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General Relativity & Cosmology

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Abstract: GR part: Robin Zegers

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1 General Relativity

1.1 SR Preliminaries

1.1.1 Affine Space

SR and classical mechanics assume the homogenity of spacetime, i.e. it's an affine space:

Definition 1.1 (Affine Space) *An affine space is a triplet* (E, V, f) *s.t.*

- E *is a set.*
- V *is a real vector space.*
- $f : E \times E \to V$ *is a map satisfying:*
  - *Charle's rule:* $f(A, B) + f(B, C) = f(A, C)$.
  - $\forall A \in E, v \in V$, $\exists! B \in E$ *s.t.* $f(A, B) = v$.

The dimision of the affine space is definede to be $\dim V$.

Example 1.1 (The Physical Spacetime) *Affine space* $(\mathbb{R}^4, \mathbb{R}^4, f)$, *where* $f(A, B) \equiv B - A$.

Definition 1.2 (Frame) *A frame in the affine space* (E, V, f) *is a tuple* (O, B), *where* $O \in E$ *is the origin of the frame, and* $B = \{e_\mu; \mu = 0, \cdots, \dim V - 1\}$ *is a set of basis of* V.

For any given frame (O,B) in the affine space (E,V,f), $\forall M\in E$, we alyways have

$$OM \equiv f(O,M) = x^\mu(M)e_\mu. \tag{1.1}$$

Hence, we can define the coordinate map $\phi : E \to \mathbb{R}^4$ so that $\phi : M \mapsto \phi(M) \equiv \{x^\mu(M)\} \in \mathbb{R}^{\dim V}$. Such a coordinate map is a bijection and is actually a homeomorphism. Moreover, frames of (E,V,f) are in one-to-one correspondence with a global coordinate system over E.

Suppose we have two frames (O,B) and $(O', B' \equiv \{e'_\mu\})$, we cam also decompose

$$O'M = x'^{\mu'}(M)e'_{\mu'}. \tag{1.2}$$

Since $e'_{\mu'}$ is a linear combination of the original e_μ, we shall denote

$$e'_{\mu'} = \Lambda_{\mu'}{}^{\mu}e_\mu, \quad \Lambda_{\mu'}{}^{\mu} \in \mathsf{GL}(4;\mathbb{R}), \tag{1.3}$$

we find

$$O'M = x'^{\mu'}(M)\Lambda_{\mu'}{}^{\mu}e_\mu. \tag{1.4}$$

Since $O'M = O'O + OM$, and denote

$$O'O = -a^\mu e_\mu, \quad a^\mu \in \mathbb{R}^4. \tag{1.5}$$

hence

$$OM = x^\mu(M)e_\mu = -O'O + OM = [a^\mu + x'^{\mu'}(M)\Lambda_{\mu'}{}^{\mu}]e_\mu, \tag{1.6}$$

$$x^\mu(M) = a^\mu + x'^{\mu'}(M)\Lambda_{\mu'}{}^{\mu}. \tag{1.7}$$

This transformation is named **affine transformation**. These transformations forms the **affine group** $\mathsf{IGL}(4;\mathbb{R}) \cong \mathbb{R}^4 \rtimes \mathsf{GL}(4;\mathbb{R})$.

The history of a point particle in (E,V,f) is a curve $\gamma : \mathbb{R} \to E$, named its **worldline**. The particle has a motion of rectilinear uniform translation iff its worldline is a straight line in E. Hence, an inertial (Galilean) frame can be identified as a frame where an isolated particle has a straight line as its worldline. Since affine transformations are the only transformations of E sending any straight line to a straight line , hence the transformations between inertial frames can be identified with a class of frames of E and a subgroup of $\mathsf{IGL}(4;\mathbb{R})$.

1.1.2 Galilean v.s. Einstein Relativity

Definition 1.3 (Principle of Relativity) *No experiment can distinguish between two Galilean frames, i.e. the laws of physics have the same form in all Galilean frames.*

In classical mechanics, we assume all inertial observers are simultaneous. Hence, the symmetry group of it is a subgroup of $\mathsf{IGL}(4;\mathbb{R})$ which preserves the time interval, i.e. the Galilean group. Galilean group giving transformations between Galilean frames preserves an absolute time used to define the classical velocity. However, the Maxwell equations is not covariant under the Galilean group.

The building block of SR amending the covariance of electromagnetism is the causality principle:

Definition 1.4 (Causality Principle) *In any Galilean frame, $\exists$ a maximal speed for transport of matter, energy, and information.*

Combine it with the principle of relativity, the maximal speed should be unique. Experimentally, the maximal speed is the speed of light c.

Now we have to find the right subgroup of the affine group preserving these properties. Let's consider photons emitted at $t = 0$ from the origin O, then they generates a lightcone defined by

$$0 = -c^2t^2 + \boldsymbol{x}^2. \tag{1.8}$$

We can define a bilinear form $\eta : V \times V \to \mathbb{R}$ so that

$$\eta_{\mu\nu} \equiv \eta(e_\mu, e_\nu) = \mathrm{diag}(-1, 1, 1, 1), \tag{1.9}$$

hence a lightcone can be charactered as

$$\eta(X, X) = 0, \quad X = x^\mu e_\mu, \ \{x^\mu\} \equiv (ct, \boldsymbol{x}). \tag{1.10}$$

This lightcone shall have the same shape in different Galilean frames. Let's consider another frame (O, B') with the same origin, then we have

$$\eta'_{\mu'\nu'} x'^{\mu'} x'^{\nu'} = \eta'_{\mu'\nu'} \Lambda_\mu{}^{\mu'} \Lambda_\nu{}^{\nu'} x^\mu x^\nu = 0. \tag{1.11}$$

Hence we shall demand

$$\eta_{\mu'\nu'} = \Lambda_{\mu'}{}^\mu \Lambda_{\nu'}{}^\nu \eta_{\mu\nu}. \tag{1.12}$$

To ensure the casuality principle, the suffices and necessary to take $\Lambda_{\mu'}{}^\mu$ in $\mathsf{O}(1,3) \subset \mathsf{GL}(4;\mathbb{R})$, which is the Lorentz group.

A change of Galilean frames (Lorentz transform) should be connectable to the identity, hence we only need the connected component of $\mathsf{O}(1,3)$, which is the special orthochronous Lorentz group $\mathsf{SO}^+(1,3)$, where

$$\det \Lambda = 1, \quad \Lambda_0^0 \geq 1. \tag{1.13}$$

As a remark, $\mathsf{O}(1,3) \cong \mathsf{SO}^+(1,3) \cup \mathcal{P}\mathsf{SO}^+(1,3) \cup \mathcal{T}\mathsf{SO}^+(1,3) \cup \mathcal{T}\mathcal{P}\mathsf{SO}^+(1,3)$.

We conclude that SR is defined on 4D Minkowskian spacetime $M_4 \equiv (E, V, f, \eta)$, where η is the Minkowskian metric, and tranformations betweeen Galilean frames are through $\mathbb{R}^4 \rtimes \mathsf{SO}^+(1,3) \equiv \mathsf{ISO}^+(1,3)$.

1.1.3 Tensors and Lorentz Invariance

The relativity principle demands laws of physics transforms according to representations of the Poincaré group $\mathsf{ISO}(1,3)$. Now we restrict ourselves to the rotation part $\mathsf{SO}^+(1,3)$.

Denote V^* the dual of V. Let $\{e_\mu\}$ be a basis in V, the corresponding dual basis on V^* is denoted as $\{e^\mu\}$ s.t.

$$e^\mu(e_\nu) = \delta^\mu_\nu. \tag{1.14}$$

Suppose $\{e'_{\mu'}\}$ is another basis on V and $\{e'^{\mu'}\}$ its dual, and $e_{\mu'} = \Lambda_{\mu'}{}^{\mu} e_\mu$, then

$$e'^{\mu'} = \Lambda^{\mu'}{}_{\mu} e^\mu. \tag{1.15}$$

It can be immediately deduced from the definition of the dual basis: Assume $e'^{\mu'} = \tilde{\Lambda}^{\mu'}_{\nu} e^\nu$, then

$$e'^{\mu'}(\Lambda_{\nu'}{}^{\nu} e_\nu) = \Lambda_{\nu'}{}^{\nu} \tilde{\Lambda}^{\mu'}_{\mu} e^\mu(e_\nu) = \Lambda_{\nu'}{}^{\mu} \tilde{\Lambda}^{\mu'}_{\mu} = \delta^{\mu'}_{\nu'}, \tag{1.16}$$

hence

$$\tilde{\Lambda}^{\mu'}_{\nu} = (\Lambda_{\mu'}{}^{\nu})^{-1} \equiv \Lambda^{\mu'}{}_{\nu}. \tag{1.17}$$

$\forall r, s \in \mathbb{N}$, we can define the tensor productede space as

$$V^{(r,s)} = V^{\otimes r} \otimes V^{*\otimes s}. \tag{1.18}$$

Elements in $V^{(r,s)}$ are called (r,s)-tensors. For any basis $\{e_\mu\}$ for V and the dual $\{e^\mu\}$ of V^*, $V^{(r,s)}$ is spanned by $\{e_{\mu_1} \otimes \cdots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \cdots \otimes e^{\nu_s}\}$, hence an (r,s)-tensor T can be expanded as

$$T = T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} e_{\mu_1} \otimes \cdots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \cdots \otimes e^{\nu_s}. \tag{1.19}$$

Given another basis $\{e'_{\mu'}\}$ and $\{e'^{\nu'}\}$ and transformation $\Lambda_{\mu'}{}^{\mu}$, we have

$$T'^{\mu'_1 \cdots \mu'_r}{}_{\nu'_1 \cdots \nu'_s} = \Lambda^{\mu'_1}{}_{\mu_1} \cdots \Lambda^{\mu'_r}{}_{\mu_r} \Lambda_{\nu'_1}{}^{\nu_1} \cdots \Lambda_{\nu'_s}{}^{\nu_s} T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}. \tag{1.20}$$

A tensorial expression for a physical law automatically inherits a manifested Lorentz covariance, hence confers with the relativity principle.

Remark

- It suffices but not necessary to use this tensorial rep for $\mathsf{SO}^+(1,3)$, since it's not simply connected. Sometimes we may need non-trivial projective representations like spinorial reps.
- Covairant tensorial expression essentially has no difference from vectorial expressions in classical mechanics. Here, we just replaced the Galilean group by $\mathsf{SO}^+(1,3)$.
- Everything here is not constrained to the Lorentz group, we can generalize to $\forall \Lambda \in \mathsf{GL}(4;\mathbb{R})$.

1.1.4 Indexology

Definition 1.5 (Musical Symbols) *The Minkowskian metric η is non-degenerate, hence it sets up a natural isomorphism $V \leftrightarrow V^*$, called the musical isomorphism $\flat : V \to V^*$, $\flat : v \mapsto v^\flat(\cdot) \equiv \eta(v, \cdot)$. The reciprocal map is denoted as $\sharp : V^* \to V$, $(\theta^\sharp)^\flat = \theta$.*

Denote $\{e_\mu\}$ a basis for V while $\{e^\mu\}$ for V^*. Let $v = v^\mu e_\mu$, then

$$v^\sharp(e_\nu) = v^\flat_\mu e^\mu(e_\nu) = v^\flat_\nu. \tag{1.21}$$

However, meanwhile we have

$$v^\sharp(e_\nu) = \eta(v, e_\nu) = v^\mu \eta_{\mu\nu}, \tag{1.22}$$

hence

$$v^\flat_\mu = \eta_{\mu\nu} v^\nu. \tag{1.23}$$

Similarly, for $\theta = \theta_\mu e^\mu$, then

$$(\theta^\sharp)^\flat(e_\nu) = \eta(\theta^\sharp, e_\nu). \tag{1.24}$$

Meanwhile,

$$(\theta^\sharp)^\flat(e_\nu) = \theta(e_\nu) = \theta_\nu, \tag{1.25}$$

hence

$$\theta_\nu = \theta^{\sharp\mu} \eta_{\mu\nu}, \tag{1.26}$$

and we find

$$\theta^{\sharp\mu} = \eta^{\mu\nu} \theta_\nu, \tag{1.27}$$

where $\theta^{\mu\nu'} \eta_{\nu'\nu} = \delta^\mu_\nu$.

The advantage of this musical symbol convention is that it labels the natural status for a vector/covector.

Remark

- From now on, we only use (V, η) to denote a spacetime. Such notation works for any symmetric bilinear form g (the metric tensor) other than η.
- Such a notation also works in Euclidean classical mechanics, but here $\eta_{\mu\nu}$ is replaced by $\delta_{\mu\nu}$, hence it's trivial and not necessary to distinguish covariant and contravariant indices.
- Musical symbols can also be applied to tensors.

Definition 1.6 (Contraction) $\forall r, s \in \mathbb{N}$, $\forall 1 \le m \le r, 1 \le n \le s$, *the contraction*

$\langle \cdot \rangle_{(m,n)} : V^{(r,s)} \to V^{(r-1,s-1)}$ *is defined by* $\forall T \in V^{(r,s)}$ *s.t.*

$$\begin{aligned}\langle T \rangle_{(m,n)} &= \langle T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} e_{\mu_1} \otimes \cdots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \cdots \otimes e^{\nu_s} \rangle_{(m,n)} \\ &= T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} \langle e_{\mu_1} \otimes \cdots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \cdots \otimes e^{\nu_s} \rangle_{(m,n)} \\ &= T^{\mu_1\cdots\mu_{m-1}\mu\mu_{m+1}\cdots\mu_r}{}_{\nu_1\cdots\nu_{n-1}\mu\nu_{n+1}\cdots\nu_s} \\ &\qquad \times e_{\mu_1} \otimes \cdots \otimes e_{\mu_{m-1}} \otimes e_{\mu_{m+1}} \otimes \cdots e_{\mu_r} \otimes e^{\nu_1} \otimes \cdots \otimes e^{\nu_{n-1}} \otimes e^{\nu_{n+1}} \otimes \cdots \otimes e^{\nu_s}.\end{aligned} \tag{1.28}$$

Definition 1.7 (Differentiation for a Function) *Let* $f : E \to \mathbb{R}$ *a* C^1 *function. Define* $\mathrm{d}f : E \to V^*$ *in any frame* $(O, \{e_\mu\})$ *with dual basis* $\{e^\mu\}$ *as*

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu} e^\mu = (\partial_\mu f) e^\mu, \tag{1.29}$$

where x^μ *is given by the coordinate map* ϕ.

$\mathrm{d}f$ is coordinate independent. Suppose we have another frame $(O', \{e'_{\mu'}\})$, then we have

$$\mathrm{d}f = \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial f}{\partial x'^{\mu'}} e^\mu = \frac{\partial f}{\partial x'^{\mu'}} \Lambda^{\mu'}{}_\mu e^\mu = \frac{\partial f}{\partial x'^{\mu'}} e'^{\mu'}. \tag{1.30}$$

Definition 1.8 (Differentiation for a Tensor) *Given a* (r, s)*-tensorial function* $T : E \to V^{(r,s)}$, *we define*

$$\mathrm{d}T = (\partial_\mu T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}) e_{\mu_1} \otimes \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \cdots \otimes e^{\nu_s} \otimes e^\mu, \tag{1.31}$$

which is an $(r, s+1)$*-tensor field. As a short-hand, we denote*

$$\partial_\mu T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} \equiv T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s,\mu}. \tag{1.32}$$

Remark

The definition for differentiation relies on E. Such definition is invalid in GR, covariant derivatives will be introduced.

1.1.5 A Free Point Particle

Let's denote curve $\gamma : \mathbb{R} \to E$ the worldline of a free point particle. Given a frame $(O, \{e_\mu\})$ of Mink_4, we have a global coordinate system $\phi = (x^0, \cdots, x^3)$ so that

$$\phi \circ \gamma = (x^0 \circ \gamma, \cdots, x^3 \circ \gamma) = (X^0, \cdots, X^3) \in \mathbb{R}^4. \tag{1.33}$$

We shall assume γ is C^1, then $\forall s \in \mathbb{R}$, we can define

$$\dot\gamma(s) = \frac{\mathrm{d}}{\mathrm{d}s}(x^\mu \circ \gamma) e_\mu \in V. \tag{1.34}$$

Under a change of frame, we have

$$\dot{\gamma}(s) = \Lambda^{\mu}{}_{\mu'}\Lambda_{\mu}{}^{\nu'}\frac{\mathrm{d}}{\mathrm{d}s}(x^{\mu'} \circ \gamma)e_{\nu'} = \frac{\mathrm{d}}{\mathrm{d}s}(x^{\mu'} \circ \gamma)e_{\mu'}, \tag{1.35}$$

it's again coordinate invariant.

Definition 1.9 (Future Orientation) *We say γ points to the future (future oriented) iff*

$$\frac{\mathrm{d}}{\mathrm{d}s}(x^0 \circ \gamma) > 0. \tag{1.36}$$

Since Lorentz transformation takes in $\mathsf{SO}^+(1,3)$, it's a well-defined notion.

For a future oriented worldline, $x^0 \circ \gamma$ is monotonically increasing, hence is a bijection $\mathbb{R} \to \mathbb{R}$, and we can re-parameterize γ by $x^0 \circ \gamma$.

The line element for the worldline writes

$$\begin{aligned}\eta(\dot{\gamma}(s), \dot{\gamma}(s)) = \eta_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}s}(x^{\mu} \circ \gamma)\frac{\mathrm{d}}{\mathrm{d}s}(x^{\nu} \circ \gamma) = -c^2\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s}\right)^2 \\ = -c^2\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2\left(1 - \frac{\dot{\boldsymbol{x}}^2}{c^2}\right).\end{aligned} \tag{1.37}$$

A causal particle's worldline have $|\dot{\boldsymbol{x}}| \leq c$, hence $\eta(\dot{\gamma}(s), \dot{\gamma}(s)) \leq 0$.

Definition 1.10 (Proper Time) *We shall say γ is parameterized by the proper time τ if $\forall \tau \in \mathbb{R}$, $\eta(\dot{\gamma}(\tau), \dot{\gamma}(\tau)) = -1$. Locally, the particle's propertime writes*

$$\mathrm{d}\tau = c\mathrm{d}t\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}. \tag{1.38}$$

τ is the coordinate time in the frame where the particle is at rest.

Definition 1.11 (Pseudo-length) *Given a causal curve $\gamma : \mathbb{R} \to E$ and two point $M, N \in E$, $M \leq N \in \gamma(\mathbb{R})$, then the pseudo-length $\ell(\gamma; M, N)$ between M and N is defined as*

$$\ell(\gamma; M, N) = \int_{s_M}^{s_N} \mathrm{d}s\sqrt{-\eta(\dot{\gamma}(s), \dot{\gamma}(s))} \tag{1.39}$$

Propsition 1.1 *The pseudo-length satisfies the following properties:*

1. *$\ell(\gamma; M, N)$ is additive: For $M \leq N \leq P \in \gamma$,*

$$\ell(\gamma; M, P) = \ell(\gamma; M, N) + \ell(\gamma; N, P). \tag{1.40}$$

2. *$\ell(\gamma; M, N)$ is independent of the chosen frame.*

3. *$\ell(\gamma; M, N)$ is re-parametrization invariant.*

4. For a proper-time-parametrized worldline, we have

$$\ell(\gamma; M, N) = \int_{\tau_M}^{\tau_N} \mathrm{d}\tau. \tag{1.41}$$

Hence the pseudo-length encodes the elapsed time along the worldline.

Suppose in some frame

$$\ell(\gamma; M, N) = \int_{s_M}^{s_N} \mathrm{d}s \sqrt{-\eta_{\mu\nu}\dot{X}^\mu(s)\dot{X}^\nu(s)}, \tag{1.42}$$

then the worldline with the shortest pseudo-length is given by variation, and we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{-\eta_{\mu\nu}\dot{X}^\nu(s)}{\sqrt{-\eta_{\mu\nu}\dot{X}^\mu(s)\dot{X}^\nu(s)}}\right) = 0. \tag{1.43}$$

It directly implies that

$$\frac{\mathrm{d}^2 X^\mu}{\mathrm{d}\tau^2} = 0. \tag{1.44}$$

A Newton-like equation of motion is recovered, and it exactly implies a straight worldline! Conversely, in Mink_4, straight worldlines correspond to motions of rectilinear uniform translation.

Intuitively, we should also obtain the equation of motion for a particle from the extreme of its action, hence we expect its action is related to its worldline length. It turns out a proper definition for it is

$$S_0[\gamma] = -mc \int \mathrm{d}s \sqrt{-\eta(\dot{\gamma}(s), \dot{\gamma}(s))} = -mc^2 \int \mathrm{d}t \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}. \tag{1.45}$$

Take the non-relativistic limit where $|\boldsymbol{v}|/c \ll 1$, the classical action for a free particle is recovered.

Notice that a photon's geodesic is null, and the previous definition for worldline length is ill-defined. In this case, we shift the definition to

$$\mathscr{E} = \int_{s_M}^{s_N} \mathrm{d}s\ \eta(\dot{\gamma}(s), \dot{\gamma}(s)). \tag{1.46}$$

Variations on it produces a null curve.

1.1.6 Covariant Formulation for Electrodynamics

Consider a massive point particle with mas m and electric charge q in an electromagnetic field described by a scalar and vector potential $(\phi, \boldsymbol{A})$. Then its action reads

$$S[\gamma; \phi, \boldsymbol{A}] = S_0[\gamma] + S_{\mathrm{int}}[\gamma; \phi, \boldsymbol{A}], \tag{1.47}$$

where the interaction part reads

$$S_{\text{int}}[\gamma; \phi, \boldsymbol{A}] = q \int_\gamma \mathrm{d}t \; (-\phi + \boldsymbol{v} \cdot \boldsymbol{A}). \tag{1.48}$$

Denote the 4-potential as $A^\mu = (\phi/c, \boldsymbol{A})$, then we find

$$S_{\text{int}}[\gamma; \phi, \boldsymbol{A}] = q \int_\gamma \mathrm{d}t \; A^\flat_\mu \frac{\mathrm{d}X^\mu}{\mathrm{d}t}. \tag{1.49}$$

Note it also manifests a re-parametrization invariance and geometrically well-behaved.

Variation on the full action gives the equation of motion for the charged particle

$$\frac{\mathrm{d}P^\flat_\mu}{\mathrm{d}t} = q(\partial_\mu A^\flat_\nu - \partial_\nu A^\flat_\mu) P^\nu \equiv q F_{\mu\nu} P^\nu, \tag{1.50}$$

where we define

$$P^\mu \equiv \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}}, \tag{1.51}$$

and introduced the electromagnetic tensor $F_{\mu\nu} = \partial_\mu A^\flat_\nu - \partial_\nu A^\flat_\mu$, whose component form is

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}. \tag{1.52}$$

It transforms as

$$F'_{\mu'\nu'} = \Lambda_{\mu'}{}^{\mu} \Lambda_{\nu'}{}^{\nu} F_{\mu\nu}. \tag{1.53}$$

Using the electromagnetic tensor, the Maxwell's equations writes

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu, \quad J^\mu \equiv (\rho c, \boldsymbol{j}), \tag{1.54}$$

and

$$\partial_{(\mu} F_{\nu\sigma)} = 0. \tag{1.55}$$

The second one is purely a structural Bianchi identity and contains no dynamics. In conclusion, the Maxwell's equations can be naturally be expressed in a manifestly Lorentz covariant form and the electrodynamics is compatible with the relativity principle. More generally, a relativistic theory should be able to written in a tensorial manifestly Lorentz covariant way.

1.2 Towards GR

1.2.1 Newton's Gravity

The classical Newtonian gravity theory neatly writes

$$\Delta\phi = 4\pi G_N \rho, \quad m_I \boldsymbol{a} = -m_G \nabla\phi. \tag{1.56}$$

where ρ is the volumetric mass density. Here, we distinguished the inertial mass m_I and the gravitational mass m_G. Although it's Galilean invariant, this gravity theory is essentially static, hence not relativistic at all.

Our goal is to obtain a relativistic theory for gravity. We can rely on the following ideas:

- Covariantize the Newtonian theory in Mink_4: Suppose we keep gravity theory a scalar theory, we can replace the Laplacian Δ by the d'Alembertian $\Box$ and try

  $$\Box\phi = 4\pi G_N \rho. \tag{1.57}$$

  However, ρ is not a Lorentz scalar and it's not covariant. An amendment to it is replacing ρ by the trace of the energy-momentum tensor $T^\mu{}_\mu$ and write

  $$\Box\phi = 4\pi G_N T^\mu{}_\mu. \tag{1.58}$$

  It was widely considered by von Laue, Nordström and Einstein, but it predicts:

  - No light diffraction.
  - Wrong Mercury precession rate.
  - No gravitational redshift.

  These inconsistencies ruled out this theory's candidacy.

- We may consider lifting the theory into a vectorial theory, like EM. We can define sort of gravitational 4-potential $A^\mu = (\phi_G/c, \boldsymbol{A}_G)$ and define an analogous field tensor $F_G^{\mu\nu}$. However, this formulation doesn't work either because:

  - Like charges repel each other, we need an opposite sign on the coupling constant for the vectorial gravitational theory, making the energy not bounded below, hence not stable.
  - Experimentally predicts no gravitional redshift.

These failed attempts implies the non-existence of covariantized Newtonian gravity in Mink_4 giving a sensible prediction.

1.2.2 The Weak Equivalence Principle

The weak equivalence principle is the ground stone for building general relativity.

Definition 1.12 (Weak Equivalence Principle) *In a given gravitational field, all bodies having the same free-fall motion, regardless of their masses and composition.*

This principle implies series of consequences:

- Consider two bodies equations of motions

$$\boldsymbol{a}_A = \frac{m_G^A}{m_I^A}\boldsymbol{g}, \quad \boldsymbol{a}_B = \frac{m_G^B}{m_I^B}\boldsymbol{g}. \tag{1.59}$$

  The weak equivalence principle expects $\boldsymbol{a}_A = \boldsymbol{a}_B$, hence we find the ratio between the gravitational mass and the inertial mass m_G/m_I is the same for all objects, and we can set it to unit. Experimentally, we can define the Eötvös parameter

$$\eta(A, B) \equiv \frac{2}{\frac{m_G^A}{m_I^A} + \frac{m_G^B}{m_I^B}}\left(\frac{m_G^A}{m_I^A} - \frac{m_G^B}{m_I^B}\right). \tag{1.60}$$

  If the weak equivalent principle holds, there should be $\eta(A, B) = 0$ for whatever pair of objects. Currently, the best result produced by the Microscope satellite gives $\eta(\mathrm{Pt}, \mathrm{Ti}) < 10^{-15}$.

- Suppose we observe a uniform gravitational field in a Galilean frame R, then we have

$$m_I \boldsymbol{a}_{m/R} = m_G \boldsymbol{g}. \tag{1.61}$$

  Consider another frame R' in free-fall with respect to R, hence

$$\boldsymbol{a}_{R'/R} = \boldsymbol{g}. \tag{1.62}$$

  In R', we observe

$$m_I \boldsymbol{a}_{m/R'} = 0 = m_G \boldsymbol{g} - m_I \boldsymbol{a}_{R'/R} = (m_G - m_I)\boldsymbol{g}. \tag{1.63}$$

  It not only implies $m_I = m_G$, but also implies the Einstein equivalence principle.

1.2.3 Einstein's Equivalence Principle and General Covariance

Definition 1.13 (Einstein's Equivalence Principle (EEP)) *No **local** experiment can distinguish between a frame in free-fall in the **local** gravitational field and a Galilean frame in the absence of gravity. Equivalently, in freely-falling frames, the laws of physics should have the same form as in special relativity without gravity.*

This principle establishes the distinguished status of free-fall frames. To transform from free-fall frames to more general frames, we need:

Definition 1.14 (Generalized Covariance Principle) *The laws of physics should have the same form in all frames, accelerate or not.*

With this principle, the Lorentz covariance under $\mathsf{SO}^+(1,3)$ is lifted to the general covariance under $\mathsf{GL}(4;\mathbb{R})$.

> **Remark**
>
> The locality of Einstein's equivalence principle can not be over-emphasized. Compared to SR states that Lorentz transformation is global, the general covariance is also locally defined. Moreover, it implies a non-uniformity for the spacetime, and discards the affine space as the underlying geometric object for it.

With the general covariance principle, locally the Minkowskian metric $\eta_{\mu\nu}$ is transformed by $\Lambda_\mu{}^\nu(x) \in \mathsf{GL}(4;\mathbb{R})$ to a more general metric tensor field

$$g_{\mu\nu}(x) = \Lambda_\mu{}^\rho(x)\Lambda_\nu{}^\sigma(x)\eta_{\rho\sigma}. \tag{1.64}$$

Such a tensor field can be assigned with a dynamics, and be a good candidate for the gravitational field.

1.3 Differential Geometry

1.3.1 Smooth Manifold

Definition 1.15 (Topological Manifold) *Let $n \in \mathbb{N}$, a **(topological) manifold** M_n is a topological space that is locally homeomorphic to $\mathbb{R}^n$, i.e., $\forall p \in M_n$, there exists:*

- *An open set $\Omega \subset M_n$ s.t. $p \in \Omega$.*
- *An open set $O \in \mathbb{R}^n$.*
- *A homeomorphism (continuous bijection with a continuous reciprocal) $\phi : \Omega \to O$.*

Definition 1.16 (Chart, Atlas, and Transition Function) *Let M_n an n-manifold:*

- *A **chart** in M_n is any pair $(\phi;\Omega)$ where $\Omega \subset M_n$ is any open set and $\phi : \Omega \to O$ is a homeomorphism to an open set $O \subset \mathbb{R}^n$.*
- *An **atlas** on M_n is any set $\mathcal{A} \equiv \{(\phi_i, \Omega_i) | i \in I\}$ of charts s.t. $\bigcup_{i\in I} \Omega_i = M_n$, where I is an index set.*
- *Let (ϕ_i, Ω_i) and (ϕ_j, Ω_j) be two charts s.t. $\Omega_i \cap \Omega_j \neq \varnothing$. Then we can define the **transition function** $\phi_{ij} \equiv \phi_j \circ \phi_i^{-1}$ restricted to $\phi_i(\Omega_i \cap \Omega_j)$. Similarly, we can define $\phi_{ji} = \phi_i \circ \phi_j^{-1}$ restricted to $\phi_j(\Omega_i \cap \Omega_j)$.*
- *We say an atlas of M_n is C^k iff all its transition functions are at least C^k.*
- *Two C^k-atlases $\mathcal{A}, \mathcal{A}'$ are called equivalent iff $\mathcal{A} \cup \mathcal{A}'$ is also a C^k-atlas of M_n. In this sense, we say a C^k n-manifold is an equivalence class of C^k-atlases on the set of n-manifolds.*
- *C^∞-manifolds are called **smooth manifolds**.*

Definition 1.17 (Differentialability) *Let M and N be two smooth manifolds, and let $F : M \to N$ be a continuous map. For $P \in M$ and (ϕ, Ω) a chart where $P \in \Omega$, let (ϕ', Ω') be a chart of N with $F(p) \in \Omega'$. We say F is C^k at P, iff $f \equiv \phi' \circ F \circ \phi^{-1}$ is C^k at $\phi(P)$. We shall say F is C^k over some open set $\tilde{\Omega} \subset M$ iff it's C^k at every $P \in \tilde{\Omega}$.*

Definition 1.18 (Diffeomorphism) *We say $F : M \to N$ is a C^k-diffeomorphism iff it's a C^k bijection with a C^k reciprocal. The set of C^k-diffeomorphisms $F : M \to M$ is denoted* $\mathrm{Diff}^k(M)$*, which is called the diffeomorphism group of M and performs as the gauge group for GR.*

Remark

A smooth (C^∞) diffeomorphism of M is given between any pair of charts (ϕ, Ω) and (ϕ', Ω') by a smooth diffeomorphism $\phi' \circ F \circ \phi^{-1}$ between open sets of $\mathbb{R}^n$. Locally, the action of a global diffeomorphism looks like a local change in coordinates according to a transition function.

Now we find the previous affine space replaced by the spacetime manifold, and the affine group replaced by the diffeomorphism group. Let's introduce some notations:

1. Take the target manifold $N = \mathbb{R}$ with its natural manifold structure, and it's covered by a single chart $(\mathrm{id}_\mathbb{R}; \mathbb{R})$. We say $F : M \to \mathbb{R}$ is C^k at $P \in M$ iff every chart $(\phi; M)$ of M with $P \in \Omega$ we havev $F \circ \phi^{-1}$ is C^k at $\phi(p)$. We denote $C^\infty(\Omega)$ the set of C^∞ functions $F : M \to \mathbb{R}$ that ar C^∞ over the open set $\Omega \subset M$. We denote $C^\infty(p)$ the set of functions $F : M \to \mathbb{R}$ that are C^∞ in an open neighborhood of $P \in M$.

2. Consider a curve $\gamma : \mathbb{R} \to M$, it is C^k iff in any chart $(\phi; \Omega)$ of M where $\gamma(\mathbb{R}) \cap \Omega \neq \varnothing$, $\phi(\gamma) \subset \mathbb{R}^n$ belongs to $C^k(\mathbb{R})$.

1.3.2 Tangent Bundle

In an affine space $(E; V; f)$, for a C^1 curve $\gamma : \mathbb{R} \to E$, we can define the velocity tangent vector as

$$\dot{\gamma}(0) \equiv \frac{\mathrm{d}}{\mathrm{d}s} f(\gamma(0), \gamma(s)) \bigg|_{s=0}. \tag{1.65}$$

The problem when switching to a manifold is the absence of f.

Let M be a smooth n-manifold, let $\gamma : \mathbb{R} \to M$ be a C^1 curve, and let $P \equiv \gamma(0)$. For any $f \in C^\infty(M)$, then $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is $C^1(\mathbb{R})$. Hence we can denote

$$\frac{\mathrm{d}}{\mathrm{d}s}(f \circ \gamma)(s) \bigg|_{s=0} = [\dot{\gamma}(0)](f). \tag{1.66}$$

Let $(\phi; \Omega)$ be a chart of M with $P \in \Omega$, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(f \circ \gamma)(s) \bigg|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}(f \circ \phi^{-1} \circ \phi \circ \gamma)(s) \bigg|_{s=0} \\ = \frac{\mathrm{d}}{\mathrm{d}s} f \circ \phi^{-1}(x^\mu \circ \gamma) \bigg|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}(x^\mu \circ \gamma) \bigg|_{s=0} \frac{\partial}{\partial x^\mu}(f \circ \phi^{-1}) \bigg|_{\phi(p)}. \end{aligned} \tag{1.67}$$

Hence, we can define the velocity vector as a differentiation

$$\dot{\gamma}(0) \equiv \frac{\mathrm{d}}{\mathrm{d}s}(x^\mu \circ \gamma)\bigg|_{s=0} \partial_\mu. \tag{1.68}$$

From this intuition, we have the following definition:

Definition 1.19 (Tangent Space) *Let M be a smooth n-manifold and $P \in M$. A derivation at P is any linear map $D : C^\infty(P) \to \mathbb{R}$ s.t. $\forall f, g \in C^\infty(P)$, we have the Leibniz rule*

$$D(fg) = D(f)g + fD(g). \tag{1.69}$$

The set of derivations at P is a $\mathbb{R}$-vector space and is called the tangent space at P, denoted as $T_P M$.

For every chart $(\phi = x^\mu, \Omega)$ of M, we define $\partial_\mu(P)$ by setting $\forall f \in C^\infty(P)$,

$$\partial_\mu f(P) \equiv \frac{\partial}{\partial x^\mu}(f \circ \phi^{-1})\bigg|_{\phi(P)}, \tag{1.70}$$

hence $\partial_\mu(P) \in T_P M$. Given a C^1 curve $\gamma : \mathbb{R} \to M$, a chart $(\phi = x^\mu, \Omega)$ with $P = \gamma(0) \in \Omega$, we have

$$\dot{\gamma}(0) = \frac{\mathrm{d}}{\mathrm{d}s}(x^\mu \circ \gamma)\bigg|_{s=0} \partial_\mu f. \tag{1.71}$$

Hence the following proposition:

Propsition 1.2 *Let M be a smooth n-manifold, let $P \in M$ and let $(\phi = x^\mu, \Omega)$ be a chart of M s.t. $P \in \Omega$. Then $\{\partial_\mu(P)\}$ is a basis of $T_P M$, in particular,*

$$\dim T_P M = n. \tag{1.72}$$

Sketch of Proof

Apparently, $\{\partial_\mu(P)\}$ is linearly independent, hence can span a vector space. Besides, $\forall f \in C^\infty(P)$, its Taylor expansion writes

$$f \circ \phi^{-1}(x^\mu) = f(P) + (x^\mu - x^\mu_P)\partial_\mu f(P). \tag{1.73}$$

Now applying $\forall D \in T_P M$ to it yields

$$D(f) = D(x^\mu)\partial_\mu f(P) \equiv D^\mu \partial_\mu f(P), \tag{1.74}$$

hence $D = D^\mu \partial_\mu$, $\{\partial_\mu\}$ is a complete basis.

Let $(\phi = x^\mu, \Omega)$ and $(\phi' = x'^\mu, \Omega)$ be two charts in M, therefore, $\forall P \in \Omega$, $\partial_\mu(P)$ and $\{\partial'_\mu\}$ are both basis for $T_P M$. Hence, there should be ${\Lambda_\mu}^\nu \in \mathsf{GL}(n; \mathbb{R})$ s.t.

$$\partial'_\mu(P) = {\Lambda_\mu}^\nu \partial_\nu(P). \tag{1.75}$$

To determine ${\Lambda_\mu}^\nu$, consider

$$\partial'_\mu x^\nu(P) = \left.\frac{\partial x^\nu}{\partial x'^\mu}\right|_P = {\Lambda_\mu}^\nu, \tag{1.76}$$

${\Lambda_\mu}^\nu$ is merely the Jacobian of the local transition function.

Remark

We replaced the single vector space V of $(E; V; f)$ by one tangent space $T_P M$ per point $P \in M$. Since $\dim T_P M = n$, all tangent spaces are necessarily isomorphic, but there are infinitely many such isomorphisms without a canonical one, unlike the case for an affine space.

Gluing tangent spaces at all points, we arrive at

Definition 1.20 (Tangent Bundle) *Let M be a smooth n-manifold, then the tangent bundle is defined as*

$$TM = \bigcup_{P \in M} \{P\} \times T_P M. \tag{1.77}$$

The tangent bundle is a smooth $2n$-dimensional manifold.

Definition 1.21 (Section) *A section (or a vector field) is any map $X : M \to TM$, $p \mapsto (p, X_p)$ where $X_p \in T_p M$. A section X is smooth iff $\forall f \in C^\infty(M)$, $X(f) \in C^\infty(M)$. The section of smooth sections of TM is denoted $\Gamma^\infty(TM)$, which is a real vector space.*

In any chart (ϕ, Ω) of M, we have $\forall P \in \Omega$, $X_p = X_p^\mu \partial_\mu(P)$ for any section $X : M \to TM$. Then $X \in \Gamma^\infty(TM)$ iff $P \mapsto X_p^\mu$ are smooth real-valued functions. Introduce any other chart (ϕ', Ω), we have

$$X_p = X_p'^{\mu'} \partial'_\mu(P) = X_p^\mu \partial_\mu(P), \tag{1.78}$$

hence

$$X_p'^\mu = \partial_\nu x'^\mu X_p^\nu. \tag{1.79}$$

Now let's exploit the structure of $\Gamma^\infty(TM)$. Let M be a smooth n-manifold, $\forall f \in C^\infty(M)$ and $X \in \Gamma^\infty(TM)$, we can define $fX \in \Gamma^\infty(M)$ by setting

$$\forall P \in M, \quad (fX)_P \equiv f(P) X_P. \tag{1.80}$$

Definition 1.22 (Lie Derivative) $\forall X, Y \in \Gamma^\infty(TM)$*, we define the Lie derivative* $[X, Y] \in \Gamma^\infty(TM)$ *as*

$$[X, Y] f \equiv X \circ Y(f) - Y \circ X(f) = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu)\partial_\mu f. \tag{1.81}$$

It can be verified that it makes $\Gamma^\infty(TM)$ a Lie algebra.

1.3.3 Cotangent Bundle and Tensor Bundles

Definition 1.23 (Cotangent Space) *Let M be a smooth n-manifold, let $P \in M$. We define the cotangent space of M at P, $T^*_P M$ as the dual space of $T_P M$.*

Definition 1.24 ((r,s)-tensor space) *$\forall r, s \in \mathbb{N}$, we define the (r,s)-tensor space at P as*

$$T^{(r,s)}_P M \equiv T_P M^{\otimes r} \otimes T^*_P M^{\otimes s}. \tag{1.82}$$

Definition 1.25 ((r,s)-tensor bundle) *$\forall r, s \in \mathbb{N}$, the (r,s)-tensor bundle then is defined as*

$$T^{(r,s)} M \equiv \bigcap_{P \in M} P \times T^{(r,s)}_P M, \tag{1.83}$$

and an (r,s)-tensor field is a section $T : M \to T^{(r,s)} M$, $P \mapsto (P, T_P)$. We denote $\Gamma^\infty(T^{(r,s)} M)$ the space of smooth (r,s)-tensor fields.

Locally, for any chart (ϕ, Ω) of M with $P \in \Omega$, we define the dual basis $\{\mathrm{d}x^\mu(p)\}$ of $\{\partial_\mu(P)\}$ by setting

$$\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu{}_\nu. \tag{1.84}$$

In any other chart (ϕ', Ω), we have

$$\mathrm{d}x'^\mu(\partial'_\nu) = \frac{\partial x^\rho}{\partial x'^\nu} \mathrm{d}x'^\mu(\partial_\rho) = \delta^\mu{}_\nu, \tag{1.85}$$

hence

$$\mathrm{d}x^\rho = \frac{\partial x^\rho}{\partial x'^\sigma} \mathrm{d}x'^\sigma. \tag{1.86}$$

Now, we can choose $\{\partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_s}\}$ a local basis for $T^{(r,s)}_P M$, hence $\forall T \in T^{(r,s)}_P M$,

$$T = T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_s}. \tag{1.87}$$

Under a local coordinate transformation, we have

$$T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} = \frac{\partial x^{\mu_1}}{\partial x'^{\mu'_1}} \cdots \frac{\partial x^{\mu_r}}{\partial x'^{\mu'_r}} \frac{\partial x'^{\nu'_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\nu'_s}}{\partial x^{\nu_s}} T^{\mu'_1 \cdots \mu'_r}{}_{\nu'_1 \cdots \nu'_s}. \tag{1.88}$$

Remark

It suffices to write the laws of physics in terms of smooth tensor fields over M to guarantee the general covariance principle.

Now we have replaced the E in an affine space by a smooth manifold and the V by TM. The rest is to find an analogue of f connecting two tangent spaces.

1.3.4 Connections, Parallel Transport, Geodesics and Curvature

Connection is the local, analytical abstraction of a straight line on a manifold. Locally, it should be a tangent vector moving along the line. In affine 3-space, two vector field $\vec{X}, \vec{Y}$

over $\mathbb{R}^3$, we have

$$\vec{Y}(0+\varepsilon\vec{X}) = \vec{Y}(0) + \varepsilon(\vec{X}\cdot\nabla)\vec{Y} + \mathcal{O}(\varepsilon^2). \tag{1.89}$$

Hence, $\vec{Y}$ is parallelly transported along $\vec{X}$ means

$$(\vec{X}\cdot\nabla)\vec{Y} = 0. \tag{1.90}$$

If $\vec{X}$ is a tangent vector of a curve, then $\vec{Y}$ is parallelly transported along it.

To generalize it to a n-manifold, if we naïvely generalize the criterion to $X^\mu\partial_\mu Y^\nu = 0$, then we won't have a generally covariant theory: A coordinate transformation induces

$$X'^\mu\partial'_\mu Y'^\nu = x^\rho\partial_\rho\left(\frac{\partial x'^\nu}{\partial x^\sigma}Y^\sigma\right) = \frac{\partial x'^\nu}{\partial x^\sigma}x^\rho\partial_\rho Y^\sigma + x^\rho Y^\sigma\partial_\rho\frac{\partial x'^\nu}{\partial x^\sigma} = 0. \tag{1.91}$$

It's not covariant as long as the Jacobian is not constant! The amendment for it is introducing connection.

Definition 1.26 (Connection) *Let M be a smooth n-manifold, an (affine) connection on M is any bilinear map $\nabla : \Gamma^\infty(TM)\times\Gamma^\infty(TM)\to\Gamma^\infty(TM)$, $(X,Y)\mapsto\nabla_X Y$ s.t.*

- *$\forall f\in C^\infty(M)$, $\forall X,Y\in\Gamma^\infty(TM)$,*

$$\nabla_{fX}Y = f\nabla_X Y, \quad \nabla_X(fY) = X(f)Y + f\nabla_X Y. \tag{1.92}$$

We refer to ∇_X as the covariant derivative w.r.t. X. In any chart $(\phi = x^\mu,\Omega)$ of M, there exists Christoffel symbols $\Gamma^\rho{}_{\mu\nu}\in C^\infty(\Omega)$ s.t. $\forall P\in M$, we have

$$(\nabla_{\partial_\mu}\partial_\nu)(P) \equiv \Gamma^\rho{}_{\mu\nu}\partial_\rho(P). \tag{1.93}$$

In any chart (ϕ,Ω) of M, $\forall X,Y\in\Gamma^\infty(TM)$, we have

$$\nabla_X Y = X^\mu\nabla_{\partial_\mu}(Y^\nu\partial_\nu) = X^\mu(\partial_\mu Y^\nu)\partial_\nu + X^\mu Y^\nu\Gamma^\rho{}_{\mu\nu}\partial_\rho, \tag{1.94}$$

hence,

$$\nabla_\mu Y \equiv Y^\nu{}_{;\mu}\partial_\nu, \quad Y^\nu{}_{;\mu} \equiv \partial_\mu Y^\nu + \Gamma^\nu{}_{\mu\rho}Y^\rho. \tag{1.95}$$

Note, $\Gamma^\rho{}_{\mu\nu}$ is not a tensor to cancel out the non-tensorial transformation term from $\partial_\mu Y^\nu$.

Remark

$\nabla_X Y(P)$ depends only on:

- X_P.
- Y_P and its first-order ordinary derivatives of its components.
- The Christoffel symbols $\Gamma^\rho{}_{\mu\nu}$ at P.

It implies ∇ can be applied to any (not necessarily smooth) section X and even to X sections defined only at the point of interest, and to C^1-sections Y. In particular, the Christoffel symbols encodes everything about ∇.

The transformation rule for the Christoffel symbols can be derived as follows: Still, consider two charts (ϕ, Ω) and (ϕ', Ω), we should expect

$$\Gamma'^\rho{}_{\mu\nu}\partial'_\rho = \nabla_{\partial'_\mu}\partial'_\nu = \nabla_{\partial'_\mu}\left(\frac{\partial x^\sigma}{\partial x'^\nu}\partial_\sigma\right) = \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}\partial_\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\alpha{}_{\rho\sigma}\partial_\alpha. \tag{1.96}$$

We then find

$$\Gamma'^\beta{}_{\mu\nu} = \frac{\partial x'^\beta}{\partial x^\alpha}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\alpha{}_{\rho\sigma} + \frac{\partial x'^\beta}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}. \tag{1.97}$$

Now we see $\Gamma^\rho{}_{\mu\nu}$ is indeed tensorial for a constant Jacobian.

Suppose we start from a coordinate system where the Christoffel symbols vanish and move to another one where there are non-trivial Christoffel symbols. Then the terms in the Christoffel symbols are second-order derivatives, similar to inertial forces induced by accelerations. Hence, the new coordinate can be interpreted as an accelerating frame, giving us a hint for gravitational or inertial forces.

Definition 1.27 (Parallel Transport) *Let M be a smooth n-manifold with connection ∇, let $\gamma : I \subset \mathbb{R} \to M$ be a C^1 curve and let Y be a smooth vector field on M. We say Y is parallel transported along γ iff $\forall s \in \mathbb{R}$,*

$$\nabla_{\dot\gamma(s)} Y = 0. \tag{1.98}$$

In any chart $(\phi = x^\mu, \Omega)$ of M s.t. $\gamma(I) \cap \Omega \neq \varnothing$, we have

$$\dot\gamma(s) = \frac{\mathrm{d}}{\mathrm{d}s}(x^\mu \circ \gamma)\frac{\partial}{\partial x^\mu}\gamma(s), \tag{1.99}$$

then denote $Y = Y^\mu \partial_\mu$, we have

$$\nabla_{\dot{(\gamma)}(s)} Y = \frac{\mathrm{d}}{\mathrm{d}s}(x^\mu \circ \gamma) Y^\nu{}_{;\mu}\partial_\nu = 0, \tag{1.100}$$

hence we should have

$$\frac{\mathrm{d}}{\mathrm{d}s}Y^\mu(\gamma(s)) + \Gamma^\mu{}_{\rho\sigma}\frac{\mathrm{d}}{\mathrm{d}s}(x^\rho \circ \gamma(s)) Y^\sigma(\gamma(s)) = 0. \tag{1.101}$$

It's only a locally defined equation, and we have to solve it chart-wise and glue them together to get a global result.

This is a linear first-order ODE. Given an initial condition, it always has a unique solution that depends linearly on the initial condition. Hence the following theorem.

Theorem 1.1 *Let M be a smooth manifold with connection ∇, let $\gamma : I \subset \mathbb{R} \to M$ be a C^1 curve. Then let $\forall P, P' \in \gamma(I)$, there exists a unique linear isomorphism $H_\nabla(P \xrightarrow{\gamma} P') : T_P M \xrightarrow{\sim} T_{P'}$ between the two tangent spaces.*

According to this theorem, we find the introduction of a connection establishes a path-dependent isomorphism between tangent spaces. That's exactly a "connection".

Remark

Unlike in affine space, isomorphism between $T_P M$ and $T_{P'} M$ depends on γ. If γ is a closed curve, hence parallel transport along it induces an automorphism on the tangent space of its base point, inspiring the following definition.

Definition 1.28 (Holonomy) *Let M be a smooth n-manifold with connection ∇. For $\forall p \in M$, we define the holonomy group of ∇ at P as*

$$\mathrm{Hol}_\nabla(P) = \{H_\nabla(P \xrightarrow{\gamma} P) | \gamma : [0,1] \to M \in C^1([0,1]), \gamma(0) = \gamma(1) = P\}. \tag{1.102}$$

Remark

$\mathrm{Hol}_\nabla(P) \subset \mathsf{GL}(T_P M)$. Besides, we can define curvature for a manifold with non-trivial $\mathrm{Hol}_\nabla(P)$.

If a closed curve $\gamma : [0,1] \to M$ is covered by a single chart $(\phi = x^\mu, \Omega)$, then we can integrate the parallel transport equation to get

$$Y^\mu(\gamma(1)) = Y^\mu(\gamma(0)) + \int_\gamma \mathrm{d}x^\rho \, \Gamma^\mu{}_{\rho\sigma} Y^\sigma(\gamma(s)). \tag{1.103}$$

If we choose the curve γ as an infinitesimal square loop $A \to B \to C \to D$, denote $\delta x^\mu \equiv B - A = C - D$ and $\delta y^\mu \equiv C - B = D - A$, linearize the above integral, we find

$$\begin{aligned} H_\nabla(A \xrightarrow{\gamma} A)^\mu{}_\nu Y^\nu_A - Y^\mu_A &= \delta x^\rho \delta y^\sigma (\Gamma^\mu{}_{\sigma\alpha} \Gamma^\alpha{}_{\rho\nu} - \Gamma^\mu{}_{\rho\alpha} \Gamma^\alpha{}_{\sigma\nu} + \partial_\sigma \Gamma^\mu{}_{\rho\nu} - \partial_\rho \Gamma^\mu{}_{\sigma\nu}) Y^\nu_A \\ &\equiv \delta x^\rho \delta y^\sigma R^\mu{}_{\nu\rho\sigma} Y^\nu_A. \end{aligned} \tag{1.104}$$

Note the holonomy group is essentially a Lie group, and the calculation above calculates the Lie algebra for it, whose generators are $R^\mu{}_\nu(\delta x, \delta y) \equiv \delta x^\rho \delta y^\sigma R^\mu{}_{\nu\rho\sigma}$. It a special case for the Ambrose-Singer theorem.

The defined $R^\mu{}_{\nu\rho\sigma}$ is exactly the Riemann curvature tensor

$$\mathrm{Riem}_\nabla \equiv R^\mu{}_{\nu\rho\sigma} \partial_\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\rho \otimes \mathrm{d}x^\sigma \in \Gamma^\infty(T^{(1,3)}(M)). \tag{1.105}$$

Its tensorial property can be proved by a coordinate-independent definition for it. Before formally defining it, we shall first introduce geodesics.

Definition 1.29 (Geodesic) *Let M be a smooth n-manifold with connection ∇. A C^2-curve $\gamma : I \subset \mathbb{R} \to M$ is a geodesic of (M, ∇) iff $\forall s \in I$,*

$$\nabla_{\dot{\gamma}(s)}\dot{\gamma}(s) = 0. \tag{1.106}$$

That is, its velocity vectors are parallel transported along itself.

Consider any chart (ϕ, Ω) intersects with γ, the parallel transport equation can be written as the local geodesic equation

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(x^\mu \circ \gamma)(s) + \Gamma^\mu{}_{\rho\sigma}\frac{\mathrm{d}}{\mathrm{d}s}(x^\rho \circ \gamma)(s)\frac{\mathrm{d}}{\mathrm{d}s}(x^\sigma \circ \gamma)(s) = 0. \tag{1.107}$$

It's a non-linear second-order ODE, hence the following theorem.

Theorem 1.2 *Let M be a smooth n-manifold with connection ∇. For $\forall p \in M$ and $\forall X \in T_P M$, there exists a unique maximal geodesic $\gamma_{P,X} : I \subset \mathbb{R} \to M$ s.t. $\gamma(0) = P$ and $\dot{\gamma}(0) = X$.*

Remark

The local geodesic equation looks like the equation of motion for an isolated point particle. The Christoffel symbol term looks like a force term inducing acceleration. Since the Christoffel symbol is not a tensor, we can find a local coordinate where they are zero, and the geodesic equation looks exactly like the Newton's second law for a free particle.
Unlike the case for Minkowskian spacetime where geodesics are straight lines extending to infinity, on a general manifold M there are maximal solutions for it.

Definition 1.30 (Geodesic Completance) *We shall say the geodesic $\gamma_{P,X}$ is complete iff I can be extened to $\mathbb{R}$. We say (M, ∇) is geodesically complete iff $\forall P \in M$, $\forall X \in T_P M$, $\gamma_{P,X}$ is complete.*

1.3.5 Covariant Derivatives for Tensor Fields

Definition 1.31 *Let M be a smooth n-manifold with connection ∇. Let $X \in \Gamma^\infty(TM)$ be a smooth vector field on the manifold, we extend the covarient derivative ∇_X to (r, s)-tensor fields by demanding:*

- *$\nabla_X : \Gamma^\infty(T^{(r,s)}M) \to \Gamma^\infty(T^{(r,s)}M)$ is linear.*
- *For $\forall f \in C^\infty(M) = \Gamma^\infty(T^{(0,0)}M)$, we expect $\nabla_X f = X(f)$.*
- *For any pair of smooth tensor fields T, S,*

$$\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S). \tag{1.108}$$

- *$\forall T \in \Gamma^\infty(T^{(r,s)}M)$, $\forall m = 1, ..., n$ and $n = 1, ..., s$, the contraction $\langle T\rangle^m_n$ between m and n indices commutes with the covariant derivative:*

$$\nabla_X \langle T\rangle^m_n = \langle \nabla_X T\rangle^m_n . \tag{1.109}$$

In any chart $(\phi = x^\mu, \Omega)$ of M, we should have

$$\nabla_{\partial_\mu} \mathrm{d}x^\nu = A_\mu{}^\nu{}_\rho \mathrm{d}x^\rho, \quad A_\mu{}^\nu{}_\rho \in C^\infty(\Omega). \tag{1.110}$$

Since

$$\begin{aligned} 0 = \partial_\rho \delta^\mu_\nu &= \nabla_{\partial_\rho}(\mathrm{d}x^\mu(\partial_\nu)) = \nabla_{\partial_\rho} \langle \partial_\nu \otimes \mathrm{d}x^\mu\rangle^1_1 \\ &= \left\langle(\nabla_{\partial_\rho}\partial_\nu) \otimes \mathrm{d}x^\mu\right\rangle^1_1 + \left\langle \partial_\nu \otimes (\nabla_{\partial_\rho}\mathrm{d}x^\mu)\right\rangle^1_1 = \Gamma^\mu{}_{\rho\nu} + A_\rho{}^\mu{}_\nu, \end{aligned} \tag{1.111}$$

we find

$$A_\rho{}^\mu{}_\nu = -\Gamma^\mu{}_{\rho\nu}, \tag{1.112}$$

and

$$\nabla_{\partial_\mu} \mathrm{d}x^\nu = -\Gamma^\nu{}_{\mu\rho}\mathrm{d}x^\rho. \tag{1.113}$$

Now we can extend covariant derivative to any tensor field: $\forall T \in \Gamma^\infty(T^{(r,s)}M)$, we have

$$\begin{aligned} \nabla_{\partial_\rho} T &= \nabla_{\partial_\rho}\left[T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}\left(\bigotimes_{i=1}^r \partial_{\mu_r}\right) \otimes \left(\bigotimes_{j=1}^s \mathrm{d}x^{\nu_j}\right)\right] \\ &= (\partial_\rho T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s})\left(\bigotimes_{i=1}^r \partial_{\mu_r}\right) \otimes \left(\bigotimes_{j=1}^s \mathrm{d}x^{\nu_j}\right) \\ &+ \sum_{k=1}^r \Gamma^{\mu_k}{}_{\rho\sigma} T^{\mu_1\cdots\mu_{k-1}\sigma\mu_{k+1}\mu_r}{}_{\nu_1\cdots\nu_s}\left(\bigotimes_{i=1}^r \partial_{\mu_r}\right) \otimes \left(\bigotimes_{j=1}^s \mathrm{d}x^{\nu_j}\right) \\ &- \sum_{\ell=1}^s \Gamma^\sigma{}_{\rho\nu_\ell} T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_{\ell-1}\sigma\nu_{\ell+1}\cdots\nu_s}\left(\bigotimes_{i=1}^r \partial_{\mu_r}\right) \otimes \left(\bigotimes_{j=1}^s \mathrm{d}x^{\nu_j}\right), \end{aligned} \tag{1.114}$$

hence

$$\begin{aligned} &T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s;\rho} = T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s,\rho} \\ &+ \sum_{k=1}^r \Gamma^{\mu_k}{}_{\rho\sigma} T^{\mu_1\cdots\mu_{k-1}\sigma\mu_{k+1}\mu_r}{}_{\nu_1\cdots\nu_s} - \sum_{\ell=1}^s \Gamma^\sigma{}_{\rho\nu_\ell} T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_{\ell-1}\sigma\nu_{\ell+1}\cdots\nu_s}. \end{aligned} \tag{1.115}$$

1.3.6 Torsion

Definition 1.32 (Torsion) *Let M be a smooth n-manifold with connection ∇, we define the torsion ∇, $\mathrm{Tor}_\nabla : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ with $\forall X, Y \in \Gamma^\infty(TM)$ we set*

$$\mathrm{Tor}_\nabla(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y]. \tag{1.116}$$

Propsition 1.3 *For torsion defined above, we have:*

1. *$\forall f \in C^\infty(M)$, we have*

$$\mathrm{Tor}_\nabla(X, fY) = \mathrm{Tor}_\nabla(fX, Y) = f\mathrm{Tor}_\nabla(X,Y). \tag{1.117}$$

2. *$\mathrm{Tor}_\nabla \in \Gamma^\infty(T^{(1,2)}M)$, and in any chart of M, we have*

$$\mathrm{Tor}_\nabla = (\Gamma^\mu{}_{\rho\sigma} - \Gamma^\mu{}_{\sigma\rho})\partial_\mu \otimes \mathrm{d}x^\rho \otimes \mathrm{d}x^\sigma. \tag{1.118}$$

The proof is trivial according to the definition.

Definition 1.33 (Torsion-Free) *We say a connection is torsion-free if* $\mathrm{Tor}_\nabla = 0$.

Remark

In GR, we always set $\mathrm{Tor}_\nabla = 0$. If we not assume torsion-free, then we will have Einstein-Cartan theory, whose dynamical evolution of spin densities always lead to a steady torsion-free connection.
Besides, the geodesic equation only feels the symmetric part for $\Gamma^\mu{}_{\rho\sigma}$, so the space-time is not torsion-free, the matter still only feels the torsion-free part.

1.3.7 Riemann Curvature Tensor

Definition 1.34 *Let M be a smooth n-manifold with connection ∇, we define the Riemann curvature tensor*

$$\mathrm{Riem}_\nabla : \Gamma^\infty(TM)^3 \to \Gamma^\infty(TM) \tag{1.119}$$

by setting $\forall X, Y, Z \in \Gamma^\infty(TM)$,

$$\mathrm{Riem}_\nabla(X,Y,Z) \equiv (\nabla_X\nabla_Y - \nabla_Y\nabla_X)Z - \nabla_{[X,Y]}Z. \tag{1.120}$$

Propsition 1.4 *$\forall f \in C^\infty(M)$ and $\forall X, Y, Z \in \Gamma^\infty(TM)$, we have:*

1. $\mathrm{Riem}_\nabla(fX,Y,Z) = \mathrm{Riem}_\nabla(X,fY,Z) = \mathrm{Riem}_\nabla(X,Y,fZ) = f\mathrm{Riem}_\nabla(X,Y,Z)$.

2. *$\mathrm{Riem}_\nabla \in \Gamma^\infty(T^{(1,3)}M)$, and in any chart of M, we have*

$$\mathrm{Riem}_\nabla = R^\mu{}_{\nu\rho\sigma}\partial_\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\rho \otimes \mathrm{d}x^\sigma, \tag{1.121}$$

where

$$R^\mu{}_{\nu\rho\sigma} = \Gamma^\mu{}_{\sigma\alpha}\Gamma^\alpha{}_{\rho\nu} - \Gamma^\mu{}_{\rho\alpha}\Gamma^\alpha{}_{\sigma\nu} + \partial_\sigma\Gamma^\mu{}_{\rho\nu} - \partial_\rho\Gamma^\mu{}_{\sigma\nu}. \tag{1.122}$$

The proof is again trivial according to the definition.

Remark

We can establish a dictionary between Newtonian gravity and GR:

- Gravitational field $\boldsymbol{g}$ corresponds to the Christoffel symbol $\Gamma^{\mu}{}_{\rho\sigma}$.
- $\partial_i g_j$ is replaced by the Riem_∇.
- $\nabla \cdot \boldsymbol{g} = -4\pi G_N \rho$ implies the left-hand side of GR field equation should be some contraction on the curvature tensor.

As a mathematical remark, the anti-symmetry of the X and Y slots on the definition for torsion and the Riemann curvature tensor implies they are related to sort of a Lie algebra structure. If we replace the vector Z in the definition for the Riemann curvature tensor by a function f, we find the expression reduces to what for the torsion tensor.

Definition 1.35 (Ricci Tensor) *Let M be a smooth manifold with connection ∇, the Ricci curvature tensor of ∇ is defined by*

$$\mathrm{Ric}_\nabla = \langle \mathrm{Riem}_\nabla \rangle^1_1 . \tag{1.123}$$

In any chart of M, we thus have

$$\mathrm{Ric}_\nabla = R^{\mu}{}_{\mu\rho\sigma} \mathrm{d}x^\rho \otimes \mathrm{d}x^\sigma . \tag{1.124}$$

It turns out it's the only non-trivial way we can contract on the Riemann curvature.

1.3.8 Pseudo-Riemannian Manifold

Definition 1.36 (The Metric Tensor) *A smooth pseudo-Riemannian n-manifold is a pair $(M; g)$ where M is a smooth n-manifold, and $g \in \Gamma^\infty(T^{(0,2)}M)$ is a symmetric smooth non-degenerate tensor field, i.e., $\forall P \in M$, $g(P) \in T^{(0,2)}_P M$ is a non-degenerate symmetric bilinear form on $T_P M$. g is called the metric tensor field (the metric).*

At every $P \in M$, there exists an orthonormal basis $\{e_i\}$ for g, in the sense that $g_P(e_i, e_j) = \eta^{(t,s)}_{ij}$, where η is the generalized Minkowskian metric where the first t-diagonal elements are -1, the rest $s = n - t$ ones are $+1$. Since g is smooth hence continuous non-degenerate tensor field, (t, s) is then constant throughout M, it's called the **signature** of g. We shall say (M, g) is:

- Riemannian: $(t, s) = (0, n)$.
- Lorentzian: $(t, s) = (1, n - 1)$.
- Pseudo-Riemannian: Otherwise.

Now we can construct musical isomorphisms associated with g in the same way as we did for the Minkowskian spacetime. For a chart where $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$, then for a contra-vector $X = X^\mu \partial_\mu$, we define

$$X^\flat_\mu = g_{\mu\nu} X^\nu. \tag{1.125}$$

Likewise, the raising for a co-vector $\theta = \theta_\mu \mathrm{d}x^\mu$ is

$$\theta^{\sharp\mu} = g^{\mu\nu}\theta_\nu, \tag{1.126}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. Musical symbols for general tensors can be similarily defined.

Definition 1.37 (Scalar Curvature) *Let (M, g) be a smooth pseudo-Riemannian n-manifold with connection ∇, we defined its scalar curvature*

$$R_\nabla = \left\langle \mathrm{Ric}_\nabla{}^{\sharp_1} \right\rangle^1_1. \tag{1.127}$$

In any chart, we have

$$R_\nabla = R^\mu{}_\mu{}^\nu{}_\nu. \tag{1.128}$$

1.3.9 The Levi-Cività Connection

If a manifold is endowed with a metric structure, then there will be a natural choice for the connection on it.

Definition 1.38 (Metric Connection) *Let (M, g) be a smooth pseudo-Riemannian n-manifold, we shall say that a connection ∇ on M is a metric connection iff $\forall X \in \Gamma^\infty(TM)$, $\nabla_X g = 0$.*

Theorem 1.3 (Fundamental Theorem of Pseudo-Riemannian Geometry) *Let (M, g) be a smooth pseudo-Riemannian manifold, then it admits a unique torsion-free metric connection ∇, called the Levi-Cività connection of (M, g).*

The proof is as follows: The data of ∇ is equivalent to the data of its Christoffel symbols $\Gamma^\mu{}_{\rho\sigma}$ in all charts of a atlas of M. Hence, it suffices to prove the metric connection condition uniquely determines the Christoffel symbols. Hence in any cahrt, there should be

$$\nabla_\mu g_{\rho\sigma} = \partial_\mu g_{\rho\sigma} - \Gamma^\nu{}_{\mu\rho} g_{\nu\sigma} - \Gamma^\nu{}_{\mu\sigma} g_{\rho\nu} = 0. \tag{1.129}$$

The torsion-free assumption implies $\Gamma^\mu{}_{\rho\sigma} = (\Gamma^\mu)_{\sigma\rho}$, plus with the symmetric condition on $g_{\mu\nu}$, we can permute the free indices and arrive at three equations:

$$\partial_\mu g_{\rho\sigma} - \Gamma^\nu{}_{\mu\rho} g_{\nu\sigma} - \Gamma^\nu{}_{\mu\sigma} g_{\rho\nu} = 0, \tag{1.130}$$
$$\partial_\rho g_{\sigma\mu} - \Gamma^\nu{}_{\rho\sigma} g_{\nu\mu} - \Gamma^\nu{}_{\rho\mu} g_{\sigma\nu} = 0, \tag{1.131}$$
$$\partial_\sigma g_{\mu\rho} - \Gamma^\nu{}_{\sigma\mu} g_{\nu\rho} - \Gamma^\nu{}_{\sigma\rho} g_{\sigma\mu} = 0. \tag{1.132}$$

Solving the equations we find the unique solution

$$\Gamma^{\mu}{}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu}(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma}). \tag{1.133}$$

Remark

If we believe the Christoffel symbols represent forces produced by the gravitional field, then $\Gamma^{\mu}{}_{\rho\sigma}$ being related to derivatives of $g_{\mu\nu}$ implies here the metric tensor in GR is analogous to the gravitational potential.

1.3.10 Riemann Normal Coordinates (RNC)

We still neeed to establish free-falling coordinates on a manifold, RNC gaurantees it.

Theorem 1.4 (The Existence of RNC) *Let (M, g) be a smooth pseudo-Riemannian manifold with a torsion-free connection ∇, then $\forall p \in M$, there exists a chart $(\phi = \xi^\mu, \Omega)$ with $P \in \Omega$ and $g_P = \eta^{(t,s)}$ and $\Gamma^{\mu}{}_{\rho\sigma}(P) = 0$. Such (ϕ, Ω) is called Riemann normal chart at P, and $\phi = \xi^\mu$ is called the Riemann normal coordinates.*

Physically, the gravitational field is cancelled out by inertial forcess in a free-falling frame, hence we expect the Christoffle symbols locally vanish. Hence, RNC can be identified as a local free-falling frame.

The proof for it focus on making $\Gamma^{\mu}{}_{\rho\sigma}(P) = 0$, as $g_P = \eta^{(t,s)}$ can always be achieved via coordinate rotations. Let's suppose $(\psi = x^\mu, \Omega)$ is a chart of M where $P \in \Omega$ and $x^\mu(P) = 0$ and $g_P = \eta^{(t,s)}$. Let's $\Gamma^{\mu}{}_{\rho\sigma}(P)$ denote the values of Christoffel symbols of ∇ at P in (ψ, Ω). Chart (ψ, Ω) always, we only need to find a new chart where all Christoffel symbols vanish. Let $(\phi = \xi^\mu, \Omega)$ satisfies

$$x^\mu(\xi) = \xi^\mu - \frac{1}{2}\Gamma^{\mu}{}_{\rho\sigma}(P)\xi^\rho\xi^\sigma. \tag{1.134}$$

Clearly, we still have $\xi^\mu(P) = 0$, and the Jacobian is

$$\frac{\partial x^\mu}{\partial \xi^\nu} = \delta^\mu_\nu - \Gamma^{\mu}{}_{\nu\sigma}\xi^\sigma. \tag{1.135}$$

At P, it gives

$$\left.\frac{\partial x^\mu}{\partial \xi^\nu}\right|_P = \delta^\mu_\nu, \tag{1.136}$$

it's a smooth Jacobian, and by the inverse function theorem, the constructed coordinate transformation is a smooth local diffeomorphism. Immediately, we find this transformation doesn't alter the form of the metric tensor. Moreover, since

$$\Gamma'^{\beta}{}_{\mu\nu} = \frac{\partial \xi^\beta}{\partial x^\alpha}\frac{\partial x^\rho}{\partial \xi^\mu}\frac{\partial x^\sigma}{\partial \xi^\nu}\Gamma^{\alpha}{}_{\rho\sigma} + \frac{\partial \xi^\beta}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \xi^\mu \partial \xi^\nu}, \tag{1.137}$$

and the second-order derivative for the coordinate transformation is

$$\left.\frac{\partial^2 x^\mu}{\partial\xi^\rho\partial\xi^\sigma}\right|_P = -\Gamma^\mu{}_{\rho\sigma}, \tag{1.138}$$

we immediately find the Christoffels in the new chart at P neatly writes

$$\Gamma'^\mu{}_{\rho\sigma}(P) = 0. \tag{1.139}$$

1.4 General Relativity

1.4.1 Basic Definitions and First Consequences

Postulate 1.1 *GR is based on the following postulates:*

- *The spacetime is a smooth Lorentzian* 4*-manifold* (M, g) *equipped with its Levi-Cività connection* ∇*.*

- *The general covariance principle: The laws of physics are written in terms of smooth tensor fields on* M*.*

- *Einstein's equivalence principle: The laws of physics are read at any* $P \in M$*, in the RNC at* P*, as in SR.*

- *The metric* g *is the solution of Einstein's field equations over* M*.*

Postulate 1.2 (Minimal Substitution Rules) *According to the 2nd and 3rd postulates, the following rules are applied to covariantize results written in SR:*

- *In SR or GR in a RNC,* $g_P = \eta_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$*, the result in GR at any chart is deduced by replacing* η *by* g*:* $g_P = g_{\mu\nu}(P)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$*. i.e., replacing the Minkowskian metric by the metric field.*

- *The semicolon rule "$,\to;$": Derivatives* ∂_μ *are substituted by covariant derivatives* ∇_μ*.*
  .

- *When integrating over a spacetime bulk, the measure is replaced by*

$$\mathrm{d}^4\xi \to \mathrm{d}^4x\sqrt{-\det g} \equiv \mathrm{d}^4x\sqrt{-g}. \tag{1.140}$$

Remark

In the presence of curvatuve, we have $[\nabla_\mu, \nabla_\nu] \neq 0$, hence when replacing derivatives by covariant derivatives we encounter the normal-ordering issue, similar to the ordering problem for quantum operators. Hence, this rule can only be applied to first-order derivatives without ambiguities: Which is enough for field theories with only first-order derivatives.

1.4.2 Causality in GR

Definition 1.39 *Let (M, g) be a smooth Lorentzian 4-manifold. $\forall P \in M$ and $\forall X_P \in T_P M$, we say*

- *X_P is time-like iff $g_P(X_P, X_P) < 0$.*
- *X_P is null (light-like) iff $g_P(X_P, X_P) = 0$.*
- *X_P is space-like iff $g_P(X_P, X_P) > 0$.*
- *X_P is causal iff $g_P(X_P, X_P) \leq 0$.*

Given a C^1 curve $\gamma : I \subset \mathbb{R} \to M$, we shall say γ is time-likee (null, space-like or causal) iff $\dot{\gamma}(s)$ is time-likee (null, space-like or causal) for all $s \in I$.

Remark

Since g is a dynamical field in GR, so the casuality is also a dynamical concept in GR. Besides, it's also a local concept and is a lot subtler.

1.4.3 Isolated Point Particles

Let γ be a time-like C^1-curve, we define its pseudo-arclength $\ell(\gamma; x_M, x_N)$ as

$$\ell(\gamma; x + M, x_N) = \int_{x_M}^{x_N} \mathrm{d}s\, \sqrt{-g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))}. \tag{1.141}$$

We say γ is parameterized by the proper time $s = \tau$, iff

$$g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s)) = -1. \tag{1.142}$$

The action for an isolated point-particle with mass m can be natural generalized to

$$S = -mc \int_{x_M}^{x_N} \mathrm{d}s\, \sqrt{-g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))} \tag{1.143}$$

according to the minimal substitution rule. In any chart $(\phi = x^\mu, \Omega)$ intersecting γ, the action writes

$$S = -mc \int \mathrm{d}s\, \sqrt{-g_{\mu\nu}(X(s))\dot{X}^\mu(s)\dot{X}^\mu(s)}, \quad X^\mu(s) \equiv x^\mu \circ \gamma(s). \tag{1.144}$$

Therefore, the Lagrangian for an isolated point particle can be taken as

$$\mathcal{L}_0[X^\mu, \dot{X}^\mu] = -mc\sqrt{-g_{\mu\nu}(X(s))\dot{X}^\mu(s)\dot{X}^\mu(s)}. \tag{1.145}$$

By variation, we find the EoM writes

$$\frac{\mathrm{d}^2 X^\mu}{\mathrm{d}s^2} + \Gamma^\mu{}_{\rho\sigma}\frac{\mathrm{d}X^\rho}{\mathrm{d}s}\frac{\mathrm{d}X^\sigma}{\mathrm{d}s} = 0, \tag{1.146}$$

it's exactly the geodesic equation, hence the following proposition:

Propsition 1.5 *The worldline for an isolated point particle with mass m in the spacetime (M, g) of GR are the time-like geodesics of its Levi-Cività connection.*

Propsition 1.6 *The worldline for a photon in the spacetime (M, g) of GR are the null geodesics of its Levi-Cività connection.*

Proof for the photon geodesic is immediate by applying the substitution rule. Clearly, non-trivial Christoffel symbols will bring light deflections.

Since we have derived the equation of motion for an isolated point particle, we should be able to take some limit to recover the equation of motions for the Newtonian gravity. We say we are in the Newtonian limit in some chart $(\phi = x^\mu, \Omega)$ of M, iff:

- $\forall i = 1, 2, 3$, $|\dot{X}^i| \ll |\dot{X}^0|$, i.e. the particle is non-relativistic.
- The gravitational field is weak over Ω, hence $\forall P \in \Omega$, we can expand

  $$g_{\mu\nu}(P) = \eta_{\mu\nu} + h_{\mu\nu}(P), \tag{1.147}$$

  where $|h_{\mu\nu}| \ll 1$ and is static.

In the Newtonian limit, the geodesic equation is dominated by

$$\ddot{X}^\mu \approx \Gamma^\mu{}_{00}\dot{X}^0\dot{X}^0, \tag{1.148}$$

since $\dot{X}^0 \gg \dot{X}^i$. The corresponding Christoffel symbols are approximated to

$$\Gamma^\mu{}_{00} = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda h_{00}. \tag{1.149}$$

The inverse of the metric to the leading order is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{1.150}$$

and the leading order geodesic equation then writes

$$\ddot{X}^\mu = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}, \tag{1.151}$$

which implies

$$\dot{X}^0 = Const \equiv c, \qquad \ddot{X}^i = \frac{1}{2}(\dot{X}^0)^2\partial^i h_{00}. \tag{1.152}$$

Hence, as long as we identify

$$h_{00} \equiv -\frac{2\Phi}{c^2}, \tag{1.153}$$

where Φ is the gravitational potential, then we recover the Newtonian gravity. As a consequence,

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right) \tag{1.154}$$

in the Newtonian limit.

Addocding to this result, gravational redshift can be deduced. Suppose a clock A is at rest in the chart where the Newtonian limit applies, whose worldline is

$$\phi \circ \gamma_A(\tau_A) = (X^0(\tau_A), \boldsymbol{X}_A). \tag{1.155}$$

Similarly, we have anothe clock B whose

$$\gamma \circ \gamma_B(\tau_B) = (X^0(\tau_B), \boldsymbol{X}_B). \tag{1.156}$$

Now we find

$$\dot{\gamma}_A(\tau_A) = \dot{X}_0(\tau_A)\partial_0, \quad \dot{\gamma}_B(\tau_B) = \dot{X}_0(\tau_B)\partial_0. \tag{1.157}$$

Given the worldline is parameterized by the proper time, we have

$$-1 = g_A(\dot{\gamma}_A(\tau_A)) = g_{00}(A)(\dot{X}_0(\tau_A))^2, \tag{1.158}$$

hence

$$\mathrm{d}\tau_A = \sqrt{-g_{00}(A)}\mathrm{d}X^0_A. \tag{1.159}$$

Similarly,

$$\mathrm{d}\tau_B = \sqrt{-g_{00}(B)}\mathrm{d}X^0_B. \tag{1.160}$$

Therefore, for a given time interval in the coordinate time ΔX^0, we find

$$\Delta X^0 = \frac{\Delta\tau_A}{\sqrt{-g_{00}(A)}} = \frac{\Delta\tau_B}{\sqrt{-g_{00}(B)}}. \tag{1.161}$$

As a consequence, different local observers feel the time elapsing in different speeds. In the Newtonian limit,

$$\frac{\Delta\tau_A}{\Delta\tau_B} = \sqrt{\frac{g_{00}(A)}{g_{00}(B)}} \approx 1 + \frac{\Phi(A) - \Phi(B)}{c^2}. \tag{1.162}$$

For example, the GPS satellites are located at the 20200 km altitude, and an accurate locating requires a timing error less that 50 ns. However, the dirft due to the gravitional redshift is about 45 μs per day, and such a drift must be corrected.

1.4.4 Maxwell Electrodynamics

We can apply the minimal substitution rules to Maxwell equations

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu, \qquad \partial_{(\mu}F_{\nu\rho)} = 0. \tag{1.163}$$

Now we can directly generalize it to a general manifold:

$$\nabla_\nu F^{\mu\nu} = \mu_0 J^\mu, \qquad \nabla_{(\mu}F_{\nu\rho)} = 0. \tag{1.164}$$

Under the eikonal approximation, this set of equations could recover null geodesics for photons.

1.4.5 Einstein's Field Eequation

Our approach building the Einstein's field eqeuation following the Lagrangian formalism, and we shall start the electromagnetic field in Mink_4. Here, we are considering a spin-1 gauge field $A = A_\mu \mathrm{d}x^\mu$, and we are thinking about coupling it to a matter field Φ. In principle, we should be able to write an action describing the while system, and we have

$$S[A,\Phi] = S_{\mathrm{EM}}[A] + S_{\mathrm{matter}}[A,\Phi]. \tag{1.165}$$

Physicists know how to write down the EM action from kindergarten:

$$S_{\mathrm{EM}}[A] = -\frac{1}{4\mu_0}\int \mathrm{d}^4x\ F_{\mu\nu}F^{\mu\nu}, \tag{1.166}$$

which manifests the gauge freedon $A \to A + \mathrm{d}f$. Varying the $S[A,\Phi]$ should recover the coupling between the EM field and charged matter. Variation w.r.t. A gives

$$\frac{1}{\mu_0}\partial_\mu F^{\mu\nu} + \frac{\delta S_{\mathrm{matter}}}{\delta A_\nu} = 0, \tag{1.167}$$

$$\partial_\nu F^{\mu\nu} = \mu_0 \frac{\delta S_{\mathrm{matter}}}{\delta A_\mu}, \tag{1.168}$$

hence we conclude the variation on S_{matter} w.r.t. A should reproduce the charge current coupling to the EM field. Conversely, we can identify the variation of the matter action w.r.t. the EM field as the definition for the current. Now we can denote the whole action as

$$S_{\mathrm{matter}}[A,\Phi] = S_{\mathrm{m}}[\Phi] + \int \mathrm{d}^4x\ J^\mu[\Phi]A_\mu. \tag{1.169}$$

The existence of matter should not break the gauge symmetry of the original A field, hence we must require

$$\partial_\mu J^\mu[\Phi] = 0, \tag{1.170}$$

hence the local charge conservation is automatically implied by the $\mathsf{U}(1)$ gauge symmetry. Note the variation of $S_{\mathrm{matter}}[A,\Phi]$ w.r.t. Φ under a gauge transformation is automatically zero for an on-shell Φ.

Now we see that gauge symmetry almost completely determines the theory, and this logic can be applied to GR, where the fundamental degree of freedom is the metric $g_{\mu\nu}$ of (M,g). Suppose the gravitational field is coupled to matter which is described by a generic tensor field Φ, then the action can be written as

$$S_{\mathrm{GR}}[g,\Phi] = S_{\mathrm{G}}[g] + S_{\mathrm{matter}}[g,\Phi]. \tag{1.171}$$

$S_{\mathrm{matter}}[g,\Phi]$ can be immediately obtained from the special-relativistic action according to the minimal substitution rule:

$$\int \mathrm{d}^4x\ \mathcal{L}[\Phi, \partial_\mu\Phi] \ \Rightarrow\ \int \mathrm{d}^4x\ \sqrt{-g}\mathcal{L}[\Phi, \nabla_\mu\Phi], \tag{1.172}$$

and now the Euler-Lagrange equation writes

$$\frac{\partial\mathcal{L}}{\partial\Phi} = \nabla_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right]. \tag{1.173}$$

Analogous to defining the charge current by varying the matter action w.r.t. A_μ, we can define the current coupling matter and the metric by variation, named as the energy-momentum tensor of matter $\Phi = T^{\mu\nu}\partial_\mu\otimes\partial_\nu$. We expect

$$\delta_g S_{\text{matter}} \equiv \frac{1}{2}\int \mathrm{d}^4x\ \sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}, \tag{1.174}$$

hence

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}}. \tag{1.175}$$

Here the factor 2 convention comes from the symmetric property of $T^{\mu\nu}$. Note we can lower the indices of $T^{\mu\nu}$ to obtian

$$\delta_g S_{\text{matter}} \equiv -\frac{1}{2}\int \mathrm{d}^4x\ \sqrt{-g}T_{\mu\nu}^{\flat_1\flat_2}\delta g^{\mu\nu}, \tag{1.176}$$

where the minus sign comes from $0 = \delta(\delta^\mu_\nu) = \delta(g^{\mu\rho}g_{\rho\nu})$, and the closed-form definition for $T_{\mu\nu}^{\flat_1\flat_2}$ can be accordingly derived.

In GR, we can always locally change the coordinate, resulting in a series of different metrics describing the same physics; hence, the gauge symmetry of GR is dentified to be the **local diffeomorphism invariance**. For a local change of coordinates $x'^\mu(x)$,

$$g_{\mu\nu} \ \Rightarrow\ g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}. \tag{1.177}$$

Now let $X \in \Gamma^\infty(TM)$ s.t. in a local chart, $X(x) = X^\mu(x)\partial_\mu$, then an infinitesimal local change of coordinates generated by this vector field reads

$$x'^\mu = x^\mu + \varepsilon X^\mu(x), \qquad 0 < \varepsilon \ll 1. \tag{1.178}$$

Therefore, the transformation for the metric induced by this coordinate transformation reads

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x'-\varepsilon X) = g_{\mu\nu} - \varepsilon(\partial_\mu X^\rho g_{\rho\nu} + \partial_\nu X^\rho g_{\mu\rho} + X^\rho\partial_\rho g_{\mu\nu}) + \mathcal{O}(\varepsilon^2). \tag{1.179}$$

Hence,

$$\delta_{\text{gauge}}g_{\mu\nu} = g'_{\mu\nu} - g_{\mu\nu} = -\varepsilon(\partial_\mu X^\rho g_{\rho\nu} + \partial_\nu X^\rho g_{\mu\rho} + X^\rho\partial_\rho g_{\mu\nu}) + \mathcal{O}(\varepsilon^2). \tag{1.180}$$

We can always change the ordinary derivatives to covariant derivatives subtracting the Christoffell symbols which cancel out in the end, and the result neatly writes

$$\delta_{\text{gauge}} g_{\mu\nu} = -\varepsilon(g_{\rho\nu}\nabla_\mu X^\rho + g_{\mu\rho}\nabla_\nu X^\rho) = -\varepsilon(X^\flat_{\nu;\mu} + X^\flat_{\mu;\nu}). \tag{1.181}$$

This implies that if we vary the metric in this way, then the variation is merely a local diffeomorphism and do not actually change the metric field. Now we can consider

$$\begin{aligned}\delta_{\text{gauge}} S_{\text{matter}}[g,\Phi] &= \frac{1}{2}\int \mathrm{d}^4x\, \sqrt{-g} T^{\mu\nu}\delta_{\text{gauge}} g_{\mu\nu} = \frac{1}{2}\int \mathrm{d}^4x\, \sqrt{-g} T^{\mu\nu}(X^\flat_{\nu;\mu} + X^\flat_{\mu;\nu}) \\ &= \int \mathrm{d}^4x\, \sqrt{-g} T^{\mu\nu} X^\flat_{\mu;\nu}. \end{aligned} \tag{1.182}$$

Here we implicitly assume the on-shellness of the matter field. Then, after an integration-by-part using

$$Y^\mu{}_{;\nu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}Y^\mu)_{,\nu}, \tag{1.183}$$

we find the gauge (local diffeomorhism) invariance requires

$$\nabla_\mu T^{\mu\nu} = 0. \tag{1.184}$$

Having established the gauge symmetry underlying the gravitational theory, we only have to look for $S_G[g] = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_G[g,\partial g]$ that inherits the expected gauge symmetry. Denote

$$\delta S_G[g] = \int \mathrm{d}^4x\, \sqrt{-g} E_{\mu\nu}\delta g^{\mu\nu}. \tag{1.185}$$

Clearly, $E^{\mu\nu}$ is symmetric. Besides, $E_{\mu\nu}$, viewed as proportional to the energy-momentum tensor of the gravitational field, should satisfy the Bianchi identity ensuring energy-momentum conservation:

$$E^{\mu\nu}{}_{;\nu} = 0. \tag{1.186}$$

Moreover, we demand $E^{\mu\nu}$ gives a field equation for g up to the 2nd order derivative, because

- In Newton's gravity, the Poisson equation is a second-order equation, and we naively expect the GR field equation is also a second-order one to bring the reasonable Newtonian limit.
- Extending Ostrogradsky's theorem to field theory says any Lagrangian theory with Euler-Lagrangian equation of order strictly greater than 2 are generally classically unstable, in the sense that they develop Ostrogradsky ghost.

Now we have the following theorem:

Theorem 1.5 (Lovelock, 1972) *In four dimension spacetime, the only action yielding an $E^{\mu\nu}$ with the two properties above is the Einstein-Hilbert action*

$$S_G[g] = \frac{1}{2\kappa}\int \mathrm{d}^4x\, \sqrt{-g}(R_\nabla - 2\Lambda), \tag{1.187}$$

where $\kappa \in \mathbb{R}_+$ *and* $\Lambda \in \mathbb{R}$.

Varying this action, we have the Einstein's field equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.188}$$

where

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_\nabla. \tag{1.189}$$

Adding the matter components, we have

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.190}$$

Remark

- κ is fixed by taking the Newtonian limit.
- Λ, the cosmological constant, is permetted by Lovelock's theorem and is responsible for the late acceleration of the expansion of the universe. Since $\Lambda \sim 1.1 \times 10^{-52}\, m^{-2}$, it has no influence on the astrophysics scale and we shall set it to zero.
- In the classical vacuum, $T_{\mu\nu} = 0$, hence $G_{\mu\nu} = 0$. However, this doesn't imply $g_{\mu\nu} = 0$! Non-trivial solutions exists, like the Schwartzschild metric.

1.4.6 Linearized Einsterin Gravity

We can consider a metric slighlty deviates from the Minkowskian metric in a local chart such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.191}$$

where $h_{\mu\nu}$ is a small perturbation. To the linear order, the Christoffel symbol reads

$$\Gamma^{\mu}{}_{\rho\sigma} = \frac{1}{2}\eta^{\mu\nu}(\partial_\rho h_{\nu\sigma} + \partial_\sigma h_{\nu\rho} - \partial_\nu h_{\rho\sigma}) + \mathcal{O}(h^2), \tag{1.192}$$

and the Riemann tensor approximates to

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \mathcal{O}(h^2). \tag{1.193}$$

After contracting indices to obtain the Ricci curvature and the scalar curvature, the Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = \cdots \tag{1.194}$$

Define

$$\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{1.195}$$

hence

$$h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma, \tag{1.196}$$

and the Einstein tensor gets simplified to

$$G_{\mu\nu} = \frac{1}{2}(\gamma^{\lambda}_{\mu,,\lambda\nu} + \gamma^{\lambda}_{\nu,\lambda\mu} - \eta_{\mu\nu}\gamma^{\lambda\sigma}{}_{,\lambda\sigma} - \Box\gamma_{\mu\nu}). \tag{1.197}$$

This expression deviates from the expected form for a wave equation; nevertheless, the extra terms can be removed by exploiting the gauge invariance in GR so that the gauge-fixing $\partial_\mu\gamma^{\mu\nu} = 0$ (Hilbert gauge) is imposed.

The viability for this gauge-fixing can be proved by considering a local diffeomorphism $x^\mu \to x^\mu + X^\mu(x)$ so that

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu X_\nu + \partial_\nu X_\mu, \tag{1.198}$$

hence

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu} + \partial_{(\mu}X_{\nu)} - \eta_{\mu\nu}\partial_\rho X^\rho. \tag{1.199}$$

If the intial $\gamma^{\mu\nu}$ doesn't satisfy the Hilbert gauge, the expected local diffeomorphism generators X^μ should satisfy

$$\Box X^\mu = -\partial_\nu\gamma^{\nu\mu}. \tag{1.200}$$

Solution for X^μ alyways exists, hence we can always find a suitable coordinate frame making $\gamma^{\mu\nu}$ in the Hilbert gauge.

Remark

The Hilbert gauge doesn't fix all the gauge degree of freedoms in the metric! We can still gauge transform the metric using a vector field satisfying $\Box X^\mu = 0$.

In the Hilbert gauge, the Einstein tensor neatly writes

$$G_{\mu\nu} = -\frac{1}{2}\Box\gamma_{\mu\nu} + \mathcal{O}(\gamma^2). \tag{1.201}$$

Then, the field equation writes

$$-\frac{1}{2}\Box\gamma_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.202}$$

In the vacuum, $T_{\mu\nu} = 0$, and

$$\Box\gamma_{\mu\nu} = 0. \tag{1.203}$$

We immediately find it propagates at c. The general solution is exactly a retarded potential

$$\gamma_{\mu\nu}(t, \boldsymbol{r}) = 2\kappa \int \mathrm{d}^3\boldsymbol{r}' \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} T_{\mu\nu}\left(t - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}, \boldsymbol{r}'\right). \tag{1.204}$$

For realistic situations, we can consider multi-pole expansion for the retarded potential. However, the energy-momentum conservation of $T^{\mu\nu}$ forbids non-trivial dipole terms, hence the leading-order contribution to the gravitational wave comes from quadruapole terms.

Remark

Such linearization can be generalized to arbitrary background spacetime, and we will find the background geometry affects the propagation of GW (which can be interpreted as the self-interaction due to non-linearity) and exactly follows the null geodesics.

Let's count the actual physical degree of freedoms of $\gamma_{\mu\nu}$. As a 4×4 symmetric matrix, it has 10 independeent components. The Hilbert gauge gives 4 constraints, and the residual gauge freedom corresponding to $\Box X^\mu = 0$ gives 4 extra unphysical degree of freedom; in the end, we find $\gamma_{\mu\nu}$ only has 2 physical degrees of freedom and corresponds to two polarizations for it.

To manifest the two polarization modes, let's consider the harmonic plane waves

$$h_{\mu\nu} = A_{\mu\nu} e^{\mathrm{i}k_\nu x^\nu}. \tag{1.205}$$

The trace-free transverse gauge from $\Box X^\mu = 0$ gauge freedom gives

$$-A_{00} + \sum_i A_{ii} = 0, \quad A_{0i} = A_{i0} = 0. \tag{1.206}$$

The Hilbert gauge gives

$$A_{\mu\nu} k^\nu = 0. \tag{1.207}$$

For $k^\mu = (\omega, 0, 0, \omega)$, we find

$$A_{\mu\nu} = A^{(+)}_{\mu\nu} + A^{(\times)}_{\mu\nu}, \tag{1.208}$$

where

$$A^{(+)}_{\mu\nu} = a_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{(\times)}_{\mu\nu} = a_\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.209}$$

If we assume the matter is static in the weak-field limit, we obtain the Newtonian limit, hence

$$T_{00} = \rho c^2, \quad T_{0i} = T_{i0} = T_{ij} = 0, \tag{1.210}$$

henc

$$\gamma_{00} = \frac{2\kappa c^4}{4\pi G_N} \frac{G_N}{c^2} \int \mathrm{d}^3 \boldsymbol{r}' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \equiv -\alpha \frac{\Phi}{c^2}, \tag{1.211}$$

where we denote

$$\alpha \equiv \frac{2\kappa c^4}{4\pi G_N}. \tag{1.212}$$

and we used the Newtonian gravity.

$$\nabla^2 \Phi = -4\pi G_N \rho, \tag{1.213}$$

From the calculated $\gamma_{\mu\nu}$, we have

$$h_{00} = -\frac{\alpha}{2}\frac{\Phi}{c^2}, \qquad h_{ij} = -\frac{\alpha}{2}\delta_{ij}\frac{\Phi}{c^2}. \tag{1.214}$$

According to the Newtonian approximation for GR, we expect

$$h_{00} = -\frac{2\Phi}{c^2}, \tag{1.215}$$

hence $\alpha = 4$ and

$$\kappa = \frac{8\pi G_N}{c^4} \approx 2.08 \times 10^{-43}\ \mathrm{N}^{-1}. \tag{1.216}$$

1.5 The Schwarzschild Solution

Definition 1.40 (Killing Vectors) *Let (M, g) be a smooth Lorentzian 4-manifold and $K \in \Gamma^\infty(TM)$ be a smooth vector field, then K is a Killing vector field iff in any chart $(\phi = x^\mu, \Omega)$,we have*

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{1.217}$$

According (1.181), the Killing vectors are generators of local diffeomorphisms preserving the metric tensor invariant, named as isometries.

Theorem 1.6 *Let (M, g) be a smooth Lorentzian manifold with a Killing vector field K. Let γ be a geodesic with 4-velocity $\dot{\gamma}$, then $K_\mu \dot{\gamma}^\mu$ is constaant along γ.*

It can be directly proven by expanding

$$\frac{\mathrm{d}}{\mathrm{d}s}(K_\mu \dot{\gamma}^\mu(s))$$

along the geodesic.

The Schwarzschild solution contains two charts: Exterior chart $\mathrm{Sch}_4^>$ where $r > r_s$ and the interior chart $\mathrm{Sch}_4^<$ where $r < r_s$. The metric on the two charts admits the same form:

$$\mathrm{d}s^2 = -c^2\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{1.218}$$

where r_s is the Schwarzschild radius

$$r_s = \frac{2G_N M}{c^2}. \tag{1.219}$$

It's a static spherical symmetric solution of the vacuum Einstein field equation $G_{\mu\nu} = 0$. Its Killing vectors are:

$$K_t = \partial_t, \tag{1.220}$$
$$K_x = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi, \tag{1.221}$$
$$K_y = \cdots, \tag{1.222}$$
$$K_z = \cdots. \tag{1.223}$$

Theorem 1.7 (Birkoff-Jebsen Theorem) *Schwarzschild solution is the unique spherically symmetric solution of $G_{\mu\nu} = 0$ up to diffeomorphism.*

For astrophysical applications, the exterior patch $\mathrm{Sch}_4^>$ is sufficient since the Schwarzschild radius

$$r_s = \frac{2G_N M}{c^2} \approx 2.95 \frac{M}{M_\odot} \text{ km} \tag{1.224}$$

is way smaller that astrophysical scales.

1.5.1 Anomalous Precession of the Mercury Perihelion

It's one of the classical tests of GR. In GR, the mercury follows a time-like geodesic in the Schwarzschild metric centered at the sun's center of mass.

In GR, we just have to work out the geodesic equation for a time-like particle

1.5.2 Light Deflection

Null geodesics in the Schwarzschild spacetime.

1.6 Blackhole Solutions

1.6.1 Schwarzschild Blackhole

If an astrophysical object's scale is less that its Schwarzschild radius, then it'll be a SSchwarzschild blackhole. In this case, the submanifold $\mathrm{Sch}_4^>$ is geodesically incomplete, which mathematically implies we have to extend the $\mathrm{Sch}_4^>$ manifold.

The origin of geodesical incompletance usually comes from the geodesic reaches the boundary of all available charts on the manifold (M, g), and it can be amended by adding charts covering the regions that were not previously reached. However, if the geodesic incompletance originates in spacetime singularity, then there exists a maximal extension and geodesics in thee extended spacetime remain incomplete and they will terminate at aa singularity.

Kruskal-Szekeres Extension We focus on null geodesics in the Schwarzschild spacetime manifesting causality structure, whose raadial geodesic writes

$$\mathrm{d}t = \pm \frac{1}{1 - \frac{r_s}{r}} \mathrm{d}r \equiv \pm \mathrm{d}r^*, \tag{1.225}$$

$$r^* \equiv \int \frac{1}{1 - \frac{r_s}{r}} \mathrm{d}r. \tag{1.226}$$

The lightcone coordinaate is defined to be

$$u \equiv t + r^*, \qquad v = t - r^*. \tag{1.227}$$

In this way, outgoing and ingoing light rays are given by $v = Const$ and $u = Const$ respectively. In this new coordinate, the Schwarzschild metric readss

$$g = -\frac{1}{2} \left(1 - \frac{r_s}{r}\right) (\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + r^2 \mathrm{d}\Omega^2, \tag{1.228}$$

where r is a function of u, v s.t.

$$\frac{u-v}{2r_s} = \frac{r}{r_s} + \ln\left(\frac{r}{r_s} - 1\right) \Rightarrow e^{r/r_s}\frac{r}{r_s}\left(1 - \frac{r}{r_s}\right) = e^{(u-v)/(2r_s)}. \tag{1.229}$$

Defining

$$U = e^{\frac{u}{2r_s}}, \qquad V = e^{-\frac{v}{r_s}}, \tag{1.230}$$

the mettric then writes

$$g = -\frac{2r_s^3}{r}e^{-\frac{r}{r_s}}(\mathrm{d}U\mathrm{d}V + \mathrm{d}V\mathrm{d}U) + r^2\mathrm{d}\Omega^2. \tag{1.231}$$

Using the Lambert function W_0,

$$\frac{r}{r_s} - 1 = W_0(-e^{-1}UV). \tag{1.232}$$

In the end, we fine

$$g = -\frac{2r_s^2 e^{-[1+W_0(-e^{-1}UV)]}}{1 + W_0(-e^{-1}UV)}(\mathrm{d}U\mathrm{d}V + \mathrm{d}V\mathrm{d}U) + r_s^2[1 + W_0(-e^{-1}UV)]^2\mathrm{d}\Omega^2, \tag{1.233}$$

the singularity at the Schwarzschild radius gets removed after such a coordinaate transformation. $\mathrm{Sch}_4^>$ cooresponds to $U > 0$, $V < 0$. However, the metric is singular when $UV = 0$ and $UV < 1$, but it's only a coordinate singularity which can be removed with a proper coordinate transformation. Nevertheless, the curvature diverges at $UV = 1$, implying it's a true singularity.

Noting outgoing and ingoing light rays corresponds to $V = Const$ and $U = Const$, hence the causal structure can be easily determined in the Kruskal-Szekeres metric.

1.6.2 General Definition for a Blackhole

Definition 1.41 (Time Orientability) *Let (M, g) be a smooth Lorentzian 4-manifold. We say it's time orientable if it admits a smooth time-like vector field $\mathscr{T}$ at every point on M.*

2 Cosmology

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \qquad \kappa^2 = 8\pi G. \tag{2.1}$$

2.1 The FRW Metric

The cosmologicl principle dictates the universe is homogeneous and isotropic, hence the spacetime should be maximally symmetric. We can always foliate the globally hyperbolic spacetime manifold into equal-time slices such that the metric writes

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\gamma_{ij}(\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j. \tag{2.2}$$

We can always choose the coordinate t as the proper time of the observer at rest, hence $N(t) = 1$. Besides, the spatial istropy implies invariance by rotation, so the angular part of the metric should be spherical:

$$\gamma_{ij}(\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j = B(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{2.3}$$

This spatial metric leads to the spatial Ricci tensor

$$R^{(3)} = \frac{2}{r^2}\left[1 - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{B}\right)\right]. \tag{2.4}$$

Since the space should be homogeneous, then $R^{(3)}$ should be a constant, hence

$$B(r) = \frac{1}{1 - Kr^2}, \qquad K \equiv \frac{R^{(3)}}{6}. \tag{2.5}$$

In the end, we find the FRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}^2 r}{1 - Kr^2} + r^2\mathrm{d}\Omega^2\right). \tag{2.6}$$

We can further take the liberty to rescale r and $a(t)$ to make K takes its value in -1, 0 and $+1$. The precise form of $a(t)$ shall be determined from the Einstein's equation. With $K = +1$, the universe is of a spherical geometry and is spatially compact; on the contrary, $K = -1$ corresponds to a hyperbolic universe and is conventionally called an open universe. Fortunately, current observations favor the flat uniderse with $K = 0$.

2.2 The Friedmann Equaations

Having obtained the metric, we only have to insert it into the Einstein's equation to get the EoM of the scale factor $a(t)$. According to the cosmological principle, the energy-momentum tensor of the unvierse should be a perfect fluid with the form

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{2.7}$$

where u^μ is the 4-velocity of the cosmological fluid. The energy density ρ and the pressure p are related by the equation of state of the matter playing the game in the evolution, and to the first order they can generally be written as

$$p = w\rho. \tag{2.8}$$

It's wel-known that non-relativistic matter has $w = 0$, while relativistic one has $w = \frac{1}{3}$, whereas the cosmological constant has $w = -1$.

The Einstein tensor of the FRW metric writes

$$G_{00} = 3H^2 + \frac{3K}{a^2}, \quad G_{0i} = 0, \quad G_{ij} = -\left(\frac{2\ddot{a}}{a} + H^2 + \frac{K}{a^2}\right)g_{ij}, \tag{2.9}$$

where H is the Hubble parameter defined to be

$$H = \frac{\dot{a}}{a}. \tag{2.10}$$

Since

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{2.11}$$

hence the 00-component gives

$$3H^2 + \frac{3K}{a^2} = \kappa^2 \rho + \Lambda \;\Rightarrow\; H^2 = \frac{1}{3}\kappa^2 \rho - \frac{\Lambda}{3} - \frac{K}{a^2}, \tag{2.12}$$

which is the first Friedmann equation. The spatial component gives

$$-\left(\frac{2\ddot{a}}{a} + H^2 + \frac{K}{a^2}\right) = p - \Lambda. \tag{2.13}$$

Substitute the H^2 given by the first Friedmann equation, we get the second Friedmann equation

$$\dot{H} = -\frac{\kappa^2}{2}(\rho + p) + \frac{K}{a^2}. \tag{2.14}$$

Note there's also an equation for energy conservation:

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.15}$$

(2.12), (2.14) and (2.15) are not linearly independent. In practice, we usuaally combine the equation of state and the energy conservation equation (2.15) to obtain the relation $\rho(a)$, then substitute it in the first Friedmann equation (2.12) to solve for the time evolution of $a(t)$. We can define

$$\rho_\Lambda \equiv \frac{\Lambda}{\kappa^2}, \qquad \rho_K \equiv= -\frac{3K}{\kappa^2 a^2}, \tag{2.16}$$

then the Friedmann equations can be neatly written as

$$H^2 = \frac{1}{3}\kappa^2 \sum_i \rho_i, \qquad \dot{H} = -\frac{\kappa^2}{2}\left(\sum_i \rho_i + \sum_i p_i\right). \tag{2.17}$$

For a matter with $p = w\rho$, the continuity equation gives

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}, \tag{2.18}$$

hence

$$\rho(a) = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}, \tag{2.19}$$

then the first Friedmann equation implies

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}, \tag{2.20}$$

hence

$$\dot{a} = \sqrt{\frac{\rho_0\kappa^2}{3a_0^{-3(1+w)}}}a^{-\frac{1+3w}{2}}, \tag{2.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{a}{a_0} = \sqrt{\frac{\rho_0\kappa^2}{3}}\left(\frac{a}{a_0}\right)^{-\frac{1+3w}{2}}. \tag{2.22}$$

The dynamics can be accordingly determined by simple integration.

2.3 Distance Measures

The FRW matric can be transformmed to the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}\chi^2 + f_K^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)], \quad f_K(\chi) = \begin{cases} \sin\chi, & K = +1, \\ \chi, & K = 0, \\ \sinh\chi, & K = -1. \end{cases} \tag{2.23}$$

Supopose an object at $(t_e, \chi, 0, 0)$ emits a photon which is recevied by us at $(t_0, 0, 0, 0)$, then the comoving distance is $d_{\text{c.m.}} = \chi$, while the physical distance writes

$$d_{\text{phy}} = a(t_0)\chi. \tag{2.24}$$

However, this physical distance cannot be directly measured. Observtions can only measure the distance either by paralex, giving the angular distance, or by luminosity measurements, giving the luminosity distance. Let's study these two distances now.

2.3.1 The Angular Distance

Suppose the physical size of the object is known to be ℓ at the emission, and let's align the direction of ℓ with the θ-axis. Then

$$\theta_e = \frac{\ell}{a(t_e)f_K(\chi)}. \tag{2.25}$$

It's the same angle measured by us today, hence the angular distance obtained it

$$d_{\text{ang}} = \frac{\ell}{\theta_e} = a(t_e)f_K(\chi). \tag{2.26}$$

We can use it to convert the angular distance to the comoving distance and obtain any result we like.

2.3.2 The Luminosity Distance

Suppose the luminosity (power) of the object is know to be L at the emission, and the measured power flux is Φ, then the lumonisity distance is defined to be

$$d_{\text{lumi}} = \sqrt{\frac{L}{4\pi\Phi}}. \tag{2.27}$$

Now let's connect this quantity to the comoving distance, i.e., theoretically compute Φ. Consider a small time interval δt_e at the emission, the energy released is $L\delta t_e$. The released energy suffers from two suppressions:

- The cosmological redshift: In the cosmological coordinate, the 4-velocity of the object is $u_e^\mu = (1, 0, 0, 0)$, and the photon with energy E released by it is of the initial 4-momentum $p_e^\mu = E_e(1, 1/a(t_e), 0, 0)$ and follows the radial null geodesic genrated by it, whose time-component reads

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{E_e a(t_e)}{a}. \tag{2.28}$$

  Therefore, for us at t_0, the photon's energy reads $E_o = p^0(t_0) = E_e a(t_e)/a(t_0) = L\delta t_e a(t_e)/a(t_0)$.

- Time dilation: To receive the amount of energy E_o costs us a different time interval from δt_e. Note the comoving length of the photon packet is $\delta\chi = \delta t_e/a(t_e)$ should be invariant, hence it costs the observer $\delta t_o = a(t_0)\delta\chi = \delta t_e a(t_0)/a(t_e)$ to recevie the photon.

Therefore, the power reaching the sphere of area $4\pi a^2(t_0) f_K^2(\chi)$ is

$$L_o = \frac{E_o}{\delta t_o} = L\frac{a^2(t_e)}{a^2(t_0)}, \tag{2.29}$$

and

$$\Phi = \frac{L_o}{4\pi a^2(t_0) f_K^2(\chi)} = \frac{L}{4\pi f_K^2(\chi)}\frac{a^2(t_e)}{a^4(t_0)} \tag{2.30}$$

In the end we find

$$d_{\text{lumi}} = \frac{a^2(t_0)}{a(t_e)} f_K(\chi) = \frac{a^2(t_0)}{a^2(t_e)} d_{\text{ang}}. \tag{2.31}$$

2.4 Horizons

There are two horizons in cosmology.

The particle horizon is the physical distance to the farthest objects that are theoretically observable, i.e., the theoretical maximal causal connection distance since the birth of the universe (if it ever exists). Suppose the universe births at $t = 0$, then the physical distance of the particle at time t_0 is

$$d_{\text{P.H.}} = a(t_0) \int_0^{t_0} \mathrm{d}t \frac{1}{a(t)}. \tag{2.32}$$

Note the size of particle horizon nowadays is different from the size of the observable universe: The horizon is integrated from the big bang singularity, while the integration for the observable universe starts from the time of recombination.

The event horizon is the maximal physical distance to the farthest object from which we may establish a casual connection in the future. Suppose the uniderse terminates at t_∞, then

$$d_{\text{E.H.}} = a(t_0) \int_{t_0}^{t_\infty} \mathrm{d}t \frac{1}{a(t)}. \tag{2.33}$$

3 Inflation Cosmology

3.1 Conformal Time

In the previous given FRW metric, the coordinate time t is the proper time of the comoving observer. However, the conformal time $\mathrm{d}\tau \equiv \mathrm{d}t/a(t)$ is more handy in inflation cosmology. In this way, the metric reads

$$\mathrm{d}s^2 = a^2(\tau)[-\mathrm{d}\tau^2 + \gamma_{ij}(x)\mathrm{d}x^i \mathrm{d}x^j]. \tag{3.1}$$

We will also define the number of e-folds as

$$\Delta N \equiv \ln \frac{a_1}{a_0}. \tag{3.2}$$

3.2 Inflation

The conventional big bang cosmology predicts the horizon size at the recombination time to be small, which contradicts with the uniform and isotropic feature of CMB. Besides, the big bang picture cannot provide us with a natural mechanism for the universe to be such flat. These perplexes can be solved by introducing inflation at the primordial stage.

The simplest inflation model is drived by a scalar field called inflaton, whose Lagrangian writes

$$\mathcal{L}[\phi, g] = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \tag{3.3}$$

whose energy-momentum tensor is given by a variation on the metric tensor w.r.t the action

$$S = \int \mathrm{d}^4 x \sqrt{-g} \mathcal{L}[\phi, g], \tag{3.4}$$

hence

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g}\mathcal{L}[\phi, g]) = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} (\partial\phi)^2 + V(\phi) \right]. \tag{3.5}$$

Assume ϕ is a homogeneous field only depends on time, we find

$$\rho = T_{00} = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{3} T^i{}_i = \frac{1}{2} \dot{\phi}^2 - V(\phi). \tag{3.6}$$

If the potential energy dominates, the kinetic term can be negelected and we find $p \approx -p$, resembles the case where the cosmological constant dominates.

Substituting the ρ and p into the Friedmann equation gives

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \qquad \dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2. \tag{3.7}$$

Besides, the EoM of the inflaton field is

$$\nabla_\mu \partial^\mu \phi - \frac{\partial V}{\partial \phi} = 0 \;\Rightarrow\; \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \tag{3.8}$$

Note under our assumption

$$\nabla_\mu \partial^\mu \phi = -\ddot{\phi} - \frac{1}{2} g^{\rho\sigma}\partial_0 \dot{\phi} = -\ddot{\phi} - 3H\dot{\phi}, \quad H \equiv \frac{\dot{a}}{a}. \tag{3.9}$$

We are still working in the standard FRW metric rather than the one formulated in conformal time.

The current standard model for inflation is the slow-roll inflation. To achieve the potential energy dominance, we expect $V(\phi)$ be flat enough and high enough. Notice the EoM is of the form of the Newton's 2nd law with the friction term $3H\dot{\phi}$, hence we expect $3H\dot{\phi} \gg \ddot{\phi}$. It's conventional to define the slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \eta \equiv -\frac{\ddot{\phi}}{\dot{\phi}H}, \quad \xi \equiv \frac{\dddot{\phi}}{\dot{\phi}H^2}. \tag{3.10}$$

In the slow-roll regime, $|\epsilon|, |\eta| \ll 1$, hence the nearly constant Hubble constant induces an exponential expansion, and the ϕ keeps rolling at a slow, nealry constant speed. In this case, we can negelect the acceleration term in the EoM of ϕ and obtain

$$\dot{\phi} \approx -\frac{1}{3H}\frac{\partial V}{\partial \phi}, \tag{3.11}$$

while one of the Friedmann equations give

$$H^2 = \frac{\kappa^2}{3} V. \tag{3.12}$$

In this way,

$$\epsilon = \frac{\kappa^2 \dot{\phi}^2}{2H^2} = \frac{1}{2\kappa^2 V^2}\left(\frac{\partial V}{\partial \phi}\right)^2, \quad \eta = -\epsilon + \frac{1}{\kappa^2 V^2}\frac{\partial^2 V}{\partial \phi^2} \approx -\epsilon. \tag{3.13}$$

Slow-roll inflation models generally fall in three categories:

- Large-field models: Field starts at large values and rolls down to zero, like the chaotic inflation where $V(\phi) = \frac{1}{4}\lambda\phi^4$.

- Small-field models: Field starts close to zero and rolls away towards a global minimum, analogous to a dynamical spontaneous symmetry breaking.

- Hybrid models: Field start like a large-field inflation, but the inflation is ended by an interruption of another fields.

Within these classifications, we can further say a model is convex if $\frac{\partial^2 V}{\partial \phi^2} > 0$, otherwise is concave.

3.3 Fluctuations of the Inflaton

Now we assign spatial inhomogeneity to the inflation field by adding a perturbation and working with the conformal time τ and setting $K = 0$:

$$\phi(\tau, \vec{x}) = \bar{\phi}(\tau) + \delta\phi(\tau, \vec{x}), \quad \delta\phi \ll \bar{\phi}. \tag{3.14}$$

Accordingly, the metric also fluctuates over the homogeneous FRW background and we denote

$$g_{\mu\nu}(\tau, \vec{x}) = \bar{g}_{\mu\nu}(\tau, \vec{x}) + \delta g_{\mu\nu}(\tau, \vec{x}). \tag{3.15}$$

Let's now consider the full Einstein-Hilbert action

$$S[\phi, g] = \int \mathrm{d}^4 x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \right), \tag{3.16}$$

we clearly can perturbatively expand is w.r.t. the fluctuations.

$g_{\mu\nu}$ contains a lot of gauge freedom, and we have to find a suitable way parameterizing it and characterizing its fluctuations. We resort to the ADM decomposition claiming that any well-behaved matric can be written as

$$\mathrm{d}s^2 = -N^2 \mathrm{d}\tau^2 + \tilde{h}_{ij}(\mathrm{d}x^i + N^i \mathrm{d}\tau)(\mathrm{d}x^j + N^j \mathrm{d}\tau) \tag{3.17}$$

We then find $\bar{g}_{\mu\nu}(x)$ corresponds to $N^2 = a^2(t)$, $\tilde{h}_{ij} = a^2(t)\delta_{ij}$. Therefore, we can perturb the ADM quantities in the following way:

$$N = a^2(t)(1 + 2\Phi), \tag{3.18}$$

$$N_i = a^2(t)(\partial_i B + S_i), \tag{3.19}$$

$$\tilde{h}_{ij} = a^2(t)(\delta_{ij} - 2\Psi\delta_{ij} + \partial_i F_j + \partial_j F_i + 2\partial_i \partial_j E + h_{ij}). \tag{3.20}$$

In this way, Φ, B, Ψ, E transform as $\mathsf{SO}(3)$ scalars, while $\vec{S}, \vec{F}$ are $\mathsf{SO}(3)$ vectors, and h_{ij} is the only $\mathsf{SO}(3)$ tensor. The advantage of this parameterization scheme is the scalar, vector, and tensor perturbations decouple at the linear order. Besides, this decomposition expects the vector modes and tensor modes transforms as irreducible vector and tensor representations of $\mathsf{SO}(3)$, therefore we demand

$$\partial_i S^i = \partial_i F^i = \partial_i h^{ij} = h^i{}_i = 0. \tag{3.21}$$

Given h_{ij} should be symmetric, we conclude that the perturbations on the metric has 4 scalar degrees of freedom, 4 vector dofs, and 2 tensor dofs. The inflaton perturbation only contributes a single scalar fluctuation.

Now let's exploit the diffeomorphism invariance of GR to further simplify the problem at hand. We consider a small local diffeomorphism $x^\mu \to x^\mu + \varepsilon^\mu$, and we find

$$\Delta\delta g_{\mu\nu} = -\bar{g}_{\mu\lambda}\partial_\nu\varepsilon^\lambda - \bar{g}_{\lambda\nu}\partial_\mu\varepsilon^\lambda - \varepsilon^\lambda\partial_\lambda\bar{g}_{\mu\nu}, \tag{3.22}$$

$$\Delta\delta\phi = -\varepsilon^\lambda\partial_\lambda\bar{\phi}. \tag{3.23}$$

By choosing approporiate ε^μ, two scalar and two vector dofs can be removed, and we fix to the longitudinal-vector gauge where

$$B = E = 0, \quad F^i = 0. \tag{3.24}$$

Denote $f' \equiv \mathrm{d}f/\mathrm{d}\tau = a\dot{f}$, and

$$\mathcal{H} \equiv \frac{1}{a}a' = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = aH. \tag{3.25}$$

Then the variation of the 2nd action gives $\Phi = \Psi$. We defind the gauge-invariant curvature perturbation as

$$\zeta = -\Phi - \frac{\mathcal{H}}{\phi'}\delta\phi = -\Phi - \frac{H}{\dot{\phi}}\delta\phi = -\Phi - \frac{\kappa}{\sqrt{2\epsilon}}\delta\Phi = -\Phi - \frac{\kappa a}{\mathfrak{z}}\delta\phi, \tag{3.26}$$

where we define $\mathfrak{z} \equiv a\sqrt{2\epsilon}$. We have

$$\frac{\mathfrak{z}'}{\mathfrak{z}} = \frac{a'}{a} + \frac{\epsilon'}{\epsilon} = \mathcal{H}(1 + \epsilon - \eta), \tag{3.27}$$

and the equation of motion of the curvature perturbation is written as

$$\zeta'' + \frac{2\mathfrak{z}'}{\mathfrak{z}}\zeta' - \nabla^2\zeta = 0. \tag{3.28}$$

We can define the Mukanow-Sasaki variable

$$q \equiv -a\frac{\phi'}{\mathcal{H}}\zeta = -\frac{\mathfrak{z}}{\kappa}\zeta = a\left(\delta\phi + \frac{\phi'}{\mathcal{H}}\Phi\right), \tag{3.29}$$

then its EoM reads

$$q'' - \frac{\mathfrak{z}''}{\mathfrak{z}}q - \nabla^2 q = 0. \tag{3.30}$$

Notice that using the Mukhanov-Sasaki variable eliminates the first-order time derivative q' in the EoM, making it properly normalized and easy to quantize.

We can use the background's spatial homogeneity and consider

$$q(\tau, \vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} q_{\vec{k}}(\tau) e^{-\mathrm{i}\vec{k}\cdot\vec{x}}, \tag{3.31}$$

then we find

$$q''_{\vec{k}} + \left(k^2 - \frac{\mathfrak{z}''}{\mathfrak{z}}\right) q_{\vec{k}} = 0, \tag{3.32}$$

where

$$\frac{\mathfrak{z}''}{\mathfrak{z}} = \mathcal{H}^2(2 + 2\epsilon - 3\eta + 2\epsilon^2 - 4\epsilon\eta + \xi) \propto a^2 H^2. \tag{3.33}$$

We may qualitatively analysis the evolution of $q_{\vec{k}}(\tau)$. At the beginning $aH \ll |k|$, hence the comoving wavelength $|k|^{-1}$ is greater than the comoving Hubble radius $(aH)^{-1}$, and this regime is called the sub-Horizon regime. The mode is asymptotically a harmonic oscillator

$$q''_{\vec{k}} + k^2 q_{\vec{k}} = 0, \tag{3.34}$$

and it turns out a suitable choice of the vacuum state gives

$$q_{\vec{k}}(\tau) = \frac{1}{\sqrt{2k}} e^{-\mathrm{i}|k|\tau}. \tag{3.35}$$

When $aH = |\vec{k}|$, we say the mode exits the horizon. When $aH \gg |\vec{k}|$, we say the mode is in the super-horizon regime where the EoM asymptotically reads

$$q''_{\vec{k}} - \frac{\mathfrak{z}''}{\mathfrak{z}} q_{\vec{k}} = 0. \tag{3.36}$$

By matching to the sub-horizon solution to fix the integral constant, we find the non-decaying mode asymptotically reads

$$q_{\vec{k}} = \frac{H(\tau_k)}{2\sqrt{|k|^3 \epsilon(\tau_k)}} \mathfrak{z}, \tag{3.37}$$

where τ_k is determined by $\mathcal{H}(\tau_k) = |k|$. Therefore, at the superhorizon scale

$$\zeta_{\vec{k}} = -\frac{\kappa H(\tau_k)}{2\sqrt{|k|^3 \epsilon(\tau_k)}} \tag{3.38}$$

remains to be a constant.

By using the mode function of the curvature perturbation one can define the corresponding quantum field operator

$$\hat{\zeta}(\vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} [\zeta_{\vec{k}} e^{\mathrm{i}\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}} + \zeta^*_{\vec{k}} e^{-\mathrm{i}\vec{k}\cdot\vec{x}} \hat{a}^\dagger_{\vec{k}}], \tag{3.39}$$

and the actual observable is the power spectrum $P_\zeta(k)$ defined from the two-point function:

$$\left\langle \hat{\zeta}(\vec{x}) \hat{\zeta}(\vec{y}) \right\rangle \equiv \int \frac{\mathrm{d}k}{k} \frac{\sin(kr)}{kr} P_\zeta(k). \tag{3.40}$$

Computation shows

$$P_\zeta(k) = \frac{k^3}{2\pi^2}|\zeta_k|^2 = \frac{\kappa^2 H^2(\tau_k)}{8\pi^2\epsilon(\tau_k)}. \tag{3.41}$$

It only depends weakly on k and follows a power-law relation, hence we can base on a pivot scale k_0 and write it as

$$P_\zeta(k) = P_\zeta(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}, \tag{3.42}$$

the spectral index n_s can be shown to be equal to $2\eta_{k_0} - 4\epsilon_{k_0}$.

The inflation doesn't source the vector or tensor perturbations. But they still have quantum fluctuations. For the tensor modes, the Einstein equation gives

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 0, \tag{3.43}$$

and the super-horizon scale solution reads

$$h_k = \frac{\sqrt{2}\kappa H(\tau_k)}{\sqrt{k^3}}, \tag{3.44}$$

giving a power spectrum

$$P_t(k) = \frac{2\kappa^2 H^2(\tau_k)}{\pi^2}. \tag{3.45}$$

The tensor spectral index n_t is defined by

$$P_t(k) = P_t(k_0)\left(\frac{k}{k_0}\right)^{n_t}, \tag{3.46}$$

and one finds $n_t = -2\epsilon(k_0)$. The tensor-to-scalar ratio is

$$r \equiv \frac{P_t(k_0)}{P_\zeta(k_0)} = 16\epsilon(k_0) = -8n_t. \tag{3.47}$$

4 Evolution of Fluctuations after Inflation

For studying this stage, we only have to replace the inflaton by the matter content of the universe. Recall that

$$T^\mu{}_\nu = (\rho + p)u^\mu u_\nu + p\delta^\mu{}_\nu, \tag{4.1}$$

and now we introduce fluctuations to the density, the pressure, and the velocity of the cosmic fluid:

$$\rho(\tau, \vec{x}) = \bar{\rho}(\tau) + \delta\rho(\tau, \vec{x}), \tag{4.2}$$

$$p(\tau, \vec{x}) = \bar{p}(\tau) + \delta p(\tau, \vec{x}), \tag{4.3}$$

$$u^\mu(\tau, \vec{x}) = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{v}(\tau, \vec{x}) \end{pmatrix}. \tag{4.4}$$

In this way,

$$\delta T^0{}_0 = -\delta\rho, \quad \delta T^0{}_i = (\bar{\rho} + \bar{p})v_i, \quad \delta T^i{}_j = \delta p \delta^i{}_j. \tag{4.5}$$

We assume the equation of state $p/\rho = w$ remains constant.

The parameterization of the metric fluctuation is the same as before, now the scalar fluctuation reads

$$\Phi'' + 3\mathcal{H}(1+w)\Phi' - w\nabla^2\Phi = 0, \quad \Psi = \Phi. \tag{4.6}$$

Now the curvture fluctuation reads $\zeta = -\Phi - (\Phi' + \mathcal{H}\Phi)/(\mathcal{H}\epsilon)$, and its EoM becomes

$$\zeta'' + 2\mathcal{H}\zeta' - w\nabla^2\zeta = 0. \tag{4.7}$$

Define $\delta \equiv \delta\rho\rho$, then the density and velocity fluctuations' EoMs are

$$\delta' + (1+w)(\nabla^2 v - 3\Phi') = 0, \tag{4.8}$$

$$v' + \mathcal{H}(1-3w)v + \frac{w}{1+w}\delta + \Phi = 0, \tag{4.9}$$

where $v_i = \partial_i v$.

In the momentum space, the curvature fluctuation's EoM reads

$$\zeta_l'' + 2\mathcal{H}\zeta_k' + wk^2\zeta_k = 0 \tag{4.10}$$

At the super-horizon scale where $|k| \ll \mathcal{H}$, the non-decaying mode ζ_k is constant, hence Φ_k is constant, they are independent from w. Therefore, these super-horizon modes ζ_k remain constant even if the matter undergoes a phase transition, and we have $\zeta_k = \zeta_k^{\text{inflation}}$. However, since when $\Phi' = 0$ we have

$$\Phi_k = -\frac{3+3w}{5+3w}\zeta_k^{\text{inflation}}, \tag{4.11}$$

and it indeed depends on the matter content of the universe. Nevertheless, it still inherits the information from the inflation.

Modes at the sub-horizon scale where $k \gg \mathcal{H}$ behaves another way. During the radiation domination, $w = 1/3$, $\mathcal{H} = 1/\tau$, and

$$\zeta_k = \zeta_k^{\text{inflation}} \frac{\sin\left(k\tau/\sqrt{3}\right)}{k\tau/\sqrt{3}}, \tag{4.12}$$

$$\Phi_k = -2\zeta_k^{\text{inflation}} \left[\frac{\sin\left(k\tau/\sqrt{3}\right)}{(k\tau/\sqrt{3})^3} - \frac{\cos\left(k\tau/\sqrt{3}\right)}{(k\tau/\sqrt{3})^2}\right], \tag{4.13}$$

both of them are decaying. While during matter domination $w = 0$, ζ_k and Φ_k remain constant at all scales.

The structure of the large-scale structure depends on the matter density fluctuation δ^m, and the CMB depends on the radiation density fluctuation δ^γ. We focuse on them in a matter domianted universe where $\mathcal{H} = 2/\tau$. We find sub-horizon matter modes are

$$\delta_k^m = -\frac{2}{3\mathcal{H}^2} k^2 \Phi_k^m, \tag{4.14}$$

while super-horizon modes reads

$$\delta_k^m = -2\Phi_k^m = Const. \tag{4.15}$$

For modes entered the horizon during the matter dominance, we have

$$\Phi_k^m = -\frac{3}{5}\zeta_k^{\text{inflation}}; \tag{4.16}$$

while for modes entered the horizon during the radiation dominance, Φ_k^m would be suppressed by $\sim (k\tau_{\text{eq}})^{-2}$ due to the decay of sub-horizon modes in the radiation domination.

For the radiation component, we have

$$\delta_k^{\gamma\prime\prime} + \frac{1}{3}k^2\delta_k^\gamma = -\frac{4}{3}k^2\Phi_k^m, \tag{4.17}$$

whose solution reads

$$\delta_k^\gamma = A_k \cos\left(\frac{k\tau}{\sqrt{3}} + \varphi_k\right) - 4\Phi_k^m. \tag{4.18}$$

4.1 CMB

The recombination happens at $z \approx 1090$ where the universe becomes transparent for photons, and the CMB are photons coming from the surface of last scattering. Their energy are distributed acccording to the Planck distribution

$$u(\omega, t) = \frac{\omega^3}{\pi^2}\frac{1}{e^{\omega/T} - 1}. \tag{4.19}$$

Nowadays it peaks at $T_0 = 2.725$ K. The total energy density $\rho^\gamma = U(T) = \frac{\pi^2}{15}T^4$.

Since $\rho^\gamma \propto T^4$, we find

$$\frac{\delta T_k}{T_0^3} = \frac{1}{4}\frac{\delta\rho_k^\gamma}{\rho} = \frac{1}{4}\delta^\gamma_{k,\mathbf{recomb}} + \Phi_k = \frac{1}{4}A_k \cos\Big(k\tau/\sqrt{3} + \varphi_k\Big). \tag{4.20}$$

Assuming adiabatic initial condition where the initial number densities of all particles fluctuate the same war, then

$$\frac{\delta n^\gamma}{n^\gamma} = \frac{\delta n^m}{n^m} \Rightarrow \delta^\gamma = \frac{4}{3}\delta^m. \tag{4.21}$$

In the end one finds

$$\frac{\delta T_k}{T_0} = \frac{1}{3}\Phi_k \cos\frac{k\tau}{\sqrt{3}}. \tag{4.22}$$

5 Observational Cosmology

1 erg $= 10^{-7}$ J

The absolute magnitude: The magnitude of the source placed at a distance of 10 pc = 32.6 ly.

$$M_X = m_X - 5\log_{10}\frac{D_L}{10\ \mathrm{pc}}. \tag{5.1}$$

The Viral theorem: For $V(r) \sim r^{-n}$,

$$2\langle T\rangle = n\langle V\rangle . \tag{5.2}$$