Renormalization Group Equations in Quantum Electrodynamics

A Detailed Technical Note

YILUN LIN

October 2025

Table of Contents

1 Introduction

Quantum Electrodynamics (QED) is a renormalizable quantum field theory describing the interaction of charged fermions (customarily $spin = \frac{1}{2}$) with the electromagnetic field. A key feature of QED is that its effective coupling depends on the renormalization scale due to vacuum polarization. The Renormalization Group Equation (REG) allows us to track how the renormalized coupling evolves with energy. In this note, we derive the REG for QED and discuss the physical implications of running couplings.

2 QED Lagrangean and Renormalization Setup

2.1 Bare and renormalized quantitties

We start with the bare Lagrangean:

$$\mathcal{L}_0 \;=\; -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \bar{\psi}_0\left(i\partial\!\!\!/ - m_0\right)\psi_0 - e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu}, \quad (2.1)$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$.

Introduce renormalization constants:

$$\psi_0 = Z_2^{1/2} \psi, \tag{2.2}$$

$$A_{0\mu} = Z_3^{1/2} A_\mu, \tag{2.3}$$

$$m_0 = Z_m m, \tag{2.4}$$

$$e_0 = Z_e e. \tag{2.5}$$

The renormalized Lagrangean can be written as:

$$\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{ct}}, \tag{2.6}$$

where the counter-term Lagrangean is:

$$\begin{aligned} \mathcal{L}_{\text{ct}} = & -\frac{1}{4} (Z_3 - 1) F_{\mu\nu} F^{\mu\nu} + (Z_2 - 1) \bar{\psi} i \partial\!\!\!/ \psi \\ & - (Z_m - 1) m \bar{\psi} \psi - (Z_1 - 1) e \bar{\psi} \psi \gamma^\mu A_\mu. \end{aligned} \tag{2.7}$$

The Ward identity implies:

$$Z_1 = Z_2. \tag{2.8}$$

3 One-Loop Corrections in QED

3.1 Vacuum Polarization

The one-loop photon self-energy diagram (electron loop) contributes:

$$\Pi^{\mu\nu}(q) = \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)\Pi\left(q^2\right), \tag{3.1}$$

where

$$\Pi\left(q^2\right) = \frac{e^2}{12\pi^2}\left(\frac{2}{\epsilon} + \ln\frac{\mu^2}{m^2} + \cdots\right). \tag{3.2}$$

Thus

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon}. \tag{3.3}$$

 Figure: Vacuum polarization in QED.

3.2 Vertex correction

By Ward identity:

$$Z_1 = Z_2. \tag{3.4}$$

4 Renormalization Group and Running Coupling

4.1 General form of Callan–Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\left(e\right)\frac{\partial}{\partial e} - n_\psi\gamma_\psi - n_A\gamma_A\right)\Gamma = 0. \tag{4.1}$$

4.2 QED beta function

Since $Z_e = Z_3^{-1/2}$,

$$\begin{aligned}\beta\left(e\right) &= \mu\frac{\mathrm{d}e}{\mathrm{d}\mu} \\ &= \frac{e^3}{12\pi^2} + \mathcal{O}\left(e^5\right).\end{aligned} \tag{4.2}$$

4.3 Running coupling

Define $\alpha = \frac{e^2}{4\pi}$:

$$\alpha\left(\mu\right) \coloneqq \frac{\alpha\left(\mu_0\right)}{1 - \frac{\alpha(\mu_0)}{3\pi}\ln\left(\mu/\mu_0\right)}. \tag{4.3}$$

5 Anomalous Dimensions

$$\gamma_\psi = \frac{e^2}{8\pi^2}, \tag{5.1}$$

$$\gamma_A = -\frac{e^2}{6\pi^2}. \tag{5.2}$$

6 Physical Interpretation

Vacuum polarization causes charge screening, making the effective electric charge scale-dependent. At higher energies, the electron cloud is probed more deeply, revealing a larger bare charge.

7 Landau Pole and Triviality

The running coupling diverges at the Landau pole:

$$\mu_{\mathrm{LP}} = \mu_0 \exp\left(\frac{3\pi}{\alpha\left(\mu_0\right)}\right). \tag{7.1}$$

This suggests QED is an effective field theory valid below extremely high scales.

8 Summary

We derived the one-loop beta function and anomalous dimensions in QED, obtained the running coupling, and discussed physical consequences such as charge screening and the Landau pole.