Complex Numbers

Problem 3
(a) Find the real and imaginary parts of $(\sqrt{3}-i)^{10}$ and $(\sqrt{3}-i)^{-7}$. For which values of n is $(\sqrt{3}-i)^n$ real?
(b) What is $\sqrt{i}$?

(a)

PART ONE: We can write $\sqrt{3}-i$ in the polar form: $2(cos\frac{\pi}{6} - isin\frac{\pi}{6})$. Then by De Moivre's Theorem, we can write $(\sqrt{3}-i)^{10}$ as $2^{10}(cos\frac{5\pi}{3} - isin\frac{5\pi}{3})$ and simplifies to $2^9 + 2^9\sqrt{3}i$ in which the real part is 2^9 and the imaginary part is $2^9\sqrt{3}i$.

PART TWO: By De Moivre's Theorem, we can write $(\sqrt{3}-i)^{-7}$ as $2^{-7}(cos\frac{-7\pi}{6} - isin\frac{-7\pi}{6})$ and simplifies to $-2^{-8} - 2^{-8}\sqrt{3}i$ in which the real part is 2^{-8} and the imaginary part is $2^{-8}\sqrt{3}i$.

PART THREE: To rewrite$(\sqrt{3}-i)^n$ we have $2^n(cos\frac{n\pi}{6} + isin\frac{n\pi}{6})$. If we want to the imaginary part to be gone, $sin\frac{n\pi}{6}$ must be 0, and subsequently $\frac{n\pi}{6}$ must be a multiple of π, Therefore, n must be a multiple of 6. ∎

(b) To rewrite the statement of the questions, we get to solve the following equation, in which the second root of unity are the solution to this question:

$$z^2 = i$$
$$z^2 = e^{\frac{\pi}{2}i}$$

Apparently $\alpha = e^{\frac{\pi}{4}i}$ is a second root of unity, and we have $(\alpha w)^2 = \alpha^2 = i$, so αw is also a second root of unity. In this case, $w = e^{\frac{2k\pi}{n}i} = e^{k\pi i}$ in which k = 0, 1. Now we plug in value of w and find the second root of unity $\alpha w = e^{\frac{\pi}{4}i}$ or $e^{\frac{5\pi}{4}i}$. ∎

Problem 5 Let z be a non-zero complex number. Prove that the three cube roots of z are the corners of an equilateral triangle in the Argand diagram.

We can write the problem as following:

$$x^3 = z$$
$$x^3 = re^{i\theta}$$

In which we let $re^{i\theta}$ denote an arbitrary complex number. Now we can solve this. First find $alpha$, which is an apparent third root of unity to this equation, and then we will find w, which will serve to encapsulate all third root of unity:

$$\begin{aligned}
\alpha &= r^{\frac{1}{3}}e^{\frac{1}{3}i\theta} \\
w &= e^{\frac{2k\pi}{n}i} = e^{\frac{2}{3}\pi i} \\
\alpha w^0 &= r^{\frac{1}{3}}e^{\frac{1}{3}\theta i} = r^{\frac{1}{3}}(cos\frac{1}{3}\theta + isin\frac{1}{3}\theta) \\
\alpha w^1 &= r^{\frac{1}{3}}e^{(\frac{1}{3}\theta+\frac{2}{3}\pi)i} = r^{\frac{1}{3}}(cos(\frac{1}{3}\theta+\frac{2}{3}\pi) + isin(\frac{1}{3}\theta+\frac{2}{3}\pi) \\
\alpha w^2 &= r^{\frac{1}{3}}e^{(\frac{1}{3}\theta+\frac{4}{3}\pi)i} = r^{\frac{1}{3}}(cos(\frac{1}{3}\theta+\frac{4}{3}\pi) + isin(\frac{1}{3}\theta+\frac{4}{3}\pi))
\end{aligned}$$

Now we have all three points. Do a distance check between all pairs of two. And for sanity's sake, we will compare the square of their distances:

$$\begin{aligned}
d^2_{\alpha w^0-\alpha w^2} &= r^{\frac{2}{3}}(-2cos\frac{1}{3}\theta cos(\frac{1}{3}\theta+\frac{4}{3}\pi)) = -2r^{\frac{2}{3}}cos(\frac{4\pi}{3}) \\
d^2_{\alpha w^1-\alpha w^2} &= r^{\frac{2}{3}}(-2cos(\frac{1}{3}\theta+\frac{2}{3}\pi)cos(\frac{1}{3}\theta+\frac{4}{3}\pi)) = -2r^{\frac{2}{3}}cos(-\frac{2\pi}{3}) \\
d^2_{\alpha w^0-\alpha w^1} &= r^{\frac{2}{3}}(-2cos\frac{1}{3}\theta cos(\frac{1}{3}\theta+\frac{2}{3}\pi)) = -2r^{\frac{2}{3}}cos(\frac{2\pi}{3})
\end{aligned}$$

Since we know that $cos(\frac{2\pi}{3}) = cos(\frac{-2\pi}{3}) = cos(\frac{4\pi}{3})$, we verify that the distances between three points are the same. Therefore, the three cube roots of z are the corners of an equilateral triangle in the Argand diagram. ∎

Problem 6 Express $\frac{1+i}{\sqrt{3}+i}$ in the form $x+iy$, where x,y $\in \mathbb{R}$. By writing each of 1+i and $\sqrt{3}+i$ in polar form, deduce that

$$cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}},\ sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

PART ONE:

$$\begin{aligned}
\frac{1+i}{\sqrt{3}+i} &= \frac{(1+i)(\sqrt{3}-i)}{4} \\
&= \frac{1+\sqrt{3}}{4} - i\frac{1-\sqrt{3}}{4}
\end{aligned}$$

PART TWO: We express the $\frac{1+i}{\sqrt{3}+i}$ in polar form:

$$\begin{aligned}
1+i &= \sqrt{2}(cos\frac{\pi}{4}+isin\frac{\pi}{4}) \\
\sqrt{3}+i &= 2(cos\frac{\pi}{6}+isin\frac{\pi}{6}) \\
\frac{1+i}{\sqrt{3}+i} &= \frac{\sqrt{2}(cos\frac{\pi}{4}+isin\frac{\pi}{4})}{2(cos\frac{\pi}{6}+isin\frac{\pi}{6})} \\
&= \frac{\sqrt{2}(cos\frac{\pi}{4}+isin\frac{\pi}{4})(cos\frac{\pi}{6}-isin\frac{\pi}{6})}{2(cos\frac{\pi}{6}+isin\frac{\pi}{6})(cos\frac{\pi}{6}-isin\frac{\pi}{6})} \\
&= \frac{\sqrt{2}}{2}(cos\frac{\pi}{4}cos\frac{\pi}{6}+sin\frac{\pi}{4}sin\frac{\pi}{6}+i(sin\frac{\pi}{4}cos\frac{\pi}{6}-cos\frac{\pi}{4}sin\frac{\pi}{6})) \\
&= \frac{\sqrt{2}}{2}(cos(\frac{\pi}{4}-\frac{\pi}{6})+isin(\frac{\pi}{4}-\frac{\pi}{6})) \\
&= \frac{\sqrt{2}}{2}cos\frac{\pi}{12}+\frac{\sqrt{2}}{2}isin\frac{\pi}{12}
\end{aligned}$$

Now we equate this to the form we obtained in PART ONE:

$$\frac{\sqrt{2}}{2}cos\frac{\pi}{12}+\frac{\sqrt{2}}{2}isin\frac{\pi}{12} = \frac{1+\sqrt{3}}{4} - i\frac{1-\sqrt{3}}{4}$$

We can now separately equate the real and imaginary parts:

$$\begin{aligned}
\frac{\sqrt{2}}{2}cos\frac{\pi}{12} &= \frac{1+\sqrt{3}}{4} \\
cos\frac{\pi}{12} &= \frac{1+\sqrt{3}}{2\sqrt{2}} \\
\frac{\sqrt{2}}{2}sin\frac{\pi}{12} &= -\frac{1-\sqrt{3}}{4} \\
sin\frac{\pi}{12} &= \frac{\sqrt{3}-1}{2\sqrt{2}}
\end{aligned}$$

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Problem 8 Find a formula for $cos4\theta$ in terms of $cos\theta$. Hence write down a quartic equation (i.e., an equation of degree 4) that has $cos\frac{\pi}{12}$ as a root. What are the other roots of your equation?

First we express $cos4\theta$ in terms of $cos\theta$, and we denote $cos\theta$ as c, $sin\theta$ as s:

$$\begin{aligned}
(cos\theta + isin\theta)^4 &= (c^2 + 2ics - s^2)^2 \\
cos4\theta + isin4\theta &= c^4 - 6c^2s^2 + 4ic^3s - 4ics^3 + s^4 \\
cos4\theta &= c^4 - 6c^2s^2 + s^4 \\
\frac{1}{2} &= x^4 - 6x^2(1 - x^2) + (1 - x^2)^2 \\
\frac{1}{2} &= 8x^4 - 8x^2 + 1
\end{aligned}$$

We know that $cos4\theta$ with $\theta = \pm\frac{\pi}{12} + \frac{2\pi k}{4}$ will yield the same result. So after eliminating the redundant ones, we are left with four distinct solutions:

$\theta = \boxed{\dfrac{\pi}{12}}, \boxed{-\dfrac{\pi}{12}}, \frac{\pi}{12} + \frac{2\pi}{4} = \boxed{\dfrac{7\pi}{12}}, -\frac{\pi}{12} + \frac{2\pi}{4} = \boxed{\dfrac{5\pi}{12}}$ ∎