DEFORMATION OF GALOIS REPRESENTATIONS

A notes for the number theory seminar at YMSC, 2022

Update on 2022-10-13 22:57

YIJUN YUAN

> *We are given either a number field K and a finite set of primes S, or a local field F, and we are given a representation of either $G_{K,S}$ or G_F into $\mathrm{GL}_n(k)$, where k is a finite field. We want to try to understand all possible lifts of this representation to $\mathrm{GL}_n(A)$, where A is a complete noetherian local ring with residue field k.*
>
> —*Fernando Q. Gouvêa (cf. [Gou01])*

Main reference: [Böc13; Maz89]
Motivation and history of deformation theory: [Maz97; Gou01]
Basics of groupoids: Appendix of [Kis09]

1. Deformations of representations of profinite groups

Notations.

p	prime number
$\mathbb{F}$	finite field of char. p
$W(\mathbb{F})$	ring of Witt vectors[1] over A
G	profinite group
$V_{\mathbb{F}}$	finite $\mathbb{F}[G]$-module with continuous G-action
d	dimeinsion of $V_{\mathbb{F}}$
$\beta_{\mathbb{F}}$	a $\mathbb{F}$-basis of $V_{\mathbb{F}}$

1.1. Deformation functors.

Notations.

$\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$	category of complete Noetherian local $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$
$\mathfrak{Ar}_{W(\mathbb{F})}$	full sub-category of finite local Artinian $W(\mathbb{F})$-algebras
$\mathfrak{m}_A$	maximal ideal of $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$

Remark 1.1 (???)**.** *Via the $W(\mathbb{F})$-structure, the residue field of any $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ is canonically isomorphic to $\mathbb{F}$.*

Definition 1.2. *Let $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.*

*(1) A **deformation** of $V_{\mathbb{F}}$ to A is a pair (V_A, ι_A), such that*
  (a) V_A is $A[G]$-module, finite free over A, with continuous G-action;
  (b) $\iota_A : V_A \otimes_A \mathbb{F} \xrightarrow{\cong} V_{\mathbb{F}}$ is G-equivariant.
*(2) A **framed deformation** of $(V_{\mathbb{F}}, \beta_{\mathbb{F}})$ to A is a triple (V_A, ι_A, β_A), where*

> We view $\mathbb{F}$ as an A-module via the canonical projection $A \to A/\mathfrak{m}_A = F$.

---

[1]$W(\mathbb{F})$ is the unique (up to unique isomorphism) complete discrete valuation ring which is absolutely unramified (uniformizer= p) and has residue field $\mathbb{F}$.

(a) (V_A, ι_A) is a deformation of $V_{\mathbb{F}}$ to A;

(b) β_A is a A-basis of V_A which reduces to $\beta_{\mathbb{F}}$ under ι_A.

Set $D_{V_{\mathbb{F}}}, D_{V_{\mathbb{F}}}^{\square} : \mathfrak{Ar}_{W(\mathbb{F})} \to \text{Set}$,

$$D_{V_{\mathbb{F}}}(A) = \{\text{deformations of } V_{\mathbb{F}} \text{ to } A\}/\cong,$$

$$D_{V_{\mathbb{F}}}^{\square}(A) = \{\text{framed deformations of } (V_{\mathbb{F}}, \beta_{\mathbb{F}}) \text{ to } A\}/\cong.$$

Remark 1.3.

(1) The **_FIXED_** basis $\beta_{\mathbb{F}}$ gives the isomorphism $V_{\mathbb{F}} \cong \mathbb{F}^d$ as vector space. Thus we can view $V_{\mathbb{F}}$ as $\bar{\rho} : G \to \text{GL}(V_{\mathbb{F}}) = \text{GL}_d(\mathbb{F})$: a d-dimensional $\mathbb{F}$-representation of G. Then

$$D_{V_{\mathbb{F}}}^{\square}(A) = \{\rho : G \to \text{GL}_d(A) \text{ lifting } \bar{\rho}\},$$

Does not guarantee $\beta_A \rightsquigarrow \beta_{\mathbb{F}}$!

$\boxed{D_{V_{\mathbb{F}}}(A)} = D_{V_{\mathbb{F}}}^{\square}(A)/\text{action by conjugates of } \ker(\text{GL}_d(A) \to \text{GL}_d(\mathbb{F})).$

Not always representable!

(2) Mazur only consdier $\boxed{D_{V_{\mathbb{F}}}}$, which describes representations lifting $V_{\mathbb{F}}$ up to isomorphism. Add "base condition" $\rightsquigarrow D_{V_{\mathbb{F}}}^{\square}$.

(3) Often consider deformation functors on $\mathfrak{Ar}_{\mathcal{O}} = $ category of local artinian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, where $\mathcal{O}$ is ring of integers of a finite totally ramified extension of $W(\mathbb{F})\left[\frac{1}{p}\right]$ ($\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}$).

For example, let K be a p-adic field with residue field $\mathbb{F}_q$, ring of integers $\mathcal{O}_K$, then $K/W(\mathbb{F}_q)\left[\frac{1}{p}\right]$ is totally ramified and $W(\mathbb{F}_q)\left[\frac{1}{p}\right]/\mathbb{Q}_p$ is unramified.

$$G \xrightarrow{\rho} \text{GL}_d(A)$$
$$\searrow_{\bar{\rho}} \quad \downarrow$$
$$\text{GL}_d(\mathbb{F})$$

1.2. Representability.

1.2.1. A finiteness condition.

Definition 1.4 (Mazur). *A profinite group G has finiteness condition Φ_p, if $\forall$ open subgroup $G' \subset G$, $\dim_{\mathbb{F}_p} \text{Hom}_{\text{cont}}(G', \mathbb{F}_p) < +\infty$.*

Remark 1.5.

(1) (Burnside basis theorem) $\dim_{\mathbb{F}_p} \text{Hom}_{\text{cont}}(G', \mathbb{F}_p) < +\infty \Leftrightarrow$ maximal pro-p quotient of G' is topologically finitely generated.

(2) $\text{Hom}_{\text{cont}}(G', \mathbb{F}_p) \cong \text{Hom}_{\text{cont}}(G'^{,\text{ab}}, \mathbb{F}_p)$.

Example 1.6 (by CFT). *The following groups have Φ_p:*

(1) The Galois group $\mathcal{G}_K = \text{Gal}(\bar{K}/K)$, with K a p-adic field.

(2) The galois group $\mathcal{G}_{F,S} = \text{Gal}(F_S/F)$, where F is a number field, S is a finite set of places of F and $F_S \subset \bar{F}$ is the maximal extension of F unramified outside S.

1.2.2. Main proposition.

Proposition 1.7 (Mazur). *If G has Φ_p, then*

(1) The functor $D_{V_{\mathbb{F}}}^{\square}$ is pro-representable by some $R_{V_{\mathbb{F}}}^{\square} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$, i.e.

$$D_{V_{\mathbb{F}}}^{\square}(A) \cong \text{Hom}_{W(\mathbb{F})}\left(\boxed{R_{V_{\mathbb{F}}}^{\square}}, A\right),$$

universal framed deformation ring

which is functorial in $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.

(2) If $\text{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then $\boxed{D_{V_{\mathbb{F}}}}$ is pro-representable by some $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$.

universal deformation ring

Maybe it's better to write $D_{V_{\mathbb{F}}}^{\square}(A) \cong \text{Hom}_{\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}}(R_{V_{\mathbb{F}}}^{\square}, A)$?

Remark 1.8. *(1) Literally, pro-representable functor = limit of representable functor. How can we realize that? [nLa22] defines pro-representable to be the filtered colimit of representables. There's a post on MSE (cf. [htt]) which discusses the difference between the two definitions. It might be ture that*

$$\operatorname{Hom}_{W(\mathbb{F})}\left(R_{V_{\mathbb{F}}}^{\square}, A\right) = \operatorname{Hom}_{\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}}\left(\varprojlim_{k} R_{V_{\mathbb{F}}}^{\square}/\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}^{k}, A\right) \cong \varinjlim_{k} \operatorname{Hom}_{\mathfrak{Ar}_{W(\mathbb{F})}}\left(R_{V_{\mathbb{F}}}^{\square}/\mathfrak{m}_{R_{V_{\mathbb{F}}}^{\square}}^{k}, A\right),$$

for any $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.

(2) (???)$R_{V_{\mathbb{F}}}^{\square}$ is unique up to unique isomorphism; the identity map in $\operatorname{Hom}\left(R_{V_{\mathbb{F}}}^{\square}, R_{V_{\mathbb{F}}}^{\square}\right)$ gives rise to a universal framed deformation over $R_{V_{\mathbb{F}}}^{\square}$.

(3) $R_{V_{\mathbb{F}}}^{\square}$ exists without Φ_p, but maybe no longer noetherian.

(4) (???)$\mathbb{F} \hookrightarrow \operatorname{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) \rightsquigarrow$ write "=" in $\operatorname{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$.

Proof of Proposition 1.7.

(1) G finite $\rightsquigarrow$ profinite.

- **(FORMAL) CONSTRUCTION:**
  Suppose G is finite. Set

$$G = \langle g_1, \cdots, g_s \mid r_1(g_1, \cdots, g_s), \cdots, r_t(g_1, \cdots, g_s)\rangle$$

a presentation. Define

$$\mathcal{R} = W(\mathbb{F})\left[X_{i,j}^k \mid i,j = 1, \cdots, d; k = 1, \cdots, s\right]/\mathcal{I},$$

where

$$\mathcal{I} = \langle r_l(X^1, \cdots, X^s) - \operatorname{id}\rangle_{1 \leq l \leq t}, X^k = \left(X_{i,j}^k\right)_{d \times d}.$$

To make $\mathcal{R}$ complete, local, noetherian, take

$$\mathcal{J} = \ker\left(\mathcal{R} \to \mathbb{F}, X^k \mapsto \bar{\rho}(g_k), k = 1, \cdots, s\right),$$

and set $R_{V_{\mathbb{F}}}^{\square} = \varprojlim_n \mathcal{R}/\mathcal{J}^n$ to be the $\mathcal{J}$-adic completion of $\mathcal{R}$. Besides that, we set $\rho_{V_{\mathbb{F}}}^{\square} : G \to \operatorname{GL}_d\left(R_{V_{\mathbb{F}}}^{\square}\right), g_k \mapsto$ image of X^k in $\operatorname{GL}_d\left(R_{V_{\mathbb{F}}}^{\square}\right)$.
  VERIFICATION:
  Take $\rho \in D_{V_{\mathbb{F}}}^{\square}(A)$, where $\rho : G \to \operatorname{GL}_d(A)$. Define

$$\mathfrak{F}_\rho \in \operatorname{Hom}_{W(\mathbb{F})}\left(R_{V_{\mathbb{F}}}^{\square}, A\right), \overline{\text{entries of } X^k} \mapsto \text{corresponding entries of } \rho(g_k), \forall k = 1, \cdots, s.$$

Then $\mathfrak{F}_\rho$ induces $\widehat{\mathfrak{F}_\rho} : \operatorname{GL}_d\left(R_{V_{\mathbb{F}}}^{\square}\right) \to \operatorname{GL}_d(A)$. It's immediate to check that $\rho = \widehat{\mathfrak{F}_\rho} \circ \rho_{V_{\mathbb{F}}}^{\square}$ and $\widehat{\mathfrak{F}_\rho}$ is unique choice to make the diagram commute. Thus, $\mathfrak{F} : \rho \mapsto \mathfrak{F}_\rho$ gives the pro-representability when G is finite.

- When G is profinite, we have $G = \varprojlim_i G/H_i$, where $H_i \subset \ker(\bar{\rho})$ are open normal subgroups. For every i, one has a universal pair $\left(R_i^{\square}, \rho_i^{\square}\right)$ by previous construction. Passing by limits, we define

$$\left(R_{V_{\mathbb{F}}}^{\square}, \rho_{V_{\mathbb{F}}}^{\square}\right) = \varprojlim_i \left(R_i^{\square}, \rho_i^{\square}\right), \text{with } R_{V_{\mathbb{F}}}^{\square} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}.$$

We will show in [Section 1.4, TBA] that $R_{V_{\mathbb{F}}}^{\square}$ is noetherian.

(2) By Schlessinger's representability criterion (cf. [Section 1.7, TBA]) or by Kisin's work (cf. [Section 2.1, TBA by Y. Chen]).

$\square$

1.7. Groupid over categories (abstract stuff...)

Definition 1.9.
(1) *A **groupoid category** is a category in which all morphisms are isomorphisms.*
(2) *Call the isomorphism classes the **connected components** of the groupoid.*

Remark 1.10.
(1) *Not necessarily all objects in a groupoid are isomorphic.*
(2) $\mathrm{Hom}_{\mathfrak{C}}(A, A)$ *forms a group for* $\forall A \in \mathrm{ob}\,\mathfrak{C}$, *where* $\mathfrak{C}$ *is a groupoid category, and the identity in* $\mathrm{Hom}_{\mathfrak{C}}(A, A)$ *is the identity morphism.*
(3) $A \cong B \Rightarrow \mathrm{Hom}_{\mathfrak{C}}(A, A) \cong \mathrm{Hom}_{\mathfrak{C}}(B, B)$ *(non-canonically)*

Definition 1.11. *Let* $\mathfrak{C}$ *be a category. Let* $\mathfrak{F}$ *be another category,* $\Theta : \mathfrak{F} \to \mathfrak{C}$ *be a functor.*
(1) *We say* $\eta \in \mathrm{ob}(\mathfrak{F})$ ***lies above*** $T \in \mathrm{ob}(\mathfrak{C})$, *if* $\Theta(\eta) = T$.
(2) *We say* $(\eta \xrightarrow{\alpha} \xi) \in \mathrm{Mor}_{\mathfrak{F}}$ ***lies above*** $(T \xrightarrow{f} S) \in \mathrm{Mor}_{\mathfrak{C}}$, *if* $\Theta(\eta) = f$.
(3) $(T \in \mathrm{ob}(\mathfrak{C}), \mathrm{id}_T)$ *is a subcategory of* $\mathfrak{C}$. *Write* $\mathfrak{F}(T)$ *the subcategory of* $\mathfrak{F}$ *over* (T, id_T).

Definition 1.12 (groupoid over $\mathfrak{C}$/category cofibered in groupoids over $\mathfrak{C}$)**.** *The triple* $(\mathfrak{F}, \mathfrak{C}, \Theta)$ *is a **groupoid over** $\mathfrak{C}$ if*

(1) *for any morphisms* $(\eta \xrightarrow{\alpha} \xi)$ *and* $(\eta \xrightarrow{\alpha'} \xi')$ *in* $\mathfrak{F}$ *over the same morphism* $T \to S$ *in* $\mathfrak{C}$, *there exists unique* $\xi \xrightarrow{u} \xi'$ *in* $\mathfrak{F}$ *over* id_S *such that* $u \circ \alpha = \alpha'$.
(2) *For any* $\eta \in \mathrm{ob}(\mathfrak{C})$ *and any* $T \xrightarrow{f} S$ *in* $\mathrm{Mor}_{\mathfrak{C}}$ *with* η *over* T, *there exists morphism* $\eta \xrightarrow{\alpha} \xi$ *in* $\mathrm{Mor}_{\mathfrak{F}}$ *over* f.

Remark 1.13.
(1) *For every* $T \in \mathrm{ob}(\mathfrak{C})$, *the category* $\mathfrak{F}(T)$ *is a groupoid. It's natural to specigy a groupoid by specifying objects in* $\mathfrak{F}(T)$ *for any* $T \in \mathrm{ob}(\mathfrak{C})$, *and specifying isomorphism class of morphisms above any* $T \xrightarrow{f} S$ *in* $\mathfrak{C}$.
(2) *Scheme and stack stuff.....*

If for each $T \in \mathrm{ob}(\mathfrak{C})$, the isomorphism classes of $\mathfrak{F}(T)$ forms a set, we associate to the category $\mathfrak{F}$ over $\mathfrak{C}$ a functor $|\mathfrak{F}| : \mathfrak{C} \to \mathrm{Set}$ by sending T to the set of isomorphism classes of $\mathfrak{F}(T)$.

Example 1.14.
(1) *Let* $\mathfrak{C} = \mathfrak{Ar}_{W(\mathbb{F})}$. *To the representation* $V_{\mathbb{F}}$ *of* G, *we define a groupoid* $\mathcal{D}_{V_{\mathbb{F}}}$ *over* $\mathfrak{C}$:
  (a) $\forall A \in \mathfrak{Ar}_{W(\mathbb{F})}$, *objects of* $\mathcal{D}_{V_{\mathbb{F}}}$ *over* A *are pairs* (V_A, ι_A) *in* $D_{V_{\mathbb{F}}}(A)$.
  (b) *A morphism* $(V_A, \iota_A) \to (V_{A'}, \iota_{A'})$ *over* $A \to A'$ *in* $\mathfrak{Ar}_{W(\mathbb{F})}$ *is a isomorphism class*

$$\left\{ \alpha : V_A \otimes_A A' \xrightarrow{\cong} V_{A'} \text{ is an isomorphism } \Big| \iota_{A'} \circ \alpha = \iota_A \right\} \Big/ (A')^*$$

(2) *Let* $\mathfrak{C} = \mathfrak{Ar}_{W(\mathbb{F})}$. *We define the groupoid* $\mathcal{D}_{V_{\mathbb{F}}}^{\square}$ *on* $\mathfrak{C}$ *as follows:*
  (a) *An object over* $A \in \mathfrak{Ar}_{W(\mathbb{F})}$ *is a triple* (V_A, ι_A, β_A), *where* $(V_A, \iota_A) \in \mathcal{D}_{V_{\mathbb{F}}}(A)$ *and* β_A *is an* A-*basis of* V_A *mapping under* ι_A *to the basis* $\beta_{\mathbb{F}}$ *of* $V_{\mathbb{F}}$.
  (b) *A morphism* $(V_A, \iota_A, \beta_A) \to (V_{A'}, \iota_{A'}, \beta_{A'})$ *over* $A \to A'$ *is an isomorphism* $\alpha : V_A \otimes_A A' \xrightarrow{\cong} V_{A'}$ *taking* β_A *to* $\beta_{A'}$.

There is an obvious morphism of groupoids $\mathcal{D}_{V_{\mathbb{F}}}^{\square} \to \mathcal{D}_{V_{\mathbb{F}}}$.

Remark 1.15.
(1) *The deformation functor* $D_{V_{\mathbb{F}}}$ *defined before is exactly* $|\mathcal{D}_{V_{\mathbb{F}}}|$ *above.*
(2) *When* $V_{\mathbb{F}}$ *has non-trivial automorphisms, then so do the object in* $D_{V_{\mathbb{F}}}(A)$. *(???) In this situation, the groipoid* $\mathcal{D}_{V_{\mathbb{F}}}$ *captures the geometry of the deformation theory of* $V_{\mathbb{F}}$ *more accurately than its functor if isomorphism classes.*

Representability of a groupoid $\Theta : \mathfrak{F} \to \mathfrak{C}$.

Definition 1.16.

(1) $\forall \eta \in \mathrm{ob}(\mathfrak{F})$, *define the category* $\widetilde{\eta}$ (***the category under*** η) *as the category with objects are morphisms with source* η *and whose morphisms from* $\eta \xrightarrow{\alpha} \xi$ *to* $\eta \xrightarrow{\alpha'} \xi'$ *are morphisms* $\xi \xrightarrow{u} \xi'$ *in* $\mathfrak{F}$ *such that* $u \circ \alpha = \alpha'$.

(2) *Groupoid* $\mathfrak{F}$ *over* $\mathfrak{C}$ *is* ***representable*** *if there exists* $\eta \in \mathfrak{F}$ *such that the canonical functor* $\widetilde{\eta} \to \mathfrak{F}$ *is an equivalence of categories.*

(3) *Similarly, we define the category* $\widetilde{T}$ *for every* $T \in \mathfrak{C}$.

> ξ and ξ' are not necessarily lying on the same object of $\mathfrak{C}$

One has a commutative diagram of categories:

$$
\begin{array}{ccc}
\widetilde{\eta} & \longrightarrow & \mathfrak{F} \\
{\scriptstyle \cong}\downarrow & & \downarrow \\
\widetilde{\Theta(\eta)} & \longrightarrow & \mathfrak{C}
\end{array}
$$

Lemma 1.17. *The left vertical homorphism above is an equivalence of categories.*

Proof. Abstract nonsense...... $\qquad\square$

Remark 1.18. *If* $\mathfrak{F}$ *is representable by* η, *then the equivalence* $\widetilde{\eta} \to \widetilde{\Theta(\eta)}$ *imples that* η, *as well as* $\Theta(\eta)$, *are well-defined up to canonical isomorphism. One says that* $\Theta(\eta)$ ***represents*** $\mathfrak{F}$ ***over*** $\mathfrak{C}$.

Lemma 1.19 (Relation with "classical" representable functor)**.**

(1) *If* $\mathfrak{F}$ *is representable by* η, *any two objects of* $\mathfrak{F}(\Theta(\eta))$ *are canonically isomorphic and there is an isomorphism of functors*

$$
\mathrm{Hom}_{\mathfrak{C}}(\Theta(\eta), -) \xrightarrow{\cong} |\mathfrak{F}|,
$$

so that $\Theta(\eta)$ *represents* $|\mathfrak{F}|$ *in the usual set theoretic sense.*

(2) *If* $|\mathfrak{F}|$ *is representable and for any* $T \in \mathrm{ob}(\mathfrak{C})$ *any two objects of* $\mathfrak{F}(T)$ *are related by a unique isomorphism, then* $\mathfrak{F}$ *is representable.*

Remark 1.20. *The groupoid* $\mathcal{D}_{V_{\mathbb{F}}}$ *in Example 1.14 is usually not representable. Extending to* $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ *is needed.*

A crucial question: why do we need to use the language of groupoids?

Let's see Kisin's motivation (just a screenshot from Bockle's notes, which is nothing but a rewrite of [Kis09, Appendix (A.6)]):

The main reason why, in some circumstances, one needs to introduce the language of groupoids, is that formation of fiber products is not compatible with the passage from a groupoid $\mathfrak{F}$ over $\mathfrak{C}$ to its associated functor $|\mathfrak{F}|$. This is a serious technical issue, since Definition 2.4.4 of relative representability depends on the formation of fiber products. We illustrate this with a simple example taken from [34, A.6].

Consider now the situation when the group G is trivial and fix $\eta = (V_A, \iota_A) \in \mathcal{D}_{V_{\mathbb{F}}}(A)$ for some $A \in \mathfrak{Ar}_{W(\mathbb{F})}$. Then $\widetilde{\eta} \times_{\mathcal{D}_{V_{\mathbb{F}}}} \mathcal{D}_{V_{\mathbb{F}}}^{\square}$ can be identified with quadruples $(V'_{A'}, \psi'_{A'}, \varphi \colon V_A \otimes_A A' \xrightarrow{\cong} V'_{A'}, \beta_{A'})$, where $(V'_{A'}, \psi'_{A'}, \beta_{A'}) \in \mathcal{D}_{V_{\mathbb{F}}}^{\square}(A')$ and morphisms over $\mathrm{id}_{A'}$ are isomorphisms of $V'_{A'}$ reducing to the identity of $V_{\mathbb{F}}$. It follows that this category is a principal homogeneous space for the formal group obtained by completing $\mathrm{PGL}_d / W(\mathbb{F})$ along its identity section. Hence $|\widetilde{\eta} \times_{\mathcal{D}_{V_{\mathbb{F}}}} \mathcal{D}_{V_{\mathbb{F}}}^{\square}|(A')$ is isomorphic to the kernel $\mathrm{Ker}(\mathrm{PGL}_d(A') \to \mathrm{PGL}_d(\mathbb{F}))$. On the other hand, $|\mathcal{D}_{V_{\mathbb{F}}}^{\square}(A')|$ is a singleton and hence the same holds for $|\widetilde{\eta}| \times_{|\mathcal{D}_{V_{\mathbb{F}}}|} |\mathcal{D}_{V_{\mathbb{F}}}^{\square}|(A')$.

REFERENCES

[Böc13] G. Böckle. "Deformations of Galois Representations". In: L. Berger et al. *Elliptic Curves, Hilbert Modular Forms and Galois Deformations*. Springer Basel, 2013, pp. 21–115. DOI: 10.1007/978-3-0348-0618-3_2.

[Gou01] F. Q. Gouvêa. "Deformations of Galois representations". In: *Arithmetic Algebraic Geometry*. Ed. by B. Conrad and K. Rubin. American Mathematical Society, 2001, pp. 235–406. URL: http://www.ams.org/books/pcms/009/05.

[htt] S. M. (https://math.stackexchange.com/users/572592/sebastian-monnet). *Relationship between two definitions of pro-representable functors*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/4013849.

[Kis09] M. Kisin. "Moduli of finite flat group schemes, and modularity". In: *Annals of Mathematics* 170.3 (Nov. 2009), pp. 1085–1180. DOI: 10.4007/annals.2009.170.1085.

[Maz89] B. Mazur. "Deforming Galois Representations". In: *Galois Groups over* $\mathbb{Q}$. Ed. by Y. Ihara, K. Ribet, and J.-P. Serre. Vol. 16. Springer US, 1989, pp. 385–437. DOI: 10.1007/978-1-4613-9649-9_7.

[Maz97] B. Mazur. "An Introduction to the Deformation Theory of Galois Representations". In: *Modular Forms and Fermat's Last Theorem*. Ed. by G. Cornell, J. H. Silverman, and G. Stevens. Springer New York, 1997, pp. 243–311. DOI: 10.1007/978-1-4612-1974-3_8.

[nLa22] nLab authors. *prorepresentable functor*. Oct. 2022. URL: https://ncatlab.org/nlab/revision/prorepresentable%20functor/7.