P8131 Spring 2021 Homework #1

Due on January 25 11:59pm

1. Show that the following distributions belong to the exponential family. Find the natural parameter θ, scale parameter ϕ and convex function $b(\theta)$. Also find the EY and $\text{Var}(Y)$ as functions of the natural parameter. Specify the canonical link functions.

   (a) Exponential distribution $Exp(\lambda)$, $f(y;\lambda) = \lambda e^{-\lambda y}$;

   (b) Binomial distribution $Bin(n,\pi)$, $f(y;\pi) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$, where n is known;

   (c) Poisson distribution $Pois(\lambda)$, $f(y;\lambda) = \frac{1}{y!}\lambda^y e^{-\lambda}$;

   (d) Chi-squared distribution $\chi^2_{(k)}$, $f(y;k) = \frac{1}{\Gamma(\frac{k}{2})2^{\frac{k}{2}}}y^{\frac{k}{2}-1}e^{-\frac{y}{2}}$ (no need to specify the canonical link function);

   (e) Negative binomial distribution $NB(m,\beta)$, $f(y;\beta) = \binom{y+m-1}{m-1}\beta^m(1-\beta)^y$, where m is known;

   (f) The Gamma distribution $Gamma(\alpha,\beta)$, $f(y;\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y}$, where the shape parameter α is known.

2. Assume $Y_1, Y_2, ..., Y_n$ are independent and follow a binomial distribution where $Y_i \sim Bin(m,\pi_i)$ and m is known. Furthermore, assume $\log\frac{\pi_i}{1-\pi_i} = X_i\beta$. What are the expressions of deviance residuals and Pearson residuals respectively (use $\hat{\beta}$ to represent the MLE)? What are the expressions of the deviance and Pearson's χ^2 statistic?

3. Consider the binary response variable $Y \sim Bernoulli$ with $P(Y=1) = \pi$ and $P(Y=0) = 1-\pi$. Observations Y_i, $i = 1, \ldots, n$, are independent and identically distributed as Y.

   (a) Find the Wald test statistic, the score test statistic, and the likelihood ratio test statistic to test hypothesis $H_0 : \pi = \pi_0$.

   (b) With large samples, the Wald test statistic, score test statistic and the likelihood ratio test statistic approximately have the $\chi^2(1)$ distribution. For $n = 10$ and data (0, 1, 0, 0, 1, 0, 0, 0, 1, 0), use these statistics to test null hypotheses on for (i) $\pi_0 = 0.1$, (ii) $\pi_0 = 0.3$, (iii) $\pi_0 = 0.5$.

   (c) Do the Wald test, score test, and the likelihood ratio test lead to the same conclusions in (b)?

4. (Optional; PhDs required) $Y_i \sim Pois(\lambda), i = 1, ..., n$. We are interested in testing $H_0 : l\log\lambda = \log\lambda_0$. What are the Wald test statistic, the score test statistic, and the likelihood ratio test statistic?