Sampler

Yihao Gu

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1 Problem setup

Consider the global-local shrinkage prior:

$$\begin{aligned}
&\beta \sim \tau\lambda\tilde{\beta}, \\
&\tilde{\beta} \sim \mathcal{N}(\mu, \sigma^2), \\
&\lambda \sim \pi_\Lambda(\cdot).
\end{aligned}$$

When using Gibbs sampler, we need to sample from the local-scale posterior:

$$\begin{aligned}
\pi(\lambda \mid \beta) &\propto \pi(\beta \mid \lambda, \tau)\pi_\Lambda(\lambda) \\
&\propto \frac{1}{\lambda}\exp\left(-\frac{(\beta - \tau\lambda\mu)^2}{2\tau^2\lambda^2\sigma^2}\right)\pi_\Lambda(\lambda) \\
&\propto \frac{1}{\lambda}\exp\left(-a^2(\frac{1}{\lambda} - c)^2\right)\pi_\Lambda(\lambda),
\end{aligned}$$

where $a = \frac{\beta}{\sqrt{2}\tau\sigma}$, $c = \frac{\tau\mu}{\beta}$. Let $\eta = \lambda^{-1}$, then

$$\begin{aligned}
\pi(\eta \mid \beta) &= \pi(\lambda \mid \beta)|d\lambda/d\eta| \\
&\propto \frac{1}{\eta}\exp\left(-a^2(\eta - c)^2\right)\pi_\Lambda\left(\frac{1}{\eta}\right). \qquad (1)
\end{aligned}$$

Our goal is to design efficient sampling algorithms for the target distribution (1) under different priors.

2 Half-cauchy prior

We first consider the half-cauchy prior $\pi_\Lambda(\lambda) \propto \frac{1}{1+\lambda^2}$. The target distribution (1) now boils down to

$$\pi(\eta \mid \beta) \propto \frac{\eta}{1+\eta^2}\exp\left(-a^2(\eta - c)^2\right). \qquad (2)$$

2.1 Slice sampler

Inspired by the slice sampler designed for the case $\mu = 0$, we propose a slice sampling algorithm for (2). More specifically, we introduce the auxiliary variable u and the augmented target distribution

$$\bar{\pi}(\eta, u) \propto \exp\left(-a^2(\eta - c)^2\right) \mathbb{I}\left(0 < u < \frac{\eta}{1+\eta^2}\right).$$

Within the general framework of slice sampling, we iteratively perform the following two steps:

1. Draw from $u \mid \eta \sim \text{Uniform}\left(0, \frac{\eta}{1+\eta^2}\right)$.
2. Draw from $\bar{\pi}(\eta \mid u) \propto \exp\left(-a^2(\eta - c)^2\right) \mathbb{I}\left(\eta_1 < \eta < \eta_2\right)$, where

$$\eta_1 = \frac{1 - \sqrt{1 - 4u^2}}{2u}, \eta_2 = \frac{1 + \sqrt{1 - 4u^2}}{2u}.$$

   $\eta \mid u$ is a normal distribution truncated between η_1 and η_2.

2.2 Rejection sampler

To design a rejection sampler for the unnormalized target density

$$f(\eta) = \frac{\eta}{1+\eta^2} \exp\left(-a^2(\eta - c)^2\right),$$

we need to find a function g that upper bounds f.

Some observation:

- $\frac{\eta}{1+\eta^2}$ can be simulated by the inverse transform method.
- When $\eta \to +\infty$, $\frac{\eta}{1+\eta^2} \approx \frac{1}{\eta} \to 0$, $\exp\left(-a^2(\eta - c)^2\right) \to 0$. Yet $\exp\left(-a^2(\eta - c)^2\right)$ decays much faster. Thus, a normal density might be an ideal candidate to upper bound the tail. Consider the normal distribution

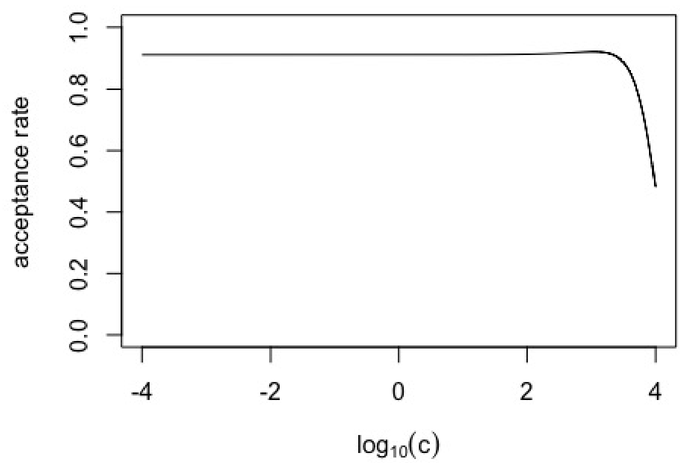
$$g(\eta) = \frac{a}{\sqrt{\pi}} \exp\left(-a^2(\eta - c)^2\right).$$

  If $a \geq \frac{\sqrt{\pi}}{2}$, g always upper bounds f, otherwise $g(\eta) \geq f(\eta)$ on $[0, \frac{\sqrt{\pi}-\sqrt{\pi-4a^2}}{2a}]$ and $[\frac{\sqrt{\pi}+\sqrt{\pi-4a^2}}{2a}, +\infty)$.

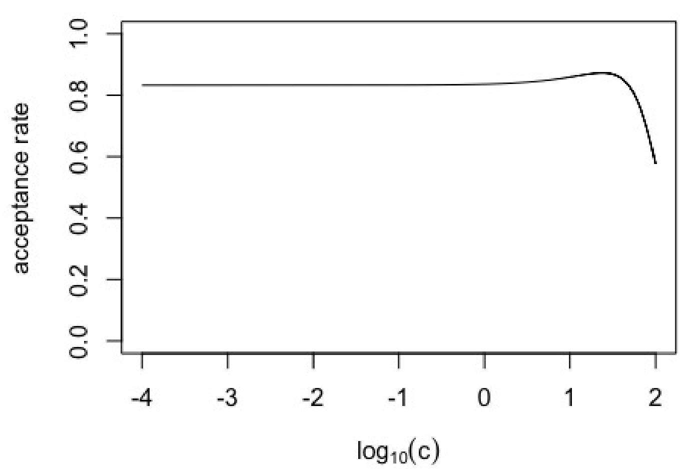
In the case where a is small (say $a < \frac{\sqrt{\pi}}{2}$) and c isn't too large (for example, $0 < c \leq \frac{1}{a}$), even if η is close to 0 (and thus far from c), the term $\exp\left(-a^2(\eta - c)^2\right)$ stays close to 1. Consider the following proposal density:

$$g(\eta) = \begin{cases} \frac{\eta}{1+\eta^2} & \text{if } \eta < m \\ \frac{a}{\sqrt{\pi}} \exp\left(-a^2(x - c)^2\right) & \text{if } \eta \geq m \end{cases},$$

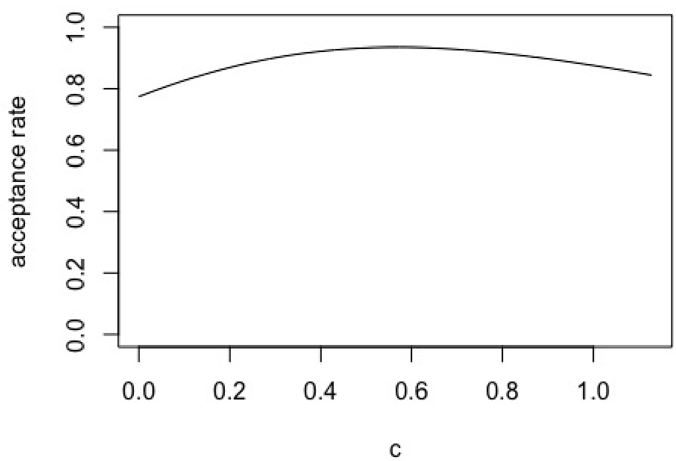
where $m = \frac{\sqrt{\pi}+\sqrt{\pi-4a^2}}{2a}$. As illustrated in Figure 1, this scheme seems to work well in terms of acceptance rate.



(a) $a = 10^{-4}, 0 \leq c \leq 10^4$.



(b) $a = 10^{-2}, 0 \leq c \leq 10^2$.



(c) $a = \frac{\sqrt{\pi}}{2}, 0 \leq c \leq \frac{2}{\sqrt{\pi}}$.

Figure 1: Acceptance rate for $0 \leq c \leq \frac{1}{a}$ under different values of a.