Renormalization Group Equations in Quantum Electrodynamics

A Detailed Technical Note

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1 Introduction

Quantum Electrodynamics (QED) is a perturbatively well-defined quantum field theory, yet its perturbative expansion necessarily produces ultraviolet (UV) divergences in loop amplitudes. These divergences arise from integrations over arbitrarily high virtual momenta in Feynman graphs and reflect the fact that QED, viewed as a continuum field theory, possesses no intrinsic physical cutoff. Renormalization is therefore required in order to define finite Green's functions and physical observables.

From a physical standpoint, renormalization reflects the fact that the parameters appearing in the bare Lagrangean (i.e. bare charge e_0, bare mass m_0, and the bare photon field normalization) are not directly measurable quantities. Instead, measurements probe effective parameters defined at a particular energy scale. The renormalization procedure systematically relates these scale-dependent effective parameters to the divergent bare parameters in such a way that all S-matrix elements remain finite.

The central observation is that divergences do not imply inconsistency. Rather, they indicate the need to reinterpret the parameters of the theory. The renormalization group (RG) then captures how these effective parameters evolve with changes of the renormalization scale. In QED, this scale dependence is physically meaningful that the running of the effective charge $e(\mu)$ describes the phenomenon of charge screening by virtual electron-positron pairs.

2 QED Lagrangean and Renormalization Setup

2.1 Bare and renormalized quantitties

We start with the bare Lagrangean:

$$\mathcal{L}_0 \;=\; -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \bar{\psi}_0\left(i\partial\!\!\!/ - m_0\right)\psi_0 - e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu}, \qquad (2.1)$$

where $\partial\!\!\!/ \;=\; \gamma^\mu\partial_\mu$.

Renormalization is implemented by expressing bare fields and parameters in terms of renormalized ones and their corresponding renormalization constants:

$$\psi_0 = Z_2^{1/2}\psi, \qquad (2.2)$$

$$A_{0\mu} = Z_3^{1/2}A_\mu, \qquad (2.3)$$

$$m_0 = Z_m m, \qquad (2.4)$$

$$e_0 = Z_e e. \qquad (2.5)$$

Gauge invariance implies the Ward identity

$$Z_1 \;=\; Z_2, \qquad (2.6)$$

where Z_1 is the vertex renormalization constant. This reduces the number of independent renormalization constants and guarantees charge renormalization is controlled entirely by the photon field renormalization

$$Z_e \;=\; Z_3^{-1/2}. \tag{2.7}$$

Thus, the renormalized QED Lagrangean becomes

$$\begin{aligned}\mathcal{L} \;=\; &-\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} \;+\; Z_2\bar{\psi}i\gamma^\mu\partial_\mu\psi \\ &-\; Z_m m\bar{\psi}\psi \;+\; Z_1 e\bar{\psi}\gamma^\mu\psi A_\mu.\end{aligned} \tag{2.8}$$

The renormalization constants Z_i are computed perturbatively by imposing renormalization conditions. In dimensional regularization with minimal subtraction, these renormalization constants take the form

$$Z_i \;=\; 1 + \Sigma_{k\geq 1}\frac{a_i^{(k)}(e)}{\epsilon^k}, \tag{2.9}$$

where $\epsilon \;=\; 4 - d$.

3 One-Loop Corrections in QED

3.1 Vacuum Polarization

The one-loop photon self-energy diagram (electron loop) contributes:

$$\Pi^{\mu\nu}(q) = \left(q^{\mu}q^{\nu} - q^{2}g^{\mu\nu}\right)\Pi\left(q^{2}\right), \tag{3.1}$$

where

$$\Pi\left(q^{2}\right) = \frac{e^{2}}{12\pi^{2}}\left(\frac{2}{\epsilon} + \ln\frac{\mu^{2}}{m^{2}} + \cdots\right). \tag{3.2}$$

Thus

$$Z_{3} = 1 - \frac{e^{2}}{6\pi^{2}}\frac{1}{\epsilon}. \tag{3.3}$$

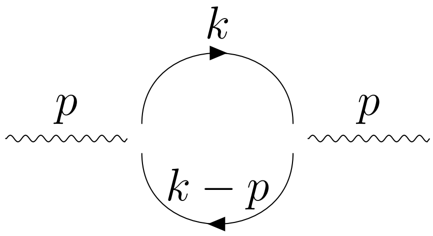

Figure: Vacuum polarization in QED.

The mathematical expression of the above Feynman diagram is

$$(-ie)^{2}\int\frac{\mathrm{d}^{4}k}{(2\pi)^{4}}\frac{i\left(2k^{\mu}-p^{\mu}\right)}{(k-p)^{2}-m^{2}+i\epsilon}\frac{i\left(2k^{\nu}-p^{\nu}\right)}{k^{2}-m^{2}+i\epsilon}. \tag{3.4}$$

3.2 Vertex correction

By Ward identity:

$$Z_1 = Z_2. \tag{3.5}$$

4 Renormalization Group and Running Coupling

4.1 General form of Callan–Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} - n_\psi\gamma_\psi - n_A\gamma_A\right)\Gamma = 0. \tag{4.1}$$

4.2 QED beta function

Since $Z_e = Z_3^{-1/2}$,

$$\begin{aligned}\beta(e) &= \mu\frac{\mathrm{d}e}{\mathrm{d}\mu} \\ &= \frac{e^3}{12\pi^2} + \mathcal{O}\left(e^5\right).\end{aligned} \tag{4.2}$$

4.3 Running coupling

Define $\alpha = \frac{e^2}{4\pi}$:

$$\alpha(\mu) \coloneqq \frac{\alpha(\mu_0)}{1 - \frac{\alpha(\mu_0)}{3\pi}\ln(\mu/\mu_0)}. \tag{4.3}$$

5 Anomalous Dimensions

$$
\begin{aligned}
\gamma_\psi &= \frac{e^2}{8\pi^2}, && (5.1)\\
\gamma_A &= -\frac{e^2}{6\pi^2}. && (5.2)
\end{aligned}
$$

6 Physical Interpretation

Vacuum polarization causes charge screening, making the effective electric charge scale-dependent. At higher energies, the electron cloud is probed more deeply, revealing a larger bare charge.

7 Landau Pole and Triviality

The running coupling diverges at the Landau pole:

$$
\mu_{\mathrm{LP}} = \mu_0 \exp\left(\frac{3\pi}{\alpha\left(\mu_0\right)}\right). \tag{7.1}
$$

This suggests QED is an effective field theory valid below extremely high scales.

8 Summary

We derived the one-loop beta function and anomalous dimensions in QED, obtained the running coupling, and discussed physical consequences such as charge screening and the Landau pole.