# 11. Tutorium: Zeitdiskrete Fourier-Trafo, DFT

## DFT

- $U_{DFT}(n) = U(jn\Delta\Omega) = \sum_{k=0}^{N-1} u(k) \cdot e^{-jkn\Delta\Omega}$

  mit $\Delta\Omega = \frac{2\pi}{N}$, $0 \leq n \leq N-1$

  $N$: Anzahl der Werte von $u(n)$

- Eingenschaft

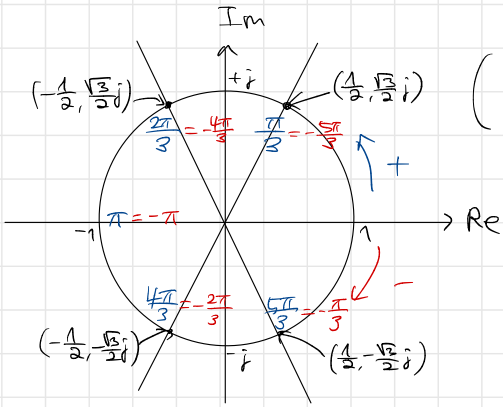
  komplex konjugiert

  $U_{DFT}(N-n) = U^{*}_{DFT}(n)$

## iDFT (inverse DFT)

- $u(k) = \frac{1}{N} \cdot \sum_{n=0}^{N-1} U_{DFT}(n) \cdot e^{jkn\Delta\Omega}$

$e^{jx} = \cos(x) + j\sin(x)$




$$\left( \begin{array}{l} e^{j\frac{\pi}{2}} = e^{-j\frac{3\pi}{2}} = j \\ e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}} = -j \end{array} \right.$$

2.1 b) **[HA]:** Überprüfe das Ergebnis durch Rücktransformation des Signals mittels inverser DFT.

aus a) $U_{DFT}(n) = \{ \underbrace{2}_{n=0}, \underbrace{-1+3j}_{n=1}, \underbrace{4}_{n=2}, \underbrace{-1-3j}_{n=3} \}$

$N = 4, \quad \Delta\Omega = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\boxed{\begin{array}{l}\text{iDFT (inverse DFT)} \\ \circ\ u(k) = \frac{1}{N} \cdot \sum_{n=0}^{N-1} U_{DFT}(n) \cdot e^{jkn\Delta\Omega}\end{array}}$$

$$u(k) = \frac{1}{4} \cdot \sum_{n=0}^{3} U_{DFT}(n) \cdot e^{jkn\frac{\pi}{2}}$$

$$u(k) = \frac{1}{4}\left(2 + (-1+3j) \cdot e^{jk\frac{\pi}{2}} + 4 \cdot e^{jk\pi} + (-1-3j) \cdot e^{jk\frac{3\pi}{2}}\right)$$

$$u(0) = \frac{1}{4}\left(2 + (-1+3j) + 4 + (-1-3j)\right)$$
$$= \frac{1}{4} \cdot 4 = 1$$

$$u(1) = \frac{1}{4}\left(2 + (-1+3j) \cdot \underbrace{e^{j\frac{\pi}{2}}}_{=j} + 4\underbrace{e^{j\pi}}_{=-1} + (-1-3j)\underbrace{e^{j\frac{3\pi}{2}}}_{-j}\right)$$
$$= \frac{1}{4}(2 + (-1+3j) \cdot j - 4 + (-1-3j) \cdot -j$$
$$= \frac{1}{4}\left(2 - \cancel{j} + \underbrace{3j^2}_{=-3} - 4 + \cancel{j} + \underbrace{3j^2}_{=-3}\right)$$
$$= \frac{1}{4} \cdot -8 = -2$$

$$u(2) = \frac{1}{4}\left(2 + (-1+3j)\underbrace{e^{j\pi}}_{=-1} + 4\underbrace{e^{j2\pi}}_{=1} + (-1-3j)\underbrace{e^{j3\pi}}_{=-1}\right.$$
$$= \frac{1}{4}(2 + (1 - \cancel{3j}) + 4 + 1 + \cancel{3j}$$
$$= \frac{1}{4} \cdot 8 = 2$$

$$u(3) = \frac{1}{4}\left(2 + (-1+3j)\underbrace{e^{j\frac{3\pi}{2}}}_{-j} + 4\underbrace{e^{j3\pi}}_{-1} + (-1-3j)\underbrace{e^{j\frac{9\pi}{2}}}_{j}\right.$$
$$= \frac{1}{4}\left(2 + (\cancel{+j}\underbrace{-3j^2}_{3}) - 4 + (\cancel{-j}\underbrace{-3j^2}_{3})\right)$$
$$= \frac{1}{4}(2 + 3 - 4 + 3) = 1$$

$\therefore u(k) = \{1, -2, 2, 1\}$

**2.2 [HA]: Berechne die Faltung $y(n) = u(n) * v(n)$, $u = \{1,0,1\}$, $v = \{0,2,1\}$ mittels Multiplikation der entsprechenden DFT-Spektren im Frequenzbereich.**

ges.: Faltung $y(n)$, Methode: Multiplikation der DFT-Spektren

geg.: $u = \{1,0,1\}$, $v = \{0,2,1\}$

Vorgehensweise: ① DFT $\rightarrow U_{DFT}(n)$, $V_{DFT}(n)$

② $Y_{DFT}(n) = U_{DFT}(n) \cdot V_{DFT}(n)$

③ iDFT $\rightarrow y(n)$

① DFT

$$U_{DFT}(n) = \sum_{k=0}^{N-1} u(k) \cdot e^{-jkn\Delta\Omega} \qquad u(n) = \{1,0,1\}$$

$$N = 3, \quad \Delta\Omega = \frac{2\pi}{N} = \frac{2\pi}{3}$$

$$U_{DFT}(n) = 1 + 0 + 1 \cdot e^{-j2n\frac{2\pi}{3}} \qquad U_{DFT}(\overset{3}{N} - n) = U^*_{DFT}(n)$$

$$U_{DFT}(3-2) = U^*_{DFT}(2)$$

$$U_{DFT}(0) = 1 + 1 = 2$$

$$U_{DFT}(1) = 1 + e^{-j\frac{4\pi}{3}} = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}j\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}j$$

$$U_{DFT}(2) = 1 + e^{-j\cdot\frac{8\pi}{3}} = 1 + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}j\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}j$$

$$\therefore U_{DFT}(n) = \left\{2, \frac{1}{2} + \frac{\sqrt{3}}{2}j, \frac{1}{2} - \frac{\sqrt{3}}{2}j\right\}$$

$$V_{DFT}(n) = 0 + 2e^{-jn\frac{2\pi}{3}} + 1 \cdot e^{-jn\frac{4\pi}{3}} \qquad v = \{0,2,1\}$$

$$V_{DFT}(0) = 2 + 1 = 3$$

$$V_{DFT}(1) = 2e^{-j\frac{2\pi}{3}} + e^{-j\frac{4\pi}{3}} = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}j\right) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}j\right) = -\frac{3}{2} - \frac{\sqrt{3}}{2}j$$

$$V_{DFT}(2) = -\frac{3}{2} + \frac{\sqrt{3}}{2}j$$

$$\therefore V_{DFT}(n) = \left\{3, -\frac{3}{2} - \frac{\sqrt{3}}{2}j, -\frac{3}{2} + \frac{\sqrt{3}}{2}j\right\}$$

② $Y_{DFT}(n) = U_{DFT}(n) \cdot V_{DFT}(n)$

$$U_{DFT} = \left\{ 2,\ \frac{1}{2} + \frac{\sqrt{3}}{2} j,\ \frac{1}{2} - \frac{\sqrt{3}}{2} j \right\}$$
$$V_{DFT} = \left\{ 3,\ -\frac{3}{2} - \frac{\sqrt{3}}{2} j,\ -\frac{3}{2} + \frac{\sqrt{3}}{2} j \right\}$$

$$\begin{aligned}
Y_{DFT}(0) &= U_{DFT}(0) \cdot V_{DFT}(0) \\
&= 2 \cdot 3 = 6 \\
Y_{DFT}(1) &= U_{DFT}(1) \cdot V_{DFT}(1) \\
&= \left(+\frac{1}{2} + \frac{\sqrt{3}}{2} j\right)\left(-\frac{3}{2} - \frac{\sqrt{3}}{2} j\right) \\
&= \cancel{-\frac{3}{4}} - \frac{\sqrt{3}}{4} j - \frac{3\sqrt{3}}{4} j \cancel{- \frac{3}{4} j^2} \\
&= -\sqrt{3} j
\end{aligned}$$

$$Y_{DFT}(1) = Y^{*}_{DFT}(2)$$

$$Y_{DFT}(2) = \sqrt{3} j$$

$$\therefore\ Y_{DFT}(n) = \{ 6,\ -\sqrt{3} j,\ \sqrt{3} j \}$$

③ iDFT → $y(n)$

$$\Delta\Omega = \frac{2\pi}{3}$$

$$u(k) = \frac{1}{N} \cdot \sum_{n=0}^{N-1} U_{DFT}(n) \cdot e^{jkn\Delta\Omega}$$

$$y(k) = \frac{1}{3} \cdot \left( 6 - \sqrt{3} j \cdot e^{jk\frac{2\pi}{3}} + \sqrt{3} j\, e^{jk\frac{4\pi}{3}} \right)$$

$$\begin{aligned}
y(0) &= \frac{1}{3}\left( 6 \cancel{- \sqrt{3} j + \sqrt{3} j} \right) \\
&= 2 \\
y(1) &= \frac{1}{3}\left( 6 - \sqrt{3} j\, \underbrace{e^{j\frac{2\pi}{3}}}_{-\frac{1}{2} + \frac{\sqrt{3}}{2} j} + \sqrt{3} j\, \underbrace{e^{j\frac{4\pi}{3}}}_{-\frac{1}{2} - \frac{\sqrt{3}}{2} j} \right) \\
&= \frac{1}{3}\left( 6 \cancel{+ \frac{\sqrt{3}}{2} j} - \frac{3}{2} j^2 \cancel{- \frac{\sqrt{3}}{2} j} - \frac{3}{2} j^2 \right) \\
&= \frac{1}{3}\left( 6 + \frac{3}{2} + \frac{3}{2} \right) = 3 \\
y(2) &= \frac{1}{3}\left( 6 - \sqrt{3} j\, \underbrace{e^{j\frac{4\pi}{3}}}_{-\frac{1}{2} - \frac{\sqrt{3}}{2} j} + \sqrt{3} j\, \underbrace{e^{j\frac{8\pi}{3}}}_{-\frac{1}{2} + \frac{\sqrt{3}}{2} j} \right) \\
&= \frac{1}{3}\left( 6 \cancel{+ \frac{\sqrt{3}}{2} j} + \frac{3}{2} j^2 \cancel{- \frac{\sqrt{3}}{2} j} + \frac{3}{2} j \right) \\
&= \frac{1}{3}(6 - 3) = 1
\end{aligned}$$

$$\therefore\ y(n) = \{ 2, 3, 1 \}$$

# AR - Aufgabe

geg.: $u = \{-1, 0, 2\}$

ges.: $U_{DFT}(n)$

$$U_{DFT} = \sum_{k=0}^{N-1} u(k) \cdot e^{-jkn\Delta\Omega} \qquad N=3 \qquad \Delta\Omega = \frac{2\pi}{N}$$

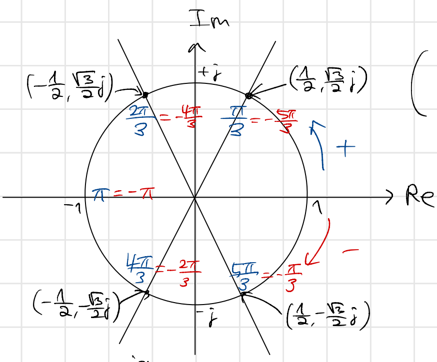
$$U_{DFT} = \left(-1 + 0 + 2e^{-jn\frac{4\pi}{3}}\right)$$

$$U_{DFT}(0) = (-1+2) = 1$$

$$U_{DFT}(1) = -1 + 2e^{-j\frac{4\pi}{3}} = -1 + 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}j\right)$$
$$= -2 + \sqrt{3}j$$

$$U_{DFT}(2) = -2 - \sqrt{3}j$$

$$\therefore U_{DFT}(n) = \{1, -2+\sqrt{3}j, -2-\sqrt{3}j\}$$




$$e^{jx} = \cos(x) + j\sin(x)$$

$$\left( \begin{array}{ll} e^{j\frac{\pi}{2}} = j & e^{-j\frac{\pi}{2}} = -j \\ e^{j\frac{3\pi}{2}} = -j & e^{-j\frac{3\pi}{2}} = j \end{array} \right.$$

$$-\frac{8\pi}{3} = -2\pi - \frac{2\pi}{3}$$