Developing Numerical Algorithms for Partial Differential Equations Using Uniform Meshes

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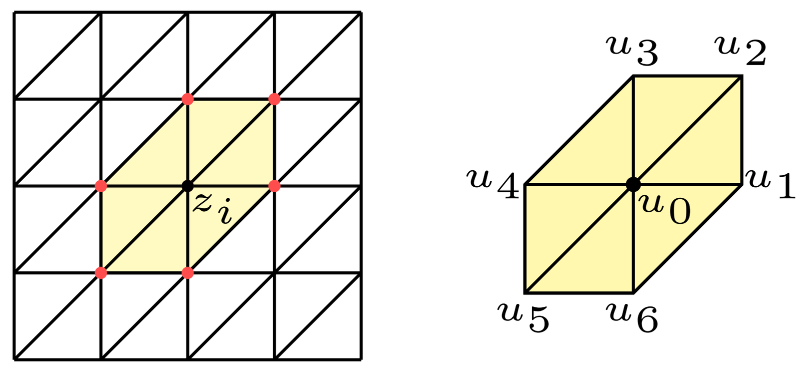
Abstract

In this project, the Gradient operator and the Hessian operator were studied on uniform meshes, and their approximations were achieved through the polynomial preserving recovery technique. A systematic approach for the formulation of finite difference schemes was proposed, adaptable to both structured and unstructured meshes. The precision of these operators is analyzed by Taylor Expansion, complemented by numerical experiments designed to authenticate the efficacy and efficiency of the proposed finite difference scheme.

Methodology

Post-processing is a fundamental approach in scientific computing. **Polynomial Preserving Recovery (PPR)** stands out as a preeminent post-processing technique, widely recognized for its capacity to recover finite element solutions [1]. In **Regular Pattern**, an interior node is considered as the solid dot point z_i,



To simplify the algorithm, coordinate (x, y) is mapped into (ξ, η) and z_i comes to origin z_0. A quadratic polynomial is defined on this patch in the sense of least-square fitting,

$$p_2(z_i) = (1, \xi, \eta, \xi^2, \xi\eta, \eta^2)(a_1, a_2, a_3, a_4, a_5, a_6)^T$$

$$\Rightarrow p_2(z_0) = (1, x, y, x^2, xy, y^2)(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5, \hat{a}_6)^T$$

$$= \hat{a}_1 + \hat{a}_2 x + \hat{a}_3 y + \hat{a}_4 x^2 + \hat{a}_5 xy + \hat{a}_6 y^2$$

Meanwhile, boundary strategy [1] is adopted to maintain the accuracy.

Gradient Recovery

The gradient recovery operator G_h is defined as the first partial derivatives of $p_2(z_0)$.

$$(G_h u)(z_0) = \begin{pmatrix} (G_h^x u)(z_0) & (G_h^y u)(z_0) \end{pmatrix} = \begin{pmatrix} \hat{a}_2 & \hat{a}_3 \end{pmatrix}$$

On regular pattern, the recovered gradient value is formulated as

$$(G_h^x u)(z_0) = \frac{1}{6h}(2u_1 + u_2 - u_3 - 2u_4 - u_5 + u_6)$$

$$(G_h^y u)(z_0) = \frac{1}{6h}(-u_1 + u_2 + 2u_3 + u_4 - u_5 - 2u_6)$$

Hessian Recovery

The Hessian recovery operator H_h is defined as the second-order partial derivatives of $p_2(z_0)$ [2].

$$(H_h u)(z_0) = \begin{pmatrix} (H_h^{xx} u)(z_0) & (H_h^{xy} u)(z_0) \\ (H_h^{yx} u)(z_0) & (H_h^{yy} u)(z_0) \end{pmatrix} = \begin{pmatrix} 2\hat{a}_4 & \hat{a}_5 \\ \hat{a}_5 & 2\hat{a}_6 \end{pmatrix}$$

On regular pattern, the recovered Hessian value is formulated as

$$(H_h^{xx} u)(z_0) = \frac{1}{h^2}(-2u_0 + u_1 + u_4)$$

$$(H_h^{yy} u)(z_0) = \frac{1}{h^2}(-2u_0 + u_3 + u_6)$$

$$(H_h^{xy} u)(z_0) = \frac{1}{2h^2}(2u_0 - u_1 + u_2 - u_3 - u_4 + u_5 - u_6)$$

$$(H_h^{yx} u)(z_0) = \frac{1}{2h^2}(2u_0 - u_1 + u_2 - u_3 - u_4 + u_5 - u_6)$$

Finite Difference Scheme

Consider the second-order elliptic equation $-\Delta u = f$ with $u = g$ on $\partial\Omega$. By employing the Hessian recovery technique for the approximation of second-order partial derivatives, Δu is approximated by $H_h^{xx} u_h + H_h^{yy} u_h$. The discretization of Laplace equation can be formulated as

$$-H_h^{xx} u_h - H_h^{yy} u_h = f$$

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Taylor Expansion

By means of Taylor expansion computed through Mathematica, the gradient and Hessian recovery in interior nodes exhibit superconvergence. Taylor expansion of gradient recovery results in the following equations, which are shown as a second-order finite difference scheme approximating $\nabla u(z_0)$.

$$(G_h^x u)(z_0) = u_x(z_0) + \frac{h^2}{6}[u_{xyy}(z_0) + u_{xxy}(z_0) + u_{xxx}(z_0)] + O(h^4)$$

$$(G_h^y u)(z_0) = u_y(z_0) + \frac{h^2}{6}[u_{yyy}(z_0) + u_{xyy}(z_0) + u_{xxy}(z_0)] + O(h^4)$$

Taylor expansion of Hessian recovery is depicted as the equations below, which is a second-order finite difference scheme approximating second-order partial derivatives.

$$(H_h^{xx} u)(z_0) = u_{xx}(z_0) + \frac{h^2}{12} u_{xxxx}(z_0) + O(h^4)$$

$$(H_h^{yy} u)(z_0) = u_{yy}(z_0) + \frac{h^2}{12} u_{yyyy}(z_0) + O(h^4)$$

$$(H_h^{xy} u)(z_0) = u_{xy}(z_0) + \frac{h^2}{12}[2u_{xyyy}(z_0) + 3u_{xxyy}(z_0) + 2u_{xxxy}(z_0)] + O(h^4)$$

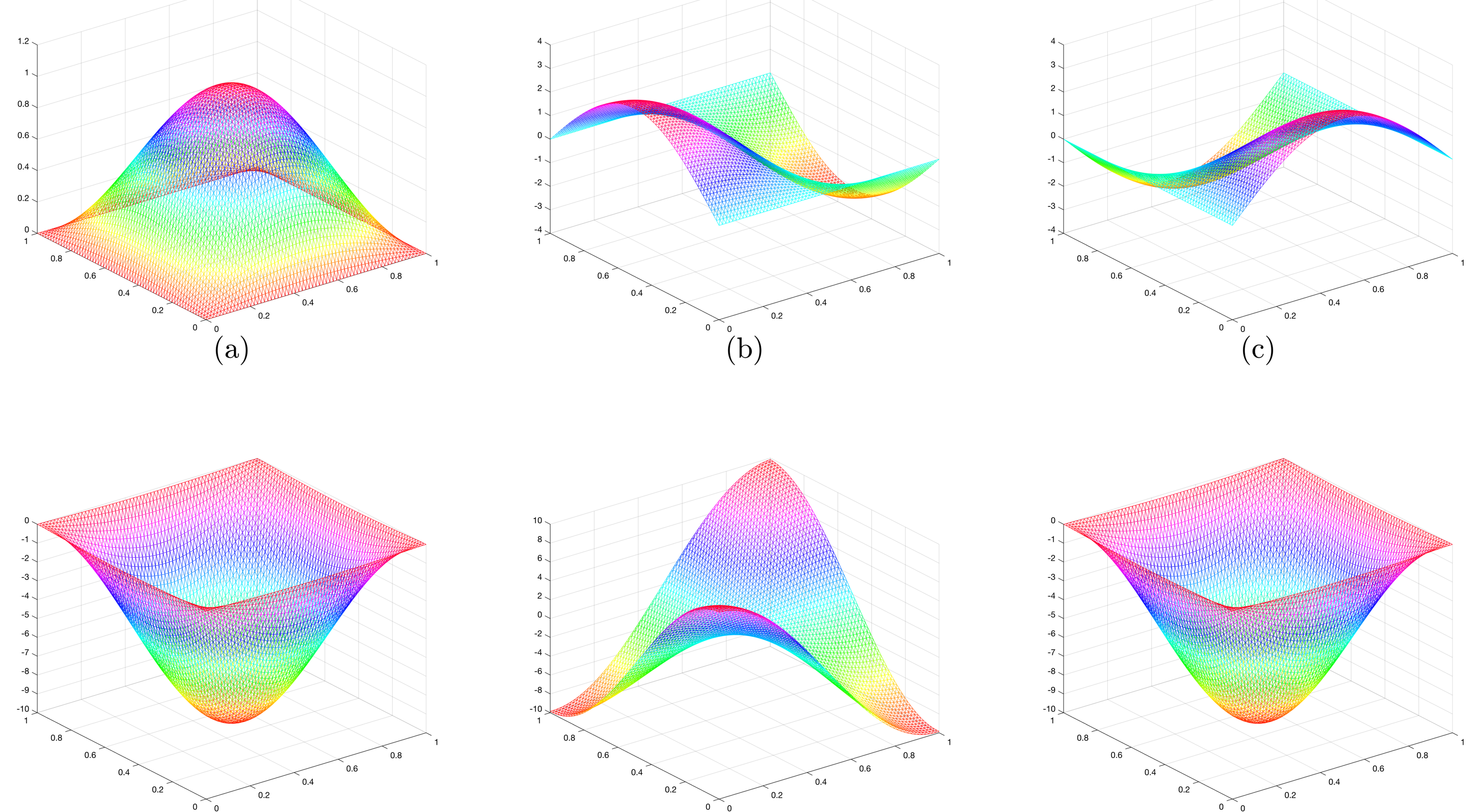
$$(H_h^{yx} u)(z_0) = u_{yx}(z_0) + \frac{h^2}{12}[2u_{xyyy}(z_0) + 3u_{xxyy}(z_0) + 2u_{xxxy}(z_0)] + O(h^4)$$

Numerical Example

Consider **Laplace equation**, with **Dirichlet condition** $u = \sin(\pi x)\sin(\pi y)$, on $\partial\Omega$.

$$-\Delta u = 2\pi^2 \sin(\pi x)\sin(\pi y), \Omega = [0, 1]^2$$

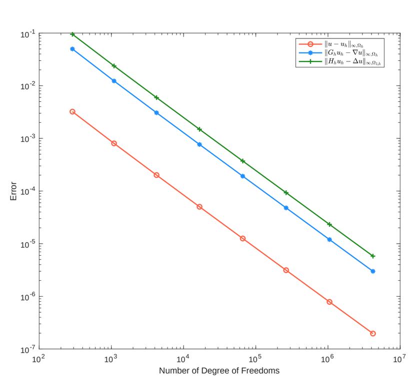
In order to track the behavior of Hessian recovery in the interior, $\Omega_{1,h} = [0.1, 0.9]^2$ is selected to avoid the influence by the boundary. Upon solving Laplace equation, the solution u_h is derived utilizing the proposed **Finite Difference Method (FDM)** on Regular Pattern triangulation. The numerical solution u_h is plotted in (a). $G_h u_h$ represents the recovered gradient shown in (b) and (c). $H_h u_h$ refers to the recovered Hessian demonstrated in (d), (e) and (f) [3].



(a) (b) (c) (d) (e) (f)

In the following table, discrete maximum errors are depicted. The table shows that the errors associated with variable u exhibit an optimal convergence rate. Notably, when excluding Hessian recovery process across all boundary nodes, PPR technique consistently demonstrates a **second-order precision**, which is superconvergent.

Dofs	$\|u - u_h\|_{\infty,\Omega_h}$	order	$\|G_h u_h - \nabla u\|_{\infty,\Omega_h}$	order	$\|H_h u_h - \Delta u\|_{\infty,\Omega_{1,h}}$	order
289	3.2190e-03	–	4.9871e-02	–	9.4821e-02	–
1089	8.0358e-04	2.00	1.2311e-02	2.02	2.3762e-02	2.00
4225	2.0082e-04	2.00	7.6622e-04	2.00	5.9442e-03	2.00
16641	5.0201e-05	2.00	7.6622e-04	2.00	1.4863e-03	2.00
66049	1.2550e-05	2.00	1.9151e-04	2.00	3.7158e-04	2.00
263169	3.1375e-06	2.00	4.7876e-05	2.00	9.2896e-05	2.00
1050625	7.8450e-07	2.00	1.1968e-05	2.00	2.3224e-05	2.00
4198401	1.9661e-07	2.00	2.9906e-06	2.00	5.8073e-06	2.00



Our future research will explore adaptive mesh refinement, using polynomial preserving recovery to approximate the operators in different types of partial differential equations.

Conclusion

In this project, polynomial preserving recovery technique is studied on uniform meshes. The gradient recovery operator G_h and Hessian recovery operator H_h are formulated in finite difference schemes. The approximating accuracy of both operators are guaranteed by Taylor expansion using Mathematica. Applying them into elliptic partial differential equations, the numerical solution u_h is obtained and the optimal convergence rate is achieved. In addition, superconvergence phenomenon is observed for the recovered Gradient and Hessian values from the numerical solution.

References

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