Hausaufgaben 11
Gruppen Nico 6

Aufgabe 11.1

i) Es gilt $x = \rho\cos\varphi \quad y = \rho\sin\varphi$ und somit gilt $\rho\cos\varphi \geq |\rho\sin\varphi|$

$$\Rightarrow \varphi \in [-\tfrac{\pi}{4}, \tfrac{\pi}{4}]$$

Die Bedingung $x^2+y^2 \leq \frac{\pi}{2}$ entspricht $\rho^2 \leq \frac{\pi}{2} \Rightarrow \rho \in [0, \sqrt{\frac{\pi}{2}}]$

Damit erhalten wir $A := \{(\rho\cos\varphi, \rho\sin\varphi) : \varphi \in [-\frac{\pi}{4}, \frac{\pi}{4}],\ 0 \leq \rho \leq \sqrt{\frac{\pi}{2}}\}$

ii)
$$\iint_A \cos(x^2+y^2)\,dx\,dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\frac{\pi}{2}}} \cos\rho^2 \cdot \rho \, d\rho\, d\varphi$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{\sqrt{\frac{\pi}{2}}} \rho\cos(\rho^2)\,d\rho$$
$$= \frac{1}{2}\pi \cdot \frac{1}{2}\sin(\rho^2)\Big|_0^{\sqrt{\frac{\pi}{2}}} = \frac{1}{4}\pi$$

Aufgabe 11.2

i) $x^2+y^2 = \rho^2 \leq z^3 \quad \Rightarrow \rho \in [0, \sqrt{z^3}]$

$0 \leq \rho\sin\varphi \leq \frac{1}{\sqrt{3}}\rho\cos\varphi \Rightarrow \tan\varphi \leq \frac{1}{\sqrt{3}} \quad \Rightarrow \varphi \in [-\cancel{\frac{\pi}{6}}\ 0, \frac{\pi}{6}]$

$M := \{(\rho\cos\varphi, \rho\sin\varphi, z) : z \in [0,1], \rho \in [0, \sqrt{z^3}], \varphi \in [-\cancel{\frac{\pi}{2}}\ 0, \frac{\pi}{6}]\}$

ii)
$$\iiint_M e^{(z^4)}\,dx\,dy\,dz = \int_0^1 \int_0^{\sqrt{z^3}} \int_{-\frac{\pi}{6}\ 0}^{\frac{\pi}{6}} e^{z^4}\ \rho\, d\varphi\, d\rho\, dz$$
$$= \int_0^1 \left[\int_0^{\sqrt{z^3}} \cancel{\tfrac{1}{3}}\tfrac{1}{6}\pi \cdot \rho\, e^{z^4}\, d\rho\right] dz$$
$$= \int_0^1 \left(\frac{2}{3}\pi \cdot e^{z^4} \cdot \frac{1}{2}\rho^2 \Big|_0^{\sqrt{z^3}}\right) dz$$
$$= \int_0^1 \frac{1}{3}\pi e^{z^4} \cdot z^3\, dz$$
$$= \frac{1}{3}\pi e^{z^4} \cdot \frac{1}{4}\Big|_0^1 = \frac{\pi}{12}\cdot(e-1) \cdot \frac{3}{2} \cdot \frac{1}{6}$$
$$= \frac{\pi}{48}(e-1)$$

Aufgabe 11.3

i) $x^2+y^2+z^2=r^2\leq 1 \quad\Rightarrow\quad r\in[0,1]$

$z=r\cos\theta\geq 0 \quad\Rightarrow \theta\in[0,\frac{\pi}{2}]$

$z=r\cos\theta > \sqrt{x^2+y^2} = \sqrt{r^2\sin^2\theta} = r\sin\theta \Rightarrow \theta\in[0,\frac{\pi}{4}]$

$C:=\{(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta): r\in[0,1], \theta\in[0,\frac{\pi}{4}], \varphi\in[0,2\pi]\}$

ii)
$$\begin{aligned}
\iiint_C x^2\,dx\,dy\,dz &= \int_0^1\int_0^{\frac{\pi}{4}}\int_0^{2\pi} r^2\sin^2\theta\cos^2\varphi\cdot r^2\sin\theta\; d\varphi\, d\theta\, dr \\
&= \int_0^1 r^4\,dr\cdot\int_0^{\frac{\pi}{4}}\sin^3\theta\,d\theta\cdot\int_0^{2\pi}\cos^2\varphi\,d\varphi \\
&= \frac{1}{5}\cdot\left(-\cos\theta+\frac{1}{3}\cos^3\theta\right)\Big|_0^{\frac{\pi}{4}}\cdot\int_0^{2\pi}\frac{1}{2}(1+\cos 2\varphi)\,d\varphi \\
&= \frac{1}{5}\cdot\left(-\frac{\sqrt{2}}{2}+\frac{1}{3}\cdot\frac{\sqrt{2}}{4}+1-\frac{1}{3}\right)\cdot\left(\frac{1}{2}\varphi+\frac{1}{4}\sin 2\varphi\right)\Big|_0^{2\pi} \\
&= \frac{1}{5}\left(\frac{2}{3}-\frac{5\sqrt{2}}{12}\right)\cdot(\pi) = \frac{8-5\sqrt{2}}{60}\pi
\end{aligned}$$

Halbkugel: $H=\{(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta): r\in[0,1], \theta\in[0,\pi/2], \varphi\in[0,2\pi]\}$

Kegel: $K=\{(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta): r\in[0,\infty[, \theta\in[0,\pi/4], \varphi\in[0,2\pi]\}$

$C=H\setminus K=\{(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta): r\in[0,1], \theta\in[\frac{\pi}{4},\frac{\pi}{2}], \varphi\in[0,2\pi]\}$