1. Let $X \in \{0,1\}$.

pdf: $f(x) = p^x(1-p)^{1-x}$

$$L(x) = f(x_1)\cdots f(x_{70})$$
$$= p^{x_1}(1-p)^{1-x_1}\cdots p^{x_{70}}(1-p)^{1-x_{70}}$$
$$= p^{\sum_{i=1}^{70} x_i}(1-p)^{70-\sum_{i=1}^{70} x_i}$$

Therefore, $l(x) = \sum_{i=1}^{70} x_i \log(p) + (70-\sum x_i)\log(1-p)$

Then we need to maximize $l(x)$. Thus:

$$\frac{\partial l}{\partial p} = \frac{\sum x_i}{p} - \frac{70-\sum x_i}{1-p} = 0$$

$$\Downarrow$$

$$p = \frac{6}{35}$$

2. If every observed value is 0 or 1

pdf for Bernouli: $f(x) = \begin{cases} p = \theta, & x=1 \\ 1-p = 1-\theta, & x=0 \end{cases}$

$\therefore$ pdf: $f(x) = \theta^k(1-\theta)^{1-k}$

$$L(x) = f(x_1)\cdots f(x_n)$$
$$= \theta^{\sum_{i=1}^{70} x_i}(1-\theta)^{n-\sum_{i=1}^{70} x_i}$$

$$l(x) = \sum_{i=1}^{n} x_i \log(\theta) + (n - \sum x_i)\log(1-\theta)$$

$$\frac{\partial l}{\partial \theta} = \frac{\sum x_i}{\theta} - \frac{n-\sum x_i}{1-\theta} \quad \text{Let } \frac{\partial l}{\partial \theta} = 0 \Rightarrow \hat{\theta} = \frac{\sum x_i}{n}$$

Therefore, if $x_1 = \cdots = x_n = 1$ or 0, $\hat{\theta}$ will be 1 or 0 which is not in the range of $(0,1)$.

So the MLE of theta does not exist.

3. (1) pdf of poission distribution : $f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$

$$L(x) = f(x_1) \cdots f(x_n)$$
$$= \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod_{i=1}^{n} (x_i!)}$$

$$l(\lambda) = -n\lambda + \sum x_i \log(\lambda) - \log\left[\prod_{i=1}^{n} (x_i!)\right]$$

$$\frac{\partial l}{\partial \lambda} = -n + \frac{\sum x_i}{\lambda} = 0 \quad \Rightarrow \quad \lambda = \frac{\sum x_i}{n}$$

(2) If all value are 0. , then $\lambda = \frac{0}{n} = 0$

Since λ is supposed to greater than 0 , the MLE does not exist

4. pdf of normal distribution : $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$L(x) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}\right]$$

$$l(x) = n\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Let $\frac{\partial l}{\partial \sigma} = 0$

Therefore, $-\frac{n}{2} \cdot \frac{1}{\sigma^2} \cdot 2\sigma + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 = 0.$

$$\Downarrow$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$$