Notes on Exercise Sheet 6

Primal/Dual Problem and KKT-Conditions

Consider an optimization problem in the **canonical** form:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f_0(\boldsymbol{x}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}) \leqslant 0, \quad i = 1, ..., m \\ & h_i(\boldsymbol{x}) = 0, \quad i = 1, ..., p \end{aligned}$$

The **Lagrange function** $\mathcal{L}\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is defined as a weighted sum of the objective and constraint functions:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}) + \sum_{i=1}^{p} \mu_i h_i(\boldsymbol{x}),$$

where $\boldsymbol{x}$ is called **primal** and $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ the **dual** variables.

The (Lagrange) **dual function** $g\colon \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is defined as:

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\boldsymbol{x} \in \text{domain}(f_0)} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}).$$

The convex optimization problem

$$\begin{aligned} \underset{(\boldsymbol{\lambda}, \boldsymbol{\mu})}{\text{maximize}} \quad & g(\boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \text{subject to} \quad & \boldsymbol{\lambda} \succeq \mathbf{0} \end{aligned}$$

is called the (Lagrange) **dual problem**.

In the Lagrange optimization framework the KKT-conditions are used to find the primal and dual optimal solutions.

Theorem 1 (Optimality Conditions) *For any optimization problem with differentiable objective and constraint functions for which strong duality obtains, any pair of primal and dual optimal $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ must satisfy KKT-conditions:*

$$\begin{aligned} & \nabla_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0, && \textit{(stationarity)} \\ & f_i(\boldsymbol{x}^*) \leqslant 0, && \textit{(primal feasibility)} \\ & h_i(\boldsymbol{x}^*) = 0, && \textit{(primal feasibility)} \\ & \lambda_i^* \geqslant 0, && \textit{(dual feasibility)} \\ & \lambda_i^* \cdot f_i(\boldsymbol{x}^*) = 0 && \textit{(complementary slackness)} \end{aligned}$$

For any convex problem, the KKT-conditions are sufficient for $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ *to be optimal with zero duality gap.*