DEFORMATION OF GALOIS REPRESENTATIONS

A notes for the number theory seminar at YMSC, 2022

Update on 2022-10-12 16:44

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> *We are given either a number field K and a finite set of primes S, or a local field F, and we are given a representation of either $G_{K,S}$ or G_F into $\mathrm{GL}_n(k)$, where k is a finite field. We want to try to understand all possible lifts of this representation to $\mathrm{GL}_n(A)$, where A is a complete noetherian local ring with residue field k.*
>
> —*Fernando Q. Gouvêa (cf. [Gou01])*

Main reference: [Böc13; Maz89]
Motivation and history of deformation theory: [Maz97; Gou01]
Basics of groupoids: Appendix of [Kis09]

1. Deformations of representations of profinite groups

Notations.

p	prime number
$\mathbb{F}$	finite field of char. p
$W(\mathbb{F})$	ring of Witt vectors[1] over A
G	profinite group
$V_{\mathbb{F}}$	finite $\mathbb{F}[G]$-module with continuous G-action
d	dimeinsion of $V_{\mathbb{F}}$
$\beta_{\mathbb{F}}$	a $\mathbb{F}$-basis of $V_{\mathbb{F}}$

1.1. Deformation functors.

Notations.

$\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$	category of complete Noetherian local $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$
$\mathfrak{Ar}_{W(\mathbb{F})}$	full sub-category of finite local Artinian $W(\mathbb{F})$-algebras
$\mathfrak{m}_A$	maximal ideal of $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$

Remark 1.1 (???)**.** *Via the $W(\mathbb{F})$-structure, the residue field of any $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ is canonically isomorphic to $\mathbb{F}$.*

Definition 1.2. *Let $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.*

*(1) A **deformation** of $V_{\mathbb{F}}$ to A is a pair (V_A, ι_A), such that*
  (a) V_A is $A[G]$-module, finite free over A, with continuous G-action;
  (b) $\iota_A : V_A \otimes_A \mathbb{F} \xrightarrow{\cong} V_{\mathbb{F}}$ is G-equivariant.

*(2) A **framed deformation** of $(V_{\mathbb{F}}, \beta_{\mathbb{F}})$ to A is a triple (V_A, ι_A, β_A), where*

> We view $\mathbb{F}$ as an A-module via the canonical projection $A \to A/\mathfrak{m}_A = F$.

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[1] $W(\mathbb{F})$ is the unique (up to unique isomorphism) complete discrete valuation ring which is absolutely unramified (uniformizer= p) and has residue field $\mathbb{F}$.

(a) (V_A, ι_A) is a deformation of $V_{\mathbb{F}}$ to A;

(b) β_A is a A-basis of V_A which reduces to $\beta_{\mathbb{F}}$ under ι_A.

Set $D_{V_{\mathbb{F}}}, D_{V_{\mathbb{F}}}^{\square} : \mathfrak{Ar}_{W(\mathbb{F})} \to \mathrm{Set}$,

$$D_{V_{\mathbb{F}}}(A) = \{deformations\ of\ V_{\mathbb{F}}\ to\ A\}/\cong,$$

$$D_{V_{\mathbb{F}}}^{\square}(A) = \{framed\ deformations\ of\ (V_{\mathbb{F}}, \beta_{\mathbb{F}})\ to\ A\}/\cong .$$

Remark 1.3.

(1) The **_FIXED_** basis $\beta_{\mathbb{F}}$ gives the isomorphism $V_{\mathbb{F}} \cong \mathbb{F}^d$ as vector space. Thus we can view $V_{\mathbb{F}}$ as $\bar{\rho} : G \to \mathrm{GL}(V_{\mathbb{F}}) = \mathrm{GL}_d(\mathbb{F})$: a d-dimensional $\mathbb{F}$-representation of G. Then

$$D_{V_{\mathbb{F}}}^{\square}(A) = \{\rho : G \to \mathrm{GL}_d(A)\ lifting\ \bar{\rho}\},$$

Does not guarantee $\beta_A \rightsquigarrow \beta_{\mathbb{F}}$!

$\boxed{D_{V_{\mathbb{F}}}(A)} = D_{V_{\mathbb{F}}}^{\square}(A)/action\ by\ conjugates\ of\ \ker(\mathrm{GL}_d(A) \to \mathrm{GL}_d(\mathbb{F})).$

Not always representable!

$$G \xrightarrow{\rho} \mathrm{GL}_d(A)$$
$$\bar{\rho} \searrow \quad \downarrow$$
$$\mathrm{GL}_d(\mathbb{F})$$

(2) Mazur only consdier $\boxed{D_{V_{\mathbb{F}}}}$, which describes representations lifting $V_{\mathbb{F}}$ up to isomorphism. Add "base condition" $\rightsquigarrow D_{V_{\mathbb{F}}}^{\square}$.

(3) Often consider deformation functors on $\mathfrak{Ar}_{\mathcal{O}} =$ category of local artinian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, where $\mathcal{O}$ is ring of integers of a finite totally ramified extension of $W(\mathbb{F})\left[\frac{1}{p}\right]$ ($\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}$).

For example, let K be a p-adic field with residue field $\mathbb{F}_q$, ring of integers $\mathcal{O}_K$, then $K/W(\mathbb{F}_q)\left[\frac{1}{p}\right]$ is totally ramified and $W(\mathbb{F}_q)\left[\frac{1}{p}\right]/\mathbb{Q}_p$ is unramified.

1.2. Representability.

1.2.1. A finiteness condition.

Definition 1.4 (Mazur). *A profinite group G has finiteness condition Φ_p, if $\forall$ open subgroup $G' \subset G$, $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < +\infty$.*

Remark 1.5.

(1) (Burnside basis theorem) $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < +\infty \Leftrightarrow$ maximal pro-p quotient of G' is topologically finitely generated.

(2) $\mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) \cong \mathrm{Hom}_{\mathrm{cont}}(G'^{,\mathrm{ab}}, \mathbb{F}_p)$.

Example 1.6 (by CFT). *The following groups have Φ_p:*

(1) The Galois group $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$, with K a p-adic field.

(2) The galois group $\mathcal{G}_{F,S} = \mathrm{Gal}(F_S/F)$, where F is a number field, S is a finite set of places of F and $F_S \subset \bar{F}$ is the maximal extension of F unramified outside S.

1.2.2. Main proposition.

$\boxed{\text{prop:1587}}$ **Proposition 1.7** (Mazur). *If G has Φ_p, then*

(1) The functor $D_{V_{\mathbb{F}}}^{\square}$ is pro-representable by some $R_{V_{\mathbb{F}}}^{\square} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$, i.e.

$$D_{V_{\mathbb{F}}}^{\square}(A) \cong \mathrm{Hom}_{W(\mathbb{F})}\left(\boxed{R_{V_{\mathbb{F}}}^{\square}} , A\right),$$

which is functorial in $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.

universal framed deformation ring

(2) If $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then $\boxed{D_{V_{\mathbb{F}}}}$ is pro-representable by some $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$.

universal deformation ring

Maybe it's better to write $D_{V_{\mathbb{F}}}^{\square}(A) \cong \mathrm{Hom}_{\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}}(R_{V_{\mathbb{F}}}^{\square}, A)$?

Remark 1.8. *(1) Literally, pro-representable functor = limit of representable functor. How can we realize that? [nLa22] defines pro-representable to be the filtered colimit of representables. There's a post on MSE (cf. [htt]) which discusses the difference between the two definitions. It might be ture that*

$$\operatorname{Hom}_{W(\mathbb{F})}\left(R_{V_{\mathbb{F}}}^{\square}, A\right) = \operatorname{Hom}_{\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}}\left(\varprojlim_{k} R_{V_{\mathbb{F}}}^{\square}/\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}^{k}, A\right) \cong \varinjlim_{k}\operatorname{Hom}_{\mathfrak{Ar}_{W(\mathbb{F})}}\left(R_{V_{\mathbb{F}}}^{\square}/\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}^{k}, A\right),$$

for any $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.

(2) (???)$R_{V_{\mathbb{F}}}^{\square}$ is unique up to unique isomorphism; the identity map in $\operatorname{Hom}\left(R_{V_{\mathbb{F}}}^{\square}, R_{V_{\mathbb{F}}}^{\square}\right)$ gives rise to a universal framed deformation over $R_{V_{\mathbb{F}}}^{\square}$.

(3) $R_{V_{\mathbb{F}}}^{\square}$ exists without Φ_p, but maybe no longer noetherian.

(4) (???)$\mathbb{F} \hookrightarrow \operatorname{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) \rightsquigarrow$ write "=" in $\operatorname{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$.

Proof of Proposition 1.7. `prop:1587`

(1) G finite $\rightsquigarrow$ profinite.

- **(FORMAL) CONSTRUCTION:**
  Suppose G is finite. Set

  $$G = \langle g_1, \cdots, g_s | r_1(g_1, \cdots, g_s), \cdots, r_t(g_1, \cdots, g_s) \rangle$$

  a presentation. Define

  $$\mathcal{R} = W(\mathbb{F})\left[X_{i,j}^k \big| i,j = 1, \cdots, d; k = 1, \cdots, s\right]/\mathcal{I},$$

  where

  $$\mathcal{I} = \langle r_l\left(X^1, \cdots, X^s\right) - \operatorname{id}\rangle_{1 \leq l \leq t}, X^k = \left(X_{i,j}^k\right)_{d \times d}.$$

  To make $\mathcal{R}$ complete, local, noetherian, take

  $$\mathcal{J} = \ker\left(\mathcal{R} \to \mathbb{F}, X^k \mapsto \bar{\rho}(g_k), k = 1, \cdots, s\right),$$

  and set $R_{V_{\mathbb{F}}}^{\square} = \varprojlim_n \mathcal{R}/\mathcal{J}^n$ to be the $\mathcal{J}$-adic completion of $\mathcal{R}$. Besides that, we set $\rho_{V_{\mathbb{F}}}^{\square} : G \to \operatorname{GL}_d\left(R_{V_{\mathbb{F}}}^{\square}\right), g_k \mapsto$ image of X^k in $\operatorname{GL}_d\left(R_{V_{\mathbb{F}}}^{\square}\right)$.
  VERIFICATION:
  Take $\rho \in D_{V_{\mathbb{F}}}^{\square}(A)$, where $\rho : G \to \operatorname{GL}_d(A)$. Define

  $$\mathfrak{F}_\rho \in \operatorname{Hom}_{W(\mathbb{F})}\left(R_{V_{\mathbb{F}}}^{\square}, A\right), \overline{\text{entries of } X^k} \mapsto \text{corresponding entries of } \rho(g_k), \forall k = 1, \cdots, s.$$

  Then $\mathfrak{F}_\rho$ induces $\widehat{\mathfrak{F}_\rho} : \operatorname{GL}_d\left(R_{V_{\mathbb{F}}}^{\square}\right) \to \operatorname{GL}_d(A)$. It's immediate to check that $\rho = \widehat{\mathfrak{F}_\rho} \circ \rho_{V_{\mathbb{F}}}^{\square}$ and $\widehat{\mathfrak{F}_\rho}$ is unique choice to make the diagram commute. Thus, $\mathfrak{F} : \rho \mapsto \mathfrak{F}_\rho$ gives the pro-representability when G is finite.

- When G is profinite, we have $G = \varprojlim_i G/H_i$, where $H_i \subset \ker(\bar{\rho})$ are open normal subgroups. For every i, one has a universal pair $\left(R_i^{\square}, \rho_i^{\square}\right)$ by previous construction. Passing by limits, we define

  $$\left(R_{V_{\mathbb{F}}}^{\square}, \rho_{V_{\mathbb{F}}}^{\square}\right) = \varprojlim_i \left(R_i^{\square}, \rho_i^{\square}\right), \text{with } R_{V_{\mathbb{F}}}^{\square} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}.$$

We will show in [Section 1.4, TBA] that $R_{V_{\mathbb{F}}}^{\square}$ is noetherian.

(2) By Schlessinger's representability criterion (cf. [Section 1.7, TBA]) or by Kisin's work (cf. [Section 2.1, TBA by Y. Chen]).

$\square$

1.7. Groupid over categories (abstract stuff...)

Definition 1.9.

(1) A **groupoid category** is a category in which all morphisms are isomorphisms.

(2) Call the isomorphism classes the **connected components** of the groupoid.

Remark 1.10.

(1) Not necessarily all objects in a groupoid are isomorphic.

(2) $\operatorname{Hom}_{\mathfrak{C}}(A, A)$ forms a group for $\forall A \in \operatorname{ob} \mathfrak{C}$, where $\mathfrak{C}$ is a groupoid category, and the identity in $\operatorname{Hom}_{\mathfrak{C}}(A, A)$ is the identity morphism.

(3) $A \cong B \Rightarrow \operatorname{Hom}_{\mathfrak{C}}(A, A) \cong \operatorname{Hom}_{\mathfrak{C}}(B, B)$ (non-canonically)

Definition 1.11. Let $\mathfrak{C}$ be a category. Let $\mathfrak{F}$ be another category, $\Theta : \mathfrak{F} \to \mathfrak{C}$ be a functor.

(1) We say $\eta \in \operatorname{ob}(\mathfrak{F})$ **lies above** $T \in \operatorname{ob}(\mathfrak{C})$, if $\Theta(\eta) = T$.

(2) We say $\left(\eta \xrightarrow{\alpha} \zeta \right) \in \operatorname{Mor}_{\mathfrak{F}}$ **lies above** $\left(T \xrightarrow{f} S \right) \in \operatorname{Mor}_{\mathfrak{C}}$, if $\Theta(\eta) = f$.

(3) $(T \in \operatorname{ob}(\mathfrak{C}), \operatorname{id}_T)$ is a subcategory of $\mathfrak{C}$. Write $\mathfrak{F}(T)$ the subcategory of $\mathfrak{F}$ over (T, id_T).

Definition 1.12 (groupoid over $\mathfrak{C}$/category cofibered in groupoids over $\mathfrak{C}$). The triple $(\mathfrak{F}, \mathfrak{C}, \Theta)$ is a **groupoid over** $\mathfrak{C}$ if

(1) for any morphisms $\left(\eta \xrightarrow{\alpha} \zeta \right)$ and $\left(\eta \xrightarrow{\alpha'} \zeta' \right)$ in $\mathfrak{F}$ over the same morphism $T \to S$ in $\mathfrak{C}$, there exists unique $\zeta \xrightarrow{u} \zeta'$ in $\mathfrak{F}$ over id_S such that $u \circ \alpha = \alpha'$.

(2) For any $\eta \in \operatorname{ob}(\mathfrak{C})$ and any $T \xrightarrow{f} S$ in $\operatorname{Mor}_{\mathfrak{C}}$ with η over T, there exists morphism $\eta \xrightarrow{\alpha} \zeta$ in $\operatorname{Mor}_{\mathfrak{F}}$ over f.

Remark 1.13. (1) $\forall T \in \operatorname{ob}(\mathfrak{C})$, the category $\mathfrak{F}(T)$ is a groupoid. It's natural to specigy a groupoid by specifying objects in $\mathfrak{F}(T)$ for any $T \in \operatorname{ob}(\mathfrak{C})$, and specifying isomorphism class of morphisms above any $T \xrightarrow{f} S$ in $\mathfrak{C}$.

(2) Scheme and stack stuff.....

If for each $T \in \operatorname{ob}(\mathfrak{C})$, the isomorphism classes of $\mathfrak{F}(T)$ forms a set, we associate to the category $\mathfrak{F}$ over $\mathfrak{C}$ a functor $|\mathfrak{F}| : \mathfrak{C} \to \operatorname{Set}$ by sending T to the set of isomorphism classes of $\mathfrak{F}(T)$.

Example 1.14. Let $\mathfrak{C} = \mathfrak{Ar}_{W(\mathbb{F})}$. To the representation $V_{\mathbb{F}}$ of G, we define a groupoid $\mathcal{D}_{V_{\mathbb{F}}}$ over $\mathfrak{C}$:

(1) $\forall A \in \mathfrak{Ar}_{W(\mathbb{F})}$, objects of $\mathcal{D}_{V_{\mathbb{F}}}$ over A are pairs (V_A, ι_A) in $D_{V_{\mathbb{F}}}(A)$.

(2) A morphism $(V_A, \iota_A) \to (V_{A'}, \iota_{A'})$ over $A \to A'$ in $\mathfrak{Ar}_{W(\mathbb{F})}$ is a isomorphism class

$$\left\{ \alpha : V_A \otimes_A A' \xrightarrow{\cong} V_{A'} \text{ is an isomorphism } \middle| \iota_{A'} \circ \alpha = \iota_A \right\} \Big/ (A')^*$$

Remark 1.15. (1) The deformation functor $D_{V_{\mathbb{F}}}$ defined before is exactly $|\mathcal{D}_{V_{\mathbb{F}}}|$ above.

(2) When $V_{\mathbb{F}}$ has non-trivial automorphisms, then so do the object in $D_{V_{\mathbb{F}}}(A)$. (???) In this situation, the groipoid $\mathcal{D}_{V_{\mathbb{F}}}$ captures the geometry of the deformation theory of $V_{\mathbb{F}}$ more accurately than its functor if isomorphism classes.

Representability of a groupoid $\Theta : \mathfrak{F} \to \mathfrak{C}$.

Definition 1.16.

(1) $\forall \eta \in \operatorname{ob}(\mathfrak{F})$, define the category $\widetilde{\eta}$ (**the category under** η) as the category with objects are morphisms with source η and whose morphisms from $\eta \xrightarrow{\alpha} \zeta$ to $\eta \xrightarrow{\alpha'} \zeta'$ are morphisms $\zeta \xrightarrow{u} \zeta'$ in $\mathfrak{F}$ such that $u \circ \alpha = \alpha'$.

(2) Groupoid $\mathfrak{F}$ over $\mathfrak{C}$ is **representable** if there exists $\eta \in \mathfrak{F}$ such that the canonical functor $\widetilde{\eta} \to \mathfrak{F}$ is an equivalence of categories.

> ζ and ζ' are not necessarily lying on the same object of $\mathfrak{C}$

(3) Similarly, we define the category $\widetilde{T}$ for every $T \in \mathfrak{C}$.

One has a commutative diagram of categories:

$$
\begin{array}{ccc}
\widetilde{\eta} & \longrightarrow & \mathfrak{F} \\
{\scriptstyle\cong}\downarrow & & \downarrow \\
\widetilde{\Theta(\eta)} & \longrightarrow & \mathfrak{C}
\end{array}
$$

Lemma 1.17. *The left vertical homorphism above is an equivalence of categories.*

Proof. Abstract nonsense...... $\qquad\square$

Remark 1.18. *If $\mathfrak{F}$ is representable by η, then the equivalence $\widetilde{\eta} \to \widetilde{\Theta(\eta)}$ imples that η, as well as $\Theta(\eta)$, are well-defined up to canonical isomorphism. One says that $\Theta(\eta)$ **represents $\mathfrak{F}$ over** $\mathfrak{C}$.*

Lemma 1.19 (Relation with "classical" representable functor)**.**

(1) If $\mathfrak{F}$ is representable by η, any two objects of $\mathfrak{F}(\Theta(\eta))$ are canonically isomorphic and there is an isomorphism of functors

$$
\operatorname{Hom}_{\mathfrak{C}}(T, -) \xrightarrow{\cong} |\mathfrak{F}|,
$$

so that T represents $|\mathfrak{F}|$ in the usual theoretic sense.

(2) If $|\mathfrak{F}|$ is representable and for any $T \in \operatorname{ob}(\mathfrak{C})$ any two objects of $\mathfrak{F}(T)$ are related by a unique isomorphism, then $\mathfrak{F}$ is representable.

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