Quasi-Elliptic Cohomology I.

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ABSTRACT. Quasi-elliptic cohomology is a variant of Tate K-theory. It is the orbifold K-theory of a space of constant loops. For global quotient orbifolds, it can be expressed in terms of equivariant K-theories. This theory reflects the geometric nature of the Tate curve and serves as a breakthrough point to solve problems on elliptic cohomology theories. In this paper we provide a systematic introduction of its construction and definition.

1. Introduction

An elliptic cohomology theory is an even periodic multiplicative generalized cohomology theory whose associated formal group is the formal completion of an elliptic curve. Most elliptic cohomology theories themselves are usually difficult to express explicitly and compute. Quasi-elliptic cohomology is constructed as an object both reflecting the geometric nature of elliptic curves and more practicable to study.

There are many famous mathematical problems on elliptic cohomology that people have been interested in for long. We cite some of them below.

It is an old idea of Witten [31] that the elliptic cohomology of a space X is related to the $\mathbb{T}-$equivariant K-theory of the free loop space $LX = \mathbb{C}^\infty(S^1, X)$ with the circle $\mathbb{T}$ acting on LX by rotating loops. There have already been relevant conclusions. Tate K-theory is the generalized elliptic cohomology associated to the Tate curve. The Tate curve $Tate(q)$ is an elliptic curve over $\mathrm{Spec}\mathbb{Z}((q))$, which is classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity [Section 2.6, [3]]. Tate K-theory is itself a distinctive subject to study. Its relation with string theory is better understood than most known elliptic cohomology theories. In addition, the definition of $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold [Section 2, [17]]. We hope to explore the physical meaning of elliptic cohomology theories and unfold the relation between elliptic cohomology theories and the free loop space via equivariant Tate K-theory. We need a bridge connecting elliptic cohomology theories and Tate K-theory.

Moreover, Morava E-theories have many properties that reflects other homotopy theories. They serve as motivating examples for the research on other cohomology theories. A classification of the level structure of its formal group is given

in [1]. Strickland proved in [48] that the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group. Stapleton proved in [43] this result for generalized Morava E-theory via transchromatic character theory [46] [47]. In each case, the power operation serves as a bridge connecting the homotopy theory and its formal group. It is conjectured that we have classification theorems of the geometric structures of each elliptic curve in the same form.

Ginzburg, Kapranov and Vasserot gave a definition of equivariant elliptic cohomology in [20] and conjectured that any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A. In his thesis [19], Gepner presented a construction of the equivariant elliptic cohomology that satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms. We are interested in answering this question from a different perspective and how to give an explicit construction of an orthogonal $G-$spectrum of each elliptic cohomology theory.

The idea of global orthogonal spectra was first inspired in Greenlees and May [21]. Many classical theories, equivariant stable homotopy theory, equivariant bordism, equivariant K-theory, etc, naturally exist not only for a single group but a specific family of groups in a uniform way. Several models of global homotopy theories have been established, including that by Bohmann [13] and Schwede [45]. The two model categories of global spectra are Quillen equivalent, as shown in [13]. But it is unclear whether global elliptic cohomology theory exists and how we should construct it.

It is difficult for elliptic cohomology theories themselves to serve as the starting point to answer the questions above, which implies the necessity of an intermediate theory. It is supposed to be a theory that is more practicable to compute and study than elliptic cohomology theories and serves as a breakthrough point to answer the questions above.

The intermediate theory we introduce in this paper is quasi-elliptic cohomology which contains all the information of Tate K-theory. The relation between the two theories can be expressed in this way:

$$(1.1) \qquad\qquad QEll^*_G(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X).$$

The idea of quasi-elliptic cohomology is motivated by Ganter's construction of Tate K-theory [14]. It is not an elliptic cohomology but from it we can recover the Tate K-theory. This new theory can be interpreted in a neat form by equivariant K-theories, which makes many constructions on it easier and more natural than those on the Tate K-theories. Some formulations can be generalized to other equivariant cohomology theories. In addition, quasi-elliptic cohomology provides a method that reduces facts such as the classification of geometric structures on the Tate curve into questions in representation theory.

It has provided some idea solving the mathematical questions we talked about above. In Section 2 of this paper, for any compact Lie group G, we construct $G-$equivariant quasi-elliptic cohomology from a loop space via bibundles, which verifies the old idea of Witten indirectly. Therefore, we in fact give a construction by loop space of $G-$equivariant Tate K-theory for compact Lie groups G. In addition, in Section 4.1 we apply the idea and construct orbifold quasi-elliptic cohomology. In Section 4.2 we give another construction motivated by Ganter's construction of orbifold Tate K-theory in [18]. The two constructions of orbifold quasi-elliptic cohomology are equivalent.

Moreover, we construct a power operation of quasi-elliptic cohomology in [**24**] via explicit formulas that interwine the power operation in K-theory and natural symmetries of the free loop space. It is closely related to the stringy power operation of Tate K-theory in [**17**]. One advantage of this power operation over that on Tate K-theory is that its construction can be generalized to other equivariant cohomology theories. Via this power operation on quasi-elliptic cohomology, we show in [**24**] that the Tate K-theory of symmetric groups modulo a certain transfer ideal, $K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)/I_{tr}^{Tate}$, classifies finite subgroups of the Tate curve. Applying the same idea and method, we prove that, for the Tate K-theory of any finite a-belian group A modulo a certain transfer ideal, $K_{Tate}(\mathrm{pt}/\!\!/A)/I_{tr}^A$, classifies $A-$Level structure of the Tate curve. This result will appear in a coming paper.

In [**25**] we give some new idea how to construct equivariant elliptic cohomology theories. We formulate a new category of orthogonal $G-$spectra and construct explicitly an orthogonal $G-$spectrum of quasi-elliptic cohomology in it. The idea of the construction can be applied to a family of equivariant cohomology theories, including Tate K-theory and generalized Morava E-theories. Moreover, this construction provides a functor from the category of global spectra to the category of orthogonal $G-$spectra. In addition, from it we obtain some new idea what global homotopy theory is right for constructing global elliptic cohomology theory.

In a conversation, Ganter indicated that quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede [**45**]. We are establishing a more flexible global homotopy theory where quasi-elliptic chomomology can fit into. We hope that it is easier to judge whether a cohomology theory, especially an elliptic cohomology theory, can be globalized in the new theory. In addition we want to show that the new global homotopy theory is equivalent to the current global homotopy theory. The construction of the new global homotopy theory has been shown in Chapter 6 and 7 of the author's PhD thesis [**26**]. In Theorem 7.2.3 [**26**] we show quasi-elliptic cohomology can be globalized in it.

This paper aims to provide a reference for this beautiful and powerful theory and a systematic introduction of its construction and definition. In Section 2 we construct four models of loop spaces, from each of which quasi-elliptic cohomology theories can be established. In Section 3 we give the definition of quasi-elliptic cohomology, set up the theory and show its properties. In Section 4, we present the construction of orbifold quasi-elliptic cohomology via the loop space of bibundles and discuss its relation with Ganter's construction of orbifold Tate K-theory.

2. Models for loop spaces

To understand $QEll^*_G(X)$, it is essential to understand the orbifold loop space. In this section, we will describe several models for the loop space of $X /\!/ G$. Lerman discussed thoroughly in Section 3, [**32**] that the strict 2-category of Lie groupoids can be embedded into a weak 2-category whose objects are Lie groupoids, 1-morphisms are bibundles and 2-morphisms equivariant diffeomorphisms between bibundles. Thus, the free loop space of an orbifold M is the category of bibundles from the trivial groupoid $S^1 /\!/ *$ to the Lie groupoid M. In Definition 2.4 we discuss $Loop_1(X /\!/ G)$ and introduce two other models, $Loop_2(X /\!/ G)$ in Definition 2.8 and $Loop_3(X /\!/ G)$ in Definition 2.9. The later two models are equivalent.

The groupoid structure of $Loop_1(X /\!/ G)$ generalizes $Map(S^1, X) /\!/ G$, which is a subgroupoid of it. Other than the $G-$action, we also consider the rotation by the circle group $\mathbb{T}$ on the objects and form the groupoids $Loop_1^{ext}(X /\!/ G)$ and $Loop_3^{ext}(X /\!/ G)$. The later one contains all the information of $Loop_1^{ext}(X /\!/ G)$.

We also construct a loop space $L_{orb}(X /\!/ G)$ by adding rotations to the orbifold loop space that Ganter used to define equivariant Tate K-theory in [**17**]. It is a subgroupoid of $Loop_3^{ext}(X /\!/ G)$. The key groupoid $\Lambda(X /\!/ G)$ in the construction of quasi-elliptic cohomology is the full subgroupoid of $L_{orb}(X /\!/ G)$ consisting of the constant loops. In order to unravel the relevant notations in the construction of $QEll^*_G(X)$, we study the orbifold loop space in Section 2.3 and Section 2.4.

Moreover, we introduce the ghost loops $GhLoop(X /\!/ G)$, which is a subgroupoid of $Loop_1^{ext}(X /\!/ G)$. It is the fourth model of loop spaces from which we can construct quasi-elliptic cohomology. It has many good features that the other three models don't have and is itself a model worth studying.

In Section 2.1 we define $Loop_1(X /\!/ G)$ and $Loop_1^{ext}(X /\!/ G)$. In Section 2.2 we recall the free loop space. In Section 2.3 we interpret the enlarged groupoid $Loop_1^{ext}(X /\!/ G)$ and introduce the groupoid $\Lambda(X /\!/ G)$ of constant loops, from which we construct quasi-elliptic cohomology. In Section 2.4 we present the model of ghost loops.

2.1. Bibundles. A standard reference for groupoids and bibundles is Section 2 and 3, [**32**]. For each pair of Lie groupoids $\mathbb{H}$ and $\mathbb{G}$, the bibundles from $\mathbb{H}$ to $\mathbb{G}$ are defined in Definition 3.25, [**32**]. The category $Bibun(\mathbb{H}, \mathbb{G})$ has bibundles from $\mathbb{H}$ to $\mathbb{G}$ as the objects and bundle maps as the morphisms.

The first question is how to define a "loop". Here we consider bibundles, i.e. the 1-morphisms in a weak 2-category of Lie groupoids defined in Section 3, [**32**]

For any manifold X, let Man_X denote the category of manifolds over X, that is, the category whose objects are manifolds Y equipped with a smooth map $Y \longrightarrow X$, and whose morphisms are smooth maps $Y \longrightarrow Y'$ making the following triangle commute.

$$
\begin{array}{ccc}
Y & \longrightarrow & Y' \\
\downarrow & \swarrow & \\
X & &
\end{array}
$$

A bibundle from $\mathbb{G}$ to $\mathbb{H}$ consists of a left principal $\mathbb{G}-$bundle P over H_0 and a right action of $\mathbb{H}$ on P via a $\mathbb{G}-$invariant map. The actions of $\mathbb{G}$ and $\mathbb{H}$ commute. Below we give the definition of bibunldes, which unravels Definition 3.25, [**32**].

DEFINITION 2.1. Let $\mathbb{G}$ and $\mathbb{H}$ be Lie groupoids. A (left principal) bibundle from $\mathbb{H}$ to $\mathbb{G}$ is a smooth manifold P together with

1. A map $\tau : P \longrightarrow \mathbb{G}_0$, and a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.
2. Action maps in $Man_{G_0 \times H_0}$

$$\mathbb{G}_1 \, {}_s\times_\tau P \longrightarrow P$$
$$P \, {}_\sigma\times_t \mathbb{H}_1 \longrightarrow P$$

which we denote on elements as $(g, p) \mapsto g \cdot p$ and $(p, h) \mapsto p \cdot h$, such that

1. $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $(g_1, g_2, p) \in \mathbb{G}_1 \, {}_s\times_t \mathbb{G}_1 \, {}_s\times_\tau P$;
2. $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$ for all $(p, h_1, h_2) \in P \, {}_\sigma\times_t \mathbb{H}_1 \, {}_s\times_t \mathbb{H}_1$;
3. $p \cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p)) \cdot p = p$ for all $p \in P$.
4. $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for all $(g, p, h) \in \mathbb{G}_1 \, {}_s\times_\tau P \, {}_\sigma\times_t \mathbb{H}_1$.
5. The map

$$\mathbb{G}_1 \, {}_s\times_\tau P \longrightarrow P \, {}_\sigma\times_\sigma P$$
$$(g, p) \mapsto (g \cdot p, p)$$

is an isomorphism.

DEFINITION 2.2. A bibundle map is a map $P \longrightarrow P'$ over $\mathbb{H}_0 \times \mathbb{G}_0$ which commutes with the $\mathbb{G}-$ and $\mathbb{H}-$actions, i.e. the following diagrams commute.

$$
\begin{array}{ccc}
\mathbb{G}_1 \, {}_s\times_\tau P & \longrightarrow & P \\
\downarrow & & \downarrow \\
\mathbb{G}_1 \, {}_s\times_\tau P' & \longrightarrow & P'
\end{array}
\qquad
\begin{array}{ccc}
P \, {}_\sigma\times_t \mathbb{H}_1 & \longrightarrow & P \\
\downarrow & & \downarrow \\
P' \, {}_\sigma\times_t \mathbb{H}_1 & \longrightarrow & P'
\end{array}
$$

For each pair of Lie groupoids $\mathbb{H}$ and $\mathbb{G}$, we have a category $Bibun(\mathbb{H}, \mathbb{G})$ with as objects bibundles from $\mathbb{H}$ to $\mathbb{G}$ and as morphisms the bundle maps. The category of smooth functors from $\mathbb{H}$ to $\mathbb{G}$ is a subcategory of $Bibun(\mathbb{H}, \mathbb{G})$.

EXAMPLE 2.3 ($Bibun(S^1 /\!\!/ *, * /\!\!/ G)$). According to the definition, a bibundle from $S^1 /\!\!/ *$ to $* /\!\!/ G$ with G a Lie group is a smooth manifold P together with two maps $\pi : P \longrightarrow S^1$ a smooth principal $G-$bundle and the constant map $r : P \longrightarrow *$. So a bibundle in this case is equivalent to a smooth principal $G-$bundle over S^1. The morphisms in $Bibun(S^1 /\!\!/ *, * /\!\!/ G)$ are bundle isomorphisms.

DEFINITION 2.4 ($Loop_1(X /\!\!/ G)$). Let G be a Lie group acting smoothly on a manifold X. We use $Loop_1(X /\!\!/ G)$ to denote the category $Bibun(S^1 /\!\!/ *, X /\!\!/ G)$, which generalizes Example 2.3. Each object consists of a smooth manifold P and two structure maps $P \xrightarrow{\pi} S^1$ a smooth principal $G-$bundle and $f : P \longrightarrow X$ a $G-$equivariant map. We use the same symbol P to denote both the object and the smooth manifold when there is no confusion. A morphism is a $G-$bundle map $\alpha : P \longrightarrow P'$ making the diagram below commute.

$$
\begin{array}{ccc}
S^1 & \xleftarrow{\ \pi\ } P \xrightarrow{\ f\ } & X \\
 & {}_{\pi'}\nwarrow \ \downarrow^\alpha \ \nearrow_{f'} & \\
 & P' &
\end{array}
$$

Thus, the morphisms in $Loop_1(X /\!\!/ G)$ from P to P' are bundle isomorphisms.

Only the $G-$action on X is considered in $Loop_1(X/\!\!/G)$. We add the rotations by adding more morphisms into the groupoid.

DEFINITION 2.5 ($Loop_1^{ext}(X/\!\!/G)$). Let $Loop_1^{ext}(X/\!\!/G)$ denote the groupoid with the same objects as $Loop_1(X/\!\!/G)$. Each morphism consists of the pair (t, α) where $t \in \mathbb{T}$ is a rotation and α is a $G-$bundle map. They make the diagram below commute.

$$
\begin{array}{ccc}
S^1 & \xleftarrow{\ \pi\ } P \xrightarrow{\ f\ } & X \\
{\scriptstyle t}\downarrow & {\scriptstyle \alpha}\downarrow \quad \nearrow & \\
S^1 & \xleftarrow{\ \pi'\ } P' \quad {\scriptstyle f'} &
\end{array}
$$

The groupoid $Loop_1(X/\!\!/G)$ is a subgroupoid of $Loop_1^{ext}(X/\!\!/G)$.

2.2. Free loop space. In this section we recall the free loop space of a $G-$space and discuss the actions on it by $Aut(S^1)$ and the loop group LG. We will also show its relation with physics.

For any space X, we have the free loop space of X

$$(2.1) \qquad\qquad LX := \mathbb{C}^\infty(S^1, X).$$

It comes with an evident action by the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by rotating the circle

$$(2.2) \qquad\qquad t \cdot \gamma := (s \mapsto \gamma(s+t)),\ t \in S^1,\ \gamma \in LX.$$

Let G be a compact Lie group. Suppose X is a right G-space. The free loop space LX is equipped with an action by the loop group LG

$$(2.3) \qquad \delta \cdot \gamma := (s \mapsto \delta(s) \cdot \gamma(s)),\ \text{for any } s \in S^1,\ \delta \in LX,\ \ \gamma \in LG.$$

Combining the action by group of automorphisms $Aut(S^1)$ on the circle and the action by LG, we get an action by the extended loop group ΛG on LX. $\Lambda G := LG \rtimes \mathbb{T}$ is a subgroup of

$$(2.4) \qquad LG \rtimes Aut(S^1),\ (\gamma, \phi) \cdot (\gamma', \phi') := (s \mapsto \gamma(s)\gamma'(\phi^{-1}(s)), \phi \circ \phi')$$

with $\mathbb{T}$ identified with the group of rotations on S^1. ΛG acts on LX by

$$(2.5) \qquad \delta \cdot (\gamma, \phi) := (t \mapsto \delta(\phi(t)) \cdot \gamma(\phi(t))),\ \text{for any } (\gamma, \phi) \in \Lambda G,\ \text{and } \delta \in LX.$$

It's straightforward to check (2.5) is a well-defined group action.

Let G^{tors} denote the set of torsion elements in G. Let $g \in G^{tors}$. Define $L_g G$ to be the twisted loop group

$$(2.6) \qquad\qquad \{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+1) = g^{-1}\gamma(s)g\}.$$

The multiplication of it is defined by

$$(2.7) \qquad (\delta \cdot \delta')(t) = \delta(t)\delta'(t),\ \text{for any } \delta, \delta' \in L_g G,\ \text{and } t \in \mathbb{R}.$$

The identity element e is the constant map sending all the real numbers to the identity element of G. Similar to ΛG, we can define $L_g G \rtimes \mathbb{T}$ whose multiplication is define by

$$(2.8) \qquad\qquad (\gamma, t) \cdot (\gamma', t') := (s \mapsto \gamma(s)\gamma'(s+t), t + t').$$

The set of constant maps $\mathbb{R} \longrightarrow G$ in $L_g G$ is a subgroup of it, i.e. the centralizer $C_G(g)$.

Before we go on, we discuss the physical meaning of the twisted loop group. Recall that the gauge group of a principal bundle is defined to be the group of its vertical automorphisms. The readers may refer [**38**] for more details on gauge groups. For a $G-$bundle $P \longrightarrow S^1$, let $L_P G$ denote its gauge group.

We have the well-known facts below.

LEMMA 2.6. *The principal $G-$bundles over S^1 are classified up to isomorphism by homotopy classes*

$$[S^1, BG] \cong \pi_0 G/\operatorname{conj.}$$

Up to isomorphism every principal $G-$bundle over S^1 is isomorphic to one of the forms $P_\sigma \longrightarrow S^1$ with $\sigma \in G$ and

$$P_\sigma := \mathbb{R} \times G/(s+1, g) \sim (s, \sigma g).$$

A complete collection of isomorphism classes is given by a choice of representatives for each conjugacy class of $\pi_0 G$.

For the gauge group $L_{P_\sigma} G$ of the bundle P_σ, we have the conclusion.

LEMMA 2.7. *For the bundle $P_\sigma \longrightarrow S^1$, $L_{P_\sigma} G$ is isomorphic to the twisted loop group $L_\sigma G$.*

PROOF. Each automorphism f of an object $S^1 \overset{\pi}{\leftarrow} P_\sigma \overset{\tilde{\delta}}{\to} X$ in $Loop_1^{ext}(X/\!\!/G)$ has the form

(2.9)
$$\begin{array}{ccc} P' & \xrightarrow{[s,g] \mapsto [s, \gamma_f(s)g]} & P \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{\quad = \quad} & S^1 \end{array}$$

for some $\gamma_f : \mathbb{R} \longrightarrow G$. The morphism is well-defined if and only if $\gamma_f(s+1) = \sigma^{-1}\gamma_f(s)\sigma$.

So we get a well-defined map

$$F : L_{P_\sigma} G \longrightarrow L_\sigma G, \ f \mapsto \gamma_f.$$

It's a bijection. Moreover, by the property of group action, F sends the identity map to the constant map $\mathbb{R} \longrightarrow G$, $s \mapsto e$, which is the trivial element in $L_\sigma G$, and for two automorphisms f_1 and f_2 at the object, $F(f_1 \circ f_2) = \gamma_{f_1} \cdot \gamma_{f_2}$. So $L_{P_\sigma} G$ is isomorphic to $L_\sigma G$.

□

2.3. Orbifold Loop Space. In this section, we present the loop spaces $Loop_2(X/\!\!/G)$ and $Loop_3(X/\!\!/G)$. Based on these models, we construct the groupoid $Loop_3^{ext}(X/\!\!/G)$ and show its relation to $Loop_1^{ext}(X/\!\!/G)$, the model by bibundles. These models, however, are not good enough to study. Instead, we consider a subgroupoid $\Lambda(X/\!\!/G)$ of $Loop_3^{ext}(X/\!\!/G)$ consisting of constant loops. It can be constructed from the orbifold loop space in Section 2.1 [**17**] that Ganter used to formulate Tate K-theory and show its relation with loop spaces.

Let G be a Lie group acting smoothly on a manifold X.

DEFINITION 2.8 ($Loop_2(X/\!\!/G)$). Let $Loop_2(X/\!\!/G)$ denote the groupoid whose objects are (σ, γ) with $\sigma \in G$ and $\gamma : [0,1] \longrightarrow X$ a continuous map such that $\gamma(1) = \gamma(0) \cdot \sigma$. A morphism $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ is a continuous map $\alpha : [0,1] \longrightarrow G$ satisfying $\gamma'(s) = \gamma(s)\alpha(s)$. Note that $\alpha(0)\sigma' = \sigma\alpha(1)$.

DEFINITION 2.9 ($Loop_3(X /\!\!/ G)$). Let $Loop_3(X /\!\!/ G)$ denote the groupoid whose objects are (σ, γ) with $\sigma \in G$ and $\gamma : \mathbb{R} \longrightarrow X$ a continuous map such that $\gamma(s+1) = \gamma(s) \cdot \sigma$, for any $s \in \mathbb{R}$. A morphism $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ is a continuous map $\alpha : \mathbb{R} \longrightarrow G$ satisfying $\gamma'(s) = \gamma(s)\alpha(s)$. Note that $\alpha(s)\sigma' = \sigma\alpha(s+1)$, for any $s \in \mathbb{R}$.

It is isomorphic to $Loop_2(X /\!\!/ G)$. We can construct the isomorphism $\phi : Loop_2(X /\!\!/ G) \longrightarrow Loop_3(X /\!\!/ G)$ in this way: an object (σ, γ) is sent to

$$\phi(\sigma, \gamma) = \gamma * \gamma\sigma * \gamma\sigma^2 * \cdots$$

where $*$ is the composition of paths. For any real number r between positive integers n and $n+1$, $\phi(\sigma, \gamma)(r) = \gamma(r-n)\sigma^n$. A morphism $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ is mapped to the morphism

$$\phi(\alpha) = \alpha * \sigma^{-1}\alpha\sigma * \sigma^{-2}\alpha\sigma^2 * \cdots .$$

For any r between positive integers n and $n+1$, $\phi(\alpha)(r) = \sigma^{-n}\alpha(r-n)\sigma^n$.

For $g \in G$, let l denote its order. If g is not a torsion element, $l = 0$. The objects of $Loop_3(X /\!\!/ G)$ can be identified with the space

$$\coprod_{g \in G} \mathcal{L}_g X$$

where

(2.10) $$\mathcal{L}_g X := \mathrm{Map}_{\mathbb{Z}/l\mathbb{Z}}(\mathbb{R}/l\mathbb{Z}, X).$$

The group $\mathbb{Z}/l\mathbb{Z}$ acts on $\mathbb{R}/l\mathbb{Z}$ by group multiplication. It is isomorphic to the cyclic group $\langle g \rangle$ by identifying the generater $[1]$ with g. So it acts on X via the $G-$action on it. The groupoid $\mathcal{L}_g X /\!\!/ L_g G$ is a full subgroupoid of $Loop_3(X /\!\!/ G)$.

Now we consider the extended loop spaces with richer morphism spaces.

DEFINITION 2.10 ($Loop_3^{ext}(X /\!\!/ G)$). Let $Loop_3^{ext}(X /\!\!/ G)$ denote the groupoid with the same objects as $Loop_3(X /\!\!/ G)$. A morphism

$$(\sigma, \gamma) \longrightarrow (\sigma', \gamma')$$

consists of the pair (α, t) with $\alpha : \mathbb{R} \longrightarrow G$ a continuous map and $t \in \mathbb{R}$ a rotation on S^1 satisfying $\gamma'(s) = \gamma(s-t)\alpha(s-t)$.

The groupoid $Loop_3(X /\!\!/ G)$ is a subgroupoid of $Loop_3^{ext}(X /\!\!/ G)$.

LEMMA 2.11. *The groupoid $Loop_1^{ext}(X /\!\!/ G)$ is isomorphic to a full subgroupoid of $Loop_3^{ext}(X /\!\!/ G)$.*

PROOF. Define a functor

$$F : Loop_1^{ext}(X /\!\!/ G) \longrightarrow Loop_3^{ext}(X /\!\!/ G)$$

by sending an object

$$S^1 \xleftarrow{\;\pi\;} P \xrightarrow{\;f\;} X$$

to (σ, γ) with $\gamma(s) := f([s, e])$ and $\sigma = \gamma(0)^{-1}\gamma(1)$ and sending a morphism

to $(\alpha, t) : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ with $\alpha(s) := F([s, e])^{-1}$.

F is a fully faithful functor but not essentially surjective.

$\square$

Therefore, $Loop_3^{ext}(X/\!\!/G)$ contains all the information of $Loop_1^{ext}(X/\!\!/G)$. Next we show a skeleton of the larger groupoid.

For each $g \in G$, $\mathcal{L}_g X /\!\!/ L_g G \rtimes \mathbb{T}$ is a full subgroupoid of $Loop_3^{ext}(X/\!\!/G)$ where $L_g G \rtimes \mathbb{T}$ acts on $\mathcal{L}_g X$ by

(2.11) $\quad \delta \cdot (\gamma, t) := (s \mapsto \delta(s+t) \cdot \gamma(s+t)), \text{ for any } (\gamma, t) \in L_g^k G \rtimes \mathbb{T}, \text{ and } \delta \in \mathcal{L}_g X.$

The action by g on $\mathcal{L}_g X$ coincides with that by $1 \in \mathbb{R}$. So we have the isomorphism

(2.12) $\qquad\qquad L_g G \rtimes \mathbb{T} = L_g G \rtimes \mathbb{R} / \langle (\overline{g}, -1) \rangle,$

where $\overline{g}$ represents the constant loop $\mathbb{T} \longrightarrow \{g\} \subseteq G$.

We have already proved Proposition 2.12.

PROPOSITION 2.12. Let G be a compact Lie group. The groupoid

$$\mathcal{L}(X/\!\!/G) := \coprod_{[g]} \mathcal{L}_g X /\!\!/ L_g G \rtimes \mathbb{T}$$

is a skeleton of $Loop_3^{ext}(X/\!\!/G)$, where the coproduct goes over conjugacy classes in $\pi_0 G$.

DEFINITION 2.13 ($Loop_3^{ext,tors}(X/\!\!/G)$). Let $Loop_3^{ext,tors}(X/\!\!/G)$ denote the full subgroupoid of $Loop_3^{ext}(X/\!\!/G)$ whose objects are the pairs (σ, γ) with $\sigma \in G^{tors}$ and $\gamma : \mathbb{R} \longrightarrow X$ a continuous map such that $\gamma(s+1) = \gamma(s) \cdot \sigma$, for any $s \in \mathbb{R}$.

The groupoid $Loop_3^{ext,tors}(X/\!\!/G)$ contains all the information we want. But is it convenient enough to study? When G is not finite, The isotropy group $L_g G \rtimes \mathbb{T}$ of an object in $\mathcal{L}_g X$ is an infinite dimensional topological group. We need even smaller groups to define a good orbifold loop space. Thus, we consider those elements $[\gamma, t] \in \Lambda_g G$ with γ a constant loop. They form a subgroup of $\Lambda_g G$ which is the quotient group of $C_G(g) \times \mathbb{R}/l\mathbb{Z}$ by the normal subgroup generated by $(g, -1)$. We denote it by $\Lambda_G(g)$. When G is a compact Lie group, $\Lambda_G(g)$ is also a compact Lie group.

Therefore, instead of $Loop_3^{ext,tors}(X/\!\!/G)$, we consider a subgroupoid $L_{orb}(X/\!\!/G)$ of it in Definition 2.15, which is closely related to Ganter's orbifold loop space in [17], and afterwards a full subgroupoid $\Lambda(X/\!\!/G)$ of $L_{orb}(X/\!\!/G)$, which we use to define quasi-elliptic cohomology in Section 3.2.

We need Definition 2.14 first.

DEFINITION 2.14. Let $C_G(g, g')$ denote the set

$$\{x \in G | gx = xg'\}.$$

Let $\Lambda_G(g, g')$ denote the quotient of $C_G(g, g') \times \mathbb{R}/l\mathbb{Z}$ under the equivalence

$$(x, t) \sim (gx, t-1) = (xg', t-1).$$

DEFINITION 2.15 ($L_{orb}(X/\!\!/G)$ and $\Lambda(X/\!\!/G)$). Let $L_{orb}(X/\!\!/G)$ denote the groupoid with the same objects as $Loop_3^{ext,tors}(X/\!\!/G)$, i.e. the space

$$\coprod_{g \in G^{tors}} \mathcal{L}_g X,$$

and with morphisms the space $\coprod\limits_{g,g'\in G^{tors}} \Lambda_G(g,g') \times X^g$.

For $\delta \in \mathcal{L}_g X$, $[a,t] \in \Lambda_G(g,g')$,

$$(2.13) \qquad \delta \cdot ([a,t],\delta) := (s \mapsto \delta(s+t) \cdot a) \in \mathcal{L}_{g'} X.$$

in the same way as (2.11).

$L_{orb}(X/\!\!/G)$ has the same objects as the orbifold loop space in [**17**] and has more morphisms with the $\mathbb{T}-$action added.

The groupoid $\Lambda(X/\!\!/G)$ defined in Example 2.13 is the full subgroupoid of $L_{orb}(X/\!\!/G)$ with constant loops as objects. In Section 3.2, we have a thorough discussion of $\Lambda(X/\!\!/G)$.

Let G^{tors}_{conj} denote a set of representatives of $G-$conjugacy classes in G^{tors}.

LEMMA 2.16. *The groupoid*

$$\coprod_{g\in G^{tors}_{conj}} \mathcal{L}_g X /\!\!/ \Lambda_G(g),$$

is a skeleton of $L_{orb}(X/\!\!/G)$ containing all the information of it. It does not depend on the choice of representatives of the $G-$conjugacy classes.

The proof is analogous to that of Lemma 2.12.

2.4. Ghost Loops. Let G be a compact Lie group and X a $G-$space. In this section we introduce a subgroupoid $GhLoop(X/\!\!/G)$ of $Loop^{ext}_1(X/\!\!/G)$, which can be computed locally.

DEFINITION 2.17 (Ghost Loops). The groupoid of ghost loops is defined to be the full subgroupoid $GhLoop(X/\!\!/G)$ of $Loop^{ext}_1(X/\!\!/G)$ consisting of objects $S^1 \leftarrow P \xrightarrow{\widetilde{\delta}} X$ such that $\widetilde{\delta}(P) \subseteq X$ is contained in a single $G-$orbit.

For a given $\sigma \in G$, define the space

$$(2.14) \qquad GhLoop_\sigma(X/\!\!/G) := \{\delta \in \mathcal{L}_\sigma X | \delta(\mathbb{R}) \subseteq G\delta(0)\}.$$

We have a corollary of Proposition 2.12 below.

PROPOSITION 2.18. $GhLoop(X/\!\!/G)$ *is equivalent to the groupoid*

$$\Lambda(X/\!\!/G) := \coprod_{[\sigma]} GhLoop_\sigma(X/\!\!/G) /\!\!/ L^1_\sigma G \rtimes \mathbb{T}$$

where the coproduct goes over conjugacy classes in $\pi_0 G$.

EXAMPLE 2.19. If G is a finite group, it has the discrete topology. In this case, LG consists of constant loops and, thus, is isomorphic to G. The space of objects of $GhLoop(X/\!\!/G)$ can be identified with X. For $\sigma \in G$ and any integer k, $L_\sigma G$ can be identified with $C_G(\sigma)$; $L_\sigma G \rtimes \mathbb{T} \cong C_G(\sigma) \times \mathbb{R}/\langle(\sigma,-1)\rangle$; and $GhLoop_\sigma(X/\!\!/G)$ can be identified with X^σ.

Unlike true loops, ghost loops have the property that they can be computed locally, as shown in the lemma below. The proof is left to the readers.

PROPOSITION 2.20. If $X = U \cup V$ where U and V are $G-$invariant open subsets, then $GhLoop(X/\!\!/G)$ is isomorphic to the fibred product of groupoids

$$GhLoop(U/\!\!/G) \cup_{GhLoop((U\cap V)/\!\!/G)} GhLoop(V/\!\!/G).$$

Thus, the ghost loop construction satisfies Mayer-Vietoris property. Moreover, it has the change-of-group property.

PROPOSITION 2.21. Let H be a closed subgroup of G. It acts on the space of left cosets G/H by left multiplication. Let pt denote the single point space with the trivial $H-$action. Then we have the equivalence of topological groupoids between $Loop_1^{ext}((G/H)/\!\!/G)$ and $Loop_1^{ext}(\mathrm{pt}/\!\!/H)$. Especially, there is an equivalence between the groupoids $GhLoop((G/H)/\!\!/G)$ and $GhLoop(\mathrm{pt}/\!\!/H)$.

PROOF. First we define a functor $F : Loop_1^{ext}((G/H)/\!\!/G) \longrightarrow Loop_1^{ext}(\mathrm{pt}/\!\!/H)$ sending an object $S^1 \leftarrow P \xrightarrow{\widetilde{\delta}} G/H$ to $S^1 \leftarrow Q \rightarrow \{eH\} = \mathrm{pt}$ where $Q \longrightarrow eH$ is the constant map, and $Q \longrightarrow S^1$ is the pull back bundle

$$
\begin{array}{ccc}
Q & \longrightarrow & \{eH\} \\
\downarrow & & \cap \\
& & \downarrow \\
P & \longrightarrow & G/H.
\end{array}
$$

It sends a morphism

$$
\begin{array}{ccccc}
P' & \longrightarrow & P & \longrightarrow & G/H \\
\downarrow & & \downarrow & & \\
S^1 & \longrightarrow & S^1 & &
\end{array}
$$

to the morphism

$$
\begin{array}{ccccc}
Q' & \longrightarrow & Q & \longrightarrow & \{eH\} \\
\downarrow & & \downarrow & & \downarrow \\
P' & \longrightarrow & P & \longrightarrow & G/H \\
\downarrow & & \downarrow & & \\
S^1 & \longrightarrow & S^1 & &
\end{array}
$$

where all the squares are pull-back.

In addition, we can define a functor $F' : Loop_1^{ext}(\mathrm{pt}/\!\!/H) \longrightarrow Loop_1^{ext}((G/H)/\!\!/G)$ sending an object $S^1 \leftarrow Q \rightarrow \mathrm{pt}$ to $S^1 \leftarrow G \times_H Q \rightarrow G \times_H \mathrm{pt} = G/H$ and sending a morphism

$$
\begin{array}{ccc}
Q' & \longrightarrow & Q \\
\downarrow & & \downarrow \\
S^1 & \longrightarrow & S^1
\end{array}
$$

to

$$
\begin{array}{ccccc}
G \times_H Q' & \longrightarrow & G \times_H Q & \longrightarrow & G \times_H \mathrm{pt} = G/H \\
\downarrow & & \downarrow & & \\
S^1 & \longrightarrow & S^1 & &
\end{array}
$$

$F \circ F'$ and $F' \circ F$ are both identity maps. So the topological groupoids $Loop_1^{ext}((G/H)/\!\!/G)$ and $Loop_1^{ext}(\mathrm{pt}/\!\!/H)$ are equivalent.

We can prove the equivalence between $GhLoop((G/H)/\!\!/G)$ and $GhLoop(\mathrm{pt}/\!\!/H)$ in the same way. $\qquad\square$

REMARK 2.22. In general, if H^* is an equivariant cohomology theory, Proposition 2.21 implies the functor

$$X/\!\!/G \mapsto H^*(GhLoop(X/\!\!/G))$$

gives a new equivariant cohomology theory. When H^* has the change of group isomorphism, so does $H^*(GhLoop(-))$.

3. Quasi-elliptic cohomology $QEll^*_G(-)$

In Section 3.2 we introduce the construction of quasi-elliptic cohomology first in terms of orbifold K-theory and then equivariant K-theory. We show the properties of the theory in Section 3.3. The main references for Section 3 are Rezk's unpublished work [40] and the author's PhD thesis [26].

Before that in Section 3.1 we show the representation ring of

$$(3.1) \qquad\qquad \Lambda_G(g) := C_G(g) \times \mathbb{R}/\langle(g,-1)\rangle,$$

which is a factor of $QEll^*_G(\mathrm{pt})$.

Moreover, in Section 3.2 we discuss the $\Lambda-$ring structure of $QEll^*_G(X)$. In Section 3.1 we introduce two groups $\Lambda^k_G(g)$ and $\Lambda_n(g)$ related closely to $\Lambda_G(g)$. We discuss some $\Lambda-$ring homomorphisms between the representation rings of them, which are essential in the construction of $\Lambda-$ring homomorphisms on quasi-elliptic cohomology in Section 3.2. A good reference for $\Lambda-$rings is the book [49].

3.1. Preliminary: representation ring of $\Lambda_G(g)$. Let $q : \mathbb{T} \longrightarrow U(1)$ be the isomorphism $t \mapsto e^{2\pi it}$. The representation ring $R\mathbb{T}$ of the circle group is $\mathbb{Z}[q^{\pm}]$.

For any compact Lie group G and a torsion element $g \in G$, we have an exact sequence

$$1 \longrightarrow C_G(g) \longrightarrow \Lambda_G(g) \overset{\pi}{\longrightarrow} \mathbb{T} \longrightarrow 0$$

where the first map is $g \mapsto [g,0]$ and the second map is

$$(3.2) \qquad\qquad \pi([g,t]) = e^{2\pi it}.$$

The map $\pi^* : R\mathbb{T} \longrightarrow R\Lambda_G(g)$ equips the representation ring $R\Lambda_G(g)$ the structure as an $R\mathbb{T}-$module.

There is a relation between the representation ring of $C_G(g)$ and that of $\Lambda_G(g)$, which is shown as Lemma 1.2 in [40] and Lemma 2.4.1 in [26].

LEMMA 3.1. $\pi^* : R\mathbb{T} \longrightarrow R\Lambda_G(g)$ *exhibits* $R\Lambda_G(g)$ *as a free* $R\mathbb{T}-$*module.*

In particular, there is an $R\mathbb{T}-$*basis of* $R\Lambda_G(g)$ *given by irreducible representations* $\{V_\lambda\}$, *such that restriction* $V_\lambda \mapsto V_\lambda|_{C_G(g)}$ *to* $C_G(g)$ *defines a bijection between* $\{V_\lambda\}$ *and the set* $\{\lambda\}$ *of irreducible representations of* $C_G(g)$.

PROOF. Let l be the order of the torsion element g. Note that $\Lambda_G(g)$ is isomorphic to

$$C_G(g) \times \mathbb{R}/l\mathbb{Z}/\langle(g,-1)\rangle.$$

Thus, it is the quotient of the product of two compact Lie groups.

Let $\lambda : C_G(g) \longrightarrow GL(n,\mathbb{C})$ be an $n-$dimensional $C_G(g)-$representation with representation space V and $\eta : \mathbb{R} \longrightarrow GL(n,\mathbb{C})$ be a representation of $\mathbb{R}$ such that $\lambda(g)$ acts on V via scalar multiplication by $\eta(1)$. Define

$$(3.3) \qquad\qquad \lambda \odot_{\mathbb{C}} \eta([h,t]) := \lambda(h)\eta(t).$$

It's straightforward to verify $\lambda \odot_{\mathbb{C}} \eta$ is a $n-$dimensional $\Lambda_G(g)-$representation with representation space V.

Any irreducible $n-$dimensional representation of the quotient group $\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle(g, -1)\rangle$ is an irreducible $n-$dimensional representation of the product $C_G(g) \times \mathbb{R}/\langle(g, -1)\rangle$. And any finite dimensional irreducible representation of the product of two compact Lie groups is the tensor product of an irreducible representation of each factor. So any irreducible representation of the quotient group $\Lambda_G(g)$ is the tensor product of an irreducible representation λ of $C_G(g)$ with representation space V and an irreducible representation η of $\mathbb{R}$. Any irreducible complex representation η of $\mathbb{R}$ is one dimensional. So the representation space of $\lambda \otimes \eta$ is still V. Let l be the order of g. $\eta(1)^l = I$. We need $\eta(1) = \lambda(g)$. So $\eta(1) = e^{\frac{2\pi i k}{l}}$ for some $k \in \mathbb{Z}$. So
$$\eta(t) = e^{\frac{2\pi i (k+lm)t}{l}}.$$
Any $m \in \mathbb{Z}$ gives a choice of η in this case. And η is a representation of $\mathbb{R}/l\mathbb{Z} \cong \mathbb{T}$.

Therefore, we have a bijective correspondence between

(1) isomorphism classes of irreducible $\Lambda_G(g)-$representation ρ, and

(2) isomorphism classes of pairs (λ, η) where λ is an irreducible $C_G(g)-$representation and $\eta : \mathbb{R} \longrightarrow \mathbb{C}^*$ is a character such that $\lambda(g) = \eta(1)I$. $\lambda = \rho|_{C_G(g)}$. $\square$

REMARK 3.2. We can make a canonical choice of $\mathbb{Z}[q^{\pm}]$-basis for $R\Lambda_G(g)$. For each irreducible G-representation $\rho : G \longrightarrow Aut(G)$, write $\rho(\sigma) = e^{2\pi i c}id$ for $c \in [0, 1)$, and set $\chi_\rho(t) = e^{2\pi i ct}$. Then the pair (ρ, χ_ρ) corresponds to a unique irreducible $\Lambda_G(g)$-representation.

EXAMPLE 3.3 ($G = \mathbb{Z}/N\mathbb{Z}$). Let $G = \mathbb{Z}/N\mathbb{Z}$ for $N \geq 1$, and let $\sigma \in G$. Given an integer $k \in \mathbb{Z}$ which projects to $\sigma \in \mathbb{Z}/N\mathbb{Z}$, let x_k denote the representation of $\Lambda_G(\sigma)$ defined by
(3.4)
$$\Lambda_G(\sigma) = (\mathbb{Z} \times \mathbb{R})/(\mathbb{Z}(N, 0) + \mathbb{Z}(k, 1)) \xrightarrow{[a,t] \mapsto [(kt-a)/N]} \mathbb{R}/\mathbb{Z} = \mathbb{T} \xrightarrow{q} U(1).$$
$R\Lambda_G(\sigma)$ is isomorphic to the ring $\mathbb{Z}[q^{\pm}, x_k]/(x_k^N - q^k)$.

For any finite abelian group $G = \mathbb{Z}/N_1\mathbb{Z} \times \mathbb{Z}/N_2\mathbb{Z} \times \cdots \times \mathbb{Z}/N_m\mathbb{Z}$, let $\sigma = (k_1, k_2, \cdots k_n) \in G$. We have
$$\Lambda_G(\sigma) \cong \Lambda_{\mathbb{Z}/N_1\mathbb{Z}}(k_1) \times_{\mathbb{T}} \cdots \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/N_m\mathbb{Z}}(k_m).$$
Then
$$R\Lambda_G(\sigma) \cong R\Lambda_{\mathbb{Z}/N_1\mathbb{Z}}(k_1) \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} R\Lambda_{\mathbb{Z}/N_m\mathbb{Z}}(k_m)$$
$$\cong \mathbb{Z}[q^{\pm}, x_{k_1}, x_{k_2}, \cdots x_{k_m}]/(x_{k_1}^{N_1} - q^{k_1}, x_{k_2}^{N_2} - q^{k_2}, \cdots x_{k_m}^{N_m} - q^{k_m})$$
where all the x_{k_j}'s are defined as x_k in (3.4).

EXAMPLE 3.4 ($G = \Sigma_3$). $G = \Sigma_3$ has three conjugacy classes represented by $1, (12), (123)$ respectively.

$\Lambda_{\Sigma_3}(1) = \Sigma_3 \times \mathbb{T}$, thus, $R\Lambda_{\Sigma_3}(1) = R\Sigma_3 \otimes R\mathbb{T} = \mathbb{Z}[X, Y]/(XY - Y, X^2 - 1, Y^2 - X - Y - 1) \otimes \mathbb{Z}[q^{\pm}]$ where X is the sign representation on Σ_3 and Y is the standard representation.

$C_{\Sigma_3}((12)) = \langle(12)\rangle = \Sigma_2$, thus, $\Lambda_{\Sigma_3}((12)) \cong \Lambda_{\Sigma_2}((12))$. So we have $R\Lambda_{\Sigma_3}((12)) \cong R\Lambda_{\Sigma_2}((12)) = \mathbb{Z}[q^{\pm}, x_1]/(x_1^2 - q) \cong \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

$C_{\Sigma_3}(123) = \langle (123) \rangle = \mathbb{Z}/3\mathbb{Z}$, thus, $\Lambda_{\Sigma_3}((123)) \cong \Lambda_{\mathbb{Z}/3\mathbb{Z}}(1)$. So we have $R\Lambda_{\Sigma_3}((123)) \cong \mathbb{Z}[q^{\pm}, x_1]/(x_1^3 - q) \cong \mathbb{Z}[q^{\pm\frac{1}{3}}]$.

Moreover, we have the conclusion below about the relation between the induced representations $Ind|_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(-)$ and $Ind|_{C_H(\sigma)}^{C_G(\sigma)}(-)$.

LEMMA 3.5. *Let H be a subgroup of G and σ an element of H. Let m denote $[C_G(\sigma) : C_H(\sigma)]$. Let V denote a $\Lambda_H(\sigma)-$representation $\lambda \odot_{\mathbb{C}} \chi$ with λ a $C_H(\sigma)-$representation, χ a $\mathbb{R}-$representation and $\odot_{\mathbb{C}}$ defined in (3.3).*
  (i)

$$(3.5) \qquad res_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \eta) = (res_{C_H(\sigma)}^{C_G(\sigma)}\lambda) \odot_{\mathbb{C}} \eta.$$

(ii) The induced representation

$$Ind_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \chi)$$

is isomorphic to the $\Lambda_G(\sigma)-$representation

$$(Ind_{C_H(\sigma)}^{C_G(\sigma)}\lambda) \odot_{\mathbb{C}} \chi.$$

Their underlying vector spaces are both $V^{\oplus m}$.
  Thus, the computation of both $Ind_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \chi)$ and $res_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \eta)$ can be reduced to the computation of representations of finite groups.

The proof is straightforward and left to the readers.

Let k be any integer. We describe the relation between

$$(3.6) \qquad \Lambda_G^k(g) := C_G(\sigma) \times \mathbb{R}/\langle (g, -k) \rangle$$

and $\Lambda_G(g)$, as well as the relation between their representation rings.
   There is an exact sequence

$$1 \longrightarrow C_G(g) \xrightarrow{g \mapsto [g,0]} \Lambda_G^k(g) \xrightarrow{\pi_k} \mathbb{R}/k\mathbb{Z} \longrightarrow 0$$

where the second map $\pi_k : \Lambda_G^k(g) \longrightarrow \mathbb{R}/k\mathbb{Z}$ is $\pi_k([g,t]) = e^{2\pi it}$.
   Let $q^{\frac{1}{k}} : \mathbb{R}/k\mathbb{Z} \longrightarrow U(1)$ denote the composition

$$\mathbb{R}/k\mathbb{Z} \xrightarrow{t \mapsto \frac{t}{k}} \mathbb{R}/\mathbb{Z} \xrightarrow{q} U(1).$$

The representation ring $R(\mathbb{R}/k\mathbb{Z})$ of $\mathbb{R}/k\mathbb{Z}$ is $\mathbb{Z}[q^{\pm\frac{1}{k}}]$. And there is a canonical isomorphism of $\Lambda-$rings

$$R\Lambda_G(g) \longrightarrow R\Lambda_G^k(g)$$

sending q to $q^{\frac{1}{k}}$.
   Analogous to Lemma 3.1, we have the conclusion about $R\Lambda_G^k(g)$ below.

LEMMA 3.6. *The map $\pi_k^* : R(\mathbb{R}/k\mathbb{Z}) \longrightarrow R\Lambda_G^k(g)$ exhibits it as a free $\mathbb{Z}[q^{\pm\frac{1}{k}}]-$module. There is a $\mathbb{Z}[q^{\pm\frac{1}{k}}]-$basis of $R\Lambda_G^k(g)$ given by irreducible representations $\{\rho_k\}$ such that the restrictions $\rho_k|_{C_G(g)}$ of them to $C_G(g)$ are precisely the $\mathbb{Z}$-basis of $RC_G(g)$ given by irreducible representations.*
   In other words, any irreducible $\Lambda_G^k(g)-$representation has the form $\rho \odot_{\mathbb{C}} \chi$ where ρ is an irreducible representation of $C_G(g)$, $\chi : \mathbb{R}/k\mathbb{Z} \longrightarrow GL(n, \mathbb{C})$ such that $\chi(k) = \rho(g)$, and

$$(3.7) \qquad \rho \odot_{\mathbb{C}} \chi([h,t]) := \rho(h)\chi(t), \text{ for any } [h,t] \in \Lambda_G^k(g).$$

$R\Lambda_G^k(g)$ is a $\mathbb{Z}[q^{\pm}]$−module via the inclusion $\mathbb{Z}[q^{\pm}] \longrightarrow \mathbb{Z}[q^{\pm\frac{1}{k}}]$.

There is a group isomorphism $\alpha_k : \Lambda_G^k(g) \longrightarrow \Lambda_G(g)$ sending $[g, t]$ to $[g, \frac{t}{k}]$. Observe that there is a pullback square of groups

(3.8)
$$\begin{array}{ccc} \Lambda_G^k(g) & \xrightarrow{\alpha_k} & \Lambda_G(g) \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi} \\ \mathbb{R}/k\mathbb{Z} & \xrightarrow{t \mapsto \frac{t}{k}} & \mathbb{R}/\mathbb{Z} \end{array}$$

By Lemma 3.6, we can make a $\mathbb{Z}[q^{\pm\frac{1}{k}}]$-basis $\{\rho \odot_{\mathbb{C}} \chi_{\rho,k}\}$ for $R\Lambda_G^k(g)$ with each $\rho : G \longrightarrow Aut(G)$ an irreducible G-representation and $\chi_{\rho,k}(t) = e^{2\pi i \frac{ct}{k}}$ with $c \in [0,1)$ such that $\rho(\sigma) = e^{2\pi i c} id$. This collection $\{\rho \odot_{\mathbb{C}} \chi_{\rho,k}\}$ gives a $\mathbb{Z}[q^{\pm\frac{1}{k}}]$−basis of $R\Lambda_G^k(g)$.

So we have the commutative square of a pushout square in the category of Λ−rings.

(3.9)
$$\begin{array}{ccc} R\Lambda_G^k(g) & \longleftarrow & R\Lambda_G(g) \\ \uparrow & & \uparrow \\ R(\mathbb{R}/k\mathbb{Z}) & \longleftarrow & R\mathbb{T} \end{array}$$

Moreover we consider

(3.10)
$$\Lambda_n(\sigma) := \Lambda_{C_G(\sigma)}(\sigma^n).$$

It is a subgroup of $\Lambda_G(\sigma^n)$. Let $\beta : \Lambda_n(\sigma) \longrightarrow \Lambda_G(\sigma^n)$ denote the inclusion. We can define

(3.11)
$$\alpha : \Lambda_n(\sigma) \longrightarrow \Lambda_G(\sigma), (g, t) \mapsto (g, nt).$$

We have the pullback square of groups

(3.12)
$$\begin{array}{ccc} \Lambda_n(\sigma) & \xrightarrow{\ \ \alpha\ \ } & \Lambda_G(\sigma) \\ \downarrow & & \downarrow \\ \mathbb{T} & \xrightarrow{e^{2\pi i t} \mapsto e^{2\pi i n t}} & \mathbb{T} \end{array}$$

In addition, $R\Lambda_n(\sigma)$ is the Λ-ring pushout of $\mathbb{Z}[q^{\pm}] \xrightarrow{R\pi} R\Lambda_G(\sigma)$ along the inclusion $\mathbb{Z}[q^{\pm}] \longrightarrow \mathbb{Z}[q^{\pm\frac{1}{n}}]$.

(3.13)
$$\begin{array}{ccc} R\Lambda_n(\sigma) & \xleftarrow{\ \ \alpha^*\ \ } & R\Lambda_G(\sigma) \\ \uparrow & & \uparrow \\ \mathbb{T} & \xleftarrow{\ R[n]\ } & \mathbb{T} \end{array}$$

In particular, there is a canonical isomorphism of Λ-rings

(3.14)
$$R\Lambda_n(\sigma) \xrightarrow{\sim} R\Lambda_G(\sigma)[q^{\pm\frac{1}{n}}].$$

3.2. Quasi-elliptic cohomology. In this section we introduce the definition of quasi-elliptic cohomology $QEll_G^*$ in terms of orbifold K-theory, and then express it via equivariant K-theory. We assume familiarity with [**44**]. The reader may read Chapter 3 in [**5**] and [**39**] for a reference of orbifold K-theory.

Quasi-elliptic cohomology is defined from the full subgroupoid $\Lambda(X/\!/G)$ of the orbifold loop space $L_{orb}(X/\!/G)$ consisting of constant loops. Before describing $\Lambda(X/\!/G)$ in detail, we recall what Inertia groupoid is. A reference for that is [**35**].

DEFINITION 3.7. Let $\mathbb{G}$ be a groupoid. The Inertia groupoid $I(\mathbb{G})$ of $\mathbb{G}$ is defined as follows.

An object a is an arrow in $\mathbb{G}$ such that its source and target are equal. A morphism v joining two objects a and b is an arrow v in $\mathbb{G}$ such that

$$v \circ a = b \circ v.$$

In other words, b is the conjugate of a by v, $b = v \circ a \circ v^{-1}$.

The torsion Inertia groupoid $I^{tors}(\mathbb{G})$ of $\mathbb{G}$ is a full subgroupoid of of $I(\mathbb{G})$ with only objects of finite order.

Let G be a compact Lie group and X a $G-$space.

EXAMPLE 3.8. The torsion inertia groupoid $I^{tors}(X/\!/G)$ of the translation groupoid $X/\!/G$ is the groupoid with

objects: the space $\coprod_{g \in G^{tors}} X^g$

morphisms: the space $\coprod_{g,g' \in G^{tors}} C_G(g, g') \times X^g$ where $C_G(g, g') = \{\sigma \in G | g'\sigma = \sigma g\} \subseteq G$.

For $x \in X^g$ and $(\sigma, g) \in C_G(g, g') \times X^g$, $(\sigma, g)(x) = \sigma x \in X^{g'}$.

DEFINITION 3.9. The groupoid $\Lambda(X/\!/G)$ has the same objects as $I^{tors}(X/\!/G)$ but richer morphisms

$$\coprod_{g,g' \in G^{tors}} \Lambda_G(g, g') \times X^g$$

where $\Lambda_G(g, g')$ is defined in Definition 2.14. For an object $x \in X^g$ and a morphism $([\sigma, t], g) \in \Lambda_G(g, g') \times X^g$, $([\sigma, t], g)(x) = \sigma x \in X^{g'}$. The composition of the morphisms is defined by

(3.15) $$[\sigma_1, t_1][\sigma_2, t_2] = [\sigma_1 \sigma_2, t_1 + t_2].$$

We have a homomorphism of orbifolds

$$\pi : \Lambda(X/\!/G) \longrightarrow \mathbb{T}$$

sending all the objects to the single object in $\mathbb{T}$, and a morphism $([\sigma, t], g)$ to $e^{2\pi i t}$ in $\mathbb{T}$.

DEFINITION 3.10. The quasi-elliptic cohomology $QEll_G^*(X)$ is defined to be $K_{orb}^*(\Lambda(X/\!/G))$.

We can unravel the definition and express it via equivariant K-theory.

Let $\sigma \in G^{tors}$. The fixed point space X^σ is a $C_G(\sigma)-$space. We can define a $\Lambda_G(\sigma)-$action on X^σ by

$$[g, t] \cdot x := g \cdot x.$$

Then we have

PROPOSITION 3.11.

$$(3.16) \qquad QEll_G^*(X) = \prod_{g \in G_{conj}^{tors}} K_{\Lambda_G(g)}^*(X^g) = \left(\prod_{g \in G^{tors}} K_{\Lambda_G(g)}^*(X^g) \right)^G.$$

Thus, for each $g \in \Lambda_G(g)$, we can define the projection

$$\pi_g : QEll_G^*(X) \longrightarrow K_{\Lambda_G(g)}^*(X^g).$$

For the single point space, we have

$$(3.17) \qquad QEll_G^0(\mathrm{pt}) \cong \prod_{g \in G_{conj}^{tors}} R\Lambda_G(g).$$

By computing the representation rings of $R\Lambda_G(g)$, we get $QEll_G^*(-)$ for contractible spaces. Then, using Mayer-Vietoris sequence, we can compute $QEll_G^*(-)$ for any G-CW complex by patching the G-cells together.

We have the ring homomorphism

$$\mathbb{Z}[q^{\pm}] = K_{\mathbb{T}}^0(\mathrm{pt}) \xrightarrow{\pi^*} K_{\Lambda_G(g)}^0(\mathrm{pt}) \longrightarrow K_{\Lambda_G(g)}^0(X)$$

where $\pi : \Lambda_G(g) \longrightarrow \mathbb{T}$ is the projection defined in (3.2) and the second is via the collapsing map $X \longrightarrow \mathrm{pt}$. So $QEll_G^*(X)$ is naturally a $\mathbb{Z}[q^{\pm}]$-algebra.

The Λ-ring structure on $QEll_G^*(X)$ is the direct product of the exterior algebra on each equivariant K-group, with componentwise multiplication.

For each $QEll_G^*$ we can equip a special family of Λ-ring homomorphisms

$$\mu^n : QEll_G^*(X) \cong \prod_{g \in G_{conj}^{tors}} K_{\Lambda_G(g)}^*(X^g) \longrightarrow \prod_{g \in G_{conj}^{tors}} K_{\Lambda_G^n(g)}^*(X^g) \cong QEll_G^*(X)[q^{\pm \frac{1}{n}}]$$

defined by

$$(3.18) \qquad QEll_G^*(X) \longrightarrow K_{\Lambda_G(g^n)}^*(X^{g^n}) \xrightarrow{\beta^*} K_{\Lambda_G^n(g)}^*(X^{g^n}) \longrightarrow K_{\Lambda_G^n(g)}^*(X^g),$$

where the first map is projection, the second is the restriction via the inclusion $\beta : \Lambda_G^n(g) \longrightarrow \Lambda_G(g^n)$, and the third is restriction along $X^g \subseteq X^{g^n}$.

In addition, we can express the Λ-ring isomorphism (3.14) and this family of Λ-ring homomorphisms $\{\mu^n\}_n$ in terms of orbifold K-theory, which are fairly neat.

Let $\Lambda_n(g, g')$ denote the quotient of $C_G(g, g') \times \mathbb{R}$ under the action of $\mathbb{Z}$ where the action of the generator of $\mathbb{Z}$ is given by $(\sigma, t) \mapsto (\sigma g^n, t - 1) = (g^n \sigma', t - 1)$. Then we can define another groupoid $\Lambda_n(X /\!\!/ G)$ with the same objects as $\Lambda(X /\!\!/ G)$ and morphisms

$$\coprod_{g, g' \in G^{tors}} \Lambda_n(g, g') \times X^g$$

such that for each $x \in X^g$, $([\sigma, t], g)(x) = \sigma x \in X^{g'}$. We can also define $\pi : \Lambda_n(X /\!\!/ G) \longrightarrow \mathbb{T}$ sending all the objects to the single object in $\mathbb{T}$ and a morphism $([\sigma, t], g)$ to the morphism $e^{2\pi i t}$ in $\mathbb{T}$.

Let $\alpha : \Lambda_n(X /\!\!/ G) \longrightarrow \Lambda(X /\!\!/ G)$ be the homomorphism of orbifolds sending an object $x \in X^g$ to $x \in X^g$ and a morphism $[\sigma, t] : x \longrightarrow x'$ to $[\sigma, nt] : x \longrightarrow x'$. Let $\beta : \Lambda_n(X /\!\!/ G) \longrightarrow \Lambda(X /\!\!/ G)$ be the functor sending an object $x \in X^g$ to $x \in X^{g^n}$ and a morphism $[\sigma, t] : x \longrightarrow x'$ to $[\sigma, t] : x \longrightarrow x'$.

Since we have the pullback square of groups (3.8) and the pushout square of groups (3.9), we have the pullback square of groupoids

(3.19)
$$
\begin{array}{ccc}
\Lambda_n(X/\!\!/G) & \xrightarrow{\ \ \alpha\ \ } & \Lambda(X/\!\!/G) \\
\downarrow & & \downarrow \\
\mathbb{T} & \xrightarrow{e^{2\pi it}\mapsto e^{2\pi nit}} & \mathbb{T},
\end{array}
$$

and the pushout square in the category of Λ-rings

(3.20)
$$
\begin{array}{ccc}
K^*_{orb}(\Lambda_n(X/\!\!/G)) & \xleftarrow{\ \ \alpha^*\ \ } & K^*_{orb}(\Lambda(X/\!\!/G)) \\
\uparrow & & \uparrow \\
K^*_{orb}(*/\!\!/\mathbb{T}) & \longleftarrow & K^*_{orb}(*/\!\!/\mathbb{T}).
\end{array}
$$

It induces a natural isomorphism

(3.21)
$$
K^*_{orb}(\Lambda(X/\!\!/G))[q^{\pm\frac{1}{n}}] \xrightarrow{\ \sim\ } K^*_{orb}(\Lambda_n(X/\!\!/G)).
$$

The Λ-ring homorphism μ^n can be constructed by

$$
\mu^n : K^*_{orb}(\Lambda(X/\!\!/G)) \xrightarrow{\ \beta^*\ } K^*_{orb}(\Lambda_n(X/\!\!/G)) \cong K^*_{orb}(\Lambda(X/\!\!/G))[q^{\pm\frac{1}{n}}].
$$

3.3. Properties. $QEll^*_G$ inherits most properties of equivariant K-theory. In this section we discuss some properties of $QEll^*_G$, including the restriction map, the Künneth map on it, its tensor product and the change of group isomorphism.

Since each homomorphism $\phi : G \longrightarrow H$ induces a well-defined homomorphism $\phi_\Lambda : \Lambda_G(\tau) \longrightarrow \Lambda_H(\phi(\tau))$ for each τ in G^{tors}, we can get the proposition below directly.

PROPOSITION 3.12. For each homomorphism $\phi : G \longrightarrow H$, it induces a ring map

$$
\phi^* : QEll^*_H(X) \longrightarrow QEll^*_G(\phi^*X)
$$

characterized by the commutative diagrams

(3.22)
$$
\begin{array}{ccc}
QEll^*_H(X) & \xrightarrow{\ \ \phi^*\ \ } & QEll^*_G(\phi^*X) \\
\pi_{\phi(\tau)} \downarrow & & \pi_\tau \downarrow \\
K^*_{\Lambda_H(\phi(\tau))}(X^{\phi(\tau)}) & \xrightarrow{\ \ \phi^*_\Lambda\ \ } & K^*_{\Lambda_G(\tau)}(X^{\phi(\tau)})
\end{array}
$$

for any $\tau \in G^{tors}$. So $QEll^*_G$ is functorial in G.

More generally, we have the restriction map below.

PROPOSITION 3.13. For any groupoid homomorphism $\phi : X/\!\!/G \longrightarrow Y/\!\!/H$, we have the groupoid homomorphism $\Lambda(\phi) : \Lambda(X/\!\!/G) \longrightarrow \Lambda(Y/\!\!/H)$ sending an object (x, g) to $(\phi(x), \phi(g))$, and a morphism $([\sigma, t], g)$ to $([\phi(\sigma), t], \phi(g))$. Thus, we get a ring map

$$
\phi^* : QEll^*(Y/\!\!/H) \longrightarrow QEll^*(X/\!\!/G)
$$

characterized by the commutative diagrams

$$
\begin{array}{ccc}
QEll^*(Y/\!\!/H) & \xrightarrow{\ \phi^*\ } & QEll^*(X/\!\!/G) \\[4pt]
\pi_{\phi(\tau)} \Big\downarrow & & \pi_\tau \Big\downarrow \\[4pt]
K^*_{\Lambda_H(\phi(\tau))}(Y^{\phi(\tau)}) & \xrightarrow{\ \phi^*_\Lambda\ } & K^*_{\Lambda_G(\tau)}(X^\tau)
\end{array}
$$

(3.23)

for any $\tau \in G^{tors}$.

Moreover, we can define Künneth map on quasi-elliptic cohomology induced from that on equivariant K-theory.

Let G and H be two compact Lie groups. X is a G-space and Y is a H-space. Let $\sigma \in G^{tors}$ and $\tau \in H^{tors}$. Let $\Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$ denote the fibered product of the morphisms

$$\Lambda_G(\sigma) \xrightarrow{\ \pi\ } \mathbb{T} \xleftarrow{\ \pi\ } \Lambda_H(\tau).$$

It is isomorphic to $\Lambda_{G \times H}(\sigma, \tau)$ under the correspondence

$$([\alpha, t], [\beta, t]) \mapsto [\alpha, \beta, t].$$

Consider the map below

$$T : K_{\Lambda_G(\sigma)}(X^\sigma) \otimes K_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_G(\sigma) \times \Lambda_H(\tau)}(X^\sigma \times Y^\tau) \xrightarrow{\ res\ } K_{\Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)}(X^\sigma \times Y^\tau)$$

$$\xrightarrow{\ \cong\ } K_{\Lambda_{G \times H}(\sigma, \tau)}((X \times Y)^{(\sigma, \tau)}).$$

where the first map is the Künneth map of equivariant K-theory, the second is the restriction map and the third is the isomorphism induced by the group isomorphism $\Lambda_{G \times H}(\sigma, \tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$.

For $g \in G^{tors}$, let 1 denote the trivial line bundle over X^g and let q denote the line bundle $1 \odot q$ over X^g. The map T above sends both $1 \otimes q$ and $q \otimes 1$ to q. So we get the well-defined map

(3.24) $\qquad K^*_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K^*_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_{G \times H}(\sigma, \tau)}((X \times Y)^{(\sigma, \tau)}).$

DEFINITION 3.14. The tensor produce of quasi-elliptic cohomology is defined by
(3.25)
$$QEll^*_G(X) \widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) \cong \prod_{\sigma \in G^{tors}_{conj},\ \tau \in H^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K^*_{\Lambda_H(\tau)}(Y^\tau).$$

The direct product of the maps defined in (3.24) gives a ring homomorphism

$$QEll^*_G(X) \widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) \longrightarrow QEll^*_{G \times H}(X \times Y),$$

which is the Künneth map of quasi-elliptic cohomology.

By Lemma 3.1 we have

$$QEll^*_G(\mathrm{pt}) \widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(\mathrm{pt}) = QEll^*_{G \times H}(\mathrm{pt}).$$

More generally, we have the proposition below.

PROPOSITION 3.15. Let X be a $G \times H-$space with trivial $H-$action and let pt be the single point space with trivial $H-$action. Then we have

$$QEll_{G \times H}(X) \cong QEll_G(X) \widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll_H(pt).$$

Especially, if G acts trivially on X, we have

$$QEll_G(X) \cong QEll(X) \widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll_G(\mathrm{pt}).$$

Here $QEll^*(X)$ is $QEll^*_{\{e\}}(X) = K^*_{\mathbb{T}}(X)$.

PROOF.

$$QEll_{G\times H}(X) = \prod_{\substack{g \in G^{tors}_{conj} \\ h \in H^{tors}_{conj}}} K_{\Lambda_{G\times H}(g,h)}(X^{(g,h)}) \cong \prod_{\substack{g \in G^{tors}_{conj} \\ h \in H^{tors}_{conj}}} K_{\Lambda_G(g)\times_{\mathbb{T}}\Lambda_H(h)}(X^g)$$

$$\cong \prod_{\substack{g \in G^{tors}_{conj} \\ h \in H^{tors}_{conj}}} K_{\Lambda_G(g)}(X^g) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_H(h)}(\mathrm{pt}) = QEll_G(X) \widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll_H(pt).$$

$\square$

PROPOSITION 3.16. If G acts freely on X,

$$QEll^*_G(X) \cong QEll^*_e(X/G).$$

PROOF. Since G acts freely on X,

$$X^\sigma = \begin{cases} \emptyset, & \text{if } \sigma \neq e; \\ X, & \text{if } \sigma = e. \end{cases}$$

Thus, $QEll^*_G(X) \cong \prod_{\sigma \in G^{tors}_{conj}} K^*_{\Lambda_G(\sigma)/C_G(\sigma)}(X^\sigma/C_G(\sigma)) \cong K^*_{\mathbb{T}}(X/G).$

Since $\mathbb{T}$ acts trivially on X, we have $K^*_{\mathbb{T}}(X/G) = QEll^*_e(X/G)$ by definition. And it is isomorphic to $K^*(X/G) \otimes R\mathbb{T}$. $\square$

We also have the change-of-group isomorphism as in equivariant K-theory.

Let H be a closed subgroup of G and X a H-space. Let $\phi : H \longrightarrow G$ denote the inclusion homomorphism. The change-of-group map $\rho^G_H : QEll^*_G(G \times_H X) \longrightarrow QEll^*_H(X)$ is defined as the composite

$$(3.26) \qquad QEll^*_G(G \times_H X) \xrightarrow{\phi^*} QEll^*_H(G \times_H X) \xrightarrow{i^*} QEll^*_H(X)$$

where ϕ^* is the restriction map and $i : X \longrightarrow G \times_H X$ is the $H-$equivariant map defined by $i(x) = [e, x]$.

PROPOSITION 3.17. The change-of-group map

$$\rho^G_H : QEll^*_G(G \times_H X) \longrightarrow QEll^*_H(X)$$

defined in (3.26) is an isomorphism.

PROOF. For any $\tau \in H^{tors}_{conj}$, there exists a unique $\sigma_\tau \in G^{tors}_{conj}$ such that $\tau = g_\tau \sigma_\tau g_\tau^{-1}$ for some $g_\tau \in G$. Consider the maps

$$(3.27) \quad \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau \xrightarrow{[[a,t],x]\mapsto[a,x]} (G \times_H X)^\tau \xrightarrow{[u,x]\mapsto[g_\tau^{-1}u,x]} (G \times_H X)^\sigma.$$

The first map is $\Lambda_G(\tau)-$equivariant and the second is equivariant with respect to the homomorphism $c_{g_\tau} : \Lambda_G(\sigma) \longrightarrow \Lambda_G(\tau)$ sending $[u,t] \mapsto [g_\tau u g_\tau^{-1}, t]$. Taking a coproduct over all the elements $\tau \in H^{tors}_{conj}$ that are conjugate to $\sigma \in G^{tors}_{conj}$ in G, we get an isomorphism

$$\gamma_\sigma : \coprod_\tau \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau \longrightarrow (G \times_H X)^\sigma$$

which is $\Lambda_G(\sigma)-$equivariant with respect to c_{g_τ}. Then we have the map
(3.28)
$$\gamma := \prod_{\sigma \in G_{conj}^{tors}} \gamma_\sigma : \prod_{\sigma \in G_{conj}^{tors}} K_{\Lambda_G(\sigma)}^*(G \times_H X)^\sigma \longrightarrow \prod_{\sigma \in G_{conj}^{tors}} K_{\Lambda_G(\sigma)}^*(\coprod_\tau \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau)$$

It's straightforward to check the change-of-group map coincide with the composite

$$QEll_G^*(G \times_H X) \xrightarrow{\gamma} \prod_{\sigma \in G_{conj}^{tors}} K_{\Lambda_G(\sigma)}^*(\coprod_\tau \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau) \longrightarrow \prod_{\tau \in H_{conj}^{tors}} K_{\Lambda_H(\tau)}^*(X^\tau)$$
$$= QEll_H^*(X)$$

with the second map the change-of-group isomorphism in equivariant $K-$theory.
$\square$

3.4. Formulas for Induction.
In this section 3.4 we introduce the induction formula for quasi-elliptic cohomology. The induction formula for Tate K-theory is constructed in Section 2.3.3, [**18**].

Let $H \subseteq G$ be an inclusion of compact Lie groups and X be a $G-$space. Then we have the inclusion of the groupoids

$$j : X/\!\!/H \longrightarrow X/\!\!/G.$$

Let $a' = \prod_{\sigma \in H_{conj}^{tors}} a'_\sigma$ be an element in $QEll_H(X) = \prod_{\sigma \in H_{conj}^{tors}} K_{\Lambda_H(\sigma)}(X^\sigma)$ where σ goes over all the conjugacy classes in H. The finite covering map

$$f' : \Lambda(G \times_H X/\!\!/G) \longrightarrow \Lambda(X/\!\!/G)$$

is defined by sending an object $(\sigma, [g, x])$ to (σ, gx) and a morphism $([g', t], (\sigma, [g, x]))$ to $([g', t], (gx, \sigma))$. The transfer of quasi-elliptic cohomology

$$\mathcal{I}_H^G : QEll_H(X) \longrightarrow QEll_G(X)$$

is defined to be the composition

(3.29) $$\qquad QEll_H(X) \xrightarrow{\cong} QEll_G(G \times_H X) \longrightarrow QEll_G(X)$$

where the first map is the change-of-group isomorphism and the second is the finite covering.

Thus

$$\mathcal{I}_H^G(a')_g = \sum_r r \cdot a'_{r^{-1}gr}$$

where r goes over a set of representatives of $(G/H)^g$, in other words, $r^{-1}gr$ goes over a set of representatives of conjugacy classes in H conjugate to g in G.

(3.30) $$\qquad \mathcal{I}_H^G(a')_g = \begin{cases} Ind_{\Lambda_H}^{\Lambda_G}(a'_g) & \text{if } g \text{ is conjuate to some element } h \text{ in } H; \\ 0 & \text{if there is no element conjugate to } g \text{ in } H. \end{cases}$$

There is another way to describe the transfer, which is shown in Rezk's unpublished work [**40**] for quasi-elliptic cohomology. The transfer of Tate K-theory can be described similarly.
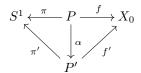
4. Orbifold quasi-elliptic cohomology

The elliptic cohomology of orbifolds involves a rich interaction between the orbifold structure and the elliptic curve. Orbifold quasi-elliptic cohomology can also be constructed from loop spaces via bibundles. We give the construction via bibundles in Section 4.1. Ganter explores this interaction in [**18**] in the case of the Tate curve, describing K_{Tate} for an orbifold X in term of the equivariant K-theory and the groupoid structure of X. We show the relation between orbifold quasi-elliptic cohomology and orbifold Tate K-theory in Section 4.2.

4.1. Definition. In this section we construct orbifold quasi-elliptic cohomology via loop space. The idea is similar to that in Section 2. For the definition of groupoid action and groupoid-principal bundles, the readers can refer to Section 3, [**32**].

Let X be an orbifold groupoid.

DEFINITION 4.1 ($Loop_1(X)$). We use $Loop_1(X)$ to denote the category $Bibun(S^1/\!\!/*, X)$, which generalizes Definition 2.4. According to Definition 2.1, each object consists of a smooth manifold P and two structure maps $P \xrightarrow{\pi} S^1$ a smooth principal $X-$bundle and $f : P \longrightarrow X_0$ an $X-$equivariant map. A morphism is an $X-$bundle map $\alpha : P \longrightarrow P'$ making the diagram below commute.

$$S^1 \xleftarrow{\;\pi\;} P \xrightarrow{\;f\;} X_0$$
$$\pi' \diagdown \quad \Big\downarrow{\scriptstyle\alpha} \quad \diagup f'$$
$$P'$$

Thus, the morphisms in $Loop_1(X)$ from P to P' are $X-$isomorphisms.

Next we add rotations to the groupoid $Loop_1(X)$ and give the definition of the groupoid $Loop_1^{ext}(X)$ which generalizes Definition 2.10.

DEFINITION 4.2 ($Loop_1^{ext}(X)$). Let $Loop_1^{ext}(X)$ denote the groupoid with the same objects as $Loop_1(X)$. Each morphism consists of the pair (t, α) where $t \in \mathbb{T}$ is a rotation and α is an $X-$bundle map. They make the diagram below commute.

$$S^1 \xleftarrow{\;\pi\;} P \xrightarrow{\;f\;} X_0$$
$$t\Big\downarrow \qquad \alpha\Big\downarrow \quad \diagup f'$$
$$S^1 \xleftarrow{\;\pi'\;} P'$$

In addition, we can define the groupoid of ghost loops for orbifolds.

DEFINITION 4.3 (Ghost Loops). The ghost loops corresponds to the full subgroupoid $GhLoop(X)$ of $Loop_1^{ext}(X)$ consisting of objects $S^1 \leftarrow P \xrightarrow{\tilde\delta} X_0$ such that $\tilde\delta(P) \subseteq X_0$ contained in a single $G-$orbit.

The groupoid constant loops $\Lambda(X)$ is a subgroupoid of $GhLoop(X)$.

DEFINITION 4.4. The groupoid $\Lambda(X)$ is the subgroupoid of $Loop_1^{ext}(X)$ consisting of objects $S^1 \leftarrow P \xrightarrow{\tilde\delta} X_0$ such that there exists a section of $s_P : P \longrightarrow S^1$ such that $f \circ s_P$ is a constant map. Let $\{x_P\}$ denote the image of $f \circ s_P$. Each object is determined by x_P and an automorphism $g \in aut(x_P)$ in X of finite order.

In each morphism

$$S^1 \xleftarrow{\ \pi\ } P \xrightarrow{\ f\ } X_0$$

with vertical maps t, α, and f' to P', $S^1 \xleftarrow{\ \pi'\ } P'$

$\alpha \in Mor_X(x_P, x_{P'})$ and the morphism $(t+1, \alpha)$ is the same as $(t, \alpha \circ g)$.

When X is a global quotient $M /\!\!/ G$, $\Lambda(X)$ is isomorphic to the groupoid $\Lambda(M /\!\!/ G)$.

DEFINITION 4.5. The orbifold quasi-elliptic cohomology of X is defined to be

(4.1) $$QEll^*(X) := K^*_{orb}(\Lambda(X)).$$

In the global quotient case,

$$QEll^*(M /\!\!/ G) = QEll^*_G(M).$$

4.2. Relation with Orbifold Tate K-theory. In this section we give another definition of orbifold quasi-elliptic cohomology equivalent to Definition 4.1. It is closely related to Ganter's construction of orbifold Tate K-theory in [**18**].

First we recall some relevant constructions and notations. The main reference is [**18**].

Consider the category of groupoids $\mathcal{G}pd$ as a 2-category with small topological groupoids as the objects and with

$$1\mathrm{Hom}(X, Y) = Fun(X, Y),$$

the groupoid of continuous functors from X to Y.

DEFINITION 4.6. The center of a groupoid X is defined to be the group

$$\mathrm{Center}(X) := 2\mathrm{Hom}(\mathrm{Id}_X, \mathrm{Id}_X) = \mathcal{N}at(\mathrm{Id}_X, \mathrm{Id}_X)$$

of natural transformations from Id_x to Id_x.

DEFINITION 4.7. Let $\mathcal{G}pd^{cen}$ denote the 2-category whose objects are pairs (X, ξ) with ξ a center element of X, and the set of morphisms from (X, ξ) to (Y, ν) is

$$1\mathrm{Hom}((X, \xi), (Y, \nu)) \subset Fun(X, Y)$$

with

$$f\xi = \nu f$$

for each morphism f.

We will assume all the center elements have finite order.

EXAMPLE 4.8. If G is a finite group, $\mathrm{Center}(pt /\!\!/ G)$ is the center of the group G.

EXAMPLE 4.9. The Inertia groupoid $I(X)$ of a groupoid X, which is defined in Definition 3.7, is isomorphic to

$$Fun(\mathrm{pt} /\!\!/ \mathbb{Z}, X).$$

Each object of $I(X)$ can be viewed as pairs (x, g) with $x \in \mathrm{ob}(X)$ and $g \in aut(x)$, $gx = x$. A morphism from (x_1, g_1) to (x_2, g_2) is a morphism $h : x_1 \longrightarrow x_2$ in X satisfying $h \circ g_1 = g_2 \circ h$ in X. So in $I(X)$,

$$\mathrm{Hom}((x_1, g_1), (x_2, g_2)) = \{h : x_1 \longrightarrow x_2 | h \circ g_1 = g_2 \circ h\}.$$

Recall $I^{tors}(X)$ is a full subgoupoid of $I(X)$ with elements (x, g) where g is of finite order. Let ξ^k denote the center element of $I^{tors}(X)$ sending (x, g) to (x, g^k). We use ξ to denote ξ^1.

For any $k \in \mathbb{Z}$, we have the 2-functor

$$\mathcal{G}pd \longrightarrow \mathcal{G}pd^{cen}$$

$$X \mapsto (I^{tors}(X), \xi^k).$$

EXAMPLE 4.10. In the global quotient case, as indicated in Example 3.8, $I^{tors}(X/\!\!/G)$ is isomorphic to $\prod\limits_{g \in G^{tors}_{conj}} X^g /\!\!/ C_G(g)$. The center element $\xi^k|_{X^g} = g^k$.

DEFINITION 4.11. Let $pt/\!\!/\mathbb{R} \times_{1 \sim \xi} I^{tors}(X)$ denote the groupoid

$$(pt/\!\!/\mathbb{R}) \times I^{tors}(X)/ \sim$$

with $\sim$ generated by $1 \sim \xi$.

LEMMA 4.12. *The groupoid $pt/\!\!/\mathbb{R} \times_{1 \sim \xi} I^{tors}(X)$ is isomorphic to $\Lambda(X)$. Thus, $QEll^*(X)$ is isomorphic to*

$$(4.2) \qquad\qquad K^*_{orb}(pt/\!\!/\mathbb{R} \times_{1 \sim \xi} I^{tors}(X)).$$

The proof of Lemma 4.12 is left to the readers.

REMARK 4.13. Orbifold quasi-elliptic cohomology $QEll(X)$ can be defined to be a subring of $K_{orb}(X)[\![q^{\pm\frac{1}{|\xi|}}]\!]$ that is the Grothendieck group of finite sums

$$\sum_{a \in \mathbb{Q}} V_a q^a$$

satisfying:

for each $a \in \mathbb{Q}$, the coefficient V_a is an $e^{2\pi i a}$ − eigenbundle of ξ.

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