this leads to the equality (4.1).

We reason by contradiction in assuming that y_A is an element of $\mathscr{L} \setminus \{\top\}$ such that, for any $x_A \in \mathscr{L} \setminus \{\top\}$, there exists $x_B \in \mathscr{L}$ satisfying

$$x_A < x_B \text{ and } \mu(x_A, x_B) \nleqslant \mu(y_A, \top).$$

In the particular case where $x_A = y_A$, we obtain that there exists $y_B \in \mathscr{L}$ such that

$$y_A < y_B \text{ and } \mu(y_A, y_B) \nleqslant \mu(y_A, \top). \tag{4.2}$$

By the condition (1), we may assume that y_A is maximal among

$$\left\{ y'_A \in \mathscr{L} \setminus \{\top\} \;\middle|\; \begin{array}{l} \forall\, x_A \in \mathscr{L} \setminus \{\top\},\ \exists\, x_B \in \mathscr{L} \text{ such that} \\ x_A < x_B \text{ and } \mu(x_A, x_B) \nleqslant \mu(y'_A, \top) \end{array} \right\}.$$

Therefore, if $y'_A \in \mathscr{L} \setminus \{\top\}$ is such that $y_A < y'_A$ and $\mu(y_A, y'_A) \nleqslant \mu(y_A, \top)$, there exists $x'_A \in \mathscr{L} \setminus \{\top\}$ such that

$$\forall\, w \in \mathscr{L}, \quad x'_A < w \Longrightarrow \mu(x'_A, w) \leqslant \mu(y'_A, \top). \tag{4.3}$$

By the condition (2), one then has $\mu(y'_A, \top) \leqslant \mu(y_A, \top)$. However, by the hypothesis on y_A, for any $x'_A \in \mathscr{L} \setminus \top$ there exists $x'_B \in \mathscr{L}$ such that $x'_A < x'_B$ and $\mu(x'_A, x'_B) \nleqslant \mu(y_A, \top)$. But then the inequality $\mu(y'_A, \top) \leqslant \mu(y_A, \top)$ cannot hold true since otherwise (4.3) applied to $w = x'_B$ would give

$$\mu(x'_A, x'_B) \leqslant \mu(y'_A, \top) \leqslant \mu(y_A, \top),$$

which leads to a contradiction. Thus, we obtain $\mu(y_A, y'_A) \leqslant \mu(y_A, \top)$ for any $y'_A \in \mathscr{L}$ such that $y_A < y'_A \leqslant \top$, which contradicts the condition (4.2) for y_A. Therefore, the equality (4.1) holds.

I'm not quite understand how the $\boxed{\text{argument}}$ works. If we set

$$\mathcal{S} := \left\{ y'_A \in \mathscr{L}\backslash\{\top\} \middle| \substack{\forall x_A \in \mathscr{L}\backslash\{\top\},\ \exists x_B \in \mathscr{L} \text{ such that} \\ x_A < x_B \text{ and } \mu(x_A, x_B) \nleq \mu(y'_A, \top)} \right\},$$

then the whole screenshot aims to show that $\mathcal{S}$ is empty by contradiction.

So after using contradiction, $\mathcal{S}$ is not empty with certain $y_A \in \mathcal{S}$. The $\boxed{\text{argument}}$ is telling us that there exists (at least) a maximal element $y_{A,\max}$ in $\mathcal{S}$, then we just take the place of y_A by $y_{A,\max}$. I believe this is what "we may assume that y_A is maximal" means. The $\boxed{\text{argument}}$ suggests us to show the existence of $y_{A,\max}$ by the Condition (1). I have two different unsuccessful attempts towards this goal as follows:

(1) If $y_{A,\max}$ does not exist, then we can recursively define a strictly increasing sequence $x_0 = y_A < x_1 < \cdots < x_n < \cdots$ in $\mathcal{S}$. By the Condition (1), there exists $N \in \mathbb{N}$ such that

$$\mu(x_N, x_{N+1}) \leq \mu(x_N, \top). \tag{a}$$

Then I don't know how to continue. By the definition of $\mathcal{S}$, there exists $x'_B \in \mathscr{L}$ such that $x_N < x'_B$ and $\mu(x_N, x'_B) \nleq \mu(x_N, \top)$. However, this does not seem to contradict (a), as x'_B is not necessarily equals to x_{N+1}.

(2) Start with $x_0 = y_A$, by (4.2) there exists $x_1 \in \mathscr{L}$ such that $x_0 < x_1$ and $\mu(x_0, x_1) \nleq \mu(x_0, \top)$. If x_1 can be chosen in $\mathcal{S}$, then we can apply (4.2) on x_1 to get $x_2 \in \mathscr{L}$ such that $x_1 < x_2$ and $\mu(x_1, x_2) \nleq \mu(x_1, \top)$. This process can not goes on forever, otherwise we get a strictly increasing sequence $x_0 < x_1 < \cdots$ in $\mathcal{S}$ such that $\mu(x_n, x_{n+1}) \nleq \mu(x_n, \top)$ for all $n \in \mathbb{N}$, which contradicts Condition (1). Therefore there exists $k \in \mathbb{N}$ such that $x_k \in \mathcal{S}$ and for every $y \in \mathcal{S}$, we have

$$\begin{aligned} &\neg(x_k < y \wedge \mu(x_k, y) \nleq \mu(x_k, \top)) \\ =&(x_k \nless y) \vee (x_k < y \wedge \mu(x_k, y) \leq \mu(x_k, \top)). \end{aligned}$$

What we want is $x_k \nless y$ (for every $y \in \mathcal{S}$), but I don't know how to rule out the case $x_k < y \wedge \mu(x_k, y) \leq \mu(x_k, \top)$. (4.2) is not applicable here, as we have another constraint that $y \in \mathcal{S}$.