The optimal regulator problem with a quadratic return function and non-linear transition functions

The objective function is:

$$W = -E_0 \left[\sum_{t=1}^{T} \beta^t [(\theta_t - a_t)' Q_t (\theta_t - a_t) + x_t' R_t x_t] \right]$$

where θ_t is defined as the logarithm of the vector of skills at time t, a_t is the targets of skill accumulation set for period t, and x_t is the vector of human capital investments made at time t. And Q_t is an $m \times m$ positive semidefinite symmetric matrix, and R_t is an $n \times n$ positive definite symmetric matrix.

For simplicity, we consider the case with only cognitive and non-cognitive skills, i.e. $m = 2$ and $\theta_t = (\theta_{C,t}, \theta_{N,t})$ and the case with only money and time investments, i.e. $n = 2$ and $x_t = (I_t, h_t)$.

Then, consider the CES human capital production functions, which are similar to the ones used in Cunha, Heckman and Schennach (2010).

$$\begin{aligned} \theta_{C,t} &= \frac{1}{\phi_{C,t}} ln\left(\gamma_{C,t,1} exp(\phi_{C,t}\theta_{C,t-1}) + \gamma_{C,t,2} exp(\phi_{C,t}\theta_{N,t-1}) + \gamma_{C,t,3} exp(\phi_{C,t} ln(I_t)) + \gamma_{C,t,4} exp(\phi_{C,t} ln(h_t))\right) + \epsilon_{C,t} \\ &\equiv \Phi_{C,t}(\theta_{C,t-1}, \theta_{N,t-1}, I_t, h_t) + \epsilon_{C,t} \qquad (1) \end{aligned}$$

$$\begin{aligned} \theta_{N,t} &= \frac{1}{\phi_{N,t}} ln\left(\gamma_{N,t,1} exp(\phi_{N,t}\theta_{C,t-1}) + \gamma_{N,t,2} exp(\phi_{N,t}\theta_{N,t-1}) + \gamma_{N,t,3} exp(\phi_{N,t} ln(I_t)) + \gamma_{N,t,4} exp(\phi_{N,t} ln(h_t))\right) + \epsilon_{N,t} \\ &\equiv \Phi_{N,t}(\theta_{C,t-1}, \theta_{N,t-1}, I_t, h_t) + \epsilon_{N,t} \qquad (2) \end{aligned}$$

where $\phi \in (-\infty, 1]$ determines the degree substitutability in the human capital production process: $\frac{1}{1-\phi}$. Notice that the substitutability of inputs in producing each skill is assumed to be the same, which can be relaxed later by using, e.g. nested CES models.

Step 1:

Start with initial guesses $\tilde{I}_1, ..., \tilde{I}_T$ and $\tilde{h}_1, ..., \tilde{h}_T$. For example, guess that $\tilde{I}_1 = ... = \tilde{I}_T = 0$ and $\tilde{h}_1 = ... = \tilde{h}_T = 0$. Set $\epsilon_{C,t} = \epsilon_{N,t} = 0$. Then with given initial endowments for each kid, the vector of skills in each period can be calculated by using Equations 1 and 2. Denote the generated level of skills by $\theta^0_{C,1}, ..., \theta^0_{C,T}$ and $\theta^0_{N,1}, ..., \theta^0_{N,T}$.

Step 2:

For each period, linearize the CES human capital production functions at $\theta^0_{C,t-1}$,$\theta^0_{N,t-1}\tilde{I}_t$, $\tilde{h}_t$:

$$\begin{aligned} \theta_{C,t} &= \Phi_{C,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t) + \frac{\partial \Phi_{C,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t)}{\partial \theta_{C,t-1}} (\theta_{C,t-1} - \theta^0_{C,t-1}) + \frac{\partial \Phi_{C,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t)}{\partial \theta_{N,t-1}} (\theta_{N,t-1} - \theta^0_{N,t-1}) \\ &+ \frac{\partial \Phi_{C,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t)}{\partial I_t} (I_t - \tilde{I}_t) + \frac{\partial \Phi_{C,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t)}{\partial h_t} (h_t - \tilde{h}_t) + \eta_{C,t} \end{aligned}$$

$$\begin{aligned} \theta_{N,t} &= \Phi_{N,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t) + \frac{\partial \Phi_{N,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t)}{\partial \theta_{C,t-1}} (\theta_{C,t-1} - \theta^0_{C,t-1}) + \frac{\partial \Phi_{N,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t)}{\partial \theta_{N,t-1}} (\theta_{N,t-1} - \theta^0_{N,t-1}) \\ &+ \frac{\partial \Phi_{N,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t)}{\partial I_t} (I_t - \tilde{I}_t) + \frac{\partial \Phi_{N,t}(\theta^0_{C,t-1}, \theta^0_{N,t-1}, \tilde{I}_t, \tilde{h}_t)}{\partial h_t} (h_t - \tilde{h}_t) + \eta_{N,t} \end{aligned}$$

Then we have:

$$\theta_t = A_t\theta_{t-1} + C_t x_t + b_t + u_t$$

where:

$$A_t \equiv \begin{bmatrix} \frac{\partial\Phi_{C,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t)}{\partial\theta_{C,t-1}} & \frac{\partial\Phi_{C,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t)}{\partial\theta_{N,t-1}} \\ \frac{\partial\Phi_{N,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t)}{\partial\theta_{C,t-1}} & \frac{\partial\Phi_{N,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t)}{\partial\theta_{N,t-1}} \end{bmatrix}$$

and

$$C_t \equiv \begin{bmatrix} \frac{\partial\Phi_{C,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t)}{\partial I_t} & \frac{\partial\Phi_{C,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t)}{\partial h_t} \\ \frac{\partial\Phi_{N,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t)}{\partial I_t} & \frac{\partial\Phi_{N,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t)}{\partial h_t} \end{bmatrix}$$

and

$$b_t \equiv \begin{bmatrix} \Phi_{C,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t) \\ \Phi_{N,t}(\theta^0_{C,t-1},\theta^0_{N,t-1},\tilde{I}_t,\tilde{h}_t) \end{bmatrix} - A_t \begin{bmatrix} \theta^0_{C,t-1} \\ \theta^0_{N,t-1} \end{bmatrix} - C_t \begin{bmatrix} \tilde{I}_t \\ \tilde{h}_t \end{bmatrix}$$

and

$$u_t \equiv \begin{bmatrix} \eta_{C,t} \\ \eta_{N,t} \end{bmatrix}$$

Compute the reduced form coefficients A_t, C_t, and b_t for $t = 1, ..., T$.

Step 3:

Define the value function in each period:

$$V_T = \beta E_{T-1}[-(\theta_T - a_T)'Q_T(\theta_T - a_T) - x'_T R_T x_T]$$

and

$$V_t = \beta E_{t-1}\left[-(\theta_t - a_t)'Q_t(\theta_t - a_t) - x'_t R_t x_t + V_{t+1}\right]$$

for $t = 1, ..., T-1$.

So the value function at time t, V_t, can be interpreted as the present value of future returns evaluated at time $t-1$.

Step 4:

Consider the last period first. In period T, the optimization problem is:

$$\max_{x_T} V_T = \beta E_{T-1}[-\theta'_T Q_T\theta_T + 2\theta'_T Q_T a_T - a'_T Q_T a_T - x'_T R_T x_T]$$

$$s.t.\ \theta_T = A_T\theta_{T-1} + C_T x_T + b_T + u_T$$

Define

$$H_T \equiv -Q_T$$

and

$$h_T \equiv -Q_T a_T$$

and

$$c_T \equiv -a'_T Q_T a_T$$

and

$$J_T \equiv -R_T$$

Compute the coefficients: $H_T,\ h_T,\ c_T,\ J_T$.

<u>Step 5:</u>

Recall that the optimization problem in period T is:

$$\begin{aligned}\max_{x_T}\ V_T = \beta E_{T-1}[\theta_T' H_T \theta_T - 2\theta_T' h_T + c_T + x_T' J_T x_T] \\ s.t.\ \theta_T = A_T\theta_{T-1} + C_T x_T + b_T + u_T\end{aligned}$$

Substituting the law of motion into the objective function, the value function in period T becomes:

$$\begin{aligned} V_T &= \beta E_{T-1}[(A_T\theta_{T-1} + C_T x_T + b_T + u_T)' H_T (A_T\theta_{T-1} + C_T x_T + b_T + u_T) - 2(A_T\theta_{T-1} + C_T x_T + b_T + u_T)' h_T + c_T + x_T' J_T x_T] \\ &= \beta E_{T-1}[x_T' C_T' H_T C_T x_T + x_T' C_T' H_T (A_T\theta_{T-1} + b_T + u_T) + (A_T\theta_{T-1} + b_T + u_T)' H_T C_T x_T + \ldots \\ &\quad + (A_T\theta_{T-1} + b_T + u_T)' H_T (A_T\theta_{T-1} + b_T + u_T) - 2x_T' C_T' h_T - 2(A_T\theta_{T-1} + b_T + u_T)' h_T + c_T + x_T' J_T x_T] \end{aligned}$$

Then, the first order condition with respect to x_T is

$$\begin{aligned} \frac{\partial V_T}{\partial x_T} &= \beta E_{T-1}[2C_T' H_T C_T \hat{x}_T + 2C_T' H_T (A_T\theta_{T-1} + b_T + u_T) - 2C_T' h_T + 2J_T \hat{x}_T] = 0 \\ & \beta[2C_T' H_T C_T \hat{x}_T + 2C_T' H_T (A_T\theta_{T-1} + b_T) - 2C_T' h_T + 2J_T x_T] \end{aligned}$$

If we assume that $E_{T-1}[u_T] = 0$, then:

$$C_T' H_T C_T \hat{x}_T + C_T' H_T (A_T\theta_{T-1} + b_T) - C_T' h_T + J_T \hat{x}_T = 0$$

So:

$$\begin{aligned} \hat{x}_T &= (C_T' H_T C_T + J_T)^{-1}(C_T' h_T) - (C_T' H_T C_T + J_T)^{-1} C_T' H_T' (A_T\theta_{T-1} + b_T) \\ &= -(C_T' H_T C_T + J_T)^{-1} C_T' H_T' A_T \theta_{T-1} + (C_T' H_T C_T + J_T)^{-1}(C_T' h_T) - (C_T' H_T C_T + J_T)^{-1} C_T' H_T' b_T \end{aligned}$$

Define:

$$\begin{aligned} G_T &\equiv -(C_T' H_T C_T + J_T)^{-1} C_T' H_T' A_T \\ g_T &\equiv (C_T' H_T C_T + J_T)^{-1}(C_T' h_T) - (C_T' H_T C_T + J_T)^{-1} C_T' H_T' b_T \end{aligned}$$

So,

$$\hat{x}_T = G_T\theta_{T-1} + g_T$$

Compute G_T and g_T.
Substitute $\hat{x}_T$ into the law of motion, we get:

$$\begin{aligned} \hat{\theta}_T &= A_T\theta_{T-1} + C_T(G_T\theta_{T-1} + g_T) + b_T + u_T \\ &= (A_T + C_T G_T)\theta_{T-1} + C_T g_T + b_T + u_T \end{aligned}$$

And substitute $\hat{\theta}_T$ into the objective function, we get:

$$\begin{aligned}
\hat{V}_T &= \beta E_{T-1}[((A_T + C_T G_T)\theta_{T-1} + C_T g_T + b_T + u_T)' H_T((A_T + C_T G_T)\theta_{T-1} + C_T g_T + b_T + u_T) + ... \\
&\quad -2((A_T + C_T G_T)\theta_{T-1} + C_T g_T + b_T + u_T)' h_T + c_T + \hat{x}_T' J_T \hat{x}_T] \\
&= \beta E_{T-1}[\theta_{T-1}'(A_T + C_T G_T)' H_T (A_T + C_T G_T)\theta_{T-1} + 2\theta_{T-1}'(A_T + C_T G_T)' H_T (C_T g_T + b_T) + ... \\
&\quad +(C_T g_T + b_T + u_T)' H_T (C_T g_T + b_T + u_T) - 2\theta_{T-1}'(A_T + C_T G_T)' h_T + ... \\
&\quad -2(C_T g_T + b_T)' h_T + c_T + (G_T \theta_{T-1} + g_T)' J_T (G_T \theta_{T-1} + g_T)] \\
&= \beta[\theta_{T-1}' \left((A_T + C_T G_T)' H_T (A_T + C_T G_T) + G_T' J_T G_T\right) \theta_{T-1} + ... \\
&\quad +2\theta_{T-1}' \left((A_T + C_T G_T)' H_T (C_T g_T + b_T) - (A_T + C_T G_T)' h_T + G_T' J_T g_T\right) + ... \\
&\quad +(C_T g_T + b_T)' H_T (C_T g_T + b_T) - 2(C_T g_T + b_T)' h_T + c_T + g_T' J_T g_T] + \beta tr(H_T E_{T-1}(u_T u_T'))
\end{aligned}$$

Here the assumption $E_{T-1}[u_T] = 0$ has been applied again.

Step 6:

The maximization problem in period $T-1$ is:

$$\max_{x_{T-1}} V_{T-1} = \beta E_{T-2}\left[-(\theta_{T-1} - a_{T-1})' Q_{T-1}(\theta_{T-1} - a_{T-1}) - x_{T-1}' R_{T-1} x_{T-1} + \hat{V}_T\right]$$
$$s.t. \;\; \theta_{T-1} = A_{T-1}\theta_{T-2} + C_{T-1} x_{T-1} + b_{T-1} + u_{T-1}$$

Substituting $\hat{V}_T$ into the objective function, the value function in period $T-1$ is:

$$\begin{aligned}
V_{T-1} &= \beta E_{T-2}\{-\theta_{T-1}' Q_{T-1}\theta_{T-1} + 2\theta_{T-1}' Q_{T-1} a_{T-1} - a_{T-1}' Q_{T-1} a_{T-1} - x_{T-1}' R_{T-1} x_T + ... \\
&\quad +\beta\theta_{T-1}' \left((A_T + C_T G_T)' H_T (A_T + C_T G_T) + G_T' J_T G_T\right) \theta_{T-1} \\
&\quad +2\beta\theta_{T-1}' \left((A_T + C_T G_T)' H_T (C_T g_T + b_T) - (A_T + C_T G_T)' h_T + G_T' J_T g_T\right) \\
&\quad +\beta \left((C_T g_T + b_T)' H_T (C_T g_T + b_T) - 2(C_T g_T + b_T)' h_T + c_T + g_T' J_T g_T\right) \\
&\quad +\beta tr(H_T E_{T-1}(u_T u_T'))\}
\end{aligned}$$

Define:

$$\begin{aligned}
H_{T-1} &\equiv -Q_{T-1} + \beta\left((A_T + C_T G_T)' H_T (A_T + C_T G_T) + G_T' J_T G_T\right) \\
h_{T-1} &\equiv -Q_{T-1} a_{T-1} - \beta\left((A_T + C_T G_T)' H_T (C_T g_T + b_T) - (A_T + C_T G_T)' h_T + G_T' J_T g_T\right) \\
c_{T-1} &\equiv -a_{T-1}' Q_{T-1} a_{T-1} + \beta\left((C_T g_T + b_T)' H_T (C_T g_T + b_T) - 2(C_T g_T + b_T)' h_T + c_T + g_T' J_T g_T\right) + \beta tr(H_T E_{T-1}(u_T u_T')) \\
J_{T-1} &\equiv -R_{T-1}
\end{aligned}$$

Note that H_{T-1} is also symmetric.
Compute $H_{T-1},\ h_{T-1},\ c_{T-1},\ J_{T-1}$ accordingly.
Then the maximization problem can be represented as follows:

$$\max_{x_{T-1}} V_{T-1} = \beta E_{T-2}[\theta_{T-1}' H_{T-1}\theta_{T-1} - 2\theta_{T-1}' h_{T-1} + c_{T-1} + x_{T-1}' J_{T-1} x_{T-1}]$$
$$s.t. \; \theta_{T-1} = A_{T-1}\theta_{T-2} + C_{T-1} x_{T-1} + b_{T-1} + u_{T-1}$$

This maximization problem is exactly in the same format as in period T. To solve it, we can simply repeat step 5.

Step 7:

Repeat steps 4 and 5 for each period, we get $H_t,\ h_t,\ C_t,\ J_t,\ G_t$ and g_t for each period t.

Step 8:

Plugging in G_t and g_t and using the initial endowments of the children, we could back out $\hat{x}_1, ..., \hat{x}_T$.

Step 9:

Use the results in step 8 as the initial guesses and repeat the whole process until $\hat{x}_1, ..., \hat{x}_T$ converge.

The optimal regulator problem with a quadratic return function and linear transition functions

Consider the Cobb-Douglas human capital production functions, which are similar to the ones used in Del Boca, Flinn and Wiswall (2012):

$$\begin{aligned} k_{C,t} &= \beta_{C,t}(k_{C,t-1})^{\alpha_{C,t,1}}(k_{N,t-1})^{\alpha_{C,t,2}}(I_t)^{\alpha_{C,t,3}}(h_t)^{\alpha_{C,t,4}}exp(\epsilon_{C,t}) \\ k_{N,t} &= \beta_{N,t}(k_{C,t-1})^{\alpha_{N,t,1}}(k_{N,t-1})^{\alpha_{N,t,2}}(I_t)^{\alpha_{N,t,3}}(h_t)^{\alpha_{N,t,4}}exp(\epsilon_{N,t}) \end{aligned}$$

Then taking logarithm on both sides, we have:

$$\theta_t = A_t\theta_{t-1} + B_t x_t + b_t + \eta_t$$

where θ_t is defined as the logarithm of the vector of skills at time t, i.e. $\theta_t = (\theta_{C,t}, \theta_{N,t}) = (lnk_{C,t}, lnk_{N,t})$. And:

$$\begin{aligned} A_t &\equiv \begin{bmatrix} \alpha_{C,t,1} & \alpha_{C,t,2} \\ \alpha_{N,t,1} & \alpha_{N,t,2} \end{bmatrix} \\ B_t &\equiv \begin{bmatrix} \alpha_{C,t,3} & \alpha_{C,t,4} \\ \alpha_{N,t,3} & \alpha_{N,t,4} \end{bmatrix} \\ b_t &\equiv \begin{bmatrix} ln\beta_{C,t} \\ ln\beta_{N,t} \end{bmatrix} \\ \eta_t &\equiv \begin{bmatrix} \epsilon_{C,t} \\ \epsilon_{N,t} \end{bmatrix} \end{aligned}$$

It is clear that this setup is simply a special case of the nonlinear setup discussed above, and thus we could apply the same procedures as above only by skipping steps 1,2 and 9.