toroidal angle

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1 Introduction

In PEST coordinates, SFL angles are cylindrical ϕ and poloidal ϑ

Our computational coordinates are an arbitrary θ and ζ, related to the SFL angles through

$$\phi = \zeta + \omega(\rho, \theta, \zeta)$$

$$\vartheta = \theta + \lambda(\rho, \theta, \zeta)$$

$$\mathbf{B} = \nabla\phi \times \nabla\chi + \nabla\Psi \times \nabla\vartheta$$

$$\mathbf{B} = \nabla\zeta \times \nabla\chi + \nabla\omega \times \nabla\chi + \nabla\Psi \times \nabla\theta + \nabla\Psi \times \nabla\lambda$$

$$B^\theta = \mathbf{B} \cdot \mathbf{e}^\theta = \mathbf{B} \cdot \nabla\theta$$

$$\begin{aligned}
&= (\nabla\zeta \times \nabla\chi) \cdot \nabla\theta && (1)\\
&+ (\nabla\omega \times \nabla\chi) \cdot \nabla\theta && (2)\\
&+ (\nabla\Psi \times \nabla\theta) \cdot \nabla\theta && (3)\\
&+ (\nabla\Psi \times \nabla\lambda) \cdot \nabla\theta && (4)
\end{aligned}$$

$$\nabla\omega = \omega_\rho \nabla\rho + \omega_\theta \nabla\theta + \omega_\zeta \nabla\zeta$$

$$\nabla\lambda = \lambda_\rho \nabla\rho + \lambda_\theta \nabla\theta + \lambda_\zeta \nabla\zeta$$

$$\nabla\Psi = \Psi_\rho \nabla\rho$$

$$\nabla\chi = \chi_\rho \nabla\rho$$

$$(\nabla\zeta \times \nabla\chi) \cdot \nabla\theta = (\nabla\zeta \times \chi_\rho \nabla\rho) \cdot \nabla\theta = \chi_\rho \frac{\mathbf{e}_\theta}{\sqrt{g}} \cdot \nabla\theta = \frac{\chi_\rho}{\sqrt{g}}$$

$$(\nabla\omega \times \nabla\chi) \cdot \nabla\theta = (\nabla\chi \times \nabla\theta) \cdot \nabla\omega = (\chi_\rho \nabla\rho \times \nabla\theta) \cdot \nabla\omega = \frac{\chi_\rho}{\sqrt{g}} \mathbf{e}_\zeta \cdot \nabla\omega = \frac{\chi_\rho}{\sqrt{g}} \omega_\zeta$$

$$(\nabla\Psi \times \nabla\theta) \cdot \nabla\theta = 0$$

$$(\nabla\Psi \times \nabla\lambda) \cdot \nabla\theta = (\nabla\theta \times \Psi_\rho \nabla\rho) \cdot \nabla\lambda = -\frac{\Psi_\rho}{\sqrt{g}} \mathbf{e}_\zeta \cdot \nabla\lambda = -\frac{\Psi_\rho}{\sqrt{g}} \lambda_\zeta$$

$$B^\theta = \frac{1}{\sqrt{g}} (\chi_\rho + \chi_\rho \omega_\zeta - \Psi_\rho \lambda_\zeta)$$

$$B^\zeta = \mathbf{B} \cdot \mathbf{e}^\zeta = \mathbf{B} \cdot \nabla\zeta$$

$$= (\nabla\zeta \times \nabla\chi) \cdot \nabla\zeta \tag{5}$$
$$+ (\nabla\omega \times \nabla\chi) \cdot \nabla\zeta \tag{6}$$
$$+ (\nabla\Psi \times \nabla\theta) \cdot \nabla\zeta \tag{7}$$
$$+ (\nabla\Psi \times \nabla\lambda) \cdot \nabla\zeta \tag{8}$$

$$(\nabla\zeta \times \nabla\chi) \cdot \nabla\zeta = 0$$

$$(\nabla\omega \times \nabla\chi) \cdot \nabla\zeta = (\nabla\chi \times \nabla\zeta) \cdot \nabla\omega = (\chi_\rho \nabla\rho \times \nabla\zeta) \cdot \nabla\omega = -\frac{\chi_\rho}{\sqrt{g}} \mathbf{e}_\theta \cdot \nabla\omega = -\frac{\chi_\rho}{\sqrt{g}} \omega_\theta$$

$$(\nabla\Psi \times \nabla\theta) \cdot \nabla\zeta = (\Psi_\rho \nabla\rho \times \nabla\theta) \cdot \nabla\zeta = \frac{\Psi_\rho}{\sqrt{g}} \mathbf{e}_\zeta \cdot \nabla\zeta = \frac{\Psi_\rho}{\sqrt{g}}$$

$$(\nabla\Psi \times \nabla\lambda) \cdot \nabla\zeta = (\nabla\zeta \times \Psi_\rho \nabla\rho) \cdot \nabla\lambda = \frac{\Psi_\rho}{\sqrt{g}} \mathbf{e}_\theta \cdot \nabla\lambda = \frac{\Psi_\rho}{\sqrt{g}} \lambda_\theta$$

$$B^\zeta = \frac{1}{\sqrt{g}} (-\chi_\rho \omega_\theta + \Psi_\rho + \Psi_\rho \lambda_\theta)$$

with $\iota = \chi_\rho / \Psi_\rho$

$$B^\theta = \frac{\Psi_\rho}{\sqrt{g}} (\iota + \iota\omega_\zeta - \lambda_\zeta) \tag{9}$$
$$B^\zeta = \frac{\Psi_\rho}{\sqrt{g}} (-\iota\omega_\theta + 1 + \lambda_\theta) \tag{10}$$

2 Current constraint

$$I(\rho) = \langle B_\theta \rangle$$

$$B_\theta = B^\theta g_{\theta\theta} + B^\zeta g_{\theta\zeta}$$

$$B_\theta = \frac{\Psi_\rho}{\sqrt{g}}(\iota + \iota\omega_\zeta - \lambda_\zeta) g_{\theta\theta} + \frac{\Psi_\rho}{\sqrt{g}}(-\iota\omega_\theta + 1 + \lambda_\theta) g_{\theta\zeta}$$

$$\langle B_\theta \rangle = \Psi_\rho \langle (\iota + \iota\omega_\zeta - \lambda_\zeta) \frac{g_{\theta\theta}}{\sqrt{g}} + (1 + \lambda_\theta - \iota\omega_\theta) \frac{g_{\theta\zeta}}{\sqrt{g}} \rangle$$

$$\langle B_\theta \rangle = \Psi_\rho \langle \iota(1 + \omega_\zeta) \frac{g_{\theta\theta}}{\sqrt{g}} - \lambda_\zeta \frac{g_{\theta\theta}}{\sqrt{g}} + (1 + \lambda_\theta) \frac{g_{\theta\zeta}}{\sqrt{g}} - \iota\omega_\theta \frac{g_{\theta\zeta}}{\sqrt{g}} \rangle$$

$$I(\rho) = \Psi_\rho \iota \langle (1 + \omega_\zeta) \frac{g_{\theta\theta}}{\sqrt{g}} - \omega_\theta \frac{g_{\theta\zeta}}{\sqrt{g}} \rangle + \Psi_\rho \langle -\lambda_\zeta \frac{g_{\theta\theta}}{\sqrt{g}} + (1 + \lambda_\theta) \frac{g_{\theta\zeta}}{\sqrt{g}} \rangle$$

$$I(\rho) + \Psi_\rho \langle \lambda_\zeta \frac{g_{\theta\theta}}{\sqrt{g}} - (1 + \lambda_\theta) \frac{g_{\theta\zeta}}{\sqrt{g}} \rangle = \Psi_\rho \iota \langle (1 + \omega_\zeta) \frac{g_{\theta\theta}}{\sqrt{g}} - \omega_\theta \frac{g_{\theta\zeta}}{\sqrt{g}} \rangle$$

$$\iota(\rho) = \frac{I(\rho)/\Psi_\rho + \langle \lambda_\zeta \frac{g_{\theta\theta}}{\sqrt{g}} - (1 + \lambda_\theta) \frac{g_{\theta\zeta}}{\sqrt{g}} \rangle}{\langle (1 + \omega_\zeta) \frac{g_{\theta\theta}}{\sqrt{g}} - \omega_\theta \frac{g_{\theta\zeta}}{\sqrt{g}} \rangle}$$