Optimization in EZ-Climate model

1 Epstein-Zin preferences

In this section we describe Epstein-Zin utility function and provide support for the maximization problem used in the EZ-Climate change model.

First we recall that expected time-separable preferences are defined as

$$\bar{U}_t := \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s u(C_{t+s})\right] \tag{1}$$

where $\bar{U}_t$ is the lifetime expected utility (sum of utilities $u(C_{t+s})$ at different points in time) and β is the discount rate.[1] Note that $\bar{U}_t$ can also be defined recursively[2]

$$\bar{U}_t = u(C_t) + \beta\mathbb{E}_t[\bar{U}_{t+1}]. \tag{2}$$

Since the utility function is invariant under affine transformations (in this case invariant under multiplication by $(1-\beta)$), the equation (2) can also be written as[3]

$$\begin{aligned} U_t &:= (1-\beta)\bar{U}_t = (1-\beta)u(C_t) + \beta\mathbb{E}_t[(1-\beta)\bar{U}_{t+1}] \\ &= (1-\beta)u(C_t) + \beta\mathbb{E}_t[U_{t+1}] \end{aligned} \tag{3}$$

The Epstein-Zin preferences generalize this idea and define preferences recursively as a function of current utility C_t and the certainty equivalent $R_t(U_{t+1})$ of future utility U_{t+1}, that is,

$$U_t = F(C_t, R_t(U_{t+1})) \tag{4}$$

where

$$R(U_{t+1}) = G^{-1}\left(\mathbb{E}_t(G(U_{t+1}))\right) \tag{5}$$

with F and G are increasing and concave, and F is homogeneous of degree one. One should notice that $R_t(U_{t+1}) = U_{t+1}$ implies no uncertainty about U_{t+1}. Also, the more concave G is, and the more uncertain U_{t+1} is, the lower is $R_t(U_{t+1})$.

Most of the literature considers the following functional forms:

$$F(c,z) = \left((1-\beta)c^\rho + \beta z^\rho\right)^{1/\rho}, \tag{6}$$

$$G(x) = \frac{x^\alpha}{\alpha}, \tag{7}$$

[1] $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ and $\mathcal{F}_t$ is the information filtration available at time t.

[2] To get this identity one should use $\mathbb{E}\left[\mathbb{E}\left[\cdot|\mathcal{F}_{t+1}\right]|\mathcal{F}_t\right] = \mathbb{E}[\cdot|\mathcal{F}_t]$ (tower property) and the definition of V_t.

[3] It is invariant in the sense that the preferences defined by $(1-\beta)u(C_t)$ are the same as the preferences defined by $u(C_t)$. Also, to be mathematically precise we should write $(1-\beta)U_t$ instead of U_t, but we do not do it because this has no effect on the definition of Epstein-Zin preferences.

which imply

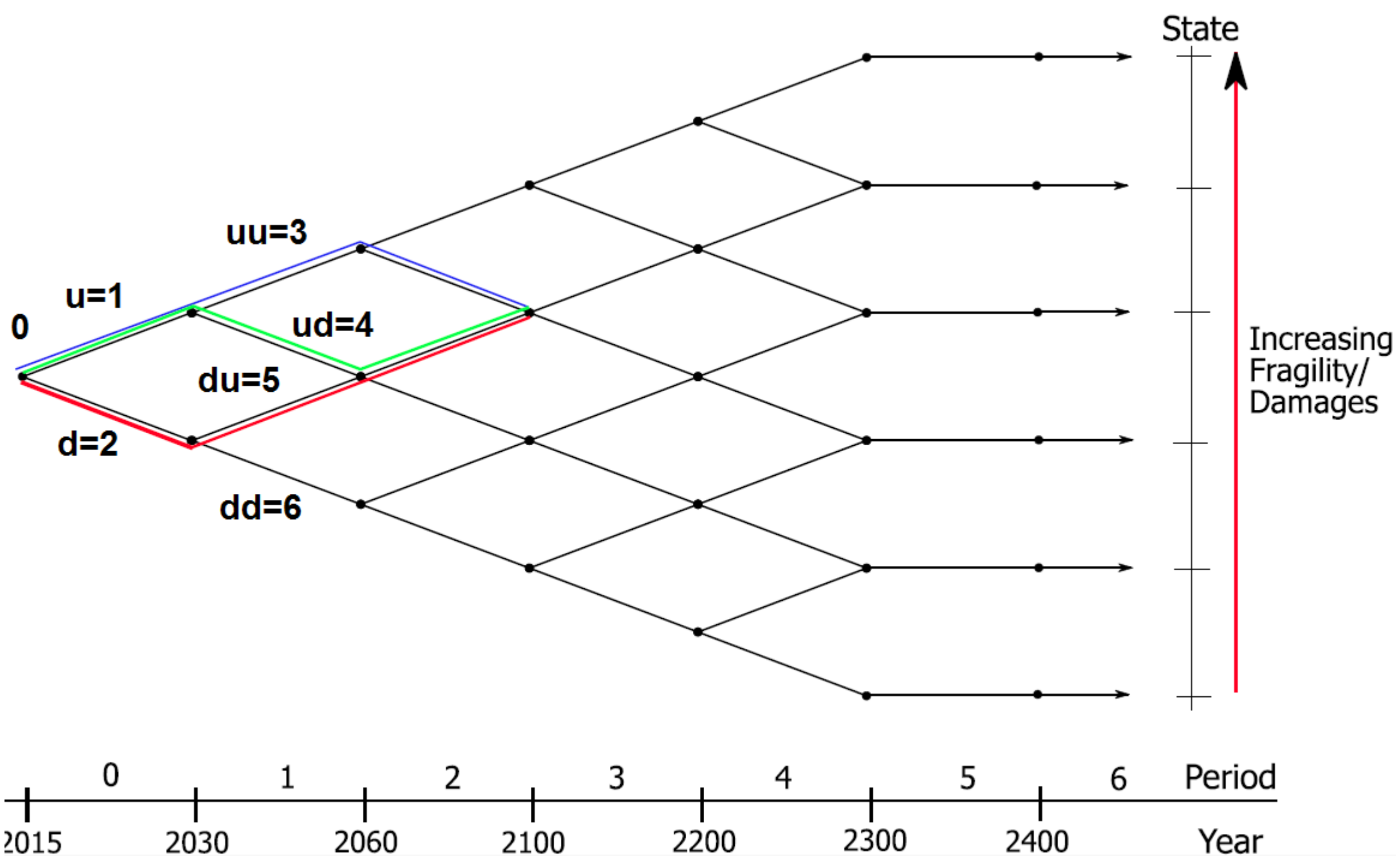
$$U_t = \left((1-\beta) C_t^{\rho} + \beta (\mathbb{E}_t[U_{t+1}^{\alpha}])^{\rho/\alpha} \right)^{1/\rho} \tag{8}$$

and this is equivalent to Epstein-Zin utility function used in the Litterman's paper. Thus, assuming that the current time is time 0, for Epstein-Zin preferences, maximizing lifetime expected utility is equivalent to maximizing U_0.

2 Maximization of U_0

In the EZ-Climate model, we have a 7-period tree, beginning 2015 (See Figure 1). Every period except period 0 has more than 1 node, where the representative agent makes mitigation decision. For an arbitrary period t, we label the nodes as state, θ_t, beginning at 0. Based on the tree model, we have 63 nodes in total and index all of them 0 - 62 from period 0 to the last period. Within each period, their number increase from top to bottom (this depends on the number of 'd' states it takes to get to the node from the start point). You can see the nodes 0 to 6 in Figure 1 to get a better understanding of the node ordering.

Figure 1: Tree Model



Essentially, the representative agent is intended to maximize his/her utility at the start point, $U(0)$ with respect to mitigation levels $\mathbf{m}$. Mathematically speaking, $\mathbf{m} \in \mathbb{R}^{63}$ is a 1×63 vector with 63 non-negative elements, x_{t,θ_t}, that stand for mitigation levels at each decision point of the tree model. This corresponds to the 63 nodes we just mentioned. For instance, $\mathbf{m}$'s third element, $x_{1,2}$ represents the mitigation level chosen for the second state of period 1.

As we know, the utility formulas are:

$$U_0 = \left[(1-\beta)c_0^\rho + \beta(E_0[U_1^\alpha])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{9}$$

$$U_1 = \left[(1-\beta)c_1^\rho + \beta(E_1[U_2^\alpha])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{10}$$

$$U_2 = \left[(1-\beta)c_2^\rho + \beta(E_2[U_3^\alpha])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{11}$$

$$U_3 = \left[(1-\beta)c_3^\rho + \beta(E_3[U_4^\alpha])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{12}$$

$$U_4 = \left[(1-\beta)c_4^\rho + \beta(E_4[U_5^\alpha])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{13}$$

$$U_5 = \left[(1-\beta)c_5^\rho + \beta(E_5[U_6^\alpha])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{14}$$

$$U_6 = \left[\frac{1-\beta}{1-\beta(1+g)^\rho}\right]^{\frac{1}{\rho}} c_6 \tag{15}$$

where $(E_t[U_{t+1}^\alpha])^{1/\alpha}$ is the certainty-equivalent of future lifetime utility, based on the agent's information at time t. Both consumption c_t and utility U_t depend on mitigation levels in $\mathbf{m}$, as they depend on mitigation actions before and after period t, so the system can be represented as

$$U_0(\mathbf{m}) = \left[(1-\beta)\mathbf{c}_0^\rho(\mathbf{m}) + \beta(E_0[U_1^\alpha(\mathbf{m})])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{16}$$

$$U_1(\mathbf{m}) = \left[(1-\beta)\mathbf{c}_1^\rho(\mathbf{m}) + \beta(E_1[U_2^\alpha(\mathbf{m})])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{17}$$

$$U_2(\mathbf{m}) = \left[(1-\beta)\mathbf{c}_2^\rho(\mathbf{m}) + \beta(E_2[U_3^\alpha(\mathbf{m})])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{18}$$

$$U_3(\mathbf{m}) = \left[(1-\beta)\mathbf{c}_3^\rho(\mathbf{m}) + \beta(E_3[U_4^\alpha(\mathbf{m})])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{19}$$

$$U_4(\mathbf{m}) = \left[(1-\beta)\mathbf{c}_4^\rho(\mathbf{m}) + \beta(E_4[U_5^\alpha(\mathbf{m})])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{20}$$

$$U_5(\mathbf{m}) = \left[(1-\beta)\mathbf{c}_5^\rho(\mathbf{m}) + \beta(E_5[U_6^\alpha(\mathbf{m})])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{21}$$

$$U_6(\mathbf{m}) = \left[\frac{1-\beta}{1-\beta(1+g)^\rho}\right]^{\frac{1}{\rho}} \mathbf{c}_6(\mathbf{m}) \tag{22}$$

Note that the above equations $U_t(\mathbf{m})$ takes into account all possible nodes at period t (due to the nature of binomial tree model). To evaluate $U_t(\mathbf{m})$, we have to use $U_{t+1}(\mathbf{m})$. This is where probabilities of our binomial tree plays an important role. EZ_Climate sets equal probabilities to the 2 branches of each node. For example, in Figure 1, $\mathbb{P}(uu|u) = \mathbb{P}(ud|u) = \frac{1}{2}$. Hence, for the node with the mitigation level x_{t,θ_t}, we calculate the expected future utility via $\mathbb{E}_t\left[U^{\alpha}_{t+1}\right] = \frac{1}{2} \times \mathbb{E}_t\left[U^{\alpha}_{t+1}\left(x_{t+1,\theta^1_{t+1}}\right)\right] + \frac{1}{2} \times \mathbb{E}_t\left[U^{\alpha}_{t+1}\left(x_{t+1,\theta^2_{t+1}}\right)\right]$, where θ^1_{t+1} and θ^2_{t+1} are the 2 child nodes of node θ_t. In other words, we take the average of possible utilities at period $t+1$.

To be more specific with the notation, the vectors $\mathbf{c}_t$ for $t \in \{0 \dots 5\}$, of dimension 1×2^t, contain the consumption of all the nodes in period t. For example: $\mathbf{c_1} = (c_{1,1}, c_{1,2})$. The same applies to $\mathbf{c}_6$, which has dimension 1×32 in our model. In order to make our demonstration clear, we focus on a single mitigation path in the form of $\widetilde{x}^p_t$ from now on. Do you mean: "Let $\widetilde{x}^p_t$ denote a mitigation path."? In particular, $\widetilde{x}^p_t \in \mathbb{R}^{t+1}$ is a $1 \times (t+1)$ vector that consists of mitigation level x_s for $s \in \{0 \dots t\}$. What tilde and p stand for? Is p numerical, or it's just to denote something, e.g., path? Then what is tilde? We recombine utilities of these individual paths to determine the certainty-equivalent of future lifetime utility $E_t[U^{\alpha}_{t+1}(\mathbf{m})]$ and utility $U_t(\mathbf{m})$ in concern. You said that you consider "a single mitigation path", but in the last sentence you say "recombining these individual paths". It's confusing. Please rephrase.

In terms of the consumption model, in each period $t \in \{0, 1, 2, 3, 4, 5, 6\}$, the agent is endowed with a certain amount of the consumption good, $\bar{c}_t$. However, the agent is not able to consume the full endowed consumption for 2 reasons: climate change and climate policy. First, in period $t \in \{1, 2, 3, 4, 5, 6\}$, some of the endowed consumption may be lost due to climate change damages. Second, in periods $t \in \{0, 1, 2, 3, 4, 5\}$, the agent may elect to spend some of the endowed consumption on mitigation to reduce his impact on the climate. The resulting consumption c_t, after damages and mitigation costs are taken into account, is given by:

$$c_0 = \bar{c}_0 \cdot (1 - \kappa(\widetilde{x}^p_0)) \tag{23}$$
$$c_1 = \bar{c}_1 \cdot (1 - D_1(\bar{x}_1(CRF_1(\widetilde{x}^p_1)), \theta_1) - \kappa(\widetilde{x}^p_1)) \tag{24}$$
$$c_2 = \bar{c}_2 \cdot (1 - D_2(\bar{x}_2(CRF_2(\widetilde{x}^p_2)), \theta_2) - \kappa(\widetilde{x}^p_2)) \tag{25}$$
$$c_3 = \bar{c}_3 \cdot (1 - D_3(\bar{x}_3(CRF_3(\widetilde{x}^p_3)), \theta_3) - \kappa(\widetilde{x}^p_3)) \tag{26}$$
$$c_4 = \bar{c}_4 \cdot (1 - D_4(\bar{x}_4(CRF_4(\widetilde{x}^p_4)), \theta_4) - \kappa(\widetilde{x}^p_4)) \tag{27}$$
$$c_5 = \bar{c}_5 \cdot (1 - D_5(\bar{x}_5(CRF_5(\widetilde{x}^p_5)), \theta_5) - \kappa(\widetilde{x}^p_5)) \tag{28}$$
$$c_6 = \bar{c}_6 \cdot (1 - D_6(\bar{x}_6(CRF_6(\widetilde{x}^p_6)), \theta_6)) \tag{29}$$

θ_t here stands for the state at period t on the path. In a specific case, suppose we have path that leads to the first state in the last period, i.e. the path of all 'u' states. Then the notation of above should be changed as following:

- $c_0 \rightarrow \mathrm{c}_{0,1}$
- $c_1 \rightarrow \mathrm{c}_{1,1}$
- $c_2 \rightarrow \mathrm{c}_{2,1}$
- $c_3 \rightarrow \mathrm{c}_{3,1}$

- $c_4 \rightarrow \mathrm{c}_{4,1}$
- $c_5 \rightarrow \mathrm{c}_{5,1}$
- $c_6 \rightarrow \mathrm{c}_{6,1}$

As we can see, both damage function $D_t(\bar{x}_t(CRF(\widetilde{x}_t^p)), \theta_t)$ and cost function $\kappa(\widetilde{x}_t^p)$ are dependent on mitigation level up to t, $\widetilde{x}_t^p$. However, $D_t(\bar{x}_t(CRF(\widetilde{x}_t^p)), \theta_t)$ and $\kappa(\widetilde{x}_t^p)$ differ in they way they depend on $\widetilde{x}_t^p$. The damage function $D_t(\bar{x}_t(CRF(\widetilde{x}_t^p)), \theta_t)$ uses $\widetilde{x}_t^p$ to determine the cumulative forcing CRF_t, which in turn help determine realized average mitigation up to time t, $\bar{x}_t$. (See Section 2.1.1 for details.) On the other hand, the cost function $\kappa(\widetilde{x}_t^p)$ uses $\widetilde{x}_t^p$ to calculates the cumulative mitigation level X_t given by:

$$X_t = \frac{\sum_{s=0}^{t} g_s \cdot x_s}{\sum_{s=0}^{t} g_s} \tag{30}$$

where g_s is the flows of GHG emissions into the atmosphere in period s, for each period up to t, absent any mitigation. It can be messy to lay out the damage and cost function here, but reader can look at Section 2.1 and Equation 39, respectively, for details.

2.1 Damage Function $D_t(\bar{x}_t(CRF_t(\widetilde{x}_t^p)), \theta_t)$

The damage function $D_t(\bar{x}_t(CRF_t(\widetilde{x}_t^p)), \theta_t)$ captures the fraction of the endowment consumption that is lost due to damages from climate change. It mainly depends on 2 variables: CRF_t, which we define as the cumulative radiative forcing up to time t, which determines global average temperature, and state θ_t, the Earth's fragility, a parameter that characterizes the uncertain relationship between the global average temperature and consumption damages.

We define damage function as a function of temperature changes, which, in turn, are a function of cumulative solar radiative forcing CRF_t, which, in our setting, are determined by the mitigation path up to that point in time t, $\widetilde{x}_t^p$. We then compare CRF_t to three baseline emissions paths, g_t, for which we have created associated damage simulations. The only way, then, to affect the level of damages is to change mitigation across time, x_t. The specification of damages has 2 components: a non-catastrophic component and a catastrophic component triggered by crossing a particular threshold. (The hazard rate associated with hitting that threshold increases with temperature.) If the threshold is crossed at any time, additional changes decreases consumption in all future periods.

We calculate the overall damage function $D_t(\bar{x}_t(CRF_t(\widetilde{x}_t^p)), \theta_t)$ for the baseline emission paths, g_t, using Monte-Carlo simulation. We run a set of simulations for each of 3 constant mitigation levels, which determine cumulative radiative forcing at each point in time. The state variable θ_t indexes the distribution resulting from these sets of simulations, and interpolation across the three mitigation level gives us a continuous function D_t across cumulative radiative forcing levels CRF_t.

Therefore, based on the simulation results, we can determine D_t as a function of CRF_t and θ_t, where CRF_t is a function of the mitigation paths g_t of x_t.

2.1.1 CRF_t as a function of x_t

We have mitigation level x_t and the GHG emission under business-as-usual case, absent any mitigation, BAU_t for $t \in \{0,1,2,3,4,5,6\}$. We look into each period by unit called **subinterval**. For each subinterval we calculate **forcing** and **absorption**, with which we get cumulative forcings CRF_t and the ghg levels GHG_t. Their values in the very beginning are:

$$\begin{cases} CRF_0 & = 4.926 \\ GHG_0 & = 400 \\ sink_0 & = 35.396 \end{cases} \tag{31}$$

Here is how we calculate the variables for each subinterval:

- For each period t:

$$\begin{cases} \text{beginning emission} & g_{t,0} = (1 - x_t) \times BAU_t \\ \text{ending emission} & g_{t,t} = \begin{cases} (1 - x_t) \times BAU_{t+1}, & \text{if } t < 5 \\ (1 - x_t) \times BAU_t, & \text{else} \end{cases} \end{cases} \tag{32}$$

- For each subinterval i:

$$\begin{cases} \text{p\_co2}_i & = 0.71 \times \left[g_{t,0} + i \times \frac{g_{t,t} - g_{t,0}}{\text{number of subintervals in period } t} \right] \\ \text{p\_c}_i & = \frac{\text{p\_co2}_i}{3.67} \\ \text{add\_p\_ppm}_i & = \text{length of subinterval} \times \frac{\text{p\_c}_i}{2.13} \\ \text{lsc}_i & = 285.6268 + \text{cum\_sink}_i \times 0.88414 \\ \text{absorption} & = 0.5 \times 0.94835 \times |GHG_i - \text{lsc}_i|^{0.741547} \\ \text{cum\_sink}_i & = \text{cum\_sink}_{i-1} + \text{absorption}_i \\ GHG_i & = GHG_{i-1} + \text{add\_p\_ppm}_i - \text{absorption}_i \\ CRF_i & = CRF_{i-1} + 0.13183 \times |GHG_i - 315.3785|^{0.607773} \end{cases}$$

2.1.2 D_t as a function of $\bar{x}_t(CRF_t)$

In order to get the desired damage D_t, we run damage simulations for the 3 base scenarios and determine the damage coefficients in concern for each. Based on the CRF_i's and GHG_i's we get in the last section, we calculate the realized average mitigation $\bar{x}_t$ up to each period and compare with the constant mitigation levels in the base cases to interpolate the damage along the given path.

The 3 base scenarios correspond to maximum final GHG levels of 450, 650, and 1000 ppm. We calculate their cumulative radiative forcings CRF_t and constant mitigation levels, or cumulative mitigation levels, $x_{450/650/1000} = \bar{x}_{450/650/1000}$, for each period. The way we infer the cumulative mitigation from cumulative forcing up to a particular point is:

$$\bar{x}(CRF_t) = w_{450}(CRF_t) \times \bar{x}_{450} + w_{650}(CRF_t) \times \bar{x}_{650} \tag{33}$$

where the weights are given by:

- If $CRF_t > CRF_{650}$, $\begin{cases} w_{650} &= \frac{CRF_{1000}-CRF_t}{CRF_{1000}-CRF_{650}} \\ w_{450} &= 0 \end{cases}$

- If $CRF_{650} > CRF_t > CRF_{450}$, $\begin{cases} w_{650} &= \frac{CRF_t-CRF_{450}}{CRF_{650}-CRF_{450}} \\ w_{450} &= \frac{CRF_{650}-CRF_t}{CRF_{650}-CRF_{450}} \end{cases}$

- Otherwise, $\begin{cases} w_{650} &= 0 \\ w_{450} &= 1+\frac{CRF_{450}-CRF_t}{CRF_{450}} \end{cases}$

We calculate an interpolated damage function using our 3 simulations where we have damage coefficients (for a given state and period) to find a smooth function that gives damages for any particular level of radiative forcing up to each point in time. For realized mitigation below that of 650 ppm, let the damage coefficients for state n at period t be $d_{t,n}^{650,1}$ and $d_{t,n}^{650,0}$. Similarly, for realized mitigation between those of 650 and 450 ppm, let the damage coefficients for state n at period t be $d_{t,n}^{450,2}$, $d_{t,n}^{450,1}$, and $d_{t,n}^{450,0}$. To find the function, we assume a linear interpolation of damages between the 650 and 1000 ppm scenarios, and a quadratic interpolation between 450 and 650 ppm. In addition, we impose a smooth pasting condition at 650 ppm, having the level and derivative of the interpolation between 650 ppm match the level and scope of the line above.

For consequent GHG level below 450 ppm, the realized mitigation is more than 100% and we assume climate damages exponentially decay toward 0. Mathematically, we let $S = \frac{d \cdot p}{l \cdot ln(0.5)}$, where d is the derivative of the quadratic damage interpolation function at 450 ppm, $p = 0.91667$ is the average mitigation in the 450 ppm simulation, and the level of damage is l. Radiative forcing at any point below 450 ppm then is y percent below that of the 450 ppm simulation, with $y = \frac{CRF_{450}-CRF}{CRF_{450}}$, where R is the radiative forcing in the 450 ppm simulation and r is the radiative forcing given the mitigation policy $\mathbf{m}$. Letting $\sigma = 60$, the extension of the damage function for $y > 0$ is defined as:

$$Damage(y) = l \cdot 0.5^{(y \cdot S)} e^{-\frac{(y \cdot p)^2}{\sigma}} \tag{34}$$

Therefore, given the cumulative forcing CRF_6 for node n up to period 6, we calculate the realized average mitigation $\bar{x}(CRF_6)$ and calculate its damage in the following way:

$$D_6(\bar{x}_6(CRF_6(\tilde{x}_6^p)), \theta_6) = \begin{cases} d_{6,n}^{650,1} \times \bar{x}(CRF_6) + d_{6,n}^{650,0}, & \text{for } \bar{x}(CRF_6) < \bar{x}_{650}; \\ d_{6,n}^{450,2} \times \bar{x}(CRF_6)^2 + d_{6,n}^{450,1} \times \bar{x}(CRF_6) + d_{6,n}^{450,0}, & \text{for } \bar{x}_{450} < \bar{x}(CRF_6) \leq \bar{x}_{650}; \\ 0.5^{\frac{\bar{x}(CRF_6)-\bar{x}_{450}+ln(d_{6,n}^{450})}{ln(0.5)}} e^{-\frac{(\bar{x}(CRF_6)-\bar{x}_{450})^2}{60}}, & \text{else} \end{cases} \tag{35}$$

The $d_{6,n}^{450}$ in the last case is the simulated damage for state n under the 450 ppm scenario.

The climate sensitivity- summarized by state of nature θ_t- is not known prior to the final period (t=6). Rather, what the representative agent knowns is the distribution of possible final states θ_6. We specify that the damage in period t, given a cumulative radiative forcing,

CRF_t, up to time t, is the probabiity weighted average of the interpolated damage function over all final states of nature reachable from that node. In particular, the damage function at time t, for the node indexed by θ_t is assumed to be:

$$D_t(\bar{x}_t(CRF_t(\widetilde{x}_t^p)), \theta_t) = \sum_{\theta_6} \mathbb{P}(\theta_6|\theta_t) \cdot D_t(CRF_t, \theta_6) \tag{36}$$

In addition, we have a penalty function for the damages of the concentrations below pre-industrial levels:

$$f(GHG_t) = \left[1 + e^{0.05\times(GHG_t - 200)}\right]^{-1} \tag{37}$$

Hence, the damage function at time t is:

$$D_t(\bar{x}_t(CRF_t(\widetilde{x}_t^p)), \theta_t) = \sum_{\theta_6} \mathbb{P}(\theta_6|\theta_t) \cdot D_t(CRF_t, \theta_6) + \left[1 + e^{0.05\times(GHG_t - 200)}\right]^{-1} \tag{38}$$

2.2 Cost Function $\kappa(x_t)$

Mitigation reduces the stock of GHGs in the atmosphere and leads to lower climate damages, and, hence, to higher future consumption. But mitigating GHG emissions is costly. Mitigating a fraction of x_t of emissions costs a fraction $\kappa(x_t)$ of the endowed consumption. In terms of the mitigation cost, we consider both the traditional cost regarding taxation and technological components. The latter one matters for x_t above the fractional-mitigation threshold, x^*, at which the backstop technology kicks in. The consequent upper and lower bound for marginal technology costs are $\tilde{\tau}$ and τ^* respectively. From these bounds we can calculate relevant parameters b and k. Notice that the cumulative mitigation X_t given in Equation 30 enters the determination of the rate of technology change, where we have constants θ_0 and θ_1. The general equation for marginal cost is given by:

$$\kappa(x_t) = [1 - \theta_0 - \theta_1 X_t] \begin{cases} \frac{g_0}{c_0} \cdot 92.08 \cdot x_t^{\alpha}, & x_t \leq x^* \\ \frac{g_0}{c_0} \cdot 92.08 \cdot x^{*\alpha} + \tilde{\tau} x_t - \tilde{\tau} x^* - \frac{b x_t (\frac{k}{x_t})^{\frac{1}{b}}}{b-1} + \frac{b x^* (\frac{k}{x^*})^{\frac{1}{b}}}{b-1}, & x_t > x^* \end{cases} \tag{39}$$