DEFORMATION OF GALOIS REPRESENTATIONS

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We are given either a number field K and a finite set of primes S, or a local field F, and we are given a representation of either $G_{K,S}$ or G_F into $\mathrm{GL}_n(k)$, where k is a finite field. We want to try to understand all possible lifts of this representation to $\mathrm{GL}_n(A)$, where A is a complete noetherian local ring with residue field k.

—Fernando Q. Gouvêa (cf. [Gou01])

Main reference: [Böc13; Maz89]
Motivation and history of deformation theory: [Maz97; Gou01]
Basics of groupoids: Appendix of [Kis09]

1. Deformations of representations of profinite groups

Notations.

p	prime number
$\mathbb{F}$	finite field of char. p
$W(\mathbb{F})$	ring of Witt vectors[1] over A
G	profinite group
$V_{\mathbb{F}}$	finite $\mathbb{F}[G]$-module with continuous G-action
d	dimeinsion of $V_{\mathbb{F}}$
$\beta_{\mathbb{F}}$	a $\mathbb{F}$-basis of $V_{\mathbb{F}}$

1.1. Deformation functors.

Notations.

$\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$	category of complete Noetherian local $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$
$\mathfrak{Ar}_{W(\mathbb{F})}$	full sub-category of finite local Artinian $W(\mathbb{F})$-algebras
$\mathfrak{m}_A$	maximal ideal of $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$

Remark 1.1 (???). *Via the $W(\mathbb{F})$-structure, the residue field of any $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ is canonically isomorphic to $\mathbb{F}$.*

Definition 1.2. *Let $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.*

(1) *A **deformation** of $V_{\mathbb{F}}$ to A is a pair (V_A, ι_A), such that*
  (a) *V_A is $A[G]$-module, finite free over A, with continuous G-action;*
  (b) *$\iota_A : V_A \otimes_A \mathbb{F} \xrightarrow{\cong} V_{\mathbb{F}}$ is G-equivariant.*

We view $\mathbb{F}$ as an A-module via the canonical projection $A \to A/\mathfrak{m}_A = F$.

(2) *A **framed deformation** of $(V_{\mathbb{F}}, \beta_{\mathbb{F}})$ to A is a triple (V_A, ι_A, β_A), where*

[1] $W(\mathbb{F})$ is the unique (up to unique isomorphism) complete discrete valuation ring which is absolutely unramified (uniformizer$=p$) and has residue field $\mathbb{F}$.

(a) (V_A, ι_A) *is a deformation of* $V_{\mathbb{F}}$ *to* A*;*
(b) β_A *is a* A*-basis of* V_A *which reduces to* $\beta_{\mathbb{F}}$ *under* ι_A*.*

Set $D_{V_{\mathbb{F}}}, D^{\square}_{V_{\mathbb{F}}} : \mathfrak{Ar}_{W(\mathbb{F})} \to \mathrm{Set}$,

$$D_{V_{\mathbb{F}}}(A) = \{\textit{deformations of } V_{\mathbb{F}} \textit{ to } A\}/\cong,$$

$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\textit{framed deformations of } (V_{\mathbb{F}}, \beta_{\mathbb{F}}) \textit{ to } A\}/\cong .$$

Remark 1.3.

(1) *The* ***FIXED*** *basis* $\beta_{\mathbb{F}}$ *gives the isomorphism* $V_{\mathbb{F}} \cong \mathbb{F}^d$ *as vector space. Thus we can view* $V_{\mathbb{F}}$ *as* $\bar{\rho} : G \to \mathrm{GL}(V_{\mathbb{F}}) = \mathrm{GL}_d(\mathbb{F})$*: a* d*-dimensional* $\mathbb{F}$*-representation of* G*. Then*

$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\rho : G \to \mathrm{GL}_d(A) \textit{ lifting } \bar{\rho}\},$$

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, "\bar{\rho}"'] & \mathrm{GL}_d(A) \arrow[d, two heads] \\ & \mathrm{GL}_d(\mathbb{F}) \end{tikzcd}$$

Does not guarantee $\beta_A \rightsquigarrow \beta_{\mathbb{F}}$!

$$D_{V_{\mathbb{F}}}(A) = D^{\square}_{V_{\mathbb{F}}}(A)/\textit{action by conjugates of } \ker(\mathrm{GL}_d(A) \to \mathrm{GL}_d(\mathbb{F})).$$

Not always representable!

(2) *Mazur only consdier* $D_{V_{\mathbb{F}}}$*, which describes representations lifting* $V_{\mathbb{F}}$ *up to isomorphism. Add "base condition"* $\rightsquigarrow D^{\square}_{V_{\mathbb{F}}}$*.*

(3) *Often consider deformation functors on* $\mathfrak{Ar}_{\mathcal{O}}$ *= category of local artinian* $\mathcal{O}$*-algebra with residue field* $\mathbb{F}$*, where* $\mathcal{O}$ *is ring of integers of a finite totally ramified extension of* $W(\mathbb{F})\left[\frac{1}{p}\right]$ $(\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F})$*.*

For example, let K *be a* p*-adic field with residue field* $\mathbb{F}_q$*, ring of integers* $\mathcal{O}_K$*, then* $K/W(\mathbb{F}_q)\left[\frac{1}{p}\right]$ *is totally ramified and* $W(\mathbb{F}_q)\left[\frac{1}{p}\right]/\mathbb{Q}_p$ *is unramified.*

1.2. Representability.

1.2.1. *A finiteness condition.*

Definition 1.4 (Mazur)**.** *A profinite group* G *has finiteness condition* Φ_p*, if* $\forall$ *open subgroup* $G' \subset G$*,* $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < +\infty$*.*

Remark 1.5.

(1) *(Burnside basis theorem)* $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < +\infty \Leftrightarrow$ *maximal pro-*p *quotient of* G' *is topologically finitely generated.*

(2) $\mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) \cong \mathrm{Hom}_{\mathrm{cont}}(G'^{,\mathrm{ab}}, \mathbb{F}_p)$*.*

Example 1.6 (by CFT)**.** *The following groups have* Φ_p*:*

(1) *The Galois group* $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$*, with* K *a* p*-adic field.*

(2) *The galois group* $\mathcal{G}_{F,S} = \mathrm{Gal}(F_S/F)$*, where* F *is a number field,* S *is a finite set of places of* F *and* $F_S \subset \bar{F}$ *is the maximal extension of* F *unramified outside* S*.*

1.2.2. *Main proposition.*

Proposition 1.7 (Mazur)**.** *If* G *has* Φ_p*, then*

(1) *The functor* $D^{\square}_{V_{\mathbb{F}}}$ *is pro-representable by some* $R^{\square}_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$*, i.e.*

$$D^{\square}_{V_{\mathbb{F}}}(A) \cong \mathrm{Hom}_{W(\mathbb{F})}\left(R^{\square}_{V_{\mathbb{F}}}, A\right),$$

universal framed deformation ring

which is functorial in $A \in \mathfrak{Ar}_{W(\mathbb{F})}$*.*

Maybe it's better to write $D^{\square}_{V_{\mathbb{F}}}(A) \cong \mathrm{Hom}_{\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}}(R^{\square}_{V_{\mathbb{F}}}, A)$?

(2) *If* $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$*, then* $D_{V_{\mathbb{F}}}$ *is pro-representable by some* $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$*.*

universal deformation ring

Remark 1.8. *(1) Literally, pro-representable functor = limit of representable functor. How can we realize that? [nLa22] defines pro-representable to be the filtered colimit of representables. There's a post on MSE (cf. [htt]) which discusses the difference between the two definitions. It might be ture that*

$$\mathrm{Hom}_{W(\mathbb{F})}\left(R^{\square}_{V_{\mathbb{F}}}, A\right) = \mathrm{Hom}_{\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}}\left(\varprojlim_k R^{\square}_{V_{\mathbb{F}}}/\mathfrak{m}^k_{R^{\square}_{V_{\mathbb{F}}}}, A\right) \cong \varinjlim_k \mathrm{Hom}_{\mathfrak{Ar}_{W(\mathbb{F})}}\left(R^{\square}_{V_{\mathbb{F}}}/\mathfrak{m}^k_{R^{\square}_{V_{\mathbb{F}}}}, A\right),$$

for any $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.

(2) (???)$R^{\square}_{V_{\mathbb{F}}}$ is unique up to unique isomorphism; the identity map in $\mathrm{Hom}\big(R^{\square}_{V_{\mathbb{F}}}, R^{\square}_{V_{\mathbb{F}}}\big)$ gives rise to a universal framed deformation over $R^{\square}_{V_{\mathbb{F}}}$.

(3) $R^{\square}_{V_{\mathbb{F}}}$ exists without Φ_p, but maybe no longer noetherian.

(4) (???)$\mathbb{F} \hookrightarrow \mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) \rightsquigarrow$ write "=" in $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$.

Proof of Proposition 1.7.

(1) G finite $\rightsquigarrow$ profinite.

- **(FORMAL) CONSTRUCTION:**
  Suppose G is finite. Set

  $$G = \langle g_1, \cdots, g_s | r_1(g_1, \cdots, g_s), \cdots, r_t(g_1, \cdots, g_s)\rangle$$

  a presentation. Define

  $$\mathcal{R} = W(\mathbb{F})\left[X^k_{i,j}\middle| i, j = 1, \cdots, d; k = 1, \cdots, s\right]/\mathcal{I},$$

  where

  $$\mathcal{I} = \langle r_l\big(X^1, \cdots, X^s\big) - \mathrm{id}\rangle_{1\le l\le t}, X^k = \big(X^k_{i,j}\big)_{d\times d}.$$

  To make $\mathcal{R}$ complete, local, noetherian, take

  $$\mathcal{J} = \ker\big(\mathcal{R} \to \mathbb{F}, X^k \mapsto \bar{\rho}(g_k), k = 1, \cdots, s\big),$$

  and set $R^{\square}_{V_{\mathbb{F}}} = \varprojlim_n \mathcal{R}/\mathcal{J}^n$ to be the $\mathcal{J}$-adic completion of $\mathcal{R}$. Besides that, we set $\rho^{\square}_{V_{\mathbb{F}}} : G \to \mathrm{GL}_d\big(R^{\square}_{V_{\mathbb{F}}}\big), g_k \mapsto$ image of X^k in $\mathrm{GL}_d\big(R^{\square}_{V_{\mathbb{F}}}\big)$.
  VERIFICATION:
  Take $\rho \in D^{\square}_{V_{\mathbb{F}}}(A)$, where $\rho : G \to \mathrm{GL}_d(A)$. Define

  $$\mathfrak{F}_\rho \in \mathrm{Hom}_{W(\mathbb{F})}\left(R^{\square}_{V_{\mathbb{F}}}, A\right), \overline{\text{entries of } X^k} \mapsto \text{corresponding entries of } \rho(g_k), \forall k = 1, \cdots, s.$$

  Then $\mathfrak{F}_\rho$ induces $\widehat{\mathfrak{F}_\rho} : \mathrm{GL}_d\big(R^{\square}_{V_{\mathbb{F}}}\big) \to \mathrm{GL}_d(A)$. It's immediate to check that $\rho = \widehat{\mathfrak{F}_\rho} \circ \rho^{\square}_{V_{\mathbb{F}}}$ and $\widehat{\mathfrak{F}_\rho}$ is unique choice to make the diagram commute. Thus, $\mathfrak{F} : \rho \mapsto \mathfrak{F}_\rho$ gives the pro-representability when G is finite.
- When G is profinite, we have $G = \varprojlim_i G/H_i$, where $H_i \subset \ker(\bar{\rho})$ are open normal subgroups. For every i, one has a universal pair $\big(R^{\square}_i, \rho^{\square}_i\big)$ by previous construction. Passing by limits, we define

  $$\left(R^{\square}_{V_{\mathbb{F}}}, \rho^{\square}_{V_{\mathbb{F}}}\right) = \varprojlim_i \left(R^{\square}_i, \rho^{\square}_i\right), \text{with } R^{\square}_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}.$$

We will show in [Section 1.4, TBA] that $R^{\square}_{V_{\mathbb{F}}}$ is noetherian.

(2) By Schlessinger's representability criterion (cf. [Section 1.7, TBA]) or by Kisin's work (cf. [Section 2.1, TBA by Y. Chen]).

$\square$

1.3. **The tangent space.**

> 首先定一个能达到的小目标，比方说我先证 $R^{\square}_{V_{\mathbb{F}}}$ 是 *Noetherian* 的.
>
> —*Wozki Shod*

What is the tangent space of a (local) ring? For $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$, we consider the affine scheme $\operatorname{Spec} A$, which consists of a single point $x = \mathfrak{m}_A$. Then the Zariski tangent space $\mathfrak{t}_A$ of A is defined to be $\operatorname{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)) = \operatorname{Hom}(\mathfrak{m}_A/\mathfrak{m}_A^2, \mathbb{F})$. By [Har77, Ex II.2.8],

$$\mathfrak{t}_A = \operatorname{Hom}(\mathfrak{m}_A/\mathfrak{m}_A^2, \mathbb{F}) = \operatorname{Hom}(\operatorname{Spec}\mathbb{F}[\varepsilon], \operatorname{Spec} A) = \operatorname{Hom}_{W(\mathbb{F})}(A, \mathbb{F}[\varepsilon]). \tag{1.1}$$

One should notice that $\mathfrak{t}_{R^{\square}_{V_{\mathbb{F}}}} = D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$.

Proposition 1.9. *If G satisfies Φ_p, then the universal deformation ring $R^{\square}_{V_{\mathbb{F}}}$ is Noetherian.*

Proof. By Lemma 1.10 (1), one only needs to prove $\dim_{\mathbb{F}} D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) < \infty$, which, by Lemma 1.10 (4), reduced to prove $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ is a finite dimensional $\mathbb{F}$-vector space. Then Lemma 1.10 (3) gives what we want. □

Lemma 1.10.

(1) If $\dim_{\mathbb{F}} D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) < \infty$, then $R^{\square}_{V_{\mathbb{F}}}$ is Noetherian.
(2) $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) \cong \operatorname{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}}) \cong \mathrm{H}^1(G, \operatorname{ad}V_{\mathbb{F}})$.
(3) If G satisfies Φ_p, then $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ is a finite-dimensional $\mathbb{F}$-vector space.
(4)

$$D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) = \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) + d^2 - h^0(G, \operatorname{ad}V_{\mathbb{F}})$$

Proof. (1) If $\dim_{\mathbb{F}} \mathfrak{t}_{R^{\square}_{V_{\mathbb{F}}}} < \infty$, we know that $\mathfrak{m}_{R^{\square}_{V_{\mathbb{F}}}}/\mathfrak{m}^2_{R^{\square}_{V_{\mathbb{F}}}}$ is of finite dimension. Consequently, by Nakayama's lemma, $\mathfrak{m}_{R^{\square}_{V_{\mathbb{F}}}}$ is a finitely generated ideal. The result follows from [Mat87, Theorem 29.4(i)].

(2) Take $\rho \in D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$. Set $M = \mathbb{F}[\varepsilon]^d$, with the action of G given by ρ. Then $\dim_{\mathbb{F}} M = 2d$ and the following sequence is exact:

$$0 \to \varepsilon M \to M \to M/\varepsilon M \to 0.$$

Notice that $\varepsilon M \cong V_{\mathbb{F}} \cong M/\varepsilon M$, one has $M \in \operatorname{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}})$.

Conversely, if E is an $2d$-dimensional $\mathbb{F}$-vector space which fits the exact sequence

$$0 \to V_{\mathbb{F}} \xrightarrow{\alpha} E \xrightarrow{\beta} V_{\mathbb{F}} \to 0,$$

we define the action of ε on E by $\alpha \circ \beta : E \xrightarrow{\beta} V_{\mathbb{F}} \xrightarrow{\alpha} E$. It's easy to check that $(\alpha \circ \beta)^2 = 0$. For α and β are homorphisms of G-modules, the $\mathbb{F}[G]$-module structure on E commutes with the action of G. This makes E into a free $\mathbb{F}[\varepsilon]$-module of rank d with action of G. It induces $\rho_E : G \to \mathrm{GL}_d(\mathbb{F}[\varepsilon])$, which is a deformation of $\bar{\rho}$. We conclude that $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) \cong \operatorname{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}})$.

For the isomorphism $\operatorname{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}}) \cong \mathrm{H}^1(G, \operatorname{ad}V)$, we give the explicit formula and leave the verification work to the reader. Given the exact sequence

$$0 \to V_{\mathbb{F}} \to M \to V_{\mathbb{F}} \to 0$$

whti $\dim_{\mathbb{F}} M = 2d$, it amount to say the map $\rho_M : G \to \mathrm{GL}_d(\mathbb{F})$ corresponding to M can be put into the block form $\begin{bmatrix} \bar{\rho}(g) & A_g \\ 0 & \bar{\rho}(g) \end{bmatrix}$, with $A_g \in \mathrm{M}_d(\mathbb{F})$. Then the map $g \mapsto A_g \bar{\rho}(g)^{-1}$ is a 1-cocycle and it induces the isomorphism we need.

(3) Let $G' = \ker(\bar{\rho})$, which is an open subgroup of G. The inflation-restriction exact sequence (cf. [Ste13, Proposition 28]) gives

$$0 \to \mathrm{H}^1(G/G', \mathrm{ad}V_{\mathbb{F}}) \to \mathrm{H}^1(G, \mathrm{ad}V_{\mathbb{F}}) \to \mathrm{H}^1(G', \mathrm{ad}V_{\mathbb{F}})^{G/G'}.$$

$\cong (\mathrm{Hom}(G', \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{ad}V_{\mathbb{F}})^{G/G'}$???

The term on the left is finite because G/G' and $V_{\mathbb{F}}$ are finite. The term on the right is finite because of Condition Φ_p for G. The result follows.

(4) Fix a deformation $V_{\mathbb{F}[\varepsilon]}$ of $V_{\mathbb{F}}$ to $\mathbb{F}[\varepsilon]$. The set of $\mathbb{F}[\varepsilon]$-basis of $V_{\mathbb{F}[\varepsilon]}$ lifting the fixed basis $\beta_{\mathbb{F}}$ of $V_{\mathbb{F}}$ is an $\mathbb{F}$-vector space of dimension d^2. Let β' and β'' be two such lifted bases. Then there is an isomorphism of framed deformations

$$(V_{\mathbb{F}[\varepsilon]}, \beta') \cong (V_{\mathbb{F}[\varepsilon]}, \beta'')$$

if and only if there exists an automorphism $1 + \varepsilon\alpha$ of $V_{\mathbb{F}[\varepsilon]}$, where $\alpha \in \mathrm{ad}V$, which takes β' to β'', so that $\alpha \in \mathrm{ad}V_{\mathbb{F}}^G$. Thus the fibers of

$$D^{\square}_{V_{\mathbb{F}}}(V_{\mathbb{F}[\varepsilon]}) \to D_{V_{\mathbb{F}}}(V_{\mathbb{F}[\varepsilon]})$$

are a principal homogeneous space under $\mathrm{ad}V_{\mathbb{F}}/(\mathrm{ad}V_{\mathbb{F}})^G$. The dimension of this space is $d^2 - h^0(G, \mathrm{ad}V_{\mathbb{F}})$.

□

Definition 1.11. *Let $\varphi : D' \to D$ be a natural transformation of functors from $\mathfrak{Ar}_{W(\mathbb{F})}$ to* **Sets***. The map φ will be called* ***formally smooth*** *if, for any surjection $A \to A' \in \mathfrak{Ar}_{W(\mathbb{F})}$, the map*

$$D'(A) \to D'(A') \times_{D(A')} D(A)$$

is surjective.

Similarly to the proof of Lemma 1.10 (4), one has

Corollary 1.12. *The natural transformation $D^{\square}_{V_{\mathbb{F}}} \to D_{V_{\mathbb{F}}}, (V_A, \beta_A) \mapsto (V_A)$ is formally smooth. Thus, if $R_{V_{\mathbb{F}}}$ is representable, then $R^{\square}_{V_{\mathbb{F}}}$ is a power series ring over $R_{V_{\mathbb{F}}}$ of relative dimension $d^2 - h^0(G, \mathrm{ad}V_{\mathbb{F}})$.*

1.3.1. ***Presentations of the universal ring $R_{V_{\mathbb{F}}}$.*** By Lemma 1.10 we have shown the first part of the following result:

Proposition 1.13. *Suppose that G satisfies Condition Φ_p and $R_{V_{\mathbb{F}}}$ is representable. Then:*

(1) $\dim \mathrm{t}_{R_{V_{\mathbb{F}}}} = h^1(G, \mathrm{ad}V_{\mathbb{F}}) =: h$ and so there is a surjection

$$\pi : W(\mathbb{F})[\![X_1, \cdots, X_h]\!] \to R_{V_{\mathbb{F}}}.$$

(2) For any π as in (1), the minimal number of generators of the ideal $\ker\pi$ is bounded above by $h^2(G, \mathrm{ad}V_{\mathbb{F}})$. More precisely, given π, one has a canonical monomorphism

$$(\ker\pi/(p, X_1, \cdots, X_h)\ker\pi)^* \to \mathrm{H}^2(G, \mathrm{ad}V_{\mathbb{F}}),$$

where, for a vector space V, we denote its dual by V^*.

The proof of the second assersion can be found in [Böc13, Lemma 5.2.2].

$V_{\mathbb{F}}$ is called ***unobstructed***

Corollary 1.14. *Assume that the hypotheses of Proposition 1.13 hold. Then, if $h^2(G, \mathrm{ad}V_{\mathbb{F}}) = 0$, the ring $R_{V_{\mathbb{F}}}$ is smooth over $W(\mathbb{F})$ of relative dimension $h^1(G, \mathrm{ad}V_{\mathbb{F}})$.*

1.4. Schlessinger's axioms.

Theorem 1.15 (Schlessinger)**.** *Let $D : \mathfrak{Ar}_{W(\mathbb{F})} \to$ **Sets** be a functor such that $D(\mathbb{F})$ is a point. For any $A, A', A'' \in \mathfrak{Ar}_{W(\mathbb{F})}$ with morphisms $A' \to A$ and $A'' \to A$, we have a map*

$$D(A' \times_A A'') \to D(A') \times_{D(A)} D(A''). \tag{1.2}$$

If D satisfies the following conditions:

- (H1) *If $A'' \to A$ is small surjective, then (1.2) is surjective.*

  kernel is principal and annihilated by $\mathfrak{m}_{A''}$
- (H2) *If $A'' \to A$ is $\mathbb{F}[\varepsilon] \to \mathbb{F}$, then (1.2) is bijective;*
- (H3) *$\dim_{\mathbb{F}}(\mathbb{F}[\varepsilon])$ is finite;*
- (H4) *If $A'' \to A$ is small surjective and $A' = A''$, then (1.2) is bijective,*

then D is pro-representable.

These conditions are called Schlessinger axioms.

1.7. Groupid over categories (abstract stuff...)

Definition 1.16.

1. *A **groupoid category** is a category in which all morphisms are isomorphisms.*
2. *Call the isomorphism classes the **connected components** of the groupoid.*

Remark 1.17.

1. *Not necessarily all objects in a groupoid are isomorphic.*
2. *$\mathrm{Hom}_{\mathfrak{C}}(A, A)$ forms a group for $\forall A \in \mathrm{ob}\,\mathfrak{C}$, where $\mathfrak{C}$ is a groupoid category, and the identity in $\mathrm{Hom}_{\mathfrak{C}}(A, A)$ is the identity morphism.*
3. *$A \cong B \Rightarrow \mathrm{Hom}_{\mathfrak{C}}(A, A) \cong \mathrm{Hom}_{\mathfrak{C}}(B, B)$ (non-canonically)*

Definition 1.18. *Let $\mathfrak{C}$ be a category. Let $\mathfrak{F}$ be another category, $\Theta : \mathfrak{F} \to \mathfrak{C}$ be a functor.*

1. *We say $\eta \in \mathrm{ob}(\mathfrak{F})$ **lies above** $T \in \mathrm{ob}(\mathfrak{C})$, if $\Theta(\eta) = T$.*
2. *We say $(\eta \xrightarrow{\alpha} \xi) \in \mathrm{Mor}_{\mathfrak{F}}$ **lies above** $(T \xrightarrow{f} S) \in \mathrm{Mor}_{\mathfrak{C}}$, if $\Theta(\alpha) = f$.*
3. *$(T \in \mathrm{ob}(\mathfrak{C}), \mathrm{id}_T)$ is a subcategory of $\mathfrak{C}$. Write $\mathfrak{F}(T)$ the subcategory of $\mathfrak{F}$ over (T, id_T).*

Definition 1.19 (groupoid over $\mathfrak{C}$/category cofibered in groupoids over $\mathfrak{C}$)**.** *The triple $(\mathfrak{F}, \mathfrak{C}, \Theta)$ is a **groupoid over** $\mathfrak{C}$ if*

1. *for any morphisms $(\eta \xrightarrow{\alpha} \xi)$ and $(\eta \xrightarrow{\alpha'} \xi')$ in $\mathfrak{F}$ over the same morphism $T \to S$ in $\mathfrak{C}$, there exists unique $\xi \xrightarrow{u} \xi'$ in $\mathfrak{F}$ over id_S such that $u \circ \alpha = \alpha'$.*
2. *For any $\eta \in \mathrm{ob}(\mathfrak{C})$ and any $T \xrightarrow{f} S$ in $\mathrm{Mor}_{\mathfrak{C}}$ with η over T, there exists morphism $\eta \xrightarrow{\alpha} \xi$ in $\mathrm{Mor}_{\mathfrak{F}}$ over f.*

Remark 1.20.

1. *For every $T \in \mathrm{ob}(\mathfrak{C})$, the category $\mathfrak{F}(T)$ is a groupoid. It's natural to specigy a groupoid by specifying objects in $\mathfrak{F}(T)$ for any $T \in \mathrm{ob}(\mathfrak{C})$, and specifying isomorphism class of morphisms above any $T \xrightarrow{f} S$ in $\mathfrak{C}$.*
2. *Scheme and stack stuff.....*

If for each $T \in \mathrm{ob}(\mathfrak{C})$, the isomorphism classes of $\mathfrak{F}(T)$ forms a set, we associate to the category $\mathfrak{F}$ over $\mathfrak{C}$ a functor $|\mathfrak{F}| : \mathfrak{C} \to \mathrm{Set}$ by sending T to the set of isomorphism classes of $\mathfrak{F}(T)$.

Example 1.21.

1. *To the representation $V_{\mathbb{F}}$ of G, we define a groupoid $\mathcal{D}_{V_{\mathbb{F}}}$ over $\mathfrak{C} = \mathfrak{Ar}_{W(\mathbb{F})}$:*
   (a) *Objects of $\mathcal{D}_{V_{\mathbb{F}}}$ over $A \in \mathfrak{Ar}_{W(\mathbb{F})}$: pairs $(V_A, \iota_A) \in D_{V_{\mathbb{F}}}(A)$.*

(b) *Morphism* $(V_A, \iota_A) \to (V_{A'}, \iota_{A'})$ *over* $A \to A'$ *in* $\mathfrak{Ar}_{W(\mathbb{F})}$*: isomorphism class*

$$\left\{\alpha : V_A \otimes_A A' \xrightarrow{\cong} V_{A'} \text{ is an isomorphism } \middle| \iota_{A'} \circ \alpha = \iota_A \right\} \Big/ (A')^*$$

(2) *We define the groupoid* $\mathcal{D}^{\square}_{V_{\mathbb{F}}}$ *on* $\mathfrak{C} = \mathfrak{Ar}_{W(\mathbb{F})}$ *as follows:*
  (a) *Objects over* $A \in \mathfrak{Ar}_{W(\mathbb{F})}$*: triples* (V_A, ι_A, β_A)*, where* $(V_A, \iota_A) \in \mathcal{D}_{V_{\mathbb{F}}}(A)$ *and* β_A *is an* A*-basis of* V_A *mapping under* ι_A *to the basis* $\beta_{\mathbb{F}}$ *of* $V_{\mathbb{F}}$*.*
  (b) *Morphism* $(V_A, \iota_A, \beta_A) \to (V_{A'}, \iota_{A'}, \beta_{A'})$ *over* $A \to A'$*: isomorphism* $\alpha : V_A \otimes_A A' \xrightarrow{\cong} V_{A'}$ *taking* β_A *to* $\beta_{A'}$*.*

There is an obvious morphism of groupoids $\mathcal{D}^{\square}_{V_{\mathbb{F}}} \to \mathcal{D}_{V_{\mathbb{F}}}$*.*

Remark 1.22.
(1) *The deformation functor* $D_{V_{\mathbb{F}}}$ *defined before is exactly* $|\mathcal{D}_{V_{\mathbb{F}}}|$ *above.*
(2) *When* $V_{\mathbb{F}}$ *has non-trivial automorphisms, then so do the object in* $D_{V_{\mathbb{F}}}(A)$*. (???) In this situation, the groipoid* $\mathcal{D}_{V_{\mathbb{F}}}$ *captures the geometry of the deformation theory of* $V_{\mathbb{F}}$ *more accurately than its functor if isomorphism classes.*

Representability of a groupoid $\Theta : \mathfrak{F} \to \mathfrak{C}$*.*

Definition 1.23.
(1) $\forall \eta \in \mathrm{ob}(\mathfrak{F})$*, define the category* $\widetilde{\eta}$ *(**the category under** η) as the category with objects are morphisms with source* η *and whose morphisms from* $\eta \xrightarrow{\alpha} \xi$ *to* $\eta \xrightarrow{\alpha'} \xi'$ *are morphisms* $\xi \xrightarrow{u} \xi'$ *in* $\mathfrak{F}$ *such that* $u \circ \alpha = \alpha'$*.* ξ and ξ' are not necessarily lying on the same object of $\mathfrak{C}$
(2) *Groupoid* $\mathfrak{F}$ *over* $\mathfrak{C}$ *is **representable** if there exists* $\eta \in \mathfrak{F}$ *such that the canonical functor* $\widetilde{\eta} \to \mathfrak{F}$ *is an equivalence of categories.*
(3) *Similarly, we define the category* $\widetilde{T}$ *for every* $T \in \mathfrak{C}$*.*

One has a commutative diagram of categories:

$$\begin{tikzcd}
\widetilde{\eta} \arrow[r] \arrow[d, "\cong"'] & \mathfrak{F} \arrow[d] \\
\widetilde{\Theta(\eta)} \arrow[r] & \mathfrak{C}
\end{tikzcd}$$

Lemma 1.24. *The left vertical homorphism above is an equivalence of categories.*

Proof. Abstract nonsense...... □

Remark 1.25. *If* $\mathfrak{F}$ *is representable by* η*, then the equivalence* $\widetilde{\eta} \to \widetilde{\Theta(\eta)}$ *imples that* η*, as well as* $\Theta(\eta)$*, are well-defined up to canonical isomorphism. One says that* $\Theta(\eta)$ ***represents*** $\mathfrak{F}$ ***over*** $\mathfrak{C}$*.*

Lemma 1.26 (Relation with "classical" representable functor)**.**
(1) *If* $\mathfrak{F}$ *is representable by* η*, any two objects of* $\mathfrak{F}(\Theta(\eta))$ *are canonically isomorphic and there is an isomorphism of functors*

$$\mathrm{Hom}_{\mathfrak{C}}(\Theta(\eta), -) \xrightarrow{\cong} |\mathfrak{F}|,$$

so that $\Theta(\eta)$ *represents* $|\mathfrak{F}|$ *in the usual set theoretic sense.*
(2) *If* $|\mathfrak{F}|$ *is representable and for any* $T \in \mathrm{ob}(\mathfrak{C})$ *any two objects of* $\mathfrak{F}(T)$ *are related by a unique isomorphism, then* $\mathfrak{F}$ *is representable.*

Remark 1.27. *The groupoid* $\mathcal{D}_{V_{\mathbb{F}}}$ *in Example 1.21 is usually not representable. Extending to* $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ *is needed.*

A crucial question: why do we need to use the language of groupoids?

Let's see Kisin's motivation (just a screenshot from Bockle's notes, which is nothing but a rewrite of [Kis09, Appendix (A.6)]):

The main reason why, in some circumstances, one needs to introduce the language of groupoids, is that formation of fiber products is not compatible with the passage from a groupoid $\mathfrak{F}$ over $\mathfrak{C}$ to its associated functor $|\mathfrak{F}|$. This is a serious technical issue, since Definition 2.4.4 of relative representability depends on the formation of fiber products. We illustrate this with a simple example taken from [34, A.6].

Consider now the situation when the group G is trivial and fix $\eta = (V_A, \iota_A) \in \mathcal{D}_{V_{\mathbb{F}}}(A)$ for some $A \in \mathfrak{Ar}_{W(\mathbb{F})}$. Then $\widetilde{\eta} \times_{\mathcal{D}_{V_{\mathbb{F}}}} \mathcal{D}^{\square}_{V_{\mathbb{F}}}$ can be identified with quadruples $(V'_{A'}, \psi'_{A'}, \varphi\colon V_A \otimes_A A' \xrightarrow{\cong} V'_{A'}, \beta_{A'})$, where $(V'_{A'}, \psi'_{A'}, \beta_{A'}) \in \mathcal{D}^{\square}_{V_{\mathbb{F}}}(A')$ and morphisms over $\mathrm{id}_{A'}$ are isomorphisms of $V'_{A'}$ reducing to the identity of $V_{\mathbb{F}}$. It follows that this category is a principal homogeneous space for the formal group obtained by completing $\mathrm{PGL}_d/W(\mathbb{F})$ along its identity section. Hence $|\widetilde{\eta} \times_{\mathcal{D}_{V_{\mathbb{F}}}} \mathcal{D}^{\square}_{V_{\mathbb{F}}}|(A')$ is isomorphic to the kernel $\mathrm{Ker}(\mathrm{PGL}_d(A') \to \mathrm{PGL}_d(\mathbb{F}))$. On the other hand, $|\mathcal{D}^{\square}_{V_{\mathbb{F}}}(A')|$ is a singleton and hence the same holds for $|\widetilde{\eta}| \times_{|\mathcal{D}_{V_{\mathbb{F}}}|} |\mathcal{D}^{\square}_{V_{\mathbb{F}}}|(A')$.

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