A Tale of Langlands Duality

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Abstract

A self-study record of the Kapustin-Witten paper [KW07] on gauge theory and geometric Langlands program.

1 Electro-magnetic Duality in Electromagnetics

We start with the very easy abelian case as a warm-up.

Convention I generally works with the Lorentzian signature $-+++$ but for most case we use Riemannian signature $++++$. Greek indices $\mu, \nu, \ldots$ runs over $0, 1, 2, 3$. Latin indices $i, j, \ldots$ runs over space-like coordinate $1, 2, 3$, while 0 always stands for time-like coordinate.

1.1 Classical Theory

The classical electrodynamics enjoys a good property. Consider the vacuum Maxwell equation

$$\begin{aligned} \operatorname{div} \vec{B} &= 0 \qquad & \operatorname{div} \vec{E} &= 0 \\ \operatorname{rot} \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \qquad & \operatorname{rot} \vec{B} &= \frac{\partial \vec{E}}{\partial t}. \end{aligned} \tag{1.1}$$

It is invariant under the transform

$$(\vec{E}, \vec{B}) \mapsto (\vec{B}, -\vec{E}). \tag{1.2}$$

This is the primitive form of electro-magnetic duality.

In the fancy bundle language, it is a $\mathrm{U}(1)$ gauge field described by connection A on a principal $\mathrm{U}(1)$ bundle E over a four-manifold X. The equation of motion is

$$\mathrm{d} \star F = 0 \tag{1.3}$$

with $F = \mathrm{d}A$ is the curvature 2-form and $\star$ is the Hodge dual. This corresponds to the left hand set of equations in (1.1). The remaining two equations indicate that $\mathrm{d}F = 0$, by Bianchi identity.

When $X = \mathbb{R}^{3,1}$, (1.3) is invariant under a transformation

$$F \mapsto F' = \star F. \tag{1.4}$$

This just resembles the electro-magnetic duality, in the sense that taking

$$E_i = F_{0i}, \quad B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}, \tag{1.5}$$

we can easily recover (1.2).

In addition, if $H^2(X) \neq 0$, F satisfies a quantization condition: its periods are integral multiples of 2π. The cohomology class of F is the Euler class of E (or alternatively the first Chern class of the associated line bundle). The the easiest example will be the Dirac string on $-z$ axis. This leads to a charge quantization

$$eg = 2\pi n,\ n \in \mathbb{Z}. \tag{1.6}$$

While charge e and g can be written as the form of integrating E and B over compact surfaces, which is of course period. And in the case of electromagnetics, the cohomology is equivalent to de Rham cohomology, which will give the Euler class by definition [BM94].

1.2 Quantization

It is the case that this duality no longer exists on arbitary manifold X other then $\mathbb{R}^{3,1}$, but when it comes to the quantum theory, some miracle happens, as I will demonstrate in the following.

Quantization of gauge field is given by the path integral

$$Z = \int \mathscr{D}A\, \mathrm{e}^{\mathrm{i}S(A)} \tag{1.7}$$

with integration over all possible topologies of the bundle E. Here the action is the celebrated

$$S(A) = \frac{1}{2e^2} \int_X F \wedge \star F + \frac{\theta}{8\pi^2} \int_X F \wedge F. \tag{1.8}$$

Note that the θ-angle is a topological invariant and relies only on the topology. It doesn't affect the classical equation of motion. So if we write explicitly the sum over all isomorphism classes of E

$$Z = \sum_E \int \mathscr{D}A \, \mathrm{e}^{\mathrm{i}S(A)}. \tag{1.9}$$

the θ-angle tells us how to weigh contributions of different E.

In order to compute it, we perform a Wick-rotation, then

$$Z = \sum_E \int \mathscr{D}A \, e^{-S_{\mathsf{E}}(A)} \tag{1.10}$$

with an Euclidean action

$$S_{\mathsf{E}}(A) = \int_X \left(\frac{1}{2e^2} F \wedge \star F - \frac{i\theta}{8\pi^2} F \wedge F \right). \tag{1.11}$$

Assuming that X is simply-connected. Since F determines the holonomy of A around all contractible loops in X, if $\pi_1(X)$ is trivial then F completely determines A, up to gauge equivalence.

We want to replace the integration over A with integration over the space of closed 2-forms F, which will make our life easier later. Here we introduce a "Lagrangian multipler"

$$Z = \int \mathscr{D}F \mathscr{D}B \exp\left(-S_{\mathsf{E}} + \mathrm{i} \int_X B \wedge \mathrm{d}F \right) \tag{1.12}$$

the new field B is a 1-form on X (or *dual connection*), so that integration over it produces the desired constraint $\delta(\mathrm{d}F) = \prod_{x \in X} \delta(\mathrm{d}F(x))$. This allows us to integrate over all (not necessarily closed) 2-forms F.

Using our favourite Gaussian integral, we get

$$Z = \int \mathscr{D}B \exp\left(-\frac{1}{2\hat{e}^2} \int_X G \wedge \star G + \frac{\mathrm{i}\hat{\theta}}{8\pi^2} \int_X G \wedge G \right) \tag{1.13}$$

where $G = \mathrm{d}B$ is the *dual curvature*. The new coupling constants $\hat{e}^2$ and $\hat{\theta}$ are defined by

$$\frac{\hat{\theta}}{2\pi} + \frac{2\pi \mathrm{i}}{\hat{e}^2} = -\left(\frac{\theta}{2\pi} + \frac{2\pi \mathrm{i}}{e^2} \right)^{-1} \tag{1.14}$$

1.3 S-duality

2 Montonen-Olive Duality

References

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