Quasi-elliptic cohomology

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Overview

Plan.

- Motivation and construction
- The power operation
- The orthogonal $G-$spectrum

Motivation

An old idea of Witten [Landweber]

The elliptic cohomology of a space X is related to the $\mathbb{T}-$equivariant K-theory of $LX = \mathbb{C}^\infty(S^1, X)$ with the circle $\mathbb{T}$ acting on LX by rotating loops.

It's surprisingly difficult to make this precise.

Why?

In application, one needs to consider the case that a group G acts on X. In this case the loop space LX has rich structures as an orbifold.

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The category of bibundles

A bibundle from $\mathbb{H}$ to $\mathbb{G}$ [Schommer-Pries][Lerman]

a smooth manifold P together with

- the structure maps:
  - $\tau : P \longrightarrow \mathbb{G}_0$;
  - a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.
- The action maps in $Man_{G_0 \times H_0}$
  - $\mathbb{G}_1 {}_s\times_\tau P \longrightarrow P$;
  - $P {}_\sigma\times_t \mathbb{H}_1 \longrightarrow P$

such that

1. $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $(g_1, g_2, p) \in \mathbb{G}_1 {}_s\times_t \mathbb{G}_1 {}_s\times_\tau P$;
2. $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$ for all $(p, h_1, h_2) \in P {}_\sigma\times_t \mathbb{H}_1 {}_s\times_t \mathbb{H}_1$;
3. $p \cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p)) \cdot p = p$ for all $p \in P$.
4. $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for all $(g, p, h) \in \mathbb{G}_1 {}_s\times_\tau P {}_\sigma\times_t \mathbb{H}_1$.
5. $\mathbb{G}_1 {}_s\times_\tau P \longrightarrow P {}_\sigma\times_\sigma P$ $\quad (g, p) \mapsto (g \cdot p, p)$ is an isomorphism.

Bibundle Map [Schommer-Pries][Lerman]

a map $P \longrightarrow P'$ over $\mathbb{H}_0 \times \mathbb{G}_0$ commuting with the $\mathbb{G}-$ and $\mathbb{H}-$action.

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Example ($Loop_1(X /\!\!/ G) := Bibun(S^1 /\!\!/ *, X /\!\!/ G)$)

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

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Example (Another Model $Loop_2(X /\!\!/ G)$)

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The Loop Spaces of Interest

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- the same objects as $Loop_1(X/\!\!/G)$;
- $(t,\alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
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Good Groupoid or Not?

A skeleton of $Loop_2^{ext}(X/\!\!/G)$

- $\mathcal{L}_g X$: the space of objects (g, γ) in $Loop_2(X/\!\!/G)$.
- $L_g G = \{\alpha : \mathbb{R} \longrightarrow G | \alpha(s+1) = g^{-1}\alpha(s)g\}$, the gauge group of the principal $G-$bundle $P_g := \mathbb{R} \times G/(s+1, a) \sim (s, ga)$ over S^1;
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$\coprod_{g \in \pi_0 G/conj} \mathcal{L}_g X /\!\!/ L_g G \rtimes \mathbb{T}$ is a skeleton of $Loop_2^{ext}(X/\!\!/G)$.

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$L_g G \rtimes \mathbb{T}$ is an infinite dimensional topological group when G is not finite.

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Quasi-elliptic cohomology

Consider the subgroupoid $\Lambda(X /\!\!/ G)$ instead

$$\Lambda(X /\!\!/ G) := \coprod_{g \in G^{tors}_{conj}} X^g /\!\!/ \Lambda_G(g)$$

G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors};

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$$

$\Lambda_G(g)$ acts on X^g by

$$[h, t] \cdot x := h \cdot x.$$

$QEll$ as equivariant $K-$theories

$$QEll_G(X) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

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Setting up the theory

Restriction

$$\phi : X /\!\!/ G \longrightarrow Y /\!\!/ H \Longrightarrow \Lambda(\phi) : \Lambda(X /\!\!/ G) \longrightarrow \Lambda(Y /\!\!/ H)$$

$$\begin{tikzcd}
QEll^*(Y /\!\!/ H) \arrow[r, "\phi^*"] \arrow[d, "\pi_{\phi(\tau)}"'] & QEll^*(X /\!\!/ G) \arrow[d, "\pi_\tau"'] \\
K^*_{\Lambda_H(\phi(\tau))}(Y^{\phi(\tau)}) \arrow[r, "\phi^*_\Lambda"] & K^*_{\Lambda_G(\tau)}(X^\tau)
\end{tikzcd}$$

Künneth Map

$K_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)}((X \times Y)^{(\sigma,\tau)}) \cong K_{\Lambda_{G\times H}(\sigma,\tau)}((X \times Y)^{(\sigma,\tau)})$ where

$\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$: pullback of $\Lambda_G(\sigma) \xrightarrow{\pi} \mathbb{T} \xleftarrow{\pi} \Lambda_H(\tau)$.

$$QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) := \prod_{\sigma\in G^{tors}_{conj} \tau\in H^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K^*_{\Lambda_H(\tau)}(Y^\tau).$$

The Künneth map: $QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) \longrightarrow QEll^*_{G\times H}(X \times Y)$.

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The change-of-group isomorphism

- H: closed subgroup of G;
- X: H-space;
- $\phi : H \longrightarrow G$ is the inclusion.

Theorem

The change-of-group map ρ_H^G is an isomorphism.

$$\rho_H^G : QEll_G^*(G \times_H X) \xrightarrow{\phi^*} QEll_H^*(G \times_H X) \xrightarrow{i^*} QEll_H^*(X)$$

- ϕ^*: *the restriction map*
- $i : X \longrightarrow G \times_H X$: $i(x) = [e, x]$.

Induced map

$\mathcal{I}_H^G : QEll(X /\!\!/ H) \xrightarrow{\cong} QEll((G \times_H X) /\!\!/ G) \longrightarrow QEll(X /\!\!/ G)$

- the first map: the change-of-group isomorphism
- the second: the finite covering $\Lambda(G \times_H X /\!\!/ G) \longrightarrow \Lambda(X /\!\!/ G)$
  obj $(\sigma, [g, x]) \mapsto (\sigma, gx)$; mor $([g', t], (\sigma, [g, x])) \mapsto ([g', t], (gx, \sigma))$.

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Power Operation of Equivariant cohomology theories

Power Operation of K-theory [Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power Operation of equivariant K-theory [Atiyah]

$$P_n : K_G(X) \longrightarrow K_{G\wr\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau)$.
Group action: $(x_1, \cdots x_n) \cdot (g_1, \cdots g_n, \sigma) := (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)})$.

Definition of Equivariant Power Operation [May][Ganter]

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Power Operation

Quasi-elliptic cohomology has power operations, which gives it the structure of an "H_∞−ring theory" [Ganter 06].

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X/\!\!/G)$.
$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^\pm]}n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)} :$$

$$QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}).$$

$$\mathbb{P}_{(\underline{g},\sigma)} : QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^\wedge_k} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$

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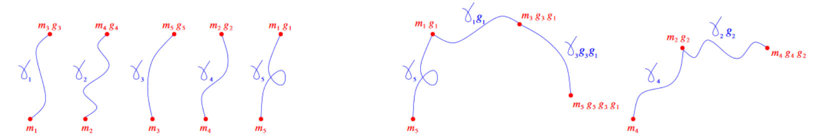
$\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$ and $\prod\limits_{k}\prod\limits_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}}X$ are $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)-$equivariant homeomorphic.

Example ($\mathcal{L}_{(g_1,\cdots g_5,(135)(24))}(X^{\times 5})$ and ${}_3\mathcal{L}_{g_5g_3g_1}(X)\times {}_2\mathcal{L}_{g_4g_2}(X)$)



This picture is from [Ganter].

$f_{(\underline{g},\sigma)}$ is the restriction of this homeomorphism to the constant loops.

the functor $f_{(\underline{g},\sigma)}$: the KEY isomorphism

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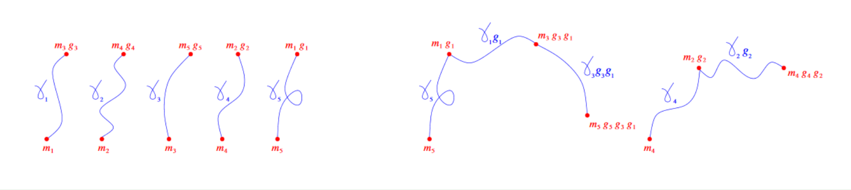

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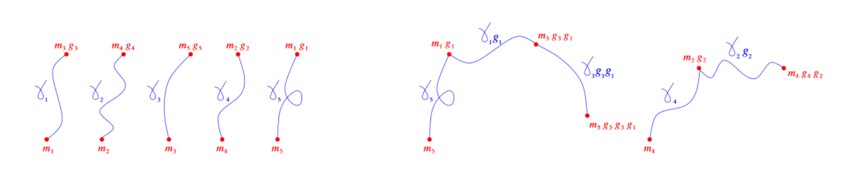

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Construction and Notation

$$C_G(g, g') := \{x \in G | gx = xg'\}.$$

$$\Lambda_G^k(g, g') := C_G(g, g') \times \mathbb{R}/(x, t) \sim (gx, t - k).$$

Note

$$\Lambda_G^1(g, g) = \Lambda_G(g).$$

A groupoid equivalent to $\Lambda(X /\!\!/ G)$

- objects $\coprod_{g \in G^{tors}} X^g$;
- morphisms $\coprod_{g, g' \in G^{tors}} \Lambda_G^1(g, g') \times X^g$

  Explicitly, (α, x) maps $x \in X^g$ to $\alpha \cdot x \in X^{g'}$.

We use the same symbol $\Lambda(X /\!\!/ G)$ to denote it.

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For $(\underline{g}, \sigma) \in G \wr \Sigma_n$, let $\Lambda_{(\underline{g},\sigma)}(X)$ denote the groupoid with

- objects: points in $\coprod\limits_{k} \coprod\limits_{(i_1,\cdots i_k)} X^{g_{i_k}\cdots g_{i_1}}$
  where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ;
- morphisms: $\coprod\limits_{k} \coprod\limits_{(i_1,\cdots i_k),(j_1,\cdots j_k)} \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}}$,
  Explicitly, (α, x) maps x to $\alpha \cdot x$.

Let

$$U : \Lambda_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda(X /\!\!/ G)$$

denote the functor sending

- an object $x \in X^{g_{i_k}\cdots g_{i_1}} \Longrightarrow x \in X^{g_{i_k}\cdots g_{i_1}}$;
- a morphism $(\alpha, x) \in \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}} \Longrightarrow$
  $(\alpha, x) \in \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}}$.

Note: $\Lambda_{(\underline{g},\sigma)}(X)$ may not be a subgroupoid of $\Lambda(X /\!\!/ G)$.

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denote the functor sending

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Example ($G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.

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$\overline{P}_N : QEll_G(X) \xrightarrow{\mathbb{P}_N} QEll_{G \wr \Sigma_N}(X^{\times N}) \xrightarrow{res} QEll_{G \times \Sigma_N}(X^{\times N}) \xrightarrow{diag^*}$
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Finite Subgroups of Elliptic Curve

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Can the classification of the finite subgroups of an elliptic curve be given by the associated elliptic cohomology theory?

Morava E-theory [Strickland]

The Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of the formal group $\mathbb{G}_E$.

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They generalized Strickland's result to generalized Morava E-theories $E_G(\mathcal{L}^h(-))$ using Stapleton's transchromatic character theory.

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$$a_4 = -5\sum_{n\geqslant 1} n^3 q^n/(1-q^n) \qquad a_6 = -\frac{1}{12}\sum_{n\geqslant 1}(7n^5+5n^3)q^n/(1-q^n).$$

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The finite subgroups of the Tate curve are the kernels of isogenies. Its finite subgroups of order N can be classified by

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Transfer Ideals

$$I_{tr}^{Tate} := \sum_{\substack{i+j=N,\\ N>j>0}} \mathrm{Image}[I_{\Sigma_i\times\Sigma_j}^{\Sigma_N} : K_{Tate}(\mathrm{pt}/\!\!/\Sigma_i \times \Sigma_j) \longrightarrow K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)]$$

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Classification of Finite Subgroups of Tate Curve

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The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{Tate}(pt/\!\!/\Sigma_N)/I^{Tate}_{tr} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e}\rangle,$$

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Motivating example: Quillen model structure on topological spaces

Represent the standard homotopy theory of CW-complexes.

- weak homotopy equivalence
- Serre fibration
- retract of relative cell complex

Morphisms

Quillen adjunction: $(L \dashv R) : \mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{D}$.

Quillen equivalence: $Ho(\mathcal{C}) \underset{\mathbb{L}}{\overset{\mathbb{R}}{\leftrightarrows}} Ho(\mathcal{D})$.

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Definition: A model structure on a category $\mathcal{C}$

Three distinguished classes of morphisms:

- Weak Equivalence
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satisfying the axioms: • Retracts • 2 of 3 • Lifting • Factorization

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Why do we care equivariant orthogonal spectra?

Right Philosophy for the Stable Homotopy Theory (1990s)

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The best model: orthogonal spectra.

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Top_G: the category of based $G-$spaces and continuous based maps.

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A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^- \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

An $\mathcal{I}_G-$FSP is an $\mathcal{I}_G-$space X with

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Global Homotopy Theory [Schwede][May]

Observation: It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

$\Rightarrow$ **global homotopy theory**

Prominent examples: equivariant stable homotopy, equivariant K-theory, equivariant bordism.

Orthogonal Space [Schwede]

The category $\mathbb{L}$

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Goerss-Hopkins-Miller theorem constructs many examples of $E_\infty-$rings which represent elliptic cohomology theories, including Tate K-theory.

Question

Can we construct $E_\infty - G-$spectrum which represents equivariant elliptic cohomology theory (e.g. $G-$equivariant Tate K-theory)?

Orthogonal $G-$spectrum of quasi-elliptic cohomology [Huan]

We construct a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$. For each faithful $G-$representation V, $E(G,V)$ weakly represents $QEll_G^V(-)$ in the sense

$$\pi_k(E(G,V)) = QEll_G^V(S^k), \text{ for each } k.$$

Can $E(G,-)$ arise from an orthogonal spectrum?

No.

For a trivial $G-$representation V, the $G-$action on $E(G,V)$ is not trivial.

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A category D_0 larger than $\mathbb{L}$

- objects: (G, V, ρ) with V an inner product vector space, G a compact group and ρ a faithful group representations $\rho : G \longrightarrow O(V)$,
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- an extension of global homotopy theory;
- classifies those theories that are almost "global";
- the restriction maps are equivariant weak equivalence.

We can define global quasi-elliptic cohomology. [Huan]

Combining the orthogonal $G-$spectra $\{E(G,-)\}$, we get an ultra-commutative global ring spectrum in the new theory.

We formulate several model structures and are formulating the one below.

Conjecture

There is a global model structure on the almost global spaces that is Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.

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We formulate several model structures and are formulating the one below.

Conjecture

There is a global model structure on the almost global spaces that is Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.

Thank you.

Some references

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