Sampler

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1 Problem setup

Consider the global-local shrinkage prior:

$$\begin{aligned}\beta &\sim \tau\lambda\tilde{\beta},\\ \tilde{\beta} &\sim \mathcal{N}(\mu, \sigma^2),\\ \lambda &\sim \pi_\Lambda(\cdot).\end{aligned}$$

When using Gibbs sampler, we need to sample from the local-scale posterior:

$$\begin{aligned}\pi(\lambda \mid \beta) &\propto \pi(\beta \mid \lambda, \tau)\pi_\Lambda(\lambda)\\ &\propto \frac{1}{\lambda}\exp\left(-\frac{(\beta-\tau\lambda\mu)^2}{2\tau^2\lambda^2\sigma^2}\right)\pi_\Lambda(\lambda)\\ &\propto \frac{1}{\lambda}\exp\left(-a^2(\frac{1}{\lambda}-c)^2\right)\pi_\Lambda(\lambda),\end{aligned}$$

where $a = \frac{\beta}{\sqrt{2}\tau\sigma}, c = \frac{\tau\mu}{\beta}$. Let $\eta = \lambda^{-1}$, then

$$\begin{aligned}\pi(\eta \mid \beta) &= \pi(\lambda \mid \beta)|d\lambda/d\eta|\\ &\propto \frac{1}{\eta}\exp\left(-a^2(\eta-c)^2\right)\pi_\Lambda\left(\frac{1}{\eta}\right).\end{aligned} \tag{1}$$

Our goal is to design efficient sampling algorithms for the target distribution (1) under different priors.

2 Half-cauchy prior

We first consider the half-cauchy prior $\pi_\Lambda(\lambda) \propto \frac{1}{1+\lambda^2}$. The target distribution (1) now boils down to

$$\pi(\eta \mid \beta) \propto \frac{\eta}{1+\eta^2}\exp\left(-a^2(\eta-c)^2\right). \tag{2}$$

2.1 Slice sampler

Inspired by the slice sampler designed for the case $\mu = 0$, we propose a slice sampling algorithm for (2). More specifically, we introduce the auxiliary variable u and the augmented target distribution

$$\bar{\pi}(\eta, u) \propto \exp\left(-a^2(\eta - c)^2\right) \mathbb{I}\left(0 < u < \frac{\eta}{1+\eta^2}\right).$$

Within the general framework of slice sampling, we iteratively perform the following two steps:

1. Draw from $u \mid \eta \sim \text{Uniform}\left(0, \frac{\eta}{1+\eta^2}\right)$.

2. Draw from $\bar{\pi}(\eta \mid u) \propto \exp\left(-a^2(\eta - c)^2\right) \mathbb{I}\left(\eta_1 < \eta < \eta_2\right)$, where

$$\eta_1 = \frac{1 - \sqrt{1 - 4u^2}}{2u}, \eta_2 = \frac{1 + \sqrt{1 - 4u^2}}{2u}.$$

   $\eta \mid u$ is a normal distribution truncated between η_1 and η_2.

2.2 Rejection sampler

To design a rejection sampler for the unnormalized target density

$$f(\eta) = \frac{\eta}{1+\eta^2} \exp\left(-a^2(\eta - c)^2\right),$$

we need to find a function g that upper bounds f.

Some observation:

- $\frac{\eta}{1+\eta^2}$ can be simulated by the inverse transform method.

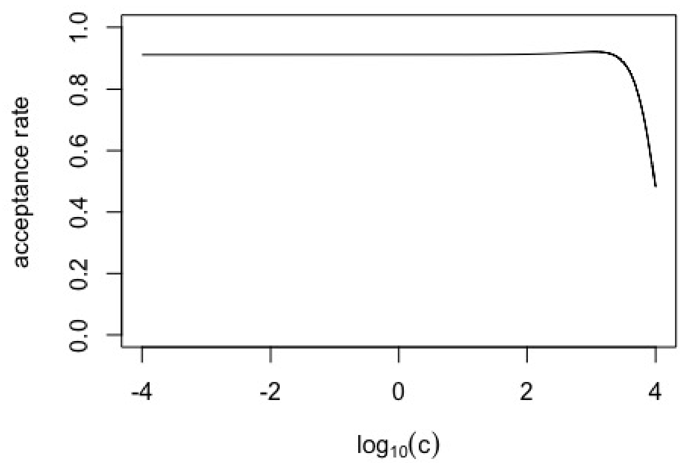
- When $\eta \to +\infty$, $\frac{\eta}{1+\eta^2} \approx \frac{1}{\eta} \to 0$, $\exp\left(-a^2(\eta - c)^2\right) \to 0$. Yet $\exp\left(-a^2(\eta - c)^2\right)$ decays much faster. Thus, a normal density might be an ideal candidate to upper bound the tail. Consider the normal distribution

$$g(\eta) = \frac{a}{\sqrt{\pi}} \exp\left(-a^2(\eta - c)^2\right).$$

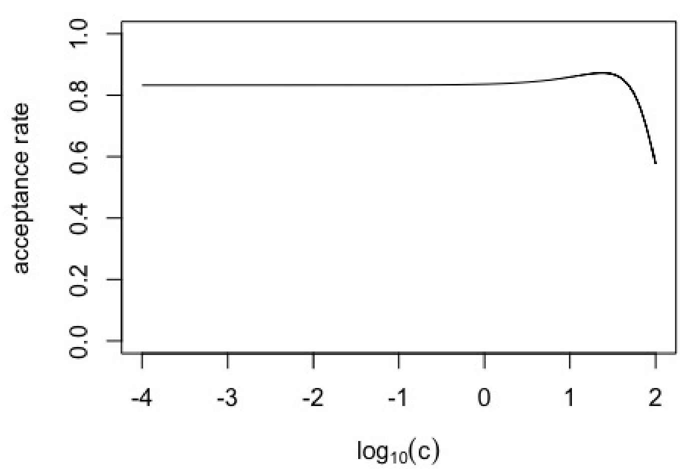
  If $a \geq \frac{\sqrt{\pi}}{2}$, g always upper bounds f, otherwise $g(\eta) \geq f(\eta)$ on $[0, \frac{\sqrt{\pi}-\sqrt{\pi-4a^2}}{2a}]$ and $[\frac{\sqrt{\pi}+\sqrt{\pi-4a^2}}{2a}, +\infty)$.

In the case where a is small (say $a < \frac{\sqrt{\pi}}{2}$) and c isn't too large (for example, $0 < c \leq \frac{1}{a}$), even if η is close to 0 (and thus far from c), the term $\exp\left(-a^2(\eta - c)^2\right)$ stays close to 1. Consider the following proposal density:

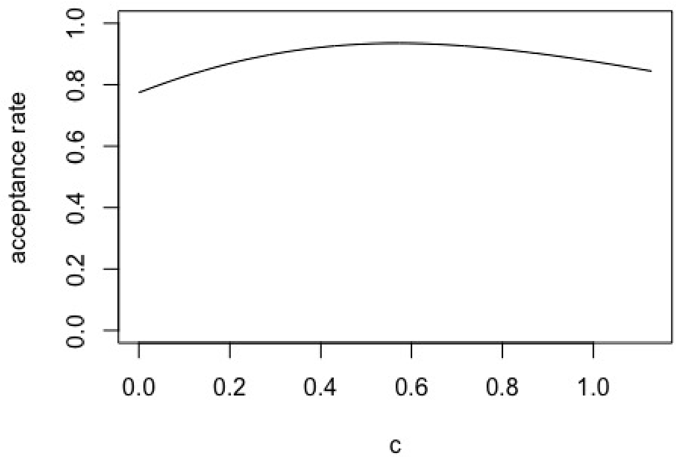
$$g(\eta) = \begin{cases} \frac{\eta}{1+\eta^2} & \text{if } \eta < m \\ \frac{a}{\sqrt{\pi}} \exp\left(-a^2(x - c)^2\right) & \text{if } \eta \geq m \end{cases},$$



(a) $a = 10^{-4}, 0 \leq c \leq 10^4$.



(b) $a = 10^{-2}, 0 \leq c \leq 10^2$.



(c) $a = \frac{\sqrt{\pi}}{2}, 0 \leq c \leq \frac{2}{\sqrt{\pi}}$.

Figure 1: Acceptance rate for $0 \leq c \leq \frac{1}{a}$ under different values of a.

where $m = \frac{\sqrt{\pi}+\sqrt{\pi-4a^2}}{2a}$. As illustrated in Figure 1, this scheme seems to work well in terms of acceptance rate.

Similarly, when a is small and $-\frac{1}{a} < c < 0$, consider the following proposal density

$$g(\eta) = \begin{cases} \frac{\eta}{1+\eta^2} \exp(-a^2 c^2) & \text{if } \eta < m \\ \frac{a}{\sqrt{\pi}} \exp\left(-a^2(x-c)^2\right) & \text{if } \eta \geq m \end{cases}.$$

2.2.1

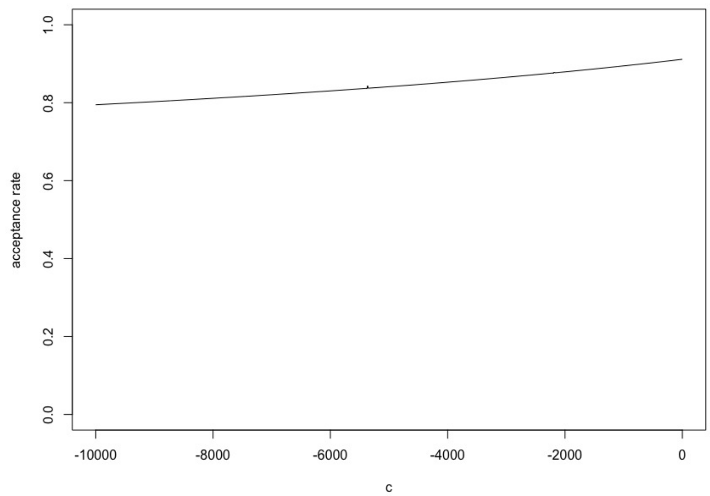
On the precision scale $x = \eta^2$, the target density can also be formulated as

$$\pi(x) \propto f(x) = \frac{1}{1+x} \exp\left(-\beta x + \alpha\sqrt{x}\right),$$

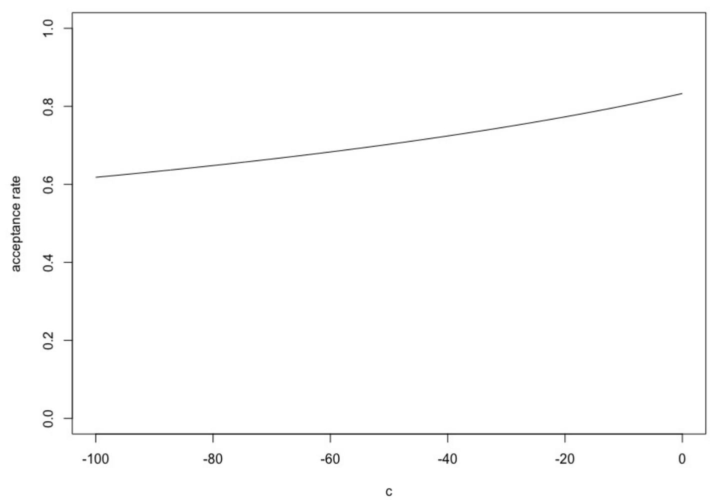
where $\beta = a^2, \alpha = 2a^2 c$. We observe three competing terms in this expression: $\frac{1}{1+x}$, $\exp(-\beta x)$ and $\exp(\alpha\sqrt{x})$. Besides, it is straightforward to show that $f(x)$ is log-convex if $\alpha \leq 0$ and log-concave if $\alpha \geq \frac{3\sqrt{3}}{4}$. The supplementary materials of xxx describes a rejection sampler that exploits the log-convexity of the target density in the case $\alpha = 0$. In this section, we want to develop a rejection sampler that is applicable to the case $\alpha \leq 0$.

Let

$$h(x) = \log f(x) = -\log(1+x) - \beta x + \alpha\sqrt{x}.$$



(a) $a = 10^{-4}, -10^4 \leq c \leq 0$.



(b) $a = 10^{-2}, -10^2 \leq c \leq 0$.

Figure 2: Acceptance rate for $-\frac{1}{a} \leq c \leq 0$ under different values of a.

The derivatives of the three terms $(-\log(1+x), -\beta x, \alpha\sqrt{x})$ are $-\frac{1}{1+x}, -\beta$ and $\frac{\alpha}{2\sqrt{x}}$, whose magnitude can be easily compared.

My initial idea is to use the term whose derivative has the largest absolute value among the three terms as the proposal density since it is hard to compare $-\log(1+x)$ with the other two terms directly. However, the sampling of $\exp(\alpha\sqrt{x})$ might cause some problems. To generate $X \sim \exp(\alpha\sqrt{x})$ on $[a, b]$, we first generate a truncated gamma random variable $Y \sim \mathcal{G}(2, -\alpha)$ on $[\sqrt{a}, \sqrt{b}]$, and then take $X = Y^2$. The R package `truncdist` can be used to generate truncated gamma distribution; however, it may fail if $\sqrt{a}$ is too large. Therefore, we use line segment to upper bound $h(x)$ when needed.

Specifically, if $\mid \alpha \mid \geq 1$ or $\mid \alpha \mid < 1, \beta \geq \frac{1+\sqrt{1-\alpha^2}}{2}$, consider

$$\tilde{g}(x) = \begin{cases} \exp(\alpha\sqrt{x}) & \text{if } x < \frac{\alpha^2}{4\beta^2} \\ h(\frac{\alpha^2}{4\beta^2}) - \beta(x - \frac{\alpha^2}{4\beta^2}) & \text{if } x \geq \frac{\alpha^2}{4\beta^2} \end{cases}.$$

If $\mid \alpha \mid < 1$ and $\frac{1-\sqrt{1-\alpha^2}}{2} < \beta < \frac{1+\sqrt{1-\alpha^2}}{2}$, consider

$$\tilde{g}(x) = \begin{cases} \frac{h(a)}{a} x & \text{if } x < a \\ h(a) - \log\left(\frac{1+x}{1+a}\right) & \text{if } a \leq x < b \\ h(b) - \beta(x - b) & \text{if } x \geq b \end{cases},$$

where

$$a = \frac{2 - \alpha^2 - 2\sqrt{1-\alpha^2}}{\alpha^2}, b = \frac{1}{\beta} - 1.$$

If $\mid \alpha \mid < 1, \beta \leq \frac{1-\sqrt{1-\alpha^2}}{2}$, consider

$$\tilde{g}(x) = \begin{cases} \frac{h(a_1)}{a_1} x & \text{if } x < a_1 \\ h(a_1) - \log\left(\frac{1+x}{1+a_1}\right) & \text{if } a_1 \leq x < a_2 \\ h(a_2) + \frac{h(b)-h(a_2)}{b-a_2}(x - a_2) & \text{if } a_2 < x \leq b \\ h(b) - \beta(x - b) & \text{if } x > b \end{cases},$$

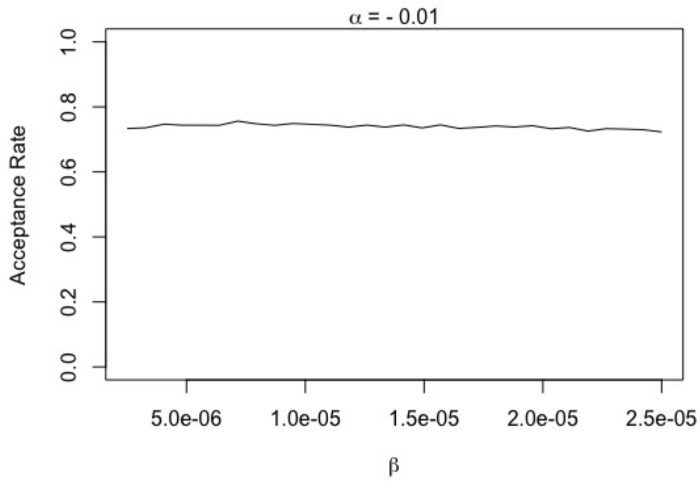
where

$$a_1 = \frac{2 - \alpha^2 - 2\sqrt{1-\alpha^2}}{\alpha^2}, a_2 = \frac{2 - \alpha^2 + 2\sqrt{1-\alpha^2}}{\alpha^2}, b = \frac{\alpha^2}{4\beta^2}.$$

Then $g(x) = \exp(\tilde{g}(x))$ upper bounds $f(x)$.

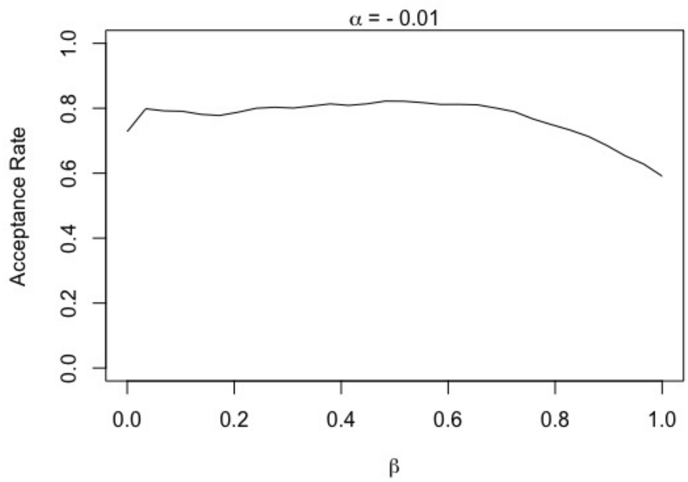
Q:

- truncated gamma distribution?
- It seems reasonable to use $\exp(\alpha\sqrt{x})$ as the proposal density on $[a_1, a_2]$ instead of $\frac{1}{1+x}$ if $\mid \alpha \mid$ is "not too small". When to use $\exp(\alpha\sqrt{x})$ / $\frac{1}{1+x}$ (/line segment)? For example, when $\alpha \approx -0.3$, $c_1 \exp(\alpha\sqrt{x})$ and $c_2 \frac{1}{1+x}$ looks "similar".

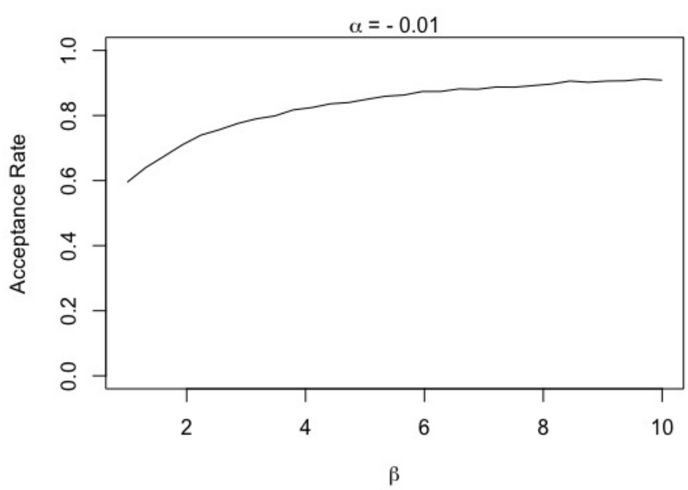


(a) $\alpha = -0.01$, $\frac{1-\sqrt{1-\alpha^2}}{20} \leq \beta \leq \frac{1-\sqrt{1-\alpha^2}}{2}$.

- One possible solution, which seems a little bit nonsense, is to additionally design a sampler for $\frac{1}{1+x}\exp(\alpha\sqrt{x})$. Two problems arise: 1. truncated gamma distribution, 2. integrate $\frac{1}{1+x}\exp(\alpha\sqrt{x})$.

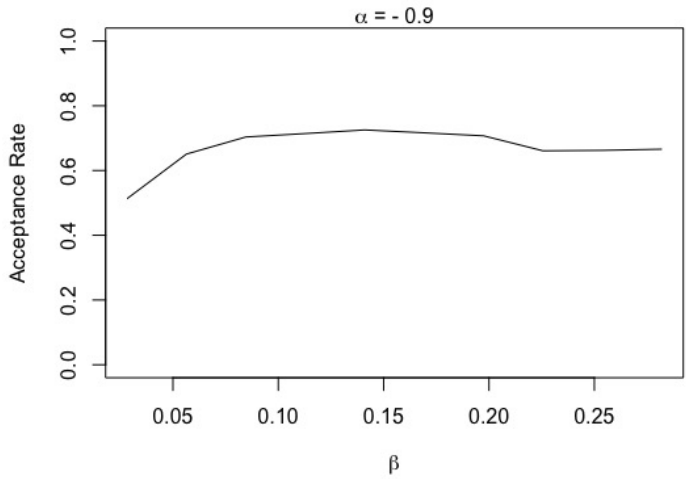


(b) $\alpha = -0.01, \frac{1-\sqrt{1-\alpha^2}}{2} \leq \beta \leq \frac{1+\sqrt{1-\alpha^2}}{2}$.

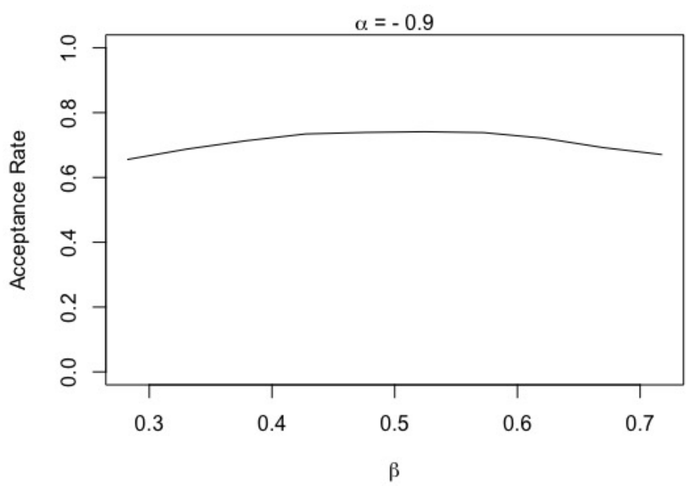


(c) $\alpha = -0.01, \beta \geq \frac{1-\sqrt{1-\alpha^2}}{2}$.

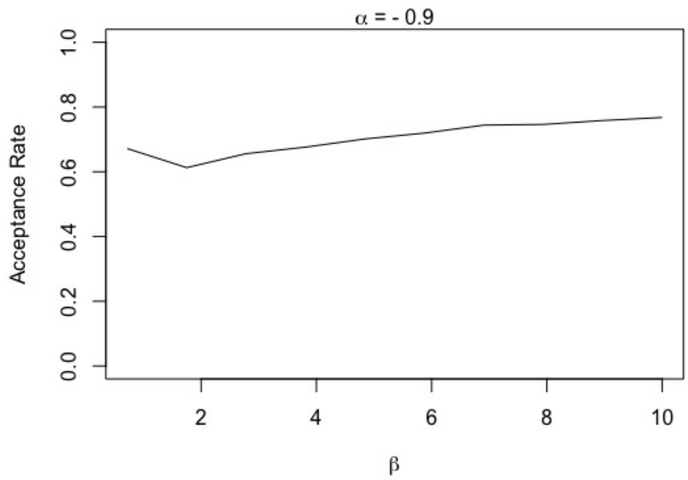
Figure 3: Acceptance rate when $\alpha = -0.01$.



(a) $\alpha = -0.9, \frac{1-\sqrt{1-\alpha^2}}{20} \leq \beta \leq \frac{1-\sqrt{1-\alpha^2}}{2}$.

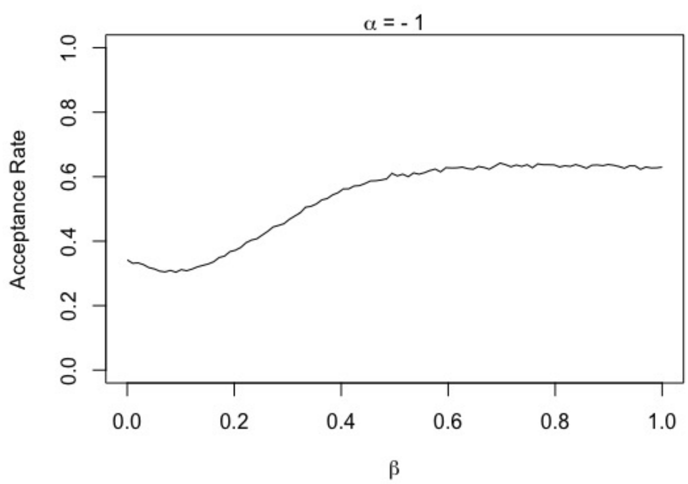


(b) $\alpha = -0.9, \frac{1-\sqrt{1-\alpha^2}}{2} \leq \beta \leq \frac{1+\sqrt{1-\alpha^2}}{2}$.

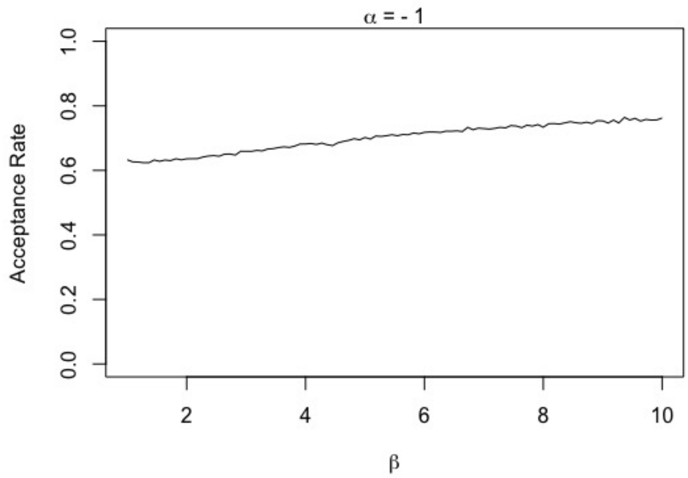


(c) $\alpha = -0.9, \beta \geq \frac{1-\sqrt{1-\alpha^2}}{2}$.

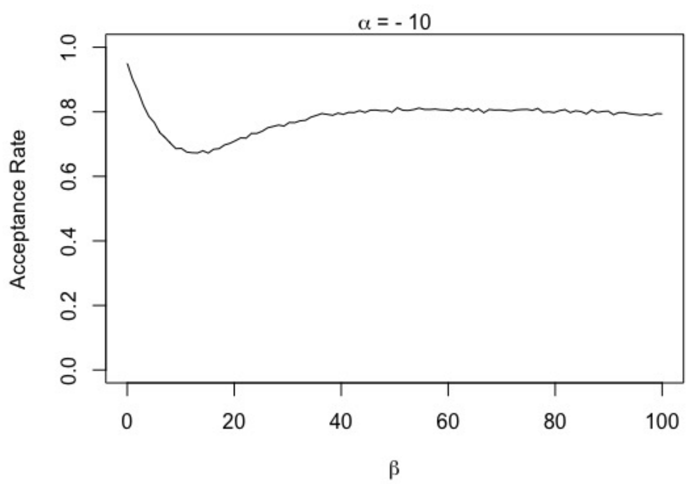
Figure 4: Acceptance rate when $\alpha = -0.9$.



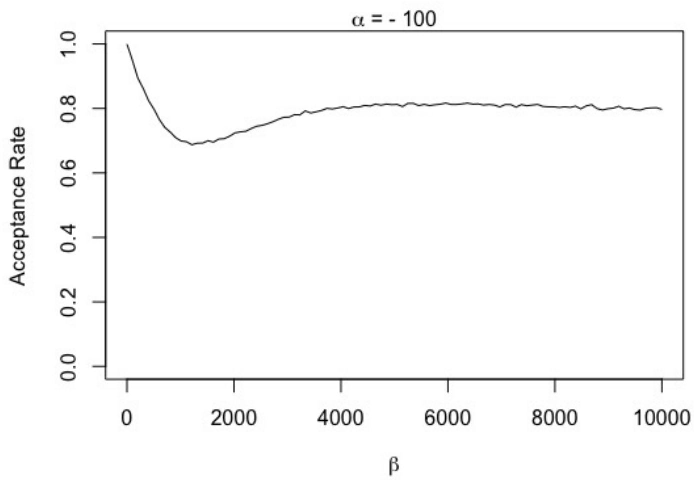
(a) $\alpha = -1, 0.0001 \leq \beta \leq 1$.



(b) $\alpha = -1, 1 \leq \beta \leq 10$.



(c) $\alpha = -10, 0.0001 \leq \beta \leq 100$.



(d) $\alpha = -100, 0.0001 \leq \beta \leq 10000$.

Figure 5: Acceptance rate when $\alpha \geq 1$.