\documentclass[12pt]{article}

\usepackage[utf8]{inputenc}

\usepackage{latexsym}

\usepackage{float}

\usepackage{parskip}

\usepackage{amsfonts}

\usepackage{caption}

\usepackage{commath}

\usepackage{amssymb,amsmath}

\usepackage{graphicx}

\usepackage[backend=bibtex,style=numeric,sorting=none]{biblatex}

\usepackage[export]{adjustbox}

\usepackage{subcaption}

\usepackage[top=1in, bottom=1in,left=1in, right=1in]{geometry}

\usepackage{minted}

\usepackage{color}

\usepackage{comment}

\newcommand{\tb}{\textcolor{blue}}

\newcommand{\tc}{\textcolor{red}}

\newcommand{\bs}{\boldsymbol}

\newenvironment{alphafootnotes}

{\par\edef\savedfootnotenumber{\number\value{footnote}}

\renewcommand{\thefootnote}{\alph{footnote}}

\setcounter{footnote}{0}}

{\par\setcounter{footnote}{\savedfootnotenumber}}

\begin{document}

\begin{center}

\Large\bf{Optimization in EZ-Climate model}

\end{center}

\normalsize

\pagenumbering{arabic}

\begin{comment}

\section{Questions on Optimization}

As the authors put it, the purpose of the optimization is to find the mitigation level, $x\_t$, at every node of the decision tree that maximizes the current utility

\begin{equation}

x^\* = \underset{x}{\mathrm{arg max }}\ U(x)

\end{equation}

Basically, the representative agent maximizes the lifetime utility at each time and for each state of nature by choosing the optimal path of mitigation, $x^\*\_t(\theta\_t)$, dependent on Earth's fragility, $\theta\_t$.

There are three approaches to solving this optimization problem: constrained optimization, dynamic programming, and greedy algorithm. They differ in the objective functions that they maximize.

\begin{enumerate}

\item The \textbf{constrained approach} finds the mitigation strategy vector \textbf{m} that maximizes the agent's utility at time 0, $U(0)$. This is what we see in the code for EZ-Climate and the underlying functions are given in the next section.

\item The \textbf{greedy algorithm approach} picks the locally optimal choice at each step. At each time $t$ and state of nature $\theta\_t$, we take mitigation levels $x\_s$ for $s \in \{0,\dots,t-1\}$ as given and find a mitigation strategy scalar $x\_t$ that maximizes the utility at this point, $U\_t(x\_t,\theta\_t)$. In this case, the representative's decision problem can be written as follows:

\begin{equation}

\mathrm{max\ }U\_t(x\_t,\theta\_t)\mathrm{\ subject\ to\ }x\leq0,\mathrm{\ for\ all\ }t\in \{0,1,2,3,4,5,6\}

\end{equation}

where the $U\_t$ functions are given in Equation \ref{u0} to \ref{u6}.

This approach to solving this problem involves breaking it apart into a sequence of smaller decisions, i.e. the nodes in the tree model. To do so, we look into the utility of each node step by step with backward induction.

\begin{align}

U\_6(\bar{c}\_6,\theta\_6)& =\left[\frac{1-\beta}{1-\beta(1+g)^{\rho}}\right]^{\frac{1}{\rho}}c\_6\\

U\_5(\bar{c}\_5,\theta\_5) & = \left[(1-\beta)c\_5^{\rho}+\beta U\_6^{\rho}(\bar{c}\_6,\theta\_{6|5})\right]^{\frac{1}{\rho}}\\

U\_4(\bar{c}\_4,\theta\_4) & = \left[(1-\beta)c\_4^{\rho}+\beta\left(\frac{1}{2}U\_5^{\alpha}(\bar{c}\_5,\theta^1\_{5|4})+\frac{1}{2}U\_5^{\alpha}(\bar{c}\_5,\theta^2\_{5|4})\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}\\

U\_3(\bar{c}\_3,\theta\_3) & = \left[(1-\beta)c\_3^{\rho}+\beta\left(\frac{1}{2}U\_4^{\alpha}(\bar{c}\_4,\theta^1\_{4|3})+\frac{1}{2}U\_4^{\alpha}(\bar{c}\_4,\theta^2\_{4|3})\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}\\

U\_2(\bar{c}\_2,\theta\_2) & = \left[(1-\beta)c\_2^{\rho}+\beta\left(\frac{1}{2}U\_3^{\alpha}(\bar{c}\_3,\theta^1\_{3|2})+\frac{1}{2}U\_3^{\alpha}(\bar{c}\_3,\theta^2\_{3|2})\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}\\

U\_1(\bar{c}\_1,\theta\_1) & = \left[(1-\beta)c\_1^{\rho}+\beta\left(\frac{1}{2}U\_2^{\alpha}(\bar{c}\_2,\theta^1\_{2|1})+\frac{1}{2}U\_2^{\alpha}(\bar{c}\_2,\theta^2\_{2|1})\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}\\

U\_0(\bar{c}\_0,\theta\_0) & = \left[(1-\beta)c\_0^{\rho}+\beta\left(\frac{1}{2}U\_1^{\alpha}(\bar{c}\_1,\theta^1\_1)+\frac{1}{2}U\_1^{\alpha}(\bar{c}\_2,\theta^2\_1)\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}

\end{align}

The consumption functions are given in Equation \ref{c0} to \ref{c5}.

\tb{This is the approach that Dr.Xu suggested. The representative agent concerns more than the start point and aims to maximize his/her utility at each time $t$ and state of nature, in his perspective.}

\item The \textbf{dynamic programming} approach is similar to the greedy one, but its objective is to maximize the expected sum of utilities looking forward at each period $t$ and state of nature $\theta\_t$. That is, the representative's decision problem can be written as follows:

\begin{align}

& \mathrm{max\ } U\_t(x\_t,\theta\_t)+\sum\_{s=t+1}^5 \mathbb{P}(\theta\_s|\theta\_t)U\_s(x\_s,\theta\_{s|t})+\sum \mathbb{P}(\theta\_6|\theta\_t)U\_6(\theta\_{6|t})\\ & \mathrm{\ subject\ to\ } x\leq0\mathrm{, for\ all \ }t\in\{0,1,2,3,4,5,6\}

\end{align}

where the utility functions $U\_s(\theta\_{s|t})$ stand for utilities for states reachable from state $\theta\_t$. The dynamic programming approach breaks the problem apart as the greedy algorithm does. Since we need to add up utilities of different periods, we define a sequence of value functions $V\_t(\bar{c}\_t,\theta\_t)$ for $t\in\{0,1,2,3,4,5,6\}$ that represents the value of of having amount $\bar{c}\_t$ at each period $t$ and state of nature $\theta\_t$. Notice that \begin{equation}

V\_6(\bar{c}\_6,\theta\_6)=\left[\frac{1-\beta}{1-\beta(1+g)^{\rho}}\right]^{\frac{1}{\rho}}\bar{c\_6}\cdot(1-D\_6(CRF\_6,\theta\_6))

\end{equation}

where $D\_6(CRF\_6,\theta\_6)$ is given in Equation \ref{damage}.

The value of any previous time periods can be calculated with backward induction. In our problem, we have

\begin{align}

V\_5(\bar{c}\_5,\theta\_5) & = \left[(1-\beta)c\_5^{\rho}+\beta V\_6^{\rho}(\bar{c}\_6,\theta\_{6|5})\right]^{\frac{1}{\rho}}+V\_6(\bar{c}\_6,\theta\_{6|5})\\

V\_4(\bar{c}\_4,\theta\_4) & = \left[(1-\beta)c\_4^{\rho}+\beta\left(\frac{1}{2}V\_5^{\alpha}(\bar{c}\_5,\theta^1\_{5|4})+\frac{1}{2}V\_5^{\alpha}(\bar{c}\_5,\theta^2\_{5|4})\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}+\sum\_{k=1}^2 V\_5(\bar{c}\_5,\theta^k\_{5|4})\\

V\_3(\bar{c}\_3,\theta\_3) & = \left[(1-\beta)c\_3^{\rho}+\beta\left(\frac{1}{2}V\_4^{\alpha}(\bar{c}\_4,\theta^1\_{4|3})+\frac{1}{2}V\_4^{\alpha}(\bar{c}\_4,\theta^2\_{4|3})\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}+\sum\_{k=1}^2 V\_4(\bar{c}\_4,\theta^k\_{4|3})\\

V\_2(\bar{c}\_2,\theta\_2) & = \left[(1-\beta)c\_2^{\rho}+\beta\left(\frac{1}{2}V\_3^{\alpha}(\bar{c}\_3,\theta^1\_{3|2})+\frac{1}{2}V\_3^{\alpha}(\bar{c}\_3,\theta^2\_{3|2})\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}+\sum\_{k=1}^2 V\_3(\bar{c}\_3,\theta^k\_{3|2})\\

V\_1(\bar{c}\_1,\theta\_1) & = \left[(1-\beta)c\_1^{\rho}+\beta\left(\frac{1}{2}V\_2^{\alpha}(\bar{c}\_2,\theta^1\_{2|1})+\frac{1}{2}V\_2^{\alpha}(\bar{c}\_2,\theta^2\_{2|1})\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}+\sum\_{k=1}^2 V\_2(\bar{c}\_2,\theta^k\_{2|1})\\

V\_0(\bar{c}\_0,\theta\_0) & = \left[(1-\beta)c\_0^{\rho}+\beta\left(\frac{1}{2}V\_1^{\alpha}(\bar{c}\_1,\theta^1\_1)+\frac{1}{2}V\_1^{\alpha}(\bar{c}\_2,\theta^2\_1)\right)^{

\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}+\sum\_{k=1}^2 V\_1(\bar{c}\_1,\theta^k\_1)

\end{align}

(See Equation \ref{c0} to \ref{c5} for more information about the consumption $c\_t$.)

\end{enumerate}

This method does not make much sense compared to the other two, because the utility function at each node takes future utilities into account in our model.

\tc{Both the greedy algorithm and dynamic programming approach have problems that we cannot figure out at this point. For example, we need to know the state of nature in order to do optimization at a certain point. How can we know a point's state of nature? In addition, we lay out the formulas as if all parent nodes have equal probabilities for children nodes of the same period. However, it should not be the case as we gain knowledge of the damage function over time.}

\end{comment}

\section{Epstein-Zin preferences}

In this section we describe Epstein-Zin utility function and provide support for the maximization problem used in the EZ-Climate change model.

First we recall that expected time-separable preferences are defined as

\begin{flalign}

\bar U\_t:=\mathbb E\_t\left[\sum\limits\_{s=0}^{\infty}\beta^{s}u(C\_{t+s})\right]

\end{flalign}

where $\bar U\_t$ is the lifetime expected utility (sum of discounted utilities $u(C\_{t+s})$ at different points in time) and $\beta$ is the discount rate.\footnote{ $\mathbb E\_t[\cdot]:=\mathbb E[\cdot | \mathcal F\_t]$ and $\mathcal F\_t$ is the information filtration available at time $t$.} Note that $\bar U\_t$ can also be written recursively\footnote{To get this identity one should use $\mathbb E\left[\mathbb E\left[\cdot | \mathcal F\_{t+1}\right] | \mathcal F\_t \right] = \mathbb E[\cdot | \mathcal F\_t]$ (tower property) and the definition of $V\_t$.}

\begin{flalign} \label{utility1}

\bar U\_t=u(C\_t)+\beta \mathbb E\_t[\bar U\_{t+1}].

\end{flalign}

Since the utility function is invariant under affine transformations (in this case invariant under multiplication by $(1-\beta)$), the equation \eqref{utility1} can also be written as\footnote{It is invariant in the sense that the preferences defined by $(1-\beta)u(C\_t)$ are the same as the preferences defined by $u(C\_t)$.}

\begin{flalign}

U\_t&:=(1-\beta)\bar U\_t=(1-\beta)u(C\_t)+\beta \mathbb E\_t[(1-\beta)\bar U\_{t+1}] \nonumber \\

&=(1-\beta)u(C\_t)+\beta \mathbb E\_t[U\_{t+1}]

\end{flalign}

The Epstein-Zin preferences generalize this idea and define preferences recursively as a function of current utility $C\_t$ and the certainty equivalent $R\_t(U\_{t+1})$ of future utility $U\_{t+1}$, that is,

\begin{flalign}

U\_t=F(C\_t,R\_t(U\_{t+1}))

\end{flalign}

where

\begin{flalign}

R(U\_{t+1})=G^{-1}\left(\mathbb E\_t(G(U\_{t+1}))\right)

\end{flalign}

with $F$ and $G$ are increasing and concave, and $F$ is homogeneous of degree one. One should notice that $R\_t(U\_{t+1})=U\_{t+1}$ implies no uncertainty about $U\_{t+1}$. Also, the more concave $G$ is, and the more uncertain $U\_{t+1}$ is, the lower is $R\_t(U\_{t+1})$.

Most of the literature considers the following functional forms:

\begin{flalign}

F(c,z)& = \left((1-\beta)c^{\rho}+\beta z^{\rho}\right)^{1/\rho}, \\

G(x)& = \frac{x^{\alpha}}{\alpha},

\end{flalign}

which imply

\begin{flalign}

U\_t=\left((1-\beta)C\_t^{\rho}+\beta(\mathbb E\_t[U\_{t+1}^{\alpha}])^{\rho/\alpha}\right)^{1/\rho}

\end{flalign}

and this is equivalent to Epstein-Zin utility function used in the Litterman's paper. Thus, assuming that the current time is time 0, for Epstein-Zin preferences, maximizing lifetime expected utility is equivalent to maximizing $U\_0$.

\section{Maximization of $U\_0$}

In the EZ-Climate model, we have a 7-period tree, beginning 2015 (See Figure \ref{tree}). Every period except period 0 has more than 1 node, where the representative agent makes mitigation decision. For an arbitrary period $t$, we label the nodes as state, $\theta\_t$, beginning at 0. Based on the tree model, we have 63 nodes in total and index all of them 0 - 62 from period 0 to the last period. Within each period, their number increase from top to bottom (this depends on the number of 'd' states it takes to get to the node from the start point). You can see the nodes 0 to 6 in Figure \ref{tree} to get a better understanding of the node ordering.

\begin{figure}[h!]

\caption{Tree Model}

\includegraphics[scale=0.6]{Tree.png}

\label{tree}

\end{figure}

Essentially, the representative agent is intended to maximize his/her utility at the start point, $U(0)$ with respect to mitigation levels \textbf{m}. Mathematically speaking, $\textbf{m}\in \mathbb{R}^{63}$ is a $1\times 63$ vector with 63 non-negative elements, $x\_{t,\theta\_t}$, that stand for mitigation levels at each decision point of the tree model. This corresponds to the 63 nodes we just mentioned. For instance, \textbf{m}'s third element, $x\_{1,2}$ represents the mitigation level chosen for the second state of period 1.

As we know, the utility formulas are:

\begin{align}

U\_0 & = \left[(1-\beta)c\_0^{\rho} + \beta (E\_0[U\_1^{\alpha}])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_1 & = \left[(1-\beta)c\_1^{\rho} + \beta (E\_1[U\_{2}^{\alpha}])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_2 & = \left[(1-\beta)c\_2^{\rho} + \beta (E\_2[U\_{3}^{\alpha}])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_3 & = \left[(1-\beta)c\_3^{\rho} + \beta (E\_3[U\_{4}^{\alpha}])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_4 & = \left[(1-\beta)c\_4^{\rho} + \beta (E\_4[U\_{5}^{\alpha}])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_5 & = \left[(1-\beta)c\_5^{\rho} + \beta (E\_5[U\_{6}^{\alpha}])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_6 & = \left[\frac{1-\beta}{1-\beta(1+g)^{\rho}}\right]^{\frac{1}{\rho}}c\_6

\end{align}

where $(E\_t[U\_{t+1}^{\alpha}])^{1/\alpha}$ is the certainty-equivalent of future lifetime utility, based on the agent's information at time $t$. Both consumption $c\_t$ and utility $U\_t$ depend on mitigation levels in \textbf{m}, as they depend on mitigation actions before and after period $t$, so the system can be represented as

\begin{align}

\label{u0}

U\_0(\mathbf{m}) & = \left[(1-\beta)\mathbf{c}\_0^{\rho}(\mathbf{m}) + \beta (E\_0[U\_{1}^{\alpha}(\mathbf{m})])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_1(\mathbf{m}) & = \left[(1-\beta)\mathbf{c}\_1^{\rho}(\mathbf{m}) + \beta (E\_1[U\_{2}^{\alpha}(\mathbf{m})])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_2(\mathbf{m}) & = \left[(1-\beta)\mathbf{c}\_2^{\rho}(\mathbf{m}) + \beta (E\_2[U\_{3}^{\alpha}(\mathbf{m})])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_3(\mathbf{m}) & = \left[(1-\beta)\mathbf{c}\_3^{\rho}(\mathbf{m}) + \beta (E\_3[U\_{4}^{\alpha}(\mathbf{m})])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_4(\mathbf{m}) & = \left[(1-\beta)\mathbf{c}\_4^{\rho}(\mathbf{m}) + \beta (E\_4[U\_{5}^{\alpha}(\mathbf{m})])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

U\_5(\mathbf{m}) & = \left[(1-\beta)\mathbf{c}\_5^{\rho}(\mathbf{m}) + \beta (E\_5[U\_{6}^{\alpha}(\mathbf{m})])^{\frac{\rho}{

\alpha}}\right]^{\frac{1}{\rho}}\\

\label{u6}

U\_6(\mathbf{m}) & = \left[\frac{1-\beta}{1-\beta(1+g)^{\rho}}\right]^{\frac{1}{\rho}}\mathbf{c}\_6(\mathbf{m})

\end{align}

Note that in the above equations $U\_t(\mathbf{m})$ takes into account all possible nodes at period $t$ (due to the nature of binomial tree model). To evaluate $U\_t(\mathbf{m})$, we have to use $U\_{t+1}(\mathbf{m})$. This is where probabilities of our binomial tree plays an important role. EZ\\_Climate sets equal probabilities to the 2 branches of each node. For example, in Figure \ref{tree}, $\mathbb{P}(uu|u)=\mathbb{P}(ud|u)=\frac{1}{2}$. Hence, for the node with the mitigation level $x\_{t,\theta\_t}$, we calculate the expected future utility via $\mathbb E\_t\left[U\_{t+1}^{\alpha}\right]=\frac{1}{2}\times \mathbb{E}\_{t}\left[U\_{t+1}^{\alpha}\left(x\_{t+1,\theta\_{t+1}^1}\right)\right]+ \frac{1}{2}\times\mathbb{E}\_{t}\left[ U\_{t+1}^{\alpha}\left(x\_{t+1,\theta\_{t+1}^2}\right)\right]$ where $\theta\_{t+1}^1$ and $\theta\_{t+1}^2$ are the 2 child nodes of node $\theta\_t$. (To continue with the notation in the last section, $\mathbb{E}\_{t}$ means the expectation based on the information filtration available at time $t$, similar for $\mathbb{E}\_{t+1}$.) In other words, we take the average of possible utilities at period $t+1$. Notice that we start with certain mitigation levels \textbf{m} in the code, so it is possible for us to calculate the expectations of future utilities at time $t$.

To be more specific with the notation, the vectors $\mathbf{c}\_t$ for $t\in\{0\dots 5\}$, of dimension $1\times 2^{t}$, contain the consumption of all the nodes in period $t$. For example: $\mathbf{c\_1} = (c\_{1,1},c\_{1,2})$. The same applies to $\mathbf{c}\_6$, which has dimension $1\times 32$ in our model. In order to make our demonstration clear, we focus on a single mitigation path from now on and denote it as $\widetilde{x}^p\_t\in\mathbb{R}^{t+1}$, a $1\times (t+1)$ vector that consists of mitigation level $x\_s$ for $s\in\{0\dots t\}$. The $p$ here denotes the path and $\widetilde{x}\_t$ distinguishes itself as a vector from the other $x\_t$'s, the mitigation levels.

Throughout our 7-period tree model, we have 32 states in the very last period and therefore 32 equally possible paths from period 0 to period 6. As mentioned before, every node of period $t<5$ has 2 child nodes, or paths, in period $t+1$. In this sense, we have $2^1=2$ paths to period 1's nodes and $2^3=8$ paths to period 3's nodes. At each node of period $t$, we look forward to its child paths til the last period to get the certainty-equivalent of future lifetime utility $E\_t[U\_{t+1}^{\alpha}(\mathbf{m})]$ and determine utility $U\_t(\mathbf{m})$ in concern. This recombination procedure makes the 32 paths compatible with our binomial tree.

In terms of the consumption model, in each period $t \in \{0,1,2,3,4,5,6\}$, the agent is endowed with a certain amount of the consumption good, $\bar{c\_t}$. However, the agent is not able to consume the full endowed consumption for 2 reasons: climate change and climate policy. First, in period $t \in \{1,2,3,4,5,6\}$, some of the endowed consumption may be lost due to climate change damages. Second, in periods $t \in \{0,1,2,3,4,5\}$, the agent may elect to spend some of the endowed consumption on mitigation to reduce his impact on the climate. The resulting consumption $c\_t$, after damages and mitigation costs are taken into account, is given by:

\begin{align}

\label{c0}

c\_0 & = \bar{c\_0}\cdot (1-\kappa(\widetilde{x}^p\_0))\\

c\_1 & = \bar{c\_1}\cdot (1-D\_1(\bar{x}\_1(CRF\_1(\widetilde{x}^p\_1)),\theta\_1)-\kappa(\widetilde{x}^p\_1))\\

c\_2 & = \bar{c\_2}\cdot (1-D\_2(\bar{x}\_2(CRF\_2(\widetilde{x}^p\_2)),\theta\_2)-\kappa(\widetilde{x}^p\_2))\\

c\_3 & = \bar{c\_3}\cdot (1-D\_3(\bar{x}\_3(CRF\_3(\widetilde{x}^p\_3)),\theta\_3)-\kappa(\widetilde{x}^p\_3))\\

c\_4 & = \bar{c\_4}\cdot (1-D\_4(\bar{x}\_4(CRF\_4(\widetilde{x}^p\_4)),\theta\_4)-\kappa(\widetilde{x}^p\_4))\\

\label{c5}

c\_5 & = \bar{c\_5}\cdot (1-D\_5(\bar{x}\_5(CRF\_5(\widetilde{x}^p\_5)),\theta\_5)-\kappa(\widetilde{x}^p\_5))\\

c\_6 & = \bar{c\_6}\cdot(1-D\_6(\bar{x}\_6(CRF\_6(\widetilde{x}^p\_6)),\theta\_6))

\end{align}

\tb{$\theta\_t$ here stands for the state at period $t$ on the path. In a specific case, suppose we have path that leads to the first state in the last period, i.e. the path of all 'u' states. Then the notation of above should be changed as following:

\begin{itemize}

\item $c\_0$ \rightarrow $\ c\_{0,1}$

\item $c\_1$ \rightarrow $\ c\_{1,1}$

\item $c\_2$ \rightarrow $\ c\_{2,1}$

\item $c\_3$ \rightarrow $\ c\_{3,1}$

\item $c\_4$ \rightarrow $\ c\_{4,1}$

\item $c\_5$ \rightarrow $\ c\_{5,1}$

\item $c\_6$ \rightarrow $\ c\_{6,1}$

\end{itemize}As we can see, both damage function $D\_t(\bar{x}\_t(CRF(\widetilde{x}^p\_t)),\theta\_t)$ and cost function $\kappa(\widetilde{x}^p\_t)$ are dependent on mitigation level up to $t$, $\widetilde{x}^p\_t$. However, $D\_t(\bar{x}\_t(CRF(\widetilde{x}^p\_t)),\theta\_t)$ and $\kappa(\widetilde{x}^p\_t)$ differ in they way they depend on $\widetilde{x}^p\_t$. The damage function $D\_t(\bar{x}\_t(CRF(\widetilde{x}^p\_t)),\theta\_t)$ uses $\widetilde{x}^p\_t$ to determine the cumulative forcing $CRF\_t$, which in turn help determine realized average mitigation up to time $t$, $\bar{x}\_t$. (See Section \ref{crf} for details.) On the other hand, the cost function $\kappa(\widetilde{x}^p\_t)$ uses $\widetilde{x}^p\_t$ to calculates the cumulative mitigation level $X\_t$ given by:

\begin{equation}

\label{cumulative}

X\_t=\frac{\sum\_{s=0}^t g\_s\cdot x\_s}{\sum\_{s=0}^t g\_s}

\end{equation}

where $g\_s$ is the flows of GHG emissions into the atmosphere in period $s$, for each period up to $t$, absent any mitigation.

It can be messy to lay out the damage and cost function here, but reader can look at Section \ref{Damage} and Equation \ref{cost}, respectively, for details.}

\subsection{Damage Function $D\_t(\bar{x}\_t(CRF\_t(\widetilde{x}^p\_t)),\theta\_t)$}

\label{Damage}

The damage function $D\_t(\bar{x}\_t(CRF\_t(\widetilde{x}^p\_t)),\theta\_t)$ captures the fraction of the endowment consumption that is lost due to damages from climate change. It mainly depends on 2 variables: $CRF\_t$, which we define as the cumulative radiative forcing up to time $t$, which determines global average temperature, and state $\theta\_t$, the Earth's fragility, a parameter that characterizes the uncertain relationship between the global average temperature and consumption damages.

We define damage function as a function of temperature changes, which, in turn, are a function of cumulative solar radiative forcing $CRF\_t$, which, in our setting, are determined by the mitigation path up to that point in time $t$, $\widetilde{x}^p\_t$. We then compare $CRF\_t$ to three baseline emissions paths, $g\_t$, for which we have created associated damage simulations. The only way, then, to affect the level of damages is to change mitigation across time, $x\_t$. The specification of damages has 2 components: a non-catastrophic component and a catastrophic component triggered by crossing a particular threshold. (The hazard rate associated with hitting that threshold increases with temperature.) If the threshold is crossed at any time, additional changes decreases consumption in all future periods.

We calculate the overall damage function $D\_t(\bar{x}\_t(CRF\_t(\widetilde{x}^p\_t)),\theta\_t)$ for the baseline emission paths, $g\_t$, using Monte-Carlo simulation. We run a set of simulations for each of 3 constant mitigation levels, which determine cumulative radiative forcing at each point in time. The state variable $\theta\_t$ indexes the distribution resulting from these sets of simulations, and interpolation across the three mitigation level gives us a continuous function $D\_t$ across cumulative radiative forcing levels $CRF\_t$.

Therefore, based on the simulation results, we can determine $D\_t$ as a function of $CRF\_t$ and $\theta\_t$, where $CRF\_t$ is a function of the mitigation paths $g\_t$ of $x\_t$.

\subsubsection{$CRF\_t$ as a function of $x\_t$}

\label{crf}

We have mitigation level $x\_t$ and the GHG emission under business-as-usual case, absent any mitigation, $BAU\_t$ for $t \in \{0,1,2,3,4,5,6\}$. We look into each period by unit called \textbf{subinterval}. For each subinterval we calculate \textbf{forcing} and \textbf{absorption}, with which we get cumulative forcings $CRF\_t$ and the ghg levels $GHG\_t$. Their values in the very beginning are:

\begin{equation}

\begin{cases}

CRF\_0 & = 4.926\\

GHG\_0 & = 400\\

sink\_0 & = 35.396

\end{cases}

\end{equation}

Here is how we calculate the variables for each subinterval:

\begin{itemize}

\item For each period $t$:

\begin{equation}

\begin{cases}

\text{beginning emission }& g\_{t,0}=(1-x\_t)\times BAU\_{t}\\

\text{ending emission}& g\_{t,t}=\begin{cases}

(1-x\_t)\times BAU\_{t+1}, &\text{if }t<5\\

(1-x\_t)\times BAU\_{t}, & \text{else}

\end{cases}

\end{cases}

\end{equation}

\item For each subinterval $i$:

\begin{cases}

\text{p\\_co2}\_i& =0.71\times\left[g\_{t,0}+i\times\frac{g\_{t,t}-g\_{t,0}}{\text{number of subintervals in period }t}\right]\\

\text{p\\_c}\_i&=\frac{\text{p\\_co2}\_i}{3.67}\\

\text{add\\_p\\_ppm}\_i&=\text{length of subinterval}\times \frac{\text{p\\_c}\_i}{2.13}\\

\text{lsc}\_i&=285.6268+\text{cum\\_sink}\_i \times0.88414\\

\text{absorption}& = 0.5\times0.94835\times\lvert GHG\_i-\text{lsc}\_i\rvert^{0.741547}\\

\text{cum\\_sink}\_i & =\text{cum\\_sink}\_{i-1}+ \text{absorption}\_i\\

GHG\_i &=GHG\_{i-1}+\text{add\\_p\\_ppm}\_i-\text{absorption}\_i\\

CRF\_i & = CRF\_{i-1} + 0.13183\times\lvert GHG\_i-315.3785\rvert ^{0.607773}

\end{cases}

\end{itemize}

\subsubsection{$D\_t$ as a function of $\bar{x}\_t(CRF\_t)$}

\label{realized average}

In order to get the desired damage $D\_t$, we run damage simulations for the 3 base scenarios and determine the damage coefficients in concern for each. Based on the $CRF\_i$'s and $GHG\_i$'s we get in the last section, we calculate the realized average mitigation $\bar{x}\_t$ up to each period and compare with the constant mitigation levels in the base cases to interpolate the damage along the given path.

The 3 base scenarios correspond to maximum final GHG levels of 450, 650, and 1000 ppm. We calculate their cumulative radiative forcings $CRF\_t$ and constant mitigation levels, or cumulative mitigation levels, $x\_{450/650/1000}=\bar{x}\_{450/650/1000}$, for each period. The way we infer the cumulative mitigation from cumulative forcing up to a particular point is:

\begin{equation}

\bar{x}(CRF\_t)=w\_{450}(CRF\_t)\times \bar{x}\_{450}+w\_{650}(CRF\_t)\times \bar{x}\_{650}

\end{equation}

where the weights are given by:

\begin{itemize}

\item If $CRF\_t>CRF\_{650}$, \begin{cases}

w\_{650}&=\frac{CRF\_{1000}-CRF\_t}{CRF\_{1000}-CRF\_{650}}\\

w\_{450}&=0

\end{cases}

\item If $CRF\_{650}>CRF\_t>CRF\_{450}$,

\begin{cases}

w\_{650}&=\frac{CRF\_{t}-CRF\_{450}}{CRF\_{650}-CRF\_{450}}\\

w\_{450}&=\frac{CRF\_{650}-CRF\_{t}}{CRF\_{650}-CRF\_{450}}

\end{cases}

\item Otherwise,

\begin{cases}

w\_{650}&=0\\

w\_{450}&=1+\frac{CRF\_{450}-CRF\_{t}}{CRF\_{450}}

\end{cases}

\end{itemize}

We calculate an interpolated damage function using our 3 simulations where we have damage coefficients (for a given state and period) to find a smooth function that gives damages for any particular level of radiative forcing up to each point in time. For realized mitigation below that of 650 ppm, let the damage coefficients for state $n$ at period $t$ be $d\_{t, n}^{650,1}$ and $d\_{t, n}^{650,0}$. Similarly, for realized mitigation between those of 650 and 450 ppm, let the damage coefficients for state $n$ at period $t$ be $d\_{t, n}^{450,2}$, $d\_{t, n}^{450,1}$, and $d\_{t, n}^{450,0}$. To find the function, we assume a linear interpolation of damages between the 650 and 1000 ppm scenarios, and a quadratic interpolation between 450 and 650 ppm. In addition, we impose a smooth pasting condition at 650 ppm, having the level and derivative of the interpolation between 650 ppm match the level and scope of the line above.

For consequent GHG level below 450 ppm, the realized mitigation is more than 100\% and we assume climate damages exponentially decay toward 0. Mathematically, we let $S=\frac{d\cdot p}{l\cdot ln(0.5)}$, where $d$ is the derivative of the quadratic damage interpolation function at 450 ppm, $p=0.91667$ is the average mitigation in the 450 ppm simulation, and the level of damage is $l$. Radiative forcing at any point below 450 ppm then is $y$ percent below that of the 450 ppm simulation, with $y=\frac{CRF\_{450}-CRF}{CRF\_{450}}$, where $R$ is the radiative forcing in the 450 ppm simulation and $r$ is the radiative forcing given the mitigation policy \textbf{m}. Letting $\sigma=60$, the extension of the damage function for $y>0$ is defined as:

\begin{equation}

Damage(y)=l\cdot 0.5^{(y\cdot S)}e^{-\frac{(y\cdot p)^2}{\sigma}}

\end{equation}

Therefore, given the cumulative forcing $CRF\_6$ for node $n$ up to period 6, we calculate the realized average mitigation $\bar{x}(CRF\_6)$ and calculate its damage in the following way:

\begin{equation}

\label{damage}

D\_6(\bar{x}\_6(CRF\_6(\widetilde{x}^p\_6)),\theta\_6)=\begin{cases}

d\_{6, n}^{650,1}\times \bar{x}(CRF\_6)+d\_{6, n}^{650,0}, &\text{for }\bar{x}(CRF\_6)<\bar{x}\_{650};\\

d\_{6, n}^{450,2}\times \bar{x}(CRF\_6)^2+d\_{6, n}^{450,1}\times \bar{x}(CRF\_6)+d\_{6, n}^{450,0}, &\text{for }\bar{x}\_{450}< \bar{x}(CRF\_6)\leq \bar{x}\_{650};\\

0.5^{\frac{\bar{x}(CRF\_6)-\bar{x}\_{450}+ln(d\_{6,n}^{450})}{ln(0.5)}}e^{-\frac{(\bar{x}(CRF\_6)-\bar{x}\_{450})^2}{60}}, &\text{else}

\end{cases}

\end{equation}

The $d\_{6,n}^{450}$ in the last case is the simulated damage for state $n$ under the 450 ppm scenario.

The climate sensitivity- summarized by state of nature $\theta\_t$- is not known prior to the final period (t=6). Rather, what the representative agent knowns is the distribution of possible final states $\theta\_6$. We specify that the damage in period $t$, given a cumulative radiative forcing, $CRF\_t$, up to time $t$, is the probabiity weighted average of the interpolated damage function over all final states of nature reachable from that node. In particular, the damage function at time $t$, for the node indexed by $\theta\_t$ is assumed to be:

\begin{equation}

D\_t(\bar{x}\_t(CRF\_t(\widetilde{x}^p\_t)),\theta\_t)=\sum\_{\theta\_6} \mathbb{P}(\theta\_6|\theta\_t)\cdot D\_t(CRF\_t,\theta\_6)

\end{equation}

In addition, we have a penalty function for the damages of the concentrations below pre-industrial levels:

\begin{equation}

f(GHG\_t)=\left[ 1+e^{0.05\times(GHG\_t-200)}\right]^{-1}

\end{equation}

Hence, the damage function at time $t$ is:

\begin{equation}

D\_t(\bar{x}\_t(CRF\_t(\widetilde{x}^p\_t)),\theta\_t)=\sum\_{\theta\_6} \mathbb{P}(\theta\_6|\theta\_t)\cdot D\_t(CRF\_t,\theta\_6)+\left[ 1+e^{0.05\times(GHG\_t-200)}\right]^{-1}

\end{equation}

\subsection{Cost Function $\kappa(x\_t)$}

Mitigation reduces the stock of GHGs in the atmosphere and leads to lower climate damages, and, hence, to higher future consumption. But mitigating GHG emissions is costly. Mitigating a fraction of $x\_t$ of emissions costs a fraction $\kappa(x\_t)$ of the endowed consumption. In terms of the mitigation cost, we consider both the traditional cost regarding taxation and technological components. The latter one matters for $x\_t$ above the fractional-mitigation threshold, $x^\*$, at which the backstop technology kicks in. The consequent upper and lower bound for marginal technology costs are $\tilde{\tau}$ and $\tau^\*$ respectively. From these bounds we can calculate relevant parameters $b$ and $k$. Notice that the cumulative mitigation $X\_t$ given in Equation \ref{cumulative} enters the determination of the rate of technology change, where we have constants $\theta\_0$ and $\theta\_1$. The general equation for marginal cost is given by:

\begin{equation}

\label{cost}

\kappa(x\_t)=\left[ 1-\theta\_0-\theta\_1X\_t\right]\begin{cases}

\frac{g\_0}{c\_0}\cdot 92.08\cdot x\_t^{\alpha}, &x\_t\leq x^\*\\

\frac{g\_0}{c\_0}\cdot 92.08\cdot x^{\*\alpha}+\tilde{\tau}x\_t-\tilde{\tau}x^\*-\frac{bx\_t(\frac{k}{x\_t})^{\frac{1}{b}}}{b-1}+\frac{bx^\*(\frac{k}{x^\*})^{\frac{1}{b}}}{b-1}, &x\_t>x^\*\\

\end{cases}

\end{equation}

\end{document}