1. Let $\theta = \int_0^1 cos(\pi x/2)dx$.
a. Find a Monte Carlo estimate $\hat{\theta}$ of θ along with its standard error using $m = 1000$ random draws by treating it as an expectation with respect to a uniform distribution.
b. Construct a stratified Monte Carlo estimate of θ using a total of $m = 1000$ random draws and 4 strata of equal width.
c. Use some form of numerical integration to approximate θ.
d. Give an upper bound for the error of the approximation of θ in part c.

2. Once again, consider $\theta = \int_0^1 cos(\pi x/2)dx$.
a. Construct an importance sampling estimate θ^* of θ using $m = 1000$ draws with importance function $\phi(x) = 3(1 - x^2)/2$.
b. Plot $\phi(x)$ and $cos(\pi x/2)$ over the interval $(0, 1)$ and comment on how they compare.

3. Repeat (1a) 100 times to get 100 observations of $\hat{\theta}$ and repeat (2a) 100 times to get 100 observations of θ^*. Compare the sample variances of the two samples.

4. Recall that $X = (X_1, X_2, ..., X_k)^T$ has a multivariate normal distribution with mean vector μ and variance-covariance matrix Σ when its pdf

$$f(x_1, x_2, ..., x_k) = (2\pi)^{-k/2}|\Sigma|^{-1/2}e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}.$$

a. Write an R function for the case when $k = 2$ to construct a Monte Carlo estimate of the integral

$$\theta = \int_c^d \int_a^b f(x_1, x_2)dx_1dx_2$$

by using Monte Carlo integration with respect to a uniform distribution in two dimensions. The function should take as arguments $(a, b, c, d, \mu, \Sigma, m)$, where m is the number of random draws.
b. Compute the estimator when $a = 0$, $b = 1$, $c = 1$, $d = 1$, $\mu_1 = \mu_2 = 0$, $\Sigma = I$, and $m = 10000$ and compare to the true value of θ
c. Now construct another function to evaluate θ, but this time directly using random draws from $f(x_1, x_2)$ and integrating the appropriate indicator function. You may use the mvtnorm package for this.
Hint($\theta = E_f[I(a < X_1 < b, c < X_2 < d)]$)
d. Compute the estimator from part c using the same paramaters as in part b.

5. Rizzo problem 5.2
6. Rizzo problem 5.3
7. Rizzo problem 5.4
8. Rizzo problem 5.14

Bonus: Repeat problem 2a, but using a numerical estimate of the integral rather than a Monte Carlo estimate.