Notes in classical field theory.

Hamiltonian (density) $\mathcal{H}$
Lagrangian (density) $\mathcal{L}$.
satisfies: $H = \int d^3x\, \mathcal{H}$, $L = \int d^3x\, \mathcal{L}$.

☆ Hamiltonian $\underset{\longleftarrow}{\xrightarrow{\text{Legendre transform.}}}$ Lagrangian

$$\mathcal{L}[\phi, \partial_t\phi] = \pi[\phi, \partial_t\phi]\cdot\partial_t\phi - \mathcal{H}[\phi, \pi].$$

$$\mathcal{H}[\phi, \pi] = \pi\cdot\partial_t\phi[\phi, \pi] - \mathcal{L}[\phi, \partial_t\phi].$$

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}.$$

Hamiltonian $\mathcal{H} \longrightarrow$ energy $(K+V) \longrightarrow$ the 00 ~~compon~~ component of a tensor $\longrightarrow$ not Lorentz invariant.

Lagrangian is Lorentz invariant.

$\mathcal{L}$ = Kinetic term + Interaction.

Kinetic terms:

① Have two fields of the same / different type.

② Include time derivatives of the field.

e.g. $\frac{1}{2}\phi\Box\phi$, $\bar{\psi}\partial\!\!\!/\psi$, $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, ...

scalar field, spinor field, tensor field

Interactions:

Have three or more fields:

e.g. $\lambda\phi^3$ (scalar field),

$g\bar{\psi}A\!\!\!/\psi$ (spinor field),

$\cancel{g\partial_\mu\psi}$ $g\partial_\mu\phi A^\mu\phi^*$

$g^2 A_\mu^2 A_\nu^2$ (vector field)

$\frac{1}{M}\partial_\mu h^{\mu\nu}\partial_\nu h_{\alpha\beta}h^{\alpha\beta}$ (tensor field)

Classical Field Theory.
The Euler-Lagrange equations & Noether's theorem.

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Definition "action" $S \stackrel{\text{def}}{=\!=} \int d^4x \, \mathcal{L}$

(Notice: integrate in the whole Minkowski space.)

Principle of Least Action (in field theory):
Just like in Lagrangian mechanics,
① Derive functional derivative with variational principle.
② Let the functional derivative vanish to get a differential equation.
③ The solution of the differential equation (with given boundary conditions) is the required ϕ (under the given type of Lagrangian).

i.e. $\delta S = \int d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi + \boxed{\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right]} \right\}$

$= 0.$

obviously $\int d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right] = 0.$

$\Longrightarrow \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0.$

Example. For a free scalar field ϕ, it is given that $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$.

Use "EL" equation: $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -m^2 \phi - \partial_\mu \partial^\mu \phi = 0$

$\longrightarrow \Box \phi + m^2 \phi = 0.$ (the famous Klein-Gordon Equation).

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Some types of Lagrangian is invariant under a specific type of variation of ϕ (i.e. $\phi \to \phi + \delta\phi$).

This property of invariance is called a $\boxed{\text{symmetry}}$ of the Lagrangian.

continuous symmetry: In the definition of this type of symmetry, the variation of ϕ can be taken very small ($\delta\phi \to 0$) infinitesimal.

If the symmetry is continuous (i.e. the parameter α labeled the variation of ϕ is continuous),

$$\frac{\delta \mathcal{L}}{\delta \alpha} = 0 \quad \text{(symmetry of } \mathcal{L}\text{)}.$$

$$\Longrightarrow \sum_n \left[\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)}\right] \frac{\delta \phi_n}{\delta \alpha} + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}\right] = 0.$$

$$\Longrightarrow \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \cdot \frac{\delta \phi_n}{\delta \alpha}\right] = 0.$$

(Since all these variation of ϕ ($\phi(\alpha)(x)$) must satisfy Euler-Lagrangian equations, the first term vanishes).

Definition Noether current: $J_\mu \overset{\text{def}}{=} \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}$.

The above derivation means, (Noether's Theorem).

$\partial_\mu J_\mu = 0$ (conservation of Noether current)

$\Longleftrightarrow$ There exists a continuous symmetry of $\mathcal{L}$ labeled by parameter α when the action attain its minimum

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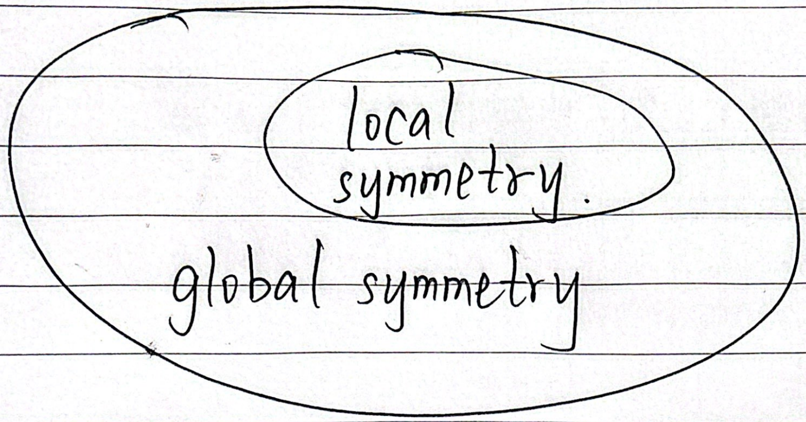
The Energy-Momentum Tensor $T_{\mu\nu}$.



The space-time translation invariance is a local symmetry in GR.

In QFT, the global space-time translation invariance (symmetry) of the action $S=\int d^4x\,\mathcal{L}$ leads to the conservation of four Noether currents:

$$T_{\mu\nu} = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - g_{\mu\nu}\mathcal{L}.$$

(actually the canonical energy-momentum tensor)

Proof. The global translation symmetry implies

$$\frac{\delta \phi}{\delta \xi^\nu} = \partial_\nu \phi \quad \text{for any different fields } \phi$$

$$\Longrightarrow \frac{\delta \mathcal{L}}{\delta \xi^\nu} = \partial_\nu \mathcal{L}, \quad \text{while } \frac{\delta \mathcal{L}}{\delta \xi^\nu} = \partial_\mu \Big(\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \xi^\nu} \Big)$$

from Euler-Lagrange equation

$$\Longrightarrow \partial_\nu \mathcal{L} = \partial_\mu \Big(\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n \Big)$$

$$\partial_\nu \mathcal{L} = g_{\mu\nu} \partial^\mu \mathcal{L}.$$

$$\longrightarrow \partial_\mu \Big(\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - g_{\mu\nu}\mathcal{L} \Big) = \partial_\mu T_{\mu\nu} = 0.$$

☒ A Typical Example of Calculation.

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Derive Coulomb's law from classical field theory.

Describe a charge e at the origin:

$$J_\mu = \begin{cases} J_0(x) = \rho(x) = e\delta^3(x) \\ J_i(x) = 0 . \end{cases}$$

while $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - A_\mu J_\mu$.

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - A_\mu J_\mu$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{1}{2}(\partial_\mu A_\mu)^2 - A_\mu J_\mu .$$

$$\Longrightarrow \frac{\partial \mathcal{L}}{\partial A_\nu} = -J_\nu , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial_\mu A_\nu + \frac{1}{2}\cdot 2(\partial_\alpha A_\alpha) \cdot g_{\beta\mu} g_{\gamma\nu} g_{\beta\gamma} .$$

(Euler – Lagrange equations)

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 .$$

$$\Longrightarrow -\partial_\mu(-\partial_\mu A_\nu) - \partial_\nu(\partial_\mu A_\mu) = J_\nu$$

$$\partial_\mu F_{\mu\nu} = J_\nu .$$

$$J_\nu = \Box A_\nu - \partial_\nu(\partial_\mu A_\mu) .$$

Lorentz gauge: $\partial_\mu A_\mu = 0$.

$$\Longrightarrow \Box A_\nu = J_\nu$$

$\xrightarrow{\text{solve}}$ $\begin{cases} A_i = 0, \\ A_0 = \frac{e}{\Box}\delta^3(x) \end{cases}$, d'Alembertian. $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$

$$A_0(x) = -\frac{e}{\Delta}\delta^3(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\cdot\frac{e}{\vec{k}^2}\cdot e^{i\vec{k}\cdot\vec{x}}$$

$$= \frac{e}{(2\pi)^3}\int_0^\infty k^2\,dk \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi\,\frac{1}{k^2}\cdot e^{ikr\cos\theta}.$$

$$= \frac{e}{8\pi^2}\cdot\frac{1}{ir}\int_{-\infty}^{\infty}\frac{e^{ikr}-e^{-ikr}}{k}\,dk$$

$$= \frac{e}{8\pi^2}\cdot\frac{1}{ir}\lim_{\delta\to 0}\int_{-\infty}^{\infty}\frac{e^{ikr}-e^{-ikr}}{k+i\delta}\cdot dk.$$

$$= \frac{e}{4\pi r}$$