HW 5.

1. (a) $f(x) = \binom{n-1}{k-1} p^{k} (1-p)^{n-k}$

$L(p) = \binom{n-1}{k-1} p^{k} (1-p)^{n-k}$

$l(p) = \log\left[\binom{n-1}{k-1}\right] + k\log(p) + (n-k)\log(1-p)$

Let $\frac{\partial l}{\partial p} = \frac{k}{p} + \frac{n-k}{1-p} = 0$

$n = 43 \quad k = 5$

So $p = \frac{5}{43}$

(b) $f(x_i) = p^{x}(1-p)^{1-x}$

Thus, $L(\theta; x_1 \cdots x_n) = p^{\sum_{i=1}^{n} x_i} (1-p)^{n-\sum_{i=1}^{n} x_i}$

$l(p) = \sum_{i=1}^{n} x_i \log(p) + (n - \sum_{i=1}^{n} x_i)\log(1-p)$

$\frac{\partial l}{\partial p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n-\sum_{i=1}^{n} x_i}{1-p} = 0 \Rightarrow \hat{p} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{3}{58}$

2. Uniform : $f(x) = \frac{1}{\theta}$

$L(\theta; x_1 \cdots x_n) = \frac{1}{\theta^n}$

$l(\theta; x_1 \cdots x_n) = -n\ln(\theta)$

$\frac{\partial l}{\partial \theta} = \quad = 0$

Notice that $x_1 \le x_2 \le \cdots x_n \le \theta$

$L(\theta; x_1 \cdots x_n) = \frac{1}{\theta^n} \le \frac{1}{x_n x_{n-1} \cdots x_1}$

$\Rightarrow \theta = x_{(n)}$

$\Rightarrow$ MLE of 0.95 quantile : $\hat{\mu} \pm 1.96\hat{\sigma}$

$$= \sum_{i=1}^{n} x_i/n + 1.96\sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}}$$

4. $v = P(x>2) = P(Z > \frac{2-\mu}{\sigma}) = 1 - \Phi(\frac{2-\mu}{\sigma})$

we know that $\hat{\mu} = \sum^{n} x_i/n$, $\hat{\sigma}^2 = \sum_{i=1}^{n}(X_i-\mu)^2/n$.

$$\hat{v} = 1 - \Phi\left[\frac{2 - \sum^{n} x_i/n}{\sqrt{\frac{\sum^{n}(x_i-\mu)^2}{n}}}\right]$$

5. Cauchy: $f(x) = \frac{1}{\pi\theta\left(1+\left(\frac{x-\theta}{\theta\theta}\right)^2\right)}$

6. $X_i \sim exp(\lambda)$

$f(x_i) = \lambda e^{-\lambda x}$

$L(\lambda; x_1 \cdots x_n) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$

$l(\lambda; x_1 \cdots x_n) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i$

$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \lambda = \frac{n}{\sum x_i} = \frac{7}{45}$

Hence, $\mu = \frac{\sum x_i}{n} = \frac{45}{7}$

7. $X_i \sim Poisson(\lambda)$

$f(x_i) = \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}$

$L(\lambda; x_1 \cdots x_n) = \lambda^{\sum_{i=1}^{n} x_i} \left[\prod_{i=1}^{n}(x_i!)\right]^{-1} e^{-n\lambda}$

$l(\lambda; x_1 \cdots x_n) = \sum_{i=1}^{n} x_i \log(\lambda) - \log\left(\prod_{i=1}^{n}(x_i!)\right) - n\lambda$

$\frac{\partial l}{\partial \lambda} = \frac{\sum x_i}{\lambda} - n = 0$

$\hat{\lambda} = \frac{\sum_{i}^{n} x_i}{n}$

8. $X \sim \exp(\beta)$

$$f(x) = \lambda e^{-\lambda x}$$

$$0.5 = \int_0^{m} \lambda e^{-\lambda x}\, dx = -e^{-\frac{m}{\lambda}} + 1$$

$$e^{-\frac{m}{\lambda}} = \frac{1}{2}$$

So we have $$e^{-\frac{m}{n}\cdot\sum_{i=1}^{n} x_i} = \frac{1}{2}$$

$$-\frac{m}{n}\sum_{i=1}^{n} x_i = -\ln 2$$

$$\hat{m} = \frac{n \ln 2}{\sum_{i=1}^{n} x_i}$$

1. We know that

$$X_1 \cdots X_n \sim B(p)$$

$$T = \sum_{i=1}^{n} x_i$$

$$f(x;p) = p^{x}(1-p)^{1-x}$$

$$L(p;x) = p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$$

$$= p^{T}(1-p)^{n-T}$$

$$\mu(x) = 1$$

$$v(t(x), p) = p^{t(x)}(1-p)^{n-t(x)}$$

2. $X_1 \cdots X_n \sim$ Geomeern (p)

Hence $f(x) = (1-p)^{x-1} p$

$$L(p;x) = p^{n}(1-p)^{\Sigma x_i - n}$$

$$= p^{n}(1-p)^{\Sigma x_i - n}$$

$$\mu(x) = 1 \qquad V(t(x), p) = p^{n}(1-p)^{t(x)-n}$$

3. $X_1 \cdots X_n \sim$ Binom (r, p)

Hence, $f(x) = \binom{x+r-1}{x-1} p^{x}(1-p)^{r}$

$$L(r,p;x) = p^{\Sigma x_i}(1-p)^{nr}\prod_{i=1}^{n}\binom{x+r-1}{x-1}$$

$$\mu(x) = \prod_{i=1}^{n}\binom{x+r-1}{x-1} \qquad v(t(x), r, p) = p^{t(x)}(1-p)^{nr}$$

4. $X_1 \dots X_n \sim \text{Gamma}(\alpha, \beta)$

$$T = \sum_{i=1}^{n} X_i \qquad f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

$$L(x; \alpha, \beta) = \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n \cdot \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \cdot e^{-\beta \Sigma x_i}$$

$$\mu(x) = \Gamma(\alpha)^{-n} \cdot \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}$$

$$V(t(x), \beta) = \beta^{n\alpha} e^{-\beta t(x)}$$

5. $\mu(r) = e^{-\beta \Sigma x_i}$

$$V(t(x), \alpha) = \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n (t(x))^{\alpha-1}$$