Yilong Wang 483728
Linda Li 458029

Task 1

1. Real symmetic matrices, upper and lower triangular matrices and diagonal matrices have only real eigenvalues.

2. The eigenvalues of triangular matrix are the elements on the diagonal of the matrix.

3. For a digonal matrix, its eigenvalues are also the elements on the diagonal

The eigenvektors of diagonal matrix $diag(a_1, \dots, a_n)$ are $a_1 \dots a_n$ and the corresponding eingenvektors are their standard basis vectors $e_1, \dots, e_n$.

Task 2.

Standard PCA

1. a) $\bar{x}_1 = \frac{1}{2}(-1+1) = 0$, $\bar{x}_2 = \frac{1}{2}(0+0) = 0$, $\bar{x}_3 = \frac{1}{2}(0+0) = 0$, $\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\hat{\Sigma} = \frac{1}{n}(X-\bar{x})(X-\bar{x})^T$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b) $$\det(\hat{\Sigma} - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 = -\lambda^2 \cdot (\lambda - 1) = 0$$

$\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$

$$(A - \lambda_1 I) \cdot W = 0 \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \Rightarrow \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow W = \left\{ x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So the first principal direction is $W = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

c) $H = W^T X$

$$= [1\ 0\ 0] \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = [-1, 1]$$

linear kernel PCA

2. a) $$K = X^T X = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

b) $$\det(K - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (-1)^2 = \lambda^2 - 2\lambda + 1 - 1 = \lambda(\lambda - 2)$$

$\lambda_1 = 0$, $\lambda_2 = 2$

$$(A - \lambda_2 I)\alpha = 0 \Rightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 + x_2 = 0$$

$$\Rightarrow \alpha = \left\{ x_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$k\alpha = \lambda \alpha$

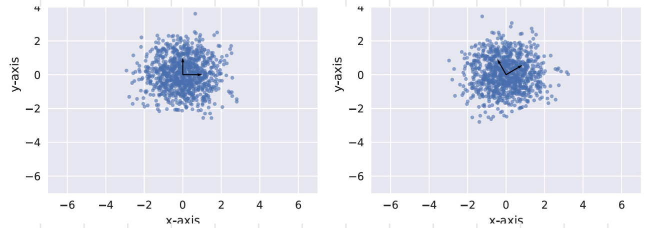
$$k \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 - \alpha_2 \\ -\alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 \\ 2\alpha_2 \end{bmatrix}$$

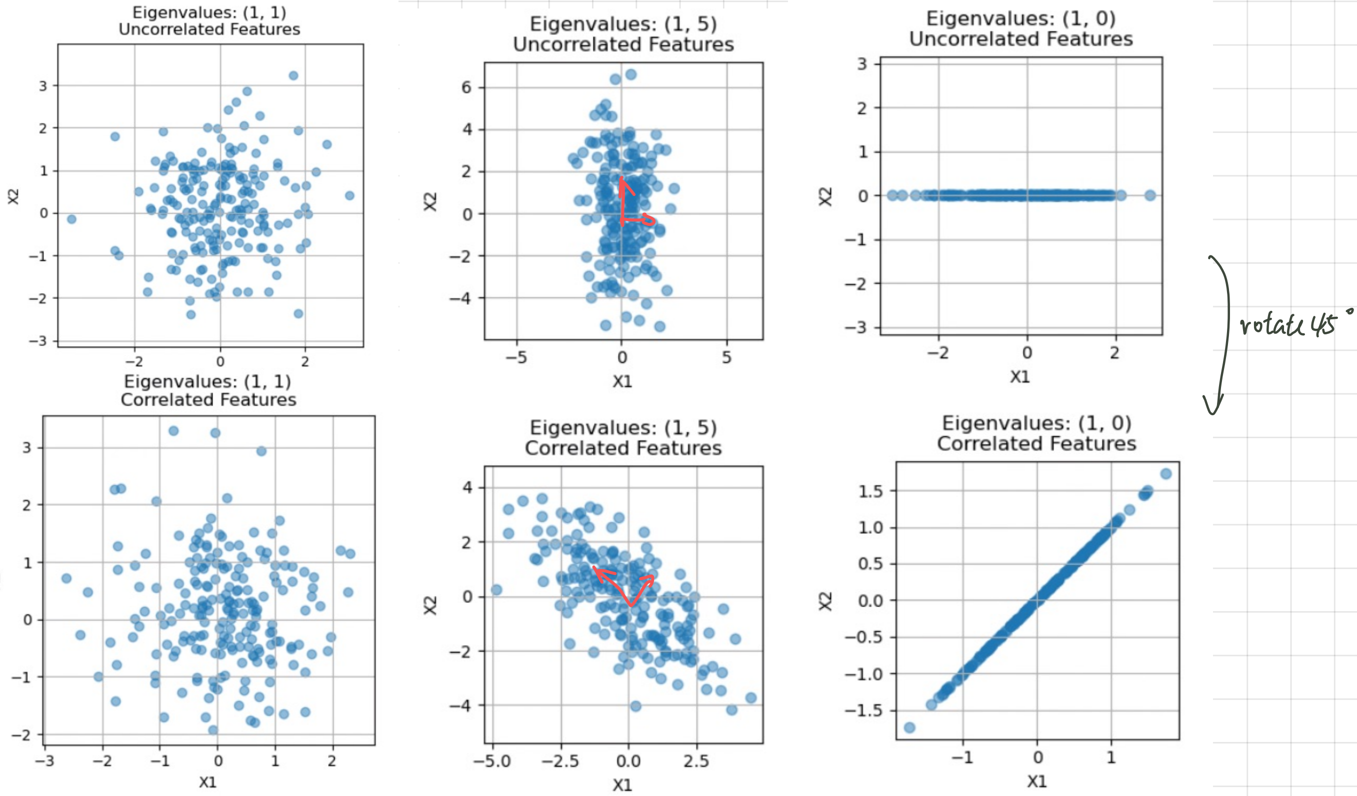
$\Rightarrow \alpha_1 + \alpha_2 = 0$

c) $W = X\alpha = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_1 = 1$
$\lambda_2 = 1$



Task 3.



Task 4

1) Since S_α is the scaled matrix, we have

$S_\alpha v = (\alpha S)v \Rightarrow S_\alpha v = \alpha(Sv)$

Since $Sv = \lambda v$, we get

$S_\alpha v = \alpha(\lambda v) \Rightarrow S_\alpha v = \alpha\lambda v$

This shows that v is also an eigenvektor of S_α with the eingenvalue $\alpha\lambda$.

$\alpha S v = \alpha \lambda v$
$S_\alpha V = \lambda_\alpha V$

2) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A^T A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ A is an orthogonal matrix

$\det(A) = 0 \cdot 0 - 1 \cdot 1 = -1 \Rightarrow$ A is not a rotation matrix.

Rotation Matrices

$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\det |R_\theta| = +1$

Task 5.

$\tilde{X} = [\hat{x}_1, \hat{x}_2, \hat{x}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$H = [h_1, h_2, h_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$H = I$
$\tilde{X} = W \cdot I = W$
$\Rightarrow \tilde{X} = W$

$\tilde{X} = WH \Rightarrow W = \tilde{X} H^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$