Space Geometry Nonlinear Total-Lagrange Timoshenko Beam Theory

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1 Rotation Vector and Matrix

First define two vector:

$$\vec{\boldsymbol{a}} = (a_1, a_2, a_3) = a_1\vec{\boldsymbol{i}} + a_2\vec{\boldsymbol{j}} + a_3\vec{\boldsymbol{k}} \tag{1}$$

$$\vec{\boldsymbol{b}} = (b_1, b_2, b_3) = b_1\vec{\boldsymbol{i}} + b_2\vec{\boldsymbol{j}} + b_3\vec{\boldsymbol{k}} \tag{2}$$

The cross product of the two vectors is:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = (a_2b_3 - a_3b_2)\vec{\boldsymbol{i}} + (a_3b_1 - a_1b_3)\vec{\boldsymbol{j}} + (a_1b_2 - a_2b_1)\vec{\boldsymbol{k}} \tag{3}$$

In matrix form:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \tilde{\boldsymbol{a}}\boldsymbol{b} \tag{4}$$

$\tilde{a}$ is the skew-symmetric matrix of a. And we can obtain from the vector multiply rule:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = -\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}} \tag{5}$$

$$\tilde{\boldsymbol{a}}\boldsymbol{b} = -\tilde{\boldsymbol{b}}\boldsymbol{a} = \tilde{\boldsymbol{b}}^T\boldsymbol{a} \tag{6}$$

$$(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \times \vec{\boldsymbol{c}} = \vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) - \vec{\boldsymbol{b}} \times (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{c}}) \tag{7}$$

$$\widetilde{\tilde{\boldsymbol{a}}\boldsymbol{b}}\boldsymbol{c} = \tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}}\boldsymbol{c} - \tilde{\boldsymbol{b}}\tilde{\boldsymbol{a}}\boldsymbol{c} \tag{8}$$

$$\widetilde{\tilde{\boldsymbol{a}}\boldsymbol{b}} = \tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}} - \tilde{\boldsymbol{b}}\tilde{\boldsymbol{a}} \tag{9}$$

The rotational spinor is:

$$\vec{\boldsymbol{w}} = (w_1, w_2, w_3) \tag{10}$$

Normalize the spinor to get the rotation vector:

$$\vec{\boldsymbol{\theta}} = \theta\vec{\boldsymbol{n}} = \frac{\theta}{w}\vec{\boldsymbol{w}} \tag{11}$$

And $w = |\vec{\boldsymbol{w}}|$,$\theta = \left|\vec{\boldsymbol{\theta}}\right|$. From Fig.1 we can see that:

$$\vec{\boldsymbol{x_\theta}} = \vec{\boldsymbol{x}} + \vec{\boldsymbol{b}} + \vec{\boldsymbol{c}} = \vec{\boldsymbol{x}} + (\vec{\boldsymbol{n}} \times \vec{\boldsymbol{x}})\sin\theta + (\vec{\boldsymbol{n}} \times (\vec{\boldsymbol{n}} \times \vec{\boldsymbol{x}}))(1 - \cos\theta) \tag{12}$$

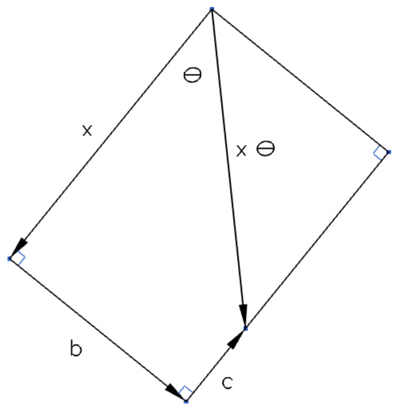

Figure 1: Rotation Vector

Because $\vec{\boldsymbol{n}} \times \vec{\boldsymbol{x}} = \frac{\tilde{\boldsymbol{w}}}{w}\boldsymbol{x}$,$\boldsymbol{x_\theta} = \boldsymbol{R}^T\boldsymbol{x}$,we can get:

$$\begin{aligned}\boldsymbol{R}^T\boldsymbol{x} &= \boldsymbol{x} + \frac{\tilde{\boldsymbol{w}}}{w}\boldsymbol{x}\sin\theta + \frac{\tilde{\boldsymbol{w}}^2}{w^2}\boldsymbol{x}(1-\cos\theta), \\ \boldsymbol{R}^T &= \boldsymbol{I} + \frac{\sin\theta}{w}\tilde{\boldsymbol{w}} + \frac{1-\cos\theta}{w^2}\tilde{\boldsymbol{w}}^2\end{aligned} \tag{13}$$

Here,$\boldsymbol{R}^T$ is the rotation matrix and $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$.Differential this equation with X we can get:

$$\dot{\boldsymbol{R}}^T\boldsymbol{R} + \boldsymbol{R}^T\dot{\boldsymbol{R}} = 0 \tag{14}$$

That means $\dot{\boldsymbol{R}}^T\boldsymbol{R}$ is skew-symmetric matrix.
Then we consider the matrix exponential:

$$Exp(\gamma\tilde{\boldsymbol{w}}) = I + \gamma\tilde{\boldsymbol{w}} + \frac{\gamma^2}{2!}\tilde{\boldsymbol{w}}^2 + \frac{\gamma^3}{3!}\tilde{\boldsymbol{w}}^3 + ... \tag{15}$$

Using $\tilde{\boldsymbol{w}}^n = -w^2\tilde{\boldsymbol{w}}^{n-2}$,we can get:

$$Exp(\gamma\tilde{\boldsymbol{w}}) = I + (\gamma - \frac{w^2}{3!} + \frac{w^4}{5!} + ...)\tilde{\boldsymbol{w}} + (\frac{\gamma^2}{2!} - \frac{\gamma^2 w^2}{4!} + \frac{\gamma^6 w^4}{6!} + ...)\tilde{\boldsymbol{w}}^2 \tag{16}$$

Given the series expansion on $\sin\theta$ and $1-\cos\theta$:

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + ... \tag{17}$$

$$1 - \cos\theta = \frac{\theta^2}{2} - \frac{\theta^4}{24} + \frac{\theta^6}{720} + ... \tag{18}$$

Substitute into the exponential equation:

$$Exp(\gamma\tilde{\boldsymbol{w}}) = \boldsymbol{I} + \frac{\sin(\gamma w)}{w}\tilde{\boldsymbol{w}} + \frac{1-\cos(\gamma w)}{w^2}\tilde{\boldsymbol{w}}^2 \tag{19}$$

This lead to the final equation:

$$\boldsymbol{R}^T = Exp(\frac{\theta}{w}\tilde{\boldsymbol{w}}) = Exp(\tilde{\boldsymbol{\theta}}) \tag{20}$$

In the same way,we can get:

$$\boldsymbol{R} = Exp(-\tilde{\boldsymbol{\theta}}) \tag{21}$$

And some more useful equations:

$$\delta\boldsymbol{R} = -\delta\tilde{\boldsymbol{\theta}}\boldsymbol{R} \tag{22}$$

$$\delta\tilde{\boldsymbol{\theta}} = -\delta\boldsymbol{R}\boldsymbol{R}^T \tag{23}$$

$$\frac{d\tilde{\boldsymbol{\theta}}}{dX_1} = -\frac{d\boldsymbol{R}}{dX_1}\boldsymbol{R}^T = \boldsymbol{R}\frac{d\boldsymbol{R}^T}{dX_1} = \tilde{\boldsymbol{k}} \tag{24}$$

Here $\boldsymbol{k}$ is the curvature matrix.

$$\begin{aligned}
\delta\tilde{\boldsymbol{k}} &= \delta\boldsymbol{R}\frac{d\boldsymbol{R}^T}{dX_1} + \boldsymbol{R}\frac{d\delta\boldsymbol{R}^T}{dX_1} \\
&= -\delta\tilde{\boldsymbol{\theta}}\boldsymbol{R}\frac{d\boldsymbol{R}^T}{dX_1} + \boldsymbol{R}\frac{d(\boldsymbol{R}^T\delta\tilde{\boldsymbol{\theta}})}{dX_1} \\
&= -\delta\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{k}} + \tilde{\boldsymbol{k}}\delta\tilde{\boldsymbol{\theta}} + \frac{d\delta\tilde{\boldsymbol{\theta}}}{dX_1}
\end{aligned} \tag{25}$$

Multiply 2 sides by $\delta\boldsymbol{\theta}$,

$$\begin{aligned}
\delta\tilde{\boldsymbol{k}}\delta\boldsymbol{\theta} = -\delta\tilde{\boldsymbol{\theta}}\delta\boldsymbol{k} &= -\delta\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{k}}\delta\boldsymbol{\theta} + \tilde{\boldsymbol{k}}\delta\tilde{\boldsymbol{\theta}}\delta\boldsymbol{\theta} + \frac{d\delta\tilde{\boldsymbol{\theta}}}{dX_1}\delta\boldsymbol{\theta} \\
&= -\delta\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{k}}\delta\boldsymbol{\theta} + 0 - \delta\tilde{\boldsymbol{\theta}}\frac{d\delta\boldsymbol{\theta}}{dX_1}, \\
\delta\boldsymbol{k} = \tilde{\boldsymbol{k}}\delta\boldsymbol{\theta} + \frac{d\delta\boldsymbol{\theta}}{dX_1}&.
\end{aligned} \tag{26}$$

2 Basic Theory on Geometry Nonlinear FEM

2.1 Strain and Stress

Given the Green-Lagrange strain:

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{G} + \boldsymbol{G}^T + \boldsymbol{G}^T\boldsymbol{G}) \tag{27}$$

$\boldsymbol{G}$ is the displacement gradient tensor:

$$\boldsymbol{G} = \begin{bmatrix}\boldsymbol{g_1} & \boldsymbol{g_2} & \boldsymbol{g_3}\end{bmatrix} \tag{28}$$

$$\boldsymbol{e} = \frac{1}{2}(\begin{bmatrix}\boldsymbol{g_1} & \boldsymbol{g_2} & \boldsymbol{g_3}\end{bmatrix} + \begin{bmatrix}\boldsymbol{g_1^T} \\ \boldsymbol{g_2^T} \\ \boldsymbol{g_3^T}\end{bmatrix} + \begin{bmatrix}\boldsymbol{g_1^T} \\ \boldsymbol{g_2^T} \\ \boldsymbol{g_3^T}\end{bmatrix}\begin{bmatrix}\boldsymbol{g_1} & \boldsymbol{g_2} & \boldsymbol{g_3}\end{bmatrix}) \tag{29}$$

Define $\boldsymbol{h_1}$,$\boldsymbol{h_2}$,$\boldsymbol{h_3}$:

$$\boldsymbol{h1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{h2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{h3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{30}$$

Using $\boldsymbol{h_1}$,$\boldsymbol{h_2}$,$\boldsymbol{h_3}$ to Expand e:

$$\boldsymbol{e} = \frac{1}{2}(\begin{bmatrix} \boldsymbol{h_1^T g_1} & \boldsymbol{h_1^T g_2} & \boldsymbol{h_1^T g_3} \\ \boldsymbol{h_2^T g_1} & \boldsymbol{h_2^T g_2} & \boldsymbol{h_2^T g_3} \\ \boldsymbol{h_3^T g_1} & \boldsymbol{h_3^T g_2} & \boldsymbol{h_3^T g_3} \end{bmatrix} + \begin{bmatrix} \boldsymbol{h_1^T g_1} & \boldsymbol{h_2^T g_1} & \boldsymbol{h_3^T g_1} \\ \boldsymbol{h_1^T g_2} & \boldsymbol{h_2^T g_2} & \boldsymbol{h_3^T g_2} \\ \boldsymbol{h_1^T g_3} & \boldsymbol{h_2^T g_3} & \boldsymbol{h_3^T g_3} \end{bmatrix} + \begin{bmatrix} \boldsymbol{g_1^T g_1} & \boldsymbol{g_1^T g_2} & \boldsymbol{g_1^T g_3} \\ \boldsymbol{g_2^T g_1} & \boldsymbol{g_2^T g_2} & \boldsymbol{g_2^T g_3} \\ \boldsymbol{g_3^T g_1} & \boldsymbol{g_3^T g_2} & \boldsymbol{g_3^T g_3} \end{bmatrix}) \tag{31}$$

$$e_{ij} = \frac{1}{2}(\boldsymbol{h_i^T g_j} + \boldsymbol{h_j^T g_i} + \boldsymbol{g_i^T g_j}) \tag{32}$$

Rearrange $\boldsymbol{e}$ and $\boldsymbol{G}$ in column form:

$$e_i = \boldsymbol{\bar{h}_i^T g} + \frac{1}{2}\boldsymbol{g}^T \boldsymbol{H_i^T g} \tag{33}$$

$$\boldsymbol{g} = \begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{g_3} \end{bmatrix} \tag{34}$$

And $\boldsymbol{\bar{h}_i}$ is the combination of $\boldsymbol{h_i}$, $\boldsymbol{H_i}$ is symmetric matrix. Next,we define:

$$\boldsymbol{C_i} = \boldsymbol{\bar{h}_i} + \frac{1}{2}\boldsymbol{H_i g} \tag{35}$$

Then:

$$e_i = \boldsymbol{C_i^T g} \tag{36}$$

The conjugate second Piola-Kirchhoff(PK2) stress is:

$$s_i = E_{ij} e_j = E_{ij} \boldsymbol{C_j^T g} \tag{37}$$

2.2 Energy Equation and Stiffness Matrix

The strain energy density is:

$$u = \frac{1}{2} s_i e_i = \frac{1}{2}\boldsymbol{g}^T \boldsymbol{C_j} E_{ij} \boldsymbol{C_i^T g} = \frac{1}{2}\boldsymbol{g}^T \boldsymbol{S^u g} \tag{38}$$

Here we get the core energy stiffness matrix:

$$\boldsymbol{S^u} = \boldsymbol{C_j} E_{ij} \boldsymbol{C_i^T} = E_{ij} \boldsymbol{C_i C_j^T} \tag{39}$$

Variation of u,we can get:

$$\begin{aligned} \delta u &= \frac{1}{2}\delta \boldsymbol{g}^T \boldsymbol{S^u g} + \frac{1}{2}\boldsymbol{g}^T \delta(\boldsymbol{S^u g}) \\ &= \frac{1}{2}\delta \boldsymbol{g}^T \boldsymbol{S^u g} + \frac{1}{2}\boldsymbol{g}^T (\delta \boldsymbol{S^u g} + \boldsymbol{S^u} \delta \boldsymbol{g}) \\ &= \delta \boldsymbol{g}^T \boldsymbol{S^u g} + \frac{1}{2}\boldsymbol{g}^T \delta \boldsymbol{S^u g} = \delta \boldsymbol{g}^T \boldsymbol{S^r g} = \delta \boldsymbol{g}^T \boldsymbol{\Phi} \end{aligned} \tag{40}$$

This implicitly define the secant stiffness matrix $\boldsymbol{S^r}$.$\boldsymbol{\Phi}$ is the core internal force vector.
Second variation of u,we can get:

$$\delta^2 u = \delta\boldsymbol{g}^T\delta\boldsymbol{\Phi} + (\delta^2\boldsymbol{g})^T\boldsymbol{\Phi} \tag{41}$$

And because $\delta\boldsymbol{\Phi} = \boldsymbol{S^t}\delta\boldsymbol{g}$,here $\boldsymbol{S^t}$ is the core tangent stiffness matrix:

$$\delta^2 u = \delta\boldsymbol{g}^T\boldsymbol{S^t}\delta\boldsymbol{g} + (\delta^2\boldsymbol{g})^T\boldsymbol{\Phi} \tag{42}$$

Next,integration of energy density u to get the equation of energy U,also in second variation form:

$$\delta^2 U = \int_{V0}\delta^2 u\, dV = \int_{V0}(\delta\boldsymbol{g}^T\boldsymbol{S^t}\delta\boldsymbol{g} + (\delta^2\boldsymbol{g})^T\boldsymbol{\Phi})\, dV \tag{43}$$

And the relation between displacement gradient $\boldsymbol{g}$ and final DOF $\boldsymbol{q}$ is:

$$\delta\boldsymbol{g} = \boldsymbol{T}\delta\boldsymbol{q} \tag{44}$$

Substitute into energy equation:

$$\delta^2 U = \int_{V0}(\delta\boldsymbol{q}^T\boldsymbol{T}^T\boldsymbol{S^t}\boldsymbol{T}\delta\boldsymbol{q} + (\delta^2\boldsymbol{g})^T\boldsymbol{\Phi})\, dV \tag{45}$$

From this equation,we can get the final tangent stiffness matrix.And the final internal force vector is the same way:

$$\delta U = \int_{V0}\delta u\, dv = \int_{V0}\delta\boldsymbol{g}^T\boldsymbol{\Phi}\, dv = \int_{V0}\delta\boldsymbol{q}^T\boldsymbol{T}^T\boldsymbol{\Phi}\, dv = \delta\boldsymbol{q}^T\int_{V0}\boldsymbol{T}^T\boldsymbol{\Phi}\, dv \tag{46}$$

Next we derive the detailed expression of stiffness and force matrices:

$$\delta u = \delta\boldsymbol{g}^T\boldsymbol{S^u}\boldsymbol{g} + \frac{1}{2}\boldsymbol{g}^T\delta\boldsymbol{S^u}\boldsymbol{g} \tag{47}$$

We know that $\boldsymbol{S^u} = E_{ij}\boldsymbol{C_i}\boldsymbol{C_j}^T$,and we can get:

$$\begin{aligned}
\frac{1}{2}\boldsymbol{g}^T\delta\boldsymbol{S^u}\boldsymbol{g} &= \frac{1}{2}\boldsymbol{g}^T\delta(E_{ij}\boldsymbol{C_i}\boldsymbol{C_j}^T)\boldsymbol{g} \\
&= \frac{E_{ij}}{2}\boldsymbol{g}^T(\delta\boldsymbol{C_i}\boldsymbol{C_j}^T + \boldsymbol{C_i}\delta\boldsymbol{C_j}^T)\boldsymbol{g} \\
&= \frac{E_{ij}}{2}\boldsymbol{g}^T(\frac{1}{2}\boldsymbol{H_i}\delta\boldsymbol{g}\boldsymbol{C_j}^T + \frac{1}{2}\boldsymbol{C_i}\delta\boldsymbol{g}^T\boldsymbol{H_j}^T)\boldsymbol{g} \\
&= \frac{E_{ij}}{4}\boldsymbol{g}^T\boldsymbol{H_i}\delta\boldsymbol{g}\boldsymbol{C_j}^T\boldsymbol{g} + \frac{E_{ij}}{4}\boldsymbol{g}^T\boldsymbol{C_i}\delta\boldsymbol{g}^T\boldsymbol{H_j}^T\boldsymbol{g} \\
&= \frac{1}{4}\boldsymbol{g}^T s_i\boldsymbol{H_i}\delta\boldsymbol{g} + \frac{1}{4}\delta\boldsymbol{g}^T s_j\boldsymbol{H_j}^T\boldsymbol{g} \\
&= \frac{1}{2}\delta\boldsymbol{g}^T s_i\boldsymbol{H_i}\boldsymbol{g}
\end{aligned} \tag{48}$$

And from Eq.39 and Eq.40,we can get:

$$\boldsymbol{S^r} = E_{ij}\boldsymbol{C_i}\boldsymbol{C_j}^T + \frac{1}{2}s_i\boldsymbol{H_i}^T \tag{49}$$

And:

$$
\begin{aligned}
\delta\boldsymbol{\Phi} &= \delta\boldsymbol{S^r g} + \boldsymbol{S^r}\delta\boldsymbol{g} \\
&= \delta(E_{ij}\boldsymbol{C_i C_j^T} + \frac{1}{2}s_i\boldsymbol{H_i^T})\boldsymbol{g} + \boldsymbol{S^r}\delta\boldsymbol{g} \\
&= (E_{ij}(\delta\boldsymbol{C_i C_j^T} + \boldsymbol{C_i}\delta\boldsymbol{C_j^T}) + \frac{1}{2}\delta\boldsymbol{s_i H_i^T})\boldsymbol{g} + \boldsymbol{S^r}\delta\boldsymbol{g} \\
&= (E_{ij}(\frac{1}{2}\boldsymbol{H_i}\delta\boldsymbol{g C_j^T} + \frac{1}{2}\boldsymbol{C_i}\delta\boldsymbol{g}^T\boldsymbol{H_j^T}) + \frac{1}{2}\delta\boldsymbol{s_i H_i^T})\boldsymbol{g} + \boldsymbol{S^r}\delta\boldsymbol{g} \\
&= \frac{E_{ij}}{2}\boldsymbol{H_i}\delta\boldsymbol{g C_j^T g} + \frac{E_{ij}}{2}\boldsymbol{C_i}\delta\boldsymbol{g}^T\boldsymbol{H_j^T g} + \frac{E_{ij}}{2}\delta e_j\boldsymbol{H_i^T g} + \boldsymbol{S^r}\delta\boldsymbol{g} \\
&= \frac{1}{2}s_i\boldsymbol{H_i}\delta\boldsymbol{g} + \frac{E_{ij}}{2}\boldsymbol{C_i g}^T\boldsymbol{H_j^T}\delta\boldsymbol{g} + \frac{E_{ij}}{2}\boldsymbol{H_i^T g}\delta(\boldsymbol{C_j^T g}) + (E_{ij}\boldsymbol{C_i C_j^T} + \frac{1}{2}s_i\boldsymbol{H_i^T})\delta\boldsymbol{g}
\end{aligned} \tag{50}
$$

And:

$$
\begin{aligned}
\frac{E_{ij}}{2}\boldsymbol{H_i^T g}\delta(\boldsymbol{C_j^T g}) &= \frac{E_{ij}}{2}\boldsymbol{H_i^T g}(\delta\boldsymbol{C_j^T g} + \boldsymbol{C_j^T}\delta\boldsymbol{g}) \\
&= \frac{E_{ij}}{4}\boldsymbol{H_i^T g}\delta\boldsymbol{g}^T\boldsymbol{H_j^T g} + \frac{E_{ij}}{2}\boldsymbol{H_i^T g C_j^T}\delta\boldsymbol{g} \\
&= \frac{E_{ij}}{4}\boldsymbol{H_i^T g g^T H_j^T}\delta\boldsymbol{g} + \frac{E_{ij}}{2}\boldsymbol{H_i^T g C_j^T}\delta\boldsymbol{g}
\end{aligned} \tag{51}
$$

Substitute Eq.51 into Eq.50,we get:

$$
\delta\boldsymbol{\Phi} = (E_{ij}(\boldsymbol{C_i} + \frac{1}{2}\boldsymbol{H_i g})(\boldsymbol{C_j^T} + \frac{1}{2}\boldsymbol{g}^T\boldsymbol{H_j^T}) + s_i\boldsymbol{H_i^T})\delta\boldsymbol{g} \tag{52}
$$

Define $\boldsymbol{B_i} = \boldsymbol{C_i} + \frac{1}{2}\boldsymbol{H_i g} = \boldsymbol{h_i} + \boldsymbol{H_i g}$, then:

$$
\boldsymbol{S^t} = E_{ij}\boldsymbol{B_i B_j^T} + s_i\boldsymbol{H_i^T} \tag{53}
$$

The core internal force matrix is:

$$
\begin{aligned}
\boldsymbol{\Phi} &= \boldsymbol{S^r g} \\
&= (E_{ij}\boldsymbol{C_i C_j^T} + \frac{1}{2}s_i\boldsymbol{H_i^T})\boldsymbol{g} \\
&= E_{ij}\boldsymbol{C_i C_j^T g} + \frac{E_{ij}}{2}\boldsymbol{C_j^T g H_i^T g} \\
&= E_{ij}\boldsymbol{C_j^T g}(\boldsymbol{C_i} + \frac{1}{2}\boldsymbol{H_i^T g}) = s_i\boldsymbol{B_i}
\end{aligned} \tag{54}
$$

3 Space Total-Lagrange Beam Theory

3.1 Beam Deformation Theory

Assuming that the beam particle at(X_1, X_2, X_3) move to:

$$
\boldsymbol{x}(\boldsymbol{X}) = \boldsymbol{x_0}(X_1) + \boldsymbol{R^T}(X_1)\boldsymbol{\zeta}(X_2, X_3) \tag{55}
$$

Where $\boldsymbol{\zeta}^T = [0\ X_2\ X_3]$ is the cross-section coordinate.And the displacement is:

$$
\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{x_0} + \boldsymbol{R^T}\boldsymbol{\zeta} - (\boldsymbol{X}(X_1) + \boldsymbol{\zeta}) = \boldsymbol{u_0} + (\boldsymbol{R}^T - \boldsymbol{I})\boldsymbol{\zeta} \tag{56}
$$

Here $\boldsymbol{u_0}$ is the centroidal displacement. Then, the displacement gradient is:

$$\boldsymbol{G} = \frac{\partial \boldsymbol{u_0}}{\partial \boldsymbol{X}} + \frac{\partial((\boldsymbol{R}^T - \boldsymbol{I})\boldsymbol{\zeta})}{\partial \boldsymbol{X}} \tag{57}$$

This can be rearrange as:

$$\boldsymbol{g_i} = \frac{\partial \boldsymbol{u_0}}{\partial X_i} + \frac{\partial((\boldsymbol{R}^T - \boldsymbol{I})\boldsymbol{\zeta})}{\partial X_i} \tag{58}$$

Then the formulation of $\boldsymbol{g_i}$ is:

$$\boldsymbol{g_1} = \frac{\partial \boldsymbol{u_0}}{\partial X_1} + \frac{\partial(\boldsymbol{R}^T\boldsymbol{\zeta})}{\partial X_1} = \frac{d\boldsymbol{u_0}}{dX_1} + \frac{d\boldsymbol{R}^T}{dX_1}\boldsymbol{\zeta} = \frac{d\boldsymbol{u_0}}{dX_1} + \boldsymbol{R}^T\boldsymbol{R}\frac{d\boldsymbol{R}^T}{dX_1}\boldsymbol{\zeta} \tag{59}$$

Using Eq.6 and Eq.25:

$$\begin{aligned}
\boldsymbol{g_1} &= \frac{d\boldsymbol{u_0}}{dX_1} + \boldsymbol{R}^T\tilde{\boldsymbol{k}}\boldsymbol{\zeta} = \frac{d\boldsymbol{u_0}}{dX_1} + \boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}, \\
\boldsymbol{g_2} &= (\boldsymbol{R}^T - \boldsymbol{I})\frac{\partial \zeta}{\partial X_2} = (\boldsymbol{R}^T - \boldsymbol{I})\boldsymbol{h_2}, \\
\boldsymbol{g_3} &= (\boldsymbol{R}^T - \boldsymbol{I})\frac{\partial \zeta}{\partial X_3} = (\boldsymbol{R}^T - \boldsymbol{I})\boldsymbol{h_3}.
\end{aligned} \tag{60}$$

Obviously that $\boldsymbol{h_2}^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $\boldsymbol{h_3}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. From Eq.32 ,we can know:

$$\begin{aligned}
e_{11} &= \boldsymbol{h_1}^T\boldsymbol{g_1} + \frac{1}{2}\boldsymbol{g_1}^T\boldsymbol{g_1}, \\
\gamma_{12} = 2e_{12} &= \boldsymbol{h_2}^T\boldsymbol{g_1} + \boldsymbol{h_1}^T\boldsymbol{g_2} + \boldsymbol{g_1}^T\boldsymbol{g_2}, \\
\gamma_{13} = 2e_{13} &= \boldsymbol{h_3}^T\boldsymbol{g_1} + \boldsymbol{h_1}^T\boldsymbol{g_3} + \boldsymbol{g_1}^T\boldsymbol{g_3}.
\end{aligned} \tag{61}$$

Using Mathematica to simplify,we can know that e_{22} and e_{23} are 0.
Substitute Eq.60 into Eq.61 and simplify:

$$\begin{aligned}
e_{11} &= e_b + e_f, \\
\gamma_{12} &= \boldsymbol{h_2}^T\boldsymbol{\phi} + \boldsymbol{h_2}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}, \\
\gamma_{13} &= \boldsymbol{h_3}^T\boldsymbol{\phi} + \boldsymbol{h_3}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}, \\
e_b &= (\frac{d\boldsymbol{u_0}}{dX_1})^T(\boldsymbol{h_1} + \frac{d\boldsymbol{u_0}}{2dX_1}), \\
e_f &= \boldsymbol{\zeta}^T\boldsymbol{k_e} + \frac{1}{2}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k} \simeq \boldsymbol{\zeta}^T\boldsymbol{k_e}, \\
\boldsymbol{\phi} &= \boldsymbol{R}(\boldsymbol{h_1} + \frac{d\boldsymbol{u_0}}{dX_1}).
\end{aligned} \tag{62}$$

The squared-curvature term of e_f is dropped.

3.2 Beam Core Equations

First we consider the core energy stiffness matrix $\boldsymbol{S^u}$,rewrite $\boldsymbol{C_i}$ of Eq.35 in matrix form:

$$\begin{aligned}
\boldsymbol{C_1} &= \begin{bmatrix} \boldsymbol{h_1} \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \boldsymbol{H} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{g_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c_1} \\ 0 \\ 0 \end{bmatrix}, \\
\boldsymbol{C_2} &= \begin{bmatrix} \boldsymbol{h_2} \\ \boldsymbol{h_1} \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & \boldsymbol{H} & 0 \\ \boldsymbol{H} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{g_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c_2} \\ \boldsymbol{c_1} \\ 0 \end{bmatrix}, \\
\boldsymbol{C_3} &= \begin{bmatrix} \boldsymbol{h_3} \\ 0 \\ \boldsymbol{h_1} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & \boldsymbol{H} \\ 0 & 0 & 0 \\ \boldsymbol{H} & 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{g_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c_3} \\ 0 \\ \boldsymbol{c_1} \end{bmatrix}.
\end{aligned} \tag{63}$$

Where $\boldsymbol{c_i} = \boldsymbol{h_i} + \frac{1}{2}\boldsymbol{H}\boldsymbol{g_i}$.$\boldsymbol{H}$ is the identity matrix.
And we know that $\boldsymbol{S^u} = E_{11}\boldsymbol{C_1}\boldsymbol{C_1^T} + E_{22}\boldsymbol{C_2}\boldsymbol{C_2^T} + E_{33}\boldsymbol{C_3}\boldsymbol{C_3^T}$,
E_{11}=E,E_{22}=E_{33}=G.Rewrite $\boldsymbol{S^u}$ in matrix form:

$$\boldsymbol{S^u} = E\begin{bmatrix} \boldsymbol{c_1}\boldsymbol{c_1^T} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + G\begin{bmatrix} \boldsymbol{c_2}\boldsymbol{c_2^T} & \boldsymbol{c_2}\boldsymbol{c_1^T} & 0 \\ \boldsymbol{c_1}\boldsymbol{c_2^T} & \boldsymbol{c_1}\boldsymbol{c_1^T} & 0 \\ 0 & 0 & 0 \end{bmatrix} + G\begin{bmatrix} \boldsymbol{c_3}\boldsymbol{c_3^T} & 0 & \boldsymbol{c_3}\boldsymbol{c_1^T} \\ 0 & 0 & 0 \\ \boldsymbol{c_1}\boldsymbol{c_3^T} & 0 & \boldsymbol{c_1}\boldsymbol{c_1^T} \end{bmatrix} \tag{64}$$

Define $\boldsymbol{S_1^U} = \boldsymbol{c_1}\boldsymbol{c_1^T}$,$\boldsymbol{S_2^U} = \boldsymbol{c_2}\boldsymbol{c_2^T}$,$\boldsymbol{S_3^U} = \boldsymbol{c_3}\boldsymbol{c_3^T}$,$\boldsymbol{S_4^U} = \boldsymbol{c_2}\boldsymbol{c_1^T}$ and $\boldsymbol{S_5^U} = \boldsymbol{c_3}\boldsymbol{c_1^T}$,and because of symmetry,we can get:

$$\boldsymbol{S^u} = \begin{bmatrix} E\boldsymbol{S_1^U} + G(\boldsymbol{S_2^U} + \boldsymbol{S_3^U}) & G\boldsymbol{S_4^U} & G\boldsymbol{S_5^U} \\ G\boldsymbol{S_4^U}^T & G\boldsymbol{S_1^U} & 0 \\ G\boldsymbol{S_5^U}^T & 0 & G\boldsymbol{S_1^U} \end{bmatrix} \tag{65}$$

And according to Eq.53,$\boldsymbol{S^t}$ is in the same form:

$$\boldsymbol{S^t} = \begin{bmatrix} E\boldsymbol{S_1} + G(\boldsymbol{S_2} + \boldsymbol{S_3}) & G\boldsymbol{S_4} & G\boldsymbol{S_5} \\ G\boldsymbol{S_4}^T & G\boldsymbol{S_1} & 0 \\ G\boldsymbol{S_5}^T & 0 & G\boldsymbol{S_1} \end{bmatrix} + \begin{bmatrix} s_{11}\boldsymbol{H} & s_{12}\boldsymbol{H} & s_{13}\boldsymbol{H} \\ s_{12}\boldsymbol{H} & 0 & 0 \\ s_{13}\boldsymbol{H} & 0 & 0 \end{bmatrix} \tag{66}$$

3.3 Transformation to Section Result

First,we need the variation of $\boldsymbol{g_i}$ based on Eq.60

$$\begin{aligned}
\delta\boldsymbol{g_1} &= \frac{d\delta\boldsymbol{u_0}}{dX_1} + \delta\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k} + \boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\delta\boldsymbol{k} \\
&= \frac{d\delta\boldsymbol{u_0}}{dX_1} + \boldsymbol{R}^T\delta\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k} + \boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\delta\boldsymbol{k} \\
&= \frac{d\delta\boldsymbol{u_0}}{dX_1} + \boldsymbol{R}^T(\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}} - \tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T)\delta\boldsymbol{\theta} + \boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T(\tilde{\boldsymbol{k}}\delta\boldsymbol{\theta} + \frac{d\delta\boldsymbol{\theta}}{dX_1}) \\
&= \frac{d\delta\boldsymbol{u_0}}{dX_1} + \boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\frac{d\delta\boldsymbol{\theta}}{dX_1} + \boldsymbol{R}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\delta\boldsymbol{\theta}, \\
\delta\boldsymbol{g_2} &= \delta\boldsymbol{R}^T\boldsymbol{h_2} = \boldsymbol{R}^T\tilde{\boldsymbol{h_2}}^T\delta\boldsymbol{\theta}, \\
\delta\boldsymbol{g_3} &= \delta\boldsymbol{R}^T\boldsymbol{h_3} = \boldsymbol{R}^T\tilde{\boldsymbol{h_3}}^T\delta\boldsymbol{\theta}.
\end{aligned} \tag{67}$$

This can be write in matrix form:

$$\delta \boldsymbol{g} = \begin{bmatrix} \delta \boldsymbol{g_1} \\ \delta \boldsymbol{g_2} \\ \delta \boldsymbol{g_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T & \boldsymbol{R}^T \tilde{\boldsymbol{k}}^T \tilde{\boldsymbol{\zeta}} \\ 0 & 0 & \boldsymbol{R}^T \tilde{\boldsymbol{h_2}}^T \\ 0 & 0 & \boldsymbol{R}^T \tilde{\boldsymbol{h_3}}^T \end{bmatrix} \begin{bmatrix} \frac{d\delta \boldsymbol{u_0}}{dX_1} \\ \frac{d\delta \boldsymbol{\theta}}{dX_1} \\ \delta \boldsymbol{\theta} \end{bmatrix} = \boldsymbol{W} \delta \boldsymbol{w} \tag{68}$$

Here $\boldsymbol{w}$ is the section gradients and will connect to the final DOF later.
And the second variation is:

$$\begin{aligned} \delta^2 \boldsymbol{g_1} &= (\delta^2 \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{k} + \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \delta^2 \boldsymbol{k}) + 2\delta \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \delta \boldsymbol{k} \\ &= (\delta^2 \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{k} + \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \delta^2 \boldsymbol{k}) + 2\boldsymbol{R}^T \delta \tilde{\boldsymbol{\theta}} \tilde{\boldsymbol{\zeta}}^T \delta \boldsymbol{k} \\ &= (\delta^2 \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{k} + \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \delta^2 \boldsymbol{k}) + 2\boldsymbol{R}^T \delta \tilde{\boldsymbol{\theta}} \tilde{\boldsymbol{\zeta}}^T (\frac{d\delta \boldsymbol{\theta}}{dX_1} + \tilde{\boldsymbol{k}} \delta \boldsymbol{\theta}), \\ \delta^2 \boldsymbol{g_2} &= \delta^2 \boldsymbol{R}^T \boldsymbol{h_2}, \\ \delta^2 \boldsymbol{g_3} &= \delta^2 \boldsymbol{R}^T \boldsymbol{h_3}. \end{aligned} \tag{69}$$

Then the section tangent stiffness matrix is:

$$\boldsymbol{S^{st}} = \int_A \boldsymbol{W}^T \boldsymbol{S^t} \boldsymbol{W} \, dA = \int_A \boldsymbol{W}^T (\boldsymbol{S_M} + \boldsymbol{S_G}) \boldsymbol{W} \, dA \tag{70}$$

The first part:

$$\boldsymbol{S^s_M} = \int_A \boldsymbol{W}^T (E\boldsymbol{B_1}\boldsymbol{B_1}^T + G\boldsymbol{B_2}\boldsymbol{B_2}^T + G\boldsymbol{B_3}\boldsymbol{B_3}^T) \boldsymbol{W} \, dA = \boldsymbol{T_1} + \boldsymbol{T_2} + \boldsymbol{T_3} \tag{71}$$

And

$$\begin{aligned} \boldsymbol{B_1} &= \begin{bmatrix} \boldsymbol{h_1} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{H} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{g_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b_1} \\ 0 \\ 0 \end{bmatrix}, \\ \boldsymbol{B_2} &= \begin{bmatrix} \boldsymbol{h_2} \\ \boldsymbol{h_1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \boldsymbol{H} & 0 \\ \boldsymbol{H} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{g_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b_2} \\ \boldsymbol{b_1} \\ 0 \end{bmatrix}, \\ \boldsymbol{B_3} &= \begin{bmatrix} \boldsymbol{h_3} \\ 0 \\ \boldsymbol{h_1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \boldsymbol{H} \\ 0 & 0 & 0 \\ \boldsymbol{H} & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{g_1} \\ \boldsymbol{g_2} \\ \boldsymbol{g_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b_3} \\ 0 \\ \boldsymbol{b_1} \end{bmatrix}. \end{aligned} \tag{72}$$

Using Mathematica to simplify $\boldsymbol{T_1}$,and because $\boldsymbol{T}$ is symmetric,we get:

$$\begin{aligned} &\boldsymbol{T_1}(1,1) = \int_A E(\boldsymbol{g_1} + \boldsymbol{h_1})(\boldsymbol{g_1} + \boldsymbol{h_1})^T \, dA, \\ &\qquad and, \boldsymbol{g_1} + \boldsymbol{h_1} = \frac{d\boldsymbol{u_0}}{dX_1} + \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{k} + \boldsymbol{h_1} = \boldsymbol{R}^T \boldsymbol{\phi} + \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{k}, \\ &\boldsymbol{T_1}(1,1) = \int_A E(\boldsymbol{R}^T \boldsymbol{\phi} + \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{k})(\boldsymbol{\phi}^T \boldsymbol{R} + \boldsymbol{k}^T \tilde{\boldsymbol{\zeta}} \boldsymbol{R}) \, dA \\ &\qquad = \int_A E(\boldsymbol{R}^T \boldsymbol{\phi}\boldsymbol{\phi}^T \boldsymbol{R} + \boldsymbol{R}^T \boldsymbol{\phi} \boldsymbol{k}^T \tilde{\boldsymbol{\zeta}} \boldsymbol{R} + \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{k} \boldsymbol{\phi}^T \boldsymbol{R} + \boldsymbol{R}^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{k}\boldsymbol{k}^T \tilde{\boldsymbol{\zeta}} \boldsymbol{R}) \, dA, \\ &\qquad and, \int_A \tilde{\boldsymbol{\zeta}} \, dA = 0, \int_A \boldsymbol{\zeta}\boldsymbol{\zeta}^T \, dA = \boldsymbol{I_s}, \\ &\boldsymbol{T_1}(1,1) = E\boldsymbol{R}^T (A\boldsymbol{\phi}\boldsymbol{\phi}^T + \tilde{\boldsymbol{k}}^T \boldsymbol{I_s} \tilde{\boldsymbol{k}}) \boldsymbol{R}. \end{aligned} \tag{73}$$

and

$$\begin{aligned}\boldsymbol{T_1}(1,2) &= \int_A E(\boldsymbol{g_1}+\boldsymbol{h_1})(\boldsymbol{g_1}+\boldsymbol{h_1})^T\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\,dA \\ &= \int_A E(\boldsymbol{R}^T\boldsymbol{\phi}\boldsymbol{\phi}^T\tilde{\boldsymbol{\zeta}}^T+\boldsymbol{R}^T\boldsymbol{\phi}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T+\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{\phi}^T\tilde{\boldsymbol{\zeta}}^T+\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T)\,dA \\ &= \int_A E(\boldsymbol{R}^T\boldsymbol{\phi}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T+\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{\phi}^T\tilde{\boldsymbol{\zeta}}^T)\,dA \\ &= E\boldsymbol{R}^T\boldsymbol{\phi}\boldsymbol{k}^T\boldsymbol{I_r}+E\boldsymbol{R}^T\tilde{\boldsymbol{k}}\boldsymbol{I_s}\tilde{\boldsymbol{\phi}}\end{aligned} \tag{74}$$

and

$$\begin{aligned}\boldsymbol{T_1}(1,3) &= \int_A E(\boldsymbol{g_1}+\boldsymbol{h_1})(\boldsymbol{g_1}+\boldsymbol{h_1})^T\boldsymbol{R}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\,dA \\ &= \int_A E(\boldsymbol{R}^T\boldsymbol{\phi}\boldsymbol{\phi}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\boldsymbol{R}^T\boldsymbol{\phi}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{\phi}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}})\,dA \\ &= \int_A E(\boldsymbol{R}^T\boldsymbol{\phi}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{\phi}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}})\,dA \\ &= \int_A E(\boldsymbol{R}^T\boldsymbol{\phi}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}})\,dA+E\boldsymbol{R}^T\tilde{\boldsymbol{k}}\boldsymbol{I_s}\tilde{\boldsymbol{k_e}}\end{aligned} \tag{75}$$

where $\boldsymbol{k_e}=\tilde{\boldsymbol{\phi}}\boldsymbol{k}$,and

$$\begin{aligned}\boldsymbol{T_1}(2,2) &= \int_A E\tilde{\boldsymbol{\zeta}}\boldsymbol{R}(\boldsymbol{g_1}+\boldsymbol{h_1})(\boldsymbol{g_1}+\boldsymbol{h_1})^T\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\,dA \\ &= \int_A E(\tilde{\boldsymbol{\zeta}}\boldsymbol{\phi}\boldsymbol{\phi}^T\tilde{\boldsymbol{\zeta}}^T+\tilde{\boldsymbol{\zeta}}\boldsymbol{\phi}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T+\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{\phi}^T\tilde{\boldsymbol{\zeta}}^T+\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T)\,dA \\ &= E(\tilde{\boldsymbol{\phi}}^T\boldsymbol{I_s}\tilde{\boldsymbol{\phi}}+\boldsymbol{I_r}\boldsymbol{k}\boldsymbol{k}^T\boldsymbol{I_r})\simeq E\tilde{\boldsymbol{\phi}}^T\boldsymbol{I_s}\tilde{\boldsymbol{\phi}}\end{aligned} \tag{76}$$

and

$$\begin{aligned}\boldsymbol{T_1}(2,3) &= \int_A E\tilde{\boldsymbol{\zeta}}\boldsymbol{R}(\boldsymbol{g_1}+\boldsymbol{h_1})(\boldsymbol{g_1}+\boldsymbol{h_1})^T\boldsymbol{R}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\,dA \\ &= \int_A E(\tilde{\boldsymbol{\zeta}}\boldsymbol{\phi}\boldsymbol{\phi}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\tilde{\boldsymbol{\zeta}}\boldsymbol{\phi}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{\phi}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}})\,dA \\ &= \int_A E(\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}})\,dA+\tilde{\boldsymbol{\phi}}^T\boldsymbol{I_s}\tilde{\boldsymbol{k_e}}\simeq\tilde{\boldsymbol{\phi}}^T\boldsymbol{I_s}\tilde{\boldsymbol{k_e}}\end{aligned} \tag{77}$$

and

$$\begin{aligned}\boldsymbol{T_1}(3,3) &= \int_A E\tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{k}}\boldsymbol{R}(\boldsymbol{g_1}+\boldsymbol{h_1})(\boldsymbol{g_1}+\boldsymbol{h_1})^T\boldsymbol{R}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\,dA \\ &= \int_A E(\tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{k}}\boldsymbol{\phi}\boldsymbol{\phi}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{k}}\boldsymbol{\phi}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{k}}\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{\phi}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}+\tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{k}}\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k}\boldsymbol{k}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}})\,dA \\ &\simeq \tilde{\boldsymbol{k_e}}^T\boldsymbol{I_s}\tilde{\boldsymbol{k_e}}\end{aligned} \tag{78}$$

Then using Mathematica to simplify $\boldsymbol{T_2}+\boldsymbol{T_3}=\boldsymbol{T_\tau}$,we get:

$$\begin{aligned}
\boldsymbol{T_\tau}(1,1) &= \int_A G(\boldsymbol{g_2}+\boldsymbol{h_2})(\boldsymbol{g_2}+\boldsymbol{h_2})^T + G(\boldsymbol{g_3}+\boldsymbol{h_3})(\boldsymbol{g_3}+\boldsymbol{h_3})^T, \\
&\quad and\, \boldsymbol{g_2}+\boldsymbol{h_2} = \boldsymbol{R}^T\boldsymbol{h_2}, \boldsymbol{g_3}+\boldsymbol{h_3} = \boldsymbol{R}^T\boldsymbol{h_3}, \\
\boldsymbol{T_\tau}(1,1) &= \int_A G\boldsymbol{R}^T(\boldsymbol{h_2}\boldsymbol{h_2}^T+\boldsymbol{h_3}\boldsymbol{h_3}^T)\boldsymbol{R}\,dA = GA\boldsymbol{R}^T\boldsymbol{I_c}\boldsymbol{R}, \\
&\quad where \boldsymbol{I_c} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{79}$$

and

$$\begin{aligned}
\boldsymbol{T_\tau}(1,2) &= \int_A G(\boldsymbol{g_2}+\boldsymbol{h_2})(\boldsymbol{g_2}+\boldsymbol{h_2})^T\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T + G(\boldsymbol{g_3}+\boldsymbol{h_3})(\boldsymbol{g_3}+\boldsymbol{h_3})^T\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T \\
&= 0
\end{aligned} \tag{80}$$

and

$$\begin{aligned}
\boldsymbol{T_\tau}(1,3) = &\int_A G((\boldsymbol{g_2}+\boldsymbol{h_2})(\boldsymbol{g_1}+\boldsymbol{h_1})^T\boldsymbol{R}^T\tilde{\boldsymbol{h_2}}^T + (\boldsymbol{g_2}+\boldsymbol{h_2})(\boldsymbol{g_2}+\boldsymbol{h_2})^T\boldsymbol{R}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}} \\
&+ (\boldsymbol{g_3}+\boldsymbol{h_3})(\boldsymbol{g_1}+\boldsymbol{h_1})^T\boldsymbol{R}^T\tilde{\boldsymbol{h_2}}^T + (\boldsymbol{g_3}+\boldsymbol{h_3})(\boldsymbol{g_2 3}+\boldsymbol{h_3})^T\boldsymbol{R}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}})\,dA \\
&= \int_A G(\boldsymbol{R}^T\boldsymbol{h_2}(\boldsymbol{R}^T\boldsymbol{\phi}+\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k})^T\boldsymbol{R}^T\tilde{\boldsymbol{h_2}}^T + \boldsymbol{R}^T\boldsymbol{h_3}(\boldsymbol{R}^T\boldsymbol{\phi}+\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k})^T\boldsymbol{R}^T\tilde{\boldsymbol{h_3}}^T)\,dA \\
&= \int_A G(\boldsymbol{R}^T\boldsymbol{h_2}\boldsymbol{h_2}^T\tilde{\boldsymbol{\phi}} + \boldsymbol{R}^T\boldsymbol{h_3}\boldsymbol{h_3}^T\tilde{\boldsymbol{\phi}})\,dA \\
&= GA\boldsymbol{R}^T\boldsymbol{I_c}\tilde{\boldsymbol{\phi}}
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
\boldsymbol{T_\tau}(2,2) &= \int_A G\tilde{\boldsymbol{\zeta}}\boldsymbol{R}((\boldsymbol{g_2}+\boldsymbol{h_2})(\boldsymbol{g_2}+\boldsymbol{h_2})^T + (\boldsymbol{g_3}+\boldsymbol{h_3})(\boldsymbol{g_3}+\boldsymbol{h_3})^T)\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\,dA \\
&= \int_A G(\tilde{\boldsymbol{\zeta}}\boldsymbol{h_2}\boldsymbol{h_2}^T\tilde{\boldsymbol{\zeta}}^T + \tilde{\boldsymbol{\zeta}}\boldsymbol{h_3}\boldsymbol{h_3}^T\tilde{\boldsymbol{\zeta}}^T)\,dA \\
&= G\boldsymbol{I_p}, \\
&where\, \boldsymbol{I_p} = \tilde{\boldsymbol{h_2}}^T\boldsymbol{I_s}\tilde{\boldsymbol{h_2}} + \tilde{\boldsymbol{h_3}}^T\boldsymbol{I_s}\tilde{\boldsymbol{h_3}}
\end{aligned} \tag{82}$$

and the equations of $\boldsymbol{T_\tau}(2,3)$ and $\boldsymbol{T_\tau}(3,3)$ are too long,so here give the results only.

$$\begin{aligned}
\boldsymbol{T_\tau}(2,3) &= G\boldsymbol{I_p}\tilde{\boldsymbol{k}}, \\
\boldsymbol{T_\tau}(3,3) &= G(A\tilde{\boldsymbol{\phi}}^T\boldsymbol{I_c}\tilde{\boldsymbol{\phi}} + \tilde{\boldsymbol{k}}^T\boldsymbol{I_p}\tilde{\boldsymbol{k}}).
\end{aligned} \tag{83}$$

The final result in matrix form is:

$$
\begin{aligned}
\boldsymbol{S}^s_{\boldsymbol{M}} = E &\begin{bmatrix} \boldsymbol{R}^T(A\boldsymbol{\phi}\boldsymbol{\phi}^T + \tilde{\boldsymbol{k}}^T\boldsymbol{I_s}\tilde{\boldsymbol{k}})\boldsymbol{R} & \boldsymbol{R}^T\tilde{\boldsymbol{k}}\boldsymbol{I_s}\tilde{\boldsymbol{\phi}} & \boldsymbol{R}^T\tilde{\boldsymbol{k}}\boldsymbol{I_s}\tilde{\boldsymbol{k_e}} \\ & \tilde{\boldsymbol{\phi}}^T\boldsymbol{I_s}\tilde{\boldsymbol{\phi}} & \tilde{\boldsymbol{\phi}}^T\boldsymbol{I_s}\tilde{\boldsymbol{k_e}} \\ symm & & \tilde{\boldsymbol{k_e}}^T\boldsymbol{I_s}\tilde{\boldsymbol{k_e}} \end{bmatrix} \\
+ G &\begin{bmatrix} A\boldsymbol{R}^T\boldsymbol{I_c}\boldsymbol{R} & 0 & A\boldsymbol{R^T}\boldsymbol{I_c}\tilde{\boldsymbol{\phi}} \\ & \boldsymbol{I_p} & \boldsymbol{I_p}\tilde{\boldsymbol{k}} \\ symm & & A\tilde{\boldsymbol{\phi}}^T\boldsymbol{I_c}\tilde{\boldsymbol{\phi}} + \tilde{\boldsymbol{k}}^T\boldsymbol{I_p}\tilde{\boldsymbol{k}} \end{bmatrix}
\end{aligned} \tag{84}
$$

Next consider the high order term $(\delta^2\boldsymbol{g})^T\boldsymbol{\Phi}$ in Eq.45,this can be split in two parts:

$$
(\delta^2\boldsymbol{g})^T\boldsymbol{\Phi} = \begin{bmatrix} \delta^2\boldsymbol{g_1}^T & \delta^2\boldsymbol{g_2}^T & \delta^2\boldsymbol{g_3}^T \end{bmatrix} \begin{bmatrix} s_{11}\boldsymbol{b_1} + s_{12}\boldsymbol{b_2} + s_{13}\boldsymbol{b_3} \\ s_{12}\boldsymbol{b_1} \\ s_{13}\boldsymbol{b_1} \end{bmatrix} \tag{85}
$$

Before giving the detailed expressions,we define some section quantities:

$$
\begin{aligned}
P =& A\sigma_b, \\
\boldsymbol{Q} =& G\int_A (\boldsymbol{h_2}\gamma_{12} + \boldsymbol{h_3}\gamma_{13})\, dA, \\
\boldsymbol{M_\sigma} =& E\int_A \boldsymbol{\zeta} e_{11}\, dA = E\int_A \boldsymbol{\zeta}\boldsymbol{\zeta}^T\boldsymbol{k_e}\, dA = E\boldsymbol{I_s}\boldsymbol{k_e}, \\
\boldsymbol{M_\tau} =& G\int_A (\tilde{\boldsymbol{\zeta}}\boldsymbol{h_2}\gamma_{12} + \tilde{\boldsymbol{\zeta}}\boldsymbol{h_3}\gamma_{13})\, dA \\
=& G\int_A (\tilde{\boldsymbol{h_2}}^T\boldsymbol{\zeta}\boldsymbol{\zeta}^T\tilde{\boldsymbol{h_2}}\boldsymbol{k} + \tilde{\boldsymbol{h_3}}^T\boldsymbol{\zeta}\boldsymbol{\zeta}^T\tilde{\boldsymbol{h_3}}\boldsymbol{k})\, dA \\
=& G\boldsymbol{I_p}\boldsymbol{k}. \\
\boldsymbol{I_p} =& \tilde{\boldsymbol{h_2}}^T\boldsymbol{I_s}\tilde{\boldsymbol{h_2}} + \tilde{\boldsymbol{h_3}}^T\boldsymbol{I_s}\tilde{\boldsymbol{h_3}}.
\end{aligned} \tag{86}
$$

Here P,$\boldsymbol{Q}$,$\boldsymbol{M_\sigma}$,$\boldsymbol{M_\tau}$ are axial force,transverse shear force,bending moments and torsional moments. Then we consider the first part of $(\delta^2\boldsymbol{g})^T\boldsymbol{\Phi}$:

$$
\begin{aligned}
\boldsymbol{L_1} &= \int_A \delta^2\boldsymbol{g_1}^T s_{11}\boldsymbol{b_{11}}\, dA \\
&= \int_A s_{11}2(\frac{d\delta\boldsymbol{\theta}^T}{dX_1} + \delta\boldsymbol{\theta}^T\tilde{\boldsymbol{k}}^T)\tilde{\boldsymbol{\zeta}}^T\delta\tilde{\boldsymbol{\theta}}^T\boldsymbol{R}(\boldsymbol{R}^T\boldsymbol{\phi} + \boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{k})\, dA \\
&= \int_A s_{11}2(\frac{d\delta\boldsymbol{\theta}^T}{dX_1} + \delta\boldsymbol{\theta}^T\tilde{\boldsymbol{k}}^T)\tilde{\boldsymbol{\zeta}}\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\phi}\, dA \\
&= \int_A s_{11}2(\frac{d\delta\boldsymbol{\theta}^T}{dX_1}\tilde{\boldsymbol{\zeta}}\delta\tilde{\boldsymbol{\phi}}\delta\boldsymbol{\theta} + \delta\boldsymbol{\theta}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\phi}}\delta\boldsymbol{\theta})\, dA \\
&= 2(\frac{d\delta\boldsymbol{\theta}^T}{dX_1}\widetilde{\boldsymbol{M_\sigma}}\delta\tilde{\boldsymbol{\phi}}\delta\boldsymbol{\theta} + \delta\boldsymbol{\theta}^T\tilde{\boldsymbol{k}}^T\widetilde{\boldsymbol{M_\sigma}}\tilde{\boldsymbol{\phi}}\delta\boldsymbol{\theta})
\end{aligned} \tag{87}
$$

And because of symmetric,this can be written in matrix form:

$$
\boldsymbol{L_1} = \begin{bmatrix} \dfrac{d\delta\boldsymbol{u_0}}{dX_1}^T & \dfrac{d\delta\boldsymbol{\theta}}{dX_1}^T & \delta\boldsymbol{\theta}^T \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \widetilde{\boldsymbol{M_\sigma}}\tilde{\boldsymbol{\phi}} \\ 0 & (\widetilde{\boldsymbol{M_\sigma}}\tilde{\boldsymbol{\phi}})^T & 2\tilde{\boldsymbol{k}}^T\widetilde{\boldsymbol{M_\sigma}}\tilde{\boldsymbol{\phi}} \end{bmatrix} \begin{bmatrix} \dfrac{d\delta\boldsymbol{u_0}}{dX_1} \\ \dfrac{d\delta\boldsymbol{\theta}}{dX_1} \\ \delta\boldsymbol{\theta} \end{bmatrix} \tag{88}
$$

The second part:

$$\boldsymbol{L_2} = \int_A s_{12}(\delta^2\boldsymbol{g_1}^T\boldsymbol{b_2} + \delta^2\boldsymbol{g_2}^T\boldsymbol{b_1}) + s_{13}(\delta^2\boldsymbol{g_1}^T\boldsymbol{b_3} + \delta^2\boldsymbol{g_3}^T\boldsymbol{b_1})\,dA \tag{89}$$

This part has 4 sub-member,we call them $\boldsymbol{L_{21}}$-$\boldsymbol{L_{24}}$,

$$\begin{aligned}
\boldsymbol{L_{21}} &= \int_A s_{12}\delta^2\boldsymbol{g_1}^T\boldsymbol{b_2}\,dA \\
&= \int_A s_{12}2(\frac{d\delta\boldsymbol{\theta}}{dX_1}^T + \delta\boldsymbol{\theta}^T\tilde{\boldsymbol{k}}^T)\tilde{\boldsymbol{\zeta}}\delta\tilde{\boldsymbol{\theta}}^T\boldsymbol{h_2}\,dA \\
&= \int_A s_{12}2(\frac{d\delta\boldsymbol{\theta}}{dX_1}^T + \delta\boldsymbol{\theta}^T\tilde{\boldsymbol{k}}^T)\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{h_2}}\delta\boldsymbol{\theta}\,dA
\end{aligned} \tag{90}$$

This can also be written in matrix form,and the central matrix is:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \int_A s_{12}\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{h_2}}\,dA \\ 0 & (\int_A s_{12}\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{h_2}}\,dA)^T & 2\int_A s_{12}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{h_2}}\,dA \end{bmatrix} \tag{91}$$

And $\boldsymbol{L_{23}}$ is the same form:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \int_A s_{13}\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{h_3}}\,dA \\ 0 & (\int_A s_{13}\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{h_3}}\,dA)^T & 2\int_A s_{13}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{h_3}}\,dA \end{bmatrix} \tag{92}$$

and the $\boldsymbol{L_{22}}$ and $\boldsymbol{L_{24}}$ have high order terms,so we drop them here.
Finally consider the section geometric stiffness matrix:

$$\boldsymbol{S_{GP}} = \int_A \boldsymbol{W}^T s_i\boldsymbol{H_i}^T\boldsymbol{W}\,dA \tag{93}$$

Using Mathematica to carry out and split the result in σ and τ part:

$$\begin{aligned}
\boldsymbol{S_{GP\sigma}}(1,1) &= \int_A s_{11}\boldsymbol{I}\,dA = P\boldsymbol{I} \\
\boldsymbol{S_{GP\sigma}}(1,2) &= \int_A s_{11}\boldsymbol{R}^T\tilde{\boldsymbol{\zeta}}^T\,dA = \boldsymbol{R}^T\widetilde{\boldsymbol{M_\sigma}}^T \\
\boldsymbol{S_{GP\sigma}}(1,3) &= \int_A s_{11}\boldsymbol{R}^T\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\,dA = \boldsymbol{R}^T\tilde{\boldsymbol{k}}^T\widetilde{\boldsymbol{M_\sigma}}^T \\
\boldsymbol{S_{GP\sigma}}(2,2) &= \boldsymbol{S_{GP\sigma}}(2,3) = \boldsymbol{S_{GP\sigma}}(3,3) = 0
\end{aligned} \tag{94}$$

And

$$\begin{aligned}
&\boldsymbol{S_{GP\tau}}(1,1) = \boldsymbol{S_{GP\tau}}(1,2) = \boldsymbol{S_{GP\tau}}(2,2) = 0 \\
&\boldsymbol{S_{GP\tau}}(1,3) = \boldsymbol{R}^T\int_A s_{12}\,dA\tilde{\boldsymbol{h_2}}^T + \boldsymbol{R}^T\int_A s_{13}\,dA\tilde{\boldsymbol{h_3}}^T, \\
&\boldsymbol{S_{GP\tau}}(2,3) = \int_A s_{12}\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{h_2}}^T\,dA + \int_A s_{13}\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{h_3}}^T\,dA, \\
&\boldsymbol{S_{GP\tau}}(3,3) = \int_A s_{12}\tilde{\boldsymbol{h_2}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\,dA + \int_A s_{13}\tilde{\boldsymbol{h_3}}\tilde{\boldsymbol{k}}^T\tilde{\boldsymbol{\zeta}}\,dA \\
&\qquad + \int_A s_{12}\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}\tilde{\boldsymbol{h_2}}^T\,dA + \int_A s_{13}\tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{k}}\tilde{\boldsymbol{h_3}}^T\,dA
\end{aligned} \tag{95}$$

Then combine $\boldsymbol{L_1}$ and $\boldsymbol{S_{GP\sigma}}$,$\boldsymbol{L_2}$ and $\boldsymbol{S_{GP\tau}}$,we get:

$$\boldsymbol{S_{GC\sigma}} = \boldsymbol{L_1} + \boldsymbol{S_{GP\sigma}} = \begin{bmatrix} P\boldsymbol{I} & \boldsymbol{R}^T \widetilde{\boldsymbol{M_\sigma}}^T & \boldsymbol{R}^T \tilde{\boldsymbol{k}}^T \widetilde{\boldsymbol{M_\sigma}}^T \\ & 0 & \widetilde{\boldsymbol{M_\sigma}} \tilde{\boldsymbol{\phi}} \\ symm & & 2\tilde{\boldsymbol{k}}^T \widetilde{\boldsymbol{M_\sigma}} \tilde{\boldsymbol{\phi}} \end{bmatrix} \tag{96}$$

$$\boldsymbol{S_{GC\tau}} = \boldsymbol{L_2} + \boldsymbol{S_{GP\tau}} = \begin{bmatrix} 0 & 0 & \boldsymbol{R}^T \tilde{\boldsymbol{Q}}^T \\ 0 & 0 & 0 \\ symm & & 0 \end{bmatrix} \tag{97}$$

And the final section tangent stiffness matrix is:

$$\boldsymbol{S_C^t} = \boldsymbol{S_M^s} + \boldsymbol{S_{GC\sigma}} + \boldsymbol{S_{GC\tau}} \tag{98}$$

3.4 Transformation to Final DOFs

First we define the linear shape function matrix $\boldsymbol{N}$:

$$\boldsymbol{N} = \frac{1}{2} \begin{bmatrix} 1-\epsilon & 0 & 0 & 1+\epsilon & 0 & 0 \\ 0 & 1-\epsilon & 0 & 0 & 1+\epsilon & 0 \\ 0 & 0 & 1-\epsilon & 0 & 0 & 1+\epsilon \end{bmatrix} \tag{99}$$

where $\epsilon = \dfrac{2X_1}{L} - 1$ is the local coordinate from -1 to 1,and we can apply the one point Gauss Integration Rule on it.
Then the section result is connected to the final DOFs:

$$\boldsymbol{w} = \begin{bmatrix} \dfrac{d\boldsymbol{N}}{dX_1} & 0 \\ 0 & \dfrac{d\boldsymbol{N}}{dX_1} \\ 0 & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{d_n} \\ \boldsymbol{a_n} \end{bmatrix} = \boldsymbol{D}\boldsymbol{v} \tag{100}$$

and $\delta\boldsymbol{w} = \boldsymbol{D}\delta\boldsymbol{v}$.The final internal force and tangent stiffness matrix are:

$$\begin{aligned} \boldsymbol{f} &= \int_0^L \boldsymbol{D}^T \boldsymbol{f_z} \, dX = \frac{L}{2} \int_{-1}^1 \boldsymbol{D}^T \boldsymbol{f_z} \, d\epsilon, \\ \boldsymbol{K^t} &= \int_0^L \boldsymbol{D}^T \boldsymbol{K_z} \boldsymbol{D} \, dX = \frac{L}{2} \int_{-1}^1 \boldsymbol{D}^T \boldsymbol{K_z} \boldsymbol{D} \, d\epsilon \end{aligned} \tag{101}$$

We can then use the one point Gauss Integration Rule to numerically get the result.