Compsci 571 HW2

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1 Classifier for Basketball Courts

(a) When running Perceptron algorithm on the dataset, it takes 7 iterations (updates) to converge. The decision boundary is $f(x_1, x_2) = -1.05 * x_1 + 1.1 * x_2$. Because after it converges, all training points are correctly classified, the error rate is 0.

Assume another linear classifier that goes through origin and achieves the same training error rate (0) as the perceptron classifier is $f(x_1, x_2) = w_1 * x_1 + w_2 * x_2$. Set $f(x_1, x_2) = 0$, we get the slope of the boundary is $-\frac{w_1}{w_2}$. From the plot of training data, we know that the boundary should go above point $[0.85, 0.80]$, and go below point $[0.85, 0.95]$. So $\frac{0.80}{0.85} < -\frac{w_1}{w_2} < \frac{0.95}{0.85}$. If we set $w_2 = 1.1$ as the perceptron boundary, we get $-1.229 < w_1 < -1.035$.

The plot of observed data, the perceptron decision boundary (the light blue line), and all other linear boundaries that achieve the same training error (the yellow area) is:
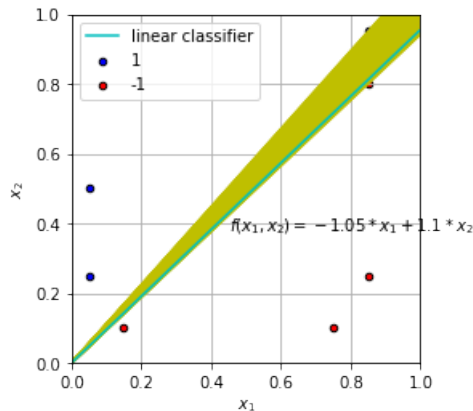

(b) The fully-grown decision tree using Gini index as splitting criterion on the observed data is:
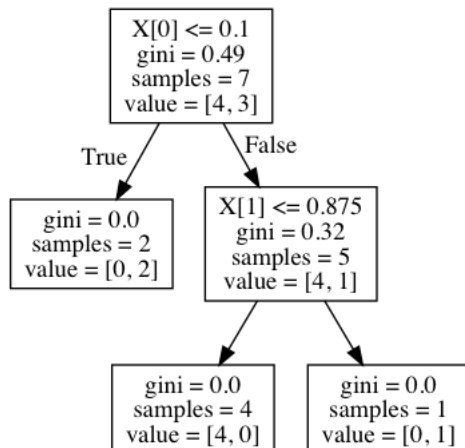
Because all training points are correctly classified by this tree, its training error is 0.

Assume another decision tree with same training error (0) splits on the same feature order but different splitting threshold (v_1 for x_1 and v_2 for x_2). Then the threshold of the first split on x_1 should be able to separate points $[0.05, 0.25], [0.05, 0.5]$ (+1) with $[0.15, 0.1]$ (-1). So v_1 should be $\in (0.05, 0.15)$. The threshold of the second split on x_2 should be able to separate points $[0.85, 0.8]$ (-1) with $[0.85, 0.95]$ (+1). So v_2 should be $\in (0.8, 0.95)$.

The plot of observed data, and the calculated decision boundary is:
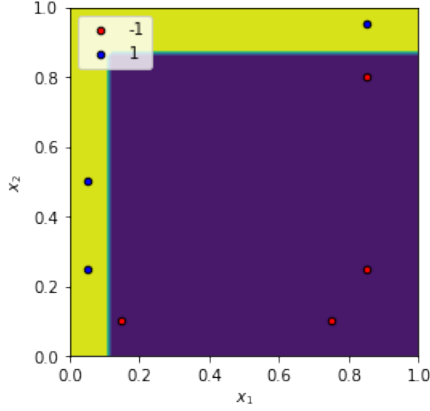
The plot of observed data, the calculated decision boundary, and all other decision boundaries that achieve the same training area (the red area) is:
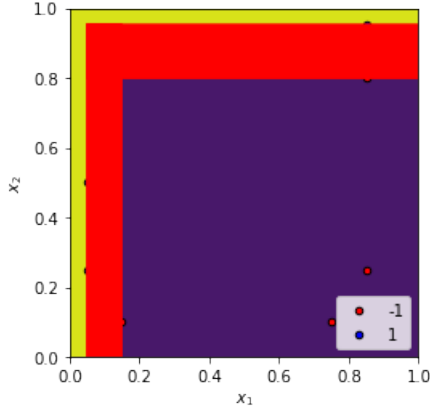
(c) Suppose the real optimal linear classifier that passes through the origin is $f(x_1, x_2) = w_1 * x_1 + w_2 * x_2$, such that it is able to minimize $R^{true}(f)$.

$$T = R^{true}(f) = \mathbb{E}_{(\mathbf{x},y)\sim D} l(f(\mathbf{x}), y) = \mathbb{E}_{(\mathbf{x},y)\sim D} \mathbf{1}_{[sign(f(\mathbf{x}))\neq y]} \tag{1}$$

$$= \mathbf{P}(sign(f(\mathbf{x})) \neq y) \tag{2}$$

$$= \mathbf{P}(y = 1, f(\mathbf{x}) \leq 0) + \mathbf{P}(y = -1, f(\mathbf{x}) \geq 0) \tag{3}$$

$$= \mathbf{P}(y = 1) * \mathbf{P}(f(\mathbf{x}) \leq 0|y = 1) + \mathbf{P}(y = -1) * \mathbf{P}(f(\mathbf{x}) \geq 0|y = -1) \tag{4}$$

$$= (1-\frac{\pi}{4}) * \mathbf{P}(w_1*x_1+w_2*x_2 \leq 0|0 \leq x_1 \leq 1, \sqrt{x_1} \leq x_2 \leq 1) + \frac{\pi}{4} * \mathbf{P}(w_1*x_1+w_2*x_2 \geq 0|0 \leq x_1 \leq 1, 0 \leq x_2 \leq \sqrt{x_1}) \tag{5}$$

$$= (1-\frac{\pi}{4}) * \mathbf{P}(x_2 \leq -\frac{w_1}{w_2}x_1|0 \leq x_1 \leq 1, \sqrt{x_1} \leq x_2 \leq 1) + \frac{\pi}{4} * \mathbf{P}(x_2 \geq -\frac{w_1}{w_2}x_1|0 \leq x_1 \leq 1, 0 \leq x_2 \leq \sqrt{x_1}) \tag{6}$$

2

Step (2) is from the property of expectation on indicator function. Step (4) is from the rule of conditional probability. In step (5), $\mathbf{P}(y = 1) = \frac{\pi}{4}$ and $\mathbf{P}(y = -1) = 1 - \frac{\pi}{4}$ because of the uniform distribution of (x_1, x_2).

Assign $p = -\frac{w_1}{w_2}$, $p \in [0, \infty)$, equation (6) becomes:

$$= (1 - \frac{\pi}{4}) * \mathbf{P}(x_2 \leq px_1 | 0 \leq x_1 \leq 1, \sqrt{x_1} \leq x_2 \leq 1) + \frac{\pi}{4} * \mathbf{P}(x_2 \geq px_1 | 0 \leq x_1 \leq 1, 0 \leq x_2 \leq \sqrt{x_1}) \quad (7)$$

So we need to find the optimized p that minimizes $R^{true}(f)$, or equivalently equation (7).

If $p \in [0, 1]$, $T = \frac{\pi}{4} * (\frac{\pi}{4} - \frac{1}{2} * 1 * p)$, and local optimized $p' = 1$ minimizes $T' = \frac{\pi}{4}[\frac{\pi}{4} - \frac{1}{2}]$.

If $p \in (1, \infty)$, based on geometry in the 2D space and integration, I get $T = \frac{1}{6}p^{-3} + (\frac{1}{2} - \frac{\pi}{8})p^{-1} + (1 - \frac{\pi}{4})$. So local optimized $p'' = (1 - \frac{\pi}{4})^{-2}$, and $T'' = \frac{1}{6}(1 - \frac{\pi}{4})^6 - \frac{1}{2}(1 - \frac{\pi}{4})^3 + (1 - \frac{\pi}{4})$.

So the global optimized $p^* = p'' = (1 - \frac{\pi}{4})^{-2} \approx \mathbf{2.158655221735395}$, and the corresponding minimal $R^{true}(f) = \frac{1}{6}(1 - \frac{\pi}{4})^6 - \frac{1}{2}(1 - \frac{\pi}{4})^3 + (1 - \frac{\pi}{4}) \approx \mathbf{0.18146363796206844}$.

This solution **is not** among the solutions that achieved the same loss (0) in part (a).

The plot of the decision boundary (blue line) on the basketball court is:
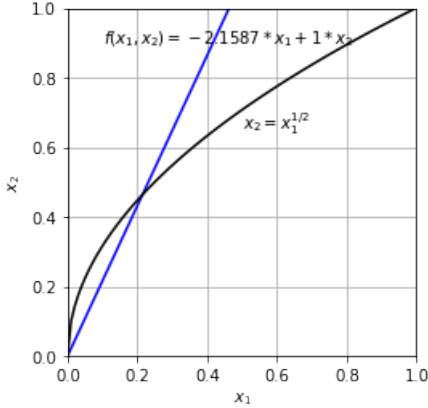
(d) The optimal depth 2 decision tree will split on $x_1 = m$ and $x_2 = n$. And $f(\mathbf{x}) = -1$ if $m \leq x_1 \leq 1$ and $0 \leq x_2 \leq n$, $f(\mathbf{x}) = 1$ otherwise.

$$T = R^{true}(f) = \mathbf{P}(y = 1) * \mathbf{P}(f(\mathbf{x}) \leq 0 | y = 1) + \mathbf{P}(y = -1) * \mathbf{P}(f(\mathbf{x}) \geq 0 | y = -1) \quad (8)$$

$$= (1 - \frac{\pi}{4}) * \mathbf{P}(f(\mathbf{x}) \leq 0 | y = 1) + \frac{\pi}{4} * \mathbf{P}(f(\mathbf{x}) \geq 0 | y = -1) \quad (9)$$

Step (8) comes from the same steps as in (c).

If $0 \leq n \leq \sqrt{m}$, $T = \frac{\pi}{4}[1 - \frac{4}{\pi}(1 - m)n] \geq \frac{\pi}{4}[1 - \frac{4}{\pi}(1 - m)\sqrt{m}]$. So the local optimized $m' = \frac{1}{3}$, local optimized $n' = \sqrt{m'} = \sqrt{\frac{1}{3}}$, and local minimal $T' = \frac{\pi}{4} - \frac{2}{3}\sqrt{\frac{1}{3}} \approx 0.4004979839376978$.

If $\sqrt{m} \leq n \leq 1$, according to geometry in 2D space and integration, $T = \frac{1}{3}n^3 + (\frac{\pi}{4} - 1)mn - \frac{\pi}{4}n + \frac{2}{3}m^{\frac{3}{2}} + \frac{\pi}{6}$. So the local optimized $m'' = \frac{(4-\pi)^2}{4^2 + 4(4-\pi) + \pi^2} \approx 0.025146138400079843$, local optimized $n'' = (1 - \frac{\pi}{4})m'' + \frac{\pi}{4} \approx 0.8892663104388737$, and the local minimal $T'' \approx 0.0574391669843608$.

So the global optimized $m^* = \frac{(4-\pi)^2}{4^2 + 4(4-\pi) + \pi^2} \approx \mathbf{0.025146138400079843}$, the global optimized $n^* = (1 - \frac{\pi}{4})m'' + \frac{\pi}{4} \approx \mathbf{0.8892663104388737}$, and the global minimal $R^{true}(f) \approx \mathbf{0.0574391669843608}$.

The real optimized tree decision boundary **is not** among those achieved in part (b).

The plot of the decision boundary (blue line) on the basketball court is:
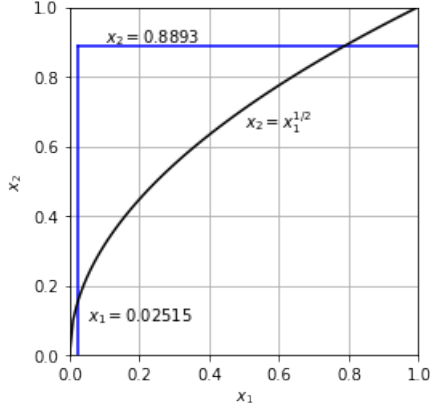
(e)

(f)

(g) For paint, assume the part inside paint ($0.5 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 0.25$) has label $y = -1$ and the part outside paint has $y = 1$.

Same as in part (c), suppose the real optimal linear classifier that passes through the origin is $f(x_1, x_2) = w_1 * x_1 + w_2 * x_2$, such that it is able to minimize $R^{true}(f)$.

$$T = R^{true}(f) = \mathbf{P}(y = 1) * \mathbf{P}(f(\mathbf{x}) \leq 0 | y = 1) + \mathbf{P}(y = -1) * \mathbf{P}(f(\mathbf{x}) \geq 0 | y = -1) \tag{10}$$

$$= \frac{7}{8} * \mathbf{P}(x_2 \leq -\frac{w_1}{w_2} x_1 | y = 1) + \frac{1}{8} * \mathbf{P}(x_2 \geq -\frac{w_1}{w_2} x_1 | y = -1) \tag{11}$$

Assign $p = -\frac{w_1}{w_2}$, $p \in [0, \infty)$,

$$= \frac{7}{8} * \mathbf{P}(x_2 \leq p x_1 | y = 1) + \frac{1}{8} * \mathbf{P}(x_2 \geq p x_1 | y = -1) \tag{12}$$

If $p \geq \frac{1}{2}$, $\mathbf{P}(x_2 \geq p x_1 | y = -1) = 0$, $T = \frac{7}{8} * \mathbf{P}(x_2 \leq p x_1 | y = 1)$.

 (i) If $1 \geq p \geq \frac{1}{2}$, $T = \frac{7}{8}[\frac{p}{2} - \frac{1}{8}]$, and local optimized $p' = \frac{1}{2}$ generates local minimal $p' = \frac{7}{64}$.
 (ii) If $p > 1$, $T = \frac{7}{8}[\frac{7}{8} - \frac{1}{2p}]$, and local optimized $p' = 1$ generates local minimal $p' = \frac{21}{64}$.

If $\frac{1}{2} > p \geq 0$,

 (i) If $\frac{1}{2} > p \geq \frac{1}{4}$, $\mathbf{P}(x_2 \geq p x_1 | y = -1) = \frac{1}{8}(\frac{1}{4p} + p - 1)$, $\mathbf{P}(x_2 \leq p x_1 | y = 1) = \frac{5p}{8} - \frac{1}{4} + \frac{1}{32p}$. $T = \frac{9p}{16} + \frac{1}{32p} - \frac{15}{64}$. So the local optimized $p' = \frac{1}{4}$ generates local minimal $T' = \frac{2}{64}$.
 (ii) If $\frac{1}{4} > p \geq 0$, $\mathbf{P}(x_2 \geq p x_1 | y = -1) = \frac{1}{4}[\frac{1}{2} - \frac{3p}{2}]$, $\mathbf{P}(x_2 \leq p x_1 | y = 1) = \frac{p}{8}$, $T = \frac{4p}{64} + \frac{1}{64}$. So the local optimal $p' = 0$ generates local minimal $T' = \frac{1}{64}$.

So in conclusion, the global optimized $-\frac{w_1}{w_2} = p = \mathbf{0}$, and the corresponding minimal true error rate $R^{true}(f) = \frac{1}{64}$.

(h) The optimal depth 2 decision tree will split on $x_1 = m$ and $x_2 = n$. And $f(\mathbf{x}) = -1$ if $m \leq x_1 \leq 1$ and $0 \leq x_2 \leq n$, $f(\mathbf{x}) = 1$ otherwise.
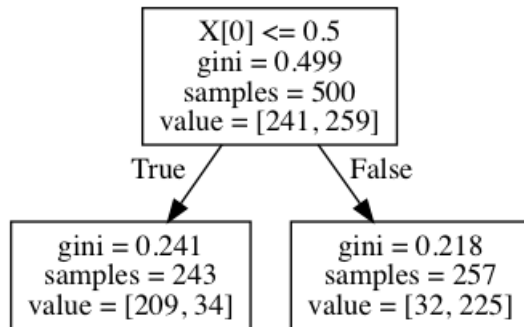
$$T = R^{true}(f) = \mathbf{P}(y = 1) * \mathbf{P}(f(\mathbf{x}) \leq 0 | y = 1) + \mathbf{P}(y = -1) * \mathbf{P}(f(\mathbf{x}) \geq 0 | y = -1) \tag{13}$$

$$= \frac{7}{8} * \mathbf{P}(f(\mathbf{x}) \leq 0 | y = 1) + \frac{1}{8} * \mathbf{P}(f(\mathbf{x}) \geq 0 | y = -1) \geq 0 \tag{14}$$

It's easy to find out that when $\mathbf{m} = \frac{1}{2}$ and $\mathbf{n} = \frac{1}{4}$, both $\mathbf{P}(f(\mathbf{x}) \leq 0 | y = 1)$ and $\mathbf{P}(f(\mathbf{x}) \geq 0 | y = -1)$ are equal to 0, and T achieves its minimal value $\mathbf{0}$.

2 Variable Importance for Trees and Random Forest

(a) (i) The decision stump based on the best split is:

```
        ┌─────────────────────┐
        │   X[0] <= 0.5        │
        │   gini = 0.499       │
        │   samples = 500      │
        │   value = [241, 259] │
        └─────────────────────┘
       True /              \ False
           ↓                  ↓
┌────────────────────┐  ┌────────────────────┐
│  gini = 0.241      │  │  gini = 0.218      │
│  samples = 243     │  │  samples = 257     │
│  value = [209, 34] │  │  value = [32, 225] │
└────────────────────┘  └────────────────────┘
```

The decision stump based on the best surrogate split is:

(ii)

(iii) The mean least-squares error of predictions on the test data of the decision stump based on the best split is 10.

The mean least-squares error of predictions on the test data of the decision stump based on the best split is

(b) (i)

(ii)

(iii)

(c) (i)

(ii)