Integraltransformationen und Partielle Differentialgleichungen
(Tutorium 9)

Vorlesungswoche: 17. – 21. Juni 2024 Sommersemester 2024

Aufgabe 25

Finde Funktionen mit den Laplacetransformierten:

$$\text{(a) } F(s) = \frac{4}{s-3}, \quad \text{(b) } F(s) = \frac{4s-5}{s^2-3s+2}, \quad \text{(c) } F(s) = \frac{5}{s^2+4}, \quad \text{(d) } F(s) = \frac{s^2+2s}{(s-2)(s^2+4)}.$$

Aufgabe 26

Mit Hilfe der Laplacetransformation löse

(a) $x'(t) + 4x(t) = 0, \quad x(0) = 2,$

(b) $x'(t) + 3x(t) = 4\mathrm{e}^{-2t}, \quad x(0) = 3,$

(c) $x''(t) + 3x'(t) + 2x(t) = 6\mathrm{e}^{t}, \quad x(0) = 3, x'(0) = -6,$

(d) $x'(t) + 3x(t) = 48t\mathrm{e}^{t}, \quad x(0) = 4.$

Laplace-Korrespondenzen

$\mathcal{L}[e^{at}](s) = \frac{1}{s-a}$ $\qquad Re(s) > Re(a)$

$\mathcal{L}[\cos(wt)](s) = \frac{s}{s^2+w^2}$ $\qquad a, w \in \mathbb{C}$

$\qquad n \in \mathbb{N}_0$

$\mathcal{L}[\sin(wt)](s) = \frac{w}{s^2+w^2}$

$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}}$

$\mathcal{L}[1](s) = \frac{1}{s}$

$\mathcal{L}[\frac{1}{n-1} e^{at} t^{n-1}](s) = \frac{1}{(s-a)^n}$

Rechenregel

Linearität: 1) $\mathcal{L}[f(t)+g(t)](s) = \mathcal{L}[f(t)](s) + \mathcal{L}[g(t)](s)$

2) $\mathcal{L}[a \cdot f(t)](s) = a\,\mathcal{L}[f(t)](s)$

Ableitungssatz: $\mathcal{L}[f'(t)](s) = s\,\mathcal{L}[f(t)](s) - f(0)$

$\mathcal{L}[f''(t)](s) = s^2\,\mathcal{L}[f(t)](s) - s f(0) - f'(0)$

$\mathcal{L}[f^{(n)}(t)](s) = s^n\,\mathcal{L}[f(t)](s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$

Aufgabe 25 Rücktransformation

Finde Funktionen mit den Laplacetransformierten:

(a) $F(s) = \frac{4}{s-3}$, (b) $F(s) = \frac{4s-5}{s^2-3s+2}$, (c) $F(s) = \frac{5}{s^2+4}$, (d) $F(s) = \frac{s^2+2s}{(s-2)(s^2+4)}$.

(a) $F(s) = \frac{4}{s-3} = 4\,\mathcal{L}[e^{3t}](s) = \mathcal{L}[4e^{3t}](s)$

$\Rightarrow f(t) = 4e^{3t}$

(b) $F(s) = \frac{4s-5}{s^2-3s+2} = \frac{3}{s-2} + \frac{1}{s-1} = \mathcal{L}[3e^{2t}](s) + \mathcal{L}[e^t](s)$

$= \mathcal{L}[3e^{2t} + e^t](s)$

$\Rightarrow f(t) = 3e^{2t} + e^t$

$= \frac{A}{s-2} + \frac{B}{s-1}$

$B = \frac{4\cdot(1)-5}{1-2} = 1$

$A = \frac{4\cdot 2-5}{2-1} = 3$

(c) $F(s) = \frac{5}{s^2+4} = \frac{5}{2} \cdot \mathcal{L}[\sin(2t)]$ $\Rightarrow f(t) = \frac{5}{2}\sin(2t)$

(d) $F(s) = \frac{s^2+2s}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4} = \frac{1}{s-2} + \frac{2}{s^2+4} = \mathcal{L}[e^{2t}](s) + \mathcal{L}[\sin(2t)](s)$

$\Downarrow$

$f(t) = e^{2t} + \sin(2t)$

$A = \frac{2^2+2\cdot 2}{2^2+4} = \frac{8}{8} = 1$

einsetzen: $s=0$: $0 = \frac{A}{-2} + \frac{C}{4} = -\frac{1}{2} + \frac{C}{4} \Rightarrow C = 2$

$s=1$: $\frac{3}{-5} = -1 + \frac{B+2}{5} \Rightarrow B = 0$

Aufgabe 26

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(a) $x'(t) + 4x(t) = 0, \quad x(0) = 2,$

(b) $x'(t) + 3x(t) = 4e^{-2t}, \quad x(0) = 3,$

(c) $x''(t) + 3x'(t) + 2x(t) = 6e^{t}, \quad x(0) = 3, x'(0) = -6,$

(d) $x'(t) + 3x(t) = 48te^{t}, \quad x(0) = 4.$

(a) $\mathcal{L}[0](s) = \mathcal{L}[1-1](s) = \mathcal{L}[1](s) - \mathcal{L}[1](s) = \frac{1}{s} - \frac{1}{s} = 0$

(b) $\mathcal{L}[4e^{-2t}](s) = 4\,\mathcal{L}[e^{-2t}](s) = 4 \cdot \frac{1}{s-(-2)} = \frac{4}{s+2} \qquad Re(s) > -2$

(c) $\mathcal{L}[6e^{t}](s) = 6\,\mathcal{L}[e^{t}](s) = \frac{6}{s-1} \qquad Re(s) > 1$

(d) $\mathcal{L}[48te^{t}](s) = 48\,\mathcal{L}[te^{t}](s) = 48 \frac{1}{(s-1)^2} \qquad Re(s) > 1$

a) ① Die DGL Laplace-transformation

$$\mathcal{L}[x'(t) + 4x(t)](s) = \mathcal{L}[0](s)$$

$$\mathcal{L}[x'(t)](s) + 4\,\mathcal{L}[x(t)](s) = 0 \qquad \Big| \text{ Ableitungssatz}$$

$$[s\,\mathcal{L}[x(t)](s) - x(0)] + 4\,\mathcal{L}[x(t)](s) = 0 \qquad \Big| \; \mathcal{L}[x(t)](s) = X(s)$$

$$sX(s) - 2 + 4X(s) = 0$$

② nach $X(s)$ umformen

$$X(s)(s+4) = 2 \quad \Rightarrow X(s) = \frac{2}{s+4}$$

③ Rücktransformation

$$x(t) = 2e^{-4t}$$

d) $\mathcal{L}[x'(t)](s) + 3\,\mathcal{L}[x(t)](s) = \mathcal{L}[48te^{t}](s)$

$$[s\,\mathcal{L}[x(t)](s) - x(0)] + 3\,\mathcal{L}[x(t)](s) = \frac{48}{(s-1)^2}$$

$$sX(s) - 4 + 3X(s) = \frac{48}{(s-1)^2}$$

$$(s+3)X(s) = \frac{48}{(s-1)^2} + 4$$

$$X(s) = \frac{48}{(s+3)(s-1)^2} + \frac{4}{s+3}$$

$$= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3} + \frac{4}{s+3}$$

$$C = \frac{48}{(-3-1)^2} = 3$$

$$B = \frac{48}{1+3} = 12$$

einsetzen: $s=0$ $\quad -A + \frac{12}{1} + 1 = \frac{48}{3\cdot 1} = 16$

$$\Rightarrow A = -3$$

$$X(s) = -\frac{3}{s-1} + \frac{12}{(s-1)^2} + \frac{7}{s+3}$$

$$x(t) = -3e^{t} + 7\cdot e^{-3t} + 12e^{t}\cdot t$$

(c) $x''(t) + 3x'(t) + 2x(t) = 6e^t, \quad x(0) = 3, x'(0) = -6,$

c) $\mathcal{L}[x''(t)](s) + 3\mathcal{L}[x'(t)](s) + 2\mathcal{L}[x(t)](s) = \frac{6}{s-1}$

$$s^2\mathcal{L}[x(t)](s) - s x(0) - x'(0) + 3\left[s\mathcal{L}[x(t)](s) - f(0)\right] + 2\mathcal{L}[x(t)](s) = \frac{6}{s-1}$$

$$s^2\cdot X(s) - 3s + 6 + 3sX(s) - 9 + 2X(s) = \frac{6}{s-1}$$

$$(s^2+3s+2)X(s) - 3s - 3 = \frac{6}{s-1}$$

$$X(s) = \frac{6}{(s+2)(s+1)(s-1)} + \frac{3(s+1)}{(s+2)(s+1)}$$

$$= \frac{6}{(s+2)(s+1)(s-1)} + \frac{3}{s+2}$$

$$\frac{6}{(s+2)(s+1)(s-1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s-1}$$

$$A = \frac{6}{-1\cdot(-3)} = 2$$

$$B = \frac{6}{1(-2)} = -3$$

$$C = \frac{6}{3\cdot 2} = 1$$

$$X(s) = \frac{2}{s+2} - \frac{3}{s+1} + \frac{1}{s-1} + \frac{3}{s+2}$$

$$= \frac{5}{s+2} - \frac{3}{s+1} + \frac{1}{s-1}$$

$$x(t) = 5e^{-2t} - 3e^{-t} + e^{t}$$