# Random matrix theory

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## 1 Problem Setup

We explore the spectral property of random matrix. We first form GOE($N$). It generates $N \times N$ real matrix $H$ with $H = H^T$. $H_{ij} \sim \mathcal{N}(0,1)$, $1 \le i < j \le N$ and $H_{ii} \sim \mathcal{N}(0,2)$, $1 \le i \le N$. The upper triangular elements are independent.
Then, we form GUE($N$). It generates $N \times N$ complex matrix $H$ with $H = H^*$. $H_{jk} \sim \mathcal{N}(0,1/2) + i\mathcal{N}(0,1/2)$, $1 \le j < k \le N$ and $H_{jj} \sim \mathcal{N}(0,1)$, $1 \le j \le N$. The upper triangular elements are independent.

## 2 Numerical experiments and analytical study about GOE($N$)

In actual implementation, to generate GOE($N$), we first generate a $N \times N$ matrix $A$ with $A_{ij} \sim \mathcal{N}(0,1)$, $1 \le i, j \le N$. The elements in $A$ are independent. We denote $A = D - L - U$, where $D$ is a diagonal matrix, $L$ is a lower triangular matrix, and $U$ is an upper triangular matrix. Then, we let $H = \sqrt{2}D - L - L^T$. $H$ satisfies the condition of GOE($N$).

### 2.1 Problem (b)

We compute eigenvalues $\lambda_n$, $1 \le n \le N$. We use *numpy.linalg.eigvalsh* to implement this. Because symmetric matrix only has real eigenvalue, we can sort them in the increasing order. Then, we define the normalized spacing

$$\hat{s}_n = (\lambda_{n+1} - \lambda_n)/\langle s_n \rangle$$

where $\langle s_n \rangle = \langle \lambda_{n+1} - \lambda_n \rangle$ is the empirical mean spacing, $1 \le n \le N-1$.

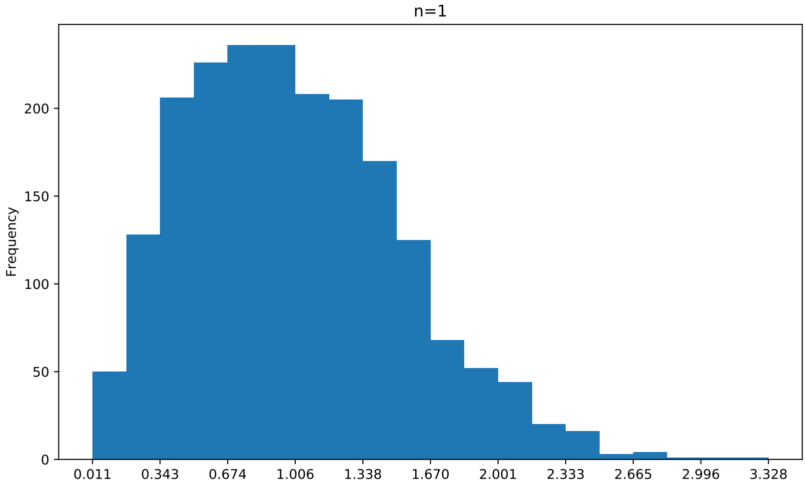


Figure 1 $N = 2$, $K = 2000$. We have $\langle s_1 \rangle = 2.49$

For $N = 2$, we sample $K = 2000$ times to compute $\langle s_n \rangle$. Then, based on the result from the previous sampling, we plot the histogram of the eigenvalue spacings in the figure 1.

For $N = 4$, we sample $K = 4000$ times to compute $\langle s_n \rangle$. Then, based on the result from the previous sampling, we plot the histogram of the eigenvalue spacings in the figure 2.

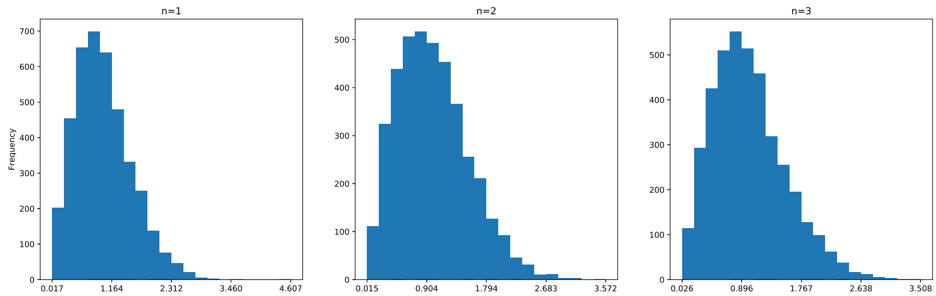


Figure 2 $N = 4$, $K = 4000$. We have $\langle s_1 \rangle = 1.89$, $\langle s_2 \rangle = 1.66$, $\langle s_3 \rangle = 1.88$

For $N = 10$, we sample $K = 10000$ times to compute $\langle \lambda_{n+1} - \lambda_n \rangle$. Then, based on the result from the previous sampling, we plot the histogram of the eigenvalue spacings in the figure 3.

We also plot the value of $\langle s_n \rangle$ in figure 4.

From figure 2 and figure 3, we guess that with fixed $N$, $\hat{s}_n$ might follow the same distribution with different $n$.

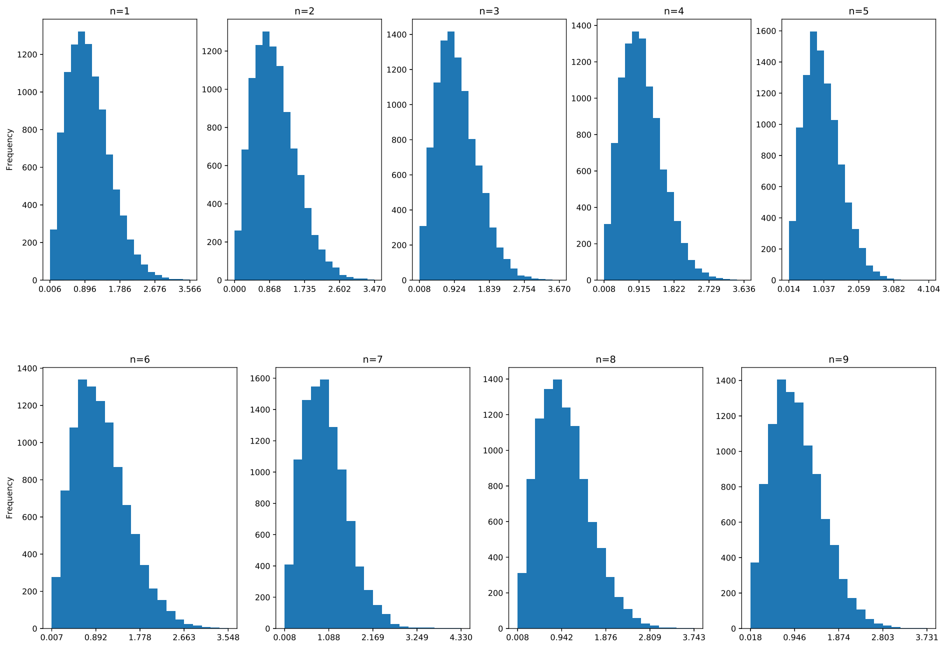


Figure 3 $N = 10,\ K = 10000$

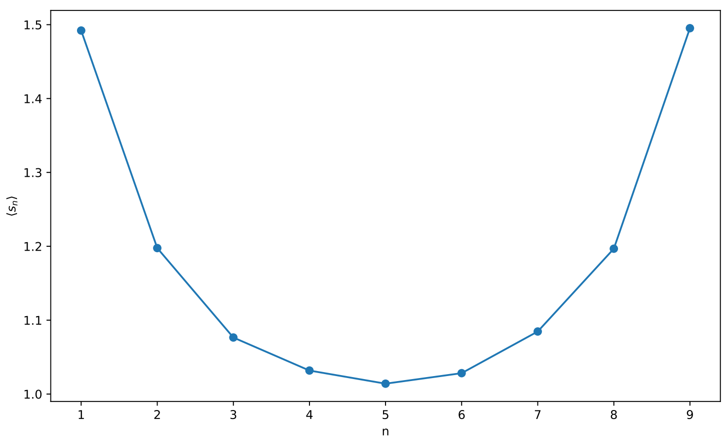


Figure 4 $N = 10,\ K = 10000$

## 2.2 Problem (c)

When $N = 2$, suppose $H$ is generated by GOE($N$). Suppose $H = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Then, the characteristic polynomial for $H$ is

$$\lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

We can directly compute the eigenvalue spacing

$$s_1 = \lambda_2 - \lambda_1 = \sqrt{(a+c)^2 - 4(ac - b^2)} = \sqrt{(a-c)^2 + 4b^2}$$

Because $a, b, c$ are independent, $a - c$ and $b$ are independent. We have $\frac{a-c}{2} \sim \mathcal{N}(0,1)$, $b \sim \mathcal{N}(0,1)$. Let us denote $d = \frac{a-c}{2}$. We have $s_1 = 2\sqrt{b^2 + d^2}$ Then, for any $r_0 > 0$,

$$p(s_1 \le r_0) = = p(b^2 + d^2 \le \frac{r_0^2}{4}) = \int_{x^2+y^2 \le \frac{r_0^2}{4}} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dxdy$$
$$= \int_0^{2\pi} \int_0^{\frac{r_0}{2}} \frac{1}{2\pi} e^{-\frac{r^2}{2}} r dr d\theta = e^{-\frac{r^2}{2}} \Big|_0^{\frac{r_0}{2}} = 1 - e^{-\frac{r_0^2}{8}}$$

Therefore, the probability density for $s_1$ is

$$p_{s_1}(x) = \frac{1}{4} x e^{-\frac{x^2}{8}}$$

Therefore, we have

$$\langle s_1 \rangle = \int_0^{\infty} \frac{1}{4} x^2 e^{-\frac{x^2}{8}} dx = \frac{\sqrt{8\pi}}{8} \int_{-\infty}^{\infty} \frac{1}{\sqrt{8\pi}} x^2 e^{-\frac{x^2}{8}} dx = \frac{\sqrt{8\pi}}{8} \cdot 4 = \sqrt{2\pi}$$

Because $s_1' = \frac{s_1}{\langle s_1 \rangle}$, we have

$$p_{s_1'}(x) = \sqrt{2\pi} p_{s_1}(\sqrt{2\pi} x) = \frac{\pi}{2} x e^{-\frac{\pi x^2}{4}}$$

We compare this analytical result with numerical experiments. We take sample number $K = 20000$. We compute $\langle s_1 \rangle$ and plot the histogram of $s_1'$ in the figure 5.

The estimated value of $\langle s_1 \rangle = 2.499$ is quite close to the analytical value $\sqrt{2\pi} \approx 2.507$. And from figure 5, our analytical result fits the numerical result closely.

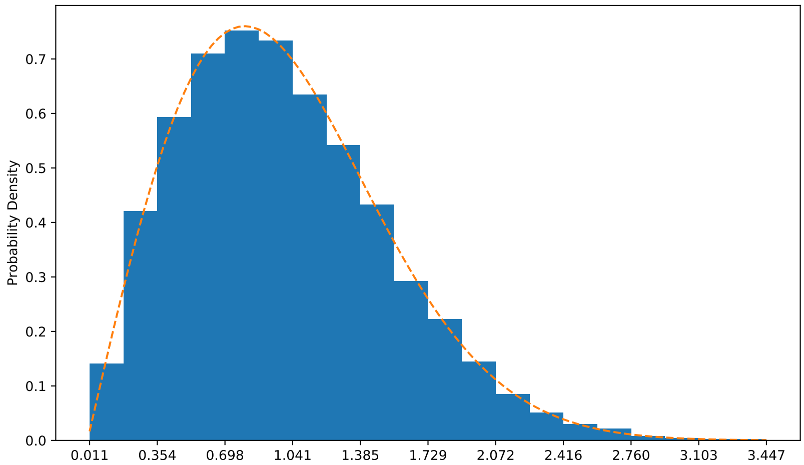


Figure 5 $N = 2,\ K = 20000$. We have $\langle s_1 \rangle = 2.499$

## 2.3 Problem(d)

A natural way to define the probability density for $H$ is

$$\begin{aligned}\rho(H) =& \prod_{i=1}^{N} \left(\frac{1}{\sqrt{4\pi}} e^{-\frac{(H_{ii})^2}{4}}\right) \prod_{1 \le i < j \le N} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(H_{ij})^2}{2}}\right) \\ =& 2^{-\frac{N}{2}} (2\pi)^{-\frac{N(N+1)}{2}} \exp\left( -\frac{1}{4} \left(\sum_{i=1}^{N} (H_{ii})^2 + \sum_{1 \le i < j \le N} 2(H_{ij})^2\right)\right) \\ =& 2^{-\frac{N}{2}} (2\pi)^{-\frac{N(N+1)}{2}} \exp\left( -\frac{1}{4} \sum_{i=1}^{N} \sum_{j=1}^{N} (H_{ij})^2 \right) \\ =& 2^{-\frac{N}{2}} (2\pi)^{-\frac{N(N+1)}{2}} \exp\left( -\frac{1}{4} \|H\|_F^2 \right)\end{aligned}$$

where $\|H\|_F$ is the Frobenius norm of $H$.

We show that $\|H\|_F^2 = \sum_{n=1}^{N} \lambda_n^2$. Suppose $H = V \Lambda V^T$ is the eigenvalue decomposition of $H$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and $V$ is $N \times N$ orthogonal matrix. Then, we have

$$\begin{aligned}\|H\|_F^2 =& \mathrm{tr}(H^T H) = \mathrm{tr}(V \Lambda V^T V \Lambda V^T) = \mathrm{tr}(V \Lambda^2 V^T) \\ =& \mathrm{tr}(\Lambda^2 V^T V) = \mathrm{tr}(\Lambda^2) = \sum_{n=1}^{N} \lambda_n^2\end{aligned}$$

And we know that for any orthogonal matrix $Q$, $Q^T H Q$ has similar eigenvalues with $H$.

Therefore, $\|Q^T HQ\|_F^2 = \|H\|_F^2$, $\rho(Q^T HQ) = \rho(H)$.
From our definition of $\rho(H)$, we observer that the value of $\rho(H)$ only depends on eigenvalues of $H$.

# 3 Numerical experiments and analytical study about GUE($N$)

In actual implementation, to generate GUE($N$), we first generate a $N \times N$ matrix $A$ with $A_{jk} \sim \mathcal{N}(0, 1/2)$, $1 \leq j, k \leq N$. The elements in $A$ are independent. We denote $A = D - L - U$, where $D$ is a diagonal matrix, $L$ is a lower triangular matrix, and $U$ is an upper triangular matrix. Then, we let $H = \sqrt{2}D - (L + L^T) - i(U + U^T)$. $H$ satisfies the condition of GUE($N$).

## 3.1 Problem (b)

We compute eigenvalues $\lambda_n$, $1 \leq n \leq N$. We use *numpy.linalg.eigvalsh* to implement this. Because hermitian matrix only has real eigenvalue, we can sort them in the increasing order. Then, we define the normalized spacing

$$\hat{s}_n = (\lambda_{n+1} - \lambda_n) / \langle s_n \rangle$$

where $\langle s_n \rangle = \langle \lambda_{n+1} - \lambda_n \rangle$ is the empirical mean spacing, $1 \leq n \leq N - 1$.
For $N = 2$, we sample $K = 2000$ times to compute $\langle s_n \rangle$. Then, based on the result from the previous sampling, we plot the histogram of the eigenvalue spacings in the figure 6.

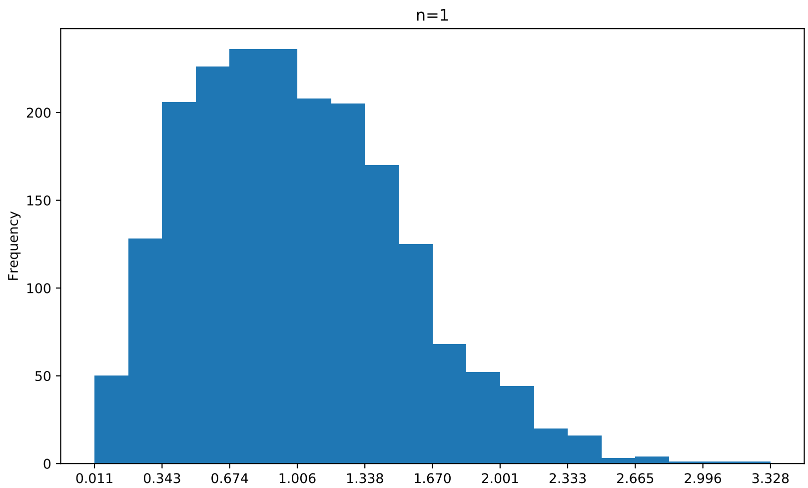


Figure 6 $N = 2$, $K = 2000$. We have $\langle s_1 \rangle = 2.24$

For $N = 4$, we sample $K = 4000$ times to compute $\langle s_n \rangle$. Then, based on the result from the previous sampling, we plot the histogram of the eigenvalue spacings in the figure 7.

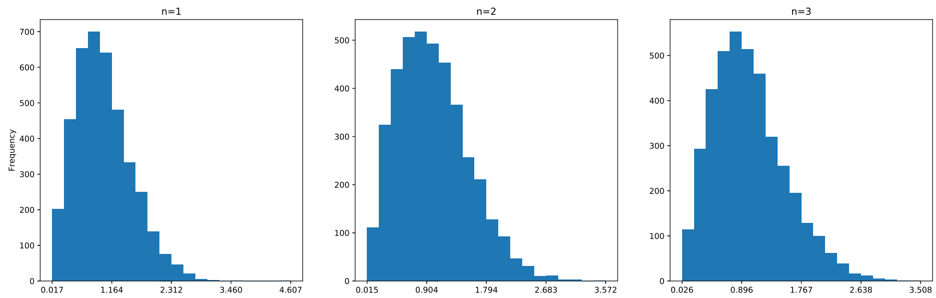

Figure 7  $N = 4$, $K = 4000$. We have $\langle s_1 \rangle = 1.75$, $\langle s_2 \rangle = 1.56$, $\langle s_3 \rangle = 1.74$

For $N = 10$, we sample $K = 10000$ times to compute $\langle \lambda_{n+1} - \lambda_n \rangle$. Then, based on the result from the previous sampling, we plot the histogram of the eigenvalue spacings in the figure 8.

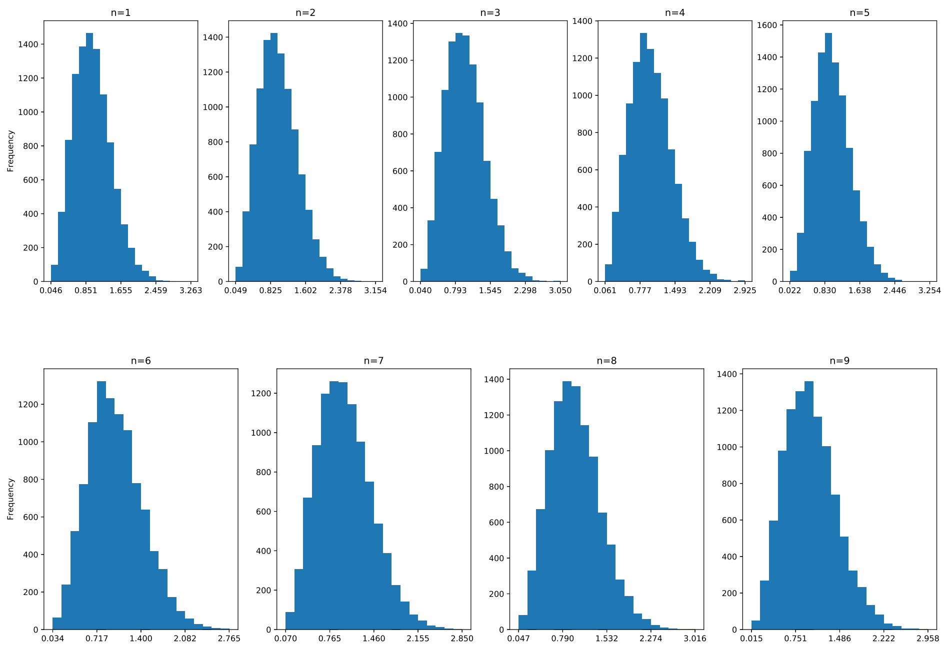

Figure 8  $N = 10$, $K = 10000$

We also plot the value of $\langle s_n \rangle$ in figure 4.

From figure 7 and figure 8, we guess that with fixed $N$, $\hat{s}_n$ might follow the same distribution with different $n$.

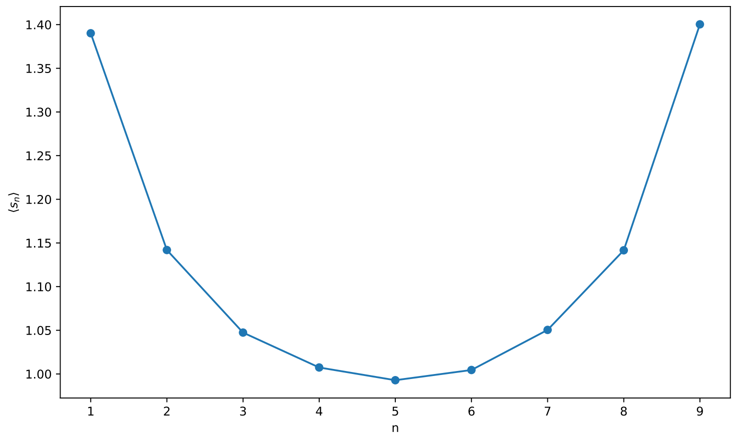


Figure 9 $N = 10,\ K = 10000$

## 3.2 Problem (c)

When $N = 2$, suppose $H$ is generated by GOE($N$). Suppose $H = \begin{bmatrix} a & b+id \\ b-id & c \end{bmatrix}$, where $a, c \sim \mathcal{N}(0,1)$ and $b, d \sim \mathcal{N}(0,1/2)$ Then, the characteristic polynomial for $H$ is

$$\lambda^2 - (a+c)\lambda + ac - b^2 - d^2 = 0$$

We can directly compute the eigenvalue spacing

$$s_1 = \lambda_2 - \lambda_1 = \sqrt{(a+c)^2 - 4(ac - b^2 - d^2)} = \sqrt{(a-c)^2 + 4(b^2 + d^2)}$$

Because $a, b, c, d$ are independent, $a - c$, $b$ and $d$ are independent. We have $a - c \sim \mathcal{N}(0,2)$, $2b, 2d \sim \mathcal{N}(0,2)$. Let us denote $e = a - c$. We have $s_1 = \sqrt{e^2 + 4b^2 + 4d^2}$. Then, for any $r_0 > 0$,

$$\begin{aligned} p(s_1 \le r_0) =& p(e^2 + 4b^2 + 4d^2 \le r_0^2) = \int_{x^2+y^2+z^2 \le \frac{r_0^2}{4}} (4\pi)^{-\frac{3}{2}} e^{-\frac{x^2+y^2+z^2}{4}} dxdydz \\ =& \int_0^{\pi} \int_0^{2\pi} \int_0^{r_0} (4\pi)^{-\frac{3}{2}} e^{-\frac{r^2}{4}} r^2 \sin\phi dr d\theta d\phi \\ =& (4\pi)^{-\frac{1}{2}} \int_0^{r_0} e^{-\frac{r^2}{4}} r^2 dr \end{aligned}$$

Therefore, the probability density for $s_1$ is

$$p_{s_1}(x) = \frac{1}{2\sqrt{\pi}} x^2 e^{-\frac{x^2}{4}}$$

Therefore, we have

$$\begin{aligned}\langle s_1 \rangle &= \int_0^\infty \frac{1}{2\sqrt{\pi}} x^3 e^{-\frac{x^2}{4}} dx = -\frac{1}{\sqrt{\pi}} \int_0^\infty x^2 (-\frac{x}{2} e^{-\frac{x^2}{4}}) dx \\ &= -\frac{1}{\sqrt{\pi}} \left[ x^2 e^{-\frac{x^2}{4}} \Big|_0^\infty - \int_0^\infty 2x e^{-\frac{x^2}{4}} dx \right] = \frac{2}{\sqrt{\pi}} \int_0^\infty x e^{-\frac{x^2}{4}} dx \\ &= -\frac{4}{\sqrt{\pi}} e^{-\frac{x^2}{4}} \Big|_0^\infty = \frac{4}{\sqrt{\pi}}\end{aligned}$$

Because $s_1' = \frac{s_1}{\langle s_1 \rangle}$, we have

$$p_{s_1'}(x) = \frac{4}{\sqrt{\pi}} p_{s_1}(\frac{4}{\sqrt{\pi}} x) = \frac{32}{\pi^2} x^2 e^{-\frac{4x^2}{\pi}}$$

We compare this analytical result with numerical experiments. We take sample number $K = 2000$. We compute $\langle s_1 \rangle$ and plot the histogram of $s_1'$ in the figure 10.

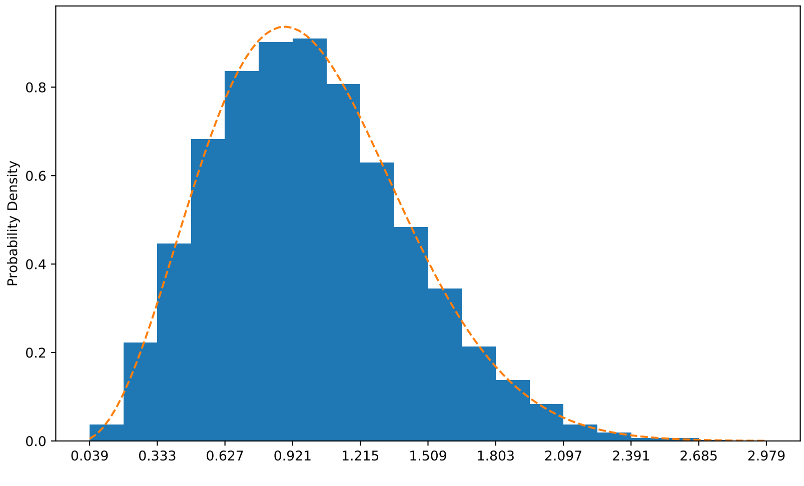


Figure 10 $N = 2$, $K = 20000$. We have $\langle s_1 \rangle = 2.251$

The estimated value of $\langle s_1 \rangle = 2.251$ is quite close to the analytical value $\frac{4}{\sqrt{\pi}} \approx 2.257$. And from figure 10, our analytical result fits the numerical result closely.

### 3.3 Problem(d)

If $X, Y \sim \mathcal{N}(0,1)$ and $X, Y$ are independent, then the probability density for $X + iY$ is $p(x+iy) = \frac{1}{\sqrt{\pi}}e^{-x^2}\frac{1}{\sqrt{\pi}}e^{-y^2} = \frac{1}{\pi}e^{-(x^2+y^2)} = \frac{1}{\pi}e^{-|x+iy|^2}$. A natural way to define the probability density for $H$ is

$$
\begin{aligned}
\rho(H) =& \prod_{j=1}^{N}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{(H_{jj})^2}{2}}\right)\prod_{1\le j<k\le N}\left(\frac{1}{\pi}e^{-\frac{|H_{jk}|^2}{2}}\right) \\
=& 2^{-\frac{N}{2}}\pi^{-\frac{N^2}{2}}\exp\left(-\frac{1}{2}\left(\sum_{j=1}^{N}(H_{jj})^2 + \sum_{1\le j<k\le N} 2|H_{jk}|^2\right)\right) \\
=& 2^{-\frac{N}{2}}\pi^{-\frac{N^2}{2}}\exp\left(-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}|H_{jk}|^2\right) \\
=& 2^{-\frac{N}{2}}\pi^{-\frac{N^2}{2}}\exp\left(-\frac{1}{2}\|H\|_F^2\right)
\end{aligned}
$$

where $\|H\|_F$ is the Frobenius norm of $H$.
Similarly, we can show that $\|H\|_F^2 = \sum_{n=1}^{N}\lambda_n^2$. Suppose $H = V\Lambda V^*$ is the eigenvalue decomposition of $H$, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ and $V$ is $N \times N$ unitary matrix. Then, we have

$$
\begin{aligned}
\|H\|_F^2 =& \operatorname{tr}(H^*H) = \operatorname{tr}(V\Lambda V^* V\Lambda V^*) = \operatorname{tr}(V\Lambda^2 V^*) \\
=& \operatorname{tr}(\Lambda^2 V^* V) = \operatorname{tr}(\Lambda^2) = \sum_{n=1}^{N}\lambda_n^2
\end{aligned}
$$

And we know that for any unitary matrix $Q$, $Q^T HQ$ has similar eigenvalues with $H$. Therefore, $\|Q^T HQ\|_F^2 = \|H\|_F^2$, $\rho(Q^T HQ) = \rho(H)$.
From our definition of $\rho(H)$, we observer that the value of $\rho(H)$ only depends on eigenvalues of $H$.

# 4 Problem (f)

### 4.1 Real Wigner matrix

We further consider real Wigner matrix $X$ with $X = X^T$ and $X_{ij} \sim \mathcal{N}(0,1)$ independently.
In actual implementation, to generate a $N \times N$ real Wigner matrix, we first generate a $N \times N$ matrix $A$ with $A_{ij} \sim \mathcal{N}(0,1)$, $1 \le i, j \le N$. The elements in $A$ are independent. We denote $A = D - L - U$, where $D$ is a diagonal matrix, $L$ is a lower triangular matrix, and $U$ is an upper triangular matrix. Then, we let $H = D - L - L^T$. $H$ is a $N \times N$ real Wigner matrix.

We study the distribution of eigenvalues of real Wigner matrices. We first take $N = 400$ and $N = 900$. The figure 11 plots the distribution of eigenvalues.

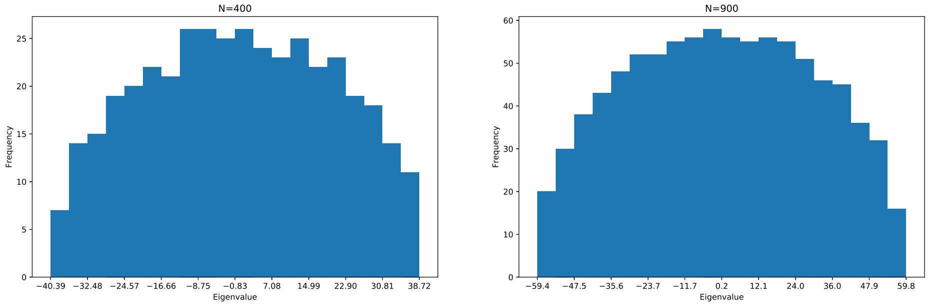


Figure 11 Distribution of eigenvalues of real Wigner matrices

From figure 11, we find that the minimal of the eigenvalues is approximately $-2\sqrt{N}$ and the maximal of the eigenvalues is approximately $2\sqrt{N}$. Therefore, we rescale the eigenvalues by $\frac{1}{\sqrt{N}}$. Then, we take $N = 1000$, $N = 2000$, $N = 3000$ and $N = 4000$. We plot the distribution of eigenvalues in figure 12 and figure 13.

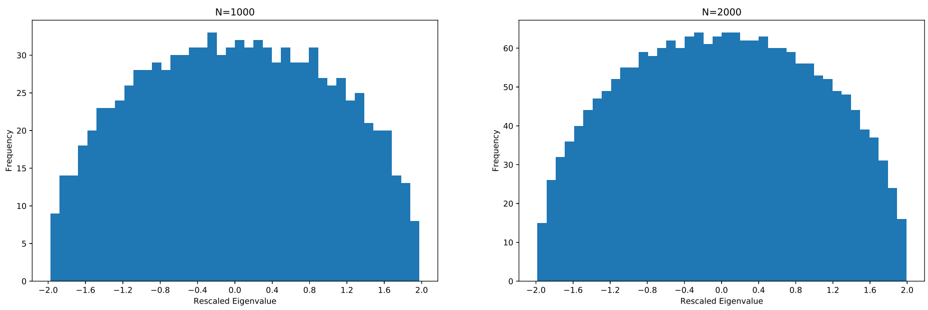


Figure 12 Distribution of rescaled eigenvalues of real Wigner matrices

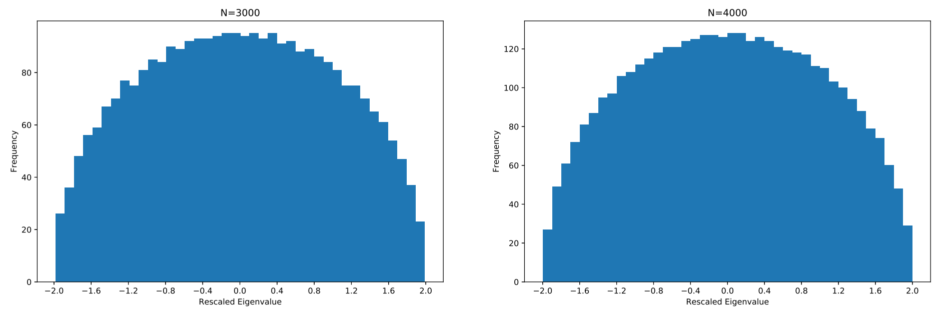


Figure 13 Distribution of rescaled eigenvalues of real Wigner matrices

From figure 12 and figure 13, the figure is like a semicircle. We guess when $N \to \infty$, the distribution density of the rescaled eigenvalues of real Wigner matrices follows:

$$\sigma(x) = \frac{1}{2\pi}\mathbf{1}_{|x|\leq 2}\sqrt{4 - x^2}$$

We take the experiment result from $N = 3000$ and $N = 4000$ to verify our guess.

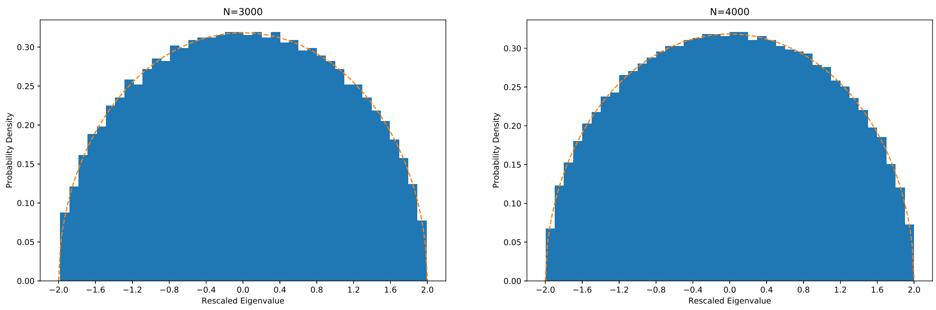


Figure 14 Distribution of rescaled eigenvalues of real Wigner matrices

From figure 26, the experiment result fits our guess.

## 4.2 GOE

We study the distribution of eigenvalues of GOE. We first take $N = 400$ and $N = 900$. The figure 15 plots the distribution of eigenvalues.

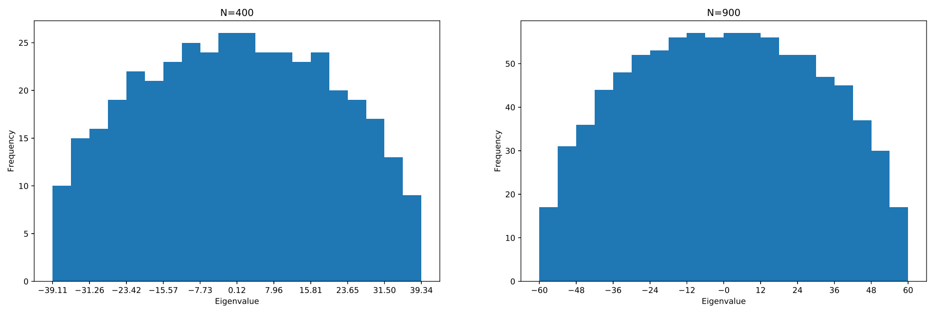


Figure 15 Distribution of eigenvalues of GOE

From figure 15, we find that the minimal of the eigenvalues is approximately $-2\sqrt{N}$ and the maximal of the eigenvalues is approximately $2\sqrt{N}$. Therefore, we rescale the eigenvalues by $\frac{1}{\sqrt{N}}$. Then, we take $N = 1000$, $N = 2000$, $N = 3000$ and $N = 4000$. We plot the distribution of eigenvalues in figure 16 and figure 17.

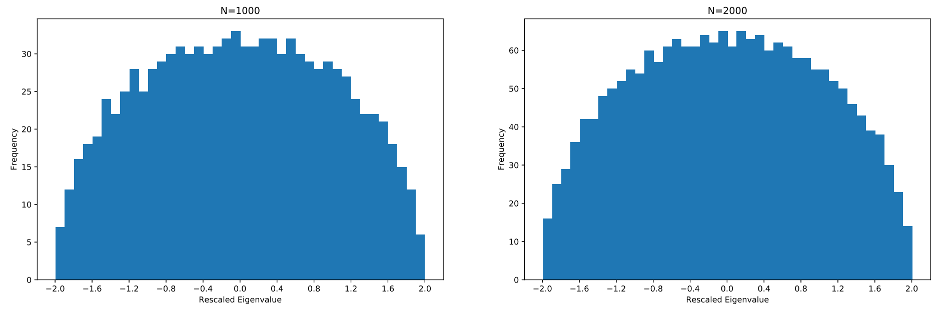


Figure 16  Distribution of rescaled eigenvalues of GOE

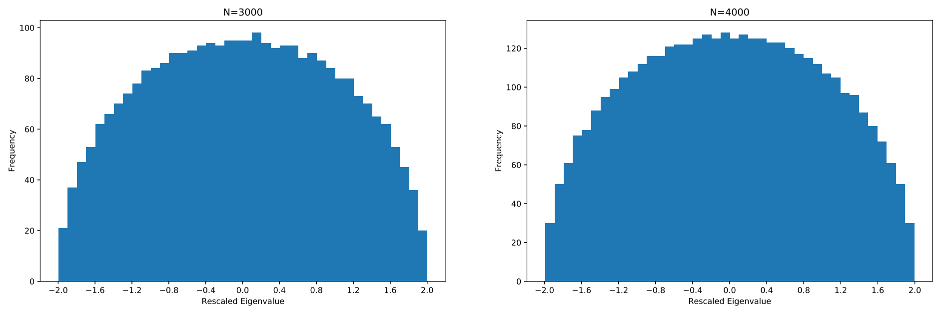


Figure 17  Distribution of rescaled eigenvalues of GOE

From figure 16 and figure 17, the figure is like a semicircle. We guess when $N \to \infty$, the distribution density of the rescaled eigenvalues of GOE follows:

$$\sigma(x) = \frac{1}{2\pi}\mathbf{1}_{|x|\leq 2}\sqrt{4 - x^2}$$

We take the experiment result from $N = 3000$ and $N = 4000$ to verify our guess.

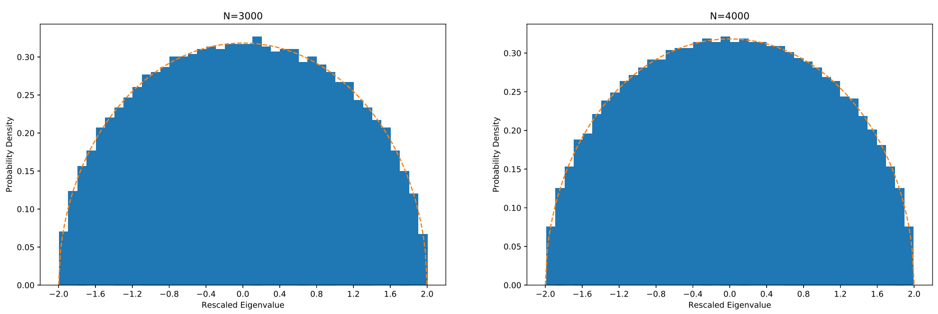


Figure 18  Distribution of rescaled eigenvalues of GOE

From figure 18, the experiment result fits our guess.

## 4.3 GUE

We study the distribution of eigenvalues of GUE. We first take $N = 400$ and $N = 900$. The figure 19 plots the distribution of eigenvalues.

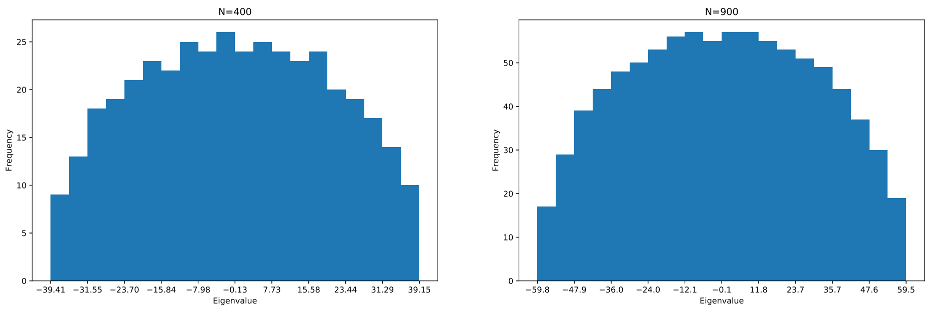


Figure 19 Distribution of eigenvalues of GUE

From figure 19, we find that the minimal of the eigenvalues is approximately $-2\sqrt{N}$ and the maximal of the eigenvalues is approximately $2\sqrt{N}$. Therefore, we rescale the eigenvalues by $\frac{1}{\sqrt{N}}$. Then, we take $N = 1000$, $N = 2000$, $N = 3000$ and $N = 4000$. We plot the distribution of eigenvalues in figure 20 and figure 21.

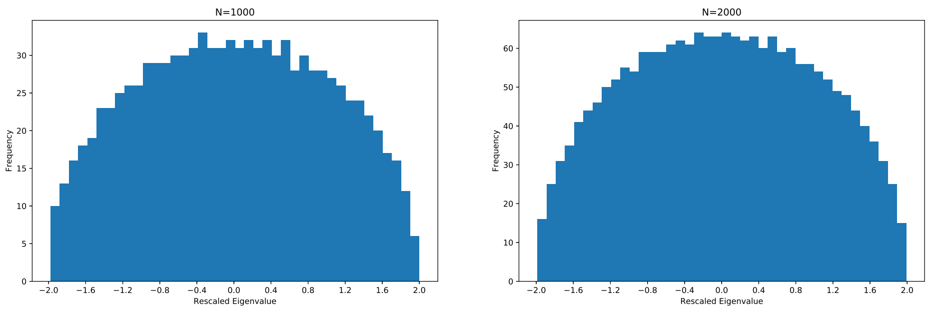


Figure 20 Distribution of rescaled eigenvalues of GUE

From figure 20 and figure 21, the figure is like a semicircle. We guess when $N \to \infty$, the distribution density of the rescaled eigenvalues of GUE follows:

$$\sigma(x) = \frac{1}{2\pi}\mathbf{1}_{|x|\leq 2}\sqrt{4 - x^2}$$

We take the experiment result from $N = 3000$ and $N = 4000$ to verify our guess.

From figure 22, the experiment result fits our guess.

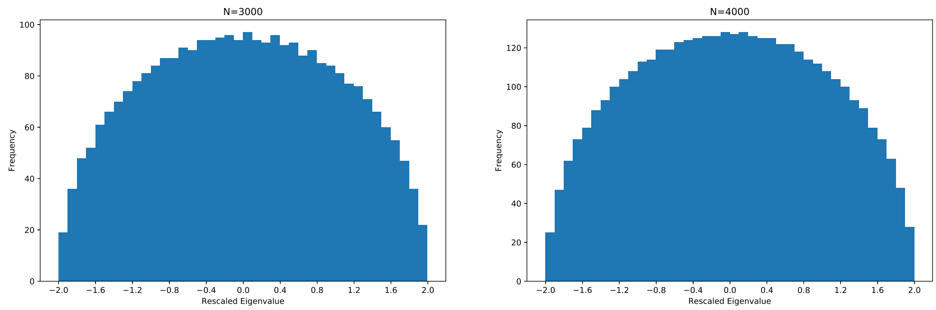


Figure 21    Distribution of rescaled eigenvalues of GUE

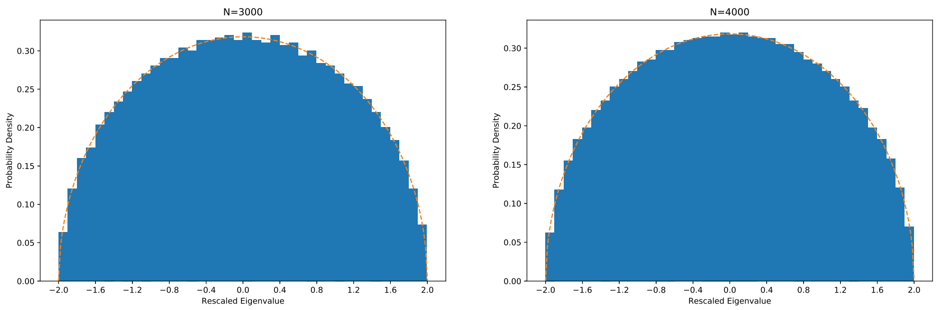


Figure 22    Distribution of rescaled eigenvalues of GUE

## 4.4 Complex Wigner matrix

We further consider complex Wigner matrix $X$ with $X = X^*$ and $X_{jk}$ has mean 0 and $\mathbb{E}|X_{jk}|^2 = 1$ independently. For any $0 < \alpha < 1$, we consider to take certain kind of complex Wigner matrix $X$ with $X = X^*$, $X_{jk} \sim \mathcal{N}(0, \alpha) + i\mathcal{N}(0, 1-\alpha)$, $1 \le j < k \le N$ and $X_{jj} \sim \mathcal{N}(0, 1)$, $1 \le j \le N$. If we take $\alpha = 0.5$, then we get GUE.

In actual implementation, to generate a $N \times N$ complex Wigner matrix, we first generate a $N \times N$ matrix $A$ with $A_{jk} \sim \mathcal{N}(0, 1)$, $1 \le j, k \le N$. The elements in $A$ are independent. We denote $A = D - L - U$, where $D$ is a diagonal matrix, $L$ is a lower triangular matrix, and $U$ is an upper triangular matrix. Then, we let $H = D - \sqrt{\alpha}(L + L^T) - i\sqrt{1-\alpha}(U + U^T)$. $H$ is a $N \times N$ complex Wigner matrix.

Because complex Wigner matrix is hermitian, all of its eigenvalues are real. We study the distribution of eigenvalues of complex Wigner matrices. We take $\alpha = 0.25$. We first take $N = 400$ and $N = 900$ in numerical experiment. The figure 23 plots the distribution of eigenvalues.

From figure 23, we find that the minimal of the eigenvalues is approximately $-2\sqrt{N}$ and the maximal of the eigenvalues is approximately $2\sqrt{N}$. Therefore, we rescale the eigenvalues by

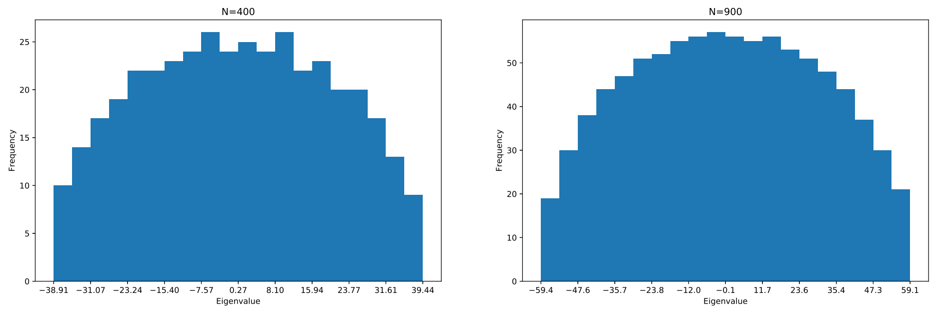


Figure 23    Distribution of eigenvalues of complex Wigner matrices, $\alpha = 0.25$

$\frac{1}{\sqrt{N}}$. Then, we take $N = 1000$, $N = 2000$, $N = 3000$ and $N = 4000$. We plot the distribution of eigenvalues in figure 24 and figure 25.

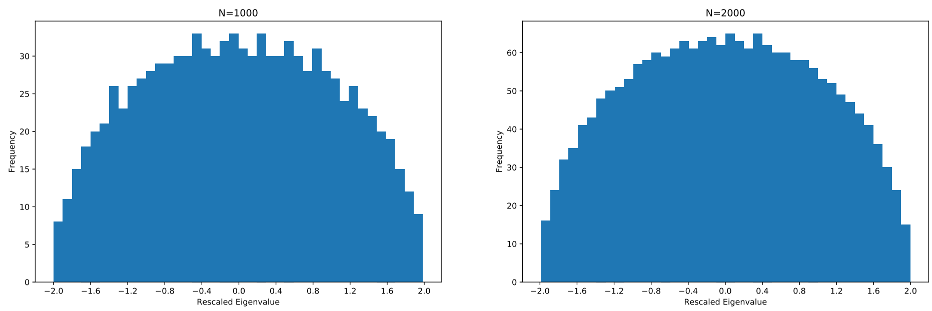


Figure 24    Distribution of rescaled eigenvalues of complex Wigner matrices, $\alpha = 0.25$

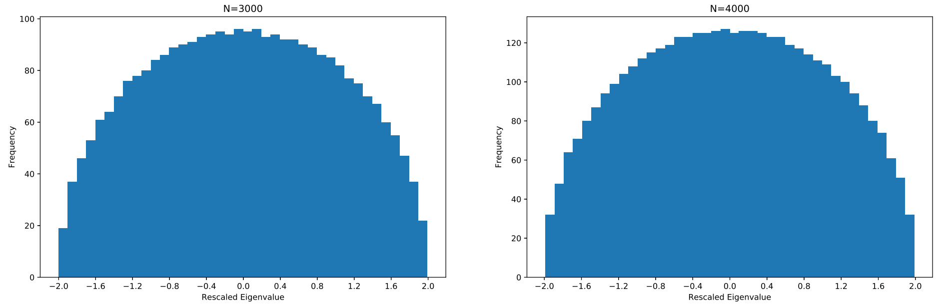


Figure 25    Distribution of rescaled eigenvalues of complex Wigner matrices, $\alpha = 0.25$

From figure 24 and figure 25, the figure is like a semicircle. We guess when $N \to \infty$, the distribution density of the rescaled eigenvalues of complex Wigner matrices follows the same distribution:

$$\sigma(x) = \frac{1}{2\pi}\mathbf{1}_{|x|\leq 2}\sqrt{4 - x^2}$$

We take the experiment result from $N = 3000$ and $N = 4000$ to verify our guess.

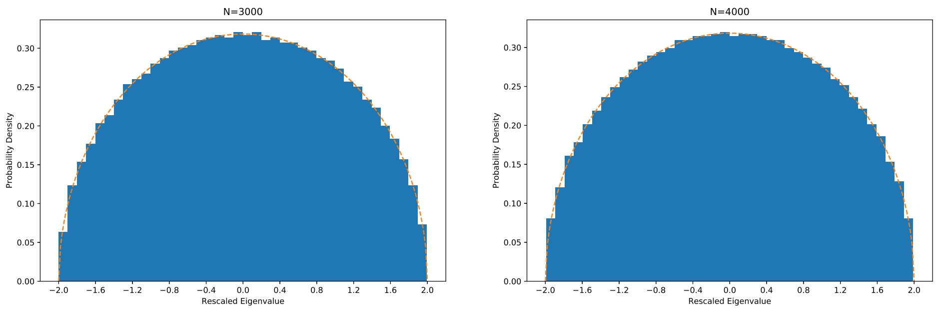


Figure 26  Distribution of rescaled eigenvalues of complex Wigner matrices, $\alpha = 0.25$

From figure 26, the experiment result fits our guess.

## 4.5 Conclusion

In random matrix theory, the definition of complex Wigner matrix is given by: we start with two independent family of i.i.d. zero mean, real or complex valued random variables $\{Z_{j,k}\}_{1\le j<k}$ and $\{Y_j\}_{1\le j}$. Consider the hermitian $N \times N$ matrix $X_N$ with entries

$$X_N(j,k) = \bar{X}_N(k,j) = \begin{cases} Z_{j,k} & j<k \\ Y_j & j=k \end{cases}$$

We call $X_N$ a Wigner matrix.

**Theorem 1 (Wigner's semicircle law)** *Let $Y_N = N^{-1/2}X_N$ be a sequence of Wigner matrix with entries satisfying $\mathbb{E}|X_{j,k}| = t$. Let $I \subset \mathbb{R}$ be an interval. Define the random variables*

$$E_n(I) = \frac{\#(\{\lambda_1(Y_N), \lambda_2(Y_N), \ldots, \lambda_N(Y_N)\} \cap I)}{N}$$

*Then, $E_n(I) \to \sigma_t(I)$ in probability as $n \to \infty$, where $\sigma_t(I) = \int_I \sigma_t(dx) = \int_I \frac{1}{2\pi t}\sqrt{4t - x^2}_+ dx$*

The previous experiment results verify Wigner's semicircle law in the case that $t = 1$.