From familiar fact in Euclidean space to a conclusion in spaces with arbitrary metrics.

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$A^T = A^{-1}$ $\Longleftarrow$ The traditional (or special) orthogonality.

$\Downarrow$ How does this condition (or definition) preserve the internal products? (in Euclidean Space).

$$x^T x = x^T (A^{-1} A) x$$
$$\longrightarrow x^T x = x^T A^T A x .$$
$$\longrightarrow x^T x = (Ax)^T Ax .$$

$\Downarrow$ rewrite with tensor expressions. (Suppose $\vec{x} = U^\alpha \vec{e}_\alpha$.)

$$U_\alpha U^\alpha = (\eta_{\alpha\beta} \Lambda^\beta{}_\nu U^\nu) \cdot \Lambda^\alpha{}_\mu U^\mu .$$

This implies $A^T = \eta_{\alpha\beta} \Lambda^\beta{}_\nu$.

While we already have $\eta_{\alpha\beta} \Lambda^\beta{}_\nu \Lambda^\alpha{}_\mu = \eta_{\mu\nu}$ derived from the invariance of the interval.

That means, in Minkowsky Space, $A^T A = diag\{-1,1,1,1\}$.

In Euclidean Space, "Orthogonality" means $A^T A = diag\{+1,+1,+1,+1\}$. implies that the identity matrix is $diag\{+1,+1,+1,+1\}$.

In Minkowsky Space, "Orthogonality" means $A^T A = diag\{-1,+1,+1,+1\}$ implies that the identity matrix is $diag\{-1,+1,+1,+1\}$

(Generalisation).

In conclusion, the identity matrix in an arbitrary space is exactly the metric tensor of the space.

$$P_\mu A^\mu = \tilde{p}(\vec{A}).$$

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3.1

(a)

$$\tilde{p} = P_\mu.$$

$$P_\mu = \tilde{p}(\vec{e}_\mu).$$

$$\tilde{p}(\vec{A}) = \tilde{p}(A^\mu \vec{e}_\mu)$$

$$\tilde{p}(\vec{A}) = A^\mu P_\mu.$$

While $\tilde{p}(\vec{A}) = P_\mu \tilde{\omega}^\mu(\vec{A})$.

$= P_\mu A^\mu$ Is it Unitary?

$\tilde{\omega}^\mu(\vec{A}) = A^\mu$. Is it Orthogonal?

$\tilde{\omega}^\mu(A^\nu \vec{e}_\nu) = A^\mu$.

$A^\nu \tilde{\omega}^\mu(\vec{e}_\nu) = A^\mu$.

Define "Unitary" again.

$\Longrightarrow \tilde{\omega}^\mu(\vec{e}_\nu) = \delta^\mu{}_\nu$.

$X^{\alpha\mu} \cancel{X^{\mu\beta}} Y^{\mu\beta} = (XY)^{\alpha\beta}$ $\quad X^\alpha{}_\mu Y^\mu{}_\beta = (XY)^\alpha{}_\beta$.

$\eta_{\alpha\beta} P^\alpha P^\beta \equiv \eta_{\alpha\beta}(\Lambda^\alpha{}_\mu P^\mu)\cdot(\Lambda^\beta{}_\nu P^\nu)$.

$\Lambda^\mu{}_\nu = \eta^\mu{}_\beta \eta^\alpha{}_\nu \Lambda_\alpha{}^\beta$ $\quad \Lambda_\Delta{}^\Delta$ $\quad \Lambda\Lambda\eta_{\alpha\beta}$ $\quad X$ $\quad \eta_{\alpha\beta}\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu$

$\eta^{\alpha\mu}\eta_{\beta\nu}\Lambda_\alpha{}^\beta$ $\quad \Lambda^\alpha{}_\beta \cdot \Lambda_\mu{}^\beta = \delta^\alpha{}_\mu$. $\quad = \eta_{\mu\nu}$? ✓

$= \Lambda^\mu{}_\nu$.

Discriminant: Whether $\Lambda^\alpha{}_\beta$ is "orthogonal" or not

$\eta_{\alpha\beta} = \mathrm{diag}\{-1,+1,+1,+1\}$.

$\delta^\alpha{}_\beta = \mathrm{diag}\{+1,+1,+1,+1\}$.

$\Longrightarrow$ Yes, since det(Λ

$-\Lambda^1{}_0\Lambda^2{}_0 + \Lambda^1{}_1\Lambda^2{}_1 + \Lambda^1{}_2\Lambda^2{}_2 + \Lambda^1{}_3\Lambda^2{}_3 = \eta_{\mu\nu}$.

$= \cancel{\eta_{\alpha\beta}\Lambda}\ \eta^{\alpha\beta}\Lambda^1{}_\alpha\Lambda^2{}_\beta$ $\quad$ Preserve the metric.

Universality of the speed of light $\Longleftrightarrow$ Preserve internal products

$\Longleftrightarrow$ Transpose = Inverse

$\Longleftrightarrow$ Determinant = ±1