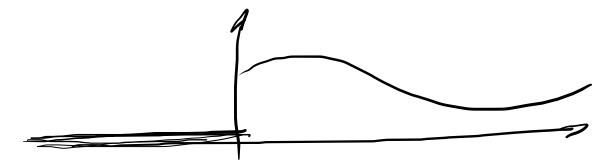
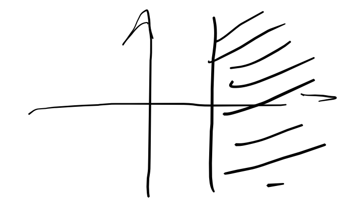
$$\hat f(\omega) = \mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$



Beziehung zur Laplacetransformation



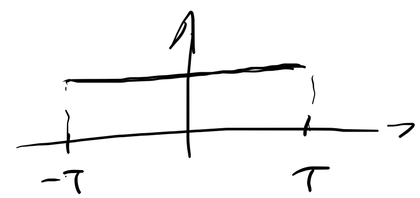
$f:[0,\infty) \to \mathbb{C}$ und setze $f(t) = 0$ für $t < 0$

Mit $s := \sigma + i\omega$

$$\mathcal{L}[f](s) = \int_0^{\infty} f(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} f(t)\, e^{-\sigma t}\, e^{-i\omega t}\, dt = \mathcal{F}[e^{-\sigma t} f(t)](\omega)$$

Beispiel 57

$$f(t) := \begin{cases} 1, & |t| \le T \\ 0, & |t| > T \end{cases}$$



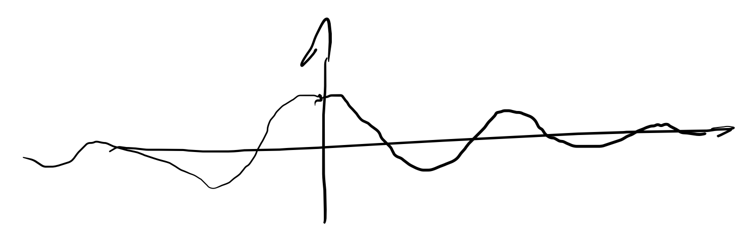
$$\mathcal{F}[f](\omega) = \int_{-T}^{T} 1 \cdot e^{-i\omega t}\, dt \overset{\omega \neq 0}{=} \left[-\frac{1}{i\omega} e^{-i\omega t} \right]_{-T}^{T} = \frac{e^{i\omega T} - e^{-i\omega T}}{i\omega} = 2T \, \frac{e^{i\omega T} - e^{-i\omega T}}{2i} \, \frac{1}{T\omega}$$

$$= 2T \, \frac{\sin(\omega T)}{T\omega} = 2T \operatorname{sinc}(\omega T)$$

Sinus cardinalis

$$\operatorname{sinc}(t) := \begin{cases} \frac{\sin(t)}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

$$\mathcal{F}[f](0) = \int_{-T}^{T} dt = 2T = 2T \operatorname{sinc}(0 \cdot T)$$

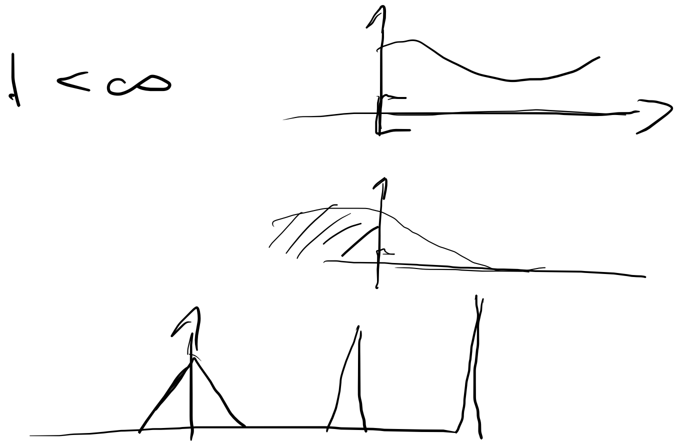


Satz 58 (Linearität)

$$\mathcal{F}[af+bg] = a\,\mathcal{F}[f] + b\,\mathcal{F}[g], \qquad a,b\in\mathbb{C},\quad f,g:\mathbb{R}\to\mathbb{C}$$

$\hat f, \hat g$ existieren.

Satz 59 $f:\mathbb{R}\to\mathbb{C}$ absolut-integrierbar $\leftarrow \int_{-\infty}^{\infty} |f(t)|\,dt < \infty$



(i) $\mathcal{F}[f(t-t_0)](\omega) = e^{-i\omega t_0}\,\mathcal{F}[f(t)](\omega), \qquad t_0\in\mathbb{R}$

(ii) $\mathcal{F}[e^{-it\omega_0} f(t)](\omega) = \mathcal{F}[f](\omega+\omega_0)$

(iii) $\mathcal{F}[f(\frac{t}{a})](\omega) = |a|\,\mathcal{F}[f(t)](a\omega) \qquad a\neq 0$

Satz 60 (Ableitung) f, f' absolut-integrierbar und $f(t)\to 0$ für $|t|\to\infty$

Dann
$$\mathcal{F}[f'](\omega) = i\omega\,\mathcal{F}[f](\omega)$$

Beweis

$$\mathcal{F}[f'](\omega) = \int_{-\infty}^{\infty} f'(\omega) e^{-i\omega t}\,dt = \underbrace{\left[f(\omega)e^{-i\omega t}\right]_{-\infty}^{\infty}}_{\to 0} + i\omega\underbrace{\int_{-\infty}^{\infty} f(\omega)\,e^{-i\omega t}\,dt}_{\hat f(\omega)} = i\omega\,\mathcal{F}[f](\omega)$$ $\square$

Satz 61 (Multiplikation)

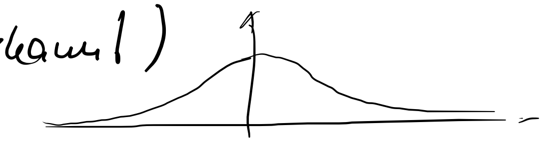
Sei $f(t)$, $t\cdot f(t)$ absolut integrierbar

Dann
$$(\mathcal{F}[f])'(\omega) = -i\,\mathcal{F}[t\,f(t)](\omega).$$

$f(t)\,e^{-i\omega t}$

- festes ω abs. int-bar nach t
- festes t ableitbar nach ω
- Ableitungen beschränkt durch abs. int-bare Funktion

Beispiel 62 $\quad f(t) = e^{-t^2/2}$ $\quad$ (Stammfunktion nicht bekannt)



$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-t^2/2} \underbrace{e^{-i0t}}_{1} \, dt = \sqrt{2\pi}$$

Für $\omega \neq 0$, betrachten wir

$$\hat{f}'(\omega) = -i \int_{-\infty}^{\infty} t \, e^{-t^2/2} e^{-i\omega t} \, dt \overset{PI}{=} i \underbrace{\left[e^{-t^2/2} e^{-i\omega t} \right]_{-\infty}^{\infty}}_{\to 0} - \omega \underbrace{\int_{-\infty}^{\infty} e^{-t^2/2} e^{-i\omega t} \, dt}_{\hat{f}(\omega)}$$

$$\Rightarrow \quad \hat{f}'(\omega) = -\omega \hat{f}(\omega), \qquad \hat{f}(0) = \sqrt{2\pi}$$

Lösung des Anfangswertproblems

$$\hat{f}(\omega) = C e^{-\omega^2/2}, \qquad C \in \mathbb{R}$$

$$\Rightarrow \quad \hat{f}(\omega) = \sqrt{2\pi} \, e^{-\omega^2/2}$$

($\leftarrow$ Eigenfunktion)

Def 63 (Faltung) $f,g: \mathbb{R} \to \mathbb{C}$

$$(f*g)(t) := \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau$$

Satz 64 (Faltungssatz) $f,g: \mathbb{R} \to \mathbb{C}$ abs. int-bar.

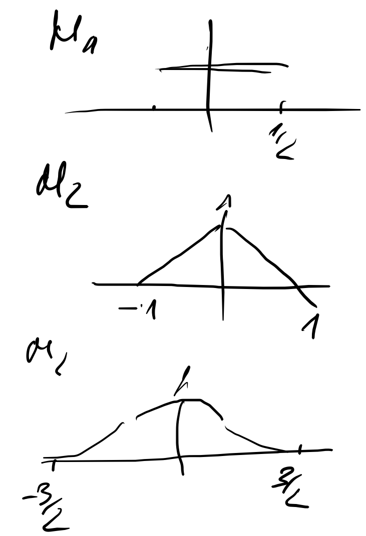
Dann $(f*g)(t)$ abs. int-bar und

$$\mathcal{F}(f*g)(\omega) = \mathcal{F}(f)(\omega) \cdot \mathcal{F}(g)(\omega).$$

Beispiel 65 (Basis-Spline)

$$M_1(t) := \begin{cases} 1, & |t| \le 1/2 \\ 0, & |t| > 1/2 \end{cases} \qquad M_n(t) := (M_{n-1} * M_1)(t)$$

$$\mathcal{F}(M_n)(\omega) = \left(\mathcal{F}(M_1)(\omega)\right)^n = \left(\operatorname{sinc}\left(\frac{\omega}{2}\right)\right)^n.$$



Satz 66 (Fourier-Inversion)

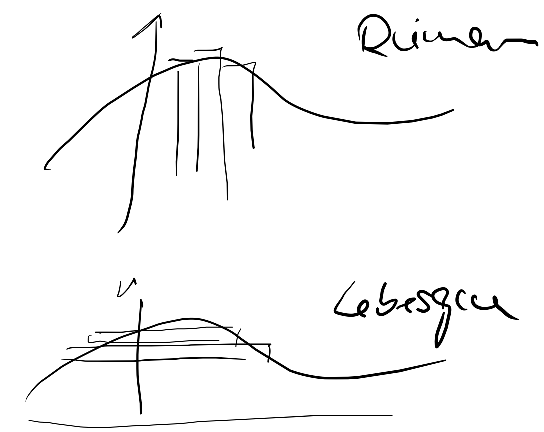
$f: \mathbb{R} \to \mathbb{C}$ stückweise stetig, $f, \hat{f}$ abs. int-bar

Dann gilt

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, dt$$

← Inverse Fouriertransformation

für alle Stetigkeitspunkte von f.

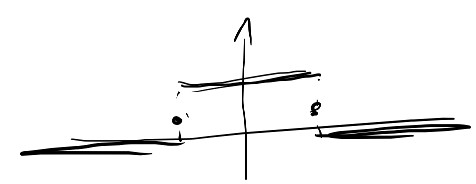


Satz 67 (Dirichlet-Jordan)

$f: \mathbb{R} \to \mathbb{C}$ stückweise stetig diff-bar, abs. int-bar.

Dann

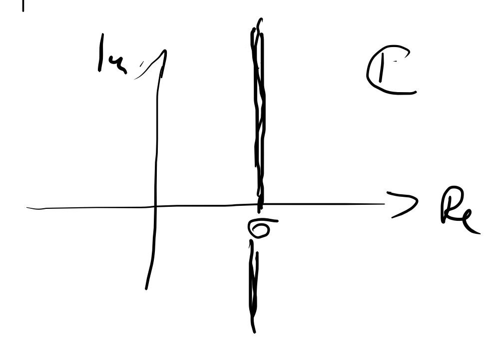
$$\lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \hat{f}(\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2}\left(f(t+) + f(t-)\right)$$



Laplace-Inversion

$f: [0, \infty) \to \mathbb{C}$ stückweise stetig exp Ord,

$\omega \mapsto \mathcal{L}[f](\sigma + i\omega)$ abs. int für festes σ



$$f(t) = \frac{e^{\sigma t}}{2\pi} \int_{-\infty}^{\infty} F(\sigma + i\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sigma + i\omega)\, e^{(\sigma + i\omega)t}\, d\omega$$

Satz 68 f, g und $\hat{f}, \hat{g}$ stückweise stetig und abs. int.

Dann

$$\mathcal{F}[f \cdot g](\omega) = \frac{1}{2\pi} \left(\mathcal{F}(f) * \mathcal{F}(g) \right)(\omega) .$$