Final project of Numerical Linear Algebra

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1 Introduction

This report is organized as follows: Section 2 describes how to use MAC scheme to discretize the Stokes equation and to attain the Saddle Point Problem. In Section 3, we introduce two types of DGS: DGS in parallel [1] and DGS in sequence (in the notes); we describe the routine of V-Cycle multi-grid method. Section 4 introduces Uzawa Iteration, Inexact Uzawa Iteration and Inexact Uzawa Iteration based on V-Cycle multi-grid. We rigorously prove that Exact Uzawa Iteration will converge in at most 2 iteration under fair assumption. The numerical results is shown in Section 5. In Appendix A, we modify V-Cycle multi-grid method based on DGS a little bit and achieve great improvement on the original one.

2 Description of the background

For 2-dimensional Stokes Equation in the region $\Omega = (0,1)^2$

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{F}, \\ -\nabla \cdot \mathbf{u} &= 0. \end{aligned} \tag{1}$$

where $\mathbf{u} = (u, v)$ is the velocity, p is the pressure, $\mathbf{F} = (f, g)$ is the external force.

The boundary condition is given by

$$\begin{aligned} &\left.\frac{\partial u}{\partial \nu}\right|_{y=0} = b, \quad \left.\frac{\partial u}{\partial \nu}\right|_{y=1} = t, \\ &\left.\frac{\partial v}{\partial \nu}\right|_{x=0} = l, \quad \left.\frac{\partial v}{\partial \nu}\right|_{x=1} = r, \\ &u|_{x=0} = u|_{x=1} = 0, \quad v|_{y=0} = v|_{y=1} = 0, \end{aligned} \tag{2}$$

where ν is the outer normal vector.

We consider the external force to be specifies as

$$\begin{aligned} f(x,y) &= -4\pi^2(2\cos(2\pi x)-1)\sin(2\pi y)+x^2, \\ g(x,y) &= 4\pi^2(2\cos(2\pi y)-1)\sin(2\pi x). \end{aligned} \tag{3}$$

The true solution is given by

$$\begin{aligned} u(x,y) &= (1-\cos(2\pi x))\sin(2\pi y), \\ v(x,y) &= -(1-\cos(2\pi y))\sin(2\pi x), \\ p(x,y) &= \frac{x^3}{3}-\frac{1}{12}. \end{aligned} \tag{4}$$

Based on the true solution, we can calculate the boundary condition as follows:

$$\begin{aligned} b(x) = -\left.\frac{\partial u}{\partial y}\right|_{y=0} &= -2\pi(1-\cos(2\pi x)), \\ t(x) = \left.\frac{\partial u}{\partial y}\right|_{y=1} &= 2\pi(1-\cos(2\pi x)), \\ l(x) = -\left.\frac{\partial v}{\partial x}\right|_{x=0} &= 2\pi(1-\cos(2\pi y)), \\ r(x) = \left.\frac{\partial v}{\partial x}\right|_{x=1} &= -2\pi(1-\cos(2\pi y)). \end{aligned} \tag{5}$$

For given scale of lattice N, and the step size $h = 1/N$, we can discretize this PDE. We define

$$\begin{aligned} &u_{i,j-\frac{1}{2}} \approx u(ih,(j-\frac{1}{2})h), && 0 \le i \le N, 1 \le j \le N, \\ &v_{i-\frac{1}{2},j} \approx u((i-\frac{1}{2})h, jh), && 1 \le i \le N, 0 \le j \le N, \\ &p_{i-\frac{1}{2},j-\frac{1}{2}} \approx p((i-\frac{1}{2})h,(j-\frac{1}{2})h), && 1 \le i \le N, 1 \le j \le N, \\ &f_{i,j-\frac{1}{2}} = f(ih,(j-\frac{1}{2})h), && 1 \le i \le N-1, 1 \le j \le N, \\ &g_{i-\frac{1}{2},j} = g((i-\frac{1}{2})h, jh), && 1 \le i \le N, 1 \le j \le N-1, \\ &b_i = b(ih), \quad t_i = t(ih), && 1 \le i \le N-1, \\ &l_j = l(jh), \quad r_j = r(jh), && 1 \le j \le N-1. \end{aligned} \tag{6}$$

Here we use $\approx$ to indicate that $u_{i,j-\frac{1}{2}}, v_{i-\frac{1}{2},j}, p_{i-\frac{1}{2},j-\frac{1}{2}}$ are the numerical solution on the corresponding site instead of the true solution. Our notation here is slightly different from the notation in the assignment.

By the boundary condition, we have $u_{0,j+\frac{1}{2}} = u_{N,j+\frac{1}{2}} = v_{i+\frac{1}{2},0} = v_{i+\frac{1}{2},N} = 0$. For the discrete form of u, v, we have $N(N-1)$ free variables respectively. For the discrete form of p, we have N^2 free variables. We denote $U \in \mathbb{R}^{(N-1)\times N}$ with $U_{i,j} = u_{i,j-\frac{1}{2}}$, $V \in \mathbb{R}^{N\times(N-1)}$ with $V_{i,j} = v_{i-\frac{1}{2},j}$, $P \in \mathbb{R}^{N\times N}$ with $P_{i,j} = p_{i-\frac{1}{2},j-\frac{1}{2}}$, $F \in \mathbb{R}^{(N-1)\times N}$ with $F_{i,j} = f_{i,j-\frac{1}{2}}$ and $G \in \mathbb{R}^{N\times(N-1)}$ with $G_{i,j} = g_{i-\frac{1}{2},j}$.

2.1 Definitions and notations

Here we state some useful definition and notations.

Definition 1 For a matrix $A \in \mathbb{R}^{m\times n}$, we define two mappings $\phi, \psi : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn\times 1}$. $\phi(A)$ is with entry $\phi(A)_{i+m(j-1)} = A_{i,j}$. $\psi(A)$ is with entry $\psi(A)_{n(i-1)+j} = A_{i,j}$. $A_{i,.}$ represents the i-th row of A; $A_{.,j}$ represents the j-th column of A.

From the above definition, we find that

$$\psi(A) = \phi(A^T), \tag{7}$$

$$\phi(A) = \begin{bmatrix} A_{.,1} \\ \vdots \\ A_{.,n} \end{bmatrix}, \quad \psi(A) = \begin{bmatrix} A_{1,.} \\ \vdots \\ A_{n,.} \end{bmatrix}. \tag{8}$$

Suppose $a_1, a_2 \dots a_n \in \mathbb{R}^{n\times 1}$. Then,

$$\phi(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}) = [a_1, a_2, \dots a_n], \quad \psi(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}) = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}. \tag{9}$$

We use $\otimes$ to represent Kronecker's product.

2.2 MAC scheme

We use MAC scheme to discretize 1.

We denote $T_{N-1} \in \mathbb{R}^{(N-1)\times(N-1)}$ with entries

$$T_{N-1} = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}, \tag{10}$$

$S_{N-1} \in \mathbb{R}^{(N-1)\times N}$ with entries

$$S_{N-1} = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}. \tag{11}$$

$S^{(2)}_{N-1} = S^T_{N-1} S_{N-1} \in \mathbb{R}^{(N-1)\times N}$. We can verify that $S^{(2)}_{N-1}$ is with entries

$$S^{(2)}_{N-1} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \tag{12}$$

We start with u. For $1 \le i \le N-1, 2 \le j \le N-1$, we have

$$-\frac{1}{h^2}(u_{i+1,j-\frac{1}{2}} + u_{i-1,j-\frac{1}{2}} + u_{i,j+\frac{1}{2}} + u_{i,j-\frac{3}{2}} - 4u_{i,j-\frac{1}{2}}) + \frac{1}{h}(p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = f_{i,j-\frac{1}{2}}. \tag{13}$$

Stacking 13 with $1 \le i \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}U_{,.j} + 2U_{,.j} - U_{,.j-1} - U_{,.j+1}\right) + \frac{1}{h}S_{N-1}P_{.,j} = F_{.,j}. \tag{14}$$

For $1 \le i \le N-1, j = 1$, we have

$$-\frac{1}{h^2}(u_{i+1,j-\frac{1}{2}} + u_{i-1,j-\frac{1}{2}} + u_{i,j+\frac{1}{2}} - 3u_{i,j-\frac{1}{2}} + hb_i) + \frac{1}{h}(p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = f_{i,j-\frac{1}{2}}. \tag{15}$$

Stacking 15 with $1 \le i \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}U_{,.j} + U_{,.j} - U_{,.j+1} - hb\right) + \frac{1}{h}S_{N-1}P_{.,j} = F_{.,j}. \tag{16}$$

For $1 \le i \le N-1, j = N$, we have

$$-\frac{1}{h^2}(u_{i+1,j-\frac{1}{2}} + u_{i-1,j-\frac{1}{2}} + u_{i,j-\frac{3}{2}} - 3u_{i,j-\frac{1}{2}} + ht_i) + \frac{1}{h}(p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = f_{i,j-\frac{1}{2}}. \tag{17}$$

Stacking 17 with $1 \le i \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}U_{,.j} + U_{,.j} - U_{,.j-1} - ht\right) + \frac{1}{h}S_{N-1}P_{.,j} = F_{.,j}. \tag{18}$$

We denote $A_N \in \mathbb{R}^{N(N-1)\times N(N-1)}$, $B_N^{(1)} \in \mathbb{R}^{N(N-1)\times N^2}$ and $F^{(bt)} \in \mathbb{R}^{N(N-1)\times 1}$

$$A_N = \begin{bmatrix} I_{N-1}+T_{N-1} & -I_{N-1} & & & \\ -I_{N-1} & 2I_{N-1}+T_{N-1} & -I_{N-1} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2I_{N-1}+T_{N-1} & -I_{N-1} \\ & & & -I_{N-1} & I_{N-1}+T_{N-1} \end{bmatrix}, \tag{19}$$

$$B_N^{(1)} = \begin{bmatrix} S_{N-1} & & \\ & \ddots & \\ & & S_{N-1} \end{bmatrix} = I_N \otimes S_{N-1}, \quad F^{(bt)} = \phi(F) + \frac{1}{h}\begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{1}{h}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ t \end{bmatrix}. \tag{20}$$

Indeed, $A_N = S_{N-1}^{(2)} \otimes I_{N-1} + I_N \otimes T_{N-1}$.

Therefore, we can combine 14, 16, 18 together in the matrix form:

$$\frac{1}{h^2}A_N\phi(U) + \frac{1}{h}B_N^{(1)}\phi(P) = F^{(bt)}. \tag{21}$$

Then, we deal with v. We denote $e_N^{(i)} \in \mathbb{R}^{N\times 1}$ to be the vector with i-th entry is 1 and other entries are 0. We denote $R_N^{(i)} \in \mathbb{R}^{(N-1)\times N^2}$.

$$R_N^{(i)} = \begin{bmatrix} -(e_N^{(i)})^T & (e_N^{(i)})^T & & \\ & \ddots & \ddots & \\ & & -(e_N^{(i)})^T & (e_N^{(i)})^T \end{bmatrix} = S_{N-1} \otimes (e_N^{(i)})^T. \tag{22}$$

It is easy to verify that

$$R_N^{(i)}\phi(P) = \begin{bmatrix} -(e_N^{(i)})^T & (e_N^{(i)})^T & & \\ & \ddots & \ddots & \\ & & -(e_N^{(i)})^T & (e_N^{(i)})^T \end{bmatrix}\begin{bmatrix} P_{.,1} \\ \vdots \\ P_{.,n} \end{bmatrix} = \begin{bmatrix} P_{i,2} - P_{i,1} \\ \vdots \\ P_{i,N} - P_{i,N-1} \end{bmatrix}. \tag{23}$$

For $2 \le i \le N-1, 1 \le j \le N-1$, we have

$$-\frac{1}{h^2}(v_{i-\frac{1}{2},j+1} + v_{i-\frac{1}{2},j-1} + v_{i+\frac{1}{2},j} + v_{i-\frac{3}{2},j} - 4v_{i-\frac{1}{2},j}) + \frac{1}{h}(p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = g_{i-\frac{1}{2},j}. \quad (24)$$

Stacking 24 with $1 \le j \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}V_{i,\cdot}^T + 2V_{i,\cdot} - V_{i+1,\cdot}^T - V_{i-1,\cdot}^T\right) + R_N^{(i)}\phi(P) = G_{i,\cdot}^T. \quad (25)$$

For $i = 1, 1 \le j \le N-1$, we have

$$-\frac{1}{h^2}(v_{i-\frac{1}{2},j+1} + v_{i-\frac{1}{2},j-1} + v_{i+\frac{1}{2},j} - 3v_{i-\frac{1}{2},j} + hl_j) + \frac{1}{h}(p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = g_{i-\frac{1}{2},j}. \quad (26)$$

Stacking 26 with $1 \le j \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}V_{i,\cdot}^T + V_{i,\cdot} - V_{i+1,\cdot}^T - hl\right) + R_N^{(i)}\phi(P) = G_{i,\cdot}^T. \quad (27)$$

For $i = N, 1 \le j \le N-1$, we have

$$-\frac{1}{h^2}(v_{i-\frac{1}{2},j+1} + v_{i-\frac{1}{2},j-1} + v_{i-\frac{3}{2},j} - 3v_{i-\frac{1}{2},j} + hr_j) + \frac{1}{h}(p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}) = g_{i-\frac{1}{2},j}. \quad (28)$$

Stacking 28 with $1 \le j \le N-1$, we have

$$\frac{1}{h^2}\left(T_{N-1}V_{i,\cdot}^T + 2V_{i,\cdot} - V_{i+1,\cdot}^T - V_{i-1,\cdot}^T - hr\right) + R_N^{(i)}\phi(P) = G_{i,\cdot}^T. \quad (29)$$

We denote $B_N^{(2)} \in \mathbb{R}^{N(N-1)\times N^2}$ and $G^{(lr)} \in \mathbb{R}^{N(N-1)\times 1}$.

$$B_N^{(2)} = \begin{bmatrix} R_N^{(1)} \\ \vdots \\ R_N^{(N)} \end{bmatrix}, \quad G^{(lr)} = \psi(G) + \frac{1}{h}\begin{bmatrix} l \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{1}{h}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ r \end{bmatrix}. \quad (30)$$

Therefore, we can combine 25, 27, 29 together in the following matrix form:

$$A_N\psi(V) + B_N^{(2)}\phi(P) = G^{(lr)}. \quad (31)$$

And we also have the following linear constraints from $\nabla \cdot \mathbf{u} = 0$. For $1 \le i \le N$ and

$1 \le j \le N$, we have

$$-\frac{1}{h}(u_{i,j-\frac{1}{2}} - u_{i-1,j-\frac{1}{2}}) - \frac{1}{h}(v_{i-\frac{1}{2},j} - v_{i-\frac{1}{2},j-1}) = 0. \tag{32}$$

We observe that

$$(B_N^{(1)})^T \phi(U) = \begin{bmatrix} S_{N-1}^T & & \\ & \ddots & \\ & & S_{N-1}^T \end{bmatrix} \begin{bmatrix} U_{.,1} \\ \vdots \\ U_{.,N} \end{bmatrix} = \begin{bmatrix} S_{N-1}^T U_{.,1} \\ \vdots \\ S_{N-1}^T U_{.,N} \end{bmatrix}, \tag{33}$$

where

$$S_{N-1}^T U_{.,j} = \begin{bmatrix} -1 & & \\ 1 & \ddots & \\ & \ddots & -1 \\ & & 1 \end{bmatrix} \begin{bmatrix} U_{1,j} \\ \vdots \\ U_{N-1,j} \end{bmatrix} = \begin{bmatrix} U_{0,j} - U_{1,j} \\ \vdots \\ U_{N-1,j} - U_{N,j} \end{bmatrix}. \tag{34}$$

Therefore, we have

$$(B_N^{(1)})^T \phi(U) = \sum_{j=1}^{N} e_N^{(j)} \otimes \begin{bmatrix} U_{0,j} - U_{1,j} \\ \vdots \\ U_{N-1,j} - U_{N,j} \end{bmatrix} \tag{35}$$

We also observe that

$$\begin{aligned} (B_N^{(2)})^T \psi(V) &= \left[(R_N^{(1)})^T, \dots (R_N^{(N)})^T\right] \begin{bmatrix} (V_{1,.})^T \\ \vdots \\ (V_{N,.})^T \end{bmatrix} \\ &= \left[S_{N-1}^T \otimes e_N^{(1)}, \dots S_{N-1}^T \otimes e_N^{(N)}\right] \begin{bmatrix} (V_{1,.})^T \\ \vdots \\ (V_{N,.})^T \end{bmatrix} = \sum_{i=1}^{N} \left(S_{N-1}^T \otimes e_N^{(i)}\right) (V_{i,.})^T, \end{aligned} \tag{36}$$

where

$$\left(S_{N-1}^T \otimes e_N^{(i)}\right) (V_{i,.})^T = \begin{bmatrix} -e_N^{(i)} & & \\ e_N^{(i)} & \ddots & \\ & \ddots & -e_N^{(i)} \\ & & e_N^{(i)} \end{bmatrix} \begin{bmatrix} V_{i,1} \\ \vdots \\ V_{i,N} \end{bmatrix} = \begin{bmatrix} V_{i,0} - V_{i,1} \\ \vdots \\ V_{i,N-1} - V_{i,N} \end{bmatrix} \otimes e_N^{(i)}. \tag{37}$$

Therefore,

$$(B_N^{(2)})^T\psi(V) = \sum_{i=1}^{N} \begin{bmatrix} V_{i,0} - V_{i,1} \\ \vdots \\ V_{i,N-1} - V_{i,N} \end{bmatrix} \otimes e_N^{(i)}. \tag{38}$$

Stacking 32 from $1 \le i \le N$ and $1 \le j \le N$, we get

$$\frac{1}{h}\left(B_N^{(1)}\right)^T \phi(U) + \frac{1}{h}\left(B_N^{(2)}\right)^T \psi(V) = 0. \tag{39}$$

In summary, we denote $\mathbf{A} \in \mathbb{R}^{2N(N-1)\times 2N(N-1)}$, $\mathbf{B} \in \mathbb{R}^{2N(N-1)\times N^2}$, $\mathbf{U} \in \mathbb{R}^{2N(N-1)\times 1}$, $\mathbf{P} \in \mathbb{R}^{N^2\times 1}$, $\mathbf{F} \in \mathbb{R}^{2N(N-1)\times 1}$

$$\mathbf{A} = \frac{1}{h^2}\begin{bmatrix} A_N & \\ & A_N \end{bmatrix}, \mathbf{B} = \frac{1}{h}\begin{bmatrix} B_N^{(1)} \\ B_N^{(2)} \end{bmatrix}, \mathbf{U} = \begin{bmatrix} \phi(U) \\ \psi(V) \end{bmatrix}, \mathbf{P} = \phi(P), \mathbf{F} = \begin{bmatrix} F^{(bt)} \\ G^{(lr)} \end{bmatrix}. \tag{40}$$

where A_N is defined in 19; $B_N^{(1)}$ and $F^{(bt)}$ is defined in 20; $B_N^{(2)}$ and $G^{(lr)}$ is defined in 30.

Then we derive the following Saddle Point Problem as the discretization of Stokes equation 1 with boundary conditions.

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ 0 \end{bmatrix}. \tag{41}$$

We can also rescale 41 by introducing $\tilde{\mathbf{A}} = h^2\mathbf{A}, \tilde{\mathbf{B}} = h^2\mathbf{B}$ and $\tilde{\mathbf{F}} = h^2\mathbf{F}$. Then, 41 turns to be

$$\begin{bmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{B}}^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{F}} \\ 0 \end{bmatrix}. \tag{42}$$

In our numerical experiment, the residual is for 42, instead of for 41.

3 V-Cycle Multi-Grid Based on DGS (Problem 1)

We consider to use V-cycle multi-grid method to solve 41. We use Distributive Gauss-Seidel (DGS) Iteration as smoother. Before we state the algorithm, we make several preparations: we calculate $\mathbf{B}^T\mathbf{B}$. Actually,

$$\mathbf{B}^T\mathbf{B} = \frac{1}{h^2}\left(\left(\mathbf{B}^{(1)}\right)^T\mathbf{B}^{(1)} + (\mathbf{B}^{(2)})^T\mathbf{B}^{(2)}\right). \tag{43}$$

From 12 and 20, $(\mathbf{B}^{(1)})^T\mathbf{B}^{(1)} = I_N \otimes (S_{N-1}^T S_{N-1}) = I_N \otimes (S_{N-1}^{(2)})$. From 30 and 22,

$$\begin{aligned}\left(\mathbf{B}^{(2)}\right)^T \mathbf{B}^{(2)} &= \sum_{i=1}^{N} \left(R_N^{(i)}\right)^T R_N^{(i)} = \sum_{i=1}^{N} \left(S_{N-1} \otimes \left(e_N^{(i)}\right)^T\right)^T \left(S_{N-1} \otimes \left(e_N^{(i)}\right)^T\right) \\ &= \sum_{i=1}^{N} S_{N-1}^{(2)} \otimes \left(e_N^{(i)} \left(e_N^{(i)}\right)^T\right) = S_{N-1}^{(2)} \otimes I_N.\end{aligned} \tag{44}$$

If we denote

$$b^{(1)} = [2, 3, \ldots, 3, 2], \quad b^{(2)} = [3, 4, \ldots, 4, 3], \tag{45}$$

then, we can write $\mathbf{B}^T\mathbf{B}$ explicitly:

$$\mathbf{B}^T\mathbf{B} = \frac{1}{h^2}\begin{bmatrix} \mathrm{diag}(b^{(1)}) & -I_N & & & \\ -I_N & \mathrm{diag}(b^{(2)}) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \mathrm{diag}(b^{(2)}) & -I_N \\ & & \ddots & -I_N & \mathrm{diag}(b^{(1)}) \end{bmatrix}. \tag{46}$$

We then introduce a diagonal matrix $\tilde{\mathbf{D}} \in \mathbb{R}^{N^2,N^2}$ as the inverse of the diagonal part of $\mathbf{B}^T\mathbf{B}$, i.e.

$$\mathbf{D} = h^2 \mathrm{diag}([d^{(1)}, d^{(2)}, \ldots d^{(2)}, d^{(1)}]), \tag{47}$$

where $d^{(1)}, d^{(2)} \in \mathbf{R}^N$ with entries

$$d^{(1)} = [1/2, 1/3, \ldots, 1/3, 1/2], \quad d^{(2)} = [1/3, 1/4, \ldots, 1/4, 1/3]. \tag{48}$$

For $1 \leq i, j \leq N$, we can classify (i, j) into three groups:

- Inner unit: $2 \leq i \leq N-1$ and $2 \leq j \leq N-1$;
- Edge unit: $i = 1, N$, $2 \leq j \leq N-1$ or $2 \leq i \leq N-1$, $j = 1, N$;
- Point unit: $i = 1, N$, $j = 1, N$.

Given a matrix $B' \in \mathbb{R}^{N\times N}$, if we let its inner unit be $1/4$, edge unit be $1/3$ and point unit be $1/2$. Then, we directly have

$$\frac{1}{h^2}\mathrm{diag}(\mathbf{D}) = \phi(B'). \tag{49}$$

3.1 DGS Iteration

Suppose the initial values are $\mathbf{U}^{(0)} = \begin{bmatrix} \psi(U^{(0)}) \\ \phi(V^{(0)}) \end{bmatrix}$ and $\mathbf{P}^{(0)} = \psi(\mathbf{P}^{(0)})$. Let $k = 0$, $\mathbf{A} = D_{\mathbf{A}} - L_{\mathbf{A}} - U_{\mathbf{A}}$. The DGS iteration is defined as follows:

Given $\mathbf{U}^{(k)}$ and $\mathbf{P}^{(k)}$, we first use Gauss-Seidel Iteration to update velocity $\mathbf{U}^{(k+1/2)}$,

$$\mathbf{U}^{(k+1/2)} = \mathbf{U}^{(k)} + (D_{\mathbf{A}} - L_{\mathbf{A}})^{-1}(F - \mathbf{B}\mathbf{P}^{(k)} - \mathbf{A}\mathbf{U}_k). \tag{50}$$

We denote $Q \in \mathbb{R}^{N\times N}$ and write $\mathbf{Q} = \phi(Q)$, then calculate $\mathbf{Q}$ as follows:

$$\mathbf{Q} = \mathbf{D}(-\mathbf{B}^T\mathbf{U}^{(k+1/2)}). \tag{51}$$

Finally, we update $\mathbf{U}^{(k+1)}$ and $\mathbf{P}^{(k+1)}$ by

$$\mathbf{U}^{(k+1)} = \mathbf{U}^{(k+1/2)} + \mathbf{B}\mathbf{Q}, \quad \mathbf{P}^{(k+1)} = \mathbf{P}^{(k)} - \mathbf{B}^T\mathbf{B}\mathbf{Q} \tag{52}$$

This update routine is introduced in [1]. We shall point out that this is quite similar to the DGS iteration in the notes. Nevertheless, this routine is performed in parallel while routine on the notes is performed in sequence.

Here we show that update routine in [1] has only minor differences to DGS (notes) if we update DGS (notes) in parallel. We denote $D' \in \mathbb{R}^{N\times N}$ with 0 on its inner unit, 1 on its edge unit and 2 on its point unit. Then we denote $\tilde{\mathbf{D}} \in \mathbb{R}^{N^2\times N^2}$ with entries $\tilde{\mathbf{D}} = \mathrm{diag}(\phi(D'))$. We assert that the update rule of DGS (notes) in parallel can be written in the matrix form as:

- Update $\mathbf{U}^{(k+1/2)}$ by 50.
- Calculate $\mathbf{Q}$ by 51.
- Update $\mathbf{U}^{(k+1)}$ and $\mathbf{P}^{(k+1)}$ by

$$\mathbf{U}^{(k+1)} = \mathbf{U}^{(k+1/2)} + \mathbf{B}\mathbf{Q}, \quad \mathbf{P}^{(k+1)} = \mathbf{P}^{(k)} - (\mathbf{B}^T\mathbf{B} + \tilde{\mathbf{D}})\mathbf{Q}. \tag{53}$$

The minor differences is that DGS (notes) in parallel has an additional $\tilde{\mathbf{D}}$.

For DGS (notes) in parallel, the calculation of $\mathbf{U}^{(k+1/2)}$ is as same as the notes. In the notes, $r_{i,j}$, the residual of the divergence equation 32 is calculated through:

$$r_{i,j} = \frac{u^{(k+1/2)}_{i,j-\frac{1}{2}} - u^{(k+1/2)}_{i-1,j-\frac{1}{2}}}{h} + \frac{v^{(k+1/2)}_{i-\frac{1}{2},j} - v^{(k+1/2)}_{i-\frac{1}{2},j-1}}{h}, \tag{54}$$

We shall point out that 54 is the major difference between updating in sequence and updating in parallel. $r_{i,j}$ in parallel only considers the value of u and v at $(k + 1/2)$ while $r_{i,j}$ in parallel considers the value of u and v at $k + 1$. If we write $R \in \mathbb{R}^{N\times N}$ with entries $r_{i,j}$ and $\mathbf{R} = \phi(R)$, then 54 is equivalent to

$$\mathbf{R} = -\mathbf{B}^T\mathbf{U}^{(k+1/2)}. \tag{55}$$

The calculation of $\delta_{i,j}$ is based on the classification of (i, j):

- (i, j) is an inner unit: $\delta_{i,j} = \frac{h}{4}r_{i,j}$;
- (i, j) is an egde unit: $\delta_{i,j} = \frac{h}{3}r_{i,j}$;
- (i, j) is an point unit: $\delta_{i,j} = \frac{h}{2}r_{i,j}$.

We write $\mathbf{Q}$ as $\mathbf{Q} = \phi(Q)$, where $Q \in \mathbb{R}^{N\times N}$ is with entries $q_{i,j}$. Directly from the definition of $\mathbf{D}$ and 49, we have

$$q_{i,j} = \frac{1}{h}\delta_{i,j}. \tag{56}$$

For the inner unit (i, j), the update of $\mathbf{U}^{(k+1)}$ is given by

$$\begin{aligned} u^{(k+1)}_{i-1,j-\frac{1}{2}} &= u^{(k+1/2)}_{i-1,j-\frac{1}{2}} + \delta_{i,j}, \quad & u^{(k+1)}_{i,j-\frac{1}{2}} &= u^{(k+1/2)}_{i,j-\frac{1}{2}} - \delta_{i,j}, \\ v^{(k+1)}_{i-\frac{1}{2},j-1} &= v^{(k+1/2)}_{i-\frac{1}{2},j-1} + \delta_{i,j}, \quad & v_{i-\frac{1}{2},j} &= v^{(k+1/2)}_{i-\frac{1}{2},j} - \delta_{i,j}. \end{aligned} \tag{57}$$

The update of $\mathbf{P}^{(k+1)}$ is given by

$$p^{(k+1)}_{i-\frac{1}{2},j-\frac{1}{2}} = p^{(k)}_{i-\frac{1}{2},j-\frac{1}{2}} - \frac{4}{h}\delta_{i,j}, \quad p^{(k+1)}_{i+\frac{1}{2},j-\frac{1}{2}} = p^{(k)}_{i+\frac{1}{2},j-\frac{1}{2}} + \frac{1}{h}\delta_{i,j}, \quad p^{(k+1)}_{i-\frac{3}{2},j-\frac{1}{2}} = p^{(k)}_{i-\frac{3}{2},j-\frac{1}{2}} + \frac{1}{h}\delta_{i,j}, \tag{58}$$

$$p^{(k+1)}_{i-\frac{1}{2},j+\frac{1}{2}} = p^{(k)}_{i-\frac{1}{2},j+\frac{1}{2}} + \frac{1}{h}\delta_{i,j}, \quad p^{(k+1)}_{i-\frac{1}{2},j-\frac{3}{2}} = p^{(k)}_{i-\frac{1}{2},j-\frac{3}{2}} + \frac{1}{h}\delta_{i,j}. \tag{59}$$

For edge unit, we take $2 \le i \le N - 1$, $j = N$ for example. The influence of $\delta_{i,N}$ on $\mathbf{U}^{(k+1)}$ is given by

$$u^{(k+1)}_{i-1,N-\frac{1}{2}} = u^{(k+1/2)}_{i-1,N-\frac{1}{2}} + \delta_{i,N},\ u^{(k+1)}_{i,N-\frac{1}{2}} = u^{(k+1/2)}_{i,N-\frac{1}{2}} - \delta_{i,N},\ v^{(k+1)}_{i-\frac{1}{2},N-1} = v^{(k+1/2)}_{i-\frac{1}{2},N-1} + \delta_{i,N}. \tag{60}$$

The update rule of $\mathbf{P}^{(k+1)}$ is given by

$$\begin{aligned} p^{(k+1)}_{i-\frac{1}{2},N-\frac{1}{2}} &= p^{(k)}_{i-\frac{1}{2},N-\frac{1}{2}} - \frac{4}{h}\delta_{i,N}, \quad & p^{(k+1)}_{i+\frac{1}{2},N-\frac{1}{2}} &= p^{(k)}_{i+\frac{1}{2},N-\frac{1}{2}} + \frac{1}{h}\delta_{i,N}, \\ p^{(k+1)}_{i-\frac{3}{2},N-\frac{1}{2}} &= p^{(k)}_{i-\frac{3}{2},N-\frac{1}{2}} + \frac{1}{h}\delta_{i,N}, \quad & p^{(k+1)}_{i-\frac{1}{2},N-\frac{3}{2}} &= p^{(k)}_{i-\frac{1}{2},N-\frac{3}{2}} + \frac{1}{h}\delta_{i,N}. \end{aligned} \tag{61}$$

For DGS [1], we shall have $p^{(k+1)}_{i-\frac{1}{2},N-\frac{1}{2}} = p^{(k)}_{i-\frac{1}{2},N-\frac{1}{2}} - \frac{3}{h}\delta_{i,N}$ instead.

For point unit, we take $i = N, j = N$ for example. The update of $\mathbf{U}^{(k+1)}$ is given by

$$u^{(k+1)}_{N-1,N-\frac{1}{2}} = u^{(k+1/2)}_{N-1,N-\frac{1}{2}} + \delta_{N,N},\ v^{(k+1)}_{N-\frac{1}{2},N-1} = v^{(k+1/2)}_{N-\frac{1}{2},N-1} + \delta_{N,N}. \tag{62}$$

The update of $\mathbf{P}^{(k+1)}$ is given by

$$\begin{aligned} &p^{(k+1)}_{N-\frac{1}{2},N-\frac{1}{2}} = p^{(k)}_{N-\frac{1}{2},N-\frac{1}{2}} - \frac{4}{h}\delta_{N,N},\ p^{(k+1)}_{N-\frac{3}{2},N-\frac{1}{2}} = p^{(k)}_{N-\frac{3}{2},N-\frac{1}{2}} + \frac{1}{h}\delta_{N,N}, \\ &p^{(k+1)}_{N-\frac{1}{2},N-\frac{3}{2}} = p^{(k)}_{N-\frac{1}{2},N-\frac{3}{2}} + \frac{1}{h}\delta_{N,N}. \end{aligned} \tag{63}$$

For DGS [1], we shall have $p^{(k+1)}_{N-\frac{1}{2},N-\frac{1}{2}} = p^{(k)}_{N-\frac{1}{2},N-\frac{1}{2}} - \frac{2}{h}\delta_{N,N}$ instead.

The minor differences between DGS [1] and DGS (notes) in parallel is the coefficient of $\delta_{i,j}$ in updating $p^{(k+1)}_{i-\frac{1}{2},j-\frac{1}{2}}$ for the edge unit and the point unit. That is the reason why we introduce the matrix $\tilde{\mathbf{D}}$. The update rule in the notes can be understood as calculating 53 column by column, i.e.

$$\mathbf{B}\mathbf{Q} = \sum_l \mathbf{B}_l\mathbf{Q}_l, \quad \left(\tilde{\mathbf{D}} + \mathbf{B}^T\mathbf{B}\right)\mathbf{Q} = \sum_l \left(\tilde{\mathbf{D}}\mathbf{B}^T\mathbf{B}\right)_l \mathbf{Q}_l. \tag{64}$$

Here we shall emphasize that $\mathbf{B} \in \mathbb{R}^{2N(N-1)\times N^2}$ and $\mathbf{Q} \in \mathbb{R}^{N^2\times 1}$.

We shall point out that the update date rule on the notes shall be performed sequentially. Namely, we calculate $r_{i,j}$ through our update process. Otherwise, we might face divergence.

3.2 V-Cycle Multi-Grid method

We then introduce the V-Cycle multi-grid method. We first consider the restriction operator.

3.2.1 Restriction

We consider the following matrix $W_N^{(1)} \in \mathbb{R}^{N\times 2N}$,

$$W_N^{(1)} = \begin{bmatrix} 1 & 1 & & & & & & \\ & & 1 & 1 & & & & \\ & & & & \cdots & \cdots & & \\ & & & & & & 1 & 1 \end{bmatrix} = I_N \otimes \begin{bmatrix} 1 & 1 \end{bmatrix}, \tag{65}$$

and $W_N^{(2)} \in \mathbb{R}^{(N-1)\times(2N-1)}$,

$$W_N^{(2)} = \begin{bmatrix} 1 & 2 & 1 & & & & & \\ & & 1 & 2 & 1 & & & \\ & & & & \cdots & \cdots & \cdots & \\ & & & & & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & W_{N-1}^{(1)} \end{bmatrix} + \begin{bmatrix} W_{N-1}^{(1)} & 0 \end{bmatrix}. \tag{66}$$

At the u- and v grid points, we consider six points restrictions, and at p-grid points, a four-point cell-centered restriction. In stencil notations, the restriction operators are (* indicates the position of the coarse-grid point).

$$R_{h/2,h}^{u} = \frac{1}{8}\begin{bmatrix} 1 & 2 & 1 \\ & * & \\ 1 & 2 & 1 \end{bmatrix}, \quad R_{h/2,h}^{v} = \frac{1}{8}\begin{bmatrix} 1 & & 1 \\ 2 & * & 2 \\ 1 & & 1 \end{bmatrix}, \quad R_{h/2,h}^{p} = \frac{1}{4}\begin{bmatrix} 1 & & 1 \\ & * & \\ 1 & & 1 \end{bmatrix}. \tag{67}$$

We can write 67 element-wise

$$U_{i,j}^{h} = \frac{1}{8}(U_{2i-1,2j-1}^{h/2} + 2U_{2i,2j-1}^{h/2} + U_{2i+1,2j-1}^{h/2} + U_{2i-1,2j}^{h/2} + 2U_{2i,2j}^{h/2} + U_{2i+1,2j}^{h/2}). \tag{68}$$

Therefore, the restriction operator for u is given by

$$R_{h/2,h}^{u} = \frac{1}{8}W_N^{(1)} \otimes W_N^{(2)}. \tag{69}$$

Namely, we have

$$\phi(U^h) = R_{h/2,h}^{u}\phi(U^{h/2}). \tag{70}$$

Similarly, for v we have

$$R_{h/2,h}^{v} = R_{h/2,h}^{u}, \quad \psi(V^h) = R_{h/2,h}^{v}\psi(V^{h/2}). \tag{71}$$

For p we have

$$R^{p}_{h/2,h} = \frac{1}{4} W^{(1)} \otimes W^{(1)}, \quad \phi(P^h) = P^{u}_{h,h/2}\phi(V^{h/2}). \tag{72}$$

The restriction of f, g is similar to u, v respectively, i.e., $R^{f}_{h/2,h} = R^{g}_{h/2,h} = R^{u}_{h/2,h}$. We denote

$$R^{U}_{h/2,h} = R^{F}_{h/2,h} = \begin{bmatrix} R^{u}_{h/2,h} & \\ & R^{u}_{h/2,h} \end{bmatrix}. \tag{73}$$

3.2.2 Lifting

The lifting operation corresponds to the transpose of restriction operation. We denote $W^{(3)} \in \mathbb{R}^{2N\times N}$ with $W^{(3)}_N = \left(W^{(1)}_N\right)^T$ and $W^{(4)}_N \in \mathbb{R}^{(2N-1)\times(N-1)}$ with $W^{(4)}_N = \left(W^{(2)}_N\right)^T$.

For u, we consider the following element-wise lifting operation

$$U^{h/2}_{2i,2j-1} = U^{h/2}_{2i,2j} = U^{h}_{i,j}, \quad U^{h/2}_{2i-1,2j-1} = U^{h/2}_{2i,2j} = \frac{1}{2}U^{h}_{i-1,j} + \frac{1}{2}U^{h}_{i,j}. \tag{74}$$

Therefore, we have

$$R^{u}_{h,h/2} = \frac{1}{2} W^{(3)}_N \otimes W^{(4)}_N, \quad R^{u}_{h,h/2}\phi(U^h) = \phi(U^{h/2}). \tag{75}$$

Similarly, we have $R^{v}_{2h,h} = R^{u}_{2h,h}$. For p, we have the following element-wise lifting operation

$$P^{h}_{2i,2j} = P^{h}_{2i,2j-1} = P^{h}_{2i-1,2j} = P^{h}_{2i-1,2j-1} = P^{2h}_{i,j}. \tag{76}$$

Then, we have

$$R^{p}_{h,h/2} = W^{(3)}_N \otimes W^{(3)}_N, \quad R^{p}_{h,h/2}\phi(P^h) = \phi(P^{h/2}). \tag{77}$$

We denote

$$R^{U}_{h,h/2} = \begin{bmatrix} R^{u}_{h,h/2} & \\ & R^{u}_{h,h/2} \end{bmatrix}. \tag{78}$$

We shall point out that $R^{u}_{h,h/2} = 4\left(R^{u}_{h/2,h}\right)^T$ and $R^{p}_{h,h/2} = 4\left(R^{p}_{h/2,h}\right)^T$. This results from that our multi-grid method is designed for the problem 41 which is not rescaled. If we change this multi-grid to the rescaled problem 42, we would have $\tilde{R}^{u}_{h,h/2} = \left(\tilde{R}^{u}_{h/2,h}\right)^T$ and $\left(\tilde{R}_{h,h/2}\right)^p = \left(\tilde{R}^{p}_{h/2,h}\right)^T$

3.3 V-Cycle

We describe the update routine of V-Cycle. We have a stopping criterion ϵ for the V-Cycle and three parameters v_1, v_2 and L for the V-Cycle. We denote k-th grid to be the grid with step size 2^{k-n}. Suppose $N = 2^n, L = 2^l$. We denote $\mathbf{A}^{(k)}, \mathbf{B}^{(k)}$ to be $\mathbf{A}, \mathbf{B}$ in the k-th grid and $\mathbf{F}^{(k)}$ to be the initial residual in the k-th grid, i.e., $\mathbf{F}^{(0)} = \mathbf{F}$.

We start with $k = 0$. We set the initial value $\mathbf{U}^{(k)} = 0, \mathbf{P}^{(k)} = 0$ and apply DGS v_1 times to get an approximate solution $\mathbf{U}^{(k)}$ and $\mathbf{P}^{(k)}$ to

$$\mathbf{A}^{(k)}\mathbf{U}^{(k)} + \mathbf{B}^{(k)}\mathbf{P}^{(k)} = \mathbf{F}^{(k)}, \quad \left(\mathbf{B}^{(k)}\right)^T \mathbf{U}^{(k)} = 0, \tag{79}$$

and record them. Then, we compute the residual $\mathbf{F}_k$

$$\mathbf{F}_k = \mathbf{F}^{(k)} - \left(\mathbf{A}^{(k)}\mathbf{U}^{(k)} + \mathbf{B}^{(k)}\mathbf{P}^{(k)}\right). \tag{80}$$

We let $r_h = h^2\mathbf{F}_k$ as the residual for the rescaled problem 42 and calculate $\|r_h\|_2$. If $\|r_h\| < \epsilon$, we stop the algorithm. Otherwise, we restrict $\mathbf{F}_k$ to $(k+1)$-th grid

$$\mathbf{F}^{(k+1)} = R^F_{2^k h, 2^{k+1} h}\mathbf{F}_k, \tag{81}$$

replace $k = 0$ with $k = 1$ and move to Stage 1.

In Stage 1, we start with $k = 1$. Given $k < n - l$, we set the initial value $\mathbf{U}^{(k)} = 0, \mathbf{P}^{(k)} = 0$ and apply DGS v_1 times to get an approximate solution $\mathbf{U}^{(k)}$ and $\mathbf{P}^{(k)}$ to 79 and record them. Then, we compute the residual $\mathbf{F}_k$ by 80, restrict $\mathbf{F}_k$ onto $(k+1)$-th grid to get $\mathbf{F}^{(k+1)}$ by 81, and replace k with $k + 1$ until $k = n - l$.

For $k = n - l$, we set the initial value $\mathbf{U}^{(k)} = 0, \mathbf{P}^{(k)} = 0$ and apply DGS v_1 times to get an approximate solution $\mathbf{U}^{(k)}$ and $\mathbf{P}^{(k)}$ to 79. We let $\mathbf{U}^{[n-l]} = \mathbf{U}^{(n-l)}$ and $\mathbf{P}^{[n-l]} = \mathbf{P}^{(n-l)}$. We move forward to Stage 2.

In Stage 2, we start with $k = n - l$. Given k, we lift $\mathbf{U}^{[k]}, \mathbf{P}^{[k]}$ to $(k-1)$-th stage, and update

$$\mathbf{U}_{k-1} = \mathbf{U}^{(k-1)} + R^U_{2^k h, 2^{k-1} h}\mathbf{U}^{[k]}, \quad \mathbf{P}_{k-1} = \mathbf{P}^{(k-1)} + R^p_{2^k h, 2^{k-1} h}\mathbf{P}^{[k]}. \tag{82}$$

where $\mathbf{U}^{(k-1)}, \mathbf{P}^{(k-1)}$ is recorded in Stage 1. Then, we use $\mathbf{U}_{k-1}$ and $\mathbf{P}_{k-1}$ as initial value, run DGS v_2 times to get the approximate solution $\mathbf{U}^{[k-1]}$ and $\mathbf{P}^{[k-1]}$ to 79 with k replaced by $k - 1$. Then we replace k by $k - 1$ until $k = 0$.

With $k = 0$, we calculate the residual $\mathbf{F}_k$ by 80 and let $r_h = h^2\mathbf{F}_k$. If $\|r_h\|_2 < \epsilon$, we stop the algorithm. Otherwise, we restrict $\mathbf{F}_k$ to the $(k+1)$-th grid, calculate $\mathbf{F}^{(1)}$ by 81, replace $k = 0$ by $k = 1$ and return to Stage 1.

We will elaborate the selection of ϵ in Section 5.

4 V-Cycle Multi-Grid Based on Uzawa (Problem 2, 3, 4)

The routine of V-Cycle is same as the one in Section 3.3.

4.1 Uzawa Iteration

We introduce the routine of Uzawa Iteration. Suppose we have the initial value $\mathbf{P}_0$. We start from $k = 0$. Given $\mathbf{P}_k$. We consider to solve $\mathbf{U}_{k+1}$ from

$$\mathbf{A}\mathbf{U}_{k+1} = \mathbf{F} - \mathbf{B}\mathbf{P}_k. \tag{83}$$

Then, we update the pressure

$$\mathbf{P}_{k+1} = \mathbf{P}_k + \alpha\mathbf{B}^T\mathbf{U}_{k+1}. \tag{84}$$

where α is a parameter. About the selection of α, because we have

$$\mathbf{P}_{k+1} = (I - \alpha\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})\mathbf{P}_k + \alpha\mathbf{B}^T\mathbf{A}^{-1}\mathbf{F}. \tag{85}$$

We prove the following proposition on notes:

Proposition 1 $\arg\min\limits_{\alpha>0} \rho(I - \alpha\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})$ *is given by*

$$\alpha^* = \frac{2}{\lambda_{min}(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}) + \lambda_{max}(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})}. \tag{86}$$

Proof We denote $\mathbf{C} = \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$. Because $\mathbf{A}$ is positive definite, $\mathbf{C}$ is semi-definite positive. Suppose λ is $\mathbf{C}$'s eigen value and $\mathbf{x} \in \mathbb{R}^{N^2}$ is its eigen-vector, i.e.

$$\lambda\mathbf{x} = (I - \alpha\mathbf{C})\mathbf{x} = \mathbf{x} - \alpha\mathbf{C}\mathbf{x}. \tag{87}$$

As a result, $\mathbf{x}$ is a eigen-vector of $\mathbf{C}$ and we have

$$\mathbf{C}\mathbf{x} = \frac{1-\lambda}{\alpha}\mathbf{x}. \tag{88}$$

Then $\frac{1-\lambda}{\alpha}$ is the eigen-value of $\mathbf{C}$. Suppose $\frac{1-\lambda}{\alpha} = \beta$, then, $\lambda = 1 - \alpha\beta$. Therefore, we have

$$\arg\min_{\alpha>0} \rho(I - \alpha\mathbf{C}) = \arg\min_{\alpha>0} \max_{\lambda_{min}(\mathbf{C})\le\beta\le\lambda_{max}(\mathbf{C})} |1 - \alpha\beta| \tag{89}$$

Directly by the properties of Chebyshev polynomial, 86 holds. Q.E.D.

We shall point out that $\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ is singular. Note that $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & 0 \end{bmatrix}$ is singular, because both

$\begin{bmatrix}\mathbf{U}\\\mathbf{P}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{U}\\\mathbf{P}+c\mathbf{1}\end{bmatrix}$ is the solution to 41, where c is any real number. Therefore, by the definition of Schur-complement, $\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ is singular and $\lambda_{min}(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}) = 0$. Our latter analysis tells us the eigen values of $\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ only consist of $0, 1$. Therefore, the optimal choice for α is 2. Nevertheless, $\alpha = 1$ achieves the best performance. We will explain this in next subsubsection.

4.1.1 Convergence of Uzawa Iteration

We observe that $\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} = \left(B_N^{(1)}\right)^T A_N^{-1}B_N^{(1)} + \left(B_N^{(2)}\right)^T A_N^{-1}B_N^{(2)}$. Notce that $A_N = S_{N-1}^{(2)} \otimes I_{N-1} + I_N \otimes T_{N-1}$. We know that T_{N-1} has spectral $\{\eta_m\}_{m=1}^{N-1}$ where $\eta_m = 2\left(1-\cos\frac{m\pi}{N}\right) = 4\sin^2\frac{m\pi}{2N}$. The corresponding eigen vector for η_m is

$$t_m = \left[\sin\frac{m\pi}{N}, \sin\frac{2m\pi}{N}, \dots \sin\frac{(N-1)m\pi}{N}\right]^T. \tag{90}$$

S_{N-1}^2 has spectral $\{\xi_n\}_{n=1}^{N}$ where $\xi_n = 2\left(1-\cos\frac{(n-1)\pi}{N}\right) = 4\sin^2\frac{(n-1)\pi}{2N}$. The corresponding eigen vector for ξ_n is

$$s_n = \left[\cos\frac{(n-1)\pi}{2N}, \cos\frac{3(n-1)\pi}{2N}, \dots \cos\frac{(2N-1)(n-1)\pi}{2N}\right]^T. \tag{91}$$

Therefore, A_N has spectral $\{\lambda_{m,n}\}_{1\le m\le N-1, 1\le n\le N}$ where $\lambda_{m,n} = \eta_m + \xi_n = 4\sin^2\frac{m\pi}{2N} + 4\sin^2\frac{(n-1)\pi}{2N}$. The corresponding eigen vector for $\lambda_{m,n}$ is $s_n \otimes t_m$. We can verify that

$$\begin{aligned}\left(S_{N-1}^{(2)}\otimes I_{N-1}\right)(s_n\otimes t_m) &= \left(S_{N-1}^{(2)}s_n\right)\otimes t_m = \xi_n s_n \otimes t_m,\\ (I_N\otimes T_{N-1})(s_n\otimes t_m) &= s_n\otimes(T_{N-1}t_m) = \eta_m s_n\otimes t_m.\end{aligned} \tag{92}$$

We denote $a^{(m,n)} = s_n \otimes t_m / l_{m,n}$, where

$$l_{m,n} = \|s_n\otimes t_m\|_2 = \begin{cases}\frac{N}{2}, & n>1\\ \frac{N}{\sqrt{2}}, & n=1\end{cases} \tag{93}$$

Therefore, we have the spectral decomposition of A_N^{-1} as follows:

$$A_N^{-1} = \sum_{m=1}^{N}\sum_{n=1}^{N-1}\lambda_{m,n}^{-1}a^{(m,n)}\left(a^{(m,n)}\right)^T. \tag{94}$$

Notice that $a_{i,j}^{(m,n)} = l_{m,n}^{-1} \sin \frac{m\pi i}{N} \cos \frac{(n-1)\pi(2j-1)}{2N}$. Therefore, based on 35 and 38 in Section 2,

$$
\begin{aligned}
(B_N^{(1)})^T a^{(m,n)} =& l_{m,n}^{-1} \sum_{j=1}^{N} e_N^{(j)} \otimes \begin{bmatrix} a_{0,j}^{(m,n)} - a_{1,j}^{(m,n)} \\ \vdots \\ a_{N-1,j}^{(m,n)} - a_{N,j}^{(m,n)} \end{bmatrix} \\
=& l_{m,n}^{-1} \sum_{j=1}^{N} \begin{bmatrix} \sin \frac{m\pi 0}{N} - \sin \frac{m\pi}{N} \\ \vdots \\ \sin \frac{m\pi(N-1)}{N} - \sin \frac{m\pi N}{N} \end{bmatrix} \otimes \cos \frac{(n-1)\pi(2j-1)}{N} e_N^{(j)} \\
=& -2 \sin \frac{m\pi}{2N} l_{m,n}^{-1} s_n \otimes \tilde{t}_m .
\end{aligned}
\tag{95}
$$

$$
\begin{aligned}
(B_N^{(2)})^T a^{(m,n)} =& l_{m,n}^{-1} \sum_{j=1}^{N} \begin{bmatrix} a_{0,j}^{(m,n)} - a_{1,j}^{(m,n)} \\ \vdots \\ a_{N-1,j}^{(m,n)} - a_{N,j}^{(m,n)} \end{bmatrix} \otimes e_N^{(j)} \\
=& l_{m,n}^{-1} \sum_{j=1}^{N} \begin{bmatrix} \sin \frac{m\pi 0}{2N} - \sin \frac{m\pi}{2N} \\ \vdots \\ \sin \frac{m\pi(N-1)}{2N} - \sin \frac{m\pi N}{2N} \end{bmatrix} \otimes \cos \frac{(n-1)\pi(2j-1)}{N} e_N^{(j)} \\
=& -2 \sin \frac{m\pi}{2N} l_{m,n}^{-1} \tilde{t}_m \otimes s_n .
\end{aligned}
\tag{96}
$$

where $\tilde{t}_m \in \mathbb{R}^N$ is with entry

$$
\tilde{t}_m = \begin{bmatrix} \cos \frac{m\pi}{2N} \\ \cos \frac{3m\pi}{2N} \\ \vdots \\ \cos \frac{m\pi(2N-1)}{2N} \end{bmatrix} = s_{m+1} . \tag{97}
$$

We denote $\delta_{m,n} = 4 \sin^2 \frac{m\pi}{2N} \lambda_{m,n+1}^{-1} = \frac{\sin^2 \frac{m\pi}{2N}}{\sin^2 \frac{m\pi}{2N} + \sin^2 \frac{n\pi}{2N}}$,

$$
S^{m,n} = (s_{m+1} \otimes s_{n+1})^T (s_{m+1} \otimes s_{n+1}) = \left(s_{m+1} s_{m+1}^T\right) \otimes \left(s_{n+1}^T s_{n+1}\right) \tag{98}
$$

Note that

$$
\begin{aligned}
\left(\sum_{n=1}^{N-1} s_{n+1}^T s_{n+1}\right)_{i,j} =& \sum_{n=1}^{N-1} \cos \frac{n\pi(2j-1)}{2N} \cos \frac{n\pi(2i-1)}{2N} \\
=& \frac{1}{2} \sum_{n=1}^{N-1} \cos \frac{(i+j-1)n\pi}{N} + \cos \frac{(i-j)n\pi}{N} .
\end{aligned}
\tag{99}
$$

Because N is even, we have

$$\sum_{n=1}^{N-1}\cos\frac{kn\pi}{N}=\begin{cases}N-1, & k=0\\ 0, & k\neq 0,2|k+1\\ -1, & k\neq 0,2|k\end{cases} \tag{100}$$

Therefore,

$$\sum_{n=1}^{N-1}s_{n+1}^{T}s_{n+1}=\frac{N}{2}I_N-\frac{1}{2}E_N. \tag{101}$$

where $E_N\in\mathbb{R}^{N\times N}$ is the matrix with all entries equal to 1. Then, we have

$$\begin{aligned}
&\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}=\left(B_N^{(1)}\right)^T A_N^{-1}B_N^{(1)}+\left(B_N^{(2)}\right)^T A_N^{-1}B_N^{(2)}\\
=&\sum_{m=1}^{N-1}\sum_{n=1}^{N}4\sin^2\frac{m\pi}{2N}\lambda_{m,n}^{-1}l_{m,n}^{-2}\left(\left(s_{m+1}s_{m+1}^T\right)\otimes\left(s_n^Ts_n\right)+\left(s_ns_n^T\right)\otimes\left(s_{m+1}^Ts_{m+1}\right)\right)\\
=&\sum_{m=1}^{N-1}\sum_{n=1}^{N-1}\frac{4\delta_{m,n}}{N^2}\left(S^{(m,n)}+S^{(n,m)}\right)+\sum_{m=1}^{N-1}\frac{2}{N^2}(S^{(0,m)}+S^{(m,0)})\\
=&\frac{4}{N^2}\sum_{m=1}^{N-1}\sum_{n=1}^{N-1}\left(\delta_{m,n}S^{(m,n)}+\delta_{n,m}S^{(m,n)}\right)+\frac{2}{N^2}\sum_{m=1}^{N-1}S^{(0,m)}+S^{(m,0)}\\
=&\frac{4}{N^2}\sum_{m=1}^{N-1}\sum_{n=1}^{N-1}S^{(m,n)}+\frac{2}{N^2}\sum_{m=1}^{N-1}S^{(0,m)}+S^{(m,0)}\\
=&\frac{1}{N^2}(NI_N-E_N)\otimes(NI_N-E_N)+\frac{1}{N^2}E_N\otimes(NI_N-E_N)+\frac{1}{N^2}(NI_N-E_N)\otimes E_N\\
=&I_N\otimes I_N-\frac{1}{N^2}E_N\otimes E_N=I_{N^2}-\left(\frac{s_1\otimes s_1}{N}\right)\left(\frac{s_1\otimes s_1}{N}\right)^T
\end{aligned} \tag{102}$$

Let us denote $\mathbf{v}=\frac{s_1\otimes s_1}{N}$. $\mathbf{v}^T\mathbf{v}=\|\mathbf{v}\|_2^2=\frac{N^2}{N^2}=1$. As a result,

$$\left(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^2=\left(I_{N^2}-\mathbf{v}\mathbf{v}^T\right)^2=I_{N^2}-2\mathbf{v}\mathbf{v}^T+\mathbf{v}\mathbf{v}^T\mathbf{v}\mathbf{v}^T=I_{N^2}-\mathbf{v}\mathbf{v}^T=\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \tag{103}$$

103 tells us $\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ is a projection matrix. Based on 103, we have the following proposition:

Proposition 2 *We take $\alpha=1$. If $\mathbf{F}\in\mathrm{range}(\mathbf{B})$, then the exact Uzawa Iteration will converge in at most 2 iteration. Namely, If $\mathbf{P}_0=0$, $\mathbf{P}_1=\mathbf{B}^T\mathbf{A}^{-1}\mathbf{F}$, $\mathbf{U}_2=\mathbf{A}^{-1}(\mathbf{F}-\mathbf{B}\mathbf{P}_1)$, $\mathbf{P}_2=(I-\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})\mathbf{P}_1+\mathbf{B}^T\mathbf{A}^{-1}\mathbf{F}$. Then $\mathbf{U}_2$ and $\mathbf{P}_2$ are the exact solution to 41.*

Proof By 103 we have $\left(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^2=\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$. Then

$$\mathbf{B}^T\mathbf{U}_2=\mathbf{B}^T\mathbf{A}^{-1}\mathbf{F}-\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\mathbf{P}_1=\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\mathbf{M}-\left(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^2\mathbf{M}=0 \tag{104}$$

$$
\begin{aligned}
\mathbf{AU}_2 + \mathbf{BP}_2 &= \mathbf{F} - \mathbf{BP}_1 + \mathbf{B}(2I - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})\mathbf{P}_1 = \mathbf{F} - \mathbf{B}(I - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})\mathbf{P}_1 \\
&= \mathbf{F} - \mathbf{B}(I - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})\mathbf{B}^T\mathbf{A}^{-1}\mathbf{F} \\
&= \mathbf{BM} - \mathbf{B}(I - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})\mathbf{B}^T\mathbf{A}^{-1}\mathbf{BM} \\
&= \mathbf{B}\left(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} - \left(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^2\right)\mathbf{M} = 0
\end{aligned}
\tag{105}
$$

Q.E.D.

4.2 Inexact Uzawa Iteration

The difference between Inexact Uzawa Iteration and Uzawa Iteration is that we do not solve the solution to 83. Instead, we find an approximate solution $\tilde{\mathbf{U}}_{k+1}$ s.t.

$$
||\mathbf{A}\tilde{\mathbf{U}}_{k+1} - (\mathbf{F} - \mathbf{BP}_k)||_2 \leq \tau \tag{106}
$$

where τ is a parameter for Inexact Uzawa Iteration. We take $\mathbf{U}_k$ as initial value and apply Conjugate Gradient method to obtain $\mathbf{U}_{k+1}$.

4.3 Inexact Uzawa Iteration Based On V-Cycle multi-grid

Here we introduce another approach to solve 41. We only use Inexact Uzawa Iteration but we solve the subproblem approximately by V-Cycle multi-grid instead of CG in 4.2. We choose Gauss-Seidel Iteration as smoother for V-Cycle multi-grid.

5 Numerical results

5.1 DGS Iteration (Problem 1)

We apply the V-Cycle multi-grid method to solve the saddle point problem 41.

- ‘DGS-s’ represents our implementation of DGS in the notes and ‘s’ is the abbreviation of ‘sequence’.
- ‘DGS-p’ represents our implementation of DGS in [1] and ‘p’ is the abbreviation of ‘parallel’ ;

We evaluate the selection of parameters L, v_1 and v_2 by three criterions: ‘time’ denotes cpu-time; ‘VC’ denotes the number of V-Cycle; e_N is the error to the true solution, i.e.

$$
e_N = h\left(\sum_{j=1}^{N}\sum_{i=1}^{N-1}\left|u_{i,j-\frac{1}{2}} - u\left(x_i, y_{j-\frac{1}{2}}\right)\right|^2 + \sum_{j=1}^{N-1}\sum_{i=1}^{N}\left|v_{i-\frac{1}{2},j} - u\left(x_{i-\frac{1}{2}}, y_j\right)\right|^2\right)^{1/2}. \tag{107}
$$

where $u(x, y)$ and $v(x, y)$ is the true solution 4 to the Stokes equation 1.

In our numerical experiment, for simplicity, we take $\nu_1 = \nu_2 = \nu = [10, 20, 40, 80, 160]$. For $N = 64, 128, 256$, we set the stopping criterion $\epsilon = 10^{-8}$ and the maximum number of V-Cycle to be 500.

Table 1 V-Cycle based on DGS, $N = 64$

L	ν	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
4	10	4.49	500	1.50e-03	2.73e-08	0.41	5	1.16e-02	3.25e-10
	20	3.85	261	1.50e-03	9.87e-09	0.07	4	1.10e-02	1.45e-09
	40	4.05	140	1.50e-03	9.97e-09	0.14	4	1.12e-02	9.33e-11
	80	3.91	69	1.50e-03	8.59e-09	0.18	3	9.52e-03	2.27e-09
	160	4.30	38	1.50e-03	8.24e-09	0.34	3	8.60e-03	1.83e-10
16	10	1.62	231	1.50e-03	9.54e-09	0.33	43	1.97e-02	9.33e-09
	20	2.17	154	1.50e-03	9.26e-09	0.28	22	1.79e-02	7.50e-09
	40	3.12	120	1.50e-03	9.04e-09	0.26	11	1.48e-02	8.06e-09
	80	3.69	72	1.50e-03	9.54e-09	0.29	6	1.11e-02	2.29e-09
	160	3.89	38	1.50e-03	8.33e-09	0.31	3	9.23e-03	2.38e-09

Table 2 V-Cycle based on DGS, $N = 128$

L	ν	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
4	10	17.72	500	4.59e-04	7.16e-05	0.25	5	1.21e-02	1.48e-10
	20	33.68	500	3.74e-04	2.06e-06	0.33	4	1.19e-02	6.75e-10
	40	64.39	500	3.74e-04	9.88e-09	0.61	4	1.26e-02	4.40e-11
	80	61.37	243	3.74e-04	9.75e-09	0.90	3	1.12e-02	1.53e-09
	160	63.20	126	3.74e-04	9.81e-09	1.80	3	1.10e-02	1.81e-10
16	10	17.61	500	3.74e-04	2.62e-06	1.41	39	2.41e-02	8.64e-09
	20	33.12	500	3.74e-04	5.29e-08	1.42	20	2.20e-02	7.01e-09
	40	56.13	443	3.74e-04	9.95e-09	1.28	10	1.81e-02	7.82e-09
	80	62.84	253	3.74e-04	9.82e-09	1.39	5	1.39e-02	8.01e-09
	160	60.52	126	3.74e-04	9.85e-09	1.73	3	1.16e-02	7.37e-10
64	10	13.89	500	3.74e-04	4.44e-06	15.41	500	1.40e-02	8.35e-07
	20	25.17	479	3.74e-04	9.82e-09	20.73	396	8.93e-03	9.88e-09
	40	24.03	239	3.74e-04	9.91e-09	19.89	198	8.86e-03	9.76e-09
	80	22.95	116	3.74e-04	9.29e-09	19.24	99	8.73e-03	9.56e-09
	160	20.83	54	3.74e-04	7.76e-09	19.68	50	8.38e-03	8.18e-09

From Table 1, 2 and 3, we observe that DGS-s takes more iterations and longer time than DGS-p to make $\|r_h\| <_2$. And DGS-s is more probable to exceed the maximum number of V-Cycle, especially when ν is small or L is large. Nevertheless, DGS-s achieves much lower e_N than DGS-p. We see that sometimes, with same parameters, $\|r_h\|$ in DGS-s is much smaller than DGS-p, but e_N in DGS-s is much smaller than DGS-p. One possible explanation is that

Table 3 V-Cycle based on DGS, $N = 256$

L	v	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|\|r_h\|\|_2$	time(s)	VC	e_N	$\|\|r_h\|\|_2$
4	10	74.93	500	2.39e-03	3.88e-04	0.75	4	1.23e-02	5.06e-09
	20	143.03	500	6.00e-04	9.22e-05	1.38	4	1.21e-02	3.27e-10
	40	280.38	500	1.65e-04	2.06e-05	1.95	3	1.30e-02	5.98e-09
	80	551.85	500	9.36e-05	9.49e-07	3.72	3	1.17e-02	7.64e-10
	160	997.67	461	9.34e-05	9.79e-09	7.40	3	1.19e-02	9.60e-11
16	10	74.96	500	1.98e-03	2.87e-04	5.38	36	2.52e-02	9.88e-09
	20	143.30	500	4.57e-04	7.26e-05	5.47	19	2.31e-02	5.80e-09
	40	279.51	500	2.10e-04	2.92e-05	5.51	10	1.90e-02	3.40e-09
	80	543.17	500	9.38e-05	1.21e-06	5.77	5	1.46e-02	3.54e-09
	160	988.71	461	9.34e-05	9.80e-09	7.31	3	1.24e-02	3.31e-10
64	10	70.79	500	2.09e-03	1.15e-04	69.27	500	2.30e-02	6.46e-08
	20	135.95	500	2.63e-04	1.79e-05	77.23	300	2.20e-02	9.66e-09
	40	267.60	500	9.35e-05	3.65e-07	74.07	150	2.18e-02	9.58e-09
	80	394.92	382	9.34e-05	9.91e-09	73.44	75	2.15e-02	9.45e-09
	160	460.33	224	9.34e-05	9.69e-09	73.82	38	2.07e-02	7.96e-09

the solution from DGS-p is easier to satisfy

$$\mathbf{AU} + \mathbf{BP} = \mathbf{F}, \tag{108}$$

while the solution from DGS-s is easier to satisfy the incompressible condition. And our stopping criterion is to stop when we have a small residual for 108. Therefore, DGS-s finds a solution that is much closer to the true solution to 1.

For $N = 512, 1024, 2048$, we set the stopping criterion $\epsilon = 10^{-6}$ and the maximum number of V-Cycle to be 100. For $N = 512, 1024$, we take $v = [20, 40, 80, 160]$. For $N = 2048$, we take $v = [40, 80, 160]$. The result of $N = 512$ is shown in Table 4; the result of $N = 1024$ is shown in Table 5; the result of $N = 2048$ is shown in Table 6.

We find that the numerical results are not ideal. DGS-p does not get the required precision of $\|r_h\|_2$ within the maximum number of V-Cycles and DGS-s does not find a solution that is close to the true solution. In Appendix A, we modify the V-Cycle multi-grid based on DGS. Our modification leads to great improvement in time and e_N for both DGS-p and DGS-s.

5.2 Uzawa Iteration (Problem 2)

We apply the V-Cycle multi-grid method to solve the saddle point problem 41. We use Uzawa Iteration as smoother instead of DGS.

We assume that $v_1 = v_2 = v$. For $N = 64, 128$, we set the stopping criterion $\epsilon = 10^{-8}$. We take $\alpha = [0.5, 0.75, 1, 1.5, 2]$. We set the maximum number of V-Cycle to be 100. We take

Table 4 V-Cycle based on DGS, $N = 512$

L	v	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
4	20	130.69	100	3.09e-03	5.61e-04	4.05	3	1.22e-02	2.37e-08
	40	251.62	100	2.21e-03	3.16e-04	5.18	2	1.31e-02	8.80e-07
	80	494.93	100	1.50e-03	1.57e-04	9.85	2	1.19e-02	2.22e-07
	160	965.09	100	1.02e-03	8.43e-05	19.54	2	1.22e-02	5.50e-08
16	20	131.24	100	6.00e-03	8.82e-04	11.21	11	2.34e-02	5.31e-07
	40	250.75	100	5.20e-03	5.63e-04	13.31	6	1.93e-02	3.10e-07
	80	498.83	100	1.62e-03	1.69e-04	13.69	3	1.48e-02	3.19e-07
	160	966.32	100	1.02e-03	8.43e-05	19.46	2	1.27e-02	6.05e-08
64	20	130.74	100	1.62e-02	1.34e-03	94.62	100	2.67e-02	1.56e-05
	40	247.70	100	1.16e-02	9.10e-04	146.57	77	2.65e-02	9.31e-07
	80	479.92	100	7.07e-03	5.51e-04	143.65	38	2.61e-02	1.00e-06
	160	940.65	100	3.35e-03	2.67e-04	142.22	19	2.52e-02	9.82e-07
256	20	101.45	100	7.07e-01	3.29e-02	73.68	100	7.07e-01	3.29e-02
	40	192.27	100	4.80e-01	1.87e-02	147.41	100	4.80e-01	1.87e-02
	80	375.33	100	2.25e-01	8.69e-03	291.34	100	2.25e-01	8.69e-03
	160	697.18	100	5.01e-02	1.93e-03	557.14	100	5.33e-02	1.91e-03

Table 5 V-Cycle based on DGS, $N = 1024$

L	v	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
4	20	569.58	100	3.47e-03	4.22e-04	16.35	3	1.22e-02	1.18e-08
	40	1128.95	100	2.64e-03	2.76e-04	22.45	2	1.32e-02	4.43e-07
	80	2230.00	100	2.03e-03	1.66e-04	44.53	2	1.20e-02	1.12e-07
	160	4388.34	100	1.65e-03	1.03e-04	82.17	2	1.23e-02	2.78e-08
16	20	574.13	100	6.78e-03	6.36e-04	41.95	9	2.35e-02	9.80e-07
	40	1123.36	100	6.02e-03	4.02e-04	47.37	5	1.93e-02	5.72e-07
	80	2240.17	100	2.12e-03	1.71e-04	60.41	3	1.49e-02	1.59e-07
	160	4411.71	100	1.65e-03	1.03e-04	79.61	2	1.28e-02	3.05e-08
64	20	571.45	100	1.96e-02	1.08e-03	428.95	100	2.70e-02	7.18e-06
	40	1125.52	100	1.64e-02	8.34e-04	574.43	68	2.70e-02	9.40e-07
	80	2229.45	100	1.26e-02	5.90e-04	564.07	34	2.66e-02	9.30e-07
	160	4324.80	100	8.41e-03	3.65e-04	501.59	17	2.57e-02	9.16e-07
256	20	543.23	100	6.40e-01	1.36e-02	408.24	100	6.40e-01	1.36e-02
	40	1058.37	100	3.98e-01	7.71e-03	788.27	100	3.98e-01	7.69e-03
	80	2135.58	100	1.55e-01	3.03e-03	1493.93	100	1.56e-01	2.98e-03
	160	4119.86	100	2.75e-02	6.48e-04	2753.62	100	3.29e-02	4.52e-04

Table 6 V-Cycle based on DGS, $N = 2048$

L	v	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|\|r_h\|\|_2$	time(s)	VC	e_N	$\|\|r_h\|\|_2$
4	40	5310.08	100	2.78e-03	1.81e-04	94.84	2	1.32e-02	2.23e-07
	80	10319.89	100	2.23e-03	1.23e-04	185.99	2	1.20e-02	5.60e-08
	160	18614.10	100	1.93e-03	8.53e-05	373.32	2	1.23e-02	1.40e-08
16	40	5579.55	100	6.25e-03	2.33e-04	199.06	5	1.93e-02	2.86e-07
	80	10418.21	100	2.32e-03	1.25e-04	258.41	3	1.49e-02	7.94e-08
	160	19096.72	100	1.93e-03	8.53e-05	362.28	2	1.28e-02	1.53e-08
64	40	5502.00	100	1.79e-02	4.39e-04	2141.09	61	2.71e-02	9.12e-07
	80	10166.60	100	1.45e-02	3.27e-04	2204.60	31	2.67e-02	8.15e-07
	160	18864.64	100	1.04e-02	2.19e-04	2106.98	15	2.57e-02	9.95e-07
256	40	5086.88	100	3.80e-01	3.70e-03	3473.32	100	3.80e-01	3.67e-03
	80	8967.20	100	1.42e-01	1.47e-03	6988.40	100	1.42e-01	1.36e-03
	160	17700.55	100	2.69e-02	5.51e-04	13512.25	100	3.08e-02	1.88e-04

$L = [4, 16]$. The result of $N = 64$ is shown in Table 7; the result of $N = 128$ is shown in Table 8. From Table 7 and Table 8, we observe that $\alpha = N^2$ is the best choice of α. With $\alpha = N^2$, after 2 Uzawa Iteration, $\|r_h\|_2$ appears to be lower than 10^{-8}. If we set $\alpha = 2$, the performance is terrible, although $\alpha = 2$ is an optimal choice by 86. We also obverse that with α closer to 1, Uzawa Iteration achieves better performance. This is consistent with our proof.

For $N = 256, 512$, we set the maximum number of V-Cycle to be 20. We take $L = [4, 16, 64]$. The result of $N = 256$ is shown in Table 9 and 10; the result of $N = 512$ is shown in Table 11 and 12. With $\alpha = 1$, Uzawa Iteration converges with 2 iterations. Therefore, the number of V-Cycle is 0.

For $N = 1024, 2048$, we set the stopping criterion $\epsilon = 10^{-6}$ and set the maximum number of V-Cycle to be 10. We take $L = [4, 16, 64]$. The result of $N = 1024$ is shown in Table 23 and 14; the result of $N = 2048$ is shown in Table 15 and 16. This numerically verifies our proposition 2.

5.3 Inexact Uzawa Iteration (Problem 3)

We apply the V-Cycle multi-grid method to solve the saddle point problem 41. We use Inexact Uzawa Iteration as smoother instead of DGS.

We assume that $v_1 = v_2 = v$. For $N = 64, 128$, we set the stopping criterion $\epsilon = 10^{-8}$. Based on the result in Section 4.2, we fix $\alpha = 1$. We take $\tau = [10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}]$. We set the maximum number of V-Cycle to be 100. The result of $N = 64$ is shown in Table 17; the result of $N = 128$ is shown in Table 18. We observe that even with inexact solution to the subproblem 79, Inexact Uzawa converges in an incredible speed with $\alpha = 1$. This is consistent with our proof.

Table 7 V-Cycle based on Uzawa, $N = 64$

α	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
0.50	2	1.06	17	1.50e-03	7.92e-09	0.61	17	1.50e-03	8.09e-09
	4	0.46	6	1.50e-03	2.20e-09	0.44	6	1.50e-03	2.20e-09
	8	0.40	2	1.50e-03	2.00e-09	0.35	2	1.50e-03	2.00e-09
	16	0.46	1	1.50e-03	3.51e-12	0.44	1	1.50e-03	3.51e-12
0.75	2	0.35	9	1.50e-03	6.18e-09	0.32	9	1.50e-03	6.20e-09
	4	0.25	3	1.50e-03	4.21e-10	0.24	3	1.50e-03	4.21e-10
	8	0.23	1	1.50e-03	2.35e-11	0.22	1	1.50e-03	2.35e-11
	16	0.18	0	1.50e-03	4.79e-12	0.18	0	1.50e-03	4.79e-12
1.00	2	0.05	0	1.50e-03	6.71e-14	0.03	0	1.50e-03	6.71e-14
	4	0.05	0	1.50e-03	6.27e-14	0.05	0	1.50e-03	6.27e-14
	8	0.10	0	1.50e-03	6.29e-14	0.09	0	1.50e-03	6.29e-14
	16	0.19	0	1.50e-03	6.23e-14	0.18	0	1.50e-03	6.23e-14
1.50	2	1.14	29	1.50e-03	7.47e-09	0.88	24	1.50e-03	8.21e-09
	4	0.45	6	1.50e-03	7.22e-09	0.53	6	1.50e-03	7.17e-09
	8	0.36	2	1.50e-03	6.01e-09	0.45	2	1.50e-03	6.01e-09
	16	0.45	1	1.50e-03	1.69e-11	0.44	1	1.50e-03	1.69e-11
2.00	2	3.69	100	2.78e-02	1.37e-02	3.27	100	2.78e-02	1.37e-02
	4	6.76	100	2.78e-02	1.37e-02	6.69	100	2.78e-02	1.37e-02
	8	13.39	100	2.78e-02	1.37e-02	12.81	100	2.78e-02	1.37e-02
	16	26.66	100	2.78e-02	1.37e-02	25.53	100	2.78e-02	1.37e-02

Table 8 V-Cycle based on Uzawa, $N = 128$

α	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
0.50	2	3.31	16	3.74e-04	8.26e-09	3.32	16	3.74e-04	8.25e-09
	4	2.31	5	3.74e-04	8.92e-09	2.21	5	3.74e-04	8.92e-09
	8	2.11	2	3.74e-04	1.02e-09	2.08	2	3.74e-04	1.02e-09
	16	2.78	1	3.74e-04	1.79e-12	2.68	1	3.74e-04	1.79e-12
0.75	2	1.91	9	3.74e-04	3.14e-09	1.91	9	3.74e-04	3.14e-09
	4	1.45	3	3.74e-04	2.15e-10	1.46	3	3.74e-04	2.15e-10
	8	1.39	1	3.74e-04	1.20e-11	1.33	1	3.74e-04	1.20e-11
	16	1.20	0	3.74e-04	2.42e-12	1.18	0	3.74e-04	2.42e-12
1.00	2	0.15	0	3.74e-04	1.05e-13	0.16	0	3.74e-04	1.05e-13
	4	0.31	0	3.74e-04	1.04e-13	0.28	0	3.74e-04	1.04e-13
	8	0.56	0	3.74e-04	1.03e-13	0.57	0	3.74e-04	1.03e-13
	16	1.50	0	3.74e-04	1.02e-13	1.14	0	3.74e-04	1.02e-13
1.50	2	6.00	28	3.74e-04	7.30e-09	5.40	27	3.74e-04	7.15e-09
	4	2.67	6	3.74e-04	3.70e-09	2.59	6	3.74e-04	3.70e-09
	8	2.17	2	3.74e-04	3.06e-09	2.14	2	3.74e-04	3.06e-09
	16	2.67	1	3.74e-04	8.64e-12	2.62	1	3.74e-04	8.64e-12
2.00	2	19.75	100	2.78e-02	6.92e-03	19.29	100	2.78e-02	6.92e-03
	4	39.60	100	2.78e-02	6.92e-03	39.04	100	2.78e-02	6.92e-03
	8	78.86	100	2.78e-02	6.92e-03	78.59	100	2.78e-02	6.92e-03
	16	156.70	100	2.78e-02	6.92e-03	158.99	100	2.78e-02	6.92e-03

Table 9 V-Cycle based on Uzawa, $N = 256$, Part 1

α	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
0.50	2	15.33	15	9.34e-05	8.62e-09	14.83	15	9.34e-05	8.62e-09
	4	10.95	5	9.34e-05	4.50e-09	10.91	5	9.34e-05	4.50e-09
	8	10.12	2	9.34e-05	5.16e-10	10.30	2	9.34e-05	5.16e-10
	16	13.02	1	9.34e-05	9.22e-13	12.86	1	9.34e-05	9.22e-13
0.75	2	8.24	8	9.34e-05	6.40e-09	8.37	8	9.34e-05	6.40e-09
	4	5.16	2	9.34e-05	6.78e-09	5.12	2	9.34e-05	6.78e-09
	8	6.39	1	9.34e-05	6.05e-12	6.52	1	9.34e-05	6.05e-12
	16	5.16	0	9.34e-05	1.23e-12	5.14	0	9.34e-05	1.23e-12
1.00	2	0.66	0	9.34e-05	2.13e-13	0.67	0	9.34e-05	2.13e-13
	4	1.36	0	9.34e-05	2.11e-13	1.33	0	9.34e-05	2.11e-13
	8	2.66	0	9.34e-05	2.10e-13	2.59	0	9.34e-05	2.10e-13
	16	5.11	0	9.34e-05	2.11e-13	5.10	0	9.34e-05	2.11e-13
1.50	2	20.15	20	9.34e-05	1.81e-07	19.66	20	9.34e-05	1.68e-07
	4	12.80	6	9.34e-05	1.87e-09	13.07	6	9.34e-05	1.87e-09
	8	10.30	2	9.34e-05	1.55e-09	10.29	2	9.34e-05	1.55e-09
	16	12.74	1	9.34e-05	4.37e-12	13.03	1	9.34e-05	4.37e-12
2.00	2	19.77	20	2.78e-02	3.48e-03	19.81	20	2.78e-02	3.48e-03
	4	40.16	20	2.78e-02	3.48e-03	39.20	20	2.78e-02	3.48e-03
	8	80.39	20	2.78e-02	3.48e-03	79.70	20	2.78e-02	3.48e-03
	16	156.49	20	2.78e-02	3.48e-03	156.19	20	2.78e-02	3.48e-03

Table 10 V-Cycle based on Uzawa, $N = 256$, Part 2

α	ν	$L = 64$			
		time(s)	VC	e_N	$\|r_h\|_2$
0.50	2	14.43	15	9.34e-05	8.85e-09
	4	10.50	5	9.34e-05	4.51e-09
	8	9.87	2	9.34e-05	5.16e-10
	16	12.73	1	9.34e-05	9.22e-13
0.75	2	7.96	8	9.34e-05	6.42e-09
	4	5.01	2	9.34e-05	6.78e-09
	8	6.29	1	9.34e-05	6.05e-12
	16	5.14	0	9.34e-05	1.23e-12
1.00	2	0.67	0	9.34e-05	2.13e-13
	4	1.32	0	9.34e-05	2.11e-13
	8	2.59	0	9.34e-05	2.10e-13
	16	5.16	0	9.34e-05	2.11e-13
1.50	2	19.46	20	9.34e-05	2.23e-08
	4	12.45	6	9.34e-05	1.86e-09
	8	9.89	2	9.34e-05	1.55e-09
	16	12.55	1	9.34e-05	4.37e-12
2.00	2	19.19	20	2.78e-02	3.48e-03
	4	37.72	20	2.78e-02	3.48e-03
	8	76.38	20	2.78e-02	3.48e-03
	16	154.29	20	2.78e-02	3.48e-03

Table 11 V-Cycle based on Uzawa, $N = 512$, Part 1

α	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
0.50	2	68.62	14	2.34e-05	9.01e-09	68.55	14	2.34e-05	9.01e-09
	4	52.78	5	2.33e-05	2.26e-09	52.69	5	2.33e-05	2.26e-09
	8	49.24	2	2.33e-05	2.59e-10	49.26	2	2.33e-05	2.59e-10
	16	61.62	1	2.33e-05	5.84e-13	61.28	1	2.33e-05	5.84e-13
0.75	2	40.90	8	2.33e-05	3.21e-09	40.27	8	2.33e-05	3.21e-09
	4	24.89	2	2.33e-05	3.41e-09	24.98	2	2.33e-05	3.41e-09
	8	30.74	1	2.33e-05	3.06e-12	31.37	1	2.33e-05	3.06e-12
	16	24.55	0	2.33e-05	7.25e-13	24.64	0	2.33e-05	7.25e-13
1.00	2	3.16	0	2.33e-05	4.25e-13	3.25	0	2.33e-05	4.25e-13
	4	6.28	0	2.33e-05	4.17e-13	6.24	0	2.33e-05	4.17e-13
	8	12.32	0	2.33e-05	4.19e-13	12.54	0	2.33e-05	4.19e-13
	16	24.76	0	2.33e-05	4.17e-13	24.48	0	2.33e-05	4.17e-13
1.50	2	95.95	20	2.34e-05	9.32e-08	96.26	20	2.34e-05	9.20e-08
	4	52.53	5	2.33e-05	7.59e-09	52.78	5	2.33e-05	7.59e-09
	8	49.96	2	2.33e-05	7.77e-10	49.66	2	2.33e-05	7.77e-10
	16	61.48	1	2.33e-05	2.27e-12	61.89	1	2.33e-05	2.27e-12
2.00	2	96.27	20	2.78e-02	1.74e-03	96.46	20	2.78e-02	1.74e-03
	4	192.11	20	2.78e-02	1.74e-03	192.71	20	2.78e-02	1.74e-03
	8	382.87	20	2.78e-02	1.74e-03	383.38	20	2.78e-02	1.74e-03
	16	768.62	20	2.78e-02	1.74e-03	738.81	20	2.78e-02	1.74e-03

Table 12 V-Cycle based on Uzawa, $N = 512$, Part 2

α	v	$L = 64$			
		time(s)	VC	e_N	$\|r_h\|_2$
0.50	2	68.84	14	2.34e-05	8.99e-09
	4	52.65	5	2.33e-05	2.26e-09
	8	49.70	2	2.33e-05	2.59e-10
	16	61.81	1	2.33e-05	5.84e-13
0.75	2	40.31	8	2.33e-05	3.21e-09
	4	24.72	2	2.33e-05	3.41e-09
	8	30.65	1	2.33e-05	3.06e-12
	16	24.63	0	2.33e-05	7.25e-13
1.00	2	3.22	0	2.33e-05	4.25e-13
	4	6.26	0	2.33e-05	4.17e-13
	8	12.44	0	2.33e-05	4.19e-13
	16	24.53	0	2.33e-05	4.17e-13
1.50	2	96.21	20	2.34e-05	6.12e-08
	4	52.41	5	2.33e-05	7.59e-09
	8	49.54	2	2.33e-05	7.77e-10
	16	60.91	1	2.33e-05	2.27e-12
2.00	2	95.83	20	2.78e-02	1.74e-03
	4	190.86	20	2.78e-02	1.74e-03
	8	381.28	20	2.78e-02	1.74e-03
	16	738.45	20	2.78e-02	1.74e-03

Table 13 V-Cycle based on Uzawa, $N = 1024$, Part 1

α	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
0.50	2	171.39	7	7.12e-05	8.03e-07	169.59	7	7.12e-05	8.03e-07
	4	136.57	2	5.41e-05	5.27e-07	136.31	2	5.41e-05	5.27e-07
	8	146.85	1	6.08e-06	1.49e-08	150.07	1	6.08e-06	1.49e-08
	16	123.72	0	5.90e-06	6.66e-09	122.86	0	5.90e-06	6.66e-09
0.75	2	103.61	4	2.65e-05	4.30e-07	103.36	4	2.65e-05	4.30e-07
	4	85.07	1	8.95e-06	9.97e-08	85.85	1	8.95e-06	9.97e-08
	8	58.94	0	6.08e-06	2.00e-08	60.48	0	6.08e-06	2.00e-08
	16	123.77	0	5.84e-06	8.35e-13	123.54	0	5.84e-06	8.35e-13
1.00	2	15.25	0	5.84e-06	8.52e-13	15.00	0	5.84e-06	8.52e-13
	4	34.03	0	5.84e-06	8.41e-13	34.01	0	5.84e-06	8.41e-13
	8	59.64	0	5.84e-06	8.35e-13	61.14	0	5.84e-06	8.35e-13
	16	124.51	0	5.84e-06	8.38e-13	124.33	0	5.84e-06	8.38e-13
1.50	2	236.31	10	1.08e-04	5.36e-06	237.10	10	1.08e-04	5.35e-06
	4	188.61	3	9.13e-06	2.48e-07	178.02	3	9.13e-06	2.48e-07
	8	155.55	1	6.08e-06	7.19e-08	155.46	1	6.08e-06	7.19e-08
	16	122.35	0	5.90e-06	2.00e-08	124.27	0	5.90e-06	2.00e-08
2.00	2	236.01	10	2.78e-02	8.72e-04	237.36	10	2.78e-02	8.72e-04
	4	506.50	10	2.78e-02	8.72e-04	505.22	10	2.78e-02	8.72e-04
	8	1009.71	10	2.78e-02	8.72e-04	1020.54	10	2.78e-02	8.72e-04
	16	2019.68	10	2.78e-02	8.72e-04	2022.32	10	2.78e-02	8.72e-04

Table 14 V-Cycle based on Uzawa, $N = 1024$, Part 2

α	v	$L = 64$			
		time(s)	VC	e_N	$\|\|r_h\|\|_2$
0.50	2	168.45	7	7.12e-05	8.03e-07
	4	136.21	2	5.41e-05	5.27e-07
	8	150.35	1	6.08e-06	1.49e-08
	16	123.88	0	5.90e-06	6.66e-09
0.75	2	102.68	4	2.65e-05	4.30e-07
	4	84.99	1	8.95e-06	9.97e-08
	8	60.95	0	6.08e-06	2.00e-08
	16	123.14	0	5.84e-06	8.35e-13
1.00	2	15.08	0	5.84e-06	8.52e-13
	4	34.60	0	5.84e-06	8.41e-13
	8	61.79	0	5.84e-06	8.35e-13
	16	124.52	0	5.84e-06	8.38e-13
1.50	2	235.67	10	1.05e-04	5.14e-06
	4	172.50	3	9.13e-06	2.48e-07
	8	157.04	1	6.08e-06	7.19e-08
	16	124.56	0	5.90e-06	2.00e-08
2.00	2	267.16	10	2.78e-02	8.72e-04
	4	504.08	10	2.78e-02	8.72e-04
	8	1012.38	10	2.78e-02	8.72e-04
	16	2011.85	10	2.78e-02	8.72e-04

Table 15 V-Cycle based on Uzawa, $N = 2048$, Part 1

α	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
0.50	2	788.83	6	1.51e-04	8.44e-07	783.56	6	1.51e-04	8.44e-07
	4	627.90	2	5.39e-05	2.64e-07	630.65	2	5.39e-05	2.64e-07
	8	302.20	0	2.17e-04	8.52e-07	298.80	0	2.17e-04	8.52e-07
	16	584.72	0	1.69e-06	3.33e-09	588.80	0	1.69e-06	3.33e-09
0.75	2	437.68	3	1.05e-04	8.66e-07	438.62	3	1.05e-04	8.66e-07
	4	392.89	1	6.94e-06	4.99e-08	398.83	1	6.94e-06	4.99e-08
	8	299.47	0	2.24e-06	9.99e-09	298.44	0	2.24e-06	9.99e-09
	16	597.01	0	1.46e-06	1.56e-12	584.76	0	1.46e-06	1.56e-12
1.00	2	79.75	0	1.46e-06	1.71e-12	79.86	0	1.46e-06	1.71e-12
	4	156.21	0	1.46e-06	1.69e-12	147.96	0	1.46e-06	1.69e-12
	8	297.99	0	1.46e-06	1.67e-12	297.00	0	1.46e-06	1.67e-12
	16	586.01	0	1.46e-06	1.68e-12	587.17	0	1.46e-06	1.68e-12
1.50	2	1268.25	10	1.08e-04	2.70e-06	1267.97	10	1.08e-04	2.69e-06
	4	594.94	2	5.56e-05	8.21e-07	597.06	2	5.56e-05	8.21e-07
	8	757.31	1	2.24e-06	3.60e-08	752.16	1	2.24e-06	3.60e-08
	16	585.89	0	1.69e-06	9.99e-09	583.54	0	1.69e-06	9.99e-09
2.00	2	1235.61	10	2.78e-02	4.36e-04	1227.84	10	2.78e-02	4.36e-04
	4	2417.00	10	2.78e-02	4.36e-04	2413.50	10	2.78e-02	4.36e-04
	8	4853.79	10	2.78e-02	4.36e-04	4791.11	10	2.78e-02	4.36e-04
	16	9379.63	10	2.78e-02	4.36e-04	9688.42	10	2.78e-02	4.36e-04

Table 16 V-Cycle based on Uzawa, $N = 2048$, Part 2

α	v	$L = 64$			
		time(s)	VC	e_N	$\|r_h\|_2$
0.50	2	796.58	6	1.51e-04	8.44e-07
	4	625.38	2	5.39e-05	2.64e-07
	8	297.99	0	2.17e-04	8.52e-07
	16	587.75	0	1.69e-06	3.33e-09
0.75	2	438.35	3	1.05e-04	8.66e-07
	4	394.91	1	6.94e-06	4.99e-08
	8	301.85	0	2.24e-06	9.99e-09
	16	585.29	0	1.46e-06	1.56e-12
1.00	2	80.42	0	1.46e-06	1.71e-12
	4	148.08	0	1.46e-06	1.69e-12
	8	299.65	0	1.46e-06	1.67e-12
	16	587.84	0	1.46e-06	1.68e-12
1.50	2	1242.80	10	1.08e-04	2.67e-06
	4	599.28	2	5.56e-05	8.21e-07
	8	761.33	1	2.24e-06	3.60e-08
	16	583.69	0	1.69e-06	9.99e-09
2.00	2	1253.92	10	2.78e-02	4.36e-04
	4	2409.35	10	2.78e-02	4.36e-04
	8	4741.62	10	2.78e-02	4.36e-04
	16	9200.85	10	2.78e-02	4.36e-04

Table 17 V-Cycle based on Inexact Uzawa, $N = 64$

τ	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
10^{-6}	2	0.39	0	1.50e-03	5.87e-10	0.04	0	1.50e-03	5.87e-10
	4	0.03	0	1.50e-03	5.44e-10	0.03	0	1.50e-03	5.44e-10
	8	0.03	0	1.50e-03	3.62e-10	0.03	0	1.50e-03	3.62e-10
	16	0.03	0	1.50e-03	5.19e-10	0.03	0	1.50e-03	5.19e-10
10^{-7}	2	0.06	0	1.50e-03	5.92e-11	0.05	0	1.50e-03	5.92e-11
	4	0.03	0	1.50e-03	5.36e-11	0.03	0	1.50e-03	5.36e-11
	8	0.03	0	1.50e-03	3.78e-11	0.03	0	1.50e-03	3.78e-11
	16	0.03	0	1.50e-03	2.67e-11	0.05	0	1.50e-03	2.67e-11
10^{-8}	2	0.09	0	1.50e-03	5.42e-12	0.05	0	1.50e-03	5.42e-12
	4	0.03	0	1.50e-03	5.05e-12	0.03	0	1.50e-03	5.05e-12
	8	0.05	0	1.50e-03	4.46e-12	0.05	0	1.50e-03	4.46e-12
	16	0.06	0	1.50e-03	3.47e-12	0.05	0	1.50e-03	3.47e-12
10^{-9}	2	0.04	0	1.50e-03	5.70e-13	0.03	0	1.50e-03	5.70e-13
	4	0.03	0	1.50e-03	5.62e-13	0.03	0	1.50e-03	5.62e-13
	8	0.05	0	1.50e-03	5.48e-13	0.03	0	1.50e-03	5.48e-13
	16	0.06	0	1.50e-03	3.38e-13	0.05	0	1.50e-03	3.38e-13

Table 18 V-Cycle based on Inexact Uzawa, $N = 128$

τ	ν	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|\|r_h\|\|_2$	time(s)	VC	e_N	$\|\|r_h\|\|_2$
10^{-6}	2	0.50	0	3.74e-04	1.42e-10	0.33	0	3.74e-04	1.42e-10
	4	0.33	0	3.74e-04	1.65e-10	0.33	0	3.74e-04	1.65e-10
	8	0.43	0	3.74e-04	1.21e-10	0.30	0	3.74e-04	1.21e-10
	16	0.37	0	3.74e-04	9.65e-11	0.63	0	3.74e-04	9.65e-11
10^{-7}	2	0.37	0	3.74e-04	1.49e-11	0.56	0	3.74e-04	1.49e-11
	4	0.36	0	3.74e-04	1.47e-11	0.31	0	3.74e-04	1.47e-11
	8	0.32	0	3.74e-04	1.18e-11	0.38	0	3.74e-04	1.18e-11
	16	0.61	0	3.74e-04	1.13e-11	0.70	0	3.74e-04	1.13e-11
10^{-8}	2	0.45	0	3.74e-04	1.74e-12	0.57	0	3.74e-04	1.74e-12
	4	0.63	0	3.74e-04	1.38e-12	0.37	0	3.74e-04	1.38e-12
	8	0.47	0	3.74e-04	1.20e-12	0.52	0	3.74e-04	1.20e-12
	16	0.75	0	3.74e-04	1.10e-12	0.42	0	3.74e-04	1.10e-12
10^{-9}	2	0.39	0	3.74e-04	8.72e-13	0.37	0	3.74e-04	8.72e-13
	4	0.45	0	3.74e-04	1.58e-13	0.45	0	3.74e-04	1.58e-13
	8	0.42	0	3.74e-04	1.32e-13	0.60	0	3.74e-04	1.32e-13
	16	0.45	0	3.74e-04	1.37e-13	0.72	0	3.74e-04	1.37e-13

For $N = 256, 512$, we set the stopping criterion $\epsilon = 10^{-7}$ and set the maximum number of V-Cycle to be 20. We take $\tau = [10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}]$, $L = [4, 16, 64]$. The result of $N = 256$ is shown in Table 19 and 20; the result of $N = 512$ is shown in Table 21 and 22.

For $N = 1024, 2048$, we set the stopping criterion $\epsilon = 10^{-6}$ and set the maximum number of V-Cycle to be 10. We take $\tau = [10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}]$, $L = [4, 16, 64, 256]$. The result of $N = 1024$ is shown in Table 23 and 24; the result of $N = 2048$ is shown in Table 25 and 26.

In summary, with $\alpha = 1$, even with inexact solution to the subproblem, Inexact Uzawa converges in 2 iterations. Namely, Inexact Uzawa does not use the structure of V-Cycle multi-grid because VCs in the numerical results are 0. We shall point out that forcing Inexact Uzawa to use the structure of V-Cycle multi-grid will greatly deteriorate the performance. This can be verified in Subsection 5.4.

5.4 Inexact Uzawa Iteration Based On V-Cycle multi-grid (Problem 4)

We apply Inexact Uzawa Iteration Based On V-Cycle multi-grid to solve the saddle point problem 41. 'iter' denotes the number of outer loop.

We assume that $\nu_1 = \nu_2 = \nu$. For $N = 64, 128$, we set the stopping criterion $\epsilon = 10^{-8}$. Based on the result in Section 4.2, we fix $\alpha = 1$. We take $\tau = [10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}]$. We set the maximum number of V-Cycle to be 100 and the maximum number of outer loop to be 100. The result of $N = 64$ is shown in Table 27; the result of $N = 128$ is shown in Table 28. We observe that if we set τ too big, i.e., $\tau \leq 10^{-7}$, Inexact Uzawa will converge slowly. But

Table 19 V-Cycle based on Inexact Uzawa, $N = 256$, Part 1

τ	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
10^{-5}	2	3.26	0	9.34e-05	3.92e-10	3.37	0	9.34e-05	3.92e-10
	4	3.21	0	9.34e-05	3.94e-10	3.28	0	9.34e-05	3.94e-10
	8	2.90	0	9.34e-05	3.55e-10	2.93	0	9.34e-05	3.55e-10
	16	4.14	0	9.34e-05	3.55e-10	3.58	0	9.34e-05	3.55e-10
10^{-6}	2	3.62	0	9.34e-05	3.81e-11	3.53	0	9.34e-05	3.81e-11
	4	3.14	0	9.34e-05	3.68e-11	3.76	0	9.34e-05	3.68e-11
	8	3.72	0	9.34e-05	3.32e-11	3.33	0	9.34e-05	3.32e-11
	16	3.48	0	9.34e-05	3.01e-11	3.52	0	9.34e-05	3.01e-11
10^{-7}	2	3.68	0	9.34e-05	4.35e-12	3.32	0	9.34e-05	4.35e-12
	4	3.37	0	9.34e-05	3.85e-12	3.42	0	9.34e-05	3.85e-12
	8	3.16	0	9.34e-05	4.23e-12	3.97	0	9.34e-05	4.23e-12
	16	3.43	0	9.34e-05	3.45e-12	4.12	0	9.34e-05	3.45e-12
10^{-8}	2	3.89	0	9.34e-05	2.37e-12	3.65	0	9.34e-05	2.37e-12
	4	4.14	0	9.34e-05	4.60e-13	4.08	0	9.34e-05	4.60e-13
	8	3.80	0	9.34e-05	4.03e-13	3.30	0	9.34e-05	4.03e-13
	16	4.33	0	9.34e-05	3.65e-13	3.77	0	9.34e-05	3.65e-13

Table 20 V-Cycle based on Inexact Uzawa, $N = 256$, Part 2

τ	v	$L = 64$			
		time(s)	VC	e_N	$\|r_h\|_2$
10^{-5}	2	2.99	0	9.34e-05	3.92e-10
	4	3.24	0	9.34e-05	3.94e-10
	8	3.42	0	9.34e-05	3.55e-10
	16	3.04	0	9.34e-05	3.55e-10
10^{-6}	2	3.31	0	9.34e-05	3.81e-11
	4	3.14	0	9.34e-05	3.68e-11
	8	3.11	0	9.34e-05	3.32e-11
	16	3.60	0	9.34e-05	3.01e-11
10^{-7}	2	3.31	0	9.34e-05	4.35e-12
	4	3.32	0	9.34e-05	3.85e-12
	8	3.04	0	9.34e-05	4.23e-12
	16	4.14	0	9.34e-05	3.45e-12
10^{-8}	2	3.33	0	9.34e-05	2.37e-12
	4	3.16	0	9.34e-05	4.60e-13
	8	4.05	0	9.34e-05	4.03e-13
	16	3.83	0	9.34e-05	3.65e-13

Table 21 V-Cycle based on Inexact Uzawa, $N = 512$, Part 1

τ	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
10^{-5}	2	21.69	0	2.33e-05	9.77e-11	21.42	0	2.33e-05	9.77e-11
	4	20.14	0	2.33e-05	9.61e-11	21.71	0	2.33e-05	9.61e-11
	8	20.80	0	2.33e-05	9.68e-11	20.93	0	2.33e-05	9.68e-11
	16	23.01	0	2.33e-05	8.91e-11	21.91	0	2.33e-05	8.91e-11
10^{-6}	2	20.84	0	2.33e-05	1.17e-11	22.49	0	2.33e-05	1.17e-11
	4	23.65	0	2.33e-05	9.38e-12	21.75	0	2.33e-05	9.38e-12
	8	21.79	0	2.33e-05	9.87e-12	23.16	0	2.33e-05	9.87e-12
	16	24.25	0	2.33e-05	9.28e-12	23.09	0	2.33e-05	9.28e-12
10^{-7}	2	23.61	0	2.33e-05	6.53e-12	22.43	0	2.33e-05	6.53e-12
	4	22.47	0	2.33e-05	1.22e-12	24.46	0	2.33e-05	1.22e-12
	8	24.00	0	2.33e-05	1.08e-12	23.14	0	2.33e-05	1.08e-12
	16	24.92	0	2.33e-05	1.19e-12	22.76	0	2.33e-05	1.19e-12
10^{-8}	2	23.24	0	2.33e-05	6.59e-12	23.01	0	2.33e-05	6.59e-12
	4	25.41	0	2.33e-05	8.48e-13	23.89	0	2.33e-05	8.48e-13
	8	22.60	0	2.33e-05	4.24e-13	25.76	0	2.33e-05	4.24e-13
	16	24.96	0	2.33e-05	7.42e-13	26.86	0	2.33e-05	7.42e-13

Table 22 V-Cycle based on Inexact Uzawa, $N = 512$, Part 2

τ	v	$L = 64$			
		time(s)	VC	e_N	$\|r_h\|_2$
10^{-5}	2	19.43	0	2.33e-05	9.77e-11
	4	19.92	0	2.33e-05	9.61e-11
	8	22.26	0	2.33e-05	9.68e-11
	16	21.98	0	2.33e-05	8.91e-11
10^{-6}	2	20.69	0	2.33e-05	1.17e-11
	4	23.03	0	2.33e-05	9.38e-12
	8	20.97	0	2.33e-05	9.87e-12
	16	22.45	0	2.33e-05	9.28e-12
10^{-7}	2	23.97	0	2.33e-05	6.53e-12
	4	23.31	0	2.33e-05	1.22e-12
	8	24.06	0	2.33e-05	1.08e-12
	16	24.95	0	2.33e-05	1.19e-12
10^{-8}	2	25.07	0	2.33e-05	6.59e-12
	4	25.08	0	2.33e-05	8.48e-13
	8	24.71	0	2.33e-05	4.24e-13
	16	25.33	0	2.33e-05	7.42e-13

Table 23 V-Cycle based on Inexact Uzawa, $N = 1024$, Part 1

τ	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
10^{-4}	2	129.13	0	5.84e-06	2.45e-10	143.11	0	5.84e-06	2.45e-10
	4	123.30	0	5.84e-06	2.48e-10	132.43	0	5.84e-06	2.48e-10
	8	126.26	0	5.84e-06	2.55e-10	125.97	0	5.84e-06	2.55e-10
	16	134.58	0	5.84e-06	2.26e-10	132.24	0	5.84e-06	2.26e-10
10^{-5}	2	142.61	0	5.84e-06	2.98e-11	149.37	0	5.84e-06	2.98e-11
	4	132.07	0	5.84e-06	2.37e-11	137.60	0	5.84e-06	2.37e-11
	8	137.58	0	5.84e-06	2.56e-11	135.87	0	5.84e-06	2.56e-11
	16	142.85	0	5.84e-06	2.15e-11	138.10	0	5.84e-06	2.15e-11
10^{-6}	2	156.89	0	5.84e-06	1.77e-11	136.46	0	5.84e-06	1.77e-11
	4	132.34	0	5.84e-06	3.27e-12	139.83	0	5.84e-06	3.27e-12
	8	143.59	0	5.84e-06	3.10e-12	142.85	0	5.84e-06	3.10e-12
	16	149.98	0	5.84e-06	2.90e-12	155.06	0	5.84e-06	2.90e-12
10^{-7}	2	149.62	0	5.84e-06	1.81e-11	152.57	0	5.84e-06	1.81e-11
	4	146.38	0	5.84e-06	2.44e-12	157.10	0	5.84e-06	2.44e-12
	8	159.65	0	5.84e-06	1.06e-12	150.80	0	5.84e-06	1.06e-12
	16	155.73	0	5.84e-06	8.76e-13	157.65	0	5.84e-06	8.76e-13

Table 24 V-Cycle based on Inexact Uzawa, $N = 1024$, Part 2

τ	v	$L = 64$				$L = 256$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
10^{-4}	2	128.49	0	5.84e-06	2.45e-10	136.86	0	5.84e-06	2.45e-10
	4	124.01	0	5.84e-06	2.48e-10	129.86	0	5.84e-06	2.48e-10
	8	128.57	0	5.84e-06	2.55e-10	136.91	0	5.84e-06	2.55e-10
	16	132.04	0	5.84e-06	2.26e-10	133.17	0	5.84e-06	2.26e-10
10^{-5}	2	149.20	0	5.84e-06	2.98e-11	143.64	0	5.84e-06	2.98e-11
	4	118.39	0	5.84e-06	2.37e-11	126.91	0	5.84e-06	2.37e-11
	8	136.68	0	5.84e-06	2.56e-11	144.08	0	5.84e-06	2.56e-11
	16	143.90	0	5.84e-06	2.15e-11	138.55	0	5.84e-06	2.15e-11
10^{-6}	2	143.87	0	5.84e-06	1.77e-11	143.25	0	5.84e-06	1.77e-11
	4	135.90	0	5.84e-06	3.27e-12	145.40	0	5.84e-06	3.27e-12
	8	146.07	0	5.84e-06	3.10e-12	141.79	0	5.84e-06	3.10e-12
	16	149.80	0	5.84e-06	2.90e-12	150.25	0	5.84e-06	2.90e-12
10^{-7}	2	144.64	0	5.84e-06	1.81e-11	151.96	0	5.84e-06	1.81e-11
	4	148.19	0	5.84e-06	2.44e-12	155.19	0	5.84e-06	2.44e-12
	8	149.64	0	5.84e-06	1.06e-12	158.31	0	5.84e-06	1.06e-12
	16	152.25	0	5.84e-06	8.76e-13	158.06	0	5.84e-06	8.76e-13

Table 25 V-Cycle based on Inexact Uzawa, $N = 2048$, Part 1

τ	v	$L = 4$				$L = 16$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
10^{-4}	2	1047.23	0	1.46e-06	7.39e-11	1021.30	0	1.46e-06	7.39e-11
	4	1023.21	0	1.46e-06	6.27e-11	1058.60	0	1.46e-06	6.27e-11
	8	999.57	0	1.46e-06	5.95e-11	981.51	0	1.46e-06	5.95e-11
	16	1048.06	0	1.46e-06	6.18e-11	1038.22	0	1.46e-06	6.18e-11
10^{-5}	2	1101.02	0	1.46e-06	4.81e-11	1083.88	0	1.46e-06	4.81e-11
	4	1105.49	0	1.46e-06	8.57e-12	1092.12	0	1.46e-06	8.57e-12
	8	1117.22	0	1.46e-06	8.85e-12	1074.91	0	1.46e-06	8.85e-12
	16	1128.42	0	1.46e-06	7.10e-12	1155.35	0	1.46e-06	7.10e-12
10^{-6}	2	1198.27	0	1.46e-06	4.93e-11	1198.43	0	1.46e-06	4.93e-11
	4	1158.74	0	1.46e-06	7.57e-12	1166.27	0	1.46e-06	7.57e-12
	8	1152.78	0	1.46e-06	3.10e-12	1110.06	0	1.46e-06	3.10e-12
	16	1146.79	0	1.46e-06	2.99e-12	1230.92	0	1.46e-06	2.99e-12
10^{-7}	2	1227.38	0	1.46e-06	5.00e-11	1211.92	0	1.46e-06	5.00e-11
	4	1240.82	0	1.46e-06	5.78e-12	1195.77	0	1.46e-06	5.78e-12
	8	1415.82	0	1.46e-06	7.55e-12	1397.70	0	1.46e-06	7.55e-12
	16	1805.99	0	1.46e-06	7.02e-12	1919.35	0	1.46e-06	7.02e-12

Table 26 V-Cycle based on Inexact Uzawa, $N = 2048$, Part 2

τ	v	$L = 64$				$L = 256$			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
10^{-4}	2	1010.59	0	1.46e-06	7.39e-11	1020.66	0	1.46e-06	7.39e-11
	4	1010.47	0	1.46e-06	6.27e-11	961.92	0	1.46e-06	6.27e-11
	8	1017.89	0	1.46e-06	5.95e-11	981.39	0	1.46e-06	5.95e-11
	16	1042.05	0	1.46e-06	6.18e-11	1043.80	0	1.46e-06	6.18e-11
10^{-5}	2	1103.14	0	1.46e-06	4.81e-11	1086.04	0	1.46e-06	4.81e-11
	4	1107.87	0	1.46e-06	8.57e-12	1071.65	0	1.46e-06	8.57e-12
	8	1114.54	0	1.46e-06	8.85e-12	1103.93	0	1.46e-06	8.85e-12
	16	1123.48	0	1.46e-06	7.10e-12	1135.06	0	1.46e-06	7.10e-12
10^{-6}	2	1164.47	0	1.46e-06	4.93e-11	1239.34	0	1.46e-06	4.93e-11
	4	1166.00	0	1.46e-06	7.57e-12	1166.77	0	1.46e-06	7.57e-12
	8	1202.36	0	1.46e-06	3.10e-12	1162.30	0	1.46e-06	3.10e-12
	16	1103.10	0	1.46e-06	2.99e-12	1128.49	0	1.46e-06	2.99e-12
10^{-7}	2	1253.61	0	1.46e-06	5.00e-11	1188.48	0	1.46e-06	5.00e-11
	4	1240.41	0	1.46e-06	5.78e-12	1236.98	0	1.46e-06	5.78e-12
	8	1432.49	0	1.46e-06	7.55e-12	1442.37	0	1.46e-06	7.55e-12
	16	2052.14	0	1.46e-06	7.02e-12	1850.57	0	1.46e-06	7.02e-12

if we select an appropriate τ, Inexact Uzawa converges in an incredibly fast speed, namely 2 iteration. This is consistent with our proof.

Table 27 Inexact Uzawa Iteration based on V-Cycle, $N = 64$

τ	v	$L = 4$				$L = 16$			
		time(s)	iter	e_N	$\|\|r_h\|\|_2$	time(s)	iter	e_N	$\|\|r_h\|\|_2$
10^{-6}	2	0.49	100	1.50e-03	4.61e-07	0.86	100	1.50e-03	1.29e-06
	4	0.21	100	1.50e-03	5.21e-07	0.35	100	1.50e-03	5.81e-07
	8	0.20	100	1.50e-03	1.69e-07	0.40	100	1.50e-03	8.68e-07
	16	0.58	100	1.50e-03	7.65e-07	0.54	100	1.50e-03	1.36e-07
10^{-7}	2	0.07	9	1.50e-03	8.17e-09	0.96	100	1.50e-03	9.21e-08
	4	0.21	100	1.50e-03	1.50e-07	0.40	100	1.50e-03	8.09e-08
	8	0.50	100	1.50e-03	6.77e-08	0.61	100	1.50e-03	9.16e-08
	16	0.05	2	1.50e-03	6.62e-09	0.77	100	1.50e-03	3.25e-08
10^{-8}	2	0.07	2	1.50e-03	6.46e-09	0.70	5	1.50e-03	5.30e-09
	4	0.03	2	1.50e-03	3.23e-09	0.34	8	1.50e-03	9.79e-09
	8	0.04	2	1.50e-03	3.64e-09	0.29	4	1.50e-03	8.74e-09
	16	0.05	2	1.50e-03	6.62e-09	0.28	4	1.50e-03	7.74e-09
10^{-9}	2	0.04	2	1.50e-03	7.39e-10	0.49	6	1.50e-03	2.80e-09
	4	0.04	2	1.50e-03	3.92e-10	0.38	4	1.50e-03	2.59e-09
	8	0.04	2	1.50e-03	9.14e-11	0.32	2	1.50e-03	2.03e-09
	16	0.06	2	1.50e-03	5.70e-11	0.30	2	1.50e-03	1.93e-09

For $N = 256, 512$, we set the stopping criterion $\epsilon = 10^{-7}$. We take $\tau = [10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}]$. The result of $N = 256$ is shown in Table 29 and 30; the result of $N = 512$ is shown in Table 31 and 32.

For $N = 1024, 2048$, we set the stopping criterion $\epsilon = 10^{-6}$. We take $\tau = [10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}]$. The result of $N = 1024$ is shown in Table 33 and 34; the result of $N = 2048$ is shown in Table 35 and 36.

With a proper choice of L and τ, i.e. $L = 4, \tau = 10^{-8}$, Ineaxt Uzawa based on V-Cycle multi-grid can converge in 2 iterations.

6 Conclusion

Our work for numerical solutions to Stokes equation ends up here, but new ideas for improvement and continuous passion for research will never fade. We give a brief summary of what we have done and an outlook of what we can improve.

We try to solve Stokes equation numerically. Through MAC scheme, we formulate a saddle point problem. However, the equation itself is undetermined. Therefore, we introduce V-Cycle multi-grid method and choose Distributive Gauss Seidel Iteration and Uzawa Iteration as smoother. For DGS, we find that whether to update in parallel or in sequence and the

Table 28 Inexact Uzawa Iteration based on V-Cycle, $N = 128$

τ	ν	$L = 4$				$L = 16$			
		time(s)	iter	e_N	$\|r_h\|_2$	time(s)	iter	e_N	$\|r_h\|_2$
10^{-6}	2	1.08	100	3.74e-04	1.37e-06	2.71	100	3.74e-04	5.40e-07
	4	1.21	100	3.74e-04	4.50e-07	2.16	100	3.74e-04	5.77e-07
	8	1.21	100	3.74e-04	5.35e-07	2.31	100	3.74e-04	5.57e-07
	16	2.93	100	3.74e-04	1.05e-06	3.64	100	3.74e-04	1.34e-06
10^{-7}	2	0.23	8	3.74e-04	9.24e-09	2.74	100	3.74e-04	5.32e-08
	4	1.19	100	3.74e-04	7.95e-08	2.39	100	3.74e-04	1.51e-07
	8	1.79	100	3.74e-04	6.19e-08	2.51	100	3.74e-04	1.13e-07
	16	0.23	2	3.74e-04	3.31e-09	3.83	100	3.74e-04	1.40e-07
10^{-8}	2	0.28	2	3.74e-04	9.30e-09	2.49	5	3.74e-04	9.45e-09
	4	0.16	3	3.74e-04	4.46e-09	1.63	4	3.74e-04	9.62e-09
	8	0.15	2	3.74e-04	1.78e-09	1.30	3	3.74e-04	9.57e-09
	16	0.21	2	3.74e-04	3.31e-09	1.16	3	3.74e-04	5.36e-09
10^{-9}	2	0.22	2	3.74e-04	1.06e-09	2.77	5	3.74e-04	3.38e-09
	4	0.18	2	3.74e-04	1.90e-10	1.84	3	3.74e-04	3.63e-09
	8	0.15	2	3.74e-04	1.78e-09	1.48	2	3.74e-04	1.85e-09
	16	0.25	2	3.74e-04	2.72e-11	1.32	2	3.74e-04	1.41e-09

Table 29 Inexact Uzawa Iteration based on V-Cycle, $N = 256$, Part 1

τ	ν	$L = 4$				$L = 16$			
		time(s)	iter	e_N	$\|r_h\|_2$	time(s)	iter	e_N	$\|r_h\|_2$
10^{-5}	4	4.75	100	9.69e-05	8.64e-06	6.84	100	9.36e-05	1.32e-05
	8	6.17	100	1.08e-04	6.83e-06	7.57	100	9.37e-05	1.78e-05
	16	6.56	100	9.35e-05	7.93e-07	12.00	100	9.46e-05	1.71e-05
	32	19.64	100	2.08e-04	1.66e-05	20.18	100	9.69e-05	1.38e-05
10^{-6}	4	4.06	100	9.34e-05	6.09e-07	7.00	100	9.34e-05	1.34e-06
	8	6.54	100	9.35e-05	5.54e-07	8.48	100	9.34e-05	1.86e-06
	16	11.37	100	9.48e-05	7.93e-07	12.46	100	9.34e-05	1.46e-06
	32	1.11	2	9.34e-05	5.36e-08	4.78	12	9.44e-05	4.62e-08
10^{-7}	4	0.42	2	9.35e-05	5.33e-08	5.04	13	9.34e-05	9.75e-08
	8	0.47	2	9.35e-05	3.58e-08	4.27	5	9.34e-05	9.19e-08
	16	0.87	2	9.34e-05	6.71e-08	4.15	3	9.34e-05	5.27e-08
	32	1.12	2	9.34e-05	5.36e-08	3.74	3	9.34e-05	4.03e-08
10^{-8}	4	0.49	2	9.34e-05	6.45e-09	5.31	8	9.34e-05	9.89e-08
	8	0.57	2	9.34e-05	8.85e-10	5.20	5	9.34e-05	1.41e-08
	16	0.79	2	9.34e-05	1.66e-09	3.78	2	9.34e-05	6.60e-08
	32	1.58	2	9.34e-05	2.17e-10	4.78	2	9.34e-05	8.07e-09

Table 30 Inexact Uzawa Iteration based on V-Cycle, $N = 256$, Part 2

τ	v	$L = 64$			
		time(s)	iter	e_N	$\|r_h\|_2$
10^{-5}	4	41.47	100	9.64e-05	1.02e-05
	8	33.61	100	9.49e-05	9.97e-06
	16	35.42	100	9.37e-05	6.00e-06
	32	39.29	100	9.52e-05	7.73e-06
10^{-6}	4	56.82	100	9.35e-05	1.01e-06
	8	49.86	100	9.34e-05	9.94e-07
	16	48.81	100	9.34e-05	9.58e-07
	32	52.27	100	9.34e-05	9.23e-07
10^{-7}	4	62.65	100	9.34e-05	5.13e-07
	8	51.84	49	9.39e-05	8.37e-08
	16	55.93	30	9.34e-05	9.86e-08
	32	55.21	21	9.34e-05	9.93e-08
10^{-8}	4	61.87	100	9.34e-05	5.13e-07
	8	49.62	49	9.39e-05	8.37e-08
	16	55.51	30	9.34e-05	9.54e-08
	32	59.74	16	9.34e-05	3.50e-08

Table 31 Inexact Uzawa Iteration based on V-Cycle, $N = 512$, Part 1

τ	v	$L = 4$				$L = 16$			
		time(s)	iter	e_N	$\|r_h\|_2$	time(s)	iter	e_N	$\|r_h\|_2$
10^{-5}	4	17.29	100	5.94e-05	8.50e-06	25.05	100	2.86e-05	1.11e-05
	8	24.10	100	1.13e-04	1.27e-05	32.37	100	2.48e-05	1.02e-05
	16	49.34	100	1.90e-04	1.06e-05	53.15	100	3.02e-05	6.01e-06
	32	94.73	100	2.04e-04	1.05e-05	89.32	100	4.87e-05	6.13e-06
10^{-6}	4	16.86	100	2.39e-05	7.00e-07	28.54	100	2.35e-05	1.22e-06
	8	27.11	100	2.56e-05	1.58e-06	36.82	100	2.34e-05	1.03e-06
	16	45.20	100	3.06e-05	1.09e-06	56.12	100	2.34e-05	5.35e-07
	32	5.26	2	2.34e-05	2.72e-08	18.87	11	3.21e-05	6.00e-08
10^{-7}	4	1.81	2	2.35e-05	2.66e-08	19.75	10	2.34e-05	9.06e-08
	8	1.97	2	2.34e-05	1.79e-08	17.52	5	2.33e-05	9.65e-08
	16	3.69	3	2.34e-05	4.87e-08	16.19	4	2.34e-05	7.50e-08
	32	5.23	2	2.34e-05	2.72e-08	15.85	2	2.36e-05	9.07e-08
10^{-8}	4	2.04	2	2.34e-05	3.22e-09	19.39	7	2.36e-05	8.60e-08
	8	2.08	2	2.34e-05	1.79e-08	18.96	4	2.34e-05	5.92e-08
	16	3.59	2	2.34e-05	8.32e-10	18.00	2	2.34e-05	3.21e-08
	32	5.71	2	2.33e-05	4.02e-09	19.14	2	2.34e-05	1.11e-08

Table 32 Inexact Uzawa Iteration based on V-Cycle, $N = 512$, Part 2

τ	ν	$L = 64$			
		time(s)	iter	e_N	$\|r_h\|_2$
10^{-5}	4	147.37	100	5.03e-05	1.01e-05
	8	131.69	100	3.74e-05	9.20e-06
	16	130.21	100	4.75e-05	5.90e-06
	32	167.52	100	3.49e-05	6.95e-06
10^{-6}	4	217.17	100	2.39e-05	1.00e-06
	8	190.83	100	2.36e-05	1.22e-06
	16	194.70	100	2.35e-05	8.26e-07
	32	225.65	100	2.36e-05	1.36e-06
10^{-7}	4	268.20	99	2.38e-05	6.69e-08
	8	200.79	42	3.19e-05	5.77e-08
	16	214.81	26	2.42e-05	2.76e-08
	32	210.97	13	2.45e-05	4.07e-08
10^{-8}	4	262.28	99	2.38e-05	6.69e-08
	8	190.76	42	3.19e-05	5.77e-08
	16	216.79	26	2.40e-05	1.14e-08
	32	208.72	13	2.45e-05	4.07e-08

Table 33 Inexact Uzawa Iteration based on V-Cycle, $N = 1024$, Part 1

τ	ν	$L = 4$				$L = 16$			
		time(s)	iter	e_N	$\|r_h\|_2$	time(s)	iter	e_N	$\|r_h\|_2$
10^{-4}	2	10.89	10	6.50e-04	9.54e-05	23.36	10	1.38e-03	8.71e-05
	4	16.98	10	2.23e-04	3.50e-05	29.16	10	1.30e-03	5.85e-05
	8	31.63	10	1.15e-04	1.66e-05	38.05	10	5.18e-03	7.01e-05
	16	68.72	10	6.98e-04	1.50e-04	75.19	10	6.29e-03	4.56e-05
10^{-5}	2	14.56	10	4.91e-04	1.17e-05	38.11	10	2.13e-04	7.12e-06
	4	22.34	10	4.04e-04	5.97e-06	38.84	10	3.11e-04	1.26e-05
	8	34.69	10	5.65e-04	1.65e-05	47.50	8	3.49e-04	9.94e-07
	16	56.70	10	5.69e-05	7.99e-06	81.49	10	8.06e-04	6.21e-06
10^{-6}	2	6.65	2	1.76e-05	3.75e-07	48.03	4	3.67e-05	9.14e-07
	4	10.92	2	6.20e-06	5.31e-08	44.81	4	3.51e-05	5.64e-07
	8	17.57	2	7.05e-06	4.28e-07	45.05	3	3.42e-05	4.87e-07
	16	38.08	3	5.75e-05	8.53e-07	55.27	3	4.32e-05	3.59e-07
10^{-7}	2	8.20	2	5.90e-06	8.93e-09	59.02	2	1.30e-05	1.71e-07
	4	10.61	2	6.20e-06	5.31e-08	56.92	2	1.09e-05	1.30e-07
	8	19.60	2	5.88e-06	1.37e-08	56.75	2	1.13e-05	1.29e-07
	16	38.83	2	5.84e-06	3.48e-09	57.10	2	1.15e-05	1.30e-07

Table 34 Inexact Uzawa Iteration based on V-Cycle, $N = 1024$, Part 2

τ	v	$L = 64$			
		time(s)	iter	e_N	$\|r_h\|_2$
10^{-4}	2	216.33	10	7.62e-04	9.50e-05
	4	245.28	10	4.34e-04	9.03e-05
	8	229.04	10	6.70e-04	8.58e-05
	16	239.70	10	3.02e-04	9.87e-05
10^{-5}	2	369.08	10	2.19e-04	1.02e-05
	4	392.86	10	8.44e-05	9.99e-06
	8	414.78	10	5.72e-05	1.00e-05
	16	421.97	10	5.89e-05	1.02e-05
10^{-6}	2	562.12	6	1.90e-04	3.80e-07
	4	644.87	6	6.84e-06	9.72e-07
	8	595.40	4	1.14e-05	9.46e-07
	16	567.34	3	1.17e-05	9.58e-07
10^{-7}	2	579.47	6	1.90e-04	3.80e-07
	4	801.35	5	2.22e-05	1.10e-07
	8	732.48	3	3.43e-05	2.32e-07
	16	761.37	2	1.94e-05	2.67e-07

Table 35 Inexact Uzawa Iteration based on V-Cycle, $N = 2048$, Part 1

τ	v	$L = 4$				$L = 16$			
		time(s)	iter	e_N	$\|r_h\|_2$	time(s)	iter	e_N	$\|r_h\|_2$
10^{-4}	2	38.38	10	6.51e-04	4.84e-05	78.45	10	2.45e-03	6.81e-05
	4	67.83	10	2.24e-04	1.79e-05	105.02	10	4.06e-03	5.92e-05
	8	121.73	10	1.15e-04	8.57e-06	152.23	10	8.33e-03	8.21e-05
	16	262.27	10	1.41e-03	1.58e-04	289.08	10	1.03e-02	3.12e-05
10^{-5}	2	54.43	10	9.62e-04	6.24e-06	126.59	10	4.02e-04	6.67e-06
	4	78.66	10	1.08e-03	1.70e-05	145.62	10	4.55e-04	3.84e-06
	8	113.89	10	1.15e-04	8.57e-06	189.04	10	1.01e-03	5.98e-06
	16	111.45	2	5.74e-05	8.70e-07	314.00	10	6.87e-04	1.20e-05
10^{-6}	2	25.76	2	1.62e-05	1.88e-07	159.59	3	7.95e-05	9.82e-07
	4	45.94	2	2.38e-06	2.66e-08	162.70	4	6.21e-05	7.27e-07
	8	73.65	3	1.15e-04	8.44e-07	169.29	3	5.69e-05	3.22e-07
	16	110.49	2	5.74e-05	8.70e-07	182.86	2	8.00e-05	5.26e-07
10^{-7}	2	29.52	2	1.86e-05	1.87e-07	207.40	2	2.55e-05	1.88e-07
	4	44.01	2	2.38e-06	2.66e-08	202.14	2	2.02e-05	1.72e-07
	8	77.92	2	1.55e-06	6.88e-09	191.94	2	1.77e-05	1.66e-07
	16	133.28	2	1.77e-06	5.52e-08	230.72	2	9.99e-06	6.47e-08

Table 36 Inexact Uzawa Iteration based on V-Cycle, $N = 2048$, Part 2

τ	v	$L = 64$			
		time(s)	iter	e_N	$\|r_h\|_2$
10^{-4}	2	699.13	10	2.50e-03	9.92e-05
	4	736.27	10	1.82e-03	8.72e-05
	8	724.34	10	2.10e-03	7.02e-05
	16	834.34	10	1.61e-03	9.32e-05
10^{-5}	2	1236.89	10	2.46e-04	1.01e-05
	4	1432.63	10	1.13e-04	8.89e-06
	8	1380.31	10	1.29e-04	8.07e-06
	16	1535.63	10	1.46e-04	5.69e-06
10^{-6}	2	2153.53	6	1.84e-04	2.09e-07
	4	2346.03	5	9.82e-06	9.82e-07
	8	1989.82	3	1.76e-05	9.48e-07
	16	2169.04	3	1.86e-05	9.41e-07
10^{-7}	2	2206.19	6	1.90e-04	1.82e-07
	4	2670.11	4	1.45e-04	8.49e-07
	8	2513.53	2	7.59e-05	5.70e-07
	16	2845.36	2	3.93e-05	2.83e-07

management of edge units and point units influence the convergence speed of the algorithm. We also have other selection of restricting and lifting operators but it affects the converge speed slightly. The numerical results of DGS are not ideal. In Appendix A, we show that projecting the residual of $\mathbf{B}^T\mathbf{U} = 0$ will greatly improve the performance of DGS, especially DGS-s. With this modification on V-Cycle, DGS-s outperforms Uzawa and Inexact Uzawa.

For Uzawa, we observe and rigorously prove that Uzawa will converge in at most 2 iterations. This is because $\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ is projection matrix, which can be proved through spectral analysis. Our numerical experiments are consistent with this proposition. Further numerical results show that even with inexact solution to the subproblem, Inexact Uzawa can still converge in at most 2 iterations. That means using Uzawa as smoother in V-Cycle is unnecessary. Uzawa itself can solve the saddle point problem efficiently. Specifically, Uzawa turns solving the original undetermined problem into solving a determined subproblem. (Because $\mathbf{A}$ is positive definite.)
in matlab and Conjugate Gradient method may not be the ideal solver for solving the subproblem. We can enhance the speed of inexact Uzawa by choosing efficient solver for positive definite problems, i.e. spectral methods.

A Appendix: Modification of V-Cycle based on DGS

From the numerical experiment, we find that V-Cycle based on DGS does not perform as good as we expected. Therefore, we consider to modify the V-Cycle multigrid a little bit. Note that we have not used the restriction operator for P. We consider to project the residual of $\mathbf{B}^T\mathbf{U} = 0$ to the coarse grid. Specifically, for the saddle point problem on the coarse grid, we do not consider to solve 41. Instead, we solve

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{R} \end{bmatrix}. \tag{109}$$

where $\mathbf{R}$ is the residual of $\mathbf{B}^T\mathbf{U} = 0$ from the finer grid. For DGS-p and DGS-s, we only need to add $R_{i,j}$ to 54. And the calculation of r_h in the V-Cycle turns to be

$$r_h = h^2 \begin{bmatrix} \mathbf{F}^{(k)} - \mathbf{A}^{(k)}\mathbf{U}^{(k)} + \mathbf{B}^{(k)}\mathbf{P}^{(k)} \\ -\left(\mathbf{B}^{(k)}\right)^T \mathbf{U}^{(k)} \end{bmatrix} \tag{110}$$

where $k = 0$. The numerical experiments show that this modification achieves great improvement.

Similarly, for $N = 64, 128, 256$, we set the stopping criterion $\epsilon = 10^{-8}$ and the maximum number of V-Cycle to be 500. We take $v_1 = v_2 = v = [10, 20, 40, 80, 160]$ and $L = [4, 16, 64]$. The result of $N = 64$ is shown in Table 37; the result of $N = 128$ is shown in Table 38; the result of $N = 256$ is shown in Table 39. We observe that with modification, DGS-s converges much faster. Though DGS-p becomes slower, it achieves lower e_N.

Table 37 V-Cycle based on DGS with modification, $N = 64$

L	v	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|\|r_h\|\|_2$	time(s)	VC	e_N	$\|\|r_h\|\|_2$
4	10	0.56	5	1.50e-03	6.02e-09	4.43	500	1.50e-03	4.82e-07
	20	0.09	5	1.50e-03	2.74e-10	6.60	500	1.50e-03	4.51e-07
	40	0.14	4	1.50e-03	7.76e-10	12.19	500	1.50e-03	4.24e-07
	80	0.17	3	1.50e-03	6.72e-09	23.04	500	1.50e-03	3.88e-07
	160	0.32	3	1.50e-03	1.87e-10	45.07	500	1.50e-03	3.42e-07
16	10	0.30	35	1.50e-03	9.15e-09	3.46	500	1.50e-03	4.27e-07
	20	0.22	18	1.50e-03	5.57e-09	5.95	500	1.50e-03	4.12e-07
	40	0.21	9	1.50e-03	5.70e-09	10.92	500	1.50e-03	4.00e-07
	80	0.23	5	1.50e-03	1.50e-09	20.79	500	1.50e-03	3.77e-07
	160	0.29	3	1.50e-03	2.14e-10	40.61	500	1.50e-03	3.40e-07

For $N = 512, 1024, 2048$, we set the stopping criterion $\epsilon = 10^{-6}$ and the maximum number of V-Cycle to be 100. We take $L = [4, 16, 64, 256]$. For $N = 512, 1024$, we take

Table 38 V-Cycle based on DGS with modification, $N = 128$

L	v	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
4	10	0.19	5	3.74e-04	4.77e-09	18.42	500	3.74e-04	1.25e-07
	20	0.30	4	3.74e-04	8.40e-09	33.51	500	3.74e-04	7.98e-08
	40	0.59	4	3.74e-04	5.32e-10	63.88	500	3.74e-04	5.97e-08
	80	0.74	3	3.74e-04	4.78e-09	119.61	500	3.74e-04	5.55e-08
	160	1.48	3	3.74e-04	5.64e-10	234.04	500	3.74e-04	5.21e-08
16	10	0.94	32	3.74e-04	7.45e-09	17.91	500	3.74e-04	1.26e-07
	20	0.97	16	3.74e-04	6.65e-09	32.45	500	3.74e-04	7.78e-08
	40	0.88	8	3.74e-04	7.57e-09	62.23	500	3.74e-04	5.76e-08
	80	1.18	5	3.74e-04	4.15e-10	116.83	500	3.74e-04	5.45e-08
	160	1.50	3	3.74e-04	5.61e-10	231.85	500	3.74e-04	5.18e-08
64	10	12.18	500	3.73e-04	1.08e-07	14.28	500	1.16e-03	2.70e-05
	20	12.86	306	3.74e-04	9.60e-09	27.03	500	3.74e-04	5.92e-07
	40	12.61	153	3.74e-04	9.42e-09	50.18	500	3.74e-04	1.71e-08
	80	12.75	76	3.74e-04	9.97e-09	97.00	500	3.74e-04	1.92e-08
	160	11.79	38	3.74e-04	9.44e-09	185.70	500	3.74e-04	2.95e-08

Table 39 V-Cycle based on DGS, $N = 256$

L	v	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
4	10	0.85	5	9.34e-05	3.52e-09	72.16	500	9.34e-05	3.13e-08
	20	1.15	4	9.34e-05	5.58e-09	132.40	500	9.34e-05	2.34e-08
	40	2.28	4	9.34e-05	3.40e-10	251.10	500	9.34e-05	1.57e-08
	80	3.46	3	9.34e-05	2.96e-09	470.66	494	9.34e-05	1.00e-08
	160	6.90	3	9.34e-05	3.49e-10	505.26	265	9.34e-05	9.98e-09
16	10	3.91	30	9.34e-05	7.93e-09	71.17	500	9.34e-05	3.17e-08
	20	3.88	15	9.34e-05	7.22e-09	131.11	500	9.34e-05	2.35e-08
	40	4.18	8	9.34e-05	3.84e-09	249.09	500	9.34e-05	1.57e-08
	80	4.47	4	9.34e-05	5.23e-09	478.46	490	9.34e-05	9.99e-09
	160	6.84	3	9.34e-05	3.46e-10	495.53	264	9.34e-05	9.99e-09
64	10	56.10	460	9.34e-05	9.96e-09	68.68	500	4.40e-04	5.29e-06
	20	54.83	231	9.34e-05	9.50e-09	124.10	500	9.35e-05	5.34e-08
	40	53.22	115	9.34e-05	9.93e-09	238.87	500	9.34e-05	1.71e-08
	80	53.11	58	9.34e-05	8.78e-09	406.60	432	9.34e-05	1.00e-08
	160	52.94	29	9.34e-05	8.48e-09	423.75	237	9.34e-05	9.98e-09

$\nu = [20, 40, 80, 160]$. For $N = 2048$, we take $\nu = [40, 80, 160]$. The result of $N = 512$ is shown in Table 40; the result of $N = 1024$ is shown in Table 41; the result of $N = 2048$ is shown in Table 42.

Table 40 V-Cycle based on DGS with modification, $N = 512$

L	ν	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|r_h\|_2$	time(s)	VC	e_N	$\|r_h\|_2$
4	20	4.73	3	2.31e-05	1.19e-07	4.23	3	2.31e-05	3.29e-08
	40	8.06	3	2.33e-05	1.45e-08	4.88	2	1.46e-05	8.63e-07
	80	10.49	2	2.06e-05	3.00e-07	9.51	2	2.07e-05	2.19e-07
	160	21.51	2	2.27e-05	7.53e-08	18.69	2	2.27e-05	5.92e-08
16	20	11.09	9	1.75e-05	6.89e-07	17.33	17	1.19e-04	7.17e-07
	40	12.20	5	2.30e-05	4.33e-07	15.53	8	1.63e-04	9.93e-07
	80	15.16	3	2.32e-05	7.62e-08	16.61	4	1.64e-04	9.98e-07
	160	21.57	2	2.26e-05	7.55e-08	25.43	3	2.58e-05	7.09e-08
64	20	110.65	100	3.24e-04	1.38e-05	96.36	100	6.21e-03	4.12e-05
	40	144.37	68	7.62e-06	8.71e-07	175.74	100	9.16e-04	5.63e-06
	80	138.49	34	7.56e-06	8.65e-07	253.55	72	1.58e-04	9.62e-07
	160	139.21	17	7.49e-06	8.55e-07	250.91	36	1.57e-04	9.51e-07
256	20	86.36	100	7.07e-01	3.29e-02	71.76	100	7.07e-01	3.29e-02
	40	166.34	100	4.80e-01	1.87e-02	136.37	100	4.80e-01	1.87e-02
	80	323.21	100	2.25e-01	8.68e-03	270.08	100	2.25e-01	8.69e-03
	160	645.30	100	4.95e-02	1.91e-03	513.07	100	5.14e-02	1.91e-03

Taking $L = 4$ and $\nu = 10, 20, 40$, DGS-s converges fast and converges to a precise solution. Notably, DGS-s is faster than Uzawa and Inexact Uzawa. This is because although Uzawa can converge in 2 iterations, solving the subproblem in Uzawa is quite time-costly. Also, because Uzawa can converge in 2 iterations, the modification of the V-Cycle multi-grid method does not affect the performance of Uzawa and Inexact Uzawa.

References

[1] Wang Ming and Chen Long. “Multigrid Methods for the Stokes Equations using Distributive Gauss–Seidel Relaxations based on the Least Squares Commutator”. In: *J Sci Comput* (2013).

Table 41 V-Cycle based on DGS with modification, $N = 1024$

L	v	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|\|r_h\|\|_2$	time(s)	VC	e_N	$\|\|r_h\|\|_2$
4	20	16.95	3	5.59e-06	6.58e-08	15.92	3	5.62e-06	1.33e-08
	40	23.48	2	5.46e-06	6.11e-07	21.20	2	7.88e-06	4.34e-07
	80	42.99	2	3.17e-06	1.51e-07	44.59	2	3.56e-06	1.10e-07
	160	82.95	2	5.16e-06	3.77e-08	80.66	2	5.18e-06	2.74e-08
16	20	43.26	9	3.19e-06	3.44e-07	65.44	15	2.31e-04	7.12e-07
	40	49.68	5	9.06e-06	2.18e-07	60.38	7	3.19e-04	9.84e-07
	80	60.03	3	5.90e-06	3.80e-08	73.96	4	1.62e-04	4.97e-07
	160	84.94	2	5.13e-06	3.78e-08	82.87	2	1.71e-04	5.25e-07
64	20	447.63	100	3.09e-04	6.27e-06	411.45	100	6.09e-03	2.00e-05
	40	572.14	62	4.27e-05	9.64e-07	792.66	100	8.75e-04	2.69e-06
	80	513.47	31	4.25e-05	9.60e-07	981.49	63	3.04e-04	9.34e-07
	160	531.45	16	2.84e-05	6.98e-07	985.54	32	2.77e-04	8.52e-07
256	20	424.79	100	6.40e-01	1.36e-02	392.79	100	6.40e-01	1.36e-02
	40	931.07	100	3.98e-01	7.69e-03	790.97	100	3.98e-01	7.69e-03
	80	1562.02	100	1.55e-01	2.98e-03	1468.00	100	1.55e-01	2.98e-03
	160	3099.83	100	2.34e-02	4.51e-04	2908.67	100	2.68e-02	4.54e-04

Table 42 V-Cycle based on DGS with modification, $N = 2048$

L	v	DGS-s (Notes)				DGS-p, [1]			
		time(s)	VC	e_N	$\|\|r_h\|\|_2$	time(s)	VC	e_N	$\|\|r_h\|\|_2$
4	40	112.39	2	9.43e-06	3.05e-07	98.97	2	1.10e-05	2.18e-07
	80	234.75	2	1.51e-06	7.43e-08	196.44	2	2.14e-06	5.50e-08
	160	410.99	2	8.01e-07	1.84e-08	372.31	2	8.98e-07	1.37e-08
16	40	195.50	4	4.77e-05	6.28e-07	240.58	6	6.32e-04	9.75e-07
	80	240.15	2	6.12e-05	6.99e-07	268.44	3	6.36e-04	9.80e-07
	160	436.21	2	7.74e-07	1.85e-08	347.41	2	1.71e-04	2.62e-07
64	40	2431.91	58	8.63e-05	8.74e-07	3634.53	100	8.65e-04	1.33e-06
	80	2557.25	29	8.61e-05	8.70e-07	3929.72	54	6.22e-04	9.55e-07
	160	2419.02	15	6.25e-05	6.34e-07	3623.19	27	6.15e-04	9.44e-07
256	40	4521.92	100	3.79e-01	3.67e-03	3696.90	100	3.80e-01	3.67e-03
	80	9194.71	100	1.41e-01	1.36e-03	7159.15	100	1.41e-01	1.36e-03
	160	16915.65	100	1.94e-02	1.87e-04	13005.16	100	2.32e-02	1.89e-04