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HW2

1. Suppose that one letter is to be selected at random from the 42 letters in the sentence, " The shortest distance between two points is a taxi." If Y denotes the number of letters in the word in which the selected letter appears, what is the value of E(Y) ?

We can see that we have 9 words

Let X denote the probability of the number of letters of selected word

$$P(X)=\begin{cases} \frac{1}{42} & Y=1 \\ \frac{2}{42} & Y=2 \\ \frac{6}{42} & Y=3 \\ \frac{4}{42} & Y=4 \\ 0 & Y=5 \\ \frac{6}{42} & Y=6 \\ \frac{7}{42} & Y=7 \\ \frac{16}{42} & Y=8 \end{cases}$$

Therefore, $E(Y) = \frac{1}{42} + \frac{2}{42}\times 2 + \frac{6}{42}\times 3 + \frac{4}{42}\times 4 + \frac{6}{42}\times 6 + \frac{7}{42}\times 7 + \frac{16}{42}\times 8$

$= 6$

2. Suppose X and Y have a continuous joint distribution where the joint pdf is $f(x,y) = 12y^2$ for $0 \le y \le x \le 1$

Find the value of E(XY)

$$E(XY) = \int_0^1 \int_y^1 xy \cdot f_Y(y) \cdot f_{X|Y=y}(x)\, dx\, dy$$

$$= \int_0^1 \int_y^1 xy \cdot 12y^2\, dx\,dy = 12 \int_0^1 \int_y^1 xy^3\, dx\,dy = 6\int_0^1 (y^3 - y^5)\,dy$$

$$= 6\left(\tfrac{1}{4}y^4 - \tfrac{1}{6}y^6\right)\Big|_0^1 = \tfrac{1}{2}$$

3. Suppose that three random variables, $X_1, X_2, \cdots X_3$ from a random sample from the uniform distribution on the interval $[0, 1]$; Find $E[(X_1 - 2X_2 + X_3)^2]$.

The pdf of X_1, X_2, X_3: $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

$$E[(X_1-2X_2+X_3)^2] = E[X_1^2 + 4X_2^2 + X_3^2 - 4X_1 \cdot X_2 - 4X_2 \cdot X_3 + 2X_1 \cdot X_3]$$

Let $g(X_1, X_2, X_3) = (X_1 - 2X_2 + X_3)^2$

Theorem:

$$E[(X_1-2X_2+X_3)^2] = \int_0^1\int_0^1\int_0^1 (X_1-2X_2+X_3)^2 \cdot 1 \, dX_1 dX_2 \cdot dX_3$$

$$= \int_0^1\int_0^1\int_0^1 (X_1^2 + 4X_2^2 + X_3^2 - 4X_1X_2 - 4X_2X_3 + 2X_1X_3) dX_1 dX_2 dX_3$$

$$= \int_0^1\int_0^1 \left(\frac{1}{3}X_1^3 + 4X_2^2X_1 + X_3^2 \cdot X_1 - 2X_2 \cdot X_1^2 - 4X_2X_3X_1 + X_3 \cdot X_1^2\right)\Big|_0^1 dX_2 \, dX_3$$

$$= \int_0^1\int_0^1 \left(\frac{1}{3} + 4X_2^2 + X_3^2 - 2X_2 - 4X_2X_3 + X_3\right) dX_2 \, dX_3$$

$$= \int_0^1 \left(\frac{2}{3} + X_3^2 - X_3\right) dX_3$$

$$= \frac{1}{2}$$

4. X has pdf, $f(x) = e^{-x}$, $x > 0$, $Y = e^{\frac{3x}{4}}$, Find $E(Y)$

$$E(Y) = \int_0^\infty Y f(x) dx = \int_0^\infty e^{\frac{3x}{4}} \cdot e^{-x} dx$$

$$= \int_0^\infty e^{-\frac{x}{4}} dx = \left(-4e^{-\frac{x}{4}}\right)\Big|_0^\infty$$

$$= 0 - (-4 \cdot 1) = 4$$

5. X is the outcome of rolling a die.

$Y = g(X) = 2X^2+1$

Find $E(Y)$

From the question we know the pdf of X is ~~f(x) = 1/6~~ $P(X=i)=\frac{1}{6}$ $i=1,2,3,4,5,6.$

$$E(Y) = \sum_i \frac{1}{6}(2x_i^2+1) = \frac{1}{6}(2\times1^2+1+2\times2^2+1+\cdots+2\times6^2+1)$$

$$= \frac{1}{6}[2\times(1^2+\cdots+6^2)+6] = \frac{1}{3}(1^2+\cdots+6^2)+1$$

$$= \frac{94}{3}$$

6. X has pdf

$f(x) = 2(1-x)$, $0<x<1$

$Y=(2x+1)$, Find $E(Y^2)$

Let $g(x) = (2x+1)^2$

$$E[(2x+1)^2] = \int_0^1 (2x+1)^2\cdot 2\cdot(1-x)\,dx = 2\int_0^1 (2x+1)^2(1-x)\,dx$$

$$= 2\int_0^1(-4x^3+3x+1)\,dx$$

$$= 2(-x^4+\tfrac{3}{2}x^2+x)\Big|_0^1$$

$$= 2(\tfrac{3}{2}-0) = 3.$$

7. Remember the bionomial theorem : $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k}y^k$, for $n\in Z^+$,

Show that $E[(ax+b)^n] = \sum_{i=0}^n \binom{n}{i} a^{n-i}b^i E(x^{n-i})$

$$E[(ax+b)^n] = \cancel{\sum (ax+b)^n f(a,b)}$$

$$= E[C_n^0 (ax)^n\cdot b^0 + C_n^1 (ax)^{n-1} b^1 + \cdots + C_n^n\cdot(ax)^0\cdot b^n]$$

$$= E(C_n^0(ax)^n b^0) + E(C_n^1(ax)^{n-1}b) + \cdots + E(C_n^n (ax)^0\cdot b^n)$$

$$= C_n^0 a^n\cdot E(x^n) + C_n^1\cdot b\cdot a^{n-1}\cdot E(x^{n-1}) + \cdots + C_n^n\cdot b^n a^0 E(x^0)$$

$$= \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i E(x^{n-i})$$

8. The proportion of defective parts in a large shipment is p. A random sample of n parts is selected from the shipment. Let X denote the number of defective parts in the sample, and Y denote the number of good pares in the sample

Find $E(X-Y)$.

If the sample size is 20 and p is 5%, what is $E(X-Y)$? Write out your answer as a complete sentence that expresses the meaning of your result.

(1) E ~~(X...)~~ we know that X follows the binomial disteributeion (n,p)

$$f(x) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Thus, $f(y) = \binom{n}{k}(1-p)^k \cdot p^{n-k}$.

Notic that $X+Y=n$, thus

$$E(X-Y) = E(X-(n-X)) = E(2X-n) = 2E(X) - n = 2np - n$$

If $n=20$, $p=5\%$, then $E[X-Y] = 20(2\times 0.05 - 1) = -18$.

This means when we have 20 samples with 5% to find the defecti parts, we expect good pare would be 18 more than the defective one