Quantum states and mode decomposition (Notes)

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Overview

These notes summarize how quantum states in quantum field theory are represented in Fock space, how field operators are mode-expanded, and how arbitrary states are expanded in the momentum eigenbasis.

1 Fock Space and Quantum States

Core idea: Fock space organizes states with varying particle number.

- **Vacuum state** $|0\rangle$: defined by

$$a_{\mathbf{p}} |0\rangle = 0, \quad \forall\, \mathbf{p}. \tag{1.1}$$

  Physical meaning: The vacuum contains no particles in any momentum mode.

- n**-particle momentum eigenstate**:

$$|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle = a^\dagger_{\mathbf{p}_1} \cdots a^\dagger_{\mathbf{p}_n} |0\rangle, \tag{1.2}$$

  satisfying

$$\hat{P}^\mu |\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle = \Big(\sum_i p_i^\mu\Big) |\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle. \tag{1.3}$$

Physical meaning: Each excitation adds a particle with momentum p_i, total four-momentum is sum of individual momenta.

2 Field Operator and Mode Decomposition

Core idea: The field operator creates or annihilates quanta at spacetime points, expanded in momentum modes.

For a free real scalar field $\hat{\phi}(x)$ in the Heisenberg picture:

$$\hat{\phi}(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \Big[a_{\mathbf{p}} e^{-i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})} + a^\dagger_{\mathbf{p}} e^{+i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})} \Big]. \tag{2.1}$$

$$[a_{\mathbf{p}},\, a^\dagger_{\mathbf{p}'}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}'). \tag{2.2}$$

Physical meaning: Each Fourier mode behaves as an independent harmonic oscillator; the field operator sums over these to create or annihilate particles at $(t, \mathbf{x})$.

3 Expansion of an Arbitrary State

Core idea: Any state in Fock space can be viewed as a superposition of n-particle momentum eigenstates.

$$|\Psi\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3} \; \Psi_n(\mathbf{p}_1, \ldots, \mathbf{p}_n) \; a^\dagger_{\mathbf{p}_1} \cdots a^\dagger_{\mathbf{p}_n} \, |0\rangle. \tag{3.1}$$

Coefficient functions:

$$\Psi_n(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \langle 0 \mid a_{\mathbf{p}_n} \cdots a_{\mathbf{p}_n} \mid \Psi \rangle. \tag{3.2}$$

Physical significance: $\Psi_n(\{\mathbf{p}_i\})$ is the n-particle wavefunction in momentum space, encoding the amplitude for finding particles with those momenta.

4 Computing the Coefficient Functions

Core idea: Projecting the state onto basis states extracts the coefficient wavefunctions.

Example: One-Particle States

$$|\Psi\rangle = \int \frac{d^3p'}{(2\pi)^3}\, f(\mathbf{p}')\, a^\dagger_{\mathbf{p}'}|0\rangle. \tag{4.1}$$

Project:

$$\Psi_1(\mathbf{p}) = \langle 0 \mid a_{\mathbf{p}}\,|\Psi\rangle = \int \frac{d^3p'}{(2\pi)^3}\, f(\mathbf{p}')\, \langle 0 \mid a_{\mathbf{p}} a^\dagger_{\mathbf{p}'} \mid 0 \rangle = f(\mathbf{p}). \tag{4.2}$$

Physical significance: The function $f(\mathbf{p})$ chosen in the state construction directly becomes the momentum-space wavefunction.

5 Position-Space Wavefunction

Core idea: Field operators can create localized excitations, giving position-space amplitudes.

Definition

$$\hat{\psi}^\dagger(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}}\, a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}. \tag{5.1}$$

Position eigenstate:

$$|\mathbf{x}\rangle = \hat{\psi}^\dagger(\mathbf{x})|0\rangle, \quad \langle \mathbf{x} \mid \mathbf{y}\rangle = \delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{5.2}$$

N-particle position wavefunction:

$$\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \langle 0 \mid \hat{\psi}(\mathbf{x}_1)\cdots\hat{\psi}(\mathbf{x}_n) \mid \Psi\rangle. \tag{5.3}$$

Physical meaning: $\Psi(\{\mathbf{x}_i\})$ encodes the joint probability amplitude for finding particles at spatial points.

6 Practical Steps

Core idea: Workflow for translating abstract states to usable wavefunctions.

1. Choose a basis (momentum, angular momentum, position, or other physical quantities) according to symmetry or measurement.
2. Apply corresponding annihilation or field operators to project $|\Psi\rangle$.
3. Extract coefficient functions (wavefunctions or wavefunctionals).

4. Use these in integrals to compute physical observables or correlation functions.