Lane - Emden Equation

$$dm = 4\pi r^2 \rho\, dr$$

$P \equiv$ pressure

$\rho \equiv$ mass density

$m \equiv$ mass

$$\nabla^2 \Phi = 4\pi G \rho \quad \left(\begin{array}{c}\text{Newtonian} \\ \text{gravitational} \\ \text{field eq}\end{array}\right)$$

$$\frac{dP}{dr} = -\frac{d\Phi}{dr}\rho \quad \left(\begin{array}{c}\text{Hydrostatic} \\ \text{eq}\end{array}\right)$$

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right)$$

(Laplacian in spherical coords)

$$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}} \quad \left(\begin{array}{c}\text{Eqn} \\ \text{of state}\end{array}\right)$$

$$\Rightarrow \frac{dP}{dr} = \gamma K \rho^{\gamma-1} \frac{d\rho}{dr}$$

$$\left(\gamma - 1 = \frac{1}{n}\right)$$

$$\gamma = 1 + \frac{1}{n}$$

$$\Rightarrow \frac{d\Phi}{dr} = -\gamma K \rho^{\gamma-2} \frac{d\rho}{dr}$$

$$\frac{d\Phi}{dr} = \frac{Gm}{r^2}$$

$$\Rightarrow \int d\Phi = -\gamma K \int \rho^{\gamma-2}\, d\rho$$

$$\Rightarrow \Phi = \frac{-\gamma K \rho^{\gamma-1}}{\gamma - 1}$$

$$= -(n+1) K \rho^{1/n}$$

($\Phi = 0$ at $\rho = 0$ on the surface)

($\Phi < 0$ at interior since $\rho > 0$)

$$\Rightarrow \left(\frac{-\phi}{(n+1) \cdot K} \right)^n = \rho$$

At $r=0$, we have $\phi = \phi_c$ and $\rho = \rho_c$

Put it into Poisson's eq

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 4\pi G \left(\frac{-\phi}{(n+1) K} \right)^n$$

$$\text{Let } \alpha^{-2} = \frac{4\pi G \, (-\phi_c)^{n}}{((n+1) K)^n}$$

Let $r = \alpha . \xi$

$$\theta = \frac{\phi}{\phi_c} = \left(\frac{\rho}{\rho_c} \right)^{1/n}$$

$\Rightarrow \theta = 1$ at the center $(r=0)$

$$\Rightarrow \frac{1}{\xi^2} \cdot \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0 \qquad \square$$

$$dm = 4\pi r^2 \rho\, dr \quad \text{where} \quad \rho = \rho_c \theta^n$$

$$\Rightarrow m(r) = \int_0^r 4\pi \tilde{r}^2 \rho\, d\tilde{r} = 4\pi\rho_c \int_0^r \tilde{r}^2 \theta^n\, d\tilde{r}$$

$$r = \alpha\xi \Rightarrow d\tilde{r} = \alpha\, d\tilde{\xi}$$

$$\Rightarrow m(r) = 4\pi\rho_c \alpha^3 \int_0^\xi \tilde{\xi}^2 \theta^n\, d\tilde{\xi}$$

$$= 4\pi\rho_c \frac{r^3}{\xi^3} \int_0^\xi \tilde{\xi}^2 \theta^n\, d\tilde{\xi}$$

(L-E eq)

$$= -4\pi\rho_c \frac{r^3}{\xi^3} \int_0^\xi d\left(\tilde{\xi}^2 \frac{d\theta}{d\xi}\right)$$

$$= -4\pi\rho_c \frac{r^3}{\xi^3} \cdot \xi^2 \cdot \theta'(\xi)$$

$$= 4\pi\rho_c r^3 \cdot \left(-\frac{\theta'(\xi)}{\xi}\right)$$

$$\Rightarrow M = m(R) = 4\pi\rho_c R^3 \left(\frac{-\theta'(\xi_n)}{\xi_n}\right).$$

$$\alpha^{*2} = \frac{4\pi G (-\phi_c)^n}{((n+1)K)^n} = \frac{4\pi G}{(n+1)K} \rho_c^{\frac{n-1}{n}}$$

$$\Rightarrow \alpha^2 = \left(\frac{r}{\xi}\right)^2 = \frac{(n+1)K \, \rho_c^{\frac{1-n}{n}}}{4\pi G}$$

$$\Rightarrow R = \xi_n \cdot \left(\frac{(n+1)K}{4\pi G}\right)^{1/2} \cdot \rho_c^{\frac{1-n}{2n}}$$

$$\Rightarrow R^{\frac{2n}{1-n}} = \xi_n^{\frac{2n}{1-n}} \cdot \left(\frac{(n+1)K}{4\pi G}\right)^{\frac{n}{1-n}} \rho_c$$

$$\Rightarrow \rho_c = \left(\frac{(n+1)K}{4\pi G}\right)^{\frac{n}{n-1}} \cdot \xi_n^{\frac{2n}{n-1}} \cdot R^{\frac{2n}{1-n}}$$

Replace ρ_c in M eq with this

$$\Rightarrow M = 4\pi \cdot \left(\frac{(n+1)K}{4\pi G}\right)^{\frac{n}{n-1}} \xi_n^{\frac{2n}{n-1}} R^{\frac{2n}{1-n}} R^3 \left(\frac{-\theta'(\xi_n)}{\xi_n}\right)$$

$$= 4\pi \left(\frac{(n+1)K}{4\pi G}\right)^{\frac{n}{n-1}} \xi_n^{\frac{2n}{n-1}} \left(\frac{-\theta'(\xi_n)}{\xi_n}\right) R^{\frac{3-n}{1-n}}$$

$$\Rightarrow \text{coeff} = 4\pi \left(\frac{(n+1)K}{4\pi G}\right)^{\frac{n}{n-1}} \cdot \xi_n^{\frac{n+1}{n-1}} (-\theta'(\xi_n)) \Big|$$

$$\log C = \log\left(4\pi \cdot \left(\frac{n+1}{4\pi G} \right)^{\frac{n}{n-1}} \xi_n^{\frac{n+1}{n-1}} \left(-\theta'(\xi_n) \right) \right) + \frac{1}{n-1} \log K$$

$$\frac{d\Phi}{dr} = -\gamma K \rho^{\gamma-2} \frac{d\rho}{dr}$$