Derive Maxwell's equations from field theory viewpoint

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This note aims to derive the Maxwell's equations in classical electrodynamics from the axiomatic field theory viewpoint. The derivation starts with the well-known expression of the Lagrangian $\mathcal{L}$ of a electromagnetic field in field theory and proceeds in three steps:

1. Assumptions and Lagrangian;

2. Variation $\longrightarrow$ Euler-Lagrange Equations $\longrightarrow$ Inhomogeneous Maxwell's equations;

3. Bianchi Identity of the Field Strength Tensor $\longrightarrow$ Homogeneous Maxwell's equations.

1 Lagrangian Density

(a) Basic field variable: The electromagnetic field is described by the four-potential $A_\mu(\vec{x})$ in field theory.

(b) Definition of the strength of field:

$$\mathbf{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \;\; \mathbf{F}^{\mu\nu} = \mathbf{g}^{\mu\alpha}\mathbf{g}^{\nu\beta}\mathbf{F}_{\alpha\beta}. \tag{1.1}$$

(c) Construct the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} - j^\mu A_\mu. \tag{1.2}$$

Note:

1. The kinetic term $-\frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}$ ensures the correct field dynamics and energy-momentum tensor.

2. The second term in (1.2) enforces coupling to sources and yields charge conservation $\partial_\mu j^\mu = 0$ by gauge invariance.

2 Variation and Inhomogeneous Maxwell's Equations

It should be noted that the action

$$\mathcal{S} = \int d^4x \, \mathcal{L} \tag{2.1}$$

must be stationary on the boundary of the region.

While the Euler-Lagrange equations read

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = 0, \tag{2.2}$$

thus

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \frac{\partial \mathcal{L}}{\partial A_\mu} = 0. \tag{2.3}$$

Firstly, we compute

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} &= -\frac{1}{4} \cdot 2 \, \mathbf{F}_{\rho\sigma} \frac{\partial \mathbf{F}^{\rho\sigma}}{\partial(\partial_\nu A_\mu)} \\ &= -\frac{1}{2} \, \mathbf{F}^{\rho\sigma} (\delta^\nu_\rho \delta^\mu_\sigma - \delta^\nu_\sigma \delta^\mu_\rho) \\ &= -\mathbf{F}^{\nu\mu}, \end{aligned} \tag{2.4}$$

and since

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -j^\mu, \tag{2.5}$$

substituting into the Euler-Lagrange equations gives

$$\begin{aligned} \partial_\nu(-\mathbf{F}^{\nu\mu}) - (-j^\mu) &= 0 \\ \partial_\nu \mathbf{F}^{\nu\mu} &= j^\mu. \end{aligned} \tag{2.6}$$

These are the inhomogeneous set of the Maxwell's equations (i.e.Gauss's law and the Ampere-Maxwell law):

$$\nabla \cdot \mathbf{E} = \rho \, , \tag{2.7}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} \, . \tag{2.8}$$

3 Bianchi Identity and Homogeneous Maxwell's Equations

Since $\mathbf{F}_{\mu\nu}$ is antisymmetric by definition (1.1), we always have the identity

$$\partial_\lambda \mathbf{F}_{\mu\nu} + \partial_\mu \mathbf{F}_{\nu\lambda} + \partial_\nu \mathbf{F}_{\lambda\mu} = 0. \tag{3.1}$$

This identity is exactly the homogeneous set of Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0 , \tag{3.2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 . \tag{3.3}$$

We have already derived the classical Maxwell's equations from field theory viewpoint.

QED.