DEFORMATION OF GALOIS REPRESENTATIONS

A notes for the number theory seminar at YMSC, 2022

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We are given either a number field K and a finite set of primes S, or a local field F, and we are given a representation of either $G_{K,S}$ or G_F into $\mathrm{GL}_n(k)$, where k is a finite field. We want to try to understand all possible lifts of this representation to $\mathrm{GL}_n(A)$, where A is a complete noetherian local ring with residue field k.

—Fernando Q. Gouvêa (cf. [Gou01])

Main reference: [Böc13; Maz89]
Motivation and history of deformation theory: [Maz97; Gou01]

1. Deformations of representations of profinite groups

Notations.

p	prime number
$\mathbb{F}$	finite field of char. p
$W(\mathbb{F})$	ring of Witt vectors[1] over A
G	profinite group
$V_{\mathbb{F}}$	finite $\mathbb{F}[G]$-module with continuous G-action
d	dimeinsion of $V_{\mathbb{F}}$
$\beta_{\mathbb{F}}$	a $\mathbb{F}$-basis of $V_{\mathbb{F}}$

1.1. Deformation functors.

Notations.

$\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$	category of complete Noetherian local $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$
$\mathfrak{Ar}_{W(\mathbb{F})}$	full sub-category of finite local Artinian $W(\mathbb{F})$-algebras
$\mathfrak{m}_A$	maximal ideal of $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$

Remark 1.1 (???)**.** *Via the $W(\mathbb{F})$-structure, the residue field of any $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ is canonically isomorphic to $\mathbb{F}$.*

Definition 1.2. *Let $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.*

(1) *A **deformation** of $V_{\mathbb{F}}$ to A is a pair (V_A, ι_A), such that*
  - (a) *V_A is $A[G]$-module, finite free over A, with continuous G-action;*
  - (b) *$\iota_A : V_A \otimes_A \mathbb{F} \xrightarrow{\cong} V_{\mathbb{F}}$ is G-equivariant.*

We view $\mathbb{F}$ as an A-module via the canonical projection $A \to A/\mathfrak{m}_A = F$.

(2) *A **framed deformation** of $(V_{\mathbb{F}}, \beta_{\mathbb{F}})$ to A is a triple (V_A, ι_A, β_A), where*
  - (a) *(V_A, ι_A) is a deformation of $V_{\mathbb{F}}$ to A;*

[1] $W(\mathbb{F})$ is the unique (up to unique isomorphism) complete discrete valuation ring which is absolutely unramified (uniformizer= p) and has residue field $\mathbb{F}$.

(b) β_A is a A-basis of V_A which reduces to $\beta_{\mathbb{F}}$ under ι_A.

Set $D_{V_{\mathbb{F}}}, D^{\square}_{V_{\mathbb{F}}} : \mathfrak{Ar}_{W(\mathbb{F})} \to \mathrm{Set}$,

$$D_{V_{\mathbb{F}}}(A) = \{\textit{deformations of } V_{\mathbb{F}} \textit{ to } A\}/\cong,$$
$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\textit{framed deformations of } (V_{\mathbb{F}}, \beta_{\mathbb{F}}) \textit{ to } A\}/\cong.$$

Remark 1.3.

(1) *The* ***FIXED*** *basis* $\beta_{\mathbb{F}}$ *gives the isomorphism* $V_{\mathbb{F}} \cong \mathbb{F}^d$ *as vector space. Thus we can view* $V_{\mathbb{F}}$ *as* $\bar{\rho} : G \to \mathrm{GL}(V_{\mathbb{F}}) = \mathrm{GL}_d(\mathbb{F})$*: a* d*-dimensional* $\mathbb{F}$*-representation of* G*. Then*

$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\rho : G \to \mathrm{GL}_d(A) \textit{ lifting } \bar{\rho}\},$$

Does not guarantee $\beta_A \rightsquigarrow \beta_{\mathbb{F}}$!

$$D_{V_{\mathbb{F}}}(A) = D^{\square}_{V_{\mathbb{F}}}(A)/\textit{action by conjugates of } \ker(\mathrm{GL}_d(A) \to \mathrm{GL}_d(\mathbb{F})).$$

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, "\bar{\rho}"'] & \mathrm{GL}_d(A) \arrow[d, two heads] \\ & \mathrm{GL}_d(\mathbb{F}) \end{tikzcd}$$

Not always representable!

(2) *Mazur only consdier* $D_{V_{\mathbb{F}}}$*, which describes representations lifting* $V_{\mathbb{F}}$ *up to isomorphism.*
   Add "base condition" $\rightsquigarrow D^{\square}_{V_{\mathbb{F}}}$.

(3) *Often consider deformation functors on* $\mathfrak{Ar}_{\mathcal{O}}$ = *category of local artinian* $\mathcal{O}$*-algebra with residue field* $\mathbb{F}$*, where* $\mathcal{O}$ *is ring of integers of a finite totally ramified extension of* $W(\mathbb{F})\left[\frac{1}{p}\right]$ $(\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F})$.
   For example, let K *be a* p*-adic field with residue field* $\mathbb{F}_q$*, ring of integers* $\mathcal{O}_K$*, then* $K/W(\mathbb{F}_q)\left[\frac{1}{p}\right]$ *is totally ramified and* $W(\mathbb{F}_q)\left[\frac{1}{p}\right]/\mathbb{Q}_p$ *is unramified.*

1.2. Representability.

1.2.1. *A finiteness condition.*

Definition 1.4 (Mazur). *A profinite group* G *has finiteness condition* Φ_p*, if* $\forall$ *open subgroup* $G' \subset G$, $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < +\infty$.

Remark 1.5.

(1) *(Burnside basis theorem)* $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < +\infty \Leftrightarrow$ *maximal pro-*p *quotient of* G' *is topologically finitely generated.*

(2) $\mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) \cong \mathrm{Hom}_{\mathrm{cont}}(G'^{,\mathrm{ab}}, \mathbb{F}_p)$.

Example 1.6 (by CFT). *The following groups have* Φ_p:

(1) *The Galois group* $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$*, with* K *a* p*-adic field.*

(2) *The galois group* $\mathcal{G}_{F,S} = \mathrm{Gal}(F_S/F)$*, where* F *is a number field,* S *is a finite set of places of* F *and* $F_S \subset \bar{F}$ *is the maximal extension of* F *unramified outside* S.

1.2.2. *Main proposition.*

prop:1587 ⟨prop:1587⟩ **Proposition 1.7** (Mazur). *If* G *has* Φ_p*, then*

(1) *The functor* $D^{\square}_{V_{\mathbb{F}}}$ *is pro-representable by some* $R^{\square}_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$*, i.e.*

$$D^{\square}_{V_{\mathbb{F}}}(A) \cong \mathrm{Hom}_{W(\mathbb{F})}\Big(R^{\square}_{V_{\mathbb{F}}}, A\Big),$$

universal framed deformation ring

which is functorial in $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.

Maybe it's better to write $D^{\square}_{V_{\mathbb{F}}}(A) \cong \mathrm{Hom}_{\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}}(R^{\square}_{V_{\mathbb{F}}}, A)$?

(2) *If* $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$*, then* $D_{V_{\mathbb{F}}}$ *is pro-representable by some* $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$.

universal deformation ring

Remark 1.8. (1) *(???)*$R^{\square}_{V_{\mathbb{F}}}$ *is unique up to unique isomorphism; the identity map in* $\mathrm{Hom}\big(R^{\square}_{V_{\mathbb{F}}}, R^{\square}_{V_{\mathbb{F}}}\big)$ *gives rise to a universal framed deformation over* $R^{\square}_{V_{\mathbb{F}}}$.

(2) $R^{\square}_{V_{\mathbb{F}}}$ *exists without* Φ_p*, but maybe no longer noetherian.*
(3) (???)$\mathbb{F} \hookrightarrow \mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) \rightsquigarrow$ *write "=" in* $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$.

Proof of prop:1587 *Proposition 1.7.*

(1) G finite $\rightsquigarrow$ profinite.
- **(FORMAL) CONSTRUCTION:**
Suppose G is finite. Set
$$G = \langle g_1, \cdots, g_s | r_1(g_1, \cdots, g_s), \cdots, r_t(g_1, \cdots, g_s) \rangle$$
a presentation. Define
$$\mathcal{R} = W(\mathbb{F})\big[X^k_{i,j} \big| i, j = 1, \cdots, d; k = 1, \cdots, s\big]/\mathcal{I},$$
where
$$\mathcal{I} = \langle r_l\big(X^1, \cdots, X^s\big) - \mathrm{id}\rangle_{1 \le l \le t}, X^k = \big(X^k_{i,j}\big)_{d \times d}.$$
To make $\mathcal{R}$ complete, local, noetherian, take
$$\mathcal{J} = \ker\big(\mathcal{R} \to \mathbb{F}, X^k \mapsto \bar{\rho}(g_k), k = 1, \cdots, s\big),$$
and set $R^{\square}_{V_{\mathbb{F}}} = \varprojlim_n \mathcal{R}/\mathcal{J}^n$ to be the $\mathcal{J}$-adic completion of $\mathcal{R}$. Besides that, we set $\rho^{\square}_{V_{\mathbb{F}}} : G \to \mathrm{GL}_d\big(R^{\square}_{V_{\mathbb{F}}}\big), g_k \mapsto$ image of X^k in $\mathrm{GL}_d\big(R^{\square}_{V_{\mathbb{F}}}\big)$.
VERIFICATION:
Take $\rho \in D^{\square}_{V_{\mathbb{F}}}(A)$, where $\rho : G \to \mathrm{GL}_d(A)$. Define
$$\mathfrak{F}_\rho \in \mathrm{Hom}_{W(\mathbb{F})}\Big(R^{\square}_{V_{\mathbb{F}}}, A\Big), \overline{\text{entries of } X^k} \mapsto \text{corresponding entries of } \rho(g_k), \forall k = 1, \cdots, s.$$
Then $\mathfrak{F}_\rho$ induces $\widehat{\mathfrak{F}_\rho} : \mathrm{GL}_d\big(R^{\square}_{V_{\mathbb{F}}}\big) \to \mathrm{GL}_d(A)$. It's immediate to check that $\rho = \widehat{\mathfrak{F}_\rho} \circ \rho^{\square}_{V_{\mathbb{F}}}$ and $\widehat{\mathfrak{F}_\rho}$ is unique choice to make the diagram commute. Thus, $\mathfrak{F} : \rho \mapsto \mathfrak{F}_\rho$ gives the pro-representability when G is finite.
- When G is profinite, we have $G = \varprojlim_i G/H_i$, where $H_i \subset \ker(\bar{\rho})$ are open normal subgroups. For every i, one has a universal pair $\big(R^{\square}_i, \rho^{\square}_i\big)$ by previous construction. Passing by limits, we define
$$\Big(R^{\square}_{V_{\mathbb{F}}}, \rho^{\square}_{V_{\mathbb{F}}}\Big) = \varprojlim_i \Big(R^{\square}_i, \rho^{\square}_i\Big), \text{with } R^{\square}_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}.$$

We will show in [Section 1.4, TBA] that $R^{\square}_{V_{\mathbb{F}}}$ is noetherian.

(2) By Schlessinger's representability criterion (cf. [Section 1.7, TBA]) or by Kisin's work (cf. [Section 2.1, TBA by Y. Chen]).

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References

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