Adaptive Information Processing

Model complexity and the MDL principle

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Global overview

Prerequisites

Additional reading

Introduction

Bishop §1.2: Probability Theory
Bishop §1.3: Model Selection
Bishop §1.4: The Curse of Dimensionality

Probabilities

Bishop §2.1: Binary Variables
Bishop §2.2: Multinomial Variables

Part 2-A

The Bayesian Information Criterion

Parameter and model estimation

Additional reading

Introduction

Bishop §3.3: Bayesian Linear Regression

Bishop §3.4: Bayesian Model Comparison

Parameter estimation

Define our variables!

Model	$\mathcal{M}_i$	model prior	$p(\mathcal{M}_i)$
Parameters	θ_i	parameter prior	$p(\theta_i \vert \mathcal{M}_i)$
Data	x^N		

The model $\mathcal{M}$ is the *structure* of the random process that generated the data.

The vector θ is the *parameter vector* of that process.

Parameter estimation

It is easy to talk about e.g.

> "the probability of the data, given the model structure but not the parameters"

$$p(x^N|\mathcal{M}) = \int_\Theta p(\theta|\mathcal{M}) p(x^N|\mathcal{M}, \theta)\, d\theta.$$

Also we can write the posterior distribution of the parameters:

$$p(\theta|x^N, \mathcal{M}).$$

We can treat the model structure $\mathcal{M}$ as a random variable.

Parameter estimation

i.i.d. Gaussian structure $\mathcal{M}_A$, $\theta_A = (m, s)$, *mean* m and *variance* s.

i.i.d. Gamma structure $\mathcal{M}_B$, $\theta_B = (\alpha, \beta)$, *shape* α and *rate* β.

$$p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

We can now write

$$p(x^N|\mathcal{M}_A) = \int_{\Theta_A} p(\theta_a|\mathcal{M}_A) p(x^N|\mathcal{M}_A, \theta_A)\, d\theta_A$$

$$P(\mathcal{M}_A|x^N) = \frac{P(\mathcal{M}_A) p(x^N|\mathcal{M}_A)}{p(x^N)},$$

where

$$p(x^N) = P(\mathcal{M}_A) p(x^N|\mathcal{M}_A) + P(\mathcal{M}_B) p(x^N|\mathcal{M}_B).$$

A-posteriori parameter distribution

$$p(\theta_i|\mathcal{M}_i, x^N) = \frac{p(\theta_i|\mathcal{M}_i)p(x^N|\mathcal{M}_i, \theta_i)}{p(x^N|\mathcal{M}_i)}$$

$$p(x^N|\mathcal{M}_i) = \int_{\Theta_i} p(\theta_i|\mathcal{M}_i)p(x^N|\mathcal{M}_i, \theta_i)\, d\theta_i$$

Parameter estimation

Maximum Likelihood

We want a point estimate for θ_i (given $\mathcal{M}_i$).

$$\hat{\theta}_i = \arg\max_{\theta_i} p(\theta_i|\mathcal{M}_i, x^N) = \arg\max_{\theta_i} p(x^N|\mathcal{M}_i, \theta_i)$$

Where we assume a uniform prior or want to work without priors.

Model estimation

A-posteriori model distribution

$$p(\mathcal{M}_i|x^N) = \frac{p(\mathcal{M}_i)p(x^N|\mathcal{M}_i)}{p(x^N)}$$

$$p(x^N) = \int_{\mathfrak{M}_i} p(\mathcal{M}_i)p(x^N|\mathcal{M}_i)\, d\mathcal{M}_i$$

Model estimation

Maximum Likelihood

We want a point estimate for $\mathcal{M}$ (MAP or ML).

$$\hat{\mathcal{M}} = \arg\max_{\mathcal{M}_i} p(\mathcal{M}_i|x^N) = \arg\max_{\mathcal{M}_i} p(x^N|\mathcal{M}_i)$$

Where we assume a uniform prior or want to work without priors.

Model estimation

We need to compute

$$p(x^N|\mathcal{M}_i) = \int_{\Theta_i} p(\theta_i|\mathcal{M}_i)p(x^N|\mathcal{M}_i,\theta_i)\,d\theta_i$$

Often $p(\theta_i|\mathcal{M}_i, x^N)$ is sharply peaked and because

$$p(\theta_i|\mathcal{M}_i, x^N) \propto p(\theta_i|\mathcal{M}_i)p(x^N|\mathcal{M}_i,\theta_i),$$

we might be able to approximate the integrand given above.

Maximum Likelihood and Overfitting

Additional reading

Overfitting
Bishop §1.1: Example: Polynomial Curve Fitting

Attempt 1 (Maximum Likelihood)

We approximate the integral by the peak of the integrand, θ_i^{MAP}. For convenience we often use θ_i^{ML}.

$$p(\theta_i|\mathcal{M}_i)p(x^N|\mathcal{M}_i,\theta_i) \approx \delta(\theta_i - \theta_i^{\text{ML}})p(\theta_i|\mathcal{M}_i)p(x^N|\mathcal{M}_i,\theta_i)$$

and find

$$p(x^N|\mathcal{M}_i) \propto p(\theta_i^{\text{ML}}|\mathcal{M}_i)p(x^N|\mathcal{M}_i,\theta_i^{\text{ML}})$$

So we end up with (with or without model prior)

$$\mathcal{M}^{\text{MAP}} = \arg\max_{\mathcal{M}_i} p(\mathcal{M}_i)p(\theta_i^{\text{MAP}}|\mathcal{M}_i)p(x^N|\mathcal{M}_i,\theta_i^{\text{MAP}})$$
$$\mathcal{M}^{\text{ML}} = \arg\max_{\mathcal{M}_i} p(x^N|\mathcal{M}_i,\theta_i^{\text{ML}})$$

Attempt 1: an example

Consider a linear regression model with, in principle, k parameters.

$$y_n = \theta^T \underline{x}_n + n_n;$$
$$y_n \in \mathbb{R}; \qquad \theta \in \mathbb{R}^k; \qquad \underline{x}_n \in \mathbb{R}^k; \qquad n_n \sim \mathcal{N}(0,\sigma^2)$$

Observe: $(y_1, \underline{x}_1), (y_2, \underline{x}_2), \ldots, (y_N, \underline{x}_N)$.
ML estimate: $\hat{\theta} = (X^TX)^{-1}X^T\underline{y}$.
Matrix: $X = [\underline{x}_1, \underline{x}_2, \ldots \underline{x}_N]^T$.
Models: $\mathcal{M} \subset \{1, 2, \ldots, k\}$. e.g.

$$\mathcal{M} = \{1, 3\}; \qquad y_n = \theta_1 x_{n1} + \theta_3 x_{n3} + n_n$$

Attempt 1: an example (continued)

$N = 50; \quad \underline{x} \in [0,1]^3; \quad \theta = (0, 0.6, 0);$

$\sigma^2 = 1$ actual $\sigma^2 = 0.799$

$\mathcal{M}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\sigma}^2$	$\ln P_{ML}$ $\sigma^2 = 1$	$\ln P_{ML}$ $\sigma^2 = \hat{\sigma}^2$
$\{\}$	0	0	0	0.949	−69.675	−69.642
$\{1\}$	0.690	0	0	0.804	−66.040	−65.485
$\{2\}$	0	0.604	0	0.799	−65.934	−65.352
$\{3\}$	0	0	0.307	0.912	−68.738	−68.635
$\{12\}$	0.379	0.361	0	0.780	−65.441	−64.728
$\{13\}$	1.171	0	−0.522	0.766	−65.099	−64.286
$\{23\}$	0	0.970	−0.472	0.766	−65.101	−64.287
$\{123\}$	0.908	0.752	−0.940	0.686	−63.097	−61.525

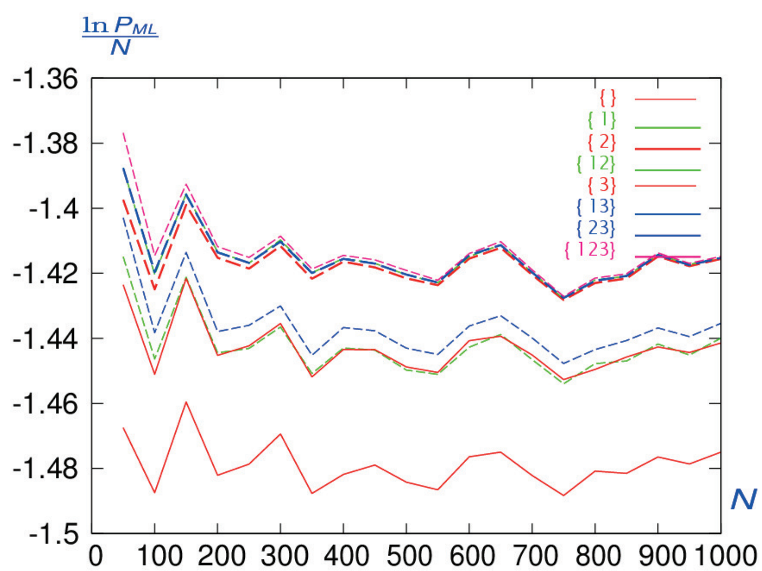
Attempt 1: an example (continued)

$N = 1000; \quad \underline{x} \in [0,1]^3; \quad \theta = (0, 0.6, 0);$

$\sigma^2 = 1$ actual $\sigma^2 = 1.015$

$\mathcal{M}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\sigma}^2$	$\ln P_{ML}$ $\sigma^2 = 1$	$\ln P_{ML}$ $\sigma^2 = \hat{\sigma}^2$
$\{\}$	0	0	0	1.144	−1491	−1486
$\{1\}$	0.435	0	0	1.083	−1460	−1459
$\{2\}$	0	0.619	0	1.015	−1426	−1426
$\{3\}$	0	0	0.507	1.058	−1448	−1447
$\{12\}$	−0.099	0.693	0	1.013	−1425	−1425
$\{13\}$	0.105	0	0.430	1.056	−1447	−1446
$\{23\}$	0	0.549	0.095	1.013	−1426	−1426
$\{123\}$	−0.173	0.622	0.167	1.010	−1424	−1424

Attempt 1: an example (continued)



Attempt 1: an example (continued)

From the graph we conclude that the "only noise" model $\mathcal{M} = \{\}$ has the worst performance, and that all models that include the actual parameter θ_2, i.e. $\{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ perform almost the same and the most complex of these, $\{1,2,3\}$, performs the best but is clearly an unwanted over-estimation.

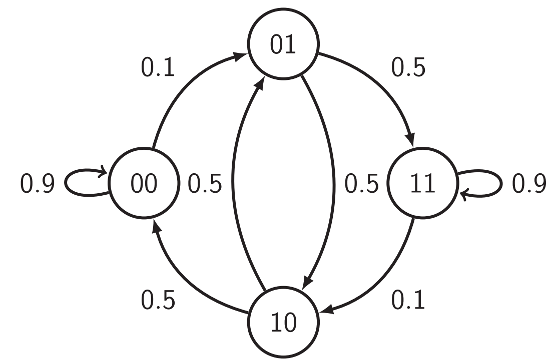
Attempt 1: another example

A discrete data example.

Consider a binary second order Markov process:
$\Pr\{X_i = 1|x^{i-1}\} = \Pr\{X_i = 1|x_{i-2}x_{i-1}\}$.
So, it is actually a set of four i.i.d. sub-sources. The ML estimate of an i.i.d. binary source:

$$\begin{aligned} n(s|x) &= \text{the number of times } s \in \mathcal{X}^* \text{ occurs in } x \\ p(x^N|\theta) &= (1-\theta)^{n(0|x^N)}\theta^{n(1|x^N)} \\ \frac{\partial}{\partial\theta}\ln p(x^N|\theta) &= \frac{n(1|x^N) - N\theta}{\theta(1-\theta)} = 0 \\ \hat{\theta} &= \frac{n(1|x^N)}{N} \end{aligned}$$

Attempt 1: another example (ctd.)



Attempt 1: another example (ctd.)

Let S be the state variable of an m-th order Markov source, so $S_i = X_{i-m}\dots X_{i-1}$ and $l(S_i) = m$ bits, then

$$\theta_s = \Pr\{X_i = 1|S_i = s\}$$

The Maximum Likelihood estimator is

$$\hat{\theta}_s = \frac{n(s1|x^N)}{n(s0|x^N) + n(s1|x^N)}$$

With this we find the ML probability for x^N (with initial state ς, see next slide)

$$p(x^N|m, \underline{\hat{\theta}}, \varsigma) = \prod_{s\in\{0,1\}^m} \left\{\hat{\theta}_s^{n(s1|x^N)}(1-\hat{\theta}_s)^{n(s0|x^N)}\right\}$$

Intermezzo: Initial state

The state variable of a m-th order Markov source is defined by the most recent m past symbols of the source. Initially, we cannot know the source state because we haven't seen m or more symbols yet.
So we assume knowledge of some initializing symbols that help in defining the first m values of the state variable. We will denote these initial symbols by ς and usually leave them unspecified.

Intermezzo: Initial state

This also implies that when we use the count function $n(s0|x^N)$ we imply the use of ς, e.g. let $x^5 = 10110$, $m = 2$, and $\varsigma = 01$. We wish to count the number of ones in state $s = 01$.

$$n(s1|x^5) = 2$$

We consider the concatenation of ς and x^5: 01 101100
and count the number of occurrences of $s1 = 011$ in that string.

Intermezzo: Initial state

s	$n(s0\|x^5)$	$n(s1\|x^5)$
00	0	0
01	0	2
10	0	1
11	2	0

So indeed, we count 2 zeros and 3 ones.

Attempt 1: another example (ctd.)

```
octave:1> mytest(50,[0.1,0.5,0.5,0.9],4)
ML models sequence logprobs:
Order 0:  logpr = -34.657359
Order 1:  logpr = -14.546445
Order 2:  logpr = -14.883390
Order 3:  logpr = -15.185437
Order 4:  logpr = -15.444986

octave:2> mytest(200,[0.1,0.5,0.5,0.9],4)
ML models sequence logprobs:
Order 0:  logpr = -137.416984
Order 1:  logpr = -102.949521
Order 2:  logpr = -87.992931
Order 3:  logpr = -84.732718
Order 4:  logpr = -80.546002
```


Attempt 1: conclusion

Obviously, this method does not work.

- Any model that includes the actual model assigns essentially the same probability to the data.

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- But high order models cannot predict well (too restricted).
- The higher order models are too well tuned.

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- We observe that (usually) the higher order models give higher probabilities to the sequence.
- But high order models cannot predict well (too restricted).
- The higher order models are too well tuned.

This is undesirable, the estimated model adapts itself to the noise and the resulting model is an over estimation of the actual model. model.

Preventing Overfitting

Additional reading Laplace Approximation

Bishop §4.4: The Laplace Approximation

Attempt 2 (Laplace approximation)

We approximate the integrand by a Gaussian around the peak. The mean and variance of the Gaussian are determined by the integrand.
This approximation turns out to give more interesting results.

Laplace approximation

Suppose we have an arbitrary non-negative real function $f(z)$, where z is a k-dimensional vector. We need an estimate of the normalizing constant Z_f.

$$Z_f = \int f(z)\, dz$$

Let z_0 be a maximum of $f(z)$. Use the Taylor expansion.

$$\ln f(z) \approx \ln f(z_0) - \frac{1}{2}(z - z_0)A(z - z_0)$$
$$A_{ij} = -\frac{\partial^2}{\partial z_i \partial z_j} \ln f(z)\Big|_{z=z_0}$$

Laplace approximation

Approximate $f(z)$ by the unnormalized Gaussian

$$g(z) = f(z_0) \exp\left(-\frac{1}{2}(z - z_0)A(z - z_0)\right)$$

A, not necessarily good, approximation of Z_f is

$$Z_f \approx Z_g = \int g(z)\, dz = f(z_0)\sqrt{\frac{(2\pi)^k}{\det A}}$$

Laplace approximation

Example

$$f(z) = \frac{1}{z^2+1} \quad \text{Has maximum at } z_0 = 0.$$

$$Z_f = \pi$$

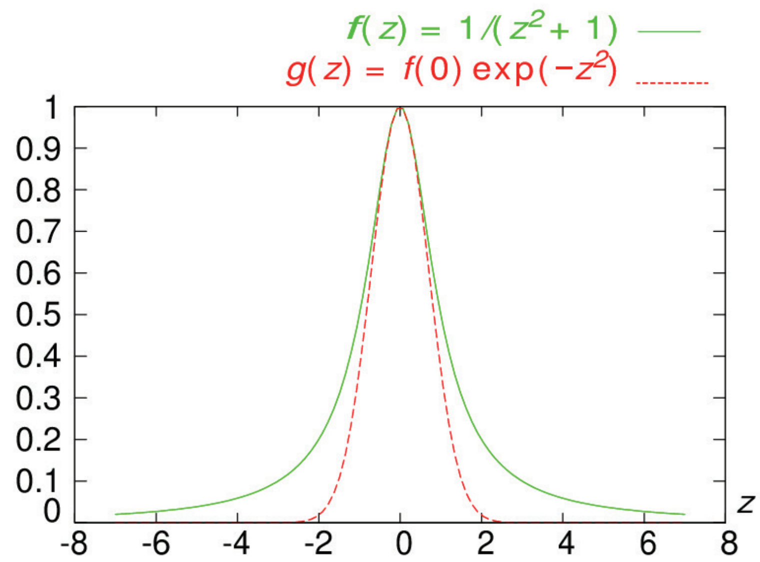
$$A = -\frac{\partial^2}{\partial z^2} \ln f = -\frac{f'' f - f'^2}{f^2}$$

$$f(0) = 1; \quad f'(0) = 0; \quad f''(0) = -2; \text{ so } A = 2$$

$$g(z) = f(0) \exp\left(-\frac{1}{2} z A z\right) = e^{-z^2}$$

$$Z_g = \sqrt{\pi}$$

Laplace approximation



Attempt 2 (Laplace approximation)

Consider again $p(x^N|\mathcal{M}_i)$.

$$p(x^N|\mathcal{M}_i) = \int_\Theta p(\theta_i|\mathcal{M}_i) p(x^N|\mathcal{M}_i, \theta_i)\, d\theta_i$$

We again use the fact that

$$p(\theta_i|\mathcal{M}_i, x^N) \propto p(\theta_i|\mathcal{M}_i) p(x^N|\mathcal{M}_i, \theta_i)$$

is often sharply peaked, say at $\hat{\theta}_i$. Using the Laplace approximation we may write

$$p(x^N|\mathcal{M}_i) \approx \sqrt{\frac{(2\pi)^k}{\det A}}\, p(\hat{\theta}_i|\mathcal{M}_i) p(x^N|\mathcal{M}_i, \hat{\theta}_i)$$

Attempt 2 (Laplace approximation)

Comparing two models give

$$\frac{p(\mathcal{M}_i|x^N)}{p(\mathcal{M}_j|x^N)} \approx \frac{p(\mathcal{M}_i)}{p(\mathcal{M}_j)} \frac{\sqrt{\frac{(2\pi)^{k_i}}{\det A_i}}\, p(\hat{\theta}_i|\mathcal{M}_i)}{\sqrt{\frac{(2\pi)^{k_j}}{\det A_j}}\, p(\hat{\theta}_j|\mathcal{M}_j)} \frac{p(x^N|\mathcal{M}_i, \hat{\theta}_i)}{p(x^N|\mathcal{M}_j, \hat{\theta}_j)}$$

Attempt 2 (Laplace approximation)

Comparing two models give

$$\frac{p(\mathcal{M}_i|x^N)}{p(\mathcal{M}_j|x^N)} \approx \frac{p(\mathcal{M}_i)}{p(\mathcal{M}_j)} \frac{\sqrt{\frac{(2\pi)^{k_i}}{\det A_i}} p(\hat{\theta}_i|\mathcal{M}_i)}{\sqrt{\frac{(2\pi)^{k_j}}{\det A_j}} p(\hat{\theta}_j|\mathcal{M}_j)} \frac{p(x^N|\mathcal{M}_i, \hat{\theta}_i)}{p(x^N|\mathcal{M}_j, \hat{\theta}_j)}$$

Initial model preference

Attempt 2 (Laplace approximation)

Comparing two models give

$$\frac{p(\mathcal{M}_i|x^N)}{p(\mathcal{M}_j|x^N)} \approx \frac{p(\mathcal{M}_i)}{p(\mathcal{M}_j)} \frac{\sqrt{\frac{(2\pi)^{k_i}}{\det A_i}} p(\hat{\theta}_i|\mathcal{M}_i)}{\sqrt{\frac{(2\pi)^{k_j}}{\det A_j}} p(\hat{\theta}_j|\mathcal{M}_j)} \frac{p(x^N|\mathcal{M}_i, \hat{\theta}_i)}{p(x^N|\mathcal{M}_j, \hat{\theta}_j)}$$

Cost of (number of) parameters

Attempt 2 (Laplace approximation)

Comparing two models give

$$\frac{p(\mathcal{M}_i|x^N)}{p(\mathcal{M}_j|x^N)} \approx \frac{p(\mathcal{M}_i)}{p(\mathcal{M}_j)} \frac{\sqrt{\frac{(2\pi)^{k_i}}{\det A_i}} p(\hat{\theta}_i|\mathcal{M}_i)}{\sqrt{\frac{(2\pi)^{k_j}}{\det A_j}} p(\hat{\theta}_j|\mathcal{M}_j)} \frac{p(x^N|\mathcal{M}_i, \hat{\theta}_i)}{p(x^N|\mathcal{M}_j, \hat{\theta}_j)}$$

Likelihood ratio

Attempt 2 (Laplace approximation)

Comparing two models give

$$\frac{p(\mathcal{M}_i|x^N)}{p(\mathcal{M}_j|x^N)} \approx \frac{p(\mathcal{M}_i)}{p(\mathcal{M}_j)} \frac{\sqrt{\frac{(2\pi)^{k_i}}{\det A_i}} p(\hat{\theta}_i|\mathcal{M}_i)}{\sqrt{\frac{(2\pi)^{k_j}}{\det A_j}} p(\hat{\theta}_j|\mathcal{M}_j)} \frac{p(x^N|\mathcal{M}_i, \hat{\theta}_i)}{p(x^N|\mathcal{M}_j, \hat{\theta}_j)}$$

Cost factors are: initial model preference, cost of (number of) parameters, likelihood ratio.

This is ML model estimation, it works because we consider the model complexity also!

BIC: Bayesian Information Criterion

A more refined approximation (Schwartz criterion or Bayesian Information Criterion) gives

$$\log \frac{p(\mathcal{M}_i|x^N)}{p(\mathcal{M}_j|x^N)} \approx \log \frac{p(\mathcal{M}_i)}{p(\mathcal{M}_j)} + \log \frac{p(x^N|\mathcal{M}_i, \hat{\theta}_i)}{p(x^N|\mathcal{M}_j, \hat{\theta}_j)} - \frac{1}{2}(k_i - k_j) \log N,$$

where k_i resp. k_j gives the number of free parameters in model $\mathcal{M}_i$ or $\mathcal{M}_j$ respectively.

This BIC is widely applied and turned out to be very useful.

What happens when we apply the correction term $\frac{k}{2} \log N$? We shall revisit the two examples.

Example 1 with BIC correction

$$N = 50; \quad \underline{x} \in [0,1]^3; \quad \theta = (0, 0.6, 0);$$
$$\sigma^2 = 1 \quad \text{actual } \sigma^2 = 0.852$$

$\mathcal{M}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\sigma}^2$	$\ln P_{BIC}$
$\{\}$	0	0	0	1.068	−72.653
$\{1\}$	0.699	0	0	0.909	−70.632
$\{2\}$	0	0.773	0	0.841	−68.923
$\{3\}$	0	0	0.572	0.944	−71.491
$\{12\}$	0.159	0.662	0	0.837	−70.790
$\{13\}$	0.553	0	0.172	0.905	−72.478
$\{23\}$	0	0.811	−0.050	0.840	−70.869
$\{123\}$	0.240	0.728	−0.159	0.834	−72.670

Example 1 with BIC correction

$$N = 1000; \quad \underline{x} \in [0,1]^3; \quad \theta = (0, 0.6, 0);$$
$$\sigma^2 = 1 \quad \text{actual } \sigma^2 = 0.977$$

$\mathcal{M}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\sigma}^2$	$\ln P_{BIC}$
$\{\}$	0	0	0	1.077	−1457.2
$\{1\}$	0.411	0	0	1.022	−1433.4
$\{2\}$	0	0.551	0	0.976	−1410.3
$\{3\}$	0	0	0.362	1.034	−1439.3
$\{12\}$	−0.017	0.564	0	0.976	−1413.7
$\{13\}$	0.315	0	0.128	1.020	−1435.6
$\{23\}$	0	0.637	−0.117	0.974	−1412.7
$\{123\}$	0.040	0.620	−0.133	0.974	−1416.1

Example 2 with BIC correction

```
octave:1> mytest(50,[0.1,0.5,0.5,0.9],4)
Parameter scaled ML log probabilities:
Order 0:  logpr = -36.613371
Order 1:  logpr = -18.458468
Order 2:  logpr = -22.707436
Order 3:  logpr = -30.833529
Order 4:  logpr = -46.741170

octave:2> mytest(200,[0.1,0.5,0.5,0.9],4)
Parameter scaled ML log probabilities:
Order 0:  logpr = -140.066143
Order 1:  logpr = -108.247838
Order 2:  logpr = -98.589565
Order 3:  logpr = -105.925987
Order 4:  logpr = -122.932541
```

BIC correction

The examples indicate that the correct model (order) is recovered, basically by using an ML selection criterion with an additional penalty term for the model complexity.

However, this BIC is derived as an approximation to the true Bayesian a-posteriori probability.

A better justification for the BIC should exist!

Part 2-B

Bayesian model estimation and the Context-tree model selection

Bayesian model estimation

Additional reading

Bishop §14.1: Bayesian Model Averaging
Bishop §14.4: Tree-based Models

Bayesian model estimation

Additional notation

i^{th} order Markov	$\mathcal{M}_i$	The state is determined by the previous i symbols.
Parameter vector	θ_i	This vector describes all probabilities $p(X_n \vert X_{n-i}, X_{n-i+1}, \ldots, X_{n-1})$.
Parameter element	$\theta_{i,s}$	$\theta_{i,s} = p(X_n \vert X_{n-i}^{n-1} = s)$.
Model state	s	s is a binary sequence of length i.
Initial state	ς	ς is also a binary sequence of length i.

Probabilities for i.i.d. sequences

Let $\mathcal{M}$, with a single real valued parameter θ, assign probabilities to binary sequences, so it generates sequences x^N, where $x_i \in \{0,1\}$ for all i, $1 \le i \le N$. The parameter $\theta \in [0,1]$ describes the probability of the i.i.d. variables X_i, or

$$\Pr\{X_i = 1\} = \theta.$$

So we define a probability function $P(x)$ as,

$$P(0) = 1 - \theta,$$
$$P(1) = \theta.$$

From the i.i.d. assumption follows

$$P(x^N|\mathcal{M},\theta) = \Pr\{X^N = x^N\},$$
$$= \prod_{i=1}^{N} P(x_i).$$

Probabilities for i.i.d. sequences

Make the knowledge of the model $\mathcal{M}$ and parameter θ explicit: $P(x^N|\mathcal{M},\theta)$.

Question: What is a good probability to assign to x^N if θ is not known?
Solution: We will take a Bayesian approach, assuming a prior distribution $p(\theta)$ over θ and calculating the resulting marginal sequence probability,

$$P(x^n|\mathcal{M}) = \int_{\Theta} p(\theta)P(x^N|\mathcal{M},\theta)\,d\theta.$$

Bayesian model estimation

Example (Revisit first lecture)

Let $\mathcal{M}_i$ be the i-th order binary Markov model (source).
Then $\theta_i \in \Theta_i = [0,1]^{2^i}$ and $\theta_{i,s} = p(X_n = 1|x_{n-i}^{n-1} = s)$. So in the binary case the parameter vector per state $\theta_{i,s}$ describes the probability of a '1' in that state.
Beta distribution for prior $p(\theta_i|\mathcal{M}_i)$, with $\alpha = \beta = \frac{1}{2}$ (Jeffreys prior).

$$p(\theta_i|\mathcal{M}_i,\varsigma) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^{2^i} \prod_{s\in\{0,1\}^i} \theta_{i,s}^{\alpha-1}(1-\theta_{i,s})^{\beta-1}$$
$$= \frac{1}{\pi^{2^i}} \prod_{s\in\{0,1\}^i} \theta_{i,s}^{-1/2}(1-\theta_{i,s})^{-1/2}$$

Bayesian model estimation

$$p(x^N|\mathcal{M}_i,\varsigma) = \int_{\Theta_i} p(\theta_i|\mathcal{M}_i)p(x^N|\mathcal{M}_i,\theta_i)\,d\theta_i$$
$$= \frac{1}{\pi^{2^i}} \int_{\Theta_i} \prod_{s\in\{0,1\}^i} \theta_{i,s}^{n(s1|x^N)-1/2}(1-\theta_{i,s})^{n(s0|x^N)-1/2}\,d\theta_i$$
$$= \frac{1}{\pi^{2^i}} \prod_{s\in\{0,1\}^i} \int_{[0,1]} \theta_{i,s}^{n(s1|x^N)-1/2}(1-\theta_{i,s})^{n(s0|x^N)-1/2}\,d\theta_{i,s}$$
$$= \prod_{s\in\{0,1\}^i} \frac{\Gamma(n(s0|x^N)+\frac{1}{2})\Gamma(n(s1|x^N)+\frac{1}{2})}{\pi\Gamma(n(s0|x^N)+n(s1|x^N)+1)}$$

Bayesian model estimation

So we must study the behaviour of

$$P_e(a,b) \triangleq \frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}{\pi\Gamma(n+1)}$$
$$a \triangleq n(0|x^N)$$
$$b \triangleq n(1|x^N)$$

It is a memoryless sub-sources of the Markov source.
x^N is generated i.i.d. with parameter θ.

The actual probability of x^N under this source is

$$p(x^N|\mathcal{M},\theta) = (1-\theta)^a\theta^b$$

Bayesian model estimation

We can write

$$P_e(a,b) = \frac{\frac{1}{2}\frac{3}{2}\cdots(a-\frac{1}{2})\cdot\frac{1}{2}\frac{3}{2}\cdots(b-\frac{1}{2})}{(a+b)!}$$

We can also consider the conditional probabilities, $P_e(1|a,b)$, the probability of a "1" following a zeros and b ones, and $P_e(0|a,b)$, the probability of a "0" following a zeros and b.

$$P_e(1|a,b) = \frac{P_e(a,b+1)}{P_e(a,b)} = \frac{b+\frac{1}{2}}{a+b+1}$$
$$P_e(0|a,b) = \frac{P_e(a+1,b)}{P_e(a,b)} = \frac{a+\frac{1}{2}}{a+b+1}$$

Bayesian model estimation

e.g. consider the sequence $x^7 = 0110010$. It contains $a=4$ zeros and $b=3$ ones, so (with some abuse of notation)

$$\begin{aligned}
P_e(x^7) = P_e(4,3) &= \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{7}{2}\frac{1}{2}\frac{3}{2}\frac{5}{2}}{7!} \\
&= \frac{5}{2048}. \\
P_e(x^7) &= P_e(x_1)P_e(x_2|x_1)\cdots P_e(x_7|x_1^6) \\
&= P_e(0|0,0)P_e(1|1,0)P_e(1|1,1)P_e(0|1,2) \\
&\quad P_e(0|2,2)P_e(1|3,2)P_e(0|3,3) \\
&= \frac{\frac{1}{2}}{1}\frac{\frac{1}{2}}{2}\frac{\frac{3}{2}}{3}\frac{\frac{3}{2}}{4}\frac{\frac{5}{2}}{5}\frac{\frac{5}{2}}{6}\frac{\frac{7}{2}}{7} = \frac{5}{2048}.
\end{aligned}$$

Bayesian model estimation

Again with the help of Stirling's approximation we can derive, for large a and b the following. (Excercise). Note: $a+b=N$.

$$\log_2\frac{p(x^N|\mathcal{M},\theta)}{P_e(a,b)} \le \frac{1}{2}\log_2 N + \frac{1}{2}\log_2\frac{\pi}{2}$$

Actually, we can prove that for all $a \ge 0$ and $b \ge 0$

$$\log_2\frac{p(x^N|\mathcal{M},\theta)}{P_e(a,b)} \le \frac{1}{2}\log_2 N + 1$$

Bayesian model estimation

Back to the i-th order Markov source.

$$p(x^N|\mathcal{M}_i,\theta_i,\varsigma) = \prod_{s\in\{0,1\}^i} \theta_{i,s}^{n(s1|x^N)}(1-\theta_{i,s})^{n(s0|x^N)}$$

$$p(x^N|\mathcal{M}_i,\varsigma) = \prod_{s\in\{0,1\}^i} P_e(n(s1|x^N), n(s0|x^N))$$

Bayesian model estimation

With the previous bound we find

$$\begin{aligned}\log_2 \frac{p(x^N|\mathcal{M}_i,\theta_i,\varsigma)}{p(x^N|\mathcal{M}_i,\varsigma)} &= \\ &\log_2 \frac{\prod_{s\in\{0,1\}^i} \theta_{i,s}^{n(s1|x^N)}(1-\theta_{i,s})^{n(s0|x^N)}}{\prod_{s\in\{0,1\}^i} P_e(n(s1|x^N), n(s0|x^N))} \\ &= \sum_{s\in\{0,1\}^i} \log_2 \frac{\theta_{i,s}^{n(s1|x^N)}(1-\theta_{i,s})^{n(s0|x^N)}}{P_e(n(s1|x^n), n(s0|x^N))} \\ &\le \sum_{s\in\{0,1\}^i} \frac{1}{2}\log_2 n(s|x^N) + 1 \overset{*1}{\le} \frac{2^i}{2}\log_2 \frac{N-i}{2^i} + 2^i\end{aligned}$$

(*1): Here we use Jensen's inequality.

Bayesian model estimation

So we conclude that for any parameter vector θ_i we have (approximately!) [From now on we do not explicitly write ς anymore]

$$\log_2 p(x^N|\mathcal{M}_i) \approx \log_2 p(x^N|\mathcal{M}_i,\theta_i) - \frac{2^i}{2}\log_2\frac{N-i}{2^i} - 2^i$$

Maximum Likelihood parameters (and $N \gg \max\{2^i, 2^j\}$)

$$\log_2 \frac{p(\mathcal{M}_i|x^N)}{p(\mathcal{M}_j|x^N)} \approx \log_2 \frac{p(\mathcal{M}_i)}{p(\mathcal{M}_j)} + \log_2 \frac{p(x^N|\mathcal{M}_i,\hat{\theta}_i)}{p(x^N|\mathcal{M}_j,\hat{\theta}_j)} - \frac{2^i-2^j}{2}\log_2 N$$

So, again we observe the parameter cost!

Context trees

Recap: Memoryless binary source: one parameter $\theta = \Pr\{X=1\}$

Recap: Markov order-k: one parameter per state. There are 2^k states. The k symbols $x_{i-k},\dots,x_{i-1}$ form the context of the symbol x_i.

Real world models: Length of context depends on its contents. e.g. Natural language (English, Dutch, $\cdots$): if context starts with $x_{i-1}=$'q' then no more symbols are needed.

Context trees

A tree source is a nice concept to describe such sources.

A tree source consists of a set $\mathcal{S}$ of suffixes that together form a tree.
To each suffix (leaf) s in the tree there corresponds a parameter θ_s.

Some more notation: By $x_{|s}^N$ we denote the sub-sequence of symbols from x^N that are preceded by the sequence s.

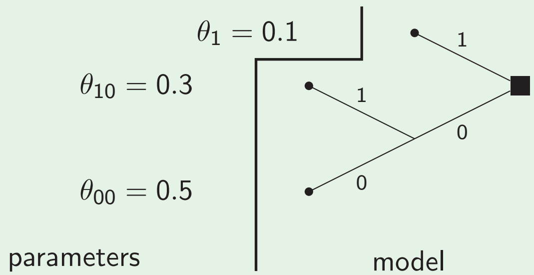
Example: $x^8 = 01011010$; $s = 01$; then $x_{|01}^8 = x_3x_5x_8 = 010$.

Context trees

Example

Let $\mathcal{S} \triangleq \{00, 10, 1\}$ and $\theta_{00} = 0.5, \theta_{10} = 0.3$, and $\theta_1 = 0.1$ then

$$\begin{aligned} \Pr\{X_i = 1 | \cdots, x_{i-2} = 0, x_{i-1} = 0\} &= 0.5, \\ \Pr\{X_i = 1 | \cdots, x_{i-2} = 1, x_{i-1} = 0\} &= 0.3, \\ \Pr\{X_i = 1 | \cdots, x_{i-1} = 1\} &= 0.1. \end{aligned}$$



Context trees

Just as before ("Bayesian model estimation") we must estimate the sequence probabilities of the memoryless subsources that correspond to the leaves of the tree (states of the source).
Let the full sequence be x^N and the subsequence for state s be written as $x_{|s}^N$. Before we wrote

$$P_e(a, b) = \frac{\Gamma(a + \frac{1}{2})\Gamma(b + \frac{1}{2})}{\pi\Gamma(a + b + 1)}$$

Context trees

We shall now use the shorthand notation for the estimated probability of the subsequence generated in state s given the full sequence x^i:

$$P_e(a_s, b_s) = \frac{\Gamma(a_s + \frac{1}{2})\Gamma(b_s + \frac{1}{2})}{\pi\Gamma(a_s + b_s + 1)}$$

$$\begin{aligned} a_s &= n(s0|x^N) = n(0|x_{|s}^N) \\ b_s &= n(s1|x^N) = n(1|x_{|s}^N) \end{aligned}$$

Context trees

Let $\mathcal{S} = \{00, 10, 1\}$. Assume we observe $x^7 = 0\,1\,0\,0\,1\,1\,0$ with an initial past ς that ends with $\cdots\,1\,1\,0$.

		ς						x^7			
	$\cdots$	1	1	0	0	1	0	0	1	1	0
	$\cdots 00$					x			x		
past	$\cdots 10$				x			x			
	$\cdots 1$						x			x	x

Now sorting the symbols of x^7 according to the relevant past symbols we get

$$\begin{aligned} P_e(0100110|\cdots 110) &= P_e(0|0{,}0)P_e(1|0{,}0)P_e(0|0{,}0)P_e(0|1{,}0)P_e(1|0{,}1)P_e(1|1{,}0)P_e(1|1{,}1) \\ &= \underbrace{P_e(00)}_{10}\,\underbrace{P_e(11)}_{00}\,\underbrace{P_e(010)}_{1} \\ &= \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6} = \frac{9}{1024} \end{aligned}$$

Context trees

See "Bayesian Estimation"

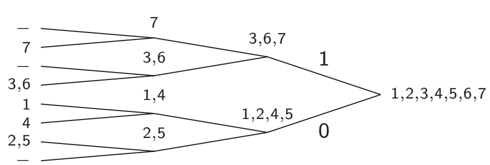
$$\log_2 \frac{p(x^N|\mathcal{S},\theta)}{\prod_{s\in\mathcal{S}} P_e(a_s, b_s)} \le \frac{|\mathcal{S}|}{2}\log_2\frac{N}{|\mathcal{S}|} + |\mathcal{S}|$$

Context trees

Problem: We do not know $\mathcal{S}$!

Context tree (of depth D)

		ς						x^7			
$\cdots$	1	1	0	0	1	0	0	1	1	0
	index			1	2	3	4	5	6	7



In every node of the tree we write the indices of the symbols that have the corresponding past symbols.
In every node s we keep track of the counts $a_s = n(s0|x^N)$ and $b_s = n(s1|x^N)$.

Context trees

We shall assign a probability to the subsequence $x^N_{|s}$ for every state s in the context tree.

We shall do this in such a way that in the root of the tree we assign a probability to the whole sequence x^N that is a mixture of all possible tree sources.

We use the following observations to build, recursively, this probability.

The probability we build is written as follows

$$P_w^s = P_w(x^N_{|s}),$$

where P_w^s is the shorthand notation we shall use.

Later we return to this and make the notation more precise.

Context trees

Suppose s is a leaf:
All we know are a_s and b_s so we assign the subsequence probability

$$P_w^s = P_e(a_s, b_s).$$

Now if s is an internal node, we have two options for the subsequence probability.

1: $P_e(a_s, b_s)$.
2: $P_w^{0s} P_w^{1s}$.

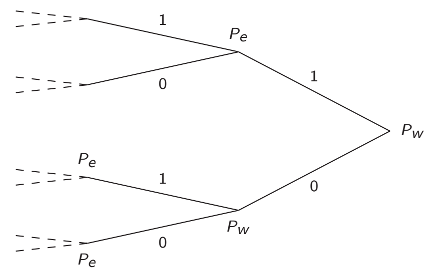
We must make a choice or better even, mix these options. So we set

$$P_w^s = \frac{P_e(a_s, b_s) + P_w^{0s} P_w^{1s}}{2}.$$

Context trees

Analysis.

Let $\mathcal{S} = \{00, 10, 1\}$ and we use a context tree with depth $D > 2$.
We look at the P_w's for different nodes.
For the nodes $s \in \mathcal{S}$ we consider (in the analysis) only the P_e's.



$$P_w^{00} \geq \frac{1}{2} P_e(a_{00}, b_{00})$$
$$P_w^{10} \geq \frac{1}{2} P_e(a_{10}, b_{10})$$
$$P_w^{1} \geq \frac{1}{2} P_e(a_{1}, b_{1})$$

Context trees

Now we consider nodes nearer to the root and take only the $P_w^{0s} P_w^{1s}$ part.

$$\begin{aligned} P_w^0 &\geq \frac{1}{2} P_w^{00} P_w^{10} \\ &\geq \frac{1}{8} P_e(a_{00}, b_{00}) P_e(a_{10}, b_{10}) \\ P_w^\lambda &\geq \frac{1}{2} P_w^0 P_w^1 \\ &\geq \frac{1}{32} P_e(a_{00}, b_{00}) P_e(a_{10}, b_{10}) P_e(a_1, b_1) \end{aligned}$$

Here λ denotes the root of the tree.

Context trees

For general trees (or suffix sets) $\mathcal{S}$ we find

$$P_w^\lambda \geq 2^{1-2|\mathcal{S}|} \prod_{s \in \mathcal{S}} P_e(a_s, b_s)$$

So

$$\log_2 P_w^\lambda \geq \log_2 p(x^N | \mathcal{S}, \theta) - \left(2|\mathcal{S}| - 1 + \frac{|\mathcal{S}|}{2} \log_2 N + |\mathcal{S}| \right).$$

Context trees

For general trees (or suffix sets) $\mathcal{S}$ we find

$$P_w^\lambda \geq 2^{1-2|\mathcal{S}|} \prod_{s \in \mathcal{S}} P_e(a_s, b_s)$$

So

$$\log_2 P_w^\lambda \geq \log_2 p(x^N|\mathcal{S},\theta) - \left(2|\mathcal{S}| - 1 + \frac{|\mathcal{S}|}{2}\log_2 N + |\mathcal{S}|\right).$$

Real sequence probability

Context trees

For general trees (or suffix sets) $\mathcal{S}$ we find

$$P_w^\lambda \geq 2^{1-2|\mathcal{S}|} \prod_{s \in \mathcal{S}} P_e(a_s, b_s)$$

So

$$\log_2 P_w^\lambda \geq \log_2 p(x^N|\mathcal{S},\theta) - \left(2|\mathcal{S}| - 1 + \frac{|\mathcal{S}|}{2}\log_2 N + |\mathcal{S}|\right).$$

Cost of describing the tree

Context trees

For general trees (or suffix sets) $\mathcal{S}$ we find

$$P_w^\lambda \geq 2^{1-2|\mathcal{S}|} \prod_{s \in \mathcal{S}} P_e(a_s, b_s)$$

So

$$\log_2 P_w^\lambda \geq \log_2 p(x^N|\mathcal{S},\theta) - \left(2|\mathcal{S}| - 1 + \frac{|\mathcal{S}|}{2}\log_2 N + |\mathcal{S}|\right).$$

Cost of the parameters

Context trees

For general trees (or suffix sets) $\mathcal{S}$ we find

$$P_w^\lambda \geq 2^{1-2|\mathcal{S}|} \prod_{s \in \mathcal{S}} P_e(a_s, b_s)$$

So

$$\log_2 P_w^\lambda \geq \log_2 p(x^N|\mathcal{S},\theta) - \left(2|\mathcal{S}| - 1 + \frac{|\mathcal{S}|}{2}\log_2 N + |\mathcal{S}|\right).$$

Contributions to the weighted probability are: Real sequence probability; Cost of describing the tree; Cost of the parameters

Context trees

This algorithm achieves the (asymptotically) optimal log-likelihood ratio (not only on the average but also individually for every data sequence).

$$\log \frac{p(x^N|\mathcal{S},\theta)}{P_w^\lambda} \leq 2|\mathcal{S}| - 1 + \frac{|\mathcal{S}|}{2}\log_2 N + |\mathcal{S}|.$$

Another essential property of the "Context-Tree Weighting" (CTW) algorithm is its efficient implementation. The number of trees squares with every increment of D and yet the amount of work is at most linear in $D \cdot N$.

Context trees

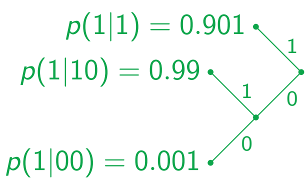
We can even write a stronger result when we realise that the method has no knowledge of a "real model".
Let $\mathbb{S}_D$ be the set of all tree models with a maximal depth of atmost D.

$$\log P_w^\lambda \geq \max_{\mathcal{S}\in\,\mathbb{S}_D} \Big\{ \log p(x^N|\mathcal{S},\theta) - \Big(2|\mathcal{S}| - 1 + \frac{|\mathcal{S}|}{2}\log_2 N + |\mathcal{S}|\Big)\Big\}.$$

This algorithm is an instantiation of the MDL principle. It finds (in the class $\mathbb{S}_D$) the model $\mathcal{S}$ that maximizes the sequence probability.

Context trees

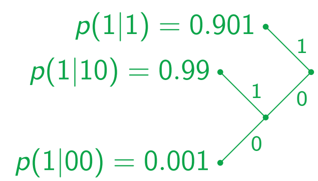
Example: Consider the following actual model.



We shall use the following models.

Context trees

Example: Consider the following actual model.



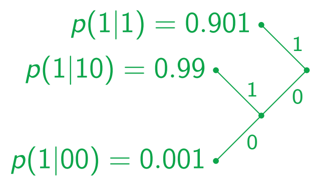
We shall use the following models.

$p(1)$ •

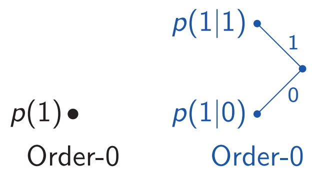
Order-0

Context trees

Example: Consider the following actual model.



We shall use the following models.



Context trees

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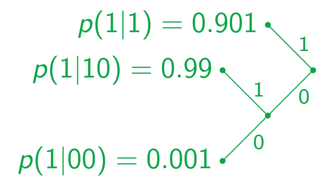

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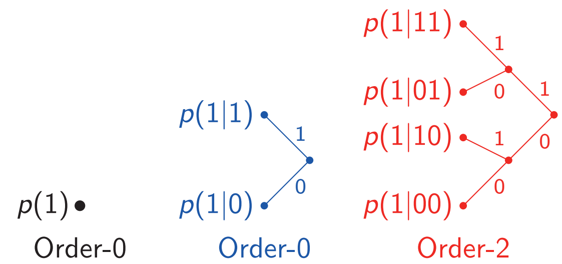
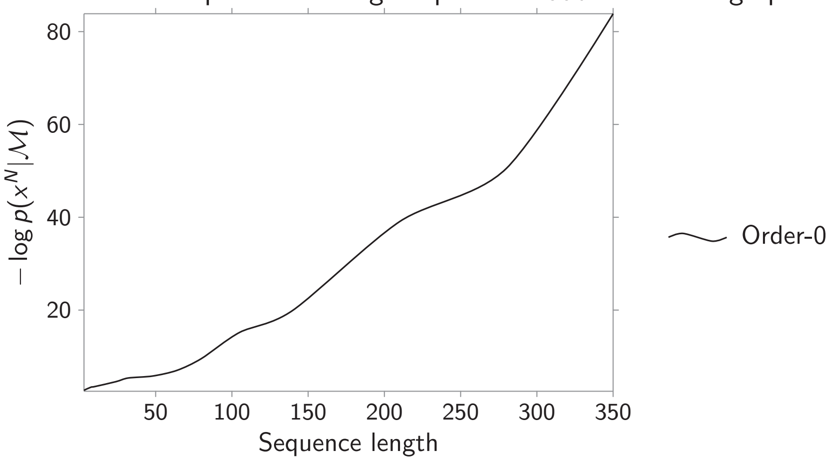


Context trees

The results for sequences of length upto $N = 350$ are shown graphically.



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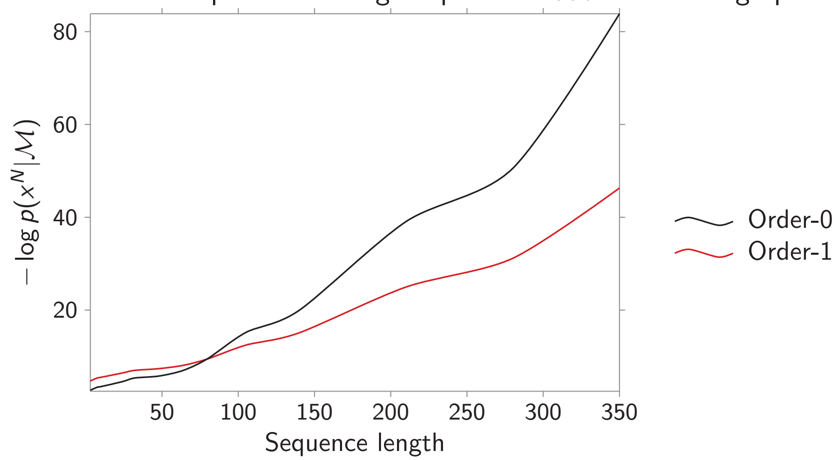

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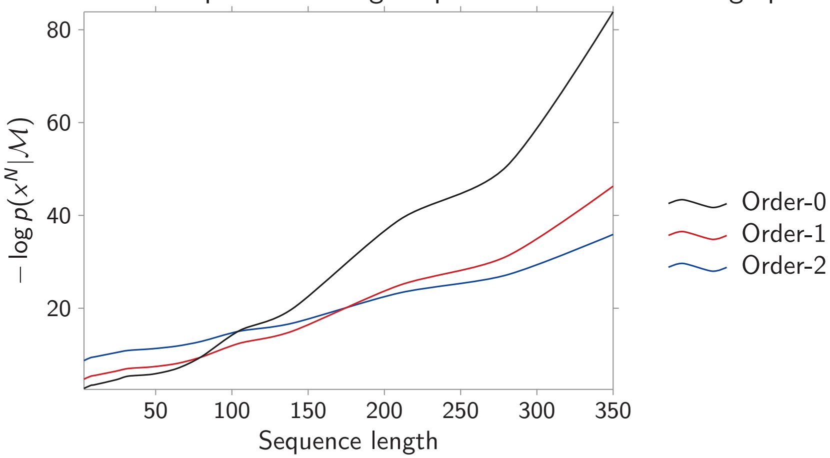

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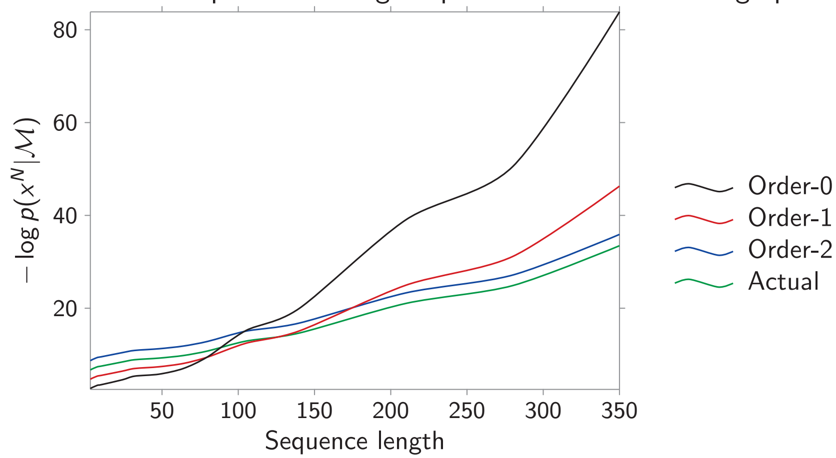

Context trees

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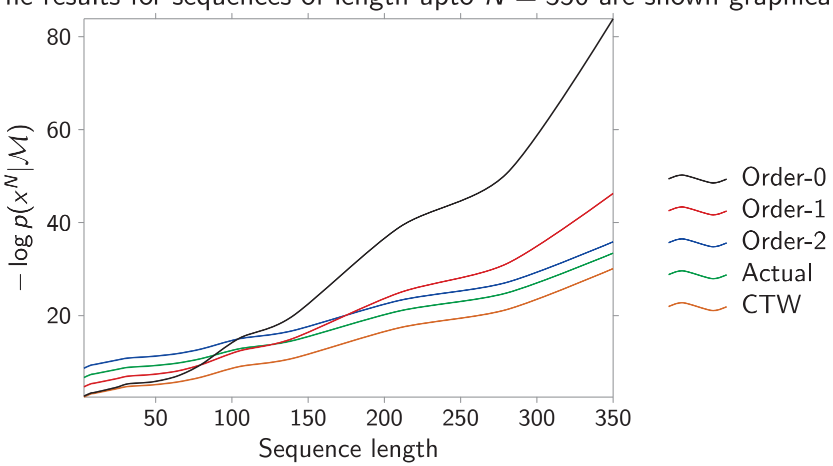


Context trees

We see that initially the memoryless (order-0) model performs even better than the actual model.

After about 80 symbols the order-1 model becomes better than both the order-0 and the actual model.

From 120 symbols on the actual model is better than the simpeler models.

The order-2 model is always worse than the actual model. It describes the same probabilities but has too many parameters.

But the CTW model outperforms all models over the whole range of sequence lengths!

Model posterior for Context trees

We shall now derive an expression, based on the previous method, for the a-posteriori model probability. We consider only binary data but the approach also works for arbitrary alphabets.

First we repeat our notation.

A model is described by a complete suffix set $\mathcal{S}$.

The suffix set can be seen as the set of leaves of a binary tree. Our model class is the set of all complete binary trees whose depth is not more than D, for a given D. We write $\mathbb{S}_D$ for the model class. So we say that $\mathcal{S} \in \mathbb{S}_D$.
The depth of a tree is the length of the longest path from the root to a leaf.

Model posterior for Context trees

Every model $\mathcal{S}$ has a set of parameters θ_s, one for every state $s \in \mathcal{S}$ of the model. θ_s gives the probability of a 1 given that the previous symbols were s.

$$\theta_s = \Pr\{X_t = 1 | X_{t-\ell}^{t-1} = s\}, \text{ where } \ell = |s|$$

Model posterior for Context trees

The probability of a sequence, given a model $\mathcal{S}$ with parameters θ_s, $s \in \mathcal{S}$ is

$$p(x^N | \mathcal{S}, \theta) = \prod_{s \in \mathcal{S}} p(x_{|s}^N | \theta_s)$$

and

$$p(x_{|s}^N | \theta_s) = (1 - \theta_s)^{n(0|x_{|s}^N)} \theta_s^{n(1|x_{|s}^N)}$$

Note (again) that $n(0|x_{|s}^N) = n(s0|x^N)$.
Actually, we silently assume that the first few symbols also have a "context". So we assume that there are some symbols preceding x^N.

Model posterior for Context trees

We must define some prior distributions. First the prior on the parameters. We use the beta distribution. (In a non-binary case this generalizes to the Dirichlet distribution.) As done before we select the parameters in the beta distribution to be $\frac{1}{2}$.
So given a model $\mathcal{S}$ then for every $s \in \mathcal{S}$ we take

$$p(\theta_s | \mathcal{S}) = \frac{1}{\pi} \theta_s^{-\frac{1}{2}} (1 - \theta_s)^{-\frac{1}{2}}$$

Model posterior for Context trees

This results in the following sequence probability, first assuming one state s only

$$p(x^N_{|s}) = \int_0^1 p(\theta_s|\mathcal{S})\theta_s^{n(s1|x^N)}(1-\theta_s)^{n(s0|x^N)}\,d\theta_s$$
$$= \frac{\Gamma(n(s0|x^N)+\frac{1}{2})\Gamma(n(s1|x^N)+\frac{1}{2})}{\pi\Gamma(n(s|x^N)+1)}$$

Now for any tree model $\mathcal{S}$ we find

$$p(x^N|\mathcal{S}) = \prod_{s\in\mathcal{S}} \int_0^1 p(\theta_s|\mathcal{S})p(x^N_{|s}|\theta_s)\,d\theta_s$$
$$= \prod_{s\in\mathcal{S}} \frac{\Gamma(n(s0|x^N)+\frac{1}{2})\Gamma(n(s1|x^N)+\frac{1}{2})}{\pi\Gamma(n(s|x^N)+1)}$$

Model posterior for Context trees

Next we need a prior on the tree models $\mathcal{S}$ in the set $\mathbb{S}_D$. We wish to use the efficient CTW method of weighting so we choose the corresponding prior.

First define
$$\Delta_D(S) \triangleq 2|\mathcal{S}| - 1 - |\{s \in \mathcal{S} : |s| = D\}|.$$
Then we take the prior
$$p(\mathcal{S}) = 2^{-\Delta_D(\mathcal{S})}$$

We prove that this is a proper prior probability.

Model posterior for Context trees

Obviously, $p(\mathcal{S}) > 0$ for all $\mathcal{S} \in \mathbb{S}_D$. We must show that it sums up to one.

We give a proof by induction.

Step 1: $D = 0$: $\mathbb{S}_0 = \{\lambda\}$, the memoryless source.

$$\Delta_0(\lambda) = 2\cdot 1 - 1 - 1$$

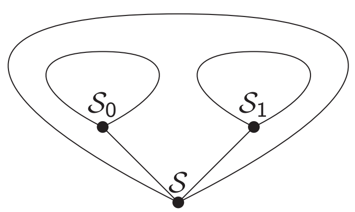
Where the last -1 comes from the fact that the single state of λ is at level $D = 0$ so $p(\lambda) = 1$.

Model posterior for Context trees

Induction: Assume it holds for $D \le D^*$. Now if $\mathcal{S} \in \mathbb{S}_{D^*+1}$ then

- $\mathcal{S} = \lambda$, i.e. root node only.
- $\mathcal{S}$ contains two trees on level 1, say $\mathcal{S}_0 \in \mathbb{S}_{D^*}$ and $\mathcal{S}_1 \in \mathbb{S}_{D^*}$. We have

$$\Delta_{D^*+1}(\mathcal{S}) = 1 + \Delta_{D^*}(\mathcal{S}_0) + \Delta_{D^*}(\mathcal{S}_1)$$



Model posterior for Context trees

- We repeat: $\mathcal{S}$ contains two trees on level 1, say $\mathcal{S}_0 \in \mathbb{S}_{D^*}$ and $\mathcal{S}_1 \in \mathbb{S}_{D^*}$. We have

$$\Delta_{D^*+1}(\mathcal{S}) = 1 + \Delta_{D^*}(\mathcal{S}_0) + \Delta_{D^*}(\mathcal{S}_1)$$

$$\begin{aligned}\sum_{\mathcal{S}\in \mathbb{S}_{D^*+1}} 2^{-\Delta_{D^*+1}(\mathcal{S})} &= 2^{-1} + \\ &\quad \sum_{\mathcal{S}_0\in \mathbb{S}_{D^*}} \sum_{\mathcal{S}_1\in \mathbb{S}_{D^*}} 2^{-1-\Delta_{D^*}(\mathcal{S}_0)-\Delta_{D^*}(\mathcal{S}_1)} \\ &= 2^{-1} + 2^{-1} \underbrace{\sum_{\mathcal{S}_0\in \mathbb{S}_{D^*}} 2^{-\Delta_{D^*}(\mathcal{S}_0)}}_{=1} \underbrace{\sum_{\mathcal{S}_1\in \mathbb{S}_{D^*}} 2^{-\Delta_{D^*}(\mathcal{S}_1)}}_{=1} \\ &= 2^{-1} + 2^{-1} = 1\end{aligned}$$

Model posterior for Context trees

We now show that the weighted sequence probability

$$p(x^N) = \sum_{\mathcal{S}\in \mathbb{S}_D} p(\mathcal{S})p(x^N|\mathcal{S}),$$

is produced by the weighting procedure of CTW, so

$$p(x^N) = P_w^\lambda.$$

Model posterior for Context trees

We shall prove this using (mathematical) induction.

First assume $D = 0$: $\mathbb{S}_0 = \{\lambda\}$, so the only tree in the set consists of a root only. Therefor $\Delta_0(\lambda) = 0$. So,

$$\begin{aligned}p(x^N) &= p(\lambda)p(x^N|\lambda) \\ &= 2^0 P_e(n(0|x^N), n(1|x^N)) \\ &= P_w^\lambda,\end{aligned}$$

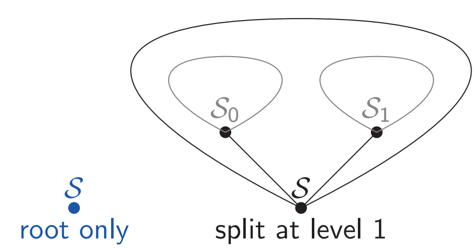
because λ is also a leaf and in a leaf $P_w = P_e$.

Model posterior for Context trees

Now assume that for all $D \le D^*$

$$\sum_{\mathcal{S}\in \mathbb{S}_D} p(\mathcal{S})p(x^N|\mathcal{S}) = P_w^\lambda$$

The tree $\mathcal{S}$ is either the root only or it consists of a root plus two trees, $\mathcal{S}_0$ and $\mathcal{S}_1$, on level one.



Model posterior for Context trees

$$\sum_{\mathcal{S}\in \mathbb{S}_{D^*+1}} p(\mathcal{S})p(x^N|\mathcal{S}) =$$
$$= 2^{-1}P_e(n(0|x^N), n(1|x^N))+$$
$$\sum_{\mathcal{S}\in \mathbb{S}_{D^*+1}:\mathcal{S}\neq\lambda} p(\mathcal{S})p(x^N|\mathcal{S})$$

Model posterior for Context trees

$$\sum_{\mathcal{S}\in \mathbb{S}_{D^*+1}:\mathcal{S}\neq\lambda} p(\mathcal{S})p(x^N|\mathcal{S}) =$$
$$\sum_{\mathcal{S}_0\in \mathbb{S}_{D^*}} \sum_{\mathcal{S}_1\in \mathbb{S}_{D^*}} \frac{1}{2}2^{-\Delta_{D^*}(\mathcal{S}_0)}2^{-\Delta_{D^*}(\mathcal{S}_1)}\times$$
$$p(x^N_{|0}|\mathcal{S}_0)p(x^N_{|1}|\mathcal{S}_1)$$
$$= \frac{1}{2}\sum_{\mathcal{S}_0\in \mathbb{S}_{D^*}} 2^{-\Delta_{D^*}(\mathcal{S}_0)}p(x^N_{|0}|\mathcal{S}_0)\times$$
$$\sum_{\mathcal{S}_1\in \mathbb{S}_{D^*}} 2^{-\Delta_{D^*}(\mathcal{S}_1)}p(x^N_{|1}|\mathcal{S}_1)$$
$$= \frac{1}{2}P_w^0 P_w^1$$

Model posterior for Context trees

And so we find

$$\sum_{\mathcal{S}\in \mathbb{S}_{D^*+1}} p(\mathcal{S})p(x^N|\mathcal{S}) =$$
$$= \frac{1}{2}P_e(n(0|x^N), n(1|x^N)) + \frac{1}{2}P_w^0 P_w^1$$
$$= P_w^\lambda$$

Model posterior for Context trees

Thus we can compute the a-posteriori model probability.

$$p(\mathcal{S}|x^N) = \frac{p(\mathcal{S})p(x^N|\mathcal{S})}{p(x^N)}$$

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Thus we can compute the a-posteriori model probability.

$$p(\mathcal{S}|x^N) = \frac{2^{-\Delta_D(\mathcal{S})}p(x^N|\mathcal{S})}{p(x^N)}$$

Model posterior for Context trees

Thus we can compute the a-posteriori model probability.

$$p(\mathcal{S}|x^N) = \frac{2^{-\Delta_D(\mathcal{S})}\prod_{s\in\mathcal{S}} P_e(n(s0|x^N), n(s1|x^N))}{p(x^N)}$$

Model posterior for Context trees

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$$p(\mathcal{S}|x^N) = \frac{2^{-\Delta_D(\mathcal{S})} \prod_{s\in\mathcal{S}} P_e(n(s0|x^N), n(s1|x^N))}{P_w^{\lambda}}$$

So, we can use the same computations as in the CTW.

An efficient way to find the Bayesian MAP model exists, but its discussion is not a part of this course.

Part 2-C

Descriptive complexity

Descriptive complexity

How difficult is it to describe this sequence?
01010101010101010101010101010101010101010101010101

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25 repetitions of “01”, simple

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10110101000001001111001100110011111110011101111001
The first 50 fractional bits of $1/\sqrt{2}$, simple

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50 coin tosses, complex

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50 coin tosses, complex

Simple sequences are "easy" to describe, complex ones must be described symbol by symbol.

Shannon complexity

Can the (Shannon) entropy be considered as a measure of complexity?

Yes, but the entropy depends on the probability of a sequence given an underlying source or stochastic data generating process.

Assuming that a source assigns probabilities $\Pr\{X = x\}$ the entropy of the source is defined as

$$H(X) = -\sum_{x \in \mathcal{X}} \Pr\{X = x\} \log_2 \Pr\{X = x\}.$$

This is the expected number of bits needed to represent X.

Shannon complexity

For a sequence x a corresponding notion is the ideal code wordlength given as

$$I(x) = -\log_2 \Pr\{X = x\}.$$

This can be interpreted as the most favorable representation length.

A disadvantage of Shannon's measures seems to be the fact that the complexity of a sequence depends on the probability of the sequence and not on the sequence itself.

Shannon complexity

Example (of the 'unreasonable' interpretation)

Let $\mathcal{X} = \{01101010000010011110, 00110111001001000100\}$ and let the source select between the two sequences with equal probability $(\frac{1}{2}, \frac{1}{2})$.

The entropy of the source is 1 bit per sequence (of 20 symbols)! However, the two strings each appear much more complex than 1 bit!!

The complexity is hidden in the source description, namely in $\mathcal{X}$, which is already known by the receiver.
We shall see that universal data compression gives a more fundamental answer to this problem.

Universal data compression

Is it also possible to find a more meaningful measure using Shannon's information measure?

Because we do not know the model and its parameter values, we must consider data compression for parametrized classes of sources.

Universal data compression

Example

Parametrized binary source (I.I.D. source class)

Alphabet:	$\mathcal{X} = \{0, 1\}$;
Sequence:	$x^N = x_1 \dots x_N$;
	(N is the block length)
Probabilities:	$\Pr\{X_i = 1\} = 1 - \Pr\{X_i = 0\} = \theta$.
	$0 \leq \theta \leq 1$.
Code:	$C : \mathcal{X}^N \to \{0, 1\}^*$
Code word:	$c(x^N) = c_1 \dots c_j \in C$
Length:	$l_C(x^N) = l(c_1 \dots c_j) = j$

Universal data compression

Ideal code wordlength

The best possible code wordlengths come from Huffman's algorithm, but these are hard to compute.

The task: minimize over the choice of lengths $l_C(x^N)$

$$\sum_{x^N \in \mathcal{X}^N} p(x^N) l_C(x^N)$$

where the lengths must satisfy Kraft's inequality

$$\sum_{x^N \in \mathcal{X}^N} 2^{-l_C(x^N)} \leq 1$$

Universal data compression

Ideal code wordlength

Ignoring the requirement that code wordlengths are integer, we find that the optimal code wordlengths are

$$l_C(x^N) = -\log_2 p(x^N)$$

The upward rounded version of these lengths still satisfy Kraft's inequality and the resulting code achieves Shannon's upper bound.

We write $l_C^*(x^N)$ for these ideal code wordlengths.

$$\begin{aligned} l_C^*(x^N) &= \left\lceil -\log_2 p(x^N) \right\rceil \\ &< -\log_2 p(x^N) + 1 \end{aligned}$$

Universal data compression

Remember $n(a|x^N)$ is the number of times the symbol a occurs in x^N.

Sequence probability: $p(x^N) = (1-\theta)^{n(0|x^N)} \theta^{n(1|x^N)}$

Expected code word length: $\bar{l}_C = \sum_{x^N \in \mathcal{X}^N} p(x^N) l_C(x^N)$

(Expected) code rate: $R_N = \frac{\bar{l}_C}{N}$

(Expected) code redundancy: $r_N = R_N - h(\theta)$

Universal data compression

First assume that we know that $\theta = \theta_1 = 0.2$ or $\theta = \theta_2 = 0.9$ but we don't know which θ generated x^N.

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We also create the code C_{12} which uses the smallest code word from C_1 and C_2 with a '0' or '1' prepended to indicate from which code the word comes.

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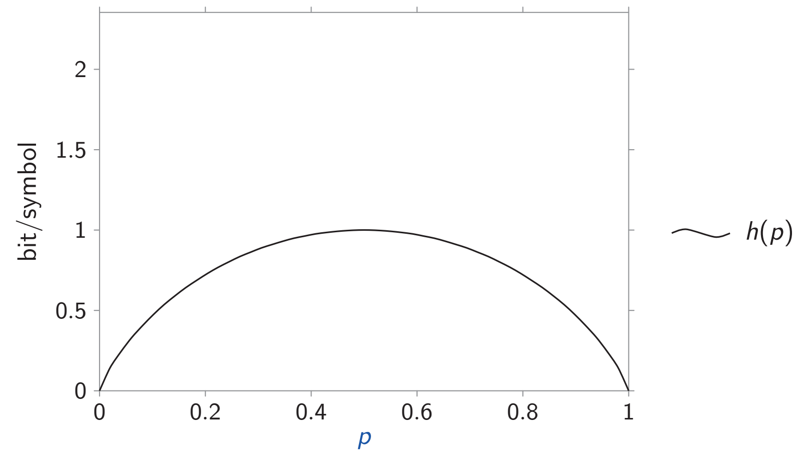
The code C_{mix} is make using the mixed (weighted) probabilities

$$p_{\text{mix}}(x^N) = \frac{p(x^N|\theta_1) + p(x^N|\theta_2)}{2}$$

Universal data compression

The results for block length $N = 6$ are shown graphically.



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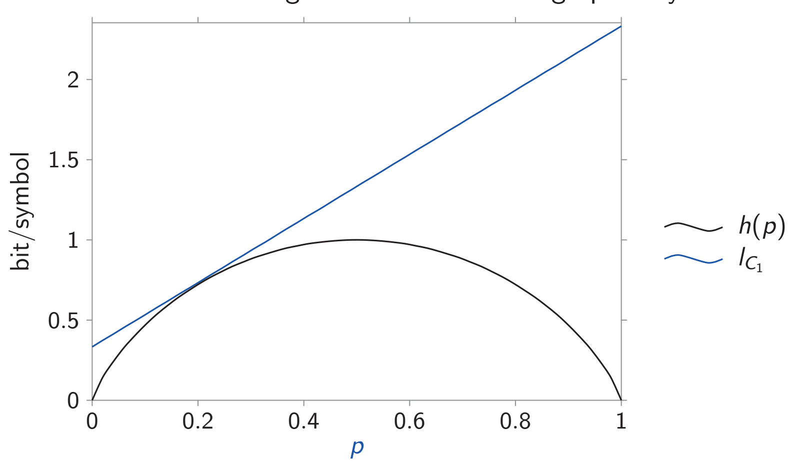

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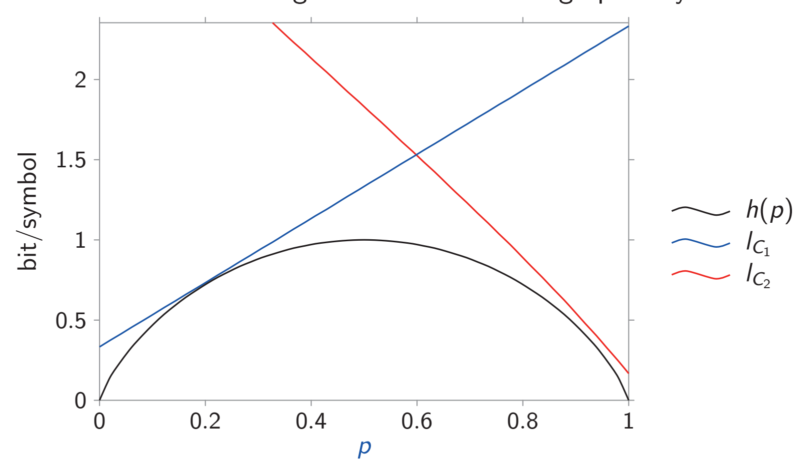

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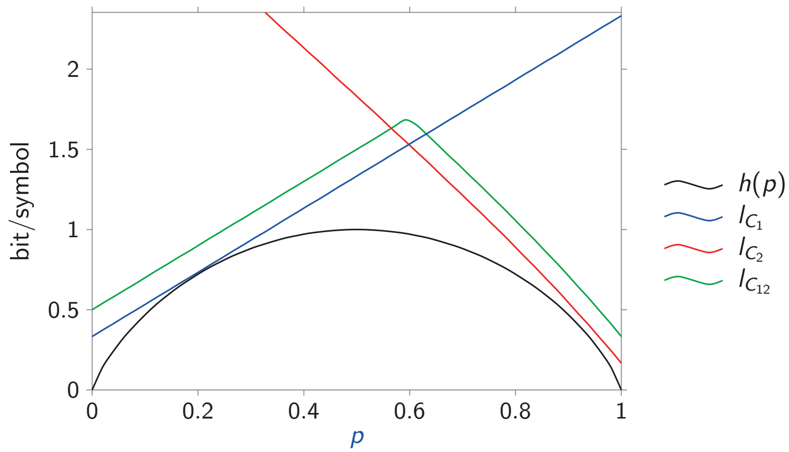

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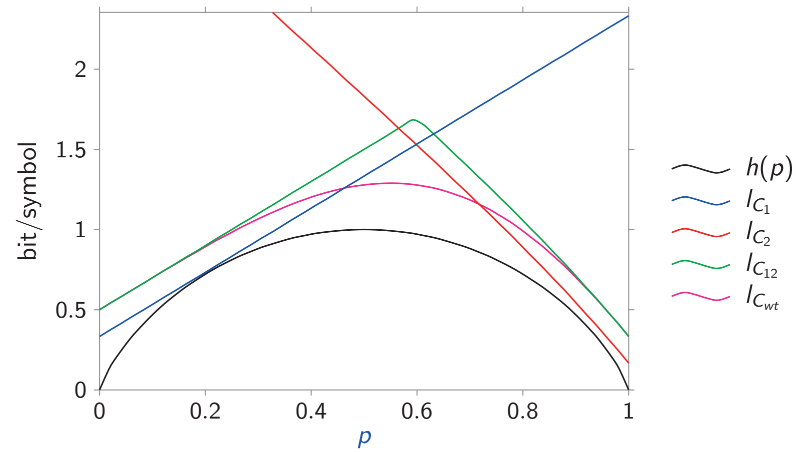

Universal data compression

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- Using an ordinary source code only works (well) if we are accurate in predicting the source probabilities.

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  First part: description of the source (parameters).
  Second part: the compressed version of the sequence assuming the given source.

Universal data compression

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- Using an ordinary source code only works (well) if we are accurate in predicting the source probabilities.
- That a two-part code works for more than one source.
  First part: description of the source (parameters).
  Second part: the compressed version of the sequence assuming the given source.
- Mixing (weighting) probabilities works at least as good as the two-part code and can be performed in one run through the data.

Universal data compression

Theorem

[Optimal number of sources] For a sequence x^N generated by an binary i.i.d. source with unknown $\Pr\{X = 1\} = \theta$ the optimal number of alternative sources is of order $\sqrt{N}$ and the achieved redundancy of the resulting code C^*, relative to any i.i.d. source, is bounded as

$$r_N(C^*) < \frac{\log_2 N}{2N}(1+\epsilon),$$

and also

$$r_N(C^*) > \frac{\log_2 N}{2N}(1-\epsilon),$$

for any $\epsilon > 0$ and N sufficiently large.

We shall not prove this theorem here.

Universal data compression

Discussion:

For the binary i.i.d. source which is described by one parameter θ, the optimal redundancy is $\frac{\log_2 N}{2N}$.

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This apparently is the cost we must pay for not knowing θ.

It also indicates that the number of discernible sources is roughly $\sqrt{N}$ in this case.

The next result will explain some of these observations.

Log-likelihood ratio and redundancy

In the Bayesian Model estimation problem we looked at the log-regret criterion:

$$\log \frac{p(x^N|\mathcal{M}_i, \theta_i)}{p(x^N|\mathcal{M}_i)},$$

regret from not knowing the parameters.

or the criterion

$$\log \frac{p(x^N|\mathcal{M}_i, \theta_i)}{p(x^N)},$$

regret from not knowing the model plus parameters.

Log-likelihood ratio and redundancy

Remember

$$p(x^N|\mathcal{M}_i) = \int_{\Theta_i} p(\theta_i|\mathcal{M}_i) p(x^N|\mathcal{M}_i, \theta_i)\, d\theta_i$$

and

$$p(x^N) = \int_{\mathfrak{M}} p(\mathcal{M}) p(x^N|\mathcal{M})\, d\mathcal{M},$$

or more often when the model class is discrete

$$p(x^N) = \sum_{\mathcal{M} \in \mathfrak{M}} p(\mathcal{M}) p(x^N|\mathcal{M}).$$

Log-likelihood ratio and redundancy

If $\mathcal{M}_i$, θ_i has actually generated x^N then

$$-\log p(x^N|\mathcal{M}_i, \theta_i)$$

is the ideal codeword length.

And

$$-\log p(x^N|\mathcal{M}_i) \text{ resp. } -\log p(x^N)$$

is the actual codeword length of a good code using these 'estimated' probabilities.

Log-likelihood ratio and redundancy

Thus

$$\log \frac{p(x^N|\mathcal{M}_i, \theta_i)}{p(x^N|\mathcal{M}_i)} \text{ resp. } \log \frac{p(x^N|\mathcal{M}_i, \theta_i)}{p(x^N)}$$

can be seen as

Data compression: The excess codeword length (individual redundancy).

Machine learning: The individual log-regret.

Log-likelihood ratio and redundancy

So now we see that the expected redundancy of a code C on sequences x^N from a source $\mathcal{M}_i$, θ_i, given by

$$r_N(C) = \sum_{x^N \in \mathcal{X}^N} p(x^N|\mathcal{M}_i, \theta_i) \log \frac{p(x^N|\mathcal{M}_i, \theta_i)}{p(x^N|\mathcal{M}_i)}$$

resp.

$$r_N(C) = \sum_{x^N \in \mathcal{X}^N} p(x^N|\mathcal{M}_i, \theta_i) \log \frac{p(x^N|\mathcal{M}_i, \theta_i)}{p(x^N)}$$

can also be seen as the expected log-regret with respect to $\mathcal{M}_i$ and θ_i.

Redundancy-capacity theorem

We again take a Bayesian approach.

But we are also dealing with variable-length codes C. We first discuss how we can relate codeword lengths $l_C(x^N)$ to probabilities $p(x^N|\mathcal{M}, \theta)$.

The answer is through the ideal codeword length

$$l_C(x^N) \sim -\log_2 p(x^N|\mathcal{M}, \theta).$$

Intermezzo: Dyadic probabilities

Let C be any binary, prefix-free code with K words, where l_i denotes the length of the i^{th} code word, that satisfies the Kraft inequality with equality, i.e.

$$\sum_{i=1}^{K} 2^{-l_i} = 1.$$

We see that 2^{-l_i} plays the role of a probability value, namely

$$\begin{aligned} 2^{-l_i} &> 0, && \text{because } 0 < l_i < \infty, \\ \sum_{i=1}^{K} 2^{-l_i} &= 1, && \text{total probability is 1.} \end{aligned}$$

Intermezzo: Dyadic probabilities

A probability vector $Q = \{q_1, q_2, \dots, q_K\}$ is called a dyadic probability vector, if for all i, $1 \le i \le K$, there exists integers n_i such that

$$q_i = 2^{-n_i}.$$

An example
$Q = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}\right\}$ is dyadic, namely

$$\begin{aligned} q_1 = q_2 = q_3 &= 2^{-2}, & q_4 &= 2^{-3}, \\ q_5 &= 2^{-4}, & q_6 = q_7 &= 2^{-5}, \end{aligned}$$
$$\sum_{i=1}^{7} q_i = 1.$$

Intermezzo: Dyadic probabilities

The code that corresponds to this probability vector has codeword lengths

$$\begin{aligned} l_1 = l_2 = l_3 &= 2, & l_4 &= 3 \\ l_5 &= 4 & l_6 = l_7 &= 5. \end{aligned}$$

An example of such a code can have the following 7 words

$$\begin{aligned} c_1 &= 00 & c_2 &= 01 \\ c_3 &= 10 & c_4 &= 110 \\ c_5 &= 1110 & c_6 &= 11110 \\ c_7 &= 11111 \end{aligned}$$

Intermezzo: Dyadic probabilities

If C is a code that satisfies the Kraft inequality with equality, then we denote the corresponding, unique dyadic probability vector by Q_C.
The set of all dyadic probability vectors of the same length K will be denoted by $\mathcal{Q}_{\mathcal{C}}$, where the vector length K is not specified explicitely. We also write $\mathcal{Q}$ for the set of all probability vectors of the same length K.
Obviously any $Q_C \in \mathcal{Q}_{\mathcal{C}}$ is also a member of $\mathcal{Q}$, so

$$\mathcal{Q}_{\mathcal{C}} \subset \mathcal{Q}.$$

Redundancy-capacity theorem

So again, let $\mathcal{Q}_\mathcal{C}$ be the set of all dyadic probabilities and $\mathcal{Q}$ be the set of all probabilities.
$\mathcal{S}$ is the set of all sources parametrized by a vector θ that takes values in a parameter space Θ.
We have seen the example of the binary i.i.d. source with a one dimensional parameter $\theta = \Pr\{X = 1\}$ and $\Theta = [0, 1]$.

Redundancy-capacity theorem

If $Q_C \in \mathcal{Q}_\mathcal{C}$ then the redundancy of of the corresponding code C is given by

$$
\begin{aligned}
r_N(C) &= \sum_{x^N \in \mathcal{X}^N} p(x^N|\theta) \log_2 \frac{p(x^N|\theta)}{Q_C(x^N)} \\
&= D(p(X^N|\theta) \| Q_C(X^N))
\end{aligned}
$$

Maximizing over the parameter values we get the maximum expected redundancy of a given code C.

$$
r_N^+(C) = \sup_{\theta \in \Theta} D(p(X^N|\theta) \| Q_C(X^N))
$$

Redundancy-capacity theorem

Now we can look for the best possible code that minimizes the maximum expected redundancy.

$$
r_N^+ = r_N^+(C^*) = \min_C r_N^+(C).
$$

So C^* is the code that minimizes the worst-case expected redundancy over all parameter values.
r_N^+ is the resulting minimax expected redundancy.

Instead of the worst-case redundancy we can also consider weighted redundancies.

Redundancy-capacity theorem

Let $w(\theta)$ be a prior distribution over θ. The Bayes redundancy is given by

$$
\mathcal{D}(w; Q_C) = \int_\Theta D(p(X^N|\theta) \| Q_C(X^N)) w(\theta)\, d\theta
$$

Because the maximum is never smaller than the average, we have

$$
r_N^+(C) \geq \mathcal{D}(w; Q_C),
$$

and likewise we obtain for the best possible code

$$
r_N^+ \geq \min_C \mathcal{D}(w; Q_C).
$$

Redundancy-capacity theorem

If we allow all probabilities, not only dyadic ones, we obtain:

$$\mathcal{D}(w; Q) = \int_{\Theta} w(\theta) D(p(X^N|\theta) \| Q(X^N))\, d\theta$$

and because we can minimize over a larger set

$$r_N^+ \geq \min_{Q} \mathcal{D}(w; Q).$$

It turns out that the Q^* that realizes the minimum is the $w(\theta)$ weighted probability

$$Q^* = \int_{\Theta} p(x^N|\theta) w(\theta)\, d\theta.$$

Redundancy-capacity theorem

And thus we can observe:

$$\begin{aligned}\mathcal{D}(w; Q^*) &= \int_{\Theta} w(\theta) D(p(X^N|\theta) \| Q^*(X^N))\, d\theta \\ &= \int_{\Theta} \sum_{x^N \in \mathcal{X}^N} w(\theta) p(x^N|\theta) \log_2 \frac{p(x^N|\theta)}{Q^*(x^N)}\, d\theta\end{aligned}$$

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Channel input θ probabilities $w(\theta)$

Redundancy-capacity theorem

And thus we can observe:

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Channel transition probabilities $p(x^N|\theta)$

Redundancy-capacity theorem

And thus we can observe:

$$\mathcal{D}(w; Q^*) = \int_\Theta w(\theta) D(p(X^N|\theta) \| Q^*(X^N))\, d\theta$$
$$= \int_\Theta \sum_{x^N \in \mathcal{X}^N} w(\theta) p(x^N|\theta) \log_2 \frac{p(x^N|\theta)}{Q^*(x^N)}\, d\theta$$

Channel output x^N probabilities $Q^*(x^N)$

Redundancy-capacity theorem

And thus we can observe:

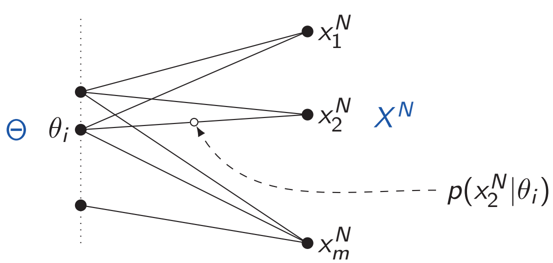
$$\mathcal{D}(w; Q^*) = \int_\Theta w(\theta) D(p(X^N|\theta) \| Q^*(X^N))\, d\theta$$
$$= \int_\Theta \sum_{x^N \in \mathcal{X}^N} w(\theta) p(x^N|\theta) \log_2 \frac{p(x^N|\theta)}{Q^*(x^N)}\, d\theta$$
$$= I(\theta; X^N)$$

Redundancy-capacity theorem

Because this last bound is independent of the prior $w(\theta)$, we can tighten the bound by maximizing over all possible priors $w(\theta)$ and find

$$r_N^+ \geq \max_{w(\theta)} I(\theta; X^N) = C_{\theta \to X^N}.$$

So the redundancy is lower bounded by (often it is equal to) the capacity of the channel from the source parameters to the source output sequence x^N.



Redundancy-capacity theorem

Redundancy: learning source parameters from data

- Source coding: we don't want this, because it causes extra codeword length, but it is unavoidable.
- Machine learning: this is what's it about, but we cannot learn faster then the channel capacity.

The meaning of model information

Efficient description of data can be split into two parts:

- Information about the 'model'

  Universal compression redundancy: The description of the parameters of the data generating process.

- Selection of one of the 'possible' sequences.

  Universal compression: One of the "typical sequences" selected and described with $NH(P_x)$ bits.

The meaning of model information

The first part describes what the model 'can do'.
The second part describes what the model 'did'.

bits of π: Almost zero complexity. The model is easy to describe and can only generate this sequence. So the first part is easy and the second part too because it is easy to compute the bits.

bits from an i.i.d. source $\theta = \frac{1}{2}$: Highly complex. The model is very simple but the set of possible sequences is large so we need a lot of bits to describe what the model did.

The meaning of model information

Occam's razor:

> *One should not increase, beyond what is necessary, the number of entities required to explain anything.*

The most useful statement of the principle for scientists is:

> *When you have two competing theories which make exactly the same predictions, the one that is simpler is the better.*

Universal source coding:

> *Take the simplest model that describes your data.*

Terminology

The two-part description separates model information from random selection

Universal coding: There is a certain unavoidable cost for parameters in a model. It is the price for learning the parameters.

Distinguishable models (parameter values): For a sequence of length N we can use (selection or weighting) about $\sqrt{N}$ distinct values.

Occam's razor: Take the simplest explanation that explains the observations.

Terminology

This results in the notion of stochastic complexity

$$-\log_2 p(x^N|\mathcal{M})$$

$$p(x^N|\mathcal{M}) = \frac{p(x^N|\mathcal{M}, \hat{\theta}(x^N))}{\sum_{x^N \in \mathcal{X}^N} p(x^N|\mathcal{M}, \hat{\theta}(x^N))}$$

is known as the NML (Normalized Maximum Likelihood).
That is must be normalized is reasonable because

$$\sum_{x^N \in \mathcal{X}^N} p(x^N|\mathcal{M}, \hat{\theta}(x^N)) \geq 1$$

And the normalizing constant determines the model cost.
Note that we assume here that the model priors $p(\mathcal{M})$ are all equal!

Terminology

Suppose I have two model classes, $\mathcal{M}_1$ and $\mathcal{M}_2$, for my data x^N and the stochastic complexity $-\log_2 p(x^N|\mathcal{M}_1)$ is smaller than $-\log_2 p(x^N|\mathcal{M}_2)$. Because the model information part is proportional to $\log_2 N$ and the "noise" part is proportional to N, a smaller complexity means "less noise". So $\mathcal{M}_1$ explains more of the data.
This leads to the Minimum Description Length Principle.

> *The best model for the data is is the model that results in the smallest stochastic complexity.*

Terminology

Stochastic complexity $\approx$ ideal codeword length.
Coding interpretation: $L(\theta) = O(\log N)$; $L(\text{noise}) = O(N)$

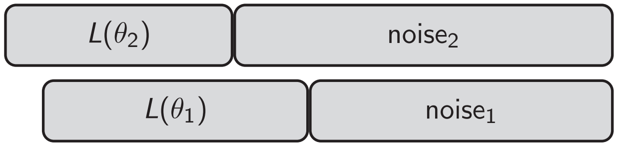
Say x^N with $N = 1000$. $L(\text{noise}_2) + L(\theta_2) = 500 + 5k_2$.

With model $\mathcal{M}_1$, x^N has smaller stochastic complexity:

$L(\text{noise}_1) > L(\text{noise}_2)$ hardly possible because $L(\theta_2) - L(\theta_1)$ cannot be large.

Terminology

Stochastic complexity $\approx$ ideal codeword length.
Coding interpretation: $L(\theta) = O(\log N)$; $L(\text{noise}) = O(N)$



Say x^N with $N = 1000$. $L(\text{noise}_2) + L(\theta_2) = 500 + 5k_2$.

With model $\mathcal{M}_1$, x^N has smaller stochastic complexity:

$L(\text{noise}_1) < L(\text{noise}_2)$ very likely.

So, $\mathcal{M}_1$ explains more of the data (less noise)

Stochastic Complexity (MDL)

"Real data model": binary 1^{th} order Markov,
$\theta_0 = \Pr\{X_i = 1 | x_{i-1} = 0\} = \frac{1}{4}$, $\theta_1 = \Pr\{X_i = 1 | x_{i-1} = 1\} = \frac{1}{2}$
Then: $\Pr\{X_i = 1\} = \frac{1}{3}$.

$\mathcal{M}_1$ is i.i.d. with $\hat{\theta}_1 \approx \frac{1}{3}$.

$\mathcal{M}_2$ is 1^{th} order Markov with $\hat{\theta}_2 \approx (\frac{1}{4}, \frac{1}{2})$.

$H(X|\mathcal{M}_1, \hat{\theta}_1) = 0.918$ bit.

$H(X|\mathcal{M}_2, \hat{\theta}_2) = 0.874$ bit.

Stochastic Complexity (MDL)

Stochastic complexity

$$S.C._1 \sim \frac{\log_2 N}{2} + 0.918N.$$
$$S.C._2 \sim \log_2 N + 0.874N.$$

For $N < 70 : S.C._1 < S.C._2$ and for $N > 70 : S.C._1 > S.C._2$.

So if there is not enough data the MDL selects a smaller model than the "true" model.

This is good!

There is not enough data to estimate properly a complex model.

Where to find information

WebPage: `http://bertdv.github.io/teaching/AIP-5SSB0/`
Prefix for all links below:
`https://github.com/bertdv/AIP-5SSB0/blob/master/lessons/`

The lecture notes: `Tjalling/AIP-part2-article.pdf`
Information Theory: `Tjalling/information-theory-background.pdf`
Markov structures and summary:
`Tjalling/summary-of-essential-content.pdf`
Exercises for part 2:
`exercises/5MB20-exercises-part-2.pdf`
Hints for the exercises
`exercises/5MB20-exercises-part-2-hints.pdf`