Hausaufgaben 10

Gruppen Nico 6

## Aufgabe 10.1

(i)

$$\int_{\vec r} \vec v \cdot d\vec s = \int_0^{2\pi} \langle \vec v(\vec r(t)), \dot{\vec r}(t)\rangle \, dt$$

$$= \int_0^{2\pi} \left\langle \begin{pmatrix} a(\cos(2t)+3\sin(2t)) \\ 6(\cos(2t)+3\sin(2t)) \\ 2e^t \end{pmatrix}, \begin{pmatrix} -2\sin(2t) \\ 2\cos(2t) \\ e^t \end{pmatrix} \right\rangle dt$$

$$= \int_0^{2\pi} -2a\left[\cos(2t)\sin(2t)+3\sin^2(2t)\right] + 12\left[\cos^2(2t)+3\sin(2t)\cos(2t)\right] + 2e^{2t} \, dt$$

$$= \int_0^{2\pi} (36-2a)\cos(2t)\sin(2t) + 12\cos^2(2t) - 6a\sin^2(2t) + 2e^{2t} \, dt$$

$$= \int_0^{2\pi} (18-a)\sin(4t) + (12+6a)\cos^2(2t) - 6a + 2e^{2t} \, dt$$

$$= \int_0^{2\pi} (18-a)\sin(4t) + (12+6a)\cdot\frac{\cos 4t+1}{2} - 6a + 2e^{2t} \, dt$$

$$= \int_0^{2\pi} (18-a)\sin(4t) + (6+3a)\cos(4t) + 6-3a + 2e^{2t} \, dt$$

$$= \frac{a-18}{4}\cos 4t + \frac{6+3a}{4}\sin(4t) + (6-3a)t + e^{2t} \Big|_0^{2\pi}$$

$$= (6-3a)2\pi + e^{4\pi} - 1 \qquad = 12\pi - 6a\pi + e^{4\pi} - 1$$

(ii)

$$\operatorname{rot} \vec v = \begin{pmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6-3a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ ist die notwendige Bedingungen für die Existenz eines Potentials

$$\Rightarrow 6-3a=0 \quad \Rightarrow a=2$$

$\vec v$ hat ein Potential $u: \mathbb{R}^3 \to \mathbb{R}^3$ bzw. eine Stammfunktion $f: \mathbb{R}^3 \to \mathbb{R}^3$ mit $\vec v = \operatorname{grad} f$

$$\frac{\partial f}{\partial x} = 2x+6y \xrightarrow[\text{nach } x]{\text{Integral}} f = x^2 + 6xy + C_1(y,z)$$

$$\frac{\partial f}{\partial y} = 6x+18y \xrightarrow[\text{nach } y]{\text{Integral}} f = 6xy + 9y^2 + C_2(x,z)$$

$$\frac{\partial f}{\partial z} = 2z \xrightarrow[\text{nach } z]{\text{Integral}} f = z^2 + C_3(x,y)$$

Vergleich: $f(x,y,z) = x^2 + 6xy + 9y^2 + z^2 + C \qquad C \in \mathbb{R}$

$$\Rightarrow u(x,y,z) = -x^2 - 6xy - 9y^2 - z^2 \text{ ist ein Potential von } \vec{v}$$

$$\int_{\vec{r}} \vec{v} \cdot d\vec{s} = u(\vec{x}(0)) - u(\vec{x}(2\pi))$$

$$= -\cos^2 0 - 6\cos 0 \cdot \sin 0 - 9\sin^2 0 - e^0$$

$$+ \cos^2 2\pi + 6\cos 2\pi \cdot \sin 2\pi + 9\sin^2 2\pi + e^{4\pi}$$

$$= -1 - 0 - 0 - 1 + 1 + 0 + 0 + e^{2\pi} = e^{4\pi} - 1$$

(iii) $$\int_{\vec{r}} z^2\, ds = \int_0^{2\pi} (e^t)^2 \left\| \begin{pmatrix} -2\sin(2t) \\ 2\cos(2t) \\ e^t \end{pmatrix} \right\| dt$$

$$= \int_0^{2\pi} e^{2t} \sqrt{4\sin^2 2t + 4\cos^2 2t + e^{2t}}\, dt$$

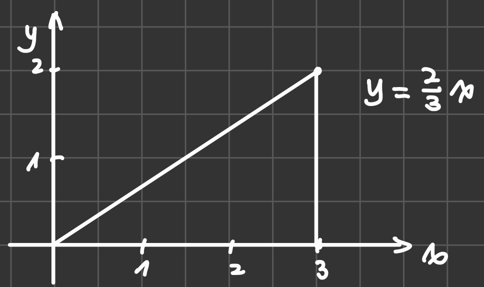
$$= \int_0^{2\pi} e^{2t} \left(4 + e^{2t}\right)^{\frac{1}{2}} dt$$

$$= \frac{1}{3}\left(4 + e^{2t}\right)^{\frac{3}{2}} \Big|_0^{2\pi}$$

$$= \frac{1}{3}\left[\left(4 + e^{4\pi}\right)^{\frac{3}{2}} - 5^{\frac{3}{2}}\right]$$

Aufgabe 10.2

(i) Skizze des Bereiches:




$$G := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 3,\ 0 \leq y \leq \tfrac{2}{3}x\}$$

$$\iint_G y^3 e^{(x^5)}\, dx\, dy = \int_0^3 \int_0^{\frac{2}{3}x} y^3 e^{x^5}\, dy\, dx = \int_0^3 \left( \frac{1}{4} y^4 e^{x^5} \Big|_0^{\frac{2}{3}x} \right) dx$$

$$= \int_0^3 \frac{1}{4} \cdot \frac{16}{81} x^4 e^{x^5}\, dx = \int_0^3 \frac{4}{81} x^4 e^{x^5}\, dx$$

$$= \frac{4}{5 \cdot 81} e^{x^5} \Big|_0^3 = \frac{4}{405}\left(e^{243} - 1\right)$$

(ii) $$\iiint_B dx\, dy\, dz = \int_0^3 \int_0^{z^2} \int_{-x^3}^{x^3} 1 \; dy\, dx\, dz$$

$$= \int_0^3 \int_0^{z^2} \left( y \Big|_{-x^3}^{x^3} \right) dx\, dz$$

$$= \int_0^3 \int_0^{z^2} 2x^3\, dx\, dz$$

$$= \int_0^3 \left( \frac{1}{2} x^4 \Big|_0^{z^2} \right) dz$$

$$= \int_0^3 \frac{1}{2} z^8\, dz = \frac{1}{18} z^9 \Big|_0^3 = \frac{3^9}{2 \cdot 3^2} = \frac{3^7}{2} = \frac{2187}{2} = 1093.5$$