NYU MATH-UA 329 Honors Analysis II Review Sheet

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0.1 Basic Inequality

Cauchy-Schwarz Inequality: $\langle x, y\rangle \leq \sqrt{\langle x, x\rangle\langle y, y\rangle} = ||x||_2||y||_2$.
Proofs of Cauchy-Schwarz: $P(t) = \sum_{i=1}^{d}(x_i - ty_i)^2 = \sum_{i=1}^{d} x_i^2 - 2t\sum_{i=1}^{d} x_i y_i + t^2\sum_{i=1}^{d} y_i^2 \geq 0$. Then $\Delta = b^2 - 4ac = (\sum_{i=1}^{d} x_i y_i)^2 - \sum_{i=1}^{d} x_i^2 \sum_{i=1}^{d} y_i^2 \leq 0$.
Triangular Inequality: $||x+y||_p \leq ||x||_p + ||y||_p$. Young's Inequality: If $a \geq 0$ and $b \geq 0$ are non-negative real numbers and if $p > 1$ and $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. Equality holds iff $a^p = b^q$. Holder's Inequality: $\langle x, y\rangle \leq ||x||_p||y||_{p'}$. Also, $||x||_p = \sup_{||y||_{p'}=1}\langle x, y\rangle$.

0.2 Open and Closed Sets

If $E \subset X$ is an open set, $\forall x \in E$, there is an open ball $B_r(x) \subset E$. Any union of open sets are still open, finite intersection of open sets in open, and the intersection of infinitely many open sets could fail to be open. Let X be any set. The discrete metric $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ if $x \neq y$. Every subset of X with discrete metric is open . A set is closed if its complement is open.
The whole space X is both open and closed. The empty set ϕ is both open and closed. U is open iff U is the union of open balls. The interior $E^\circ := \bigcup_{U \subset E, U \text{open}} U$ is the largest open set contained in E, where $E \subset X$. $\overline{E} := \bigcap_{E \subset F, F \text{closed}} F$ is the smallest closed super-set of E.
Alternative definition: $x \in \overline{A} \iff \forall$ open sets U with $x \in U$, $U \cap A \neq \phi$. Twin facts: $(\overline{E})^c = (E^c)^\circ$. $(E^\circ)^c = \overline{(E^c)}$. Boundary $\partial E := \overline{E} \cap \overline{E^c} = \overline{E} \backslash E^\circ$. Notice that $\partial\mathbb{Q} = \mathbb{R}$. E dense in X means $\overline{E} = X$.

0.3 Compactness

BW Thm: Every bounded sequence in R has a convergent subseq. In general, $E \subset X$ is said to be bounded if $E \subset B_R(p)$, $R < \infty$.
$K \subset X$ is said to be sequentially compact if every sequence in K has a convergent subseq with limit in K. seq compact implies closed and bdd, cannot be reversed in general. For R, seq compact iff clsoed and bdd. compact iff seq compact iff totally bounded and complete.

0.4 Continuity

Continuity means $f(B_\delta(a)) \subset B_\epsilon(f(a)) \implies B_\delta(a) \subset f^{-1}(B_\epsilon(f(a)))$.
Proposition: $f : X \to Y$ is continuous $\iff$ preimage of every open (closed) set in Y is open (closed) in X.
Fact: f is continous at $a \iff$ whenever $x_n \to a$ in X, $f(x_n) \to f(a)$ in Y.
Uniformly continuous: $\forall\epsilon > 0 \exists\delta > 0 s.t. \forall a \in X, d(x, a) < \delta \implies d(f(x), f(a)) < \epsilon$. So one δ works uniformly for all a.
If $f : K \to Y$ is a cts bijection (K is compact), then it is a homeomophism, i.e. f^{-1} is also cts.
If $f : K \to Y$ is cts, K is compact, then f is uniformly cts.

0.5 Sets of functions

$d_{\sup}(f,g) = \sup_x |f(x) - g(x)|$. $\mathcal{B}(E;\mathbb{R}) := \{f : E \to \mathbb{R} | f \text{ is bounded}\}$ is complete.
Uniform convergence preserves continuity, uniform continuity etc. Pointwise convergence might not preserve continuity.
A set of functions $\mathcal{F} := \{f : X \to \mathbb{R}\}$ is said to be equicontinuous at a if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $\forall f \in \mathcal{F}, d(x,a) < \delta \implies d(f(x), f(a)) < \epsilon$.
Proposition: If $f_n \to f$ pointwise and (f_n) is equicts at a, f is cts at a
Proposition: K a compact space. If $(f_n) \subset C(K)$, $f_n \to f$ pointwise and (f_n) is equicts on K, then $f_n \to f$ uniformly on K.
Uniform Convergence: $\forall \epsilon > 0 \exists N > 0$ s.t. $\forall n \geq N, x \in E, |f_n(x) - f(x)| < \epsilon$.
Uniformly Cauchy: $\forall \epsilon > 0 \exists N > 0$ s.t. $\forall n, m \geq N, x \in E, |f_n(x) - f_m(x)| < \epsilon$.
Ascoli-Arzela: K compact. $\mathcal{F} \subset C(K)$ is totally bdd iff it is equicts and p.w. bdd.

0.6 Contraction Map

A map $T : X \to X$ is said to be a contraction mapping if $\exists \alpha < 1$ such that $d(T(x), T(y)) \leq \alpha d(x,y)$ $\forall x, y \in X$.
Fact: Any contraction is Lipschitz continuous with Lipschitz constant $\leq \alpha$.
Contraction Mapping Theorem: X be complete. Then any contraction map on X has a (unique) fixed point.

0.7 Connectedness

X is said to be pathwise connected if $\forall x, y \in X$ $\exists$a continuous curve $f : [0,1] \to X$ such that $f(0) = x, f(1) = y$.
X is disconnected if it can be written as the union of two non-empty disjoint open (or closed) sets, i.e. if $\exists U, V$ open s.t. $X = U \cup V$, $U \cap V = \phi$, $U \neq \phi, V \neq \phi$. X is connected if it is not disconnected. X is disconnected if $\exists$ a non-empty proper subset which is both open and closed.
X is disconnected means $X = U \cup V$. U, V are open, non-empty, disjoint. X is connected iff 1. whenever $X = U \cup V$(open, disjoint), we must have $U = \phi$ or $V = \phi$, or 2. whenever $X = U \cup V$(open, non-empty), we must have $U \cap V \neq \phi$.

0.8 Differentiability

$f : E \to \mathbb{R}^m, E \subset \mathbb{R}^n$, is Frechet differentiable if $f(a+h) = f(a) + L(h) + r(h)$, i.e. $f(a+h) = f(a) + f'(a)h + o(|h|)$.
f is Gateaux differentiable if $(D_u f)(a) = \lim_{t\to\infty} \frac{f(a+tu)-f(a)}{t}$ exists for all $u \in \mathbb{R}^n$.
If f is Frechet differentiable, $(D_u f)(a) = L_{f;a}(u) = f'(a)u$. Frechet implies Gateaux.

0.9 Mean Value Theorem

Basic MVT on $\mathbb{R}$: If $f : [a,b] \to \mathbb{R}$ is cts on [a,b] and diff'l on (a,b) then $\exists \xi \in (a,b)$ such that $f(b) - f(a) = f'(\xi)(b-a)$.
Extension to $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}^n$ is convex and open: $f(b) - f(a) = \nabla f(\xi) \cdot (b-a)$ for some $\xi \in (a,b)$.
- Corollary: $f' = 0 \implies f$ is constant on the domain
Extension to $f : E \to \mathbb{R}^m$, where $E \subset \mathbb{R}^n$ is convex and open: $[f(b) - f(a)] \cdot u = [f'(\xi) \cdot (b-a)] \cdot u$

0.10 Continuous Partials Implies Differentiability

Let $f : E \to \mathbb{R}^m$, $E \subset \mathbb{R}^n$ open, $a \in E$. Suppose $\partial_1 f, ..., \partial_n f$ exist in a nbd of a and continuous at a. Then f is diff'l at a.

0.11 Order of Differentiation

Let E be open, $f : E \to \mathbb{R}^m$ be s.t. $\partial_i f$, $\partial_j f$, and $\partial_i \partial_j f$ exist in a nbd of $a \in E$. If $\partial_i \partial_j f$ is cts at a, then $\partial_j \partial_i f$ exists and equals $\partial_i \partial_j f$.

0.12 Multi-Index Notation

$\partial^\alpha f := \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} f$. $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. $\binom{r}{\alpha} = \frac{r!}{\alpha!}$. $(\sum_{i=1}^n x_i)^r = \sum_{|\alpha|=r} \binom{r}{\alpha} x^\alpha$.

0.13 Taylor's Formula for $f \in C^{r+1}(a, b)$

For any $x, x_0 \in (a, b)$, $\exists \xi$ between x_0 and x such that

$$f(x) = \sum_{k=0}^{r} f^{(k)}(x_0) \frac{(x - x_0)^k}{k!} + f^{(r+1)}(\xi) \frac{(x - x_0)^{r+1}}{(r + 1)!}$$

0.14 Multivariate Taylor

$$f(x) = \sum_{0 \le |\alpha| \le r} \partial^\alpha f(x_0) \frac{h^\alpha}{\alpha!} + \sum_{|\alpha|=r+1} \partial^\alpha f(z) \frac{h^\alpha}{\alpha!}$$

where $z \in [x_0, x]$. Or alternatively,

$$f(x) = \sum_{k=0}^{r} \frac{1}{k!} D_h^k f(x_0) + \frac{1}{(r + 1)!} D_h^{r+1} f(z)$$

If $f \in C^{r+1}(\Omega)$, then the remainder theorem satisfies $D_h^{r+1} f(z) = o(|h|^{r+1})$ as $h \to 0$.

0.15 Extrema

If x_0 is a local extrema of f, then $\nabla f(x_0) = 0$.
In one variable, assume $f'(a) = 0$. If $f''(a) > 0$, it is local min. If $f''(a) < 0$, it is local max. If $f''(a) = 0$, it is inconclusive. In higher-dimension, assume $\nabla f(a) = 0$. If $H(a)$ is positive/negative definite, then a is a local min/ max. If $H(a)$ is semidefinite, we cannot tell without further information.

0.16 Convexity

f is convex if $f(x) \ge f(a) + \nabla f(a) \cdot (x - a)$. f is convex iff the Hessian is positive semi-definite.
For strict convexity, positive definite hessian implies strictly convex, but strictly convex does not implies positive definite hessian. Consider $f(x) = x^4$.

0.17 Inverse Function Theorem

Side note: If $f : I \to J$ is continuous and bijective, then it is strictly monotonic on I,m as a consequence of the intermediate value theorem.
Theorem: Let $\Omega \subset \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}^n$ be C^1. Then $\forall a \in \Omega$ s.t. $f'(a)$ is invertible, $\exists$ a neighborhood U of a such that
1) $V := f(U)$ is open and $f : U \to V$ is bijective;
2) the resulting $f^{-1} : V \to U$ is C^1 (This means f is a "local diffeomorphism" on Ω, meaning C^1 with a C^1 local inverse everywhere on Ω).
We say that a square matrix is invertible if and only if the determinant is not equal to zero

0.18 Implicit Function Theorem and Lagrange Multiplier

The system of euqations $f_i(x_1, ..., x_n, y_1, ..., y_m) = 0, i = 1, ..., n$ implicitly defines $x_1, ..., x_n$ as a function of $y_1, ..., y_m$ near a point (a, b) such that $f(a, b) = 0$, provided the Jacobain $(\partial_j f_i)_{i=1,...,n; j=1,...,n}$ is invertible at (a, b).
$\min_x f(x)$ subjected to $g(x) = 0$. By Lagrange Multiplier, 1) $\nabla f(x) = \lambda \nabla g(x)$; 2) $g(x) = 0$. Solve for x by plugging the solved λ.

0.19 Jordan Measure

A is (Jordan) measurable if $\underline{\mu}(A) = \overline{\mu}(A)$, i.e. the inner measure is equal to the outer measure.
Alternatively, A is (Jordan) measurable if and only if $\overline{\mu}(\partial A) = 0$, i.e. ∂A can be covered with simple sets of arbitrarily small measure / ∂A is a null set (that is, of measure zero).
Theorem: $\overline{\mu}(A) = \underline{\mu}(A) + \overline{\mu}(\partial A)$.

0.20 Jordan Null vs. Lebesgue Null

A is Jordan null if $\forall \epsilon > 0 \exists R_1, ..., R_m$ finite collections of rectangles such that $A \subset \bigcup_{i=1}^m R_i$ and $\sum_{i=1}^m \mu(R_i) < \epsilon$. A is Lebesgue null if $\forall \epsilon > 0 \exists R_1, R_2, ...$ (at most countable) such that $A \subset \bigcup_{i=1}^\infty R_i$ and $\sum_{i=1}^\infty \mu(R_i) < \epsilon$.

0.21 Integration

f is Riemann/Darboux integrable if $L(f) = U(f) = \int_{[a,b]} f$. Here $L(f) = \sup_\Pi L(f, \Pi)$, $U(f) = \inf_\Pi U(f, \Pi)$. $L(f, \Pi) := \sum_{R \in \mathcal{R}(\Pi)} m(f; R)\mu(R)$, $U(f, \Pi) := \sum_{R \in \mathcal{R}(\Pi)} M(f; R)\mu(R)$. $m(f; R) := \inf_R f$, $M(f; R) := \sup_R f$.
If f is continuous, then f is integrable. If f is integrable, then f is bounded.
Lebesgue's Theorem: $f : [a, b] \to \mathbb{R}$ is bdd. $[a, b] \subset \mathbb{R}^d$. f is integrable $\iff$ $\{x : f \text{ is discts at } x\}$ is Lebesgue null.
Fubini's Theorem: If $\int\int_{X \times Y} |f(x, y)| d(x, y) < +\infty$, i.e. f is Riemann/Darboux integrable, we have

$$\int\int_{X \times Y} f(x, y) d(x, y) = \int_X \left(\int_Y f(x, y) dy \right) dx = \int_Y \left(\int_X f(x, y) dx \right) dy$$

Proposition: If $f \geq 0$ and $\int_I f = 0$, then $\{f > 0\}$ is null.

0.22 Differentiation under the Integral Sign

Let $f(x, t)$ and $\partial_2 f(x, t)$ be continuous functions of two variables where x is in the range of integration and t is in some interval aronud t_0. Then $\frac{d}{dt} \int_a^b f(x, t) dx = \int_a^b \frac{\partial}{\partial t} f(x, t) dx$ is valid at $t = t_0$. In other words, $F(t) := \int_a^b f(x, t) dx$ and $G(t) := \int_a^b \partial_2 f(x, t) dx$ and we have $F'(t_0) = G(t_0)$.