. Lemma 3.3.10 is a special case of Theorem 3.1.9 that the element $\sigma \in \Sigma_k$ has only one cycle. We can define a functor

$$A_k : \mathrm{pt}//\mathbb{R}\times_{1\sim\phi_k} \Phi_k(X) \longrightarrow \mathrm{pt}//\mathbb{R}\times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X)[\xi^{\frac{1}{k}}]$$

by sending an object $(\underline{x}, \underline{g}, (12\cdots k))$ to $(x_1, g_k\cdots g_1)$ and sending a morphism $[\underline{h}, (12\cdots k)^m, t]$ to

$$[h_k g^{-1}_{1-m}\cdots g^{-1}_{k-1} g^{-1}_k, m+t].$$

Recall $h_k g_{1-m}^{-1} \cdots g_{k-1}^{-1} g_k^{-1}$ conjugates $g_k \cdots g_1$ to itself. It is $\beta^{\underline{h},\mathrm{Id}}_{(12\cdots k),(12\cdots k)}$ defined in (3.6), which is used in the proof of Theorem 3.1.9. A_k is an isomorphism, as implied in the proof of Theorem 3.1.9.

□

Remark 3.3.11. *As an immediate corollary, we have the equivalence between the disjoint union of groupoids* $\coprod_k pt//\mathbb{R} \times_{1\sim\phi_k} \Phi_k(X)$ *and* $\coprod_k pt//\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X)[\xi^{\frac{1}{k}}]$ *where k goes over all the integers.*

Moreover, the symmetric power $S(\coprod_k pt//\mathbb{R} \times_{1\sim\phi_k} \Phi_k(X))$ *is equivalent to the groupoid* $S(\coprod_k pt//\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X)[\xi^{\frac{1}{k}}])$*. The formula of the equivalence is straightforward to obtain from the functors A_ks.*

Theorem 3.1.9 can be reinterpreted as the lemma below, where the t part makes a difference and we need the product S^R.

Lemma 3.3.12. *The groupoid* $S^R(\coprod_k pt//\mathbb{R}\times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X)[\xi^{\frac{1}{k}}])$ *is equivalent to* $pt//\mathbb{R}\times_{1\sim S(\xi)} I^{tors}(S(X))$.

Most details in the proof of Lemma 3.3.12 come from that of Theorem 3.1.9.

Proof. I'll define a functor F from $\mathrm{pt}//\mathbb{R}\times_{1\sim S(\xi)} I^{tors}(S(X))$ to $S^R(\coprod_k \mathrm{pt}//\mathbb{R}\times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X)[\xi^{\frac{1}{k}}])$.

Let

$$(\underline{x}, \underline{g}, \sigma)$$

be an object in the category $\mathrm{pt}//\mathbb{R} \times_{1\sim S(\xi)} I^{tors}(S(X))$ where $\sigma \in \Sigma_n$. F sends it to

$$\prod_k \prod_{(i_1,\cdots i_k)} (x_{i_1}, g_{i_k} \cdots g_{i_1})$$

where the second product goes over all the $k-$cycles of σ.

Let

$$[\underline{h}, \tau; t]$$

be a morphism starting at the object $(\underline{x}, \underline{g}, \sigma)$.

Let τ send the $k-$cycle $i = (i_1, \cdots i_k)$ of σ to a $k-$cycle $j = (j_1, \cdots j_k)$ of σ and $\tau(i_1) = j_{1+m_i}$.

F sends $[\underline{h}, \tau; t]$ to

$$\times_k \times_{i\in\theta_k} ([\beta^{\underline{h},\tau}_{\tau(1),1}, m_1 + t], \cdots [\beta^{\underline{h},\tau}_{\tau(M_i^\sigma), M_i^\sigma}, m_{M_i^\sigma} + t], \tau|_{W_i^\sigma})$$

where $\tau|_{W_i^\sigma}$ denotes the permutation induced by τ on the set $W_i^\sigma = \{\alpha_1^i, \alpha_2^i, \cdots \alpha^i_{M_i^\sigma}\}$, $\tau^{-1}(j)$ is short for $\tau^{-1}(\alpha^i_j)$, and $\tau(j_l) = \tau(j)_{l+m_j}$.

F is fully faithful and essentially surjective, as implied in the proof of Theorem 3.1.9. $\square$

Lemma 3.3.13. *We have an equivalence of groupoids*

$$Q^R : S^R(pt//\mathbb{R} \times_{1\sim\phi} \Phi(X)) \longrightarrow pt//\mathbb{R} \times_{1\sim S(\xi)} I^{tors}(S(X)),$$

which is natural in X and satisfies

$$Q^R S^R(\phi) = S(\xi) Q^R.$$

Proof. Let I be the inclusion

$$\mathrm{pt}//\mathbb{R} \times_{1\sim\phi} \Phi(X) \longrightarrow \mathrm{pt}//\mathbb{R} \times_{1\sim S(\xi)} I^{tors}(S(X)).$$

Let ϵ be the counit of the adjunction $(S, *, (\ )) \dashv$ forget. Let Q be the composition

$$Q : S(\mathrm{pt}//\mathbb{R} \times_{1\sim\phi} \Phi(X)) \xrightarrow{S(I)} S(\mathrm{pt}//\mathbb{R} \times_{1\sim S(\xi)} I^{tors}(S(X))) \xrightarrow{\epsilon} \mathrm{pt}//\mathbb{R} \times_{1\sim S(\xi)} I^{tors}(S(X)).$$

Let Q^R be the restriction of Q to the subgroupoid $S^R(\mathrm{pt}//\mathbb{R} \times_{1\sim\phi} \Phi(X))$, i.e. the composition

$$Q^R : S^R(\mathrm{pt}//\mathbb{R} \times_{1\sim\phi} \Phi(X)) \xrightarrow{S^R(I)} S^R(\mathrm{pt}//\mathbb{R} \times_{1\sim S(\xi)} I^{tors}(S(X)))$$
$$\xrightarrow{\text{restriction of } \epsilon} \mathrm{pt}//\mathbb{R} \times_{1\sim S(\xi)} I^{tors}(S(X)).$$

The essential image of I consists exactly of the indecomposable objects of $\mathrm{pt}//\mathbb{R} \times_{1\sim S(\xi)} I^{tors}(S(X))$, thus, both Q and Q^R are essentially surjective.

Q is not fully faithful but Q^R is. That's why we need the product S^R instead of S.

$\square$

3.4 Power Operation for orbifold quasi-elliptic cohomology

In this section I first recall the definition of power operation for cohomology theories for orbifolds and construct Atiyah's power operation for quasi-elliptic cohomology in Example 3.4.3. Then I construct the total power operation for the orbifold quasi-elliptic cohomology

$$P^{Ell} : QEll(X) \longrightarrow QEll(SX)$$

for any orbifold groupoid X, as shown in (3.38). The power operation I constructed in Section 3.1 is a special case of it for $G-$spaces.

First Let's recall the definition of the total power operation for orbifold theories given in [25].

Definition 3.4.1. *Let E be a cohomology theory for orbifolds with product and let S be the symmetric power product on the orbifolds. A total power operation for E is a (non-linear) natural transformation*

$$P : E \Rightarrow E \circ S$$

satisfying

- *Comodule property: P makes E a comodule over the comonad $(-) \circ S$.*
- *Exponentiality: The map*

$$P : E(\emptyset) \longrightarrow E(pt)$$

sends 0 to 1, and

$$P : E(X \sqcup Y) \longrightarrow E(SX \times SY)$$

sends (a, b) to the external product $P(a) \otimes P(b)$.

We can write P as $P = (P_n)_{n\geq 0}$, with $P_n := l_n^* P$. Then P_n is called the nth power operation of P.

Example 3.4.2. *Let's recall the Atiyah's power operations for equivariant K-theories defined in [5]. In real or complex K-theory, let $x = [V]$ be represented by a G-equivariant vector bundle V over a $G-$space X. Then the nth exterior product $P_n(V) := V^{\boxtimes n}$ over $X^{\times n}$ has an obvious $G \wr \Sigma_n-$structure.*

Then let's consider the case when X is a topological orbifold. Let V be an orbifold vector bundle over X. Then $P_n(V) := V^{\boxtimes n}$ is an orbifold vector bundle over S_nX. $P = (P_n)_{n\geq 0}$ satisfies the axioms of a total power operation.

Example 3.4.3. *Let's consider the case when E is quasi-elliptic cohomology. Let X be a topological orbifold. As defined in Section 3.3.1, ξ^k is the center element on $I^{tors}(X)$ sending (x, g) to (x, g^k). Let V be an orbifold vector bundle over the orbifold*

$$pt//\mathbb{R} \times_{1\sim\xi} I^{tors}(X),$$

thus, V represents an element in $QEll(X)$.

Recall $QEll(X)$ is a $\mathbb{Z}[q^{\pm}]-$algebra via the inclusion

$$\mathbb{Z}[q^{\pm}] \cong QEll(pt) \longrightarrow QEll(X),$$

and the exterior product $\boxtimes$ in quasi-elliptic cohomology is defined as the tensor product of $\mathbb{Z}[q^{\pm}]-$algebras. Then

$$P_n(V) := V^{\boxtimes n}$$

is an orbifold vector bundle over

$$S^R(pt//\mathbb{R} \times_{1\sim\xi} I^{tors}(X)),$$

which is equivalent to

$$pt//\mathbb{R} \times_{1\sim\xi} SI^{tors}(X).$$

So $P_n(V)$ is in $QEll^*(S(X))$.

$P = (P_n)_{n\geq 0}$ satisfies the axioms of a total power operation.

Before the construction of the power operation of $QEll$, I introduce several more necessary constructions.

Let X be an orbifold groupoid and $k \geq 1$ an integer. We define the map

$$s_k : K_{orb}(\mathrm{pt}//\mathbb{R} \times_{1\sim\xi} I^{tors}(X)) \longrightarrow K_{orb}(\mathrm{pt}//\mathbb{R} \times_{k\sim\xi} I^{tors}(X)) \quad (3.35)$$

$$[\sum V_a q^a] \mapsto [\sum V_a q^{\frac{a}{k}}] \quad (3.36)$$

and

$$\coprod_k s_k : K_{orb}(\mathrm{pt}//\mathbb{R} \times_{1\sim\xi} I^{tors}(X)) \longrightarrow K_{orb}(\coprod_k (\mathrm{pt}//\mathbb{R} \times_{k\sim\xi} I^{tors}(X))). \tag{3.37}$$

The functor

$$(\ )_k : \Lambda_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda^1_{(\underline{g},\sigma)}(X)$$

defined in (3.22) is a special local case of s_k when X is a $G-$space and $(\underline{g}, \sigma)$ is fixed.

Let $\theta : QEll(X) \longrightarrow K_{orb}(\mathrm{pt}//\mathbb{R} \times_{1\sim\phi} \Phi(X))$ be the additive operation whose $k-$th component is $A_k^* \circ s_k$, where A_k is the equivalence defined in Lemma 3.3.10.

Now we are ready to define the total power operation of $QEll^*$:

$$P^{Ell} : QEll(X) \xrightarrow{\theta} K_{orb}(\mathrm{pt}//\mathbb{R}\times_{1\sim\phi}\Phi(X) \xrightarrow{P} K_{orb}(S^R(\mathrm{pt}//\mathbb{R}\times_{1\sim\phi}\Phi(X))) \overset{(Q^{R*})^{-1}}{\longrightarrow} QEll(SX). \tag{3.38}$$

Theorem 3.4.4. *P^{Ell} satisfies the axioms of a total power operation in Definition 3.4.1.*

Proof. From the definition of P^{Ell}, we can see it is a well-defined natural transformation $QEll \Rightarrow QEll \circ S$ and is a comodule over the comonad $(-) \circ S$.

In addition, the functor θ has the property of additivity

$$\theta : QEll(X \sqcup Y) \longrightarrow QEll(\Phi(X) \sqcup \Phi(Y))$$
$$(a, b) \mapsto (\theta(a), \theta(b)).$$

And the power operation P defined in Example 3.4.3 has the exponential property. Therefore, P^{Ell} has the property of exponential. So P^{Ell} is a total power operation. $\square$

Remark 3.4.5. *Let $X//G$ be a quotient orbifold. The power operation I constructed in Section 3.1 for quotient orbifolds is in fact the one below, if expressed totally:*

$\mathbb{P} : QEll^*(X//G) \xrightarrow{\coprod_k s_k} K^*_{orb}(\coprod_k pt//\mathbb{R}\times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X//G)[\xi^{\frac{1}{k}}]) \xrightarrow{P} K^*_{orb}(S^R(\coprod_k pt//\mathbb{R}\times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X//G)[\xi^{\frac{1}{k}}])) \xrightarrow{J^*} QEll^*(S(X//G))$ where J is constructed from the functors $J_{(\underline{g},\sigma)}$ in the proof of Theorem 3.1.9.

Here is a little explanation about $\mathbb{P}$. Each orbifold vector bundle V over the groupoid

$$\coprod_k pt//\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X//G)[\xi^{\frac{1}{k}}],$$

it's a $\mathbb{Z}[q^{\pm}]-$algebra. The $n-$th exterior product $P_n(V) := V^{\boxtimes n}$ is the tensor product as $\mathbb{Z}[q^{\pm}]-$algebras. It is an orbifold vector bundle over $S^R(\coprod_k pt//\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X//G)[\xi^{\frac{1}{k}}])$. And $P = (P_n)_{n\geq 0}$ satisfies the axioms for total power operation.

For quotient orbifolds, P^{Ell} and $\mathbb{P}$ are the same up to isomorphism. The diagram

$$\begin{tikzcd}
QEll^*(X//G) \arrow[d, "\theta"] & QEll^*(S(X//G)) \\
K_{orb}(pt//\mathbb{R} \times_{1\sim\phi} I^{tors}(\Phi(X//G))) \arrow[r, "P"] & K_{orb}(S^R(pt//\mathbb{R} \times_{1\sim\phi} \Phi(X//G))) \arrow[u, "(Q^{R*})^{-1}"'] \\
K_{orb}(\coprod_k pt//\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X//G)[\xi^{\frac{1}{k}}]) \arrow[u, "\coprod_k A_k^*"] \arrow[r, "P"] & K_{orb}(S^R(\coprod_k pt//\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I^{tors}(X//G)[\xi^{\frac{1}{k}}])) \arrow[u, "S^R(\coprod_k A_k^*)"']
\end{tikzcd}$$

commutes. The vertical maps $\coprod_k A_k^*$ and $S^R(\coprod_k A_k^*)$ are both equivalences of groupoids. And the horizontal maps are the power operation defined in Example 3.4.3.

3.5 Finite Subgroups of Tate Curve

Strickland showed in [60] that the quotient of the Morava E-theory of the symmetric group by a certain transfer ideal can be identified with the product of rings $\prod_{k\geq 0} R_k$ where each R_k classifies subgroup-schemes of degree p^k in the formal group associated to $E^0\mathbb{C}P^\infty$. I prove similar conclusions for Tate K-theory and quasi-ellitpic cohomology. The main conclusion for Section 3.5 is Theorem 3.5.1.

3.5.1 Background and Main Conclusion

An elliptic curve over the complex numbers $\mathbb{C}$ is a connected Riemann surface, i.e. a connected compact 1-dimensional complex manifold, of genus 1. By the uniformization theorem every elliptic curve over $\mathbb{C}$ is analytically isomorphic to a 1-dimensional complex torus, and can be expressed as

$$\mathbb{C}^*/q^{\mathbb{Z}}$$

with $q \in \mathbb{C}$ and $0 < |q| < 1$, where $\mathbb{C}^*$ is the multiplicative group $\mathbb{C}\backslash\{0\}$.

Tate curve $\mathrm{Tate}(q)$ is the elliptic curve

$$E_q : y^2 + xy = x^3 + a_4 x + a_6$$

whose coefficients are given by the formal power series in $\mathbb{Z}((q))$

$$a_4 = -5\sum_{n\geqslant 1} n^3 q^n/(1-q^n) \qquad a_6 = -\frac{1}{12}\sum_{n\geqslant 1}(7n^5 + 5n^3)q^n/(1-q^n).$$

The points of order N in $\mathbb{C}^*/q^{\mathbb{Z}}$ are

$$\begin{aligned}(\mathbb{C}^*/q^{\mathbb{Z}})[N] &= \{z \in \mathbb{C}^* | z^N \in q^{\mathbb{Z}}\}/q^{\mathbb{Z}} \\ &= \{z \in \mathbb{C}^* | z^N = q^k \text{ for some } k \in \mathbb{Z}\}/q^{\mathbb{Z}} \\ &= \{z \in \mathbb{C}^* | z^N = q^k, k = 0, 1, \cdots N-1\}/q^{\mathbb{Z}}.\end{aligned}$$

$\mathrm{Tate}(q)[N]$, the scheme of points of order N in $\mathrm{Tate}(q)$, is defined to be

$$\mathrm{Spec}(K_{Tate}(\mathrm{pt}//C_N)).$$

It is isomorphic to

$$\coprod_{k=0}^{N-1} \mathrm{Spec}\mathbb{Z}((q))[x]/(x^N - q^k).$$

In addition, there is a question how to classify all the finite subgroups of $\mathbb{C}^*/q^{\mathbb{Z}}$. The answer is already known, as shown below.

To give a subgroup for each order N, pick a pair of integers (d, e) and a nonzero complex number q' such that $N = de$ and $d, e \geq 1$. Let q' be a nonzero complex number such that $q^d = q'^e$. Consider the map

$$\begin{aligned}\psi_d : \mathbb{C}^*/q^{\mathbb{Z}} &\longrightarrow \mathbb{C}^*/q'^{\mathbb{Z}} \\ x &\mapsto x^d.\end{aligned}$$

It's well-defined since $\psi_d(q^{\mathbb{Z}}) \subseteq q'^{\mathbb{Z}}$.

We can check that $\mathrm{Ker}\psi_d$ has order N. Explicitly, it's

$$\{\mu_d^n q^{\frac{m}{e}} q^{\mathbb{Z}} | n, m \in \mathbb{Z}\}$$

where μ_d is a $d-$th primitive root of 1 and $q^{\frac{1}{e}}$ is a $e-$th primitive root of q. In fact

$$\{\mathrm{Ker}\psi_d |\ d \text{ divides } N \text{ and } d \geq 1\}$$

gives all the subgroups of $\mathbb{C}^*/q^{\mathbb{Z}}$ of order N.

Theorem 3.5.1 gives a classification of finite subgroups of Tate curve and a similar conclusion for the quasi-elliptic cohomology. In Section 3.5.2 I show the proof of Theorem 3.5.1.

Let N be an integer. Analogous to the transfer ideal I_{tr} of equivariant K-theory defined in (2.19), we can define the transfer ideal for Tate K-theory

$$I_{tr}^{Tate} := \sum_{\substack{i+j=N, \\ N>j>0}} \mathrm{Image}[I_{\Sigma_i \times \Sigma_j}^{\Sigma_N} : K_{Tate}(\mathrm{pt}//\Sigma_i \times \Sigma_j) \longrightarrow K_{Tate}(\mathrm{pt}//\Sigma_N)] \tag{3.39}$$

where I_H^G is the transfer map of K_{Tate} along $H \hookrightarrow G$ defined in (2.48), and the transfer ideal for quasi-elliptic cohomology

$$\mathcal{I}_{tr}^{QEll} := \sum_{\substack{i+j=N, \\ N>j>0}} \mathrm{Image}[\mathcal{I}_{\Sigma_i \times \Sigma_j}^{\Sigma_N} : QEll(\mathrm{pt}//\Sigma_i \times \Sigma_j) \longrightarrow QEll(\mathrm{pt}//\Sigma_N)] \tag{3.40}$$

with $\mathcal{I}_H^G$ the transfer map of $QEll$ along $H \hookrightarrow G$ defined in (2.50).

Theorem 3.5.1. *The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve. Explicitly,*

$$K_{Tate}(pt//\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[q']/\langle q^d - q'^e \rangle, \tag{3.41}$$

where I_{tr}^{Tate} is the transfer ideal defined in (3.39) and q' is the image of q under the power operation P^{Tate} constructed in [25]. The product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

We have the analogous conclusion for quasi-elliptic cohomology.

$$QEll(pt//\Sigma_N)/\mathcal{I}_{tr}^{QEll} \cong \prod_{N=de} \mathbb{Z}[q,q^{-1}][q']/\langle q^d - q'^e\rangle, \tag{3.42}$$

where $\mathcal{I}_{tr}^{QEll}$ *is the transfer ideal defined in (3.40) and* q' *is the image of* q *under the power operation* $\mathbb{P}_N$ *constructed in Section 3.2. The product goes over all the ordered pairs of positive integers* (d,e) *such that* $N = de$ *whose order matters.*

3.5.2 The proof

I'll show the proof of (3.42). The proof of (3.41) is similar.

Let's start with several simple examples.

Example 3.5.2 ($N = 1$ **and** $N = 2$**).** *When* $N = 1$, $q' = q$, $QEll(pt//\Sigma_1) = \mathbb{Z}[q,q^{-1}]$, *and* $\mathcal{I}_{tr}^{QEll}$ *is trivial. (3.42) holds obviously.*

The second simplest case is when $N = 2$.

$$\begin{aligned} QEll(pt//\Sigma_2) &= K_{\Lambda_{\Sigma_2}(1)}(pt) \times K_{\Lambda_{\Sigma_2}((12))}(pt) \cong R\Lambda_{\Sigma_2}(1) \times R\Lambda_{\Sigma_2}((12)) \\ &\cong \mathbb{Z}[q,q^{-1}][1,s]/(s^2-1) \times \mathbb{Z}[q,q^{-1}][x]/(x^2-q) \end{aligned}$$

where s *is the sign representation of* Σ_2.

By the formula (3.30), $q' = \mathbb{P}_{(1)(1)}(q) \times \mathbb{P}_{(12)}(q)$ *with* $\mathbb{P}_{(1)(1)}(q) = q^2$ *and* $\mathbb{P}_{(12)}(q) = q^{\frac{1}{2}}$. *We may notice that* $q^{\frac{1}{2}}$ *is the only representation in* $\mathbb{Z}[q,q^{-1}][x]/(x^2-q)$ *solving the equation* $x^2 - q = 0$.

The transfer ideal $\mathcal{I}_{tr}^{QEll}$ *is the image of the transfer* $\mathcal{I}_{\Sigma_1\times\Sigma_1}^{\Sigma_2}$, *namely,*

$$Ind_{\Lambda_{\Sigma_1\times\Sigma_1}(1)}^{\Lambda_{\Sigma_2}(1)} K_{\Lambda_{\Sigma_1\times\Sigma_1}(1)}(pt) \subseteq K_{\Lambda_{\Sigma_2}(1)}(pt).$$

By Lemma 2.4.7, $Ind_{\Lambda_{\Sigma_1\times\Sigma_1}(1)}^{\Lambda_{\Sigma_2}(1)} 1 = 1+s$ *and generally* $Ind_{\Lambda_{\Sigma_1\times\Sigma_1}(1)}^{\Lambda_{\Sigma_2}(1)} q^n = (1+s)\otimes q^n$. *Thus, we have*

$$K_{\Lambda_{\Sigma_2}(1)}(pt)/\mathcal{I}_{tr}^{QEll} \cong \mathbb{Z}[q,q^{-1}][q']/(q'-q^2).$$

And

$$K_{\Lambda_{\Sigma_2}(12)}(pt) \cong \mathbb{Z}[q,q^{-1}][q']/(q'^2-q).$$

So

$$QEll(pt//\Sigma_2)/\mathcal{I}_{tr}^{QEll} \cong \mathbb{Z}[q,q^{-1}][q']/(q'-q^2) \times \mathbb{Z}[q,q^{-1}][q']/(q'^2-q).$$

Thus, when $N=2$, the isomorphism (3.42) holds.

Example 3.5.3 **($N=3$).** *Now let's consider the case when $N=3$. As shown in Example 2.4.4,*

$$QEll(pt//\Sigma_3) = K_{\Lambda_{\Sigma_3}(1)}(pt) \times K_{\Lambda_{\Sigma_3}(12)}(pt) \times K_{\Lambda_{\Sigma_3}(123)}(pt) \cong R\Lambda_{\Sigma_3}(1) \times R\Lambda_{\Sigma_3}(12) \times R\Lambda_{\Sigma_3}(123)$$
$$\cong R\Sigma_3 \otimes \mathbb{Z}[q,q^{-1}] \times \mathbb{Z}[q,q^{-1}][x]/(x^2-q) \times \mathbb{Z}[q,q^{-1}][y]/(y^3-q).$$

By (3.30),

$$q' = (\mathbb{P}_{(1)(1)(1)}, \mathbb{P}_{(12)(1)}, \mathbb{P}_{(123)}) = (q^3, q^{\frac{1}{2}} \otimes q, q^{\frac{1}{3}}) \tag{3.43}$$

Then let's consider the transfer ideal. In this case $\mathcal{I}_{tr}^{QEll}$ is the image of $\mathcal{I}_{\Sigma_2\times\Sigma_1}^{\Sigma_3}$, namely

$$Ind_{\Lambda_{\Sigma_2\times\Sigma_1}(1)}^{\Lambda_{\Sigma_3}(1)} K_{\Lambda_{\Sigma_2\times\Sigma_1}(1)}(pt) \times Ind_{\Lambda_{\Sigma_2\times\Sigma_1}(12)}^{\Lambda_{\Sigma_3}(12)} K_{\Lambda_{\Sigma_2\times\Sigma_1}(12)}(pt) \subseteq K_{\Lambda_{\Sigma_3}(1)}(pt) \times K_{\Lambda_{\Sigma_3}(12)}(pt).$$

Since $\Lambda_{\Sigma_3}(12) = \Lambda_{\Sigma_2\times\Sigma_1}(12)$, the second part

$$Ind_{\Lambda_{\Sigma_2\times\Sigma_1}(12)}^{\Lambda_{\Sigma_3}(12)} K_{\Lambda_{\Sigma_2\times\Sigma_1}(12)}(pt)$$

is $K_{\Lambda_{\Sigma_2\times\Sigma_1}(12)}(pt) = K_{\Lambda_{\Sigma_3}(12)}(pt)$.

As indicated in Proposition 1.1 and Corollary 1.5 in [5], for each n,

$$\{Ind_{\Sigma_\alpha}^{\Sigma_n} 1 \mid \alpha = (\alpha_1, \alpha_2, \cdots \alpha_r) \text{ is a partition of } n \text{ and } \Sigma_\alpha = \Sigma_{\alpha_1} \times \cdots \times \Sigma_{\alpha_r}.\} \tag{3.44}$$

is a base for $R\Sigma_n$. Thus, by Lemma 2.4.1 and Lemma 2.4.7, as a $\mathbb{Z}[q,q^{-1}]-$module,

$$Ind_{\Lambda_{\Sigma_2\times\Sigma_1}(1)}^{\Lambda_{\Sigma_3}(1)} K_{\Lambda_{\Sigma_2\times\Sigma_1}(1)}(pt)$$

contains all the base elements given in (3.44) except the one corresponding to the partition (123)*, which is the trivial representation of $\Lambda_{\Sigma_3}(1)$. So*

$$K_{\Lambda_{\Sigma_3}(1)}(pt)/Ind_{\Lambda_{\Sigma_2\times\Sigma_1}(1)}^{\Lambda_{\Sigma_3}(1)} K_{\Lambda_{\Sigma_2\times\Sigma_1}(1)}(pt)$$

is $\mathbb{Z}[q, q^{-1}]$*, which is obviously* $\mathbb{Z}[q, q^{-1}][q']/(q' - q^3)$.

And $K_{\Lambda_{\Sigma_3}(123)}(pt) = \mathbb{Z}[q, q^{-1}][q']/(q'^3 - q)$. $q^{\frac{1}{3}} = \mathbb{P}_{(123)}(q)$ *is the only element in the representation ring* $R\Lambda_{\Sigma_3}(123)$ *solving the equation* $x^3 - q = 0$.

Combining the above cases we get

$$QEll(pt//\Sigma_3)/\mathcal{I}_{tr}^{QEll} \cong \mathbb{Z}[q, q^{-1}][q']/(q' - q^3) \times \mathbb{Z}[q, q^{-1}][q']/(q'^3 - q).$$

Thus, the isomorphism (3.42) holds when $N = 3$.

Example 3.5.4 ($N = 4$)**.** *Now let's consider the case when* $N = 4$*, which is more complicated than the previous examples.*

$$QEll(pt//\Sigma_4) = K_{\Lambda_{\Sigma_4}(1)}(pt) \times K_{\Lambda_{\Sigma_4}(12)}(pt) \times K_{\Lambda_{\Sigma_4}(123)}(pt) \times K_{\Lambda_{\Sigma_4}(1234)}(pt) \times K_{\Lambda_{\Sigma_4}(12)(34)}(pt).$$

There are many similarities between Example 3.5.4 and the previous examples.

In this case the transfer ideal $\mathcal{I}_{tr}^{QEll}$ *is*

$$\begin{aligned}
&Ind_{\Lambda_{\Sigma_3 \times \Sigma_1}(1)}^{\Lambda_{\Sigma_4}(1)} K_{\Lambda_{\Sigma_3 \times \Sigma_1}(1)}(pt) \times Ind_{\Lambda_{\Sigma_2 \times \Sigma_2}(1)}^{\Lambda_{\Sigma_4}(1)} K_{\Lambda_{\Sigma_2 \times \Sigma_2}(1)}(pt) \times \\
&Ind_{\Lambda_{\Sigma_1 \times \Sigma_3}(1)}^{\Lambda_{\Sigma_4}(1)} K_{\Lambda_{\Sigma_1 \times \Sigma_3}(1)}(pt) \times Ind_{\Lambda_{\Sigma_2 \times \Sigma_2}(12)}^{\Lambda_{\Sigma_4}(12)} K_{\Lambda_{\Sigma_2 \times \Sigma_2}(12)}(pt) \times \\
&Ind_{\Lambda_{\Sigma_3 \times \Sigma_1}(12)}^{\Lambda_{\Sigma_4}(12)} K_{\Lambda_{\Sigma_3 \times \Sigma_1}(12)}(pt) \times Ind_{\Lambda_{\Sigma_3 \times \Sigma_1}(123)}^{\Lambda_{\Sigma_4}(123)} K_{\Lambda_{\Sigma_3 \times \Sigma_1}(123)}(pt) \times Ind_{\Lambda_{\Sigma_2 \times \Sigma_2}(12)(34)}^{\Lambda_{\Sigma_4}(12)(34)} K_{\Lambda_{\Sigma_2 \times \Sigma_2}(12)(34)}(pt)
\end{aligned}$$

which is contained in

$$K_{\Lambda_{\Sigma_4}(1)}(pt) \times K_{\Lambda_{\Sigma_4}(12)}(pt) \times K_{\Lambda_{\Sigma_4}(123)}(pt) \times K_{\Lambda_{\Sigma_4}(12)(34)}(pt).$$

By Lemma 2.4.1 and Lemma 2.4.7,

$$Ind_{\Lambda_{\Sigma_3 \times \Sigma_1}(1)}^{\Lambda_{\Sigma_4}(1)} K_{\Lambda_{\Sigma_3 \times \Sigma_1}(1)}(pt) \times Ind_{\Lambda_{\Sigma_2 \times \Sigma_2}(1)}^{\Lambda_{\Sigma_4}(1)} K_{\Lambda_{\Sigma_2 \times \Sigma_2}(1)}(pt) \times Ind_{\Lambda_{\Sigma_1 \times \Sigma_3}(1)}^{\Lambda_{\Sigma_4}(1)} K_{\Lambda_{\Sigma_1 \times \Sigma_3}(1)}(pt) \tag{3.45}$$

contains all the base elements given in (3.44) with $N = 4$ *except the one corresponding to the partition* (1234)*, which is the trivial representation of* $\Lambda_{\Sigma_4}(1)$*. So the summand* $K_{\Lambda_{\Sigma_4}(1)}(pt)$ *modulo the ring (3.45) gives* $\mathbb{Z}[q, q^{-1}]$*, which is obviously isomorphic to* $\mathbb{Z}[q, q^{-1}][q']/(q' - q^4)$*. By the formula (3.30),* $\mathbb{P}_{(1)(1)(1)(1)}(q) = q^4$*. So when* $d = 4$ *and* $e = 1$ *in (3.42),* $\mathbb{Z}[q, q^{-1}][q']/(q' - q^4)$ *is isomorphic to* $K_{\Lambda_{\Sigma_4}(1)}(pt)$ *modulo the ring (3.45).*

$K_{\Lambda_{\Sigma_4}(12)}(pt)$ *is completely contained in the image of* $K_{\Lambda_{\Sigma_2 \times \Sigma_2}(12)}(pt)$ *under the transfer* *since* $\Lambda_{\Sigma_4}(12) = \Lambda_{\Sigma_2\times\Sigma_2}(12)$*. So* $K_{\Lambda_{\Sigma_4}(12)}(pt)$ *modulo*

$$Ind^{\Lambda_{\Sigma_4}(12)}_{\Lambda_{\Sigma_2\times\Sigma_2}(12)} K_{\Lambda_{\Sigma_2\times\Sigma_2}(12)}(pt) \times Ind^{\Lambda_{\Sigma_4}(12)}_{\Lambda_{\Sigma_3\times\Sigma_1}(12)} K_{\Lambda_{\Sigma_3\times\Sigma_1}(12)}(pt)$$

is trivial. And since $\Lambda_{\Sigma_4}(123) = \Lambda_{\Sigma_3\times\Sigma_1}(123)$, $K_{\Lambda_{\Sigma_4}(123)}(pt)$ *and* $Ind^{\Lambda_{\Sigma_4}(123)}_{\Lambda_{\Sigma_3\times\Sigma_1}(123)} K_{\Lambda_{\Sigma_3\times\Sigma_1}(123)}(pt)$ *are equal. So* $K_{\Lambda_{\Sigma_4}(123)}(pt)$ *modulo* $Ind^{\Lambda_{\Sigma_4}(123)}_{\Lambda_{\Sigma_3\times\Sigma_1}(123)} K_{\Lambda_{\Sigma_3\times\Sigma_1}(123)}(pt)$ *is also trivial.*

In addition, $K_{\Lambda_{\Sigma_4}(1234)}(pt) \cong R\Lambda_{\Sigma_4}(1234) \cong \mathbb{Z}[q,q^{-1}][q']/(q'^4 - q)$*. By (3.30),* $\mathbb{P}_{(1234)} = q^{\frac{1}{4}}$ *which is the only solution of the equation* $x^4 - q = 0$ *in the ring* $R\Lambda_{\Sigma_4}(1234)$*. So when* $d = 1$ *and* $e = 4$ *in (3.42),* $\mathbb{Z}[q,q^{-1}][q']/(q'^4 - q)$ *is isomorphic to* $K_{\Lambda_{\Sigma_4}(1234)}(pt)$*.*

Now let's look at the part for the cycle $(12)(34)$*, which is a new case different from the previous cases.* $C_{\Sigma_4}((12)(34))$ *is the wreath product* $(\mathbb{Z}/2\mathbb{Z})\wr\Sigma_2$*.* $\Lambda_{\Sigma_4}((12)(34))$ *is isomorphic to* $\Lambda_{\Sigma_2}((12)) \wr_{\mathbb{T}} \Sigma_2$ *with the isomorphism defined by*

$$[(g_1, g_2; \sigma), t] \mapsto ([g_1, t], [g_2, t]; \sigma).$$

We have conclusions for the representations of wreath product like $\Lambda_{\Sigma_2}((12)) \wr_{\mathbb{T}} \Sigma_2$ *by replacing the tensor product* $\otimes$ *between the* ρ_i*s in Theorem 3.1.7 by* $\otimes_{\mathbb{Z}[q,q^{-1}]}$*. As shown in Example 2.4.3, a base for the representation ring* $R\Lambda_{\Sigma_2\times\Sigma_2}((12)(34))$ *is*

$$\{q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{b}{2}} \mid a, b \in \mathbb{Z}\}. \tag{3.46}$$

Then by Theorem 3.1.7,

$$\{Ind^{\Lambda_{\Sigma_2}((12))\wr_{\mathbb{T}}\Sigma_2}_{\Lambda_{\Sigma_2\times\Sigma_2}((12)(34))} q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{b}{2}} \otimes D_1 \mid a, b \in \mathbb{Z}\} \bigcup \{(q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{a}{2}})^{\sim} \otimes D_\tau \mid a \in \mathbb{Z}, \tau \in R\Sigma_2\} \tag{3.47}$$

contains all the irreducible representations of $\Lambda_{\Sigma_2}((12)) \wr_{\mathbb{T}} \Sigma_2$*, where* D_τ *corresponding the representation* τ *is defined in (3.15) and* D_1 *is the representation corresponding to the trivial representation* 1*.*

Now let's consider the image of

$$K_{\Lambda_{\Sigma_2\times\Sigma_2}((12)(34))}(pt) \cong R\Lambda_{\Sigma_2\times\Sigma_2}((12)(34))$$

under the transfer. The image is generated by the image of the basis given in (3.46), namely,

$$\{Ind_{\Lambda_{\Sigma_2\times\Sigma_2}((12)(34))}^{\Lambda_{\Sigma_4}((12)(34))} q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{b}{2}} \otimes D_1 \mid a, b \in \mathbb{Z}\}, \tag{3.48}$$

which is the first set in (3.47). Since $q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{a}{2}}$ *is 1-dimensional representation, we have*

$$Ind_{\Lambda_{\Sigma_2\times\Sigma_2}((12)(34))}^{\Lambda_{\Sigma_4}((12)(34))} q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{a}{2}} \otimes D_1 = (q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{a}{2}})^{\sim} \otimes D_{Ind_{\Sigma_1\times\Sigma_1}^{\Sigma_2} 1} = (q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{a}{2}})^{\sim} \otimes D_{1+s}$$

where s is the sign representation of Σ_2. They gives all the elements in the second set in (3.47) except

$$\{(q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{a}{2}})^{\sim} \otimes D_1 \mid a \in \mathbb{Z}\}. \tag{3.49}$$

So $K_{\Lambda_{\Sigma_4}(12)(34)}(pt)$ *modulo* $Ind_{\Lambda_{\Sigma_2\times\Sigma_2}(12)(34)}^{\Lambda_{\Sigma_4}(12)(34)} K_{\Lambda_{\Sigma_2\times\Sigma_2}(12)(34)}(pt)$ *is a* $\mathbb{Z}[q, q^{-1}]-$*module generated by (3.49).*

Since

$$(q^{\frac{a}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{a}{2}})^{\sim} \otimes D_1 = q^{2b}(q^{\frac{a}{2}-b} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{a}{2}-b})^{\sim}$$

where b is largest integer smaller than $\frac{a}{2}$. As a $\mathbb{Z}[q, q^{-1}]-$*module,*

$$K_{\Lambda_{\Sigma_4}(12)(34)}(pt) \text{ modulo } Ind_{\Lambda_{\Sigma_2\times\Sigma_2}(12)(34)}^{\Lambda_{\Sigma_4}(12)(34)} K_{\Lambda_{\Sigma_2\times\Sigma_2}(12)(34)}(pt)$$

is generated by

$$\{1, (q^{\frac{1}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{1}{2}})^{\sim}\},$$

which is a basis for it. The solutions to $x^2 - q^2 = 0$ *in the ring* $R\Lambda_{\Sigma_4}(12)(34)$ *are q and* $(q^{\frac{1}{2}} \otimes_{\mathbb{Z}[q,q^{-1}]} q^{\frac{1}{2}})^{\sim}$*. So*

$$K_{\Lambda_{\Sigma_4}(12)(34)}(pt) \text{ modulo } Ind_{\Lambda_{\Sigma_2\times\Sigma_2}(12)(34)}^{\Lambda_{\Sigma_4}(12)(34)} K_{\Lambda_{\Sigma_2\times\Sigma_2}(12)(34)}(pt)$$

is isomorphic to

$$\mathbb{Z}[q, q^{-1}][q']/(q'^2 - q^2),$$

corresponding to the case $d = 2$ and $e = 2$ in (3.42). So for $N = 4$, the isomorphism (3.42) also holds.

I'll continue the discussion further after proving the isomorphism (3.42).

Now let's see the proof of (3.42) for any N.

Proof of (3.42). Let's start with two extreme cases.

Case I:

σ is 1.

By Lemma 2.4.1 and Lemma 2.4.7, $K_{\Lambda_{\Sigma_N}(1)}(\mathrm{pt}) \cong R\Lambda_{\Sigma_N}(1) \cong R\Sigma_N \otimes \mathbb{Z}[q, q^{-1}]$. Thus, we can get a basis for the $\mathbb{Z}[q, q^{-1}]-$module $K_{\Lambda_{\Sigma_N}(1)}(\mathrm{pt})$ from a a basis of $R\Sigma_N$. By Proposition 1.1 and Corollary 1.5 in [5], the set (3.44) gives a basis for $R\Sigma_N$.

All the irreducible representations of $K_{\Lambda_{\Sigma_N}(1)}(\mathrm{pt})$ are

$$\{(Ind_{\Sigma_\alpha}^{\Sigma_N} 1) \otimes q^k \mid k \in \mathbb{Z};\ \alpha = (\alpha_1, \alpha_2, \cdots \alpha_r) \text{ is a partition of } n \text{ and } \Sigma_\alpha = \Sigma_{\alpha_1} \times \cdots \times \Sigma_{\alpha_r}.\}$$

They are all contained in

$$\sum_{\substack{i+j=N,\\ N>j>0}} Ind_{\Lambda_{\Sigma_i \times \Sigma_j}(1)}^{\Lambda_{\Sigma_N}(1)} K_{\Lambda_{\Sigma_i \times \Sigma_j}(1)}(\mathrm{pt})$$

except $1 \otimes q^k$ with $k \in \mathbb{Z}$. The trivial Σ_N-representation 1 is corresponding to the partition (N). It can not be obtained as induced representation.

Thus, $K_{\Lambda_{\Sigma_N}(1)}(\mathrm{pt})$ modulo

$$\sum_{\substack{i+j=N,\\ N>j>0}} Ind_{\Lambda_{\Sigma_i \times \Sigma_j}(1)}^{\Lambda_{\Sigma_N}(1)} K_{\Lambda_{\Sigma_i \times \Sigma_j}(1)}(\mathrm{pt})$$

is isomorphic to $\mathbb{Z}[q, q^{-1}]$. By the formula (3.30), $\mathbb{P}_1 = q^N$ where 1 denote the identity element in $1 \wr \Sigma_N$. On the right hand of (3.42), when $d = N$ and $e = 1$, $\mathbb{Z}[q, q^{-1}][q']/(q' - q^N)$ is equal to $\mathbb{Z}[q, q^{-1}]$. Thus, $K_{\Lambda_{\Sigma_N}(1)}(\mathrm{pt})$ modulo

$$\sum_{\substack{i+j=N,\\ N>j>0}} Ind_{\Lambda_{\Sigma_i \times \Sigma_j}(1)}^{\Lambda_{\Sigma_N}(1)} K_{\Lambda_{\Sigma_i \times \Sigma_j}(1)}(\mathrm{pt})$$

is isomorphic to it.

Case II:

σ is $(12 \cdots N)$.

By Lemma 2.4.1 and Lemma 2.4.7, $K_{\Lambda_{\Sigma_N}(12\cdots N)}(\mathrm{pt}) \cong R\Lambda_{\Sigma_N}(12\cdots N) \cong \mathbb{Z}[q,q^{-1}][x]/(x^N - q)$. The image of the transfer to this summand is trivial. And by (3.30) $\mathbb{P}_{(12\cdots N)}(q) = q^{\frac{1}{N}}$.

By sending x to $q' = q^{\frac{1}{N}}$, $K_{\Lambda_{\Sigma_N}(12\cdots N)}(\mathrm{pt})$ is isomorphic to $\mathbb{Z}[q,q^{-1}][q']/(q'^N - q)$, which is the summand on the right hand of (3.42) with $d = 1$ and $e = N$,

Case III:

σ is in $\Sigma_r \times \Sigma_{N-r}$ with $r > 0$ such that all the cycles of the same length are either in Σ_r or Σ_{N-r}. For example, the cycle

$$(1\ 2)(3\ 4)(5\ 6)(7\ 8\ 9\ 10)(11\ 12\ 13\ 14)(15\ 16\ 17) \in \Sigma_{17}$$

is in this case. It's in $\Sigma_6 \times \Sigma_{11}$; the cycles $(1\ 2)(3\ 4)(5\ 6)$, $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)$ are not in this case.

Most elements in Σ_N belong to Case III. σ is not in this case if and only if it consists of cycles of the same length, such as $(1\ 2)(3\ 4)$, $(1\ 2\ 3)$, 1, $(1\ 2\ 3)(4\ 5\ 6)$.

For those σ that belong to Case III, $\Lambda_{\Sigma_N}(\sigma) = \Lambda_{\Sigma_r\times\Sigma_{N-r}}(\sigma)$, so $Ind^{\Lambda_{\Sigma_N}(\sigma)}_{\Lambda_{\Sigma_r\times\Sigma_{N-r}}(\sigma)}$ is the identity map, so $K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt})$ is equal to $Ind^{\Lambda_{\Sigma_N}(\sigma)}_{\Lambda_{\Sigma_r\times\Sigma_{N-r}}(\sigma)} K_{\Lambda_{\Sigma_r\times\Sigma_{N-r}}(\sigma)}(\mathrm{pt})$. So the summand corresponding to σ in $QEll(\mathrm{pt}//\Sigma_N)$ is completely cancelled.

Case IV:

σ consists of cycles of the same length, in other words, it consists of d $e-$cycles with $N = de$. For example, $(1\ 2)(3\ 4) \in \Sigma_4$ and $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12) \in \Sigma_{12}$ are both in this case.

The two extreme cases, Case I and Case II, are both special cases of Case IV. But we don't need to discuss the representation ring of wreath product in Case I and Case II.

The centralizer $C_{\Sigma_N}(\sigma) \cong C_e \wr \Sigma_d$, where $C_e = \mathbb{Z}/e\mathbb{Z}$ is the cyclic group with order e. Recall

$$\Lambda_{\Sigma_N}(\sigma) \cong \Lambda_{\Sigma_e}(12\cdots e) \wr_{\mathbb{T}} \Sigma_d$$

is the subgroup of $\Lambda_{\Sigma_e}(12\cdots e) \wr \Sigma_d$ with elements of the form

$$([a_1,t],[a_2,t],\cdots [a_d,t];\tau).$$

$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt})$ is the representation ring $R\Lambda_{\Sigma_N}(\sigma)$. According to Theorem 3.1.8, as a $\mathbb{Z}[q,q^{-1}]-$module, it has the basis

$$\{Ind_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_{(d)}}^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d_1}\otimes_{\mathbb{Z}[q,q^{-1}]}\cdots\otimes_{\mathbb{Z}[q,q^{-1}]}(q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d_r}\otimes D_\tau \mid$$

$(d)=(d_1,d_2,\cdots d_r)$ is a partition of d.

$a_1,a_2,\cdots a_r$ are in $\{0,1,\cdots e-1\}$. And $\tau\in R\Sigma_{(d)}$ is irreducible.$\}$

where for each $a\in\mathbb{Z}$, $q^{\frac{a}{e}}:\Lambda_{C_e}((12\cdots e))\longrightarrow U(1)$ is the map

$$q^{\frac{a}{e}}([(12\cdots e)^j,t])=e^{2\pi i a\frac{j+t}{e}}. \tag{3.50}$$

Namely, it's the map x_1^a in the sense of Example 2.4.3.

For each partition (d) of d, if it has more than one cycle, $\Sigma_{(d)}$ is a subgroup of some $\Sigma_{d_1}\times\Sigma_{d-d_1}$ for some positive integer $0<d_1<d$. So for each

$$Ind_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_{(d)}}^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d_1}\otimes_{\mathbb{Z}[q,q^{-1}]}\cdots\otimes_{\mathbb{Z}[q,q^{-1}]}(q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d_r}\otimes D_\tau$$

with $r\geq 2$, it's equal to $Ind_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_{d_1}\times\Sigma_{d-d_1})}^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d}(Ind_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_{(d)}}^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_{d_1}\times\Sigma_{d-d_1})}$ $(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d_1}\otimes_{\mathbb{Z}[q,q^{-1}]}\cdots\otimes_{\mathbb{Z}[q,q^{-1}]}(q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d_r}\otimes D_\tau)$ by the property of induced representation. Note that

$$\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_{d_1}\times\Sigma_{d-d_1})\cong\Lambda_{\Sigma_{d_1e}\times\Sigma_{N-d_1e}}(\sigma).$$

So

$$Ind_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_{(d)}}^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_{d_1}\times\Sigma_{d-d_1})}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d_1}\otimes_{\mathbb{Z}[q,q^{-1}]}\cdots\otimes_{\mathbb{Z}[q,q^{-1}]}(q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d_r}\otimes D_\tau$$

is in $K_{\Lambda_{\Sigma_{d_1e}\times\Sigma_{N-d_1e}}(\sigma)}(\text{pt})$, Thus, each base element with $r\geq 2$ is contained in the transfer ideal.

When $r=1$, consider

$$(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}\otimes D_\tau$$

with $\tau\in R\Sigma_d$. As indicated in Proposition 1.1 and Corollary 1.5 in [5], each τ, except the trivial representation of Σ_d, can be induced from a representation τ' in some $R(\Sigma_i\times\Sigma_{d-i})$ with $d>i>0$.

Claim: the representation

$$Ind_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})}^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}i}\otimes_{\mathbb{Z}[q,q^{-1}]}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d-i}\otimes D_{\tau'}$$

is isomorphic to

$$(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}\otimes D_{Ind_{\Sigma_i\times\Sigma_{d-i}}^{\Sigma_d}\tau'},$$

which is

$$(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}\otimes D_{\tau}.$$

To prove this, let's consider a set $\{\tau_\alpha\}_{\alpha\in\Sigma_d/\Sigma_i\times\Sigma_{d-i}}$ of coset representatives. Then

$$\{\eta_\alpha := (1,\cdots 1;\tau_\alpha)\}_{\alpha\in\Sigma_d/\Sigma_i\times\Sigma_{d-i}}$$

is a set of coset representatives of

$$\big(\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d\big)/\big(\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})\big).$$

Let W be a representation space of $\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})$, Then $Ind_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})}^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d}W$ is the direct product of $[\Sigma_d:\Sigma_i\times\Sigma_{d-i}]$ copies of W. For any element

$$H=(g_1,\cdots g_d;\beta)\in\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d,$$

and each $\alpha\in\Sigma_d/\Sigma_i\times\Sigma_{d-i}$, there is a unique $\alpha'\in\Sigma_d/\Sigma_i\times\Sigma_{d-i}$ and a unique

$$J_\alpha=(g_1',\cdots g_d';\gamma_\alpha)\in\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})$$

such that $H\eta_\alpha=\eta_{\alpha'}J_\alpha$. Note that

$$g_1',\cdots g_d'$$

is a permutation of

$$g_1,\cdots g_d.$$

So $(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}(g_1',\cdots g_d')=(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}(g_1,\cdots g_d)$. In addition, $\beta\tau_\alpha=\tau_{\alpha'}\gamma_\alpha$. Let

$$\prod_\alpha w_\alpha$$

be an element in

$$Ind_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})}^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d} W.$$

We have

$$\left(Ind_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})}^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}i}\otimes_{\mathbb{Z}[q,q^{-1}]}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d-i}\otimes D_{\tau'}\right)(H)(\prod_{\alpha}w_\alpha)$$
$$=\prod_{\alpha}J_\alpha w_{\beta(\alpha)}=\prod_{\alpha}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}(g_1,\cdots g_d)D_{\tau'}(1,\cdots 1;\gamma_\alpha)(w_{\beta\alpha})$$
$$=(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}(g_1,\cdots g_d)\prod_{\alpha}\tau'(\gamma_\alpha)(w_{\beta\alpha})$$
$$=(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}(g_1,\cdots g_d)(Ind_{\Sigma_i\times\Sigma_{d-i}}^{\Sigma_d}\tau')(\beta)(\prod_{\alpha}w_\alpha)$$
$$=(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}(g_1,\cdots g_d;\beta)D_{Ind_{\Sigma_i\times\Sigma_{d-i}}^{\Sigma_d}\tau'}(g_1,\cdots g_d;\beta)(\prod_{\alpha}w_\alpha)$$
$$=\left((q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}\otimes D_{Ind_{\Sigma_i\times\Sigma_{d-i}}^{\Sigma_d}\tau'}\right)(g_1,\cdots g_d;\beta)(\prod_{\alpha}w_\alpha)$$

So the claim is proved.

Since

$$\{Ind_{\Sigma_i\times\Sigma_{d-i}}^{\Sigma_d}\tau' \mid \tau'\in R(\Sigma_i\times\Sigma_{d-i}) \text{ and } i=1,2,\cdots d-1.\}$$

contains all the irreducible representation of Σ_d except the trivial representation, which is corresponding to the partition (d), thus, by the claim, $K_{\Lambda_{\Sigma_N}(\sigma)}(\text{pt})$ modulo the image of the transfer, is a $\mathbb{Z}[q,q^{-1}]-$module generated by the equivalent classes represented by

$$\{((q^{\frac{a}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d})^{\sim} \mid a=0,1,\cdots e-1\}. \tag{3.51}$$

For any a, $(q^{\frac{a}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}$ is $(q^{\frac{1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}$ to the $a-$th power. Note that, by (3.30), $(q^{\frac{1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}$ is

$$q' := \mathbb{P}_\sigma(q).$$

To get the isomorphism (3.42), consider a map $\Psi:\mathbb{Z}[q,q^{-1}][x]\longrightarrow K_{\Lambda_{\Sigma_N}(\sigma)}(\text{pt})/\mathcal{I}_{tr}^{QEll}$ by sending q to q and x to q', which is a well-defined $\mathbb{Z}[q,q^{-1}]-$homomorphism.

Since $q'^e=q^d$, $K_{\Lambda_{\Sigma_N}(\sigma)}(\text{pt})/\mathcal{I}_{tr}^{QEll}$ is a $\mathbb{Z}[q,q^{-1}]-$module generated by

$$1, q', \cdots q'^{e-1}.$$

So any element in it can be expressed as

$$\sum_{j=0}^{e-1} f_j(q)q'^j$$

where each $f_j(q)$ is in the polynomial ring $\mathbb{Z}[q,q^{-1}]$. It's the image of

$$\sum_{j=0}^{e-1} f_j(q)x^j$$

in $\mathbb{Z}[q,q^{-1}][x]$. So Ψ is surjective.

Now let's study its kernel. If

$$F := \sum_{j=0}^{e-1} f_j(q)q'^j$$

is in $\mathcal{I}_{tr}^{QEll}$, then it's in $\mathbb{Z}[q,q^{-1}]$. So we can assume $F=0$.

For each element $[(a_1,\cdots a_d;\beta),t]$ in $\Lambda_{\Sigma_N}(\sigma)$ with $(a_1,\cdots a_d;\beta)\in C_{\Sigma_N}(\sigma)\cong C_e\wr\Sigma_d$,

$$q([(a_1,\cdots a_d;\beta),t]) = e^{2\pi it}, \tag{3.52}$$

and

$$q'([(a_1,\cdots a_d;\beta),t]) = e^{\frac{2\pi i(a_1+\cdots a_d+dt)}{e}}. \tag{3.53}$$

$$\begin{aligned}
F([(a_1,\cdots a_d;\beta),t]) &= \sum_{j=0}^{e-1} f_j(q)q'^j([(a_1,\cdots a_d;\beta),t]) \\
&= \sum_{j=0}^{e-1} f_j(e^{2\pi it})e^{\frac{2\pi ij(a_1+\cdots+a_d+dt)}{e}} \\
&= \sum_{j=0}^{e-1} f_j(e^{2\pi it})e^{\frac{2\pi ijdt}{e}}e^{\frac{2\pi ij(a_1+\cdots+a_d)}{e}}.
\end{aligned}$$

Let

$$F_j(t) := f_j(e^{2\pi it})e^{\frac{2\pi ijdt}{e}}$$

be the complex-valued function in the variable t. And let α denote the number $e^{\frac{2\pi i}{e}}$. The integers

$$(a_1+\cdots+a_d)$$

go over $0, 1, \cdots e-1$. Consider the e equations

$$\sum_{j=0}^{e-1} F_j(t)\alpha^{jk} = 0, \text{ for } k = 0, 1, \cdots e-1.$$

In other words,

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{e-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(e-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha^{e-1} & \alpha^{2(e-1)} & \cdots & \alpha^{(e-1)^2} \end{pmatrix} \begin{pmatrix} F_0(t) \\ F_1(t) \\ F_2(t) \\ \vdots \\ F_{e-1}(t) \end{pmatrix} = 0$$

The determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{e-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(e-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha^{e-1} & \alpha^{2(e-1)} & \cdots & \alpha^{(e-1)^2} \end{pmatrix}$$

is

$$\prod_{j=0}^{e-2} \prod_{k=j+1}^{e-1} (\alpha^k - \alpha^j). \tag{3.54}$$

When $\alpha = e^{\frac{2\pi i}{e}}$, each $(\alpha^k - \alpha^j)$ in the product (3.54) is nonzero, so for any e, the determinant is nonzero and the matrix is non-singular. So we get $F_j(t) = 0$ for any $t \in \mathbb{R}$ and $j = 0, 1, 2, \cdots e-1$.

So each $f_j(q)$ in F is the zero polynomial.

The kernel of Ψ is the ideal generated by $q'^e - q^d$.

□

Below is a conclusion which has no relation with (3.42).

Lemma 3.5.5. *Let $\sigma \in \Sigma_N$ consist of d $e-$cycles with $N = de$. Then the equation*

$$x^e - q^d = 0 \tag{3.55}$$

has totally (d,e) solutions in $K_{\Lambda_{\Sigma_N}(\sigma)}(pt)$ modulo the image of transfer, i.e.

$$\sum_{\substack{i+j=N,\\ N>j>0,\\ \sigma\in\Sigma_i\times\Sigma_j}} Image[\mathcal{I}^{\Sigma_N}_{\Sigma_i\times\Sigma_j} : QEll(pt//\Sigma_i\times\Sigma_j) \longrightarrow QEll(pt//\Sigma_N)]$$
$$= \sum_{\substack{i+j=N,\\ N>j>0,\\ \sigma\in\Sigma_i\times\Sigma_j}} Ind^{\Lambda_{\Sigma_N}(\sigma)}_{\Lambda_{\Sigma_i\times\Sigma_j}(\sigma)} K_{\Lambda_{\Sigma_i\times\Sigma_j}(\sigma)}(pt).$$

And each solution is in the polynomial ring $\mathbb{Z}[q,q^{-1}][q']$ where $q' = \mathbb{P}_\sigma(q)$. Here (d,e) is the greatest common divisor of d and e.

Proof. First let's see whether we can find a representation of $\Lambda_{\Sigma_N}(\sigma)$ solving the equation (3.55).

Since q^d is a 1-dimensional representation, x has to be 1-dimensional, thus, irreducible. Theorem 3.1.8 gives us all the irreducible representations of $\Lambda_{\Sigma_N}(\sigma)$. For any subgroup $\Sigma_i\times\Sigma_j$ of Σ_N containing σ with $i+j=N$, $N>j>0$, for any $\Lambda_{\Sigma_i\times\Sigma_j}(\sigma)-$representation V with positive dimension, the induced representations $Ind^{\Lambda_{\Sigma_N}(\sigma)}_{\Lambda_{\Sigma_i\times\Sigma_j}(\sigma)}V$ has dimension larger than 1. By Case IV in the proof of Theorem 3.5.1, x is of the form

$$q^m(q^{\frac{a}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}\otimes D_\tau$$

with $m\in\mathbb{Z}$, $a\in\{0,1,\cdots e-1\}$ and τ a 1-dimensional representation of Σ_d. It's equivalent to

$$q^m(q^{\frac{a}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}\otimes D_1.$$

$$[q^m(q^{\frac{a}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d}]^e = q^{ad+me} = q^d.$$

Then $ad+me=d$. So $(a,m) = (k\frac{e}{(d,e)}+1, -k\frac{d}{(d,e)})$ with $k\in\mathbb{Z}$. To make $a\in\{0,1,\cdots e-1\}$, the value k should be in $\{1,2,\cdots(d,e)-1\}$. Each k gives exactly one pair (a,m). And each solution

$$q^m(q^{\frac{a}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]}d} = q^m q'^a \tag{3.56}$$

is in the polynomial ring $\mathbb{Z}[q,q^{-1}][q']$.

Next I show this (d,e) solutions are the only solutions.

Consider any element $x=\sum\limits_{a\in\mathbb{Z}} V_a q^a$ in $R\Lambda_{\Sigma_N}(\sigma)$, with each V_a a $\Lambda_{\Sigma_N}(\sigma)-$representation.

Let a_0 be the smallest integer such that V_{a_0} is nonzero. Let $x^e = \sum\limits_{b\in\mathbb{Z}} W_b q^b$. The nonzero term in x^e with the smallest index is $(V_{a_0})^{\otimes e} q^{ea_0}$, which is equal to q^d. Thus, $V_{a_0}q^{a_0}$ is one of the (d, e) solutions above, and $a_0 = m$.

$$0 = W_{d+1} = eV_m^{\otimes_{\mathbb{Z}[q,q^{-1}]}e-1} \otimes_{\mathbb{Z}[q,q^{-1}]} V_{m+1},$$

so $V_{m+1} = 0$.

Assume $V_a = 0$ for $m < a \leq n$. $0 = W_{m(e-1)+n+1} = eV_m^{\otimes_{\mathbb{Z}[q,q^{-1}]}e-1} \otimes_{\mathbb{Z}[q,q^{-1}]} V_{n+1}$. Thus $V_{n+1} = 0$.

Then by induction, V_a with $a > m$ are all zero. So $x = V_m q^m$.

Thus, we get the lemma.

□

From the power operation of quasi-elliptic cohomology, we can construct the Adams operation for quasi-elliptic cohomology.

Proposition 3.5.6. *The composition*

$$\begin{aligned}\overline{P}_N :& QEll_G(X) \xrightarrow{\mathbb{P}_N} QEll_{G\wr\Sigma_N}(X^{\times N}) \xrightarrow{res} QEll_{G\times\Sigma_N}(X^{\times N}) \\ &\xrightarrow{diag^*} QEll_{G\times\Sigma_N}(X) \cong QEll_G(X) \otimes_{\mathbb{Z}[q^\pm]} QEll_{\Sigma_N}(pt) \\ &\longrightarrow QEll_G(X) \otimes_{\mathbb{Z}[q^\pm]} QEll_{\Sigma_N}(pt)/\mathcal{I}^{QEll}_{tr} \cong QEll_G(X) \otimes_{\mathbb{Z}[q^\pm]} \prod_{N=de} \mathbb{Z}[q,q^{-1}][q']/\langle q^d - q'^e\rangle\end{aligned}$$

defines an additive operation, where res is the restriction map by the inclusion

$$G \times \Sigma_N \hookrightarrow G \wr \Sigma_N,\ (g, \sigma) \mapsto (g, \cdots g; \sigma),$$

diag is the diagonal map

$$X \longrightarrow X^{\times N},\ x \mapsto (x, \cdots x)$$

and the last map is the isomorphism in Theorem 3.5.1. Moreover, it is a homomorphism between the $\mathbb{Z}[q^\pm]-$*algebras.*

Proof. Let $V = \prod\limits_{g\in G^{tors}_{conj}} V_g \in QEll_G(X)$. Apply the explicit formula of the power operation in (3.30), the composition $diag^* \circ res \circ \mathbb{P}_N$ sends V to

$$\prod_{\substack{g \in G^{tors}_{conj} \\ \sigma \in \Sigma_N{}^{tors}_{conj}}} \otimes_k \otimes_{(i_1, \cdots i_k)} V_{g^k} q^{\frac{1}{k}}$$

where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ, and the tensor products are those of the $\mathbb{Z}[q^{\pm}]-$algebras. Each factor has a decomposition

$$\bigoplus_{\rho} V_\rho \otimes_{\mathbb{Z}[q^\pm]} \rho \in K_{\Lambda_G(g)}(X^g) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt})$$

where ρ goes over all the irreducible $\Lambda_{\Sigma_N}(\sigma)-$representation and V_ρ is the $\Lambda_G(\sigma)-$bundle uniquely determined by ρ and V. Then after taking the quotient by the transfer ideal $\mathcal{I}^{QEll}_{tr}$, all the factors in $diag^* \circ res \circ \mathbb{P}_N(V)$ are cancelled except those corresponding to the elements in $\Sigma_N{}^{tors}_{conj}$ with cycles of the same length. And for the bundles in the factor corresponding to the element $\sigma \in \Sigma_N$ with d $e-$cycles and $de = N$, the nontrivial part is the subbundles $V_{g^e, \rho} \otimes_{\mathbb{Z}[q^\pm]} \rho$ corresponding to $\rho = q'^a$ with

$$q'_{d,e} = \mathbb{P}_\sigma(q) = (q^{\frac{1}{e}})^{\otimes_{\mathbb{Z}[q, q^{-1}]} d}$$

and $a = 0, 1, \cdots e-1$. The restriction of q' to the subgroup $C_{\Sigma_N}(\sigma)$ is trivial. V_{g^e, q'^a} is the fixed point space of $V_{g^e}^{\otimes_{\mathbb{Z}[q,q^{-1}]} d}$ by the permutations Σ_d, i.e. the pull-back bundle via the diagonal map $i_{g,d,e} : X^{g^e} \hookrightarrow (X^{g^e})^{\times d}$.

Thus,

$$\overline{P}_N(V) = \prod_{\substack{(d,e) \\ N = de}} i^*_{(g,d,e)} V_{g^e}^{\otimes_{\mathbb{Z}[q^\pm]} d} \otimes_{\mathbb{Z}[q^\pm]} q'_{d,e} = \prod_{\substack{(d,e) \\ N = de}} V_{g^e, q'} \otimes_{\mathbb{Z}[q^\pm]} q'_{d,e}. \tag{3.57}$$

By this formula, we can see immediately that $\overline{P}_N$ is a homomorphism between the $\mathbb{Z}[q^\pm]-$algebras.

Let V, W be two elements in $QEll_G(X)$. We have $(V \oplus W)_{g^e, q'} = V_{g^e, q'} \oplus W_{g^e, q'}$. So we have

$$\begin{aligned} \overline{P}_N(V \oplus W) &= \prod_{\substack{(d,e) \\ N = de}} (V \oplus W)_{g^e, q'} \otimes_{\mathbb{Z}[q^\pm]} q'_{d,e} = \prod_{\substack{(d,e) \\ N = de}} (V_{g^e, q'} \oplus W_{g^e, q'}) \otimes_{\mathbb{Z}[q^\pm]} q'_{d,e} \\ &= \Big(\prod_{\substack{(d,e) \\ N = de}} V_{g^e, q'} \otimes_{\mathbb{Z}[q^\pm]} q'_{d,e} \Big) \oplus \Big(\prod_{\substack{(d,e) \\ N = de}} W_{g^e, q'} \otimes_{\mathbb{Z}[q^\pm]} q'_{d,e} \Big) = \overline{P}_N(V) \oplus \overline{P}_N(W). \end{aligned}$$

$\square$

Remark 3.5.7. *$\overline{P}_n$ is the Adams operation for quasi-elliptic cohomology, as explained below.*

Tate K-theory is an elliptic cohomology with a Hopkins-Kuhn-Ravenel theory. And using the stringy power operation $P^{string}_n = \mathbb{P}_n \otimes Id$ cited in Remark 3.2.3 we can define Hecke operators for Tate K-theory. From P^{string}_n we can define the Hecke operators T_n, as shown in Section 5.4 in [24]. T_n is the Adams operations for Tate K-theory. $\overline{P}_n$ uniquely extends to an additive operator

$$\overline{P^{string}}_n = \overline{P}_n \otimes Id : K_{Tate}(X//G) \longrightarrow K_{Tate}(X//G) \otimes_{\mathbb{Z}((q))} K_{Tate}(pt//\Sigma_N)/I^{Tate}_{tr}.$$

For any $x \in K_{Tate}(X//G)$, the trace of $\overline{P^{string}}_n(x)$ is equal to $nT_n(x)$.

Chapter 4

Spectra

Goerss, Hopkins and Miller have proved that the moduli stack of elliptic curves can be covered by E_∞ elliptic spectra. It is not known whether this result can be extended to global elliptic cohomology theories and global ring spectra. In Chapter 4 we construct an orthogonal $G-$spectrum for each compact Lie group G which weakly represents quasi-elliptic cohomology. However, we show that it cannot arise from an orthogonal spectrum. Instead, in Chapter 6 we construct a new global homotopy theory and in Chapter 7 we show there is a global orthogonal spectrum in this new global homotopy theory that weakly represents orthogonal quasi-elliptic cohomology.

Let X be a $G-$space. Let $KU_{G,n}$ denote the space representing the $n-$th $G-$equivariant $K-$theory. Recall

$$QEll_G^*(X) = \prod_{\sigma \in G^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma).$$

In Section 4.3, for each compact Lie group G and each integer n, we construct a space $QEll_{G,n}$ representing $QEll^n_G(-)$ in the sense of (4.1).

$$\pi_0(QEll_{G,n}) = QEll^n_G(S^0), \text{ for each } k. \tag{4.1}$$

First I construct in Theorem 4.3.7 a homotopical right adjoint R_g for the functor $X \mapsto X^g$ from the category of G-spaces to the category of $\Lambda_G(g)$-spaces. Then we get

$$\prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{\Lambda_G(g)}(X^g, KU_{\Lambda_G(g),n})$$

is weakly equivalent to

$$\mathrm{Map}_G(X, \prod_{g \in G^{tors}_{conj}} R_g(KU_{\Lambda_G(g),n})),$$

as stated in Theorem 4.3.8.

So $QEll_{G,n} := \prod_{g \in G^{tors}_{conj}} R_g(KU_{\Lambda_G(g),n})$ is one choice of the classifying space we want.

In Section 4.5, based on the construction of $QEll_{G,n}$, we construct for each faithful $G-$representation V a space $E(G,V)$ that weakly represents $QEll^V_G(-)$ in the sense of (4.2),

$$\pi_k(E(G,V)) = QEll^V_G(S^k), \text{ for each } k. \tag{4.2}$$

Moreover, in Section 4.5.4 we construct the structure maps making E an orthogonal $G-$spectra and an $\mathcal{I}_G-$FSP.

In Section 4.5.5 we construct the restriction maps $E(G,V) \longrightarrow E(H,V)$ for each group homomorphism $H \longrightarrow G$. This map is not a homeomorphism, but an $H-$weak equivalence.

The orthogonal $G-$spectra $E(G,-)$ cannot arise from an orthogonal spectrum, as indicated in Remark 4.5.29. This fact motivates us to construct a new global homotopy theory.

4.1 Basics in equivariant homotopy theory

In this section I introduce the basic notions and concepts in equivariant homotopy theory that I need in further sections. The main reference is [44].

Let G be a compact Lie group. Let $\mathcal{T}$ denote the category of topological spaces and continuous maps. Let $G\mathcal{T}$ denote the category of $G-$spaces, namely, spaces X equipped with continuous $G-$action

$$G \times X \longrightarrow X$$

and continuous $G-$maps.

Let H be a closed subgroup of G. Let X be a $G-$space and Y an $H-$space. Define

$$X^H := \{x | hx = x, \forall h \in H\}. \tag{4.3}$$

For $x \in X$, the isotropy group of x

$$G_x := \{h | hx = x\}. \tag{4.4}$$

The induced $G-$space

$$G \times_H Y$$

is a quotient space of $G \times Y$ with (gh, x) and (g, hx) identified for $g \in G$, $h \in H$. The coinduced $G-$space

$$\mathrm{Map}_H(G, Y)$$

is the space of $H-$maps $G \longrightarrow Y$ with a left action by G induced by the right action of G on itself, namely

$$(g \cdot f)(g') = f(g'g).$$

We have the adjunctions

$$G\mathcal{T}(G \times_H Y, X) \cong H\mathcal{T}(Y, X) \tag{4.5}$$

and

$$H\mathcal{T}(X, Y) \cong G\mathcal{T}(X, \mathrm{Map}_H(G, Y)). \tag{4.6}$$

We have the $G-$homeomorphisms

$$G \times_H Y \cong (G/H) \times Y. \tag{4.7}$$

Fix a family of representatives $\{g_\alpha\}_{\alpha \in G/H}$ of the left cosets of H in G. For any $g \in G$, there is a unique g_α and $h \in H$ such that $g = g_\alpha h$. The homeomorphism is defined by

$$[g, y] \mapsto (g_\alpha H, hy).$$

The left $G-$action on $(G/H) \times X$ is defined by

$$a(g_\alpha H, y) = (g_\beta H, h'y) \tag{4.8}$$

where $g_\beta H$ is the unique left coset containing ag_α and $ag_\alpha = g_\beta h'$.

We have the $G-$homeomorphism

$$\mathrm{Map}_H(G, Y) \cong \prod_{H \backslash G} Y. \tag{4.9}$$

Fix a family of representatives $\{b_\tau\}_{\tau \in H \backslash G}$ of the right cosets of H in G. For any $g \in G$, there is a unique b_τ and $h'' \in H$ such that $g = h'' b_\tau$. The homeomorphism is defined by

$$f \mapsto (f(b_\tau))_{\tau \in H \backslash G}.$$

The left $G-$action on $\prod_{H\backslash G} Y$ is defined by

$$g \cdot (y_\tau)_{\tau \in H\backslash G} = (h_{\tau'} y_{\tau'})_{\tau \in H\backslash G}, \tag{4.10}$$

where for each $\tau \in H\backslash G$, there is a unique $\tau' \in H\backslash G$ and unique $h_{\tau'} \in H$ s.t. $Hb_\tau g = Hb_{\tau'}$, $b_\tau g = h_{\tau'} b_{\tau'}$.

Definition 4.1.1. *A $G-$homotopy between $G-$maps $X \rightrightarrows Y$ is a $G-$map*

$$X \times I \longrightarrow Y$$

where $I = [0,1]$ is a trivial $G-$space.

This gives as a homotopy category $hG\mathcal{T}$ whose objects are $G-$spaces and morphisms are $G-$homotopy classes of continuous $G-$maps. Recall that a map of spaces is a weak equivalence if it induces an isomorphism of all homotopy groups.

Definition 4.1.2. *A $G-$map $f : X_1 \longrightarrow X_2$ is said to be weak equivalence if $f^H : X_1^H \longrightarrow X_2^H$ is a weak equivalence for all the subgroups H of G.*

Let $\overline{h}G\mathcal{T}$ denote the category constructed from $hG\mathcal{T}$ by adjoining formal inverses to the weak equivalences. $\overline{h}G\mathcal{T}$ is the desired homotopy category which contains all the algebraic invariants of $G-$spaces we are interested in.

Another approach to study equivariant homotopy theory is we start with the orbit category $\mathcal{O}_G$.

Example 4.1.3. *$\mathcal{O}_G$ has objects G/H indexed by the subgroups H of G, and morphisms $G-$maps $G/H \longrightarrow G/K$. Note that there is a $G-$map $f : G/H \longrightarrow G/K$ if and only if H is subconjugate to K since if $f(eH) = gK$, then $g^{-1}Hg \subseteq K$. Obviously $\mathcal{O}_G$ is a full subcategory of $G\mathcal{T}$.*

Each $G-$space X determines a contravariant functor

$$X^{(-)} : \mathcal{O}_G \longrightarrow \mathcal{T}$$

defined by

$$X^{G/H} := X^H$$

and for each morphism

$$f : G/H \longrightarrow G/K,\ aH \mapsto agK,$$

the map $X(f) : X^K \longrightarrow X^H$ *is given by* $x \mapsto gx$.

Given two functors $X_1, X_2 : \mathcal{O}_G^{op} \longrightarrow \mathcal{T}$, a natural transformation $F : X_1 \longrightarrow X_2$ is a weak equivalence if it is an objectwise equivalence. We can observe that a $G-$map $X_1 \longrightarrow X_2$ is a weak equivalence if and only if the corresponding natural transformation $X_1^{(-)} \longrightarrow X_2^{(-)}$ is an objectwise weak equivalence.

Let $h\mathcal{T}^{\mathcal{O}_G^{op}}$ denote the homotopy category of $\mathcal{O}_G-$shaped diagrams in $\mathcal{T}$ by formally inverting the weak equivalences. We have a well-defined functor

$$\Phi : G\mathcal{T} \longrightarrow \mathcal{T}^{\mathcal{O}_G^{op}},\ X \mapsto X^{(-)}.$$

It preserves weak equivalences, so we have an induced functor

$$\Phi : \overline{h}G\mathcal{T} \longrightarrow \overline{h}\mathcal{T}^{\mathcal{O}_G^{op}}.$$

Moreover, we can define a functor

$$\Psi : \mathcal{T}^{\mathcal{O}_G^{op}} \longrightarrow G\mathcal{T},$$

as indicated in [20]. I sketch the construction below. For any small topological category $\mathcal{D}$, let $B_n(\mathcal{D})$ be the set of n-tuples $\underline{f} = (f_1, \cdots f_n)$ of composable arrows of $\mathcal{D}$, depicted

$$d_0 \xleftarrow{f_1} d_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} d_n.$$

$B_0(\mathcal{D})$ is the set of objects of $\mathcal{D}$ and $B_n(\mathcal{D})$ is topologized as a subspace of the $n-$fold product of the total morphism space $\coprod \mathcal{D}(d, d')$. With the zeroth and last face given by deleting the zeroth or last arrow of $n-$tuples $\underline{f}$ and with the remaining face and degeneracy operations given by composition or by insertion of identity maps in the appropriate position, $B_*(\mathcal{D})$ is a simplicial set, i.e. the nerve of $\mathcal{D}$. Its geometric realization is the classifying space $B\mathcal{D}$. Let $S : \mathcal{O}_G^{op} \longrightarrow G\mathcal{T}$ be the covariant functor sending an object G/H to the $G-$space G/H, and it sends a morphism $G/H \longrightarrow G/K$ to the same map $G/H \longrightarrow G/K$. For each $\mathcal{O}_G^{op}-$space T, define ΨT to be the $G-$space $B(T, \mathcal{O}_G^{op}, S)$ to be the geometric realization of

$B_*(T, \mathcal{O}_G^{op}, S)|$ where $B_*(T, \mathcal{O}_G^{op}, S)$ is the simplicial space whose set of $n-$simplices is

$$\{(t, \underline{f}, s) | t \in T(d_0),\ \underline{f} \in B_n(\mathcal{O}_G^{op}),\ s \in S(d_n)\}.$$

The theorem below shows the two approaches are equivalent.

Theorem 4.1.4 (Elemendorf's Theorem). *The functor* $\Psi : \overline{h}\mathcal{T}^{\mathcal{O}_G^{op}} \longrightarrow \overline{h}G\mathcal{T}$ *is a right adjoint of the functor* $\Phi : \overline{h}G\mathcal{T} \longrightarrow \overline{h}\mathcal{T}^{\mathcal{O}_G^{op}}$*. They give an equivalence of the two categories.*

Next let's look at how equivariant CW-complex is constructed.

Let X be a space of the homotopy type of a $G-$CW complex. Let X_n denotes the $n-$skeleton of X. X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching $G-$cells $G/H \times D^{n+1}$ along attaching $G-$maps $G/H \times S^n \longrightarrow X^n$.

We have the homotopy pushout for each k

$$\begin{tikzcd} \coprod G/H \times S^k \arrow[r] \arrow[d] & \coprod G/H \times D^k \arrow[d] \\ X_k \arrow[r] & X_{k+1} \end{tikzcd} \tag{4.11}$$

X is the homotopy colimit of the diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[drr] & X_1 \arrow[r] \arrow[dr] & X_2 \arrow[r] \arrow[d] & \cdots \\ & & X & \end{tikzcd} \tag{4.12}$$

with each map in it an inclusion.

Let $G\mathcal{C}$ denote the category of $G-$CW complexes and celluar maps.

Proposition 4.1.5 is a conclusion needed for the construction of the rest of the section.

Proposition 4.1.5. *Let D be a complete category. Let $i : \mathcal{O}_G^{op} \longrightarrow G\mathcal{C}^{op}$ be the inclusion of subcategory. If $F_1, F_2 : G\mathcal{C}^{op} \longrightarrow D$ are two functors sending homotopy colimit to homotopy limit and if we have a natural transformation $p : F_1 \longrightarrow F_2$, which gives a weak equivalence at orbits, then it also gives a weak equivalence on $G\mathcal{C}$.*

Especially, if p gives a retract at each orbit, F_1 is a retract of F_2 at each $G-CW$ complexes.

Proof. Apply the functor F_1 and F_2 to the homotopy pushout diagram (4.11), and get the homotopy pullback diagrams

$$\begin{tikzcd}
F_i(\coprod G/H \times S^k) & F_i(\coprod G/H \times D^k) \arrow[l] \\
F_i(X_k) \arrow[u] & F_i(X_{k+1}) \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.13}$$

for $i = 1, 2,\ k = 0, 1, 2 \cdots$.

Since F_1 and F_2 are weak equivalent on $\mathcal{O}_G^{op}$, for each closed subgroup H of G, we have homotopy equivalence $p_H : F_1(G/H) \longrightarrow F_2(G/H)$.

For $0-$dimensional $G-$cells, we have

$$F_1(\coprod G/H \times S^0) \simeq F_2(\coprod (G/H) \times S^0) \tag{4.14}$$

$$F_1(X_0) \simeq F_2(X_0). \tag{4.15}$$

Since $G/H \times D^k$ is $G-$homotpic to G/H, we have

$$F_1(\coprod G/H \times D^k) \simeq F_2(\coprod (G/H) \times D^k) \text{ for any } k. \tag{4.16}$$

So we have the commutative diagram

$$\begin{tikzcd}
 & & F_1(X_1) & & \\
F_1(X_0) \arrow[urr] \arrow[d] & & F_1(\coprod (G/H) \times S^0) \arrow[ll] \arrow[rr] \arrow[d] & & F_1(\coprod (G/H) \times D^1) \arrow[ull] \arrow[d] \\
F_2(X_0) \arrow[drr] & & F_2(\coprod (G/H) \times S^0) \arrow[ll] \arrow[rr] & & F_2(\coprod (G/H) \times D^1) \arrow[dll] \\
 & & F_2(X_1) & &
\end{tikzcd}$$

which gives the weak equivalence

$$F_1(X_1) \simeq F_2(X_1). \tag{4.17}$$

Then we consider (4.13) when $k = 1$, similarly we can get the commutative diagram

$$\begin{tikzcd}
 & F_1(X_2) & \\
F_1(X_1) \arrow[d] \arrow[ur] & F_1(\coprod(G/H) \times S^1) \arrow[l] \arrow[r] \arrow[d] & F_1(\coprod(G/H) \times D^2) \arrow[d] \arrow[ul] \\
F_2(X_1) \arrow[dr] & F_2(\coprod(G/H) \times S^1) \arrow[l] \arrow[r] & F_2(\coprod(G/H) \times D^2) \arrow[dl] \\
 & F_2(X_2) &
\end{tikzcd}$$

thus, the weak equivalence

$$F_1(X_2) \simeq F_2(X_2). \tag{4.18}$$

Note that we have weak equivalences

$$F_1(\coprod(G/H) \times S^k) \simeq F_2(\coprod(G/H) \times S^k) \tag{4.19}$$

for any $k = 0, 1, 2 \cdots$. So by induction, for each skeleton X_k, we have the weak equivalence

$$F_1(X_k) \simeq F_2(X_k). \tag{4.20}$$

Passing to the colimit of the diagram (4.12), we get the weak equivalence

$$F_1(X) \simeq F_2(X) \tag{4.21}$$

since F_1 and F_2 send colimit to limit.

If we have natural transformations $i : F_1 \longrightarrow F_2$ and $r : F_2 \longrightarrow F_1$ between the functors $F_1, F_2 : \mathcal{O}_G^{op} \longrightarrow D$ such that $r \circ i$ is the identity natural transformation, by the above argument, for each $k = 0, 1, 2 \cdots$, we have the weak equivalences

$$r_k : F_2(X_k) \longrightarrow F_1(X_k)$$

$$i_k : F_1(X_k) \longrightarrow F_2(X_k)$$

and

$$r_k \circ i_k = id_{F_1(X_k)}.$$

Passing to the colimit of (4.12), we can get the weak equivalences

$$r_X : F_2(X) \longrightarrow F_1(X)$$
$$i_X : F_1(X) \longrightarrow F_2(X)$$

and

$$r_X \circ i_X = id_{F_1(X)}.$$

Thus, $F_1(X)$ is a retract of $F_2(X)$.

□

4.2 Equivariant Obstruction theory

Equivariant obstruction theory works exactly as it does nonequivariantly. I will give a sketch. The references for this theory include [12], [44], [61].

First let's recall several basic concepts.

Let $h\mathcal{O}_G$ be the homotopy category of $\mathcal{O}_G$. A coefficient system is a contravariant functor from $h\mathcal{O}_G$ to the category $\mathcal{A}b$ of abelian groups. One example of coefficient system is the system $\underline{\pi}_n(X)$ of a based $G-$space X with

$$\underline{\pi}_n(X)(G/H) := \pi_n(X^H).$$

More generally, any covariant functor $h\mathcal{T} \longrightarrow \mathcal{A}b$, such as π_n, composed with the functor $X^{(-)} : \mathcal{O}_G \longrightarrow \mathcal{T}$ gives a coefficient system.

Equivariant obstruction theory is established in terms of ordinary cohomology theories with coefficients in such coefficient system.

Let X be a $G-$CW complex. We have a coefficient system

$$\underline{C}_n(X) = \underline{H}_n(X^n, X^{n-1}; \mathbb{Z}). \tag{4.22}$$

Its value on G/H is

$$H_n((X^n)^H, (X^{n-1})^H).$$

The connecting homomorphisms of the triples $((X^n)^H, (X^{n-1})^H, (X^{n-2})^H)$ specify a map

$d : \underline{C}_n(X) \longrightarrow \underline{C}_{n-1}(X)$ of coefficient systems, and $d^2 = 0$. In this way we get a chain complex of coefficient systems $\underline{C}_*(X)$.

Let $Hom_{\mathcal{O}_G}(M, M')$ denote the Abelian group of maps of coefficient system $M \longrightarrow M'$. Define

$$C^n_G(X; M) = Hom_{\mathcal{O}_G}(\underline{C}_n(X), M), \text{ with } \delta = Hom_{\mathcal{O}_G}(d, id). \tag{4.23}$$

$C^*_G(X; M)$ is a cochain complex of Abelian groups. Its homology is the Bredon cohomology of X, denoted by $H^*_G(X; M)$.

Definition 4.2.1. *A connected space X is said to be $n-$simple if $\pi_1(X)$ is Abelian and acts trivially on $\pi_q(X)$ for $q \leq n$.*

Let (X, A) be a relative $G-$CW complex and let Y be a $G-$space such that Y^H is non-empty, connected, and $n-$simple if H occurs as an isotropy subgroup of $X \setminus A$.

Given a $G-$map $f : X^n \cup A \longrightarrow Y$, we ask when f can be extended to X^{n+1}. Composing the attaching maps $G/H \times S^n \longrightarrow X$ of cells of $X \setminus A$ with f gives elements of $\pi_n(Y^H)$, which together specify a well-defined cocycle

$$c_f \in C^{n+1}_G(X, A; \underline{\pi}_n(Y)).$$

f can extend to X^{n+1} if and only if $c_f = 0$.

If f and f' are maps $X^n \cup A \longrightarrow Y$ and h is a homotopy rel A of the restrictions of f and f' to $X^{n-1} \cup A$, then f, f', and h together define a map

$$h(f, f') : (X \times [0, 1])^n \longrightarrow Y.$$

Applying the cocycle $c_{h(f,f')}$ to cells $j \times [0, 1]$ we obtain a deformation cochain

$$d_{f,f',h} \in C^n_G(X, A; \underline{\pi}_n(Y))$$

such that $\delta d_{f,f',h} = c_f - c_{f'}$.

Theorem 4.2.2. *(i) For $f : X^n \cup A \longrightarrow Y$, the restriction of f to $X^{n-1} \cup A$ extends to a map $X^{n+1} \cup A \longrightarrow Y$ if and only if $[c_f] = 0$ in $H^{n+1}_G(X, A; \underline{\pi}_n(Y))$.*

(ii) Given maps $f, f' : X^n \longrightarrow Y$ and homotopy rel A of their restrictions to $X^{n-1} \cup A$, there is an obstruction in $H^n_G(X, A; \underline{\pi}_n(Y))$ that vanishes if and only if the restriction of the

given homotopy to $X^{n-2} \cup A$ extends to a homotopy $f \simeq f'$ rel A.

4.3 The Construction of $QEll_{G,n}$

In this section, for each integer n, each compact Lie group G, I construct a space $QEll_{G,n}$ representing the $n-$th $G-$equivariant quasi-elliptic cohomology $QEll^n_G$ up to weak equivalence.

Before constructing the spectra, I explain what a good "weak equivalence" means.

Definition 4.3.1 (homotopical adjunction). *Let H and G be two compact Lie groups. Let*

$$L : G\mathcal{T} \longrightarrow H\mathcal{T} \text{ and } R : H\mathcal{T} \longrightarrow G\mathcal{T} \tag{4.24}$$

be two functors. A left-to-right homotopical adjunction is a natural map

$$Map_H(LX, Y) \longrightarrow Map_G(X, RY), \tag{4.25}$$

which is a weak equivalence of spaces when X is a $G-CW$ complex.

Analogously, a right-to-left homotopical adjunction is a natural map

$$Map_G(X, RY) \longrightarrow Map_H(LX, Y) \tag{4.26}$$

which is a weak equivalence of spaces when X is a $G-CW$ complex.

L is called a homotopical left adjoint and R a homotopical right adjoint.

Let's see an example.

Example 4.3.2. *Let $G = \mathbb{Z}/2\mathbb{Z}$ and g be a generator of G. We want to find a homotopical right adjoint R of the functor $X \mapsto X^g$ from the category $G\mathcal{T}$ of G-spaces to the category $\mathcal{T}$ of topological spaces.*

Let Y be a topological space. Suppose we have

$$Map(X^g, Y) \simeq Map_G(X, RY).$$

G has two subgroups, e and G.

$$RY^e = Map_G(G/e, RY) \simeq Map((G/e)^g, Y) \simeq pt;$$
$$RY^G = Map_G(G/G, RY) \simeq Map((G/G)^g, Y) = Y.$$

If Y is the empty set, $R\emptyset$ is EG. And generally for any Y, one choice of RY is the join $Y * EG$.

By Elmendorf's theorem 4.1.4, the space RY is unique up to $G-$homotopy. By definition, the functor R is a homotopical right adjoint to the fixed point functor $X \mapsto X^g$.

This definition below is Definition 4.5 in Chapter V of [44].

Definition 4.3.3. *A family $\mathcal{F}$ in G is a set of subgroups of G that is closed under subconjugacy: if $H \in \mathcal{F}$ and $g^{-1}Kg \subset H$, then $K \in \mathcal{F}$. An $\mathcal{F}$-spaces is a G-space all of whose isotropy groups are in $\mathcal{F}$. Define a functor $\underline{\mathcal{F}} : h\mathcal{O}_G \longrightarrow Sets$ by sending G/H to the 1-point set if $H \in \mathcal{F}$ and to the empty set if H is not in $\mathcal{F}$. Define the universal $\mathcal{F}$-space X of the homotopy type of a G-CW complex, there is one and, up to homotopy, only one G-map $X \longrightarrow E\mathcal{F}$. Define the classifying space of the family $\mathcal{F}$ to be the orbit space $B\mathcal{F} = E\mathcal{F}/G$.*

For any compact Lie group G, let $\langle g \rangle$ denote the cyclic subgroup of G generated by $g \in G^{tors}$ and $*$ denote the join. Let

$$S_{G,g} := \mathrm{Map}_{\langle g \rangle}(G, *_K E(\langle g \rangle/K))$$

where K goes over all the maximal subgroups of $\langle g \rangle$ and $E(\langle g \rangle/K)$ is the universal space of the cyclic group $\langle g \rangle/K$. The action of $\langle g \rangle/K$ on $E(\langle g \rangle/K)$ is free.

For this space $S_{G,g}$, it's classified up to $G-$homotopy, as shown in the following lemma.

Lemma 4.3.4. *For any closed subgroup $H \leqslant G$, $S_{G,g}$ satisfies*

$$S^H_{G,g} \simeq \begin{cases} pt, & \textit{if for any } b \in G,\ b^{-1}\langle g \rangle b \nleqslant H; \\ \emptyset, & \textit{if there exists a } b \in G \textit{ such that } b^{-1}\langle g \rangle b \leqslant H. \end{cases} \tag{4.27}$$

Proof. For any closed subgroup H of G.

$$S^H_{G,g} = \mathrm{Map}_{\langle g \rangle}(G/H, *_K E(\langle g \rangle/K)) \tag{4.28}$$

where K goes over all the cyclic groups $\langle g^m \rangle$ with $\frac{|g|}{m}$ a prime.

If there exists an $b \in G$ such that $b^{-1}\langle g \rangle b \leqslant H$, it's equivalent to say that there exists points in G/H that can be fixed by g. But there are no points in $*_K E(\langle g \rangle / K)$ that can be fixed by g. So there is no $\langle g \rangle-$equivariant map from G/H to $*_K E(\langle g \rangle / K)$. In this case $S^H_{G,g}$ is empty.

If for any $b \in G$, $b^{-1}\langle g \rangle b \nleq H$, it's equivalent to say that there are no points in G/H that can be fixed by g. And for any subgroup $\langle g^m \rangle$ which is not $\langle g \rangle$ itself, $(*_K E(\langle g \rangle / K))^{\langle g^m \rangle}$ is the join of several contractible spaces $E(\langle g \rangle / K)^{\langle g^m \rangle}$, thus, contractible. So all the homotopy groups $\pi_n((*_K E(\langle g \rangle / K))^{\langle g^m \rangle})$ are trivial. For any $n \geq 1$ and any $\langle g \rangle-$equivariant map

$$f : (G/H)^n \longrightarrow *_K E(\langle g \rangle / K)$$

from the $n-$skeleton of G/H, the obstruction cocycle $[c_f]$, which is formulated in Section 4.2, is zero.

Then by Theorem 4.2.2, f can be extended to the $(n+1)-$cells of G/H, and any two extensions f and f' are $\langle g \rangle-$homotopic.

So in this case $S^H_{G,g}$ is contractible. □

Remark 4.3.5. *Let S_g denote the set of all the proper subgroups of $\langle g \rangle$ and M_g denote the set of all the maximal subgroups of $\langle g \rangle$. The space*

$$T_{G,g} := Map_{\langle g \rangle}(G, *_{K \in S_g} E(\langle g \rangle / K))$$

also satisfies the conditions in Lemma 4.3.4, so it's $G-$homotopy equivalent to $S_{G,g}$. In fact, for any subset S'_g of S_G containing M_g, the space

$$T'_{G,g} = Map_{\langle g \rangle}(G, *_{K \in S'_G} E(\langle g \rangle / K))$$

is a choice satisfying the conditions in Lemma 4.3.4, which is straightforward to check. The weak equivalence $S_{G,g} \longrightarrow T'_{G,g}$ can be constructed as

$$f \mapsto i^* \circ f$$

where $i : *_{K \in M_g} E(\langle g \rangle / K)) \longrightarrow *_{K \in S'_G} E(\langle g \rangle / K))$ is the inclusion $x \mapsto (1x, 0, \cdots 0)$.

In Theorem 4.3.6 I show another example of homotopical right adjoint, which is crucial to the construction of $QEll_G^n$.

Theorem 4.3.6. *Let G be a compact Lie group and $g \in G^{tors}$. Consider the functor*

$$L_g : G\mathcal{T} \longrightarrow C_G(g)\mathcal{T},\ X \mapsto X^g.$$

A homotopical right adjoint of it is $R_g : C_G(g)\mathcal{T} \longrightarrow G\mathcal{T}$ with

$$R_g Y = Map_{C_G(g)}(G, Y * S_{C_G(g),g}). \tag{4.29}$$

Proof. Let H be any closed subgroup of G.

First I show given a $C_G(g)-$equivariant map $f : (G/H)^g \longrightarrow Y$, it extends uniquely up to $C_G(g)-$homotopy to a $C_G(g)-$equivariant map $\widetilde{f} : G/H \longrightarrow Y * S_{C_G(g),g}$. f can be viewed as a map $(G/H)^g \longrightarrow Y * S_{C_G(g),g}$ by composing with the inclusion of one end of the join

$$Y \longrightarrow Y * S_{C_G(g),g},\ y \mapsto (1y, 0).$$

If $bH \in (G/H)^g$, define $\widetilde{f}(bH) := f(bH)$.

If bH is not in $(G/H)^g$, its stabilizer group does not contain g. By Lemma 4.3.4, for any subgroup K of it, $S^K_{C_G(g),g}$ is contractible. So $(Y * S_{C_G(g),g})^K = Y^K * S^K_{C_G(g),g}$ is contractible. In other words, if K occurs as the isotropy subgroup of a point outside $(G/H)^g$, $\pi_n((Y * S_{C_G(g),g})^K)$ is trivial. By Theorem 4.2.2, f can extend to a $C_G(g)-$equivariant map $\widetilde{f} : G/H \longrightarrow Y * S_{C_G(g),g}$, and any two extensions are $C_G(g)-$homotopy equivalent. In addition, $S^g_{C_G(g),g}$ is empty. So the image of the restriction of any map $G/H \longrightarrow Y * S_{C_G(g),g}$ to the subspace $(G/H)^g$ is contained in the end Y of the join.

Thus, $\mathrm{Map}_{C_G(g)}((G/H)^g, Y)$ is weak equivalent to $\mathrm{Map}_{C_G(g)}(G/H, Y * S_{C_G(g),g})$.

Moreover, by the adjunction (4.6) we have the equivalence

$$\mathrm{Map}_G\Big(G/H,\ \mathrm{Map}_{C_G(g)}(G, Y * S_{C_G(g),g})\Big) \cong \mathrm{Map}_{C_G(g)}(G/H,\ Y * S_{C_G(g),g}) \tag{4.30}$$

So we get

$$R_g Y^H = \mathrm{Map}_G(G/H, R_g Y) \simeq \mathrm{Map}_{C_G(g)}((G/H)^g, Y) \tag{4.31}$$

Let X be of the homotopy type of a $G-$CW complex. Let X^k denote the $k-$skeleton of

X. Consider the functors

$$\mathrm{Map}_G(-, R_gY) \text{ and } \mathrm{Map}_{C_G(g)}((-)^g, Y)$$

from $G\mathcal{T}$ to $\mathcal{T}$. Both of them sends homotopy colimit to homotopy limit. In addition, we have a natural map from $\mathrm{Map}_G(-, R_gY)$ to $\mathrm{Map}_{C_G(g)}((-)^g, Y)$ by sending a $G-$map $F : X \longrightarrow R_gY$ to the composition

$$X^g \xrightarrow{F^g} (R_gY)^g \longrightarrow Y^g \subseteq Y \tag{4.32}$$

with the second map $f \mapsto f(e)$. Note that for any $f \in (R_gY)^g$, $f(e) = (g \cdot f)(e) = f(eg) = f(g) = g \cdot f(e)$ so $f(e) \in (Y * S_{C_G(g),g})^g = Y^g$ and the second map is well-defined. It gives weak equivalence on orbits, as shown in (4.31). Thus, by Proposition 4.1.5, R_g is a homotopical right adjoint of L.

□

Theorem 4.3.7. *Let G be a compact Lie group, $g \in G^{tors}$, and Y a $\Lambda_G(g)-$space. The subgroup*

$$\{[(1, t)] \in \Lambda_G(g) | t \in \mathbb{R}\}$$

of $\Lambda_G(g)$ is isomorphic to $\mathbb{R}$. Let's use the same symbol $\mathbb{R}$ to denote it. Consider the functor $\mathcal{L}_g : G\mathcal{T} \longrightarrow \Lambda_G(g)\mathcal{T}, X \mapsto X^g$ where $\Lambda_G(g)$ acts on X^g by

$$[g, t] \cdot x = gx.$$

The functor $\mathcal{R}_g : \Lambda_G(g)\mathcal{T} \longrightarrow G\mathcal{T}$ with

$$\mathcal{R}_gY = Map_{C_G(g)}(G, Y^{\mathbb{R}} * S_{C_G(g),g}) \tag{4.33}$$

is a homotopical right adjoint of $\mathcal{L}_g$.

Proof. Let X be a $G-$space. Let H be any closed subgroup of G. Note for any $G-$space X, $\mathbb{R}$ acts trivially on X^g, thus, the image of any $\Lambda_G(g)-$equivariant map $X^g \longrightarrow Y$ is in $Y^{\mathbb{R}}$. So we have

$$\mathrm{Map}_{\Lambda_G(g)}(X^g, Y) = \mathrm{Map}_{C_G(g)}(X^g, Y^{\mathbb{R}}).$$

First I show $f : (G/H)^g \longrightarrow Y^{\mathbb{R}}$ extends uniquely up to $C_G(g)-$homotopy to a $C_G(g)-$equivariant map $\widetilde{f} : G/H \longrightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$. f can be viewed as a map $(G/H)^g \longrightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$ by composing with the inclusion as the end of the join

$$Y^{\mathbb{R}} \longrightarrow Y^{\mathbb{R}} * S_{C_G(g),g},\ y \mapsto (1y, 0).$$

If $bH \in (G/H)^g$, define $\widetilde{f}(bH) = f(bH)$.

If bH is not in $(G/H)^g$, its stabilizer group does not contain g. By Lemma 4.3.4, for any subgroup K of it, $S^K_{C_G(g),g}$ is contractible. So $(Y^{\mathbb{R}} * S_{C_G(g),g})^K = (Y^{\mathbb{R}})^K * S^K_{C_G(g),g}$ is contractible. In other words, if K occurs as the isotropy subgroup of a point in G/H outside $(G/H)^g$, $\pi_n((Y^{\mathbb{R}} * S_{C_G(g),g})^K)$ is trivial. By Theorem 4.2.2, f can extend to a $C_G(g)-$equivariant map $\widetilde{f} : G/H \longrightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$, and any two extensions are $C_G(g)-$homotopy equivalent. In addition, $S^g_{C_G(g),g}$ is empty. So the image of the restriction of any map $G/H \longrightarrow Y^{\mathbb{R}} * S_{C_G(g),g}$ to the subspace $(G/H)^g$ is contained in the end $Y^{\mathbb{R}}$ of the join.

Thus, $\mathrm{Map}_{C_G(g)}((G/H)^g, Y^{\mathbb{R}})$ is weak equivalent to $\mathrm{Map}_{C_G(g)}(G/H, Y^{\mathbb{R}} * S_{C_G(g),g})$.

Moreover, by the adjunction (4.6) we have the equivalence

$$\mathrm{Map}_G\Big(G/H,\ \mathrm{Map}_{C_G(g)}(G, Y^{\mathbb{R}} * S_{C_G(g),g})\Big) \cong \mathrm{Map}_{C_G(g)}(G/H,\ Y^{\mathbb{R}} * S_{C_G(g),g}) \qquad (4.34)$$

So we get

$$\mathcal{R}_g Y^H = \mathrm{Map}_G(G/H, \mathcal{R}_g Y) \backsimeq \mathrm{Map}_{C_G(g)}((G/H)^g, Y) \qquad (4.35)$$

Let X be a space of the homotopy type of a $G-$CW complex. Consider the functors

$$\mathrm{Map}_G(-, \mathcal{R}_g Y) \text{ and } \mathrm{Map}_{C_G(g)}((-)^g, Y)$$

from $G\mathcal{T}$ to $\mathcal{T}$. Both of them sends homotopy colimit to homotopy limit. In addition, we have a natural map from $\mathrm{Map}_G(-, \mathcal{R}_g Y)$ to $\mathrm{Map}_{C_G(g)}((-)^g, Y)$ by sending a $G-$map $F : X \longrightarrow \mathcal{R}_g Y$ to the composition

$$X^g \xrightarrow{F^g} (\mathcal{R}_g Y)^g \longrightarrow Y^g \qquad (4.36)$$

with the second map $f \mapsto f(e)$. Note that for any $f \in (\mathcal{R}_g Y)^g$, $f(e) = (g \cdot f)(e) = f(eg) = f(g) = g \cdot f(e)$ so $f(e) \in (Y^{\mathbb{R}} * S_{C_G(g),g})^g = Y^{\mathbb{R}}$ and the second map is well-defined. It gives weak equivalence on orbits, as shown in (4.35). Thus, by Proposition 4.1.5, $\mathcal{R}_g$ is a homotopical right adjoint of $\mathcal{L}_g$.

□

Theorem 4.3.7 implies Theorem 4.3.8 directly.

Theorem 4.3.8. *For any compact Lie group G and any integer n, Let $KU_{G,n}$ denote the space representing the $n-th$ $G-$equivariant $KU-$theory. The $n-th$ quasi-elliptic cohomology*

$$QEll^n_G(X) \cong \prod_{g\in G^{tors}_{conj}} [X^g, KU_{\Lambda_G(g),n}]^{\Lambda_G(g)}$$

is weakly represented by the space

$$QEll_{G,n} := \prod_{g\in G^{tors}_{conj}} \mathcal{R}_g(KU_{\Lambda_G(g),n})$$

in the sense of (4.1) where $\mathcal{R}_g(KU_{\Lambda_G(g),n})$ is the space

$$Map_{C_G(g)}(G, KU^{\mathbb{R}}_{\Lambda_G(g),n} * S_{C_G(g),g}).$$

Moreover, for Real quasi-elliptic cohomology

$$QEllR^*(X) := \prod_{g\in G^{tors}_{conj}} KR^*_{\Lambda_G(g)}(X^g),$$

and real quasi-elliptic cohomology

$$QEllr^*(X) := \prod_{g\in G^{tors}_{conj}} KO^*_{\Lambda_G(g)}(X^g),$$

we also have similar conclusions by applying the homotopical right adjoint $\mathcal{R}_g$ in Theorem 4.3.7.

Theorem 4.3.9. *Let $KR_{G,n}$ denote the space representing the $n-th$ $G-$equivariant $KR-$theory. The $G-$space*

$$QEllR_{G,n} := \prod_{g\in G^{tors}_{conj}} Map_{C_G(g)}(G, KR^{\mathbb{R}}_{\Lambda_G(g),n} * S_{C_G(g),g})$$

weakly represents the Real quasi-elliptic cohomology $QEllR^n_G(-)$ in the sense of

$$\pi_0(QEllR_{G,n}) = QEllR^n_G(S^0). \tag{4.37}$$

Theorem 4.3.10. *Let $KO_{G,n}$ denote the space representing the $n-th$ $G-$equivariant $KO-$theory. The $G-$space*

$$QEllr_{G,n} := \prod_{g\in G^{tors}_{conj}} Map_{C_G(g)}(G, KO^{\mathbb{R}}_{\Lambda_G(g),n} * S_{C_G(g),g})$$

weakly represents the real quasi-elliptic cohomology $QEllr^n_G(-)$ in the sense of

$$\pi_0(QEllr_{G,n}) = QEllr^n_G(S^0). \tag{4.38}$$

4.4 Global homotopy theory

In this section, I recall the concepts and construction of orthogonal spaces and orthogonal spectra needed for my construction in Section 4.5 and Chapter 7. There are many references for this section, [11], [56], [43], etc.

4.4.1 Orthogonal G-spectra

First Let's briefly recall some basic concepts for orthogonal G-spectra, which are usually defined in terms of diargrams on the category of orthogonal representations of G. The main reference for this part is [11] and [41].

Let G be a compact Lie group. Let $\mathcal{I}_G$ denote the category whose objects are pairs $(\mathbb{R}^n, \rho)$ with ρ a homomorphism from G to $O(n)$ giving $\mathbb{R}^n$ the structure of a $G-$representation. Morphisms $(\mathbb{R}^m, \mu) \longrightarrow (\mathbb{R}^n, \rho)$ are linear isometric isomorphisms $\mathbb{R}^m \longrightarrow \mathbb{R}^n$.

Let Top_G denote the category with objects based $G-$spaces and morphisms continuous based maps.

Definition 4.4.1. *An $\mathcal{I}_G$−space is a G−continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$. Morphisms between $\mathcal{I}_G$−spaces are natural G−transformations.*

Example 4.4.2. *The sphere $\mathcal{I}_G$−space S is the functor $V \mapsto S^V$ which sends a representation to its one-point compactification.*

Definition 4.4.3. *An orthogonal G−spectrum is an $\mathcal{I}_G$−space X together with a natural transformation of functors $\mathcal{I}_G \times \mathcal{I}_G \longrightarrow Top_G$*

$$X(-) \wedge S^{-} \longrightarrow X(- \oplus -)$$

satisfying appropriate associativity and unitality diagrams. In other words, an orthogonal G−spectrum is an $\mathcal{I}_G$−space with an action of the sphere $\mathcal{I}_G$−space.

Definition 4.4.4. *For $\mathcal{I}_G$−spaces X and Y, define the "external" smash product $X\overline{\wedge}Y$ by*

$$X\overline{\wedge}Y = \wedge \circ (X \times Y) : \mathcal{I}_G \times \mathcal{I}_G \longrightarrow Top_G; \tag{4.39}$$

thus $(X\overline{\wedge}Y)(V, W) = X(V) \wedge Y(W)$.

We have an equivariant notion of a functor with smash product (FSP).

Definition 4.4.5. *An $\mathcal{I}_G$−FSP is an $\mathcal{I}_G$−space X with a unit G−map $\eta : S \longrightarrow X$ and a natural product G−map $\mu : X\overline{\wedge}X \longrightarrow X \circ \bigoplus$ of functors $\mathcal{I}_G \times \mathcal{I}_G \longrightarrow Top_G$ such that the evident unit, associativity and centrality of unit diagram also commutes.*

A $\mathcal{I}_G$−FSP is commutative if the diagram below commutes.

$$\begin{array}{ccc} X(V) \wedge X(W) & \xrightarrow{\mu} & X(V \oplus W) \\ {\scriptstyle\tau}\Big\downarrow & & \Big\downarrow{\scriptstyle X(\tau)} \\ X(W) \wedge X(V) & \xrightarrow{\mu} & X(W \oplus V). \end{array}$$

Note that this diagram commutes implies the centrality of unit diagram commutes.

Lemma 4.4.6. *An $\mathcal{I}_G$−FSP has an underlying $\mathcal{I}_G$−spectrum with structure G−map*

$$\sigma = \mu \circ (id\overline{\wedge}\eta) : X\overline{\wedge}S \longrightarrow X \circ \oplus.$$

4.4.2 Orthogonal spectra

The global homotopy theory is established to better describe certain theories naturally exists not only for a particular group, but for all groups of certain type in a compatible way. Some good examples of this are equivariant stable homotopy, equivariant K-theory, and equivariant bordism.

The idea of global orthogonal spectra was first inspired in the paper [28] by Greenlees and May where they introduce the concept of global $\mathcal{I}_*-$functors with smash product. The idea is developed by Mandell and May [41] and Bohmann [11]. Schwede develops another modern approach of global homotopy theory using a different categorical framework in [56], which is the main reference for Section 4.4.2.

For definition of orthogonal spectra in detail, please refer [43], [42], [56].

First let's see the definition of orthogonal spaces. Let $\mathbb{L}$ denote the category whose objects are inner product real spaces and whose morphism set between two objects V and W are the linear isometric embeddings $L(V, W)$.

Definition 4.4.7. *An orthogonal space is a continuous functor $Y : \mathbb{L} \longrightarrow \mathcal{T}$ to the category of topological spaces. A morphism of orthogonal spaces is a natural transformation. We denote by spc the category of orthogonal spaces.*

For each compact Lie group G, let $\mathcal{U}_G$ denote a fixed complete $G-$universe, and let $s(\mathcal{U}_G)$ denote the poset, under inclusion, of finite dimensional $G-$subrepresentations of $\mathcal{U}_G$.

The $G-$equivariant path components

$$\pi_0^G(Y) := colim_{V\in s(\mathcal{U}_G)}\pi_0(Y(V)^G).$$

Let G and K be compact Lie groups. And let V be a finite-dimensional $G-$representation and U a $K-$ representation. Then we can define a left $K-$action and right $G-$action:

$$\forall \phi \in L(V, U), k \in K, g \in G, v \in V, ((k, g) \cdot \phi)(v) := k\phi(g^{-1}v).$$

Definition 4.4.8. *The global classifying space $B_{gl}G$ of a compact Lie group G is the free orthogonal space*

$$B_{gl}G = L(V, -)/G$$

where V is any faithful $G-$representation.

$B_{gl}G(\mathcal{U}_K)$ classifies $(K, G)-$bundles. For the trivial universe $\mathbb{R}^\infty$, $B_{gl}G(\mathbb{R}^\infty) = BG$; $EG = L(V, \mathbb{R}^\infty)$ for any faithful $G-$representation.

Orthogonal spectra is the stabilization of orthogonal spaces.

Let $\mathbb{O}$ denote the category whose objects are inner product real spaces and the morphisms $O(V, W)$ between two objects V and W is the Thom space of the total space

$$\xi(V, W) := \{(w, \phi) \in W \times L(V, W) | W \perp \phi(V)\}$$

of the orthogonal complement vector bundle, whose structure map $\xi(V, W) \longrightarrow L(V, W)$ is the projection to the second factor.

Definition 4.4.9. *An orthogonal spectrum is a based continuous functor from $\mathbb{O}$ to the category of based compactly generated weak Hausdorff spaces. A morphism is a natural transformation of functors. Let Sp denote the category of orthogonal spectrum.*

For each linear isometric embedding $\phi : V \longrightarrow W$, consider the one-point compactification of the inclusion of the fiber over ϕ of the bundle $\xi(V, W)$. We can define a continuous map

$$(-, \phi) : S^{W-\phi(V)} \longrightarrow O(V, W),\ w \mapsto (w, \phi)$$

Let X be an orthogonal spectrum. The structure map of X associated to ϕ is defined to be the composite

$$\phi_\star := X \circ (X(V) \wedge (-, \phi)) : X(V) \wedge S^{W-\phi(V)} \xrightarrow{X(V)\wedge(-,\phi)} X(V) \wedge O(V, W) \xrightarrow{X} X(W).$$

If $\phi_0 : V \longrightarrow V \oplus U,\ v \mapsto (v, 0)$ is the direct summand inclusion, Let

$$\sigma_{V,U} := \phi_{0\star} : X(V) \wedge S^U \longrightarrow X(V \oplus U)$$

denote the associated structure map.

Remark 4.4.10 (Coordinatrized orthogonal spectra). *Every real inner product space is isometrically isomorphic to $\mathbb{R}^n$ with the standard inner product for some $n \geq 0$. This leads to a more explicit "coordinatized" description of orthogonal spectra.*

Up to isomorphism, an orthogonal spectrum X is determined by the values $X_n := X(\mathbb{R}^n)$ and the following additional data relating these values:

- *a based continuous left action of the orthogonal group $O(n)$ on X_n for each $n \geq 0$;*
- *based $O(n)-$maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$. This data is subject to the following condition: for all $n, m \geq 0$, the iterated structure map*

$$\sigma_{n,m} : X_n \wedge S^m \longrightarrow X_{n+m}$$

defined as the composition

$$X_n \wedge S^m \xrightarrow{\sigma_n \wedge S^{m-1}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge S^{m-2}} \cdots \xrightarrow{\sigma_{n+m-2} \wedge S^1} X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}.$$

is $O(n) \times O(m)-$equivariant. $O(m)$ acts on S^m since this is the one-point compactification of $\mathbb{R}^m$, and $O(n) \times O(m)$ acts on the target by restriction, along orthogonal sum, of the $O(n+m)-$action.

If G is a compact Lie group and V is a $G-$representation, then $X(V)$ is a $G-$ space via the action of $O(V)$ on $X(V)$, i.e. by restriction along the representation homomorphism

$$G \longrightarrow \boldsymbol{O}(V, V), \; g \mapsto (0, g \cdot -).$$

If V and W are $G-$representations, then the structure map

$$\sigma_{V,W} : X(V) \wedge S^W \longrightarrow X(V \oplus W)$$

is $G-$equivariant where the group G also acts on the representation sphere S^W. $\sigma_{V,W}$ is also $O(V) \times O(W)-$equivariant, so altogether it is equivariant for the semi-direct product group $G \ltimes (O(V) \times O(W))$ formed from the conjugation action of G on $O(V)$ and $O(W)$.

Definition 4.4.11. *Given an orthogonal spectrum X and a compact Lie group G, the collection of $G-$spaces $X(V)$, for V a $G-$representation, and the equivariant structure maps $\sigma_{V,W}$ form an orthogonal $G-$spectrum. This orthogonal $G-$spectrum*

$$X\langle G \rangle = \{X(V), \sigma_{V,W}\}$$

is called the underlying orthogonal $G-$spectrum of X.

An essential invariant of an orthogonal spectrum X is its *integer graded equivariant homotopy groups* $\pi_k^G(X)$.

If $\psi : V \longrightarrow W$ is a linear isometric embedding and $f : S^V \longrightarrow X(V)$ a continuous based map, we define $\psi_* f : S^W \longrightarrow X(W)$ as the composite

$$\psi_* f : S^W \cong S^V \wedge S^{W-\psi(V)} \xrightarrow{f \wedge S^{W-\psi(V)}} X(V) \wedge S^{W-\psi(V)} \xrightarrow{\psi_*} X(W).$$

The construction is continuous in both variables, i.e. the map

$$L(V,W) \times \mathrm{Map}(S^V, X(V)) \longrightarrow \mathrm{Map}(S^W, X(W)),\ (\psi, f) \mapsto \psi_* f$$

is continuous.

We obtain a functor from the poset $s(\mathcal{U}_G)$ to sets by sending $V \in s(\mathcal{U}_G)$ to

$$[S^V, X(V)]^G,$$

the set of $G-$equivariant homotopy classes of based $G-$maps from S^V to $X(V)$. For $V \subseteq W$ in $s(\mathcal{U}_G)$, the inclusion $i : V \longrightarrow W$ is sent to the map

$$[S^V, X(V)]^G \longrightarrow [S^W, X(W)]^G,\ [f] \mapsto [i_* f].$$

The 0-th equivariant homotopy group $\pi_0^G(X)$ is then defined as

$$\pi_0^G(X) = \mathrm{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(V)]^G.$$

We can also define the global version of loop spectrum and suspension, which are functors from Sp to Sp. For inner product space V,

$$(\Omega X)(V) := \mathrm{Map}(S^1, X(V));$$

$$(S^1 \wedge X)(V) := S^1 \wedge X(V).$$

We have the adjunction

$$Sp(X, \Omega Y) \cong Sp(S^1 \wedge X, Y)$$

taking a morphism $f : X \longrightarrow \Omega Y$ to the morphism $\widehat{f} : S^1 \wedge X \longrightarrow Y$ whose $V-$th level is

$$f(V) : S^1 \wedge X(V) \longrightarrow Y(V),\ s \wedge x \mapsto f(V)(x)(a).$$

Let k be a positive integer. We set

$$\pi_k^G(X) = \pi_0^G(\Omega^k X) \text{ and } \pi_{-k}^G(X) = \pi_0^G(S^k \wedge X). \tag{4.40}$$

Definition 4.4.12. *A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a global equivalence if the induced map*

$$\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$$

is an isomorphism for all compact Lie groups G and all integers k.

The global equivalences are the weak equivalences of the global model structure on the category of orthogonal spectra.

4.4.3 Unitary spectra

Real global homotopy theory is a richer theory than the global homotopy theory based on compact Lie groups, which is talked about in Section 4.4.2. It can describe Real phenomenon from a global perspective. And it's the natural place where topological K-theory and complex bordism reside.

Real global homotopy theory is formalized by defining unitary space and unitary spectra. Since the class of closed subgroups of orthogonal groups coincides with the class of closed subgroups of unitary groups, the global homotopy theory of unitary spaces is Quillen equivalent to the global homotopy theory of orthogonal spaces.

Let W be a complex inner product space, i.e. a finite dimensional $\mathbb{C}-$vector space equipped with a hermitian inner product $(-,-)$. Let rW denote the underlying real inner product spaces of W, i.e. the underlying finite dimensional $\mathbb{R}-$vector space equipped with the euclidean inner product

$$\langle v, w\rangle = \mathrm{Re}(v, w).$$

Let

$$C = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{Id}_{\mathbb{C}}, \tau\}$$

denote the Galois groups of $\mathbb{C}$ over $\mathbb{R}$ where $\tau : \mathbb{C} \longrightarrow \mathbb{C}$ is the complex conjugation $\tau(\lambda) = \overline{\lambda}$.

Definition 4.4.13. *$L^{\mathbb{C}}$ is the complex isometries category whose objects are finite dimensional complex inner product spaces and morphism space between two objects V and W is the space of the pairs*

$$(\psi, \sigma) \in L(rV, rW) \times C$$

that satisfy $\psi(\lambda \cdot v) = \sigma(\lambda) \cdot \psi(v)$ and $(\psi(v), \psi(v')) = \sigma((v, v'))$. The composition in $L^{\mathbb{C}}$ is defined by

$$(\psi, \sigma) \circ (\psi', \sigma') = (\psi\psi', \sigma\sigma')$$

and the identity morphism of V is $(Id_V, Id_{\mathbb{C}})$.

The extra piece of the unitary structure gives a richer global homotopy theory that is indexed not only on compact Lie groups, but on the larger class of augmented Lie groups.

Definition 4.4.14. *An augmented Lie group is a compact Lie group G equipped with a continuous homomorphism $\epsilon : G \longrightarrow C$, called the augmentation, to the Galois group of $\mathbb{C}$ over $\mathbb{R}$.*

Let $G_{ev} = \epsilon^{-1}(Id_{\mathbb{C}})$ denote the even part of G and $G_{odd} = \epsilon^{-1}(\tau)$ the odd part of G. The product in the category of augmented Lie groups is defined to be the fiber product over C. Explicitly, the product of two augmented Lie groups G and K is the augmented Lie group $G \times_C K$ with $(G \times_C K)_{ev} = G_{ev} \times K_{ev}$ and $(G \times_C K)_{odd} = G_{odd} \times K_{odd}$.

Example 4.4.15 (extended unitary group). *The endomorphism group of a complex inner product space W in the category $L^{\mathbb{C}}$ is defined to be*

$$\widetilde{U}(W) := L^{\mathbb{C}}(W, W).$$

The augmentation $\epsilon_W : \widetilde{U}(W) \longrightarrow C$ is defined to be $\epsilon_W(\varphi, c) = c$.

The extended unitary group is a closed subgroup of $O(rW)$.

The augmented Lie group contain compact Lie groups as the ones with trivial augmentation. It also contain the Real Lie groups of Atiyah and Segal in [8], which are defined as compact Lie groups equipped with an involution.

Example 4.4.16 (Split augmented Lie groups). *Let G be a compact Lie group equipped with an involution $\tau : G \longrightarrow G$ on it, namely a Real Lie group in the sense of [6]. We can* *construct an augmented Lie group from it. The semi-direct product $G \rtimes_\tau C$ is an augmented Lie group with the augmentation*

$$G \rtimes_\tau C \longrightarrow C,\ (g, \sigma) \mapsto \sigma.$$

A real representation of $G \rtimes_\tau C$ amounts to a unitary representation V of G with a real structure $\tau : V \longrightarrow V$ such that

$$\tau(g \cdot v) = \tau(g) \cdot \tau(v),\ \forall g \in G, v \in V.$$

For the opposite direction, given an augmented Lie group, it is isomorphic to a $G \rtimes_\tau C$ for some Real Lie group G if and only if its augmentation has a multiplicative section, i.e., it has an odd element of order 2.

For any complex inner product space W, there is a canonical involution τ on $U(W)$ by complex conjugation. We have the isomorphism of augmented Lie groups

$$U(W) \rtimes_\tau C \longrightarrow \widetilde{U}(W),\ (\psi, \tau) \mapsto \psi \circ \tau_W. \tag{4.41}$$

Definition 4.4.17. *A real representation of an augmented Lie group G is a finite-dimensional complex inner product space V and a continuous homomorphism $\rho : G \longrightarrow \widetilde{U}(V)$, i.e., such that $\epsilon_V \circ \rho = \epsilon$.*

Definition 4.4.18. *Let G be an augmented Lie group. An augmented right $G-$space is a right $G-$space A equipped with a continuous map $\epsilon : A \longrightarrow C$ such that*

$$\epsilon(a \cdot g) = \epsilon(a) \cdot \epsilon(g)$$

for all $a \in A$ and all $g \in G$.

Example 4.4.19 (Product augmented Lie groups). *Given any compact Lie group G, we can augment the product $G \times C$ by the projection to the second factor. We denote it by $G^\sharp = (G \times C, proj)$ and call it the product augmented Lie group corresponding to G. Then the even part of $G^\sharp$ can be identified with G via*

$$G \cong (G^\sharp)_{ev},\ g \mapsto (g, Id_{\mathbb{C}}). \tag{4.42}$$

A real representation of $G^\sharp$ is equivalent to an underlying unitary representation V of G equipped with a $G-$equivariant real structure $\tau : V \longrightarrow V$ i.e. the action of $(1,\tau) \in G \times C$. So the fixed point subspace V^τ is $G-$invariant, hence an orthogonal $G-$representation, and the canonical isomorphism

$$C \otimes_{\mathbb{R}} V^\tau \cong V \tag{4.43}$$

is $G-$equivariant, i.e. the $G-$action on the left hand side is the complexification of $G-$action on V^τ. The fixed point functor $V \mapsto V^\tau$ is thus an equivalence from the category of real representation of $G^\sharp$ to the category of orthogonal representations of G. In particular, isomorphism classes of real representations of $G^\sharp$ biject with isomorphism classes of orthogonal representation of G, and the map

$$RR(G^\sharp) \cong RO(G), [V] \mapsto [V^\tau] \tag{4.44}$$

is a ring isomorphism. The inver isomorphism is given by complexification

$$RO(G) \longrightarrow RR(G^\sharp), [W] \mapsto [\mathbb{C} \otimes_{\mathbb{R}} W]. \tag{4.45}$$

Example 4.4.20 (Products of augmented Lie groups). *The category of augmented Lie groups has products. They are given by fiber product over C. More explicitly, the product of two augmented Lie groups G and K is the augmented Lie group $G \times_C K$ with*

$$(G \times_C K)_{ev} = G_{ev} \times K_{ev} \text{ and } (G \times_C K)_{odd} = G_{odd} \times K_{odd}.$$

Generally, the augmented product $G \times_C K$ is not the product of underlying groups G and K.

Definition 4.4.21. *A unitary space is a continuous functor from the complex isometries category $L^{\mathbb{C}}$ to the category of spaces, where $L^{\mathbb{C}}$ is the category in Definition 4.4.13. A morphism of unitary spaces is a natural transformation of functors. We denote the category of unitary spaces by spc^U.*

Orthogonal and unitary spaces are related by various functors

$$\begin{tikzcd} spc^U \arrow[r, bend left, "u"] \arrow[r, bend right, "\psi"'] & spc \arrow[l, "c"] \end{tikzcd}$$

where c is the "complexification", u the "underlying" and ψ the "fixed point" functor. c and u arise by precomposition with continuous functors

$$\begin{tikzcd} L \arrow[r, bend left, "(-)_{\mathbb{C}}"] & L^{\mathbb{C}} \arrow[l, bend left, "r"] \end{tikzcd}$$

relating the real and complex isometries categories.

The complexification functor $(-)_{\mathbb{C}}$ sends a euclidean inner product space V to its complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ equipped with the unique hermitian inner product $(-,-)$ that satisfies

$$(1 \otimes v, 1 \otimes w) = \langle v, w \rangle$$

for all $v, w \in V$. On the morphism spaces, $(-)_{\mathbb{C}}$ sends $\phi \in L(V,W)$ to $(\phi_{\mathbb{C}}, \mathrm{Id}_{\mathbb{C}}) \in L^{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}})$.

The realification functor r sends a hermitian inner product space W to its underlying $\mathbb{R}-$vector space equipped with the euclidean inner product defined by

$$\langle v, w \rangle = \mathrm{Re}(v, w).$$

On morphisms, r sends (ϕ, σ) to ϕ.

Then,

$$uY := Y \circ (-)_{\mathbb{C}} \text{ and } cX := X \circ r$$

where Y is a unitary space and X is an orthogonal space.

Moreover, we can define an involution $\psi : uY \longrightarrow uY$ on the underlying orthogonal space of a unitary space Y. For any euclidean inner product space V, let $\tau_V \in \widetilde{U}(V_{\mathbb{C}})$ denote the canonical real structure determined by

$$\tau_V(\lambda \otimes v) = \tau(\lambda) \otimes v = \overline{\lambda} \otimes v.$$

Define

$$\psi_V = Y(\tau_V) : Y(V_{\mathbb{C}}) \longrightarrow Y(V_{\mathbb{C}}).$$

And the fixed points

$$(Y^{\psi})(V) := Y(V_{\mathbb{C}})^{\psi_V} = \{y \in Y(V_{\mathbb{C}}) | Y(\tau_V)(y) = y\}.$$

Definition 4.4.22. *The real global classifying space $B_{gl}^R G$ of an augmented Lie group G is the unitary space*

$$B_{gl}^R G = L^{\mathbb{C}}(V, -)/G$$

where V is any faithful real $G-$representation.

Unitary spectra is the stabilization of unitary space.

Let $\mathbb{U}$ denote the category whose objects are complex inner product spaces and the morphisms $U(V, W)$ between two objects V and W is the Thom space of the total space

$$\xi^{\mathbb{C}}(V, W) := \{(w, \phi) \in W \times L^{\mathbb{C}}(V, W) | W \perp \phi(V)\}$$

of the orthogonal complement vector bundle, whose structure map $\xi^{\mathbb{C}}(V, W) \longrightarrow L^{\mathbb{C}}(V, W)$ is the projection to the second factor.

Definition 4.4.23. *A unitary spectrum is a based continuous functor from $\mathbb{U}$ to the category $\boldsymbol{T}$ of based compactly generated weak Hausdorff spaces. A morphism is a natural transformation of functors. Let Sp^U denote the category of orthogonal spectrum.*

For each $\mathbb{C}-$linear isometric embedding $\phi : V \longrightarrow W$, consider the one-point compactification of the inclusion of the fiber over ϕ of the bundle $\xi(V, W)$. We can define a continuous map

$$(-, \phi) : S^{W-\phi(V)} \longrightarrow U(V, W),\ w \mapsto (w, \phi)$$

Let Y be a unitary spectrum. The structure map of Y associated to ϕ is defined to be the composite

$$\phi_\star := Y \circ (Y(V) \wedge (-, \phi)) :\ Y(V) \wedge S^{W-\phi(V)} \xrightarrow{Y(V)\wedge(-,\phi)} Y(V) \wedge U(V, W) \xrightarrow{Y} Y(W).$$

If $\phi_0 : V \longrightarrow V \oplus W,\ v \mapsto (v, 0)$ is the direct summand inclusion, Let

$$\sigma_{V,W} := \phi_{0\star} : Y(V) \wedge S^W \longrightarrow Y(V \oplus W)$$

denote the associated structure map.

The complexification functor c, underlying functor u and fixed point functors ψ between orthogonal and unitary spaces all have stable analogs.

$$\begin{tikzcd} Sp^U \arrow[r, bend left, "u"] \arrow[r, bend right, "\psi"'] & Sp \arrow[l, "c"] \end{tikzcd}$$

Let X be an orthogonal spectrum and Y a unitary spectrum.

$$cX := X \circ r.$$

$$(uY)(V) := \mathrm{Map}(S^{iV}, Y(V_{\mathbb{C}}))$$

where V is a complex inner product space and $iV = i\mathbb{R} \otimes_{\mathbb{R}} V \subset V_{\mathbb{C}}$ is the imaginary part of V.

We can define a natural involution $\psi : uY \longrightarrow uY$. Define

$$\psi_V := \mathrm{Map}(S^{\tau}, Y(\tau_V)) : \mathrm{Map}(S^{iV}, Y(V_{\mathbb{C}})) \longrightarrow \mathrm{Map}(S^{iV}, Y(V_{\mathbb{C}}))$$

where $\tau : iV \longrightarrow iV$ is multiplication by -1.

$$(Y^{\psi})(V) := \mathrm{Map}^C(S^{iV}, Y(V_{\mathbb{C}})).$$

4.4.4 Global K-theory

A classical example of orthogonal spectra is global K-theory. Quasi-elliptic cohomology can be expressed in terms of equivariant K-theory. And this example is especially important for our construction.

In [35] Joachim constructs G-equivariant K-theory as an orthogonal G-spectrum for any compact Lie group G. In fact it is the only known $E^{\infty}-$version of equivariant complex K-theory when G is a compact Lie group.

Let G be a compact Lie group. For any real $G-$representation V, let $\mathbb{C}l_V$ be the Clifford algebra of V and $\mathcal{K}_V$ be the $G-C^*-$algebra of compact operators on $L^2(V)$. Let $s := C_0(\mathbb{R})$ be the graded $G-C^*-$algebra of continuous functions on $\mathbb{R}$ vanishing at infinity with trivial $G-$action. Then the orthogonal $G-$spectrum for equivariant K-theory defined by Joachim is the lax monoidal functor given by

$$\mathbb{K}_G(V) = Hom_{C^*}(s, \mathbb{C}l_V \otimes \mathcal{K}_V)$$

of $\mathbb{Z}/2-$graded $*-$homomorphisms from s to $\mathbb{C}l_V \otimes \mathcal{K}_V$.

Bohmann showed in her paper [11] that Joachim's model is "global", i.e. the lax monoidal functor $\mathbb{K}$ is an orthogonal $\mathcal{G}-$spectrum. For more detail, please read [11] for reference.

Schwede's construction of global K-theory KR in [56] is a unitary analog of the construction by Joachim. It is an ultra-commutative ring spectrum whose $G-$homotopy type realizes Real $G-$equivariant periodic K-theory.

For any complex inner product space W, let $\Lambda(W)$ be the exterior algebra W and $Sym(W)$ the symmetric algebra of it. The tensor product

$$\Lambda(W) \otimes Sym(W)$$

inherits a hermitian inner product from W and it's $\mathbb{Z}/2-$graded by even and odd exterior powers. Let $\mathcal{H}_W$ denote the Hilbert space completion of $\Lambda(W) \otimes Sym(W)$. Let $\mathcal{K}_W$ be the C^*-algebra of compact operators on $\mathcal{H}_W$. The orthogonal spectrum KR is defined to be the lax monoidal functor

$$KR(W) = Hom_{C^*}(s, \mathcal{K}_W).$$

Let uW denote the underlying euclidean vector space of W. There is an isomorphism of $\mathbb{Z}/2-$graded C^*-algebras

$$Cl(uW) \otimes_{\mathbb{R}} \mathcal{K}(L^2(W)) \cong \mathcal{K}_W.$$

So we get a homeomorphism

$$KR(W) \cong Hom_{C^*}(s, Cl(uW) \otimes_{\mathbb{R}} \mathcal{K}(L^2(W))) = \mathbb{K}(uW).$$

In [56], Schwede shows that the spaces in the orthogonal spectrum KR represent real equivariant K-theory.

Theorem 4.4.24. *For an augmented Lie group G, a "large" real $G-$representation and a compact $G-$space B, there is a bijection $\Psi_{G,B,V} : K_G(B) \longrightarrow [B_+, KR(V)]^G$ that is natural in B. The left hand side is the real $G-$equivariant $K-$group of B.*

We have the relations below between the global Real K-theory KR, periodic unitary K-theory KU and periodic orthogonal real K-theory KO.

$$KU = u(KR);\ \ KO = KR^{\psi} \tag{4.46}$$

We will use the orthogonal spectra KU in the construction of orthogonal quasi-elliptic cohomology.

Definition 4.4.25. *An orthogonal $G-$representation is called ample if its complexified symmetric algebra is complete complex $G-$universe.*

Theorem 4.4.26. *(i) Let G be a compact Lie group and V an orthogonal $G-$representation. For every ample $G-$representation W, the adjoint structure map*

$$\widetilde{\sigma}^K_{V,W} : KU(V) \longrightarrow Map(S^W, KU(V \oplus W))$$

is a $G-$weak equivalence.

(ii) Let G be an augmented Lie group and V a real $G-$representation such that $Sym(V)$ is a complete real $G-$universe. For every real $G-$representation W the adjoint structure map

$$\widetilde{\sigma}^K_{V,W} : KR(V) \longrightarrow Map(S^W, KR(V \oplus W))$$

is a $G-$weak equivalence.

4.5 Orthogonal $G-$spectra of $QEll_G$

In Section 4.5.3 and 4.5.4, via the spaces I construct in Section 4.3, I construct a $G-$orthogonal spectra for quasi-elliptic cohomology up to weak equivalence (4.1), which is a commutative orthogonal $G-$spectra.

4.5.1 Preliminaries: faithful representations of $\Lambda_G(g)$

Before the construction of the $G-$orthogonal spectra, I discuss about $\Lambda_G(\sigma)-$representations.

Let G be a compact Lie group. As shown in Theorem 4.4.24, $KU(V)$ represents $G-$equivariant complex K-theory when V is a faithful $G-$representation. Before the construction in Section 4.5.3, we construct a faithful $\Lambda_G(\sigma)-$representation from a faithful $G-$representation.

Lemma 4.5.1. *Let G be an augmented Lie group with augmentation ϵ_G.*

1. *if $\sigma \in G_{odd}$, for any augmentation ϵ_Λ on $\Lambda_G(\sigma)$, its restriction on $C_G(\sigma)$ can not be $\epsilon_G|_{C_G(\sigma)}$. One choice of an augmentation ϵ_Λ on $\Lambda_G(\sigma)$ is the trivial augmentation.*
2. *if $\sigma \in G_{even}$, we have a nontrivial augmentation ϵ on $\Lambda_G(\sigma)$ defined by $\epsilon([\sigma, t]) = \epsilon_G(\sigma)$. ϵ is the only augmentation whose restriction on $C_G(\sigma)$ is $\epsilon_G|_{C_G(\sigma)}$.*

The proof of the lemma is straightforward, noting the fact that the only augmentation on $\mathbb{R}$ is trivial since it's dense.

Let $\sigma \in G^{tors}$ with order l. We construct a functor $(-)_\sigma$ from the category of $G-$representations to the category of $\Lambda_G(\sigma)-$representations.

Let ρ be a complex $G-$representation with underlying space V. Let $i : C_G(\sigma) \hookrightarrow G$ denote the inclusion of groups. The restriction i^*V is a complex $C_G(\sigma)-$representation.

Let $\{\lambda\}$ denote all the irreducible complex representations of $C_G(\sigma)$. As said in [22], we have the decomposition of a representation into its isotypic components

$$i^*V \cong \bigoplus_\lambda V_\lambda \tag{4.47}$$

where V_λ denotes the sum of all subspaces of V isomorphic to λ. Each

$$V_\lambda = Hom_{C_G(\sigma)}(\lambda, V) \otimes_{\mathbb{C}} \lambda$$

is unique as a subspace. Note that σ acts on each V_λ as a diagonal matrix.

Let's equip each V_λ a $\Lambda_G(\sigma)-$action, as shown below.

Each $\lambda(\sigma)$ is of the form $e^{\frac{2\pi i m_\lambda}{l}} I$ with $0 < m_\lambda \leq l$ and I the identity matrix. As shown in Lemma 2.4.1, $V_\lambda \odot_{\mathbb{C}} q^{\frac{m_\lambda}{l}}$ is a well-defined complex $\Lambda_G(\sigma)-$representation. Define

$$(V_\lambda)_\sigma := V_\lambda \odot_{\mathbb{C}} q^{\frac{m_\lambda}{l}} \tag{4.48}$$

and

$$(V)_\sigma := \bigoplus_\lambda V_\lambda \odot_{\mathbb{C}} q^{\frac{m_\lambda}{l}} \tag{4.49}$$

Each $(V_\lambda)_\sigma$ is the isotypic component of $(V)_\sigma$ corresponding to the irreducible representation $\lambda \odot_{\mathbb{C}} q^{\frac{m}{l}}$.

The complex $\Lambda_G(\sigma)-$representation $(V)_\sigma$ has the same dimension as V.

Proposition 4.5.2. *Let V be a faithful $G-$representation. And let $\sigma \in G^{tors}$.*

(i) If V contains a trivial subrepresentation, $(V)_\sigma$ is a faithful $\Lambda_G(\sigma)-$representation.

(ii) $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$ is a faithful $\Lambda_G(\sigma)-$representation.

(iii) $(V)_\sigma \oplus V^\sigma$ is a faithful $\Lambda_G(\sigma)-$representation.

Proof. (i) Let $[a,t] \in \Lambda_G(\sigma)$ be an element acting trivially on $(V)_\sigma$. Assume $t \in [0,1)$. On $(V_1)_\sigma$, $[a,t]v_0 = e^{2\pi i t}v_0 = v_0$. So $t=0$. Then on the whole space V_σ, since $C_G(\sigma)$ acts faithfully on it and for any $v \in V_\sigma$, $[a,0]\cdot v = a \cdot v = v$, then $a = e$.

So $(V)_\sigma$ is a faithful $\Lambda_G(\sigma)-$representation.

(ii) Let $[a,t] \in \Lambda_G(\sigma)$ be an element acting trivially on V_σ. Consider the subrepresentation $(V_\lambda)_\sigma$ and $(V_\lambda)_\sigma \otimes_{\mathbb{C}} q^{-1}$ of $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$ respectively. Let v be an element in the underlying vector space V_λ. On $(V_\lambda)_\sigma$, $[a,t]\cdot v = e^{\frac{2\pi i m_\lambda t}{l}} a \cdot v = v$; and on $(V_\lambda)_\sigma \otimes_{\mathbb{C}} q^{-1}$, $[a,t]\cdot v = e^{\frac{2\pi i m_\lambda t}{l} - 2\pi i t} a \cdot v = v$. So we get $e^{2\pi i t}\cdot v = v$. Thus, $t=0$.

$C_G(\sigma)$ acts faithfully on V, so it acts faithfully on $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$. Since $[a,0]\cdot w = w$, for any $w \in (V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$, so $a = e$.

Thus, $(V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}$ is a faithful $\Lambda_G(\sigma)-$representation.

(iii) Note that V^σ with the trivial $\mathbb{R}-$action is the representation $(V^\sigma)_\sigma \otimes_{\mathbb{C}} q^{-1}$. The representation $(V)_\sigma \oplus V^\sigma$ contains a subrepresentation $(V^\sigma)_\sigma \oplus (V^\sigma)_\sigma \otimes_{\mathbb{C}} q^{-1}$, which is a faithful $\Lambda_G(\sigma)-$representation by the second conclusion of Proposition 4.5.2. So $(V)_\sigma \oplus V^\sigma$ is faithful.

□

Lemma 4.5.3. *For any $\sigma \in G^{tors}$, $(-)_\sigma$ defined in (4.49) is a functor from the category of $G-$spaces to the category of $\Lambda_G(\sigma)-$spaces.*

Moreover, $(-)_\sigma \oplus (-)_\sigma \otimes_{\mathbb{C}} q^{-1}$ and $(-)_\sigma \oplus (-)^\sigma$ in Proposition 4.5.2 are also well-defined functors from the category of $G-$spaces to the category of $\Lambda_G(\sigma)-$spaces.

Proof. Let $f : V \longrightarrow W$ be a $G-$equivariant map. Then f is $C_G(\sigma)-$equivairant for each $\sigma \in G^{tors}$. For each irreducible complex $C_G(\sigma)-$representation λ, $f : V_\lambda \longrightarrow W_\lambda$ is $C_G(\sigma)-$equivairant. And

$$f_\sigma : (V_\lambda)_\sigma \longrightarrow (W_\lambda)_\sigma,\ v \mapsto f(v)$$

with the same underlying spaces is well-defined and is $\Lambda_G(\sigma)-$equivariant.

It is straightforward to check if we have two $G-$equivariant maps $f : V \longrightarrow W$ and $g : U \longrightarrow V$, then

$$(f \circ g)_\sigma = f_\sigma \circ g_\sigma.$$

So $(-)_\sigma$ gives a well-defined functor from the category of $G-$representations to the category of $\Lambda_G(\sigma)-$representation.

Similarly, we can see $(-)_\sigma \otimes_{\mathbb{C}} q^{-1}$ is also a well-defined functor from the category of $G-$representations to the category of $\Lambda_G(\sigma)-$representation, so $(-)_\sigma \oplus (-)_\sigma \otimes_{\mathbb{C}} q^{-1}$ is.

Since the fixed point functor $(-)^\sigma$ is also a functor from the category of $G-$spaces to the category of $\Lambda_G(\sigma)-$spaces, $(-)_\sigma \oplus (-)^\sigma$ is as well.

□

Proposition 4.5.4. *Let H and G be two compact Lie groups. Let $\sigma \in G$ and $\tau \in H$. Let V be a $G-$representation and W a $H-$representation.*

(i) We have the isomorphisms of representations below.

$$(V \oplus W)_{(\sigma,\tau)} = (V_\sigma \oplus W_\tau)$$

as $\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)-$representations.

$$(V \oplus W)_{(\sigma,\tau)} \oplus (V \oplus W)_{(\sigma,\tau)} \otimes_{\mathbb{C}} q^{-1} = ((V)_\sigma \oplus (V)_\sigma \otimes_{\mathbb{C}} q^{-1}) \oplus ((W)_\tau \oplus (W)_\tau \otimes_{\mathbb{C}} q^{-1})$$

as $\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)-$representations.

And

$$(V \oplus W)_{(\sigma,\tau)} \oplus (V \oplus W)^{(\sigma,\tau)} = ((V)_\sigma \oplus V^\sigma) \oplus ((W)_\tau \oplus W^\tau)$$

as $\Lambda_{G\times H}(\sigma,\tau)\cong\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)-$*representations.*

(ii) Let $\phi : H \longrightarrow G$ *be a group homomorphism. Let* $\phi_\tau : \Lambda_H(\tau) \longrightarrow \Lambda_G(\phi(\tau))$ *denote the group homomorphism obtained from* ϕ*. Then we have*

$$\phi_\tau^*(V)_{\phi(\tau)} = (V)_\tau,$$

$$\phi_\tau^*((V)_{\phi(\tau)} \oplus (V)_{\phi(\tau)} \otimes_{\mathbb{C}} q^{-1}) = (V)_\tau \oplus (V)_\tau \otimes_{\mathbb{C}} q^{-1},$$

and

$$\phi_\tau^*((V)_{\phi(\tau)} \oplus V^{\phi(\tau)}) = (V)_\tau \oplus V^\tau$$

as $\Lambda_H(\tau)-$*representations.*

Proof. (i) Let

$$\{\lambda_G\} \text{ and } \{\lambda_H\}$$

denote the sets of all the irreducible $C_G(\sigma)-$representations and all the irreducible $C_H(\tau)-$representations.

λ_G and λ_H are irreducible representations of $C_{G\times H}(\sigma,\tau)$ via the inclusion $C_G(\sigma) \longrightarrow C_{G\times H}(\sigma,\tau)$ and $C_H(\tau) \longrightarrow C_{G\times H}(\sigma,\tau)$.

The $\mathbb{R}-$representation assigned to each $C_{G\times H}(\sigma,\tau)-$irreducible representation in $V\oplus W$ is the same as that assigned to the irreducible representations of V and W.

So we have

$$(V\oplus W)_{(\sigma,\tau)} = (V_\sigma \oplus W_\tau)$$

as $\Lambda_{G\times H}(\sigma,\tau)\cong\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)-$representations.

Similarly we can prove the other two conclusions in (i).

(ii) Let $\sigma=\phi(\tau)$. If $(\phi_\tau^*V)_{\lambda_H}$ is a $C_H(\tau)-$subrepresentation of $\phi_\tau^* V_{\lambda_G}$, the $\mathbb{R}-$representation assigned to it is the same as that to V_{λ_G}.

So we have

$$\phi_\tau^*(V)_{\phi(\tau)} = (V)_\tau$$

as $\Lambda_H(\tau)-$representations.

Similarly we can prove the other two conclusions in (ii).

□

4.5.2 Real $\Lambda_G(\sigma)-$representation

In this section I discuss real $\Lambda_G(\sigma)-$representation and its relation with complex $\Lambda_G(\sigma)-$representation introduced in Lemma 2.4.1.

Let G be a compact Lie group, $\sigma \in G^{tors}$. For real representations of $\Lambda_G(\sigma)$, the case is a little complicated. First let's recall some definitions and conclusions in real representation theory. The main reference is [13] and [22].

Definition 4.5.5. *A complex representation $\rho : G \longrightarrow Aut_{\mathbb{C}}(V)$ is said to be self dual if it is isomorphic to its complex dual $\rho^* : G \longrightarrow Aut_{\mathbb{C}}(V^*)$ where $V^* := Hom_{\mathbb{C}}(V, \mathbb{C})$ and $\rho^*(g) = \rho(g^{-1})^*$.*

An irreducible complex representation $\rho : G \longrightarrow Aut_{\mathbb{C}}(V)$ is said to be of ***real type*** *if satisfies the equivalent conditions (1-3).*

(1) $V = \mathbb{C} \otimes U$ is the complexification of a real representation $G \longrightarrow Aut_{\mathbb{R}}(U)$.

(2) V admits an equivariant real structure, i.e. an antilinear map $S : V \longrightarrow V$ such that $S^2(v) = v$.

(3) There is an equivariant isomorphism $B : V \longrightarrow V^*$ such that $B^* = B$.

An irreducible complex representation is said to be of ***quaternionic type*** *if it satisfies the equivalent conditions (4-6).*

(4) $V = W_{\mathbb{C}}$ is obtained from a quaterionic representation $G \longrightarrow Aut_{\mathbb{H}}(W)$ by restriction of scalars $\mathbb{C} \subset \mathbb{H}$.

(5) V admits an equivariant quaternionic structure, i.e. an antilinear map $S : V \longrightarrow V$ such that $S^2(v) = -v$.

(6) There is an equivariant isomorphism $B : V \longrightarrow V^*$ such that $B^* = -B$.

An irreducible complex representation is said to be of ***complex type*** *if it's not self dual.*

Definition 4.5.6. *A complex representation is said to have an* ***irreducible real form*** *if it is the complexification of an irreducible real representation.*

Lemma 4.5.7. *An irreducible complex representation V is of*

(I) real type if and only if V has irreducible real form.

(II) complex type if and only if $V \oplus V^*$ has irreducible real form.

(III) quaternionic type if and only if $V \oplus V$ has irreducible real form.

Example 4.5.8. *Let $G = \mathbb{Z}/p\mathbb{Z}$ for some integer p. Let $\rho_1 : \mathbb{Z}/p\mathbb{Z} \longrightarrow O(2)$ send m to the matrix*

$$\begin{bmatrix} \cos\frac{2\pi m}{p} & -\sin\frac{2\pi m}{p} \\ \sin\frac{2\pi m}{p} & \cos\frac{2\pi m}{p} \end{bmatrix}$$

Let $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $u_1 = e_1 + ie_2$, $u_2 = e_1 - ie_2 \in \mathbb{C}^2$. Let $V_1 = \mathbb{C}\langle u_1\rangle$, and $V_2 = \mathbb{C}\langle u_2\rangle$. Then the complexification of the real $\mathbb{Z}/p\mathbb{Z}-$representation $\mathbb{R}^2$ have the decomposition into irreducible $\mathbb{Z}/p\mathbb{Z}-$ representations

$$\mathbb{C}^2 \cong V_1 \oplus V_2. \tag{4.50}$$

V_1 and V_2 are both not self dual, so they are of complex type. And $V_1 = V_2^*$.

Let $\sigma = [n] \in \mathbb{Z}/p\mathbb{Z}$ be non-trivial. Then let χ_1 and χ_2 be complex $\mathbb{R}-$representations with $\chi_1(t) = e^{\frac{2\pi int}{p}}$ and $\chi_2(t) = e^{-\frac{2\pi int}{p}}$.

Applying Lemma 2.4.1, we have irreducible complex $\Lambda_G(\sigma)-$representations $(V_1 \odot_{\mathbb{C}} \chi_1)$ and $(V_2 \odot_{\mathbb{C}} \chi_2)$, which are of complex type. And $(V_1 \odot_{\mathbb{C}} \chi_1) = (V_2 \odot_{\mathbb{C}} \chi_2)^*$. So $(V_1 \odot_{\mathbb{C}} \chi_1) \oplus (V_2 \odot_{\mathbb{C}} \chi_2)$ is an irreducible real $\Lambda_G(\sigma)-$representation.

$[m, t]$ is mapped to

$$\begin{bmatrix} e^{\frac{2\pi(m+nt)i}{p}} & 0 \\ 0 & e^{-\frac{2\pi(m+nt)i}{p}} \end{bmatrix}$$

which is conjugate to the real matrix

$$\begin{bmatrix} \cos\frac{2\pi(m+nt)}{p} & -\sin\frac{2\pi(m+nt)}{p} \\ \sin\frac{2\pi(m+nt)}{p} & \cos\frac{2\pi(m+nt)}{p} \end{bmatrix}$$

Example 4.5.9. *Let G be any compact Lie group. Let V be an irreducible real $C_G(\sigma)-$representation. Let's consider its complexfication $V \otimes_{\mathbb{R}} \mathbb{C}$. If it is irreducible complex representation, by Lemma 2.4.1, we can get an irreducible complex $\Lambda_G(\sigma)-$representation $V \odot_{\mathbb{C}} \eta$ where the character $\eta(1)$ acts the same as σ on V. And as the discussion in Example 4.5.11, we can get an irreducible real $\Lambda_G(\sigma)-$representation.*

If the $C_G(\sigma)-$representation $V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus V_k$ is reducible with each V_k an irreducible *complex $C_G(\sigma)-$ subrepresentation, then each V_k is either of complex type or of quaternion type. So $V_k + V_k$ or $V_k + V_k^*$ has irreducible real form. Since $V \otimes_{\mathbb{R}} \mathbb{C}$ has irreducible real form, the decomposition is either $V \otimes_{\mathbb{R}} \mathbb{C} = V_1 \oplus V_1$ if V_1 is of quaternion type or $V \otimes_{\mathbb{R}} \mathbb{C} = V_1 \oplus V_1^*$ if V_1 is of complex type. Let $\sigma = e^{2\pi ai}I$ be a nontrivial element. Let χ be an irreducible complex $\mathbb{R}-$representation with $\chi(t) = e^{2\pi ait}$. Then $V_1 \odot_{\mathbb{C}} \chi$ is an irreducible complex $\Lambda_G(\sigma)-$representation. And $(V \odot_{\mathbb{C}} \eta) \oplus (V \odot_{\mathbb{C}} \eta)^*$ is a real $\Lambda_G(\sigma)-$representation. If σ is trivial and η is the trivial representation, $(V \odot_{\mathbb{C}} \eta)$ and $(V \odot_{\mathbb{C}} \eta)^*$ are isomorphic real representations, thus, $(V \odot_{\mathbb{C}} \eta) \oplus (V \odot_{\mathbb{C}} \eta)^*$ is also of real type.*

Example 4.5.10 (real representation ring of circle). *We know the complex representation ring $R\mathbb{T}$ of a circle is $\mathbb{Z}[q, q^{-1}]$ where $q : \mathbb{T} \longrightarrow U(1)$ is the isomorphism class of irreducible complex representation sending $e^{2\pi it}$ to $(e^{2\pi it})$. The real representation ring $RO(\mathbb{T})$ is the subring of $R\mathbb{T}$ fixed by the involution on it given by $q \mapsto q^{-1}$.*

Let $f(q)$ be any polynomial in q and let $f(q) = f_+(q) + f_-(q) + n \cdot 1$ where $f_+(q)$ is the part in $f(q)$ with positive power in q, $f_-(q)$ is the part with negative power and 1 is the trivial representation. $f(q)$ represents an element in $RO(\mathbb{T})$ if and only if f_+ and f_- are the same polynomial. Let V be the representation space of f_+. Then V^* is the representation space of f_-. So a real representation of $\mathbb{T}$ is always of the form

$$V \oplus V^* \oplus n\mathbb{R}$$

for some complex representation V of $\mathbb{T}$ and nonnegative integer n.

Example 4.5.11. *Let $\rho : C_G(g) \longrightarrow Aut_{\mathbb{R}}(V)$ be an irreducible complex $C_G(g)-$representation. Then as in Lemma 2.4.1, there exists a character $\eta : \mathbb{R} \longrightarrow \mathbb{C}$ such that $\rho(g) = \eta(1)I$. And $\rho \odot_{\mathbb{C}} \eta$ is an irreducible complex representation of $\Lambda_G(g)$. Since $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t)$, it's not self-dual if η is nontrivial. In this case it's of complex type. By Lemma 4.5.7, $(V \odot_{\mathbb{C}} \eta) \oplus (V \odot_{\mathbb{C}} \eta)^*$ has irreducible real form.*

If V is of real type, it is the complexification of a real $C_G(g)-$representation W. If $g = e$ and the character η we choose is trivial, $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t) = \rho(\alpha^{-1})^T = \rho(\alpha) = (\rho \odot_{\mathbb{C}} \eta)[\alpha, t]$ since V is self-dual. In this case W is a real $\Lambda_G(g)-$representation via $[\alpha, t] \cdot w = \alpha w$. And $V \odot_{\mathbb{C}} \eta$ is of real type since it's the complexification of W. For any nontrivial element g in G^{tors}, the $\Lambda_G(g)-$representation $V \odot_{\mathbb{C}} \eta$ is of complex type, then $(V \odot_{\mathbb{C}} \eta) \oplus (V \odot_{\mathbb{C}} \eta)^*$ is of the real type.

If V is of quaternion type, then $V = U_{\mathbb{C}}$ can be obtained from a quaternion $C_G(g)-$representation U by restricting the scalar to $\mathbb{C}$. If $g = e$ and η is trivial, $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t) = \rho(\alpha^{-1})^T = \rho(\alpha) = (\rho \odot_{\mathbb{C}} \eta)[\alpha, t]$ since V is self-dual. In this case W is a quaternion $\Lambda_G(g)-$representation with $[\alpha, t] \cdot w = \alpha w$. So $V \odot_{\mathbb{C}} \eta$ is of quaternion type.

Consider the case that V is of complex type. If $g = e$ and η is trivial, $(\rho \odot_{\mathbb{C}} \eta)^*([\alpha, t]) = \rho \odot_{\mathbb{C}} \eta([\alpha^{-1}, -t])^T = \rho(\alpha^{-1})^T \eta(-t) = \rho(\alpha^{-1})^T = \rho(\alpha) \neq (\rho \odot_{\mathbb{C}} \eta)[\alpha, t]$ since V is not self-dual. So $V \odot_{\mathbb{C}} \eta$ is of complex type.

For any compact Lie group, let's use $RO(G)$ denote the real representation ring of G. In light of the analysis in Example 4.5.9 and 4.5.10, we have the following conclusion.

Lemma 4.5.12. *Let $\sigma \in G^{tors}$. Then the map $\pi^* : RO\mathbb{T} \longrightarrow RO\Lambda_G(\sigma)$ exhibits $RO\Lambda_G(\sigma)$ as a free $RO\mathbb{T}-$module.*

In particular there is an $RO\mathbb{T}-$basis of $RO\Lambda_G(\sigma)$ given by irreducible real representations $\{V_\Lambda\}$. There is a bijection between $\{V_\Lambda\}$ and the set $\{\lambda\}$ of irreducible real representations of $C_G(\sigma)$. When σ is trivial, V_Λ has the same underlying space V as λ. When σ is nontrivial, $V_\Lambda = ((\lambda \otimes_{\mathbb{R}} \mathbb{C}) \odot_{\mathbb{C}} \eta) \oplus ((\lambda \otimes_{\mathbb{R}} \mathbb{C}) \odot_{\mathbb{C}} \eta)^*$ where η is a complex $\mathbb{R}-$representation such that $(\lambda \otimes_{\mathbb{R}} \mathbb{C})(\sigma)$ acts on $V \otimes_{\mathbb{R}} \mathbb{C}$ via the scalar multiplication by $\eta(1)$. The dimension of V_Λ is twice as that of λ.

As in (4.49), we can construct a functor $(-)^{\mathbb{R}}_\sigma$ from the category of real $G-$representations to the category of real $\Lambda_G(\sigma)-$representations with

$$(V)^{\mathbb{R}}_\sigma = (V \otimes_{\mathbb{R}} \mathbb{C})_\sigma \oplus (V \otimes_{\mathbb{R}} \mathbb{C})^*_\sigma, \tag{4.51}$$

which we saw in Example 4.5.9 is of real type.

Similarly we have the conclusion below.

Proposition 4.5.13. *Let V be a faithful real $G-$representation. And let $\sigma \in G^{tors}$ and l denote its order. Then $(V)^{\mathbb{R}}_\sigma$ is a faithful real $\Lambda_G(\sigma)-$representation.*

Proof. Let $[a, t] \in \Lambda_G(\sigma)$ be an element acting trivially on $(V)^{\mathbb{R}}_\sigma$. Assume $t \in [0, 1)$. Let $v \in (V \otimes_{\mathbb{R}} \mathbb{C})_\sigma$ and let v^* denote its correspondence in $(V \otimes_{\mathbb{R}} \mathbb{C})^*_\sigma$. Then $[a, t] \cdot (v + v^*) = (ae^{2\pi imt} + ae^{-2\pi imt})(v + v^*) = v + v^*$ where $0 < m \leq l$ is determined by σ. Thus a is equal to both $e^{2\pi imt}I$, and $e^{-2\pi imt}I$. Thus $t = 0$ and a is trivial.

So $(V)^{\mathbb{R}}_{\sigma}$ is a faithful real $\Lambda_G(\sigma)-$representation.

$\square$

Moreover, we have

Proposition 4.5.14. *Let H and G be two compact Lie groups. Let $\sigma \in G$ and $\tau \in H$. Let V be a real $G-$representation and W a real $H-$representation.*

(i) We have the isomorphisms of representations below.

$$(V \oplus W)^{\mathbb{R}}_{(\sigma,\tau)} = (V^{\mathbb{R}}_{\sigma} \oplus W^{\mathbb{R}}_{\tau})$$

as $\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)-$representations.

(ii) Let $\phi : H \longrightarrow G$ be a group homomorphism. Let $\phi_\tau : \Lambda_H(\tau) \longrightarrow \Lambda_G(\phi(\tau))$ denote the group homomorphism obtained from ϕ. Then we have

$$\phi^*_\tau(V)^{\mathbb{R}}_{\phi(\tau)} = (V)^{\mathbb{R}}_{\tau},$$

as $\Lambda_H(\tau)-$representations.

4.5.3 The Construction of $E(G, V)$

Let G be any compact Lie group. In Section 4.3, I construct a $G-$space $QEll_G$ representing $QEll^*_G(-)$.

In Section 4.5.3 I show that there is a $\mathcal{I}_G-$space $E(G,-)$ such that for each faithful real $G-$representation V, $E(G, V)$ weakly represents $QEll^V_G(-)$.

I show in Section 4.5.4 that these spaces $E(G, V)$ together give an orthogonal $G-$spectra and a $\mathcal{I}_G-$FSP.

First we need an orthogonal version of the space $S_{G,g}$ satisfying the condition in Lemma 4.3.4.

Let $g \in G^{tors}$ and V a real $G-$representation. Let $Sym^n(V)$ denote the $n-$th symmetric power $V^{\otimes n}$, which has an evident $G \wr \Sigma_n-$action on it. And let

$$Sym(V) := \bigoplus_{n\geq 0} Sym^n(V).$$

If V is an ample $G-$representation, $Sym(V)$ is a $G-$representation containing all the irreducible $G-$representations. Since V is faithful $G-$representation, for any closed subgroup H of G, $Sym(V)$ is a faithful $H-$representation and, thus, complete $H-$universe.

Let $S(G,V)_g$ be the space

$$Sym(V) \setminus Sym(V)^g. \tag{4.52}$$

It has an involution induced by the complex conjugation on V. For any subgroup H of G containing g, $S(H,V)_g$ has the same underlying space as $S(G,V)_g$.

Proposition 4.5.15. *Let V be an orthogonal $G-$representation. For any closed subgroup $H \leqslant C_G(g)$, $S(G,V)_g$ satisfies*

$$S(G,V)_g^H \simeq \begin{cases} pt, & \text{if } \langle g\rangle \nleqslant H; \\ \emptyset, & \text{if } \langle g\rangle \leqslant H. \end{cases} \tag{4.53}$$

Proof. If $\langle g\rangle \leqslant H$, $Sym(V)^H$ is a subspace of $Sym(V)^g$, so $(Sym(V)\setminus Sym(V)^g)^H$ is empty.

If $\langle g\rangle \nleqslant H$, g is not in H. To simplify the symbol, let $Sym^{n,\perp}$ denote the orthogonal complement of $Sym^n(V)^g$ in $Sym^n(V)$.

$$\begin{aligned}(Sym(V) \setminus Sym(V)^g)^H &= colim_{n\longrightarrow\infty} Sym^n(V)^H \setminus (Sym^n(V)^g)^H \\ &= colim_{n\longrightarrow\infty} (Sym^n(V)^g)^H \times \big((Sym^{n,\perp})^H \setminus \{0\}\big)\end{aligned}$$

Let k_n denote the dimension of $(Sym^{n,\perp})^H$. Then

$$(Sym^{n,\perp})^H \setminus \{0\} \backsimeq S^{k_n-1}. \tag{4.54}$$

As n goes to infinity, k_n goes to infinity. When k_n is large enough, S^{k_n-1} is contractible. So

$$(Sym(V) \setminus Sym(V)^g)^H$$

is contractible.

So we have proved the conclusion. $\square$

If V is a faithful $G-$representation, by Proposition 4.5.13, $(V)_g^{\mathbb{R}}$ is a faithful $\Lambda_G(g)-$representation. And we consider V^g as a $\Lambda_G(g)-$representation with trivial $\mathbb{R}-$action. Then by Theorem

4.4.24, $KU((V)^{\mathbb{R}}_g \oplus V^g)$ represents $K^{(V)^{\mathbb{R}}_g \oplus V^g}_{\Lambda_G(g)}(-)$. $(V)^{\mathbb{R}}_g$ is not always an ample orthogonal $\Lambda_G(g)-$representation, thus,

$$\mathrm{Map}(S^{(V)^{\mathbb{R}}_g}, KU((V)^{\mathbb{R}}_g \oplus V^g))$$

is not $\Lambda_G(g)-$weak equivalent to $KU(V^g)$. But it does represent $K^{V^g}_{\Lambda_G(g)}(-)$, as shown below.

Let X be a $G-$space.

$$\begin{aligned}[X^g, \mathrm{Map}(S^{(V)^{\mathbb{R}}_g}, KU((V)^{\mathbb{R}}_g \oplus V^g))]^{\Lambda_G(g)} &= [X^g \wedge S^{(V)^{\mathbb{R}}_g}, KU((V)^{\mathbb{R}}_g \oplus V^g)]^{\Lambda_G(g)} \\ &= K^{(V)^{\mathbb{R}}_g \oplus V^g}_{\Lambda_G(g)}(X^g \wedge S^{(V)^{\mathbb{R}}_g}) = K^{V^g}_{\Lambda_G(g)}(X^g).\end{aligned}$$

To simplify the symbol, let's use

$$F_g(G, V)$$

to denote the space $\mathrm{Map}_{\mathbb{R}}(S^{(V)^{\mathbb{R}}_g}, KU((V)^{\mathbb{R}}_g \oplus V^g))$. The basepoint c_0 of it is the constant map from $S^{(V)^{\mathbb{R}}_g}$ to the basepoint of $KU((V)^{\mathbb{R}}_g \oplus V^g)$.

For $F_g(G, V)$, we have the conclusions below.

Proposition 4.5.16. *Let G and H be compact Lie groups. Let V be a real $G-$representation and W a real $H-$representation. Let $g \in G^{tors}$, $h \in H^{tors}$.*

(i) $F_g : (G, V) \mapsto F_g(G, V)$ is a functor from $\mathcal{I}_G$ to the category $C_G(g)\mathcal{T}$ of $C_G(g)-$spaces.

(ii) We have the unit map

$$\eta_g(G, V) : S^{V^g} \longrightarrow F_g(G, V)$$

and the multiplication

$$\mu^F_{(g,h)}((G, V), (H, W)) : F_g(G, V) \wedge F_h(H, W) \longrightarrow F_{(g,h)}(G \times H, V \oplus W)$$

making the unit, associativity and centrality of unit diagram commute.

And $\eta_g(G, V)$ is $C_G(g)-$equivariant and $\mu^F_{(g,h)}((G, V), (H, W))$ is $C_{G\times H}(g, h)-$equivariant.

(iii)Let Δ_G denote the diagonal map

$$G \longrightarrow G \times G,\ g \mapsto (g, g).$$

Let $\widetilde{\sigma}_g(G,V,W): F_g(G,V) \longrightarrow Map(S^{W^g}, F_g(G, V\oplus W))$ *denote the map*

$$x \mapsto \Big(w \mapsto \big(\Delta_G^* \circ \mu^F_{(g,h)}((G,V),(G,W))\big)\big(x, \eta_g(G,W)(w)\big)\Big).$$

Then $\widetilde{\sigma}_g(G,V,W)$ *is a* $\Lambda_G(g)-$*weak equivalence when* V *is an ample* $G-$*representation.*

(iv) We have

$$\mu^F_{(g,h)}((G,V),(H,W))(x\wedge y) = \mu^F_{(h,g)}((H,V),(G,V))(y\wedge x) \tag{4.55}$$

for any $x \in F_g(G,V)$ *and* $y \in F_h(H,W)$.

Proof. (i) Let V_1 and V_2 be orthogonal $G-$representations and $f: V_1 \longrightarrow V_2$ be a linear isometric isomorphism. f gives the linear isometric isomorphisms $f_1: (V_1)^{\mathbb{R}}_g \longrightarrow (V_2)^{\mathbb{R}}_g$, and $f_2: (V_1)^{\mathbb{R}}_g \oplus V_1^g \longrightarrow (V_2)^{\mathbb{R}}_g \oplus V_2^g$. Then define $F_g(f): F_g(V_1) \longrightarrow F_g(V_2)$ in this way: for any $\mathbb{R}-$equivariant map $\alpha: S^{(V_1)^{\mathbb{R}}_g} \longrightarrow KU((V_1)^{\mathbb{R}}_g \oplus V_1^g)$, $F_g(f)(\alpha)$ is the composition

$$S^{(V_2)^{\mathbb{R}}_g} \xrightarrow{S(f_1^{-1})} S^{(V_1)^{\mathbb{R}}_g} \xrightarrow{\alpha} KU((V_1)^{\mathbb{R}}_g \oplus V_1^g) \xrightarrow{KU(f_2)} KU((V_2)^{\mathbb{R}}_g \oplus V_2^g) \tag{4.56}$$

which is still $\mathbb{R}-$equivariant.

It's straightforward to check $F_g(Id)$ is the identity map, and for morphisms $V_1 \xrightarrow{f} V_2 \xrightarrow{f'} V_3$ in $\mathcal{I}_G$, we have $F_g(f'\circ f) = F_g(f')\circ F_g(f)$.

So we have a well-defined functor $F_g: \mathcal{I}_G \longrightarrow C_G(g)\mathcal{T}$.

(ii) Define the unit map $\eta_g(G,V): S^{V^g} \longrightarrow F_g(G,V)$ by

$$v \mapsto (v' \mapsto \eta^K_{(V)^{\mathbb{R}}_g \oplus V^g}(v \wedge v')) \tag{4.57}$$

where $\eta^K_{(V)^{\mathbb{R}}_g \oplus V^g}: S^{(V)^{\mathbb{R}}_g \oplus V^g} \longrightarrow KU((V)^{\mathbb{R}}_g \oplus V^g)$ is the unit map for global K-theory. Since $(V)^{\mathbb{R}}_g \oplus V^g$ is a $\Lambda_G(g)-$representation, $\eta^K_{(V)^{\mathbb{R}}_g \oplus V^g}$ is $\Lambda_G(g)-$equivariant. So $\eta_g(G,V)$ is well-defined and $\Lambda_G(g)-$equivariant.

Define the multiplication $\mu^F_{(g,h)}((G,V),(H,W)): F_g(G,V)\wedge F_h(H,W) \longrightarrow F_{(g,h)}(G\times H, V\oplus W)$ by

$$\alpha\wedge\beta \mapsto (v\wedge w \mapsto \mu^K_{V,W}(\alpha(v)\wedge\beta(w))) \tag{4.58}$$

where $\mu^K_{V,W}$ is the multiplication for global K-theory.

Since $\mu^K_{V,W}$ is $\Lambda_G(g)\times\Lambda_H(h)-$equivariant, $\mu^F_{(g,h)}((G,V),(H,W))$ is $C_{G\times H}(g,h)-$equivariant.

It's straightforward to check the unit map and multiplication make the unit, associativity and centrality of unit diagram commute.

(iii) Since V is a faithful $G-$representation, by Proposition 4.5.2, $(V)_g^{\mathbb{R}} \oplus V^g$ is a faithful $\Lambda_G(g)-$representation. By Theorem 4.4.26, we have the $\Lambda_G(g)-$weak equivalence

$$KU((V)_g^{\mathbb{R}} \oplus V^g) \xrightarrow{\widetilde{\sigma}^K} \mathrm{Map}(S^{(W)_g^{\mathbb{R}} \oplus W^g}, KR((V \oplus W)_g^{\mathbb{R}} \oplus (V \oplus W)^g))$$

where $\widetilde{\sigma}^K$ is the right adjoint of the structure map of the global complex K-theory KU. Thus we have the $\Lambda_G(g)-$weak equivalence

$$\begin{aligned}\mathrm{Map}(S^{(V)_g^{\mathbb{R}}}, KU((V)_g^{\mathbb{R}} \oplus V^g)) &\longrightarrow \mathrm{Map}(S^{(V)_g^{\mathbb{R}}}, \mathrm{Map}(S^{(W)_g^{\mathbb{R}} \oplus W^g}, KR((V \oplus W)_g^{\mathbb{R}} \oplus (V \oplus W)^g)))\\ &= \mathrm{Map}(S^{W^g}, \mathrm{Map}(S^{(V \oplus W)_g^{\mathbb{R}}}, KR((V \oplus W)_g^{\mathbb{R}} \oplus (V \oplus W)^g))),\end{aligned}$$

i.e. $F_g(G, V) \simeq_{C_G(g)} \mathrm{Map}(S^{W^g}, F_g(G, V \oplus W))$.

(iv) (4.55) comes directly from the commutativity of the orthogonal spectrum KU. $\square$

In Theorem 4.3.8 I construct a $G-$space $QEll_G$ representing $QEll_G^*(-)$. With $F_g(G, V)$ and $S(G, V)_g$ just constructed, we can go further than that. Apply Theorem 4.3.7, we get the conclusion below.

Proposition 4.5.17. *Let V be a faithful orthogonal $G-$representation. Let $B'(G, V)$ denote the space*

$$\prod_{g \in G_{conj}^{tors}} Map_{C_G(g)}(G, F_g(G, V) * S(G, V)_g).$$

$QEll_G^V(-)$ is weakly represented by $B'(G, V)$ in the sense

$$\pi_0(B'(G, V)) = QEll_G^V(S^0). \tag{4.59}$$

The proof of Proposition 4.5.17 is analogous to that of Theorem 4.3.8 step by step.

Below is the main theorem in Section 4.5.3. Let's use formal linear combination

$$t_1 a + t_2 b \text{ with } 0 \leqslant t_1, t_2 \leqslant 1, t_1 + t_2 = 1$$

to denote points in join, as talked in Appendix A.1.

Proposition 4.5.18. *Let $E_g(G,V)$ denote*

$$\{t_1a+t_2b\in F_g(G,V)*S(G,V)_g|\|b\|\leqslant t_2\}/\{t_1c_0+t_2b\}.$$

It is the quotient space of a closed subspace of the join $F_g(G,V)*S(G,V)_g$ with all the points of the form $t_1c_0+t_2b$ collapsed to one point, which I pick as the basepoint of $E_g(G,V)$, where c_0 is the basepoint of $F_g(G,V)$. $E_g(G,V)$ has the evident $C_G(g)-$action. And it is $C_G(g)-$weak equivalent to $F_g(G,V)*S(G,V)_g$. As a result, $\prod\limits_{g\in G^{tors}_{conj}} Map_{C_G(g)}(G,E_g(G,V))$ is $G-$weak equivalent to $\prod\limits_{g\in G^{tors}_{conj}} Map_{C_G(g)}(G,F_g(G,V)*S(G,V)_g)$. So when V is a faithful $G-$representation,

$$E(G,V):=\prod_{g\in G^{tors}_{conj}} Map_{C_G(g)}(G,E_g(G,V)) \tag{4.60}$$

weakly represents $QEll^V_G(-)$ in the sense

$$\pi_0(E(G,V))\cong QEll^V_G(S^0). \tag{4.61}$$

Proof. First I show $F_g(G,V)*S(G,V)_g$ is $C_G(g)-$homotopy equivalent to

$$E'_g(G,V):=\{t_1a+t_2b\in F_g(G,V)*S(G,V)_g|\|b\|\leqslant t_2\}.$$

Note that $b\in S(G,V)_g$ is never zero. Let $j:E'_g(G,V)\longrightarrow F_g(G,V)*S(G,V)_g$ be the inclusion. Let $p:F_g(G,V)*S(G,V)_g\longrightarrow E'_g(G,V)$ be the $C_G(g)-$map sending t_1a+t_2b to $t_1a+t_2\frac{min\{\|b\|,t_2\}}{\|b\|}b$. Both j and p are both continuous and $C_G(g)-$equivariant. $p\circ j$ is the identity map of $E'_g(G,V)$. We can define a $C_G(g)-$homotopy

$$H:(F_g(G,V)*S(G,V)_g)\times I\longrightarrow F_g(G,V)*S(G,V)_g$$

from the identity map on $F_g(G,V)*S(G,V)_g$ to $j\circ p$ by shrinking. For any $t_1a+t_2b\in F_g(G,V)*S(G,V)_g$, Define

$$H(t_1a+t_2b,t):=t_1a+t_2((1-t)b+t\frac{min\{\|b\|,t_2\}}{\|b\|}b). \tag{4.62}$$

Then I show $E'_g(G,V)$ is $G-$weak equivalent to $E_g(G,V)$.

Let $q : E'_g(G,V) \longrightarrow E_g(G,V)$ be the quotient map. Let H be a closed subgroup of $C_G(g)$.

If g is in H, since $S(G,V)_g^H$ is empty, so $E_g(G,V)^H$ is in the end $F_g(G,V)$ and can be identified with $F_g(G,V)^H$. In this case q^H is the identity map.

If g is not in H, $E'_g(G,V)^H$ is contractible. The cone $\{c_0\} * S(G,V)_g^H$ is contractible, so $q\big((\{c_0\} * S(G,V)_g)^H\big) = q(\{c_0\} * S(G,V)_g^H)$ is contractible. Note that the subspace of all the points of the form $t_1c_0 + t_2b$ for any t_1 and b is $q\big((\{c_0\} * S(G,V)_g)^H\big)$. Therefore, $E_g(G,V)^H = E'_g(G,V)^H / q(\{c_0\} * S(G,V)_g)^H$ is contractible.

Therefore, $E'_g(G,V)$ is $G-$weak equivalent to $F_g(G,V) * S(G,V)_g$. $\square$

In fact, for any based $C_G(g)-$space Y, we have the general conclusion below.

Proposition 4.5.19. *Let $g \in G^{tors}$. Let Y be a based $\Lambda_G(g)-$space. Let $\widetilde{Y}_g$ denote the $C_G(g)-$space*

$$\{t_1a + t_2b \in Y^{\mathbb{R}} * S(G,V)_g | \|b\| \leqslant t_2\} / \{t_1y_0 + t_2b\}.$$

It is the quotient space of a closed subspace of $Y^{\mathbb{R}} * S(G,V)_g$ with all the points of the form $t_1y_0 + t_2b$ collapsed to one point, i.e the basepoint of $\widetilde{Y}_g$, where y_0 is the basepoint of Y. $\widetilde{Y}_g$ is $C_G(g)-$weak equivalent to $Y^{\mathbb{R}} * S(G,V)_g$. As a result, the functor $R_g : C_G(g)\mathcal{T} \longrightarrow G\mathcal{T}$ with

$$R_g\widetilde{Y} = Map_{C_G(g)}(G, \widetilde{Y}_g) \tag{4.63}$$

is a homotopical right adjoint of $L : G\mathcal{T} \longrightarrow C_G(g)\mathcal{T}$, $X \mapsto X^g$.

The proof of Proposition 4.5.19 is analogous to that of Theorem 4.3.7 and Proposition 4.5.18.

Remark 4.5.20. *We can consider $E_g(G,V)$ as a quotient space of a subspace of $F_g(G,V) \times Sym(V) \times I$*

$$\{(a,b,t) \in F_g(G,V) \times Sym(V) \times I | \|b\| \leqslant t;\ \text{and } b \in S(G,V)_g \text{ if } t \neq 0\} \tag{4.64}$$

by identifying points $(a,b,1)$ with $(a',b,1)$, and collapsing all the points (c_0,b,t) for any b and t. In other words, the end $F_g(G,V)$ in the join $F_g(G,V) * S(G,V)_g$ is identified with the points of the form $(a,0,0)$ in (4.64).

In Section 4.5.4 we need this identification of $E_g(G,V)$ in mind to prove the structure maps are well-defined and continuous.

Proposition 4.5.21. *For each $g \in G^{tors}$,*

$$E_g : \mathcal{I}_G \longrightarrow C_G(g)\mathcal{T},\ (G,V) \mapsto E_g(G,V)$$

is a well-defined functor. As a result,

$$E : \mathcal{I}_G \longrightarrow G\mathcal{T},\ (G,V) \mapsto \prod_{g \in G^{tors}_{conj}} Map_{C_G(g)}(G, E_g(G,V))$$

is a well-defined functor.

Proof. Let V and W be $G-$ representations and $f : V \longrightarrow W$ a linear isometric isomorphism. Then f induces a $C_G(g)-$homeomorphism $F_g(f)$ from $F_g(G,V)$ to $F_g(G,W)$ and a $C_G(g)-$homeomorphism $S_g(f)$ from $S(G,V)_g$ to $S(G,W)_g$. We have the well-defined map

$$E_g(f) : E_g(G,V) \longrightarrow E_g(G,W)$$

sending a point represented by $t_1 a + t_2 b$ in the join to that represented by $t_1 F_g(f)(a) + t_2 S_g(f)(b)$.

It's straightforward to check $E_g(Id)$ is the identity map and the composition law holds.

$E(f) : E(G,V) \longrightarrow E(G,W)$ is defined by

$$\prod_{g \in G^{tors}_{conj}} \alpha_g \mapsto \prod_{g \in G^{tors}_{conj}} E_g(f) \circ \alpha_g.$$

It's straightforward to check that it's well-defined, $E(Id)$ is identity and the composition law holds. □

4.5.4 Structure Maps

In Section 4.5.4 I construct a unit map η^E and a multiplication μ^E so that we get an orthogonal $G-$spectrum and $\mathcal{I}_G-$FSP that represents quasi-elliptic cohomology.

Let G and H be compact Lie groups, V an orthogonal $G-$representation and W an orthogonal $H-$representation. Let's use x_g to denote the basepoint of $E_g(G,V)$, which is defined in Proposition 4.5.18.

Let $g \in G^{tors}$. For each $v \in S^V$, there are $v_1 \in S^{V^g}$ and $v_2 \in S^{(V^g)^\perp}$ such that $v = v_1 \wedge v_2$.

Let $\eta_g^E(G,V) : S^V \longrightarrow E_g(G,V)$ be the map

$$\eta_g^E(G,V)(v) := \begin{cases} (1 - \|v_2\|)\eta_g(G,V)(v_1) + \|v_2\|v_2, & \text{if } \|v_2\| \leqslant 1; \\ x_g, & \text{if } \|v_2\| \geqslant 1. \end{cases} \tag{4.65}$$

where $\eta_g(G,V)$ is the unit map defined in Proposition 4.5.16.

Lemma 4.5.22. *The map $\eta_g^E(G,V)$ defined in (4.65) is well-defined, continuous and $C_G(g)-$equivariant.*

Proof. When v_1 is infinity, $\eta_g(G,V)(v_1)$ is the basepoint of $F_g(V)$. So by the construction of $E_g(G,V)$ in Proposition 4.5.18, $v = v_1 \wedge v_2$ is mapped to the basepoint of $E_g(G,V)$.

When v_2 is infinity, $\eta_g^E(G,V)(v)$ is the basepoint by definition. So $\eta_g^E(G,V)$ is well-defined. And since $\eta_g(G,V)$ is $C_G(g)-$equivariant, $\eta_g^E(G,V)$ is $C_G(g)-$equivariant.

Next I prove $\eta_g^E(G,V)$ is continuous by showing for each point in $E_g(G,V)$, there is an open neighborhood of it whose preimage is open in S^V.

Consider a point x in the image of $\eta_g^E(G,V)$ represented by $t_1a + t_2b$.

Case I: $0 < t_2 < 1$ and a is not the basepoint of $F_g(G,V)$.

Let A be an open neighborhood of a in $F_g(G,V)$ not including the basepoint. We can find such an A since $F_g(G,V)$ is Hausdorff. Let $\delta > 0$ be a small enough value. Let $U_{x,\delta}$ be the open neighborhood of x

$$U_{x,\delta} := \{[s_1\alpha + s_2\beta] \in E_g(G,V) | \alpha \in A, |s_2 - t_2| < \delta, \|\beta - b\| < \delta\}.$$

Then $\eta_g^E(G,V)^{-1}(U_{x,\delta})$ is the smash product of $\eta_g(G,V)^{-1}(A)$, which is open in S^{V^g}, and an open subset of $S^{(V^g)^\perp}$

$$\{w \in S^{(V^g)^\perp} | t_2 - \delta < \|w\| < t_2 + \delta, \|w - b\| < \delta\}.$$

So it's open in S^V.

Case II: $t_2 = 0$ and a is not the basepoint of $F_g(G,V)$.

Let A be an open neighborhood of a in $F_g(G,V)$ not including the basepoint. Let $\delta > 0$ be a small enough value. Let $W_{x,\delta}$ be the open neighborhood of x

$$W_{x,\delta} := \{[s_1\alpha + s_2\beta] \in E_g(G,V) | \alpha \in A, |s_2| < \delta, \|\beta - b\| < \delta\}.$$

Then $\eta_g^E(G,V)^{-1}(W_{x,\delta})$ is the smash product of $\eta_g(G,V)^{-1}(A)$, which is open in S^{V^g}, and an open subset of $S^{(V^g)^\perp}$

$$\{w \in S^{(V^g)^\perp} | \|w\| < \delta, \|w - b\| < \delta\}.$$

So it's open in S^V.

Case III: x is the basepoint x_g of $E_g(G,V)$.

Let A_0 be an open neighborhood of the basepoint c_0.

For any point w of the form $t_1c_0 + t_2b$ in the the space $E'_g(G,V)$ with $0 < t_2 < 1$, let U_{w,δ_w} denote the open subset of $E_g(G,V)$

$$\{[s_1\alpha + s_2\beta] \in E_g(G,V) | \alpha \in A_0, |s_2 - t_2| < \delta_w, \|\beta - b\| < \delta_w\}$$

with δ_w small enough.

Let W_δ denote the open subset of $E_g(G,V)$

$$\{[s_1\alpha + s_2\beta] \in E_g(G,V) | \alpha \in A_0, |s_2| < \delta, \|\beta - b\| < \delta\}$$

with δ small enough.

For any $b \in S(G,V)_g$ with $\|b\| \leqslant 1$, let V_{b,δ_b} denote the open subsect of $E_g(G,V)$

$$\{[s_1\alpha + s_2\beta] \in E_g(G,V) | s_2 > 1 - \delta_b, \|\beta - b\| < \delta_b\}$$

with δ_b small enough.

Let's consider the open neighborhood U of x that is the union of the spaces defined above

$$U := (\bigcup_w U_{w,\delta_w}) \cup W_\delta \cup (\bigcup_b V_{b,\delta_b})$$

where w goes over all the points of the form $[t_1c_0 + t_2b]$ in $E_g(G,V)$ with $0 < t_2 < 1$, and b

goes over all the points in $S(G,V)_g$ with $\|b\| \leqslant 1$.

The preimage of each U_{w,δ_w} and W_δ is open, the proof of which is analogous to Case I and II. The preimage of V_{b,δ_b} is the smash product of S^{V^g} and the open set of $S^{(V^g)^\perp}$

$$\{w_2 \in S^{(V^g)^\perp} | \|w_2\| > 1-\delta_b, \|w_2 - b\| < \delta_b\},$$

thus, is open.

The preimage of U is the union of open subsets in S^V, thus, open.

Therefore, The map $\eta^E_g(G,V)$ defined in (4.65) is continuous.

□

Remark 4.5.23. *For any $g \in G^{tors}$, it's straightforward to check the diagram below commutes.*

$$\begin{array}{ccc} S^{V^g} & \xrightarrow{\eta_g(G,V)} & F_g(G,V) \\ \downarrow & & \downarrow \\ S^V & \xrightarrow{\eta^E_g(G,V)} & E_g(G,V) \end{array}$$

where both vertical maps are inclusions.

By Lemma 4.5.22, the map

$$\eta^E(G,V) : S^V \longrightarrow \prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, E_g(G,V))$$

defined by

$$v \mapsto \prod_{g \in G^{tors}_{conj}} (\alpha \mapsto \eta^E_g(G,V)(\alpha \cdot v)), \tag{4.66}$$

is well-defined and continuous. Moreover, $\eta^E : S \longrightarrow E$ with $E(G,V)$ defined in (4.60) is a well-defined functor.

Next, let's construct the multiplication map μ^E. First we define a map $\mu^E_{(g,h)}((G,V),(H,W))$: $E_g(G,V) \wedge E_h(H,W) \longrightarrow E_{(g,h)}(G \times H, V \oplus W)$ by sending a point $[t_1a_1+t_2b_1] \wedge [u_1a_2+u_2b_2]$

to

$$\begin{cases} [(1-\sqrt{t_2^2+u_2^2})\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2)+\sqrt{t_2^2+u_2^2}(b_1+b_2)], & \text{if } t_2^2+u_2^2\le 1 \text{ and } t_2u_2\neq 0;\\ [(1-t_2)\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2)+t_2b_1], & \text{if } u_2=0 \text{ and } 0<t_2<1;\\ [(1-u_2)\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2)+u_2b_2], & \text{if } t_2=0 \text{ and } 0<u_2<1;\\ [1\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2)+0], & \text{if } u_2=0 \text{ and } t_2=0;\\ x_{g,h}, & \text{Otherwise.} \end{cases} \tag{4.67}$$

where $\mu^E_{(g,h)}((G,V),(H,W))$ is the one defined in (4.58) and $x_{g,h}$ is the basepoint of $E_{(g,h)}(G\times H, V\oplus W)$.

Lemma 4.5.24. *The map $\mu^E_{(g,h)}((G,V),(H,W))$ defined in (4.67) is well-defined and continuous.*

Proof. Note that when either a_1 is the basepoint of $F_g(G,V)$, or a_2 is the basepoint of $F_h(H,W)$, or $t_2=1$, or $u_2=1$, the point $[t_1a_1+t_2b_1]\wedge[u_1a_2+u_2b_2]$ is mapped to the basepoint $x_{g,h}$.

The spaces $S(G,V)_g$ have the following properties:

(i) There is no zero vector in any $S(G,V)_g$ by its construction;

(ii) For any $b_1\in S(G,V)_g$, $b_2\in S(H,W)_h$, b_1, b_2 and b_1+b_2 are all in $S(G\times H,V\oplus W)_{(g,h)}$. b_1 and b_2 are orthogonal to each other, so $\|b_1+b_2\|^2=\|b_1\|^2+\|b_2\|^2$. Thus, if $t_2u_2\neq 0$, $\|b_1+b_2\|\leqslant\sqrt{t_1^2+t_2^2}$.

Therefore, $\mu^E_{(g,h)}((G,V),(H,W))$ is well-defined.

Let

$$x=[s_1\alpha+s_2\beta]$$

be a point in the image of $\mu^E_{(g,h)}((G,V),(H,W))$. If s_2 is nonzero, there is unique $\beta_1\in S(G,V)_g\cup\{0\}$ and unique $\beta_2\in S(H,W)_h\cup\{0\}$ such that $\beta=\beta_1+\beta_2$.

For each point in the image, I pick an open neighborhood of it so that its preimage in $E_g(G,V)\wedge E_h(H,W)$ is open.

Case I: x is not the basepoint, $0<s_1,s_2<1$ and β_1 and β_2 are both nonzero.

Let $A(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G\times H,V\oplus W)$ not containing the basepoint. Let $\delta>0$ be some small enough value. We consider the open neighborhood $U_{x,\delta}$

of x

$$U_{x,\delta} := \{[r_1a+r_2d] \in E_{(g,h)}(G\times H, V\oplus W) | \|d_1-\beta_1\| < \delta, \|d_2-\beta_2\| < \delta, a \in A(\alpha), |r_2^2-s_2^2| < \delta\}$$

where $d = d_1 + d_2$ with $d_1 \in S(G,V)_g \cup \{0\}$ and $d_2 \in S(H,W)_h \cup \{0\}$.

The preimage of $U_{x,\delta}$ is

$$\{[t_1a_1 + t_2d_1] \wedge [u_1a_2 + u_2d_2] \in E_g(G,V) \wedge E_h(H,W) | a_1 \wedge a_2 \in \mu^F_{(g,h)}((G,V),(H,W))^{-1}(A(\alpha)),$$
$$\|d_1 - \beta_1\| < \delta, \|d_2 - \beta_2\| < \delta, |t_2^2 + u_2^2 - s_2^2| < \delta\},$$

where $\mu^F_{(g,h)}((G,V),(H,W))$ is the multiplication defined in (4.58).

Note that $E_g(G,V) \wedge E_h(H,W)$ is the quotient space of a subspace of the product of spaces

$$F_g(G,V) \times S(G,V)_g \times [0,1] \times F_h(H,W) \times S(H,W)_h \times [0,1]$$

and $U_{x,\delta}$ is the quotient of an open subset of this product. So it is open in $E_g(G,V) \wedge E_h(H,W)$.

Case II: x is not the basepoint, $0 < s_1, s_2 < 1$ and $\beta \in S(H,W)_h$.

Let $A(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G \times H, V \oplus W)$ not containing the basepoint. Let $\delta > 0$ be some small enough value.

Consider the open neighborhood $W_{x,\delta}$ of x

$$W_{x,\delta} := \{[r_1a+r_2d] \in E_{(g,h)}(G\times H, V\oplus W) | \|d_1-\beta_1\| < \delta, \|d_2\| < \delta, a \in A(\alpha), |r_2^2-s_2^2| < \delta\}$$

where $d = d_1 + d_2$ with $d_1 \in S(G,V)_g \cup \{0\}$ and $d_2 \in S(H,W)_h \cup \{0\}$.

The preimage of $W_{x,\delta}$ is

$$\{[t_1a_1 + t_2d_1] \wedge [u_1a_2 + u_2d_2] \in E_g(G,V) \wedge E_h(H,W) | a_1 \wedge a_2 \in \mu^F_{(g,h)}((G,V),(H,W))^{-1}(A(\alpha)),$$
$$\|d_1 - \beta_1\| < \delta, \|d_2\| < \delta, |t_2^2 + u_2^2 - s_2^2| < \delta\}.$$

It is the quotient of an open subspace of the product

$$F_g(G,V) \times S(G,V)_g \times [0,1] \times F_h(H,W) \times S(H,W)_h \times [0,1].$$

So the preimage of $W_{x,\delta}$ is open in $E_g(G,V) \wedge E_h(H,W)$.

Case III: x is not the basepoint, $0 < s_1, s_2 < 1$ and $\beta \in S(G,V)_g$.

We can show the map is continuous at such points in a way analogous to Case II.

Case IV x is not the basepoint and s_2 is zero.

Let $A(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G \times H, V \oplus W)$ not containing the basepoint. Let $\delta > 0$ be some small enough value.

Consider the open neighborhood of x

$$B_{x,\delta} := \{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) | a \in A(\alpha), \|d_1\| < \delta, \|d_2\| < \delta, 0 \leqslant r_2^2 < \delta\}$$

where $d = d_1 + d_2$ with $d_1 \in S(G,V)_g \cup \{0\}$ and $d_2 \in S(H,W)_h \cup \{0\}$.

The preimage of $B_{x,\delta}$ is

$$\{[t_1 a_1 + t_2 d_1] \wedge [u_1 a_2 + u_2 d_2] \in E_g(G,V) \wedge E_h(H,W) | a_1 \wedge a_2 \in \mu^F_{(g,h)}((G,V),(H,W))^{-1}(A(\alpha)),$$
$$\|d_1\| < \delta, \|d_2\| < \delta, 0 \leqslant t_2^2 + u_2^2 < \delta\}.$$

It is the quotient of an open subspace of the product

$$F_g(G,V) \times S(G,V)_g \times [0,1] \times F_h(H,W) \times S(H,W)_h \times [0,1].$$

So the preimage of $B_{x,\delta}$ is open in $E_g(G,V) \wedge E_h(H,W)$.

Case V: $x = [s_1\alpha + s_2\beta]$ is the base point.

Let $A_0(\alpha)$ be an open neighborhood of α in $F_{(g,h)}(G \times H, V \oplus W)$.

For any point w in $E'_{(g,h)}(G \times H, V \oplus W)$ of the form $t_1 c_0 + t_2 b$ with $0 < t_2 < 1$ and b_1 b_2 both nonzero, let U_{w,δ_w} be the open subset of $E_{(g,h)}(G \times H, V \oplus W)$

$$\{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) | \|d_1 - b_1\| < \delta_w, \|d_2 - b_2\| < \delta_w, a \in A_0(\alpha), |r_2^2 - t_2^2| < \delta_w\}$$

with δ_w small enough.

For each point y in $E'_{(g,h)}(G \times H, V \oplus W)$ of the form $t_1 c_0 + t_2 b$ with $0 < t_2 < 1$ and $b \in S(H,W)_h$, let W_{y,δ_y} be the open subset

$$\{[r_1 a + r_2 d] \in E_{(g,h)}(G \times H, V \oplus W) | \|d_1 - b_1\| < \delta_y, \|d_2\| < \delta_y, a \in A_0(\alpha), |r_2^2 - t_2^2| < \delta_y\}$$

with δ_y small enough.

For each point z in $E'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and $b\in S(G,V)_g$, let V_{z,δ_z} be the open subset

$$\{[r_1a+r_2d]\in E_{(g,h)}(G\times H, V\oplus W)|\|d_2-b_2\|<\delta_z, \|d_1\|<\delta_z, a\in A_0(\alpha), |r_2^2-t_2^2|<\delta_z\}$$

with δ_z small enough.

Let $B_{x_0,\delta}$ denote the open set

$$\{[r_1a+r_2d]\in E_{(g,h)}(G\times H, V\oplus W)|\|d_2\|<\delta, \|d_1\|<\delta, a\in A_0(c_0), 0\leqslant r_2<\delta\}$$

with δ small enough,

For each θ in $E'_{(g,h)}(G\times H, V\oplus W)$ of the form $0+1b$, let D_{θ,δ_θ} be the open subset

$$\{[r_1a+r_2d]\in E_{(g,h)}(G\times H, V\oplus W)|\|d-b\|<\delta_\theta, 1\geqslant r_2\geqslant 1-\delta_\theta\}$$

with δ_θ small enough.

Let's consider the open neighborhood of x in $E_{(g,h)}(G\times H, V\oplus W)$ that is the union of the spaces above

$$U:=(\bigcup_w U_{w,\delta_w})\cup(\bigcup_y W_{y,\delta_y})\cup(\bigcup_z V_{z,\delta_z})\cup B_{x_0,\delta}\cup(\bigcup_\theta D_{\theta,\delta_\theta})$$

where w goes over all the points in $E'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and b_1, b_2 both nonzero, y goes over all the points in $E'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and $b\in S(H,W)_h$, z goes over all the points in $E'_{(g,h)}(G\times H, V\oplus W)$ of the form $t_1c_0+t_2b$ with $0<t_2<1$ and $b\in S(G,V)_g$, and θ goes over all the points of the form $0+1b$ in $E'_{(g,h)}(G\times H, V\oplus W)$.

The preimage of each U_{x,δ_x}, W_{y,δ_y}, V_{z,δ_z}, $B_{x_0,\delta}$ is open, the proof of which are analogous to that of Case I, II, III and IV. The preimage of D_{θ,δ_θ} is

$$\{[t_1a_1+t_2d_1]\wedge[u_1a_2+u_2d_2]\in E_g(G,V)\wedge E_h(H,W)|\|d_1+d_2-b\|<\delta_\theta, 1-\sqrt{t_2^2+u_2^2}<\delta_\theta\},$$

which is open. Therefore, the preimage of U is open.

Combining all the cases above, the multiplication $\mu^E_{(g,h)}((G,V),(H,W))$ defined in (4.67) is continuous.

$\square$

Remark 4.5.25. *For any $g \in G^{tors}$, $h \in H^{tors}$, we have the diagram below commutes.*

$$\begin{array}{ccc} F_g(G,V) \wedge F_h(H,W) & \xrightarrow{\mu^F_{(g,h)}((G,V),(H,W))} & F_{(g,h)}(G \times H, V \oplus W) \\ \downarrow & & \downarrow \\ E_g(G,V) \wedge E_h(H,W) & \xrightarrow{\mu^E_{(g,h)}((G,V),(H,W))} & E_{(g,h)}(G \times H, V \oplus W) \end{array}$$

where the vertical maps are both inclusion into the end of the join.

The basepoint of $E(G,V)$ is the product of the basepoint of each factor $\mathrm{Map}_{C_G(g)}(G, E_g(G,V))$, i.e. the product of the constant map to the base point of each $E_g(G,V)$.

We can define the multiplication

$$\mu^E((G,V),(H,W)) : E(G,V) \wedge E(H,W) \longrightarrow E(G \times H, V \oplus W) \tag{4.68}$$

by the composition

$$\begin{aligned} &\prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, E_g(G,V)) \wedge \prod_{h \in H^{tors}_{conj}} \mathrm{Map}_{C_H(h)}(H, E_g(H,W)) \longrightarrow \\ &\prod_{g \in G^{tors}_{conj}, h \in H^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, E_g(G,V)) \wedge \mathrm{Map}_{C_H(h)}(H, E_h(H,W)) \longrightarrow \\ &\prod_{g \in G^{tors}_{conj}, h \in H^{tors}_{conj}} \mathrm{Map}_{C_{G\times H}(g,h)}\big(G \times H, E_g(G,V) \wedge E_h(H,W)\big) \longrightarrow \\ &\prod_{g \in G^{tors}_{conj}, h \in H^{tors}_{conj}} \mathrm{Map}_{C_{G\times H}(g,h)}\big(G \times H, E_{(g,h)}(G \times H, V \oplus W)\big) \end{aligned}$$

where the first map sends

$$\prod_{g \in G^{tors}_{conj}} \alpha_g \wedge \prod_{h \in H^{tors}_{conj}} \beta_h$$

to

$$\prod_{g \in G^{tors}_{conj}, h \in H^{tors}_{conj}} \alpha_g \wedge \beta_h,$$

the second map sends a point

$$\prod_{g \in G^{tors}_{conj}, h \in H^{tors}_{conj}} \alpha_g \wedge \beta_h$$

to

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}}\Big((a,b)\mapsto \alpha_g(a)\wedge\beta_h(b)\Big),$$

the third map sends

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}} f_{(g,h)}$$

to

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}} \mu^E_{(g,h)}((G,V),(H,W))\circ f_{(g,h)}.$$

More explicitly, $\mu^E((G,V),(H,W))$ sends

$$(\prod_{g\in G^{tors}_{conj}}\alpha_g)\wedge(\prod_{h\in H^{tors}_{conj}}\beta_h)$$

to

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}}\Big((g',h')\mapsto \mu^E_{(g,h)}((G,V),(H,W))\big(\alpha_g(g')\wedge\beta_h(h')\big)\Big). \tag{4.69}$$

Lemma 4.5.26. *Let G, H, K be compact Lie groups. Let V be an orthogonal $G-$representation, W an orthogonal $H-$representation, and U an orthogonal $K-$representation. Let $g\in G^{tors}$, $h\in H^{tors}$, and $k\in K^{tors}$. Then we have the commutative diagrams below.*

$$\begin{array}{ccc} S^V\wedge S^W & \xrightarrow{\eta^E_g(G,V)\wedge\eta^E_h(H,W)} & E_g(G,V)\wedge E_h(H,W) \\ \big\downarrow\cong & & \big\downarrow \mu^E_{(g,h)}((G,V),(H,W)) \\ S^{V\oplus W} & \xrightarrow{\eta^E_{(g,h)}(G\times H,V\oplus W)} & E_{(g,h)}(G\times H,V\oplus W) \end{array} \tag{4.70}$$

$$\begin{array}{ccc} E_g(G,V)\wedge E_h(H,W)\wedge E_k(K,U) & \xrightarrow{\mu^E_g((G,V),(H,W))\wedge Id} & E_{(g,h)}(G\times H,V\oplus W)\wedge E_k(K,U) \\ \big\downarrow Id\wedge\mu^E_{(h,k)}(H\times K,W\oplus U) & & \big\downarrow \mu^E_{((g,h),k)}((G\times H,V\oplus W),(K,U)) \\ E_g(G,V)\wedge E_{(h,k)}(H\times K,W\oplus U) & \xrightarrow{\mu^E_{(g,(h,k))}((G,V),(H\times K,W\oplus U))} & E_{(g,h,k)}(G\times H\times K,V\oplus W\oplus U) \end{array} \tag{4.71}$$

$$\begin{array}{ccccc} S^V\wedge E_h(H,W) & \xrightarrow{\eta^E_g(G,V)\wedge Id} & E_g(G,V)\wedge E_h(H,W) & \xrightarrow{\mu^E_{(g,h)}((G,V),(H,W))} & E_{(g,h)}(G\times H,V\oplus W) \\ \big\downarrow\tau & & & & \big\downarrow E_{(g,h)}(\tau) \\ E_h(H,W)\wedge S^V & \xrightarrow{Id\wedge\eta^E_g(G,V)} & E_h(H,W)\wedge E_g(G,V) & \xrightarrow{\mu^E_{(h,g)}((H,W),(G,V))} & E_{(h,g)}(H\times G,W\oplus V) \end{array} \tag{4.72}$$

Moreover, we have

$$\mu^E_{(g,h)}((G,V),(H,W))(x\wedge y)=\mu^E_{(h,g)}((H,W),(G,V))(y\wedge x) \tag{4.73}$$

for any $x\in E_g(G,V)$ and $y\in E_h(H,W)$.

Proof. In this proof, I identify the end $F_g(G,V)$ in the space $E_g(G,V)$ with the points of the form $(a,0,0)$, i.e. $1a+00$, in the space (4.64) as indicated in Remark 4.5.20. And if the coordinate t_2 in a point t_1a+t_2b is zero, then b is the zero vector.

(i) Unity.

Let $v\in S^V$ and $w\in S^W$. Let

$$v=v_1\wedge v_2,\text{ with } v_1\in S^{V^g},\text{ and } v_2\in S^{(V^g)^\perp},$$

$$w=w_1\wedge w_2,\text{ with } w_1\in S^{W^h},\text{ and } w_2\in S^{(W^h)^\perp},$$

$$\mu^E_{(g,h)}\big((G,V),(H,W)\big)\circ\big(\eta^E_g(G,V)\wedge\eta^E_h(H,W)\big)(v\wedge w)$$

is the basepoint if $\|v_2\|^2+\|w_2\|^2\geqslant 1$. If $\|v_2\|^2+\|w_2\|^2\leqslant 1$, it equals

$$\big[(1-\sqrt{\|v_2\|^2+\|w_2\|^2})\eta_g(G,V)(v_1)\wedge\eta_h(H,W)(w_1)+\sqrt{\|v_2\|^2+\|w_2\|^2}(v_2+w_2)\big]. \tag{4.74}$$

On the other direction, $\eta^E_{(g,h)}(G\times H,V\oplus W)(v\wedge w)$ is the basepoint if $\|v_2+w_2\|\geqslant 1$. Note that since v_2 and w_2 are orthogonal to each other, $\|v_2+w_2\|^2=\|v_2\|^2+\|w_2\|^2$.

If $\|v_2+w_2\|\leqslant 1$, it is

$$[(1-\sqrt{\|v_2\|^2+\|w_2\|^2})\eta_g(G,V)(v_1)\wedge\eta_h(H,W)(w_1)+\sqrt{\|v_2\|^2+\|w_2\|^2}(v_2+w_2)], \tag{4.75}$$

which is equal to the term in (4.74) by Proposition 4.5.16 (ii).

(ii) Associativity.

Let $x=[t_1a_1+t_2b_1]$ be a point in $E_g(G,V)$, $y=[s_1a_2+s_2b_2]$ a point in $E_h(H,W)$, and $z=[r_1a_3+r_2b_3]$ a point in $E_k(K,U)$.

$$\mu^E_{((g,h),k)}((G\times H,V\oplus W),(K,U))\circ(\mu^E_{(g,h)}((G,V),(H,W))\wedge Id)(x\wedge y\wedge z)$$

is the basepoint if $t_2^2+s_2^2+r_2^2\geqslant 1$.

If $t_2^2+s_2^2+r_2^2\leqslant 1$,

$$\begin{aligned}
&\mu^E_{((g,h),k)}((G\times H,V\oplus W),(K,U))\circ(\mu^E_{(g,h)}((G,V),(H,W))\wedge Id)(x\wedge y\wedge z)\\
=&\mu^E_{((g,h),k)}((G\times H,V\oplus W),(K,U))\\
&([(1-\sqrt{t_2^2+s_2^2})\mu^F_{g,h}((G,V),(H,W))(a_1\wedge a_2)+\sqrt{t_2^2+s_2^2}(b_1+b_2)]\wedge z)\\
=&[(1-\sqrt{t_2^2+s_2^2+r_2^2})\mu^F_{((g,h),k)}((G\times H,V\oplus W),(K,U))(\mu^F_{g,h}((G,V),(H,W))(a_1\wedge a_2)\wedge a_3)\\
&+\sqrt{t_2^2+s_2^2+r_2^2}(b_1+b_2+b_3)]
\end{aligned}$$

Note that

$$(\sqrt{t_2^2+s_2^2})^2+u_2^2=(\sqrt{t_2^2+s_2^2+u_2^2})^2$$

Then let's consider the other direction.

$$\mu^E_{(g,(h,k))}((G,V),(H\times K,W\oplus U))\circ(Id\wedge\mu^E_{(h,k)}(H\times K,W\oplus U))(x\wedge y\wedge z)$$

is the basepoint if $t_2^2+s_2^2+r_2^2\geqslant 1$.

If $t_2^2+s_2^2+r_2^2\leqslant 1$,

$$\begin{aligned}
&\mu^E_{(g,(h,k))}((G,V),(H\times K,W\oplus U))\circ(Id\wedge\mu^E_{(h,k)}(H\times K,W\oplus U))(x\wedge y\wedge z)\\
=&\mu^E_{(g,(h,k))}((G,V),(H\times K,W\oplus U))\\
&(x\wedge[(1-\sqrt{r_2^2+s_2^2})\mu^F_{(h,k)}((H,W),(K,U))(a_2\wedge a_3)+\sqrt{r_2^2+s_2^2}(b_2+b_3)])\\
=&[(1-\sqrt{t_2^2+s_2^2+r_2^2})\mu^F_{(g,(h,k))}((G,V),(H\times K,W\oplus U))(a_1\wedge\mu^F_{(h,k)}((H,W),(K,U))(a_2\wedge a_3))\\
&+\sqrt{t_2^2+s_2^2+r_2^2}(b_1+b_2+b_3)],
\end{aligned}$$

which is equal to

$$\begin{aligned}
&[(1-\sqrt{t_2^2+s_2^2+r_2^2})\mu^F_{((g,h),k)}((G\times H,V\oplus W),(K,U))(\mu^F_{(g,h)}((G,V),(H,W))(a_1\wedge a_2)\wedge a_3)\\
&+\sqrt{t_2^2+s_2^2+r_2^2}(b_1+b_2+b_3)]
\end{aligned}$$

by Proposition 4.5.16 (ii).

(iii) Centrality of unit.

Let $v\in S^V$ and $x=[t_1a+t_2b]$ a point in $E_h(H,W)$.

$$E_{(g,h)}(\tau) \circ \mu^E_{(g,h)}((G,V),(H,W)) \circ (\eta^E_g(G,V) \wedge Id)(v \wedge x)$$

is the base point if $\|v_2\|^2 + t_2^2 \geqslant 1$. If $\|v_2\|^2 + t_2^2 \leqslant 1$, it's

$$\begin{aligned}
&[(1-\sqrt{\|v_2\|^2+t_2^2})\mu^F_{(g,h)}((G,V),(H,W))(\eta_g(G,V)(v_1)\wedge a)+\sqrt{\|v_2\|^2+t_2^2}(v_2+b)]\\
=&[(1-\sqrt{\|v_2\|^2+t_2^2})\mu^F_{(h,g)}((H,W),(G,V))(a\wedge\eta_g(G,V)(v_1))+\sqrt{\|v_2\|^2+t_2^2}(v_2+b)],
\end{aligned}$$

by Proposition 4.5.16 (ii).

$$\mu^E_{(h,g)}((H,W),(G,V)) \circ (Id \wedge \eta^E_h(H,W)) \circ \tau(v \wedge x)$$

is the base point if $\|v_2\|^2 + t_2^2 \geqslant 1$. If $\|v_2\|^2 + t_2^2 \leqslant 1$, it's

$$[(1-\sqrt{\|v_2\|^2+t_2^2})\mu^F_{(h,g)}((H,W),(G,V))(a\wedge\eta_g(G,V)(v_1))+\sqrt{\|v_2\|^2+t_2^2}(v_2+b)].$$

So the centrality of unit diagram commutes.

(iv) According to the formula of $\mu^E_{(g,h)}((G,V),(H,W))$ and Proposition 4.5.16 (iv), $\mu^E_{(g,h)}((G,V),(H,W))(x\wedge y) = [(1-\sqrt{t_2^2+s_2^2})\mu^F_{g,h}((G,V),(H,W))(a_1\wedge a_2)+\sqrt{t_2^2+s_2^2}(b_1+b_2)] = [(1-\sqrt{t_2^2+s_2^2})\mu^F_{h,g}((H,W),(G,V))(a_2\wedge a_1)+\sqrt{t_2^2+s_2^2}(b_2+b_1)] = \mu^E_{(h,g)}((H,W),(G,V))(y\wedge x)$.

□

Theorem 4.5.27. *Let $\Delta_G : G \longrightarrow G\times G$ be the diagonal map $g \mapsto (g,g)$. For $G-$representations V and W, let*

$$(\Delta_G)^*_{V\oplus W} : E(G\times G, V\oplus W) \longrightarrow E(G, V\oplus W)$$

denote the restriction map defined by the formula (4.93). Then

$$E : \mathcal{I}_G \longrightarrow G\mathcal{T}$$

together with the unit map η^E defined in (4.66) and the multiplication $\Delta^*_G\circ\mu^E((G,-),(G,-))$ gives a commutative $\mathcal{I}_G-FSP$ that weakly represents $QEll^*_G(-)$.

Proof. Let G, H, K be compact Lie groups, V an orthogonal $G-$representation , W an orthogonal $H-$representation and U an orthogonal $K-$representation.

Let

$$X = \prod_{g\in G^{tors}_{conj}} \alpha_g \in E(G,V);\ Y = \prod_{h\in H^{tors}_{conj}} \beta_h \in E(H,W);\ Z = \prod_{k\in K^{tors}_{conj}} \gamma_k \in E(K,U).$$

First let's check the diagram of unity commutes.

Let $v \in S^V$ and $w \in S^W$. $\mu^E((G,V),(H,W)) \circ (\eta^E(G,V) \wedge \eta^E(H,W))(v \wedge w)$ is

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}} \Big((g',h') \mapsto \mu^E_{(g,h)}((G,V),(H,W)) \circ (\eta^E_g(G,V) \wedge \eta^E_h(H,W))(g'\cdot v \wedge h'\cdot w)\Big). \tag{4.76}$$

$\eta^E(G\times H, V\oplus W)(v\wedge w)$ is

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}} \Big((g',h') \mapsto \eta^E_{(g,h)}(G\times H, V\oplus W)(g'\cdot v \wedge h'\cdot w)\Big), \tag{4.77}$$

which is equal to the term (4.76) by Lemma 4.5.26.

Next let's check the diagram of associativity commutes.

$$\mu^E((G\times H, V\oplus W),(K,U)) \circ (\mu^E((G,V),(H,W)) \wedge Id)(X \wedge Y \wedge Z)$$

is

$$\prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj},k\in G^{tors}_{conj}} \Big((g',h',k') \mapsto$$
$$\mu^E_{((g,h),k)}((G\times H, V\oplus W),(K,U)) \circ (\mu^E_{(g,h)}((G,V),(H,W)) \wedge Id)(\alpha_g(g') \wedge \beta_h(h') \wedge \gamma_k(k'))\Big)$$

And

$$\mu^E((G,V),(H\times K, W\oplus U)) \circ (Id \wedge \mu^E(H\times K, W\oplus U))(X\wedge Y\wedge Z)$$

is

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}, k\in G^{tors}_{conj}} \Big((g',h',k')\mapsto$$

$$\mu^E_{(g,(h,k))}((G,V),(H\times K, W\oplus U))\circ (Id\wedge \mu^E_{(h,k)}(H\times K, W\oplus U))(\alpha_g(g')\wedge\beta_h(h')\wedge\gamma_k(k'))\Big)$$

By Lemma 4.5.26, the two terms are equal.

Then let's check the diagram of centrality of unit commutes.

$$E(\tau)\circ\mu^E((G,V),(H,W))\circ(\eta^E(G,V)\wedge Id)(v\wedge X)$$

is

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \Big((h',g')\mapsto E_{(g,h)}(\tau)\circ\mu^E_{(g,h)}((G,V),(H,W))\circ(\eta^E_g(G,V)\wedge Id)((g'\cdot v)\wedge\beta_h(h')\Big)$$

$$\mu^E((H,W),(G,V))\circ(Id\wedge\eta^E(H,W))\circ\tau(v\wedge X)$$

is

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \Big((h',g')\mapsto \mu^E_{(h,g)}((H,W),(G,V))\circ(Id\wedge\eta^E_h(H,W))\circ\tau((g'\cdot v)\wedge\beta_h(h')\Big)$$

The two terms are equal. So the centrality of unit diagram commutes.

Moreover, let's check

$$\mu^E((G,V),(H,W))(X\wedge Y)=\mu^E((H,W),(G,V))(Y\wedge X). \tag{4.78}$$

By Lemma 4.5.26,

$$\begin{aligned}\mu^E((G,V),(H,W))(X\wedge Y) &= \prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}} \Big((g',h')\mapsto \mu^E_{(g,h)}((G,V),(H,W))(\alpha_g(g')\wedge\beta_h(h'))\Big)\\ &= \prod_{g\in G^{tors}_{conj},h\in H^{tors}_{conj}} \Big((h',g')\mapsto \mu^E_{(h,g)}((H,W),(G,V))(\beta_h(h')\wedge\alpha_g(g'))\Big)\\ &= \mu^E((H,W),(G,V))(Y\wedge X).\end{aligned}$$

Therefore we have the commutativity of E.

□

Remark 4.5.28. *For each $g \in G^{tors}_{conj}$, consider*

$$E(G,V)_g := Map_{C_G(g)}(G, E_g(G,V)).$$

It has the relation with $E(G,V)$

$$E(G,V) = \prod_{g\in G^{tors}_{conj}} E(G,V)_g.$$

Moreover, we can define a unit map $\eta^{E'}_g(G,V)$ and a multiplication map $\mu^{E'}_{(g,h)}((G,V),(H,W))$. Let

$$\eta^{E'}_g(G,V): S^V \longrightarrow E(G,V)_g$$

be the map defined by

$$v \mapsto (\alpha \mapsto \eta^E_g(G,V)(\alpha\cdot v)). \tag{4.79}$$

Let $\mu^{E'}_{(g,h)}((G,V),(H,W)) : E(G,V)_g \wedge E(H,W)_h \longrightarrow E(G\times H, V\oplus W)_{(g,h)}$ denote the map sending

$$\alpha_g \wedge \beta_h$$

to

$$(g',h') \mapsto \mu^E_{(g,h)}((G,V),(H,W))\big(\alpha_g(g')\wedge\beta_h(h')\big). \tag{4.80}$$

We have the relations

$$\eta^E(G,V) = \prod_{g\in G^{tors}_{conj}} \eta^{E'}_g(G,V), \quad \mu^E((G,V),(H,W)) = \prod_{g\in G^{tors}_{conj}} \mu^{E'}_g((G,V),(H,W)).$$

The proof of Theorem 4.5.27 also shows that $E(G,-)_g$ is a $\mathcal{I}_G-$FSP with the unit map $\eta_g^{E'}(G,-)$ and the multiplication $\Delta_G^* \circ \mu_{(g,g)}^{E'}((G,-),(G,-))$.

Remark 4.5.29. *Definition 4.4.11 gives a way how an orthogonal spectrum gives orthogonal $G-$spectra. Then we may ask under what condition a given orthogonal $G-$spectrum is part of a global family, i.e. arise from an orthogonal spectrum in the way indicated in Definition 4.4.11. There is a criterion. The two conditions below are equivalent:*

(a) The $G-$spectrum Y is isomorphic to an orthogonal $G-$spectrum of the form $X\langle G\rangle$ for some orthogonal spectrum X;

(b) for every trivial $G-$representation V the $G-$action on $Y(V)$ is trivial.

It's straightforward to check E doesn't satisfy the condition (b), so it cannot arise from an orthogonal spectrum.

Proposition 4.5.30. *Let G be any compact Lie group. Let V be an ample orthogonal $G-$representation and W an orthogonal $G-$representation.*

Let $\sigma^E_{G,V,W} : S^W \wedge E(G,V) \longrightarrow E(G, V\oplus W)$ denote the structure map of E defined by the unit map $\eta^E(G,V)$. Let $\widetilde{\sigma}^E_{G,V,W}$ denote the right adjoint of $\sigma^E_{G,V,W}$. Then

$$\widetilde{\sigma}^E_{G,V,W} : E(G,V) \longrightarrow Map(S^W, E(G,V\oplus W))$$

is a $G-$weak equivalence.

Proof. From the formula of $\eta^E(G,V)$, we can get an explicit formula for

$$\widetilde{\sigma}^E_{G,V,W} : E(G,V) \longrightarrow \mathrm{Map}(S^W, E(G,V\oplus W)).$$

For any element

$$\alpha := \prod_{g\in G^{tors}_{conj}} \alpha_g$$

in

$$E(G,V) = \prod_{g\in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, E_g(G,V)).$$

Let w be an element in S^W. For each $g \in G^{tors}_{conj}$, w has a unique decomposition

$$w = w^1_g \wedge w^2_g$$

with $w_g^1 \in S^{W^g}$ and $w_g^2 \in S^{(W^g)^\perp}$. $\widetilde{\sigma}^E_{G,V,W}$ sends α to

$$w \mapsto \Big(\prod_{g \in G^{tors}_{conj}} g' \mapsto \Delta_G^* \circ \mu^E_{(g,g)}((G,V),(G,W))(\alpha_g(g'), \eta^E_g(G,W)(g' \cdot w)) \Big).$$

It suffices to show that for each $g \in G^{tors}_{conj}$, the map

$$\widetilde{\sigma}^E_{G,g,V,W} : E_g(G,V) \longrightarrow \mathrm{Map}_{C_G(g)}(S^W, E_g(G, V \oplus W)) \tag{4.81}$$

$$x \mapsto \Big(w \mapsto \Delta_G^* \circ \mu^E_{(g,g)}((G,V),(G,W))(x, \eta^E_g(G,W)(w)) \Big) \tag{4.82}$$

is a $C_G(g)-$weak equivalence.

I check for each closed subgroup H of $C_G(g)$, the map $(\widetilde{\sigma}^E_{G,g,V,W})^H$ on the fixed point space is a homotopy equivalence.

Case I: $g \in H$.

$E_g(G,V)^H$ is the space $F_g(G,V)^H$. By Proposition 4.5.16,

$$\widetilde{\sigma}_g(G,V,W)^H : F_g(G,V)^H \longrightarrow \mathrm{Map}_H(S^{W^g}, F_g(G, V \oplus W))$$

is a weak equivalence.

By Theorem 4.3.7,

$$\mathrm{Map}_H(S^W, E_g(G, V \oplus W)) \longrightarrow \mathrm{Map}_H(S^{W^g}, F_g(G, V \oplus W)), \ f \mapsto f|_{S^{W^g}}$$

is a homotopy equivalence.

And we have the diagram below whose commutativity can be checked directly by applying the formula (4.82).

$$\begin{tikzcd} F_g(G,V)^H \arrow[r, "\simeq"] \arrow[dr] & \mathrm{Map}_H(S^{W^g}, F_g(G, V \oplus W)) \\ & \mathrm{Map}_H(S^W, E_g(G, V \oplus W)) \arrow[u, "\simeq"'] \end{tikzcd} \tag{4.83}$$

So

$$\widetilde{\sigma}^E_{G,g,V,W} : F_g(G,V)^H \longrightarrow \mathrm{Map}_H(S^{W^g}, F_g(G, V \oplus W))$$

is a homotopy equivalence.

Case II: g is not in H.

In this case, $E_g(G,V)^H$ is contractible. It suffices to show that $\mathrm{Map}_H(S^W, E_g(G, V \oplus W)$ is also contractible.

Note that for any closed subgroup H' of H, $E_g(G, V \oplus W)^{H'}$ is contractible. So for each $n-$cell $H/H' \times D^n$ of S^W, it's mapped to $E_g(G, V \oplus W)^{H'}$ unique up to homotopy.

So $\mathrm{Map}_H(S^W, E_g(G, V \oplus W)$ is contractible.

Therefore $\widetilde{\sigma}^E_{G,g,V,W}$ is a $C_G(g)-$weak equivalence. So $\widetilde{\sigma}^E_{G,V,W}$ is a $G-$weak equivalence.

□

By Proposition 4.5.18 and Proposition 4.5.30 we can get the conclusion below.

Corollary 4.5.31. *For any compact Lie group G, any faithful $G-$representation V, $E(G,V)$ represents $QEll^V_G(-)$ weakly in the sense*

$$\pi_k(E(G,V)) = QEll^V_G(S^k). \tag{4.84}$$

4.5.5 The Restriction map

Let G be a compact Lie group. For any complex inner product space V, $KU(V)$ has an $O(V)-$action inheriting from that on V. Any real representation $\rho : G \longrightarrow O(V)$ defines a $G-$action on $E(G,V)$. In this section I construct the restriction maps $E(G,V) \longrightarrow E(H,V)$ for group homomorphisms $H \longrightarrow G$.

Let $\phi : H \longrightarrow G$ be a group homomorphism and V a $G-$representation. From the change of group isomorphism, for any homomorphism of compact Lie groups $\phi : H \longrightarrow G$ and $H-$space X, we have

$$QEll^*_G(G \times_H X)$$

isomorphic to

$$QEll^*_H(X).$$

Thus, for any subgroup K of H, we have the isomorphism

$$QEll^n_G(G/K) = QEll^n_G(G \times_H H/K) \cong QEll^n_H(H/K).$$

So by Proposition 4.5.18 the space $E(G,V)^K$ is homotopy equivalent to $E(H,V)^K$ when V is a faithful $G-$representation. It implies when we consider $E(G,V)$ as an $H-$space, it is $H-$weak equivalent to $E(H,V)$.

As indicated in Remark 4.5.29, the orthogonal $G-$spectrum $E(G,-)$ cannot arise from an orthogonal spectrum. As a result, the restriction map $E(G,V) \longrightarrow E(H,V)$ cannot be a homeomorphism. We construct in this section a restriction map ϕ_V^* that is $H-$weak equivalence such that the diagram below commutes.

$$\begin{array}{ccc} \pi_k(E(G,V)) & \xrightarrow{\cong} & QEll_G^V(S^k) \\ \Big\downarrow{\scriptstyle \pi^k(\phi_V^*)} & & \Big\downarrow{\scriptstyle \phi^*} \\ \pi_k(E(H,V)) & \xrightarrow{\cong} & QEll_H^V(S^k) \end{array} \tag{4.85}$$

where ϕ^* is the restriction map of quasi-elliptic cohomology.

Now let's start the construction of the restriction map ϕ_V^*. Let X be a $G-$space. Let $g \in G^{tors}$ and $h \in H^{tors}$.

The group homomorphism $\phi : H \longrightarrow G$ sends $C_H(h)$ to $C_G(g)$ and also gives

$$\phi_* : \Lambda_H(h) \longrightarrow \Lambda_G(\phi(h)),\ [h', t] \mapsto [\phi(h'), t].$$

ϕ induces an $H-$action on X. Especially, $X^g = X^h$ and ϕ_* induces a $\Lambda_H(h)-$action on it for each $h \in H^{tors}$.

We can define a homotopy equivalence

$$P_g(G,V) : \mathrm{Map}_G(X, \mathrm{Map}_{C_G(g)}(G, E_g(G,V))) \longrightarrow \mathrm{Map}_{C_G(g)}(X^g, F_g(G,V))$$

similar to that defined in (4.36), as shown below.

Let

$$\widetilde{f} : X \longrightarrow \mathrm{Map}_{C_G(g)}(G, E_g(G,V))$$

be a $G-$map. $P_g(G,V)(\widetilde{f})$ is defined as the composition

$$X^g \xrightarrow{f^g} \mathrm{Map}_{C_G(g)}(G, E_g(G,V)) \xrightarrow{\alpha \mapsto \alpha(e)} F_g(G,V) \tag{4.86}$$

Let's first consider the equivalent definition of quasi-elliptic cohomology

$$QEll^*_G(X) = \prod_{g\in G^{tors}} K^*_{\Lambda_G(g)}(X^g)$$

with the product over all the torsion elements of G. With this definition, the restriction map can have a relatively simple form.

For each $g \in G^{tors}$, we first define the map

$$Res_{\phi,g} : \mathrm{Map}_{C_G(g)}(G, E_g(G,V)) \longrightarrow \prod_{\tau} \mathrm{Map}_{C_H(\tau)}(H, E_\tau(H,V))$$

in the form

$$\prod_{\tau}\Big(R_{\phi,\tau} : \mathrm{Map}_{C_G(g)}(G, E_g(G,V)) \longrightarrow \mathrm{Map}_{C_H(\tau)}(H, E_\tau(H,V))\Big)$$

where τ goes over all the elements τ in H^{tors} such that $\phi(\tau) = g$. Then we will combine all the $Res_{\phi,g}$s to define the restriction map ϕ^*_V.

The restriction map

$$\phi^*_V : E(G,V) \longrightarrow E(H,V)$$

to be defined should make the diagram (4.87) commute, which implies that (4.85) commutes.

$$\begin{array}{ccccccc} X^g & \longrightarrow & X & \xrightarrow{\widetilde{f}} & \mathrm{Map}_{C_G(g)}(G, E_g(G,V)) & \xrightarrow{\alpha\mapsto\alpha(e)} & F_g(G,V) \\ \Big\downarrow{=} & & \Big\downarrow{=} & & \Big\downarrow{R_{\phi,\tau}} & & \Big\downarrow{res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}} \\ X^\tau & \longrightarrow & X & \xrightarrow{R_{\phi,\tau}\circ\widetilde{f}} & \mathrm{Map}_{C_H(\tau)}(H, E_\tau(H,V)) & \xrightarrow{\beta\mapsto\beta(e)} & F_\tau(H,V) \end{array} \tag{4.87}$$

where $res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}$ is the restriction map defined in (4.88).

Let $\tau \in H^{tors}$ and $g = \phi(h)$. Then we have the isomorphism

$$a_\tau : (V)^{\mathbb{R}}_g \oplus V^g \longrightarrow (V)^{\mathbb{R}}_\tau \oplus V^\tau$$

sending v to v. For any $[b,t] \in \Lambda_H(h)$, $a_\tau([\phi(b),t]v) = [b,t]a_\tau(v)$.

In addition, we have the restriction map $res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} : F_g(G,V) \longrightarrow F_\tau(H,V)$ defined as below. Let $\beta : S^{(V)^{\mathbb{R}}_g} \longrightarrow KU((V)^{\mathbb{R}}_g \oplus V^g)$ be a $\mathbb{R}-$equivariant map. Note that $S^{(V)^{\mathbb{R}}_\tau}$ and $S^{(V)^{\mathbb{R}}_g}$ have the same underlying space, and $(V)^{\mathbb{R}}_g \oplus V^g$ and $(V)^{\mathbb{R}}_\tau \oplus V^\tau$ have the same underlying vector space.

$res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}(\beta)$ is defined to be the composition

$$S^{(V)^{\mathbb{R}}_\tau} \xrightarrow{x\mapsto x} S^{(V)^{\mathbb{R}}_g} \xrightarrow{\beta} KU((V)^{\mathbb{R}}_g \oplus V^g) \xrightarrow{KU(a_\tau)} KU((V)^{\mathbb{R}}_\tau \oplus V^\tau) \tag{4.88}$$

which is the identity map on the underlying spaces.

Let $\psi : K \longrightarrow H$ be another group homomorphism and $\psi(k) = h$ for some $k \in K$. Then we have

$$res|^{\Lambda_H(h)}_{\Lambda_K(k)} \circ res|^{\Lambda_G(g)}_{\Lambda_H(h)} = res|^{\Lambda_G(g)}_{\Lambda_K(k)} \tag{4.89}$$

Note $S(G,V)_g$ has the same underlying space as $S(H,V)_\tau$. Consider the join of maps

$$res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * Id : F_g(G,V) * S(G,V)_g \longrightarrow F_\tau(H,V) * S(H,V)_\tau \tag{4.90}$$

It is the identity map on the underlying space and has the equivariant property: for any $a \in C_H(\tau)$, $x \in H$,

$$res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * Id(\phi(a)\cdot x) = a \cdot res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * Id(x). \tag{4.91}$$

$res|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * b_\tau$ gives a well-defined map on the quotient space $E_g(G,V) \longrightarrow E_\tau(H,V)$. Let's use $r_{\phi,\tau}$ to denote this map. It also has the equivariant property as (4.91).

For any ρ in $\mathrm{Map}_{C_G(g)}(G, E_g(G,V))$, let $R_{\phi,\tau}(\rho)$ be the composition

$$H \xrightarrow{\phi} G \xrightarrow{\rho} E_g(G,V) \xrightarrow{r_{\phi,\tau}} E_\tau(H,V). \tag{4.92}$$

$R_{\phi,\tau}(\rho)$ is $C_H(\tau)-$equivariant:

$$\begin{aligned} R_{\phi,\tau}(\rho)(ah) &= r_{\phi,\tau}(\rho(\phi(ah))) = r_{\phi,\tau}(\rho(\phi(a)\phi(h))) \\ &= ar_{\phi,\tau}(\rho(\phi(h))) = a\cdot R_{\phi,\tau}(\rho)(h), \end{aligned}$$

for any $a \in C_H(\tau)$, $h \in H$.

For any $g \in Im\phi$, $Res_{\phi,g}$ is defined to be

$$\prod_\tau R_{\phi,\tau}$$

where τ goes over all the $\tau \in H^{tors}$ such that $\phi(\tau) = g$. The restriction map is defined to be

$$\phi_V^* := \prod_g Res_g : E(G, V) \longrightarrow E(H, V) \tag{4.93}$$

where g goes over all the elements in G^{tors} in the image of ϕ.

Lemma 4.5.32. *(i) R_τ defined in (4.92) is the restriction map making the diagram*

$$\begin{tikzcd} Map_{C_G(g)}(G, E_g(G,V)) \arrow[r, "\alpha\mapsto\alpha(e)"] \arrow[d, "R_{\phi,\tau}"'] & F_g(G,V) \arrow[d, "res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}"] \\ Map_{C_H(\tau)}(H, E_\tau(H,V)) \arrow[r, "\beta\mapsto\beta(e)"] & F_\tau(H,V) \end{tikzcd} \tag{4.94}$$

commute. So the restriction map ϕ_V^* makes the diagram (4.87) commute.

(ii)Let $\phi : H \longrightarrow G$ and $\psi : K \longrightarrow H$ be two group homomorphism and V a $G-$representation. Then

$$\psi_V^* \circ \phi_V^* = (\phi \circ \psi)_V^*.$$

The composition is associative.

(iii) $Id_V^* : E(G, V) \longrightarrow E(G, V)$ is the identity map.

Proof. It's straightforward to check by the formula (4.92) and that the restriction map of the global K-theory is associative.

(i) $R_{\phi,\tau}(\alpha)(e) = r_{\phi,\tau} \circ \alpha(e) = res|^{\Lambda_G(g)}_{\Lambda_H(\tau)}\alpha(e)$. So the diagram (4.94) commutes, which implies (4.87) commutes.

(ii) Let $\rho_g : G \longrightarrow E_g(G, V)$ be a $C_G(g)-$equivariant map for each $g \in G^{tors}$. Note that if we have $\psi(\sigma) = \tau$ and $\phi(\tau) = g$, then $r_{\phi,\tau} \circ r_{\psi,\sigma} = r_{\phi\circ\psi,\sigma}$ since both sides are identity maps on the underlying spaces.

Then we have for any $k \in K$,

$$\begin{aligned}\psi_V^* \circ \phi_V^*(\prod_{g\in G^{tors}} \rho_g) &= \prod_g \prod_\tau \prod_\sigma r_{\psi,\sigma} \circ r_{\phi,\tau}\rho_g(\phi(\psi(k))) \\ &= \prod_g \prod_\tau \prod_\sigma r_{\psi\circ\phi,\sigma}\rho_g(\phi \circ \psi(k)) = (\phi \circ \psi)_V^*(\prod_{g\in G^{tors}} \rho_g)\end{aligned}$$

where τ goes over all the elements in H^{tors} with $\phi(\tau) = g$ and σ goes over all the elements in K^{tors} with $\psi(\sigma) = \tau$. So

$$\psi_V^* \circ \phi_V^* = (\phi \circ \psi)_V^*.$$

(iii) For the identity map $Id : G \longrightarrow G$, by the formula of the restriction map, $Id_V^*(\prod_{g\in G^{tors}} \rho_g) = \prod_{g\in G^{tors}} \rho_g$, thus, is the identity. $\square$

We also show the construction of the restriction map for the equivalent definition of quasi-elliptic cohomology

$$QEll_G^*(X) = \prod_{g\in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g),$$

which is unique up to $G-$homeomorphism.

For each $g \in G^{tors}_{conj}$, we first define the map

$$Resc_{\phi,g} : \mathrm{Map}_{C_G(g)}(G, E_g(G,V)) \longrightarrow \prod_\tau \mathrm{Map}_{C_H(\tau)}(H, E_\tau(H,V))$$

in the form

$$\prod_\tau \Big(Rc_{\phi,\tau} : \mathrm{Map}_{C_G(g)}(G, E_g(G,V)) \longrightarrow \mathrm{Map}_{C_H(\tau)}(H, E_\tau(H,V)) \Big)$$

where τ goes over all the elements τ in H^{tors}_{conj} such that $\phi(\tau)$ is conjugate to g in G. Then we combine all the $Resc_{\phi,g}$s to define the restriction map $\widetilde{\phi}^*_V$.

Let $\tau \in H^{tors}_{conj}$ and $g = g_\tau^{-1}\phi(h)g_\tau$ for some $g_\tau \in G$. Then we have the isomorphism

$$ac_{\phi,\tau} : (V)^{\mathbb{R}}_g \oplus V^g \longrightarrow (V)^{\mathbb{R}}_\tau \oplus V^\tau$$

sending v to $g_\tau \cdot v$. For any $b \in C_H(h)$, $ac_{\phi,\tau}(\phi(b)v) = (g_\tau b g_\tau^{-1}) ac_{\phi,\tau}(v)$.

In addition, we have the restriction map $res_c|^{\Lambda_G(g)}_{\Lambda_H(\tau)} : F_g(G,V) \longrightarrow F_\tau(H,V)$ defined as below. Let $\beta : S^{(V)^{\mathbb{R}}_g} \longrightarrow KU((V)^{\mathbb{R}}_g \oplus V^g)$ be a $\mathbb{R}-$equivariant map. Note that $S^{(V)^{\mathbb{R}}_\tau}$ and $S^{(V)^{\mathbb{R}}_g}$ have the same underlying space, and $(V)^{\mathbb{R}}_g \oplus V^g$ and $(V)^{\mathbb{R}}_\tau \oplus V^\tau$ have the same underlying vector space.

$res_c|^{\Lambda_G(g)}_{\Lambda_H(\tau)}(\beta)$ is defined to be the composition

$$S^{(V)^{\mathbb{R}}_\tau} \xrightarrow{x \mapsto g_\tau^{-1}x} S^{(V)^{\mathbb{R}}_g} \xrightarrow{\beta} KU((V)^{\mathbb{R}}_g \oplus V^g) \xrightarrow{KU(ac_{\phi,\tau})} KU((V)^{\mathbb{R}}_\tau \oplus V^\tau) \tag{4.95}$$

Note that different choice of g_τ leads to a difference by a $C_G(g)-$homeomorphism on the domain.

Let $\psi : K \longrightarrow H$ be another group homomorphism and $h_k^{-1}\psi(k)h_k = \tau$ for some $k \in K$ and $h_k \in H$. Then we have

$$g_\tau^{-1}\phi(h_k)^{-1}\phi \circ \psi(k)\phi(h_k)g_\tau = g. \tag{4.96}$$

With this choice of g_τ and h_k fixed, we have

$$\begin{aligned} &res_c|^{\Lambda_H(h)}_{\Lambda_K(k)} \circ res_c|^{\Lambda_G(g)}_{\Lambda_H(h)}(\beta)(x) = KU(ac_{\psi,k})KU(ac_{\phi,\tau})\beta(g_\tau^{-1}\phi(h_k)^{-1}x) \\ =&KU(ac_{\phi\circ\psi,k})\beta((\phi(h_k)g_\tau)^{-1} \cdot x) = res_c|^{\Lambda_G(g)}_{\Lambda_K(k)}(\beta)(x). \end{aligned}$$

Thus,

$$res_c|^{\Lambda_H(h)}_{\Lambda_K(k)} \circ res_c|^{\Lambda_G(g)}_{\Lambda_H(h)} = res_c|^{\Lambda_G(g)}_{\Lambda_K(k)} \tag{4.97}$$

Moreover, we have the map $b_{\phi,\tau} : S(G,V)_g \longrightarrow S(H,W)_\tau$ sending w to $g_\tau w$. Note $S(G,V)_g$ has the same underlying space as $S(H,V)_\tau$. It's straightforward to check

$$b_{\psi,k} \circ b_{\phi,\tau} = b_{\phi\circ\psi,k}. \tag{4.98}$$

Consider the join of maps

$$res_c|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * b_{\phi,\tau} : F_g(G,V) * S(G,V)_g \longrightarrow F_\tau(H,V) * S(H,V)_\tau \tag{4.99}$$

Note that $g_\tau^{-1}C_H(\phi(h))g_\tau$ is a subgroup of $C_G(g)$. It has the equivariant property: for any $a \in C_H(\tau)$, $x \in H$,

$$res_c|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * b_{\phi,\tau}(g_\tau^{-1}\phi(a)g_\tau \cdot x) = a \cdot res_c|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * b_{\phi,\tau}(x). \tag{4.100}$$

$res_c|^{\Lambda_G(g)}_{\Lambda_H(\tau)} * b_{\phi,\tau}$ gives a well-defined map on the quotient space $E_g(G,V) \longrightarrow E_\tau(H,V)$. Let's use $rc_{\phi,\tau}(V)$ to denote this map. It also has the equivariant property as (4.100).

For any ρ in $\mathrm{Map}_{C_G(g)}(G, E_g(G,V))$, let $Rc_{\phi,\tau}(\rho)$ be the composition

$$H \xrightarrow{g_\tau^{-1}\phi(-)g_\tau} G \xrightarrow{\rho} E_g(G,V) \xrightarrow{r_{\phi,\tau}} E_\tau(H,V). \tag{4.101}$$

$Rc_{\phi,\tau}(\rho)$ is $C_H(\tau)-$equivariant:

$$\begin{aligned}
Rc_{\phi,\tau}(\rho)(ah) &= rc_{\phi,\tau}\big(\rho(g_\tau^{-1}\phi(ah)g_\tau)\big) = rc_{\phi,\tau}(\rho((g_\tau^{-1}\phi(a)g_\tau)(g_\tau^{-1}\phi(h)g_\tau)))\\
&= rc_{\phi,\tau}\left(g_\tau^{-1}\phi(a)g_\tau\rho(g_\tau^{-1}\phi(h)g_\tau)\right) = a\cdot rc_{\phi,\tau}(\rho(g_\tau^{-1}\phi(h)g_\tau))\\
&= a\cdot Rc_{\phi,\tau}(\rho)(h),
\end{aligned}$$

for any $a \in C_H(\tau)$, $h \in H$.

For any $g \in Im\phi$, $Resc_g$ is defined to be

$$\prod_\tau Rc_{\phi,\tau}$$

where τ goes over all the $\tau \in H^{tors}_{conj}$ such that $\phi(\tau)$ is conjugate to g in G. The restriction map is defined to be

$$\widetilde{\phi}^*_V := \prod_g Resc_g : E(G,V) \longrightarrow E(H,V) \tag{4.102}$$

where g goes over all the elements in G^{tors}_{conj} that are $G-$conjugate to some element in $\phi(H)$.

Lemma 4.5.33. *(i) Rc_τ defined in (4.92) is the restriction map making the diagram*

$$\begin{tikzcd}
Map_{C_G(g)}(G, E_g(G,V)) \arrow[r, "\alpha\mapsto\alpha(e)"] \arrow[d, "Rc_{\phi,\tau}"'] & F_g(G,V) \arrow[d, "res_c|^{\Lambda_G(g)}_{\Lambda_H(\tau)}"] \\
Map_{C_H(\tau)}(H, E_\tau(H,V)) \arrow[r, "\beta\mapsto\beta(e)"] & F_\tau(H,V)
\end{tikzcd} \tag{4.103}$$

commute. So the restriction map ϕc^*_V makes the diagram (4.104)commute.

$$\begin{tikzcd}
X^g \arrow[r] \arrow[d, "="'] & X \arrow[r, "\widetilde{f}"] \arrow[d, "="'] & Map_{C_G(g)}(G, E_g(G,V)) \arrow[r, "\alpha\mapsto\alpha(e)"] \arrow[d, "Rc_{\phi,\tau}"] & F_g(G,V) \arrow[d, "res_c|^{\Lambda_G(g)}_{\Lambda_H(\tau)}"] \\
X^\tau \arrow[r] & X \arrow[r, "Rc_{\phi,\tau}\circ\widetilde{f}"] & Map_{C_H(\tau)}(H, E_\tau(H,V)) \arrow[r, "\beta\mapsto\beta(e)"] & F_\tau(H,V)
\end{tikzcd} \tag{4.104}$$

So the restriction map ϕc^*_V also makes the diagram (4.85) commute.

(ii)Let $\phi : H \longrightarrow G$ and $\psi : K \longrightarrow H$ be two group homomorphism and V a $G-$representation. Then

$$\widetilde{\psi}^*_V \circ \widetilde{\phi}^*_V = \widetilde{\phi\circ\psi}^*_V.$$

The composition is associative.

(iii) Let $Id : G \longrightarrow G$ be the identity map. Then $\widetilde{Id}_V^* : E(G,V) \longrightarrow E(G,V)$ is the identity map.

Proof. It's straightforward to check by the formula (4.101).

(i) $Rc_{\phi,\tau}(\alpha)(e) = rc_{\phi,\tau}\circ\alpha(e) = res_c|^{\Lambda_G(g)}_{\Lambda_H(\tau)}\alpha(e)$. So the diagram (4.103) commutes, which implies (4.104) commutes.

(ii) Let $\rho_g : G \longrightarrow E_g(G,V)$ be a $C_G(g)-$equivariant map for each $g \in G^{tors}$. Note that if we have $h_\sigma^{-1}\psi(\sigma)h_\sigma = \tau$ and $g_\tau^{-1}\phi(\tau)g_\tau = g$ for some $h_\sigma \in H$ and $g_\tau \in G$, then we have $g_\tau^{-1}\phi(h_\sigma)^{-1}\phi\psi(\sigma)\phi(h_\sigma)g_\tau = g$. Moreover, we have $rc_{\phi,\tau}\circ rc_{\psi,\sigma} = rc_{\phi\circ\psi,\sigma}$ since (4.97) and (4.98) hold.

Then we have for any $k \in K$,

$$\begin{aligned}\widetilde{\psi}_V^*\circ\widetilde{\phi}_V^*(\prod_{g\in G^{tors}_{conj}}\rho_g) &= \prod_g\prod_\tau\prod_\sigma r_{\psi,\sigma}\circ r_{\phi,\tau}\rho_g(g_\tau^{-1}\phi(h_\sigma^{-1}\psi(k)h_\sigma)g_\tau)\\ &= \prod_g\prod_\tau\prod_\sigma r_{\psi\circ\phi,\sigma}\rho_g(g_\tau^{-1}\phi(h_\sigma)^{-1}\phi\psi(k)\phi(h_\sigma)g_\tau) = \widetilde{\phi\circ\psi}_V^*(\prod_{g\in G^{tors}_{conj}}\rho_g)\end{aligned}$$

where τ goes over all the elements in H^{tors}_{conj} with $g_\tau^{-1}\phi(\tau)g_\tau = g$ and σ goes over all the elements in K^{tors} with $h_\sigma^{-1}\psi(\sigma)h_\sigma = \tau$. So

$$\widetilde{\psi}_V^*\circ\widetilde{\phi}_V^* = \widetilde{\phi\circ\psi}_V^*.$$

(iii) For the identity map $Id : G \longrightarrow G$, by the formula of the restriction map, $\widetilde{Id}_V^*(\prod_{g\in G^{tors}_{conj}}\rho_g) = \prod_{g\in G^{tors}_{conj}}\rho_g$, thus, is the identity. ∎

4.6 Global Idea

As indicated in Remark 4.5.29, the orthogonal $G-$spectrum $E(G,-)$ cannot arise from an orthogonal spectrum. In Section 4.6 we express my thinking how to view these orthogonal $G-$spectra globally, on the way to that we introduce a new perspective to view the theory, quasi-elliptic cohomology.

Let's consider two categories first. For each compact Lie group G, we can construct a category I'_G with objects $\sigma \in G^{tors}$ and morphisms $Mor(\sigma,\sigma') = \Lambda_G(\sigma,\sigma')$, which is the quotient of $C_G(\sigma, \sigma') \times \mathbb{R}$ under the action of $\mathbb{Z}$

$$(g, t) \mapsto (g\sigma, t+1) = (\sigma' g, t+1).$$

I'_G is a groupoid. A skeleton $sk(I'_G)$ of I'_G is the category with objects $\sigma \in G^{tors}_{conj}$ with $Mor(\sigma, \sigma) = \Lambda_G(\sigma)$.

Each group homomorphism $\rho : H \longrightarrow G$ gives a functor

$$\rho_* : I'_H \longrightarrow I'_G \tag{4.105}$$

sending $\tau \in H^{tors}$ to $\rho(\tau)$. If ρ is a group isomorphic, ρ_* is an isomorphism of categories.

Then we define I' with objects I'_G for each compact Lie group G, and morphisms $\rho_* : I'_H \longrightarrow I'_G$ given by group homomorphisms ρ. We have well-defined functor from the category of compact Lie groups to I' sending G to I'_G.

I' is a symmetric strict monoidal category. We have the isomorphism of groupoids $I'_G \times_{\mathbb{T}} I'_H \longrightarrow I'_{G\times H}$ sending $(\sigma, \tau) \mapsto (\sigma, \tau)$, and on morphisms $([g, t], [h, t]) \mapsto [(g, h), t]$. The unit object is I'_e, which has one object e and the morphisms $\Lambda_e(e) = \mathbb{T}$, i.e. the circle.

Example 4.6.1. *For each $G-$space X, we have the functor*

$$e(X) : I'_G \longrightarrow G\mathcal{T}$$

sending all the objects σ to X, and a morphism $\alpha = [g, t] \in Mor(\sigma, \sigma')$ to the identity map.

Example 4.6.2. *Let X be a $G-$space. We can define a I'_G-space*

$$L_G(X) : I'_G \longrightarrow \mathcal{T},\ g \mapsto X^g.$$

Let $\alpha = [g, t] \in Mor(\sigma, \sigma')$. $L_G(X)(\alpha) : X^\sigma \longrightarrow X^{\sigma'}$ is defined by $x \mapsto gx$.

For each compact Lie group G, we have the functor

$$L_G : G\mathcal{T} \longrightarrow I'_G\mathcal{T}, X \mapsto L_G(X).$$

A morphism $\phi : X \longrightarrow X'$ in $G\mathcal{T}$ gives the morphism between I'_G-spaces

$$\phi^* = \{\phi_\sigma : X^\sigma \longrightarrow X'^\sigma, x \mapsto \phi(x)\}_\sigma.$$

Let $\rho : H \longrightarrow G$ be any group homomorphism. We have the restriction $\rho^* : G\mathcal{T} \longrightarrow H\mathcal{T}, X \mapsto \rho^* X$ and $\rho^* : I'_G\mathcal{T} \longrightarrow I'_H\mathcal{T},\ F \mapsto F \circ \rho_*$ with ρ_* defined in (4.105). The diagram

$$\begin{tikzcd} L_G(X) \arrow[r, "\phi^*"] \arrow[d, "\rho^*"] & L_G(X') \arrow[d, "\rho^*"] \\ L_H(\rho^* X) \arrow[r, "\phi^*"] & L_H(\rho^* X') \end{tikzcd}$$

commutes.

So for each $G-$space X, we have a natural transformation, the restriction

$$\rho^* : L_G(X) \longrightarrow L_H(\rho^* X).$$

Especially, for each $G-$representation V, we have define the I'_G-space $L_G(S^V)$.

Let $\mathcal{I}_G\mathcal{S}$ denote the category of $\mathcal{I}_G-$spectra, i.e. orthogonal $G-$spectra.

Example 4.6.3. *Let V be a $G-$representation. We can define another I'_G-space*

$$F(G, V) : I'_G \longrightarrow \mathcal{T}$$

sending the object σ to $F_\sigma(G, V)$. For a morphism $\alpha = [g, t] \in Mor(\sigma, \sigma')$, it's sent to

$$F_\sigma(G, V) \longrightarrow F_{\sigma'}(G, V),\ x \mapsto \alpha x$$

where αx is $res|^{\Lambda_G(\sigma)}_{\Lambda_G(\sigma')}(x)$ defined in (4.88), i.e.

$$S^{(V)_{\sigma'}} \xrightarrow{x \mapsto g^{-1}x} S^{(V)_\sigma} \xrightarrow{\beta} KU((V)_\sigma \oplus V^\sigma) \xrightarrow{KU(a_{\sigma'})} KU((V)_{\sigma'} \oplus V^{\sigma'}) \tag{4.106}$$

Let $i_G : sk(I'_G) \longrightarrow I'_G$ be the inclusion of the subcategory, which is an equivalence of categories.

$$Map_{I'_G\mathcal{T}}(L_G(X), F(G, V)) \cong Map_{sk(I'_G)\mathcal{T}}(i_G^* L_G(X), i_G^* F(G, V)) \cong \prod_{\sigma \in G^{tors}_{conj}} Map_{C_G(\sigma)}(X^\sigma, KU_{\Lambda_G(\sigma)}). \tag{4.107}$$

Example 4.6.4. *First let's see for each $G-$representation V, we can define*

$$\mathcal{E}(G,V) : I'_G \longrightarrow \mathcal{T} \tag{4.108}$$

sending an object g to $E(G,V)_g$ which is defined in Remark 4.5.28.

It's straightforward to check $\mathcal{E}(G,V)$ is a well-defined I'_G-space: Let $\alpha = [g,t] \in \Lambda_G(\sigma,\sigma')$ and $f \in E(G,V)_\sigma$. Then $\mathcal{E}(G,V)(\alpha)$ send each $\rho \in E(G,V)_\sigma$ to $R_{\sigma'}(\rho)$ defined in (4.92), i.e.

$$G \xrightarrow{g'\mapsto gg'} G \xrightarrow{\rho} E_\sigma(G,V) \xrightarrow{r_{\sigma'}(V)} E_{\sigma'}(G,V). \tag{4.109}$$

By Lemma 4.5.32 (ii), the composition law holds and for each $Id_\sigma = [e,0] \in \Lambda_G(\sigma,\sigma)$, $\mathcal{E}(G,V)(Id)$ is the identity.

By Theorem 4.3.7, $\prod\limits_{\sigma\in G^{tors}_{conj}} Map_{C_G(\sigma)}(X^\sigma, F(G,V)(\sigma))$ is weakly equivalent to

$$\prod_{\sigma\in G^{tors}_{conj}} Map_G(X, \mathcal{E}(G,V)(\sigma)),$$

which is not

$$Map_{I'_G\mathcal{T}}(e(X), \mathcal{E}(G,V)).$$

But we do have a natural transformation $\mathcal{P}(G,V) : \mathcal{E}(G,V) \longrightarrow F(G,V)$. For each $\sigma \in G^{tors}$, $\mathcal{P}(G,V)(\sigma) : \mathcal{E}(G,V)(\sigma) \longrightarrow F(G,V)(\sigma)$ is defined by

$$Map_{C_G(\sigma)}(G, E_\sigma(G,V)) \xrightarrow{f\mapsto f(e)} F_\sigma(G,V). \tag{4.110}$$

4.7 The category $\mathbb{S}$

Let's consider the full subcategory $\mathbb{S}$ of the homotopy category Ho(Top) consisting of the objects

$$\{S_{G,g}|\ G \text{ is a compact Lie group and } g \in G\}.$$

Let $\mathbb{G}$ be the category with objects $\{(G,g)|G$ is a compact Lie group and $g \in G\}$ and a morphisms $\phi : (H,h) \longrightarrow (G,g)$ a group homomorphism from H to G sending h to g.

$\mathbb{G}$ is a lax symmetric monoidal category with product $(G,g) \times (H,h) = (G \times H, (g,h))$ and unit object $(\{e\}, e)$.

We have a contravariant functor $\rho : \mathbb{G} \longrightarrow \mathbb{S}$ sending an object (G, g) in $\mathbb{G}$ to $S_{G,g}$.

For a morphism $\phi : (H, h) \longrightarrow (G, g)$, the restriction $\phi|_{\langle h \rangle} : \langle h \rangle \longrightarrow \langle g \rangle$ is surjective. For any subgroup K of $\langle g \rangle$, we have isomorphism of groups $\phi_K : \langle h \rangle / \phi^{-1}(K) \longrightarrow \langle g \rangle / K$. If K is a maximal subgroup of $\langle g \rangle$, $\phi^{-1}(K)$ is a maximal subgroup of $\langle h \rangle$.

Then for any $f \in S_{G,g}$, $\rho(\phi)(f)$ is the composition

$$H \xrightarrow{\phi} G \xrightarrow{f} *_K E(\langle g \rangle / K) \xrightarrow{*_K(\phi_K^{-1})_*} *_K E(\langle h \rangle / \phi^{-1}(K)) \xrightarrow{i} *_{K'} E(\langle h \rangle / K')$$

where K goes over all the maximal subgroups of $\langle g \rangle$, K' goes over all the maximal subgroup of $\langle h \rangle$, and the last map i is inclusion.

ρ is not symmetric monoidal. Let's consider the contravariant functor

$$\rho_N : \mathbb{G} \longrightarrow \mathbb{S},\ (G, g) \mapsto S_{N_G(g), g}, \tag{4.111}$$

where $N_G(g)$ denotes the normalizer of g in G.

For each morphism $\phi : (H, h) \longrightarrow (G, g)$, it gives a homomorphism $\phi : (N_H(h), h) \longrightarrow (N_G(g), g)$. For any $f \in S_{G,g}$, $\rho_N(\phi)(f)$ is the composition

$$N_H(h) \xrightarrow{\phi} N_G(g) \xrightarrow{f} *_K E(\langle g \rangle / K) \xrightarrow{*_K(\phi_K^{-1})_*} *_K E(\langle h \rangle / \phi^{-1}(K)) \xrightarrow{i} *_{K'} E(\langle h \rangle / K')\ .$$

I'll show $S_{N_{G \times H}(h,g),(g,h)}$ is $N_G(g) \times N_H(h)-$ weak equivalent to $S_{N_G(g),g} * S_{N_H(h),h}$.

First let's look at the fixed point spaces. Let K be a subgroup of $N_{G \times H}(h, g) = N_G(g) \times N_H(h)$. For any $(a, b) \in K$, $(a, b)\langle (g, h) \rangle (a, b)^{-1}$ is contained in K if and only if $(g, h) \in K$. Thus, by Lemma 4.3.4,

$$\left(S_{N_G(g),g} * S_{N_H(h),h}\right)^K \simeq \emptyset \tag{4.112}$$

if and only if $(g, h) \in K$; and

$$S^K_{N_G(g) \times N_H(h),(g,h)} \simeq \emptyset \tag{4.113}$$

if and only if $(g, h) \in K$. Thus,

$$S_{N_G(g),g} * S_{N_H(h),h} \simeq_{N_G(g) \times N_H(h)} S_{N_G(g) \times N_H(h),(g,h)}. \tag{4.114}$$

Let's see an easy fact. Let u denote the least common multiple of $|h|$ and $|g|$. Then the maximal subgroups of $\langle (h, g) \rangle$ are of the form $\langle (h, g)^{\frac{u}{p}} \rangle$ for some prime p dividing u. And

$\langle h^{\frac{u}{p}} \rangle$ (resp. $\langle g^{\frac{u}{p}} \rangle$) is either a maximal subgroup of $\langle h \rangle$ (resp. $\langle g \rangle$) or the trivial group.

For each prime p dividing u, if the cyclic group $\langle h^{\frac{u}{p}} \rangle$ is not trivial, it is isomorphic to $\langle (h,g)^{\frac{u}{p}} \rangle$

By Milnor's construction [45], for any topological group G, the universal space $EG = colim G * G * \cdots * G$. We have a $G-$weak equivalence

$$\kappa_G : EG * EG \longrightarrow EG$$

as the colimit of the maps

$$(*_1^n G) * (*_1^m G) \longrightarrow *_1^{m+n} G$$

sending

$$(t_1 a_1, \cdots t_n a_n), (t_{n+1} b_1, \cdots t_{n+m} b_{n+m})$$

to

$$(t_1 a_1, t_{n+1} b_1, t_2 a_2, t_{n+2} b_2, \cdots t_n a_n, t_{n+n} b_n, t_{n+n+1} b_{n+1}, \cdots t_{n+m} b_m)$$

if $n \leqslant m$; to

$$(t_1 a_1, t_{n+1} b_1, t_2 a_2, t_{n+2} b_2, \cdots t_m a_m, t_{n+m} b_m, t_{m+1} a_{m+1}, \cdots t_n a_n)$$

if $n \geqslant m$, where each $t_j \geqslant 0$ and $\sum_{j=1}^{n+m} t_j = 1$.

Let $\{p_g\}$ denote all the primes dividing $|g|$ and $\{p_h\}$ denote all the primes dividing $|h|$. Let $A = \{p \in \mathbb{Z} | p$ divides both $|h|$ and $|g|$ and $g^{\frac{u}{p}}$ and $h^{\frac{u}{p}}$ are both nontrivial$\}$. For any $r \in A$, we have isomorphisms

$$\langle g \rangle / \langle g^{\frac{|g|}{r}} \rangle \cong \langle (g,h) \rangle / \langle (g,h)^{\frac{u}{r}} \rangle, \text{ and } \langle h \rangle / \langle h^{\frac{|h|}{r}} \rangle \cong \langle (g,h) \rangle / \langle (g,h)^{\frac{u}{r}} \rangle,$$

thus, homeomorphisms

$$f_r : E\Big(\langle g \rangle / \langle g^{\frac{|g|}{r}} \rangle \Big) \cong E\Big(\langle (g,h) \rangle / \langle (g,h)^{\frac{u}{r}} \rangle \Big), \text{ and } g_r : E\Big(\langle h \rangle / \langle h^{\frac{|h|}{r}} \rangle \Big) \cong E\Big(\langle (g,h) \rangle / \langle (g,h)^{\frac{u}{r}} \rangle \Big).$$

Let κ_r denote the composition

$$E(\langle g \rangle / \langle g^{\frac{|g|}{r}} \rangle) * E(\langle h \rangle / \langle g^{\frac{|h|}{r}} \rangle) \xrightarrow{f_r * g_r} E(\langle (g,h) \rangle / \langle (g,h)^{\frac{u}{r}} \rangle) * E(\langle (g,h) \rangle / \langle (g,h)^{\frac{u}{r}} \rangle) \longrightarrow E(\langle (g,h) \rangle / \langle (g,h)^{\frac{u}{r}} \rangle)$$

where the last map is

$$\kappa_{\langle (g,h)\rangle / \langle (g,h)^{\frac{u}{r}}\rangle}.$$

Let

$$\kappa : \left(*_{p_g}\ E(\langle g\rangle / \langle g^{\frac{|g|}{p_g}}\rangle)\right) * \left(*_{p_h}\ E(\langle h\rangle / \langle h^{\frac{|h|}{p_h}}\rangle)\right) \longrightarrow *_{p|u} E(\langle (g,h)\rangle / \langle (g,h)^{\frac{u}{p}}\rangle)$$

sends

$$*_{p_g \notin A} a_{p_g}\alpha_{p_g} *_{p_h \notin A} b_{p_h}\beta_{p_h} *_{r\in A} d_r(t_{r1}\alpha_{r1}, \beta_{r2} f_{r2})$$

in

$$*_{p_g \notin A} E(\langle g\rangle / \langle g^{\frac{|g|}{p_g}}\rangle) *_{p_h \notin A} E(\langle h\rangle / \langle h^{\frac{|h|}{p_h}}\rangle) *_{r\in A} \big(E(\langle g\rangle / \langle g^{\frac{|g|}{r}}\rangle) * E(\langle h\rangle / \langle h^{\frac{|h|}{r}}\rangle)\big),$$

to

$$*_{p_g \notin A} a_{p_g}\alpha_{p_g} *_{p_h \notin A} b_{p_h}\beta_{p_h} *_{r\in A} d_r \kappa_r(t_{r1}\alpha_{r1}, \beta_{r2} f_{r2}).$$

Then, we define $\Phi : S_{N_G(g),g} * S_{N_H(h),h} \longrightarrow S_{N_G(g)\times N_H(h),(g,h)}$ as the composition $\kappa_* \circ \times$, sending $(t_1 f_1, t_2 f_2)$ to a map

$$(a,b) \mapsto \kappa(t_1 f_1(a), t_2 f_2(b)).$$

Φ is $N_G(g)\times N_H(h)-$weak equivalence.

Chapter 5

Real and real Quasi-elliptic cohomology

In this chapter we construct Real and real quasi-elliptic cohomology.

Based on the definitions in [3], we formulate the definition of Real $\mathcal{G}-$space and Real orbifold vector bundle. From them, we give a reasonable definition of Real quasi-elliptic cohomology. Moreover, we discuss the involution on the spectra.

5.1 Real K-theory

First let's recall Real K-theory and its equivariant version in the sense of [6].

Definition 5.1.1. *A Real space X is a compact space X with an involution $x \mapsto \overline{x}$.*

Definition 5.1.2. *Let X be a Real space. A Real vector bundle E over X is a complex vector bundles E over X with a given involution $E \to \overline{E}$ which takes the fibre E_x antilinearly to $E_{\overline{x}}$, and such that $\overline{p(e)} = p(\overline{e})$, where $e \in E$ and $p : E \to X$ is the projection map.*

$KR(X)$ is defined to be the Grothendieck group of the isomorphism classes of Real vector bundles over X.

Moreover, we can define equivariant space and bundle.

Definition 5.1.3. *A Real Lie group G is a Lie group G with a Lie group involution $g \mapsto \overline{g}$ on it.*

Definition 5.1.4. *Let G be a Real Lie group with involution $g \mapsto \overline{g}$ on it. A Real $G-$space X is a G-space with an involution on it such that $\overline{g \cdot x} = \overline{g} \cdot \overline{x}$.*

Definition 5.1.5. *Let G be a Real Lie group. A Real $G-$vector bundle over a Real $G-$space X is a real vector bundle which is also a $G-$bundle, and is also a Real $G-$space.*

The equivariant version $KR^*_G(X)$ with G a Real Lie group and X a Real $G-$space is defined to be the Grothendieck group of the isomorphism classes of Real $G-$vector bundles over X.

5.2 Basic Constructions and Real quasi-elliptic cohomology

In this section I will explain what Real quasi-elliptic cohomology should be.

We define the notion "involution" for Lie groupoid $\mathcal{G}$ first.

Definition 5.2.1. *Let $\mathcal{G}$ be a Lie groupoid. An involution on $\mathcal{G}$ is a covariant Lie groupoid functor $F : \mathcal{G} \mapsto \mathcal{G}$ such that $F \circ F$ is the identity functor. A Lie groupoid with an involution on it is called a Real Lie groupoid.*

In the rest of this chapter, let G be a Real Lie group and X a Real $G-$space. There are some simple facts about Real Lie groups: the involution of the identity element of G is itself. And for any subgroup $H \leq G$, its image under the involution, denoted by $\overline{H}$, is also a subgroup of G.

Example 5.2.2. *For the subspace X^H of X, the involution on X induced a homeomorphism between $X^H \longrightarrow X^{\overline{H}}$ by*

$$\sigma_H : x \mapsto \overline{x}.$$

This map is well-defined: if we have $hx = x$, $\forall h \in H$ and $\forall x \in X^H$, then $\overline{x} = \overline{hx} = \overline{h}\overline{x}$. And $\sigma_{\overline{H}} \circ \sigma_H$ is the identity map on X^H. Especially, we have

$$\sigma_g : X^g \longrightarrow X^{\overline{g}},\ x \mapsto \overline{x},$$

$\forall g \in G$.

Obviously, the centralizer $C_G(\overline{g})$ is exactly $\overline{C_G(g)}$, and $C_G(\overline{g}, \overline{g}')$ is $\overline{C_G(g, g')}$. And we have the bijection

$$\sigma_{g,g'} : \Lambda_G(g, g') \times X^g \longrightarrow \Lambda_G(\overline{g}, \overline{g}') \times X^{\overline{g}}$$

sending $([h, t], x)$ to $([\overline{h}, -t], \overline{x})$. $\sigma_{\overline{g},\overline{g}'} \circ \sigma_{g,g'}$ is the identity map on $\Lambda_G(g, g')$.

By σ_g and $\sigma_{g,g'}$, we get an involution on the orbifold groupoid $\Lambda(X//G)$. Let's denote the involution by σ_X.

Given a Real Lie groupoid $\mathcal{G}$, we also define Real $\mathcal{G}-$space and Real orbifold vector bundle, i.e. Real vector bundle over $\mathcal{G}$.

Let's recall the definition of $\mathcal{G}-$space first. Definition 5.2.3 is from the book [3].

Definition 5.2.3. *Let $\mathcal{G}$ be an orbifold groupoid. A left $\mathcal{G}-$space is a manifold E equipped with an action by $\mathcal{G}$. Such an action is given by two maps: an anchor $\pi : E \to G_0$, and an action $\mu : G_1 \times_{G_0} E \longrightarrow E$. The latter map is defined on pairs (g, e) with $\pi(e) = s(g)$, and written $\mu(g, e) = g \cdot e$. It satisfies the usual identities for an action: $\pi(g \cdot e) = t(g)$, $1_x \cdot e = e$, and $g \cdot (h \cdot e) = (gh) \cdot e$, for $x \xrightarrow{h} y \xrightarrow{g} z$ in G_1 and $e \in E$ with $\pi(e) = x$.*

I give the definition of Real $\mathcal{G}-$space below.

Definition 5.2.4. *Let $\mathcal{G}$ be an orbifold groupoid. A left Real $\mathcal{G}-$space is a Real space E which is a left $\mathcal{G}-$space such that the anchor π and the action μ preserve the involutions. Explicitly, $\pi(\overline{e}) = \overline{\pi(e)}$ and $\overline{g \cdot e} = \overline{g} \cdot \overline{e}$.*

Let's also recall the definition of $\mathcal{G}-$vector bundle, which is from the book [3].

Definition 5.2.5. *A $\mathcal{G}-$vector bundle over an orbifold groupoid $\mathcal{G}$ is a $\mathcal{G}-$space E for which $\pi : E \to G_0$ is a vector bundle, such that the action of $\mathcal{G}$ on E is fiberwise linear. Namely, any arrow $g : x \to y$ induces a linear isomorphism $g : E_x \to E_y$. In particular, E_x is a linear representation of the isotropy group G_x for each $x \in G_0$.*

And I give the definition for Real $\mathcal{G}-$vector bundle below.

Definition 5.2.6. *A Real $\mathcal{G}-$vector bundle over a Real orbifold groupoid $\mathcal{G}$ is a $\mathcal{G}-$vector bundle over $\mathcal{G}$, a Real vector bundle over G_0, and also a Real $\mathcal{G}-$space, so that the involutions and the $\mathcal{G}-$structure are compatible with each other.*

Now let's consider Real $\Lambda(X//G)-$vector bundles E over the Real groupoid $\Lambda(X//G)$ with the involution σ_X.

For any $g \in G$, consider the subgroupoid $X^g//\Lambda_G(g)$ of $\Lambda(X//G)$. The involution on E maps the restriction of E over $X^g//\Lambda_G(g)$ to a $X^{\overline{g}}//\Lambda_G(\overline{g})-$vector bundle over $X^{\overline{g}}//\Lambda_G(\overline{g})$, each fibre of which over $\overline{x}$ is antilinearly isomorphic to that of E over x.

Thus, the involution of E induces an isomorphism $K_{\Lambda_G(g)}(X^g) \to K_{\Lambda_G(\overline{g})}(X^{\overline{g}})$.

Definition 5.2.7. *$QEllR^*_G(X)$ is defined to be the Grothendieck group of the isomorphism classes of Real $\Lambda(X//G)-$vector bundles over the Real groupoid $\Lambda(X//G)$ with the involution σ_X constructed in Example 5.2.2.*

Below is an interpretation of this map, which gives an involution of $QEllR^*_G(X)$.

First let's recall the definition of universe for $G-$representations.

Definition 5.2.8. *A $G-$universe $\mathcal{U}$ is a countable infinite direct sum $\oplus_{n=1}^{\infty}\mathcal{U}'$ of a $G-$inner product space $\mathcal{U}'$ satisfying the following:*

(1) the one-dimensional trivial $G-$representation is contained in $\mathcal{U}'$;

(2) $\mathcal{U}$ is topologized as the union of all finite dimensional $G-$subspaces of $\mathcal{U}$ (each with the norm topology);

(3) the $G-$action on all finite dimensional $G-$subspaces V of $\mathcal{U}$ factors through a compact Lie group quotient of G.

Definition 5.2.9. *If the $G-$action on $\mathcal{U}$ is trivial, then $\mathcal{U}$ is called a trivial universe. If each finite dimensional $G-$representations is isomorphic to a $G-$subspace of $\mathcal{U}$, then $\mathcal{U}$ is called a complete $G-$universe.*

We have seen that the involution of the Real group G induces an isomorphism δ_g from $\Lambda_G(\overline{g})$ to $\Lambda_G(g)$, which sends $[h,t]$ to $[\overline{h},-t]$. If V is a $\Lambda_G(g)-$representation, then the complex conjugate $\overline{V}$ of V is a representation of $\Lambda_G(\overline{g})$ with the action defined by

$$[\overline{h},t]\cdot\overline{v} := \overline{[h,-t]\cdot v}, \tag{5.1}$$

where $[\overline{h},t]\in\Lambda_G(\overline{g})$, $v\in V$, and $\overline{v}$ is the complex conjugate of v.

So the complex conjugate of each $\Lambda_G(g)-$inner product space is a $\Lambda_G(\overline{g})-$inner product space. And if V is a one-dimensional trivial $\Lambda_G(g)-$representation, $\overline{V}$ is a one-dimensional trivial $\Lambda_G(\overline{g})-$representation. The opposite is also true.

Let $\mathcal{U}_g$ be a complete complex $\Lambda_G(g)-$universe. Let $\overline{\mathcal{U}}_g$ denote the same space as $\mathcal{U}_g$ with the same topology but for each $V\in\mathcal{U}_g$, we consider it as a $\Lambda_G(\overline{g})-$representation with the action defined in (5.1). Then $\overline{\mathcal{U}}_g$ is a $\Lambda_G(\overline{g})-$universe.

If W is a finite dimensional $\Lambda_G(\overline{g})-$representation, its complex conjugate $\overline{W}$ is a finite dimensional $\Lambda_G(g)-$representation, so $\overline{W}$ is isomorphic to a $\Lambda_G(g)-$subspace of $\mathcal{U}_g$, which implies W is isomorphic to a $\Lambda_G(\overline{g})-$subspace of $\overline{\mathcal{U}}_g$. So $\overline{\mathcal{U}}_g$ is a complete $\Lambda_G(\overline{g})-$universe.

Let $BU_G(n,V)$ be the Grassmannian $G-$space of complex $n-$planes in a complex inner product $G-$space V and $EU_G(n,V)$ be the $G-$space consisting of pairs (π,v) where$\pi\in BU_G(n,V)$ and $v\in\pi$. If V is sufficiently large, for example if V contains a complete complex $G-$universe, $BU_G(n,V)$ classifies complex $n-$dimensional $G-$vector bundles, and the projection $p: EU_G(n,V)\longrightarrow BU_G(n,V)$, $(\pi,v)\mapsto\pi$ is a universal complex $n-$plane $G-$bundle.

We can construct a $G-$spectrum representing equivariant K-theory. Let $\mathcal{U}$ be a complete $G-$universe. For any $V \subseteq W \subset \mathcal{U}$ and $q \geqslant 0$, we have an inclusion

$$BU_G(q, V \oplus \mathcal{U}) \longrightarrow BU_G(q + |W - V|, W \oplus \mathcal{U})$$

sending a plane A to $A + (W - V)$. Let $BU_G(V)$ be the space

$$\coprod_{q \geqslant 0} BU_G(q, V \oplus \mathcal{U}).$$

The plane $V \in BU(|V|, V \oplus \mathcal{U})$ is a $G-$fixed point. Take it to be the canonical basepoint of $BU_G(V)$. We have an inclusion $BU_G(V)$ in $BU_G(W)$ of based $G-$spaces.

Define BU_G to be the colimit of the $BU_G(V)$.

By sending a plane A to its complex conjugate, we get a homeomorphism from $BU_{\Lambda_G(g)}(n, V)$ to $BU_{\Lambda_G(\overline{g})}(n, \overline{V})$. Passing to the colimit, we get a homeomorphism

$$BU_{\Lambda_G(g)} \longrightarrow BU_{\Lambda_G(\overline{g})},$$

and thus, a homeomorphism

$$\Delta_g : BU_{\Lambda_G(g)} \times \mathbb{Z} \longrightarrow BU_{\Lambda_G(\overline{g})} \times \mathbb{Z},$$

which send (A, n) to $(\overline{A}, n)$. For any $[h, t] \in \Lambda_G(g)$, Δ_g sends the space $[h, t] \cdot V$ to $[\overline{h}, -t] \cdot \overline{V}$. And $\Delta_{\overline{g}} \circ \Delta_g$ is the identity map.

We can consider $K_{\Lambda_G(g)}(X^g)$ as a space homeomorphic to $\mathrm{map}_{\Lambda_G(g)}(X^g, BU_{\Lambda_G(g)} \times \mathbb{Z})$. If $E|_{X^g//\Lambda_G(g)}$ is classified by a $\Lambda_G(g)-$map $f : X^g \to BU_{\Lambda_G(g)} \times \mathbb{Z}$, then $\overline{E}|_{X^{\overline{g}}//\Lambda_G(\overline{g})}$ is classified by the composition of the maps

$$\overline{f} : \; X^{\overline{g}} \xrightarrow{\sigma_{\overline{g}}} X^g \xrightarrow{f} BU_{\Lambda_G(g)} \times \mathbb{Z} \xrightarrow{\Delta_g} BU_{\Lambda_G(\overline{g})} \times \mathbb{Z} \,. \tag{5.2}$$

Thus, $\overline{f}(x) = \Delta_g(f(\overline{x}))$. It's straightforward to check $\overline{f}$ is a $\Lambda_G(\overline{g})-$map:

$\forall [\overline{h}, -t] \in \Lambda_G(\overline{g})$, $x \in X^{\overline{g}}$, we have $\overline{f}([\overline{h}, -t] \cdot x) = \Delta_g(f(\overline{[\overline{h}, -t] \cdot x})) = \Delta_g(f([h, t] \cdot \overline{x})) = \Delta_g([h, t] f(\overline{x})) = [\overline{h}, -t] \Delta_g(f(\overline{x})) = [\overline{h}, -t] \overline{f}(x)$.

Thus, $\overline{f}$ represents a map in $\mathrm{Map}_{\Lambda_G(\overline{g})}(X^{\overline{g}}, BU_{\Lambda_G(\overline{g})} \times \mathbb{Z})$. And $\overline{\overline{f}}$ goes back to f.

If $\{g\}$ is a family of representatives for the conjugacy classes of elements in G^{tors}, so is

$\{\overline{g}\}$. As a whole, we get an involution on

$$QEllR_G(X) \cong \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)} X^g \cong \prod_{g \in G^{tors}_{conj}} [X^g, BU_{\Lambda_G(g)} \times \mathbb{Z}]^{\Lambda_G(g)},$$

by sending

$$f \in \prod_{g \in G^{tors}_{conj}} [X^g, BU_{\Lambda_G(g)} \times \mathbb{Z}]^{\Lambda_G(g)}$$

to

$$\prod_{\overline{g}} \overline{f} \in \prod_{\overline{g}} [X^{\overline{g}}, BU_{\Lambda_G(\overline{g})} \times \mathbb{Z}]^{\Lambda_G(\overline{g})} = QEllR_G(X).$$

5.3 Construction of the Real Spectra

5.3.1 $QEll_G$ as a diagram space

Recall the conclusion in Theorem 4.3.7, there is a $G-$spectrum representing $QEll^*_G$, which is unique up to $G-$homotopy. In the rest, for any compact Lie group G, $BU_G \times \mathbb{Z}$ is denoted by KU_G.

Let's recall the orbit category $\mathcal{O}_G$ of G, introduced in Example 4.1.3.

Let Λ_G be a subcategory of $\mathcal{O}_G$ with objects $[G/\langle g \rangle]_{g \in G}$. If g and g' are conjugate in G, the space of morphism $[G/\langle g \rangle] \to [G/\langle g' \rangle]$ is that of $\mathcal{O}_G$; if g and g' are not conjugate, it's empty.

Proposition 5.3.1. *$QEll_G$ is a functor from the category Λ_G to the category of space, sending $[G/\langle g \rangle]$ to $R_g(KU_{\Lambda_G(g)})$. Let's call it R_G.*

Proof. If g' is conjugate to g in G, $R_g(KU_{\Lambda_G(g)})$ is $G-$homeomorphic to $R_{g'}(KU_{\Lambda_G(g')})$, as explained below. Thus, if we choose another family $\{g'\}$ of representatives of the conjugacy classes in G^{tor}, $\coprod_{g \in G^{tors}_{conj}} R_g(KU_{\Lambda_G(g)})$ is $G-$homeomorphic to $\coprod_{g' \in G^{tors}_{conj}} R_{g'}(KU_{\Lambda_G(g')})$.

If $g' = h^{-1}gh$, the centralizer $C_G(g') = h^{-1}C_G(g)h$, and each $E(\langle g' \rangle / K')$ in the formula of $S_{C_G(g'),g'}$ is the same space as a $E(\langle g \rangle/(h^{-1}K'h))$ but with different group action, $g' \cdot x := gx$,i.e. $hg'h^{-1}x$, for any $x \in E(\langle g \rangle/(h^{-1}K'h))$.

Let $f : C_G(g) \longrightarrow *_K E(\langle g \rangle/K)$ be a $\langle g \rangle-$equivariant map. We can define a $\langle g' \rangle-$map $\widetilde{f} : C_G(g') \longrightarrow *_{K'} E(\langle g' \rangle/K')$ by $\widetilde{f}(b) := f(hbh^{-1})$. This map $S_{C_G(g),g} \longrightarrow S_{C_G(g'),g'}$, $f \mapsto \widetilde{f}$, is a homeomorphism.

Generally, for any subgroup $H < G$ containing g, we can define a homeomorphism

$S_{H,g} \longrightarrow S_{h^{-1}Hh,g'}$ in this way. In addition, for any $a \in H$, $b \in h^{-1}Hh$, $\widetilde{a \cdot f}(b) = f(hbh^{-1}a) = f(h(bh^{-1}ah)h^{-1}) = \widetilde{f}(bh^{-1}ah) = ((h^{-1}ah) \cdot \widetilde{f})(b)$.

Thus, $\widetilde{a \cdot f} = (h^{-1}ah) \cdot \widetilde{f}$.

A special case is when the subgroup is G itself, we get a homeomorphism $j_h : S_{G,g} \longrightarrow S_{G,g'}$.

Similarly, $KU_{\Lambda_G(g')}$ is the same space as $KU_{\Lambda_G(g)}$ but with the group action defined by $[a,t] \cdot x := [hah^{-1}, t]x$. Together with the map $S_{C_G(g),g} \longrightarrow S_{C_G(g'),g'}$, we get a map

$$\tau_h : KU^{\mathbb{R}}_{\Lambda_G(g)} * S_{C_G(g),g} \longrightarrow KU^{\mathbb{R}}_{\Lambda_G(g')} * S_{C_G(g'),g'}.$$

For $a \in C_G(g)$, $x \in KU^{\mathbb{R}}_{\Lambda_G(g)} * S_{C_G(g),g}$, $\tau_h(a \cdot x) = h^{-1}ah \cdot (\tau_h(x))$.

Given a $C_G(g)-$map $F : G \longrightarrow KU^{\mathbb{R}}_{\Lambda_G(g)} * S_{C_G(g),g}$, we can define a $C_G(g')-$map $\widetilde{F} : G \longrightarrow KU^{\mathbb{R}}_{\Lambda_G(g')} * S_{C_G(g'),g'}$ by $\widetilde{F}(b) := (\tau_h \circ F)(hbh^{-1})$.

For any $b \in G$, $a' \in C_G(g')$ $\widetilde{F}(a'b) = \tau_h \circ F((ha'h^{-1})(hbh^{-1})) = \tau_h((ha'h^{-1})F(hbh^{-1})) = a'\tau_h(F(hbh^{-1})) = a'\widetilde{F}(b)$. So $\widetilde{F}$ is $C_G(g')-$equivariant.

And, for $\alpha, b \in G$, $\widetilde{\alpha \cdot F}(b) = (\tau_h \circ F)(hbh^{-1}\alpha) = (\tau_h \circ F)(h(bh^{-1}\alpha h)h^{-1}) = \widetilde{F}(b(h^{-1}\alpha h)) = ((h^{-1}\alpha h) \cdot \widetilde{F})(b)$.

Thus, $\widetilde{\alpha \cdot F} = (h^{-1}\alpha h) \cdot \widetilde{F}$.

Let $\nu_h : G \longrightarrow G$ be the isomorphism $\nu_h(b) = h^{-1}bh$. So finally we get a $G-$homeomorphism between $[G, KU^{\mathbb{R}}_{\Lambda_G(g)} * S_{C_G(g),g}]^{C_G(g)}$ and $[G, KU^{\mathbb{R}}_{\Lambda_G(g')} * S_{C_G(g'),g}]^{C_G(g')}$ by sending F to $\nu_h^* \widetilde{F}$. Join with the G-homeomorphism $\nu_h^* \circ j_h$ on $S_{G,g}$, we get a $G-$homeomorphism J_h between $R_g(KU_{\Lambda_G(g)})$ and $R_{g'}(KU_{\Lambda_G(g')})$.

Thus, we get a functor R_G from Λ_G to the category of based space, sending the object $[G/\langle g \rangle]$ to $R_g(KU_{\Lambda_G(g)})$, and sending the morphism $[G/\langle g \rangle] \longrightarrow [G/\langle g' \rangle]$, $a\langle g \rangle \mapsto ah\langle g' \rangle$ to J_h.

$\square$

For each $g \in G$, I'll define a map $\Phi_g : R_g(KU_{\Lambda_G(g)}) \longrightarrow R_{\overline{g}}(KU_{\Lambda_G(\overline{g})})$ in this section. Combining the Φ_gs together, we can get an involution on the diagram spectra R_G.

5.3.2 Thoughts on involution on diagram spaces

Before I construct the involution of $QEllR^*_G$ I talk a little about my thoughts on the involution on diagram spectra. There may be more general definition than these I give, which are based on the involution on the diagram $\mathcal{D}$. The reference for diagram spaces and diagram spectra is [42].

Let $\mathcal{D}$ be a skeletally small category. Let

$$\beta_D : \mathcal{D} \longrightarrow \mathcal{D}$$

be an involution on D, i.e. a functor such that $\beta_D \circ \beta_D$ is the identity functor. Obviously, for any $\mathcal{D}-$space X, $X \circ \beta_D$ is also a $\mathcal{D}-$space. If $\mathcal{D}$ is symmetric monoidal, an involution β_D on it needs to be symmetric monoidal.

There are several examples we will use in the constructions in the later sections.

Example 5.3.2. *Let $\mathcal{O}_G$ denote the orbit category of a Lie group G with objects $[G/H]$ and the space of morphisms $[G/H] \to [G/K]$ is the space of $G-$maps $G/H \to G/K$, which is nonempty if and only if H is subconjugate to K.*

When G is a Real Lie group, we can define an involution on $\mathcal{O}_G$ by sending $[G/H]$ to $[G/\overline{H}]$. As for the morphisms, if $f : G/H \longrightarrow G/K$ is a morphism in $\mathcal{O}_G$ with $f(eH) = gK$, we can define a map $\overline{f} : G/\overline{H} \longrightarrow G/\overline{K}$ by $\overline{f}(e\overline{H}) = \overline{g}\overline{K}$, which is well-defined. In this way we get a functor between $\mathcal{O}_G$. I'll denote this involution by $\overline{(\cdot)}_{\mathcal{O}}$. When there's no confusion, it's denoted by $\overline{(\cdot)}$.

Example 5.3.3. *Let G be a Real Lie group with involution $\overline{(\cdot)}$. Then we can define an involution on Λ_G by sending the object $[G/\langle g\rangle]$ to $[G/\langle \overline{g}\rangle]$. For the morphisms, let $f : G/\langle g\rangle \longrightarrow G/\langle g'\rangle$ be a morphism in Λ_G with $f(e\langle g\rangle) = h\langle g'\rangle$. Then we can define a morphism $\overline{f} : G/\langle \overline{g}\rangle \longrightarrow G/\langle \overline{g}'\rangle$ by $\overline{f}(e\langle \overline{g}\rangle) = \overline{h}\langle \overline{g}'\rangle$, which is well-defined. Then we get a functor between Λ_G. I'll use $\overline{(\cdot)}_{\Lambda}$ to denote it. When there's no confusion, it's denoted by $\overline{(\cdot)}$.*

Example 5.3.4. *Let G be a Real Lie group with involution $\overline{(\cdot)}$. We can define an involution on the product $\Lambda_G \times \mathcal{O}_G$ by sending (a, b) to $(\overline{a}, \overline{b})$.*

Other than this one, $Id \times \overline{(\cdot)}_{\mathcal{O}}$ and $\overline{(\cdot)}_{\Lambda} \times Id$ also give involution on the product.

Let $\mathcal{DT}$ be the category of $\mathcal{D}-$spaces and natural maps between them.

I define the involution on a $\mathcal{D}-$space associated to the involution β_D.

Definition 5.3.5. *An involution on a $\mathcal{D}-$space X is a morphism Ψ_D in $\mathcal{DT}$ from X to $X \circ \beta_D$ such that $\Psi_D \circ \Psi_D$ is the identity functor between X.*

Example 5.3.6. *A trivial case is that the involution on $\mathcal{D}$ is just the identity functor and the involution Ψ_D on X is also the identity.*

Now let's consider the case when $\mathcal{D}$ is symmetric monoidal with unit u and product $\Box$. Let R be a monoid in $\mathcal{D}\mathcal{T}$ with unit λ and product ϕ. Let

$$\alpha_R : (R, \phi) \longrightarrow (R, \phi)$$

be an involution on R, i.e. an involution on the $\mathcal{D}-$space R respect the monoid structure.

Here is the definition I give for an involution on a $\mathcal{D}-$spectrum (X, σ) over R where $X : \mathcal{D} \longrightarrow T$ is a $\mathcal{D}-$space and $\sigma : X(d) \wedge R(e) \longrightarrow X(d \Box e)$ are the structure maps.

Definition 5.3.7. *An involution on a $\mathcal{D}-$ spectrum (X, σ) is an involution Φ_D on the $\mathcal{D}-$space X such that for any objects d, e in $\mathcal{D}$, the diagram commutes*

$$\begin{tikzcd}
X(d) \wedge R(e) \arrow[r, "\sigma"] \arrow[d, "\Phi_D(d) \wedge \alpha_R(e)"] & X(d \Box e) \arrow[d, "\Phi_D(d \Box e)"] \\
X(\beta_D(d)) \wedge R(\alpha_R(e)) \arrow[r, "\sigma"] & X(\beta_D(d \Box e))
\end{tikzcd}$$

Then the involution preserves the associativity diagram.

5.3.3 The construction of Φ_{Λ_G}

Now let's start constructing the involution Ψ_{Λ_G} for the spectrum R_G.

First let's define a map $\psi_{H,h} : S_{H,h} \to S_{\overline{H},\overline{h}}$ for any $H \leq G$ and $h \in H$, which has the property $\psi_{\overline{H},\overline{h}} \circ \psi_{H,h} = \mathrm{Id}_{S_{H,h}}$.

If K is a maximal subgroup of $H \leq G$, then $\overline{K}$ is a maximal subgroup $\overline{H}$ and

$$\{\overline{K} | K \text{ is a maximal subgroup of } H\}$$

are all the maximal subgroup of $\overline{H}$. So the involution induces a bijection on the quotients $\langle h \rangle / K \to \langle \overline{h} \rangle / \overline{K}$. This isomorphism induce a homeomorphism $E(\langle h \rangle / K) \to E(\langle \overline{h} \rangle / \overline{K})$ sending (V, v) to the complex conjugate $(\overline{V}, \overline{v})$, where V is a complex $\langle h \rangle / K-$ representation space and $v \in V$. The complex conjugate $\overline{V}$ is a $\langle \overline{h} \rangle / \overline{K}-$representation space via

$$\overline{\alpha} \cdot \overline{w} := \overline{\alpha \cdot w}$$

where $\alpha \in \langle h \rangle / K$ and $\overline{w} \in \overline{V}$.

In this way we get a homeomorphism

$$\delta_h : *_K E(\langle h \rangle / K) \longrightarrow *_{\overline{K}} E(\langle \overline{h} \rangle / \overline{K}).$$

$*_K E(\langle h \rangle / K)$ can be viewed as a $\langle h \rangle-$space while $*_{\overline{K}} E(\langle \overline{h} \rangle / \overline{K})$ can be viewed as a $\langle \overline{h} \rangle-$space. For any $a \in \langle h \rangle$, $x \in *_K E(\langle h \rangle / K)$, we have $\delta_h(a \cdot x) = \overline{a} \cdot \delta_h(x)$.

Let $f : H \to *_K E(\langle h \rangle / K)$ be a $\langle h \rangle-$map. Define $\psi_{H,h}(f) : \overline{H} \to *_{\overline{K}} E(\langle \overline{h} \rangle / \overline{K})$ to be the composition

$$\overline{H} \xrightarrow{\overline{(\cdot)}} H \xrightarrow{f} *_K E(\langle h \rangle / K) \xrightarrow{\delta_h} *_{\overline{K}} E(\langle \overline{h} \rangle / \overline{K}).$$

In other words, $\psi_{H,h}(f)(\alpha) = \delta_h(f(\overline{\alpha}))$, for $\alpha \in \overline{H}$.

Let $h' \in \langle h \rangle$. $\psi_{H,h}(f)(\overline{h'} \cdot \alpha) = \delta_h(f(\overline{\overline{h'} \cdot \alpha})) = \delta_h(f(h' \cdot \overline{\alpha})) = \delta_h(h' \cdot f(\overline{\alpha})) = \overline{h'} \cdot \delta_h(f(\overline{\alpha})) = \overline{h'} \cdot \psi_{H,h}(f)(\alpha)$. Thus, $\psi_{H,h}(f)$ is a $\langle \overline{h} \rangle-$map. $\psi_{H,h} : S_{H,h} \to S_{\overline{H},\overline{h}}$ is well-defined. And $\psi_{\overline{H},\overline{h}} \circ \psi_{H,h}(f)$ goes back to the map f.

$S_{H,h}$ is a $H-$space with the action defined by $(a \cdot f)(x) := f(x \cdot a)$ where $a, x \in H$.

$\psi_{H,h}(a \cdot f) = \overline{a} \cdot \psi_{H,h}(f)$ since for any $x \in \overline{H}$, $\psi_{H,h}(a \cdot f)(x) = \delta_h((a \cdot f)(\overline{x})) = \delta_h(f(\overline{x} \cdot a)) = \delta_h(f(\overline{x \cdot \overline{a}})) = \psi_{H,h}(f)(x \cdot \overline{a}) = (\overline{a} \cdot \psi_{H,h}(f))(x)$.

From the maps

$$\Delta_g : KU_{\Lambda_G(g)} \to KU_{\Lambda_G(\overline{g})},\ V \mapsto \overline{V}$$

and $\psi_{C_G(g),g}$, we get a well-defined map between the joins

$$\phi_g : KU_{\Lambda_G(g)} * S_{C_G(g),g} \longrightarrow KU_{\Lambda_G(\overline{g})} * S_{C_G(\overline{g}),\overline{g}}.$$

$\phi_{\overline{g}} \circ \phi_g$ is the identity map. And for any $b \in C_G(g)$, $x \in KU_{\Lambda_G(g)} * S_{C_G(g),g}$, $\phi_g(bx) = \overline{b}\phi_g(x)$.

Next, use ϕ_g to construct a map

$$\Phi_g : \mathrm{Map}_{C_G(g)}(G, KU_{\Lambda_G(g)} * S_{C_G(g),g}) \longrightarrow \mathrm{Map}_{C_G(\overline{g})}(G, KU_{\Lambda_G(\overline{g})} * S_{C_G(\overline{g}),\overline{g}}).$$

Given a $C_G(g)-$map $F : G \longrightarrow KU_{\Lambda_G(g)} * S_{C_G(g),g}$, let $\Phi_g(F)$ be the composition of maps:

$$G \xrightarrow{\overline{(\cdot)}} G \xrightarrow{F} KU_{\Lambda_G(g)} * S_{C_G(g),g} \xrightarrow{\phi_g} KU_{\Lambda_G(\overline{g})} * S_{C_G(\overline{g}),\overline{g}}.$$

We can check that $\Phi_g(F)$ is $C_G(\overline{g})-$equivariant. For $\alpha \in G$ and $h \in C_G(\overline{g})$, $\Phi_g(F)(h \cdot \alpha) = \phi_g(F(\overline{h \cdot \alpha})) = \phi_g(F(\overline{h} \cdot \overline{\alpha})) = \phi_g(\overline{h} \cdot F(\overline{\alpha})) = \overline{\overline{h}} \cdot \phi_g(F(\overline{\alpha})) = h \cdot \Phi_g(F)(\alpha)$. Thus

$$\Phi_g : \mathrm{Map}_{C_G(g)}(G, KU_{\Lambda_G(g)} * S_{C_G(g),g}) \longrightarrow \mathrm{Map}_{C_G(\overline{g})}(G, KU_{\Lambda_G(\overline{g})} * S_{C_G(\overline{g}),\overline{g}})$$

is well-defined.

$\mathrm{Map}_{C_G(g)}(G, KU_{\Lambda_G(g)} * S_{C_G(g),g})$ is a $G-$space with the action defined by $a \cdot F(\alpha) = F(\alpha \cdot a)$ for $a, \alpha \in G$.

$\Phi_g(a \cdot F) = \overline{a} \cdot \Phi_g(F)$ since $\Phi_g(a \cdot F)(\alpha) = \phi_g((a \cdot F)(\overline{\alpha})) = \phi_g(F(\overline{\alpha} \cdot a)) = \phi_g(F(\overline{\alpha \cdot \overline{a}})) = \Phi_g(F)(\alpha \cdot \overline{a}) = (\overline{a} \cdot \phi_g(F))(\alpha)$.

$\Phi_{\overline{g}} \circ \Phi_g$ is the identity map. And for any $b \in G$, $x \in \mathrm{Map}_{C_G(g)}(G, KU_{\Lambda_G(g)} * S_{C_G(g),g})$, $\Phi_g(bx) = \overline{b}\Phi_g(x)$.

Moreover, if $g' = h^{-1}gh$, then we have $\overline{g}' = \overline{h}^{-1}\overline{g}\overline{h}$, and the commutative diagram below

$$\begin{tikzcd}
R_g(KU_{\Lambda_G(g)}) \arrow[r, "J_h"] \arrow[d, "\Phi_g"] & R_{g'}(KU_{\Lambda_G(g')}) \arrow[d, "\Phi_{g'}"] \\
R_{\overline{g}}(KU_{\Lambda_G(\overline{g})}) \arrow[r, "J_{\overline{h}}"] & R_{\overline{g}'}(KU_{\Lambda_G(\overline{g}')})
\end{tikzcd} ,$$

where J_h is the $G-$homeomorphism defined in the proof of Proposition 5.3.1. It's straightforward to check it commute.

Therefore, combining the Φ_gs, we get an involution

$$\Phi_{\Lambda_G} : R_G \longrightarrow R_G \circ \overline{(\cdot)}_\Lambda$$

on $QEll_G$.

We'll go a little further on this.

Recall a $\mathcal{O}_G-$space is defined to be a continuous contravariant functor $\mathcal{O}_G \to \mathcal{T}$ where $\mathcal{T}$ is the category of topological spaces. Let $\mathcal{O}_G\mathcal{T}$ denote the category of $\mathcal{O}_G-$spaces.

Let $G\mathcal{U}$ denote the category of $G-$spaces. Let $\Theta : G\mathcal{U} \to \mathcal{O}_G\mathcal{U}$ be the functor send a $G-$space X to its fixed points $G/H \mapsto X^H$.

Moreover, we can construct $\Lambda_G \times \mathcal{O}_G-$space R by sending $([G/\langle g \rangle], [G/H])$ to $(R_g KU_{\Lambda_G(g)})^H$.

$R_G = R([G/\langle e\rangle])$. We can also get an involution

$$\Phi : R \longrightarrow R \circ (\overline{(\cdot)}_\Lambda \times \overline{(\cdot)}_{\mathcal{O}}). \tag{5.3}$$

For each $([G/\langle g\rangle], [G/H])$, $\Phi_{[G/\langle g\rangle],[G/H]}$ is $\Theta(\Phi_g)([G/H])$. Let's denote it by $\Phi_{g,H}$.

5.3.4 Construction of the involution Δ

In this section I'll show a functor $T : \Lambda_G \times \mathcal{O}_G \longrightarrow \mathcal{T}$ and give an involution on it.

For each $g \in G$, consider the $\mathcal{O}_G-$space

$$T_g : G/H \mapsto \mathrm{Map}_{\Lambda_G(g)}((G/H)^g, KU_{\Lambda_G(g)}).$$

$T_g(G/H)$ is homotopic to $(R_g BU_{\Lambda_G(g)})^H$.

For any $a \in G$, T_g and $T_{a^{-1}ga}$ are naturally isomorphic, as shown below.

For any two subgroup H and K of G, consider

$$\begin{tikzcd}
\mathrm{Map}_{\Lambda_G(g)}((G/H)^g, KU_{\Lambda_G(g)}) \arrow[r] \arrow[d, "\eta_H"] & \mathrm{Map}_{\Lambda_G(g)}((G/K)^g, KU_{\Lambda_G(g)}) \arrow[d, "\eta_K"] \\
\mathrm{Map}_{\Lambda_G(a^{-1}ga)}((G/a^{-1}Ha)^{a^{-1}ga}, KU_{\Lambda_G(a^{-1}ga)}) \arrow[r] & \mathrm{Map}_{\Lambda_G(a^{-1}ga)}((G/a^{-1}Ka)^{a^{-1}ga}, KU_{\Lambda_G(a^{-1}ga)})
\end{tikzcd}.$$

Let the upper horizontal map be induced by a map $(G/K)^g \to (G/H)^g, \alpha K \mapsto \alpha\beta H$. Then the bottom horizontal map is induced by $(G/a^{-1}Ka)^{a^{-1}ga} \to (G/a^{-1}Ha)^{a^{-1}ga}, a^{-1}\alpha K a \mapsto a^{-1}\alpha\beta H a$. η_H maps f to the composition of maps

$$\eta_H(f) : \qquad (G/a^{-1}Ha)^{a^{-1}ga} \longrightarrow (G/H)^g \xrightarrow{f} KU_{\Lambda_G(g)} \longrightarrow KU_{\Lambda_G(a^{-1}ga)}$$

where the first map is $\alpha a^{-1}Ha \mapsto a\alpha a^{-1}H$ and the third one is $V \mapsto a^{-1}Va$.

It's straightforward to check the diagram commutes.

Thus, we get a functor T from the diagram $\Lambda_G \times \mathcal{O}_G$ to the category of based spaces, sending $([G/\langle g\rangle], [G/H])$ to $T_g(G/H)$.

For any subgroup H and K of G, consider diagram

$$\begin{tikzcd}
\mathrm{Map}_{\Lambda_G(g)}((G/H)^g, KU_{\Lambda_G(g)}) \arrow[r] \arrow[d] & \mathrm{Map}_{\Lambda_G(g)}((G/K)^g, KU_{\Lambda_G(g)}) \arrow[d] \\
\mathrm{Map}_{\Lambda_G(\overline{g})}((G/\overline{H})^{\overline{g}}, KU_{\Lambda_G(\overline{g})}) \arrow[r] & \mathrm{Map}_{\Lambda_G(\overline{g})}((G/\overline{K})^{\overline{g}}, KU_{\Lambda_G(\overline{g})}).
\end{tikzcd}$$

Let the upper horizontal map be induced by a map $(G/K)^g \to (G/H)^g, \alpha K \mapsto \alpha\beta H$. Then the bottom horizontal map is induced by $(G/\overline{K})^{\overline{g}} \to (G/\overline{H})^{\overline{g}}, \overline{\alpha}\overline{K} \mapsto \overline{\alpha\beta H}$. The vertical maps are the involution $f \mapsto \overline{f}$ sending f to the composition of maps

$$\overline{f}: \qquad (G/\overline{H})^{\overline{g}} \xrightarrow{\overline{(\cdot)}} (G/H)^g \xrightarrow{f} KU_{\Lambda_G(g)} \xrightarrow{\Delta_g} KU_{\Lambda_G(\overline{g})} .$$

$\overline{f}$ is $\Lambda_G(\overline{g})-$equivariant and for $[h,t] \in \Lambda_G(g)$, $[\overline{h},t] \cdot \overline{f} = \overline{[h,t] \cdot f}$. So $f \to \overline{f}$ is well-defined. $\overline{\overline{f}}$ goes back to f.

It's straightforward to check this diagram commutes. Thus, we get an involution

$$\Delta : T \longrightarrow T \circ (\overline{(\cdot)}_\Lambda \times \overline{(\cdot)}_{\mathcal{O}}),$$

where $\overline{(\cdot)}_\Lambda \times \overline{(\cdot)}_{\mathcal{O}}$ is the involution defined in Example 5.3.4.

5.3.5 Relation between Φ and Δ

It's straightforward to check the diagram

$$\begin{tikzcd}
T_g(G/H) \arrow[r, "f\mapsto f(e)", "\simeq"'] \arrow[d, "\Delta_{g,H}"] & (R_g KU_{\Lambda_G(g)})^H \arrow[d, "\Phi_{[G/\langle g\rangle],[G/H]}"] \\
T_{\overline{g}}(G/\overline{H}) \arrow[r, "f\mapsto f(e)", "\simeq"'] & (R_{\overline{g}} KU_{\Lambda_G(\overline{g})})^{\overline{H}}
\end{tikzcd} \tag{5.4}$$

commutes.

The involution is unique up to $G-$equivariance in this sense: if Φ' is another involution, then $\Phi \circ \Phi'$ is a $G-$equivalence: In diagram (5.5),

$$
\begin{array}{ccc}
(R_g KU_{\Lambda_G(g)})^H & \xrightarrow[\simeq]{f\mapsto f(e)} & T_g(G/H) \\
\Big\downarrow{\scriptstyle \Phi'_{g,H}} & & \Big\downarrow{\scriptstyle \Delta_{g,H}} \\
(R_{\overline{g}} KU_{\Lambda_G(\overline{g})})^{\overline{H}} & \xrightarrow[\simeq]{f\mapsto f(e)} & T_{\overline{g}}(G/\overline{H}) \\
\Big\downarrow{\scriptstyle \Phi_{\overline{g},\overline{H}}} & & \Big\downarrow{\scriptstyle \Delta_{\overline{g},\overline{H}}} \\
(R_g KU_{\Lambda_G(g)})^H & \xrightarrow[\simeq]{f\mapsto f(e)} & T_g(G/H)
\end{array}
\tag{5.5}
$$

the composition $\Delta_{\overline{g},\overline{H}} \circ \Delta_{g,H}$ is the identity map. So $\Phi \circ \Phi'$ is weak equivalence on the fixed point sets. By the equivariant Whitehead theorem, $\Phi \circ \Phi'$ is a $G-$equivalence.

The involution Φ is what desired.

Chapter 6

A new global homotopy theory

Not every equivariant cohomology theory has a global version in the sense of the definitions in [41], [43], [56], etc. Unlike many classical global theories, global K-theory, global cobordism, etc., quasi-elliptic cohomology cannot fit into most existent global homotopy theories smoothly as indicated in Remark 4.5.29.This motivates us to establish a more flexible global homotopy theory in this chapter. In Chapter 7 I show quasi-elliptic chomomology can fit into this new theory.

An orthogonal space is a continuous functor from the category $\mathbb{L}$ of inner product real spaces to the category of topological spaces. We enlarge the category $\mathbb{L}$ by adding restriction maps to it, which are identity morphisms on the underlying vector spaces in $\mathbb{L}$, and form a category D. Instead of orthogonal spaces, we study the subcategory D_0 of it corresponding to finite groups and the category D_0T of D_0-spaces. There is a fully faithful functor from the category of $\Sigma-$spaces to the category of D_0-spaces where Σ is the category of finite sets and injective maps. In other words, D_0T contains all the information of $\mathbb{L}T$.

We establish several model structures on D_0T. As a category of diagram spaces, it's equipped the level model structure, as shown in [42]. Moreover, D_0 is a generalized Reedy category in the sense of [10]. Thus, there is a Reedy model structure on D_0T, as shown in Section 6.4.

It's conjectured that there is a model structure on D_0T Quillen equivalent to the global model structure on the category of orthogonal spaces constructed in [56]. We construct a global model structure on D_0T in Section 6.5. In Section 6.6 we show this model structure on a full subcategory D_0^wT of D_0T is Quillen equivalent to the global model structure on the orthogonal spaces.

In Section 6.7, we introduce the unitary D_0-space and unitary D_0-spectra.

6.1 The category D_0 and D_0-spaces

Definition 6.1.1. *Let D be a category with objects (G, V, ρ) where V is an inner product vector space, G a compact group G and ρ a faithful group representations*

$$\rho : G \longrightarrow O(V).$$

A morphism in D $\phi = (\phi_1, \phi_2) : (G, V, \rho) \longrightarrow (H, W, \tau)$ consists of a linear isometric embedding $\phi_2 : V \longrightarrow W$ and a group homomorphism $\phi_1 : \tau^{-1}(O(\phi_2(V))) \longrightarrow G$, which make the diagram commute.

$$\begin{tikzcd} G \arrow[r, "\rho"] & O(V) \arrow[d, "\phi_{2*}"] \\ \tau^{-1}(O(\phi_2(V))) \arrow[u, "\phi_1"] \arrow[r, "\tau"] & O(W) \end{tikzcd} \tag{6.1}$$

In other words, the group action of H on $\phi_2(V)$ is induced from that of G.

The composition of two morphisms

$$(G, V, \rho) \overset{(\phi_1,\phi_2)}{\longrightarrow} (H, W, \tau) \overset{(\psi_1,\psi_2)}{\longrightarrow} (K, U, \beta)$$

is defined to be

$$(\phi_1 \circ \psi_1|_{\beta^{-1}(O(\psi_2\circ\phi_2(V)))}, \psi_2 \circ \phi_2).$$

The identity morphism in $D((G, V, \rho), (G, V, \rho))$ is (Id, Id). And the composition is associative.

All the maps in (6.1) are injective. Given a linear isometric embedding $\phi_2 : V \longrightarrow W$, $\tau^{-1}(\phi_{2*}(\rho(G)))$ is always nonempty since the identity element is in it. But the group homomorphism ϕ_1 may not always exist. If it exists, it's unique and injective.

Lemma 6.1.2. *Two objects (G, V, ρ) and (H, W, τ) in D are isomorphic if and only if there's an isomorphism $G \longrightarrow H$ which makes V and W isomorphic as representations.*

The proof is straightforward.

Remark 6.1.3. *Since the representation ρ in an object (G, V, ρ) of D is faithful, we may consider the group G as a closed subgroup of $O(V)$. Then the diagram (6.1) is in fact*

$$\begin{tikzcd}
G \arrow[r] & O(V) \arrow[d] \\
H \cap O(V) \arrow[u] \arrow[r] & O(W)
\end{tikzcd}$$

where we consider $O(V)$ as a closed subgroup of $O(W)$ as well. All the maps in the diagram are inclusions.

Example 6.1.4. *Let (G, V, ρ) be an object of D and (ϕ_1, ϕ_2) be a morphism in $D((G, V, \rho), (G, V, \rho))$. $\phi_{2*} : O(V) \longrightarrow O(V)$ is always of the form $A \longrightarrow TAT^{-1}$ for some linear transformation T in $O(V)$. By the commutativity of the diagram*

$$\begin{tikzcd}
G \arrow[r, "\rho"] & O(V) \arrow[d, "\phi_{2*}"] \\
G \arrow[u, "\phi_1"] \arrow[r, "\rho"] & O(V)
\end{tikzcd}$$

$\phi_1(g) = T^{-1}gT$.

We have the bijection

$$F : N_{O(V)}(G) \longrightarrow D((G, V, \rho), (G, V, \rho))$$
$$T \mapsto \phi_T$$

where ϕ_T is the morphism

$$(g \mapsto T^{-1}gT,\ v \mapsto \rho(T)v).$$

So the automorphism group of (G, V, ρ) is $N_{O(V)}(G)$.

To define a reasonable topology on $D((G, V, \rho), (H, W, \tau))$ we need the category L defined below.

Definition 6.1.5. *Let L be the category with objects inner product spaces and morphisms the linear isometric embeddings. Let $L(V, W)$ be the space of all the linear isometric embeddings $V \longrightarrow W$.*

This category is a topological category. $L(V, W)$ is homeomorphic to the Stiefel manifold of $dim(V)-$frames in W and has a CW structure.

Consider the map

$$P : D((G,V,\rho),(H,W,\tau)) \longrightarrow L(V,W)$$
$$(\phi_1,\phi_2) \mapsto \phi_2$$

It's injective. We define the topology on $D((G,V,\rho),(H,W,\tau))$ by requiring P to be continuous so $D((G,V,\rho),(H,W,\tau))$ is a subspace of $L(V,W)$. The map F in Example 6.1.4 is continuous under this topology.

Remark 6.1.6. *The space $D((G,V,\rho),(H,W,\tau))$ of morphisms inherits an $H-$action and $G-$action on it. For $\phi = (\phi_1,\phi_2) : (G,V,\rho) \longrightarrow (H,W,\tau)$ and $(g,h) \in G \times H$,*

$$(g,h)\cdot(\phi_1,\phi_2) = (g\phi_1(h^{-1} - h)g^{-1}, h\phi_2(g^{-1}\cdot -)) \tag{6.2}$$

The $H-$action on ϕ_2 is left and the $G-$action on it is right, whereas the $H-$action on ϕ_1 is right and the $G-$action on ϕ_1 is left.

We can see the map P is $G \times H-$equivariant. So $D((G,V,\rho),(H,W,\tau))$ is a $G \times H-$subspace of $L(V,W)$.

Proposition 6.1.7. *The category D is a symmetric monoidal category.*

Proof. The tensor product $+ : D \times D \longrightarrow D$ is defined by

$$((G,V,\rho),(H,W,\tau)) \mapsto (G \times H, V \oplus W, \rho \oplus \tau). \tag{6.3}$$

The unit object is $u = (e, 0, *)$ where e is the trivial group and $*$ is the unique map from e. From the property of product of representations, the tensor product is associative. And we have the isomorphism $(G \times H, V \oplus W, \rho \oplus \tau) \longrightarrow (H \times G, W \oplus V, \tau \oplus \rho)$.

It's straightforward to check it satisfies all the required diagrams.

□

Definition 6.1.8. *A $D-$space is a continuous functor $X : D \longrightarrow T$ to the category of compactly generated weak Hausdorff spaces. A morphism of $D-$spaces is a natural transformation. I will use DT to denote the category of $D-$spaces.*

Definition 6.1.9. *A monoid $D-$space is a $D-$space R equipped with unit morphism $\eta : 1 \longrightarrow R$ and a multiplication morphism $\mu : R \boxtimes R \longrightarrow R$ that are unital and associative in* *the sense that the square*

$$\begin{tikzcd}
(R \boxtimes R) \boxtimes R \arrow[r, "\cong"] \arrow[d, "\mu \boxtimes R"'] & R \boxtimes (R \boxtimes R) \arrow[r, "R \boxtimes \mu"] & R \boxtimes R \arrow[d, "\mu"] \\
R \boxtimes R \arrow[rr, "\mu"'] & & R
\end{tikzcd}$$

commutes.

A monoid $D-$space R is commutative if moreover $\mu \circ \tau_{R,R} = \mu$, where $\tau_{R,R} : R \boxtimes R \longrightarrow R \boxtimes R$ is the symmetry isomorphism of the box product.

A morphism of monoid $D-$spaces is a morphism of $D-$spaces $f : R \longrightarrow S$ such that $f \circ \mu^R = \mu^S \circ (f \boxtimes f)$ and $f \circ \eta_R = \eta_S$.

Note that each object (G, V, ρ) is isomorphic to $(\rho(G), V, i)$ with $i : \rho(G) \hookrightarrow O(V)$ the inclusion of closed subgroup. We won't lose any information if consider the full subcategory D' of D with objects (G, V, i) with G a closed subgroup of $O(V)$ and i the inclusion. Let's denote (G, V, i) by (G, V). In fact we are more interested in the full subcategory D_0 of D' defined below.

Definition 6.1.10. *D_0 is a full subcategory of D', whose object (G, V) with finite group G. An D_0-space is a continuous functor $X : D_0 \longrightarrow T$ to the category of compactly generated weak Hausdorff spaces. A morphism of D_0-spaces is a natural transformation. I will use D_0T to denote the category of D_0-spaces.*

D_0 is also a symmetric monoid category with the tensor product and unit defined in the proof of Proposition 6.1.7.

Definition 6.1.11. *A monoid D_0-space is a D_0-space R equipped with unit morphism $\eta : 1 \longrightarrow R$ and a multiplication morphism $\mu : R \boxtimes R \longrightarrow R$ that are unital and associative in the sense that the square*

$$\begin{tikzcd}
(R \boxtimes R) \boxtimes R \arrow[r, "\cong"] \arrow[d, "\mu \boxtimes R"'] & R \boxtimes (R \boxtimes R) \arrow[r, "R \boxtimes \mu"] & R \boxtimes R \arrow[d, "\mu"] \\
R \boxtimes R \arrow[rr, "\mu"'] & & R
\end{tikzcd}$$

commutes.

A monoid D_0-space R is commutative if moreover $\mu \circ \tau_{R,R} = \mu$, where $\tau_{R,R} : R \boxtimes R \longrightarrow R \boxtimes R$ is the symmetry isomorphism of the box product.

A morphism of monoid D_0-spaces is a morphism of D_0-spaces $f : R \longrightarrow S$ such that $f \circ \mu^R = \mu^S \circ (f \boxtimes f)$ and $f \circ \eta_R = \eta_S$.

Example 6.1.12. *We can define the D_0-sphere. For each object (G, V) in D_0, define*

$$S^{(G,V)} := S^V. \tag{6.4}$$

S^V inherits a $G-$action from that on V.

Let $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$ be a morphism in D_0.

$$S(\phi) := S^{\phi_2} : S^V \longrightarrow S^W. \tag{6.5}$$

$S(Id)$ is the identity map.

Definition 6.1.13. *Let R be a monoid in $D_0\mathcal{T}$ with unit η and product μ. A D_0-spectrum over R is a D_0-space $X : D_0 \longrightarrow \mathcal{T}$ together with continuous maps*

$$\sigma : X(d) \wedge R(b) \longrightarrow X(d + b),$$

natural in d and b, such that the composite

$$X(d) \cong X(d) \wedge S^0 \xrightarrow{id \wedge \eta} X(d) \wedge R(u) \xrightarrow{\sigma} X(d + u) \cong X(d) \tag{6.6}$$

is the identity and the following diagram commutes:

$$\begin{tikzcd} X(d) \wedge R(b) \wedge R(f) \arrow[r, "\sigma \wedge id"] \arrow[d, "id \wedge \mu"'] & X(d+b) \wedge R(f) \arrow[d, "\sigma"] \\ X(d) \wedge R(b+f) \arrow[r, "\sigma"'] & X(d+b+f) \end{tikzcd} \tag{6.7}$$

D_0 is a topological category. And we can define a degree function on D_0 by

$$deg(G, V) = |G| \dim V,$$

the order of the group G times the dimension of the vector space V.

We have this lemma:

Lemma 6.1.14. *Two objects (G, V) and (H, W) are isomorphic if and only if there's an* *isomorphism $G \longrightarrow H$ which makes V and W isomorphic as representations.*

But the objects with the same degree may not be isomorphic.

We will show D_0 is a generalized Reedy category in the sense of Definition 6.1.15 below, which is from [10]. Recall that a subcategory S of a category R is called wide if S has the same objects as R. Let $\mathrm{Iso}(R)$ denote the maximal subgroupoid of R.

Definition 6.1.15. *A generalized Reedy structure on a small category R consists of wide subcategories R^+, R^-, and a degree-function $d : Ob(R) \longrightarrow \mathbb{N}$ satisfying the following four axioms:*

(i) non-invertible morphisms in R^+ (resp. R^-) raise (resp. lower) the degree; isomorphisms in R preserve the degree;

(ii) $R^+ \cap R^- = Iso(R)$;

(iii) every morphism f of R factors as $f = gh$ with $g \in R^+$ and $h \in R^-$, and this factorization is unique up to isomorphism;

(iv) If $\theta f = f$ for $\theta \in Iso(R)$ and $f \in R^-$, then θ is an identity.

A generalized Reedy structure is dualizable if in addition the following axiom holds:

(iv)' If $f\theta = f$ for $\theta \in Iso(R)$ and $f \in R^+$, then θ is an identity.

A (dualizable) generalized Reedy category is a small category equipped with a (dualizable) generalized Reedy structure.

A morphism of generalized Reedy categories $R \longrightarrow S$ is a functor which takes R^+ (resp. R^-) to S^+ (resp. S^-) and which preserves the degree.

Proposition 6.1.16. *D_0 is a dualizable generalized Reedy category.*

Proof. Let D_0^- be the subcategory of D_0 with the same objects as D_0 and morphisms $(G, V) \overset{(\alpha_1,\alpha_2)}{\longrightarrow} (H, V')$ where $\alpha_1 : H \longrightarrow G$ a group homomorphism and α_2 an isometric isomorphism. In other words, the morphisms of D_0^- are all restrictions. We have the commutative diagrams

$$\begin{tikzcd} G \arrow[r] & O(V) \arrow[d, "\alpha_{2*}"] \\ H \arrow[u, "\alpha_1"] \arrow[r] & O(V') \end{tikzcd} \tag{6.8}$$

The left vertical map is injective. So the order of H is no larger than that of G. So

$$deg(G,V) \geq deg(H,V').$$

By Lemma 6.1.14 we know that if $(G,V) \stackrel{(\alpha_1,\alpha_2)}{\longrightarrow} (H,V')$ is not an isomorphism, $deg(G,V) > deg(H,V')$.

Let D_0^+ be the subcategory of D_0 with the same objects as D_0 and morphisms

$$(G,V) \stackrel{(\phi_1,\phi_2)}{\longrightarrow} (H,W)$$

where $\phi_1 : H \cap O(\phi_{2*}V) \longrightarrow G$ is a group isomorphism. In other words, the morphisms in D_0^+ are linear isometric embeddings. Since G is isomorphic to a subgroup of H and $\dim V \leq \dim W$, so $deg(G,V) \leq deg(H,W)$. $deg(G,V) = deg(H,W)$ if and only if (G,V) is isomorphic to (H,W).

Any morphism $(\phi_1,\phi_2) : (G,V) \longrightarrow (H,W)$ has the decomposition

$$(G,V) \stackrel{(i,Id)}{\longrightarrow} (\phi_1(H \cap \phi_{2*}(G)),V) \stackrel{(\phi_1,\phi_2)}{\longrightarrow} (H,W) \tag{6.9}$$

where $i : \phi_1(H \cap \phi_{2*}(G)) \longrightarrow G$ is the inclusion. Note that in the second morphism (ϕ_1,ϕ_2), $\phi_1 : H \cap \phi_{2*}(G) \longrightarrow \phi_1(H \cap \phi_{2*}(G))$ is a group isomorphism.

If $(\phi_1,\phi_2) = (f_1,f_2) \circ (\alpha_1,\alpha_2)$ with $(\alpha_1,\alpha_2) : (G,V) \longrightarrow (G',V')$ in D_0^- and $(f_1,f_2) : (G',V') \longrightarrow (H,W)$ in D_0^+, $\alpha_2 : V \longrightarrow V'$ is an isometric isomorphism and $f_1 : H \cap O(f_{2*}(V')) \longrightarrow G'$ is a group isomorphism. The group homomorphisms α_1 (resp. f_1) is uniquely determined by α_2 (resp. f_2). Note that $O(f_2(V')) \cap H = O(\phi_2(V)) \cap H$. We have the commutative diagram

$$\begin{tikzcd}
\phi_1(H \cap O(\phi_2(V))) \arrow[r] & O(V) \arrow[d, "\alpha_2"] \\
G' \arrow[u, "\phi_1 \circ f_1^{-1}"] \arrow[r] & O(V')
\end{tikzcd}$$

So we have the isomorphism in D_0

$$(\phi_1 \circ f_1^{-1}, \alpha_2) : (\phi_1(H \cap O(\phi_2(V))),V) \longrightarrow (G',V').$$

The morphism $(i, Id) : (G, V) \longrightarrow (\phi_1(H \cap O(\phi_2(V))), V)$ equals to the composition $(\phi_1 \circ f_1^{-1}, \alpha_2)^{-1} \circ (\alpha_1, \alpha_2)$ and $(\phi_1, \phi_2) : (\phi_1(H \cap O(\phi_2(V))), V) \longrightarrow (H, W, \tau)$ is the composition $(f_1, f_2) \circ (\phi_1 \circ f_1^{-1}, \alpha_2)$. So the decomposition (6.9) is unique up to isomorphism.

Let $(\beta_1, \beta_2) : (G, V) \longrightarrow (G', V')$ be a morphism in both D_0^+ and D_0^-. Then β_1 and β_2 are both isomorphism. So (β_1, β_2) is an isomorphism. By Lemma 6.1.14, $D_0^+ \cap D_0^- = \mathrm{Iso}(D_0)$.

Let $\alpha = (\alpha_1, \alpha_2) : (G, V) \longrightarrow (G', V')$ be a morphism in D_0^- and $\theta' = (\theta'_1, \theta'_2) : (G', V') \longrightarrow (G', V')$ be a morphism in $\mathrm{Iso}(D_0)$. If $\theta' \circ \alpha = \alpha$, θ'_2 is the identity map. So θ'_1 is the identity.

Let $f = (f_1, f_2) : (G, V) \longrightarrow (H, W)$ be a morphism in D_0^+ and $\theta = (\theta_1, \theta_2) : (G, V) \longrightarrow (G, V)$ be a morphism in $\mathrm{Iso}(D_0)$. If $f \circ \theta = f$, θ_2 is the identity map. So θ_1 is the identity.

So D_0 is a dualizable generalized Reedy category in the sense of Definition 6.1.15.

□

We have the fully faithful functors

$$l : \mathbb{L} \longrightarrow D$$

sending V to $(O(V), V, Id)$, and

$$l_0 : \Sigma \longrightarrow D_0$$

from the category Σ of finite sets and injective maps to D_0, sending $\underline{n}$ to $(\Sigma_n, \mathbb{R}^n)$. They give the forgetful functors

$$\begin{aligned} \mathbb{U} : DT &\longrightarrow \mathbb{L}T \\ X &\mapsto X \circ l \end{aligned}$$

and

$$\begin{aligned} \mathbb{U}_0 : D_0T &\longrightarrow \Sigma T \\ Y &\mapsto Y \circ l_0 \end{aligned}$$

By the proposition below from [42], both forgetful functors have left adjoints and the unit of each adjunction is a natural isomorphism.

Proposition 6.1.17. *Let $l : C \longrightarrow D$ be a continuous functor between topological categories.*

If C is skeletally small, then $\mathbb{U} : DT \longrightarrow CT$ has a left adjoint prolongation functor $\mathbb{P} : CT \longrightarrow DT$. If $l : C \longrightarrow D$ is fully faithful, then the unit $\eta : Id \longrightarrow \mathbb{UP}$ of the adjunction is a natural isomorphism.

Proof. Please see Section 23 of [42]. □

6.2 Global Classifying Space

Example 6.2.1. *(i) Let (G, V) be an object of D_0, $\alpha : H \longrightarrow G$ be a group homomorphism from a finite group, then H acts on $(\alpha(H), V)$.*

(ii) Let (G, V) and (G, W) be two objects of D_0. Then $(G \times G, V \oplus W)$ is an objects in D_0. Let $\Delta : G \longrightarrow G \times G$ be the diagonal map. Then G acts on $(\Delta(G), V \oplus W)$ via its action on the $G-$representation $V \oplus W$.

(iii) Let $\mathcal{U}_G$ be a $G-$universe.

$$\mathcal{U}_G \cong \bigoplus_{\lambda \in \Lambda} \bigoplus_{1}^{\infty} \lambda$$

where Λ contains representatives of all irreducible $G-$representations. Let

$$\rho_\infty : G \longrightarrow O(\mathcal{U}_G)$$

denote the group homomorphism of the representation. Then G acts on $(\rho_\infty(G), \mathcal{U}_G)$ via its action on the $G-$universe.

We have the fundamental contractibility property below.

Proposition 6.2.2. *Let G be a compact Lie group, (V, ρ) a $G-$representation and $(\mathcal{U}, \rho_\infty)$ a $G-$universe such that V embeds into $\mathcal{U}$. Then the space $D_0((\rho(G), V), (\rho_\infty(G), \mathcal{U}))$, equipped with the $G-$action*

$$g \cdot (\phi_1, \phi_2) = (\rho(g)\phi_1(\rho_\infty(g)^{-1} - \rho_\infty(g))\rho(g)^{-1}, \rho_\infty(g)\phi_2(\rho(g)^{-1} -)), \tag{6.10}$$

is weakly $G-$contractible.

Proof. Let (U, τ) be a $G-$representation of finite or countably infinite dimension. Let K be a closed subgroup of G and $\Delta : G \longrightarrow G \times G$ the diagonal map. We have the continuous well-defined map

$$
\begin{aligned}
H : [0,1] \times D((\rho(G), V), (\tau(G), U))^K &\longrightarrow D((\rho(G), V), ((\tau \oplus \rho) \circ \Delta(G), U \oplus V))^K \\
H(t, \phi)_2(v) &= (t\phi_2(v), \sqrt{1-t^2}v); \\
H(t, \phi)_1 &= Id,
\end{aligned}
$$

which is a homotopy from the constant map

$$(Id, i_2 : V \longrightarrow U \oplus V)$$

to postcomposition with the map

$$(Id, i_1 : U \longrightarrow U \oplus V).$$

Since $(\mathcal{U}, \rho_\infty)$ is a $G-$universe such that V embeds into $\mathcal{U}$, we have $(\mathcal{U}, \rho_\infty) = (U \oplus V^\infty, \tau \oplus \rho_\infty)$ for some $G-$representation (U, τ).

$$
\begin{aligned}
D((\rho(G), V), (\rho_\infty(G), \mathcal{U}))^K &= D((\rho(G), V), ((\tau \oplus \rho^\infty) \circ \Delta(G), U \oplus V^\infty))^K \\
&= colim_{n \geqslant 0} D((\rho(G), V), ((\tau \oplus \rho^n) \circ \Delta(G), U \oplus V^n))^K,
\end{aligned}
$$

where the colimit is formed along the postcomposition maps with

$$((\tau \oplus \rho^n) \circ \Delta(G), U \oplus V^n) \longrightarrow ((\tau \oplus \rho^{n+1}) \circ \Delta(G), U \oplus V^{n+1}).$$

Each map in the colimit system is a closed embedding and homotopic to a constant map. So the colimit is weakly contractible. So $D((\rho(G), V), (\rho_\infty(G), \mathcal{U}))$ is weakly $G-$contractible.

□

Given an object $d = (G, V)$ in D_0, we can define the evaluation functor

$$ev_d : D_0T \longrightarrow GT$$

sending each X to $X(d)$, and its left adjoint $D_d : GT \longrightarrow D_0T$ sending a $G-$space A to

$$D_dA : b \mapsto D(d, b) \times A/(g \cdot \phi, a) \backsim (\phi, g \cdot a). \tag{6.11}$$

If we consider ev_d as the evaluation functor from D_0T to the category T, its left adjoint is the shift desuspension functor

$$F_d : T \longrightarrow D_0T$$

with

$$F_d(A)(b) = D_0(d, b) \wedge A. \tag{6.12}$$

Proposition 6.2.3. *Let (V, ρ) be a faithful $G-$representation, and $(\mathcal{U}_H, \tau_\infty)$ a complete $H-$universe. Let $d = (\rho(G), V)$ and $b = (\tau_\infty(H), \mathcal{U}_H)$ be objects in D_0.*

(i) The $(G \times H)-$space $D((\rho(G), V), (\tau_\infty(H), \mathcal{U}_H))$ is a universal $\mathcal{F}(H; G)-$space of the homotopy type of a $G \times H-CW$ complex, where $\mathcal{F}(H; G)$ consists of the subgroups Γ of $H \times G$ that intersect $G \times 1$ only in the identity element (e, e).

(ii) If (W, ρ') be another $G-$representation such that $d' = (\rho'(G), W)$ is an object in D_0, then the restriction map

$$\rho_{V,W}(\tau_\infty(H), \mathcal{U}_H) : D(((\rho \oplus \rho') \circ \Delta(G), V \oplus W), (\tau_\infty(H), \mathcal{U}_H)) \longrightarrow D((\rho(G), V), (\tau_\infty(H), \mathcal{U}_H))$$

is a $(G \times H)-$homotopy equivalence.

For each $G-$space A, the map

$$(\rho_{V,W} \times_G A)(b) : D_{((\rho \oplus \rho') \circ \Delta(G), V \oplus W)}(b) \longrightarrow (D_{(\rho(G), V)}A)(b)$$

is an $H-$homotopy equivalence and the morphism of D_0-spaces

$$\rho_{V,W} \times_G A : D_{((\rho \oplus \rho') \circ \Delta(G), V \oplus W)}A \longrightarrow D_{(\rho(G), V)}A$$

is a global equivalence.

Proof. (i) Let Γ be any closed subgroup of $G \times H$. If Γ intersects $G \times 1$ nontrivially, since V is a faithful $G-$representation, $D((\rho(G), V), (\tau_\infty(H), \mathcal{U}_H))^\Gamma$ is empty. If Γ intersects $G \times 1$ only in (e, e), Γ is the graph of a unique continuous homomorphism $\alpha : K \longrightarrow G$ where K is the projection of Γ to H.

$D((\rho(G),V),(\tau_\infty(H),\mathcal{U}_H))^\Gamma$ and $D((\rho\circ\alpha(K),V),(\tau_\infty(H),\mathcal{U}_H))^K$ are the same spaces, which can be checked directly from (6.2). Since $\mathcal{U}_H$ is a complete $H-$universe, the underlying $K-$universe is also complete, so $D((\rho\circ\alpha(K),V),(\tau_\infty(H),\mathcal{U}_H))^K$ is contractible by the proof of Proposition 6.2.2. So $D((\rho(G),V),(\tau_\infty(H),\mathcal{U}_H))^K$ is a universal $\mathcal{F}(H;G)-$space.

(ii) Since $V\oplus W$ is also a faithful $G-$representation, so $D(((\rho\oplus\rho')\circ\Delta(G),V\oplus W),(\tau_\infty(H),\mathcal{U}_H))$ is also a universal $\mathcal{F}(H;G)-$space. So the map

$$\rho_{V,W}(\tau_\infty(H),\mathcal{U}_H):D(((\rho\oplus\rho')\circ\Delta(G),V\oplus W),(\tau_\infty(H),\mathcal{U}_H))\longrightarrow D((\rho(G),V),(\tau_\infty(H),\mathcal{U}_H))$$

defined by postcomposition with the map $(Id,i:V\longrightarrow V\oplus W)$ is a $(G\times H)-$homotopy equivalence. The functor $-\times_G A$ preserves homotopy, so the restriction map $(\rho_{V,W}\times_G A)(\tau_\infty(H),\mathcal{U}_H)$ is an $H-$homotopy equivalence.

The D_0-spaces $D_{((\rho\oplus\rho')\circ\Delta(G),V\oplus W)}A$ and $D_{(\rho(G),V)}A$ are closed, so $\rho_{V,W}\times_G A$ is a global equivalence. □

For each $G-$space A, the map

$$\rho_{V,W}(\tau_\infty(H),\mathcal{U}_H)\times A/\sim:$$
$$D(((\rho\times\rho')\circ\Delta(G),V\oplus W),(\tau_\infty(H),\mathcal{U}_H))\times A/\sim\longrightarrow D((\rho(G),V),(\tau_\infty(H),\mathcal{U}_H))\times A/\sim$$

is an $H-$homotopy equivalence. Then $\rho_{V,W}\times A/\sim$ is a global equivalence. And we can define the $D-$global classifying space.

$$B^D_{gl}G=D((\rho(G),V),-)/G. \tag{6.13}$$

6.3 Level Model Structure

In [42] a "level model structure" is given to the category of $\mathcal{D}-$spaces for any skeletally small category $\mathcal{D}$.

I recall and apply the conclusions directly from [42].

Definition 6.3.1. *Let $f:X\longrightarrow Y$ be a morphism in the category D_0T.*

(i) f is level equivalence if for each object d in D_0, $f(d):X(d)\longrightarrow Y(d)$ is a weak equivalence.

(ii) f is level fibration if for each object d in D_0, $f(d) : X(d) \longrightarrow Y(d)$ is a Serre fibration.

(iii) f is level acyclic fibration if it is both a level equivalence and a level fibration.

(iv) f is a $q-$cofibration if it satisfies the left lifting property with respect to the level acyclic fibrations.

(v) f is a level acyclic $q-$cofibration if it is both a level equivalence and a $q-$cofibration.

Definition 6.3.2. *Let I denote the set of $h-$cofibrations $S^{n-1}_+ \longrightarrow D^n_+$, where $n \geq 0$ (when $n = 0$ interpreted as $* \longrightarrow S^0$).*

Let J be the set of $h-$cofibrations $i_0 : D^n_+ \longrightarrow (D^n \times I)_+$.

Define FI to be the set of all maps $F_d i$ with d an object in D_0 and $i \in I$. F_d is the functor defined by (6.12).

Define FJ to be the set of all maps $F_d j$ with d an object in D_0 and $j \in J$.

We have this model structure on the category T.

Theorem 6.3.3. *The category T is a compactly generated proper topological model category with respect to the weak equivalences, Serre fibrations, and retracts of relative $I-$cell complexes. The sets I and J are the generating $q-$cofibrations and the generating acyclic $q-$cofibrations.*

The level structure on D_0T inherits the model structure in Theorem 6.3.3.

Theorem 6.3.4. *The category of D_0-spaces is a compactly generated topological model category with respect to the level equivalences, level fibrations and $q-$cofibrations. It is right proper, and it is left proper. The sets FI and FJ are the generating $q-$cofibrations and generating acyclic $q-$cofibrations, and the following identifications hold.*

(i) The level fibrations are the maps that satisfy the right lifting property with respect to FJ or, equivalently, with respect to retracts of relative FJ-cell complexes, and all D_0-spaces are level fibrant.

(ii) The level acyclic fibrations are the maps that satisfy the right lifting property with respect to FI or, equivalently, with respect to retracts of relative FI-cell complexes.

(iii) The q-cofibrations are the retracts of relative FI-cell complexes.

(iv) The level acyclic $q-$cofibrations are the retracts of relative FJ-cell complexes.

(v) Any cofibrant D_0-space is non-degeneratly based.

6.4 Reedy Model Structure

In this section I construct the Reedy Model Structure of D_0T. The main principle is from Theorem 1.6 in [10]. But in my case, for each object d of D_0, the chosen projective model structure on $T^{Aut(d)}$ is the projective model structure in which the weak equivalences and fibrations are defined not forgetting the group action.

Let's recall some constructions on Reedy category first. Let R be any generalized Reedy category as in Definition 6.1.15. For each object r of R, let $R^+(r)$ denote the category with objects the non-invertible morphisms in R^+ with codomain r and morphisms from $\phi : b \longrightarrow r$ to $\phi' : b' \longrightarrow r$ all the morphisms $u : b \longrightarrow b'$ such that $\phi = \phi' \circ u$. The automorphism group $Aut(r)$ acts on $R^+(r)$ by composition.

For an $R-$space X The $r-$th latching object $L_r(X)$ of X is defined to be

$$L_r(X) = \varinjlim_{b \longrightarrow r} X_b \tag{6.14}$$

where the colimit is taken over the category $R^+(r)$. $L_r(X)$ has an $Aut(r)-$action.

Dually, we can define $R^-(r)$ to be the category with objects the non-invertible morphisms in R^- with domain r and morphisms from $\psi : r \longrightarrow b$ to $\psi' : r \longrightarrow b'$ all the morphisms $w : r \longrightarrow b'$ such that $\psi = w \circ \psi'$. $Aut(r)$ acts on $R^-(r)$ by precomposition.

For a $R-$space X, the $r-$th matching object $M_r(X)$ of X is defined to be

$$M_r(X) = \varprojlim_{r \longrightarrow b} X_b \tag{6.15}$$

where the limit is taken over the category $R^-(r)$. $Aut(r)$ acts on $M_r(X)$.

Note that for any object r of R and $R-$space X, the maps

$$L_r(X) \longrightarrow X_r \longrightarrow M_r(X)$$

is $Aut(r)-$equivariant. For a map $f : X \longrightarrow Y$ of $R-$spaces, these give the relative latching map

$$X_r \cup_{L_r(X)} L_r(Y) \longrightarrow Y_r$$

and the relative matching map

$$X_r \longrightarrow M_r(X) \times_{M_r(Y)} Y_r.$$

$L_r(f)$ and $M_r(f)$ are both $Aut(r)-$equivariant, so the two maps are also $Aut(r)-$equivariant.

Moreover, there is an alternative definition of latching and matching objects in [10], which is more global.

Definition 6.4.1. *Let R be a generalized Reedy category.*

For each natural number n, let $R_{\leq n}$ denote the full subcategory of R of objects of degree $\leq n$.

Let $G_n(R)$ denote the full subgroupoid of $Iso(R)$ with objects of degree n.

Let R_n denote the discrete category of objects of R of degree n.

Let $R^+((n))$ be the category with objects the non-invertible morphisms $u : b \longrightarrow r$ in R^+ such that $deg(r) = n$ and morphisms from u to u' the commutative square

$$\begin{tikzcd} b \arrow[r, "f"] \arrow[d, "u"] & b' \arrow[d, "u'"] \\ r \arrow[r, "g"] & r' \end{tikzcd}$$

such that $f \in R^+$ and $g \in G_n(R)$.

Let $R^+(n)$ denote the wide subcategory of $R^+((n))$ with morphisms for which g is an identity.

We have the relation

$$R^+(n) = \coprod_{deg(r)=n} R^+(r) \tag{6.16}$$

and the commutative diagram

$$\begin{tikzcd} R & R^+((n)) \arrow[l, "a_n"'] \arrow[r, "c_n"] & G_n(R) \arrow[r, "j_n"] & R \\ & R^+(n) \arrow[u, "k_n"] \arrow[r, "b_n"'] & R_n \arrow[u, "i_n"'] & \end{tikzcd} \tag{6.17}$$

where a_n is the domain-functor, b_n and c_n are codomain-functors, and i_n, j_n and k_n are inclusion-functors. c_n is cofibered and the square is a pullback.

Let X be a $R-$space and

$$X_n := j_n^*(X) = X|_{G_n(R)}. \tag{6.18}$$

For any functor $\alpha : A \longrightarrow B$ between small categories, let $\alpha! : T^A \longrightarrow T^B$ denote the left Kan extension along α.

Then the latching object

$$L_n(X) = (c_n)!a_n^*(X) \in T^{G_n(R)} \tag{6.19}$$

We have

$$L_n(X)_r = L_r(X). \tag{6.20}$$

Definition 6.4.2. *For each natural number n, let $R^-((n))$ be the category with objects the non-invertible morphisms $u : r \longrightarrow b$ in R^- such that $deg(r) = n$ and morphisms from u to u' the commutative squares*

$$\begin{tikzcd} r \arrow[r, "g"] \arrow[d, "u"] & r' \arrow[d, "u'"] \\ b \arrow[r, "f"] & b' \end{tikzcd}$$

such that $f \in R^-$ and $g \in G_n(R)$.

Let $D^-(n)$ be the wide subcategory of $D^-((n))$ with morphisms for which g is an identity.

We have the relation

$$R^-(n) = \coprod_{deg(r)=n} R^-(r) \tag{6.21}$$

and the commutative diagram

$$\begin{tikzcd} R & R^-((n)) \arrow[l, "\gamma_n"'] \arrow[r, "\delta_n"] & G_n(R) \arrow[r, "j_n"] & R \\ & R^-(n) \arrow[u, "\beta_n"] \arrow[r, "\kappa_n"'] & R_n \arrow[u, "i_n"'] & \end{tikzcd} \tag{6.22}$$

where γ_n is the codomain-functor, β_n and δ_n are domain-functors, and i_n, j_n and κ_n are inclusion-functors. δ_n is cofibered and the square is a pullback.

Then the matching object

$$M_n(X) = (\delta_n)_*\gamma_n^*(X) \in T^{G_n(R)} \tag{6.23}$$

We have

$$M_n(X)_r = M_r(X). \tag{6.24}$$

Each $T^{Aut(r)}$ admits a projective model structure with

• weak equivalences given by continuous $Aut(r)-$maps $f : A \longrightarrow B$ that induce weak homotopy equivalences $f^H : A^H \longrightarrow B^H$ on the $H-$fixed point spaces for each closed subgroup H of $Aut(r)$.

• fibrations given by continuous $Aut(r)-$maps $f : A \longrightarrow B$ that induce Serre fibration $f^H : A^H \longrightarrow B^H$ on the $H-$fixed point spaces for each closed subgroup H of $Aut(r)$.

This model structure is topological, proper and cofibrantly generated.

Definition 6.4.3. *Let R be a generalized Reedy category. A map $f : X \longrightarrow Y$ in RT is called a*

-Reedy cofibration if for each r, the relative latching map $X_r \coprod_{L_r(X)} L_r(Y) \longrightarrow Y_r$ is a $Aut(r)-$cofibration.

-Reedy weak equivalence if for each r,

$$f(r)^H : X(r)^H \longrightarrow Y(r)^H$$

is a weak equivalence for each closed subgroup H of $Aut(r)$.

-Reedy fibration if for each r, the relative matching map

$$X_r^H \longrightarrow M_r(X)^H \times_{M_r(Y)^H} Y_r^H$$

is a Serre fibration for each closed subgroup H of $Aut(r)$.

Definition 6.4.4. *A trivial (co)fibration is a Reedy (co)fibration which is also a Reedy weak equivalence.*

We have

Theorem 6.4.5. *Let R be a generalized Reedy category. The Reedy cofibrations, Reedy weak equivalences and Reedy fibrations form a model structure, the Reedy model structure, on the category of $R-$spaces.*

The proof of Theorem 6.4.5 is a slight variation of Theorem 1.6 in [10]. Before the proof, I show several lemmas needed for it.

Lemma 6.4.6. *A map $f : X \longrightarrow Y$ in RT is a Reedy cofibration (resp. a Reedy weak* *equivalence, resp. a Reedy fibration) if and only if, for each natural number n, the map*

$$X_n \cup_{L_n(X)} L_n(Y) \longrightarrow Y_n \ (resp.\ X_n \longrightarrow Y_n,\ resp.\ X_n \longrightarrow M_n(X) \times_{M_n(Y)} Y_n)$$

is a cofibration (resp. a Reedy weak equivalence, resp. a Reedy fibration) in $T^{G_n(R)}$.

Proof. This follows from the equivalence of categories

$$T^{G_n(R)} \xrightarrow{\sim} \prod_r T^{Aut(r)}$$

where r goes over a set of representatives for the connected components of the groupoid $G_n(R)$. □

Lemma 6.4.7. *Let $f : A \longrightarrow B$ be a trivial Reedy cofibration; suppose that, for each n, the induced map $L_n(f)_r : L_n(A)_r \longrightarrow L_n(B)_r$ is a trivial cofibration in $T^{Aut(r)}$ for each object r of R of degree n. Then $f : A \longrightarrow B$ has the left lifting property with respect to Reedy fibrations.*

Proof. Consider a commutative square in RT

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ B \arrow[r, "\beta"'] & X \end{tikzcd}$$

where f is a trivial Reedy cofibration and g is a Reedy fibration and for each object r of R of degree n, the induced map $L_n(f)_r : L_n(A)_r \longrightarrow L_n(B)_r$ is a trivial cofibration in $T^{Aut(r)}$. I construct a diagonal filler $\gamma : B \longrightarrow Y$ by constructing inductively a filler $\gamma_{\leq n} : B_{\leq n} \longrightarrow Y_{\leq n}$ on the full subcategory $R_{\leq n}$ of objects of R of degree $\leq n$.

For $n = 0$, we get a diagonal filler $\gamma_0 : B_0 \longrightarrow Y_0$ in

$$\begin{tikzcd} A_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & Y_0 \arrow[d, "g_0"] \\ B_0 \arrow[r, "\beta_0"'] & X_0 \end{tikzcd}$$

since $R_{\leq 0}$ is the groupoid $G_0(R)$, and $L_0(A) = 0$, and $M_0(X) = 1$, so that by hypothesis f_0 is a trivial cofibration in $T^{G_0(R)}$ and g_0 is a fibration in $T^{G_0(R)}$.

Assume by induction that a filler $\gamma_{\leq n-1} : B_{\leq n-1} \longrightarrow Y_{\leq n-1}$ has been found for

$$\begin{tikzcd}
A_{\leq n-1} \arrow[r, "\alpha_{\leq n-1}"] \arrow[d, "f_{\leq n-1}"'] & Y_{\leq n-1} \arrow[d, "g_{\leq n-1}"] \\
B_{\leq n-1} \arrow[r, "\beta_0"'] & X_{\leq n-1}.
\end{tikzcd}$$

This yields composite maps in $T^{G_n(R)}$

$$L_n(B) \longrightarrow L_n(Y) \longrightarrow Y_n \text{ and } B_n \longrightarrow M_n(B) \longrightarrow M_n(Y)$$

and the commutative square

$$\begin{tikzcd}
A_n \cup_{L_n(A)} L_n(B) \arrow[r] \arrow[d, "v_n"'] & Y_n \arrow[d, "w_n"] \\
B_n \arrow[r] & X_n \times_{M_n(X)} M_n(Y).
\end{tikzcd}$$

On each object r, the maps are all $Aut(r)-$equivariant.

Now we construct a $G_n(R)-$equivariant filler to complete the inductive step. By hypothesis v_n is a cofibration and w_n is a fibration in $T^{G_n(R)}$. It sufficient to check v_n is a weak equivalence. Consider the diagram below in $T^{Aut(r)}$. The square in it is a pushout.

$$\begin{tikzcd}
L_n(A)_r \arrow[r] \arrow[d, "L_n(f)_r"'] & A_r \arrow[r, "f_r"] \arrow[d] & B_r \\
L_n(B)_r \arrow[r] & (A \cup_{L_n(A)} L_n(B))_r. \arrow[ur, "(v_n)_r"'] &
\end{tikzcd}$$

Since by hypothesis $L_n(f)_r$ is a trivial cofibration, $A_r \longrightarrow (A \cup_{L_n(A)} L_n(B))_r$ is a weak equivalence in $T^{Aut(r)}$. And f_r is a weak equivalence in $T^{Aut(r)}$ by hypothesis, so v_n is a weak equivalence in $T^{Aut(r)}$.

$\square$

Lemma 6.4.8. *Let $f : A \longrightarrow B$ be a Reedy cofibration such that for all objects r of R of degree $< n$, $f_r : A_r \longrightarrow B_r$ is a weak equivalence in $T^{Aut(r)}$. Then, for each object b of R of degree n, the induced map $L_n(f)_b : L_n(A)_b \longrightarrow L_n(B)_b$ is a trivial cofibration in $T^{Aut(b)}$.*

Proof. Let's prove the lemma by induction.

For $n = 0$, the conclusion is obvious.

Assume by induction that $L_k(f)_b : L_k(A)_b \longrightarrow L_k(B)_b$ is a trivial cofibration of $k < n$ and any object b with degree k. We want to show that $i_n^* L_n(f)$ is a trivial cofibration in T^{R_n} where $i_n : R_n \longrightarrow G_n(R)$ is the inclusion. To this end we have to find a filler for any commutative square

$$\begin{tikzcd} i_n^* L_n(A) \arrow[r] \arrow[d, "i_n^* L_n(f)"'] & Y \arrow[d, "g"] \\ i_n^* L_n(B) \arrow[r] & X \end{tikzcd} \tag{6.25}$$

in T^{R_n} in which $g : Y \longrightarrow X$ is a fibration. As in (6.19), $i_n^* L_n = i_n^*(c_n)!a_n^* = (b_n)!k_n^* a_n^*$, a filler for the square (6.25) is the same as a filler for the square below in $T^{R^+(n)}$

$$\begin{tikzcd} k_n^* a_n^*(A) \arrow[r] \arrow[d, "k_n^* a_n^*(f)"'] & b_n^*(Y) \arrow[d, "b_n^*(g)"] \\ k_n^* a_n^*(B) \arrow[r] & b_n^*(X). \end{tikzcd}$$

The category $S = R^+(n)$ is a generalized Reedy category for which $S^+ = S$, $S^- = \mathrm{Iso}(S)$. In particular, the Reedy fibrations in ST are the same as Reedy fibrations in RT. So $b_n^*(g)$ is a Reedy fibration and $k_n^* a_n^*(f)$ is a Reedy cofibration whose induced maps on latching objects of degree $< n$ are trivial cofibrations.

Let $\phi = a_n k_n$. The functor $\phi_k^* : G^{G_k(R)} \longrightarrow T^{G_k(S)}$ induces a canonical isomorphism

$$L_k(\phi^*(A)) \cong \phi_k^*(L_k(A)).$$

Thus, the relative latching map

$$\phi^*(A) \cup_{L_k(\phi^*(A))} L_k(\phi^*(B)) \longrightarrow \phi^*(B)$$

may be identified with ϕ_k^* of the relative latching map $A_k \cup_{L_k(A)} L_k(B) \longrightarrow B_k$. Since $\phi_k : G_k(S) \longrightarrow G_k(R)$ is a faithful functor between groupoids, ϕ_k^* preserves cofibrations, thus $k_n^* d_n^*(f)$ is a Reedy cofibration. Since $L_k(A) \longrightarrow L_k(B)$ is a trivial cofibration for $k < n$, $L_k(\phi^*(A)) \longrightarrow L_k(\phi^*(B))$ is a trivial cofibration for $k < n$.

□

Lemma 6.4.9. *Let $g : Y \longrightarrow X$ be a trivial Reedy fibration. Suppose for each n, each object r of degree n, the induced map $M_n(g)_r : M_n(Y)_r \longrightarrow M_n(X)_r$ is a trivial fibration in*

$T^{Aut(r)}$.

Proof. The proof is dual to that of Lemma 6.4.7.

□

Lemma 6.4.10. *Let $g : Y \longrightarrow X$ be a Reedy fibration such that $g_r : Y_r \longrightarrow X_r$ is a weak equivalence for all objects r of R of degree $< n$. Then for each object b of degree n, the induced map $M_n(Y)_b \longrightarrow M_n(X)_b$ is a trivial fibration in $T^{Aut(b)}$.*

Proof. The proof is dual to that of Lemma 6.4.8.

□

Proof of Theorem 6.4.5. Limits and colimits in RT are constructed pointwise. The class of Reedy weak equivalences has the 2-out-of-3 property. Moreover, all three classes are closed under retract. It remains to be shown that the lifting and factorization axioms of a Quillen model category hold.

For the lifting axiom, by Lemma 6.4.8, trivial Reedy cofibrations fulfill the hypothesis of Lemma 6.4.7, and therefore have the left lifting property with respect to Reedy fibrations. Dually, Lemma 6.4.10 and 6.4.9 imply that trivial Reedy fibration have the right lifting property with respect to Reedy cofibrations.

Now let's show the factorization axiom holds. Let $f : X \longrightarrow Y$ be a map in RT, I will construct inductively a factorization $X \longrightarrow A \longrightarrow Y$ of f into a trivial Reedy cofibration followed by a Reedy fibration.

For $n = 0$, obviously f_0 can be factored in $T^{G_0(R)}$ as $X_0 \longrightarrow A_0 \longrightarrow Y_0$ into a trivial cofibration followed by a fibration.

Then assume $X_{\leq n-1} \longrightarrow A_{\leq n-1} \longrightarrow Y_{\leq n-1}$ is a factorization of $f_{\leq n-1}$ into trivial Reedy cofibration followed by Reedy fibration in $T^{R_{\leq n-1}}$ in $R_{\leq n-1}T$. Then we have the commutative diagram in $T^{G_n(R)}$.

$$\begin{tikzcd}
L_n(X) \arrow[r] \arrow[d] & L_n(A) \arrow[r] & L_n(Y) \arrow[d] \\
X_n \arrow[d] & & Y_n \arrow[d] \\
M_n(X) \arrow[r] & M_n(A) \arrow[r] & M_n(Y).
\end{tikzcd}$$

This diagram induces a map $X_n \cup_{L_n(X)} L_n(A) \longrightarrow M_n(A) \times_{M_n(Y)} Y_n$ which can be factored as a trivial cofibration followed by a fibration in $T^{G_n(R)}$

$$X_n \cup_{L_n(X)} L_n(A) \xrightarrow{\sim} A_n \longrightarrow M_n(A) \times_{M_n(Y)} Y_n.$$

The object A_n of $T^{G_n(R)}$ together with the maps $L_n(A) \longrightarrow A_n \longrightarrow M_n(A)$ define an extension of $A_{\leq n-1}$ to an object $A_{\leq n}$ in $R_{\leq n}T$ together with a factorization of $f_{\leq n}$: $X_{\leq n} \longrightarrow Y_{\leq n}$ into a Reedy cofibration $X_{\leq n} \longrightarrow A_{\leq n}$ followed by a Reedy fibration $A_{\leq n} \longrightarrow Y_{\leq n}$. The former map is a trivial Reedy cofibration because $X_n \longrightarrow A_n$ decomposes into $X_n \longrightarrow X_n \cup_{L_n(X)} L_n A \longrightarrow A_n$, the first map of which is a weak equivalence by Lemma 6.4.8, the second is by construction. So we get the required factorization of $f_{\leq n}$ in $R_{\leq n}T$.

The factorization of f into a Reedy cofibration followed by a trivial Reedy fibration is constructed dually using Lemma 6.4.10.

□

Next we describe explicit sets of generating cofibrations and generating acyclic cofibrations of the Reedy model category. First let's recall some relevant concepts and notions. The reference is [31].

Let d be an object of D_0 and A a $G-$space. Let F_A^d be the free diagram on A generated at d, i.e., the D_0-space defined by

$$F_A^d(b) = D_0(d, b) \times_G A.$$

Let $\partial D_0(d, b)$ be the subspace of $D_0(d, b)$ consisting of $g : d \longrightarrow b$ for which there is a factorization

$$g = \overrightarrow{g}\overleftarrow{g}$$

with $\overrightarrow{g} \in D_0^+$, $\overleftarrow{g} \in D_0^-$ and $\overleftarrow{g} \neq Id_d$. In other words, $\partial D_0(d, b)$ is the set of maps from d to b that factor through an object of degree less than that of d.

Let ∂F_*^d denote the boundary of F_*^d, i.e., a sub-diagram of F_*^d defined by

$$(\partial F_A^d)(b) = \partial D_0(d, b) \times_G A.$$

For a set of maps K in the category of topological spaces, let $RF_K^{D_0}$ denote the set of maps in D_0T of the form $F^d(i) : F^d_{A_k} \coprod_{\partial F^d_{A_k}} \partial F^d_{B_k} \longrightarrow F^d_{B_k}$ for d an object of D_0 and $i : A_k \longrightarrow B_k$ an element in K.

Proposition 6.4.11. *A map $f : A \longrightarrow B$ in RT is a trivial Reedy cofibration if and only if, for each object r of degree n, the relative latching map $(A_n \cup_{L_n(A)} L_n(B))_r \longrightarrow (B_n)_r$ is a trivial cofibration in $T^{Aut(r)}$.*

A map $g : Y \longrightarrow X$ in RT is a trivial Reedy fibration if and only if for each object r of degree n, the relative matching map $(Y_n)_r \longrightarrow (X_n \times_{M_n(X)} M_n(Y))_r$ is a trivial fibration in $T^{Aut(r)}$.

Proof. For each object r of degree n, the induced map $f_r : A_r \longrightarrow B_r$ in $T^{Aut(r)}$ factors as

$$A_r \xrightarrow{u_r} A_r \cup_{L_r(A)} L_r(B) \xrightarrow{v_r} B_r.$$

If f is a trivial Reedy cofibration then f_r is a weak equivalence, so by Lemma 6.4.8, u_r is a weak equivalence and hence v_r is a trivial cofibration.

Conversely, if each v_r is a trivial cofibration, then an induction on n based on Lemma 6.4.8 shows that u_r is a weak equivalence and hence f is a trivial Reedy cofibration.

The proof for the trivial Reedy fibration $g : Y \longrightarrow X$ is dual, using Lemma 6.4.10 instead of Lemma 6.4.8.

□

Proposition 6.4.12. *Let R be any generalized Reedy category and M a model category. If $A \longrightarrow B$ is a map in M and $X \longrightarrow Y$ is a map of $R-$diagrams in M, then for every object α of R the following are equivalent:*

(1) The dotted arrow exists in every solid arrow diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & Y_\alpha \times_{M_\alpha Y} M_\alpha X \end{tikzcd}$$

(2) The dotted arrow exists in every solid arrow diagram of the form

$$\begin{tikzcd} F^{\alpha}_{A}\coprod_{\partial F^{\alpha}_{A}}\partial F^{\alpha}_{B} \arrow[r] \arrow[d] & X \arrow[d] \\ F^{\alpha}_{B} \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

Corollary 6.4.13. *The Reedy model structure on D_0T is cofibrantly generated.*

Proof. We get the conclusion from Proposition 6.1.16, Proposition 6.4.11, Proposition 6.4.12 and that for any compact Lie group G, the projective model structure T^G is cofibrantly generated.

We describe the set of generating cofibrations and generating acyclic cofibrations explicitly.

For each object r in R, let I_r be the set of generating cofibrations in the projective model structure on the category $T^{Aut(r)}$ of $Aut(r)-$spaces and J_r be the set of generating acyclic cofibrations for the projective model structure on the category of $Aut(r)-$spaces.

Let $RF_I^{D_0}$ denote the set of maps in D_0T of the form $F^d(i) : F^d_{A_d}\coprod_{\partial F^d_{A_d}}\partial F^d_{B_d} \longrightarrow F^d_{B_d}$ for d an object of D_0 and $i : A_d \longrightarrow B_d$ an element in I_d.

Let $RF_J^{D_0}$ denote the set of maps in D_0T of the form $F^d(i) : F^d_{A_d}\coprod_{\partial F^d_{A_d}}\partial F^d_{B_d} \longrightarrow F^d_{B_d}$ for d an object of D_0 and $i : A_d \longrightarrow B_d$ an element in J_d.

$RF_I^{D_0}$ detects the acyclic fibrations in the Reedy model structure and $RF_J^{D_0}$ detects the fibrations in the Reedy model structure.

□

The *h-cofibrations* are the morphisms with the homotopy extension property and used in various categories, such as the category of $G-$spaces, orthogonal spaces and orthogonal spectra. The category of D_0-spaces is a category enriched over the category of spaces. First we recall the definition of h-cofibration in the standard way and some basic properties of it.

Definition 6.4.14. *A homotopy in D_0T is a morphism*

$$H : [0,1]\times A \longrightarrow X$$

defined on the pairing of the unit interval with a D_0-space.

A morphism $f : X \longrightarrow Y$ D_0T is a homotopy equivalence if there is a morphism $g : Y \longrightarrow X$ such that gf and fg are homotopic to the respective identity morphisms.

Remark 6.4.15. *If $f : X \longrightarrow Y$ is a homotopy equivalence, then for each $d = (G, V)$ and closed subgroup H of $Aut(d)$, $f(d)^H : X(d)^H \longrightarrow Y(d)^H$ is a homotopy equivalence. So each homotopy equivalence is a Reedy weak equivalence.*

Definition 6.4.16. *A morphism of D_0−spaces $f : A \longrightarrow B$ is an h-cofibration if it has the homotopy extension property, i.e. given a morphism $\beta : B \longrightarrow X$ and a homotopy $H : [0,1] \times A \longrightarrow X$ such that $H_0 = \beta \circ f$, there is a homotopy $\overline{H} : [0,1] \times B \longrightarrow X$ such that $\overline{H} \circ ([0,1] \times f) = H$ and $\overline{H}_0 = \beta$ where for $t \in [0,1]$, $H_t : A \longrightarrow X$ is the composition*

$$A \cong \{t\} \times A \overset{incl \times A}{\longrightarrow} [0,1] \times A \overset{H}{\longrightarrow} X.$$

We have the closure properties below of h-cofibrations.

Proposition 6.4.17. *Let $\mathcal{C}$ be a cocomplete category tensored and cotensored over the category of spaces.*

(i) the class of h-cofibrations in $\mathcal{C}$ is closed under retracts, cobase change, coproducts, sequential compositions and transfinite compositions.

(ii) Let $\mathcal{C}'$ be another category tensored and cotensored over the category of spaces, and $F : C \longrightarrow C'$ a continuous functor that commutes with colimits and tensors with $[0,1]$. Then F takes h−cofibrations in $\mathcal{C}$ to h−cofibration in $\mathcal{C}'$.

(iii) If $\mathcal{C}$ is topological model category in which every object is fibrant, then every cofibration is an h−cofibration.

Proof. See the proof of Corollary 4.14 in [56]. □

Proposition 6.4.18. *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of D_0−spaces such that g is Reedy weak equivalence. If in addition f or g is an h-cofibration, then the morphism k is a Reedy weak equivalence.

Proof. Let $d = (G, V, \rho)$ be an object of D_0. Then the square

$$\begin{tikzcd} A(d) \arrow[r, "f(d)"] \arrow[d, "g(d)"'] & B(d) \arrow[d, "h(d)"] \\ C(d) \arrow[r, "k(d)"'] & D(d) \end{tikzcd}$$

is a pushout square of $G-$spaces such that $f(d)$ or $h(d)$ is an h-cofibration of $G-$spaces by Proposition 6.4.17 (ii).

It yields a pushout square of spaces of $H-$fixed points

$$\begin{tikzcd} A(d)^H \arrow[r, "(f(d))^H"] \arrow[d, "(g(d))^H"'] & B(d)^H \arrow[d, "(h(d))^H"] \\ C(d)^H \arrow[r, "(k(d))^H"'] & (D(d))^H \end{tikzcd}$$

where H is any closed subgroup of G. Again by Proposition 6.4.17 (ii), $(f(d))^H$ or $(h(d))^H$ is an $h-$cofibration of spaces and $(f(d))^H$ is a weak equivalences. Thus, by the gluing lemma for weak equivalences and pushout along h-cofibrations shows that then $(k(d)^H$ is a weak equivalence. Hence the morphism k is a Reedy weak equivalence.

□

Definition 6.4.19. *Let $\mathcal{M}$ be a model category that is a category enriched, tensored and cotensored over the category of compactly generated weak Hausdorff spaces. We denote the tensor by $\times$. Given a continuous map of spaces $f : A \longrightarrow B$ and a morphism $g : X \longrightarrow Y$ in $\mathcal{M}$. Let $f\Box g$ denote the pushout product morphism defined by*

$$f\Box g = (f \times Y) \cup (A \times g) : A \times Y \cup_{A\times X} B \times X \longrightarrow B \times Y.$$

$\mathcal{M}$ is called topological if the following two conditions hold:

- *if f is a cofibration of spaces and g is a cofibration in $\mathcal{M}$, then the pushout product morphism $f\Box g$ is also a cofibration;*
- *if in addition f or g is a weak equivalence, then so is the pushout product morphism $f\Box g$.*

Let $i_k : \partial D^k \longrightarrow D^k$ and $j_k : D^k \times \{0\} \longrightarrow D^k \times [0, 1]$ be the inclusions. $\{i_k\}_{k\geqslant 0}$ is the standard set of generating cofibrations for the Quillen model structure on the category of spaces, and $\{j_k\}_{k\geqslant 0}$ is the standard set of generating acyclic cofibrations.

Proposition 6.4.20. *Let $\mathcal{M}$ be a model category that is a category enriched, tensored and cotensored over the category of spaces. Suppose that there is a set of objects $\mathcal{G}$ of $\mathcal{M}$ with the following properties:*

(a) The acyclic fibrations are characterized by the right lifting property with respect to the morphisms of the form $i_k \times K$ for all $k \geqslant 0$ and $K \in \mathcal{G}$.

(b) The fibrations are characterized by the right lifting property with respect to the morphisms of the form $j_k \times K$ for all $k \geqslant 0$ and $K \in \mathcal{G}$.

Then the model structure is topological.

We have

Theorem 6.4.21. *The Reedy model structure is topological and proper.*

Proof. Each D_0-space is fibrant, so the Reedy cofibrations are h-cofibrations by Proposition 6.4.17 (iii), and the Reedy model category is right proper. Left properness is a special case of Proposition 6.4.18.

The proof that the Reedy model category is topological is formal.

□

Proposition 6.4.22. *We have the Quillen pair between the strong level model structure on the category of symmetric spaces and the Reedy model structure of the D_0-spaces, which is not a Quillen equivalence*

$$P : \Sigma T \rightleftarrows D_0 T : \mathbb{U}_0 \tag{6.26}$$

$\mathbb{U}_0$ is fully faithful.

Proof. There is no matching object in ΣT. $\mathbb{U}_0$ sends Reedy weak equivalence to the strong level weak equivalence and sends Reedy fibrations to strong level fibrations. So $\mathbb{U}_0$ preserves both fibrations and trivial fibrations. So $(P, \mathbb{U}_0)$ is a Quillen pair. $\mathbb{U}_0$ is fully faithful by Proposition 6.1.17. □

6.5 Global Model Structure

In this section we construct the global model structure on the category of D_0-spaces.

Definition 6.5.1. *A morphism $f : X \longrightarrow Y$ of D_0-spaces is a global equivalence if for any object (G, V) in D_0, every closed subgroup H of G, every $k \geqslant 0$ and all continuous maps $\alpha : \partial D^k \longrightarrow X(G, V)^H$ and $\beta : D^k \longrightarrow Y(G, V)^H$ such that $f(G, V)^H \circ \alpha = \beta|_{\partial D^k}$, there is a $G-$representation W, a morphism $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (G, W)$ with $\phi_1 = Id_G$ and ϕ_2 the inclusion of $G-$subrepresentation, and a continuous map $\lambda : D^k \longrightarrow X(G, W)^H$ such that $\lambda|_{\partial D^k} = X(\phi)^H \circ \alpha$ and such that $f(G, W)^H \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\phi)^H \circ \beta$.*

In other words, for every commutative square

$$\begin{tikzcd}
\partial D^k \arrow[r] \arrow[d, "incl"] & X(G,V)^H \arrow[d, "f(G,V)^H"] \\
D^k \arrow[r] & Y(G,V)^H,
\end{tikzcd}$$

there exists an object (G, W) and a morphism $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (G, W)$ with $\phi_1 = Id_G$ and ϕ_2 the inclusion of $G-$subrepresentation, and a continuous map $\lambda : D^k \longrightarrow X(G, W)^H$ such that the diagram below commutes.

$$\begin{tikzcd}
\partial D^k \arrow[r] \arrow[d, "incl"] & X(G,V)^H \arrow[r, "X(\phi)^H"] & X(G,W)^H \arrow[d, "f(G,W)^H"] \\
D^k \arrow[r] \arrow[urr, dotted, "\lambda"] & Y(G,V)^H \arrow[r, "Y(\phi)^H"] & Y(G,W)^H
\end{tikzcd}$$

Definition 6.5.2. *A D_0-space Y is closed if it takes each morphism $\phi : (G, V) \longrightarrow (H, W)$ to a closed embedding $Y(\phi) : Y(G, V) \longrightarrow Y(H, W)$.*

Proposition 6.5.3. *Let $f : X \longrightarrow Y$ be a morphism between closed D_0-spaces. Then f is a global equivalence if and only if for any finite group G, any subgroup H of G, any $G-$universe $\mathcal{U}_G$ with the representation map*

$$\rho_\infty : G \longrightarrow O(\mathcal{U}_G),$$

the map

$$f(\rho_\infty(G), \mathcal{U}_G)^H : X(\rho_\infty(G), \mathcal{U}_G)^H \longrightarrow Y(\rho_\infty(G), \mathcal{U}_G)^H$$

is a weak equivalence.

Proof. The poset $s(\mathcal{U}_G)$ has a cofinal subsequence, so all colimits over $s(\mathcal{U}_G)$ can be realized as sequential colimits. The claim is then a straightforward consequence of the fact that compact spaces such as D^k and ∂D^k are finite with respect to sequences of closed embeddings. Points in compactly generated weak Hausdorff spaces are always closed, so the T_1-separation property holds.

$\square$

Proposition 6.5.4. *(i) Every Reedy weak equivalence is a global equivalence.*

(ii) The composite of two global equivalences is a global equivalence.

(iii) If f, g, and h are composable morphisms of D_0-spaces such that gf and hg are global equivalences, then f, g, h and hgf are also global equivalences.

(iv) Every retract of a global equivalence is a global equivalence.

(v) A coproduct of any set of global equivalences is a global equivalence.

(vi) A finite product of global equivalences is a global equivalence.

(vii) Let $f_n : Y_n \longrightarrow Y_{n+1}$ be a global equivalence of D_0-spaces that is objectwise a closed embedding, for $n \leq 0$. Then the canonical morphism $f_\infty : Y_0 \longrightarrow Y_\infty$ to the colimit of the sequence $\{f_n\}_{n\leq 0}$ is a global equivalence.

Proof. (i) Let $f : X \longrightarrow Y$ be a Reedy weak equivalence and $d = (G, V)$ an object in D_0. Let H be a closed subgroup of G. Let $\alpha : \partial D^k \longrightarrow X(d)^H$ and $\beta : D^k \longrightarrow Y(d)^H$ continuous maps such that $f(d)^H \circ \alpha = \beta|_{\partial D^k}$. Since f is a Reedy weak equivalence, the map $f(d)^H : X(d)^H \longrightarrow Y(d)^H$ is a weak equivalence, so there is a continuous map $\lambda : D^k \longrightarrow X(d)^H$ such that $\lambda|_{\partial D^k} = \alpha$ and $f(d)^H \circ \lambda$ is homotopic to β relative to ∂D^k. So the pair (Id_d, λ) solves the lifting problem, and hence f is a global equivalence.

(ii) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be global equivalences, $d = (G, V)$ be an object in D_0, and H a closed subgroup of G. Let $\alpha : D^k \longrightarrow X(d)^H$ and $\beta : D^k \longrightarrow Z(d)^H$ continuous maps such that $(gf)(d)^H \circ \alpha = \beta|_{\partial D^k}$. Since g is a global equivalence, the lifting problem $(f(d)^H \circ \alpha, \beta)$ has a solution $(\psi : d \longrightarrow b, \lambda : D^k \longrightarrow Y(b)^H)$ such that

$$\lambda|_{\partial D^k} = Y(\psi)^H \circ f(d)^H \circ \alpha = f(b)^H \circ X(\psi)^H \circ \alpha,$$

and $g(b)^G \circ \lambda$ is homotopic to $Z(\psi)^H \circ \beta$ relative ∂D^k. Since f is a global equivalence, the lifting problem $(X(\psi)^H \circ \alpha, \lambda)$ has a solution $(\phi : b \longrightarrow b', \lambda' : D^k \longrightarrow X(b')^H)$ such that

$$\lambda'|_{\partial D^k} = X(\phi)^H \circ X(\psi)^H \circ \alpha = X(\phi\psi)^H \circ \alpha$$

and such that $f(b')^H \circ \lambda'$ is homotopic to $Y(\phi)^H \circ \lambda$ relative ∂D^k. Then $(gf)(b')^H \circ \lambda'$ is homotopic, relative ∂D^k, to

$$g(b')^H \circ Y(\phi)^H \circ \lambda = Z(\phi)^H \circ g(b)^H \circ \lambda$$

which in turn is homotopic to $Z(\phi\psi)^H \circ \beta$, also relative ∂D^k. So the pair $(\phi\psi, \lambda')$ solves the original lifting problem for the morphism $gf : X \longrightarrow Z$.

(iii)

Claim 1: If $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow Q$ are composable morphisms such that $gf : X \longrightarrow Z$ and $hg : Y \longrightarrow Q$ are global equivalences, then f is a global equivalence.

Let $d = (G, V)$ be an object of D_0 and H any closed subgroup of G. Let $\alpha : \partial D^k \longrightarrow X(d)^H$ and $\beta : D^k \longrightarrow Y(d)^H$ continuous maps such that $f(d)^H \circ \alpha = \beta|_{\partial D^k}$. Since gf is a global equivalence and

$$(gf)(d)^H \circ \alpha = g(d)^H \circ f(d)^H \circ \alpha = g(d)^H \circ \beta|_{\partial D^k} = (g(d)^H \circ \beta)|_{\partial D^k},$$

the lifting problem $(\alpha, g(d)^H \circ \beta)$ has a solution $(\phi : d \longrightarrow b, \lambda : D^k \longrightarrow X(b)^H)$ such that $\lambda|_{\partial D^k} = X(\phi)^H \circ \alpha$ and $(gf)(b)^H \circ \lambda$ is homotopic to $Z(\phi)^H \circ g(d)^H \circ \beta$ relative ∂D^k. We let

$$H : D^k \times [0,1] \longrightarrow Z(b)^H$$

be a homotopy between $(gf)(b)^H \circ \lambda$ and $Z(\phi)^H \circ g(d)^H \circ \beta$ relative ∂D^k. Let $K : \partial D^k \times [0,1] \longrightarrow Y(b)^H$ be the constant homotopy of the map $f(b)^H \circ X(\phi)^H \circ \alpha = Y(\phi)^H \circ f(d)^H \circ \alpha$. We have the commutative diagram:

$$\begin{tikzcd}
(\partial D^k \times [0,1]) \cup (D^k \times \{0,1\}) \arrow[r, "K \cup (f(b)^H \circ \lambda) \cup (Y(\phi)^H \circ \beta)"] \arrow[d] & Y(b)^H \arrow[d, "(hg)(b)^H"] \\
D^k \times [0,1] \arrow[r, "h(b)^H \circ H"'] & Q(b)^H
\end{tikzcd}$$

Since the pair $(D^k \times [0,1], \partial D^k \times [0,1] \cup D^k \times \{0,1\})$ is homeomorphic to the pair $(D^{k+1}, \partial D^{k+1})$ and since hg is a global equivalence, there is a morphism $\psi : b \longrightarrow b'$ and a continuous map $\lambda' : D^k \times [0,1] \longrightarrow Y(b')^H$ such that

$$\lambda'|_{\partial D^k \times [0,1] \cup D^k \times \{0,1\}} = Y(\psi)^H \circ (K \cup (f(b)^H \circ \lambda) \cup (Y(\phi)^H \circ \beta)).$$

So λ' is a homotopy, relative ∂D^k, from

$$Y(\psi)^H \circ f(b)^H \circ \lambda = f(b')^H \circ X(\psi)^H \circ \lambda$$

to $Y(\psi\phi)^H \circ \beta$. So the pair $(\psi\phi, X(\psi)^H \circ \lambda)$ solves the original lifting problem for the morphism $f : X \longrightarrow Y$, and thus f is a global equivalence.

Claim 2: If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two composable morphisms such that gf and f are global equivalences, then g is a global equivalence.

Let $d = (G, V)$ be an object of D_0 and H a closed subgroup of G. Let $\alpha : \partial D^k \longrightarrow Y(d)^H$ and $\beta : D^k \longrightarrow Z(d)^H$ be two continuous maps such that $g(d)^H \circ \alpha = \beta|_{\partial D^k}$. For $k = 0$ the lifting data is determined by the point $\beta(0) \in Z(d)^H$. Since gf is a global equivalence, there is an object $b = (G, W)$ of D_0 and a morphism $\phi : d \longrightarrow b$ and a point $x \in X(b)^H$ such that $(gf)(b)^H(x)$ is in the same path component of $Z(b)^H$ as $Z(\phi)(\beta(0))$. The pair $(\phi, f(b)^H(x))$ then solves the original lifting problem.

Now suppose $k \leq 1$ and choose a point $x \in \partial D^k$. Since f is a global equivalence there is a morphism $\phi : d \longrightarrow b$ and a point $\widetilde{x} \in X(W)^G$ such that $f(b)(\widetilde{x})$ is in the same path component of $Y(b)^H$ as the point $Y(\phi)^H(\alpha(x))$. The homotopy extension property of the pair $(\partial D^k, \{x\})$ lets us replace $Y(\phi \circ \alpha)$ by a homotopic map $\overline{\alpha} : D^k \longrightarrow Y(b)^H$ such that $f(b)(\widetilde{x}) = \overline{\alpha}(x)$.

Let's choose a continuous surjection $\epsilon : D^{k-1} \longrightarrow \partial D^k$ such that ϵ factors over a homeomorphism $D^k/\partial D^{k-1} \cong \partial D^k$. Since f is a global equivalence, the lifting problem

$$\begin{tikzcd}
\partial D^{k-1} \arrow[r, "\mathrm{const}_{\widetilde{x}}"] \arrow[d] & X(b)^H \arrow[d, "f(b)^H"] \\
D^{k-1} \arrow[r, "\overline{\alpha}\circ\epsilon"'] & Y(b)^H
\end{tikzcd}$$

has a solution. There is an object $b' = (G, U)$ and a morphism $\psi : b \longrightarrow b'$ in D_0 and a continuous map $\lambda : D^{k-1} \longrightarrow X(b')^H$ such that $\lambda|_{\partial D^{k-1}}$ is the constant map with value $X(\psi)^H(\widetilde{x})$ and $f(b')^H \circ \lambda$ is homotopic to $Y(\psi)^H \circ \overline{\alpha} \circ \epsilon$ relative ∂D^{k-1}. Since λ is constant on ∂D^{k-1}, it factors uniquely as

$$\lambda = \widetilde{\alpha} \circ \epsilon$$

for a continuous map $\widetilde{\alpha} : \partial D^k \longrightarrow X(b')^H$, and then $f(b')^H \circ \widetilde{\alpha}$ is homotopic to $Y(\psi)^H \circ \overline{\alpha}$, and hence homotopic to $Y(\psi\phi)^H \circ \alpha$. Instead of solving the original lifting problem (α, β) for $g(d)^H$, it suffices to solve the lifting problem $(Y(\psi\phi)^H \circ \alpha, Z(\psi\phi)^H \circ \beta)$ for $g(b')^H$. We can prove in the same way that the original lifting problem for $g(d)^H$, the map $\alpha : \partial D^k \longrightarrow Y(d)^H$ is homotopic to a map of the form $f(d)^H \circ \widetilde{\alpha}$ for some continuous map $\widetilde{\alpha} : \partial D^k \longrightarrow X(d)^H$.

We use the homotopy extension property of the pair $(D^k, \partial D^k)$ to replace β by a homotopic map $\overline{\beta}$ such that $\overline{\beta}|_{\partial D^k} = (gf)(b)^H \circ \widetilde{\alpha}$. Since (α, β) is pair-homotopic to $(f(d)^H \circ \widetilde{\alpha}, \overline{\beta})$, it suffices to solve the lifting problem $(f(d)^H \circ \widetilde{\alpha}, \overline{\beta})$.

Since gf is a global equivalence, the lifting problem $(\widetilde{\alpha}, \overline{\beta})$ for $(gf)(d)^H$ has a solution $(\phi : d \longrightarrow b, \lambda : D^k \longrightarrow X(b)^H)$ such that $\lambda|_{\partial D^k} = X(\phi)^H \circ \widetilde{\alpha}$ and $(gf)(b)^H \circ \lambda$ is homotopic to $Z(\phi)^H \circ \overline{\beta}$ relative ∂D^k. Then

$$(f(b)^H \circ \lambda)|_{\partial D^k} = f(b)^H \circ X(\phi)^H \circ \widetilde{\alpha} = Y(\phi) \circ f(d)^H \circ \widetilde{\alpha},$$

so the pair $(\phi, f(b)^H \circ \lambda)$ solves the lifting problem $(f(d)^H \circ \widetilde{\alpha}, \overline{\beta})$, and thus g is a global equivalence.

Finally we prove the 2-out-of-6 property. Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, and $h : Z \longrightarrow Q$ be the three composable morphisms such that $gf : X \longrightarrow Z$ and $hg : Y \longrightarrow Q$ are global equivalences. Then f is a global equivalence by Claim 1, so g is a global equivalence by Claim 2. Apply Claim 2 again and we get h is a global equivalence. Therefore, hgf is a global equivalence by part (ii).

(iv) Let g be global equivalence and f a retract of g. So there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "j"'] & B \arrow[r, "s"'] & Y \end{tikzcd}$$

such that $ri = Id_X$ and $sj = Id_Y$. Let $d = (G, V)$ be an object of D_0 and $\alpha : \partial D^k \longrightarrow X(d)^H$ and $\beta : D^k \longrightarrow Y(d)^H$ continuous maps such that $f(d)^H \circ \alpha = \beta|_{\partial D^k}$. Since g is a global equivalence and

$$g(d)^H \circ i(d)^H \circ \alpha = j(d)^H \circ f(d)^H \circ \alpha = (j(d)^H \circ \beta)|_{\partial D^k},$$

there is a morphism $\phi : d \longrightarrow b$ and a continuous map $\lambda : D^k \longrightarrow A(b)^H$ such that $\lambda|_{\partial D^k} = A(\phi)^H \circ i(d)^H \circ \alpha$ and $g(b)^H \circ \lambda$ is homotopic to $B(\phi)^H \circ j(d)^H \circ \beta$ relative ∂D^k. Then

$$(r(b)^H \circ \lambda)|_{\partial D^k} = r(b)^H \circ A(\phi)^H \circ i(V) \circ \alpha = X(\phi) \circ r(d)^H \circ i(d)^H \circ \alpha = X(\phi) \circ \alpha$$

and

$$f(b)^H \circ r(b)^H \circ \lambda = s(b)^H \circ g(b)^H \circ \lambda$$

is homotopic to

$$s(b)^H \circ B(\phi)^H \circ j(d)^H \circ \beta = Y(\phi)^H \circ s(d)^H \circ j(d)^H \circ \beta = Y(\phi)^H \circ \beta$$

relative ∂D^k. So the pair $(\phi, r(b)^H \circ \lambda)$ solves the original lifting problem for the morphism $f : X \longrightarrow Y$. So f is a global equivalence.

(v) is true since D^k is connected so any lifting problem for a coproduct of D_0-spaces is located in one of the summands.

(vi) It suffices to consider a product of two global equivalences $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$. Since by (ii) global equivalences are closed under composition and

$$f \times f' = (f \times Y') \circ (X \times f'), \tag{6.27}$$

it suffices to show that for every global equivalence $f : X \longrightarrow Y$ and every D_0-space Z the morphism $f \times Z : X \times Z \longrightarrow Y \times Z$ is a global equivalence. Let (G, V) be any object in D_0, H any subgroup of G, and $\alpha : \partial D^k \longrightarrow (X \times Z)(G, V)^H$ and $\beta : D^k \longrightarrow Y \times Z(G, V)^H$ continuous maps such that $(f \times Z)(G, V)^H \circ \alpha = \beta|_{\partial D^k}$. For D_0-spaces we have the equality for fixed points

$$(X \times Z)(G, V)^H = X(G, V)^H \times Z(G, V)^H. \tag{6.28}$$

Therefore, we have $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ for continuous maps $\alpha_1 : \partial D^k \longrightarrow X(G, V)^H$, $\alpha_2 : \partial D^k \longrightarrow Z(G, V)^H$, $\beta_1 : D^k \longrightarrow Y(G, V)^H$ and $\beta_2 : D^k \longrightarrow Z(G, V)^H$. The relation $(f \times Z)(G, V)^H \circ (\alpha_1, \alpha_2) = (\beta_1, \beta_2)|_{\partial D^k}$ shows that $\alpha_2 = \beta_2|_{\partial D^k}$. Since f is a global equivalence, the lifting problem (α_1, β_1) for $f(G, V)$ has a solution $(\phi = (Id_G, \phi_2) :$

$(G,V) \longrightarrow (G,W), \lambda : D^k \longrightarrow X(G,W)^H)$, with ϕ_2 the inclusion of $G-$subrepresentation, such that $\lambda|_{\partial D^k} = X(\phi)^H \circ \alpha_1$ and $f(G,W)^H \circ \lambda$ is homotopic to $Y(\phi)^H \circ \beta_1$ relative ∂D^k. Then the pair $(\phi, (\lambda, Z(\phi) \circ \beta_2))$ solves the original lifting problem, so $f \times Z$ is a global equivalence.

(vii) Let (G,V) be any object in D_0 and $\alpha : \partial D^k \longrightarrow Y_0(G,V)^H$ and $\beta : D^k \longrightarrow Y_\infty(G,V)^H$ continuous maps such that $f_\infty(G,V)^H \circ \alpha = \beta|_{\partial D^k} : \partial D^k \longrightarrow Y_\infty(G,V)^H$. Since D^k is compact and $Y_\infty(G,V)$ is a colimit of a sequence of closed embeddings, the map β factors through a map $\overline{\beta} : D^k \longrightarrow Y_n(G,V)$ for some $n \leq 0$. Since the canonical map $Y_n(G,V) \longrightarrow Y_\infty(G,V)$ is injective, $\overline{\beta}$ lands in the $H-$fixed points and restricts to $((f_{n-1} \circ \cdots \circ f_0)(G,V))^H \circ \alpha$ on ∂D^k.

The composite $f_{n-1} \circ \cdots \circ f_0 : Y_0 \longrightarrow Y_k$ is a global equivalence by part (ii), so there is a morphism $\phi = (Id_G, \phi_2) : (G,V) \longrightarrow (G,W)$, with ϕ_2 the inclusion of $G-$subrepresentation, and a continuous map $\lambda : D^k \longrightarrow Y_0(G,W)^H$ such that $\lambda|_{\partial D^k} = Y_0(\phi) \circ \alpha$ and $((f_{n-1} \circ \cdots \circ f_0)(G,W))^H \circ \lambda$ is homotopic to $\overline{\beta}$ relative ∂D^k. So the pair (ϕ, λ) is also a solution for the original lifting problem and hence $f_\infty : Y_0 \longrightarrow Y_\infty$ is a global equivalence. □

Definition 6.5.5. *A morphism $f : X \longrightarrow Y$ in D_0T is a global fibration if it is a Reedy fibration and for each object $d = (G,V)$, $b = (G,W)$ of D_0, any closed subgroup H of G and any morphism $\phi = (\phi_1, \phi_2) : d \longrightarrow b$ with $\phi_1 = Id_G$, ϕ_2 an inclusion of $G-$subrepresentation, the map*

$$(f(d)^H, X(\phi)^H) : X(d)^H \longrightarrow Y(d)^H \times_{Y(b)^H} X(b)^H$$

is a weak equivalence.

In other words, a morphism f is a global fibration if and only if f is a Reedy fibration and for each object $d = (G,V)$, $b = (G,W)$ of D_0, any closed subgroup H of G and any morphism $\phi = (\phi_1, \phi_2) : d \longrightarrow b$ with $\phi_1 = Id_G$, the square of $H-$fixed point spaces

$$\begin{tikzcd} X(d)^H \arrow[r, "X(\phi)^H"] \arrow[d, "f(d)^H"'] & X(b)^H \arrow[d, "f(b)^H"] \\ Y(d)^H \arrow[r, "Y(\phi)^H"'] & Y(b)^H \end{tikzcd} \tag{6.29}$$

is a homotopy cartesian.

Definition 6.5.6. *A D_0-space X is static if for every object (G,V) of D_0, every morphism*

$\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (G, W)$ with $\phi_1 = Id_G$, ϕ_2 an inclusion of $G-$subrepresentation, the map $X(\phi) : X(G, V) \longrightarrow X(G, W)$ is a $G-$weak equivalence.

Proposition 6.5.7. *A D_0-space X is static if and only if the unique morphism from X to a terminal D_0-space is a global fibration.*

Proof. Straightforward from the definitions. □

Theorem 6.5.8. *The global equivalences, global fibrations and Reedy cofibrations form a model structure, the global model structure on the category of D_0-spaces. The fibrant objects in the global model structure are the static D_0-spaces. The global structure is proper, topological and compactly generated.*

Before the proof of Theorem 6.5.8, I state and prove several properties of global equivalences needed for the proof.

Proposition 6.5.9. *(i) Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of D_0-spaces such that f is a global equivalence. If in addition f or g is an $h-$cofibration, then the morphism k is a global equivalence.

(ii) Let

$$\begin{tikzcd} C & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ C' \arrow[from=u, "\gamma"'] & A' \arrow[l, "g'"] \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a commutative diagram of D_0-spaces such that g and g' are $h-$cofibartions. If the morphisms α, β and γ are global equivalences, then so is the induced morphism of pushouts

$$\gamma \cup \beta : C \cup_A B \longrightarrow C' \cup_{A'} B'.$$

(iii) Let

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"] & Y \end{tikzcd}$$

be a pullback square of D_0-spaces in which f is a global equivalence. If in addition one of the morphisms f or h is a Reedy fibration, then the morphism g is also a global equivalence.

(iv) Every global equivalence that is also a global fibration is a Reedy weak equivalence.

Proof. We use the Reedy model structure (Theorem 6.4.5) to construct a cofibrant replacement of the pushout square. More explicitly, we construct a Reedy weak equivalence $\alpha : A^b \longrightarrow A$ from a Reedy cofibrant D_0-space A^b. Then we factor

$$f \circ \alpha = \beta \circ f^b \text{ and } g \circ \alpha = \gamma \circ g^b$$

for some Reedy cofibrations $f^b : A^b \longrightarrow B^b$ and $g^b : A^b \longrightarrow C^b$ and Reedy weak equivalence $\beta : B^b \longrightarrow B$ and $\gamma : C^b \longrightarrow C$. We then have the commutative diagram

$$\begin{tikzcd} C^b \arrow[d, "\gamma"', "\sim"] & A \arrow[l, "g^b"'] \arrow[r, "f^b"] \arrow[d, "\sim"', "\alpha"] & B^b \arrow[d, "\sim"', "\beta"] \\ C & A \arrow[l, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

in which all vertical maps are Reedy weak equivalences, the D_0-spaces in the upper row are Reedy cofibrant and the morphisms g^b and f^b are Reedy cofibrations. As Reedy cofibrations, f^b and g^b are also $h-$cofibrations by Proposition 6.4.17 (iii).

Claim: The induced morphism

$$\gamma \cup \beta : C^b \cup_{A^b} B^b \longrightarrow C \cup_A B = E \tag{6.30}$$

is a Reedy weak equivalence.

Assume that the morphism g is an $h-$cofibration. Since f^b is an $h-$cofibration, so is its cobase change, the upper map in the commutative square below.

$$\begin{tikzcd} C^b \arrow[r] \arrow[d, "\gamma"', "\sim"] & C^b \cup_{A^b} B^b \arrow[d, "\gamma \cup B^b"] \\ C \arrow[r] & C \cup_{A^b} B^b \end{tikzcd}$$

Since γ is a Reedy weak equivalence, so is its cobase change by Proposition 6.4.18.

Since f^b is an $h-$cofibration and α is a Reedy weak equivalence, its cobase change $B^b \longrightarrow A \cup_{A^b} B^b$ is also Reedy weak equivalence by Proposition 6.4.18. Since $\beta : B^b \longrightarrow B$

is a Reedy weak equivalence, so is the morphism $f \cup \beta : A \cup_{A^b} B^b \xrightarrow{\sim} B$, by 2-out-of-3. The commutative square

$$\begin{tikzcd}
A \cup_{A^b} B^b \arrow[r] \arrow[d, "f\cup\beta"', "\sim"] & C \cup_{A^b} B^b \arrow[d, "C\cup\beta"] \\
B \arrow[r] & C \cup_A B
\end{tikzcd}$$

is a pushout where $f \cup \beta$ is a Reedy weak equivalence and the upper horizontal morphism is a $h-$cofibration since it's the cobase change of the $h-$cofibration $g : A \longrightarrow C$. So by Proposition 6.4.18, $C \cup \beta$ in the square is a Reedy weak equivalence.

Since the morphism (6.30) factors as the composite

$$C^b \cup_{A^b} B^b \overset{\gamma \cup B^b}{\longrightarrow} C \cup_{A^b} B^b \overset{C\cup\beta}{\longrightarrow} C \cup_A B$$

with both factors Reedy weak equivalences. So the claim is proved.

Now consider the commutative square

$$\begin{tikzcd}
C^b \arrow[r, "k^b"] \arrow[d, "\gamma"', "\sim"] & C^b \cup_{A^b} B^b \arrow[d, "\sim"', "\gamma\cup\beta"] \\
C \arrow[r, "k"'] & C \cup_A B = E
\end{tikzcd}$$

where both vertical morphisms are Reedy weak equivalences, hence global equivalences. So to show k is a global equivalence, we should show k^b is a global equivalence.

Assume without loss of generality that the D_0-spaces A, B and C are Reedy cofibrant and f is a Reedy cofibration. So the cobase change k is also a Reedy cofibration, and in particular, E is also Reedy cofibrant. For each $(G,\mathcal{U})$ with $\mathcal{U}$ a $G-$universe, the square

$$\begin{tikzcd}
A(G,\mathcal{U}) \arrow[r, "f(\mathcal{U})"] \arrow[d, "g(\mathcal{U})"'] & B(\mathcal{U}) \arrow[d, "h(\mathcal{U})"] \\
C(\mathcal{U}) \arrow[r, "k(\mathcal{U})"'] & D(\mathcal{U})
\end{tikzcd}$$

is a pushout. Since f is a Reedy cofibration, it's an $h-$cofibration of D_0-spaces by Proposition 6.4.18 (iii). So $f(\mathcal{U})$ is an $h-$cofibration of $G-$spaces. Since A and B are Reedy cofibrant, they are closed and $f(\mathcal{U})$ is a $G-$weak equivalence by Proposition 6.5.3.

The map $f(\mathcal{U})$ is then an $h-$cofibration and a $G-$weak equivalence, hence so is its cobase change $k(\mathcal{U})$. Since C and D are Reedy cofibrant, thus closed, by Proposition 6.5.3, so k is a global equivalence.

(ii) Choose a factorization $f = \overline{\beta}\overline{f}$ as the composite of an $h-$cofibration $\overline{f} : A \longrightarrow \overline{B}$ followed by a Reedy weak equivalence $\overline{\beta} : \overline{B} \longrightarrow B$ by the Reedy model structure, Theorem 6.4.5, and that all Reedy cofibrations are $h-$cofibrations by Proposition 6.4.18 (iii). The canonical morphism $\overline{B} \longrightarrow C \cup_A \overline{B}$ is an $h-$cofibration as a cobase change of the $h-$cofibartion g. The pushout square

$$\begin{tikzcd} \overline{B} \arrow[r, "\overline{\beta}"] \arrow[d] & B \arrow[d] \\ C \cup_A \overline{B} \arrow[r, "C\cup\overline{\beta}"'] & C \cup_A B \end{tikzcd}$$

then shows, using part (i), that $C\cup\overline{\beta}$ is a global equivalence as the cobase change of a global equivalence along an $h-$cofibration.

By replacing (B, f) by $(\overline{B}, \overline{f})$ we can thus assume without loss of generality that in addition to g and g', the morphism f is also an $h-$cofibration. The morphism $\gamma \cup \beta$ factors as the composite

$$C \cup_A B \xrightarrow{\gamma\cup B} C' \cup_A B \xrightarrow{C'\cup\beta} C' \cup_{A'} B'.$$

The morphism $\gamma \cup B$ participates in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[r, "\gamma"] \arrow[d] & C' \arrow[d] \\ B \arrow[r] & C \cup_A B \arrow[r, "\gamma\cup B"'] & C' \cup_A B \end{tikzcd}$$

in which both squares are pushouts. The canonical morphism from C to $C \cup_A B$ is an $h-$cofibration as a cobase change of the $h-$cofibration f. So $\gamma \cup B$ is a global equivalence, by part (i), as the cobase change of the global equivalence γ along an $h-$cofibration.

The canonical morphism $B \longrightarrow A' \cup_A B$ is a global equivalence, by part (i), as the cobase change of the global equivalence α along the $h-$cofibration f. The composite

$$B \longrightarrow A' \cup_A B \xrightarrow{f'\cup\beta} B'$$

is the global equivalence β, so the second morphism $f' \cup \beta$ is also a global equivalence. The morphism $C' \cup \beta$ participates in the commutative diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "g'"'] & A' \cup_A B \arrow[r, "f' \cup \beta"] \arrow[d, "g' \cup \beta"'] & B' \arrow[d] \\ C' \arrow[r] & C' \cup_A B \arrow[r, "C' \cup \beta"'] & C' \cup_A B' \end{tikzcd}$$

in which both squares are pushouts. The vertical morphism $g' \cup B$ is an $h-$cofibration as a cobase change of the $h-$cofibration g. So $C' \cup \beta$ is a global equivalence as the cobase change of a global equivalence along and $h-$cofibration. Now we have shown that both $\gamma \cup B$ and $C' \cup \beta$ are global equivalences, hence so is the composite $\gamma \cup \beta$.

(iii) Let $d = (G, V)$ be an object of D_0 and H a subgroup of G. Let $\alpha : \partial D^k \longrightarrow P(d)^H$ and $\beta : D^k \longrightarrow Z(d)^H$ be continuous maps such that $g(d)^H \circ \alpha = \beta|_{\partial D^k}$. Since f is a global equivalence, there is a morphism $\phi = (\phi_1, \phi_2) : d \longrightarrow b$ in D_0 with $\phi_1 = Id_G$ and ϕ_2 the inclusion of a $G-$subrepresentation, and a continuous map $\lambda : D^k \longrightarrow X(b)^H$ such that $\lambda_{\partial D^k} = X(\phi)^G \circ k(d)^H \circ \alpha$ and such that $f(b)^H \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\phi) \circ h(d)^H \circ \beta$. Let $H : D^k \times [0,1] \longrightarrow Y(b)^H$ be a relative homotopy from $Y(\phi) \circ h(d)^H \circ \beta = h(b)^H \circ Z(\phi) \circ \beta$ to $f(b)^H \circ \lambda$. Now we distinguish two cases.

Case 1: The morphism h is a Reedy fibration. Thenthe relative matching map

$$Z_b^H \longrightarrow M_b(Z)^H \times_{M_b(Z)^H} Z_b^H$$

is a Serre fibration. So $h(b)^H : Z(b)^H \longrightarrow Y(b)^H$ is a Serre fibration. We can choose a lift $\overline{H}$ in the square

$$\begin{tikzcd} D^k \times 0 \cup_{\partial D^k \times 0} \partial D^k \times [0,1] \arrow[r, "Z(\phi)\circ\beta\cup K"] \arrow[d, "\sim"'] & Z(b)^H \arrow[d, "h(b)^H"] \\ D^k \times [0,1] \arrow[r, "H"'] \arrow[ur, dotted, "\overline{H}"] & Y(b)^H \end{tikzcd}$$

where $K : \partial D^k \times [0,1] \longrightarrow Z(b)^H$ is the constant homotopy from $g(b)^H \circ P(\phi)^H \circ \alpha$ to itself. Since the square is a pullback and $h(b)^H \circ \overline{H}(-,1) = H(-,1) = f(b)^H \circ \lambda$, there is a unique continuous map $\overline{\lambda} : D^k \longrightarrow P(b)^H$ that satisfies

$$g(b)^H \circ \overline{\lambda} = \overline{H}(-,1) \text{ and } k(b)^H \circ \overline{\lambda} = \lambda.$$

The restriction of $\overline{\lambda}$ to ∂D^k satisfies

$$g(b)^H \circ \overline{\lambda}|_{\partial D^k} = \overline{H}(-,1)|_{\partial D^k} = g(b)^H \circ P(\phi)^H \circ \alpha \text{ and}$$
$$k(b)^H \circ \overline{\lambda}|_{\partial D^k} = \lambda|_{\partial D^k} = X(\phi)^H \circ k(d)^H \circ \alpha = k(b)^H \circ P(\phi)^H \circ \alpha.$$

The pullback property thus implies that $\overline{\lambda}|_{\partial D^k} = P(\phi)^H \circ \alpha$.

Finally the composite $g(b)^H \circ \overline{\lambda}$ is homotopic, relative ∂D^k and via $\overline{H}$, to $\overline{H}(-,0) = Z(\phi)^H \circ \beta$. This is the required lifting data, and we have verified the defining property of a global equivalence for the morphism g.

Case 2: If the morphism f is a Reedy fibration. The argument is similar to that of Case 1. Now $f(b)^H : X(b)^H \longrightarrow Y(b)^H$ is a Serre fibration. We can choose a lift H' in the square

$$\begin{tikzcd}
D^k \times 1 \cup_{\partial D^k \times 1} \partial D^k \times [0,1] \arrow[r, "\lambda \cup K'"] \arrow[d, "\sim"'] & Z(b)^H \arrow[d, "f(b)^H"] \\
D^k \times [0,1] \arrow[r, "H"'] \arrow[ur, dotted, "H'"'] & Y(b)^H
\end{tikzcd}$$

where $K' : \partial D^k \times [0,1] \longrightarrow X(b)^H$ is the constant homotopy from $X(\phi)^H \circ k(d)^H \circ \alpha$ to itself. Since the square is a pullback and $f(b)^H \circ H'(-,0) = H(-,0) = H(b)^H \circ Z(\phi)^H \circ \beta$, there is a unique continuous map $\overline{\lambda} : D^k \longrightarrow P(b)^H$ that satisfies

$$g(b)^H \circ \overline{\lambda} = Z(\phi)^H \circ \beta \text{ and } k(b)^H \circ \overline{\lambda} = H'(-,0).$$

The restriction of $\overline{\lambda}$ to ∂D^k satisfies

$$g(b)^H \circ \overline{\lambda}|_{\partial D^k} = Z(\phi)^H \circ g(d)^H \circ \alpha = g(b)^H \circ P(\phi)^H \circ \alpha \text{ and}$$
$$k(b)^H \circ \overline{\lambda}|_{\partial D^k} = H'(-,0)|_{\partial D^k} = X(\phi)^H \circ k(d)^H \circ \alpha = k(b)^H \circ P(\phi)^H \circ \alpha.$$

The pullback property implies that $\overline{\lambda}|_{\partial D^k} = P(\phi)^H \circ \alpha$. Since $g(b)^H \circ \overline{\lambda} = Z(\phi)^H \circ \beta$, this is the required lifting data, and we have verified the defining property of a global equivalence for the morphism g.

(iv) Let $f : X \longrightarrow Y$ be a morphism of D_0-spaces that is both a global fibration and a global equivalence. Let $d = (G, V)$ be an object in D_0 and H a closed subgroup of G.

Let $\alpha : \partial D^k \longrightarrow X(d)^H$ and $\beta : D^k \longrightarrow Y(d)^H$ be continuous maps such that the diagram below commutes.

$$\begin{tikzcd}
\partial D^k \arrow[r, "\alpha"] \arrow[d, "i_k"'] & X(d)^H \arrow[d, "f(d)^H"] \\
D^k \arrow[r, "\beta"'] & Y(d)^H
\end{tikzcd}$$

We will exhibit a continuous map $\mu : D^k \longrightarrow X(d)^H$ such that $\mu|_{\partial D^k} = \alpha$ and $f(d)^H \circ \mu$ is homotopic to β, relative ∂D^k. Then $f(d)^H$ is a weak equivalence, so f is Reedy weak equivalence.

Since f is a global equivalence, there is an object $b = (G, W)$ in D_0 and a morphism $\phi = (\phi_1, \phi_2) : d \longrightarrow b$ with $\phi_1 = Id_G$ and ϕ_2 the inclusion of a $G-$subrepresentation, and a continuous map $\lambda : D^k \longrightarrow X(b)^H$ such that $\lambda|_{\partial D^k} = X(\phi)^H \circ \alpha : \partial D^k \longrightarrow X(b)^H$ and such that $f(b)^H \circ \lambda : D^k \longrightarrow Y(b)^H$ is homotopic to $Y(\phi) \circ \beta$, relative to ∂D^k. Since f is a Reedy fibration, so $f(b)^H : X(b)^H \longrightarrow Y(b)^H$ is a Serre fibration. So λ can be improved into a continuous map $\lambda' : D^k \longrightarrow X(b)^H$ such that $\lambda'|_{\partial D^k} = \lambda|_{\partial D^k} = X(\phi)^H \circ \alpha$ and such that $f(b)^H \circ \lambda'$ is equal to $Y(\phi)^H \circ \beta$.

Since f is a global fibration the morphism

$$(f(d)^H, X(\phi)^H) : X(d)^H \longrightarrow Y(d)^H \times_{Y(b)^H} X(b)^H$$

is a weak equivalence. So there is a continuous map $\mu : D^k \longrightarrow X(d)^H$ such that $\mu|_{\partial D^k} = \alpha$ and $(f(d)^H, X(\phi)^H) \circ \mu$ is homotopic, relative ∂D^k to $(\beta, \lambda') : D^k \longrightarrow Y(d)^H \times_{Y(b)^H} X(b)^H$:

$$\begin{tikzcd}
\partial D^k \arrow[r, "\alpha"] \arrow[d, "i_k"'] & X(d)^H \arrow[d, "{(f(d)^H, X(\phi)^H)}"] \\
D^k \arrow[r, "{(\beta,\lambda')}"'] \arrow[ur, "\mu"] & Y(d)^H \times_{Y(b)^H} X(b)^H
\end{tikzcd}$$

This is the μ we want.

$\square$

Definition 6.5.10. *Let $j : A \longrightarrow B$ be a morphism in a topological model category. j can be factored as the composite through the mapping cylinder*

$$A \xrightarrow{c(j)} Z(j) = ([0,1] \times A) \cup_j B \xrightarrow{r(j)} B,$$

where $c(j)$ is the front mapping cylinder inclusion and $r(j)$ is the projection, which is homotopy equivalence. If both A and B are cofibrant, the morphism $c(j)$ is a cofibration by the pushout product property. Let j be a morphism between cofibrant objects. Define $Z(j)$ to be the set of all pushout product maps

$$i_k \square c(j) : D^k \times A \cup_{\partial D^k \times A} \partial D^k \times Z(j) \longrightarrow D^k \times Z(j)$$

for $k \leq 0$ and $i_k : \partial D^k \longrightarrow D^k$ is the inclusion.

Proposition 6.5.11. *Let $\mathcal{C}$ be a topological model category, $j : A \longrightarrow B$ a morphism between cofibrant objects and $f : X \longrightarrow Y$ a fibration. Then the following two conditions are equivalent:*

(i) The square of spaces

$$\begin{tikzcd} map(B,X) \arrow[r, "map(j{,}X)"] \arrow[d, "map(B{,}f)"'] & map(A,X) \arrow[d, "map(A{,}f)"] \\ map(B,Y) \arrow[r, "map(j{,}Y)"'] & map(A,Y) \end{tikzcd} \tag{6.31}$$

is homotopy cartesian.

(ii) The morphism f has the right lifting property with respect to the set $Z(j)$.

Proof. The proof is that of Proposition 5.5, Chapter I, in [56].

□

Let J denote the set of generating trivial cofibrations in the Reedy model structure, i.e. $RF_J^{D_0}$ described in the proof of Corollary 6.4.13. Let K denote the set of morphisms that detect the squares (6.29) are homotopy cartesian. Let (V, ρ) and (W, ρ') be $G-$representations such that $d = (\rho(G), V)$ and $b = ((\rho \oplus \rho') \circ \Delta(G), V \oplus W)$ be two objects in D_0 and let $\phi : d \longrightarrow b$ be the morphism (Id_G, i_V) with $i_V : V \longrightarrow V \oplus W$ the inclusion. Then we have the restriction morphism

$$\rho_{G,V,W} = \phi^* : D_b* \longrightarrow D_d*$$

where $*$ is the one-point $G-$space. This morphism is a global equivalence by Proposition 6.2.3 (ii). Let

$$K := \bigcup_{G,V,W} Z(\rho_{G,V,W}), \tag{6.32}$$

the set of all pushout products of boundary inclusions $\partial D^k \longrightarrow D^k$ with the mapping cylinder inclusions of the morphisms $\rho_{G,V,W}$, where the union goes over a set of representatives of the isomorphism classes of triples (G, V, W) with (G, V) and (G, W) objects of D_0. By Proposition 6.5.11 the right lifting property with respect to the union $J \cup K$ characterizes the global fibrations.

Proposition 6.5.12. *A morphism of D_0-spaces is a global fibration if and only if it has the right lifting property with respect to the set $J \cup K$.*

Proof of Theorem 6.5.8. The numbering of the model category axioms is as that in Definition 3.3, [18].

The category of D_0-spaces is complete and cocomplete, so axiom MC1 holds. By Proposition 6.5.4 (iii), global equivalences have the 2-out-of-6 property, so they satisfy the model category axiom 2-out-of-3(MC2). By Proposition 6.5.4(iv), global equivalences are closed under retracts. It's straightforward that Reedy cofibrations and global fibrations are closed under retracts, so MC3 holds.

By the Reedy model structure, each morphism in D_0T can be factored as a Reedy cofibration followed by a Reedy weak equivalence. Since by Proposition 6.5.4 Reedy weak equivalences are global equivalences, we get one of the factorizations required by MC5. For the other half of MC5, apply the small object argument to the set $J \cup K$. All morphisms in J are Reedy cofibrations and Reedy weak equivalence. Since D_b* and D_d* are Reedy cofibrant, the morphisms in K are also Reedy cofibrations, and they are global equivalences because the morphisms $\rho_{G,V,W}$ are. The small object argument provides a functorial factorization of every morphism $\phi : X \longrightarrow Y$ of D_0-spaces as composite

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in K and q has the right lifting property with respect to $J \cup K$. Since all morphisms in K are flat cofibrations and global equivalences, the morphism i is a Reedy cofibration by the closure properties of Proposition 6.5.9. Moreover, q is a global fibration by Proposition 6.5.12.

Now we show the lifting properties of MC4. By Proposition 6.5.9 (iv) a morphism that is both a global fibration and a global equivalence is a Reedy weak equivalence, and hence an acyclic fibration in the strong level model structure. So every morphism that is simultaneously global fibrations and a global equivalence has the right lifting property with respect to Reedy cofibrations. Now let $j : A \longrightarrow B$ be a Reedy cofibration that is also a global equivalence. I show that it has the left lifting property with respect to all global fibrations. Factor $j = q \circ i$, via the small object argument for $J \cup K$, where $i : A \longrightarrow W$ is an $J \cup K-$cell complex and $q : W \longrightarrow B$ a global fibration. Then q is a global equivalence since j and i are, and hence an acyclic fibration in the strong level model structure by Proposition 6.5.9 (iv). Since j is a Reedy cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, "="] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus j is a retract of the morphism i that has the left lifting property with respect to all global fibrations. But then j itself has the lifting property. Thus we verified all the model category axioms. Meanwhile We have also specified sets of generating Reedy cofibrations I and generating acyclic cofibrations $J \cup K$. Sources and targets of all morphisms in these sets are small with respect to sequential colimits of Reedy cofibrations. So the global model structure is compactly generated.

Left properness of the global model structure follows from Proposition 6.5.9 (i) and the fact that Reedy cofibrations are $h-$cofibrations by Proposition 6.4.17 (iii). Right properness follows from Proposition 6.5.9 (iii) since global fibrations are Reedy fibrations.

The proof the the global model structure is topological is formal.

□

Proposition 6.5.13. *Let $f : A \longrightarrow B$ be a morphism of D_0-spaces. Then the following conditions are equivalent.*

(i) The morphism f is a global equivalence.

(ii) For some (hence any) Reedy cofibrant approximation $f^b : A^b \longrightarrow B^b$ in the Reedy model structure and every fibrant object X the induced map

$$[f^b, X] : [B^b, X] \longrightarrow [A^b, X]$$

on homotopy classes of morphism is a bijection.

Proof. By Proposition 6.5.4 (i), the morphism f is a global equivalenc if and only if the

Reedy cofibrant approximation $f^b : A^b \longrightarrow B^b$ is a global equivalence. Since A^b and B^b are both Reedy cofibrant, they are cofibrant in the global model structure. So f^b is a global equivalence if and only the induced map $[f^b, X]$ is bijective for every fibrant object in the global model structure. By Theorem 6.5.8 these fibrant objects are precisely the static D_0-spaces.

□

6.6 Relation between D_0-spaces and $I-$spaces

In this section I show the global model structure of $I-$spaces and the global model structure on a subcategory of D_0-spaces are Quillen equivalent.

Before that let's recall basic concepts and properties of $I-$spaces, the main reference of which is [55]. I omit most of the proofs.

Definition 6.6.1 ($I-$spaces). *Let I denote the category of finite sets and injective maps. An $I-$ space is a functor from the category I to the category T of spaces.*

The category I is symmetric monoidal under disjoint union. The category of $I-$spaces thus inherits a Day convolution product $\boxtimes$.

Definition 6.6.2. *(i) A morphism $f : Y \longrightarrow Z$ of $I-$spaces is a strong level equivalence if for every finite set A the map $f(A) : Y(A) \longrightarrow Z(A)$ is a Σ_A-weak equivalence.*

(ii) A morphism $f : Y \longrightarrow Z$ of $I-$spaces is a strong level fibration if for every finite set A the map $f(A) : Y(A) \longrightarrow Z(A)$ is a Σ_A-fibration.

(iii) A morphism $f : Y \longrightarrow Z$ of $I-$spaces is a flat cofibration if for every finite set A the latching morphism $Y(A) \cup_{L_A Y} L_A Z \longrightarrow Z(A)$ is a Σ_A-cofibration.

Proposition 6.6.3. *The strong level equivalences, strong level fibrations and flat cofibrations form a model structure, the strong level model structure, on the category of $I-$spaces. The strong level model structure is proper, topological, cofibrantly generated and satisfies the pushout product property with respect to the box product.*

The global model structure for $I-$spaces is obtained from the strong level model structure by Bousfield localization with the appropriated class of static $I-$spaces as local objects.

Definition 6.6.4. *An $I-$space Z is static if for every finite group G, every faithful finite $G-$set A and every monomorphism of finite $G-$sets $\phi : A \longrightarrow B$ the structure map $Z(\phi) : Z(A) \longrightarrow Z(B)$ is a $G-$weak equivalence.*

Definition 6.6.5 (global equivalence of $I-$spaces). *A morphism of $f : X \longrightarrow Y$ of $I-$spaces is a global equivalence if for some (hence any) flat approximation $f^b : X^b \longrightarrow Y^b$ in the strong level model structure and every static $I-$space Z the induced map*

$$[f^b, Z] : [Y^b, X] \longrightarrow [X^b, Z]$$

on homotopy classes of morphisms is bijective.

Definition 6.6.6 (global fibration of $I-$spaces). *A morphism $f : X \longrightarrow Y$ of $I-$spaces is a global fibration if it is a strong level fibration and for every finite group G, every faithful finite $G-$set A and every monomorphism of finite $G-$sets $\phi : A \longrightarrow B$ the square of $G-$fixed point spaces*

$$\begin{tikzcd} X(A)^G \arrow[r, "X(\phi)^G"] \arrow[d, "f(A)^G"'] & X(B)^G \arrow[d, "f(B)^G"] \\ Y(A)^G \arrow[r, "Y(\phi)^G"'] & Y(B)^G \end{tikzcd}$$

is homotopy cartesian, in other words,

$$(f(A)^G, X(\phi)^G) : X(A)^G \longrightarrow Y(A)^G \times_{Y(B)^G} X(B)^G$$

is a weak equivalence.

Theorem 6.6.7 (global model structure for $I-$spaces). *The global equivalences, global fibrations and flat cofibrations form the global model structure on the category of $I-$spaces. The fibrant objects in the global model structure are the static $I-$spaces. The global model structure is proper, topological, compactly generated, and it satisfies the pushout product and monoid axioms with respect to the box product of $I-$spaces.*

The global model structure of $I-$spaces is Quillen equivalent to a model structure on the category of orthogonal spaces, $\mathcal{F}in-$global model structure. I briefly introduce it below. My reference is [55].

Definition 6.6.8. *Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is*

- *an $\mathcal{F}-$level equivalence if for every compact Lie group G in $\mathcal{F}$ and every $G-$representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence;*

- *an $\mathcal{F}-$level fibration if for every compact Lie group G in $\mathcal{F}$ and every $G-$representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a Serre fibration;*
- *an $\mathcal{F}-$cofibration if the latching morphism $X(\mathbb{R}^m) \cup_{L_{\mathbb{R}^m}X} L_{\mathbb{R}^m}Y \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F} \cap O(m)-$cofibration for all $m \leq 0$.*

Proposition 6.6.9. *Let $\mathcal{F}$ be a global family. The $\mathcal{F}-$level equivalences, $\mathcal{F}-$level fibrations and $\mathcal{F}-$cofibrations form a model structure, the $\mathcal{F}-$level model structure, on the category of orthogonal spaces. The $\mathcal{F}-$level model structure is proper, topological and cofibrantly generated.*

Definition 6.6.10. *Let $\mathcal{F}$ be a global family.*

(i) A morphism $f : X \longrightarrow Y$ of orthogonal spaces is an $\mathcal{F}-$global equivalence if for every compact Lie group G in $\mathcal{F}$, every $G-$representation V, every $k \leq 0$ and all maps $\alpha : \partial D^k \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ such that $f(V)^G \circ \alpha = \beta|_{\partial D^k}$ there is a $G-$representation W, a $G-$equivariant linear isometric embedding $\phi : V \longrightarrow W$ and a continuous map $\lambda : D^k \longrightarrow X(W)^G$ such that $\lambda_{\partial D^k} = X(\phi)^G \circ \alpha$ and such that $f(W)^G \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\phi) \circ \beta$.

(ii) A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a $\mathcal{F}-$global fibration if it is an $\mathcal{F}-$level fibration and for every compact Lie group G in the family $\mathcal{F}$, every faithful $G-$representation V and every equivariant linear isometric embedding $\phi : V \longrightarrow W$ of $G-$representations, the map

$$(f(V)^G, X(\phi)^G) : X(V)^G \longrightarrow Y(V)^G \times_{Y(W)^G} X(W)^G$$

is a weak equivalence.

Theorem 6.6.11. *Let $\mathcal{F}$ be a global family. The $\mathcal{F}-$global equivalence, $\mathcal{F}-$global fibrations and $\mathcal{F}-$cofibrations form a model structure, the $\mathcal{F}-$global model structure, on the category of orthogonal spaces. The $\mathcal{F}-$global model structure is cofibrantly generated, proper and topological.*

Let $\mathcal{F}in$ be the global family of all finite groups.

Let $\mathbb{R} : I \longrightarrow L$ denote the functor sending a finite set A to the free $\mathbb{R}-$vector space $\mathbb{R}A$ generated by A with A an orthonormal basis. The injective maps between finite sets then linearize to isometric embeddings. Let $U_F : LT \longrightarrow IT$ denote the forgetful functor

$$X \mapsto X \circ \mathbb{R}.$$

It has a left adjoint L_F.

Theorem 6.6.12. *The global model structure of $I-$spaces is Quillen equivalent to the $\mathcal{F}in-$global model structure on the category of orthogonal spaces. We have the Quillen equivalence*

$$L_F : IT \rightleftarrows LT : U_F \tag{6.33}$$

The lemma below is from [33]. We need it later in this section.

Lemma 6.6.13. *Let $P : A \longrightarrow B$ and $U : B \longrightarrow A$ be a Quillen adjoint pair. If U creates the weak equivalences of B and $\eta : \alpha \longrightarrow UP\alpha$ is a weak equivalence for all cofibrant objects A, then (P, U) is a Quillen equivalence.*

Let $l_0 : I \longrightarrow D_0$ be the inclusion sending $\underline{\underline{n}}$ to $(\Sigma_n, \mathbb{R}^n)$. Let U be the restriction of $l_0^* : D_0T \longrightarrow IT$ to D_0T. By Proposition 6.1.17 U has a left adjoint L. U also has a right adjoint R.

Let p be a functor from D_0 to L sending an object (G, V) to V, and a morphism $\phi = (\phi_1, \phi_2)$ to the linear isometric embedding ϕ_2. Let P denote the functor from the category of orthogonal spaces to the category of D_0-spaces sending X to the orthogonal space

$$(G, V) \mapsto X(V).$$

P has a left adjoint Q that sends a D_0-space X to the orthogonal space

$$V \mapsto X(e, V).$$

$X(e, V)$ is the colimit of all the restriction maps, i.e. the diagram with all the maps like

$$X(G, V) \longrightarrow X(H, V).$$

We have the commuting diagram

$$\begin{tikzcd} & LT \arrow[dl, "P"'] \arrow[dr, "U_F"] & \\ D_0^w T \arrow[rr, "U"'] & & IT \end{tikzcd} \tag{6.34}$$

$U_F = U_D \circ P$. So the left adjoints commutes: $L_F = Q \circ L$.

Theorem 6.6.14. *(i) The adjoint functors*

$$Q : D_0 T \rightleftarrows LT : P \tag{6.35}$$

is a Quillen pair between the global model structure on D_0-spaces and the $\mathcal{F}in-$global model structure on the orthogonal spaces.

(ii) And the adjoint functors

$$Q : D_0 T \rightleftarrows LT : P \tag{6.36}$$

is a Quillen pair between the global model structure on $I-$spaces and the global model structure on D_0-spaces.

Proof. (i) First let's check (Q, P) is a Quillen pair. By the definition of global equivalences in $D_0 T$ and LT, we can check directly that P sends global equivalence to global equivalence and it creates the global equivalence in LT.

For any object $r = (G, V)$ in D_0, any orthogonal space X, the matching object $M_r(PX)$ is

$$PX(r) = X(V).$$

Let $f : X \longrightarrow Y$ be a $\mathcal{F}in-$global fibration in the category of orthogonal spaces. Then for any subgroup H of G, $f(V)^H : X(V)^H \longrightarrow Y(V)^H$ is a Serre fibration. And for each equivariant linear isometric embedding $\phi : V \longrightarrow W$ of $H-$representations, the map

$$(f(V)^H, X(\phi)^H) : X(V)^H \longrightarrow Y(V)^H \times_{Y(W)^H} X(W)^H \tag{6.37}$$

is a weak equivalence.

So the map

$$(Pf(G,V)^H, PX(\psi)^H) : PX(G,V)^H \longrightarrow PY(G,V)^H \times_{PY(G,W)^H} PX(G,W)^H \qquad (6.38)$$

is a weak equivalence, where $\psi = (\psi_1, \psi_2)$ a morphism in D_0 with ψ_1 the identity map, ψ_2 the inclusion of $G-$representation.

The space $M_r(PX) \times_{M_r(PY)} PY_r$ is exactly $X(V) \times_{Y(V)} Y(V)$. The identity map $X(V)^H \longrightarrow X(V)^H$ and

$$Pf(G,V)^H : PX(G,V)^H \longrightarrow PY(G,V)^H$$

are both Serre fibrations. Since $Y(V)^H$ is connected, $Pf(G,V)^H$ is surjective if $X(V)^H$ is nonempty. So the map

$$X(V)^H \longrightarrow (X(V) \times_{Y(V)} Y(V))^H \qquad (6.39)$$

is a Serre fibration. Note that when $X(V)^H$ is empty, the map (6.39) is $f(V)^H$. Thus, $P(f)$ is a $\mathcal{F}in-$global fibration.

So (Q, P) is a Quillen pair.

(ii) Let $l_0 : I \longrightarrow D_0$ be the inclusion sending $\underline{\underline{n}}$ to $(\Sigma_n, \mathbb{R}^n)$. Let $U = l_0^* : D_0T \longrightarrow IT$. By Proposition 6.1.17 U has a left adjoint P. And since l_0 is fully faithful, the unit $\eta : Id \longrightarrow UP$ of the adjunction is a natural isomorphism.

Note that in I there is no matching object. U sends the Reedy fibrations to strong level fibrations. If $f : X \longrightarrow Y$ is a morphism of D_0-spaces, for each object $d = (G,V)$, $b = (G,W)$ of D_0, any closed subgroup H of G and any morphism $\phi = (\phi_1, \phi_2) : d \longrightarrow b$ with $\phi_1 = Id_G$, ϕ_2 an inclusion of $G-$subrepresentation, the map

$$(f(d)^H, X(\phi)^H) : X(d)^H \longrightarrow Y(d)^H \times_{Y(b)^H} X(b)^H$$

is a weak equivalence, then for $j : \underline{\underline{n}} \longrightarrow \underline{\underline{m}}$, the restriction

$$(f(l_0(\underline{\underline{n}})^H, X(l_0(j))^H) : X(l_0(\underline{\underline{n}}))^H \longrightarrow Y(l_0(\underline{\underline{n}}))^H \times_{Y(l_0(\underline{\underline{m}}))^H} X(l_0(\underline{\underline{m}}))^H$$

is also a weak equivalence. So U sends global fibrations to global fibrations. It also direct to check it sends global equivalence to global equivalence.

So we have the Quillen pair

$$L : IT^{gl} \rightleftarrows D_0T^{gl} : U \ . \tag{6.40}$$

□

Let $\eta : Id \longrightarrow PQ$ be the unit of the Quillen pair (P, Q). For any D_0-space X, PQX sends (G, V) to $X(e, V)$.

Definition 6.6.15. *Let D_0^wT be the full subcategory of D_0T with objects $X : D_0 \longrightarrow T$ such that the unit $\eta(X) : X \longrightarrow PQ(X)$ is a global equivalence.*

Remark 6.6.16. *By the transfer theorem, D_0^wT inherits the global model structure on D_0T in Theorem 6.5.8.*

And we have the Quillen pair

$$Q : D_0^wT \rightleftarrows LT : P \tag{6.41}$$

between the global model structure on D_0-spaces and the $\mathcal{F}in-$global model structure on the orthogonal spaces., and the adjoint functors

$$Q : D_0^wT \rightleftarrows LT : P \tag{6.42}$$

between the global model structure on $I-$spaces and the global model structure on D_0-spaces.

Moreover, apply Lemma 6.6.13, these two Quillen pairs are both Quillen equivalences.

However this is not so exciting since QEllR discussed in the next Chapter is not in the subcategory D_0^wT.

6.7 Unitary D_0-spectra

In this section we construct the Real version of D_0-space and spectra.

Definition 6.7.1. *$D^{\mathbb{C}}$ is the category whose objects are (G, V, ρ) with G an augmented Lie group and (V, ρ) a faithful real representation of G, and the morphism space $D((G, V, \rho), (H, W, \tau))$ is the space of the pairs (ϕ_1, ϕ_2) with $\phi_2 \in L^{\mathbb{C}}(V, W)$ and $\phi_1 : \tau^{-1}(\widetilde{U}(\phi_2(V)))\rho(G) \longrightarrow G$ a* *group homomorphism, which make the diagram commute.*

$$\begin{tikzcd} G \arrow[r, "\rho"] & \widetilde{U}(V) \arrow[d, "\phi_{2*}"] \\ \tau^{-1}(\widetilde{U}(\phi_2(V))) \arrow[u, "\phi_1"] \arrow[r, "\tau"] & \widetilde{U}(W) \end{tikzcd} \tag{6.43}$$

In other words, the action of the augmented Lie group H on $\phi_2(V)$ is induced from that of G.

Let $D_0^{\mathbb{C}}$ denote the full subcategory with objects (G, V, i) where G is a finite group and i is the inclusion of augmented Lie subgroup into $\widetilde{U}(V)$. We can omit i from the symbol.

Definition 6.7.2. *A $D_0^{\mathbb{C}}-$space is a continuous functor from the category $D_0^{\mathbb{C}}$ to the category of spaces. A morphism of $D_0^{\mathbb{C}}-$spaces is a natural transformation of functors. We denote the category of $D_0^{\mathbb{C}}-$spaces by $D_0^{\mathbb{C}}T$.*

$D-$spaces and $D^{\mathbb{C}}-$spaces are related by various functors

$$\begin{tikzcd} D^{\mathbb{C}}T \arrow[r, bend left, "u"] \arrow[r, bend right, "\psi"'] & DT \arrow[l, "c"] \end{tikzcd} \tag{6.44}$$

where c is the "complexification", u the "underlying" and ψ the "fixed point" functor. c and u arise by precomposition with continuous functors

$$\begin{tikzcd} D \arrow[r, bend left, "(-)_{\mathbb{C}}"] & D^{\mathbb{C}} \arrow[l, bend left, "r"] \end{tikzcd}$$

relating the real and complex isometries categories.

The complexification functor $(-)_{\mathbb{C}}$ sends the compact Lie group G to the product augmented Lie group $G^{\sharp}$, and sends a euclidean inner product space V to its complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ equipped with the unique hermitian inner product $(-,-)$ that satisfies

$$(1 \otimes v, 1 \otimes w) = \langle v, w \rangle$$

for all $v, w \in V$. On the morphism spaces, $(-)_{\mathbb{C}}$ sends $\phi_2 \in L(V, W)$ to $((\phi_2)_{\mathbb{C}}, \mathrm{Id}_{\mathbb{C}}) \in L^{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}})$, and ϕ_1 to $\phi_1 \times \mathrm{Id}_C$.

The realification functor r sends an augmented Lie group G to its underlying Lie group, and sends a hermitian inner product space W to its underlying $\mathbb{R}-$vector space equipped with the euclidean inner product defined by

$$\langle v, w\rangle = \mathrm{Re}(v, w).$$

On morphisms, r sends $\phi_2 = (\phi, \sigma)$ to ϕ and ϕ_1 to ϕ_1.

Then, the functors in (6.44) are defined by

$$uY := Y \circ (-)_{\mathbb{C}} \text{ and } cX := X \circ r \tag{6.45}$$

where Y is a $D^{\mathbb{C}}-$space and X an $D-$space.

Moreover, we can define an involution $\psi : uY \longrightarrow uY$ on the underlying orthogonal space of a unitary space Y. For any $(G, V) \in ob(D_0)$, let $\tau_V \in \widetilde{U}(V_{\mathbb{C}})$ denote the canonical real structure determined by

$$\tau_V(\lambda \otimes v) = \tau(\lambda) \otimes v = \overline{\lambda} \otimes v.$$

Define

$$\psi_V = Y(Id, \tau_V) : Y(G^\sharp, V_{\mathbb{C}}) \longrightarrow Y(G^\sharp, V_{\mathbb{C}}).$$

And the fixed points

$$(Y^\psi)(G, V) := Y(G^\sharp, V_{\mathbb{C}})^{\psi_V} = \{y \in Y(G^\sharp, V_{\mathbb{C}}) | Y(Id, \tau_V)(y) = y\}.$$

Proposition 6.7.3. *The category $D^{\mathbb{C}}$ is a symmetric monoidal category.*

Proof. Let (G, V, ρ) and (H, W, τ) be two objects in $D^{\mathbb{C}}$. The tensor product of (G, V, ρ) and (H, W, τ) is defined to be $(G \times_C H, V \oplus W, \rho \times_C \tau)$. The product is obviously associative and commutative. The unit is $(1, , 1)$ where 1 is the trivial group equipped with the trivial augmentation.

□

Corollary 6.7.4. *The categories $D_0^{\mathbb{C}}$, $(D_0^{\mathbb{C}})^w$, D_0^w are all symmetric monoidal categories.*

Similarly we can also define monoid $D^{\mathbb{C}}-$space and $D_0^{\mathbb{C}}-$space.

Proposition 6.7.5. *$D_0^{\mathbb{C}}$ is a generalized Reedy model category in the sense of Definition*

6.1.15.

Definition 6.7.6. *The real global classifying space $B^R_{gl}G$ of an augmented Lie group G is the $D^{\mathbb{C}}_0-$space*

$$B^R_{gl}G = D^{\mathbb{C}}_0((G,V),-)/G$$

for any object (G,V) in $D^{\mathbb{C}}_0$.

$D^{\mathbb{C}}_0-$spectra is the stabilization of $D^{\mathbb{C}}_0-$space.

Example 6.7.7. *We can define the $D^{\mathbb{C}}_0-$sphere. For each object (G,V) in $D^{\mathbb{C}}_0$, we can define*

$$S^{(G,V)} := S^V. \tag{6.46}$$

S^V inherits a $G-$action from that on V.

Let $\phi = (\phi_1, \phi_2) : (G,V) \longrightarrow (H,W)$ be a morphism in $D^{\mathbb{C}}_0$.

$$S(\phi) := S^{\phi_2} : S^V \longrightarrow S^W. \tag{6.47}$$

$S(Id)$ is the identity map.

Definition 6.7.8. *Let R be a monoid in $D^{\mathbb{C}}_0\mathcal{T}$ with unit η and product μ. A $D^{\mathbb{C}}_0-$spectrum over R is a $D^{\mathbb{C}}_0-$space $X : D^{\mathbb{C}}_0 \longrightarrow \mathcal{T}$ together with continuous maps*

$$\sigma : X(d) \wedge X(b) \longrightarrow X(d+b),$$

natural in d and b, such that the composite

$$X(d) \cong X(d) \wedge S^0 \overset{id\wedge\eta}{\longrightarrow} X(d) \wedge R(b) \overset{\sigma}{\longrightarrow} X(d+u) \cong X(d) \tag{6.48}$$

is the identity and the following diagram commutes:

$$\begin{tikzcd} X(d) \wedge R(b) \wedge R(f) \arrow[r, "\sigma\wedge id"] \arrow[d, "id\wedge\phi"'] & X(d+b) \wedge R(f) \arrow[d, "\sigma"] \\ X(d) \wedge R(d+f) \arrow[r, "\sigma"'] & X(d+b+f) \end{tikzcd} \tag{6.49}$$

The complexification functor c, underlying functor u and fixed point functors ψ between

D_0–spaces and $D_0^{\mathbb{C}}$–spaces all have stable analogs.

$$\begin{tikzcd} D_0^{\mathbb{C}}\mathcal{S} \arrow[r, bend left, "u"] \arrow[r, bend right, "\psi"'] & D_0\mathcal{S} \arrow[l, "c"] \end{tikzcd}$$

Let X be a D_0–spectrum and Y a $D_0^{\mathbb{C}}$–spectrum.

$$cX := X \circ r.$$

$$(uY)(V) = \mathrm{Map}(S^{iV}, Y(G^{\sharp}, V_{\mathbb{C}}))$$

where V is a complex inner product space and $iV = i\mathbb{R} \otimes_{\mathbb{R}} V \subset V_{\mathbb{C}}$ is the imaginary part of V.

We can define a natural involution $\psi : uY \longrightarrow uY$. Define

$$\psi_V := \mathrm{Map}(S^{\tau}, Y(Id, \tau_V)) : \mathrm{Map}(S^{iV}, Y(G^{\sharp}, V_{\mathbb{C}})) \longrightarrow \mathrm{Map}(S^{iV}, Y(G^{\sharp}, V_{\mathbb{C}}))$$

where $\tau : iV \longrightarrow iV$ is multiplication by -1.

$$(Y^{\psi})(V) := \mathrm{Map}^{C}(S^{iV}, Y(V_{\mathbb{C}})).$$

Chapter 7

Global Quasi-elliptic Cohomology

In this chapter we construct a global spectrum representing Quasi-elliptic cohomology in the sense developed in Chapter 6.

Based on the orthogonal $G-$spectrum $(E(G,V), \eta^E(G,V), \mu^E(G,V))$ constructed in Section 4.5, we show the construction of the Real version ER and the real version EO in Section 7.1. They weakly represent the Real and real quasi-elliptic cohomology respectively. We show ER is a unitary $D^{\mathbb{C}}-$space and, in Section 7.2, a $D^{\mathbb{C}}-$FSP over S. Similarly one can show that E and EO are both D_0-spaces and D_0-FSP over S.

In Section 7.3, we show the relation between Real, complex and real global quasi-elliptic cohomologies. We have the global equivalences

$$E \backsimeq u(ER); \text{ and } EO \backsimeq (ER)^{\psi}.$$

7.1 Construction of $QEllR$

In Section 4.5.3 and 4.5.4 I construct an orthogonal $G-$spectrum $(E(G,V), \eta^E(G,V), \mu^E(G,V))$ weakly representing quasi-elliptic cohomology. In Section 7.1.1, I show there is a Real and a real version of that orthogonal $G-$spectrum representing the Real and the real quasi-elliptic cohomology respectively. In Section 4.5.5 I construct the restriction maps. In Section 7.1.2 I show that data makes up a $D^{\mathbb{C}}-$space.

7.1.1 Construction

We have the Real version ER and real version EO of the family of spaces E in Proposition 4.5.18. I present them respectively in this section.

Let G and H be augmented Lie groups. Let V be a real $G-$representation and W a real $H-$representation. Let $g \in G^{tor}$, $h \in H^{tors}$.

Let's use $FR_g(G,V)$ denote the space $\mathrm{Map}_{\mathbb{R}}(S^{(V)_g}, KR((V^g)_g \oplus V^g)$ where KR represents the global Real K-theory. The basepoint cr_0 of it is the constant map from $S^{(V)_g}$ to the basepoint of $KR((V)_g \oplus V^g)$.

For $FR_g(G,V)$, we have the conclusions similar to Proposition 4.5.16.

Define the unit map $\eta R_g(G,V) : S^{V^g} \longrightarrow FR_g(G,V)$ by

$$v \mapsto (v' \mapsto \eta R^K_{(V)_g \oplus V^g}(v \wedge v')) \tag{7.1}$$

where $\eta R^K_{(V)_g \oplus V} : S^{(V)_g \oplus V^g} \longrightarrow KR((V)_g \oplus V^g)$ is the unit map for global Real K-theory. Since $(V)_g \oplus V^g$ is a $\Lambda_G(g)-$representation, $\eta R^K_{(V)_g \oplus V^g}$ is $\Lambda_G(g)-$equivariant. So $\eta R_g(G,V)$ is well-defined.

Define the multiplication $\mu R^F_{(g,h)}((G,V),(H,W)) : FR_g(G,V) \wedge FR_h(H,W) \longrightarrow FR_{(g,h)}(G \times H, V \oplus W)$ by

$$\alpha \wedge \beta \mapsto (v \wedge w \mapsto \mu R^K(\alpha(v) \wedge \beta(w))) \tag{7.2}$$

where μR^K is the multiplication for global Real K-theory. Since μR^K here is $\Lambda_G(g) \times \Lambda_H(h)-$equivariant, $\mu R^F_{(g,h)}((G,V),(H,W))$ is $C_{G \times H}(g,h)-$equivariant.

$FR_g(G,V)$ weakly represents $KR^{V^g}_{\Lambda_G(g)}(-)$. The unit map $\eta R_g(G,V) : S^{V^g} \longrightarrow FR_g(G,V)$ and the multiplication $\mu R^F_{(g,h)}((G,V),(H,W)) : FR_g(G,V) \wedge FR_h(H,W) \longrightarrow FR_{(g,h)}(G \times H, V \oplus W)$ make the unit, associativity and centrality of unit diagram commute. Moreover, $\eta R_g(G,V)$ is $C_G(g)-$equivariant and $\mu R^F_{(g,h)}((G,V),(H,W))$ is $C_{G \times H}(g,h)-$equivariant. And both maps are real.

The proofs of these conclusions are similar to that of Proposition 4.5.16. And we also have the conclusion below.

Proposition 7.1.1. *Let $ER_g(G,V)$ be the $C_G(g)-$space*

$$\{t_1 a + t_2 b \in FR_g(G,V) * S(G,V)_g \| \|b\| \leqslant t_2\} / \{t_1 cr_0 + t_2 b\}.$$

It is the quotient space of a closed subspace of $FR_g(G,V) * S(G,V)_g$ with all the points of the form $t_1 cr_0 + t_2 b$ collapsed to one point, i.e the basepoint of $ER_g(G,V)$, where cr_0 is the basepoint of $FR_g(G,V)$. $ER_g(G,V)$ is $C_G(g)-$weak equivalent to $FR_g(G,V) * S(G,V)_g$. As a result, $\prod_{g \in G^{tors}_{conj}} Map_{C_G(g)}(G, ER_g(G,V))$ is $G-$weak equivalent to $\prod_{g \in G^{tors}_{conj}} Map_{C_G(g)}(G, FR_g(G,V) * S(G,V)_g)$. So

$$ER(G,V) := \prod_{g \in G^{tors}_{conj}} Map_{C_G(g)}(G, ER_g(G,V)) \tag{7.3}$$

weakly represents $QEllR^V_G(-)$.

Let G and H be compact Lie groups. Let V be an orthogonal $G-$representation and W an orthogonal $H-$representation. Let $g \in G^{tors}$, $h \in H^{tors}$.

Let's use $FO_g(G,V)$ denote the space $\mathrm{Map}_{\mathbb{R}}(S^{(V)^{\mathbb{R}}_g}, KO((V)^{\mathbb{R}}_g \oplus V^g))$ where KO represents the global real K-theory. The basepoint co_0 of it is the constant map from $S^{(V)^{\mathbb{R}}_g}$ to the basepoint of $KO((V)^{\mathbb{R}}_g \oplus V^g)$.

For $FO_g(G,V)$, we have the conclusions similar to Proposition 4.5.16.

It weakly represents $KO^{V^g}_{\Lambda_G(g)}(-)$. And we have the unit map $\eta O_g(G,V) : S^{V^g} \longrightarrow FO_g(G,V)$ and the multiplication $\mu O^F_{(g,h)}((G,V),(H,W)) : FO_g(G,V) \wedge FO_h(H,W) \longrightarrow FO_{(g,h)}(G \times H, V \oplus W)$ that making the unit, associativity and centrality of unit diagram commute. Moreover, $\eta O_g(G,V)$ is $C_G(g)-$equivariant and $\mu O^F_{(g,h)}((G,V),(H,W))$ is $C_{G\times H}(g,h)-$equivariant. And both maps are real.

The proofs of these conclusions are similar to that of Proposition 4.5.16. And we also have the conclusion below.

Define the unit map $\eta O_g(G,V) : S^{V^g} \longrightarrow FO_g(G,V)$ by

$$v \mapsto (v' \mapsto \eta O^K_{(V)^{\mathbb{R}}_g \oplus V^g}(v \wedge v')) \tag{7.4}$$

where $\eta O^K_{(V)^{\mathbb{R}}_g \oplus V^g} : S^{(V)^{\mathbb{R}}_g \oplus V^g} \longrightarrow KO((V)^{\mathbb{R}}_g \oplus V^g)$ is the unit map for global real K-theory. Since $(V)^{\mathbb{R}}_g \oplus V^g$ is a $\Lambda_G(g)-$representation, $\eta O^K_{(V)^{\mathbb{R}}_g \oplus V^g}$ is $\Lambda_G(g)-$equivariant. So $\eta O_g(G,V)$ is well-defined.

Define the multiplication $\mu O^F_{(g,h)}((G,V),(H,W)) : FO_g(G,V) \wedge FO_h(H,W) \longrightarrow FO_{(g,h)}(G \times H, V \oplus W)$ by

$$\alpha \wedge \beta \mapsto (v \wedge w \mapsto \mu O^K(\alpha(v) \wedge \beta(w))) \tag{7.5}$$

where μO^K is the multiplication for global Real K-theory. Since μO^K here is $\Lambda_G(g) \times \Lambda_H(h)-$equivariant, $\mu O^F_{(g,h)}((G,V),(H,W))$ is $C_{G\times H}(g,h)-$equivariant.

Proposition 7.1.2. *Let $EO_g(G,V)$ be the $C_G(g)-$space*

$$\{t_1 a + t_2 b \in FO_g(G,V) * S(G,V)_g \,|\, \|b\| \leqslant t_2\}/\{t_1 co_0 + t_2 b\}.$$

It is the quotient space of a closed subspace of $FO_g(G,V) * S(G,V)_g$ with all the points of the form $t_1 co_0 + t_2 b$ collapsed to one point, i.e the basepoint of $EO_g(G,V)$, where co_0 is the base-

point of $FO_g(G,V)$. $EO_g(G,V)$ is $C_G(g)-$weak equivalent to $FO_g(G,V) * S(G,V)_g$. As a result, $\prod_{g\in G^{tors}_{conj}} Map_{C_G(g)}(G, EO_g(G,V))$ is $G-$weak equivalent to $\prod_{g\in G^{tors}_{conj}} Map_{C_G(g)}(G, FO_g(G,V)* S(G,V)_g)$. So

$$EO(G,V) := \prod_{g\in G^{tors}_{conj}} Map_{C_G(g)}(G, EO_g(G,V)) \tag{7.6}$$

weakly represents $QEllO^V_G(-)$.

7.1.2 ER is a $D^{\mathbb{C}}-$space

Proposition 7.1.3. *Let G be a augmented Lie group and V a real $G-$representation such that for each $g \in G^{tors}_{conj}$, V^g is a faithful $C_G(g)/\langle g\rangle-$representation. Consider G as an augmented Lie subgroup of $\widetilde{U}(V)$. Let $N_{\widetilde{U}(V)}(G)$ denote the normalizer of G in $\widetilde{U}(V)$. Then there is a well-defined $N_{\widetilde{U}(V)}(G)-$action on $ER(G,V)$.*

Remark 7.1.4. *The conclusion of Proposition 7.1.3 is a corollary of Theorem 7.1.5 when G is a finite group, by Example 6.1.4. I show explicitly what the group action is in the proof below. It has similar form as the restriction map.*

Proof of Proposition 7.1.3. There is a right $N_{\widetilde{U}(V)}(G)-$ action on G by conjugation

$$a \cdot g = a^{-1}ga \text{ for any } g \in G, a \in N_{\widetilde{U}(V)}(G). \tag{7.7}$$

Let $g \in G^{tors}$ and $\rho : G \longrightarrow ER_g(G,V)$ be a $C_G(g)-$equivariant map. Define the map $a\rho : G \longrightarrow E_{aga^{-1}}(G,V)$ to be the composition

$$G \xrightarrow{g\mapsto a^{-1}ga} G \xrightarrow{\rho} ER_g(G,V) \xrightarrow{x\mapsto a\cdot x} ER_{aga^{-1}}(G,V). \tag{7.8}$$

Let $\alpha \in G$ and $b \in C_G(aga^{-1}) = aC_G(g)a^{-1}$.

$$\begin{aligned}(a\rho)(b\alpha) = a(\rho(a^{-1}ba \cdot a^{-1}\alpha a)) &= a(a^{-1}ba \cdot \rho(a^{-1}\alpha a)) \\ = ba \cdot \rho(a^{-1}\alpha a) &= b(a\rho)(\alpha)\end{aligned}$$

So $a\rho \in \mathrm{Map}_{C_G(aga^{-1})}(G, ER_{aga^{-1}}(G,V))$ and we get a well-defined map

$$a\cdot - : ER(G,V) \longrightarrow ER(G,V), \quad \prod_{g\in G^{tors}_{conj}} \rho_g \mapsto \prod_{g\in G^{tors}_{conj}} a\rho_g \tag{7.9}$$

It's straightforward to check for any a, $a' \in N_{\widetilde{U}(V)}(G)$, $a(a' \prod_{g\in G^{tors}} \rho_g) = (aa') \prod_{g\in G^{tors}} \rho_g$ and $e \cdot -$ is the identity map.

So (7.9) defines a $N_{\widetilde{U}(V)}(G)-$action on $ER(G,V)$.

□

Theorem 7.1.5. *ER is a $D_0^{\mathbb{C}}-$space.*

Proof. Let (G,V) be an object in $D_0^{\mathbb{C}}$.

Let $\phi = (\phi_1, \phi_2) : (G,V) \longrightarrow (H,W)$ be a morphism in $D_0^{\mathbb{C}}$. Let H' denote $H \cap \widetilde{U}(\phi_2(V))$. Recall both ϕ_1 and ϕ_2 are injective. And the linear isometric embedding ϕ_2 gives linear isometric embedding

$$\phi_{2*} : (V)_h \oplus V^{\phi_1(h)} \longrightarrow (W)_h \oplus W^h,\ (v_1, v_2) \mapsto (\phi_2(v_1), \phi_2(v_2)). \tag{7.10}$$

It is equivariant in the sense that

$$\phi_{2*}([\phi_1(a), t] \cdot x) = [a,t] \cdot \phi_{2*}(x)$$

for any $x \in (V)_h \oplus V^{\phi_1(h)}$, $[a,t] \in \Lambda_{H'}(h)$, and $[\phi_1(a), t] \in \Lambda_G(\phi_1(h))$.

Let

$$\beta : S^{(V)_h} \longrightarrow KR((V)_h \oplus V^{\phi_1(h)})$$

be an element in $FR_{\phi_1(h)}(G,V)$. We can define $FR_g(\phi_{2*})(\beta) : S^{(W)_h} \longrightarrow KR((W)_h \oplus W^h)$ by the composition

$S^{(W)_h} = S^{(W)_h - (V)_{\phi_1(h)}} \wedge S^{(V)_{\phi_1(h)}} \xrightarrow{Id \wedge \beta} S^{(W)_h - (V)_{\phi_1(h)}} \wedge KR((V)_{\phi_1(h)} \oplus V^{\phi_1(h)}) \longrightarrow KR((W)_h \oplus V^{\phi_1(h)})$
$\xrightarrow{KR(Id \oplus \phi_2)} KR((W)_h \oplus W^h)$

where the third map is the structure map of KR and $Id \oplus \phi_2$ is the evident linear isometric embedding

$$(W)_h \oplus V^{\phi_1(h)} \longrightarrow (W)_h \oplus W^h.$$

It is $\mathbb{R}-$linear.

It's straightforward to check for morphisms

$$(G,V) \xrightarrow{\phi=(\phi_1,\phi_2)} (H,W) \xrightarrow{\psi=(\psi_1,\psi_2)} (K,U)$$

we have

$$FR_g(\psi_{2*}) \circ FR_g(\phi_{2*})(\beta) = FR_g((\psi \circ \phi)_{2*})(\beta). \tag{7.11}$$

and if ϕ is the identity map, $FR_g(\phi_{2*})$ is identity.

ϕ_2 also gives embeddings

$$Sym(V) \longrightarrow Sym(W) \text{ and } Sym(V)^{\phi_1(h)} \longrightarrow Sym(W)^h,$$

So we have well-defined

$$S(\phi_2) : S(G, V)_{\phi_1(h)} \longrightarrow S(H, W)_h, \tag{7.12}$$

which is equivariant in the sense:

$$S(\phi_2)(\phi_1(h') \cdot y) = h' \cdot S(\phi_2)(y), \text{ for any } h' \in C_{H'}(h).$$

Then we have the join $\phi_{2*} * S(\phi_2) : FR_{\phi_1(h)}(G, V) * S(G, V)_{\phi_1(h)} \longrightarrow FR_h(H, W) * S(H, W)_h$.

Let

$$\phi_{2\star} : ER_{\phi_1(h)}(G, V) \longrightarrow ER_h(H, W)$$

denote the quotient of the restriction $\phi_{2*} * S(\phi_2)|_{ER'_{\phi_1(h)}(G,V)}$.

Let $h \in H^{tors}$. Let $f \in \mathrm{Map}_{C_G(\phi_1(h))}(G, ER_{\phi_1(h)}(G, V))$. Define $\widetilde{\phi_\star f} : C_{\widetilde{U}(W)}(h) \times_{C_{H'}(h)} H' \longrightarrow \mathrm{Map}_{C_H(h)}(G, ER_h(H, W))$ by

$$\widetilde{\phi_\star f}([\alpha, h']) = \alpha(\phi_{2\star} \circ f \circ \phi_1)(h') \tag{7.13}$$

Define

$$ER(\phi)_h : \mathrm{Map}_{C_G(\phi_1(h))}(G, ER_{\phi_1(h)}(G, V)) \longrightarrow \mathrm{Map}_{C_H(h)}(H, ER_h(H, V))$$

by

$$ER(\phi)_h(f)(g) := \begin{cases} \widetilde{\phi_\star f}(g), & \text{if } g \in C_{\widetilde{U}(W)}(h) \times_{C_{H'}(h)} H'; \\ cr_0, & \text{otherwise.} \end{cases} \tag{7.14}$$

where cr_0 denotes the basepoint of $E_h(H, V)$.

$$ER(\phi) := \prod_{h \in H^{tors}} ER(\phi)_h. \tag{7.15}$$

If ϕ is the identity map, $ER(\phi)$ is the identity map. If $\phi : (G, V) \longrightarrow (H, W)$ and $\psi : (H, W) \longrightarrow (K, U)$ are two morphisms in $D_0^{\mathbb{C}}$, we have

$$ER(\psi \circ \phi) = ER(\psi) \circ ER(\phi).$$

So ER defines a functor from $D_0^{\mathbb{C}}$ to the category $\mathcal{T}$ of spaces.

□

7.2 ER is a $D_0^{\mathbb{C}}-$FSP over S

In Section 7.2 we show ER is a $D_0^{\mathbb{C}}-$FSP over S.

We construct the unit map $\eta^{ER} : S \longrightarrow ER$ the multiplication

$$\mu^{ER} : ER \wedge ER \longrightarrow ER.$$

More explicitly, for any objects $d = (G, V)$ and $b = (H, W)$ in $D_0^{\mathbb{C}}$, we have $\eta^{ER}(G, V) : S^V \longrightarrow ER(G, V)$

$$\mu^{ER}((G, V), (H, W)) : ER(G, V) \wedge ER(H, W) \longrightarrow ER(G \times H, V \oplus W)$$

Recall

$$ER(G, V) = \prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, ER_g(G, V)), \tag{7.16}$$

which is defined in (7.3), with $ER_g(G, V)$ defined in Proposition 7.1.1.

Let G and H be augmented Lie groups, V a real $G-$representation and W a real $H-$representation. Let's use xr_g to denote the basepoint of $ER_g(G, V)$.

Let $g \in G^{tors}$. For each $v \in S^V$, there is $v_1 \in S^{V^g}$ and $v_2 \in S^{(V^g)^{\perp}}$ such that $v = v_1 \wedge v_2$.

We construct η^{ER} similar to η^E defined in (4.66).

Let $\eta^{ER}_g(G,V) : S^V \longrightarrow ER_g(G,V)$ be the map

$$\eta^{ER}_g(G,V)(v) := \begin{cases} (1-\|v_2\|)\eta R_g(G,V)(v_1) + \|v_2\|v_2, & \text{if } \|v_2\| \leqslant 1; \\ xr_g, & \text{if } \|v_2\| \geqslant 1. \end{cases} \tag{7.17}$$

where $\eta R_g(G,V)$ is the map defined in (7.1). $\eta^{ER}_g(G,V)$ is well-defined, continuous and $C_G(g)-$equivariant.

$\eta^{ER}(G,V) : S^V \longrightarrow \prod_{g\in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, ER_g(G,V))$ is defined by

$$v \mapsto \prod_{g\in G^{tors}_{conj}} (\alpha \mapsto \eta R^E_g(G,V)(\alpha\cdot v)), \tag{7.18}$$

which is well-defined, continuous and $G-$equivariant. And $\eta^{ER} : S \longrightarrow ER$ is a well-defined functor.

We construct the multiplication map μ^{ER} analogous to the construction of μ^E in (4.68).

Define a map $\mu^{ER}_{g,h}((G,V),(H,W)) : ER_g(G,V) \wedge ER_h(H,W) \longrightarrow ER_{(g,h)}(G\times H, V\oplus W)$ by sending a point $[t_1a_1+t_2b_1]\wedge[u_1a_2+u_2b_2]$ to

$$\begin{cases} [(1-\sqrt{t_2^2+u_2^2})\mu R^F_{g,h}((G,V),(H,W))(a_1\wedge a_2) + \sqrt{t_2^2+u_2^2}(b_1+b_2)], & \text{if } t_2^2+u_2^2 \leq 1 \text{ and } t_2u_2 \neq 0; \\ [(1-t_2)\mu R^F_{g,h}((G,V),(H,W))(a_1\wedge a_2) + t_2b_1], & \text{if } u_2 = 0 \text{ and } 0 < t_2 < 1; \\ [(1-u_2)\mu R^F_{g,h}((G,V),(H,W))(a_1\wedge a_2) + u_2b_2], & \text{if } t_2 = 0 \text{ and } 0 < u_2 < 1; \\ [1\mu R^F_{g,h}((G,V),(H,W))(a_1\wedge a_2) + 0], & \text{if } u_2 = 0 \text{ and } t_2 = 0; \\ xr_{g,h}, & \text{Otherwise.} \end{cases} \tag{7.19}$$

where $\mu R^F_{g,h}((G,V),(H,W))$ is the one defined in (7.2) and $xr_{g,h}$ is the basepoint of $ER_{(g,h)}(G\times H, V\oplus W)$.

Note that the basepoint of $ER(G,V)$ is the product of the basepoint of each factor $\mathrm{Map}_{C_G(g)}(G, ER_g(G,V))$, i.e. the product of constant map to the base point of each $ER_g(G,V)$.

We can define the multiplication

$$\mu^{ER}((G,V),(H,W)) : ER(G,V)\wedge ER(H,W) \longrightarrow ER(G\times H, V\oplus W) \tag{7.20}$$

by the composition

$$\prod_{g\in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, ER_g(G,V)) \wedge \prod_{h\in H^{tors}_{conj}} \mathrm{Map}_{C_H(h)}(H, ER_g(H,W)) \longrightarrow$$

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, ER_g(G,V)) \wedge \mathrm{Map}_{C_H(h)}(H, ER_h(H,W)) \longrightarrow$$

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \mathrm{Map}_{C_{G\times H}(g,h)}(G\times H, ER_g(G,V)\wedge ER_h(H,W)) \longrightarrow$$

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \mathrm{Map}_{C_{G\times H}(g,h)}(G\times H, ER_{(g,h)}(G\times H, V\oplus W))$$

where the first map sends

$$\prod_{g\in G^{tors}_{conj}} \alpha_g \wedge \prod_{h\in H^{tors}_{conj}} \beta_h$$

to

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \Big(\alpha_g\wedge\beta_h : (a,b)\mapsto \alpha_g(a)\wedge\beta_h(b)\Big),$$

the second map sends a point

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \Big(\alpha_g\wedge\beta_h : (a,b)\mapsto \alpha_g(a)\wedge\beta_h(b)\Big)$$

to

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \Big((a,b)\mapsto \alpha_g(a)\wedge\beta_h(b)\Big),$$

the third map sends

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} f_{(g,h)}$$

to

$$\prod_{g\in G^{tors}_{conj}, h\in H^{tors}_{conj}} \mu^{ER}_{(g,h)}((G,V),(H,W))\circ f_{(g,h)}.$$

Proposition 7.2.1. *(i) $\mu^{ER}_{(G,V),(H,V)}$ defined in (7.20) is $G\times_C H-$equivariant.*

(ii) $\mu^{ER}_{(g,h)}$ makes the diagram (7.21) commute for any $(g,h)\in G\times_C H$.

$$\begin{array}{ccc}
Map_{C_G(g)}(G, ER_g(G,V))\wedge Map_{C_H(h)}(H, ER_h(H,W)) & \longrightarrow & F_g(G,V)\wedge F_h(H,W) \\
\Big\downarrow{\scriptstyle \mu^{ER}_{(g,h)}((G,V),(H,W))} & & \Big\downarrow{\scriptstyle \mu R^F_{g,h}((G,V),(H,W))} \\
Map_{C_{G\times_C H}(g,h)}(G\times_C H, ER_{(g,h)}(G\times_C H, V\oplus W)) & \longrightarrow & F_{(g,h)}(G\times H, V\oplus W)
\end{array} \tag{7.21}$$

The left vertical map is $(f_1, f_2) \mapsto (f_1(e), f_2(e))$ *and the right vertical map is* $f \mapsto f(e)$.

Proof. It's straightforward to check. $\square$

We have the conclusion similar to Lemma 4.5.26.

Lemma 7.2.2. *Let* G, H, K *be augmented Lie groups. Let* V *be a real* $G-$*representation,* W *a real* $H-$*representation,* U *a real* $K-$*representation. Let* $(*, \emptyset)$ *be the unit in* $D_0^{\mathbb{C}}$ *where* $*$ *is the trivial group. Let* $g \in G^{tors}$, $h \in H^{tors}$, $k \in K^{tors}$. *Then*

$$\mu_{(*,\emptyset),(G,V)} = Id_{ER(G,V)} = \mu_{(G,V),(*,\emptyset)}. \tag{7.22}$$

And we have the commutative diagrams below.

$$\begin{tikzcd} S^V \wedge S^W \arrow[r, "\eta_g^{ER}(G,V)\wedge\eta_h^{ER}(H,W)"] \arrow[d, "\cong"] & ER_g(G,V) \wedge ER_h(H,W) \arrow[d, "\mu_{(g,h)}^{ER}((G,V),(H,W))"] \\ S^{V\oplus W} \arrow[r, "\eta_{(g,h)}^{ER}(G\times H,V\oplus W)"] & ER_{(g,h)}(G\times H, V\oplus W) \end{tikzcd} \tag{7.23}$$

$$\begin{tikzcd} ER_g(G,V) \wedge ER_h(H,W) \wedge ER_k(K,U) \arrow[r] \arrow[d, "Id\wedge\mu_{(h,k)}^{ER}(H\times K,W\oplus U)"'] & ER_{(g,h)}(G\times H, V\oplus W)\wedge ER_k(K,U) \arrow[d, "\mu_{((g,h),k)}^{ER}((G\times H,V\oplus W),(K,U))"'] \\ ER_g(G,V)\wedge ER_{(g,h)}(H\times K, W\oplus U) \arrow[r] & ER_{(g,h,k)}(G\times H\times K, V\oplus W\oplus U) \end{tikzcd} \tag{7.24}$$

where the top horizontal map is $\mu_g^{ER}((G,V),(H,W))\wedge Id$ *and the bottom map is* $\mu_{(g,(h,k))}^{ER}((G,V),(H\times K, W\oplus U))$,

$$\begin{tikzcd} S^V \wedge ER_h(H,W) \arrow[r, "\tau"] \arrow[d, "\eta_g^{ER}(G,V)\wedge Id"'] & ER_h(H,W)\wedge S^V \arrow[d, "Id\wedge\eta_h^{ER}(H,W)"'] \\ ER_g(G,V)\wedge ER_h(H,W) \arrow[r] \arrow[d, "\mu_{(g,h)}^{ER}((G,V),(H,W))"'] & ER_h(H,W)\wedge ER_g(G,V) \arrow[d, "\mu_{(h,g)}^{ER}((H,W),(G,V))"'] \\ ER_{(g,h)}(G\times H, V\oplus W) \arrow[r, "ER_{(g,h)}(\tau)"'] & ER_{(h,g)}(H\times G, W\oplus V) \end{tikzcd} \tag{7.25}$$

Proof. Analogous to that of Lemma 4.5.26.

$\square$

So we get the main conclusion in this section.

Theorem 7.2.3. ER *is a* $D_0^{\mathbb{C}}-FSP$ *weakly representing* $QEllR$.

Proof. The proof is analogous to that of Theorem 4.5.27. □

In addition, we can construct the structure maps for $QEll$ and $QEllr$ analogously and .show they are D_0-FSP over S.

One more word is generally we have the construction below.

Construction If A is any topological space and X an orthogonal spectrum, we can define a new orthogonal spectra $A * X$ by joining with A; the structure maps and actions of the orthogonal groups do not interact with A. More explicitly, we set

$$(A * X)(V) = A * X(V)$$

for an inner product space V. The structure maps are given by the composite

$$(A*X)(V)\wedge S^W = (A*X(V))\wedge S^W \xrightarrow{\rho} A*(X(V)\wedge S^W) \xrightarrow{\sigma_{V,W}} A*X(V\oplus W) = (A*X)(V\oplus W)$$

where the first map ρ is the map constructed in Appendix A.2.

7.3 The global Real, complex and real quasi-elliptic cohomology

The global Real, complex and real quasi-elliptic cohomology have similar relations as those for global K-theory.

Let G be any compact Lie group. Let K be a subgroup of it and V a real inner product space. On fixed point spaces

$$\begin{aligned}
E(G,V)^K &\backsimeq \prod_{g\in G^{tors}_{conj}} \mathrm{Map}_{\Lambda_G(g)}((G/K)^g, \mathrm{Map}(S^{iV}, u(KR)((V)_g))) \\
&\cong \prod_{g\in G^{tors}_{conj}} \mathrm{Map}_{\Lambda_G(g)}((G/K)^g \times S^{iV}, u(KR)((V)_g)) \\
&\cong \prod_{g\in G^{tors}_{conj}} \mathrm{Map}(S^{iV}, \mathrm{Map}_{\Lambda_G(g)}((G/K)^g, u(KR)((V)_g))) \\
&\backsimeq u(ER)(G,V)^K
\end{aligned}$$

and

$$
\begin{aligned}
EO(G,V)^K &\backsimeq \prod_{g\in G^{tors}_{conj}} \mathrm{Map}_{\Lambda_G(g)}((G/K)^g, \mathrm{Map}^c(S^{iV}, u(KR)(((V)_g)))) \\
&\cong \prod_{g\in G^{tors}_{conj}} \mathrm{Map}^c_{\Lambda_G(g)}((G/K)^g \times S^{iV}, u(KR)((V)_g)) \\
&\cong \prod_{g\in G^{tors}} \mathrm{Map}^c(S^{iV}, \mathrm{Map}_{\Lambda_G(g)}((G/K)^g, u(KR)((V)_g))) \\
&\backsimeq u(ER)^{\psi}(G,V)^K
\end{aligned}
$$

It's $G-$weak equivalence.

Thus, we have the global equivalences below:

$$E \backsimeq u(ER);\ \text{and}\ EO \backsimeq (ER)^{\psi}.$$

Remark 7.3.1. *I believe there are smarter constructions of spectra and global spectra of quasi-elliptic cohomology than that we give in the thesis, which should have better properties. I'm still trying some of my ideas on this.*

7.4 An example of D_0-spectra and D_0-FSP over S

Let's consider an equivariant cohomology theory which is similar to $QEll^*$ but a little simpler than that. Let G be a compact Lie group and X a $G-$space. For each $g \in G^{tors}$, $C_G(g)$ acts on X^g.

$$E^*_G(X) := \prod_{g\in G^{tors}_{conj}} K^*_{C_G(g)}(X^g). \tag{7.26}$$

By Theorem 4.3.6, E^*_G can be represented weakly by the space

$$E_G = \prod_{g\in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, KU_{C_G(g)} * S_{C_G(g),g}) \tag{7.27}$$

And we can construct the D_0-space representing the theory, the tensor product of it in a way similar to the construction for $QEllR^*$.

Theorem 7.4.1. *E is a D_0-space.*

Proof. Let (G, V) be an object in D_0.

$$E(G, V) := \prod_{g \in G^{tors}} \mathrm{Map}_{C_G(g)}(G, KU(V) * S(G, V)_g). \tag{7.28}$$

Let $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$ be a morphism in D_0. Let H' denote $H \cap O(\phi_2(V))$. Both ϕ_1 and ϕ_2 are injective.ϕ_2 is equivariant in the sense that

$$\phi_2(\phi_1(h) \cdot x) = h \cdot \phi_2(x), \text{ for any } h \in C_{H'}(h).$$

So ϕ_2 also gives an embedding

$$Sym(V) \longrightarrow Sym(W) \text{ and } Sym(V)^{\phi_1(h)} \longrightarrow Sym(W)^h,$$

So we have well-defined

$$S(\phi_2) : S(G, V)_{\phi_1(h)} \longrightarrow S(H, W)_h, \tag{7.29}$$

which is equivariant in the sense:

$$S(\phi_2)(\phi_1(h') \cdot y) = h' \cdot S(\phi_2)(y), \text{ for any } h' \in C_{H'}(h).$$

Let $h \in H^{tors}$. Let $f \in \mathrm{Map}_{C_G(\phi_1(h))}(G, KR_{\phi_1(h)}(V)^{\mathbb{R}} * S(G, V)_{\phi_1(h)})$. Define

$$\phi_{2\star} : KU(V) * S(G, V)_{\phi_1(h)} \longrightarrow KU(W) * S(H, W)_h$$

to be the join of $KU(\phi_2)$ and $S(\phi_2)$. Define $\widetilde{\phi_\star f} : C_{O(W)}(h) \times_{C_{H'}(h)} H' \longrightarrow \mathrm{Map}_{C_H(h)}(G, KU(W) * S(H, W)_h)$ by

$$\widetilde{\phi_\star f}([\alpha, h']) = \alpha(\phi_{2\star} \circ f \circ \phi_1)(h') \tag{7.30}$$

Define

$$E(\phi)_h : \mathrm{Map}_{C_G(\phi_1(h))}(G, KU(V) * S(G, V)_{\phi_1(h)}) \longrightarrow \mathrm{Map}_{C_H(h)}(H, KU(V) * S(H, V)_h)$$

by

$$E(\phi)_h(f)(g) := \begin{cases} \widetilde{\phi_\star f}(g), & \text{if } g \in C_{O(W)}(h) \times_{C_{H'}(h)} H'; \\ *, & \text{otherwise.} \end{cases} \tag{7.31}$$

where $*$ denotes the basepoint in $KU(V)$.

$$E(\phi) := \prod_{h \in H^{tors}} E(\phi)_h. \tag{7.32}$$

If ϕ is the identity map, $E(\phi)$ is the identity map. If $\phi : (G,V) \longrightarrow (H,W)$ and $\psi : (H,W) \longrightarrow (K,U)$ are two morphisms in D_0, we have

$$E(\psi \circ \phi) = E(\psi) \circ E(\phi).$$

So E defines a functor from D_0 to the category $\mathcal{T}$ of spaces.

□

Theorem 7.4.2. *E is a monoid D_0-space.*

Proof. The multiplication morphism and unit morphism are constructed analogously to those of $EllR$ in Section 7.2.

Let (G,V) and (H,W) be objects in D_0. Then for any $g \in G$, any $h \in H$, V is a faithful $C_G(g)-$representation and W is a faithful $C_H(h)-$representation.

Let

$$\mu^{KU}_{V,W,g,h} : KU(V) \wedge KU(W) \longrightarrow KU(V \oplus W)$$

denote the multiplication of global complex K-theory.

To simplify the symbol, I use $R_g KU(G,V)$ denote the $G-$space $\mathrm{Map}_{C_G(g)}(G, KU(V) * S(G,V)_g)$.

Define $p_{d,b,g,h} : R_g KU(G,V) \times R_h KU(H,W) \longrightarrow \mathrm{Map}_{C_{G \times_C H}(g,h)}(G \times H, \Big(KU(V) * S(G,V)_g\Big) \times \Big(KU(W) * S(H,W)_h\Big))$ be the map $(\alpha, \beta) \mapsto \Big(\alpha \times \beta : (a,b) \mapsto (\alpha(a), \beta(b))\Big)$.

Define

$$S_{d,b,g,h} : S(G,V)_g * S(H,W)_h \longrightarrow S(G \times H, V \oplus W)_{(g,h)}, \ (t_1 v, t_2 w) \mapsto t_1 v + t_2 w. \tag{7.33}$$

Here $t_1, t_2 \geq 0$ and $t_1 + t_2 = 1$. This is the same map as that defined in (**??**). $S_{d,b,g,h}$ is $C_{G \times H}(g,h)-$equivariant.

Let $\xi_{d,b,g,h}$ be the map from

$$\Big(KU(V) * S(G,V)_g\Big) \times \Big(KU(W) * S(H,W)_h\Big)$$

to

$$\big(KU(V) \times KU(W)\big) * S(G,V)_g * S(H,W)_h$$

by sending

$$((u_1 x_g, u_2 s_g), (t_1 x_h, t_2 s_h))$$

to

$$(\frac{u_1 t_1}{u_1 t_1 + u_2 + t_2}(x_g, x_h), \frac{u_2}{u_1 t_1 + u_2 + t_2} s_g, \frac{t_2}{u_1 t_1 + u_2 + t_2} s_h).$$

Here $t_1, t_2, u_1, u_2 \geq 0$ and $t_1 + t_2 = 1$, $u_1 + u_2 = 1$. Note that the sum $u_1 t_1 + u_2 + t_2$ is nonzero.

$s_{d,b}$ is $C_{G\times H}(g,h)-$equivariant. And for any objects a, b, d in D_0, we have the associativity.

Let

$$\lambda_{d,b} : \prod_{g\in G^{tors}} R_g KU(G,V) \times \prod_{h\in H^{tors}} R_h KU(H,W) \longrightarrow \prod_{(g,h)\in(G\times_C H)^{tors}} R_g KU(G,V) \times R_h KU(H,W)$$

be the map sending

$$(\prod_{g\in G^{tors}} \alpha_g, \prod_{h\in H^{tors}} \beta_h)$$

to

$$\prod_{(g,h)\in(G\times_C H)^{tors}} \alpha_g \times \beta_h$$

as discussed in Example A.1.5.

Then $\mu_{d,b,g,h} : R_g KU(G,V) \times R_h KU(H,W) \longrightarrow R_{(g,h)} KU(G\times H, V\oplus W)$ is defined to be the composition

$$(\mu^{KU}_{V,W,h,g} * S_{d,b,g,h})_* \circ \xi_{d,b,g,h*} \circ p_{d,b,g,h}. \tag{7.34}$$

The multiplication morphism of E is defined by

$$\mu_{d,b} := \Big(\prod_{(g,h)\in(G\times_C H)^{tors}} \mu_{d,b,g,h}\Big) \circ \lambda_{d,b} \tag{7.35}$$

For any object $d=(G,V)$ of D_0 and $g\in G$, let $1_{d,g}$ denote the constant map sending G to the basepoint of $KU(V)$. And let 1_d denote

$$\prod_{g\in G^{tors}} 1_{d,g}.$$

It's straightforward to check, as I did in the proof of Proposition7.2.1, that with the multiplication μ and unit morphism 1, E is a monoid D_0-space.

□

Let's recall the definition of diagram FSP in [42]. Let S denote the D_0-sphere discussed in Example 6.1.12.

Definition 7.4.3. *A D_0-FSP over S is a D_0-space X together with a unit map $\eta : S \longrightarrow X$ of D_0-spaces and a continuous natural product map $\mu : X \boxtimes X \longrightarrow X \circ +$ of functors $D_0 \times D_0 \longrightarrow \mathcal{T}$ such that the composite*

$$X(d) \cong X(d) \wedge S(e,0) \xrightarrow{id\wedge\eta} X(d)\wedge X(e,0) \xrightarrow{\mu} X(d+(e,0)) \cong X(d)$$

is the identity and the following unity, associativity, and centrality of unit diagrams commute:

$$\begin{tikzcd}
S(d)\wedge S(b) \arrow[r, "\eta\wedge\eta"] \arrow[d, "\mu"'] & X(d)\wedge X(b) \arrow[d, "\mu"] \\
S(d+b) \arrow[r, "\eta"'] & X(d+b)
\end{tikzcd}$$

$$\begin{tikzcd}
X(d)\wedge X(b)\wedge X(f) \arrow[r, "\mu\wedge id"] \arrow[d, "id\wedge\mu"'] & X(d+b)\wedge X(f) \arrow[d, "\mu"] \\
X(d)\wedge X(b+f) \arrow[r, "\mu"'] & X(d+b+f)
\end{tikzcd}$$

and

$$\begin{tikzcd} S(d) \wedge X(b) \arrow[r, "\eta\wedge id"] \arrow[d, "\tau"'] & X(d) \wedge X(b) \arrow[r, "\mu"] & X(d+b) \arrow[d, "X(\tau)"] \\ X(b) \wedge S(d) \arrow[r, "id\wedge\eta"'] & X(b) \wedge X(d) \arrow[r, "\mu"'] & X(b+d) \end{tikzcd}$$

A D_0-FSP is commutative if the following diagram commutes, in which case the centrality of unit diagram just given commutes automatically.

$$\begin{tikzcd} X(d) \wedge X(b) \arrow[r, "\mu"] \arrow[d, "\tau"'] & X(d+b) \arrow[d, "X(\tau)"] \\ X(b) \wedge X(d) \arrow[r, "\mu"'] & X(b+d) \end{tikzcd}$$

For $QEllR$, so far I cannot define a unit map to make it a $D_0^{\mathbb{C}}-$FSP. But for E, we have the theorem below.

Theorem 7.4.4. *E is a commutative D_0-FSP.*

Proof. I will define $\eta : S \longrightarrow E$ as the product

$$\prod_{g\in G} \eta_g$$

with each $\eta_g(G,V) : S(G,V) \longrightarrow R_gKU(G,V)$. First we have the $C_G(g)-$equivariant map $\eta'_g(G,V) : S(G,V) \longrightarrow KU(V) * S(G,V)_g$ sending s to $(1\eta^{KU}(V)(s),0)$ where $\eta^{KU}(V) : S^V \longrightarrow KU(V)$ is the unit map for global complex $K-$theory.

Then we can extend $\eta'_g(G,V)$ to $\eta_g(G,V) : S(G,V) \longrightarrow R_gKU(G,V)$ by sending s to

$$f : G \longrightarrow KU(V) * S(G,V)_g,\ g \mapsto (1g\eta^{KU}(V)(s),0).$$

Then $\eta := \prod_{g\in G} \eta_g$.

It's straightforward to check η, together with the multiplication morphism μ defined in Theorem 7.4.2, equip E the structure of a D_0-FSP over S.

$\square$

Appendices

Appendix A

Some Basics about Join

A.1 Join

A.1.1 Definition

Definition A.1.1. *In topology, the join $A * B$ of two topological spaces A and B is defined to be the quotient space*

$$(A \times B \times [0,1])/R,$$

where R is the equivalence relation generated by

$$(a, b_1, 0) \sim (a, b_2, 0) \text{ for all } a \in A \text{ and } b_1, b_2 \in B,$$

$$(a_1, b, 1) \sim (a_2, b, 1) \text{ for all } a_1, a_2 \in A \text{ and } b \in B.$$

At the endpoints, this collapses $A \times B \times \{0\}$ to A and $A \times B \times \{1\}$ to B.

The join $A * B$ is the homotopy colimit of the diagram (whose maps are projections)

$$A \longleftarrow A \times B \longrightarrow B\ .$$

A nice way to write points of $A * B$ is as formal linear combination $t_1 a + t_2 b$ with $0 \leq t_1, t_2 \leq 1$ and $t_1 + t_2 = 1$, subject to the rules $0a + 1b = b$ and $1a + 0b = a$. The coordinates correspond exactly to the points in $A * B$.

We need the topology on $A * B$ to make the four maps below continous

$$A * B \longrightarrow A,\ t_1 a + t_2 b \mapsto a$$

$$A * B \longrightarrow \mathbb{R},\ t_1 a + t_2 b \mapsto t_1$$

$$A * B \longrightarrow B,\ t_1 a + t_2 b \mapsto b$$

$$A * B \longrightarrow \mathbb{R},\ t_1 a + t_2 b \mapsto t_2.$$

Proposition A.1.2. *Join is associative and commutative. Explicitly, $A * (B * C)$ is homeomorphic to $(A * B) * C$, and $A * B$ is homeomrphic to $B * A$.*

Proof. Consider the map $\rho_1 : A * (B * C) \longrightarrow (A * B) * C$ defined by

$$\rho_1(s_1(t_1 a + t_2 b) + s_2 c) = \begin{cases} s_1 t_1 a + (s_1 t_2 + s_2)(\frac{s_1 t_2}{s_1 t_2 + s_2} b + \frac{s_2}{s_1 t_2 + s_2} c) & \text{if } s_1 t_2 + s_2 \text{ is nonzero;} \\ 1a & \text{if } s_1 t_2 + s_2 = 0. \end{cases}$$

This is a continuous map in the topology we want. We can also define a map analogously $\rho_2 : (A * B) * C \longrightarrow A * (B * C)$ defined by

$$\rho_2(s_1 a + s_2(t_1 b + t_2 c) = \begin{cases} (s_1 + s_2 t_1)(\frac{s_1}{s_1 + s_2 t_1} a + \frac{s_2 t_1}{s_1 + s_2 t_1} b) + s_2 t_2 c & \text{if } s_1 t_2 + s_2 \text{ is nonzero;} \\ 1c & \text{if } s_1 t_2 + s_2 = 0. \end{cases}$$

It's straightforward to check the composition $\rho_1 \circ \rho_2$ and $\rho_2 \circ \rho_1$ are both identity maps. Thus, $A * (B * C)$ is homeomorphic to $(A * B) * C$.

Since there is a homeomorphism

$$f : A * B \longrightarrow B * A$$

$$t_1 a + t_2 b \mapsto t_2 b + t_1 a,$$

join is commutative.

$\square$

A.1.2 Construction of some maps

Example A.1.3. *Let A, B, C, D be topological spaces. If we have two continuous maps $f : A \longrightarrow C$ and $g : B \longrightarrow D$, we can define the join of the two maps*

$$f * g : A * B \longrightarrow C * D.$$

Let $t_1 a + t_2 b$ denote a point in $A * B$ with $a \in A$, $b \in B$, and $t_1, t_2 \geq 0$, $t_1 + t_2 = 1$.

$$(f * g)(t_1 a + t_2 b) := t_1 f(a) + t_2 g(b).$$

By the topology we choose for join, $f * g$ is continuous.

Example A.1.4. *We can also define several types for join of spaces as below. Let A B C D be topological spaces*

(1) We can define a map

$$\alpha : (A \times B) * C \longrightarrow (A * C) \times (B * C)$$

by sending $(t_1(a, b), t_2 c)$ to $((t_1 a, t_2 c), (t_1 b, t_2 c))$. Under the topology of the join we choose, it's well-defined and continuous.

(2) We can define a map

$$\beta : (A \times B) * (C \times D) \longrightarrow (A * C) \times (B * D)$$

by sending $(t_1(a, b), t_2(c, d))$ to $((t_1 a, t_2 c), (t_1 b, t_2 d))$. It's well-defined and continuous.

Example A.1.5. *For the product of spaces, let's take the product topology on it, namely, the coarsest topology.*

For the products of spaces

$$A := \prod_{i \in \Lambda_1} A_i \text{ and } B := \prod_{j \in \Lambda_2} B_j,$$

we can define a map

$$\lambda_{A,B} : \prod_{i \in \Lambda_1} A_i * \prod_{j \in \Lambda_2} B_j \longrightarrow \prod_{(i,j) \in \Lambda_1 \times \Lambda_2} A_i * B_j$$

by sending an element

$$t_1 \prod_{i\in\Lambda_1} a_i + t_2 \prod_{j\in\Lambda_2} b_j$$

to

$$\prod_{(i,j)\in\Lambda_1\times\Lambda_2} t_1 a_i + t_2 b_j$$

with $t_1, t_2 \geq 0$ *and* $t_1 + t_2 = 1$. *It is well-defined and continuous.*

Let $C := \prod_{k\in\Lambda_3} C_k$ *be another product of spaces. We can also define*

$$\lambda_{A,B,C} : \prod_{i\in\Lambda_1} A_i * \prod_{j\in\Lambda_2} B_j * \prod_{k\in\Lambda_3} C_k \longrightarrow \prod_{(i,j,k)\in\Lambda_1\times\Lambda_2\times\Lambda_3} A_i * B_j * C_k$$

by sending

$$t_1 \prod_{i\in\Lambda_1} a_i + t_2 \prod_{j\in\Lambda_2} b_j + t_3 \prod_{k\in\Lambda_3} c_k$$

to

$$\prod_{(i,j,k)\in\Lambda_1\times\Lambda_2\times\Lambda_3} t_1 a_i + t_2 b_j + t_3 c_k$$

with $t_1, t_2, t_3 \geq 0$ *and* $t_1 + t_2 + t_3 = 1$. *It is also well-defined and continuous.*

Let's use $A * B$ *denote the space*

$$\prod_{(i,j)\in\Lambda_1\times\Lambda_2} A_i * B_j.$$

In addition, we have the associativity:

$$\lambda_{A,B,C} = \lambda_{A*B,C} \circ (\lambda_{A,B} * Id_c) = \lambda_{A,B*C} \circ (Id_A * \lambda_{B,C}),$$

which is from the associativity of the join.

A.1.3 Group Action on the Join

Example A.1.6. *Let* G *be a compact Lie group. Let* A, B *be* $G-$*spaces. Then* $A * B$ *has a* $G-$*structure on it by*

$$g \cdot (t_1 a + t_2 b) := t_1(g \cdot a) + t_2(g \cdot b), \text{ for any } g \in G, a \in A, b \in B, \text{ and } t_1, t_2 \geq 0, t_1 + t_2 = 1. \tag{A.1}$$

It's straightforward to check (A.1) defines a continuous group action.

Example A.1.7. *Let G and H be compact Lie groups. Let A be a $G-$space and B a $H-$space. Then $A * B$ has a $G \times H-$structure on it by*

$$(g,h)\cdot(t_1a+t_2b) := t_1(g\cdot a)+t_2(h\cdot b), \text{ for any } g \in G, a \in A, b \in B, \text{ and } t_1, t_2 \geq 0, t_1+t_2 = 1. \tag{A.2}$$

It's straightforward to check (A.2) defines a continuous group action.

A.2 Reduced suspension

Let X be a topological space. The unreduced suspension SX is the quotient space

$$(X \times I)/\{(x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X\}.$$

SX is homeomorphic to the join $S^0 * X$, and each point in it can be written as a formal linear combination $s_1x + s_2p_\pm$ in a unique way where $S^0 = \{p_+, p_-\}$, $s_1, s_2 \geq 0$ and $s_1 + s_2 = 1$. And we have the equivalence $0x + 1p_\pm = p_\pm$ and $1x + 0p_\pm = x$.

If X is a pointed space with basepoint x_0, The reduced suspension ΣX is obtained from SX by collapsing the line $x_0 \times I$. So each point in ΣX can be expressed as a linear combination $s_1x + s_2p_\pm$ as above but with one more equivalence that $s_1x_0 + s_2p_\pm = 1p_+ = 1p_-$ for any $s_1, s_2 \geq 0$ and $s_1 + s_2 = 1$. And it is homeomorphic to the smash product $S^1 \wedge X$.

Example A.2.1. *Let A be a pointed space with basepoint a_0 and B a topological space. Then $A * B$ is a pointed space with basepoint $1a_0$. We can define a continuous map*

$$\rho_{AB} : S^1 \wedge (A * B) \longrightarrow (S^1 \wedge A) * B$$

by sending

$$s_1p_\pm + s_2(t_1a + t_2b)$$

to

$$(s_1 + s_2t_1)(\frac{s_1}{s_1 + s_2t_1}p_\pm + \frac{s_2t_1}{s_1 + s_2t_1}a) + s_2t_2b$$

when $s_1 + s_2t_1$ is nonzero, and to b when $s_1 + s_2t_1$ is zero, where $s_1 + s_2 = 1$ and $t_1 + t_2 = 1$ and $s_1, s_2, t_1, t_2 \geq 0$.

As we have seen in the proof of Proposition A.1.2, the formula of ρ_{AB} gives a bijection between $S^0 * (A * B)$ and $(S^0 * A) * B$. Each point in the subspace $I \times \{a_0\}$ of $S^0 * (A * B)$ is of the form $s_1 p_\pm + s_2(1a_0 + 0b)$. It is mapped to a point $(s_1 p_\pm + s_2 a_0) + 0b$ in $SA * B$. Thus, the image $\rho_{AB}(I \times \{a_0\})$ is the subspace $(I \times \{a_0\}) \times \{1\}$ of $SA * B$ at the endpoint of the join, and it is contained in the subspace

$$(I \times \{a_0\}) \times B \times I \subset SA \times B \times I$$

to collapse when we get the reduced suspension $(S^1 \wedge A) * B$.

So ρ_{AB} is well-defined map. When B is the single point space, ρ_{AB} is bijective.

Moreover, if A is a based $G-$space and B is a $H-$spaces for some compact Lie groups G and H, then ρ_{AB} is a based $G \times H-$equivariant map.

Example A.2.2. *We can also define a map $S^2 \wedge (A * B) \longrightarrow (S^2 \wedge A) * B$ by*

$$S^2 \wedge (A * B) \cong S^1 \wedge ((S^1 \wedge A) * B) \longrightarrow (S^1 \wedge (S^1 \wedge A)) * B \cong (S^2 \wedge A) * B.$$

For any finite-dimensional vector space V, $S^V \wedge (A * B)$ is homeomorphic to the smash product of

$$\wedge_1^{\dim V} S^1 \wedge (A * B)$$

By induction we can also define $S^V \wedge (A * B) \longrightarrow (S^V \wedge A) * B$.

In addition, let's consider a special case of Example A.1.5.

Example A.2.3. *Let $\{A_i\}_\Lambda$ be topological spaces. We can define a map as λ_{AB} in Example A.1.5*

$$S^1 \wedge (\prod_{i \in \Lambda} A_i) \longrightarrow \prod_{i \in \Lambda} (S^1 \wedge A_i)$$

by sending

$$s_1 p_\pm + s_2 \prod_{i \in \Lambda} a_i$$

with each $a_i \in A_i$ to

$$\prod_{i \in \Lambda} s_1 p_\pm + s_2 a_i$$

for any $s_1, s_2 \geq 0$ and $s_1 + s_2 = 1$.

More generally, for a vector space W, we have

$$f_W : S^W \wedge (\prod_{i \in \Lambda} A_i) \longrightarrow \prod_{i \in \Lambda} (S^W \wedge A_i)$$

by sending

$$s_1 p + s_2 \prod_{i \in \Lambda} a_i$$

with $p \in S^W$ and each $a_i \in A_i$ to

$$\prod_{i \in \Lambda} s_1 p + s_2 a_i$$

for any $s_1, s_2 \geq 0$ and $s_1 + s_2 = 1$.

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