Sampler

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1 Problem setup

Consider the global-local shrinkage prior:

$$\begin{aligned}
&\beta \sim \tau\lambda\tilde{\beta},\\
&\tilde{\beta} \sim \mathcal{N}(\mu, \sigma^2),\\
&\lambda \sim \pi_\Lambda(\cdot).
\end{aligned}$$

When using Gibbs sampler, we need to sample from the local-scale posterior:

$$\begin{aligned}
\pi(\lambda \mid \beta) &\propto \pi(\beta \mid \lambda, \tau)\pi_\Lambda(\lambda)\\
&\propto \frac{1}{\lambda}\exp\left(-\frac{(\beta-\tau\lambda\mu)^2}{2\tau^2\lambda^2\sigma^2}\right)\pi_\Lambda(\lambda)\\
&\propto \frac{1}{\lambda}\exp\left(-a^2(\frac{1}{\lambda}-c)^2\right)\pi_\Lambda(\lambda),
\end{aligned}$$

where $a = \frac{\beta}{\sqrt{2}\tau\sigma}, c = \frac{\tau\mu}{\beta}$. Let $\eta = \lambda^{-1}$, then

$$\begin{aligned}
\pi(\eta \mid \beta) &= \pi(\lambda \mid \beta)|d\lambda/d\eta|\\
&\propto \frac{1}{\eta}\exp\left(-a^2(\eta-c)^2\right)\pi_\Lambda\left(\frac{1}{\eta}\right). \qquad (1)
\end{aligned}$$

Our goal is to design efficient sampling algorithms for the target distribution (1) under different priors.

2 Half-cauchy prior

We first consider the half-cauchy prior $\pi_\Lambda(\lambda) \propto \frac{1}{1+\lambda^2}$. The target distribution (1) now boils down to

$$\pi(\eta \mid \beta) \propto \frac{\eta}{1+\eta^2}\exp\left(-a^2(\eta-c)^2\right). \qquad (2)$$

2.1 Slice sampler

Inspired by the slice sampler designed for the case $\mu = 0$, we propose a slice sampling algorithm for (2). More specifically, we introduce the auxiliary variable u and the augmented target distribution

$$\bar{\pi}(\eta, u) \propto \exp\left(-a^2(\eta - c)^2\right) \mathbb{I}\left(0 < u < \frac{\eta}{1+\eta^2}\right).$$

Within the general framework of slice sampling, we iteratively perform the following two steps:

1. Draw from $u \mid \eta \sim \text{Uniform}\left(0, \frac{\eta}{1+\eta^2}\right)$.
2. Draw from $\bar{\pi}(\eta \mid u) \propto \exp\left(-a^2(\eta - c)^2\right) \mathbb{I}\left(\eta_1 < \eta < \eta_2\right)$, where

$$\eta_1 = \frac{1 - \sqrt{1-4u^2}}{2u}, \eta_2 = \frac{1 + \sqrt{1-4u^2}}{2u}.$$

   $\eta \mid u$ is a normal distribution truncated between η_1 and η_2.

2.2 Rejection sampler

To construct a rejection sampler for the unnormalized target density

$$f(\eta) = \frac{\eta}{1+\eta^2} \exp\left(-a^2(\eta - c)^2\right), \tag{3}$$

we need to find a function g that upper bounds f. The efficiency of the rejection sampler depends on the proximity between f and g. To enhance this, a common approach is to employ a piecewise proposal density the closely mirrors the form of f.

We first give some observation based on (3) that could be instrumental in carfting our rejection sampler:

- $g(\eta) = \frac{\eta}{1+\eta^2}$ can be sampled via the inverse transform method.
- When $\eta \to +\infty$, $\frac{\eta}{1+\eta^2} \approx \frac{1}{\eta} \to 0$, $\exp\left(-a^2(\eta - c)^2\right) \to 0$. Yet the exponential term decays much faster. Thus, a truncated normal density might be an ideal candidate to upper bound the tail. We use the R package `truncnorm` to generate truncated normal random variables.
- For $\eta > c$, there is a simple trick to sample from the density

$$g(\eta) = \eta \exp\left(-a^2(\eta - c)^2\right).$$

  Rewrite it as

$$g(\eta) = (\eta - c) \exp\left(-a^2(\eta - c)^2\right) + c \exp\left(-a^2(\eta - c)^2\right),$$

  then with probability $\frac{1}{1+ac\sqrt{\pi}}$ we sample from a transformed Chi distribution, and with probability $\frac{ac\sqrt{\pi}}{1+ac\sqrt{\pi}}$ we sample from a truncated normal distribution.

We partition the area $\{a > 0, c \in \mathbb{R}\}$ into eight distinct regimes, each with a corresponding rejection sampler:

1. $\{0 < a < 1, 0 \le c < \frac{1}{a}\}$
2. $\{0 < a < 0.25, c \ge \frac{1}{a}\}$
3. $\{0.25 \le a < 1, c \ge \frac{1}{a}\} \cup \{a \ge 1, c \ge 1\}$
4. $\{a \ge 1, 0.5 \le c < 1\}$
5. $\{a \ge 1, 0 \le c < 0.5\}$
6. $\{0 < a < 1, \max\{-\frac{1}{a}, -\frac{1}{2a^2}\} \le c < 0\}$
7. $\{0 < a \le 1, c < -\frac{1}{2a^2}\} \cup \{a \ge 1\}$
8. $\{0 < a < \frac{1}{2}, -\frac{1}{2a^2} \le c < -\frac{1}{a}\}$

2.2.1 Regime $\{0 < a < 1, 0 \le c < \frac{1}{a}\}$

In this regime, we consider the following proposal density:

$$g(\eta) = \begin{cases} \frac{\eta}{1+\eta^2} & \text{if } 0 \le \eta < m \\ \frac{a}{2} \exp\left(-a^2(\eta - c)^2\right) & \text{if } \eta \ge m \end{cases},$$

where $m = \frac{1+\sqrt{1-a^2}}{a}$.

Empirically, the acceptance rate is more than $\exp(-1)$. Figure 1 shows how the acceptance rate changes as a and c changes.

2.2.2 Regime $\{0 < a < 0.25, c \ge \frac{1}{a}\}$

In this regime, we consider the following proposal density:

$$g(\eta) = \begin{cases} \frac{\eta}{1+\eta^2} \exp\left(-\frac{9}{16}a^2c^2\right) & \text{if } 0 \le \eta < m_1 \\ \frac{m_1}{1+m_1^2} \exp\left(-a^2(\eta - c)^2\right) & \text{if } m_1 \le \eta < m_2 \\ \frac{m_2}{1+m_2^2} \exp\left(-a^2(\eta - c)^2\right) & \text{if } m_2 \le \eta < m_3 \\ \frac{m_3}{1+m_3^2} \exp\left(-a^2(\eta - c)^2\right) & \text{if } \eta \ge m_3 \end{cases},$$

where $m_1 = \frac{c}{4}, m_2 = \frac{c}{2}, m_3 = c$.

Analysis of the acceptance rate. For a given a, the acceptance rate converges to 0.67 as $c \to \infty$.

2.2.3 Regime $\{0.25 \le a < 1, c \ge \frac{1}{a}\} \cup \{a \ge 1, c \ge 1\}$

In this regime, we consider the following proposal density:

$$g(\eta) = \begin{cases} \frac{1}{2} \exp\left(-a^2(\eta - c)^2\right) & \text{if } 0 \le \eta < m \\ \frac{m}{1+m^2} \exp\left(-a^2(\eta - c)^2\right) & \text{if } \eta \ge m \end{cases},$$

where $m = \max\{1, c - \frac{\log c}{\sqrt{2}a}\}$.

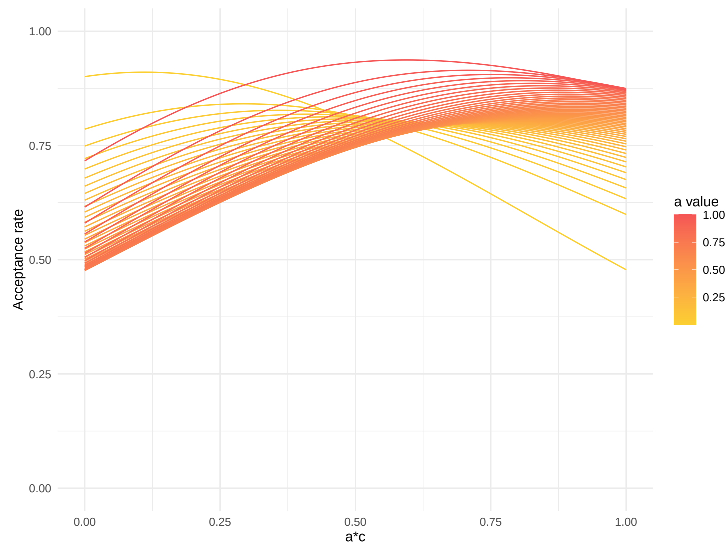

Figure 1: Acceptance rate for different values of a and $a \cdot c$ in regime $\{10^{-4} < a < 1, 0 \leq c < \frac{1}{a}\}$.

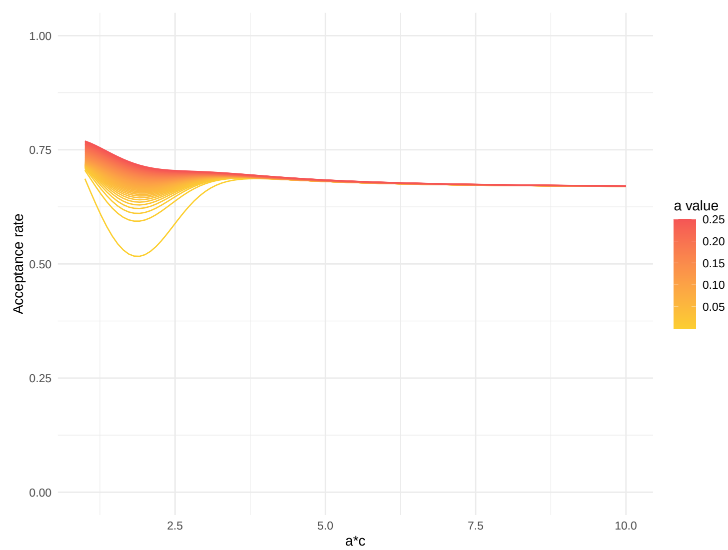

Figure 2: Acceptance rate for different values of a and $a \cdot c$ in regime $\{10^{-4} < a < 0.25, c \geq \frac{1}{a}\}$.

2.2.4 Regime $\{a \geq 1, 0.5 \leq c < 1\}$

In this regime, consider the following proposal density:

$$g(\eta) = \frac{1}{2} \exp\left(-a^2(\eta - c)^2\right).$$

2.2.5 Regime $\{a \geq 1, 0 \leq c < 0.5\}$

In this regime, we consider the following proposal density:

$$g(\eta) = \begin{cases} \frac{c}{1+c^2} \exp\left(-a^2(\eta - c)^2\right) & \text{if } 0 \leq \eta < c \\ \eta \exp\left(-a^2(\eta - c)^2\right) & \text{if } \eta \geq c \end{cases}.$$

2.2.6 Regime $\{0 < a < 1, \max\{-\frac{1}{a}, -\frac{1}{2a^2}\} \leq c < 0\}$

In this regime, we consider the following proposal density:

$$g(\eta) = \begin{cases} \frac{\eta}{1+\eta^2} \exp\left(-a^2 c^2\right) & \text{if } 0 \leq \eta < m \\ \frac{a}{2} \exp\left(-a^2(\eta - c)^2\right) & \text{if } \eta \geq m \end{cases},$$

where $m = \frac{1+\sqrt{1-a^2}}{a}$.

2.2.7 Regime $\{0 < a \leq 1, c < -\frac{1}{2a^2}\} \cup \{a \geq 1\}$

On the precision scale $x = \eta^2$, the target density can also be formulated as

$$\pi(x) \propto f(x) = \frac{1}{1+x} \exp\left(-\beta x + \alpha\sqrt{x}\right),$$

where $\beta = a^2, \alpha = 2a^2 c$. Then the regime $\{0 < a \leq 1, c < -\frac{1}{2a^2}\} \cup \{a \geq 1\}$ is equivalent to $\{\alpha \leq -1\} \cup \{\beta \geq 1\}$. Denote

$$h(x) = \log f(x) = -\log(1+x) - \beta x + \alpha\sqrt{x},$$

and we exploit the convexity of $h(x)$ to devise a rejection sampler. Let

$$\tilde{g}(x) = \begin{cases} \alpha\sqrt{x} & \text{if } x < \frac{\alpha^2}{4\beta^2} \\ h(\frac{\alpha^2}{4\beta^2}) - \beta(x - \frac{\alpha^2}{4\beta^2}) & \text{if } x \geq \frac{\alpha^2}{4\beta^2} \end{cases},$$

which upper bounds $h(x)$. Then $g(x) = \exp(\tilde{g}(x))$ upper bounds $f(x)$. Note that to sample $X \sim \exp(\alpha\sqrt{x})$ on $[0, \frac{\alpha^2}{4\beta^2}]$, we first generate a truncated gamma random variable $Y \sim \mathcal{G}(2, -\alpha)$ on $[0, \frac{-\alpha}{2\beta}]$, and then take $X = Y^2$.

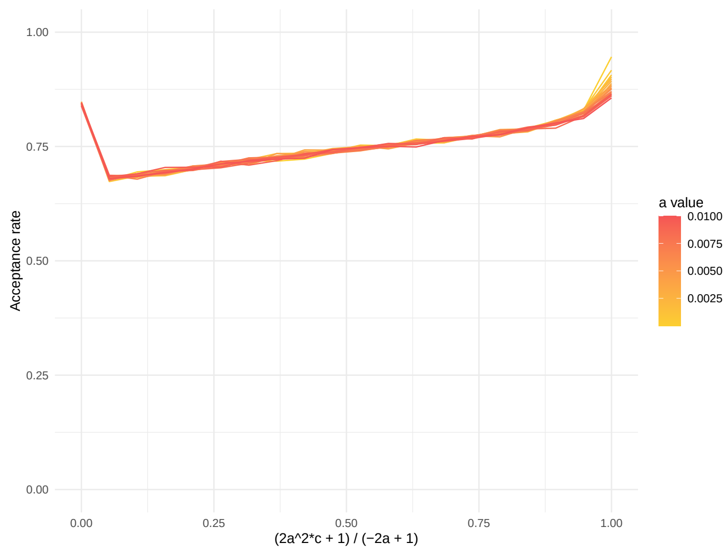

Figure 3: Acceptance rate in regime $\{10^{-5} \le a \le 10^{-2}, -\frac{1}{2a^2} \le c < -\frac{1}{a}\}$ obtained via simulation. The x-lab is $(c + \frac{1}{2a^2})/(-\frac{1}{a} + \frac{1}{2a^2})$.

2.2.8 Regime $\{0 < a < \frac{1}{2}, -\frac{1}{2a^2} \le c < -\frac{1}{a}\}$

Based on the target density (3), we can construct the following proposal:

$$g(\eta) = \begin{cases} \frac{\eta}{1+\eta^2} \exp(-a^2c^2) & \text{if } x < m_1 \\ \frac{1}{1+\eta^2} m \exp(-a^2(m-c)^2) & \text{if } m_1 \le m_2 \\ \frac{m_2}{1+m_2^2} \exp(-a^2(\eta-c)^2) & \text{if } x \ge m_2 \end{cases},$$

where

$$\begin{aligned} m &= \frac{ac + \sqrt{a^2c^2+2}}{2a}, \\ m_1 &= m \exp(-a^2(m^2 - 2mc)), \\ m_2 &= \max\{c + \frac{1}{2a^2}, 1\}. \end{aligned}$$

m is chosen as the mode of $\eta \exp(-a^2(\eta - c)^2)$, m_1 is where $\frac{\eta}{1+\eta^2} \exp(-a^2c^2)$ and $\frac{1}{1+\eta^2} m \exp(-a^2(m-c)^2)$ intersects, and $(-a^2(\eta-c)^2)' \mid_{\eta=m_2} = -1$.