IO Class Notes: Single Agent Dynamics.

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November 8, 2004

The notes on single agent dynamics are in sections. One section provides mathematical background. This is not meant for you to read independently of the other lectures. I will introduce it when I need it in the first example lecture. After watching me go over this example you should go over the math notes. At that point the reason we need all this and what you should take away from the math should be clear. The mathematical background section has two subsections.

1. Definitions. This subsection is meant to give you easy access to some definitions you might need to understand the theorems of the next subsection.

2. Relevant Theorems. This section should be thought of as a reference for the mathematical details that underlie much of the analysis. It contains the contraction mapping theorem and some of its corollaries, as well as Blackwell's sufficient conditions for a contraction. These are the theorems that underlie almost all of the analysis done in the examples. We also provide a few additional results that make it easy to use the contraction mapping theorems and interpret their results. The additional results may be necessary in order to use this mode of analysis in your own research.

We actually will begin the lectures with an example. The first example is a standard investment example modified to allow for stochastic outcomes to the investment process, and to allow for exit. I use it to introduce the relevant mathematical material. The rest of the examples are taken directly from articles in the literature. What I have done is; put them in the framework we are using for analysis, tried to simplify the complex articles, and tried to draw out the implications relevant for I.O. in more detail. The other examples are

1. Monopolist Learning About Unknown Demand. This is a simplified version of a model presented in Aghion, Bolton, Harris and Julien (RESTUD, 1991), discussed as a problem in Stokey Lucas and Prescott. It both deals with a problem of direct interest to I.O., and allows us to introduce the notion of Bayseian learning. I have added a discussion of entry and exit and of the "sample paths" of the prices of firms we ought to see in a data set consisting of firms following optimal policies. Here we obtain the value function using Blackwell's theorem.

2. Entry and Exit Decisions. This is a discreetized version of Dixit's original paper (QJE 1989). It allows us to introduce the idea of an $s-S$ policy in a setting that is particularly important for I-O. Here we obtain the value function by checking the conditions which define a contraction mapping directly. This example is turned into a computer problem set at the end of this section. The problem set is designed to teach you how to move between the theoretical model and actual patterns in the data.

The final section of the single agent dynamic notes contains empirical examples. These are reviews of the papers that have had an impact on I.O. either through the substance of their analysis, or because of the techniques they introduce.

1. Renewing Patents (Pakes, 1986 *Econometrica*).
2. Replacing Engines (Rust, 1987,*Econometrica*).
3. Water Depletion in California (Timmins, 2000,*Econometrica*)

A Simple Investment Example.

We begin with a simple investment example. This is similar to problems you have seen in macro, except that we will allow for stochastic outcomes to the investment process, and an exit decision.

Needed Primitives.

- $\pi(\omega, z) : \Omega \times Z \to R$, provides the profits that accrue given an (ω, z) couple. $s \equiv (\omega, z)$ is called the state of the system. It determines profits.
- ω is the state variable whose evolution we can effect using investment (knowledge, advertising or physical capital, or the stock of a resource we are exploring). It "lives on" (takes values in) Ω (usually either the integers or the real numbers). The stochastic process generating the sequence of realizations $\{\omega_t\}$ is a "controlled" Markov process. By this we mean that the distribution of ω_{t+1} is determined by ω_t, the Markov component, and x_t, where x_t is investment (or the control). Hence to complete the specification for the

stochastic process we need a set of distribution functions, one for each possible (ω_t, x_t) couple. This determines the distribution of ω_{t+1} given (ω_t, x_t);

$$\mathcal{P}_\omega = \{P(\cdot|\omega, x); \omega \in \Omega, x \in X \subset R^+\}.$$

The family $\mathcal{P}_\omega$ is assumed to be stochastically increasing in the natural order of both Ω and X. That is the higher is current ω, ceteris paribus, the higher future ω is likely to be; and the higher is current x, ceteris paribus, the higher future ω is likely to be.

- We need a cost of investment, which we will usually designate as $c(x)$ or just x.

- z symbolizes an exogenous variable that helps determine profits. Examples include factor costs, the general state of demand .... The difference between z and ω is we have no control over the sequence $\{z_t\}$. It evolves as an exogenous stochastic process with family

$$\mathcal{P}_z = \{P(\cdot|z), z \in Z \in R\},$$

  assumed stochastically increasing in z.

The "Sequence" Problem.

There are two reasons for writing this problem down. One is that it will give you some appreciation of the simplification allowed us by going to the "Bellman" equation. The other is that sometimes this rather long winded form of the problem can prove helpful in designing estimators, or in proving theorems. Consequently you will see versions of it in the literature.

Let superscript t denote the history of a variable until t, and subscript t denote and actual year, so the history of states is

$$s^t \equiv (s_1, \ldots, s_t) \in S^t (\equiv S \times S \ldots \times S)$$

Call $\psi_t(S^t) : S^t \rightarrow X$ for $t = 1, 2, \ldots$ a policy function for period t. This maps the observed history until that t into a choice for investment at that t. It is a function because it is defined for all possible histories. The policy is a sequence of such policy functions, one for each possible future t, and will be denoted by ψ (no subscript). Note how complicated this object is. It is an infinite sequence of functions, and as the sequence increases the number of arguments in each function increases, growing without bound.

We will say that the policy ψ is feasible from s_1 if and only if $\psi_t(s^t) \in X, \ \forall t$. Let $\Psi(s_1)$ be the set of all feasible policies. We will say that the goal of the decision maker is to choose a $\psi \in \Psi(s_1)$ to maximize the expected discounted value of future net cash flows. To do this we have to calculate the EDV of each ψ and that is given by

$$V_\psi(s_1) \equiv E_\psi\{\sum_{t=0}^{\infty} \beta^t (\pi(s_t) - c(\psi_t(s^t)))|s_1\}$$

where it is understood that E_ψ refers to the expectation given that policy ψ is followed, but this

$$= \sum_{t=0}^{\infty} \beta^t E_\psi\{\pi(s_t) - c(\psi_t(s^t))|s_1\}$$

$$= \sum_{t=0}^{\infty} \beta^t \sum_{s_t} [\pi(s_t) - c(\psi_t(s^t)] p(s_t|\psi, s_1),$$

where we calculate $p(s_t|\psi, s_1)$ from

$$p(s_t|\psi, s_1) =$$

$$\sum_{s(t-1),\ldots,s(2)} p(s_t|s_{t-1}, \psi_{t-1}(s^{t-1}))p(s_{t-1}|s_{t-2}, \psi_{t-2}(s^{t-2})), \ldots, p(s_2|s_1, \psi_1(s^1)).$$

Note that just calculating the value and policy associated with any given policy is very difficult.

Now define

$$V^*(s_1) = sup_{\psi \in \Psi(s_1)} V_\psi(s_1), \tag{1}$$

with $V_\psi(s_1)$ as defined above, and $\pi^*(s_1)$ to be the corresponding policy (there may be more than one of the latter that brings us to the same value). Our problem is to find $V^*(s_1)$ and $\psi^*(s_1)$; usually both as functions, i.e. for every possible value of $s \in S$. Looks very difficult, if not impossible, to do.

The "Functional" Equation.

We are now going to rewrite the problem in terms of a single unknown function; the value function generated by the optimal policy, say $v(s)$, where

$$v(s) = sup_{x \in X}\{\pi(s) - c(x) + \beta \int_{\omega',z'} v(\omega', z')p(\omega'|\omega, s)p(z'|z). \quad (2)$$

This equation, which solves for a function in terms of itself, or which specifies a "fixed point" in a space of functions, is often called Bellman's equation (though apparently he was not the first to use it). It is very intuitive; it just says the value today is current returns plus the expected discounted value of tomorrow given optimal behavior in the interim. The book by Stokey Lucas and Precott goes into a great deal of detail on the conditions under which;

- Any $v(s)$ which solves (2) equals $V^*(s)$ as defined by (1).
- Any $x(s)$ which solves (2) is an optimal policy for (1).

(they also give conditions for the converse, i.e. any $V^*(s)$ which solves (1) satisfies the functional equation in (2), etc.). A sufficient condition, and one that will do for us, is that $V^*(\cdot)$ is bounded. So we do not go over the details.

Note that we have transformed the problem from a problem of finding an infinite sequence of functions of increasing complexity to a problem of finding a single function. Given the value function we can directly calculate optimal policies.

However to proceed further we need rules for operating in a "function" space; i.e. we need to know what it means when we say one function is "the same" as an other, for defining operators on functions (like the r.h.s. of equation (2)), and for determining when we are at a fixed point for that operator (the usual rules apply to numbers). For this we go to section 1.1 of the Mathematical Preliminaries (entitled, some definitions). After defining what we mean by a closed space of functions we move on to the contraction mapping theorem.

Using Tools from the theory of Complete Metric Spaces to Analyze our Example

To make our problem slightly more realistic let the firm have the option of exiting. If it exits then it receives a one time payoff of ϕ dollars and never reappears again. The functional equation for our problem then becomes

$$v(s) = max\{\phi, sup_{x \in X}[\pi(s) - c(x) + \beta \int_{s'} v(s')p(s'|x,s)]\} \quad (3)$$

where the above integral is written out in full as

$$\int_{\omega', z'} v(\omega', z')p(\omega'|\omega, x)p(z'|z).$$

The Bellman equation in (3), is a fixed point to the operator T where $T : B(S) \to B(S)$ and is defined pointwise by

$$Tf_{(s)} = max\{\phi, sup_{x \in X}[\pi(s) - c(x) + \beta \int_{\omega', z'} f(\omega', z')p(\omega'|\omega, x)p(z'|z)]\}.$$

Prove to yourself that T does in fact take $B(S)$ into itself. Note for this we are going to need that $\pi(s)$ is bounded from above, so this is an assumption we will make (we do not need to assume that $\pi(s)$ is bounded from below, why?). Now *assume* that T is a contraction. From the contraction mapping theorem then we know that there is a unique value function and we know how to compute it.

In computing the value function we will need to start somewhere. The usual starting point, i.e. the $f(s)$ for the first iteration is $\pi(s)$. If you do it in this way then

$$v^1(s) == max\{\phi, sup_{x \in X}[\pi(s) - c(x) + \beta \int_{\omega', z'} \pi(\omega', z')p(\omega'|\omega, x)p(z'|z)]\},$$

is the value function for a two period problem, and can be used to analyze "two-period" problems. More generally $v^t(s)$ is the value function for the t period problem, and we know $v^t(s)$ converges (at a geometric rate) to v^*. We will of course need to stop the computer somewhere. What we generally do is set an ϵ and insure that $\rho(v^k, v^*) \leq \epsilon$ by setting $\rho(v^k, v^{k-1}) \leq \epsilon(1-\beta)$ (see the third condition of the contraction mapping theorem).

Before going to the actual calculations you should check to see whether you can use the corollary to prove the following properties of both each iteration of the iterative process, and of the "limit" function from your calculations (i.e. of the value function).

- Use the first part of the corollary to prove that each iteration, and indeed the value function is weakly increasing in both ω and z.

- Is the value function strongly increasing in these variables?

- Is the continuation value strongly increasing in these variables.?

- Draw the value function as the max of the continuation value and the stopping value for a given z (that is plot it as a function of ω)

- You now should know the form of the "optimal stopping" rule for the firm – what is it? It is easiest to describe this by a diagram in the (z, ω) plane.

- Say we had two groups of firms, one with $\omega = \omega_0$ and one with $\omega = \omega_1 > \omega_0$ but that all firms started with the same initial z. Also assume that $i(\omega_0, z) = i(\omega_1, z) = 0$.

Say now there were different realizations of z' for each firm each drawn randomly from $P(\cdot|z)$, and as a result in each group the appropriate firms exit. Look now at only the surviving firms in each group. What do you expect to be the relationship of the average z of the surviving firms in each group to the initial ω in the groups? Now think back to the Olley-Pakes article.

Here is a start on these questions. I begin by showing that $v(\cdot)$ is weakly increasing in ω. Assume that $f(\cdot)$ is increasing in ω and consider any $(\omega_1, z) > (\omega_2, z)$. Then

$$[Tf]_{(\omega_1,z)} = max\{\phi, sup_{x\in X}[\pi(\omega_1, z) - c(x) + \beta \int_{\omega',z'} v(\omega', z')p(\omega'|\omega_1, x)p(z'|z)]\}$$

$$\geq max\{\phi, [\pi(\omega_1, z) - c(x_2) + \beta \int_{\omega',z'} v(\omega', z')p(\omega'|\omega_1, x_2)p(z'|z)]\}$$

where x_2 is defined as the optimal value of x when $(\omega, z) = (\omega_2, z)$, and the inequality follows because x_2 is a feasible choice for the agent with (ω_1, z),

$$\geq max\{\phi, [\pi(\omega_2, z) - c(x_2) + \beta \int_{\omega',z'} v(\omega', z')p(\omega'|\omega_2, x_2)p(z'|z)]\}$$

provided $\pi(\cdot)$ is increasing in ω for each z, and $P(\cdot|\omega, z)$ is stochastically increasing in ω (both of which are assumed in our assumptions on primitives),

$$= max\{\phi, sup_{x\in X}[\pi(\omega_2, z) - c(x_2) + \beta \int_{\omega',z'} v(\omega', z')p(\omega'|\omega_2, x_2)p(z'|z)]\}$$

$$= [Tf]_{(\omega(2),z)} \ \bullet$$

So corollary 1 proves that the value function is weakly increasing in ω. The proof that it is increasing in z is analogous. To get strongly increasing we would have to show that if $f(\cdot)$

were weakly increasing $Tf(\cdot)$ would be strongly increasing (apply corollary 2). That is not true and the reason is that we can not replace either of the weak inequality signs above with strong inequality signs. That is provided we exit at (ω_1, z) we will exit at any $(\omega_2 \leq \omega_1, z)$, in which case Tf evaluated at the two values is the same $(= \phi)$. It is easy to show however that the continuation value, the value after the *sup* operator, is strictly increasing (the sum of a weakly increasing and strongly increasing function is strongly increasing). This in turn implies that no matter z there will be a unique $\underline{\omega}(z)$ s.t. we exit *if* $\omega \leq \underline{\omega}(z)$. I leave filling in the rest of the details to you.

A More Detailed Example.

We now proceed with the actual calculations for a rather simple case for which it is easy to write down all the needed expressions, and hence which is easy to discuss in class. We assume there is no z, that ω takes values on the integers, that $\omega_{t+1} = \omega_t + 1$ with probability $p(x)(1-\delta)$ and it equals $\omega_t - 1$ with probability $\delta(1 - p(x))$, and it equals ω_t with the remaining probability. Further assume $c(x) = x$.

Then you should be able to show that at any iteration the f.o.c. for $x = x^k(\omega)$ is the Kuhn-Tucker condition

$$x \left\{ \beta \left([v^k(\omega + 1) - v^k(\omega)](1 - \delta) + [v^k(\omega) - v^k(\omega - 1)]\delta \right) \frac{\partial p(x)}{\partial x} - 1 \right\}$$
$$= 0.$$

Since we know that each iterate of the value function is increasing in ω we know that the first term inside the round brackets is positive. Our regularity conditions insure $\frac{\partial p(x)}{\partial x} \geq 0$ also. There may, of course, be more than one x which satisfies this f.o.c. for a given ω. Note that to insure that there isn't it suffices to assume that the derivative of $p(x)$ is decreasing in x (then the derivative of the f.o.c. w.r.t. falls in x; a special case of this is $p(x) = ax/(1 + ax)$ for $a > 0$).

Then either the f.o.c. when evaluated at $x = 0$ has expression in the interior of the brackets negative, in which case $x = 0$, or we have a unique interior x which solves this f.o.c. So this gives us $x(\omega)$ for each ω. Note that this $x(\omega)$ will increase in the "slope" of the value function (what an increase in x does is increase the probability of attaining a higher ω, and the value of that is given by this slope).

We have already shown that $v(\omega)$ is bounded and increasing in ω. Thus it must be the case that there is an $\overline{\omega}$ such that if

$$\omega \geq \overline{\omega} \ \Rightarrow \beta[v(\omega + 1) - v(\omega)]\frac{\partial p(x = 0)}{\partial x} < 1.$$

So if, $\omega \geq \overline{\omega}$, $x(\omega) = 0$. Consequently provided we assume that $p(0) = 0$, i.e. if you do not invest you cannot improve ω at all, we have $\omega_{t+1} \leq \overline{\omega}$.

Further we have already shown that there is an ω low enough to induce exit. Hence we have just shown that provided we start with an $\omega < \overline{\omega}$, we will never observe an ω outside of the region $[\underline{\omega}.\overline{\omega}]$, where $\underline{\omega}$ is the value at which you exit. Consequently the Markov process the model produces for ω is a finite state Markov chain.

You should be able to put z back in, and provided that z can only take values in a finite set, you get the same types of result (i.e. a finite state Markov chain). First write down the first order (really Kuhn-Tucker) condition determining the optimal x given $v(\cdot)$ and formulate an argument similar to that above

[Hint: let $w(\tau|\omega, z) \equiv \int v(\omega+\tau, z')P(dz'|z)$, and then express the first order condition in terms of the difference $w(1|\cdot) - w(0|\cdot)$.] Then show that for each z there is an $\overline{\omega}$ such that for $\omega > \overline{\omega}$, $x(\omega, z) = 0$, and since there are a finite number of z there are a finite number of (z, ω) couples.

Now go back to learn Blackwell's theorem. After you learn Blackwell's theorem use it to show that the operator defining our value function is a contraction mapping. This implies everything we have done above as correct. Here are some additional questions that will insure that you can apply what you have just learned.

- For the simple problem can you now draw sample paths in (ω, t) space (begin by drawing the boundary points)? For the more complicated problem that includes z can you draw sample paths in $\omega, z)$ space. A recurrent subset of the points in S is a set of points that, for any given firm, will be "hit" infinitely often (more than a finite number of times) with probability one. Are there any recurrent points for this problem.

- Say you could observed the discrete and continuous control (the discrete control is just whether the firm is active and the continuous control is x). Further assume that: you observed (ω_t, z_t), you knew the form of the profit function, and you observed x_t up to an error which was independent of the form of the profit function. Provide a way of estimating (β, a, ϕ). Why did I assume there was investment was observed subject to error (i.e. that we did not observe it exactly).