Homework 1: Advection-Diffusion Equation, Finite-Difference Schemes

AST560 2025

Due: Thursday March 13th 2025

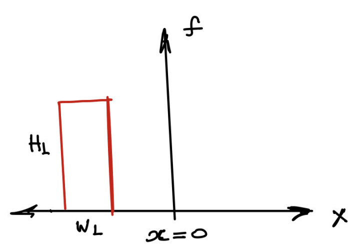
Problem 1: Anisotropic Diffusion

Consider the advection-diffusion equation, but generalize the diffusion term to be *anisotropic*

$$\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{u} f) = \nabla \cdot (\mathbf{D} \cdot \nabla f). \tag{1}$$

Here $\mathbf{D}$ is the second-order *diffusion tensor*. Anisotropic diffusion occurs in all magnetized plasmas: the magnetic field provides a *preferred* direction, along which the heat flows much more easily than perpendicular to it. Assume that the flow is incompressible. Repeat the proof for L_2 conservation but for this system and derive the condition on the tensor $\mathbf{D}$ for the monotonic decay of the L_2 norm of the solution.

Problem 2: L_2 Norm for Compressible Flow



The proof for L_2 conservation requires that the flow be incompressible. In this problem you will derive an explicit solution to a *compressible* flow problem, and show that L_2-norm is indeed not conserved. Consider the 1D advection equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(uf) = 0 \tag{2}$$

where $u = u_L$ when $x < 0$ and $u = u_R$ when $x \geq 0$. Assume both u_L and u_R are positive. Consider the initial condition (a rectangular pulse of height H_L and width W_L) as shown in the figure. Derive the exact solution at a later time $t > 0$ when the pulse has completely moved to the right half of the domain. Using this compute the L_2 norms of the initial and final solutions, showing that the L_2 norm can increase *or* decrease depending on the ratio u_L/u_R.

Problem 3: Upwind-Biased Scheme for Advection

Consider the 1D advection equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(uf) = 0 \tag{3}$$

where $u > 0$ is a constant speed. Derive an *upwind-biased*, semi-discrete scheme for this equation. This scheme should look like

$$\frac{df_0}{dt} + \frac{G^R - G^L}{\Delta x} = 0. \tag{4}$$

where G^R and G^R are the numerical fluxes at the right/left interfaces respectively. Use Fig. [4] from the notes as a guidance, and the third-order edge extrapolation formulas listed in Section 3.2

Use a Taylor series expansion to determine the order of convergence for this scheme. Finally, derive the numerical dispersion relation using Fourier analysis, and plot the real and imaginary parts of the dispersion relation.

Problem 4: Five-Point Stencil Scheme for Diffusion

Consider the 1D diffusion equation

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2} \tag{5}$$

where α is constant. We have already derived the semi-discrete second-order, three-point stencil scheme for this as

$$\frac{\partial f_h}{\partial t} = \alpha \frac{f_R - 2f_0 + f_L}{\Delta x^2}. \tag{6}$$

Derive a higher-order scheme using *five* values $f_{LL}, f_L, f_0, f_R, f_{RR}$ (where LL and RR are values in cells to the left of L and right of R respectively.

Use a Taylor series expansion to determine the order of convergence for this scheme. Finally, derive the numerical dispersion relation using Fourier analysis, and plot it.

Problem 5: Numerical Diffusion in First-Order Updwind Scheme

Consider the semi-discrete, first-order upwind scheme for linear advection:

$$\frac{df_j}{dt} + \frac{f_j^n - f_{j-1}^n}{\Delta x} = 0.$$

Though this scheme preserves positivity and monotonicity, it is very diffusive. Rewrite this scheme as a combination of a centered-difference scheme for the first derivative and a centered-difference scheme for the second derivative. Hence, explicitly derive the *numerical diffusion* coefficient of the first-order scheme.

Problem 6: Flux form of Semi-Discrete Scheme, and Discrete L_2 Conservation

We can write a generic semi-discrete scheme in the form

$$\frac{df_i}{dt} + \frac{G_{i+1/2} - G_{i-1/2}}{\Delta x} = 0 \tag{7}$$

where $G_{i\pm 1/2}$ are the numerical fluxes. (See Section 4 where we wrote the scheme for advection-diffusion equation in this form). Then, the specific scheme we use (upwind, central, etc) all can be written by an appropriate choice of numerical fluxes. Write down the numerical fluxes (for example $G_{i-1/2}$) for (a) first-order upwind scheme for advection equation, (b) second-order central scheme for constant diffusion equation.

To understand how L_2 evolves for a generic semi-discrete scheme written above, multiply by f_i and sum over i, and use index-shifting to rewrite the second term. Use this generic form and the numerical fluxes for the second-order central scheme for constant diffusion to show that the L_2 norm decays, as it should.

(As you showed in the previous problem that a first-order upwind scheme can be written as a combination of central schemes for advection and diffusion, this problem shows that the L_2 norm for first-order upwind for advection decays monotonically).

Problem 7: Stability of Implicit Time-Stepping Schemes and the Simple-Harmonic Oscillator

Consider the simple first-order ODE

$$\dot{f} = (\lambda - i\omega) f \tag{8}$$

where $\lambda \leq 0$. For this we will construct two discrete schemes: the *backward-Euler scheme*:

$$\frac{f^{n+1} - f^n}{\Delta t} = (\lambda - i\omega) f^{n+1} \tag{9}$$

and the *time-centered scheme*

$$\frac{f^{n+1} - f^n}{\Delta t} = (\lambda - i\omega) \frac{1}{2}(f^{n+1} + f^n). \tag{10}$$

Derive the stability properties of these schemes and show that they are *unconditionally stable*. What is the advantage of using the time-centered scheme? (Hint: Consider the properties of $|f|^2$ in the limit $\lambda = 0$).

Apply these implicit schemes to the harmonic oscillator

$$\frac{d^2 z}{dt^2} = -\omega^2 z \tag{11}$$

by rewriting as a system of first-order ODEs

$$\frac{dz}{dt} = v; \quad \frac{dv}{dt} = -\omega^2 z. \tag{12}$$

Derive the exact conserved quadratic invariant for this system and show that the time-centered scheme conserves this invariant exactly.

Introduce energy-angle coordinates:

$$\omega z = E \sin\theta; \quad v = E \cos\theta \tag{13}$$

As a challenge, derive an expression for the discrete phase-errors for the time-centered scheme, that is, attempt to derive a Taylor series expression for $\tan\theta^{n+1} = \tan(\theta^n + \omega\Delta t)$, and see how many terms it matches, thus deriving the phase-error.

(The Boris algorithm used in particle-in-cell (PIC) codes is a time-centered scheme for the $\mathbf{v} \times \mathbf{B}$ term, and has structure and properties very similar to the one for the harmonic oscillator. Standard PIC codes use time-centering for Maxwell equations as well as for the particle push to allow stepping over the cyclotron frequency, but must resolve the plasma-frequency, and of course, the speed of light).