HA 12

Aufgabe 12.1

(i) $G = \{(x,y,g(x,y)) \in \mathbb{R}^3 : (x,y) \in B\} = \{(x,y,x^3+y) : (x,y) \in \mathbb{R}^2, x \in [0,1], 0 < y < x^3\}$

$\vec{\eta}(x,y) := (x,y,g(x,y)) = (x,y,x^3+y) \quad (x,y) \in B$

$$\frac{\partial \vec{\eta}(x,y)}{\partial x} \times \frac{\partial \vec{\eta}(x,y)}{\partial y} = \begin{pmatrix} 1 \\ 0 \\ 3x^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3x^2 \\ -1 \\ 1 \end{pmatrix}$$

$$\left| \frac{\partial \vec{\eta}(x,y)}{\partial x} \times \frac{\partial \vec{\eta}(x,y)}{\partial y} \right| = \sqrt{9x^4+2}$$

$$F(G) = \iint_G 1 \, dO = \int_0^1 \int_0^{x^3} \sqrt{9x^4+2} \, dy dx$$
$$= \int_0^1 x^3 \sqrt{9x^4+2} \, dx$$
$$= \frac{2}{36 \cdot 3} (9x^4+2)^{\frac{3}{2}} \Big|_0^1$$
$$= \frac{1}{54}\left[(11)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right]$$

(ii) $$\iint_G (2+x^3+y-z)\, dO = \iint_B (2+\cancel{x^3}+\cancel{y}-\cancel{x^3}-\cancel{y}) \cdot \sqrt{9x^4+2} \, dx dy$$
$$= \int_0^1 \int_0^{x^3} 2 \sqrt{9x^4+2} \, dy dx$$
$$= \frac{1}{27}\left[11^{\frac{3}{2}} - 2^{\frac{3}{2}}\right]$$

Aufgabe 12.2

$M = \{ \vec{\eta}(\varphi, x) := (x, \cos\varphi, \sin\varphi) : x \in [0,3], \varphi \in [0,\pi] \}$

$$\frac{\partial \vec{\eta}(\varphi,x)}{\partial \varphi} \times \frac{\partial \vec{\eta}(\varphi,x)}{\partial x} = \begin{pmatrix} 0 \\ -\sin\varphi \\ \cos\varphi \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\iint_M \vec{v} \cdot \vec{dO} = \iint_M \left\langle \vec{v}(\vec{\eta}(\varphi,x)), \vec{dO} \right\rangle dx dy$$
$$= \int_0^3 \int_0^\pi \left\langle \begin{pmatrix} e^{x^3+\cos^3\varphi+\sin^3\varphi} \\ e^{2x}(\cos\varphi + 2\sin\varphi) \\ e^{2x}(2\cos\varphi - \sin\varphi) \end{pmatrix}, \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix} \right\rangle d\varphi dx$$
$$= \int_0^3 \int_0^\pi e^{2x}(\cos^2\varphi + 4\sin\varphi\cos\varphi - \sin^2\varphi) \, d\varphi dx$$

$$= \frac{1}{2}e^{2x}\Big|_0^3 \cdot \int_0^{\pi} \cos 2\varphi + 2\sin 2\varphi \; d\varphi$$

$$= \frac{1}{2}(e^6 - 1) \cdot \left(\frac{1}{2}\sin 2\varphi - \frac{1}{2}\cos 2\varphi\right)\Big|_0^{\pi}$$

$$= \frac{1}{2}(e^6 - 1) \cdot \left(-\frac{1}{2} + \frac{1}{2}\right) = 0$$