

DEFORMATION OF GALOIS REPRESENTATIONS

A notes for the number theory seminar at YMSC, 2022

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We are given either a number field K and a finite set of primes S , or a local field F , and we are given a representation of either $G_{K,S}$ or G_F into $\mathrm{GL}_n(k)$, where k is a finite field. We want to try to understand all possible lifts of this representation to $\mathrm{GL}_n(A)$, where A is a complete noetherian local ring with residue field k .

—Fernando Q. Gouvêa (cf. [Gou01])

Main reference: [Böc13; Maz89]

Motivation and history of deformation theory: [Maz97; Gou01]

Basics of groupoids: Appendix of [Kis09]

1. DEFORMATIONS OF REPRESENTATIONS OF PROFINITE GROUPS

Notations.

p	prime number
\mathbb{F}	finite field of char. p
$W(\mathbb{F})$	ring of Witt vectors ¹ over A
G	profinite group
$V_{\mathbb{F}}$	finite $\mathbb{F}[G]$ -module with continuous G -action
d	dimeinsion of $V_{\mathbb{F}}$
$\beta_{\mathbb{F}}$	a \mathbb{F} -basis of $V_{\mathbb{F}}$

1.1. Deformation functors.

Notations.

$\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$	category of complete Noetherian local $W(\mathbb{F})$ -algebra with residue field \mathbb{F}
$\mathfrak{Ar}_{W(\mathbb{F})}$	full sub-category of finite local Artinian $W(\mathbb{F})$ -algebras
\mathfrak{m}_A	maximal ideal of $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$

Remark 1.1 (???). *Via the $W(\mathbb{F})$ -structure, the residue field of any $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ is canonically isomorphic to \mathbb{F} .*

Definition 1.2. *Let $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.*

- (1) *A **deformation** of $V_{\mathbb{F}}$ to A is a pair (V_A, ι_A) , such that*
 - (a) *V_A is $A[G]$ -module, finite free over A , with continuous G -action;*
 - (b) *$\iota_A : V_A \otimes_A \mathbb{F} \xrightarrow{\cong} V_{\mathbb{F}}$ is G -equivariant.*
- (2) *A **framed deformation** of $(V_{\mathbb{F}}, \beta_{\mathbb{F}})$ to A is a triple (V_A, ι_A, β_A) , where*

¹ $W(\mathbb{F})$ is the unique (up to unique isomorphism) complete discrete valuation ring which is absolutely unramified (uniformizer = p) and has residue field \mathbb{F} .

We view \mathbb{F} as an A -module via the canonical projection $A \rightarrow A/\mathfrak{m}_A = \mathbb{F}$.

- (a) (V_A, ι_A) is a deformation of $V_{\mathbb{F}}$ to A ;
- (b) β_A is a A -basis of V_A which reduces to $\beta_{\mathbb{F}}$ under ι_A .

Set $D_{V_{\mathbb{F}}}, D_{V_{\mathbb{F}}}^{\square} : \mathfrak{A}\mathfrak{r}_{W(\mathbb{F})} \rightarrow \text{Set}$,

$$D_{V_{\mathbb{F}}}(A) = \{\text{deformations of } V_{\mathbb{F}} \text{ to } A\} / \cong,$$

$$D_{V_{\mathbb{F}}}^{\square}(A) = \{\text{framed deformations of } (V_{\mathbb{F}}, \beta_{\mathbb{F}}) \text{ to } A\} / \cong.$$

Remark 1.3.

- (1) The **FIXED** basis $\beta_{\mathbb{F}}$ gives the isomorphism $V_{\mathbb{F}} \cong \mathbb{F}^d$ as vector space. Thus we can view $V_{\mathbb{F}}$ as $\bar{\rho} : G \rightarrow \text{GL}(V_{\mathbb{F}}) = \text{GL}_d(\mathbb{F})$: a d -dimensional \mathbb{F} -representation of G . Then

$$D_{V_{\mathbb{F}}}^{\square}(A) = \{\rho : G \rightarrow \text{GL}_d(A) \text{ lifting } \bar{\rho}\},$$

Does not guarantee $\beta_A \rightsquigarrow \beta_{\mathbb{F}}!$

$$D_{V_{\mathbb{F}}}(A) = D_{V_{\mathbb{F}}}^{\square}(A) / \text{action by conjugates of } \ker(\text{GL}_d(A) \rightarrow \text{GL}_d(\mathbb{F})).$$

Not always representable!

- (2) Mazur only consider $D_{V_{\mathbb{F}}}$, which describes representations lifting $V_{\mathbb{F}}$ up to isomorphism.

Add "base condition" $\rightsquigarrow D_{V_{\mathbb{F}}}^{\square}$.

- (3) Often consider deformation functors on $\mathfrak{A}\mathfrak{r}_{\mathcal{O}} =$ category of local artinian \mathcal{O} -algebra with residue field \mathbb{F} , where \mathcal{O} is ring of integers of a finite totally ramified extension of $W(\mathbb{F}) \left[\frac{1}{p} \right]$ ($\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}$).

For example, let K be a p -adic field with residue field \mathbb{F}_q , ring of integers \mathcal{O}_K , then $K/W(\mathbb{F}_q) \left[\frac{1}{p} \right]$ is totally ramified and $W(\mathbb{F}_q) \left[\frac{1}{p} \right] / \mathbb{Q}_p$ is unramified.

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}_d(A) \\ & \searrow \bar{\rho} & \downarrow \\ & & \text{GL}_d(\mathbb{F}) \end{array}$$

1.2. Representability.

1.2.1. A finiteness condition.

Definition 1.4 (Mazur). A profinite group G has finiteness condition Φ_p , if \forall open subgroup $G' \subset G$, $\dim_{\mathbb{F}_p} \text{Hom}_{\text{cont}}(G', \mathbb{F}_p) < +\infty$.

Remark 1.5.

- (1) (Burnside basis theorem) $\dim_{\mathbb{F}_p} \text{Hom}_{\text{cont}}(G', \mathbb{F}_p) < +\infty \Leftrightarrow$ maximal pro- p quotient of G' is topologically finitely generated.
- (2) $\text{Hom}_{\text{cont}}(G', \mathbb{F}_p) \cong \text{Hom}_{\text{cont}}(G'^{\text{ab}}, \mathbb{F}_p)$.

Example 1.6 (by CFT). The following groups have Φ_p :

- (1) The Galois group $\mathcal{G}_K = \text{Gal}(\bar{K}/K)$, with K a p -adic field.
- (2) The Galois group $\mathcal{G}_{F,S} = \text{Gal}(F_S/F)$, where F is a number field, S is a finite set of places of F and $F_S \subset \bar{F}$ is the maximal extension of F unramified outside S .

1.2.2. Main proposition.

Proposition 1.7 (Mazur). If G has Φ_p , then

- (1) The functor $D_{V_{\mathbb{F}}}^{\square}$ is pro-representable by some $R_{V_{\mathbb{F}}}^{\square} \in \widehat{\mathfrak{A}\mathfrak{r}}_{W(\mathbb{F})}$, i.e.

$$D_{V_{\mathbb{F}}}^{\square}(A) \cong \text{Hom}_{W(\mathbb{F})}(R_{V_{\mathbb{F}}}^{\square}, A),$$

which is functorial in $A \in \mathfrak{A}\mathfrak{r}_{W(\mathbb{F})}$.

↑ universal framed deformation ring

- (2) If $\text{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then $D_{V_{\mathbb{F}}}$ is pro-representable by some $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{A}\mathfrak{r}}_{W(\mathbb{F})}$.

↑ universal deformation ring

Maybe it's better to write $D_{V_{\mathbb{F}}}^{\square}(A) \cong \text{Hom}_{\widehat{\mathfrak{A}\mathfrak{r}}_{W(\mathbb{F})}}(R_{V_{\mathbb{F}}}^{\square}, A)$?

Remark 1.8. (1) Literally, pro-representable functor = limit of representable functor. How can we realize that? [nLa22] defines pro-representable to be the filtered colimit of representables. There's a post on MSE (cf. [htt]) which discusses the difference between the two definitions. It might be true that

$$\mathrm{Hom}_{W(\mathbb{F})}(R_{V_{\mathbb{F}}}^{\square}, A) = \mathrm{Hom}_{\widehat{\mathfrak{A}r_{W(\mathbb{F})}}}\left(\varprojlim_k R_{V_{\mathbb{F}}}^{\square}/\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}^k, A\right) \cong \varprojlim_k \mathrm{Hom}_{\mathfrak{A}r_{W(\mathbb{F})}}\left(R_{V_{\mathbb{F}}}^{\square}/\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}^k, A\right),$$

for any $A \in \mathfrak{A}r_{W(\mathbb{F})}$.

- (2) $(\text{???})R_{V_{\mathbb{F}}}^{\square}$ is unique up to unique isomorphism; the identity map in $\mathrm{Hom}(R_{V_{\mathbb{F}}}^{\square}, R_{V_{\mathbb{F}}}^{\square})$ gives rise to a universal framed deformation over $R_{V_{\mathbb{F}}}^{\square}$.
- (3) $R_{V_{\mathbb{F}}}^{\square}$ exists without Φ_p , but maybe no longer noetherian.
- (4) $(\text{???})\mathbb{F} \hookrightarrow \mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) \rightsquigarrow$ write "=" in $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$.

Proof of Proposition 1.7.

- (1) G finite \rightsquigarrow profinite.

• **(FORMAL) CONSTRUCTION:**

Suppose G is finite. Set

$$G = \langle g_1, \dots, g_s | r_1(g_1, \dots, g_s), \dots, r_t(g_1, \dots, g_s) \rangle$$

a presentation. Define

$$\mathcal{R} = W(\mathbb{F})[X_{i,j}^k | i, j = 1, \dots, d; k = 1, \dots, s] / \mathcal{I},$$

where

$$\mathcal{I} = \langle r_l(X^1, \dots, X^s) - \mathrm{id} \rangle_{1 \leq l \leq t}, X^k = (X_{i,j}^k)_{d \times d}.$$

To make \mathcal{R} complete, local, noetherian, take

$$\mathcal{J} = \ker(\mathcal{R} \rightarrow \mathbb{F}, X^k \mapsto \bar{\rho}(g_k), k = 1, \dots, s),$$

and set $R_{V_{\mathbb{F}}}^{\square} = \varprojlim_n \mathcal{R}/\mathcal{J}^n$ to be the \mathcal{J} -adic completion of \mathcal{R} . Besides that, we set $\rho_{V_{\mathbb{F}}}^{\square} : G \rightarrow \mathrm{GL}_d(R_{V_{\mathbb{F}}}^{\square})$, $g_k \mapsto$ image of X^k in $\mathrm{GL}_d(R_{V_{\mathbb{F}}}^{\square})$.

VERIFICATION:

Take $\rho \in D_{V_{\mathbb{F}}}^{\square}(A)$, where $\rho : G \rightarrow \mathrm{GL}_d(A)$. Define

$$\mathfrak{F}_{\rho} \in \mathrm{Hom}_{W(\mathbb{F})}(R_{V_{\mathbb{F}}}^{\square}, A), \overline{X^k} \mapsto \text{corresponding entries of } \rho(g_k), \forall k = 1, \dots, s.$$

Then \mathfrak{F}_{ρ} induces $\widehat{\mathfrak{F}_{\rho}} : \mathrm{GL}_d(R_{V_{\mathbb{F}}}^{\square}) \rightarrow \mathrm{GL}_d(A)$. It's immediate to check that $\rho = \widehat{\mathfrak{F}_{\rho}} \circ \rho_{V_{\mathbb{F}}}^{\square}$ and $\widehat{\mathfrak{F}_{\rho}}$ is unique choice to make the diagram commute. Thus, $\mathfrak{F} : \rho \mapsto \mathfrak{F}_{\rho}$ gives the pro-representability when G is finite.

- When G is profinite, we have $G = \varprojlim_i G/H_i$, where $H_i \subset \ker(\bar{\rho})$ are open normal subgroups. For every i , one has a universal pair $(R_i^{\square}, \rho_i^{\square})$ by previous construction. Passing by limits, we define

$$(R_{V_{\mathbb{F}}}^{\square}, \rho_{V_{\mathbb{F}}}^{\square}) = \varprojlim_i (R_i^{\square}, \rho_i^{\square}), \text{ with } R_{V_{\mathbb{F}}}^{\square} \in \widehat{\mathfrak{A}r_{W(\mathbb{F})}}.$$

We will show in [Section 1.4, TBA] that $R_{V_{\mathbb{F}}}^{\square}$ is noetherian.

- (2) By Schlessinger's representability criterion (cf. [Section 1.7, TBA]) or by Kisin's work (cf. [Section 2.1, TBA by Y. Chen]).

□

1.3. The tangent space.

首先定一个能达到的小目标, 比方说我先证 $R_{V_{\mathbb{F}}}^{\square}$ 是 Noetherian 的.

— Wozki Shod

What is the tangent space of a (local) ring? For $A \in \widehat{\mathfrak{A}r}_{W(\mathbb{F})}$, we consider the affine scheme $\text{Spec } A$, which consists of a single point $x = \mathfrak{m}_A$. Then the Zariski tangent space \mathfrak{t}_A of A is defined to be $\text{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)) = \text{Hom}(\mathfrak{m}_A/\mathfrak{m}_A^2, \mathbb{F})$. By [Har77, Ex II.2.8],

$$(1.1) \quad \mathfrak{t}_A = \text{Hom}(\mathfrak{m}_A/\mathfrak{m}_A^2, \mathbb{F}) = \text{Hom}(\text{Spec } \mathbb{F}[\varepsilon], \text{Spec } A) = \text{Hom}_{W(\mathbb{F})}(A, \mathbb{F}[\varepsilon]).$$

One should notice that $\mathfrak{t}_{R_{V_{\mathbb{F}}}^{\square}} = D_{V_{\mathbb{F}}}^{\square}(\mathbb{F}[\varepsilon])$.

Proposition 1.9. *If G satisfies Φ_p , then the universal deformation ring $R_{V_{\mathbb{F}}}^{\square}$ is Noetherian.*

Proof. By Lemma 1.10 (1), one only needs to prove $\dim_{\mathbb{F}} D_{V_{\mathbb{F}}}^{\square}(\mathbb{F}[\varepsilon]) < \infty$, which, by Lemma 1.10 (4), reduced to prove $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ is a finite dimensional \mathbb{F} -vector space. Then Lemma 1.10 (3) gives what we want. \square

Lemma 1.10.

- (1) *If $\dim_{\mathbb{F}} D_{V_{\mathbb{F}}}^{\square}(\mathbb{F}[\varepsilon]) < \infty$, then $R_{V_{\mathbb{F}}}^{\square}$ is Noetherian.*
- (2) *$D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) \cong \text{Ext}_{\mathbb{F}[G]}^1(V_{\mathbb{F}}, V_{\mathbb{F}}) \cong H^1(G, \text{ad } V_{\mathbb{F}})$.*
- (3) *If G satisfies Φ_p , then $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ is a finite-dimensional \mathbb{F} -vector space.*
- (4)

$$D_{V_{\mathbb{F}}}^{\square}(\mathbb{F}[\varepsilon]) = \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) + d^2 - h^0(G, \text{ad } V_{\mathbb{F}})$$

Proof. (1) If $\dim_{\mathbb{F}} \mathfrak{t}_{R_{V_{\mathbb{F}}}^{\square}} < \infty$, we know that $\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}/\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}^2$ is of finite dimension. Consequently, by Nakayama's lemma, $\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}$ is a finitely generated ideal. The result follows from [Mat87, Theorem 29.4(i)].

(2) Take $\rho \in D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$. Set $M = \mathbb{F}[\varepsilon]^d$, with the action of G given by ρ . Then $\dim_{\mathbb{F}} M = 2d$ and the following sequence is exact:

$$0 \rightarrow \varepsilon M \rightarrow M \rightarrow M/\varepsilon M \rightarrow 0.$$

Notice that $\varepsilon M \cong V_{\mathbb{F}} \cong M/\varepsilon M$, one has $M \in \text{Ext}_{\mathbb{F}[G]}^1(V_{\mathbb{F}}, V_{\mathbb{F}})$.

Conversely, if E is an $2d$ -dimensional \mathbb{F} -vector space which fits the exact sequence

$$0 \rightarrow V_{\mathbb{F}} \xrightarrow{\alpha} E \xrightarrow{\beta} V_{\mathbb{F}} \rightarrow 0,$$

we define the action of ε on E by $\alpha \circ \beta : E \xrightarrow{\beta} V_{\mathbb{F}} \xrightarrow{\alpha} E$. It's easy to check that $(\alpha \circ \beta)^2 = 0$. For α and β are homomorphisms of G -modules, the $\mathbb{F}[G]$ -module structure on E commutes with the action of G . This makes E into a free $\mathbb{F}[\varepsilon]$ -module of rank d with action of G . It induces $\rho_E : G \rightarrow \text{GL}_d(\mathbb{F}[\varepsilon])$, which is a deformation of $\bar{\rho}$. We conclude that $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) \cong \text{Ext}_{\mathbb{F}[G]}^1(V_{\mathbb{F}}, V_{\mathbb{F}})$.

For the isomorphism $\text{Ext}_{\mathbb{F}[G]}^1(V_{\mathbb{F}}, V_{\mathbb{F}}) \cong H^1(G, \text{ad } V)$, we give the explicit formula and leave the verification work to the reader. Given the exact sequence

$$0 \rightarrow V_{\mathbb{F}} \rightarrow M \rightarrow V_{\mathbb{F}} \rightarrow 0$$

whit $\dim_{\mathbb{F}} M = 2d$, it amount to say the map $\rho_M : G \rightarrow \text{GL}_d(\mathbb{F})$ corresponding to M can be put into the block form $\begin{bmatrix} \bar{\rho}(g) & A_g \\ 0 & \bar{\rho}(g) \end{bmatrix}$, with $A_g \in \text{M}_d(\mathbb{F})$. Then the map $g \mapsto A_g \bar{\rho}(g)^{-1}$ is a 1-cocycle and it induces the isomorphism we need.

- (3) Let $G' = \ker(\bar{\rho})$, which is an open subgroup of G . The inflation-restriction exact sequence (cf. [Ste13, Proposition 28]) gives

$$0 \rightarrow H^1(G/G', \text{ad}V_{\mathbb{F}}) \rightarrow H^1(G, \text{ad}V_{\mathbb{F}}) \rightarrow H^1(G', \text{ad}V_{\mathbb{F}})^{G/G'}. \quad \uparrow \cong (\text{Hom}(G', \mathbb{F}_p) \otimes_{\mathbb{F}_p} \text{ad}V_{\mathbb{F}})^{G/G'} ???$$

The term on the left is finite because G/G' and $V_{\mathbb{F}}$ are finite. The term on the right is finite because of Condition Φ_p for G . The result follows.

- (4) Fix a deformation $V_{\mathbb{F}[\varepsilon]}$ of $V_{\mathbb{F}}$ to $\mathbb{F}[\varepsilon]$. The set of $\mathbb{F}[\varepsilon]$ -basis of $V_{\mathbb{F}[\varepsilon]}$ lifting the fixed basis $\beta_{\mathbb{F}}$ of $V_{\mathbb{F}}$ is an \mathbb{F} -vector space of dimension d^2 . Let β' and β'' be two such lifted bases. Then there is an isomorphism of framed deformations $(V_{\mathbb{F}[\varepsilon]}, \beta') \cong (V_{\mathbb{F}[\varepsilon]}, \beta'')$ if and only if there exists an automorphism $1 + \varepsilon\alpha$ of $V_{\mathbb{F}[\varepsilon]}$, where $\alpha \in \text{ad}V$, which takes β' to β'' , so that $\alpha \in \text{ad}V_{\mathbb{F}}^G$. Thus the fibers of $D_{V_{\mathbb{F}}}^{\square}(V_{\mathbb{F}[\varepsilon]}) \rightarrow D_{V_{\mathbb{F}}}(V_{\mathbb{F}[\varepsilon]})$ are a principal homogeneous space under $\text{ad}V_{\mathbb{F}}/(\text{ad}V_{\mathbb{F}})^G$. The dimension of this space is $d^2 - h^0(G, \text{ad}V_{\mathbb{F}})$. \square

1.7. Groupoid over categories (abstract stuff...)

Definition 1.11.

- (1) A **groupoid category** is a category in which all morphisms are isomorphisms.
- (2) Call the isomorphism classes the **connected components** of the groupoid.

Remark 1.12.

- (1) Not necessarily all objects in a groupoid are isomorphic.
- (2) $\text{Hom}_{\mathfrak{C}}(A, A)$ forms a group for $\forall A \in \text{ob } \mathfrak{C}$, where \mathfrak{C} is a groupoid category, and the identity in $\text{Hom}_{\mathfrak{C}}(A, A)$ is the identity morphism.
- (3) $A \cong B \Rightarrow \text{Hom}_{\mathfrak{C}}(A, A) \cong \text{Hom}_{\mathfrak{C}}(B, B)$ (non-canonically)

Definition 1.13. Let \mathfrak{C} be a category. Let \mathfrak{F} be another category, $\Theta : \mathfrak{F} \rightarrow \mathfrak{C}$ be a functor.

- (1) We say $\eta \in \text{ob}(\mathfrak{F})$ **lies above** $T \in \text{ob}(\mathfrak{C})$, if $\Theta(\eta) = T$.
- (2) We say $(\eta \xrightarrow{\alpha} \xi) \in \text{Mor}_{\mathfrak{F}}$ **lies above** $(T \xrightarrow{f} S) \in \text{Mor}_{\mathfrak{C}}$, if $\Theta(\alpha) = f$.
- (3) $(T \in \text{ob}(\mathfrak{C}), \text{id}_T)$ is a subcategory of \mathfrak{C} . Write $\mathfrak{F}(T)$ the subcategory of \mathfrak{F} over (T, id_T) .

Definition 1.14 (groupoid over \mathfrak{C} /category cofibered in groupoids over \mathfrak{C}). The triple $(\mathfrak{F}, \mathfrak{C}, \Theta)$ is a **groupoid over \mathfrak{C}** if

- (1) for any morphisms $(\eta \xrightarrow{\alpha} \xi)$ and $(\eta \xrightarrow{\alpha'} \xi')$ in \mathfrak{F} over the same morphism $T \rightarrow S$ in \mathfrak{C} , there exists unique $\xi \xrightarrow{u} \xi'$ in \mathfrak{F} over id_S such that $u \circ \alpha = \alpha'$.
- (2) For any $\eta \in \text{ob}(\mathfrak{C})$ and any $T \xrightarrow{f} S$ in $\text{Mor}_{\mathfrak{C}}$ with η over T , there exists morphism $\eta \xrightarrow{\alpha} \xi$ in $\text{Mor}_{\mathfrak{F}}$ over f .

Remark 1.15.

- (1) For every $T \in \text{ob}(\mathfrak{C})$, the category $\mathfrak{F}(T)$ is a groupoid. It's natural to specify a groupoid by specifying objects in $\mathfrak{F}(T)$ for any $T \in \text{ob}(\mathfrak{C})$, and specifying isomorphism class of morphisms above any $T \xrightarrow{f} S$ in \mathfrak{C} .
- (2) Scheme and stack stuff....

If for each $T \in \text{ob}(\mathfrak{C})$, the isomorphism classes of $\mathfrak{F}(T)$ forms a set, we associate to the category \mathfrak{F} over \mathfrak{C} a functor $|\mathfrak{F}| : \mathfrak{C} \rightarrow \text{Set}$ by sending T to the set of isomorphism classes of $\mathfrak{F}(T)$.

Example 1.16.

- (1) To the representation $V_{\mathbb{F}}$ of G , we define a groupoid $\mathcal{D}_{V_{\mathbb{F}}}$ over $\mathfrak{C} = \mathfrak{Atr}_{W(\mathbb{F})}$:
 - (a) Objects of $\mathcal{D}_{V_{\mathbb{F}}}$ over $A \in \mathfrak{Atr}_{W(\mathbb{F})}$: pairs $(V_A, \iota_A) \in D_{V_{\mathbb{F}}}(A)$.

(b) Morphism $(V_A, \iota_A) \rightarrow (V_{A'}, \iota_{A'})$ over $A \rightarrow A'$ in $\mathfrak{Art}_{W(\mathbb{F})}$: isomorphism class

$$\left\{ \alpha : V_A \otimes_A A' \xrightarrow{\cong} V_{A'} \text{ is an isomorphism } \mid \iota_{A'} \circ \alpha = \iota_A \right\} / (A')^*$$

(2) We define the groupoid $\mathcal{D}_{V_{\mathbb{F}}}^{\square}$ on $\mathfrak{C} = \mathfrak{Art}_{W(\mathbb{F})}$ as follows:

(a) Objects over $A \in \mathfrak{Art}_{W(\mathbb{F})}$: triples (V_A, ι_A, β_A) , where $(V_A, \iota_A) \in \mathcal{D}_{V_{\mathbb{F}}}(A)$ and β_A is an A -basis of V_A mapping under ι_A to the basis $\beta_{\mathbb{F}}$ of $V_{\mathbb{F}}$.

(b) Morphism $(V_A, \iota_A, \beta_A) \rightarrow (V_{A'}, \iota_{A'}, \beta_{A'})$ over $A \rightarrow A'$: isomorphism $\alpha : V_A \otimes_A A' \xrightarrow{\cong} V_{A'}$ taking β_A to $\beta_{A'}$.

There is an obvious morphism of groupoids $\mathcal{D}_{V_{\mathbb{F}}}^{\square} \rightarrow \mathcal{D}_{V_{\mathbb{F}}}$.

Remark 1.17.

- (1) The deformation functor $D_{V_{\mathbb{F}}}$ defined before is exactly $|\mathcal{D}_{V_{\mathbb{F}}}|$ above.
- (2) When $V_{\mathbb{F}}$ has non-trivial automorphisms, then so do the object in $D_{V_{\mathbb{F}}}(A)$. (???) In this situation, the groupoid $\mathcal{D}_{V_{\mathbb{F}}}$ captures the geometry of the deformation theory of $V_{\mathbb{F}}$ more accurately than its functor of isomorphism classes.

Representability of a groupoid $\Theta : \mathfrak{F} \rightarrow \mathfrak{C}$.

Definition 1.18.

- (1) $\forall \eta \in \text{ob}(\mathfrak{F})$, define the category $\tilde{\eta}$ (**the category under η**) as the category with objects are morphisms with source η and whose morphisms from $\eta \xrightarrow{\alpha} \xi$ to $\eta \xrightarrow{\alpha'} \xi'$ are morphisms $\xi \xrightarrow{u} \xi'$ in \mathfrak{F} such that $u \circ \alpha = \alpha'$.
- (2) Groupoid \mathfrak{F} over \mathfrak{C} is **representable** if there exists $\eta \in \mathfrak{F}$ such that the canonical functor $\tilde{\eta} \rightarrow \mathfrak{F}$ is an equivalence of categories.
- (3) Similarly, we define the category \tilde{T} for every $T \in \mathfrak{C}$.

ξ and ξ' are not necessarily lying on the same object of \mathfrak{C}

One has a commutative diagram of categories:

$$\begin{array}{ccc} \tilde{\eta} & \longrightarrow & \mathfrak{F} \\ \downarrow \cong & & \downarrow \\ \widetilde{\Theta(\eta)} & \longrightarrow & \mathfrak{C} \end{array}$$

Lemma 1.19. The left vertical homomorphism above is an equivalence of categories.

Proof. Abstract nonsense..... □

Remark 1.20. If \mathfrak{F} is representable by η , then the equivalence $\tilde{\eta} \rightarrow \widetilde{\Theta(\eta)}$ implies that η , as well as $\Theta(\eta)$, are well-defined up to canonical isomorphism. One says that $\Theta(\eta)$ **represents \mathfrak{F} over \mathfrak{C}** .

Lemma 1.21 (Relation with "classical" representable functor).

- (1) If \mathfrak{F} is representable by η , any two objects of $\mathfrak{F}(\Theta(\eta))$ are canonically isomorphic and there is an isomorphism of functors

$$\text{Hom}_{\mathfrak{C}}(\Theta(\eta), -) \xrightarrow{\cong} |\mathfrak{F}|,$$

so that $\Theta(\eta)$ represents $|\mathfrak{F}|$ in the usual set theoretic sense.

- (2) If $|\mathfrak{F}|$ is representable and for any $T \in \text{ob}(\mathfrak{C})$ any two objects of $\mathfrak{F}(T)$ are related by a unique isomorphism, then \mathfrak{F} is representable.

Remark 1.22. The groupoid $\mathcal{D}_{V_{\mathbb{F}}}$ in Example 1.16 is usually not representable. Extending to $\widehat{\mathfrak{Art}_{W(\mathbb{F})}}$ is needed.

A crucial question: why do we need to use the language of groupoids?

Let's see Kisin's motivation (just a screenshot from Bockle's notes, which is nothing but a rewrite of [Kis09, Appendix (A.6)]):

The main reason why, in some circumstances, one needs to introduce the language of groupoids, is that formation of fiber products is not compatible with the passage from a groupoid \mathfrak{F} over \mathfrak{C} to its associated functor $|\mathfrak{F}|$. This is a serious technical issue, since Definition 2.4.4 of relative representability depends on the formation of fiber products. We illustrate this with a simple example taken from [34, A.6].

Consider now the situation when the group G is trivial and fix $\eta = (V_A, \iota_A) \in \mathcal{D}_{V_{\mathbb{F}}}(A)$ for some $A \in \mathfrak{A}_{W(\mathbb{F})}$. Then $\tilde{\eta} \times_{\mathcal{D}_{V_{\mathbb{F}}}} \mathcal{D}_{V_{\mathbb{F}}}^{\square}$ can be identified with quadruples $(V'_{A'}, \psi'_{A'}, \varphi: V_A \otimes_A A' \xrightarrow{\cong} V'_{A'}, \beta_{A'})$, where $(V'_{A'}, \psi'_{A'}, \beta_{A'}) \in \mathcal{D}_{V_{\mathbb{F}}}^{\square}(A')$ and morphisms over $\text{id}_{A'}$ are isomorphisms of $V'_{A'}$, reducing to the identity of $V_{\mathbb{F}}$. It follows that this category is a principal homogeneous space for the formal group obtained by completing $\text{PGL}_d/W(\mathbb{F})$ along its identity section. Hence $|\tilde{\eta} \times_{\mathcal{D}_{V_{\mathbb{F}}}} \mathcal{D}_{V_{\mathbb{F}}}^{\square}|(A')$ is isomorphic to the kernel $\text{Ker}(\text{PGL}_d(A') \rightarrow \text{PGL}_d(\mathbb{F}))$. On the other hand, $|\mathcal{D}_{V_{\mathbb{F}}}^{\square}(A')|$ is a singleton and hence the same holds for $|\tilde{\eta}| \times_{|\mathcal{D}_{V_{\mathbb{F}}}|} |\mathcal{D}_{V_{\mathbb{F}}}^{\square}|(A')$.

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