DEFORMATION OF GALOIS REPRESENTATIONS

A notes for the number theory seminar at YMSC, 2022

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We are given either a number field K and a finite set of primes S, or a local field F, and we are given a representation of either $G_{K,S}$ or G_F into $\mathrm{GL}_n(k)$, where k is a finite field. We want to try to understand all possible lifts of this representation to $\mathrm{GL}_n(A)$, where A is a complete noetherian local ring with residue field k.

—Fernando Q. Gouvêa (cf. [Gou01])

Main reference: [Böc13; Maz89]
Motivation and history of deformation theory: [Maz97; Gou01]

1. Deformations of representations of profinite groups

Notations.

p	prime number
$\mathbb{F}$	finite field of char. p
$W(\mathbb{F})$	ring of Witt vectors[1] over A
G	profinite group
$V_{\mathbb{F}}$	finite $\mathbb{F}[G]$-module with continuous G-action
d	dimeinsion of $V_{\mathbb{F}}$
$\beta_{\mathbb{F}}$	a $\mathbb{F}$-basis of $V_{\mathbb{F}}$

1.1. Deformation functors.

Notations.

$\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$	category of complete Noetherian local $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$
$\mathfrak{Ar}_{W(\mathbb{F})}$	full sub-category of finite local Artinian $W(\mathbb{F})$-algebras
$\mathfrak{m}_A$	maximal ideal of $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$

Remark 1.1 (???)**.** *Via the $W(\mathbb{F})$-structure, the residue field of any $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ is canonically isomorphic to $\mathbb{F}$.*

Definition 1.2. *Let $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.*

*(1) A **deformation** of $V_{\mathbb{F}}$ to A is a pair (V_A, ι_A), such that*
   - *(a) V_A is $A[G]$-module, finite free over A, with continuous G-action;*
   - *(b) $\iota_A : V_A \otimes_A \mathbb{F} \xrightarrow{\cong} V_{\mathbb{F}}$ is G-equivariant.*

We view $\mathbb{F}$ as an A-module via the canonical projection $A \to A/\mathfrak{m}_A = F$.

*(2) A **framed deformation** of $(V_{\mathbb{F}}, \beta_{\mathbb{F}})$ to A is a triple (V_A, ι_A, β_A), where*
   - *(a) (V_A, ι_A) is a deformation of $V_{\mathbb{F}}$ to A;*

[1] $W(\mathbb{F})$ is the unique (up to unique isomorphism) complete discrete valuation ring which is absolutely unramified (uniformizer$= p$) and has residue field $\mathbb{F}$.

(b) β_A is a A-basis of V_A which reduces to $\beta_{\mathbb{F}}$ under ι_A.

Set $D_{V_{\mathbb{F}}}, D^{\square}_{V_{\mathbb{F}}} : \mathfrak{Ar}_{W(\mathbb{F})} \to \mathrm{Set}$,

$$D_{V_{\mathbb{F}}}(A) = \{\textit{deformations of } V_{\mathbb{F}} \textit{ to } A\}/\cong,$$

$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\textit{framed deformations of } (V_{\mathbb{F}}, \beta_{\mathbb{F}}) \textit{ to } A\}/\cong .$$

Remark 1.3.

(1) *The **FIXED** basis $\beta_{\mathbb{F}}$ gives the isomorphism $V_{\mathbb{F}} \cong \mathbb{F}^d$ as vector space. Thus we can view $V_{\mathbb{F}}$ as $\bar{\rho} : G \to \mathrm{GL}(V_{\mathbb{F}}) = \mathrm{GL}_d(\mathbb{F})$: a d-dimensional $\mathbb{F}$-representation of G. Then*

$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\rho : G \to \mathrm{GL}_d(A) \textit{ lifting } \bar{\rho}\},$$

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, "\bar{\rho}"'] & \mathrm{GL}_d(A) \arrow[d] \\ & \mathrm{GL}_d(\mathbb{F}) \end{tikzcd}$$

Does not guarantee $\beta_A \rightsquigarrow \beta_{\mathbb{F}}$!

$$D_{V_{\mathbb{F}}}(A) = D^{\square}_{V_{\mathbb{F}}}(A)/\textit{action by conjugates of } \ker(\mathrm{GL}_d(A) \to \mathrm{GL}_d(\mathbb{F})).$$

Not always representable!

(2) *Mazur only consdier $D_{V_{\mathbb{F}}}$, which describes representations lifting $V_{\mathbb{F}}$ up to isomorphism. Add "base condition" $\rightsquigarrow D^{\square}_{V_{\mathbb{F}}}$.*

(3) *Often consider deformation functors on $\mathfrak{Ar}_{\mathcal{O}}$ = category of local artinian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, where $\mathcal{O}$ is ring of integers of a finite totally ramified extension of $W(\mathbb{F})\left[\frac{1}{p}\right]$ ($\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}$).*

For example, let K be a p-adic field with residue field $\mathbb{F}_q$, ring of integers $\mathcal{O}_K$, then $K/W(\mathbb{F}_q)\left[\frac{1}{p}\right]$ is totally ramified and $W(\mathbb{F}_q)\left[\frac{1}{p}\right]/\mathbb{Q}_p$ is unramified.

1.2. Representability.

1.2.1. *A finiteness condition.*

Definition 1.4 (Mazur)**.** *A profinite group G has finiteness condition Φ_p, if $\forall$ open subgroup $G' \subset G$, $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < +\infty$.*

Remark 1.5.

(1) *(Burnside basis theorem) $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < +\infty \Leftrightarrow$ maximal pro-p quotient of G' is topologically finitely generated.*

(2) $\mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) \cong \mathrm{Hom}_{\mathrm{cont}}(G'^{,\mathrm{ab}}, \mathbb{F}_p)$.

Example 1.6 (by CFT)**.** *The following groups have Φ_p:*

(1) *The Galois group $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$, with K a p-adic field.*

(2) *The galois group $\mathcal{G}_{F,S} = \mathrm{Gal}(F_S/F)$, where F is a number field, S is a finite set of places of F and $F_S \subset \bar{F}$ is the maximal extension of F unramified outside S.*

1.2.2. *Main proposition.*

prop:1587

Proposition 1.7 (Mazur)**.** *If G has Φ_p, then*

(1) *The functor $D^{\square}_{V_{\mathbb{F}}}$ is pro-representable by some $R^{\square}_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$, i.e.*

$$D^{\square}_{V_{\mathbb{F}}}(A) \cong \mathrm{Hom}_{W(\mathbb{F})}\left(R^{\square}_{V_{\mathbb{F}}}, A\right),$$

universal framed deformation ring

which is functorial in $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.

Maybe it's better to write $D^{\square}_{V_{\mathbb{F}}}(A) \cong \mathrm{Hom}_{\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}}(R^{\square}_{V_{\mathbb{F}}}, A)$?

(2) *If $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then $D_{V_{\mathbb{F}}}$ is pro-representable by some $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$.*

universal deformation ring

Remark 1.8. *(1) Literally, pro-representable functor = limit of representable functor. How can we realize that? [nLa22] defines pro-representable to be the filtered colimit of representables. There's a post on MSE (cf. [htt]) which discusses the difference between the two definitions. It might be ture that*

$$\mathrm{Hom}_{W(\mathbb{F})}\left(R^{\square}_{V_{\mathbb{F}}}, A\right) = \mathrm{Hom}_{\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}}\left(\varprojlim_k R^{\square}_{V_{\mathbb{F}}}/\mathfrak{m}^k_{R^{\square}_{V_{\mathbb{F}}}}, A\right) \cong \varinjlim_k \mathrm{Hom}_{\mathfrak{Ar}_{W(\mathbb{F})}}\left(R^{\square}_{V_{\mathbb{F}}}/\mathfrak{m}^k_{R^{\square}_{V_{\mathbb{F}}}}, A\right),$$

for any $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.

(2) (???)$R^{\square}_{V_{\mathbb{F}}}$ is unique up to unique isomorphism; the identity map in $\mathrm{Hom}\big(R^{\square}_{V_{\mathbb{F}}}, R^{\square}_{V_{\mathbb{F}}}\big)$ gives rise to a universal framed deformation over $R^{\square}_{V_{\mathbb{F}}}$.

(3) $R^{\square}_{V_{\mathbb{F}}}$ exists without Φ_p, but maybe no longer noetherian.

(4) (???)$\mathbb{F} \hookrightarrow \mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) \rightsquigarrow$ write "=" in $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$.

prop:1587

Proof of Proposition 1.7.

(1) G finite $\rightsquigarrow$ profinite.

- **(FORMAL) CONSTRUCTION:**
  Suppose G is finite. Set

  $$G = \langle g_1, \cdots, g_s | r_1(g_1, \cdots, g_s), \cdots, r_t(g_1, \cdots, g_s)\rangle$$

  a presentation. Define

  $$\mathcal{R} = W(\mathbb{F})\big[X^k_{i,j}\big| i,j = 1, \cdots, d; k = 1, \cdots, s\big]/\mathcal{I},$$

  where

  $$\mathcal{I} = \langle r_l\big(X^1, \cdots, X^s\big) - \mathrm{id}\rangle_{1\le l\le t}, X^k = \big(X^k_{i,j}\big)_{d\times d}.$$

  To make $\mathcal{R}$ complete, local, noetherian, take

  $$\mathcal{J} = \ker\big(\mathcal{R} \to \mathbb{F}, X^k \mapsto \bar{\rho}(g_k), k = 1, \cdots, s\big),$$

  and set $R^{\square}_{V_{\mathbb{F}}} = \varprojlim_n \mathcal{R}/\mathcal{J}^n$ to be the $\mathcal{J}$-adic completion of $\mathcal{R}$. Besides that, we set $\rho^{\square}_{V_{\mathbb{F}}} : G \to \mathrm{GL}_d\big(R^{\square}_{V_{\mathbb{F}}}\big), g_k \mapsto$ image of X^k in $\mathrm{GL}_d\big(R^{\square}_{V_{\mathbb{F}}}\big)$.
  VERIFICATION:
  Take $\rho \in D^{\square}_{V_{\mathbb{F}}}(A)$, where $\rho : G \to \mathrm{GL}_d(A)$. Define

  $$\mathfrak{F}_\rho \in \mathrm{Hom}_{W(\mathbb{F})}\left(R^{\square}_{V_{\mathbb{F}}}, A\right), \overline{\text{entries of } X^k} \mapsto \text{corresponding entries of } \rho(g_k), \forall k = 1, \cdots, s.$$

  Then $\mathfrak{F}_\rho$ induces $\widehat{\mathfrak{F}_\rho} : \mathrm{GL}_d\big(R^{\square}_{V_{\mathbb{F}}}\big) \to \mathrm{GL}_d(A)$. It's immediate to check that $\rho = \widehat{\mathfrak{F}_\rho} \circ \rho^{\square}_{V_{\mathbb{F}}}$ and $\widehat{\mathfrak{F}_\rho}$ is unique choice to make the diagram commute. Thus, $\mathfrak{F} : \rho \mapsto \mathfrak{F}_\rho$ gives the pro-representability when G is finite.
- When G is profinite, we have $G = \varprojlim_i G/H_i$, where $H_i \subset \ker(\bar{\rho})$ are open normal subgroups. For every i, one has a universal pair $\big(R^{\square}_i, \rho^{\square}_i\big)$ by previous construction. Passing by limits, we define

  $$\left(R^{\square}_{V_{\mathbb{F}}}, \rho^{\square}_{V_{\mathbb{F}}}\right) = \varprojlim_i \left(R^{\square}_i, \rho^{\square}_i\right), \text{with } R^{\square}_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}.$$

We will show in [Section 1.4, TBA] that $R^{\square}_{V_{\mathbb{F}}}$ is noetherian.

(2) By Schlessinger's representability criterion (cf. [Section 1.7, TBA]) or by Kisin's work (cf. [Section 2.1, TBA by Y. Chen]).

□

1.7. Groupid over categories (abstract stuff...)

Definition 1.9.

(1) A ***groupoid category*** *is a category in which all morphisms are isomorphisms.*
(2) Call the isomorphism classes the ***connected components*** *of the groupoid.*

Remark 1.10.

(1) Not necessarily all objects in a groupoid are isomorphic.
(2) $\mathrm{Hom}_{\mathscr{C}}(A, A)$ *forms a group for* $\forall A \in \mathrm{ob}\,\mathscr{C}$*, where* $\mathscr{C}$ *is a groupoid category, and the identity in* $\mathrm{Hom}_{\mathscr{C}}(A, A)$ *is the identity morphism.*
(3) $A \cong B \Rightarrow \mathrm{Hom}_{\mathscr{C}}(A, A) \cong \mathrm{Hom}_{\mathscr{C}}(B, B)$ *(non-canonically)*

Definition 1.11. *Let* $\mathscr{C}$ *be a category. Let* $\mathscr{F}$ *be another category,* $\Theta : \mathscr{F} \to \mathscr{C}$ *be a functor.*

(1) We say $\eta \in \mathrm{ob}(\mathscr{F})$ ***lies above*** $T \in \mathrm{ob}(\mathscr{C})$*, if* $\Theta(\eta) = T$*.*
(2) We say $\left(\eta \xrightarrow{\alpha} \zeta\right) \in \mathrm{Mor}_{\mathscr{F}}$ ***lies above*** $\left(T \xrightarrow{f} S\right) \in \mathrm{Mor}_{\mathscr{C}}$*, if* $\Theta(\eta) = f$*.*
(3) $(T \in \mathrm{ob}(\mathscr{C}), \mathrm{id}_T)$ *is a subcategory of* $\mathscr{C}$*. Write* $\mathscr{F}(T)$ *the subcategory of* $\mathscr{F}$ *over* (T, id_T)*.*

Definition 1.12 (groupoid over $\mathscr{C}$/category cofibered in groupoids over $\mathscr{C}$)**.** *The triple* $(\mathscr{F}, \mathscr{C}, \Theta)$ *is a* ***groupoid over*** $\mathscr{C}$ *if*

(1) for any morphisms $\left(\eta \xrightarrow{\alpha} \zeta\right)$ *and* $\left(\eta \xrightarrow{\alpha'} \zeta'\right)$ *in* $\mathscr{F}$ *over the same morphism* $T \to S$ *in* $\mathscr{C}$*, there exists unique* $\zeta \xrightarrow{u} \zeta'$ *in* $\mathscr{F}$ *over* id_S *such that* $u \circ \alpha = \alpha'$*.*
(2) For any $\eta \in \mathrm{ob}(\mathscr{C})$ *and any* $T \xrightarrow{f} S$ *in* $\mathrm{Mor}_{\mathscr{C}}$ *with* η *over* T*, there exists morphism* $\eta \xrightarrow{\alpha} \zeta$ *in* $\mathrm{Mor}_{\mathscr{F}}$ *over* f*.*

Remark 1.13. *(1)* $\forall T \in \mathrm{ob}(\mathscr{C})$*, the category* $\mathscr{F}(T)$ *is a groupoid over* $\mathscr{C}$*.*
(2) Scheme and stack stuff.....

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