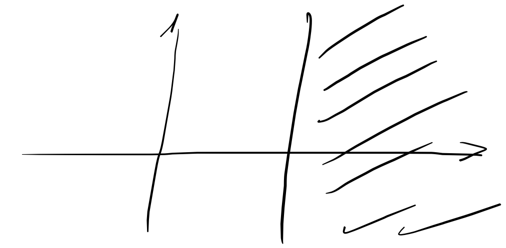
$f : [0,\infty) \to \mathbb{C}$



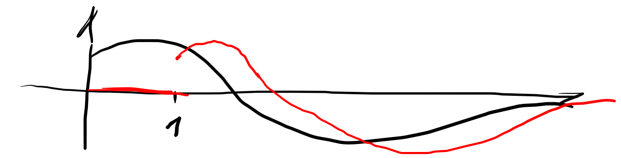
$$F(s) := \mathcal{L}[f](s) := \int_0^\infty f(t)\, e^{-st}\, dt$$

Satz 46 (Verschiebung)

$f : [0,\infty) \to \mathbb{C}$. Definier $(\Theta \geq 0)$

$$g(t) := \begin{cases} f(t-\Theta), & t \geq \Theta \\ 0 & \text{sonst} \end{cases}$$

$$\mathcal{L}[g](s) := e^{-\Theta s} \mathcal{L}[f](s)$$

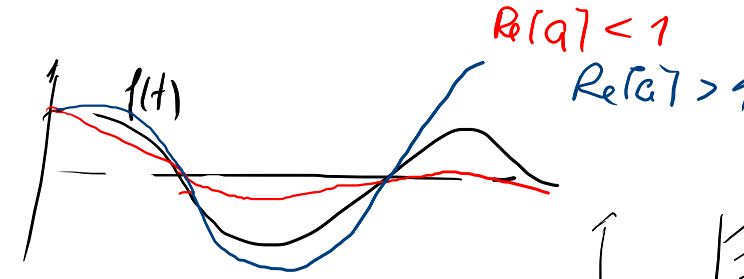


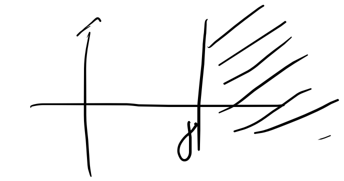
$dt \mapsto dt$
$t \mapsto t+\Theta$
$t-\Theta \mapsto t$

Beweis

$$\mathcal{L}[g](s) = \int_0^\infty g(t)\, e^{-st}\, dt = \int_\Theta^\infty f(t-\Theta)\, e^{-st}\, dt = \int_0^\infty f(t)\, e^{-s(t+\Theta)}\, dt$$

$$= e^{-s\Theta} \int_0^\infty f(t)\, e^{-st}\, dt = e^{-s\Theta} \mathcal{L}[f](s) \qquad \square$$





Satz 47 (Dämpfung)

Sei $f:[0,\infty)\to\mathbb{C}$, $\quad a\in\mathbb{C}$

$$\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)$$

Beweis:

$$\mathcal{L}[e^{at}f(t)](s) = \int_0^\infty e^{at} f(t)\, e^{-st}\, dt = \int_0^\infty f(t)\, e^{-(s-a)t}\, dt = \mathcal{L}[f](s-a)$$

Satz 48 (Ableitungen)

Sei $f:[0,\infty)\to\mathbb{C}$ stetig diff'bar und f' von exp. Ord. γ.

$|f'(t)| \le C e^{\gamma t}$

$|g+D| \le \tilde{C} e^{\gamma t}$

Dann f exp Ord γ und

$$\mathcal{L}[f'](s) = s\,\mathcal{L}[f] - f(0), \qquad Re[s]>\gamma.$$

Beweis

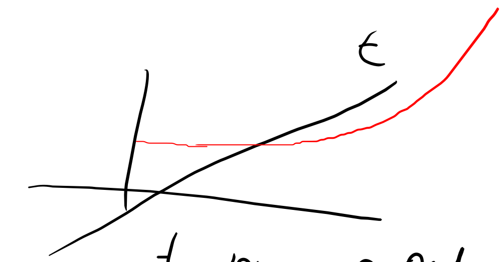
$$|f(t)| = \left| f(0) + \int_0^t f'(t)\, dt \right|$$

$$\left|\int_0^t f'(t)\, dt\right| \le \int_0^t |f'(\tau)|\, d\tau \le \int_0^t C\, e^{\gamma\tau} d\tau = \frac{C}{\gamma}\left(e^{\gamma t} - e^{\gamma 0}\right) \le \frac{C}{\gamma} e^{\gamma t}$$

$$\mathcal{L}[f'](s) = \int_0^\infty f'(t)\, e^{-st}\, dt \overset{\text{Partielle Integration}}{=} \left[f(t)\, e^{-st}\right]_0^\infty + \int_0^\infty f(t)\, s\, e^{-st}\, dt$$

$$\underset{Re[s]>\gamma}{=} -f(0) + s\,\mathcal{L}[f](s)$$

$\square$



f von exp Ord
$\varepsilon > 0$

Korollar 49

$$\mathcal{L}[f^{(n)}](s) = -s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) + s^n\mathcal{L}[f](s)$$

Satz 50 (Multiplikation) Sei $f:[0,\infty) \to \mathbb{C}$ stetig von exp Ord γ.

Dann ist $t f(t)$ von exp Ord $\gamma+\varepsilon$, $\varepsilon > 0$.

$$\mathcal{L}[t f(t)](s) = -\frac{d}{ds}\mathcal{L}[f](s) \qquad \mathrm{Re}[s] > \gamma.$$

Beweis

$$\frac{d}{ds}\mathcal{L}[f](s) = \frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\, dt \overset{\text{Leibniz-Regel}}{=} \int_0^\infty f(t)\frac{d}{ds} e^{-st}\, dt$$

$$= -\int_0^\infty t f(t)\, e^{-st}\, dt = -\mathcal{L}[t f(t)](s)$$

$\square$

Beispiel (Monome)

$$\mathcal{L}[1](s) = \mathcal{L}[e^{0\cdot t}](s) = \frac{1}{s-0} = \frac{1}{s}$$

$$\mathcal{L}[t^n](s) = \mathcal{L}[t^n\cdot 1](s) = (-1)^n\frac{d^n}{ds^n}\mathcal{L}[1](s) = (-1)^n\frac{d^n}{ds^n}\frac{1}{s} = \frac{(-1)^{2n}n!}{1}\frac{1}{s^{n+1}}$$

$$= n!\frac{1}{s^{n+1}} \qquad \mathrm{Re}[s] > 0$$

$$\mathcal{L}[t^n](s) = n! \frac{1}{s^{n+1}}$$

Gebrochenrationale Funktionen

$$\mathcal{L}\left[\frac{A}{(\ell-1)!} e^{s_0 t} t^{\ell-1}\right](s) = \frac{A}{(s-s_0)^\ell}$$

<u>Beispiel 52</u>

$$x''(t) = 3x'(t) - 2x(t), \qquad x(0)=2, \quad x'(0) = -1$$

Mit $X(s) := \mathcal{L}[x(t)](s)$ erhalten wir

$$\mathcal{L}[x'(t)](s) = s\,\mathcal{L}[x(t)](s) - x(0)$$
$$\mathcal{L}[x''(t)](s) = s^2\,\mathcal{L}[x(t)](s) - s\,x(0) - x'(0)$$

und

$$s^2 X(s) - 2s + 1 = 3s\,X(s) - 3 \cdot 2 - 2X(s)$$

$$\rightarrow (s^2 - 3s + 2)\,X(s) = 2s - 7$$

$$\rightarrow \quad X(s) = \frac{2s-7}{s^2-3s+2} = \frac{2s-7}{(s-2)(s-1)}$$

$$+\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2}$$
$$= \frac{3}{2} \pm \frac{1}{2}$$

$$X(s) = \frac{2s-7}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1}$$

NB $\quad 2s - 7 = A(s-1) + B(s-2)$

$\longrightarrow s=1: \quad -5 = -B \quad \Rightarrow \underline{B=5}$

$s=2: \quad \underline{-3 = A}$

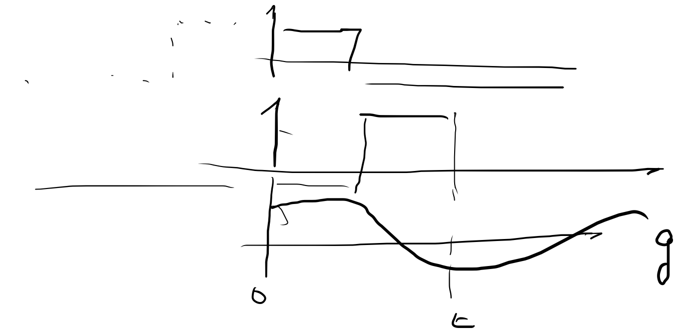
$$\Rightarrow \quad X(s) = -\frac{3}{s-2} + \frac{5}{s-1} = \mathcal{L}[5e^{t}] - \mathcal{L}[3e^{2t}]$$

$$\Rightarrow \quad x(t) = 5e^{t} - 3e^{2t} \qquad \text{(Satz von Lerch)}$$

Def (Faltung) $f, g : [0,\infty) \to \mathbb{C}$

$$(f * g)(t) := "\int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau"$$

$$= \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$



Satz 54 (Faltung) $f, g : [0,\infty) \to \mathbb{C}$ stückweise stetig, exp Ord γ

Dann ist $(f * g)$ von exp Ord $\gamma + \varepsilon$, $\varepsilon > 0$

und

$$\mathcal{L}[f * g](s) = \mathcal{L}[f](s) \cdot \mathcal{L}[g](s)$$