Universal Finite Subgroup of Tate Curve

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Abstract. In [**6**] Katz and Mazur define the universal finite subgroup of an elliptic curve. In this paper we give an explicit construction of the universal finite subgroup of the Tate curve via quasi-elliptic cohomology and its power operation.

1. Introduction

An elliptic curve $E \longrightarrow S$ over a base S is an abelian group $S-$scheme E whose fiber at every geometric point is an elliptic curve. Katz and Mazur discussed the moduli problem [N-Isog] in Chapter 6 [**6**]. [N-Isog](E/S) is defined to be the set of finite locally free subgroup commutative $S-$subgroup-schemes $G < E[N]$ which are rank N over S. In other words, [N-Isog](E/S) is the set of subgroup schemes of degree N in E. In [**6**] Katz and Mazur prove that this moduli problem is relatively representable in Proposition 6.5.1 and it is finite and flat over (Ell) in Theorem 6.8.1, which amounts to the following.

Let E be an elliptic curve over a commutative ring A. Fix $N \geq 0$. There exists a ring homomorphism $A \longrightarrow B$ and a subgroup $G < E_B$ of rank N in E_B where E_B is the pullback of E over $\mathrm{Spec}(B)$. The pair $(B, G < E_B)$ is the universal subgroup of rank N of $E \longrightarrow \mathrm{Spec}(A)$ in the sense that: given a ring homomorphism $A \longrightarrow C$ and a subgroup H of rank N in E_C, there exists a unique ring homomorphism $g : B \longrightarrow C$ compatible with the maps from A such that $H = g^*(G)$, i.e.there is a pullback diagram in schemes of the form

$$\begin{tikzcd} H \arrow[r, tail] \arrow[d] & E_C \arrow[r] \arrow[d] & \mathrm{Spec}(C) \arrow[d] \\ G \arrow[r, tail] & E_B \arrow[r] \arrow[d] & \mathrm{Spec}(B) \arrow[d] \\ & E \arrow[r] & \mathrm{Spec}(A). \end{tikzcd} \tag{1.1}$$

In [**8**] and [**9**] Strickland gives an explicit construction of the universal subgroup of order p^k of the formal group of Morava E-theory. However, there is no known

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constructions of the universal finite subgroup of any elliptic curve. In this paper we construct the universal subgroup of rank N of the Tate curve $Tate(q)$.

The Tate curve $Tate(q)$ is an elliptic curve over $\mathrm{Spec}\mathbb{Z}((q))$, which is classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity. A good reference for $Tate(q)$ is Section 2.6 of [**1**]. Tate K-theory is the generalized elliptic cohomology associated to the Tate curve. The relation between Tate K-theory and string theory is better understood than for most known elliptic cohomology theories. The definition of $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold, which is formulated explicitly in Section 2, [**3**]. Its relation with string theory and loop space makes Tate K-theory itself a distinctive subject to study.

In Section 8.7.1 [**6**] Katz and Mazur discuss the torsion points of order N of $Tate(q)$, which can be classified by the Tate K-theory of the cyclic group $\mathbb{Z}/N\mathbb{Z}$, shown by Ganter in [**3**]. In [**4**] we give a classification theorem of the finite subgroups of $Tate(q)$, which is analogous to the principal result in Strickland [**9**]. More explicitly, the Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve. The application of an intermediate theory, quasi-elliptic cohomology, is essential in the proof of this classification theorem, as shown in Section 6.3 in [**4**].

These two classifications are essential ingredients in the construction of the universal group of order N.

Quasi-elliptic cohomology also plays a key role in the construction of the universal finite subgroup of the Tate curve. It is motivated by Ganter in [**3**], set up by Rezk in his unpublished manuscript [**7**], and written in detail in the author's PhD thesis [**5**]. This theory is a variant of Tate K-theory. It is the orbifold K-theory of a space of constant loops. For global quotient orbifolds, it can be expressed in terms of equivariant K-theories. Quasi-elliptic cohomology serves as an object both reflecting the geometric nature of elliptic curves and more practicable to study.

We construct in [**4**] a power operation $\{\mathbb{P}_N\}_N$ of quasi-elliptic cohomology relating to the level structure of the Tate curve. The construction of it mixes power operation in K-theory with natural operation of dilating and rotating loops and can be generalized to other equivariant cohomology theories. Moreover, from the power operation, we construct in Proposition 6.5 [**4**] a ring homomorphism

$$\overline{P}_N : QEll(X/\!\!/G) \longrightarrow QEll(X/\!\!/G) \otimes_{\mathbb{Z}[q^{\pm}]} QEll(\mathrm{pt}/\!\!/\Sigma_N)/\mathcal{I}_{tr}^{\Sigma_N}$$

Ando, Hopkins and Strickland discuss the additive power operation of Morava E-theories

$$E^0 \longrightarrow E(B\Sigma_{p^k})/I_{tr}$$

in [**2**]. Applying the Strickland's theorem in [**9**] they show that it has a nice algebra-geometric interpretation in terms of the formal group and it takes the quotient by the universal subgroup. In this paper we show that the additive operation $\overline{P}_N$ plays an essential part in the construction of the universal subgroup of order N of the Tate curve.

In Section 2, we give a sketch of the Tate curve and its finite subgroups. The main reference is [**6**]. In Section 3, we introduce quasi-elliptic cohomology and its relation with Tate K-theory. We also recall the classification theorems of the Tate curve and the operations on quasi-elliptic cohomology that we need later in Section 4. The main references are [**4**] and [**7**]. In Section 4.1, 4.2 and 4.3 we construct

G_{univ}. In Section 4.4 we prove that G_{univ} is the universal subgroup of order N of $Tate(q)$.

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2. Background on finite subgroups of the Tate curve

In this section we introduce the Tate curve and its finite subgroups. The main references are Section 2.6 in [**1**] and Section 8.7, 8.8 in [**6**].

An elliptic curve over the complex numbers $\mathbb{C}$ is a connected Riemann surface, i.e. a connected compact 1-dimensional complex manifold, of genus 1. By the uniformization theorem every elliptic curve over $\mathbb{C}$ is analytically isomorphic to a 1-dimensional complex torus, and can be expressed as

$$\mathbb{C}^*/q^{\mathbb{Z}}$$

with $q \in \mathbb{C}$ and $0 < |q| < 1$, where $\mathbb{C}^*$ is the multiplicative group $\mathbb{C}\backslash\{0\}$.

The Tate curve $Tate(q)$ is the elliptic curve

$$E_q : y^2 + xy = x^3 + a_4 x + a_6$$

whose coefficients are given by the formal power series in $\mathbb{Z}((q))$

$$a_4 = -5\sum_{n\geqslant 1} n^3 q^n/(1-q^n) \qquad a_6 = -\frac{1}{12}\sum_{n\geqslant 1}(7n^5+5n^3)q^n/(1-q^n).$$

Before we talk about the torsion part of $Tate(q)$, we recall a smooth one-dimensional commutative group scheme T over $\mathbb{Z}[q^{\pm}]$. It sits in a short exact sequence of group-schemes over $\mathbb{Z}[q^{\pm}]$

$$0 \longrightarrow \mathbb{G}_m \longrightarrow T \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

T is a functor from the category of $\mathbb{Z}[q^{\pm}]-$algebras to the category of abelian groups. For each $\mathbb{Z}[q^{\pm}] \longrightarrow R$,

$$T(R) = \frac{R^* \times \mathbb{Q}}{\langle (q,-1)\rangle}.$$

Note $\mathbb{G}_m(R) = R^*$. And we have the exact sequence

$$0 \longrightarrow R^* \longrightarrow T(R) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

The $N-$torsion points $T[N]$ of it is the disjoint union of N schemes $T_0[N]$, $\cdots$ $T_{N-1}[N]$, where

$$T_i[N] = \mathrm{Spec}(\mathbb{Z}[q^{\pm}][x]/(x^N - q^i)).$$

It fits into a short exact sequence

$$0 \longrightarrow \mu_N \xrightarrow{a_N} T[N] \xrightarrow{b_N} \mathbb{Z}/N\mathbb{Z} \longrightarrow 0.$$

For a $\mathbb{Z}[q^{\pm}]-$algebra R,

$$T(R)[N] = \{[a,x] \in T(R) \mid N[a,x] = 0\}.$$

The canonical extension structure on $T(N)$ is compatible with an alternating paring of $\mathbb{Z}[q^{\pm}]-$group schemes $e_N : T(N) \times T(N) \longrightarrow \mu_N$ in the sense that

$$e_N(a_N(x), y) = x^{b_N(y)}, \text{ for any } \mathbb{Z}[q^{\pm}] - \text{algebra } R \text{ and any } x \in \mu_N(R).$$

And we have the conclusion below, which is Theorem 8.7.5 in [**6**].

THEOREM 2.1. *There exists a faithfully flat* $\mathbb{Z}[q^{\pm}]-$*algebra* R*, an elliptic curve* E/R*, and an isomorphism of ind-group-schemes over* R

$$T_{torsion}\otimes_{\mathbb{Z}[q^{\pm}]}R\xrightarrow{\sim}E_{tors},$$

such that for every $N\geq 1$*, the isomorphism on* $N-$*torsion points* $T[N]\otimes R\xrightarrow{\sim}E[N]$ *is compatible with* e_N-*pairings.*

Thus, we have the unique isomorphism of ind-group-schemes on $\mathbb{Z}((q))$

$$T_{torsion}\otimes_{\mathbb{Z}[q^{\pm}]}\mathbb{Z}((q))\xrightarrow{\sim}Tate(q)_{tors}.$$

The isomorphism is compatible with the canonical extension structure: for each $N\geq 1$,

$$\begin{tikzcd} 0 \arrow[r] & \mu_N \arrow[r] \arrow[d, "="'] & T[N] \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z}/N\mathbb{Z} \arrow[r] \arrow[d, "="'] & 0 \\ 0 \arrow[r] & \mu_N \arrow[r] & Tate(q)[N] \arrow[r] & \mathbb{Z}/N\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Therefore, $Tate(q)[N]$ is isomorphic to the disjoint union

$$\coprod_{k=0}^{N-1}\mathrm{Spec}(\mathbb{Z}((q))[x]/(x^N-q^k)).$$

In addition, we have the question how to classify all the finite subgroups of $Tate(q)$. As shown in Proposition 6.5.1 in [**6**], the ring O_{Sub_N} that classifies subgroups of $Tate(q)$ of order N exists.

To give a description of O_{Sub_N}, first we describe the isogenies for the analytic Tate curve over $\mathbb{C}$.

To give a subgroup for each order N, pick a pair of integers (d,e) and a nonzero complex number q' such that $N=de$ and $d,e\geq 1$. Let q' be a nonzero complex number such that $q^d=q'^e$. Consider the map

$$\psi_d:\mathbb{C}^*/q^{\mathbb{Z}}\longrightarrow\mathbb{C}^*/q'^{\mathbb{Z}}$$
$$x\mapsto x^d.$$

It is well-defined since $\psi_d(q^{\mathbb{Z}})\subseteq q'^{\mathbb{Z}}$.

We can check that $\mathrm{Ker}\psi_d$ has order N. Explicitly, it is

$$\{\mu_d^n q^{\frac{m}{e}}q^{\mathbb{Z}}|n,m\in\mathbb{Z}\}$$

where μ_d is a $d-$th primitive root of 1 and $q^{\frac{1}{e}}$ is a $e-$th primitive root of q. In fact

$$\{\mathrm{Ker}\psi_d|\ d\text{ divides }N\text{ and }d\geq 1\}$$

gives all the subgroups of $\mathbb{C}^*/q^{\mathbb{Z}}$ of order N.

PROPOSITION 2.2. For each pair of number (d,e), there exists an isogeny

$$\Psi_{d,e}:Tate(q)\longrightarrow Tate(q')$$

of the elliptic curves over O_{Sub_N} such that its kernel is the universal finite subgroup.

We have

$$O_{Sub_N}\otimes\mathbb{C}=\prod_{N=de}\mathbb{C}((q))[q']/\langle q^d-q'^e\rangle.$$

Moreover, we have the conclusion below.

PROPOSITION 2.3. The finite subgroups of the Tate curve are the kernels of isogenies.

3. Quasi-elliptic cohomology

In this section we recall the definition of quasi-elliptic cohomology in term of equivariant K-theory and state the conclusions that we need in Section 4. For more details on quasi-elliptic cohomology, please refer [**4**] and [**7**].

Let X be a $G-$space. Let $G^{tors} \subseteq G$ be the set of torsion elements of G. Let $\sigma \in G^{tors}$. The fixed point space X^σ is a $C_G(\sigma)-$space. We can define a $\Lambda_G(\sigma)-$action on X^σ by

$$[g,t]\cdot x := g\cdot x.$$

Then quasi-elliptic cohomology of the orbifold $X/\!\!/G$ is defined by

DEFINITION 3.1.

$$QEll^*(X/\!\!/G) := \prod_{g\in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g) = \Big(\prod_{g\in G^{tors}} K^*_{\Lambda_G(g)}(X^g)\Big)^G, \tag{3.1}$$

where G^{tors}_{conj} is a set of representatives of $G-$conjugacy classes in G^{tors}.

We have the ring homomorphism

$$\mathbb{Z}[q^\pm] = K^0_{\mathbb{T}}(\mathrm{pt}) \xrightarrow{\pi^*} K^0_{\Lambda_G(g)}(\mathrm{pt}) \longrightarrow K^0_{\Lambda_G(g)}(X)$$

where $\pi : \Lambda_G(g) \longrightarrow \mathbb{T}$ is the projection $[a,t] \mapsto e^{2\pi i t}$ and the second is via the collapsing map $X \longrightarrow \mathrm{pt}$. So $QEll^*_G(X)$ is naturally a $\mathbb{Z}[q^\pm]-$algebra.

PROPOSITION 3.2. The relation between quasi-elliptic cohomology and Tate K-theory is

$$QEll^*(X/\!\!/G)\otimes_{\mathbb{Z}[q^\pm]}\mathbb{Z}((q)) = K^*_{Tate}(X/\!\!/G). \tag{3.2}$$

We have the computation via representation theory that

$$QEll(\mathrm{pt}/\!\!/(\mathbb{Z}/N\mathbb{Z})) = \prod_{k=0}^{N-1}\mathbb{Z}[q^\pm][x_k]/(x_k^N - q^k), \tag{3.3}$$

where each x_k is the representation of $\Lambda_{\Sigma_N}(k)$ defined by

$$\Lambda_{\Sigma_N}(k) = (\mathbb{Z}\times\mathbb{R})/(\mathbb{Z}(N,0)+\mathbb{Z}(1,k)) \xrightarrow{[a,t]\mapsto[(kt-a)/N]} \mathbb{R}/\mathbb{Z} = \mathbb{T} \xrightarrow{\ q\ } U(1). \tag{3.4}$$

We have the isomorphism below.

PROPOSITION 3.3. $T[N] \cong \mathrm{Spec}(QEll(\mathrm{pt}/\!\!/(\mathbb{Z}/N\mathbb{Z})))$.

In [**4**], we construct a power operation $\mathbb{P}_N$ for quasi-elliptic cohomology. Via it we show by representation theory the conclusion below.

PROPOSITION 3.4.

$$QEll(\mathrm{pt}/\!\!/\Sigma_N)/\mathcal{I}^{\Sigma_N}_{tr} \cong \prod_{N=de}\mathbb{Z}[q^\pm][q']/\langle q^d - q'^e\rangle, \tag{3.5}$$

where q' is the image of q under the power operation $\mathbb{P}_N$ and

$$\mathcal{I}^{\Sigma_N}_{tr} := \sum_{\substack{i+j=N,\\ N>j>0}} \mathrm{Image}[\mathcal{I}^{\Sigma_N}_{\Sigma_i\times\Sigma_j} : QEll(\mathrm{pt}/\!\!/\Sigma_i\times\Sigma_j) \longrightarrow QEll(\mathrm{pt}/\!\!/\Sigma_N)] \tag{3.6}$$

is the transfer ideal for quasi-elliptic cohomology. The product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

Applying the relation (3.2), we can get the conclusion below as a corollary of Proposition 3.4.

THEOREM 3.5. *The Tate K-theory of symmetric groups modulo the transfer ideal I_{tr}^{Tate} classifies the finite subgroups of the Tate curve. Explicitly,*

$$K_{Tate}(pt/\!\!/\Sigma_N)/I_{tr}^{\Sigma_N} \cong \prod_{N=de} \mathbb{Z}((q))[q']/\langle q^d - q'^e \rangle, \tag{3.7}$$

where q' is the image of q under the power operation P^{Tate} constructed in Definition 3.15, [?] *and*

$$I_{tr}^{\Sigma_N} := \sum_{\substack{i+j=N,\\ N>j>0}} Image[I_{\Sigma_i\times\Sigma_j}^{\Sigma_N} : K_{Tate}(pt/\!\!/\Sigma_i \times \Sigma_j) \longrightarrow K_{Tate}(pt/\!\!/\Sigma_N)] \tag{3.8}$$

is the transfer ideal of the Tate K-theory. The product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

Moreover, via the power operation $\overline{P}_N$ we construct a new operation

$$\overline{P}_N : QEll_G(X) \longrightarrow QEll(X/\!\!/G) \otimes_{\mathbb{Z}[q^\pm]} QEll(\mathrm{pt}/\!\!/\Sigma_N)/\mathcal{I}_{tr}^{\Sigma_N}$$

of quasi-elliptic cohomology. It is essential in the construction of the universal finite subgroup of order N of $Tate(q)$.

PROPOSITION 3.6. The composition

$$\begin{aligned}
\overline{P}_N : &QEll(X/\!\!/G) \xrightarrow{\mathbb{P}_N} QEll(X^{\times N}/\!\!/G \wr \Sigma_N) \xrightarrow{res} QEll(X^{\times N}/\!\!/G \times \Sigma_N) \\
&\xrightarrow{diag^*} QEll(X/\!\!/G \times \Sigma_N) \cong QEll(X/\!\!/G) \otimes_{\mathbb{Z}[q^\pm]} QEll(\mathrm{pt}/\!\!/\Sigma_N) \\
&\longrightarrow QEll(X/\!\!/G) \otimes_{\mathbb{Z}[q^\pm]} QEll(\mathrm{pt}/\!\!/\Sigma_N)/\mathcal{I}_{tr}^{\Sigma_N} \\
&\cong QEll(X/\!\!/G) \otimes_{\mathbb{Z}[q^\pm]} \prod_{N=de} \mathbb{Z}[q^\pm][q']/\langle q^d - q'^e \rangle
\end{aligned}$$

defines a ring homomorphism, where res is the restriction map by the inclusion

$$G \times \Sigma_N \hookrightarrow G \wr \Sigma_N,\ (g, \sigma) \mapsto (g, \cdots g; \sigma),$$

$diag$ is the diagonal map

$$X \longrightarrow X^{\times N},\ x \mapsto (x, \cdots x).$$

The operation $\overline{P}_N$ sends q to q'.

4. The universal finite subgroup of the Tate curve

In this section we construct the universal finite subgroup G_{univ} of of order N of $Tate(q)$. In Section 4.1, 4.2 and 4.3 we show the construction of $G_{univ} := (O_{Sub_{\Sigma_N}}, Ker\psi < s^*Tate[N])$. In Section 4.4 we give an explicit description of G_{univ} and prove the main conclusion that G_{univ} is universal.

4.1. The scheme $s^*Tate[N]$. In this section we construct the scheme $s^*Tate[N]$. Let $O_{Sub_{\Sigma_N}}$ denote the ring

$$\prod_{N=de} \mathbb{Z}[q^{\pm}][q']/\langle q^d - q'^e\rangle.$$

As shown in Theorem 3.5, $K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)/I_{tr}^{\Sigma_N}$ is isomorphic to

$$O_{Sub_N} \cong O_{Sub_{\Sigma_N}} \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)),$$

the ring classifying finite subgroups of $Tate(q)$ of order N. Let $O_{Sub_{d,e}}$ denote the factor $\mathbb{Z}[q^{\pm}][q']/\langle q^d - q'^e\rangle$ in $O_{Sub_{\Sigma_N}}$.

Let 1 denote the trivial group. Consider the pushout

$$\begin{tikzcd} QEll(\mathrm{pt}/\!\!/1) \arrow[r, "\pi^*"] \arrow[d] & QEll(\mathrm{pt}/\!\!/\Sigma_N)/\mathcal{I}_{tr}^{\Sigma_N} \arrow[d] \\ QEll(\mathrm{pt}/\!\!/(\mathbb{Z}/N\mathbb{Z})) \arrow[r, "\pi^*"] & QEll(\mathrm{pt}/\!\!/(\mathbb{Z}/N\mathbb{Z}) \times \Sigma_N)/\mathcal{I}_{tr}^{\Sigma_N}. \end{tikzcd} \tag{4.1}$$

In other words, we have the pushout square

$$\begin{tikzcd} \mathbb{Z}[q^{\pm}] \arrow[r, "s^*"] \arrow[d] & O_{Sub_{\Sigma_N}} \arrow[d] \\ O_{T[N]} \arrow[r] & O_{s^*Tate[N]} \end{tikzcd} \tag{4.2}$$

where s^* is the inclusion. It sends q to q.

Let $s^*_{d,e}$ denote the composition

$$\mathbb{Z}[q^{\pm}] \xrightarrow{s^*} O_{Sub_{\Sigma_N}} \longrightarrow O_{Sub_{d,e}}$$

where the second map is the projection of the product. We have the relation

$$s^*Tate[N] = \coprod_{N=de} s^*_{d,e}Tate[N]. \tag{4.3}$$

Moreover, we have the pullback diagrams

$$\begin{tikzcd} s^*Tate[N] \arrow[r] \arrow[d] & T[N] \arrow[d] \\ Sub_{\Sigma_N} \arrow[r, "s"] & \mathrm{Spec}\mathbb{Z}[q^{\pm}] \end{tikzcd} \tag{4.4}$$

and

$$\begin{tikzcd} s^*_{d,e}Tate[N] \arrow[r] \arrow[d] & T[N] \arrow[d] \\ Sub_{d,e} \arrow[r, "s"] & \mathrm{Spec}\mathbb{Z}[q^{\pm}]. \end{tikzcd} \tag{4.5}$$

By the pullback square (4.2),

$$O_{s^*Tate[N]} \cong QEll(\mathrm{pt}/\!\!/(\mathbb{Z}/N\mathbb{Z})) \otimes_{\mathbb{Z}[q^{\pm}]} QEll(\mathrm{pt}/\!\!/\Sigma_N)/I_{tr}^{\Sigma_N}.$$

The ring $O_{s^*_{d,e}Tate[N]} \cong QEll(\mathrm{pt}/\!\!/(\mathbb{Z}/N\mathbb{Z})) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}[q^\pm][q'_{d,e}]/\langle q^d - q'^e_{d,e}\rangle$. We have

$$O_{s^*Tate[N]} = \prod_{N=de} O_{s^*_{d,e}Tate[N]}.$$

4.2. The scheme $t^*Tate[N]$. Moreover, we construct another scheme $t^*Tate[N]$. We consider the pushout

$$\begin{tikzcd} \mathbb{Z}[q^\pm] \arrow[r, "t^*"] \arrow[d] & O_{Sub_{\Sigma_N}} \arrow[d] \\ O_{T[N]} \arrow[r] & O_{t^*Tate[N]} \end{tikzcd} \tag{4.6}$$

where t^* sends q to q' and the pullback

$$\begin{tikzcd} t^*Tate[N] \arrow[r] \arrow[d] & T[N] \arrow[d] \\ Sub_{\Sigma_N} \arrow[r, "t"] & \mathrm{Spec}\mathbb{Z}[q^\pm]. \end{tikzcd} \tag{4.7}$$

Let $t^*_{d,e}$ denote the composition

$$\mathbb{Z}[q^\pm] \xrightarrow{t^*} O_{Sub_{\Sigma_N}} \longrightarrow O_{Sub_{d,e}}$$

where the second map is the projection. We have the relation

$$t^*Tate[N] = \prod_{N=de} t^*_{d,e}Tate[N]. \tag{4.8}$$

In addition, by the pushout diagram (4.6),

$$O_{t^*Tate[N]} \cong QEll(\mathrm{pt}/\!\!/(\mathbb{Z}/N\mathbb{Z})) \otimes QEll(\mathrm{pt}/\!\!/\Sigma_N)/I^{\Sigma_N}_{tr} / \sim$$

where $\sim$ is the equivalence $q \otimes 1 \sim 1 \otimes q'$.

The ring $O_{t^*_{d,e}Tate[N]} \cong QEll(\mathrm{pt}/\!\!/(\mathbb{Z}/N\mathbb{Z})) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}[q^\pm][q'_{d,e}]/\langle q^d - q'^e_{d,e}\rangle / \sim$. Here $\sim$ is the equivalence $q \otimes 1 \sim 1 \otimes q'_{d,e}$.

We have

$$O_{t^*Tate[N]} = \prod_{N=de} O_{t^*_{d,e}Tate[N]}.$$

4.3. The map ψ^*. By the universal property of pushout, there is a unique map $\psi^* : O_{t^*Tate[N]} \longrightarrow O_{s^*Tate[N]}$ making the diagrams commute.

$$\begin{tikzcd} \mathbb{Z}[q^\pm] \arrow[r, "t^*"] \arrow[d] & O_{Sub_{\Sigma_N}} \arrow[d] \arrow[ddr, bend left] & \\ O_{T[N]} \arrow[r] \arrow[drr, bend right, "\overline{P}_N"] & O_{t^*Tate[N]} \arrow[dr, dotted, "\psi^*"] & \\ & & O_{s^*Tate[N]}. \end{tikzcd} \tag{4.9}$$

Thus, we have the commutative diagrams

$$\begin{tikzcd}
s^*Tate[N] \arrow[dr, "\psi"] \arrow[drr, bend left, "\overline{P}_N^*"] \arrow[ddr, bend right] & & \\
& t^*Tate[N] \arrow[r] \arrow[d] & T[N] \arrow[d] \\
& Sub_{\Sigma_N} \arrow[r, "t"'] & \mathrm{Spec}\mathbb{Z}[q^{\pm}]
\end{tikzcd} \tag{4.10}$$

Next we show the explicit formula for ψ^*.

Recall we defined an element $x_k \in O_{T[N]}$ by (3.4). For any $\mathbb{Z}[q^{\pm}]-$algebra R, $x_k : T(R)[N] \longrightarrow R$ is the map

$$x_k([a,t]) = \begin{cases} a, & \text{if } t = \frac{k}{N} \text{ with } k = 0, 1, \cdots N-1; \\ 0, & \text{if } [a,t] \neq [a', \frac{k}{N}] \text{ for any } a'. \end{cases}$$

Note in $T(R)[N]$, $[a, t+1] = [aq, t]$.

By the formula of the operation

$$\overline{P}_N : O_{T[N]} \longrightarrow O_{s^*Tate[N]},$$

it sends $\prod_{m=0}^{N-1} q$ to $\prod_{m=0}^{N-1} 1 \otimes q'_{d,e} = q'$, and sends each x_k to $\prod_{\substack{N=de \\ e|k}} \prod_{\alpha_m=0}^{e-1} x_m^d {q'_{d,e}}^{-\alpha_m}$ where $m = \frac{k}{e} + \alpha_m d$.

Therefore, the map ψ^* sends both $q \otimes 1$ and $1 \otimes q'$ to $1 \otimes q'$, and $1 \otimes q$ to $q \otimes 1$.

In addition, for any $O_{Sub_{d,e}}-$algebra $O_{Sub_{d,e}} \longrightarrow A$,

$$s^*_{d,e}Tate[N](A) = \frac{A^* \times \mathbb{Q}}{\langle (q, -1) \rangle} \text{ and } t^*_{d,e}Tate[N](A) = \frac{A^* \times \mathbb{Q}}{\langle (q'_{d,e}, -1) \rangle}.$$

We can define a map $s^*_{d,e}Tate[N](A) \longrightarrow t^*_{d,e}Tate[N](A)$ sending $[a, x]$ to $[a^d, ex]$. This map is well-defined: $[q^d, -e] = [q'^e, -e] = 0$ in $t^*_{d,e}Tate[N](A)$.

The map $\psi : s^*Tate[N] \longrightarrow t^*Tate[N]$ defined in the diagram (4.9) can be constructed as the coproduct of the maps

$$\psi_{d,e} : s^*_{d,e}Tate[N] \longrightarrow t^*Tate[N].$$

Note $\psi^*_{d,e}x_k \in O_{s^*Tate[N]}$ and $(\psi^*_{d,e}x_k)[a,x] = x_k(\psi_{d,e}[a,x]) = x_k([a^d, ex])$.

4.4. The main theorem. Now we are ready to state the main conclusion of this paper.

THEOREM 4.1. *The universal finite subgroup of order N of $Tate(q)$ is the pair*

$$G_{univ} := (O_{Sub_{\Sigma_N}}, Ker\psi < s^*Tate[N])$$

in the sense that for any $\mathbb{Z}[q^{\pm}]-$algebra $\mathbb{Z}[q^{\pm}] \longrightarrow R$, there is a 1-1 correspondence

$$\begin{array}{c} \{\mathbb{Z}[q^{\pm}] - \textit{algebra maps } Sub_{\Sigma_N} \longrightarrow R\} \\ \Big\updownarrow \\ \{\textit{finite subgroup schemes } G \leqslant T[N]_R \textit{ of degree } N\} \end{array} \tag{4.11}$$

where $T[N]_R$ *is the pullback*

$$\begin{tikzcd} T[N]_R \arrow[r] \arrow[d] & T[N] \arrow[d] \\ SpecR \arrow[r] & Spec\mathbb{Z}[q^{\pm}] \end{tikzcd}.$$

It is a group scheme over SpecR. In other words, given a subgroup scheme $G \leqslant T[N]_R$ *of degree* N*, there exists a unique pullback square*

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, hook] & Ker\psi \arrow[d, hook] \\ T[N]_R \arrow[r] \arrow[d] & s^*Tate[N] \arrow[d] \\ SpecR \arrow[r] & SpecO_{Sub_{\Sigma_N}} \end{tikzcd}. \tag{4.12}$$

Proof of Theorem 4.1. We prove the conclusion by three steps.

Step I: We show given a finite subgroup $H < T[N]_R$ of order N, for each (d, e) with $N = de$, we can construct a map $F^*_{d,e} : \mathrm{Spec}R \longrightarrow \mathrm{Spec}O_{Sub_{\Sigma_N}}$.

Consider the exact sequence

$$0 \longrightarrow R^* \longrightarrow T(R) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow R^* \cap H \longrightarrow H \longrightarrow H/(R^* \cap H) \longrightarrow 0.$$

Let $d = |R^* \cap H|$ and $e = |H/(R^* \cap H)|$. Then $H/(R^* \cap H) = \mathbb{Z}[\frac{1}{e}]/\mathbb{Z} \cong \mathbb{Z}/e\mathbb{Z}$. The group $R^* \cap H$ is the kernel of the projection

$$H \longrightarrow H/(R^* \cap H),\ [x, \frac{1}{e}] \mapsto \frac{1}{e}.$$

Consider

$$\begin{tikzcd} 0 \arrow[r] & \mu_d \arrow[r, hook] \arrow[d, equal] & H \arrow[r] \arrow[d, hook] & \mathbb{Z}[\frac{1}{e}]/\mathbb{Z} \arrow[r] \arrow[d, dotted, "\exists!\ r"] & 0 \\ 0 \arrow[r] & \mu_d \arrow[r, hook] & Tate(q)_R \arrow[r, "\phi_d"] & Tate(q^d)_R \arrow[r] & 0 \end{tikzcd}$$

where $Tate(q)_R$ is the pullback

$$\begin{tikzcd} Tate(q)_R \arrow[r] \arrow[d] & Tate(q) \arrow[d] \\ \mathrm{Spec}R \arrow[r] & \mathrm{Spec}\mathbb{Z}((q)) \end{tikzcd}$$

and $\phi_d([x, \lambda] = [x^d, \lambda]$.

Define $q' \in R^*$ by

$$r([\frac{1}{e}]) = [q', \frac{1}{e}] \tag{4.13}$$

where $[\frac{1}{e}] \in \mathbb{Z}[\frac{1}{e}]/\mathbb{Z}$ is a generator. Note that $0 = e \cdot [q', \frac{1}{e}] = [q'^e, 1] = [q'^e q^{-d}, 0]$ and $q'^e = q^d$. Then we can construct a well-defined ring map

$$F_{d,e} : \mathbb{Z}[q, q']/\langle q^d - q'^e \rangle \longrightarrow R$$

which sends q to q and sends q' to the q' defined in (4.13).

Then $F^*_{d,e} : \mathrm{Spec}R \longrightarrow \mathrm{Spec}O_{Sub_{\Sigma_N}}$ is the map we want.

Step II: We show that given a ring homomorphism $O_{Sub_{\Sigma_N}} \xrightarrow{g} R$, we can construct a finite subgroup $G < T[N]_R$ of order N, via the diagram (4.12).

Since $\mathbb{Z}[q^{\pm}] \longrightarrow \mathrm{Spec}R$ is finite and flat, there exists a locally constant map $\delta : \mathrm{Spec}R \longrightarrow \mathbb{Z}_{\geq 0}$. Then $\mathrm{Spec}R$ is the disjoint union $\coprod\limits_{d \in \mathbb{Z}_{\geq 0}} \{x \in \mathrm{Spec}R \mid \delta(x) = d\}$. Let R_d denote $\{x \in \mathrm{Spec}R \mid \delta(x) = d\}$.

Thus the ring homomorphism g can be viewed as the product of maps

$$g_{d,e} : O_{Sub_{d,e}} \longrightarrow R_{d,e}.$$

If no $x \in \mathbb{Z}[q, q']/\langle q^d - q'^e \rangle$ s.t $x^e = q^d$, $g_{d,e} = 0$. Then, the contribution of $g_{d,e}$ to the subgroup G is 0. There is only one $g_{d,e}$ that is non-trivial. Then we can just consider the pull-back

$$\begin{tikzcd} G \arrow[r] \arrow[d] & Ker\psi \cap \mathrm{Spec}O_{Sub_{d,e}} \arrow[d] \\ T[N] \arrow[r] \arrow[d] & s^*_{d,e}T[N] \arrow[d] \\ \mathrm{Spec}R_{d,e} \arrow[r] & \mathrm{Spec}O_{Sub_{d,e}} \end{tikzcd}$$

Then we have an explicit formula for $(Ker\psi)(R)$.

(4.14)

$$(Ker\psi)(R) = \coprod_{N=de} ker\psi_{d,e} = \coprod_{N=de} \{[a, x] \in s^*_{d,e}Tate[N](R_{d,e}) \mid [a^d, ex] = [1, 0]\}.$$

And $G(R)$ is the factor $\{[t, x] \in s^*_{d,e}Tate[N](R_{d,e}) \mid [t^d, ex] = [1, 0]\}$.

Step III: It is straightforward to check the two maps in Step I and Step II are the inverse of each other. Then, we have the 1-1 correspondence (4.11). □

References

1. Matthew Ando, Michael J. Hopkins, and Neil P. Strickland: *Elliptic spectra, the Witten genus and the theorem of the cube*, Invent. Math. , 146(3):595–687, 2001.
2. Matthew Ando, Michael J. Hopkins, and Neil P. Strickland: *The sigma orientation is an H_∞ map*, Amer. J. Math., 126(2):247–334, 2004.
3. Nora Ganter: *Stringy power operations in Tate K-theory*, (2007), available at arXiv: math/0701565.
4. Zhen Huan: *Quasi-Elliptic Cohomology and its Power Operations*, J. Homotopy Relat. Struct. (2018). https://doi.org/10.1007/s40062-018-0201-y.
5. Zhen Huan: *Quasi-elliptic cohomology*, Thesis (Ph.D.)University of Illinois at Urbana-Champaign. 2017. 290 pp. http://hdl.handle.net/2142/97268.
6. Nicholas M. Katz and Barry Mazur: *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985.
7. Charles Rezk: *Quasi-elliptic cohomology*, unpublished manucript, 2011.
8. N. P. Strickland: *Finite subgroups of formal groups*, J. Pure Appl. Algebra 121 (1997), no. 2, 161–208.
9. N. P. Strickland: *Morava E-theory of symmetric groups*, Topology 37 (1998), no. 4, 757–779.

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