Def 72 (Diffusion)

Gesucht $u(x,t) : \mathbb{R}\times[0,\infty) \to \mathbb{R}$

sodass $$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$$

$$u(x,0) = u_0(x)$$

$$\left(u_0 * \left(\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} e^{-\frac{(\cdot)^2}{4t}}\right) \right)(x)$$

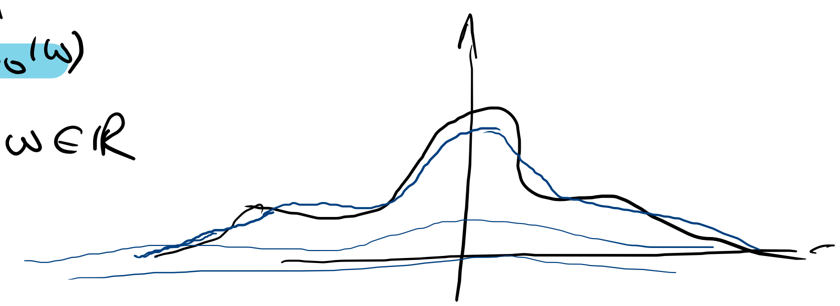
Fouriertransformation bezüglich x:

$$\hat{u}(\omega,t) := \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx.$$

Transformation der DGL:

$$\frac{\partial}{\partial t} \hat{u}(\omega,t) = -\omega^2 \hat{u}(\omega,t), \qquad \hat{u}(\omega,0) = \hat{u}_0(\omega)$$

$\longrightarrow$ Gewöhnliche DGL in t für jedes $\omega\in\mathbb{R}$



Lösung für $\omega\in\mathbb{R}$:

$$\hat{u}(\omega,t) = \hat{u}_0(\omega)\, e^{-\omega^2 t}$$

Für Rücktransformation

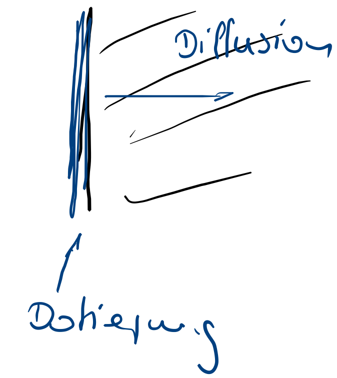
$$\mathcal{F}\left[e^{-x^2/2}\right](\omega) = \sqrt{2\pi}\, e^{-\omega^2/2} \xrightarrow[\frac{1}{\sqrt{2t}}]{\text{skaling}} \mathcal{F}\left[\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} e^{-\left(\frac{x}{\sqrt{2t}}\right)^2/2}\right] = e^{-\omega^2 t}$$

wird breiter für $t\to\infty$

$$\longrightarrow u(x,t) = \underbrace{u_0(x) * \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}}}_{\text{Faltung bezüglich } x} = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} u_0(\xi)\, e^{-\frac{(x-\xi)^2}{4t}}\, d\xi$$

Faltung mit Glockenkurve

Definition 73 (Diffusion im Halbleiter)



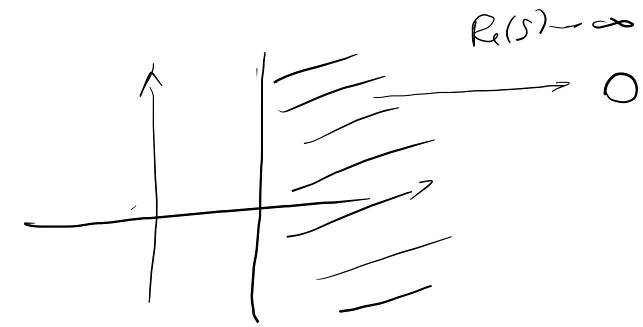
Gesucht $u : [0,\infty) \times [0,\infty) \to \mathbb{R}$

so dass $$\frac{\partial}{\partial t} u(x,t) = \gamma \frac{\partial^2}{\partial x^2} u(x,t), \qquad \gamma > 0$$

$$u(x,0) = 0, \qquad u(0,t) = T$$

Laplacetransformation nach t:

$$U(x,s) := \int_0^\infty u(x,t)\, e^{-st}\, dt \qquad \text{für festes } x \in [0,\infty)$$



Transformation der DGL:

$$s\, U(x,s) = \gamma \frac{\partial^2}{\partial x^2} U(x,s), \qquad U(0,s) = \frac{T}{s}$$

$\to$ Gewöhnliche DGL

$\to$ Polynom $\gamma \lambda^2 - s \overset{!}{=} 0, \qquad \lambda_{1,2} = \pm\sqrt{\frac{s}{\gamma}}$

$\to$ Fundamentalsystem

$$U_1(x,s) = e^{-\sqrt{s/\gamma}\cdot x} \qquad U_2(x,s) = e^{\sqrt{s/\gamma}\, x}$$

$$\lightning \quad \lim_{Re(s)\to\infty} U_2(x,s) = \infty$$

Wir beschränken uns auf die erste Lösung

$$U(x,s) = \frac{T}{s} e^{-\sqrt{s/\gamma}\, x}$$

$$\mathcal{L}\left[\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right](s) = \frac{1}{s} e^{-a\sqrt{s}} \qquad a \geq 0$$

konjugierte gaußsche Fehlerfunktion

$$\operatorname{erfc}(t) := \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-\tau^2}\, d\tau \qquad \tau \rightsquigarrow (\tau + x)/2\sqrt{\gamma t}$$

setze $a = \sqrt{x^2/\gamma}$

$\tau \mapsto \tau + x/2\sqrt{\gamma t}$

$$u(x,t) = T \operatorname{erfc}\left(\frac{x}{2\sqrt{t\gamma}}\right) = \frac{2T}{\sqrt{\pi}} \int_{x/2\sqrt{t\gamma}}^\infty e^{-\tau^2}\, d\tau = \frac{2T}{\sqrt{\pi}} \int_0^\infty e^{-(\tau + x/2\sqrt{\gamma t})^2}\, d\tau$$

$\tau \rightsquigarrow \tau/2\sqrt{\gamma t}$

$$= \frac{T}{\sqrt{\gamma t \pi}} \int_0^\infty e^{-(\tau + x)^2/4\gamma t}\, d\tau$$