§3.9.4. Finite flat group schemes

R: commutative ring. (+Noetherian???)

Def: An (affine) group scheme over $\mathrm{Spec}\, R$ is a representable functor from the category of schemes over $\mathrm{Spec}\, R$ to the category of groups, with a choice of the representing object.

($\Leftrightarrow$ an affine ($\mathrm{Spec}\,R$)-scheme G with a morphism of schemes $m: G\times_{\mathrm{Spec}R} G \longrightarrow G$ over $\mathrm{Spec}\,R$, satisfying: $\forall$ scheme $T/\mathrm{Spec}\,R$, $(G(T), m)$ is a group.)

By Yoneda embedding, we have an equivalent definition:

Def': An affine group scheme G over $\mathrm{Spec}\,R$ is an $\mathrm{Spec}\,R$-scheme $\pi: G\to \mathrm{Spec}\,R$ together with $\mathrm{Spec}\,R$-morphisms $m: G\times_{\mathrm{Spec}R} G\to G$, $i: G\to G$ (inverse), and $e: \mathrm{Spec}\,R\longrightarrow G$ (identity section) s.t.

(1) $m\circ(m\times id_G) = m\circ(id_G\times m)$

(2) $m\circ(e\times id_G) = j_1: \mathrm{Spec}R\times_{\mathrm{Spec}R} G \xrightarrow{\sim} G$

(3) $m\circ(id_G\times e) = j_2: G\times_{\mathrm{Spec}R}\mathrm{Spec}R \xrightarrow{\sim} G$

(4) $e\circ\pi = m\circ(id_G\times i)\circ\Delta_{G/\mathrm{Spec}R} = m\circ(i\times id_G)\circ\Delta_{G/\mathrm{Spec}R}: G\longrightarrow G$

Def: (1) A homomorphism of $\mathrm{Spec}R$-group schemes $G_1\to G_2$ is a morphism of $\mathrm{Spec}R$-schemes $f: G_1\longrightarrow G_2$ s.t. $f\circ m_1 = m_2\circ(f\times f): G_1\times_{\mathrm{Spec}R} G_1\longrightarrow G_2$

(2) A $\mathrm{Spec}R$-group scheme $\pi: G=\mathrm{Spec}\,A\longrightarrow \mathrm{Spec}\,R$ is

finite if π is finite (as morphism of schemes), i.e. (for an open cover $\{V_i=\mathrm{Spec}\,B_i\}$, $f^{-1}(V_i)=U_i$ is an open affine subscheme $\mathrm{Spec}\,A_i$, which induces a ring homomorphism $B_i\longrightarrow A_i$, makes A_i a finitely generated B_i-mod.;)

flat if π is flat (as morphism of schemes) ($f: X\to Y$ is flat $\Leftrightarrow \forall p\in X$, $f_p: \mathcal{O}_{Y,f(p)}\longrightarrow \mathcal{O}_{X,p}$ is flat, i.e. $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{Y,f(p)}$-module)

Let G be a finite flat group scheme $/R$.

[Stacks Project, Lemma 29.48.2] $\Rightarrow$ R noetherian $+ G$ finite flat $\Rightarrow G$ locally free of finite rank

Def: The rank (order) of G is its local rank.

Reference: lecture notes of Ben Moonen, Chapter 3 Basic theory of group schemes

* Hopf algebra

- If $G = \operatorname{Spec} A$ is an affine group scheme over $\operatorname{Spec} R$, then A is a commutative Hopf algebra

→ Boeckle writes "cocommutative", but cocommutative Hopf algebra corresponds to commutative group scheme.

Def: (Hopf algebra). R-algebra with R-linear maps $\mu: A \to A \otimes_R A$ (comultiplication)

$\varepsilon: A \to R$ (counit)

$\iota: A \to A$ (inverse)

satisfying:

Example (1) Γ: an abstract finite group. $\mathcal{A} = \operatorname{Hom}(\Gamma, R)$ is R-algebra.

define •(comultiplication) $\mu: \mathcal{A} \to \mathcal{A} \otimes_R \mathcal{A}$, $\mu(f)(a \otimes b) = f(ab)$, $\forall a, b \in \Gamma$

•(counit) $\varepsilon: \mathcal{A} \to R$, $\varepsilon(f) = 1$

•(coinverse) $\iota: \mathcal{A} \to \mathcal{A}$, $\iota(f)(s) = f(s^{-1})$ $\forall s \in \Gamma$

$\rightsquigarrow$ $\mathcal{A}$ is a cocommutative Hopf algebra.

Rmk. Γ is finite $\rightsquigarrow$ $\mathcal{A} \otimes \mathcal{A}$ is naturally isomorphic to $\operatorname{Hom}(\Gamma \times \Gamma, R)$

(2). $\mathcal{A} = R[X]/(X^m - 1)$, with $\mu(X) = X \otimes X$, $\iota(X) = X^{-1}$ and $\varepsilon(X) = 1$.

This defines the multiplicative group scheme μ_m. It is étale over R iff. m is invertible in R.

étale morphism $\Leftrightarrow$ flat + unramified $\Leftrightarrow$ smooth + unramified.

unramified: $f: X \to Y$, locally of finite presentation,
$\forall x \in X$ and $y = f(x)$, one has ① $k(x)$ is finite sep. ext. of $k(y)$
② $f^{\#}(m_y)\mathcal{O}_{x,X} = m_x$, where $f^{\#}: \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$ and m_y, m_x are maximal ideals of the local rings.

• K: finite extension of $\mathbb{Q}_p$.

Def: A flat representation of G_K is a continuous representation of G_K on a finite abelian group V s.t. $\exists$ finite group scheme $\mathcal{G}$ over $\mathcal{O}_K$ so that $V \cong \mathcal{G}(\bar{K})$ as a $\mathbb{Z}[G_K]$-module.

[G_K-action on $\mathcal{G}(\bar{K})$: $\mathcal{G}(\bar{K}) = \overset{f}{\overset{\cap}{\mathrm{Hom}}}(\mathrm{Spec}\,\bar{K}, \mathcal{G})$, $\forall \sigma \in \mathrm{Gal}(\bar{K}/K)$, $\sigma(f) = f \circ \mathrm{Spec}(\sigma)$, where $\mathrm{Spec}(\sigma): \mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(\bar{K})$]

FACT. Suppose $\mathcal{G}$ is a finite flat group scheme over $\mathcal{O}_K$ of order prime to p. Then ~~the~~ following 3 equivalent conditions hold:

(1) $\mathcal{G}$ is étale

(2) The action of G_K on $\mathcal{G}(\bar{K})$ is via $\pi_1(\mathrm{Spec}\,\mathcal{O}_K)$

(3) The action of G_K is unramified.

Conversely, any unramified cont. rep. of G_K on a finite abelian group is flat.

Assumption: V is of p-power order.

Lemma: V: cont. lin. rep of G_K on a finite abelian p-group. Then V is flat over K^{ur} $\Rightarrow$ V is flat over K.

Pf: V is flat over $K^{ur} \Rightarrow V|G_{K^{ur}} \cong \underset{\sim}{\mathcal{G}^{ur}}(\overline{K^{ur}})$ as a $\mathbb{Z}[G_{K^{ur}}]$-module

$\hookrightarrow \mathcal{G}^{ur} = \operatorname{Spec}(\mathcal{A}^{ur})$ is a finite group scheme $/ \mathcal{O}_{K^{ur}}$

For any finite unram. ext. L/K with G_L acts trivially on V,

$\mathcal{O}_L \hookrightarrow \mathcal{O}_{K^{ur}} \rightsquigarrow \mathcal{G}^{ur} \longrightarrow \operatorname{Spec} \mathcal{O}_{K^{ur}} \longrightarrow \operatorname{Spec} \mathcal{O}_L$ is a finite flat group scheme $/\mathcal{O}_L$

Let $\mathcal{A}_{\mathcal{O}_L}$ be the Hopf algebra of the above group scheme $\mathcal{G}_{\mathcal{O}_L}$, with $\mathcal{G}_{\mathcal{O}_L}(\overline{K}) \cong V$ as $\mathbb{Z}[G_L]$-modules.

- $\mathcal{A}_L := \mathcal{A}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} L$

$\underbrace{\text{Ex 3.10.5(c)}} \longrightarrow \mathcal{A}_L^{Gal(L/K)}$ is a finite Hopf algebra $\mathcal{A}_K$ over K s.t. $\mathcal{G}_K(\overline{K}) \cong V$ as $\mathbb{Z}[G_K]$-modules for the group scheme $\operatorname{Spec} \mathcal{A}_K$

- $\mathcal{A}_{\mathcal{O}_K} := \mathcal{A}_{\mathcal{O}_L}^{Gal(L/K)}$.

Going to prove: $\mathcal{A}_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathcal{O}_L \cong \mathcal{A}_{\mathcal{O}_L}$ $\qquad (*)$

($\mathcal{A}_{\mathcal{O}_L}$ satisfies the Galois descent for $Gal(L/K)$)

[Thm (Galois descent) Let L/K be a finite Galois extension and V be a fin. dim. L-v.s. with a semi-linear $Gal(L/K)$-action, (i.e. $\sigma(ax) = \sigma(a)\sigma(x)$, $\forall \sigma \in Gal(L/K)$, $a \in L$, $x \in V$). Then we have

$$V^{Gal(L/K)} \otimes_K L \xrightarrow{\simeq} V$$
]

[Corollary: the Hopf structure on $\mathcal{A}_{\mathcal{O}_L}$ descents to $\mathcal{A}_{\mathcal{O}_K} = \mathcal{A}_K \cap \mathcal{A}_{\mathcal{O}_L}$]

Strategy: Dévissage!

[Dévissage: ① Prove the case of mod p
② Prove case of mod $p^n \Rightarrow$ case of mod p^{n+1}
③ taking inverse limit]

- π: uniformizer of K (and L, unramified)

To prove $(*)$, it's reduced to prove:

$$(\mathcal{A}_{\mathcal{O}_L}/\pi^n\mathcal{A}_{\mathcal{O}_L})^{\mathrm{Gal}(L/K)} \otimes_{\mathcal{O}_K/\pi^n\mathcal{O}_K} \mathcal{O}_L/\pi^n\mathcal{O}_L \cong \mathcal{A}_{\mathcal{O}_L}/\pi^n\mathcal{A}_{\mathcal{O}_L} \quad (\#)$$

then $(*)$ follows from taking inverse limit.

$\boxed{\text{Step ①}}$

<u>(#) for $n=1$</u>: normal basis theorem $\rightsquigarrow k_L \cong k_K[\mathrm{Gal}(L/K)]$

$\mathcal{A}_{\mathcal{O}_L}$ is a free $\mathcal{O}_L$-module of rank $r \Rightarrow \mathcal{A}_{\mathcal{O}_L}/\pi\mathcal{A}_{\mathcal{O}_L} \cong k_L^r$

$$\begin{aligned} &\cong k_K[\mathrm{Gal}(L/K)]^r \\ &\cong k_K^r \otimes_{k_K} k_K[\mathrm{Gal}(L/K)] \\ &\text{as } k_K[G]\text{-modules} \end{aligned}$$

$$\text{L.H.S of } (\#)^{n=1} = (\mathcal{A}_{\mathcal{O}_L}/\pi\mathcal{A}_{\mathcal{O}_L})^{\mathrm{Gal}(L/K)} \otimes_{\mathcal{O}_K/\pi\mathcal{O}_K} \mathcal{O}_L/\pi\mathcal{O}_L$$
$$\cong k_K^r \otimes_{k_K} k_L \cong k_L^r \cong \mathcal{A}_{\mathcal{O}_L}/\pi\mathcal{A}_{\mathcal{O}_L} = \text{R.H.S of } (\#)^{n=1} \quad \checkmark$$

<u>(#) for n implies (#) for $n+1$</u> $\boxed{\text{Step ②}}$

We have the exact seq:

Write $G = \mathrm{Gal}(L/K)$

* By (generalized) Hilbert 90, $H^1(G, \mathcal{A}_{\mathcal{O}_L}/\pi)$ vanishes.

$\Longrightarrow$ taking G-invariant is exact.

By applying $(-)^G \otimes_{\mathcal{O}_K} \mathcal{O}_L$ on $(*)$, we obtain

$$\begin{tikzcd}
& (\mathcal{A}_{\mathcal{O}_L}/\pi)^G \otimes_{\mathcal{O}_K} \mathcal{O}_L \arrow[r] \arrow[d, "\cong"'] & (\mathcal{A}_{\mathcal{O}_L}/\pi^{n+1})^G \otimes_{\mathcal{O}_K} \mathcal{O}_L \arrow[r] \arrow[d] & (\mathcal{A}_{\mathcal{O}_L}/\pi^n)^G \otimes_{\mathcal{O}_K} \mathcal{O}_L \arrow[d, "\cong"'] \\
0 \arrow[r] & \mathcal{A}_{\mathcal{O}_L}/\pi \arrow[r] & \mathcal{A}_{\mathcal{O}_L}/\pi^{n+1} \arrow[r] & \mathcal{A}_{\mathcal{O}_L}/\pi^n
\end{tikzcd}$$

The first and the third vertical are iso. by ind. hypo.
then the result follows from snake lemma.

$\boxed{\text{Step ③}}$ $(*) = \varprojlim (\#)$

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Coro: If M is a fin. cont. $\mathbb{Z}[Gal(K^{ur}/K)]$-module and $\mathcal{G}$ is a finite flat group scheme over $\mathcal{O}_K$, ~~the~~ the rep $M\otimes\mathcal{G}(\bar{K})$ arises from a finite flat group scheme. In particular, M arises from a finite flat group scheme.

Pf: (1) V: rep of G_K on $\mathcal{G}(\bar{K})\otimes M$.

M is discrete and unramified $\Rightarrow \exists$ finite unram. ext. L/K over which M becomes trivial

$\mathcal{G}$ is flat on $\mathcal{O}_K \Rightarrow$ flat on $\mathcal{O}_L$, we apply the above lemma on V.

(2) $\forall n\in\mathbb{N}$, the trivial G_K-~~module~~ $\mathbb{Z}/(p^n)$ arises from a finite flat group scheme $\mathcal{G}/\mathcal{O}_K$ (cf. Ex 3.10.5(d)) #

- $\Gamma_K = Gal(K^{ur}/K)$