# PHYS 150 TERM PAPER - YOUNG TABLEAUX AND ITS APPLICATIONS

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## 1. Introduction

brief discussion on young tableaux's significance

1.1. **Partition and Young Diagram.** First of all, let us consider a partition $\lambda$ on a set of $n$ elements, which can be written as an ordered set of numbers $\{\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 0\}$ and whose size $|\lambda| \equiv \sum_i \lambda_i = n$. To visualize this partition, we can use a Young diagram, a set of left justified cells with $k$ rows, each $i$th row with $\lambda_i$ cells, where $\lambda_i$ is weakly decreasing as we go down the rows. An example of the partition $\{4, 2, 2, 1\}$ is shown below in **Figure 1**.

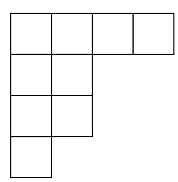

Figure 1. Young diagram for partition $\{4, 2, 2, 1\}$

A Young tableau is a Young diagram with numbers within each cell. A semi-standard Young tableau is one whose number in the cells are weakly decreasing to the right to the bottom. A semi-standard tableau of size $|\lambda| = n$ is considered standard if its filling is a bijective assignment from $\{1, 2, 3...n\}$. Example of both cases are shown below for partition $\{4, 2, 2, 1\}$.

| | | | |
|---|---|---|---|
| 1 | 2 | 2 | 3 |
| 3 | 5 | | |
| 5 | 7 | | |
| 9 | | | |

(A)

| | | | |
|---|---|---|---|
| 1 | 3 | 5 | 7 |
| 2 | 4 | | |
| 6 | 9 | | |
| 8 | | | |

(B)

Figure 2. (A). Semi-standard Young tableau for partition $\{4, 2, 2, 1\}$; (B). Standard Young tableau for partition $\{4, 2, 2, 1\}$ with $|\lambda| = n = 9$

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**Definition 1.1.** A hook $H(i,j)$ is a set of cells to the right or below the cell at $i$th row and $j$th column, plus the $(i,j)$ cell itself.

**Definition 1.2.** Hook length $h_\lambda(i,j)$ of a Young diagram of shape $\lambda$ is the number cells in the hook $H_\lambda(i,j)$.

| | | | |
|---|---|---|---|
| 7 | 5 | 2 | 1 |
| 4 | 2 | | |
| 3 | 1 | | |
| 1 | | | |

FIGURE 3. A tableaux of $\lambda = \{4, 2, 2, 1\}$ filled by each cell's hook length

With the hook length defined, we can find the total number of standard tableaux of a given shape $\lambda$. There are a total of $n!$ ways to place the $\{1, 2, 3...n\}$ in each cell. Since the tableau is standard, we know that for each possible filling, the number in cell $(i,j)$ cannot be permuted to the right or below, thereby decreasing the multiplicity by $h_\lambda(i,j)$, and we have arrived at the hook length formula, giving the total number of standard tableaux $d_\lambda$ in shape $\lambda$:

$$d_\lambda = \frac{|\lambda|!}{\prod_{i,j} h_\lambda(i,j)} \tag{1.1}$$

**Lemma 1.3.** *Let $f, g \in A(X)$ and let $E$, $F$ be cozero sets in $X$.*

1. *If $f$ is $E$-regular and $F \subseteq E$, then $f$ is $F$-regular.*
2. *If $f$ is $E$-regular and $F$-regular, then $f$ is $E \cup F$-regular.*
3. *If $f(x) \geq c > 0$ for all $x \in E$, then $f$ is $E$-regular.*

The following is an example of a proof.

*Proof.* Set $j(\nu) = \max(I \backslash a(\nu)) - 1$. Then we have

$$\sum_{i \notin a(\nu)} t_i \sim t_{j(\nu)+1} = \prod_{j=0}^{j(\nu)} (t_{j+1}/t_j).$$

Hence we have

$$\begin{aligned} \prod_\nu \Big( \sum_{i \notin a(\nu)} t_i \Big)^{|a(\nu-1)|-|a(\nu)|} &\sim \prod_\nu \prod_{j=0}^{j(\nu)} (t_{j+1}/t_j)^{|a(\nu-1)|-|a(\nu)|} \\ &= \prod_{j \geq 0} (t_{j+1}/t_j)^{\sum_{j(\nu) \geq j} (|a(\nu-1)|-|a(\nu)|)}. \end{aligned} \tag{1.2}$$

By definition, we have $a(\nu(j)) \supset c(j)$. Hence, $|c(j)| = n - j$ implies (5.4). If $c(j) \notin a$, $a(\nu(j))c(j)$ and hence we have (5.5). □

> This is an example of an 'extract'. The magnetization $M_0$ of the Ising model is related to the local state probability $P(a) : M_0 = P(1) - P(-1)$. The equivalences are shown in Table 1.

Table 1

| | $-\infty$ | $+\infty$ |
|---|---|---|
| $f_+(x,k)$ | $e^{\sqrt{-1}kx} + s_{12}(k)e^{-\sqrt{-1}kx}$ | $s_{11}(k)e^{\sqrt{-1}kx}$ |
| $f_-(x,k)$ | $s_{22}(k)e^{-\sqrt{-1}kx}$ | $e^{-\sqrt{-1}kx} + s_{21}(k)e^{\sqrt{-1}kx}$ |

**Definition 1.4.** This is an example of a 'definition' element. For $f \in A(X)$, we define

$$\mathcal{Z}(f) = \{E \in Z[X] : f \text{ is } E^c\text{-regular}\}. \tag{1.3}$$

*Remark* 1.5. This is an example of a 'remark' element. For $f \in A(X)$, we define

$$\mathcal{Z}(f) = \{E \in Z[X] : f \text{ is } E^c\text{-regular}\}. \tag{1.4}$$

**Example 1.6.** This is an example of an 'example' element. For $f \in A(X)$, we define

$$\mathcal{Z}(f) = \{E \in Z[X] : f \text{ is } E^c\text{-regular}\}. \tag{1.5}$$

**Exercise 1.7.** This is an example of the `xca` environment. This environment is used for exercises which occur within a section.

The following is an example of a numbered list.

FIGURE 4. This is an example of a figure caption with text.

FIGURE 5

(1) First item. In the case where in $G$ there is a sequence of subgroups

$$G = G_0, G_1, G_2, \ldots, G_k = e$$

such that each is an invariant subgroup of $G_i$.

(2) Second item. Its action on an arbitrary element $X = \lambda^\alpha X_\alpha$ has the form

$$[e^\alpha X_\alpha, X] = e^\alpha \lambda^\beta [X_\alpha X_\beta] = e^\alpha c^\gamma_{\alpha\beta} \lambda^\beta X_\gamma, \tag{1.6}$$

  (a) First subitem.

$$-2\psi_2(e) = c^\delta_{\alpha\gamma} c^\gamma_{\beta\delta} e^\alpha e^\beta.$$

  (b) Second subitem.
    (i) First subsubitem. In the case where in $G$ there is a sequence of subgroups

$$G = G_0, G_1, G_2, \ldots, G_k = e$$

such that each subgroup $G_{i+1}$ is an invariant subgroup of $G_i$ and each quotient group $G_{i+1}/G_i$ is abelian, the group $G$ is called *solvable*.

    (ii) Second subsubitem.
  (c) Third subitem.

(3) Third item.

Here is an example of a cite. See [**?**].

**Theorem 1.8.** *This is an example of a theorem.*

**Theorem 1.9** (Marcus Theorem)**.** *This is an example of a theorem with a parenthetical note in the heading.*

## 2. Some more list types

This is an example of a bulleted list.

- $\mathcal{J}_g$ of dimension $3g-3$;
- $\mathcal{E}_g^2 = \{$Pryms of double covers of $C = \square$ with normalization of $C$ hyperelliptic of genus $g-1\}$ of dimension $2g$;
- $\mathcal{E}_{1,g-1}^2 = \{$Pryms of double covers of $C = \square_{P^1}^H$ with $H$ hyperelliptic of genus $g-2\}$ of dimension $2g-1$;
- $\mathcal{P}_{t,g-t}^2$ for $2 \leq t \leq g/2 = \{$Pryms of double covers of $C = \square_{C''}^{C'}$ with $g(C') = t-1$ and $g(C'') = g-t-1\}$ of dimension $3g-4$.

This is an example of a 'description' list.

**Zero case:** $\rho(\Phi) = \{0\}$.

**Rational case:** $\rho(\Phi) \neq \{0\}$ and $\rho(\Phi)$ is contained in a line through 0 with rational slope.

**Irrational case:** $\rho(\Phi) \neq \{0\}$ and $\rho(\Phi)$ is contained in a line through 0 with irrational slope.

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