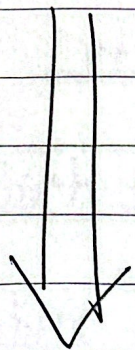


From familiar fact in Euclidean space to a conclusion in spaces with arbitrary metrics.

$A^T = A^{-1} \iff$ The traditional (or special) orthogonality.

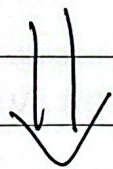


How does this condition (or definition) preserve the internal products?
(in Euclidean Space).

$$x^T x = x^T (A^{-1} A) x$$

$$\rightarrow x^T x = x^T A^T A x$$

$$\rightarrow x^T x = (Ax)^T Ax$$



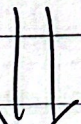
rewrite with tensor expressions.
(Suppose $\vec{x} = U^\alpha \vec{e}_\alpha$.)

$$U_\alpha U^\alpha = (\eta_{\alpha\beta} \Lambda^\beta_\nu U^\nu) \cdot \Lambda^\alpha_\mu U^\mu$$

This implies $A^T = \eta_{\alpha\beta} \Lambda^\beta_\nu$

While we already have $\eta_{\alpha\beta} \Lambda^\beta_\nu \Lambda^\alpha_\mu = \eta_{\mu\nu}$
derived from the invariance of the interval.

That means, in Minkowsky Space, $A^T A = \text{diag}\{-1, 1, 1, 1\}$.



In Euclidean Space,

"Orthogonality" means

$$A^T A = \text{diag}\{+1, +1, +1, +1\}$$

implies that the identity matrix is $\text{diag}\{+1, +1, +1, +1\}$.

In Minkowsky Space,

"Orthogonality" means

$$A^T A = \text{diag}\{-1, +1, +1, +1\}$$

implies that the identity matrix is $\text{diag}\{-1, +1, +1, +1\}$.

(Generalisation)

In conclusion, the identity matrix in an arbitrary space is exactly the metric tensor of the space.

$$P_\mu A^\mu = \tilde{P}(\vec{A})$$

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3.1

$$\tilde{P} = P_\mu$$

(a)

$$P_\mu = \tilde{P}(\vec{e}_\mu)$$

$$\tilde{P}(\vec{A}) = \tilde{P}(A^\mu \vec{e}_\mu)$$

$$\tilde{P}(\vec{A}) = A^\mu P_\mu$$

While

$$\tilde{P}(\vec{A}) = P_\mu \tilde{\omega}^\mu(\vec{A})$$

$= P_\mu A^\mu$ Is it Unitary?

$\tilde{\omega}^\mu(\vec{A}) = A^\mu$ Is it Orthogonal?

$$\tilde{\omega}^\mu(A^\nu \vec{e}_\nu) = A^\mu$$

$$A^\nu \tilde{\omega}^\mu(\vec{e}_\nu) = A^\mu$$

Define "Unitary" again.

$$\tilde{\omega}^\mu(\vec{e}_\nu) = \delta^\mu_\nu$$

$$X^{\alpha\mu} Y^{\mu\beta} = (XY)^{\alpha\beta}$$

$$X^\alpha_\mu Y^\mu_\beta = (XY)^\alpha_\beta$$

$$\eta_{\alpha\beta} P^\alpha P^\beta = \eta_{\alpha\beta} (\Lambda^\alpha_\mu P^\mu) (\Lambda^\beta_\nu P^\nu)$$

$$\Lambda^\mu_\nu = \eta^\mu_\beta \eta^\alpha_\nu \Lambda^\beta_\alpha$$

$$\eta^{\alpha\mu} \eta^{\beta\nu} \Lambda^\beta_\alpha$$

$$\Lambda^\alpha_\beta \Lambda^\beta_\mu = \delta^\alpha_\mu$$

$$= \Lambda^\mu_\nu$$

$$\eta_{\alpha\beta} = \text{diag}\{-1, +1, +1, +1\}$$

$$\delta^\alpha_\beta = \text{diag}\{+1, +1, +1, +1\}$$

Discriminant:
Whether Λ^α_μ is "orthogonal" or not
Yes, since $\det(\Lambda) = \pm 1$

$$-\Lambda^0_0 \Lambda^2_0 + \Lambda^1_0 \Lambda^2_1 + \Lambda^2_1 \Lambda^3_2 + \Lambda^3_2 \Lambda^4_3 = \eta_{\mu\nu}$$

$$= \eta_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu$$

Preserve the metric

Universality of the speed of light \iff Preserve internal products

\iff Transpose = Inverse

\iff Determinant = ± 1