

DEFORMATION OF GALOIS REPRESENTATIONS

A notes for the number theory seminar at YMSC, 2022

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We are given either a number field K and a finite set of primes S , or a local field F , and we are given a representation of either $G_{K,S}$ or G_F into $\mathrm{GL}_n(k)$, where k is a finite field. We want to try to understand all possible lifts of this representation to $\mathrm{GL}_n(A)$, where A is a complete noetherian local ring with residue field k .

—Fernando Q. Gouvêa (cf. [Gou01])

Main reference: [Böc13; Maz89]

Motivation and history of deformation theory: [Maz97; Gou01]

1. DEFORMATIONS OF REPRESENTATIONS OF PROFINITE GROUPS

Notations.

p	prime number
\mathbb{F}	finite field of char. p
$W(\mathbb{F})$	ring of Witt vectors ¹ over A
G	profinite group
$V_{\mathbb{F}}$	finite $\mathbb{F}[G]$ -module with continuous G -action
d	dimeinsion of $V_{\mathbb{F}}$
$\beta_{\mathbb{F}}$	a \mathbb{F} -basis of $V_{\mathbb{F}}$

1.1. Deformation functors.

Notations.

$\widehat{\mathfrak{Art}}_{W(\mathbb{F})}$	category of complete Noetherian local $W(\mathbb{F})$ -algebra with residue field \mathbb{F}
$\mathfrak{Art}_{W(\mathbb{F})}$	full sub-category of finite local Artinian $W(\mathbb{F})$ -algebras
\mathfrak{m}_A	maximal ideal of $A \in \widehat{\mathfrak{Art}}_{W(\mathbb{F})}$

Remark 1.1 (???). *Via the $W(\mathbb{F})$ -structure, the residue field of any $A \in \widehat{\mathfrak{Art}}_{W(\mathbb{F})}$ is canonically isomorphic to \mathbb{F} .*

Definition 1.2. *Let $A \in \mathfrak{Art}_{W(\mathbb{F})}$.*

- (1) *A **deformation** of $V_{\mathbb{F}}$ to A is a pair (V_A, ι_A) , such that*
 - (a) *V_A is $A[G]$ -module, finite free over A , with continuous G -action;*
 - (b) *$\iota_A : V_A \otimes_A \mathbb{F} \xrightarrow{\cong} V_{\mathbb{F}}$ is G -equivariant.*
- (2) *A **framed deformation** of $(V_{\mathbb{F}}, \beta_{\mathbb{F}})$ to A is a triple (V_A, ι_A, β_A) , where*
 - (a) *(V_A, ι_A) is a deformation of $V_{\mathbb{F}}$ to A ;*

¹ $W(\mathbb{F})$ is the unique (up to unique isomorphism) complete discrete valuation ring which is absolutely unramified (uniformizer = p) and has residue field \mathbb{F} .

We view \mathbb{F} as an A -module via the canonical projection $A \rightarrow A/\mathfrak{m}_A = \mathbb{F}$.

(b) β_A is a A -basis of V_A which reduces to $\beta_{\mathbb{F}}$ under ι_A .

Set $D_{V_{\mathbb{F}}}, D_{V_{\mathbb{F}}}^{\square} : \mathfrak{Art}_W(\mathbb{F}) \rightarrow \text{Set}$,

$$D_{V_{\mathbb{F}}}(A) = \{\text{deformations of } V_{\mathbb{F}} \text{ to } A\} / \cong,$$

$$D_{V_{\mathbb{F}}}^{\square}(A) = \{\text{framed deformations of } (V_{\mathbb{F}}, \beta_{\mathbb{F}}) \text{ to } A\} / \cong.$$

Remark 1.3.

(1) The **FIXED** basis $\beta_{\mathbb{F}}$ gives the isomorphism $V_{\mathbb{F}} \cong \mathbb{F}^d$ as vector space. Thus we can view $V_{\mathbb{F}}$ as $\bar{\rho} : G \rightarrow \text{GL}_d(\mathbb{F}) = \text{GL}_d(\mathbb{F})$: a d -dimensional \mathbb{F} -representation of G . Then

$$D_{V_{\mathbb{F}}}^{\square}(A) = \{\rho : G \rightarrow \text{GL}_d(A) \text{ lifting } \bar{\rho}\},$$

Does not guarantee $\beta_A \rightsquigarrow \beta_{\mathbb{F}}!$

$$D_{V_{\mathbb{F}}}(A) = D_{V_{\mathbb{F}}}^{\square}(A) / \text{action by conjugates of } \ker(\text{GL}_d(A) \rightarrow \text{GL}_d(\mathbb{F})).$$

Not always representable!

(2) Mazur only consider $D_{V_{\mathbb{F}}}$, which describes representations lifting $V_{\mathbb{F}}$ up to isomorphism.

Add "base condition" $\rightsquigarrow D_{V_{\mathbb{F}}}^{\square}$.

(3) Often consider deformation functors on $\mathfrak{Art}_{\mathcal{O}} =$ category of local artinian \mathcal{O} -algebra with residue field \mathbb{F} , where \mathcal{O} is ring of integers of a finite totally ramified extension of $W(\mathbb{F}) \left[\frac{1}{p} \right]$ ($\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}$).

For example, let K be a p -adic field with residue field \mathbb{F}_q , ring of integers \mathcal{O}_K , then $K/W(\mathbb{F}_q) \left[\frac{1}{p} \right]$ is totally ramified and $W(\mathbb{F}_q) \left[\frac{1}{p} \right] / \mathbb{Q}_p$ is unramified.

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}_d(A) \\ & \searrow \bar{\rho} & \downarrow \\ & & \text{GL}_d(\mathbb{F}) \end{array}$$

1.2. Representability.

1.2.1. A finiteness condition.

Definition 1.4 (Mazur). A profinite group G has finiteness condition Φ_p , if \forall open subgroup $G' \subset G$, $\dim_{\mathbb{F}_p} \text{Hom}_{\text{cont}}(G', \mathbb{F}_p) < +\infty$.

Remark 1.5.

- (1) (Burnside basis theorem) $\dim_{\mathbb{F}_p} \text{Hom}_{\text{cont}}(G', \mathbb{F}_p) < +\infty \Leftrightarrow$ maximal pro- p quotient of G' is topologically finitely generated.
- (2) $\text{Hom}_{\text{cont}}(G', \mathbb{F}_p) \cong \text{Hom}_{\text{cont}}(G'^{\text{ab}}, \mathbb{F}_p)$.

Example 1.6 (by CFT). The following groups have Φ_p :

- (1) The Galois group $\mathcal{G}_K = \text{Gal}(\bar{K}/K)$, with K a p -adic field.
- (2) The Galois group $\mathcal{G}_{F,S} = \text{Gal}(F_S/F)$, where F is a number field, S is a finite set of places of F and $F_S \subset \bar{F}$ is the maximal extension of F unramified outside S .

1.2.2. Main proposition.

prop:1587

Proposition 1.7 (Mazur). If G has Φ_p , then

- (1) The functor $D_{V_{\mathbb{F}}}^{\square}$ is pro-representable by some $R_{V_{\mathbb{F}}}^{\square} \in \widehat{\mathfrak{Art}}_W(\mathbb{F})$, i.e.

$$D_{V_{\mathbb{F}}}^{\square}(A) \cong \text{Hom}_{W(\mathbb{F})} \left(R_{V_{\mathbb{F}}}^{\square}, A \right),$$

which is functorial in $A \in \mathfrak{Art}_W(\mathbb{F})$.

universal framed deformation ring

- (2) If $\text{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then $D_{V_{\mathbb{F}}}$ is pro-representable by some $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Art}}_W(\mathbb{F})$.

universal deformation ring

Maybe it's better to write $D_{V_{\mathbb{F}}}^{\square}(A) \cong \text{Hom}_{\widehat{\mathfrak{Art}}_W(\mathbb{F})}(R_{V_{\mathbb{F}}}^{\square}, A)$?

Remark 1.8. (1) Literally, pro-representable functor = limit of representable functor. How can we realize that? [nLa22] defines pro-representable to be the filtered colimit of representables. There's a post on MSE (cf. [htt]) which discusses the difference between the two definitions. It might be true that

$$\mathrm{Hom}_{W(\mathbb{F})}(R_{V_{\mathbb{F}}}^{\square}, A) = \mathrm{Hom}_{\widehat{\mathfrak{A}r_{W(\mathbb{F})}}}\left(\varprojlim_k R_{V_{\mathbb{F}}}^{\square}/\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}^k, A\right) \cong \varprojlim_k \mathrm{Hom}_{\mathfrak{A}r_{W(\mathbb{F})}}\left(R_{V_{\mathbb{F}}}^{\square}/\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}^k, A\right),$$

for any $A \in \mathfrak{A}r_{W(\mathbb{F})}$.

- (2) $(\rho_{V_{\mathbb{F}}}^{\square})$ is unique up to unique isomorphism; the identity map in $\mathrm{Hom}(R_{V_{\mathbb{F}}}^{\square}, R_{V_{\mathbb{F}}}^{\square})$ gives rise to a universal framed deformation over $R_{V_{\mathbb{F}}}^{\square}$.
- (3) $R_{V_{\mathbb{F}}}^{\square}$ exists without Φ_p , but maybe no longer noetherian.
- (4) $(\rho_{V_{\mathbb{F}}}^{\square}) \hookrightarrow \mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) \rightsquigarrow$ write " $=$ " in $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$.

Proof of Proposition 1.7.

- (1) G finite \rightsquigarrow profinite.

• **(FORMAL) CONSTRUCTION:**

Suppose G is finite. Set

$$G = \langle g_1, \dots, g_s | r_1(g_1, \dots, g_s), \dots, r_t(g_1, \dots, g_s) \rangle$$

a presentation. Define

$$\mathcal{R} = W(\mathbb{F})[X_{i,j}^k | i, j = 1, \dots, d; k = 1, \dots, s] / \mathcal{I},$$

where

$$\mathcal{I} = \langle r_l(X^1, \dots, X^s) - \mathrm{id} \rangle_{1 \leq l \leq t}, X^k = (X_{i,j}^k)_{d \times d}.$$

To make \mathcal{R} complete, local, noetherian, take

$$\mathcal{J} = \ker(\mathcal{R} \rightarrow \mathbb{F}, X^k \mapsto \bar{\rho}(g_k), k = 1, \dots, s),$$

and set $R_{V_{\mathbb{F}}}^{\square} = \varprojlim_n \mathcal{R}/\mathcal{J}^n$ to be the \mathcal{J} -adic completion of \mathcal{R} . Besides that, we set $\rho_{V_{\mathbb{F}}}^{\square} : G \rightarrow \mathrm{GL}_d(R_{V_{\mathbb{F}}}^{\square})$, $g_k \mapsto$ image of X^k in $\mathrm{GL}_d(R_{V_{\mathbb{F}}}^{\square})$.

VERIFICATION:

Take $\rho \in D_{V_{\mathbb{F}}}^{\square}(A)$, where $\rho : G \rightarrow \mathrm{GL}_d(A)$. Define

$$\mathfrak{F}_{\rho} \in \mathrm{Hom}_{W(\mathbb{F})}(R_{V_{\mathbb{F}}}^{\square}, A), \text{ entries of } X^k \mapsto \text{corresponding entries of } \rho(g_k), \forall k = 1, \dots, s.$$

Then \mathfrak{F}_{ρ} induces $\widehat{\mathfrak{F}}_{\rho} : \mathrm{GL}_d(R_{V_{\mathbb{F}}}^{\square}) \rightarrow \mathrm{GL}_d(A)$. It's immediate to check that $\rho = \widehat{\mathfrak{F}}_{\rho} \circ \rho_{V_{\mathbb{F}}}^{\square}$ and $\widehat{\mathfrak{F}}_{\rho}$ is unique choice to make the diagram commute. Thus, $\mathfrak{F} : \rho \mapsto \mathfrak{F}_{\rho}$ gives the pro-representability when G is finite.

- When G is profinite, we have $G = \varprojlim_i G/H_i$, where $H_i \subset \ker(\bar{\rho})$ are open normal subgroups. For every i , one has a universal pair $(R_i^{\square}, \rho_i^{\square})$ by previous construction. Passing by limits, we define

$$(R_{V_{\mathbb{F}}}^{\square}, \rho_{V_{\mathbb{F}}}^{\square}) = \varprojlim_i (R_i^{\square}, \rho_i^{\square}), \text{ with } R_{V_{\mathbb{F}}}^{\square} \in \widehat{\mathfrak{A}r_{W(\mathbb{F})}}.$$

We will show in [Section 1.4, TBA] that $R_{V_{\mathbb{F}}}^{\square}$ is noetherian.

- (2) By Schlessinger's representability criterion (cf. [Section 1.7, TBA]) or by Kisin's work (cf. [Section 2.1, TBA by Y. Chen]).

□

1.7. Groupoid over categories (abstract stuff...)

Definition 1.9.

- (1) A **groupoid category** is a category in which all morphisms are isomorphisms.
- (2) Call the isomorphism classes the **connected components** of the groupoid.

Remark 1.10.

- (1) Not necessarily all objects in a groupoid are isomorphic.
- (2) $\text{Hom}_{\mathcal{C}}(A, A)$ forms a group for $\forall A \in \text{ob } \mathcal{C}$, where \mathcal{C} is a groupoid category, and the identity in $\text{Hom}_{\mathcal{C}}(A, A)$ is the identity morphism.
- (3) $A \cong B \Rightarrow \text{Hom}_{\mathcal{C}}(A, A) \cong \text{Hom}_{\mathcal{C}}(B, B)$ (non-canonically)

Definition 1.11. Let \mathcal{C} be a category. Let \mathcal{F} be another category, $\Theta : \mathcal{F} \rightarrow \mathcal{C}$ be a functor.

- (1) We say $\eta \in \text{ob}(\mathcal{F})$ **lies above** $T \in \text{ob}(\mathcal{C})$, if $\Theta(\eta) = T$.
- (2) We say $\left(\eta \xrightarrow{\alpha} \zeta\right) \in \text{Mor}_{\mathcal{F}}$ **lies above** $\left(T \xrightarrow{f} S\right) \in \text{Mor}_{\mathcal{C}}$, if $\Theta(\eta) = f$.
- (3) $(T \in \text{ob}(\mathcal{C}), \text{id}_T)$ is a subcategory of \mathcal{C} . Write $\mathcal{F}(T)$ the subcategory of \mathcal{F} over (T, id_T) .

Definition 1.12 (groupoid over \mathcal{C} /category cofibered in groupoids over \mathcal{C}). The triple $\mathcal{F}, \mathcal{C}, \Theta$ is a **groupoid over \mathcal{C}** if

- (1) for any morphisms $\left(\eta \xrightarrow{\alpha} \zeta\right)$ and $\left(\eta \xrightarrow{\alpha'} \zeta'\right)$ in \mathcal{F} over the same morphism $T \rightarrow S$ in \mathcal{C} , there exists unique $\zeta \xrightarrow{u} \zeta'$ in \mathcal{F} over id_S such that $u \circ \alpha = \alpha'$.
- (2) For any

REFERENCES

- | | |
|------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Bockle2013 | [Böc13] G. Böckle. “Deformations of Galois Representations”. In: L. Berger et al. <i>Elliptic Curves, Hilbert Modular Forms and Galois Deformations</i> . Springer Basel, 2013, pp. 21–115. DOI: 10.1007/978-3-0348-0618-3_2. |
| Gouvea2001 | [Gou01] F. Q. Gouvêa. “Deformations of Galois representations”. In: <i>Arithmetic Algebraic Geometry</i> . Ed. by B. Conrad and K. Rubin. American Mathematical Society, 2001, pp. 235–406. URL: http://www.ams.org/books/pcms/009/05 . |
| MSE4013849 | [htt] S. M. (https://math.stackexchange.com/users/572592/sebastian-monnet). <i>Relationship between two definitions of pro-representable functors</i> . Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/4013849 . |
| Mazur1989 | [Maz89] B. Mazur. “Deforming Galois Representations”. In: <i>Galois Groups over \mathbb{Q}</i> . Ed. by Y. Ihara, K. Ribet, and J.-P. Serre. Vol. 16. Springer US, 1989, pp. 385–437. DOI: 10.1007/978-1-4613-9649-9_7. |
| Mazur1997 | [Maz97] B. Mazur. “An Introduction to the Deformation Theory of Galois Representations”. In: <i>Modular Forms and Fermat’s Last Theorem</i> . Ed. by G. Cornell, J. H. Silverman, and G. Stevens. Springer New York, 1997, pp. 243–311. DOI: 10.1007/978-1-4612-1974-3_8. |
| nlab | [nLa22] nLab authors. <i>prorepresentable functor</i> . Oct. 2022. URL: https://ncatlab.org/nlab/revision/prorepresentable%20functor/7 . |