$$\theta_{r+1} = \left[\alpha I_r^{\phi} + (1-\alpha)\,\theta_r^{\phi}\right]^{\frac{\rho}{\phi}} \qquad (1)$$

$$\frac{\partial \theta_{r+1}}{\partial I_r} = \frac{\rho}{\cancel{\phi}}\left[\alpha I_r^{\phi} + (1-\alpha)\,\theta_r^{\phi}\right]^{\frac{\rho}{\phi}-1} \cancel{\phi}\,\alpha I_r^{\phi-1}$$

$$= \rho\left[\alpha I_r^{\phi} + (1-\alpha)\,\theta_r^{\phi}\right]^{\frac{\rho}{\phi}-1} \alpha I_r^{\phi-1}$$

$$\frac{\partial^2 \theta_{r+1}}{\partial I_r \partial \theta_r} = \rho\left[\alpha I_r^{\phi} + (1-\alpha)\,\theta_r^{\phi}\right]^{\frac{\rho}{\phi}-2} \phi\left(\frac{\rho}{\phi}-1\right)(1-\alpha)\,\theta_r^{\phi-1}\,\alpha I_r^{\phi-1}$$

$$= \rho(\rho-\phi)\left[\alpha I_r^{\phi} + (1-\alpha)\,\theta_r^{\phi}\right]^{\frac{\rho}{\phi}-2}(1-\alpha)\,\theta_r^{\phi-1}\,\alpha I_r^{\phi-1}$$

< 0 if $\phi > \rho$ (edg. substitutes)

> 0 if $\phi < \rho$ (edg. complements)

$$\frac{\partial^2 \theta_{r+1}}{\partial I_r \partial I_r} = \rho \left[\alpha I_r^{\phi} + (1-\alpha) \theta_r^{\phi} \right]^{\frac{\rho}{\phi}-2} \phi \left(\frac{\rho}{\phi} - 1 \right) \alpha I_r^{\phi-1} \alpha I_r^{\phi-1}$$

$$+ \rho \left[\alpha I_r^{\phi} + (1-\alpha) \theta_r^{\phi} \right]^{\frac{\rho}{\phi}-1} \alpha (\phi - 1) I_r^{\phi-2}$$

$$= \rho (\rho - \phi) \left[\alpha I_r^{\phi} + (1-\alpha) \theta_r^{\phi} \right]^{\frac{\rho}{\phi}-2} \alpha^2 I_r^{2\phi-2}$$

$$+ \rho \left[\alpha I_r^{\phi} + (1-\alpha) \theta_r^{\phi} \right]^{\frac{\rho}{\phi}-1} \alpha (\phi - 1) I_r^{\phi-2}$$

$$= \left(\rho \left[\alpha I_r^{\phi} + (1-\alpha) \theta_r^{\phi} \right]^{\frac{\rho}{\phi}-1} \alpha I_r^{\phi-1} \right) \cdot$$

$$\left[(\rho - \phi) \underbrace{\frac{\alpha I_r^{\phi}}{\alpha I_r^{\phi} + (1-\alpha) \theta_r^{\phi}}}_{= \Gamma < 1} \frac{1}{I_r} - (1-\phi) \frac{1}{I_r} \right]$$

$$= \left(\rho \left[\alpha I_r^{\phi} + (1-\alpha) \theta_r^{\phi} \right]^{\frac{\rho}{\phi}-1} \alpha I_r^{\phi-1} \right) \cdot$$

$$\left[\frac{1}{I_r} \left(\Gamma (\rho - \phi) - (1 - \phi) \right) \right]$$

we need :

$$I \left(\rho - \phi\right) < \left(1 - \phi\right)$$

given $\rho \leq 1$

$\phi \leq 1$

$$\Rightarrow \left(1 - \phi\right) \geq 0$$

I believe the above is always true independently if $\rho \geq \phi$.