The Real Klein-Gordon Field.

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Start with Classical Klein-Gordon Equation $(\Box + m^2)\phi = 0$ $\xrightarrow{\text{quantize it}}$ Quantized K-G Equation.

$\Downarrow$

d'Alembertian,

$$\Box = \partial^\mu \partial_\mu = \frac{d^2}{dt^2} - \vec{\nabla}^2$$

Commutation Relations:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y}),$$

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0.$$

Dirac delta function $\delta^{(3)}(\vec{x}-\vec{y}) = \begin{cases} +\infty, & \vec{x}=\vec{y} \\ 0, & \vec{x} \neq \vec{y}. \end{cases}$

while $\int_{-\infty}^{+\infty} d^3x\, \delta^{(3)}(\vec{x}) = 1.$

Expand the K-G Field to Fourier integral with respect to momentum $\vec{p}$:

$$\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p},t).$$

$$-\vec{\nabla}^2 \phi(\vec{x},t) = |\vec{p}|^2 \phi(\vec{p},t).$$

$$\Longrightarrow (\Box + m^2)\phi = 0 \longrightarrow \left(\frac{\partial^2}{\partial t^2} + |\vec{p}|^2 + m^2\right)\phi(\vec{p},t) = 0$$

equation of motion for a hamonic oscillator

with $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$

$\rightarrow H = \frac{1}{2}\vec{P}^2 + \frac{1}{2}\omega^2\phi^2 = \omega\cdot(a^\dagger a + \frac{1}{2})$

While $[\phi, P] = i \overset{\text{should}}{\Longleftrightarrow} [a, a^\dagger] = 1$

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$\Longrightarrow$ Spectrum should be described by eigenstates: $|n\rangle = (a^\dagger)^n |0\rangle$, eigenvalues: $(n+\frac{1}{2})\omega$.

For K-G Field,

$$\begin{cases} \phi = \frac{1}{\sqrt{2\omega}}(a+a^\dagger) \Longrightarrow \phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{1}{\sqrt{2\omega_p}}(a_p + a^\dagger_{-p}) e^{i\vec{p}\cdot\vec{x}} . \\ p = -i\cdot\sqrt{\frac{\omega}{2}}\cdot(a-a^\dagger) \Longrightarrow \pi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i)\cdot\sqrt{\frac{\omega_p}{2}} \cdot (a_p - a^\dagger_{-p}) \cdot e^{i\vec{p}\cdot\vec{x}} . \end{cases}$$

While $[\phi(\vec{x}), \pi(\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')$ is given,

$$\Longrightarrow [a_p, a^\dagger_{p'}] = (2\pi)^3 \delta^3(\vec{p}-\vec{p}') .$$

Derive the Hamiltonian: $H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(m\phi)^2\right]$ (in K-G Field).

$$\Rightarrow \cancel{\mathcal{H} = \frac{1}{2}\pi(\vec{x})^2}\ \mathcal{H} = \frac{1}{2}\pi(\vec{x})^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$= \int \frac{d^3\vec{p}\, d^3\vec{p}'}{(2\pi)^6} \left[-\frac{\sqrt{\omega_p \omega_{p'}}}{4} \cdot (a_p - a^\dagger_{-p})(a_{p'} - a^\dagger_{-p'}) + \frac{-\vec{p}\cdot\vec{p}' + m^2}{4\sqrt{\omega_p \omega_{p'}}} \cdot (a_p + a^\dagger_{-p}) \cdot (a_{p'} + a^\dagger_{-p'}) \right] \cdot e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{x})}$$

$$\Longrightarrow H = \int d^3x\, \mathcal{H} = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_p \cdot (a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p])$$

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Since $H = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger]\right)$

we have $\Longrightarrow [H, a_p^\dagger] = \omega_p a_p^\dagger$, $[H, a_p] = -\omega_p a_p$.

For each value of momentum $\vec{p}$,
its ground state: $|0\rangle$: s.t. $a_p|0\rangle = 0$.
with energy $E_{ground} = 0$.

Notice: The eigenstate $(a_{p_1}^\dagger)^{k_1}(a_{p_2}^\dagger)^{k_2}(a_{p_3}^\dagger)^{k_3}\cdots$ ~~$(a_{p_k}$~~ $(a_{p_n}^\dagger)^{k_n}|0\rangle$
describes the state where k_i particles ~~are~~ have momentum $\vec{p}_i$ $(i = 1, \cdots, n)$.

A single mode $\vec{p}$ can contain arbitrarily many particles. K-G particles (spin=0, neutral) obey Bose-Einstein statistics.

When ground state $|0\rangle$ is normalized (so that $\langle 0|0\rangle = 1$), since $|\vec{p}\rangle \propto a_p^\dagger|0\rangle$, in order to make $\langle p|q\rangle$ to be Lorentz invariant, define $|\vec{p}\rangle \overset{def}{=} \sqrt{2E_p}\, a_p^\dagger|0\rangle$.

(since the term $E_p\delta^3(\vec{p}-\vec{q})$ is Lorentz invariant while $\delta^3(\vec{p}-\vec{q})$ is not).

☆ Important fact: $\int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2E_p}$ is Lorentz invariant

$\Longrightarrow$ if $f(p)$ is Lorentz invariant $\Longrightarrow \int d^3p \cdot \frac{f(p)}{2E_p}$ is Lorentz invariant

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Summary: The method we just used to quantize a Klein-Gordon Field is called "canonical quantization". Since we use the Canonical Commutation Relations ($[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$ & $[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$) and commutation relation $[a_p, a^\dagger_{p'}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')$ to promote ϕ and π to operators ☆.

Notice that $[a_p, a^\dagger_{p'}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')$
$\Longleftrightarrow [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$.

Or in case of other quantities,
e.g. $[\phi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x}-\vec{y})$
$\Longleftrightarrow [a_{klm}, a^\dagger_{k'l'm'}] = \delta(k-k')\delta_{ll'}\delta_{mm'}$.

☆ The canonical quantization doesn't appear to be as axiomatic as path integral quantization, since we are simply imposing the canonical commutation relations when defining the operators, without a systematic justification for doing so.

※ The Klein-Gordon Propagator:

$$[\phi(\vec{x}), \phi(\vec{y})] = \langle 0|[\phi(\vec{x}), \phi(\vec{y})]|0\rangle = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2E_p} \cdot \left(e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\right).$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \cdot \frac{-1}{\vec{p}^2 - m^2} \cdot e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}$$

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Definition. $D_R(\vec{x}-\vec{y}) \overset{\text{def}}{=\!=} \theta(x^0-y^0)\langle 0|[\phi(\vec{x}),\phi(\vec{y})]|0\rangle.$

Since K-G equation is $(\Box+m^2)\phi=0$, we call the operator $\Box+m^2 = g_{\mu\nu}\partial^\mu\partial^\nu+m^2$ "Klein-Gordon ~~equ~~ operator".
(Just like $\Delta=\vec{\nabla}^2$ is Laplacian and $\Delta u=0$ is called Laplace equation.)

Apply the Klein-Gordon operator to $D_R(\vec{x}-\vec{y})$,

$$
\begin{aligned}
(\Box+m^2)D_R(\vec{x}-\vec{y}) &= \Box\theta(x^0-y^0)\langle 0|[\phi(\vec{x}),\phi(\vec{y})]|0\rangle \\
&+2\partial_\mu\theta(x^0-y^0)\partial^\mu\langle 0|[\phi(\vec{x}),\phi(\vec{y})]|0\rangle+\theta(x^0-y^0) \\
&\cdot(\Box+m^2)\langle 0|[\phi(\vec{x}),\phi(\vec{y})]|0\rangle = -i\delta^{(4)}(\vec{x}-\vec{y}).
\end{aligned}
$$

$\Longrightarrow$ $D_R(\vec{x}-\vec{y})$ is the retarded Green's function of K-G operator.

Definition. $D_F(\vec{x}-\vec{y}) \overset{\text{def}}{=\!=} \int \frac{d^4p}{(2\pi)^4}\frac{i}{\vec{p}^2-m^2+i\epsilon}e^{-i(\vec{p}\cdot\vec{x}-\vec{p}\cdot}$

$$
= \begin{cases} D(\vec{x}-\vec{y}), & x^0>y^0 \\ D(\vec{y}-\vec{x}), & x^0<y^0 \end{cases} = \langle 0|T\phi(\vec{x})\phi(\vec{y})|0\rangle
$$

$$
\begin{aligned}
(\Box+m^2)D_F(\vec{x}-\vec{y}) &= (\Box+m^2)\langle 0|T\phi(\vec{x})\phi(\vec{y})|0\rangle \\
&= -i\delta^{(4)}(\vec{x}-\vec{y}).
\end{aligned}
$$

$D_F(\vec{x}-\vec{y})$ is also a Green's function of the K-G operator.
$D_F(\vec{x}-\vec{y})$ is the Feynman propagator for a Klein-Gordon particle (spin = 0 & neutral).

If the K-G Field is coupled to an external field $j(\vec{x})$,

the field equation becomes $(\Box+m^2)\phi = j(\vec{x})$.

Lagrangian: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + j(\vec{x})\phi(\vec{x})$.

Before $j(\vec{x})$ is turned on $\phi_0 = \int \frac{d^3p}{(2\pi)^3}\cdot\frac{1}{\sqrt{2E_p}}\cdot(a_p^\dagger e^{i\vec{p}\cdot\vec{x}} + a_p\cdot e^{-i\vec{p}\cdot\vec{x}})$.

Then with an external source $j(\vec{x})$,

$$\phi(\vec{x}) = \phi_0(\vec{x}) + i\int d^4y\, D_R(\vec{x}-\vec{y})\, j(\vec{y})$$

$$= \phi_0(\vec{x}) + i\int d^4y \int \frac{d^3p}{(2\pi)^3}\cdot\frac{1}{2E_p}\cdot\theta(x^0-y^0)\cdot(e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{i\vec{p}\cdot(\vec{x}-\vec{y})})\cdot j(\vec{y})$$

$$= \int \frac{d^3p}{(2\pi)^3}\cdot\frac{1}{\sqrt{2E_p}}\cdot\left[e^{-i\vec{p}\cdot\vec{x}}\cdot\left(a_p + \frac{i}{\sqrt{2E_p}}\tilde{j}(\vec{p})\right) + e^{i\vec{p}\cdot\vec{x}}\cdot\left(a_p^\dagger - \frac{i}{\sqrt{2E_p}}\tilde{j}^*(\vec{p})\right)\right]$$

$$\Longrightarrow H = \int \frac{d^3p}{(2\pi)^3} E_p\left(a_p^\dagger - \frac{i}{\sqrt{2E_p}}\tilde{j}^*(\vec{p})\right)\left(a_p + \frac{i}{\sqrt{2E_p}}\tilde{j}(\vec{p})\right)$$