Exercises. Dec. 26th. 2024.

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1. Special Relativity

1.1 S.I. $\longrightarrow$ Natural Units.

(b). $100\,W = 100\,J \cdot s^{-1} = 100\,kg \cdot m^2 \cdot s^{-3} = \frac{100}{27 \times 10^{24}}\,kg \cdot m^{-1}$.

$= 3.7 \times 10^{-24}\,kg \cdot m^{-1}$.

(c). $\hbar = 1.05 \times 10^{-34}\,J \cdot s = 1.05 \times 10^{-34}\,kg \cdot m^2 \cdot s^{-1}$

$= \frac{1.05 \times 10^{-34}}{3 \times 10^8}\,kg \cdot m = 3.5 \times 10^{-43}\,kg \cdot m$.

(d). $v = 30\,m \cdot s^{-1} = \frac{30}{3 \times 10^8} = 1 \times 10^{-7}$.

(e). $3 \times 10^4\,kg \cdot m \cdot s^{-1} = \frac{3 \times 10^4}{3 \times 10^8}\,kg = 1 \times 10^{-4}\,kg$.

(f). $10^5\,Pa = 10^5\,N \cdot m^{-2} = 10^5\,kg \cdot m^{-1} \cdot s^{-2} = \frac{10^5}{9 \times 10^{16}}\,kg \cdot m^{-3}$

$= 1.1 \times 10^{-12}\,kg \cdot m^{-3}$.

(g). $10^3\,kg \cdot m^{-3} = 10^3\,kg \cdot m^{-3}$.

(h). $10^6\,J \cdot s^{-1} \cdot cm^{-2} = 10^{10}\,kg \cdot m^2 \cdot s^{-3} \cdot m^{-2} = \frac{10^{10}}{27 \times 10^{24}}\,kg \cdot m^{-3}$.

$= 3.7 \times 10^{-16}\,kg \cdot m^{-3}$.

1.2 Natural Units $\rightarrow$ S.I.

(a) $v = 10^{-2} = 3 \times 10^6\,m \cdot s^{-1}$.

(b) $10^{19}\,kg \cdot m^{-3} = 10^{19}\,kg \cdot m \cdot m^{-2} \cdot m^{-2} = 9 \times 10^{35}\,kg \cdot m^{-1} \cdot s^{-2}$.

(c). $t = 10^{18}\,m = \frac{10^{18}}{3 \times 10^8}\,s = 3.3 \times 10^9\,s$.

(d) $u = 1\,kg \cdot m^{-3} = 1\,kg \cdot m^2 \cdot m^{-2} \cdot m^{-3} = 9 \times 10^{16}\,J \cdot m^{-3}$.

(e) $10\,m^{-1} = 10\,m \cdot m^{-2} = 9 \times 10^{17}\,m \cdot s^{-2}$.

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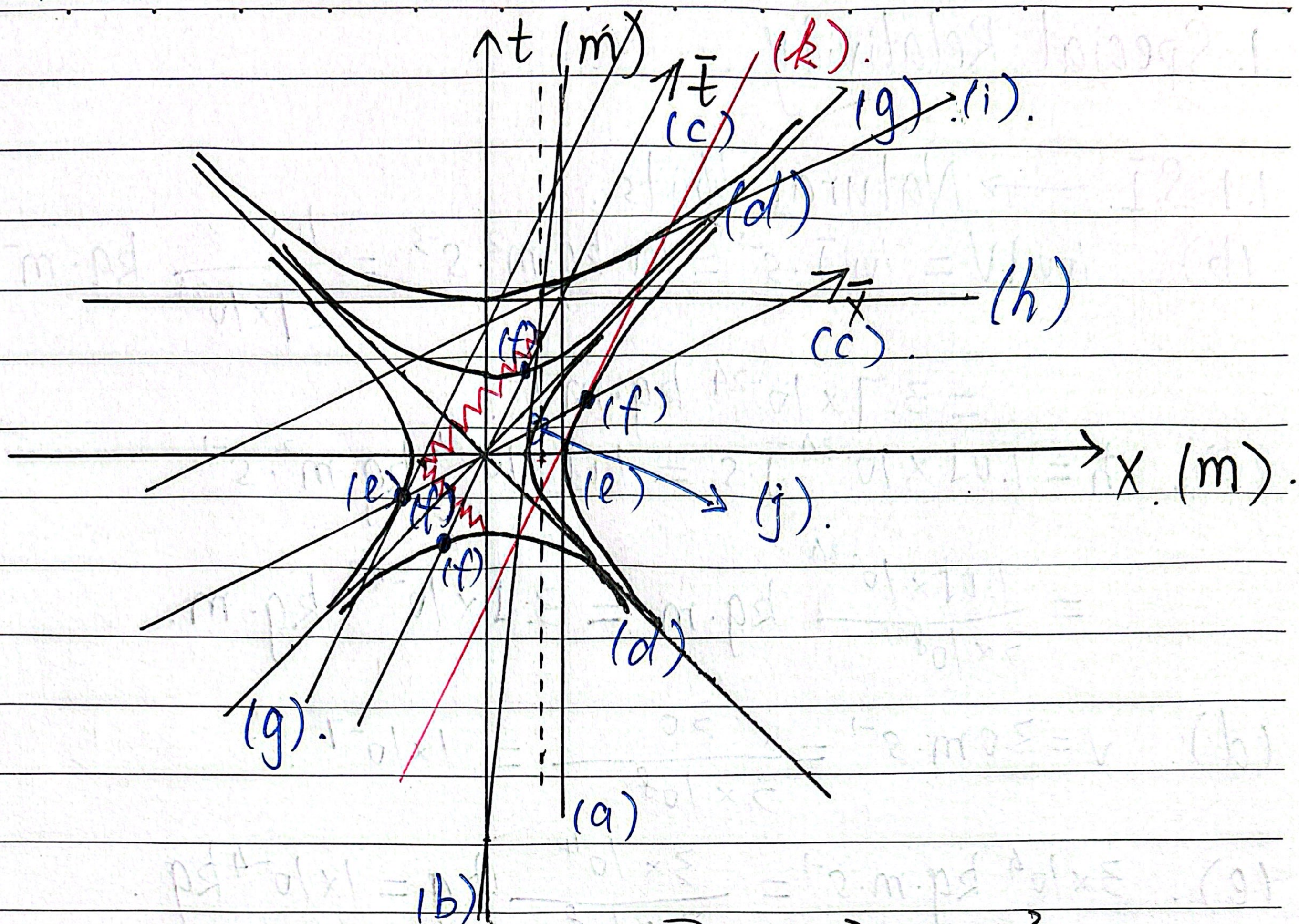
1.3



(d) $-(\Delta t)^2+(\Delta x)^2=-1$. (e) $-(\Delta t)^2+(\Delta x)^2=1$.

(g) $t=\pm x$.

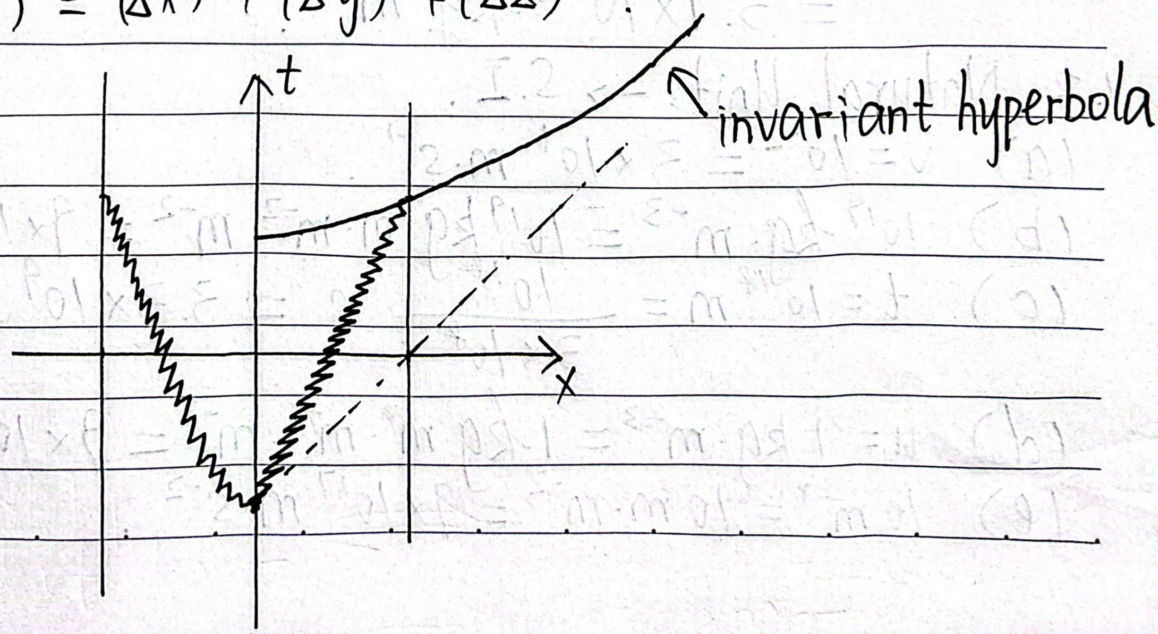
1.4

(a) $\sum_{\alpha=0}^{3} V_\alpha \Delta x^\alpha = V_0 \Delta t + V_1 \Delta x + V_2 \Delta y + V_3 \Delta z$.

(b) $\sum_{i=1}^{3} (\Delta x^i)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$.

1.5

(a)



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1.7 ① $\Delta\bar{s}^2 = \Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

$M_{00} = -1$, $M_{11} = M_{22} = M_{33} = 1$.
$M_{\alpha\beta} = 0$ when $\alpha \neq \beta$.

② $\Delta\bar{s}^2 = -(\Delta\bar{t})^2 + (\Delta\bar{x})^2 + (\Delta\bar{y})^2 + (\Delta\bar{z})^2$
$= -(\alpha\Delta t + \beta\Delta x)^2 + (\mu\Delta t + \nu\Delta x)^2 + (a\Delta y)^2 + (b\Delta z)^2$
$= (\mu^2 - \alpha^2)\cdot(\Delta t)^2 + (\nu^2 - \beta^2)\cdot(\Delta x)^2 + 2(\mu\nu - \alpha\beta)\cdot(\Delta t)\cdot(\Delta x)$
$+ a^2\cdot(\Delta y)^2 + b^2\cdot(\Delta z)^2$

$$\therefore M = \begin{pmatrix} \mu^2-\alpha^2 & \mu\nu-\alpha\beta & 0 & 0 \\ \mu\nu-\alpha\beta & \nu^2-\beta^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & b^2 \end{pmatrix}.$$

1.10

(a) $(0,0,0,0)$ and $(-1,1,0,0)$
$\Delta s^2 = -1+1=0$, hence they are null separated.

(b) $(1,1,-1,0)$ and $(-1,1,0,2)$.
$\Delta s^2 = -4+0+1+4 = 1 > 0$, hence they are spacelike separated.

(c) $(6,0,1,0)$ and $(5,0,1,0)$.
$\Delta s^2 = -(\Delta t)^2 = -1 < 0$, hence they are timelike separated.

(d) $(-1,1,-1,1)$ and $(4,1,-1,6)$.
$\Delta s^2 = -25+0+0+25=0$, hence they are null separated

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3.17 (a) Proof.

Let $\tilde{q}$ be a function of contravectors into numbers, which satisfies that $\tilde{q}(\vec{A}) = h(\vec{C}, \vec{A})$,

where $\vec{C}$ is a given $\binom{1}{0}$ tensor and $\vec{A}$ is an arbitrary $\binom{1}{0}$ tensor. $(\vec{C} \neq \vec{0})$.

Now prove the linearity of $\tilde{q}$:

Obviously, $\tilde{q}(k\vec{A} + j\vec{B}) = k h(\vec{C}, \vec{A}) + j h(\vec{C}, \vec{B})$
$= k\tilde{q}(\vec{A}) + j\tilde{q}(\vec{B})$.

Hence, $\tilde{q}$ is a one-form.

Since $\dfrac{h(\ ,\vec{A})}{h(\ ,\vec{B})}$ only depends on $\vec{A}$ and $\vec{B}$,

that means $\dfrac{h(\vec{D},\vec{A})}{h(\vec{D},\vec{B})} = \dfrac{h(\vec{C},\vec{A})}{h(\vec{C},\vec{B})}$ for arbitrary $\vec{A}, \vec{B}, \vec{D}$.

$$\Longrightarrow \frac{h(\vec{D},\vec{A})}{\tilde{q}(\vec{A})} = \frac{h(\vec{D},\vec{B})}{\tilde{q}(\vec{B})} \Rightarrow \frac{h(\vec{D},\vec{A})}{\tilde{q}(\vec{A})} \text{ only depends on } \vec{D}$$

Define $\tilde{p}(\vec{D}) := \dfrac{h(\vec{D},\vec{A})}{\tilde{q}(\vec{A})}$ as a function of vectors into numbers

The linearity of $\tilde{p}$ is obvious because of the linearity of h.

In this way, $\tilde{p}(\vec{D}) \cdot \tilde{q}(\vec{A}) = h(\vec{D}, \vec{A})$ for $\forall \vec{D}, \vec{A}$.

Therefore, $\tilde{p} \otimes \tilde{q} = h$.

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3.31 Proof.

(a). Since $\vec{U}$ is a timelike unit vector,

$$\eta_{\alpha\beta} U^\alpha U^\beta = -1 .$$

While $U_\beta = \eta_{\beta\gamma} U^\gamma$ is obviously in Minkowski space,

$\Longrightarrow$ we have $\eta_{\beta\gamma} U^\gamma - U_\beta = \eta_{\beta\gamma} U^\gamma + (\eta_{\alpha\gamma} U^\alpha U^\gamma) U_\beta = 0$,

And because $\eta_{\beta\gamma} = \eta_{\alpha\gamma} \eta^\alpha{}_\beta$ (since $\eta^\alpha{}_\beta = \delta^\alpha{}_\beta$),

$$\Longrightarrow \eta_{\alpha\gamma} (\eta^\alpha{}_\beta + U^\alpha U_\beta) U^\gamma = 0 .$$

Hence, $\eta_{\alpha\gamma} (\eta^\alpha{}_\beta + U^\alpha U_\beta) U^\gamma V^\beta = \cancel{V_\perp^\alpha}\ \eta_{\alpha\gamma} V_\perp^\alpha U^\gamma = 0$ for arbitrary V.

$\square$.

(b). $P^\alpha{}_\beta = \eta^{\alpha\mu} P_{\mu\beta}$.

$\therefore P^\alpha{}_\beta = \eta^{\alpha\mu} \eta_{\mu\beta} + \eta^{\alpha\mu} U_\mu U_\beta$.

While $V_\perp^\beta = (\eta^\beta{}_\nu + U^\beta U_\nu) \cdot V^\nu$,

Then $P^\alpha{}_\beta V_\perp^\beta = (\eta^{\alpha\mu} \eta_{\mu\beta} + \eta^{\alpha\mu} U_\mu U_\beta) \cdot (\eta^\beta{}_\nu + U^\beta U_\nu) \cdot V^\nu$.

$= (\eta^{\alpha\mu} \eta_{\mu\beta} \eta^\beta{}_\nu + \eta^{\alpha\mu} \eta^\beta{}_\nu U_\mu U_\beta + \eta^{\alpha\mu} \eta_{\mu\beta} U^\beta U_\nu$

$+ \eta^{\alpha\mu} U_\mu U_\beta U^\beta U_\nu) \cdot V^\nu$.

And $U_\beta U^\beta = \eta_{\alpha\beta} U^\alpha U^\beta = -1$ because $\vec{U}$ is a timelike unit vector.

Therefore, $P^\alpha{}_\beta V_\perp^\beta = (\delta^\alpha{}_\nu + 2 U^\alpha U_\nu - \eta^{\alpha\mu} U_\mu U_\nu) \cdot V^\nu$.

The validity of $U^\alpha U_\nu = \eta^{\alpha\mu} U_\mu U_\nu$ is obviously (index raising).

$$\Longrightarrow \delta^\alpha{}_\nu + 2 U^\alpha U_\nu - \eta^{\alpha\mu} U_\mu U_\nu = \eta^\alpha{}_\nu + U^\alpha U_\nu .$$

Thus, for arbitrary vector $\vec{V}$, $P^\alpha{}_\beta V_\perp^\beta = (\eta^\alpha{}_\nu + U^\alpha U_\nu) \cdot V^\nu = V_\perp^\alpha$

QED.

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(c). Suppose $\vec{q}$ is an arbitrary contravector with $\eta_{\alpha\beta} q^\alpha q^\beta \neq 0$.

What we need to prove is that

$$\eta_{\gamma\mu} q^\gamma [\eta^\mu{}_\nu - q^\mu q_\nu/(q^\alpha q_\alpha)] V^\nu = 0 \text{ for arbitrary } \vec{V}.$$

While $\eta_{\gamma\mu} q^\gamma [\eta^\mu{}_\nu - q^\mu q_\nu/(q^\alpha q_\alpha)]$

$$= \eta_{\gamma\nu} q^\gamma - q_\nu$$

$$= 0.$$

The validity of the conclusion has been proved.

For a null vector $\vec{q}$, $q^\alpha q_\alpha = 0$, thus the expression $\eta_{\mu\nu} - q_\mu q_\nu/(q^\alpha q_\alpha)$ doesn't make sense.

This failure when it comes to null vectors directly relate to the definition of the projection tensor $P_{\mu\nu}$.

If we just change the definition to be $P_{\mu\nu} := \lambda(q^\alpha q_\alpha \eta_{\mu\nu} - q_\mu q_\nu)$ where λ can be an arbitrary constant scalar, then the orthogonality of $P^\mu{}_\nu V^\nu$ to $\vec{q}$ is obviously valid for arbitrary $\vec{V}$ and $\vec{q}$, no matter whether $\vec{q}$ is a null contravector or not.

(d). Suppose $\vec{V}$ and $\vec{W}$ are two arbitrary vectors perpendicular to $\vec{U}$, i.e. $\eta_{\alpha\beta} V^\alpha U^\beta = 0$ and $\eta_{\mu\nu} W^\mu U^\nu = 0$.

$$U_\mu U_\nu V^\mu W^\nu = (U_\mu V^\mu)\cdot(U_\nu W^\nu) = 0.$$

Hence $P_{\mu\nu} V^\mu W^\nu = (\eta_{\mu\nu} + U_\mu U_\nu) V^\mu W^\nu = \eta_{\mu\nu} V^\mu W^\nu = g(\vec{V}, \vec{W})$.

Furthermore, the expression $P_{\mu\nu} V^\mu W^\nu = \eta_{\alpha\beta} V^\alpha W^\beta$ is valid once either $\vec{V}$ or $\vec{W}$ is perpendicular to $\vec{U}$.