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Lie Groups and Lie Algebras

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1 Groups, Rings and Fields

Definition 1.1 (Group)**.** *A group G is defined as a set G with a binary operation $''\cdot''$: $G \times G \to G$ satisfying:*

1. *Associativity:* $\forall a, b, c \in G,\ a \cdot (b \cdot c) = (a \cdot b) \cdot c$
2. *There exists an unique identity element $e \in G$ such that $\forall a \in G,\ a \cdot e = e \cdot a = a$*
3. $\forall a \in G$*, there exists an unique inverse element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$*

Generally, the binary operation on G is not communtative. If the group operation is communtative for all elements in G, then G is an **abelian group**.

For additive groups, we may use "+" instead of "$\cdot$" to denote group operation. Meanwhile, sometimes we just omit the symbol of group operation:

$$g \cdot h \Rightarrow gh$$

Definition 1.2 (Congujacy Class)**.** $\forall a \in G$*, its congujacy class is defined to be the set $g(a)$:*

$$g(a) := \{h^{-1}ah | h \in G\}$$

Definition 1.3 (Subgroup)**.** *A subgroup of G is a subset $H \subset G$ and group operation on G is also the group operation on H.*

Definition 1.4 (Normal Subgroup)**.** *The normal subgroup N of group G is defined to be the set of $h \in G$ such that $\forall g \in G,\ ghg^{-1} \in N$*

Definition 1.5 (Quotient Group)**.** *For any normal subgroup N of G, the quotient group G/N is defined to be the set of cosets*

$$aN = \{a \cdot h | h \in N\}, \quad a \in G$$

The group operation on G/N is defined as follows:

$$(a_1 N) \cdot (a_2 N) = (a_1 \cdot a_2) N$$

Definition 1.6 (Center)**.** *The center of group G is defined to be the set of elements in G which commute with all elements of G.*

Definition 1.7 (Product)**.** *The product group $G \times H$ is defined to be the set of ordered pairs (g, h), where $g \in G,\ h \in H$. The group operation on $G \times H$ is defined as*

$$(g, h)(g', h') = (gg', hh')$$

Definition 1.8 (Homomorphism/Isomorphism)**.** *A linear map between two groups $\phi : G \to H$ is a homomorphism if*

$$\forall g, h \in G,\ \phi(gh) = \phi(g)\phi(h)$$

If the inverse of a homomorphism is also a homomorphism, then the map is called an isomorphism.

For a homomorphism, it's easy to verify that

$$\phi(e_G) = e_H \quad \phi(g^{-1}) = \phi(g)^{-1}$$

We can treat isomorphic groups as the same, and classification of groups is usually up to isomorphism.

Definition 1.9 (Group Action)**.** *The group action of G on a set X is a map $\psi : G \times X \to X$ such that $\forall a, b \in G, x \in X$, we have:*

1. $\psi(e, x) = x$
2. $\psi(a, \psi(b, x)) = \psi(a \cdot b, x)$

If G is a $n \times n$ matrix group, then elements of G are automatically actions on n dimensional vector space.

Here are some example of groups:

Example 1.1.

1. *Cyclic group $\mathbb{Z}_N$. Its identity element is denoted by $e = 1$ and oother elements are generated by an element a with $a^N = 1$. That is, the group is:*

   $$\mathbb{Z}_N = \{1, a, a^2, ..., a^{N-1}\}$$

   As a finite group, the number N is called the ***order*** *of group $\mathbb{Z}_N$.*

2. *Addition Group of integers $\mathbb{Z}$. Group operation is denoted by "+", the identity element is 0 and the whole group is generated by the element 1.*

   $$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$$

   Apparently it's an infinite discrete group.

3. *Additive group on rational numbers $\mathbb{Q}$: We can define suitbale topological structures on $\mathbb{Q}$ making it a continuous group, but $\mathbb{Q}$ is not smooth and thus not a Lie group.*

4. *Additive group on $\mathbb{R}$: A Lie group.*

Definition 1.10 (Ring)**.** *A ring R is defined to be a set with two binary operations $''\cdot''$: $R \times R \to R$ and $''+''$: $R \times R \to R$ satisfying:*

1. *R is an abelian group under $''+''$.*
2. *R is monoid under $''\cdot''$*
3. *Distribution law:*

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

For example, set of integers $\mathbb{Z}$ and set of all $n \times n$ real matrices $M_n(\mathbb{R})$ can form a ring.

Definition 1.11 (Field)**.** *A field F is a ring whose operation " $\cdot$ " form an abelian group structure on $F - \{0\}$.*

2 Matrix Groups

Definition 2.1 (Matrix Group)**.** *A matrix group is a set of invertible groups and the group operation is matrix multiplication, inverse elements are inverse matrices and identity element is identity matrix I.*

Definition 2.2 (Dimension)**.** *The dimension of a matrix group $\dim G$ can be defiend as the number of independent real parameters.*

Here are some example:

Example 2.1.

1. *General linear group on field F: $\mathsf{GL}(n; F)$ is defined as the set of all $n \times n$ invertible matrices whose elements are in F.*

   Since restrication on $\det M \neq 0$ doesn's provide an equation, it won't eliminate free parameters and we have:

   (a) $\dim \mathsf{GL}(n; \mathbb{R}) = n^2$
   (b) $\dim \mathsf{GL}(n; \mathbb{C}) = 2n^2$

   Obviously, all matrix groups are subgroups of $\mathsf{GL}(n; F)$.

2. *Special linear group on field F: $\mathsf{SL}(n; F)$ is defined as all invertible $n \times n$ matrices on F with determinants equal to 1. Since $\det M = 1$ reduces one degree of freedom for real matrices and two degree of freedom for complex matrices (equations on real part and imaginary part), thus:*

   (a) $\dim \mathsf{SL}(n; \mathbb{R}) = n^2 - 1$
   (b) $\dim \mathsf{SL}(n; \mathbb{C}) = 2n^2 - 2$

3. *Unitary group:* $\mathsf{U}(n)$ *is defined to be the set of all* $n \times n$ *unitary matrices, i.e.*

$$M^\dagger M = I$$

For the dimension of $\mathsf{U}(n)$*, notice that if we decompose* M *as*

$$M = A + \mathrm{i}B,\ A, B \in M_n(\mathbb{R})$$

Then

$$M^\dagger M = (A^T - \mathrm{i}B^T)(A + \mathrm{i}B) = A^T A + \mathrm{i}A^T B - \mathrm{i}B^T A + B^T B$$

Then $M^\dagger M = I$ *indicating* $A^T A + B^T B = I$ *and* $A^T B = B^T A$*.* $A^T B = B^T A$ *indicating the real part and imaginary part are related, thus although* $M^\dagger M = I$ *gives* $2n^2$ *real equations, only* n^2 *equations are independent. Therefore,* $\mathsf{U}(n)$ *has* n^2 *independent parameters and*

$$\dim \mathsf{U}(n) = n^2$$

Notice that the determinant of $M \in \mathsf{U}(n)$ *is indefinite:* $M^\dagger M = I$ *indicating*

$$\overline{\det M} \det M = 1 \Rightarrow |\det M|^2 = 1$$

Thus we only know $\det M$ *is a complex number with unit magnitude.*

4. *Special unitary group:* $\mathsf{SU}(n)$ *is set of elements in* $\mathsf{U}(n)$ *whose determinant is 1. Obviously we have*

$$\dim \mathsf{SU}(n) = n^2 - 1$$

Elements of $\mathsf{U}(n)$ *or* $\mathsf{SU}(n)$ *can be viewed as actions on* $\mathbb{C}^n$ *preserving* $\|x\|^2$*.*

For instance, $\mathsf{SU}(2)$ *is a 3 dimensional group whose elements can be written as*

$$\begin{pmatrix} a + b\mathrm{i} & c + d\mathrm{i} \\ -c + d\mathrm{i} & a - b\mathrm{i} \end{pmatrix}, \quad a^2 + b^2 + c^2 + d^2 = 1$$

Thus we can topologically indentify $\mathsf{SU}(2)$ *and* S^3*. We can allso use 3 parameters to represent elements of* $\mathsf{SU}(2)$*, according the following substitutions:*

$$a = \cos\alpha \quad \beta = \sin\alpha\cos\beta \quad c = \sin\alpha\sin\beta\cos\gamma \quad d = \sin\alpha\sin\beta\sin\gamma$$

5. *Indefinite unitary group and indefinite special unitary group:* $\mathsf{U}(p, q)$ *is defined to be set of* $(p + q) \times (p + q)$ *matrices, preserving following metric:*

$$g = \mathrm{diag}(\underbrace{1, ..., 1}_{p}, \underbrace{-1, ..., -1}_{q}) \qquad M^\dagger g M = g$$

$\mathsf{SU}(p, q)$ *is set of elements in* $\mathsf{U}(p, q)$ *with determinants equal to 1.*

6. *Orthogonal group:* $\mathsf{O}(n)$ *is defined to be set of* $n \times n$ *real orthogonal matrices. That is,* $\forall M \in \mathsf{O}(n)$ *satisfies*
$$M^T M = I$$
Since $M^T M$ *is symmetric, thus* $M^T M = I$ *only gives us* $n(n+1)/2$ *independent equations and we have*
$$\dim \mathsf{O}(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$
Unlike $\mathsf{U}(n)$, $M^T M = I$ *gives much stronger restrcation on the determinants of matrices in* $\mathsf{O}(n)$*:* $\forall M \in \mathsf{O}(n)$ *is real matrix, thus* $\det M \in \mathbb{R}$ *and*
$$M^T M = I \Rightarrow (\det M)^2 = 1 \Rightarrow \det M = \pm 1$$
Therefore, $\mathsf{U}(n)$ *is composed by two disconnected parts, and only the part with* $\det M = +1$ *contains identity element so that it forms a subgroup* $\mathsf{SO}(n)$*. The we denote the subset of* $\mathsf{O}(n)$ *with* $\det M = -1$ *as* $\mathsf{O}(n)^-$*, and we claim* $\forall M \in \mathsf{O}(n)^-$ *can be represented as* $M = RN$*, where* $N \in \mathsf{SO}(n)$ *and* R *is spatial reflection.*

According to the definition of $\mathsf{O}(n)$*, it's clear that action of* $\mathsf{O}(n)$ *on* n *dimensional Euclidean space* $\mathbb{R}^n$ *preserves Euclidean norm. Action of* $\mathsf{SO}(n)$ *corresponds to rotation in* $\mathbb{R}^n$*, while action of* $\mathsf{O}(n)^-$ *corresponds to rotation combined with reflection.*

Since restricting $\det M = +1$ *on* $\mathsf{O}(n)$ *doesn't eliminate a continuous degree of freedom, thus*
$$\dim \mathsf{SO}(n) = \dim \mathsf{O}(n) = \frac{n(n-1)}{2}$$
As a trivial example, elements of $\mathsf{SO}(2)$ *can be represented as*
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi)$$
Then it's clear that $\mathsf{SO}(2) \cong \mathsf{U}(1) \cong S^1$*.*

Relation between $\mathsf{SO}(3)$ *and* $\mathsf{SU}(2)$ *is more interesting. Recall that elements of* $\mathsf{SU}(2)$ *can be represented as*
$$\begin{pmatrix} a + b\mathrm{i} & c + d\mathrm{i} \\ -c + d\mathrm{i} & a - b\mathrm{i} \end{pmatrix}, \quad a^2 + b^2 + c^2 + d^2 = 1$$
While elements of $\mathsf{SO}(3)$ *can be written as*
$$\begin{pmatrix} 1 - 2c^2 - 2d^2 & 2bc - 2ad & 2bd + 2ac \\ 2bc + 2ad & 1 - 2b^2 - 2d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & 1 - 2b^2 - 2c^2 \end{pmatrix}, \quad a^2 + b^2 + c^2 + d^2 = 1$$
We can verify $\mathsf{SU}(2)$ *and* $\mathsf{SO}(3)$ *are homomorphic, but they aren't isomorphic: Clearly, parameter tuples* (a, b, c, d) *and* $(-a, -b, -c, -d)$ *correspond to the same element in*

$\mathsf{SO}(3)$ *but different elements in* $\mathsf{SU}(2)$. *Thus the homomorphism* $\phi : \mathsf{SU}(2) \to \mathsf{SO}(3)$ *is 2 to 1 map. Actually, it can be proved that*

$$\mathsf{PSU}(2) := \mathsf{SU}(2)/\mathbb{Z}_2 \cong \mathsf{SO}(3)$$

More generally, let's consider $\mathsf{PSU}(N) := \mathsf{SU}(N)/\mathbb{Z}_N$. $\mathbb{Z}_N$ *is defined to be*

$$\mathbb{Z}_N = \{1, e^{2\pi \mathrm{i}/N} I, ..., e^{2\pi \mathrm{i}(N-1)/N} I\}$$

Then $\mathbb{Z}_N$ *is clearly the center of* $\mathsf{SU}(N)$ *and elements of* $\mathsf{SU}(N)/\mathbb{Z}_N$ *are cosets labelled by* $a \in \mathsf{SU}(N)$, *i.e.* $a \in \mathsf{SU}(N)$ *mod out equivalence relation below:*

$$a \sim e^{2\pi \mathrm{i}/N} a$$

7. *Indefinite orthogonal group (Pseudo orthogonal group):* $\mathsf{O}(p,q)$ *is defined to be set of* $(p+q) \times (p+q)$ *matrices satisfying*

$$MgM^T = g, \quad g = \mathrm{diag}(\underbrace{1, ..., 1}_{p}, \underbrace{-1, ..., -1}_{q})$$

   $\mathsf{SO}(p,q)$ *is set of elements in* $\mathsf{O}(p,q)$ *with determinant is 1. Lorentz group is exactly* $\mathsf{O}(1,3)$, *preserving Minkowski metric* $\mathrm{diag}(1,-1,-1,-1)$.

8. *Euclidean group:* $\mathsf{E}(n)$, *also denoted by* $\mathsf{ISO}(n)$, *is defined to be the group of saptial rotation and translation in* $\mathbb{R}^n$. *Elements of* $\mathsf{E}(n)$ *can be expressed as an* $(n+1) \times (n+1)$ *matrix:*

$$\begin{pmatrix} \mathbf{R}_{n\times n} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

   Here $\mathbf{R}_{n\times n} \in \mathsf{O}(n)$ *and* $\mathbf{a}$ *is* n *dimensional column vector,* $\mathbf{0}$ *is* n *dimensional zero row vector.* $\mathsf{E}(n)$ *acts on elements of* $(n+1)$ *dimensional vector space with the following form:*

$$\mathbf{v} = (x_1, ..., x_n, 1)^T$$

   Direct computation shows

$$\begin{pmatrix} \mathbf{R}_{n\times n} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_{n\times n}\mathbf{x} + \mathbf{a} \\ 1 \end{pmatrix}$$

   Clearly, $\mathsf{E}(n)$ *is a subgroup of* $\mathsf{GL}(n+1;\mathbb{R})$. *Besides,* $\mathsf{O}(n)$ *is a subgroup of* $\mathsf{E}(n)$. *If we introduce translation group* $\mathsf{T}(n)$ *whose elements can be expressed as*

$$\begin{pmatrix} \mathbf{I}_{n\times n} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

   Then the intutiation that $\mathsf{O}(n)$ *is* $\mathsf{E}(n)$ *excluding translation leads to the relation below:*

$$\mathsf{O}(n) \cong \mathsf{E}(n)/\mathsf{T}(n)$$

Actually, $\mathsf{T}(n)$ is a normal subgroup of $\mathsf{E}(n)$, we can verify it by direct calculation: For any element g of $\mathsf{E}(n)$, we have

$$g = \begin{pmatrix} \mathbf{R}_{n\times n} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

It's inverse is

$$g^{-1} = \begin{pmatrix} \mathbf{R}_{n\times n}^{-1} & -\mathbf{R}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

For any element of $\mathsf{T}(n)$, we express it as

$$h = \begin{pmatrix} \mathbf{I}_{n\times n} & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix}$$

Then

$$\begin{aligned} ghg^{-1} &= \begin{pmatrix} \mathbf{R} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{R} & \mathbf{a}+\mathbf{R}\mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{R}\mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \in \mathsf{T}(n) \end{aligned}$$

9. *Poincaré group: $\mathsf{ISO}(1, n-1)$ is defined to be the indefinite version of $\mathsf{E}(n)$. For $n = 4$, we have*

$$\dim \mathsf{ISO}(1,3) = \dim \mathsf{O}(1,3) + \dim \mathsf{T}(4) = 6 + 4 = 10$$

10. *Symplectic group: $\mathsf{Sp}(2n; F)$ is defined to be the set of $2n \times 2n$ matrices M whose elements belong to field F, satisfying*

$$M\Omega M^T = \Omega, \quad \Omega = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{n\times n} \\ -\mathbf{I}_{n\times n} & \mathbf{0} \end{pmatrix}$$

We can express elements of $\mathsf{Sp}(2n)$ as blocked matrices:

$$M = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

Then group action preserving symplectic form indicates

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{D}^T \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$$

Thus we have following independent matrix equations:

$$\begin{aligned} \mathbf{A}\mathbf{B}^T - \mathbf{B}\mathbf{A}^T &= 0 \\ \mathbf{A}\mathbf{D}^T - \mathbf{B}\mathbf{C}^T &= \mathbf{I} \\ \mathbf{C}\mathbf{D}^T - \mathbf{D}\mathbf{C}^T &= 0 \end{aligned}$$

The first and the third equation give $n(n-1)/2$ scalar equations respectively, and the second eqaution gives us n^2 independent scalar equations. Therefore, the dimension of a symplectic group is

$$\dim \mathsf{Sp}(2n) = 4n^2 - n(n-1) - n^2 = n(2n+1)$$

$\mathsf{Sp}(2n;\mathbb{R})$ *naturally acts on $\mathbb{R}^{2n}$, whose coordinates is usually denoted as $(q^1, ..., q^n, p_1, ..., p_n)$. We can view $\mathsf{Sp}(2n;\mathbb{R})$ as a group composed by operations on $\mathbb{R}^{2n}$ preserving Poisson brackets:*

$$\{f, g\} = \sum_{\ell=1}^{n} \left(\frac{\partial f}{\partial q^\ell} \frac{\partial g}{\partial p_\ell} - \frac{\partial f}{\partial p_\ell} \frac{\partial g}{\partial q^\ell} \right)$$

Through the definition of $\mathsf{Sp}(2n, F)$, we immediately know the determinant of a symplectic group element must be $+1$ or -1. However, it can be proved that $\forall M \in \mathsf{Sp}(2n, F)$, $\det M = +1$.

11. *Unitary symplectic group, also called compact symplectic group: $\mathsf{USp}(2n)$ is defiend to be $\mathsf{Sp}(2n;\mathbb{C}) \cap \mathsf{SU}(2n)$, therefore any element M of $\mathsf{USp}(2n)$ simultaneously satisfies*

$$M\Omega M^T = \Omega, \qquad M^\dagger M = I$$

Its dimension is

$$\dim \mathsf{USp}(2n) = 2n(2n+1) - 2n^2 - n = n(2n+1)$$

For the proof, please refer to Homework 1.

3 Subgroups and Some Interesting Relations in Matrix Groups

At the beginning, let's review the definition of direct product: For two groups G and H, their direct product $G \times H$ is defined to be the set $\{(g, h) | g \in G,\ h \in H\}$ and group multiplication is defined to be $(g, h) \cdot (g', h') = (gg', hh')$. Based on group product, we have the following definition[1]:

Definition 3.1. *Consider a group K with subgroups G and H. Then $G \times H \cong K$ if:*

1. *Map $\phi : G \times H \to K$, $(g, h) \mapsto gh$ is isomorphism.*
2. *$\forall g \in G, h \in H$, $gh = hg$ in K.*
3. *$G \cap H$ is a trivial subgroup of K, i.e. $G \cap H = \{e_K\}$.*

If ϕ in the first criteria is not an isomorphism but a homomorphism, then we can only deduce $G \times H$ is isomorphic to a subgroup of K, i.e. $G \times H \subset K$.

Propsition 3.1. *If $G \times H \cong K$, then G and H are normal subgroups of K.*

[1] We can make an analogy between these criterion and direct sum between vector spaces.

Proof. When viewed as subgroups of K, we can denote elements of G, H as (g, e_H) and (e_G, h). For any element of K, it can be denoted as (a, b), where $a \in G$ and $b \in H$. Thus we have

$$(a, b)(g, e_H)(a^{-1}, b^{-1}) = (aga^{-1}, e_H) \in G$$
$$(a, b)(e_G, h)(a^{-1}, b^{-1}) = (e_G, bhb^{-1}) \in H$$

□

Here we have some propsitions (as well as some example):

Propsition 3.2.

$$\mathsf{GL}(m; F) \times \mathsf{GL}(n-m; F) \subset \mathsf{SL}(n; F)$$
$$\mathsf{SL}(m; F) \times \mathsf{SL}(n-m; F) \subset \mathsf{SL}(n; F)$$
$$\mathsf{U}(m) \times \mathsf{U}(n-m) \subset \mathsf{U}(n)$$
$$\mathsf{SU}(m) \times \mathsf{SU}(n-m) \subset \mathsf{SU}(n)$$

Proof. Juts notice the elements of the product groups on the LHS can be represented as the blocked diagonal matrices below:

$$\begin{pmatrix} \mathbf{M}_{n\times n} & 0 \\ 0 & \mathbf{N}_{(n-m)\times(n-m)} \end{pmatrix}$$

Appraently this set of matrices can be viewed as elements in the corresponding group with dimension n. □

Propsition 3.3. $\mathsf{SL}(n; \mathbb{R}) \subset \mathsf{GL}(n; \mathbb{R})$

Proof. Obvious. □

Propsition 3.4. *If n is odd, then* $\mathsf{SL}(n; \mathbb{R}) \times \mathsf{GL}(1; \mathbb{R}) \cong \mathsf{GL}(n; \mathbb{R})$.

Proof. First of all, if n is odd, then we can construct a bijection $\phi : \mathsf{SL}(n; \mathbb{R}) \times \mathsf{GL}(1; \mathbb{R}) \to \mathsf{GL}(n; \mathbb{R})$ as follows:

$$\phi : (M, aI) \mapsto aM$$

Secondly, when embeded in $\mathsf{GL}(n; \mathbb{R})$, elements of $\mathsf{GL}(1; \mathbb{R})$ is aI, where $a \in \mathbb{R} - \{0\}$. Clearly, every matrices in $\mathsf{SL}(n; \mathbb{R})$ commute with aI.

Finally, we have to check if $\mathsf{SL}(n; \mathbb{R}) \cap \mathsf{GL}(1; \mathbb{R})$ trivial in $\mathsf{GL}(n; \mathbb{R})$. For odd n, the only element of $\mathsf{SL}(n; \mathbb{R}) \cap \mathsf{GL}(1; \mathbb{R})$ is I, and it's apparently trivial.

If n is even, then things will be different: On one hand, $\phi : (M; aI) \mapsto aM$ will no longer cover $\mathsf{GL}(n; \mathbb{R})$, as $\det(aM) = a^n$ is always postive. On the other hand, $\mathsf{SL}(n; \mathbb{R}) \cap \mathsf{GL}(1; \mathbb{R})$ won't be trivial, since $\det(-I) = \det(I) = 1$ when n is even and thus $\mathsf{SL}(n; \mathbb{R}) \cap \mathsf{GL}(1; \mathbb{R}) = \{+I, -I\}$. □

Propsition 3.5. $\mathsf{SU}(n) \subset \mathsf{U}(n)$

Proof. Obvious. □

Propsition 3.6. $\mathsf{SU}(n) \times \mathsf{U}(1) \not\subset \mathsf{U}(n)$, *while* $(\mathsf{SU}(n) \times \mathsf{U}(1))/\mathbb{Z}_n \cong \mathsf{U}(n)$.

Proof. First of all, elements of $\mathsf{SU}(n)$ and $\mathsf{U}(1)$ are clearly commutative in $\mathsf{U}(n)$, thus the second criteria is satisfied.

However, if we consider map $\phi : (M, e^{\mathrm{i}\theta}) \mapsto e^{\mathrm{i}\theta} M$ between $\mathsf{SU}(n) \times \mathsf{U}(1)$ and $\mathsf{U}(n)$, then we will find that it's not bijective: For any matrix $S \in \mathsf{U}(n)$, we denote its determinant as $e^{\mathrm{i}\Psi}$. Thus there exists a matrix $Q \in \mathsf{SU}(n)$ such that

$$S = e^{\mathrm{i}\psi} Q, \quad e^{\mathrm{i}n\psi} = e^{\mathrm{i}\Psi}$$

Thus there are actually n elements correspond to a single element in $\mathsf{U}(n)$:

$$(Q, e^{\mathrm{i}\Psi/n}),\ (Q, e^{\mathrm{i}\Psi/n + 2\pi\mathrm{i}/n}), ...,\ (e^{\mathrm{i}\Psi/n + 2\pi\mathrm{i}(n-1)/n})$$

Therefore the first criteria is violated, not to mention the third one: $\mathsf{SU}(n) \cap \mathsf{U}(1)$ is exactly isomorphic to $\mathbb{Z}_n$:

$$\mathsf{SU}(n) \cap \mathsf{U}(1) = \{I, e^{2\pi\mathrm{i}/n}, e^{4\pi\mathrm{i}/n}, ..., e^{2(n-1)\pi\mathrm{i}/n}\} \cong \mathbb{Z}_n$$

In the quotient group $(\mathsf{SL}(n;\mathbb{R}) \times \mathsf{U}(1))/\mathbb{Z}_n$, we have cosets as group elements and there's a one-to-one correspondence between $S \in \mathsf{U}(n)$ and

$$(Q, e^{\mathrm{i}\Psi/n})\mathbb{Z}_n = \{(Q, e^{\mathrm{i}\Psi/n}),\ (Q, e^{\mathrm{i}\Psi/n + 2\pi\mathrm{i}/n}), ...,\ (e^{\mathrm{i}\Psi/n + 2\pi\mathrm{i}(n-1)/n})\} \in (\mathsf{SL}(n;\mathbb{R}) \times \mathsf{U}(1))/\mathbb{Z}_n$$

There are two other ways taking quotient group: First, we can take $\mathsf{U}(1)/\mathbb{Z}_n$, in this way we identify $e^{\mathrm{i}\theta}$ and $e^{\mathrm{i}\theta + e\pi\mathrm{i}\ell/n}$ and clearly $\mathsf{SU}(n) \times (\mathsf{U}(1)/\mathbb{Z}_n) \cong \mathsf{U}(n)$. Or we can take $\mathsf{PSU}(n) = \mathsf{SU}(n)/\mathbb{Z}_n$, in this way $\mathsf{PSU}(n) \cap \mathsf{U}(1)$ is obviously trivial in $\mathsf{U}(n)$, and $(Q, e^{\mathrm{i}\Psi/n})$, $(Q, e^{\mathrm{i}\Psi/n + 2\pi\mathrm{i}/n}), ...,$ $(e^{\mathrm{i}\Psi/n + 2\pi\mathrm{i}(n-1)/n})$ will again be identified, making there exist bijection between $\mathsf{PSU}(n) \times \mathsf{U}(1)$ and $\mathsf{U}(n)$, thus $\mathsf{PSU}(n) \times \mathsf{U}(1) \cong \mathsf{U}(n)$. □

Propsition 3.7. $(\mathsf{SU}(m) \times \mathsf{SU}(n-m) \times \mathsf{U}(1))/\mathbb{Z}_{\mathrm{lcm}(m,n-m)} \subset \mathsf{SU}(n)$

Proof. Using blocked diagonal matrices we can immediately know $\mathsf{SU}(m) \times \mathsf{SU}(n-m)$ is subgroup of $\mathsf{SU}(n)$. However, elements of $(\mathsf{SU}(m) \times \mathsf{SU}(n-m)) \cap \mathsf{U}(1)$ have the following form:

$$\mathrm{diag}(\underbrace{e^{2\pi\mathrm{i}k/m}, ..., e^{2\pi\mathrm{i}k/m}}_{m}, \underbrace{e^{-2\pi\mathrm{i}k/(n-m)}, ..., e^{-2\pi\mathrm{i}k/(n-m)}}_{n-m}), \quad k \in [0, \mathrm{lcm}(m, n-m))$$

Thus it's clear $\mathsf{SU}(m) \times \mathsf{SU}(n-m) \times \mathsf{U}(1)$ is not a well-defined subgroup of $\mathsf{SU}(n)$.

But $(\mathsf{SU}(m) \times \mathsf{SU}(n-m) \times \mathsf{U}(1))/\mathbb{Z}_{\mathrm{lcm}(m,n-m)}$. We can view $\mathbb{Z}_{\mathrm{lcm}(m,n-m)}$ as the cyclic group generated by

$$\mathrm{diag}(\underbrace{e^{2\pi\mathrm{i}/m}, ..., e^{2\pi\mathrm{i}/m}}_{m}, \underbrace{e^{-2\pi\mathrm{i}/(n-m)}, ..., e^{-2\pi\mathrm{i}/(n-m)}}_{n-m})$$

Taking $\mathsf{U}(1)/\mathbb{Z}_{\mathrm{lcm}(m,n-m)}$ will identify $k \sim k+1$. □

Proposition 3.6 is of significance in physics: The gauge group of standard model is $\mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$, and now we know from mathematics that it's not a subgroup of the proposed GUT gauge group $\mathsf{SU}(5)$, thus $\mathsf{SU}(5)$ GUT theory is falsified mathematically.

4 Basic Topology and Topological Properties of Lie Groups

Definition 4.1 (Topological Space). *A topological space is a set X with a collection of open set $\mathscr{T} = \{U_i\}$ satisfying:*

1. *$\varnothing$ and X itself belong to $\mathscr{T}$.*
2. *For any subcollection (finite or infinite) $S = \{U_j | j \in J\} \subset \mathscr{T}$, $\bigcup_J U_j \in \mathscr{T}$.*
3. *For any finite sbucollection $S = \{U_j | j \in J\} \subset \mathscr{T}$, $\bigcap_J U_j \in \mathscr{T}$.*

We can define various topologies for a set to make it a topological space. For example, in $\mathbb{R}^n$ we can use open cubes to define usual topology by choosing

$$\mathscr{T} = \{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)\}$$

We can even define $\mathscr{T}$ is the collection of all susbets of X and get trivial topology. The following definition make it possible for a subset of a topological space to inherit topology from the mother space.

Definition 4.2 (Relative Topology). *Suppose a topological space $(X, \mathscr{T})$ and a subset $Y \subset X$, the relative topology $\mathscr{S}$ on Y is defined to be*

$$\mathscr{S} = \{Y \cap U | U \in \mathscr{T}\}$$

Many sets we interested in are metric spaces, i.e. sets with metrics:

Definition 4.3 (Metric Space). *A metric space is a set X with a metric map $d : X \times X \to \mathbb{R}$, satisfying:*

1. *$\forall x, y \in X,\ d(x, y) = d(y, x)$*
2. *$d(x, y) \geq 0$, the equality holds if and only if $x = y$.*
3. *$\forall x, y, z \in X,\ d(x, y) + d(y, z) \geq d(x, z)$*

Only in metric spaces can we define discreteness:

Definition 4.4 (Discreteness). *A metric space X is discrete if $\forall x \in X$ and $\forall Y \in X - \{x\}$, $\exists s \in \mathbb{R}^+$ such that $d(x, y) > s$ always holds.*

We can define a rather natural topology on metric spaces:

Definition 4.5 (Metric Topology). *For a metric space (X, d), we define its matric topology by defining open subsets are open balls $U_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\}$ and their unions (finite or infinite).*

Definition 4.6 (Product Topology). *Condiser two topological spaces $(X, \mathscr{T})$ and $(Y, \mathscr{O})$, the product topology on $X \times Y$ is defined by*

$$\mathscr{T} \times \mathscr{O} = \{U^x \times U^y | U^x \in \mathscr{T}, U^y \in \mathscr{O}\}$$

Recall the definition of closed sets in analysis: A subset $Y \subset X$ is closed if and only if any sequence $A_m \in Y$ converges to a single point in Y. Now we can give an equivalent definition via toppogy:

Definition 4.7. *Closed Set A subset $Y \subset X$ is closed is closed if and only if its complement $Y_c = X - Y$ is open.*

For example. whatever the set X is, X itself and $\varnothing$ are both open and closed. For $\mathbb{R}$, if we define its topology by defining all open intervals and their unions to be open, then closed sets in $\mathbb{R}$ are all closed intervals and their finite unions. Note: Intervals like $[a, b)$ are neither open nor closed. On a metric space with metric topology, $\overline{U_\varepsilon(x)} = \{y \in X | d(x, y) \leq \varepsilon\}$ is closed.

Continuity of maps can be redefined through topology too:

Definition 4.8 (Continuity)**.** *A map between two topological spaces $f : X \to Y$ is continuous if and only if for any open set $U_Y \in Y$, $f^{-1}(U_Y)$ is an open set in X.*

In topology, an important equivalence relation is homeomorphism:

Definition 4.9 (Homeomorphism)**.** *Two topological spaces X and Y are homeomorphic if and only if there exists a continuous mao $f : X \to Y$ such that $f^{-1} : Y \to X$ exists and is continuous.*

Usually, people prove two topological spaces are homeomorphic by constructive prooves. However, it's much harder to prove two topological spaces are not homeomorphic and one way to do this is using topological invariants, which we will introduce later.

Definition 4.10 (Topological Groups)**.** *A topological group is a topological space with a group structure, satisfying:*

1. *Group operation $\cdot : G \times G \to G$ and inverse $^{-1} : G \to G$ are continuous.*
2. *$G \times G$ has a product topology.*
3. *Isomorphism of topological groups is also group isomorphism and topollogical homeomorphism.*

A topological group is not necessarily a continuous group, even for point groups we can attach topologies to them, like trivial topology or meric topology. For the latter one, we can consider $\mathbb{Z}_N$ and express it as points on a circle so that we can measure distance between discrete group elements and have a metric topology.

Now we can define matrix Lie groups in topological sense:

Definition 4.11 (Matrix Lie Group, GTM 222, Page 4)**.** *A matrix Lie group G is a subgroup of $\mathsf{GL}(n; \mathbb{C})$ satisfying for any sequence of matrices $\{A_m\}$ in G which converges to A, A is either in G or non-invertible. In other word, a matrix Lie group is a closed subgroup of $\mathsf{GL}(n; \mathbb{C})$.*

Definition 4.12 (Compactness). *If $X \subset \mathbb{R}^n$, then X is compact if and only if X is closed and bounded. Bounded means there exists $s \in \mathbb{R}$ such that $\forall x, y \in X$, $d(x,y) < s$.*

For example, $\mathbb{R}$ is non-compact, while $S^1 \cong \mathsf{U}(1) \cong \mathsf{SO}(2)$ and $\mathsf{SU}(2) \cong S^3$ are compact. $\mathsf{U}(n), \mathsf{SU}(n), \mathsf{O}(n), \mathsf{SO}(n), \mathsf{USp}(2n)$ are compact, too.

On the contrary, $\mathsf{GL}(n;\mathbb{R}), \mathsf{GL}(n;\mathbb{C}), \mathsf{SL}(n;\mathbb{R}), \mathsf{SL}(n;\mathbb{C}), \mathsf{Sp}(2n;\mathbb{R}), \mathsf{Sp}(2n;\mathbb{C}), \mathsf{O}(1,3)$ are non-compact.

Compactness is of significance in physics: In quantum gauge field theory, gauge group G is unitary only when G is compact.

Definition 4.13 (Connectedness). *A topological space X is connected if it cannot be written as $X = X_1 \cup X_2$, X_1, X_2 are open subsets in X and $X_1 \cap X_2 = \varnothing$, otherwise X is disconnected.*

In practice, we usually check a generally stronger condition: path-connectedness. However, it can be proved for matrix groups that connectedness is equivalent to path-connectedness.

For example, $\mathsf{GL}(n;\mathbb{C}), \mathsf{SL}(n;\mathbb{C}), \mathsf{U}(n), \mathsf{SU}(n), \mathsf{SO}(n)$ are connected, while $\mathsf{O}(n)$ is disconnected: $\mathsf{O}(n) = \mathsf{SO}(n) \cup \mathsf{O}(n)^-$.

Definition 4.14 (Connected Components). *Consider an topological space X and fix a point $x \in X$, the connected component containing x is the union of all connected open sets that contain x.*

Propsition 4.1. *If a topological space X has finitely many connected components, then each connected component is open and closed.*

For example, $\mathsf{O}(p,q)$ has 4 connected components, $\mathsf{SO}(p,q)$ has 2 connected components.

Definition 4.15 (Simply Connectedness). *A topological space X is simply connected if and only if every loop in X can be continuously shrunk to a point. The definiiton of a loop is a map $f : [0,1] \to X$ satisfying $f(0) = f(1)$.*

For example, $\mathsf{U}(1)$ is not simply connected, and we can further infer for any Lie group G, $G \times \mathsf{U}(1)$ can't be simply connected, due to the $\mathsf{U}(1)$ part. $\mathsf{SU}(2) \cong S^3$ is simply connected, howeverr, since

$$\mathsf{SO}(3) \cong \mathsf{SU}(2)/\mathbb{Z}_2 \cong \mathbb{R}P^3$$

Thus $\mathsf{SO}(3)$ is not connected: In $\mathbb{R}P^3$, a point on unit sphere and its antipodal point are indetified, for example, $(1,0,0)$ and $(-1,0,0)$, and a curve connecting the two points is a loop in $\mathbb{R}P^3$. However, it's obvious that the loop can't be shrunk to a point. Moreover, it can be proved that all $\mathsf{SO}(N)$ are not simply connected for $N > 1$.

5 Differential Manifold

Definition 5.1 (Differential Manifold)**.** *A n-dimensional differential manifold M is a topological space staisfying:*

1. *M has a colloection of pairs $\{(U_i, \phi_i)\}$, $\{U_i\}$ is a collection of open sets covering M, and $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism from U_i to an open subset $U_i' \in \mathbb{R}^n$.*
2. *Given U_i, U_j, $U_i \cap U_j \neq \varnothing$, then the map $\psi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is infinitely differentiable.*

A pair (U_i, ϕ_i) is called a coordinate chart, ϕ is named as coordinate (map), ψ_{ij} is transition map. An atlas is defined to be the collection of all charts on a manifold.

Suppose we have a map $f : M \to N$, $\dim M = m$, $\dim N = n$, then $\forall p \in M$ is mapped to $f(p) \in N$. We can take a chart (U, ϕ) on M and (V, ψ), then f has a coordinate representation:

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$$

Denote x^μ is coordinate induced by (U, ϕ) on M while y^α is coordinate induced by (V, ψ) on N, then f can be represented as $y^\alpha(x^\mu)$. If $y^\alpha(x^\mu)$ is infinitely differentiable at p, then f is smooth at this point. Moreover, if f is smooth at every point of M, then f is smooth.

Definition 5.2 (Diffeomorphism)**.** *If a map $f : M \to N$ and its inverse are both smooth, then f is a diffeomorphism and M, N are diffeomorphic.*

Definition 5.3 (Product Manifold)**.** *Suppose a m dimensional manifold M with atlas $\{(U_i, \phi_i)\}$ and a n dimensional manifold N with atlas $\{(V_i, \psi_i)\}$, then $M \times N$ is a $(m+n)$ dimensional manifold, a point in $M \times N$ is an ordered pair (p, q), $p \in M$, $q \in N$, its atlas is $\{((U_i, V_j), (\phi_i, \psi_j))\}$. The coordinate map is $(p, q) \to (\phi(p), \psi(q)) \in \mathbb{R}^{m+n}$.*

Definition 5.4 (Embeded Submanifold)**.** *M is a n dimensional manifold, a k dimensional embeded submanifold of M is a set $S \subset M$, and $\forall p \in S$, there exists a chart (U, ψ) contains p such that $\phi(S \cap U)$ is the intersecton of a k dimensional plane with $\phi(U)$.*

Now we can talk about Lie groups in manifold terminology.

Definition 5.5 (Lie Group)**.** *A Lie group is a differential manifold G with a group action $\cdot : G \times G \to G$ and inverse ${}^{-1} : G \to G$, satisfying group axioms and are smooth and differentiable.*

Product group corresponds to product manifold, so $\mathsf{U}(1)^n \cong T^n \cong (S^1)^n$. While for subgroups, we have the following theorem:

Theorem 5.1 (Cartan's Colsed Subgroup Theorem)**.** *Any colsed subgroup $H \subset G$ is a embeded submanifold of G.*

The definition of Lie group homomorphism is the same as homomorphism for topological groups. It can be proved for maps on matix groups, smoothness is equivalent to continuity.

Definition 5.6 (Group Action)**.** *Given a Lie group G, its group action on an manifold M is a smooth map $\cdot : G \times M \to M$ satisfying*

1. *For identity element $e \in G$, $\forall p \in M$, $e \cdot p = p$.*
2. $\forall g_1, g_2 \in G,\ p \in M, g_1(g_2(p)) = (g_1 g_2)(p)$.

6 Lie Algebras

6.1 Lie Algebra ABC

Definition 6.1 (Lie Algebra)**.** *A finite dimensional Lie algebra $\mathfrak{g}$ is a finite dimensional vector space together with a map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying*

1. *Bilinearity.*
2. *Antisymmetry:* $[X, Y] = -[Y, X],\ \forall X, Y \in \mathfrak{g}$.
3. *Jacobi Identity:*$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Since the bilinear map $[\ ,\ ]$ is not associative, thus $\mathfrak{g}$ isn't a ring. Generators of Lie algebra can be viewed as basis of the vector space $\mathfrak{g}$. If $[X, Y] = 0$, we say X, Y are **commute**. If $\forall X, Y \in \mathfrak{g}$, we have $[X, Y] = 0$, then the Lie algebra is **abelian**.

Definition 6.2 (Real/Complex Lie algebra)**.** *If $\mathfrak{g}$ is a vector space over $\mathbb{R}$, then $\mathfrak{g}$ is a real Lie algebra. If $\mathfrak{g}$ is a vector space over $\mathbb{C}$, then $\mathfrak{g}$ is a complex Lie algebra.*

Definition 6.3 (Complexification)**.** *Take a real Lie algebra $\mathfrak{g}$, we can introduce complex coefficients to basis of $\mathfrak{g}$ and get its complexification $\mathfrak{g}_{\mathbb{C}}$. Elements of $\mathfrak{g}_{\mathbb{C}}$ has the form $v_1 + \mathrm{i} v_2$, where $v_1, v_2 \in \mathfrak{g}$.*

Definition 6.4 (Dimension of Lie Algebra)**.** *The dimension of a Lie algebra is defined to be its dimension as a vector space.*

Definition 6.5 (Lie Subalgebra)**.** *A Lie subalgebra of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ in the sense of linear algebra which is also closed under $[\ ,\ ]$ operation.*

Definition 6.6 (Center)**.** *Center is defined to be the set of all elements commuting with all elements in $\mathfrak{g}$. It's a Lie subalgebra.*

Definition 6.7 (Lie Algebra Homomorphism)**.** *A Lie algebra homomorphism is a linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ satisfying*

$$\phi([g, h]) = [\phi(g), \phi(h)], \quad \forall g, h \in \mathfrak{g}_1$$

If ϕ is also bijective, then it's an isomorphism.

Definition 6.8 (Direct Sum)**.** *For Lie algebra $\mathfrak{g}_1, \mathfrak{g}_2$, their direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the direct sum of vector spaces $\mathfrak{g}_1, \mathfrak{g}_2$. Lie algebra on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is defined to be*

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2])$$

Definition 6.9 (Structure Constant). *Suppose a n dimensional Lie aglebra $\mathfrak{g}$ with basis $X_1, ..., X_N$, its structure constant is defined through*

$$[X_i, X_j] = \sum_{k=1}^{N} c_{ijk} X_k$$

Since $[\ ,\ ]$ is anti-symmetric and satisfies Jacobi identity, we have

1. $c_{ijk} = -c_{jik}$
2. $\sum_{\ell=1}^{N} (c_{jk\ell} c_{i\ell m} + c_{ij\ell} c_{k\ell m} + c_{ki\ell} c_{j\ell m}) = 0$

Here are some examples:

1. $\mathfrak{u}(1)$: It has a single generator, and $\mathfrak{u}(1) \cong \mathbb{R}$. Clearly it's abelian. What's more, acccording to the definition of direct sum, $\bigoplus_{\ell=1}^{n} \mathfrak{u}(1)$ is always abelian.
2. 3-dimensional vectors in $\mathbb{R}^3$ can also be a Lie algebra, Lie bracket is cross product and $c_{ijk} = \varepsilon_{ijk}$.
3. All $n \times n$ real/compelx matrices form a Lie algebra $\mathfrak{gl}(n;\mathbb{R})/\mathfrak{gl}(n;\mathbb{C})$. Lie bracket is defined to be $[X, Y] = XY - YX$.
4. Define $\mathfrak{sl}(n;\mathbb{R})/\mathfrak{sl}(n;\mathbb{C})$ to be all traceless $n \times n$ real/compelx matrices, it's a Lie subalgebra of $\mathfrak{gl}(n;\mathbb{R})/\mathfrak{gl}(n;\mathbb{C})$.

6.2 Lie Groups and Lie Algebras

Intuitively, Lie algebras are infinitesimal generators of Lie groups. And according to the following theorem, there is an intimate connection between Lie algebras and Lie groups:

Theorem 6.1 (Lie's Third Theorem). *For any finite dimensional Lie algebra $\mathfrak{g}$, there exists a connected Lie subgroup G of $\mathsf{GL}(n;\mathbb{C})$ whose Lie algebra is isomorphic to $\mathfrak{g}$.*

However,G is not unique. For example, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, but obviously $\mathsf{SU}(2)$ is not isomorphic to $\mathsf{SO}(3)$.

To derive matrix Lie algebra from matrix Lie group, we'd better introduce matrix exponential first:

Definition 6.10 (Matrix Exponential). *A matrix ponential is defined to be the series:*

$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n \qquad X^0 := I$$

It can be proved e^{tX} is well-defined and has the following properties:

1. $e^0 = I$

2. $e^{aX}e^{bX} = e^{(a+b)X} \Rightarrow e^X e^{-X} = T$

3. $\left(e^X\right)^\dagger = e^{X^\dagger}$

4. If $[X,Y]=0$, then $e^X e^Y = e^{X+Y}$

5. For any $C \in \mathsf{GL}(n;\mathbb{C})$, $e^{CXC^{-1}} = Ce^X C^{-1}$.

6. $\dfrac{\mathrm{d}}{\mathrm{d}t} e^{tX} = Xe^{tX}$

Here we have a new definition for matrix Lie algebra:

Definition 6.11. *The associated Lie algebra of matrix group G is the set of all matrices X such that $e^{tX} \in G, \forall t \in \mathbb{R}$. The Lie bracket is $[X,Y] = XY - YX$.*

Theorem 6.2 (Lie Product Formula)**.** *Consider a Lie group G and its Lie algebra $\mathfrak{g}$, then $\forall X, Y \in \mathfrak{g}$,*

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{X/m}e^{Y/m}\right)^m$$

According to Definition 6.11, the associated algebra of a matrix Lie group satisfies the following properties:

Propsition 6.1. *Associated Lie algebra is well-defined. i.e. Consider a matrix group G with associated Lie algebra $\mathfrak{g}$, then*[1]

1. $\forall s \in \mathbb{R},\ X \in \mathfrak{g},\ sX \in \mathfrak{g}$.

2. $\forall C \in G,\ X \in \mathfrak{g},\ CXC^{-1} \in \mathfrak{g}$.

3. $\forall X, Y \in \mathfrak{g},\ X + Y \in \mathfrak{g}$.

4. $\forall X, Y \in \mathfrak{g},\ [X,Y] = XY - YX \in \mathfrak{g}$.

Proof. The first proposition is trivial, we may just focus on the rest ones. For the second proposition, we can just use the fifth property in Definition 6.10:

$$e^{CXC^{-1}} = Ce^X C^{-1}$$

As $C, C^{-1}, e^X \in G$, according to the axiom of groups, we have

$$e^{CXC^{-1}} = Ce^X C^{-1} \in G \Rightarrow CXC^{-1} \in \mathfrak{g}$$

For the third proposition, use Lie product formula and we have

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{X/m}e^{Y/m}\right)^m$$

As for any finite m, $e^{X/m}e^{Y/m} \in G$, besides the definition of matrix Lie groups assures the colseness of G under limition operations, thus

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{X/m}e^{Y/m}\right)^m \in G \Rightarrow X + Y \in \mathfrak{g}$$

[1] The second property is not only a lemma needed to prove the fourth property, but it's also related to gauge transformation and is of physical interest in gauge theory.

The fourth one relies on the second one, we have to choose any two elements $X, Y \in \mathfrak{g}$ and consider $e^{tX}Ye^{-tX} \in \mathfrak{g}$. Since $\mathfrak{g}$, as a vector space, is colsed under limitation operations, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX}Ye^{-tX}\right)\bigg|_{t=0} = XY - YX = [X, Y] \in \mathfrak{g}$$

$\square$

Here are some examples on associated Lie algebras:

Example 6.1.

1. *$\mathfrak{gl}(n;\mathbb{R})$ and $\mathfrak{gl}(n;\mathbb{C})$: Lie algebras associated to $\mathsf{GL}(n;\mathbb{R})$ and $\mathsf{GL}(n;\mathbb{C})$. According to matrix exponential, we clearly have*

$$\mathfrak{gl}(n;\mathbb{C}) = M_n(\mathbb{C}) \qquad \mathfrak{gl}(n;\mathbb{R}) = M_n(\mathbb{R})$$

Note that $\mathfrak{gl}(n;\mathbb{C})$ should be considered as a complex algebra and we intuitively have

$$\mathfrak{gl}(n;\mathbb{C}) = \mathfrak{gl}(n;\mathbb{R})_{\mathbb{C}}$$

Deonte $E_{k,\ell}$ is a matrix whose the only nozero element is 1 at position (k, ℓ), then we can construct a basis (find a set of generators) for $\mathfrak{gl}(n)$ (no matter it's on $\mathbb{R}$ or $\mathbb{C}$):

$$E_{k,\ell}, \quad k, \ell = 1, 2, ..., n$$

Therefore, $\mathfrak{gl}(n)$ is a n^2 dimensional vectoor space. The following job is to find commutation relations:

$$\begin{aligned}
[E_{k,\ell}, E_{m,n}]_{i,j} &= (E_{k,\ell})_{i,p}(E_{m,n})_{p,j} - (E_{m,n})_{i,p}(E_{k,\ell})_{p,j} \\
&= \delta_{ki}\delta_{\ell p}\delta_{mp}\delta_{nj} - \delta_{mi}\delta_{np}\delta_{kp}\delta_{j\ell} \\
&= \delta_{ik}\delta_{m\ell}\delta_{nj} - \delta_{im}\delta_{nk}\delta_{\ell j}
\end{aligned}$$

Thus

$$[E_{k,\ell}, E_{m,n}] = \delta_{m,\ell}E_{k,n} - \delta_{n,k}E_{m,j}$$

2. *$\mathfrak{sl}(n;\mathbb{R})$ and $\mathfrak{sl}(n;\mathbb{C})$: Lie algebras associated to $\mathsf{SL}(n;\mathbb{R})$ and $\mathsf{SL}(n;\mathbb{C})$. Compared to $\mathsf{GL}(n)$, there's only one more constraint: $\det M = +1$. According to*

$$\det e^X = e^{\operatorname{tr} X}$$

Thus $\mathfrak{sl}(n)$ is composed by $n \times n$ matrices with zero trace. We can easily find generators for it:

$$E_{k,\ell},\ k, \ell = 1, 2, ..., n, k \neq \ell, \qquad E_{k,k} - E_{k+1,k+1},\ k = 1, 2, ..., n-1$$

Counting the number of generators, we have

$$\dim \mathfrak{sl}(n) = n^2 - n + n - 1 = n^2 - 1$$

3. $\mathfrak{u}(n)$: *Lie algebra associated to* $\mathsf{U}(n)$. *As* $\forall X \in \mathfrak{u}(n)$, $\forall t \in \mathbb{R}$, $e^{tX} \in \mathsf{U}(n)$, *thus*

$$\left(e^{tX}\right)^{\dagger} e^{tX} = e^{tX^{\dagger}} e^{tX} = I$$

Differentiating by t *then set* $t = 0$, *we have the constraint giving us* $\mathfrak{u}(n)$:

$$X^{\dagger} + X = 0,\ \forall X \in \mathfrak{u}(n)$$

That is, $\mathfrak{u}(n)$ *is composed by anti-hermitian matrices of order* n. *To explicitly write down a set if its generators, let's consider two cases:*

- *If elements of* X *are real numbers, then* $X^T + X = 0$, *thus the corresponding generators are*

$$E_{k,\ell} - E_{\ell,k},\ k,\ell = 1,2,...,n,\ k < \ell$$

- *If elements of* X *are all complex numbers, then* $X^{\dagger} + X = 0$ *corresponds to* $-X^T + X = 0$, *thus the generators are*

$$\mathrm{i}(E_{k,\ell} + E_{\ell,k}),\ k,\ell = 1,2,...,n,\ k < \ell \qquad \mathrm{i}E_{k,k},\ k = 1,2,...,n$$

$\mathfrak{u}(n)$ *itself is a real vector space, however, if we introduce complex coefficients to complexify it, we can easily have*

$$\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}(n;\mathbb{C})$$

4. $\mathfrak{su}(n)$: *Lie algebra associated to* $\mathsf{SU}(n)$. *Clearly it's composed by elements of* $\mathfrak{u}(n)$ *with trace zero. Thus we can immediately write down its generators:*

$$\begin{gathered} E_{k,\ell} - E_{\ell,k},\ k,\ell = 1,2,...,n,\ k < \ell \\ \mathrm{i}(E_{k,\ell} + E_{\ell,k}),\ k,\ell = 1,2,...,n,\ k < \ell \\ \mathrm{i}(E_{k,k} - E_{k+1,k+1}),\ k = 1,2,...,n-1 \end{gathered}$$

It can be showed that

$$\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathfrak{u}(1) \qquad \mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n;\mathbb{C})$$

Note that after complexification, we can build relations between seemingly unrelated Lie algebras. That's the reason why complexification is very useful: Classifying complex Lie algebras is much easier than classifying real Lie algebras.

5. $\mathfrak{so}(n)$: *Lie algebra associated to* $\mathsf{SO}(n)$. *Using matrix exponentials, it's obvious that*

$$X^T + X = 0,\ \forall X \in \mathfrak{so}(n)$$

Thus one choice for its generators is

$$E_{k,\ell} - E_{\ell,k},\ k,\ell = 1,2,...,n,\ k < \ell$$

It can be proved that

$$\mathfrak{so}(3) \cong \mathfrak{su}(2)$$

6. $\mathfrak{o}(p,q)$*: Lie algebra associated to* $\mathsf{O}(p,q)$*. Denote*

$$g = \mathrm{diag}(\underbrace{1,...,1}_{p},\underbrace{-1,...,-1}_{q})$$

Then $\forall X \in \mathfrak{o}(p,q)$*, we should have*

$$e^{tX^T} g e^{tX} = g$$

Thus

$$X^T g + gX = 0$$

Denote

$$X = \begin{pmatrix} \mathbf{A}_{p\times p} & \mathbf{B}_{p\times q} \\ \mathbf{C}_{q\times p} & \mathbf{D}_{q\times q} \end{pmatrix}$$

Then

$$\begin{pmatrix} \mathbf{A}^T & -\mathbf{C}^T \\ \mathbf{B}^T & -\mathbf{D}^T \end{pmatrix} = -\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & -\mathbf{D} \end{pmatrix}$$

Thus

$$\mathbf{A} + \mathbf{A}^T = 0 \quad \mathbf{B} = \mathbf{C}^T \quad \mathbf{D} + \mathbf{D}^T = 0$$

And we have

$$\begin{aligned} \mathbf{A} &= E_{k,\ell} - E_{\ell,k},\ k,\ell = 1,2,...,p,\ k<\ell \\ \mathbf{D} &= E_{k,\ell} - E_{\ell,k},\ k,\ell = p+1,...,p+q,\ k<\ell \\ \mathbf{B}^T = \mathbf{C} &= E_{k,\ell},\ k = 1,2,...,p,\ \ell = 1,2,...,q \end{aligned}$$

Just counting the number of generators, we have

$$\dim \mathfrak{so}(p,q) = \dim \mathfrak{so}(p+q)$$

$\forall X \in \mathfrak{o}(p,q)$*, clearly* $\mathrm{tr}\, X = 0$*, thus* $\mathfrak{o}(p,q)$ *has the same Lie algebra generators as* $\mathfrak{so}(p,q)$*.*

Now consider $\mathfrak{so}(2,1)$*, we can explicitly write its generators down as*

$$L_1' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_2' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad L_3' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Direct calculation shows

$$\begin{aligned} [L_1', L_2'] &= -L_3' \\ [L_2', L_3'] &= L_1' \\ [L_3', L_1'] &= -L_2' \end{aligned}$$

Meanwhile, generators of $\mathfrak{so}(3)$ *can be written as*

$$L_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Then the commutation relation is

$$\begin{aligned}[L_1, L_2] &= L_3\\ [L_2, L_3] &= L_1\\ [L_3, L_1] &= L_2\end{aligned}$$

At a first glance, $\mathfrak{so}(2,1)$ and $\mathfrak{so}(3)$ have different commutation relations, implying they are not likely to be isomorphic in the real sense, and it can be proved that we cannot construct a real isomorphic map between $\mathfrak{so}(2,1)$ and $\mathfrak{so}(3)$. However, if we consider the same problem on $\mathbb{C}$, it can be proved that $\mathfrak{so}(2,1)_{\mathbb{C}} \cong \mathfrak{so}(3)_{\mathbb{C}}$. Again, we have witnessed the power of complexification in bridging diffferent Lie algebras together.

7. $\mathfrak{sp}(2n;\mathbb{R})$*: The associated Lie algebra of* $\mathsf{Sp}(2n;\mathbb{R})$*. For $\forall X \in \mathfrak{sp}(2n;\mathbb{R})$, we have*

$$e^{tX}\Omega e^{tX^T} = \Omega \Rightarrow X\Omega + \Omega X^T = 0 \Rightarrow X = \Omega X^T \Omega$$

Denote

$$X = \begin{pmatrix} \mathbf{A}_{n\times n} & \mathbf{B}_{n\times n} \\ \mathbf{C}_{n\times n} & \mathbf{D}_{n\times n} \end{pmatrix}$$

Then we have

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} -\mathbf{D}^T & \mathbf{B}^T \\ \mathbf{C}^T & -\mathbf{A}^T \end{pmatrix} \Rightarrow \begin{cases} \mathbf{A} = -\mathbf{D}^T \\ \mathbf{B} = \mathbf{B}^T \\ \mathbf{C} = \mathbf{C}^T \end{cases}$$

Therefore a set of generators of $\mathfrak{sp}(2n;\mathbb{R})$ can be expressed as

$$\begin{aligned}&E_{k,\ell} - E_{\ell+n,k+n},\ k,\ell = 1,...,n\\ &E_{k,\ell+n} + E_{\ell,k+n},\ k,\ell = 1,...,n,\ k<\ell\\ &E_{k,k+n},\ k = 1,...,n\\ &E_{k+n,\ell} + E_{\ell+n,k},\ k,\ell = 1,...,n,\ k<\ell\\ &E_{k+n,k},\ k = 1,...,n\end{aligned}$$

6.3 Lie Algebras: A Differential Manifold Viewpoint

As we know, a Lie group G can be viewed as a differential manifold with smooth group structure. In this section, we will show how Lie algebra and Lie bracket naturally emerge from manifold structure.

Definition 6.12 (Tangent Vector)**.** *A tangent vector at a point $p \in M$ is defined to be an equivalence class of curves $c(t)$ in M. We define $c_1(t) \sim c_2(t)$ if*

1. $c_1(0) = c_2(0) = p$

2. *Assum p is in coordinate patch (U, x^μ), then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}x^\mu(c_1(t))\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}x^\mu(c_2(t))\right|_{t=0}$$

The set of all tangent vectors at a point p has the structure of a vector space, and is named as tangent space, denoted as T_pM. Its basis is denoted as $e_\mu = \partial_\mu$, and

$$\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} T_pM$$

The subscript $\mathbb{R}$ indicates dimension in the real sense.

Definition 6.13 (Lie Agebra)**.** *The Lie algebra of a Lie group G is defined to be the tangent space T_eM at identity element e.*

Take $\mathsf{SU}(2) \cong S^3$ as example, as we know its elements can be expressed as

$$\begin{pmatrix} a+b\mathrm{i} & c+d\mathrm{i} \\ -c+d\mathrm{i} & a-b\mathrm{i} \end{pmatrix}, \quad a^2+b^2+c^2+d^2=1$$

The identity element corresponds to $(a,b,c,d) = (1,0,0,0)$, a curve in $\mathsf{SU}(2)$ is

$$(a(t), b(t), c(t), d(t)), \quad a^2(t)+b^2(t)+c^2(t)+d^2(t)=1$$

Thus the tangent vector induced by the curve is

$$(a'(t), b'(t), c'(t), d'(t)), \quad a(t)a'(t)+b(t)b'(t)+c(t)c'(t)+d(t)d'(t)=0$$

As $e = (1,0,0)$, it can be deduced that at e, the elements of T_eM has the form $(0, b'(t), c'(t), d'(t))$, where $b'(t), c'(t), d'(t)$ are unconstrained. Thus we can select a basis as $(0,1,0,0), (0,0,1,0), (0,0,0,1)$, corresponding to the generators of $\mathfrak{su}(2)$ as below:

$$\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

The emerging of Lie bracket is more tricky: It's related to Lie derivative of a vector field relative to another one.

Definition 6.14 (Vector Field)**.** *A vector field v is a section of tangent bundle TM which is smooth and for any differentiable function $f : M \to \mathbb{R}$, $v(f)$ is also a differentiable function.*

Definition 6.15 (Flow)**.** *A flow $\sigma(t,x) : \mathbb{R} \to M \to M$ is an action of real additive group on an manifold M, satisfying :*

1. *For fixed $t \in \mathbb{R}$, $\sigma(t,\cdot) : M \to M$ is a diffeomorphism and is called one-parameter group.*

2. *Under infinitisimal action $\sigma(\varepsilon,\cdot)$, x^μ is changed to $\sigma^\mu(\varepsilon,x) = x^\mu + \varepsilon X^\mu(x)$, where $X^\mu(x)$ is a vector field and is called infinitisimal generator of the flow.*

According to the definition of flow $\sigma(t,x)$, it can be deduced that $\sigma^\mu(0,x) = x^\mu$ and thus we have the equation that determines a flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^\mu(t,x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\sigma^\mu(t+\varepsilon,x) - \sigma^\mu(t,x)] = \lim_{\varepsilon\to 0}[\sigma^\mu(\varepsilon,\sigma(t,x)) - \sigma^\mu(t,x)] = X^\mu(\sigma(t,x))$$

Now, we consider two vector fields $X^\mu(x)$ and $Y^\mu(x)$ and we want to investigate the change of vector field $Y(x)$ along the flow generated by field $X(x)$. It's actually a tricky job, since on a general manifold different tangent spaces at different points are not identified and it makes no sense to make definition like

$$\lim_{\epsilon\to 0}\frac{1}{\varepsilon}[Y(\sigma(t+\epsilon,x)) - Y(\sigma(t,x))]$$

As the manifold does not necessarily have metric structure, we cannot define a way to translate a vector at point $\sigma(t,x)$ to the point $\sigma(t+\varepsilon,x)$.

Now, the only thing we can do is to recall the definition of tangent vectors. Let's generally consider a map between two manifolds $f: M \to N$. At a point $p \in M$, a tangent vector here is defined to be an equivalence class of curves. Meanwhile, each curve in this class is mapped to a curve in N passing through $f(p)$ by f. If we are so lucky that these curves in N also form an equivalence class, then we will have a correspondence between two vectors in dofferent tangent spaces. According to Definition 6.12, we are indeeed very lucky! The first criteria is obviously satisfied, we only have to check the second one: Suppose two curves $c_1(t), c_2(t)$ in M satisfying the second criteria, then in N, assuming the coordinate map is $y^\mu = f^\mu(x)$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}y^\mu(f(c_1(t)))\right|_{t=0} = \frac{\partial y^\mu}{\partial x^\nu}\left.\frac{\mathrm{d}}{\mathrm{d}t}x^\nu(c_1(t))\right|_{t=0} = \frac{\partial y^\mu}{\partial x^\nu}\left.\frac{\mathrm{d}}{\mathrm{d}t}x^\nu(c_2(t))\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}y^\mu(f(c_2(t)))\right|_{t=0}$$

This motivates us to formally define push forward map:

Definition 6.16 (Push Forward). *Consider a map $f: M \to N$, the push forward map $f_*: T_pM \to T_{f(p)}N$ is defined by*

$$[f_*v](g) = v(g\circ f)$$

Where g is any differential function on N

This definition is compatible with our previous definition, since if we denote $v = \mathrm{d}c(t)/\mathrm{d}t$, then

$$v(g\circ f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}g(f(c(t)))\right|_{t=0}$$

Thus f_*v is exactly identical to the equivalence class $[f(c(t))]$. To write f_*v explicitly, consider $[f_*v](y^\mu)$:

$$(f_*v)^\mu = [f_*v](y^\mu) = v(y^\mu\circ f) = v(y^\mu(x)) = \frac{\partial y^\mu}{\partial x^\nu}v^\nu = \frac{\partial f^\mu}{\partial x^\nu}v^\nu$$

Now we have prepared to find a way to represent the change of vector field $Y(x)$ along the flow generated by field $X(x)$. $X(x)$ generates a flow $\sigma(t,x)$ on M and it can be viewed as a map $\sigma_t: M \to M$, therefore it induces a push forward map $(\sigma_t)_*: T_xM \to T_{\sigma_t(x)}M$. Therefore we have the following definition:

Definition 6.17 (Lie Derivative)**.** *The Lie derivative of vector field $Y(x)$ relative to the field $X(x)$ is defined to be*

$$\mathcal{L}_X Y = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[(\sigma_{-\varepsilon})_* Y(\sigma_\varepsilon(x)) - Y(x)] = (X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu)\partial_\nu := [X,Y]$$

Now let's spend some time on proving the explicit formula: Deonting the two vector fields as

$$Y(x) = Y^\mu\partial_\mu \quad X(x) = X^\mu\partial_\mu$$

Then

$$Y(\sigma_\varepsilon(x)) = Y(x^\mu + \varepsilon X^\mu(x)) = Y^\mu\partial_\mu + \varepsilon X^\nu\partial_\nu Y^\mu\partial_\mu + \mathcal{O}(\varepsilon^2)$$

As

$$\sigma^\mu_{-\varepsilon}(x) = x^\mu - \varepsilon X^\mu(x)$$

We have

$$(\sigma_{-\varepsilon})_* Y(\sigma_\varphi(x)) = (\delta^\nu_\mu - \varepsilon\partial_\mu X^\nu)(Y^\mu + \varepsilon X^\rho\partial_\rho Y^\mu)\partial_\nu = Y(x) + \varepsilon(X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu)\partial_\nu + \mathcal{O}(\varepsilon^2)$$

Thus

$$\mathcal{L}_X Y = (X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu)\partial_\nu$$

Propsition 6.2.

$$f_*([X,Y]) = [f_*X, f_*Y]$$

Proof.

$$\begin{aligned}
[f_*X, f_*Y] &= \frac{\partial f^\mu}{\partial x^\nu} X^\nu \overline{\partial}_\mu \left(\frac{\partial f^\rho}{\partial x^\sigma} Y^\sigma\right)\overline{\partial}_\rho - \frac{\partial f^\mu}{\partial x^\nu} Y^\nu \overline{\partial}_\mu \left(\frac{\partial f^\rho}{\partial x^\sigma} X^\sigma\right)\overline{\partial}_\rho \\
&= (X^\nu\partial_\nu Y^\sigma - Y^\nu\partial_\nu X^\sigma)\frac{\partial f^\rho}{\partial x^\sigma}\overline{\partial}_\rho \\
&= f_*([X,Y])
\end{aligned}$$

□

We can also verify that Lie bracket satisfies Jacobi identity.

Now consider a Lie group manifold G, we have the following definition based on the group structure:

Definition 6.18 (Left-translation)**.** *For fixed $a \in G$, left-translation map $L_a : G \to G$ is defined as*

$$L_a g = ag,\ \forall g \in G$$

According to Lie group axiom, L_a is a smooth map and thus induces a push forward map $L_{a*} : T_g G \to T_{ag} G$. We then have the concept of left-invariance:

Definition 6.19 (Left-invariant Vector Field)**.** *For a vector field X on G, it's left-invariant if*

$$L_{a*} X(g) = X(ag),\ \forall g \in G$$

For a Lie group manifold G, an element $v \in T_eG$ uniquely determines a left-invariant vector field as follows:

$$X_v|_g = L_{g*}v,\ g \in G$$

As

$$L_{a*}X_v|_g = L_{a*}L_{g*}v = L_{(ag)*}v = X_v|_{ag}$$

It's indeed a well-defined left-invariant vector field. Conversely, any left-invariant vector field also defines a unique vector in T_eG: Just consider $L_{g^{-1}*}X_g$. Thus T_eG is ispmorphic to the set of left-invariant vector fields on G and we can alternatively define Lie algebra as the set of all left-invariant vector fields. This definition is convenient for defining Lie brackets:

According to Proposition 6.2, for left-invariant vector fields X, Y, we have

$$L_{a*}([X,Y]_g) = [L_{a*}X_g, L_{a*}Y_g] = [X_{ag}, Y_{ag}] = [X,Y]_{ag}$$

Thus $[X,Y]$ is also left-invariant and the set of left-invariant vector fields is colsed under Lie derivatives. Then we can finally give a formal definition of Lie algebra:

Definition 6.20 (Lie Algebra)**.** *The Lie algebra of a Lie group G is defined to be the set of left-invariant vector fields $\mathfrak{g}$ with the Lie bracket:*

$$[X,Y] = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu$$

7 Basics on Representation Theory

7.1 Linear Represnetations

Definition 7.1 (Lie Group Representation)**.** *A finite-dimensional complex representation of a Lie group G is a Lie group homomorphism $\Pi : G \to \mathsf{GL}(V)$, where V is a finite-dimensional complex vector space isomorphic to $\mathbb{C}^n$. The dimension of the representation Π is defined to be*

$$\dim \Pi = \dim V$$

$\forall g \in G$, it's action on V through representation Π is denoted as $\Pi(g)$.

Definition 7.2 (Lie Algebra Representation)**.** *A finite-dimensional complex representation of Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is again a finite-dimensional complex vector space. We also define*

$$\dim \pi = \dim V$$

Definition 7.3 (Faithful Representation)**.** *For Lie group or Lie algebra representationds, Π or π is faithful if it's injective.*

Above dfinitions are made for complex representations, based on complex vector space. If we demand V to be a real vector space isomorphic to $\mathbb{R}^n$, then we can make similar statements for real representations, whose dimension is also defined to be $\dim V$.

An important issue in group theory is doing calssifications. For Lie group/Lie algebra representations, we focus on classifying irreducible representations:

Definition 7.4 (Irreducible Representation). *Consider a representation of Lie group* $\Pi : G \to \mathsf{GL}(V)$*. Define a subspace* $W \subset V$ *to be an invariant subspace if* $\forall w \in W$*,* $\forall g \in G$*,* $\Pi(g)w \in W$*. If the only invariant subspaces for the representation are* V *and* $\{0\}$*, then* Π *is irreducible.*

We can make similar definition for Lie algebra representations. Irriduciable representations are like bricks to build other representations, and is of great importance in classifying Lie groups. To do classifications, we have to define equivalence relations: In mathematics, equivalence is tied to isomorphism:

Definition 7.5 (Intertwining Map). *Consider two represnetations* $\Pi : G \to \mathsf{GL}(V)$*,* $\Sigma : G \to \mathsf{GL}(W)$*, then* Π, Σ *are defined to be homomorphic if there exists a linear map* $\phi : V \to W$ *such that*

$$\phi(\Pi(g)v) = \Sigma(g)\phi(v), \quad \forall g \in G,\ v \in V$$

ϕ *is called the inertwining map.*

Definition 7.6 (Isomorphism). *If the intertwining map* ϕ *is invertiable, the it's a representation isomorphism and they are isomorphic. We can make a similar definition for Lie algebra representations.*

Classifications of Lie group representations are up to isomorphism in this sense.

Theorem 7.1 (Schur's Lemma).

1. *Let* $\Pi : G \to \mathsf{GL}(V)$ *and* $\Sigma : G \to \mathsf{GL}(W)$ *be irreducible real/complex representations of Lie group* G *or Lie algebra* $\mathfrak{g}$*.* ϕ *is intertwining map* $\phi : V \to W$*, then* ϕ *is either 0 or an isomorphism.*

2. *Let* $\Pi : G \to \mathsf{GL}(V)$ *be an irreducible complex representation for a Lie group or Lie algebra, let* $\phi_1 : V \to V$ *be an intertwining map to* V *itself, then* $\phi = \lambda I,\ \lambda \in \mathbb{C}$*.*

3. *Let* $\Pi : G \to \mathsf{GL}(V)$*,* $\Sigma : G \to \mathsf{GL}(W)$ *be irreducible complex representations of a Lie group or Lie algebra and let* $\phi_1, \phi_2 : V \to W$ *be nonzero intertwining maps. Then* $\phi_1 = \lambda\phi_2, \lambda \in \mathbb{C}$*.*

Proof. For the first clause: $\phi = 0$ is obvious. As ϕ satisfies

$$\phi(\Pi(g)v) = \Sigma(g)\phi(v), \quad \forall g \in G,\ v \in V$$

If ϕ is not an representation isomorphism, then $\ker\phi \neq \{0\}.\forall v_1 \neq 0 \in \ker\phi$,we have $\phi(v_1) = 0$. Then for any $g \in G$, we have

$$\phi(\Pi(g)v) = \Sigma(g)\phi(v) = \Sigma(g)0 = 0$$

Thus $\Pi(g)v \in \ker\phi$ and $\ker\phi$ is invariant subspace of representation Π, contradicating with the satatement that "Π is irreducible". Therefore, ϕ is either 0 or an isomorphism for Lie group representations. Following the same process, obviously it's the same for Lie algebra representations.

For the second clause, if $\phi : V \to V$ is the intertwining map to itself, then $\forall g \in G$, we have

$$\Pi(g)\phi = \phi\Pi(g)$$

Since we are considering complex representations, ϕ has at least one eigenvalue κ and corresponds to an eigenspace K. However, for any vector $k \in K$, we have

$$\phi(\Pi(g)k) = \Pi(g)\phi(k) = \kappa\Pi(g)k \Rightarrow \Pi(g)k \in K,\ \forall g \in G$$

Thus K is an invariant subspace for the representation. Since Π is irreducible, thus K is either $\{0\}$, which is impossible, or V, which is the only possible condition. If ϕ has the whole V as an eigenspace, then it must be proportional to identity matrix. Thus $\phi = \lambda I, \lambda \in \mathbb{C}$.

For the third clause: If $\phi_2 \neq 0$, then it's an isomorphism and $\phi_1 \circ \phi_2^{-1}$ is a nonzero intertwining map on W itself. Thus $\phi_1 \circ \phi_2^{-1} = \lambda I$ and $\phi_1 = \lambda\phi_2$. □

It it the first clause of Schur's lemma that assures us the legitimacy of classifying irreducible representations: Two irreducible representations are either isomorphic, or has nothing to do woth each other at all.

Corollary 7.1. *Two irreducible representations of the same Lie group/algebra can't be isomorphic if* $\dim V \neq \dim W$

Proof. According to the fundamental theorem of linear maps, there doesn't exist a bijection between vector spaces with different dimension. Thus if $\Pi : G \to \mathsf{GL}(V)$ and $\Sigma : G \to \mathsf{GL}(W)$ are two irreducible representations and $\dim V \neq \dim W$, then potential intertwining map $\phi : V \to W$ can only be trivial zero map. □

Corollary 7.2. *If* Π *is an irreducible representation of* G *and* A *is the center of* G*, then* $\Pi(A) = \lambda I,\ \lambda \in \mathbb{C}$*. It's the same for Lie algebra representations.*

Proof. $\forall X \in A,\ g \in G$, we have

$$\Pi(X)\Pi(g) = \Pi(Xg) = \Pi(gX) = \Pi(g)\Pi(X)$$

Thus $\Pi(X)$ is a intertwining map on the vector space itself, thus $\Pi(A) = \lambda I$. □

If we have a Lie group representation Π, then $\forall g \in G$, $\Pi(g) \in \mathsf{GL}(V)$ and thus can be represented in matrix exponential, indicating that it induces a Lie algebra representation via

$$\pi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Pi(e^{tX})\bigg|_{t=0}$$

This is actually one way to calculate Lie algebra representations based on Lie group representations.

Definition 7.7 (Unitary Representation)**.** *Lie group representation* $\Pi : G \to \mathsf{GL}(V)$ *is unitary, if* $\forall g \in G$*,*

$$\Pi(g)^\dagger \Pi(g) = I$$

Then unitary Lie algebra representation should satisfy

$$\pi(X)^\dagger + \pi(X) = 0$$

Which could be proved by differentiating $e^{t\pi(X)} = \Pi(e^{tX})$ *by* t*.*

Theorem 7.2. *A Lie group* G *has a faithful finite-dimensional unitary representation if and only if* G *is compact.*

For example, Lorentz group is non-compact and doesn't have a finite-dimensional faithful unitary representation. $\mathsf{GL}(n;\mathbb{R}), \mathsf{GL}(n;\mathbb{C}), \mathsf{SL}(n;\mathbb{R}), \mathsf{SL}(n;\mathbb{C})$ are non-compact and also has no faithful unitary representations. While $\mathsf{U}(1)$ has faithful unitary representations:

$$\forall e^{\mathrm{i}\theta} \in \mathsf{U}(1),\ \Pi_a(e^{\mathrm{i}\theta}) := e^{\mathrm{i}a\theta},\ a \in \mathbb{Z}$$

Note that $a \in \mathbb{Z}$ is necessary, as we should have

$$\Pi_a(e^{\mathrm{i}\theta} e^{\mathrm{i}(2\pi - \theta)}) = \Pi_a(1) = 1 = e^{2\pi a \mathrm{i}}$$

The integer a is called $\mathsf{U}(1)$ charge and is related to the quantization of electric charge.

As $\mathsf{U}(1)$ is abelian, according to Schur's lemma, only one dimensional representation of it is irreducible, as $\mathsf{U}(1)$ is center of itself and thus $\Pi(e^{\mathrm{i}\theta}) = \lambda I$, then any subspace of V is automatically invariant. Similarly, $\mathfrak{u}(1)$ is irreducible only when the dimension is 1. And unlike $\Pi_a(e^{\mathrm{i}\theta})$, we have $\pi_a(\mathrm{i}\theta) = \mathrm{i}a\theta,\ a \in \mathbb{R}$ and there's no 1-1 correspondence between $\pi(\mathfrak{u}(1))$ and $\Pi(\mathsf{U}(1))$, which is rlelated to the non-simply-connectness of $\mathsf{U}(1)$.

Propsition 7.1. *For a connected Lie group* G*, consider its Lie group rpresentation* $\Pi : G \to \mathsf{GL}(V)$ *and Lie algebra representation* $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$*, then:*

1. π *is irreducible if and only if* Π *is irreducible.*
2. π_1, π_2 *are isomorphic if and only if* Π_1, Π_2 *are isomorphic.*
3. *If* G *is simply connected, then every Lie algebra representation* π *can be derived from* Π *by* $\left.\dfrac{\mathrm{d}}{\mathrm{d}t}\Pi(e^{tX})\right|_{t=0}$.

Proof. Just consider matrix exponentials. □

Now let's take a glance at examples on Lie group/algebra representations:

Example 7.1.

1. *Standard Representation* (*Fundamental Representation*)*: For a matrix Lie group* G*, it's standard representation is just the inclusion map* $\Pi : G \to \mathsf{GL}(n;\mathbb{C})$*.* $\mathsf{U}(n), \mathsf{SU}(n)$ *have unitary representations as standard representations.* $\mathsf{SO}(n)$*, as subgroup of* $\mathsf{SU}(n)$*, is also unitary. While* $\mathsf{Sp}(2n;\mathbb{R}/\mathbb{C})$ *is non-compact, having no finite-dimensional faithful unitary representations.* $\mathsf{USp}(2n)$ *is compact, has finite-dimensional faithful unitary representations.*

2. Trivial Representation: For a Lie group G, its tirvial representation is a homomorphism $\Pi : G \to \mathsf{GL}(1;\mathbb{C})$ satisfying

$$\Pi(g) = 1, \quad \forall g \in G$$

The corresponding Lie algebra representation is

$$\pi(X) = 0, \quad \forall X \in \mathfrak{g}$$

Trivial representation is obviously non-faithful.

Definition 7.8 (Adjoint Representation)**.** *Consider a Lie group G, its adjoint representation is defined to be a homomorphism $\mathrm{Ad} : G \to \mathsf{GL}(\mathfrak{g})$ satisfying*

$$\mathrm{Ad}_A X = AXA^{-1}, \quad \forall A \in G,\ X \in \mathfrak{g}$$

Ad *naturally induces a Lie algebra representation* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$*:*

$$\mathrm{ad}_X Y = [X, Y], \quad \forall X, Y \in \mathfrak{g}$$

Obviously, we have

$$\dim(\mathrm{Ad}) = \dim(\mathfrak{g}) = \dim(G) = \dim(\mathrm{ad})$$

And

$$\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X}$$

Propsition 7.2. $\mathrm{Ad} : G \to \mathsf{GL}(\mathfrak{g})$ *and* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *are well-defined.*

Proof. Consider $\forall A, B \in G$ and $\forall X \in \mathfrak{g}$, we have

$$\mathrm{Ad}_A \mathrm{Ad}_B X = \mathrm{Ad}_A(BXB^{-1}) = ABXB^{-1}A^{-1} = \mathrm{Ad}_{AB} X$$

Thus Ad is indeed a group homomorphism. Meanwhile, $\forall X, Y, Z \in \mathfrak{g}$, we have

$$\mathrm{ad}_{[X,Y]} Z = [[X, Y], Z] = -[[Z, X], Y] - [[Y, Z], X] = [X, [Y, Z]] - [Y, [X, Z]] = [\mathrm{ad}_X, \mathrm{ad}_Y] Z$$

i.e. $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$ and ad is a Lie algebra homomorphism. $\square$

To make the concept of adjoint representation concrete, let's calculate the adjoint representation of $\mathfrak{su}(2)$: As we know, $\dim \mathfrak{su}(2) = 3$ and we can choose a basis $\{X_1, X_2, X_3\}$ satisfying

$$[X_i, X_j] = \varepsilon_{ijk} X_k$$

Thus

$$\mathrm{ad}_{X_i} X_j = \varepsilon_{ijk} X_k$$

and thus

$$(\mathrm{ad}_{X_i})_{k,\ell} = \varepsilon_{i\ell k} = -\varepsilon_{ik\ell}$$

That is

$$\mathrm{ad}_{X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathrm{ad}_{X_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \mathrm{ad}_{X_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can immediately notice the similarity between above matrices and generators of $\mathfrak{so}(3)$. Actually, the adjoint representation of $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$. In fact, this phenomenon comes from the relation between $\mathfrak{su}(2)$ and its adjoint representation: They are isomorphic, as the center of $\mathfrak{su}(2)$ is trivial.

Besides, notice that $\varepsilon_{ik\ell}$ is exactly the structure constant of $\mathfrak{su}(2)$. Motivated by this phenomenon, we propose the following proposition:

Propsition 7.3. *For a general Lie algebra whose structure constant $c_{ij,k}$ is defined by*

$$[X_i, X_j] = c_{ij,k} X_k$$

The matrix elements of its adjoint representation is

$$(\mathrm{ad}_{X_i})_{j,k} = c_{ik,j}$$

In QCD, there are 8 kinds of gluons. It's related to the 8 dimensional adjoint representation of $\mathfrak{su}(3)$.

7.2 Constructing New Representations from the Old

Definition 7.9 (Direct Sum Representation)**.** *$\Pi_1, ..., \Pi_k$ are representations of G acting on $V_1, ..., V_k$, the direct sum representation $\Pi_1 \oplus ... \oplus \Pi_k : G \to \mathsf{GL}(V_1 \oplus ... \oplus V_k)$ is defined by*

$$(\Pi_1 \oplus ... \oplus \Pi_k)(A)(v_1, ..., v_k) = (\Pi_1(A)v_1, ..., \Pi_k(A)v_k), \quad \forall A \in G$$

Similarily, for Lie algebra $\mathfrak{g}$ and its representations $\pi_1, ..., \pi_k$ on $V_1, ..., V_k$, the direct sum representation is defined by

$$(\pi_1 \oplus ... \oplus \pi_k)(X)(v_1, ..., v_k) = (\pi_1(X)v_1, ..., \pi_k(X)v_k), \quad \forall X \in \mathfrak{g}$$

According to the definition of the dimension of a representation, clearly

$$\dim(\Pi_1 \oplus ... \oplus \Pi_k) = \dim(\pi_1 \oplus ... \oplus \pi_k) = \dim(V_1 \oplus ... \oplus V_k) = \dim V_1 + ... + \dim V_k$$

If we have a bunch of irreducible representations, we can construct new representations by direct sums. Conversely, for a Lie group/algebra representation, whether it can be decomposed into direct sums of irreducible representations is related to a thing called complete reducibility:

Definition 7.10 (Complete Reducibility)**.** *A finite-dimensional representation of G or $\mathfrak{g}$ is completely reducible if it's isomorphic to a direct sum of irreducible representations. A Lie group/algebra has complete reducibility if any finite-dimensional representation of it is completely reducible.*

Propsition 7.4. *If G is a matrix Lie group and Π is a finite-dimensional unitary representation, then Π is completely reduciable. It's the same for the representation of a real Lie algebra $\mathfrak{g}$.*

Theorem 7.3. *If G is a compact matrix Lie group, then every finite-dimensional representation of it is completely reducible.*

Theorem 7.4. *If a complex Lie algebra $\mathfrak{g}$ is semi-simple, then it's completely reducible.*

Definition 7.11 (Tensor Product Representation)**.** *There are two kinds of tensor product representations:*

1. *Let Π_1 be a representation of G_1 acting on V_1 and Π_2 is a representation of G_2 on V_2. The tensor product representation $\Pi_1 \otimes \Pi_2 : G_1 \times G_2 \to \mathsf{GL}(V_1 \otimes V_2)$ is defined by*
$$(\Pi_1 \otimes \Pi_2)(A, B) = \Pi_1(A) \otimes \Pi_2(B),\ \forall A \in G_1, B \in G_2$$
For Lie algebra representations $\pi_1 : \mathfrak{g}_1 \to \mathfrak{gl}(V_1)$ and $\pi_2 : \mathfrak{g}_2 \to \mathfrak{gl}(V_2)$, the tensor product representation is $\pi_1 \otimes \pi_2 : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{gl}(V_1 \otimes V_2)$:
$$(\pi_1 \otimes \pi_2)(X_1, X_2) = \pi_1(X_1) \otimes I_2 + I_1 \otimes \pi_2(X_2), \quad \forall X_1 \in \mathfrak{g}_1,\ X_2 \in \mathfrak{g}_2$$
Which could be derived by
$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_1(e^{tX_1}) \otimes \Pi_2(e^{tX_2})\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}e^{t\pi_1(X_1)} \otimes e^{t\pi_2(X_2)}\bigg|_{t=0}$$

2. *If Π_1 and Π_2 are representations of the same group G acting on different vector spaces V_1, V_2, then the tensor product representation is defined to be $\Pi_1 \otimes \Pi_2 : G \to \mathsf{GL}(V_1 \otimes V_2)$:*
$$(\Pi_1 \otimes \Pi_2)(A) = \Pi_1(A) \otimes \Pi_2(A), \quad \forall A \in G$$
The corresponding Lie algebra representation is $\pi_1 \otimes \pi_2 : \mathfrak{g} \to \mathfrak{gl}(V_1 \otimes V_2)$:
$$\pi_1 \otimes \pi_2(X) = \pi_1(X) \otimes I_2 + I_1 \otimes \pi_2(X), \quad \forall X \in \mathfrak{g}$$

Generally, $\pi_1 \otimes \pi_2$ is a reducible representation. Decomposing it into direct sums of irreducible representations is tackled by Clebsch-Gordon theory to be introduced later on.

Definition 7.12 (Dual Representation)**.** *Consider a representation $\Pi : G \to \mathsf{GL}(V)$, its dual representation Π^* is a representation of G on the dual space of V, denoted as V^*. In linear algebra, the dual of an operator $T : V \to V$ is defined to be $T^* : V^* \to V^*$ such that $\forall v \in V, w^* \in V^*$, we have*
$$(T^* w^*)(v) = w^*(Tv)$$
Then we have $T^* = T^T$. However, we can't define $\Pi^*(g) = [\Pi(g)]^T$, as we will have
$$\Pi^*(g_1)\Pi^*(g_2) = [\Pi(g_2)\Pi(g_1)]^T = [\Pi(g_2 g_1)]^T \neq \Pi^*(g_1 g_2)$$

Instead, to preserve group operation we have to define

$$\Pi^*(g) = [\Pi(g^{-1})]^T$$

The corresponding Lie algebra representation can be derived form

$$\pi^*(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Pi^*(e^{tX})\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}e^{-t\pi(X)^T}\bigg|_{t=0} = -\pi(X)^T$$

It's easy to check that $\pi^* : G \to \mathfrak{gl}(V^*)$ is a well-defined Lie algebra homomorphism:

$$\pi^*([X,Y]) = -[X,Y]^T = -[Y^T, X^T] = [X^T, Y^T] = [\pi^*(X), \pi^*(Y)]$$

For an unitary representation, we have

$$[\Pi(g^{-1})]^T = [\Pi(g)^\dagger]^T = \overline{\Pi(g)}$$

Thus sometimes this kind of adjoint representation is called conjugate representation. For fundamental representation of $\mathsf{SU}(n)$, denoted as $\mathtt{n}$, the conjugate representation $\bar{\mathtt{n}}$ is sometimes called anti-fundamental representation.

If a representation's dual representation is isomorphic to the representation, then it is called a self-dual representation. A real representation is the one whose $\Pi(g)$ and $\pi(g)$ are all real, then we have the following proposition:

Propsition 7.5. *Every real represnetation is sefl-dual.*

Proof. To be completed... □

If a self-dual representation is not real, then it's called pseudo-real representation. Consider the proposition below:

Propsition 7.6. *The standard 2 dimensional representation of* $\mathsf{SU}(2)$ *is self-dual (pseudo-real).*

Proof. $\forall U \in \mathsf{SU}(2)$, its standard representation is $\Pi_s : \mathsf{SU}(2) \to \mathsf{GL}(\mathbb{C}^2)$:

$$\Pi_s(U) = \begin{pmatrix} a+b\mathrm{i} & c+d\mathrm{i} \\ -c+d\mathrm{i} & a-b\mathrm{i} \end{pmatrix}, \quad a^2+b^2+c^2+d^2 = 1$$

Its dual representation is $\Pi_s^* : \mathsf{SU}(2) \to \mathsf{GL}(\mathbb{C}^{2*})$:

$$\Pi_s^*(U) = [\Pi_s(U^{-1})]^T = \begin{pmatrix} a-b\mathrm{i} & c-d\mathrm{i} \\ -c+d\mathrm{i} & a+b\mathrm{i} \end{pmatrix}$$

Let's consider the map $\phi : \mathbb{C}^2 \to \mathbb{C}^{2*}$:

$$\phi : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Then we have

$$\Pi_s^*(U)\phi = \begin{pmatrix} c-d\mathrm{i} & -a+b\mathrm{i} \\ a+b\mathrm{i} & c-d\mathrm{i} \end{pmatrix} = \phi\Pi_s(U)$$

Therefore ϕ is an intertwining map. Besides, it's clear that ϕ is a bijection, thus ϕ is an isomorphism and the standard 2 dimensional representation of $\mathsf{SU}(2)$ is self-dual. □

Since $\mathsf{SU}(2)$ is sefl-dual, it's intuative that $\mathfrak{su}(2)$ is also self-dual. (As Lie algebra can be viewed as linearized Lie group near the identity element.) However, it can be proved that for a general $SU(n)$ with $n > 2$, it's nor self-dual.

We have introduced complexification in definition 6.3, the proposition below establishs the importance of this method in studying representations of a real Lie algebra:

Propsition 7.7. *If $\mathfrak{g}$ is a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ is its complexification, then for every finite dimensional complex representation π of $\mathfrak{g}$, there is a unique extension of π into a complex representation $\pi_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, defiend by*

$$\pi_{\mathbb{C}}(X + \mathrm{i}Y) = \pi(X) + \mathrm{i}\pi(Y), \quad \forall X, Y \in \mathfrak{g}$$

$\pi_{\mathbb{C}}$ is irreducible if and only if π is irreducible.

This proposition assures the power of complexification in Lie algebra representation theory. Suppose we have a real Lie algebra $\mathfrak{g}$ with generators $X_1, ..., X_n$ whose representation is hard to construct, we can consider it's complexification $\mathfrak{g}_{\mathbb{C}}$ with generators $X'_1, ..., X'_n$ and it's representation is usually easy to construct. As long as we know how to linearly combine $X'_1, ..., X'_n$ to get $X_1, ..., X_n$, we will easily get representation of $\mathfrak{g}$ out of representation of $\mathfrak{g}_{\mathbb{C}}$.

7.3 Representations of Common Non-abelian Lie Groups

7.3.1 SU(2)

Let V_m be the sapce of homogenous ploynomials of degree m in two complex variables z_1, z_2, elements of V_m can be expressed as

$$f(z) = f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + a_2 z_1^{m-2} z_2^2 + \cdots a_{m-1} z_1 z_2^{m-1} + a_m z_2^m, \quad a_i \in \mathbb{C}$$

Clearly

$$\dim V_m = m + 1$$

It can be verified $\mathsf{SU}(2)$ has a representation Π_m on V_m defined by

$$[\Pi_m(U)] f(z) = f(U^{-1} z), \quad \forall U \in \mathsf{SU}(2)$$

There are two ways of interpreting the seemingly unnatural inverse matrix U^{-1} in the above definition:

1. It's the restriction imposed by the definition of representation. $\Pi_m(U)$ should be viewed as an action on f in $f(z)$ rather than action on z. Therefore if we define $\Pi_m(U) f(z) = f(Uz)$, we will have

   $$\Pi_m(U_1)\Pi_m(U_2) f(z) = \Pi_m(U_2) f(U_1 z) = f(U_2 U_1 z) = \Pi_m(U_2 U_1 z)$$

   It contradicts with the definition of Lie group representation. On the other hand, according to the correct definition, we have

   $$\Pi_m(U_1)\Pi_m(U_2) f(z) = \Pi_m(U_2) f(U_1^{-1} z) = f(U_2^{-1} U_1^{-1} z = f((U_1 U_2)^{-1} z)) = \Pi_m(U_1 U_2) f(z)$$

   Which is consistent with the definition of group representation.

2. U^{-1} represents a sense of duality. We can think z as sort of contra-vectors while f is something like a co-vector dual to z. $\Pi_m(U)$ acting on f can thus be treated as something like operators acting on dual vectors.

   To make our statements more rigorous, let's consider representations of $\mathsf{SU}(2)$ on V_1. First note that any element in V_1 can be written as

$$f(z) = a_0 z_1 + a_1 z_2 = \begin{pmatrix} a_0 & a_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

   $\forall U \in \mathsf{SU}(2)$, its action on a contra-vector $z = (z_1, z_2)^T$ can be written in matrix form :

$$Uz = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

   Then representation of $\mathsf{SU}(2)$ on V_1 is

$$\begin{aligned} [\Pi_1(U)]f(z) &= f(U^{-1}z) = a_0(U_{11}^{-1}z_1 + U_{12}^{-1}z_2) + a_1(U_{21}^{-1}z_1 + U_{22}^{-1}z_2) \\ &= (U_{11}^{-1}a_0 + U_{21}^{-1}a_1)z_1 + (U_{12}^{-1}a_0 + U_{22}^{-1}a_1)z_2 \\ &= \begin{pmatrix} a_0 & a_1 \end{pmatrix} \begin{pmatrix} U_{11}^{-1} & U_{12}^{-1} \\ U_{21}^{-1} & U_{22}^{-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned}$$

   If we denote

$$[\Pi_1(U)]f(z) = \begin{pmatrix} a_0' & a_1' \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

   Then we should have

$$\begin{pmatrix} a_0' & a_1' \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \end{pmatrix} \begin{pmatrix} U_{11}^{-1} & U_{12}^{-1} \\ U_{21}^{-1} & U_{22}^{-1} \end{pmatrix}$$

   However, row vectors represent dual vectors, it's contra-vector form corresponds to the Hermitian conjugate. Since the matrix is in $\mathsf{SU}(2)$ and satisfies $U = \left(U^{-1}\right)^\dagger$, we have

$$\begin{pmatrix} a_0'^* \\ a_1'^* \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} a_0^* \\ a_1^* \end{pmatrix}$$

Let's back to consider the general representation $\Pi_m : \mathsf{SU}(2) \to V_m$. As the induced Lie algebra representation can be derived from

$$\pi_m(X) = \frac{\mathrm{d}}{\mathrm{d}t} \Pi_m(e^{tX}) \bigg|_{t=0}$$

Thus $\pi_m : \mathfrak{su}(2) \to \mathfrak{gl}(V_m)$ can be derived by considering

$$[\pi_m(X)f](z) = \frac{\mathrm{d}}{\mathrm{d}t}[\Pi_m(e^{tX})]f(z)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(e^{-tX}z)\bigg|_{t=0}$$

Given that

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = e^{-tX} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = -X \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

We thus have

$$
\begin{aligned}
[\pi_m(X)f](z) &= \frac{\partial f}{\partial z_1}\left.\frac{\mathrm{d}z_1}{\mathrm{d}t}\right|_{t=0} + \frac{\partial f}{\partial z_2}\left.\frac{\mathrm{d}z_2}{\mathrm{d}t}\right|_{t=0} \\
&= -(X_{11}z_1 + X_{12}z_2)\frac{\partial f}{\partial z_1} - (X_{21}z_1 + X_{22}z_2)\frac{\partial f}{\partial z_2}
\end{aligned}
$$

Therefore, the Lie algerba representation of $\mathfrak{su}(2)$ induced by Π_m on V_m can be expressed with differential operators[1] To write $\pi_m(X)$ explicitly, let's consider a basis of $\mathfrak{su}(2)$:

$$
J_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}
$$

Then we have a basis of $\pi_m(\mathfrak{su}(2))$:

$$
\begin{aligned}
\pi_m(J_1) &= -\mathrm{i}z_2\frac{\partial}{\partial z_1} - \mathrm{i}z_1\frac{\partial}{\partial z_2} \\
\pi_m(J_2) &= -z_2\frac{\partial}{\partial z_1} + z_1\frac{\partial}{\partial z_2} \\
\pi_m(J_3) &= -\mathrm{i}z_1\frac{\partial}{\partial z_1} + \mathrm{i}z_2\frac{\partial}{\partial z_2}
\end{aligned}
$$

Before we moving forward, let's take a break and investigate the relaton between $\pi_1(J_i)$ and the standard 2 dimensional representation of $\mathfrak{su}(2)$: Denote $f(z_1, z_2) = a_0z_1 + a_1z_2$, then direct calculation shows

$$
\begin{aligned}
\pi_1(J_1)f(z_1, z_2) &= -\mathrm{i}a_0z_1 - \mathrm{i}a_1z_1 = \begin{pmatrix} a_0 & a_1 \end{pmatrix}(-J_1^T)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
\pi_1(J_2)f(z_1, z_2) &= -a_0z_2 + a_1z_1 = \begin{pmatrix} a_0 & a_1 \end{pmatrix}(-J_2^T)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
\pi_1(J_3)f(z_1, z_2) &= -\mathrm{i}z_1a_0 + \mathrm{i}z_2a_2 = \begin{pmatrix} a_0 & a_1 \end{pmatrix}(-J_3^T)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\end{aligned}
$$

Therefore, π_1 is isomorphic to the dual of the standard 2 dimensional representation of $\mathfrak{su}(2)$.

For convenience, sometimes we don't study the original Lie algebra $\mathfrak{su}(2)$ but study its complexification $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2;\mathbb{C})$. According to proposition 7.7, determining the representations of complexified Lie algebra allows us to determine representations of the original algebra. For $\mathfrak{sl}(2;\mathbb{C})$, we can choose generators as follows:

$$
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
$$

Then we have

$$
\pi_m(X) = -z_2\frac{\partial}{\partial z_1} \quad \pi_m(Y) = -z_1\frac{\partial}{\partial z_2} \quad \pi_m(H) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}
$$

[1] It's not hard to aware that a differential operator can be viewed as a linear map on ploynomial space.

(Note that by the time now we are studying representation of $\mathfrak{sl}(2;\mathbb{C})$ on V_m, but can easily recover representation of $\mathfrak{su}(2)$ by linearly reorginazing generaotrs.) Applying $\pi_m(X), \pi_m(Y), \pi_m(H)$ to $z_1^{m-k}z_2^k \in V_m$ shows

$$\begin{aligned}
\pi_m(X)(z_1^{m-k}z_2^k) &= -(m-k)z_1^{m-k-1}z_2^{k+1} \\
\pi_m(Y)(z_1^{m-k}z_2^k) &= -kz_1^{m-k+1}z_2^{k-1} \\
\pi_m(H)(z_1^{m-k}z_2^k) &= (2k-m)z_1^{m-k}z_2^k
\end{aligned}$$

Thus $z_1^{m-k}z_2^k$ is eigenstate of $\pi_m(H)$, corresponding to eigenvalue $(2k-m)$. In representation theory, $(2k-m)$ has a fancy name: weight. Clearly, in V_m there are $(m+1)$ eigenstates of $\pi_m(H)$ and the corresponding weight ranges from $-m$ to $+m$. Meanwhile, it's obvious that $\pi_m(X)(z_1^{m-k}z_2^k)$ has weight $(2k+2-m)$, while $\pi_m(Y)(z_1^{m-k}z_2^k)$ has weight $(2k-2-m)$. Therefore $\pi_m(X)$ and $\pi_m(Y)$ shifts the weight by ± 2 respectively, and they correspond to creation and annhailation operators in physics.

z_2^m has the highest weight $+m$ and is named as the highest weight state, while z_1^{-m} has the lowest weight $-m$ and is named as the lowest weight state. Later we will learn a systematic method to construct a group representation and the key to the method is to find the highest/lowest state at the beginning, then we can construct the whole representation by acting ladder operators. By the way, $\pi_m(H)$ can be interpreted as an angular momentum operator on a direction, but the corresponding magnetic quantum number s should be taken as half of the number m here. i.e. $s = m/2$.

Actually, the Lie algebra representation π_m is irreducible for all $m \in \mathbb{N}_+$, proof is as follows:

Proof. In the previous text we have used π_m to denote representations of $\mathfrak{su}(2)$ as well as $\mathfrak{sl}(2;\mathbb{C})$, but it doesn't matter, as $\mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{sl}(2;\mathbb{C})$ so that $\mathfrak{su}(2)$ and $\mathfrak{sl}(2;\mathbb{C})$ have the same irreducibility. Thus, we can just focus on proving π_m is irreducible representation of $\mathfrak{sl}(2;\mathbb{C})$, as it's much more convenient to handle.

We have to show every non-zero invariant subspace of $\pi_m : \mathfrak{sl}(2;\mathbb{C}) \to \mathfrak{gl}(V_m)$ is equal to V_m. Assume W is a non-zero invariant subspace, then any element $w \in W$ has the form

$$w = a_0 z_1^m + a_1 z_1^{m-1} z_2 + \cdots + a_{m-1} z_1 z_2^{m-1} + a_m z_2^m$$

Of course not all a_i are nonzero, but there exist at least one integer k such that $a_k \neq 0$, let's denote k_0 as the minium k such that $a_k \neq 0$. Therefore, k_0 corresponds to the term $a_{k_0} z_1^{m-k_0} z_2^{k_0}$. Sine W is a invariant subspace of π_m, then we should have

$$\pi_m^{m-k_0}(X)^{m-k_0}(a_{k_0} z_1^{m-k_0} z_2^{k_0}) \propto z_2^m \in W$$

Then

$$\pi_m^j(Y)(z_2^m) \in W, \quad j = 0, 1, 2, ..., m$$

As j ranges from 0 to m, $\pi_m^j(Y)(z_2^m)$ generates $z_2^m, z_2^{m-1}z_1, ..., z_2 z_1^{m-1}, z_1^m$ and all of them belong to W, so we have $W = V_m$. Therefore, π_m irreducible for all $m \in \mathbb{N}_+$. $\square$

In fact, it can be proved that every irreducible complex represnetation of $\mathfrak{sl}(2;\mathbb{C})$ is isomorphic to one of the π_m. Hence if an irreducible complex representation of $\mathfrak{sl}(2;\mathbb{C})$ is of dimension $(m+1)$, then it must isomorphic to π_m here.

Besides, as $\mathsf{SU}(2)$ is simply-connected, we can confirm there is a one-to-one correspondence between representation of $\mathfrak{su}(2)$ and Lie group representation of $\mathsf{SU}(2)$. Since π_m is irreducible, we conclude that Π_m is irreducible, too.

7.3.2 $\mathsf{SO}(3)$

The only irreducible representations of $\mathsf{SO}(3)$ are realized on the space of homogeneous sphere harmonics[2] on $\mathbb{R}^3$ of degree ℓ, denoted as V_ℓ. $\Pi_\ell : \mathsf{SO}(3) \to \mathsf{GL}(V_\ell)$ is defined to be

$$[\Pi_\ell(R)]f(\vec{x}) = f(R^{-1}\vec{x}), \quad R \in \mathsf{SO}(3),\ \vec{x} = (x, y, z)^T$$

In quantum mechanics, the representation Π_ℓ of $\mathsf{SO}(3)$ describes a system with angular momentum quantum number $j = \ell$, while the magent quantum number is related to the Lie algebra representation $\pi_{2\ell}(H)/2$.

Propsition 7.8. $\mathsf{SO}(3)$ *has no even-dimensional irreducible representations.*

Proof. Consider to prove the Lie algebra representations π_m of $\mathfrak{su}(2)$ cannot be uplifted to a representation Σ_m of $\mathsf{SO}(3)$ when m is odd. (Remember that $\dim \pi_m = m + 1$, thus it corresponds to even dimensional representations.)

Let's take the following generators X_i for $\mathfrak{su}(2)$ and L_i for $\mathfrak{so}(3)$:

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad X_3 = \frac{1}{2}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$
$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It's easy to verify they have the same commutation relations:

$$[X_i, X_j] = \varepsilon_{ijk}X_k \qquad [L_i, L_j] = \varepsilon_{ijk}L_k$$

As $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, a representatiom σ_m of $\mathfrak{so}(3)$ should isomorphic to a representaton π_m of $\mathfrak{su}(2)$. Without loss of generality, we have

$$\sigma_m(L_3) = \pi_m(X_3)$$

As

$$X_3 = -\frac{\mathrm{i}}{2}H, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[2]WHY?

As there is a basis $u_0, u_1, ..., u_m$ for V_m such that u_j is an eigenvector for $\pi_m(H)$ with eigenvalue $(m-2j)$, so u_j should be an eigenvector for $\sigma_m(L_3)$ with eigenvalue $-\mathrm{i}(m-2j)/2$. Thus in the basis of u_j, $\sigma_m(L_3)$ is a diagonal matrix:

$$\sigma_m(L_3) = \mathrm{diag}\left(-\frac{\mathrm{i}}{2}m, -\frac{\mathrm{i}}{2}(m-2), ..., \frac{\mathrm{i}}{2}(m-2), \frac{\mathrm{i}}{2}m\right)$$

For the corresponding Lie group representation, we should have $\Sigma_m(e^{tL_3}) = \exp(t\sigma_m(L_3))$. Since

$$e^{2\pi L_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi) & -\sin(2\pi) \\ 0 & \sin(2\pi) & \cos(2\pi) \end{pmatrix} = I$$

We should expect to have $\Sigma_m(e^{2\pi L_3}) = \Sigma_m(I) = I$. However, calculation shows

$$e^{2\pi\sigma_m(L_3)} = \begin{cases} I, & m \text{ is even} \\ -I, & m \text{ is odd} \end{cases}$$

Thus for odd m we get a contradiction and $\mathsf{SO}(3)$ has no even-dimensional irreducible representations. □

Therefore $\mathsf{SO}(3)$ only has odd dimensional irreducible representations with dimension $2\ell+1$, the corresponding Lie algebra representation is $\pi_{2\ell}$ on V_ℓ. $\mathsf{SO}(3)$ is an example that not every representation of its Lie algebra comes from the Lie group representation, due to $\mathsf{SO}(3)$ is not simply-connected.

In fact, all representations of $\mathsf{SO}(3)$ are real and thus all odd-dimensional irreducible representations of $\mathsf{SU}(2)$ and $\mathfrak{su}(2)$ are real (as they are isomorphic to representations of $\mathsf{SO}(3)$). On the other hand, even-dimensional irreducible representations of $\mathsf{SU}(2)$ and $\mathfrak{su}(2)$ are pseudo-real (they are not isomorphic to a real representation, but they are self-dual).

8 Semisimple Lie Algebras

8.1 Levi's Decomposition

Definition 8.1 (Lie Algebra Ideal). *A Lie algebra ideal* $\mathfrak{i}$ *of Lie algebra* $\mathfrak{g}$ *is a Lie subalgebra such that* $\forall X \in \mathfrak{g}$ *and* $\forall Y \in \mathfrak{i}$, $[X, Y] \in \mathfrak{i}$.

Definition 8.2 (Semisimple Lie Algebra). *A Lie algebra is semisimple if and only if it has no non-zero abelian ideals.*

Definition 8.3 (Simple Lie Algebra). *A Lie algebra* $\mathfrak{g}$ *is simple if it's not abelian and the only ideals are* $\{0\}$ *and* $\mathfrak{g}$ *itself.*

According to definition, a simple Lie algebra surely has no non-zero abelian ideals, thereby it's semi-simple. $\mathfrak{su}(2)$ and $\mathfrak{sl}(2;\mathbb{C})$ are simple, and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is semi-simple.

Definition 8.4 (Solvable Lie Algebra). *Consider the derived series*

$$\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}] \supset [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supset \cdots$$

If the sequence therminates at $\{0\}$, *then* $\mathfrak{g}$ *is a solvable Lie algebra.*

For example, $\mathfrak{su}(2)$ is not solvable, as $[\mathfrak{su}(2), \mathfrak{su}(2)] \cong \mathfrak{su}(2)$; but $\mathfrak{u}(1)$ is solvable, since $\mathfrak{u}(1)$ is abelian and $[\mathfrak{u}(1), \mathfrak{u}(1)] = \{0\}$.

Propsition 8.1. *A Lie algebra cannot be semisimple and solvable at the same time.*

Proof. Denote

$$\mathfrak{g}_0 = \mathfrak{g} \quad \mathfrak{g}_1 = [\mathfrak{g}_0, \mathfrak{g}_0] \quad \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] \ \cdots$$

Then $\mathfrak{g}_i$ are obviously Lie subalgebras of $\mathfrak{g}$. If $\mathfrak{g}$ is solvable, then there exists $\mathfrak{g}_\ell$ such that $[\mathfrak{g}_\ell, \mathfrak{g}_\ell] = 0$, i.e. $\mathfrak{g}_\ell$ is an abelian subalgebra of $\mathfrak{g}$. Besides, any element of $\mathfrak{g}_i$ is a commutator of two elements in $\mathfrak{g}_{i-1}$ and also belongs to $\mathfrak{g}_{i-1}$, therefore $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}_{i-1}$. Hence $\mathfrak{g}_\ell$ is a non-zero abelian ideal of $\mathfrak{g}_{\ell-1}$ and $\mathfrak{g}_{\ell-1}$ is not a semisimple Lie algebra. I'm not sure whether we can deduce $\mathfrak{g}$ is not semisimple based on one of its subalgebras $\mathfrak{g}_{\ell-1}$ is not semisimple. □

Of course there are Lie algebras which is neither semisimple nor solvable, for example, $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

Theorem 8.1 (Levi's Decomposition Theorem)**.** *Any finite dimensional Lie algebra $\mathfrak{g}$ can be decomposed as a semi-direct sum of a solvable Lie algebra $\mathfrak{s}$ and a semisimple Lie algebra $\mathfrak{l}$, $\mathfrak{g} = \mathfrak{s} \oplus_s \mathfrak{l}$. The semi-direct sum $\mathfrak{s} \oplus_s \mathfrak{l}$ of $\mathfrak{s}$ and $\mathfrak{l}$ is a Lie algebra $\mathfrak{s} \cup \mathfrak{l}$ where $\mathfrak{s} \cap \mathfrak{l} = \{0\}$, $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}$, $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}$, $[\mathfrak{s}, \mathfrak{l}] \subset \mathfrak{s}$.*

Above definitions and theorems are appliable for both real and complex Lie algebras. Later on we will introduce the classification of complex simple Lie algebras.

8.2 Root systems

From now on, all Lie algebras we mentioned will be semisimple unless we specifically stress its non-semisimpleness. Anyway, Lie algebras of physical interest are all semisimple.

Given a real Lie algebra $\mathfrak{r}$, consider its complexification $\mathfrak{g} \equiv \mathfrak{r} \oplus i\mathfrak{r}$, then elements of $\mathfrak{g}$ would be compelx linear combinations of elements of $\mathfrak{r}$. As $\mathfrak{g}$ is a complex vector space, we could associate it with a usual complex inner product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. It's worth noting that the inner product is anti-linear for the first slot but linear for the second one:

$$\langle \lambda X, Y \rangle = \overline{\lambda} \langle X, Y \rangle = \left\langle X, \overline{\lambda} Y \right\rangle, \quad \forall X, Y \in \mathfrak{g}, \lambda \in \mathbb{C}$$

We claim that there always exists certain inner products such that $\forall X \in \mathfrak{r}$ and $\forall Y, Z \in \mathfrak{g}$,

$$\langle \mathrm{ad}_X Y, Z \rangle = - \langle Y, \mathrm{ad}_X Z \rangle$$

and the restriction of inner product on $\mathfrak{r}$ always takes real value. As for matrix Lie algebras $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, we have $\mathfrak{r} = \mathfrak{g} \cap \mathfrak{u}(n)$, and the inner product can be defined by

$$\langle X, Y \rangle \equiv \mathrm{tr}\left(X^\dagger Y\right)$$

For $\forall X \in \mathfrak{r}$, $X^\dagger = -X$ and we indeed have

$$\langle \mathrm{ad}_X Y, Z \rangle = \mathrm{tr}\left(Y^\dagger X^\dagger Z\right) - \mathrm{tr}\left(X^\dagger Y^\dagger Z\right) = - \mathrm{tr}\left(Y^\dagger [X, Z]\right) = - \langle Y, \mathrm{ad}_X Z \rangle$$

For later convenience, it's worthwhile to define so called "matrix adjoint" for elements of $\mathfrak{g}$:

Definition 8.5. *For* $\forall X \in \mathfrak{g} \cong \mathfrak{r} \oplus \mathrm{i}\mathfrak{r}$*,* $\exists X_1, X_2 \in \mathfrak{r}$ *such that*

$$X = X_1 + \mathrm{i}X_2$$

Therefore $\forall Y, Z \in \mathfrak{g}$*, we have*

$$\langle \mathrm{ad}_X Y, Z\rangle = -\langle Y, \mathrm{ad}_{X_1} Z\rangle + \mathrm{i}\langle Y, \mathrm{ad}_{X_2} Z\rangle = \langle Y, \mathrm{ad}_{-X_1+\mathrm{i}X_2} Z\rangle$$

Hence we define

$$X^* = (X_1 + \mathrm{i}X_2)^* = -X_1 + \mathrm{i}X_2$$

And

$$\langle \mathrm{ad}_X Y, Z\rangle = \langle Y, \mathrm{ad}_{X^*} Z\rangle$$

Besides the preferred inner product, we can also define another billinear form:

Definition 8.6 (Killing form)**.** $\forall X, Y \in \mathfrak{g}$*, the Killing form is defined by*

$$K(X, Y) = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y)$$

For a basis $\{X_i\}$ *of* $\mathfrak{g}$*, recall the definition of structural constants*

$$[X_i, X_j] = c_{ijk} X_k$$

Hence

$$(\mathrm{ad}_{X_i})_{k\ell} = c_{i\ell k}$$

And we then have

$$K(X_i, X_j) = c_{ik\ell} c_{j\ell k}$$

For the introduction of root system, we also need to define Cartan subalgebra:

Definition 8.7 (Cartan subalgebra)**.** *The Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is the maximal abealian subalgebra of* $\mathfrak{g}$*. That is,*

- $\forall H_1, H_2 \in \mathfrak{h}$*,* $[H_1, H_2] = 0$
- *If* $X \in \mathfrak{g}$ *commutes with every element of* $\mathfrak{h}$*, then* $X \in \mathfrak{h}$
- *The rank of Lie algebra* $\mathfrak{g}$ *is defined to be the diemnsion of its Cartan subalgebra:*

$$\mathrm{rank}\,\mathfrak{g} \equiv \dim \mathfrak{h}$$

According to definition, the matrices of adjoint representation of elements of $\mathfrak{h}$ are mutally commute, hence they inherits common eigenvectors in $\mathfrak{g}$. Therefore, $\exists X \in \mathfrak{g}$ such that $\forall H \in \mathfrak{h}$,

$$\mathrm{ad}_H X = \lambda(H) X$$

Besides, $\mathfrak{h}$ is a vector space indicating $\forall H_1, H_2 \in \mathfrak{h}$, $H_1 + kH_2 \in \mathfrak{h}$, and we have

$$\mathrm{ad}_{H_1+kH_2} X = \lambda(H_1 + kH_2) X = \mathrm{ad}_{H_1} X + k\mathrm{ad}_{H_2} X = [\lambda(H_1) + k\lambda(H_2)] X$$

Therefore $\lambda(H)$ is actually a linear functional on $\mathfrak{h}$ and elementary linear algebra tells us there exists an unique non-zero element $\alpha \in \mathfrak{h}$ such that

$$\langle \alpha, H \rangle = \lambda(H)$$

These considerations lead to the formal definition of root vectors:

Definition 8.8 (Root). *A non-zero element $\alpha \in \mathfrak{h}$ is a root if there exists a non-zero $X \in \mathfrak{g}$ such that*

$$\mathrm{ad}_H X = \langle \alpha, H \rangle X, \quad \forall H \in \mathfrak{h}$$

We denote the eigenspace associated with α as $\mathfrak{g}_\alpha$ and call it root space of root α, whose non-zero element is called a root vector.

The preferred choice of inner product on $\mathfrak{g}$ could significantly constrain α. As $\mathfrak{h} \cap \mathfrak{r} \neq \emptyset$, then $\forall H \in \mathfrak{h} \cap \mathfrak{r}$ and for the eigenvector $X \in \mathfrak{g}$ associated with α, we have

$$\langle \mathrm{ad}_H X, X \rangle = \overline{\langle \alpha, H \rangle} \langle X, X \rangle = -\langle X, \mathrm{ad}_H X \rangle = -\langle \alpha, H \rangle \langle X, X \rangle$$

Thus $\overline{\langle \alpha, H \rangle} = -\langle \alpha, H \rangle$, it must be pure imaginary, indicating α can only be element of $i\mathfrak{r}$.

Corollary 8.1. *For complexified Lie algebra $\mathfrak{g} = \mathfrak{r} \oplus i\mathfrak{r}$, its root α must belong to the subspace $i\mathfrak{r}$.*

It can be proved that simultaneous eigenvectors of ad_H, $H \in \mathfrak{h}$ forms a complete basis of $\mathfrak{g} - \mathfrak{h}$. Hence the vector space $\mathfrak{g}$ could be decomposed as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$$

What's more, for semisimple Lie algebra these eigenvalues are all non-degenerate, and

$$\dim \mathfrak{g}_\alpha = 1, \quad \forall \alpha$$

Propsition 8.2. *For any root $\alpha, \beta \in \mathfrak{h}$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$*

Proof. $\forall X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, H \in \mathfrak{h}$, we should have

$$\mathrm{ad}_H X_\alpha = \langle \alpha, H \rangle X, \quad \mathrm{ad}_H Y_\beta = \langle \beta, H \rangle Y_\beta$$

Jacobi identity implies

$$\begin{aligned}
\mathrm{ad}_H [X_\alpha, Y_\beta] = [H, [X_\alpha, Y_\beta]] &= -[X_\alpha, [Y_\beta, H]] - [Y_\beta, [H, X_\alpha]] \\
&= [X_\alpha, \mathrm{ad}_H Y_\beta] - [Y_\beta, \mathrm{ad}_H X_\alpha] \\
&= \langle \alpha, H \rangle [X_\alpha, Y_\beta] + \langle \beta, H \rangle [X_\alpha, Y_\beta] \\
&= \langle \alpha + \beta, H \rangle [X_\alpha, Y_\beta]
\end{aligned}$$

Therefore $[X_\alpha, Y_\beta] \in \mathfrak{g}_{\alpha+\beta}$. ☐

Propsition 8.3. *If α is a root whose corresponding root vector is X, then $-\alpha$ is also a root and its associated root vector is X^*.*

Proof. For $X \in \mathfrak{g}$, $\exists X_1, X_2 \in \mathfrak{r}$ such that $X = X_1 + \mathrm{i}X_2$. Hence

$$\mathrm{ad}_H X = \langle \alpha, H \rangle X = \langle \alpha, H \rangle X_1 + \mathrm{i} \langle \alpha, H \rangle X_2$$

If $H \in \mathfrak{h} \cap \mathfrak{r}$, then

$$(\mathrm{ad}_H X)^* = \mathrm{ad}_H X^* = (\langle \alpha, H \rangle X_1 + \mathrm{i} \langle \alpha, H \rangle X_2)^*$$

Note that $\alpha \in \mathrm{i}\mathfrak{r}$ and $\langle \alpha, H \rangle$ is pure imaginary, hence

$$\mathrm{ad}_H X^* = \langle \alpha, H \rangle X_1 - \mathrm{i} \langle \alpha, H \rangle X_2 = \langle -\alpha, H \rangle X^*$$

The range of choosing H can be expanded to $\mathfrak{h}$ and a full proof can be obtained. □

Now the merit of investigating roots and root vectors emerges: For each root α, we can always choose

$$H_\alpha = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

Such that $\forall X_\alpha \in \mathfrak{g}_\alpha$, we have $Y_\alpha \equiv X_\alpha^* \in \mathfrak{g}_{-\alpha}$ and

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha$$

What's more, it can be proved that a suitable $X_\alpha \in \mathfrak{g}_\alpha$ can always be found such that

$$[X_\alpha, Y_\alpha] = H_\alpha$$

In this way, $\mathfrak{sl}(2; \mathbb{C})$ commutation relation appears:

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha, \quad [X_\alpha, Y_\alpha] = H_\alpha$$

Definition 8.9 (Root system)**.** *A root system R is a finite set of non-zero elements of a real inner product space E (for our cases $E = \mathrm{i}\mathfrak{t} \subset \mathfrak{h}$) satisfying*

- *R spans E.*
- *If $\alpha \in R$, then $-\alpha \in R$, and the only multiples of α in R are $\pm\alpha$.*
- *If $\alpha, \beta \in R$, define the reflection $s_\alpha : E \to E$ with respect to the plane perpendicular to α by*

  $$s_\alpha \beta \equiv \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

  Then $s_\alpha \beta \in R$. All such s_α with $\alpha \in R$ generates a finite group, called the Weyl group of R.
- *$\forall \alpha, \beta \in R$,*

  $$\langle \beta, H_\alpha \rangle = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

It can be proved that the set of roots for a semisimple Lie algebra is indeed a root system in $i\mathfrak{r}$. As a root system is stringently constrained by its symmetry group, therefore the classification of semisimple Lie algebras can be reduced to the classification of possible root systems.

Definition 8.10 (Simple roots)**.** *For a root system $R \subset E$, the set of simple roots is a subset $\Delta \subset R$ given that*

- *Δ is a linearly independent basis of E.*
- *Any root $\alpha \in R$ can be written as a linear combination of elements of Δ with integer coefficients, and these coefficients are either all non-negative or all non-positive. We can categorize roots in R into positive roots and negative roots in this way.*

We claim that a root system always has simple roots, proof can be found in Hall.

Definition 8.11 (Chevalley basis)**.** *With simple roots $\alpha_i (i = 1, ..., r)$, we can construct so called Chevalley basis for semisimple Lie algebra $\mathfrak{g}$: The basis for Cartan subalgebra are*

$$H_{\alpha_i} \equiv \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

And we choose the basis for $\mathfrak{g} - \mathfrak{h}$ as root vectors E_α and $E_{-\alpha}$ such that

$$\begin{aligned}
[H_{\alpha_i}, H_{\alpha_j}] &= 0 \\
[H_{\alpha_i}, E_{\alpha_j}] &= \frac{2 \langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} E_{\alpha_j} \\
[E_{\alpha_i}, E_{-\alpha_i}] &= H_{\alpha_i} \\
[E_{\alpha_i}, E_{\alpha_j}] &= N_{ij} E_{\alpha_i + \alpha_j}, \quad N_{ij} \in \mathbb{Z}
\end{aligned}$$

As an exercies, consider $\mathfrak{sl}(n+1; \mathbb{C})$. Its Cartan subalgebra elements are those trace zero diagonal matrices

$$H = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{pmatrix}, \quad \sum_{i=1}^{n+1} \lambda_i = 0$$

A basis of $\mathfrak{sl}(n+1; \mathbb{C})$ can be taken as $E_{j,k} (j, k = 1, ..., n+1)$, and we have

$$\mathrm{ad}_H E_{j,k} = (\lambda_j - \lambda_k) E_{j,k}$$

Hence $E_{j,k}$ with $j \neq k$ are indeed simultaneous eigenvectors for elements of Cartan subalgebra, therefore they are root vectors. For finding roots α_{jk}, define inner product by

$$\langle X, Y \rangle = \mathrm{tr}\Big(X^\dagger Y\Big)$$

Then we find

$$\alpha_{jk} = E_{j,j} - E_{k,k} \equiv e_j - e_k$$

Meanwhile, we have

$$\langle \alpha_{jk}, \alpha_{jk} \rangle = 2$$

Hence the roots are of equal length $\sqrt{2}$. We can thus take the simple roots as

$$\Delta = \{e_1 - e_2, e_2 - e_3, \cdots, e_n - e_{n+1}\}$$

This root system is also noted as A_n for some reason we'll introduce later. The Weyl group element is

$$s_{\alpha_{jk}} = 1 - \alpha_{jk} \langle \alpha_{j,k}, \cdot \rangle$$

Note that for $A = A^i e_i$, $s_{\alpha_{jk}}$ interchanges A^j and A^k, hence the Weyl group of A_n is isomorphic to the permutation group S_{n+1}. The Chevalley basis is

$$H_i = e_i - e_{i+1}, \quad E_{\alpha_i} = E_{i,i+1}, \quad E_{-\alpha_i} = E_{i+1,i}$$

8.3 Classification of rank-2 Lie algebras

Different root systems give rise to different semisimple Lie algebras, here we will classify all rank-2 semisimple Lie algebras by classifying all rank-2 root systems.

As we know, classifications in mathematicas are all up to certain sense of "isomorphism", there's no exception here:

Definition 8.12. *Two root systems (E, R) and (F, S) are isomorphic if there exists a linear bijection $\phi : E \to F$ such that $\phi(R) = S$ and $\forall \alpha \in R, \beta \in E$,*

$$\phi(s_\alpha \beta) = s_{\phi\alpha} \phi\beta$$

That is, root system isomorphism should preserve the Weyl group structure.

Propsition 8.4. *If α, β are roots in R, α is not a multiple of β and $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$, then one of the following holds:*

- $\langle \alpha, \beta \rangle = 0$
- $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ *and the angle between α and β is $\pi/3$ or $2\pi/3$.*
- $\langle \alpha, \alpha \rangle = 2 \langle \beta, \beta \rangle$ *and the angle between α and β is $\pi/4$ or $3\pi/4$.*
- $\langle \alpha, \alpha \rangle = 3 \langle \beta, \beta \rangle$ *and the angle between α and β is $\pi/6$ or $5\pi/6$.*

Proof. Denote

$$m_1 = \frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}, \qquad m_2 = \frac{2 \langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$$

Then the axioms of root system suggest

$$m_1 m_2 = 4 \frac{\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = 4 \cos^2 \theta \in \mathbb{Z}$$

Therefore the only possible values for θ are

$$\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{6}, \frac{5\pi}{6}$$

□

Propsition 8.5. *For two simple roots $\alpha, \beta \in \Delta$, $\langle \alpha, \beta \rangle \le 0$*

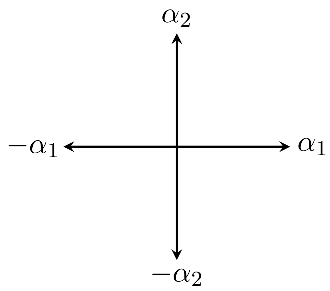
Proof. If $\langle \alpha, \beta \rangle > 0$, the root

$$s_\alpha \beta = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

contradicts the definition of simple roots. □

With all the preperations, we can just enumerate all possible simple roots' configurations for rank-2 semi-simple Lie algebra:

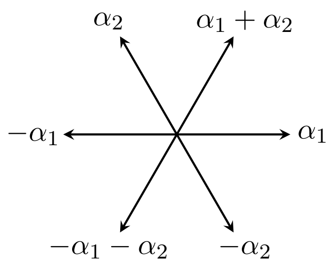
1. $\langle \alpha_1, \alpha_2 \rangle = 0$, the root system can be depicted as



   This root system is denoted as $A_1 \oplus A_1$ and corresponds to Lie algebra $\mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C}$, as we can see each root gives rise to a set of $\mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{sl}(2;\mathbb{C})$ Lie algebra and these two sets of Lie algebras are orthogonal. Notice here we have two simple roots for $\mathfrak{h}$ while four roots to be assigned with four 1-D root spaces, hence

$$\dim(A_1 \oplus A_2) = 6$$

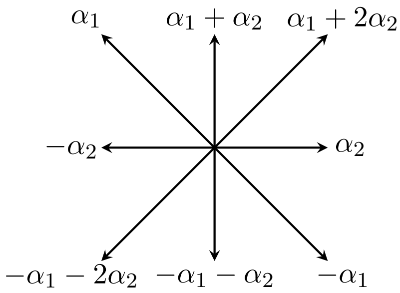
2. $\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle$, the angle is $2\pi/3$. The root system is



   This root system is denoted as A_2, which corresponds to the Lie algebra $\mathfrak{su}(3)_\mathbb{C} \cong \mathfrak{sl}(3;\mathbb{C})$. From root diagram we can directly read out

$$\dim A_2 = 2 + 6 = 8$$

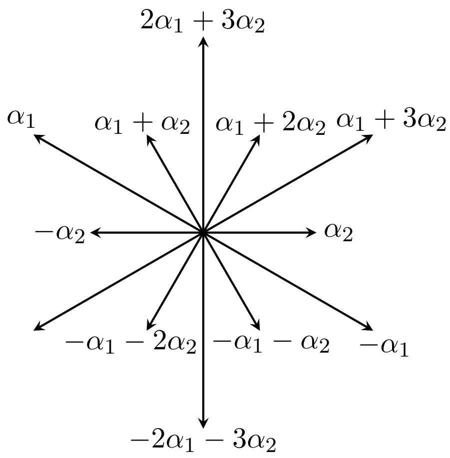
3. $\langle \alpha_1, \alpha_1 \rangle = 2 \langle \alpha_2, \alpha_2 \rangle$ and the angle is $3\pi/4$, the root system is



This root system is denoted as B_2, corresponding to $\mathfrak{so}(5)_{\mathbb{C}} \cong \mathfrak{sp}(4;\mathbb{C})$. Obviously,

$$\dim B_2 = 10$$

4. $\langle \alpha_1, \alpha_1 \rangle = 3 \langle \alpha_2, \alpha_2 \rangle$, the angle is $5\pi/6$ and the diagram is



This root system is G_2, corresponding to the compelx exceptional Lie algebra $\mathfrak{g}_2$ which has no matrix Lie algebra manifestation, clearly we have

$$\dim G_2 = 14$$

As a remark, when generating these root diagrams we basically only used R is closed under Weyl group, the clause 4 in root system axioms is more a consistency condition whose merits are manifestated in proving theorems.

8.4 Classification of simple Lie algebras and Dynkin Diagrams

The classification of rank-2 Lie algebras is a “toy model” of the classification of general semisimple Lie algebras which we’ll introduce now. As semisimple Lie algebras are built from simple Lie algebras which are in one-to-one correspondence to their root systems, our question can be reduced to classifying the basic building blocks of root systems, which has everything to do with reducibability.

Definition 8.13. *A root system (E, R) is reducible, if there exists an orthogonal direct sum decomposition $E = E_1 \oplus E_2$ with $\dim E_1, \dim E_2 > 0$ such that every element of R is either in E_1 or in E_2. If such decomposition doesn't exist, then (E, R) is irreducible.*

A simple Lie algebra is in one-to-one correspondence to an irreducible root system. Inspired by our method of classifying rank-2 Lie algebras, especially Proposition 8.4, we consider using a diagrammatic way to represent irreducible roots of simple Lie algebras:

Definition 8.14 (Dynkin diagram)**.** *A Dynkin diagram is a graph defined as follows:*

- *Nods v_i are in one-to-one correspondence to simple roots α_i.*
- *Between two nodes v_i, v_j, we can draw 0, 1, 2 or 3 lines respectively representating the angle between α_i and α_j are $\pi/2$, $2\pi/3$, $3\pi/4$ or $5\pi/6$ respectively.*
- *Besides, if α_i and α_j are of different length, we draw an arrow from the longer one to the shorter one.*

In this way, the Dynkin diagrams for those rank-2 roots we've already encountered are:

- $A_1 \oplus A_1$: $\bullet \quad \bullet$
- A_2: $\bullet\!-\!\bullet$
- B_2: $\bullet\!\Rightarrow\!\bullet$
- G_2: $\bullet\!\Rrightarrow\!\bullet$

Note that $A_1 \oplus A_1$ is a reducible root system, and its Dynkin diagram is disconnected. Acutally we have the follwing theorem:

Theorem 8.2.

1. *A root system is irreducible if and only if its Dynkin diagram is connected, and the corresponding Lie algebra is simple.*
2. *Two root systems are isomorphic if and only if their Dynkin diagrams are isomorphic as graphs, that is, if there's a bijection map from the vertices of one to the vertices of the other preserving the number of bonds and the direction of the arrows.*

Proof. Refer to B. Hall page 217-218. □

According to this theorem, we can classify complex simple Lie algebras by classifying inequivalent Dynkin diagrams.

Definition 8.15 (Cartan matrix)**.** *The Cartan matrix $\mathcal{C}_{ij}$ of a root system is defined as*

$$\mathcal{C}_{ij} = \frac{2\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_j, \alpha_j\rangle} \equiv \langle\alpha_i, H_{\alpha_j}\rangle$$

The axioms of root system ensures elements of $\mathcal{C}_{ij}$ are all integers and diagonal elements are all 2. As we are considering simple roots, the off-diagonal elements are all non-positive. If two Dynkin diagrams are isomorphic, their Cartan matrices are similar.

For example, the Cartan matrices of rank-2 Lie algebras are

$$A_1 \oplus A_1 : \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad A_2 : \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad B_2 : \quad \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \qquad G_2 : \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

Propsition 8.6. *The Cartan matrix of a semisimple Lie algebra is positive-definite.*

Proof. Multiplying each row by $\langle \alpha_j, \alpha_j \rangle$, the resulting matrix is $M_{ij} = 2\langle \alpha_i, \alpha_j \rangle$, which is a positive definite inner product matrix on $\mathbb{R}^{\operatorname{rank}\mathfrak{g}}$. □

Corollary 8.2. $\det \mathcal{C}_{ij} > 0$

Now we have sufficient tools for classifying simple Lie algebras by classifying irreducible root systems via categorizing all connected Dynkin diagrams.

1. A_n: Corresponds to $\mathfrak{sl}(n+1;\mathbb{C}) \cong \mathfrak{su}(n+1)_{\mathbb{C}}$ whose Dynkin diagram is

   The Cartan matrix is
   $$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$
   The simple roots of A_n can be taken as
   $$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \cdots \alpha_n = e_n - e_{n+1}$$
   Where $\{e_1, \cdots, e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$. The dimension of this Lie algebra is
   $$\dim A_n = n(n+2)$$

2. B_n: Corresponds to $\mathfrak{so}(2n+1)_{\mathbb{C}}$, the Dynkin diagram is

   The Cartan matrix is
   $$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -2 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$
   With $\mathbb{R}^n$ standard basis $\{e_1, \cdots, e_n\}$, we can choose simple roots as
   $$\alpha_1 = e_1 - e_2, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n$$
   The dimension of this Lie algebra is
   $$\dim B_n = n(2n+1)$$

3. C_n: Corresponds to $\mathfrak{sp}(2n;\mathbb{C})$ or $\mathfrak{usp}(2n)$, Dynkin diagram is

The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{pmatrix}$$

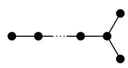
With $\mathbb{R}^n$ standard basis $\{e_1, \cdots, e_n\}$, we can choose simple roots as

$$\alpha_1 = e_1 - e_2, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n$$

Note that we obviously have $B_2 \cong C_2$. The dimension of this Lie algebra is

$$\dim C_n = n(2n+1)$$

4. D_n: Corresponds to $\mathfrak{so}(2n)_{\mathbb{C}}$, whose Dynkin diagram is



The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix}$$

The simple roots can be taken as

$$\alpha_1 = e_1 - e_2, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n$$

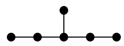
From Dynkin diagrams, we directly have

$$D_3 = \mathfrak{so}(6)_{\mathbb{C}} \cong A_3 = \mathfrak{su}(4)_{\mathbb{C}}, \quad D_2 = \mathfrak{so}(4)_{\mathbb{C}} \cong A_1 \oplus A_1 = \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$$

The dimension of this Lie algebra is

$$\dim D_n = n(2n-1)$$

5. E_6: The Dynkin diagram is



The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

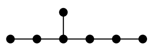
Simple roots of E_6 can be manisfested by standard basis of $\mathbb{R}^6$ via

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_4 + e_5,$$
$$\alpha_5 = -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5) + \frac{\sqrt{3}}{2}e_6, \quad \alpha_6 = e_4 - e_5$$

The dimension of this Lie algebra is

$$\dim E_6 = 78$$

6. E_7: The Dynkin diagram is



The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

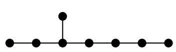
The simple roots can be chosen as

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_4 + e_5,$$
$$\alpha_5 = -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) + \frac{\sqrt{2}}{2}e_7, \quad \alpha_6 = \frac{4}{3}e_6 - \frac{\sqrt{2}}{3}e_7, \quad \alpha_7 = e_4 - e_5$$

The dimension of this Lie algebra is

$$\dim E_7 = 133$$

7. E_8: The Dynkin diagram is



The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

The simple roots are

$$e_1 = e_1 - e - 2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_4 + e_5,$$
$$\alpha_5 = -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8), \quad \alpha_6 = e_6 - e_7, \quad \alpha_7 = e_7 - e_8, \quad \alpha_8 = e_4 - e_5$$

The dimension of this Lie algebra is

$$\dim E_8 = 248$$

8. F_4: The Dynkin diagram is

The Cartan Matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

The simple roots are

$$\alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4, \quad \alpha_3 = e_4, \quad \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$$

The dimension of this Lie algebra is

$$\dim F_4 = 52$$

9. G_2:

The Cartan matrix is

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

The simple roots have already been demonstrated in the last section. The dimension of this Lie algebra is

$$\dim G_2 = 14$$

Definition 8.16. *If a (semi)simple Lie algebra $\mathfrak{g}$ satisfies one of these equivalent conditions:*

1. *The simple roots of $\mathfrak{g}$ are of the same length.*
2. *There's no arrrow in the Dynkin diagram of $\mathfrak{g}$.*
3. *The Cartan matrix of $\mathfrak{g}$ is symmetric.*

Then $\mathfrak{g}$ is called a simply-laced Lie algebra. Otherwise $\mathfrak{g}$ is a non-simply-laced Lie algebra.

Thereofre, A_n, D_n, E_n typed Lie algebras are simply-laced, while B_n, C_n, F_4 and G_2 are non-simply-laced. The structure of simply-laced Lie algebras often appear in different branches of mathematics with the name "ADE classification". For example, the finite subgroups of $\mathsf{SO}(3)$ (or $\mathsf{SU}(2)$) can be classified in terms of

$$\begin{aligned}&A_n \leftrightarrow \text{cyclic group } \mathbb{Z}_n, \quad D_n \leftrightarrow \text{dihedral group}, \quad E_6 \leftrightarrow \text{tetrahedral group},\\ &E_7 \leftrightarrow \text{octahedral group}, \quad E_8 \leftrightarrow \text{icosahedral group}\end{aligned}$$

It has something to do with McKay correspondence.

8.5 Criteria of Subalgebras

For a given semisimple Lie algebra $\mathfrak{g}$, its subalgebra $\mathfrak{s}$ is of significant physical interest, because in spontaneous symmetry breaking, a gauge group could be broken into a smaller gauge group which is a subgroup of the original one. Here $\mathfrak{s}$ could be a direct sum semisimple Lie algebras with a number of $\mathfrak{u}(1)$ factors.

As a semisimple Lie algrbra can be decomposed into a direct sum of simple Lie algebras

$$\mathfrak{g} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

Therefore it's sufficient to study the subalgebras $\mathfrak{s}_i$ of each simple Lie algebra $\mathfrak{g}_i$, and naturally we have

$$\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \subset \mathfrak{g}$$

But here's the question: For two given semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{s}$, how to justify whether $\mathfrak{s} \subset \mathfrak{g}$?

It's obvious that if $\mathfrak{g}$ and $\mathfrak{s}$ corresponds matrix Lie group G and S, then if S is a subgroup of G, then $\mathfrak{s} \subset \mathfrak{g}$ and $\mathfrak{s}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$. For example:

- $\mathsf{SU}(k) \times \mathsf{SU}(n-k) \times \mathsf{U}(1)/\mathbb{Z}_{\mathrm{lcm}(k,n-k)} \subset \mathsf{SU}(n)$ indicates

$$A_{k-1} \oplus A_{n-k-1} \oplus \mathfrak{u}(1)_\mathbb{C} \subset A_{n-1}$$

- $\mathsf{SO}(2n-1) \subset \mathsf{SO}(2n) \subset \mathsf{SO}(2n+1)$, hence

$$B_{n-1} \subset D_n \subset B_n$$

- $\mathsf{USp}(2n) \cong \mathsf{Sp}(n) \subset \mathsf{SU}(2n)$, therefore

$$C_n \subset A_{2n-1}$$

As we have introduced root system, it seems that it's practical to explore subalgebra criterion interms of root system. Actually we have the follwoing theorem:

Theorem 8.3 (R-type subalgebra). *If semisimple Lie algebra $\mathfrak{g}$ has Cartan subalgebra $\mathfrak{h}$ and root system (E, R) and the vector space $\mathfrak{s}$ can be dedcomposed as*

$$\mathfrak{s} = \mathfrak{t} \oplus \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$$

where $\mathfrak{t} \subset \mathfrak{h}$ and S is a sub-root-system of R, then $\mathfrak{s}$ is an R-type subalgebra of $\mathfrak{g}$.

For example, $\mathfrak{A}_1$ is a sub-root-system of any semisimple Lie algebra $\mathfrak{g}$, hence $A_1 \subset \mathfrak{g}$. Besides, it's clear that $A_1 \oplus A_1 \subset B_2$ and $A_1 \oplus A_1 \subset G_2$, but $A_1 \oplus A_1 \subsetneq A_2$. Meanwhile, $A_2 \subset G_2$ but $A_2 \subsetneq B_2$, $B_2 \subsetneq G_2$.

We can move forward to abstractness. As the Dynkin diagram encodes all the information about the structure of this Lie algebra's simple roots, we can deduce that if the Dynkin diagram of $\mathfrak{s}$ is isomorphic to a sub-diagrm of what of $\mathfrak{g}$, then we have $\mathfrak{s} \subset \mathfrak{g}$. Furthermore, if there are nodes corresponding to $\alpha_{i_1}, \cdots, \alpha_{i_\ell}$ which are not included in $\mathfrak{s}$, where $\ell = \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{s}$, then we have

$$\mathfrak{s} \oplus \mathfrak{u}(1)_{\mathbb{C}}^{\oplus \ell} \subset \mathfrak{g}$$

Proof. The fist part claiming $\mathfrak{s} \subset \mathfrak{g}$ is trivial. For the second part, consider orthogonally decompose $\mathfrak{g}$'s Cartan subalgebra $\mathfrak{h}_\mathfrak{g}$ into

$$\mathfrak{h}_\mathfrak{g} = \mathfrak{h}_\mathfrak{s} \oplus \mathfrak{h}_\mathfrak{s}^\perp$$

where $\mathfrak{h}_\mathfrak{s}$ is the Cartan subalgebra of $\mathfrak{s}$. We claim it induces a well-defined Lie algebra direct sum $\mathfrak{s} \oplus \mathfrak{h}_\mathfrak{s}^\perp$, because for any root vector $X_\alpha \in \mathfrak{s}$ and any element $H \in \mathfrak{h}_\mathfrak{s}^\perp$, we have

$$[H, X_\alpha] = \langle \alpha, H \rangle X_\alpha = 0$$

As a vector space, we trivially have

$$\mathfrak{h}_\mathfrak{s}^\perp \cong \mathfrak{u}(1)_{\mathbb{C}}^{\oplus \ell}$$

Hence

$$\mathfrak{s} \oplus \mathfrak{u}(1)_{\mathbb{C}}^{\oplus \ell} \subset \mathfrak{g}$$

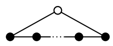
□

For example, $A_{n-1} \oplus \mathfrak{u}(1)_\mathbb{C} \subset B_n,\ C_n,\ D_n$, corresponding to

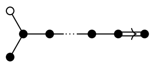
$$\mathfrak{su}(n)_\mathbb{C} \oplus \mathfrak{u}(1)_\mathbb{C} \subset \mathfrak{so}(2n+1)_\mathbb{C},\ \mathfrak{sp}(n)_\mathbb{C},\ \mathfrak{so}(2n)_\mathbb{C}$$

Definition 8.17 (Extended Dynkin diagram, affine Dynkin diagram)**.** *The extended Dynkin diagram of Lie algebra $\mathfrak{g}$, denoted as $\hat{\mathfrak{g}}$, are defined as follows:*

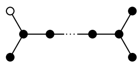
$\hat{A}_1$: 

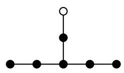
$\hat{A}_n$: 

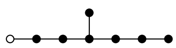
$\hat{B}_2 \cong \hat{C}_2$:

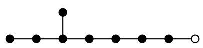
$\hat{B}_n$: 

$\hat{C}_n$:

$\hat{D}_n$: 

$\hat{E}_6$: 

$\hat{E}_7$: 

$\hat{E}_8$: 

$\hat{F}_4$:

$\hat{G}_2$:

Actually, $\hat{\mathfrak{g}}$ is called the affine Lie algebra of $\mathfrak{g}$, which is an infinite dimensional Lie algebra containing $\mathfrak{g}$ as its subalgebra, and the extended Dynkin diagram of $\mathfrak{g}$ depicts $\hat{\mathfrak{g}}$. As a comment, the Cartan matrices of extended Dynkin diagrams always satisfy $\det \mathcal{C} = 0$.

Once $\mathfrak{g}$ is given, we can construct its R-type subalgebras in one of these ways:

1. Delete one node from the extended Dynkin diagram of $\mathfrak{g}$ and the result is the Dynkin diagram of a non-abealian R-type subalgebra of $\mathfrak{g}$.

2. Delete k nodes from the extended Dynkin diagram of $\mathfrak{g}$, the result is the Dynkin diagram of a non-abealian Lie algebra $\mathfrak{s}$ and $\mathfrak{s} \oplus \mathfrak{u}(1)_{\mathbb{C}}^{\oplus(k-1)}$ is an R-type subalgebra of $\mathfrak{g}$.

3. To get all R-type subalgebras of $\mathfrak{g}$, we just have to recursively apply this alogrithm.

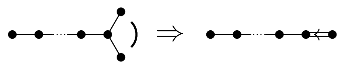
For example, $A_5 \oplus A_1 \subset E_6$, $A_7 \subset E_7$, $A_8 \subset E_8$, $E_6 \oplus A_2 \subset E_8$. Here a trivial corollary is that a non-simply-laced algebra cannot be a subalgebra of a simply-laced one.

Note that being an R-type subalgebra is just a sufficient condition of being a subalgebra, subalgebras that are not R-type are called S-type, its classification is very complicated and we shall not waste too much time here but just give a simple introduction: Obviously a S-type subalgebra should satisfy $\operatorname{rank}\mathfrak{s} < \operatorname{rank}\mathfrak{g}$, and a subclass of S-type subalgebra can be constrcuted from "folding" the original Dynkin diagram, such as:

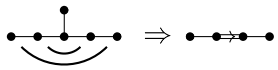
- $B_n \subset A_{2n-1}$: Since all roots of A_{2n-1} are of the same length while B_n do not, the roots of B_n cannot form a sub-root-system of A_{2n-1}, but we can "fold" A_{2n-1} to get

 $\Rightarrow$

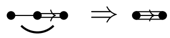
- $C_n \subset D_{n+1}$:



- $F_4 \subset E_6$:



- $G_2 \subset B_3$:



The sufficient and necessary condition for a Lie algebra $\mathfrak{s}$ being a subalgebra of another Lie algebra $\mathfrak{g}$ is highly non-trivial, here we just present some trivial necessary conditions:

- $\dim \mathfrak{s} \leq \dim \mathfrak{g}$
- $\operatorname{rank} \mathfrak{s} \leq \operatorname{rank} \mathfrak{g}$
- If $\operatorname{rank} \mathfrak{s} = \operatorname{rank} \mathfrak{g}$, then $\mathfrak{s}$ has to be an R-type subalgebra of $\mathfrak{g}$, hence the Dynkin diagram of $\mathfrak{s}$ can be embeded into an extended Dynkin diagram of one of $\mathfrak{g}$'s subalgebras.

9 Representation of Semisimple Lie Algebras

9.1 Highest Weight Representations

As a general finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ is isomorphic to a direct sum of its irreducible representations, we could just focus on constructing irreps of semisimple Lie algebras. The goal of this subsection is to make it clear that each irrep of a semisimple $\mathfrak{g}$ is in one-to-one correspondence to so-called "highest weight representation".

Before investigating weight in detail, we will just enumerate the key points of this section as follows:

- Each irrp π_μ of $\mathfrak{g}$ is uniquely characterized by the highest weight

$$\mu = (\mu_1, \cdots, \mu_r), \quad \mu_i \geq 0$$

- From the highest weight μ, we can recursively subtract roots (i.e. rows of Cartan matrix) from it to generate the whole weight system of π_μ.
- For a rep π_μ with weights $(\lambda_1, \cdots, \lambda_r)$, its dual rep has weights $(-\lambda_1, \cdots, -\lambda_r)$.
- The trivial rep always has highest weight $\mu = (0, \cdots, 0)$.
- The adjoint rep is always self-dual and its highest weight can be read off using the extended Dynkin diagram $\hat{g}$.

Definition 9.1 (Weight)**.** *Let (π, V) be a representation of semisimple Lie algebra $\mathfrak{g}$, denote its Cartan subalgebra as $\mathfrak{h}$. Clearly the representation matrix $\pi(H)$ of $H \in \mathfrak{h}$ are simultaneously diagonalizable on V and the corresponding eigenvalue serves as a linear functional of H, hence there exists a non-zero vector $v \in V$ and $\lambda \in \mathfrak{h}$ such that*

$$\pi(H)v = \langle \lambda, H \rangle v, \quad \forall H \in \mathfrak{h}$$

Such λ is called a weight while v is called a weight vector. The eigenspace corresponding to weight λ is called weight space and whose dimension is the multiplicity of λ. The set of all weights is called the weight system of (π, V).

Clearly, if we take (π, V) as $(\mathrm{ad}, \mathfrak{g})$, then the definition of weights coincides with what of roots. Denote the simple roots of $\mathfrak{h}$ as $\{\alpha_i\}$ and the corresponding normalized basis is

$$H_{\alpha_i} = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

Then a weight λ can be represented as a tuple

$$\lambda = (\langle \lambda, H_{\alpha_1} \rangle, \cdots, \langle \lambda, H_{\alpha_r} \rangle)^T$$

As a remark, a simple root α_i can also be expressed in terms of its components on H_{α_j}:

$$\alpha_i = (\mathcal{C}_{i1}, \cdots, \mathcal{C}_{ir})$$

Where $\mathcal{C}_{ij}$ is the element of Cartan matrix

$$\mathcal{C}_{ij} \equiv \langle \alpha_i, H_{\alpha_j} \rangle = \frac{2 \langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$$

Definition 9.2 (The ordering of weights)**.** *A weight μ_1 is "higher" that μ_2, denoted as $\mu_1 \succ \mu_2$, iff*

$$\mu_1 - \mu_2 = \sum_{i=1}^{r} a_i \alpha_i$$

where $a_i \geq 0$ and not identically equal to 0.

With such an ordering, the definition of "the highest weight" and "the lowest weight" is obivous.

Like a root system, a weight system is stringently constrained and inherits many properties:

Propsition 9.1. *If (π, V) is a finite dimensional representation of $\mathfrak{g}$, then every weight λ of π is integral, i.e. $\forall H_{\alpha_i}$, $\langle \lambda, H_{\alpha_i} \rangle \in \mathbb{Z}$.*

Proof. As we know, for each simple root α_i, there's a subalgebra $\mathfrak{s}_{\alpha_i} = \{X_{\alpha_i}, Y_{\alpha_i}, H_{\alpha_i}\} \cong \mathfrak{sl}(2; \mathbb{C})$. If v is a weight vector of weight λ, then

$$\pi(H_{\alpha_i})v = \langle \lambda, H_{\alpha_i} \rangle v$$

As v could also be viewed as a vector acted by the $\mathfrak{sl}(2; \mathbb{C})$ Lie algebra $\mathfrak{s}_{\alpha_i}$, hence the corresponding eigenvalues are all integers and $\langle \lambda, H_{\alpha_i} \rangle \in \mathbb{Z}$. □

Theorem 9.1 (B. Hall, p. 243). *A weight λ is a dominant integral if for all H_{α_i}, $\langle \lambda, H_{\alpha_i} \rangle \geq 0$. Then we have the following theorems:*

1. *Every finite dimensional irrep of $\mathfrak{g}$ has a highest weight μ and μ is a dominant integral.*

2. *Two finite dimensional irreps of $\mathfrak{g}$ with the same highest weight are isomorphic.*

3. *If μ is a dominant integral, then there exists an finite dimensional irrep of $\mathfrak{g}$ with highest weight μ.*

As an irrep π is nearly in one-to-one correspondence to its highest weight μ, it's reasonable that we can somehow generate all the weights λ from μ, it can be achieved given the following proposition:

Propsition 9.2. *For a root $\alpha = (a_1, \cdots, a_r) \in \mathfrak{h}$ with root vector $Z_\alpha \in \mathfrak{g}_\alpha$ and a representation π with a weight $\mu = (m_1, \cdots, m_r)$ whose corressponding weight vector is denoted as $v \neq 0$. Then we have*

$$\pi(H_{\alpha_i})\pi(Z_\alpha) = (m_i + a_i)\pi(Z_a)v$$

indicating that either $\pi(Z_\alpha)v = 0$ or $\pi(Z_\alpha)v$ corresponds to a new weight vector with weight

$$\mu + \alpha = (m_1 + a_1, \cdots, m_r + a_r)$$

Proof. According to the definition of a root, we have

$$[H_{\alpha_i}, Z_\alpha] = \langle \alpha, H_{\alpha_i} \rangle Z_\alpha = a_i Z_\alpha$$

Hence

$$\pi(H_{\alpha_i})\pi(Z_\alpha)v = \pi(Z_\alpha)\pi(H_{\alpha_i})v + a_i\pi(Z_\alpha)v = (m_i + a_i)\pi(Z_\alpha)v$$

□

Theorem 9.2. *For every irrep (π, V) of $\mathfrak{g}$, V can be decomposed as a direct sum of its weight spaces.*

Proof. Denote the direct sum of weight space as W, clearly $W \subseteq V$. Clearly W is an invariant subspace under the action of H_{α_i}. If Z_α is a root vector corresponding to root α, $\pi(Z_\alpha)$ maps the weight space of μ into what of $\mu + \alpha$. Hence W is also invariant under the action of all root vectors— since $\mathfrak{g}$ is spanned by $\mathfrak{h}$ and all root vectors, W is invariant under $\mathfrak{g}$. According to the irreducibility of π, we must have $W = V$. □

Here's an important lemma for calculating all weights:

Lemma 9.1. *Suppose λ is a weight of (π, V) and $\langle \lambda, \alpha \rangle > 0$ for some root α, define*

$$j = \langle \lambda, H_\alpha \rangle = \frac{2 \langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

Then $\lambda - k\alpha, (k = 1, \cdots, j)$ gives us all weights of $(2\pi, V)$.

Now we can propose a systematic way of generating all weights of a representation π: Starting from the highest weight

$$\mu = (\mu_1, \cdots, \mu_r)$$

For any $\mu_i > 0$, we add $\mu - k\alpha_i, (k = 1, \cdots, \mu_i)$ to the weight system of π (α_i are the simple roots). For each new weight, we could apply the same procedure till no new weight is added.

Propsition 9.3. *If (π, V) is a representation of $\mathfrak{g}$, then μ is a weight of π iff $-\mu$ is a weight of its dual representation $\pi^*(X) = -\pi(X)^T$.*

Proof. According to the definition of weight μ

$$\pi(H)v = \langle \mu, H \rangle v$$

Then

$$\pi^*(H)^T v = -\langle \mu, H \rangle v$$

As transpose doesn't affect the eigenvalues, we thus have

$$\pi^*(H)v = -\langle \lambda, H \rangle v$$

implying that $-\mu$ is a weight of π^*. ☐

Now let's check out some examples:

1. For the trivial representation $\pi(X) = 0$, there's only one weight $\mu = (0, \cdots, 0)$ and the dimension of such representation is 1.

2. Consider $\mathfrak{g} = A_1 = \mathfrak{su}(2)$, then the only simple root is $\alpha = 2e_1$. For a representation π_m with highest weight m, the weight system is $\{m, m-2, \cdots, -(m-2), -m\}$, each weight has multiplicity 1 and the dimension of π_m is $(m+1)$.

3. For $\mathfrak{g} = \mathfrak{su}(3) = A_3$, the simple roots can be decomposed on the basis of $\{H_{\alpha_i}\}$ as

   $$\alpha_1 = (2, -1), \qquad \alpha_2 = (-1, 2)$$

   Now we can discuss various representations of $\mathfrak{su}(3)$:

   - Fundamental representation **3**: The highest weight is

     $$\mu \equiv (\langle \mu, H_{\alpha_1} \rangle, \langle \mu, H_{\alpha_2} \rangle) = (1, 0)$$

     Then we have

     $$(1,0) \xrightarrow{-\alpha_1} (-1,1) \xrightarrow{-\alpha_2} (0,-1)$$

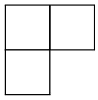
     This kind of diagram is called a weight diagram.

- Anti-fundamental representation $\bar{\mathbf{3}}$: The highest weight is $\mu = (0,1)$ and the weight diagram is
$$(0,1) \xrightarrow{-\alpha_2} (1,-1) \xrightarrow{-\alpha_1} (-1,0)$$
Note that the lowest weight of a representation is the opposite of its dual representation.
- Adjoint representation $\mathbf{8}$: The highest weight is $(1,1)$ and the weight diagram is
$$\begin{tikzcd}
 & (1,1) \arrow[dl, "-\alpha_1"'] \arrow[dr, "-\alpha_2"] & \\
(-1,2) \arrow[d, "-2\alpha_2"'] \arrow[dr, "-\alpha_2"] & & (2,-1) \arrow[dl, "-\alpha_1"'] \arrow[d, "-2\alpha_1"] \\
(1,-2) \arrow[dr, "-\alpha_1"] & (0,0) & (-2,1) \arrow[dl, "-\alpha_2"'] \\
 & (-1,-1) &
\end{tikzcd}$$
Actually it's identical to the root system of A_2 and the dual representation of this representation is itself, hence it's a real representation. Besides the weight $(0,0)$ has multiplicity 2, its weight space is spanned by $\pi(Z_{-\alpha_1})\pi(Z_{-\alpha_2})v_{(1,1)}$ and $\pi(Z_{-\alpha_2})\pi(Z_{-\alpha_1})v_{(1,1)}$, where $v_{(1,1)}$ is a weight vector of $(1,1)$. It can be proved that the weight space of $(0,0)$ is equal to the Cartan subalgebra of A_2.
- Rank-2 symmetric representation $\mathbf{6}$ with highest weight $(2,0)$:
$$\begin{tikzcd}
 & (2,0) \arrow[dl, "-\alpha_1"'] \arrow[dr, "-2\alpha_1"] & \\
(0,1) \arrow[d, "-\alpha_2"'] & & (-2,2) \arrow[dl, "-\alpha_2"'] \arrow[d, "-2\alpha_2"] \\
(1,-1) \arrow[r, "-\alpha_1"] & (-1,0) & (0,-2)
\end{tikzcd}$$

4. $\mathfrak{g} = A_{n-1} = \mathfrak{su}(n)$: The fundamental representation $\boldsymbol{n}$ has highest weight $(1,0,\cdots,0)$ while the anti-fundamental representation $\bar{\boldsymbol{n}}$ has highest weight $(0,\cdots,0,1)$, the adjoint representation has highest weight $(1,0,\cdots,0,1)$.

   In general, an irreducible representation with highest weight $(\mu_1,\cdots,\mu_{n-1})$ can be equivalently represented by a Young tableux $[l_1;l_2;\cdots;l_{n-1}]$, where the i-th column has l_i-boxes, and
$$l_i = \mu_i + \mu_{i+1} + \cdots + \mu_{n-1}$$
   For example, the fundamental representation with highest weight $(1,0,\cdots,0)$ is
$$\square$$

While the adjoint representation of $\mathfrak{su}(3)$ with highest weight $(1,1)$ is



For a representation π with highest weight $(\mu_1, \mu_2, \cdots, \mu_{n-1})$, its dual representation has highest weight $(\mu_{n-1}, \mu_{n-2}, \cdots, \mu_1)$, as the lowest weight of π is

$$(-\mu_{n-1}, -\mu_{n-2}, \cdots, -\mu_1)$$

implying the highest weight of its dual rep is $(\mu_{n-1}, \mu_{n-2}, \cdots, \mu_1)$.

5. $\mathfrak{g} = \mathfrak{so}(2n+1) = B_n$. The vector representation has highest weight $(1, 0, \cdots, 0)$ and its dimension is $(2n+1)$ with weight diagram

$$\begin{aligned}&(1,0,0,\cdots,0) \to (-1,1,0,\cdots,0) \to (0,-1,1,\cdots,0) \to \cdots \to (0,\cdots,-1,1,0)\\ &\to (0,\cdots,0,-1,2) \to (0,\cdots,0) \to (0,\cdots,0,1,-2) \to (0,\cdots,1,-1,0)\\ &\to \cdots \to (-1,0\cdots,0)\end{aligned}$$

B_n also has a spinor representation with highest weight $(0, \cdots, 0, 1)$ and dimension 2^n.

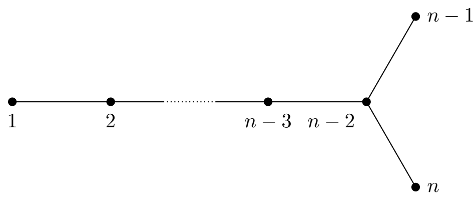
The adjoint representation of $B_n (n > 0)$ has highest weight $(0, 1, 0, \cdots, 0)$ and dimension $n(2n+1)$, while for B_2 the adjoint rep is $(0, 2)$. All representations of B_n are self-dual.

6. $\mathfrak{g} = \mathfrak{sp}(n) = C_n$. Its vector representation has highest weight $(1, 0, \cdots, 0)$ and its dimension is $2n$. The weight diagram is

$$\begin{aligned}&(1,0,\cdots,0) \to (-1,1,0,\cdots,0) \to (0,\cdots,0,-1,1) \to (0,\cdots,0,1,-1)\\ &\to (1,-1,0,\cdots,0) \to (-1,0,\cdots,0)\end{aligned}$$

The adjoint representation of C_n has highest weight $(2, 0, \cdots, 0)$ and dimension $n(2n+1)$. All representations of C_n are self-dual.

7. $\mathfrak{g} = \mathfrak{so}(2n) = D_n$. The simple roots are labeled as



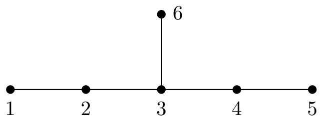
The vector rep has highest weight $(1, 0, \cdots, 0)$ whose dimension is $2n$. The weight diagram is

$$\begin{aligned}&(1,0,\cdots,0) \to (-1,1,0,\cdots,0) \to (0,-1,1,\cdots,0) \to \cdots \to (0,\cdots,1,-1,-1)\\ &\to (0,\cdots,1,-1,0,0) \to (1,-1,\cdots,0) \to (-1,0,\cdots,0)\end{aligned}$$

There are two inequivalent spinor representations $(0, \cdots, 1, 0)$ and $(0, \cdots, 0, 1)$, both of them are of dimension 2^{n-1} and mutually conjugate. Actually, they correspond to the chiral and anti-chiral spinor rep of Lorentz group in physics.

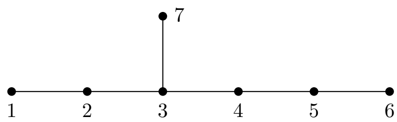
The adjoint rep is of highest weight $(0, 1, \cdots, 0)$ and dimension $n(2n-1)$. In general, the dual representation of the one with highest weight $(\mu_1, \cdots, \mu_{n-1}, \mu_n)$ is of highest weight $(\mu_1, \cdots, \mu_n, \mu_{n-1})$.

8. $\mathfrak{g} = E_6$, the simple roots are labeled as



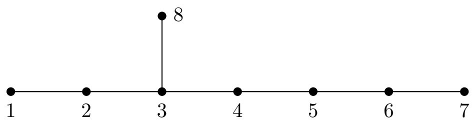
The fundamental rep **27** has highest weight $(1, 0, 0, 0, 0, 0)$, its conjugate $\overline{\mathbf{27}}$ has highest weight $(0, 0, 0, 0, 1, 0)$. The adjoint rep **78** has highest weight $(0, 0, 0, 0, 0, 1)$. In general, the dual representation of the one with highest weight $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6)$ is one with highest weight $(\mu_5, \mu_4, \mu_3, \mu_2, \mu_1, \mu_6)$.

9. $\mathfrak{g} = E_7$, we take its simple roots as



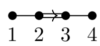
The fundamental rep **56** has highest weight $(1, 0, 0, 0, 0, 0, 0)$, the adjoint rep **133** has highest weight $(0, 0, 0, 0, 0, 1, 0)$. All reps of E_7 are self-dual.

10. $\mathfrak{g} = E_8$, we label the simple roots as



The adjoint rep **248** has highest weight $(1, 0, \cdots, 0)$, which is also the smallest non-trivial rep of E_8. All reps of E_8 are real.

11. $\mathfrak{g} = F_4$, the simple roots are labeled as



The fundamental rep **26** has highest weight $(0, 0, 0, 1)$; the adjoint rep **52** has highest weight $(1, 0, 0, 0)$. All reps of F_4 are real.

12. $\mathfrak{g} = G_2$, the fundamental rep **7** has highest weight $(0, 1)$, and the adjoint rep **14** has highest weight $(1, 0)$. All reps of G_2 are real.

13. $\mathfrak{g} = A_1 \oplus A_1$. The Cartan matrix is

$$\mathcal{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Thus the simple roots are $\alpha_1 = (2,0), \alpha_2 = (0,2)$. The irreducible rep having highest weight $\mu = (\mu_1, \mu_2)$ is isomorphic to the tensor product of A_1 rep π_{μ_1} and π_{μ_2}. For example, rep with highest weight $(1,1)$ has the weight diagram

$$\begin{tikzcd}
 & (1,1) \arrow[dl, "-\alpha_1"'] \arrow[dr, "-\alpha_2"] & \\
(-1,1) \arrow[dr, "-\alpha_2"'] & & (1,-1) \arrow[dl, "-\alpha_1"] \\
 & (-1,-1) &
\end{tikzcd}$$

It's just the bifundamental rep $(\mathbf{2},\mathbf{2}) \cong \mathbf{2} \otimes \mathbb{1}_{2\times 2} + \mathbb{1}_{2\times 2} \otimes \mathbf{2}$ of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Note the adjoint rep of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is

$$(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \cong (\mathbf{3} \otimes \mathbb{1}_{1\times 1} + \mathbb{1}_{3\times 3} \otimes 0) \oplus (\mathbb{1}_{1\times 1} \otimes \mathbf{3} + 0 \otimes \mathbb{1}_{3\times 3})$$

9.2 Weyl Character Formula and Dimensions

Here we will encounter some simple ways to compute the dimension of a rep (π, V) of $\mathfrak{g}$ without resorting to drawing the whole weight diagram.

Definition 9.3 (Character)**.** *The character of π is a function $\chi_\pi : \mathfrak{g} \to \mathbb{C}$ defined by*

$$\chi_\pi(X) = \operatorname{tr}\left(e^{\pi(X)}\right), \quad \forall X \in \mathfrak{g}$$

As V can be decomposed into a direct sum of weight spaces V_λ with multiplicity $\operatorname{mult}(\lambda)$, then we obviously have

$$\chi_\pi(H) = \sum_\lambda \operatorname{mult}(\lambda) e^{\langle \lambda, H \rangle}$$

A trivial corollary is that the dimensional of V is just the value of χ_π at the origin:

$$\dim \pi \equiv \dim V = \sum_\lambda \operatorname{mult}(\lambda) = \chi_\pi(0)$$

For example, let π to be the $(m+1)$-dimensional irrep of $\mathfrak{sl}(2;\mathbb{C})$ and take

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$\chi_\pi(aH) = \chi_\pi \left[\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right] = \operatorname{tr} \begin{pmatrix} e^{ma} & & & \\ & e^{(m-2)a} & & \\ & & \ddots & \\ & & & e^{-ma} \end{pmatrix} = \frac{\sinh[(m+1)a]}{\sinh a}$$

Take the limit $a \to 0$, we find $\chi_\pi(0) = m+1$ and $\dim \pi = \chi_\pi(0)$ is thus verified.

Theorem 9.3 (Weyl Character Formula)**.** *For a rep (π, V) of $\mathfrak{g}$ with highest weight μ, we have*

$$\chi_\pi(H) = \frac{\sum_{w\in W} \epsilon(w) e^{\langle w\cdot(\mu+\delta), H\rangle}}{\sum_{w\in W} \epsilon(w) e^{\langle w\cdot\delta, H\rangle}}$$

Here the sum is over all elements of Weyl group W. $\epsilon(w)$ is the determinent of the Weyl group action, or equivalently,

$$\epsilon(w) \equiv (-1)^{s(w)}$$

where $s(w)$ is the minimal number of s_{α_i} elements in the multiplicative decomposition of w. δ is defined to be half of the sum of positive roots of $\mathfrak{g}$:

$$\delta \equiv \frac{1}{2} \sum_{\alpha_+ \in R_+} \alpha_+$$

Corollary 9.1.

$$\dim \pi = \prod_{\alpha \in R_+} \frac{\langle \mu + \delta, \alpha\rangle}{\langle \delta, \alpha\rangle}$$

Proof. Set $H = t\delta$, according to Weyl character formula, we would have

$$\chi_\pi(t\delta) = \prod_{\alpha\in R_+} \frac{\sinh\langle \alpha, (\mu+\delta)t/2\rangle}{\sinh\langle \alpha, \delta t/2\rangle}$$

Take $t \to 0$, we then have

$$\dim \pi = \prod_{\alpha \in R_+} \frac{\langle \mu + \delta, \alpha\rangle}{\langle \delta, \alpha\rangle}$$

□

Now let's apply this formula to several examples:

1. Representation of $\mathfrak{su}(3)$ with highest weight (m_1, m_2). The simple roots are $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3$, and the other positive root is $\alpha_3 = \alpha_1 + \alpha_2 = e_1 - e_3$. Hence

$$\delta = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 + \alpha_2$$

Therefore

$$\begin{aligned}&\langle \alpha_1, \mu+\delta\rangle = m_1 + 1, \quad \langle \alpha_2, \mu+\delta\rangle = m_2 + 1, \quad \langle \alpha_3, \mu+\delta\rangle = m_1 + m_2 + 2\\ &\langle \alpha_1, \delta\rangle = 1, \quad \langle \alpha_2, \delta\rangle = 1, \quad \langle \alpha_3, \delta\rangle = 2\end{aligned}$$

Therefore

$$\dim(m_1, m_2) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$$

2. $\mathfrak{su}(4)$: The simple roots are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4$$

The other positive roots are

$$\alpha_1 + \alpha_2, \quad \alpha_2 + \alpha_3, \quad \alpha_1 + \alpha_2 + \alpha_3$$

Hence

$$\delta = \frac{3}{2}\alpha_1 + 2\alpha_2 + \frac{3}{2}\alpha_3$$

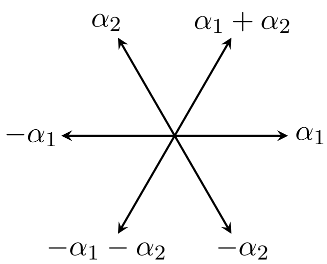
For the rep with highest weight $\mu = (m_1, m_2, m_3)$, we thus have

$$\begin{aligned}
&\langle \alpha_1, \mu + \delta \rangle = m_1 + 1, \quad \langle \alpha_2, \mu + \delta \rangle = m_2 + 1, \quad \langle \alpha_3, \mu + \delta \rangle = m_3 + 1 \\
&\langle \alpha_1 + \alpha_2, \mu + \delta \rangle = m_1 + m_2 + 2, \quad \langle \alpha_2 + \alpha_3, \mu + \delta \rangle = m_2 + m_3 + 2 \\
&\langle \alpha_1 + \alpha_2 + \alpha_3, \mu + \delta \rangle = m_1 + m_2 + m_3 + 3 \\
&\langle \alpha_1, \delta \rangle = 1, \quad \langle \alpha_2, \delta \rangle = 1, \quad \langle \alpha_3, \delta \rangle = 1, \langle \alpha_1 + \alpha_2, \delta \rangle = 2 \\
&\langle \alpha_2 + \alpha_3, \delta \rangle = 2, \quad \langle \alpha_1 + \alpha_2 + \alpha_3, \delta \rangle = 3
\end{aligned}$$

Therefore

$$\dim(m_1, m_2, m_3) = \frac{1}{12}(m_1{+}1)(m_2{+}1)(m_3{+}1)(m_1{+}m_2{+}2)(m_2{+}m_3{+}2)(m_1{+}m_2{+}m_3{+}3)$$

Here's an example using Weyl character formula to compute the multiplicity of a weight λ in the highest weight representation. Take $\mathfrak{g} = \mathfrak{su}(3)$, two simple roots are $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3$ and the root system is



Half of the summation of positive roots is

$$\delta = \alpha_1 + \alpha_2 = e_1 - e_3$$

Then in general, an element of the Cartan subalgebra $\mathfrak{h}$ can be written as

$$H = aH_{\alpha_1} + bH_{\alpha_2}$$

The action of Weyl group is just the permutation group S_3 acting on three elements $\{e_1, e_2, e_3\}$, therefore $w \in W$ and the corresponding $\epsilon(w)$ are as follows: Now in principle it's possible to compute each term in

$$\chi_\pi(H) = \frac{\sum_{w \in W} \epsilon(w) e^{\langle w \cdot (\mu + \delta), H \rangle}}{\sum_{w \in W} \epsilon(w) e^{\langle w \cdot \delta, H \rangle}}$$

w	$\epsilon(w)$
1	1
$e_1 \leftrightarrow e_2$	-1
$e_2 \leftrightarrow e_3$	-1
$e_3 \leftrightarrow e_1$	-1
$e_1e_2e_3 \leftrightarrow e_2e_3e_1$	1
$e_1e_2e_3 \leftrightarrow e_3e_1e_2$	1

For a rep with highest weight $\mu = (m_1, m_2)$, note $m_i \equiv \langle \mu, H_{\alpha_i} \rangle$. We'll just cite the character formula for a generic $\mathfrak{su}(3)$ highest weight represnetation:

$$\chi_{(m_1,m_2)}\left(aH_{\alpha_1} + bH_{\alpha_2}\right) = \left[\left(e^{(m_1+1)a+(m_2+1)b} + e^{(m_1+1)b-(m_1+m_2+2)a} + e^{(m_2+1)a-(m_1+m_2+2)b}\right.\right.$$
$$\left.-e^{(m_1+m_2+2)b-(m_1+1)a} - e^{(m_1+m_2+2)a-(m_2+1)b} - e^{-(m_2+1)a-(m_1+1)b}\right]$$
$$/\left(e^{a+b} + e^{b-2a} + e^{a-2b} - e^{2a-b} - e^{2b-a} - e^{-a-b}\right)$$

For $(m_1, m_2) = (0, 0)$, we have $\chi_{(0,0)}\left(aH_{\alpha_1} + bH_{\alpha_2}\right) = 1$; if $(m_1, m_2) = (1, 0)$, we then have

$$\chi_{(1,0)}\left(aH_{\alpha_1} + bH_{\alpha_2}\right) = e^a + e^{b-a} + e^b$$

The three terms respectively correspond to weights $(1, 0), (-1, 1)$ and $(0, -1)$. While for $(m_1, m_2) = (1, 1)$, we would have

$$\chi_{(1,1)}\left(aH_{\alpha_1} + bH_{\alpha_2}\right) = e^{a+b} + e^{-a+2b} + e^{-b+2a} + 2 + e^{a-2b} + e^{b-2a} + e^{-a-b}$$

Now we can see the weight $(0, 0)$ does have multiplicity 2.

9.3 Casimir Operators

Now we introduce another characterization of a representation (π, V) of a semisimple Lie algebra $\mathfrak{g}$, which is the Casimir.

Definition 9.4 (Tensor algebra)**.** *The tensor algebra of $\mathfrak{g}$ is defined to be*

$$\mathcal{T}(\mathfrak{g}) \equiv \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$$

which is equipped with the associative mulplication

$$(v_1 \otimes \cdots \otimes v_k) \cdot (w_1 \otimes \cdots \otimes w_\ell) \equiv v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell$$

Now we consider an associative subalgebra (two-sided ideal) $\mathcal{J} \subset \mathcal{T}(\mathfrak{g})$, generated by elements of the form $x \otimes y - y \otimes x - [x, y]$:

$$\mathcal{J} \equiv \{A \otimes (x \otimes y - y \otimes x - [x, y]) \otimes B | x, y, \in \mathfrak{g},\ A, B \in \mathcal{T}(\mathfrak{g})\}$$

Definition 9.5 (Universal enveloping algebra)**.** *The universal enveloping algebra of* $\mathfrak{g}$ *is defined to be the quotient algebra*

$$U(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{J} = \mathcal{T}(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])$$

Definition 9.6 (Casimir operator)**.** *The Casimir operators are the generators of the center* $Z(U(\mathfrak{g}))$*, which commute with any element of* $U(\mathfrak{g})$*.*

Of course, the above definition of Casimir operators is soooo abstract and can hardly be used for calculation. Fortunately, every semisimple Lie algebra inherits a special Casimir operator:

Definition 9.7 (Quadratic Casimir operator)**.** *For all semisimple Lie algebra* $\mathfrak{g}$*, its quadratic Casimir operator is defined by*

$$C_2 = -\sum_i X_i^2$$

Where $\{X_i\}$ *is a set of orthonormal basis of* $\mathfrak{r}$*, the real part of the complex Lie algebra* $\mathfrak{g}$*.*

Proof. As $\forall X \in \mathfrak{g}$ can be written as $X_1 + \mathrm{i}X_2$, where $X_1, X_2 \in \mathfrak{r}$, it suffices to prove for any X_i, $[X_i, C_2] = 0$. Direct calculation shows

$$\begin{aligned} -[X_j, C_2] &= \sum_k [X_j, X_k^2] = \sum_k ([X_j, X_k]X_k + X_k[X_j, X_K]) = \sum_{k,\ell} c_{jk\ell} X_\ell X_k + \sum_{k,\ell} c_{jk\ell} X_k X_\ell \\ &= \sum_{k,\ell} (c_{jk\ell} + c_{j\ell k}) X_\ell X_k \end{aligned}$$

On the basis $\{X_j\}$, the matrix elements of ad_{X_j} are

$$(\mathrm{ad}_{X_j})_{k\ell} = c_{j\ell k}$$

According to the definition of our inner product on Lie algebra, we know that

$$\langle \mathrm{ad}_X Y, Z \rangle = -\langle Y, \mathrm{ad}_X Z \rangle$$

Hence $(\mathrm{ad}_X)^T = -\mathrm{ad}_X$ and $c_{jk\ell} = -c_{j\ell k}$, therefore it's obvious that $[X_i, C_2] = 0$. $\square$

Now consider a representation (π, V) with highest weight μ, then $\pi(C_2)$ commutes with any $\pi(X)$. Since (π, V) is an irrep, by Schur's lemma, $\pi(C_2)$ should be proportional to $\mathbb{1}$ with a constant coefficient only depends on the weight μ, thus

$$\pi(C_2) = -\sum_i \pi(X_i^2) = c_\mu \mathbb{1}$$

Propsition 9.4. *The number* c_μ *can be computed in terms of*

$$c_\mu = \langle \mu, \mu + 2\delta \rangle$$

where

$$\delta = \frac{1}{2} \sum_{\alpha_+ \in R_+} \alpha_+$$

Proof. Consider taking an orthonormal basis of $\mathfrak{r} \subset \mathfrak{g}$ ass follows: First take an orthonormal basis H_i for the real part of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{h} \subset g$. Then for each $\alpha \in R_+$ we can always choose a unit vector $X_\alpha \in \mathfrak{g}_\alpha$ such that X_α^* is a unit vector in $\mathfrak{g}_{-\alpha}$, then we define

$$Y_\alpha = \frac{1}{\mathrm{i}\sqrt{2}}(X_\alpha + X_\alpha^*), \qquad Z_\alpha = \frac{1}{\sqrt{2}}(X_\alpha - X_\alpha^*)$$

They satisfy $Y_\alpha^* = Y_{-\alpha}$ and $Z_\alpha^* = Z_{-\alpha}$, besides they are unit vectors orthogonal to each other. Now an orthonormal basis of $\mathfrak{r}$ can be taken as H_i $(i = 1, \cdots, r)$ and Y_α, Z_α for $\alpha \in R_+$.

Consider

$$-Y_\alpha^2 - Z_\alpha^2 = X_\alpha X_\alpha^* + X_\alpha^* X_\alpha = 2X_\alpha^* X_\alpha + [X_\alpha, X_\alpha^*$$

Hence

$$C_2 = \sum_{\alpha \in R_+} (2X_\alpha^* X_\alpha + [X_\alpha, X_\alpha^*]) - \sum_{j=1}^{r} H_j^2$$

Suppose v is a highest weight vector with weight μ, we then have

$$\pi(C_2)v = -\sum_{i=1}^{r} \pi(H_j^2)v + \sum_{\alpha \in R_+} (2\pi(X_\alpha^*)\pi(X_\alpha) + \pi[X_\alpha, X_\alpha^*])v$$

Note the first term is

$$-\sum_{j=1}^{r} \pi(H_j)^2 v = -\sum_{i=1}^{r} \langle \mu, H_j \rangle^2 v = \sum_{j=1}^{r} |\langle \mu, H_j \rangle|^2 v = \langle \mu, \mu \rangle v$$

Note the second equality comes from $\langle \mu, H_j \rangle$ is purly imagionary: ad_H is anti-symmetric, all $\langle \alpha, H_i \rangle$ are imaginary. Besides, $\langle \alpha, \mu \rangle$ are always real for any weight μ, hence $\langle \mu, H_j \rangle$should be real.

The second term vanishes, as for highest weight vector, $\pi(X_\alpha)v = 0$. For the third term, we have

$$\langle [X_\alpha, X_\alpha^*], H_\alpha \rangle = \langle \mathrm{ad}_{X_\alpha} X_\alpha^*, H_\alpha \rangle = \left\langle X_\alpha^*, \mathrm{ad}_{X_\alpha^*} H_\alpha \right\rangle = -\langle X_\alpha^*, \langle -\alpha, H_\alpha \rangle X_\alpha^* \rangle = \langle \alpha, H_\alpha \rangle$$

implying that

$$[X_\alpha, X_\alpha^*] = \alpha$$

Thus

$$\sum_{\alpha \in R_+} \pi([X_\alpha, X_\alpha^*])v = \sum_{\alpha \in R_+} \langle \mu, \alpha \rangle v$$

Hence

$$c_\mu = \langle \mu, \mu + 2\delta \rangle$$

□

Here's an example: The generators of $\mathfrak{so}(3)$ in its fundamental representation can be taken as

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The inner product on it can be taken as

$$\langle X, Y \rangle \equiv \frac{1}{2} \operatorname{tr}\left(X^\dagger Y \right)$$

In this way, $\{L_i\}$ is a set of orthonormal basis. The quadratic Casimir can then be taken as $C_2 = -(L_1^2 + L_2^2 + L_3^2)$. For the $(2\ell+1)$-dimensional rep labelled by highest weight μ, where $\langle \mu, H_\alpha \rangle = 2\ell, \mu = \ell\alpha$, the eigenvalue of $\pi_{2\ell+1}(C_2)$ then is $c_\mu = \langle \mu, \mu + 2\delta \rangle = \ell(\ell+1) \langle \alpha, \alpha \rangle$. From the definition of root α $[H_\alpha, X_\alpha] = \langle \alpha, H_\alpha \rangle X_\alpha = 2X_\alpha$, we can take $X_\alpha = L_1 + \mathrm{i}L_2$, $H_\alpha = 2\mathrm{i}L_3$ and $\alpha = \mathrm{i}L_3$, then $\langle \alpha, \alpha \rangle = 1$ and

$$c_\mu = \ell(\ell+1)$$

It just matches the eigenvalue of total angular momentum operator L^2 in QM. For a consistency check, take $\ell = 1$ which corresponds to the fundamental rep $\pi(L_i) = L_i$, we now have

$$\pi_3(C_2) = C_2 = 2\mathbb{1}$$

In fact, for a general semisimple Lie algebra $\mathfrak{g}$, il y a other Casimir operators other than C_2 which are invariant tensors of $\mathfrak{g}$ as well. For a degree-n Casimir operator $C_n \in \mathfrak{g}^{\otimes n}$, it also satisfies $[X_j, C_n] = 0$, $\forall X_j \in \mathfrak{g}$. For possible later reference, we list the degrees of Casimir operators for semisimple Lie algebras here:

$\mathfrak{g}$	Possible degree of its Casimirs
A_n	$2, 3, \cdots, n+1$
B_n	$2, 4, \cdots, 2n$
C_n	$2, 4, \cdots, 2n$
D_n	$2, 4, \cdots, 2n-2, n$
E_6	2,5,6,8,9,12
E_7	2,6,8,10,12,14,18
E_8	2,8,12,14,18,20,24,30
F_4	2,6,8,12
G_2	2,6

10 Tensor Product Representation

10.1 General Procedure

Given a Lie algebra $\mathfrak{g}$, consider two irreducible representations (π_1, V_1) and (π_2, V_2), then we may consider decomposing their tensor product representation into a direct sum of irreps:

$$\pi_1 \otimes \pi_2 \cong \bigoplus_i \pi_i^{\oplus m_i}$$

Denote the weight system of π_1 and π_2 as W_1 and W_2. As V_1 and V_2 can be decomposed into direct sums of weight spaces in terms of

$$V_1 \cong \bigoplus_{\lambda \in W_1} V_1^{\lambda}, \qquad V_2 \cong \bigoplus_{\mu \in W_2} V_2^{\mu}$$

Hence we expect the vector space $V_1 \otimes V_2$, whom the tensor product rep $\pi_1 \otimes \pi_2$ is acting on, can be decomposed as

$$V_{\pi_1 \otimes \pi_2} = V_1 \otimes V_2 = \bigoplus_{\lambda \in W_1, \mu \in W_2} V_1^{\lambda} \otimes V_2^{\mu}$$

For any weight vector $v_1 \in V_1^{\lambda}, v_2 \in V_2^{\mu}$, according to the definition of tensor product rep, we have for $\forall H \in \mathfrak{h}$,

$$(\pi_1 \otimes \pi_2)(H)(v_1 \otimes v_2) = (\pi_1(H)v_1) \otimes v_2 + v_1 \otimes (\pi_2(H)v_2) = \langle \lambda + \mu, H \rangle \, v_1 \otimes v_2$$

Therefore, for any weight $\lambda \in W_1, \mu \in W_2$, $\lambda + \mu$ is a weight for $\pi_1 \otimes \pi_2$.

Following the above discussion, we can propose a general procedure for determining direct sum decomposition:

1. Write down the weight system of π_1 and π_2, say, W_1 and W_2 respectively.
2. Write down all the possible sum of weights $\lambda + \mu$, $\forall \lambda \in W_1, \mu \in W_2$. As weights may have non-trivial multiplicities, hence the weight $\lambda + \mu$ for $\pi_1 \otimes \pi_2$ has multiplicity $\mathrm{mult}(\lambda) \times \mathrm{mult}(\mu)$. More pricisely, as a weight Λ for $\pi_1 \otimes \pi_2$ can be decomposed into $\lambda + \mu$ in many ways, we then have

$$\mathrm{mult}(\Lambda) = \sum_{\lambda \in W_1, \mu \in W_2, \lambda + \mu = \Lambda} \mathrm{mult}(\lambda) \times \mathrm{mult}(\mu)$$

3. Try to re-organize these weights into weights of irreps of $\mathfrak{g}$.

Here are some examples:

- $\mathfrak{su}(2), \pi_1 = \mathbf{m}, \pi_2 = \mathbf{n}$. The weights of π_1 are

$$m-1, m-3, \cdots - (m-1)$$

The weights of π_2 are

$$n-1, n-3, \cdots, -(n-1)$$

Hence the weights of $\pi_1 \otimes \pi_2$ are the sums of the weights above, the highest one is $m+n-2$ while the lowest is $-(m+n-2)$. All the other weights have a non-trivial multiplicity, and we could get the decomposition of

$$\mathbf{m} \otimes \mathbf{n} \cong (\mathbf{m+n-1}) \oplus (\mathbf{m+n-3}) \oplus \cdots \oplus (\mathbf{m-n+1})$$

- $\mathfrak{su}(3), \pi_1 = \mathbf{3}, \pi_2 = \bar{\mathbf{3}}$. The weights of π_1 are

$$(1,0), \quad (-1,1), \quad (0,-1)$$

While the weights of π_2 are

$$(0,1), \quad (1,-1), \quad (-1,0)$$

So the weights of $\pi_1 \otimes \pi_2$ are

$$(1,1), \quad (-1,2), \quad 3\times(0,0), \quad (2,-1), \quad (1,-2), \quad (-2,1), \quad (-1,-1)$$

Now apart from one weight $(0,0)$, the rest just form the adjoint representation $\mathbf{8}$, so we have

$$\mathbf{3}\otimes\bar{\mathbf{3}} \cong \mathbf{8}\oplus\mathbf{1}$$

- $\mathfrak{su}(3), \pi_1 = \mathbf{3}, \pi_2 = \mathbf{3}$. The weights of π_1 and π_2 are

$$(1,0), \quad (-1,1), \quad (0,-1)$$

Hence the weights of $\pi_1 \otimes \pi_2$ are

$$(2,0), \quad 2\times(0,1), \quad 2\times(1,-1), \quad 2\times(-1,0), \quad (-2,2), \quad (0,-2)$$

We can identify these weights to $\mathbf{6}$ with highest weight $(2,0)$ and $\bar{\mathbf{3}}$ with highest weight $(0,1)$, so

$$\mathbf{3}\otimes\mathbf{3} \cong \mathbf{6}\oplus\bar{\mathbf{3}}$$

Such tensor product decomposition is of vital significance in particle physics. For example, consider the $\mathsf{SU}(3)$ flavor symmetry in the massless limit of QCD, it rotates the three lightest quarks u, d, s. If we indentify representation $\mathbf{3}$ as $\mathsf{SU}(3)$ acting on the 3D quark space (u, d, s) while $\bar{\mathbf{3}}$ is $\mathsf{SU}(3)$ acting on the anti-quark space $(\bar{u}, \bar{d}, \bar{s})$, then $\mathsf{SU}(3)$ tensor product representation can be used to classify particles composed of quarks:

- A meson consists a quark and an anti-quark, so it's transform under $\mathsf{SU}(3)$ can be classified according to

$$\mathbf{3}\otimes\bar{\mathbf{3}} \cong \mathbf{8}\oplus\mathbf{1}$$

The rep $\mathbf{1}$ correspond to "the meson singlet", while $\mathbf{8}$ corresponds to "the meson octet". The meson octet can be diagrammatically represented as Figure 1; while the meson singlet is just denoted as η'.

- A baryon consists 3 quarks, and we have

$$\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} \cong \mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}$$

Here the rep $\mathbf{10}$ corresponds to "the baryon decuplet" (Figure 2), the two $\mathbf{8}$ reps correspond to "the baryon octet" (Figure 3) including the proton and neutron.

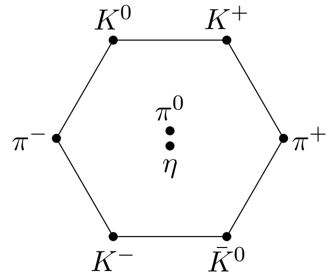


Figure 1. The meson octet.

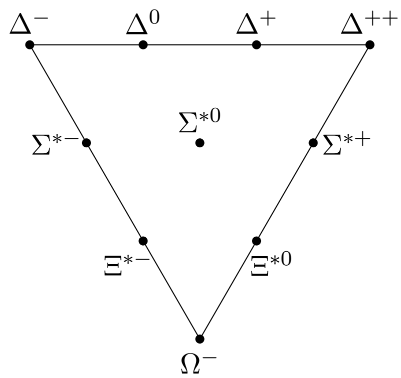


Figure 2. The baryon decuplet.

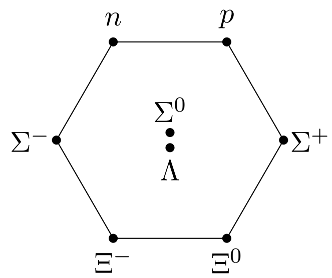


Figure 3. The baryon octet.

10.2 Tensor Representations of $\mathsf{SU}(N)$ and Young Diagram

Now let's consider the irreducible tensor representations of $\mathsf{SU}(N)$ group, whose Lie algebra just corresponds to A_{N-1}. Our discussion is based on the tensor product decomposition of fundamental and anti-fundamental representations.

Let's denote a representation (Π, V) of $\mathsf{SU}(N)$ acting on the N-dimensional vector space V as

$$v' = \Pi(g)v, \quad \forall g \in \mathsf{SU}(N), v \in V$$

Then for the fundamental representation $\mathbf{N}$, we could write it down in components as

$$v'_i = [\Pi(g)]_{ij} v_j$$

Similarly, for anti-fundamental representation $\bar{\mathbf{N}}$, we have

$$v_i'^* = [\Pi^*(g)]_{ij} v_j^* = [\bar{\Pi}(g)]_{ij} v_j^*$$

Following the convention in differential geometry, we assign upper indices for vectors under the fundamental representation (contravariant vectors) while lower indices for vectors under the anti-fundamental representation (covariant vectors), that is

$$v_i \to v^i, \quad v_i^* \to v_i, \quad [\Pi(g)]_{ij} \to U^i{}_j, \quad [\bar{\Pi}(g)]_{ij} \to U_i{}^j$$

Here U should be an $N \times N$ unitary matrix. The group action can thus be expressed as

$$v'^i = U^i{}_j v^j, \quad v_i' = U_i{}^j v_j$$

Sicne $UU^\dagger = \mathbb{1}$, we should have

$$U^i{}_j U_k{}^j = \delta^i_k$$

Now we consider a rank (n, m) tensor product space $\underbrace{V \otimes \cdots \otimes V}_{n} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{m}$, then its element just transform as

$$(v')^{i_1 \cdots i_n}_{j_1 \cdots j_m} = (U^{i_1}{}_{k_1} \cdots U^{i_n}{}_{k_n})(U_{j_1}{}^{\ell_1} \cdots U_{j_m}{}^{\ell_m}) v^{k_1 \cdots k_n}_{\ell_1 \cdots \ell_m}$$

In fact, this tensor representation is just the tensor product representation $\mathbf{N}^{\otimes n} \otimes \bar{\mathbf{N}}^{\otimes m}$, which is usually reducible. For example, consider rep on $(2, 0)$ tensor product space whose element is v^{ij}, it transforms as

$$v'^{ij} = U^i{}_k U^j{}_\ell v^{k\ell}$$

This representation can be reduced into symmetric and anti-symmetric part:

$$S^{ij} \equiv \frac{v^{ij} + v^{ji}}{2}, \quad A^{ij} \equiv \frac{v^{ij} - v^{ji}}{2}$$

They themselves are respectively irreps of $\mathsf{SU}(N)$ and form two invariant subspaces of the rank $(2, 0)$ tensor representation $\mathbf{N} \otimes \mathbf{N}$, besides we have the tensor product decomposition

$$\mathbf{N} \otimes \mathbf{N} \cong \frac{\mathbf{N(N-1)}}{\mathbf{2}} \oplus \frac{\mathbf{N(N+1)}}{\mathbf{2}}$$

For example, for $\mathsf{SU}(2)$ it means

$$\mathbf{2} \otimes \mathbf{2} \cong \mathbf{1} \oplus \mathbf{3}$$

While for $\mathsf{SU}(3)$, we have

$$\mathbf{3} \otimes \mathbf{3} \cong \bar{\mathbf{3}} \oplus \mathbf{6}$$

Note that the 3-dimensional rank-2 anti-symmetric tensor representation A^{ij} is equivalent to the anti-fundamental representation v_i in the $\mathsf{SU}(3)$ case, as we can deduce it from investigating the weight system.

We can take a further generalization: Define rank-n symmetric tensor representation of $\mathsf{SU}(N)$ as

$$v^{(i_1 \cdots i_n)} = \frac{1}{n!} \sum_{\sigma \in S_n} v^{i_{\sigma(1)} \cdots i_{\sigma(n)}}$$

Here σ is an element of the permutation group S_n, describing a permutation of n elements $\{1, \cdots, n\} \to \{\sigma(1), \cdots, \sigma(n)\}$. Similarly, we can define the rank-n anti-symmetric tensor representation of $\mathsf{SU}N$ by

$$v^{[i_1 \cdots i_n]} = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) v^{i_{\sigma(1)} \cdots i_{\sigma(n)}}$$

We define $\epsilon(\sigma) = 1$ for an even permutation while -1 for an odd premutation. The dimension of rank-n anti-symmetric representation of $\mathsf{SU}(N)$ equals to

$$C_N^n = \frac{N!}{n!(N-n)!}$$

In general, we can construct new tensor representations from old ones using the invariant tensors of $\mathsf{SU}(N)$, they are defined to be invariant under $\mathsf{SU}(N)$ actions, including

1. δ^i_j, clearly

$$U^i{}_k U_j{}^\ell \delta^k_\ell = U^i{}_k U_j{}^k = \delta^i_j$$

2. The Levi-Civita symbol $\epsilon_{i_1 \cdots i_N}$ and $\epsilon^{i_1 \cdots i_N}$, defined by

$$\epsilon_{i_1 \cdots i_N} = \epsilon^{i_1 \cdots i_N} = \begin{cases} 1 & (i_1, \cdots, i_N) \text{ differs an even permutation from } (1, \cdots, N) \\ -1 & (i_1, \cdots, i_N) \text{ differs an odd permutation from } (1, \cdots, N) \\ 0 & \text{otherwise} \end{cases}$$

Under $\mathsf{SU}(N)$ action, we have

$$\epsilon_{i_1 \cdots i_N} = U_{i_1}{}^{j_1} \cdots U_{i_N}{}^{j_N} \epsilon_{j_1 \cdots j_N}$$

Since

$$U_{i_1}{}^{j_1} \cdots U_{i_N}{}^{j_N} \epsilon_{j_1 \cdots j_N} = \det(U) \epsilon_{i_1 \cdots i_N}$$

With Levi-Civita symbol, we can transform a rank-$(N-1)$ anti-symmetric tensor representation into the anti-fundamental representation:

$$v_{i_1} = \epsilon_{i_1 \cdots i_N} v^{[i_2 \cdots i_N]}$$

Therefore, we can deduce the rank-2 anti-symmetric representation of $\mathsf{SU}(3)$ is equivalent to $\bar{\mathbf{3}}$ without considering the weight system.

Now let's just forget stuffs about decomposing tensor product reps with weight systems but just with these tensor product vector spaces. For example, let's derive $\mathbf{3} \otimes \bar{\mathbf{3}} \cong \mathbf{8} \oplus \mathbf{1}$.

The fundamental rep $\mathbf{3}$ acts on a vector space spanned by contravariant vectors u^i, while the anti-fundamental rep $\bar{\mathbf{3}}$ acts on covariant vector space spanned by v_j. Then $u^i v_j$ can be viewed as a linear map which can be decomposed into traceless part and identity part

$$\underbrace{u^i v_j}_{\mathbf{3} \otimes \mathbf{3}} = \underbrace{\left(u^i v_j - \frac{1}{3} \delta^i_j u^k v_k \right)}_{\mathbf{8}} + \underbrace{\frac{1}{3} \delta^i_j u^k v_k}_{\mathbf{1}}$$

As the trace part is manifestly invariant under $\mathsf{SU}(3)$ transformation, so does the traceless part which describes a traceless tensor $T^i{}_j$. Actually it's just an element of the Lie algebra $\mathfrak{su}(3)$ under the adjoint representation $\mathbf{8}$.

We can apply this technique to discover the physics of meson octet. We can denote the 3-dimensional vector acted by $\mathbf{3}$ as $(u, d, s)^T$, where u, d, s just respectively correspond to an element in the weight spaces of $(1, 0), (-1, 1)$ and $(0, -1)$. Likewise, vectors acted by $\bar{\mathbf{3}}$ can be denoted as $(\bar{u}, \bar{d}, \bar{s})$, where $\bar{u}, \bar{d}, \bar{s}$ correspond to the weight spaces of $(-1, 0), (1, -1)$ and $(0, 1)$. After doing tensor product, the weights of $\mathbf{8}$ are mapped to different kinds of mesons according to the dictionary below:

Weight	Quark combiantion	Meson type
$(1, 1)$	$u\bar{s}$	K^+
$(2, -1)$	$u\bar{d}$	π^+
$(-2, 1)$	$d\bar{u}$	π^-
$(-1, -1)$	$s\bar{u}$	K^-
$(1, -2)$	$s\bar{d}$	$\bar{K}^0$
$(-1, 2)$	$d\bar{s}$	K^0
$(0, 0)$	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$	π^0
	$\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$	η

For the quark combination corresponding to weight $(0, 0)$, notice that if we directly carry out tensor product, we will get a quark combination of

$$u\bar{u}, \quad d\bar{d}, \quad s\bar{s}$$

We assume an inner product has been defined such that $u\bar{u}, d\bar{d}$ and $s\bar{s}$ are orthonormal, then as the meson singlet $\mathbf{1}$ corresponds to the trace part proporthonal to $u\bar{u} + d\bar{d} + s\bar{s}$, we need the weight $(0, 0)$ part in $\mathbf{8}$ otrhogonal to it and mutually orthonormal, so we would take

$$\begin{aligned} \pi^0 &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\ \eta &= \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \end{aligned}$$

With the inner product assumed, the meson singlet is normalized to be

$$\eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

We can label such quark combination on the meson octet in Figure 4.

We can generalize our story to $\mathsf{SU}(N)$ group: It can be proved that

$$\mathbf{N} \otimes \bar{\mathbf{N}} \cong \text{adjoint rep} \oplus \mathbf{1}$$

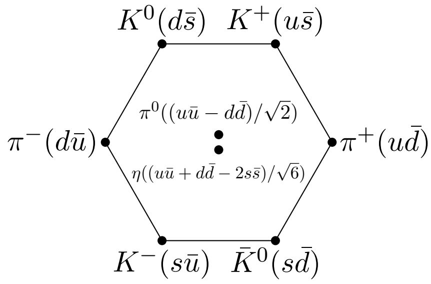
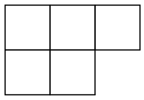


Figure 4. The meson octet labelled with quark state.

We can get a feel of it by considering

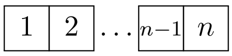
$$\underbrace{u^i v_j}_{\mathbf{N}\otimes\bar{\mathbf{N}}} = \underbrace{\left(u^i v_j - \frac{1}{N}\delta^i_j u^k v_k\right)}_{\mathbf{N^2-1}} + \underbrace{\frac{1}{N}\delta^i_j u^k v_k}_{\mathbf{1}}$$

In order to concisely describe an arbitrary irreducible tensor represnetation of $\mathsf{SU}(N)$ (or in other words, A_{N-1}), we can again resort to Young diagram. For example, in the Young diagram



each box represents a contravariant tensor index, each column indicates the anti-symmetrization imposed on the indices while each row means the symmetrization of the indices.

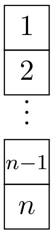
In this sense,



can be understood as describing a rank-n symmetric tensor representation of $\mathsf{SU}(N)$:

$$v^{(i_1\cdots i_n)} \equiv \frac{1}{n!}\sum_{\sigma\in S_n} v^{i_{\sigma(1)}}\cdots v^{i_{\sigma(n)}}$$

While



corresponds to a rank-n anti-symmetric tensor representation of $\mathsf{SU}(N)$:

$$v^{[i_1\cdots i_n]} \equiv \frac{1}{n!}\sum_{\sigma\in S_n} \epsilon(\sigma) v^{i_{\sigma(1)}}\cdots v^{i_{\sigma(n)}}$$

More generally, for an irreducible representation of $\mathfrak{su}(n)$ with highest weight $(\mu_1,\cdots,\mu_{n-1})$, it can be equivalently denoted by a Young diagram $[\ell_1;\ell_2;\cdots;\ell_{n-1}]$, where the i-th row has ℓ_i boxes, and

$$\ell_i = \mu_i + \mu_{i+1} + \cdots + \mu_{n-1}$$

For example, the fundamental rep with highest weight $(1, 0, \cdots, 0)$ is

$$\begin{array}{|c|}\hline \\ \hline \end{array}$$

The adjoint rep of $\mathfrak{su}(3)$ whose highest weight is $(1, 1)$ can be denoted as

$$\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

In general, a rank-n symmetric representation of $\mathsf{SU}(N)$ has Young diagram

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \ldots \begin{array}{|c|c|}\hline n-1 & n \\ \hline \end{array}$$

and it corresponds to the highest weight $(n, 0, \cdots, 0)$. On the other hand, a rank-n anti-symmetric rep of $\mathsf{SU}(N)$ with Young diagram

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline n-1 \\ \hline n \\ \hline \end{array}$$

corresponds to the highest weight $(0, \cdots, 0, 1, 0, \cdots, 0)$, here the 1 is in the n-th slot.

To keep tracking the dimension of the representation, it's good to use the Young tableau of a Young diagram. For a Young diagram of a representation of $\mathsf{SU}(N)$, we can fill integers from 1 to N into the boxes under these rules:

1. For each row, the integers are non-decreasing from left to right.

2. For each column, the integers are increasing from top to bottom.

The number of Young tableaux associated to a Young diagram is then equal to the dimension of the irrep. Let's go throught several examples:

- Rank-3 symmetric rep $\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array}$ of $\mathfrak{su}(3)$: The Young tableaux are

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 3 & 3 & 3 \\ \hline \end{array}$$

  Thus the rank-3 symmetric rep of $\mathfrak{su}(3)$ is 10-dimensional. Note each Young tableau $\begin{array}{|c|c|c|}\hline i & j & k \\ \hline \end{array}$ corresponds to a symmetric tensor component $v^{(ijk)}$, such as

$$v^{(111)} = v^{111}, \ v^{(112)} = \frac{1}{3}(v^{112}+v^{121}+v^{211}), \ v^{123} = \frac{1}{6}(v^{123}+v^{132}+v^{213}+v^{231}+v^{321}+v^{312})$$

- Rank-2 anti-symmetric rep of $\mathfrak{su}(4)$:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array},\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array},\ \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array},\ \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}$$

  Each Young tableau $\begin{array}{|c|}\hline i\\\hline j\\\hline\end{array}$ corresponds to an anti-symmetric tensor component $v^{[ij]}$.

- Adjoint representation of $\mathfrak{su}(3)$:

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

  Again we verified that the dimension of the adjoint rep is 8. For the mixed tensor components of an $\mathbf{8}$, we can write down two different components:

$$\mathbf{8}_1 : \begin{array}{|c|c|}\hline i&j\\\hline k\\\cline{1-1}\end{array} = \frac{1}{4}(v^{ijk} + v^{jik} - v^{kij} - v^{kji})$$

$$\mathbf{8}_2 : \begin{array}{|c|c|}\hline i&k\\\hline j\\\cline{1-1}\end{array} = \frac{1}{4}(v^{ijk} + v^{kji} - v^{jik} - v^{jki})$$

  Actually, these are exactly the ones appearing in the tensor product decomposition

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \cong \mathbf{1} \oplus \mathbf{8}_1 \oplus \mathbf{8}_2 \oplus \mathbf{10}$$

10.3 Clebsch-Gordon Coefficients

Suppose we have the tensor product decomposition

$$\pi_1 \otimes \pi_2 = \bigoplus_{\mu} \pi_\mu^{\oplus m_\mu}$$

where μ is the highest weight of an irrep π_μ and m_μ is the multiplicity of π_μ in such an decomoposition. For the vector spaces V_1, V_2 whom π_1 and π_2 is acting on, we have a similar decomposition

$$V_1 \otimes V_2 = \bigoplus_{\mu,\beta} V_{\mu,\beta}$$

Here $\beta = 1, 2, \cdots, m_\mu$ is a label to distinguish different copies of π_μ.

Now consider taking an orthonormal basis of V_1 and V_2 to be

$$\left| \begin{matrix} \mu_1 \\ \lambda_1\kappa_1 \end{matrix} \right\rangle, \quad \left| \begin{matrix} \mu_2 \\ \lambda_2\kappa_2 \end{matrix} \right\rangle$$

Here μ_i represents the highest weight of the representation π_i, λ_i is a weight in the weight system of π_i and κ_i is a label distinguishing different weight vectors in the weight space of λ_i. We deniote an orthonormal basis for $V_{\mu,\beta}$ as

$$\left| \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \right\rangle$$

Here λ and κ are understood in the same way.

Clearly, the tesor product space $V_1 \otimes V_2$ manifestly inherits a basis composed by

$$\left| \begin{matrix} \mu_1 \\ \lambda_1\kappa_1 \end{matrix} \right\rangle \otimes \left| \begin{matrix} \mu_2 \\ \lambda_2\kappa_2 \end{matrix} \right\rangle \equiv \left| \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \right\rangle$$

Since $\left| \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \right\rangle$ should be a complete basis on $V_1 \otimes V_2$, we then have the relations below:

$$\left| \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \right\rangle = \sum_{\beta,\mu,\lambda,\kappa} \left| \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \right\rangle \left\langle \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \middle| \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \right\rangle$$

$$\left| \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \right\rangle = \sum_{\lambda_1,\lambda_2,\kappa_1,\kappa_2} \left| \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \right\rangle \left\langle \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \middle| \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \right\rangle$$

So-called Clebsch-Gordon coefficients are just the coefficients

$$\left\langle \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \middle| \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \right\rangle, \quad \left\langle \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \middle| \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \right\rangle$$

Obviously, they satisfy the orthonormal relations

$$\sum_{\beta,\mu,\lambda,\kappa} \left\langle \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1'\kappa_1' & \lambda_2'\kappa_2' \end{matrix} \middle| \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \right\rangle \left\langle \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \middle| \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \right\rangle = \delta_{\lambda_1'\lambda_1}\delta_{\kappa_1'\kappa_1}\delta_{\lambda_2'\lambda_2}\delta_{\kappa_2'\kappa_2}$$

$$\sum_{\lambda_1,\lambda_2,\kappa_1,\kappa_2} \left\langle \beta', \begin{matrix} \mu' \\ \lambda'\kappa' \end{matrix} \middle| \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \right\rangle \left\langle \begin{matrix} \mu_1 & \mu_2 \\ \lambda_1\kappa_1 & \lambda_2\kappa_2 \end{matrix} \middle| \beta, \begin{matrix} \mu \\ \lambda\kappa \end{matrix} \right\rangle = \delta_{\beta'\beta}\delta_{\mu'\mu}\delta_{\lambda'\lambda}\delta_{\kappa'\kappa}$$

Besides, a CG coefficient is non-zero if and only if

$$\lambda = \lambda_1 + \lambda_2$$

It depicts a selection rule.

Let's take spin $\frac{1}{2}$ angular momentum addition in quantum mechanics for a concrete example. In the language of Lie algebra here, it's just the tensor product decomposition of $\mathfrak{su}(2)$:

$$\mathbf{2} \otimes \mathbf{2} \cong \mathbf{3} \oplus \mathbf{1}$$

Let's use the following 2-dimensional representation of $\mathfrak{sl}(2;\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The 2-dimensional vector space it's acting on is spanned by orthonormal basis $|\uparrow\rangle, |\downarrow\rangle$, and we have

$$H|\uparrow\rangle = |\uparrow\rangle, \quad H|\downarrow\rangle = -|\downarrow\rangle, \quad X|\uparrow\rangle = 0, \quad X|\downarrow\rangle = |\uparrow\rangle, \quad Y|\uparrow\rangle = |\downarrow\rangle, \quad Y|\downarrow\rangle = 0$$

Now consider two spin $\frac{1}{2}$ systems, denote their representations as π_1 and π_2, which act on the 2D vector space in the way above.

In the tensor product repersentation $\pi_1 \otimes \pi_2$, the 4-dimensional vector space manifestly has a basis

$$|\uparrow\uparrow\rangle\,, \quad |\uparrow\downarrow\rangle\,, \quad |\downarrow\uparrow\rangle\,, \quad |\downarrow\downarrow\rangle$$

The action of $\mathfrak{sl}(2;\mathbb{C})$ generators are

$$(\pi_1 \otimes \pi_2)(H) = \pi_1(H) \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \pi_2(H) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$(\pi_1 \otimes \pi_2)(X) = \pi_1(X) \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \pi_2(X) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\pi_1 \otimes \pi_2)(Y) = \pi_1(Y) \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \pi_2(Y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Now we consider decomposing the 4D vector space into the direct sum of 3D irrep and the 1D irrep. For the triplet, the vector space $V_{\mathbf{3}}$ is spanned by

$$|\uparrow\uparrow\rangle\,, \quad \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |\downarrow\downarrow\rangle$$

Under this basis, the action of $\mathfrak{sl}(2;\mathbb{C})$ generators is

$$\pi_{\mathbf{3}}(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \pi_{\mathbf{3}}(X) = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi_{\mathbf{3}}(Y) = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

Note that out basis vectors are eigenstates of $\pi_{\mathbf{3}}(H)$ corresponding to eigenvalues $+2, 0, -2$, which are exactly equal to the weights of rep **3**. The vector space of the singlet is spanned by

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Since it's a trivial reprersentation, we have

$$\pi_{\mathbf{1}}(H) = \pi_{\mathbf{1}}(X) = \pi_{\mathbf{1}}(Y) = 0$$

This state just corresponds to weight 0 of rep **1**.

Now let's rewrite the previous procedure in terms of our notation here:

$$|\uparrow\rangle \equiv \left| \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle, \quad |\downarrow\rangle \equiv \left| \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle$$

Hence

$$|\uparrow\uparrow\rangle \equiv \left|\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}\right\rangle, \; |\uparrow\downarrow\rangle \equiv \left|\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\right\rangle, \; |\downarrow\uparrow\rangle \equiv \left|\begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}\right\rangle, \; |\downarrow\downarrow\rangle \equiv \left|\begin{matrix} 1 & 1 \\ -1 & -1 \end{matrix}\right\rangle$$

and we have

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \equiv \left|\begin{matrix} 0 \\ 0 \end{matrix}\right\rangle$$

$$|\uparrow\uparrow\rangle \equiv \left|\begin{matrix} 2 \\ 2 \end{matrix}\right\rangle, \quad \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \equiv \left|\begin{matrix} 2 \\ 0 \end{matrix}\right\rangle, \quad |\downarrow\downarrow\rangle \equiv \left|\begin{matrix} 2 \\ -2 \end{matrix}\right\rangle$$

Then we have the table of CG coefficients:

	$\left\|\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}\right\rangle$	$\left\|\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\right\rangle$	$\left\|\begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}\right\rangle$	$\left\|\begin{matrix} 1 & 1 \\ -1 & -1 \end{matrix}\right\rangle$
$\left\langle\begin{matrix} 0 \\ 0 \end{matrix}\right\|$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
$\left\langle\begin{matrix} 2 \\ 2 \end{matrix}\right\|$	1	0	0	0
$\left\langle\begin{matrix} 2 \\ 0 \end{matrix}\right\|$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$\left\langle\begin{matrix} 2 \\ -2 \end{matrix}\right\|$	0	0	0	1

Now we consider the general pocedure of deriving CG coefficients by the example of $\mathfrak{so}(3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2;\mathbb{C})$. Consider the rescaled generators

$$J_0 = \frac{1}{2}H, \quad J_+ = \frac{1}{\sqrt{2}}X, \quad J_- = -\frac{1}{\sqrt{2}}Y$$

In this way, they satisfy the commutation relation

$$[J_+, J_-] = -J_0, \quad [J_0, J_+] = J_+, \quad [J_0, J_-] = J_-$$

Note we have

$$J_0^\dagger = J_0, \quad J_\pm^\dagger = -J_\mp$$

So the quadratic Casimir operator is

$$C_2 = J_0 J_0^\dagger + J_+ J_+^\dagger + J_- J_-^\dagger = J_0^2 - J_+ J_- J_- J_+$$

We can use the standard notation of angular momentum j for an irrep of $\mathfrak{so}(3)$, which is related to the highest weight μ by

$$j = \frac{1}{2}\mu$$

j can either be an integer or a half-integer. The orthonormal weight vectors are denoted as

$$|j, m\rangle\,, \quad m = j, j-1, \cdots, -(j-1), -j$$

The Lie algebra generators act on $|j, m\rangle$ in terms of

$$J_0\,|j, m\rangle = m\,|j, m\rangle\,, \quad J_\pm\,|j, m\rangle = C_\pm(j, m)\,|j, m \pm 1\rangle$$

Here $C_\pm(j, m)$ is a constant depending on j and m. Of course for the highest weight and the lowest weight, we have

$$J_+\,|j, j\rangle = 0, \quad J_-\,|j, -j\rangle = 0$$

As $[J_+, J_-] = -J_0$, we then have

$$\langle j, m|J_+J_-|j, m\rangle = \langle j, m|J_-J_+|j, m\rangle - \langle j, m|J_0|j, m\rangle$$

Inserting the complete relation, we have

$$\langle j, m|J_+|j, m-1\rangle\,\langle j, m-1|J_-|j, m\rangle = \langle j, m|J_-|j, m+1\rangle\,\langle j, m+1|J_+|j, m\rangle + \langle j, m|J_0|j, m\rangle$$

Since $J_\pm^\dagger = -J_\mp$, we have

$$|\,\langle j, m-1|J_-|j, m\rangle\,|^2 = |\,\langle j, m|J_-|j, m+1\rangle\,|^2 + m$$

Consider the Casimir operator C_2, we have

$$\begin{aligned}\langle j, m|C_2|j, m\rangle &= \langle j, m|J_0^2|j, m\rangle - \langle j, m|J_+J_-|j, m\rangle - \langle j, m|J_-J_+|j, m\rangle \\ &= m^2 + |\,\langle j, m-1|J_-|j, m\rangle\,|^2 + |\,\langle j, m|J_-|j, m+1\rangle\,|^2\end{aligned}$$

Given that

$$\langle j, m|C_2|j, m\rangle = j(j+1)$$

We can solve to get

$$|\,\langle j, m-1|J_-|j, m\rangle\,|^2 = \frac{1}{2}[j(j+1) - m(m-1)]$$

For the phase factor, we take the convention that

$$\begin{aligned}C_-(j, m) &= \langle j, m-1|J_-|j, m\rangle \equiv \sqrt{\frac{(j+m)(j-m+1)}{2}} \\ C_+(j, m) &= \langle j, m+1|J_+|j, m\rangle \equiv -\sqrt{\frac{(j-m)(j+m+1)}{2}}\end{aligned}$$

Notice that due to different conventions, our result here may differ a factor $1/\sqrt{2}$ or so from the standard quantum mechanics literature.

Now consider tensor product decomposition

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus |j_1 - j_2|$$

The natural basis for $V_1 \otimes V_2$ is composed by

$$|j_1, m_1; j_2, m_2\rangle \equiv |j_1, m_1\rangle\, |j_2, m_2\rangle$$

Let's denote the states in irrep as $|j, m\rangle$, then there should be relations

$$\sum_{j,m} |j, m\rangle\, \langle j, m|j_1, m_1; j_2, m_2\rangle = |j_1, m_1; j_2, m_2\rangle$$

$$\sum_{m_1,m_2} |j_1, m_1; j_2, m_2\rangle\, \langle j_1, m_1; j_2, m_2|j, m\rangle = |j, m\rangle$$

$$\sum_{m_1,m_2} \left\langle j', m'\middle|j_1, m_1; j_2, m_2\right\rangle \langle j_1, m_1; j_2, m_2|j, m\rangle = \delta_{jj'}\delta_{mm'}$$

$$\sum_{j,m} \left\langle j_1', m_1'; j_2', m_2'\middle|j, m\right\rangle \langle j, m|j_1, m_1; j_2, m_2\rangle = \delta_{j_1j_1'}\delta_{j_2j_2'}\delta_{m_1m_1'}\delta_{m_2m_2'}$$

Now consider the expression

$$\langle j, m|J_-|j_1, m_1; j_2, m_2\rangle = \langle j, m|J_-|j, m+1\rangle\, \langle j, m+1|j_1, m_1; j_2, m_2\rangle$$

Meanwhile,

$$\begin{aligned}
&\langle j, m|J_-|j_1, m_1; j_2, m_2\rangle \\
&= \sum_{m_1',m_2'} \left\langle j, m\middle|j_1, m_1'; j_2, m_2'\right\rangle \left\langle j_1, m_1'; j_2, m_2'\middle|J_-\middle|j_1, m_1; j_2, m_2\right\rangle \\
&= \sum_{m_1',m_2'} \left\langle j, m\middle|j_1, m_1'; j_2, m_2'\right\rangle \left[\left\langle j_1, m_1'\middle|J_-\middle|j_1, m_1\right\rangle \left\langle j_2, m_2'\middle|j_2, m_2\right\rangle + \left\langle j_1, m_1'\middle|j_1, m_1\right\rangle \left\langle j_2, m_2'\middle|J_-\middle|j_2, m_2\right\rangle\right] \\
&= \langle j, m|j_1, m_1-1; j_2, m_2\rangle\, \langle j_1, m_1-1|J_-|j_1, m_1\rangle + \langle j, m|j_1, m_1; j_2, m_2-1\rangle\, \langle j_2, m_2-1|J_-|j_2, m_2\rangle
\end{aligned}$$

Then we could get the equation

$$\begin{aligned}
&\sqrt{(j-m)(j+m+1)}\,\langle j_1, m_1; j_2, m_2|j, m+1\rangle \\
&= \sqrt{(j_1+m_1)(j_1-m_1+1)}\,\langle j_1, m_1-1; j_2, m_2|j, m\rangle + \sqrt{(j_2+m_2)(j_2-m_2+1)}\,\langle j_1, m_1; j_2, m_2-1|j, m\rangle
\end{aligned}$$

For J_+, we similarly have

$$\begin{aligned}
&\sqrt{(j+m)(j-m+1)}\,\langle j_1, m_1; j_2, m_2|j, m-1\rangle \\
&= \sqrt{(j_1-m_1)(j_1+m_1+1)}\,\langle j_1, m_1+1; j_2, m_2|j, m\rangle + \sqrt{(j_2-m_2)(j_2+m_2+1)}\,\langle j_1, m_1; j_2, m_2+1|j, m\rangle
\end{aligned}$$

From these equations, we can solve all CG coefficients recursively.

10.4 Branching Rules

Spontaneous symmetry breaking in particle physics breaks a gauge group G into a subgroup S. Branching rule depicts given a rep of G, what reps of S will we get, and what are the charges of these S reps if there are remaining $\mathsf{U}(1)$ electromagnetic sectors.

For a semi-simple Lie algebra $\mathfrak{g}$, consider the embedding of a Lie algebra $\mathfrak{s} \subset \mathfrak{g}$. For a representation (π, V) of $\mathfrak{g}$, we want to decompose it as a direct sum of (π_i, V_i) of the subalgebra $\mathfrak{s}$. Note that $\mathfrak{s}$ can be a direct sum of simple Lie algebras and multiple $\mathfrak{u}(1)$.

For simplicity, we just focus on the R-type subalgebra and we first take the example of

$$\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{u}(1)^{\oplus \ell}$$

where $\mathfrak{s}'$ is a non-abelian subalgebra of $\mathfrak{g}$ whose simple roots are chosen from the simple roots of $\mathfrak{g}$. That is, the Dynkin diagram of $\mathfrak{s}'$ is generated by deleting ℓ nodes from what of $\mathfrak{g}$.

Consider a weight λ in the weight system of (π, V) with weight vector v defined as

$$\pi(H)v = \langle \lambda, H \rangle\, v, \quad \forall H \in \mathfrak{h} \subset \mathfrak{g}$$

Since the Cartan subalgebra of $\mathfrak{s}$ is a subalgebra of $\mathfrak{h}$, so λ should also be a weight of $\bigoplus_i \pi_i$ with weight vector v.

Let's denote the simple roots of $\mathfrak{g}$ by $(\alpha_1, \cdots, \alpha_r)$ while the simple roots for $\mathfrak{s}'$ as a subset $(\alpha_{i_1}, \cdots, \alpha_{i_{r'}})$. For a weight λ of π written as

$$\lambda = (\langle \lambda, H_{\alpha_1} \rangle, \cdots, \langle \lambda, H_{\alpha_r} \rangle)$$

it then should be a weight of $\mathfrak{s}'$ denoted as

$$\lambda = (\left\langle \lambda, H_{\alpha_{i_1}} \right\rangle, \cdots, \left\langle \lambda, H_{\alpha_{i_{r'}}} \right\rangle)$$

Now we can give the branching rules:

1. Write down all the weights of the representation (π, V).

2. Ignore the ℓ nodes deleted from the Dynkin diagram of $\mathfrak{g}$, and one can naturally see how the weights are organized into different irreps (π_i, V_i) of $\mathfrak{s}'$. For example, one can pick out the dominant integral weights λ (remember it's a weight with $\langle \lambda, H_\alpha \rangle \geq 0$ for all simple roots α of $\mathfrak{s}'$), and they give rise to irreducible reps π_λ of $\mathfrak{s}'$ with highest weight λ.

3. For the charges of each irrep (π_i, V_i) under $\mathfrak{u}(1)^{\oplus \ell}$, we can denote it as $(q_{i,1}, q_{i,2}, \cdots, q_{i,\ell})$. We can construct ℓ linearly independent elements $Q_j (j = 1, \cdots, \ell) \in \mathfrak{h} \subset \mathfrak{g}$ as the $\mathfrak{u}(1)$ generators. Besides, we demand them to satisfy the orthogonality conditions:

   (a) $\langle Q_i, Q_j \rangle = 0$ for $i \neq j$, usually it's a physical requirement.
   (b) $\langle Q_i, \alpha_{j_k} \rangle = 0$ for all the simple roots α_{j_k} in $\mathfrak{s}'$.

   Then we can prove that for each irrep (π_i, V_i), each charge

   $$q_{i,j} = \langle Q_j, w \rangle$$

   is a constant number for every weight w in (π_i, V_i). (Note that here w is still an element of $\mathfrak{h} \subset \mathfrak{g}$.) Because the weight system of (π_i, V_i) is generated by subtracting simple roots from the highest weight.

Let's study some examples:

1. $\mathfrak{su}(3) \to \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. The Dynkin diagram is embedded as

The fundamental rep of $\mathfrak{su}(3)$ has weights

$$(1,0), \quad (-1,1), \quad (0,-1)$$

Hence we find the first two weights can generate the fundamental rep $\mathbf{2}$ of $\mathfrak{su}(2)$, while the third weight gives the trivial rep $\mathbf{1}$ of $\mathfrak{su}(2)$, we then have

$$\mathbf{3} \to \mathbf{2} \oplus \mathbf{1}$$

For the charge under $\mathfrak{u}(1)$, the generator of $\mathfrak{u}(1)$ can be taken as

$$Q = H_{\alpha_1} + 2H_{\alpha_2}$$

Clearly we have $\langle Q, \alpha_1 \rangle = 0$. Here all the weights in the rep $\mathbf{2}$ has charge 1 under Q, while the weight in rep $\mathbf{1}$ has charge -2 under Q, so the final branching rule is

$$\mathbf{3} \to \mathbf{2}_1 \oplus \mathbf{1}_{-2}$$

More generally, for $\mathfrak{su}(n+1) \to \mathfrak{su}(n) \oplus \mathfrak{u}(1)$ we have

$$\mathbf{n+1} \to \mathbf{n}_1 \oplus \mathbf{1}_{-n}$$

Such branching rule can be applied successively. For example, consider

$$\mathfrak{su}(n+1) \to \mathfrak{su}(n) \oplus \mathfrak{u}(1) \to [\mathfrak{su}(n-1) \oplus \mathfrak{u}(1)] \oplus \mathfrak{u}(1)$$

Then we have

$$\mathbf{n+1} \to \mathbf{n}_1 \oplus \mathbf{1}_{-n} \to (\mathbf{n-1})_{(1,1)} \oplus \mathbf{1}_{(-(n-1),1)} \oplus \mathbf{1}_{(0,n)}$$

2. $\mathfrak{su}(5) \to \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. It's the assumed breaking of $\mathfrak{su}(5)$ GUT gauge algebra into the Standard Model gauge algebra $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. The Dynkin diagram of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ can be enmedded as

For the fundamental rep of $\mathfrak{su}(5)$, we have its weights

$$(1,0,0,0),\ (-1,1,0,0),\ (0,-1,1,0),\ (0,0,-1,1),\ (0,0,0,-1)$$

We can organize them into the representations $(\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$. For the $\mathfrak{u}(1)$ generator, we can choose it as

$$Q = 2H_{\alpha_1} + 4H_{\alpha_2} + 6H_{\alpha_3} + 3H_{\alpha_4}$$

Then the rep $(\mathbf{3},\mathbf{1})$ has charge 2 , while the rep $(\mathbf{1},\mathbf{2})$ has charge -3, hence the branching rule is given by

$$\mathbf{5} \to (\mathbf{3},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{2})_{-3}$$

We can also consider the rank-2 anti-symmetric rep $\mathbf{10} = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ of $\mathfrak{su}(5)$, its weights are

$$\begin{aligned}&(0,1,0,0),\ (1,-1,1,0),\ (-1,0,1,0),\ (1,0,-1,1),\ (-1,1,-1,1),\ (1,0,0,-1)\\&(0,-1,0,1),\ (-1,1,0,-1),\ (0,-1,1,-1),\ (0,0,-1,0)\end{aligned}$$

We can organize them into reps of $(\bar{\mathbf{3}},\mathbf{1}),(\mathbf{3},\mathbf{2})$ and $(\mathbf{1},\mathbf{1})$. After computing charges under Q, we get the branching rule

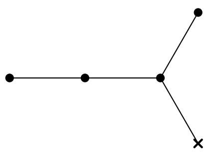
$$\mathbf{10} \to (\bar{\mathbf{3}},\mathbf{1})_4 \oplus (\mathbf{3},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{1})_{-6}$$

In fact, the two branching rules we get here are exactly from the quark and lepton contents of the Standard Model known as QUDLE. The $\mathsf{U}(1)$ charges shown here are six times the values of hypercharge Y in physics literature. The hypercharge is defined as

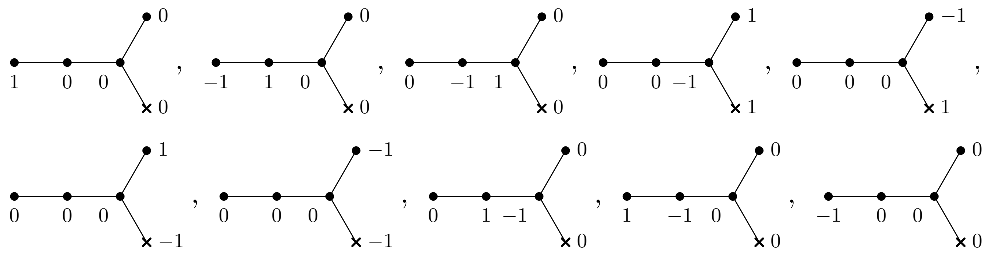
$$Y \equiv Q - T_3$$

Here T_3 is the third component of weak isospin.

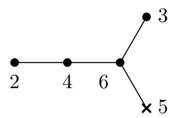
3. $\mathfrak{so}(10) \to \mathfrak{su}(5)$. The Dynkin diagram is embedded as



For the representation $\mathbf{10}$, the weights are



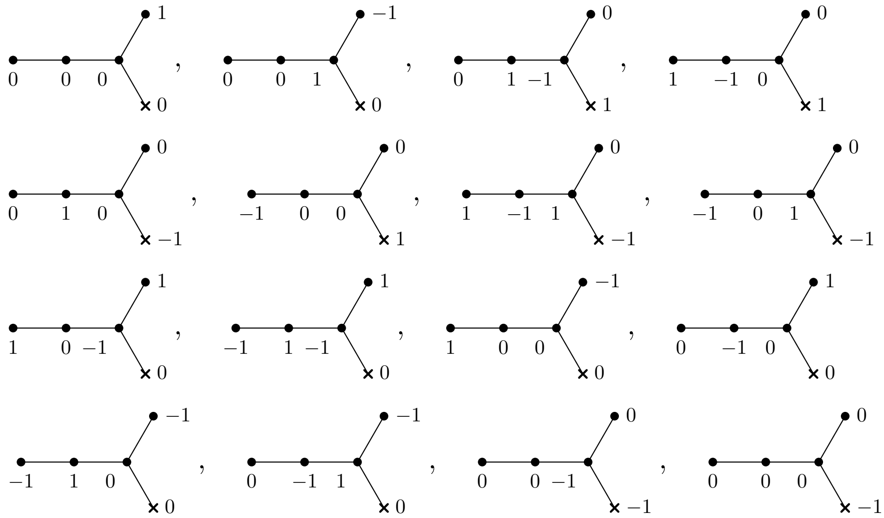
We can organize these weights into the rep $\mathbf{5} \oplus \bar{\mathbf{5}}$, the $\mathsf{U}(1)$ charge can be taken as



So the branching rule is

$$\mathbf{10} \to \mathbf{5}_2 \oplus \bar{\mathbf{5}}_{-2}$$

For the spinor rep **16** with the highest weight $(0,0,0,1,0)$, the weights are



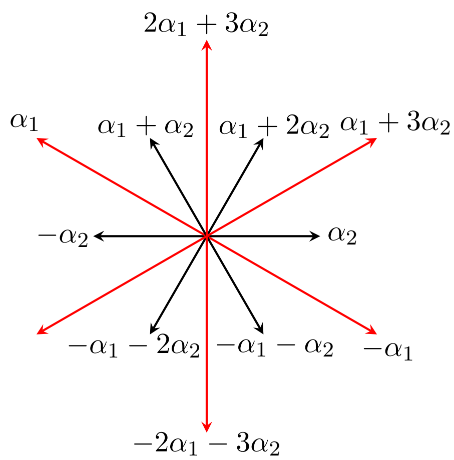
The branching rule is given by

$$\mathbf{16} \to \bar{\mathbf{5}}_3 \oplus \mathbf{10}_{-1} \oplus \mathbf{1}_{-5}$$

In the $\mathsf{SO}(10)$ GUT theory, the representation **16** contains the full QUDLE and a right-handed nutrino in the representation $\mathbf{1}_{-5}$ after breaking to $\mathsf{SU}(5) \times \mathsf{U}(1)$.

4. Now it's a case where the Dynkin diagram of $\mathfrak{s}'$ cannot be embedded into the Dynkin diagram of $\mathfrak{g}$, but the root system of $\mathfrak{s}'$ is still a subsystem of $\mathfrak{g}$. Our example is $G_2 \to \mathfrak{su}(3)$.

   Let's recall the root system of G_2:



   The simple roots of G_2 are α_1 and α_2, while the simple roots of $\mathfrak{su}(3)$ can be taken as α_1 and $\alpha_1 + 3\alpha_2$. For the coroots, we have

$$H_{\alpha_1+3\alpha_2} = \frac{2(\alpha_1 + 3\alpha_2)}{\langle \alpha_1 + 3\alpha_2, \alpha_1 + 3\alpha_2 \rangle} = H_{\alpha_1} + H_{\alpha_2}$$

Hence for each weight $\lambda = (\lambda_1, \lambda_2) = (\langle\lambda, H_{\alpha_1}\rangle, \langle\lambda, H_{\alpha_2}\rangle)$ of a representation π of G_2, it corresponds to a weight of a rep of $\mathfrak{su}(3)$:

$$\lambda = (\langle\lambda, H_{\alpha_1}\rangle, \langle\lambda, H_{\alpha_1+3\alpha_2}\rangle) = (\langle\lambda, H_{\alpha_1}\rangle, \langle\lambda, H_{\alpha_1} + H_{\alpha_2}\rangle) = (\lambda_1, \lambda_1 + \lambda_2)$$

Hence for example, for the fundamental rep **7** of G_2 whose weights are

$$(0,1),\ (1,-1),\ (-1,2),\ (0,0),\ (1,-2),\ (-1,1),\ (0,-1)$$

They just correspond to the weight of $\mathfrak{su}(3)$:

$$(0,1),\ (1,0),\ (-1,1),\ (0,0),\ (1,-1),\ (-1,0),\ (0,-1)$$

Organize them into irreps of $\mathfrak{su}(3)$, the branching rule then is

$$\mathbf{7} \to \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}$$

11 Futher Applications in Physics and Advanced Topics

11.1 Lorentz Group

The role Lorentz group playing is well acknowledged in modern physics. Recall that the full 4-dimensional group $\mathsf{O}(1,3)$ is defined as the group of 4×4 real matrices M preserving

$$M^T g M = g,$$

where g is the Minkowski metric

$$g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

If we denote a vector as $(t, x, y, z)^T$ and denote action of M as

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = M \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Then the 4-dimensional distance is invariant:

$$(t')^2 - (x')^2 - (y')^2 - (z')^2 = t^2 - x^2 - y^2 - z^2$$

First let's investigate some global properties of $\mathsf{O}(1,3)$. It can be easily verified that

$$L(\beta) \equiv \begin{pmatrix} \cosh\beta & \sinh\beta & & \\ \sinh\beta & \cosh\beta & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \beta \in \mathbb{R},$$

belongs to $\mathsf{O}(1,3)$ and its elements are unbounded. In fact, $L(\beta)$ corresponds to a Lorentz boost in the x-direction. Nonetheless, $\mathsf{O}(1,3)$ has a compact $\mathsf{O}(3)$ subgroup consisting elemets like

$$\begin{pmatrix} 1 & \\ & M_{3\times 3} \end{pmatrix}, \quad M_{3\times 3} \in \mathsf{SO}(3)$$

Apart from non-compactness, $\mathsf{O}(1,3)$ is also disconnected with 4 connected components. As $M^T g M = g$, we then have

$$(M^T g M)_{11} = M_{11}^2 - M_{21}^2 - M_{31}^2 - M_{41}^2 = 1$$

Thus

$$|M_{11}| \geq 1$$

Now we can define 4 connected subsets of $\mathsf{O}(1,3)$ as

1. $L^{\uparrow +}: \ M_{11} \geq 1, \det M = +1;$
2. $L^{\uparrow -}: \ M_{11} \geq 1, \det M = -1;$
3. $L^{\downarrow +}: \ M_{11} \leq -1, \det M = +1;$
4. $L^{\downarrow -}: \ M_{11} \leq -1, \det M = -1$

The subset $L^{\uparrow +}$ is a connected subgroup of $\mathsf{O}(1,3)$ and is often called as the restricted Lorentz group $\mathsf{SO}^+(1,3)$. Elements of it are generated by Lorentz boosts in x, y, z directions and 3D Euclidean space rotations, where the latter opeartions form the $\mathsf{SO}(3)$ subgroup of $\mathsf{SO}^+(1,3)$.

We can define the parity transformation as

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

and define the time reversal as

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Then we have

$$L^{\uparrow -} = PL^{\uparrow +}, \quad L^{\downarrow +} = PTL^{\uparrow +}, \quad L^{\downarrow -} = TL^{\uparrow +}$$

The union of $L^{\uparrow +}$ and $L^{\downarrow +}$ forms the group $\mathsf{SO}(1,3)$, which is also disconnected.

Now focus on the restricted Lorentz group $\mathsf{SO}^+(1,3)$. Just analogous to $\mathsf{SO}(3)$, it's also not simply-connected. The ssimply-connected matrix group is the double cover of $\mathsf{SO}^+(1,3)$:

$$\mathsf{SO}^+(1,3) \cong \mathsf{SL}(2;\mathbb{C})/\{\pm \mathbb{1}\}$$

$\mathsf{SL}(2;\mathbb{C})$ is also called $\mathsf{Spin}(1,3)$, because it's closely related to spinors. Now we can explicitly discuss the surjective $\mathsf{SL}(2;\mathbb{C}) \to \mathsf{SO}^+(1,3)$.

The standard coordinates on $\mathbb{R}^{1,3}$ can be rewritten as a 2×2 Hermitian matrix

$$X \equiv \begin{pmatrix} t+z & x-\mathrm{i}y \\ x+\mathrm{i}y & t-z \end{pmatrix} = t\mathbb{1} + \vec{x}\cdot\vec{\sigma}$$

Here $\vec{\sigma}$ is the vector of Pauli matrices. Note that

$$\det X = t^2 - x^2 - y^2 - z^2$$

just reproduces the invariant distance. We can take a group element $S \in \mathsf{SL}(2;\mathbb{C})$ and act it on X as

$$X' \equiv SXS^\dagger$$

In this way $\det X$ is invariant and its Hermiticity is preserved, hence we can viewed it as a $\mathbb{R}^{1,3}$ isometry and a homomorphism between $\mathsf{SL}(2;\mathbb{C})$ and $\mathsf{SO}^+(1,3)$. Since the kernel of the homomorphism is $\{\pm\mathbb{1}\}$, hence $\mathsf{SO}^+(1,3) \cong \mathsf{SL}(2;\mathbb{C})/\{\pm\mathbb{1}\}$. For

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2;\mathbb{C})$$

We can explicitly write down the corresponding $\mathsf{SO}^+(1,3)$ element as

$$\frac{1}{2}\begin{pmatrix} a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d} & a\bar{b}+\bar{a}b+\bar{c}d+c\bar{d} & \mathrm{i}(\bar{a}b-a\bar{b}+\bar{c}d-c\bar{d}) & a\bar{a}-b\bar{b}+c\bar{c}-d\bar{d} \\ a\bar{c}+\bar{a}c+b\bar{d}+\bar{b}d & a\bar{d}+\bar{a}d+b\bar{c}+\bar{b}c & \mathrm{i}(\bar{a}d-a\bar{d}+b\bar{c}-c\bar{b}) & a\bar{c}+\bar{a}c-b\bar{d}-\bar{b}d \\ \mathrm{i}(a\bar{c}-\bar{a}c+b\bar{d}-\bar{b}d) & \mathrm{i}(a\bar{d}-\bar{a}d+b\bar{c}-\bar{b}c) & \bar{a}d+a\bar{d}-b\bar{c}-c\bar{b} & \mathrm{i}(a\bar{c}-\bar{a}c-b\bar{d}+\bar{b}d) \\ a\bar{a}+b\bar{b}-c\bar{c}-d\bar{d} & a\bar{b}+\bar{a}b-\bar{c}d-c\bar{d} & \mathrm{i}(\bar{a}b-a\bar{b}+c\bar{d}-\bar{c}d) & a\bar{a}-b\bar{b}-c\bar{c}+d\bar{d} \end{pmatrix}$$

Now we can consider the reprsentation of Lorentz group. As usual, we study representations of a group by studying its Lie algebra. Since the subset simply-connected to the identity element of Lorentz group is $\mathfrak{SO}^+(1,3)$, hence we could just study its Lie algebra, which is denoted as $\mathfrak{so}(1,3)$, whose generators are taken to be

$$J^{01} = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad J^{02} = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad J^{03} = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}.$$

$$J^{12} = \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad J^{23} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{pmatrix}, \quad J^{31} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&-1&0&0 \end{pmatrix}.$$

From these generators we can define a rank-2 anti-symmetric tensor

$$J^{\mu\nu} = -J^{\nu\mu}, \quad \mu,\nu = 0,1,2,3,$$

and they satisfy this commutation relation:

$$[J^{\mu\nu}, J^{\rho\sigma}] = g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}.$$

A generic element of $\mathfrak{so}(1,3)$ can be expressed as

$$\frac{1}{2}\omega_{\mu\nu} J^{\mu\nu},$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is an anti-symmetric 4×4 matrix parametrizing this element.

Now consider complexifying $\mathfrak{so}(1,3)$, we will prove

$$\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}).$$

The proof is straight forward: Constructing

$$\begin{aligned}
\mathcal{C}^+ &\equiv \frac{1}{2}(J^{31} + \mathrm{i}J^{23} + \mathrm{i}J^{02} - J^{01}), & \mathcal{D}^+ &\equiv \frac{1}{2}(J^{31} + \mathrm{i}J^{23} - \mathrm{i}J^{02} + J^{01}),\\
\mathcal{C}^- &\equiv \frac{1}{2}(-J^{31} + \mathrm{i}J^{23} - \mathrm{i}J^{02} - J^{01}), & \mathcal{D}^- &\equiv \frac{1}{2}(-J^{31} + \mathrm{i}J^{23} + \mathrm{i}J^{02} + J^{01}),\\
\mathcal{C}^3 &\equiv -\mathrm{i}J^{12} + J^{03}, & \mathcal{D}^3 &\equiv -\mathrm{i}J^{12} - J^{03}.
\end{aligned}$$

It can be verified that

$$[\mathcal{C}^3, \mathcal{C}^\pm] = \pm 2\mathcal{C}^\pm, \quad [\mathcal{C}^+, \mathcal{C}^-] = \mathcal{C}^3,, \quad [\mathcal{D}^3, \mathcal{D}^\pm] = \pm 2\mathcal{D}^\pm, \quad [\mathcal{D}^+, \mathcal{D}^-] = \mathcal{D}^3, \quad [\mathcal{C}, \mathcal{D}] = 0.$$

So far we have reshaped studying representations of $\mathfrak{so}(1,3)$ into studying representations of $\mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}) = A_1 \oplus A_1$. As we know, its irreps are $(\mathbf{m}, \mathbf{n})$. In physics, we often denote these reps as (s_1, s_2), where

$$s_1 \equiv \frac{m_1 - 1}{2}, \quad s_2 \equiv \frac{m_2 - 1}{2}.$$

We physicists often relate different representations to particles of different spins. To mathematically describe the notion of *spin* accurately, we need to consider the $\mathfrak{so}(3)$ subalgebra of $\mathfrak{so}(1,3)$, which is generated by J^{12}, J^{23}, J^{31}, in other words, by

$$\mathcal{C}^+ + \mathcal{D}^+, \quad \mathcal{C}^- + \mathcal{D}^-, \quad \mathcal{C}^3 + \mathcal{D}^3.$$

Then we can consider the branching rule of an $\mathfrak{so}(1,3)$ rep (s_1, s_2) into irreps of $\mathfrak{so}(3)$:

$$(s_1, s_2) \to (s_1 + s_2) \oplus (s_1 + s_2 - 1) \oplus \cdots \oplus |s_1 - s_2|.$$

These numbers labelling the irreps of $\mathfrak{so}(3)$ are defined as the spins of corresponding states.For a general representation (s_1, s_2), we can decompose it into reps with different spins, and some degrees of freedom are unphysical once we impose the eqeuations of motion.

Here we enumerate some commonly used reps in physics:

1. Scalar: $(0, 0)$, spin-0.

2. Spinor: Spin-$\frac{1}{2}$. There are three commonly used reps for spinors:

   (a) Left-handed Weyl spinor $\left(\frac{1}{2},0\right)$.

   (b) Right-handed Weyl spinor $\left(0,\frac{1}{2}\right)$.

   (c) Dirac spinor $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$.

3. Vector: $\left(\frac{1}{2},\frac{1}{2}\right)$, spin-1. Clearly this representation is 4-dimensional. However, we find branching rule indicates such representation can be decomposed into $1\oplus 0$, and only the spin-1 part (3-dimensional representation $\mathbf{3}$) is physical.

4. Rarita-Schwinger field: $\left(1,\frac{1}{2}\right)\oplus\left(\frac{1}{2},1\right)$.

5. Graviton: $(1,1)$, spin-2, which can be embedded into a traceless symmetric tensor.

6. Rank-2 anti-symmetric tensor field: $(1,0)\oplus(0,1)$. It corresponds to the adjoint rep of $A_1\oplus A_1$, an example is $F_{\mu\nu}$.

Having investigated reps of $\mathfrak{so}(1,3)$, we can lift them up to reps of $\mathsf{SO}^+(1,3)$, and here we will encounter two subtleties:

- $\mathsf{SO}^+(1,3)$ is non-compact, so it has no finite dimensional faithful unitary rep, so those non-trivial reps mentioned above are all non-unitary.

- Representations with half integer spins are all projective reps of $\mathsf{SO}^+(1,3)$ rather than actual reps, such a rep (Π,V) satisfies

$$\Pi(g_1)\Pi(g_2)=\pm\Pi(g_1g_2).$$

  Actually, such (Π,V) is a rep of the double cover group $\mathsf{SL}(2;\mathbb{C}):=\mathsf{Spin}(1,3)$.

Anyway, we can explicitly study the action of $\mathsf{SO}^+(1,3)$ for a few finite dimensional reps. Let's deote a generic element of $\mathsf{SO}^+(1,3)$ as

$$g=\exp\left(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}\right),$$

then we have:

1. Scalar field: It corresponds to the trivial rep

$$\phi'=1\cdot\phi.$$

2. Vector field: Corresponds to the standard rep, and the group acts as

$$V'^{\mu}=\left[\exp\left(\frac{1}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right)\right]^{\mu}{}_{\nu}V^{\nu}.$$

3. Left-handed Weyl spinor:
$$\psi_\alpha = \left[\exp\left(\frac{1}{2}\omega_{\mu\nu}S_L^{\mu\nu}\right)\right]_\alpha^{\ \beta}\psi_\beta$$
Here $S_L^{\mu\nu}$ are 2×2 matrices such that
$$S_L^{ij} = -\frac{\mathrm{i}}{2}\epsilon^{ijk}\sigma^k, \quad S_L^{0i} = -\frac{1}{2}\sigma^i, \quad i,j = 1,2,3.$$
Here σ^k are Pauli matrices and $S_L^{\mu\nu}$ are exactly the left-handed Weyl spinor rep of $\mathfrak{so}(1,3)$.

Note that it's a projective rep. If we take the Lorentz algebra element
$$A = 2\pi J^{12}$$
It corresponds to the Lorentz group element $g = e^A = \mathbb{1}_4$ in standard rep, but in the spinor rep we have
$$\pi(g) = e^{2\pi S_L^{12}} = -\mathbb{1}_2.$$
We can similarly argue that right-handed Weyl spinor and Dirac spinor are also under projective reps.

For the double cover group $\mathsf{Spin}(1,3)$, the spinor reps are well-defined reps as $\pi(g)$ do not correspond the identity element in $\mathsf{Spin}(1,3)$. A common interpretation is that when we rotate a spinor by 2π angle, it doesn't go back to itself but goes to its reverse.

4. Right-handed Weyl spinor:
$$\psi_{\dot\alpha} = \left[\exp\left(\frac{1}{2}\omega_{\mu\nu}S_R^{\mu\nu}\right)\right]_{\dot\alpha}^{\ \dot\beta}\psi_{\dot\beta}.$$
Here $S_R^{\mu\nu}$ are defined as
$$S_R^{ij} = -\frac{\mathrm{i}}{2}\epsilon^{ijk}\sigma^k, \quad S_R^{0i} = \frac{1}{2}\sigma^i, \quad i,j = 1,2,3.$$
They are just $\mathfrak{so}(1,3)$ matrices under $\left(0,\frac{1}{2}\right)$ rep.

5. Dirac spinor:
$$\Psi_a = \left[\exp\left(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\right]_a^{\ b}\Psi_b.$$
Here $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\gamma^\nu]$ and γ^μ are Dirac matrices defined by
$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$$
We can take Weyl representation for γ-matrices
$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0\end{pmatrix}, \quad \gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3,$$
here γ^5 can project Dirac spinors to left/right-handed Weyl spinors:
$$\psi_L = \frac{1-\gamma^5}{2}\Psi, \quad \psi_R = \frac{1+\gamma^5}{2}\Psi.$$

11.2 Poincaré Group

Poincaré group is the basic tool of classifying particles in field theory, it's the semi-direct product of $\mathsf{SO}^+(1,3)$ with the space-time translation group $\mathbb{R}^4$:

$$\mathsf{ISO}^+(1,3) \equiv \mathbb{R}^4 \rtimes \mathsf{SO}^+(1,3).$$

$\mathbb{R}^4$ is a normal subgroup of Poincaré group. The Lie algebra of Poincaré group has generators $J^{\mu\nu} = -J^{\nu\mu}$ and P^μ with the commutation relations

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}, \\
[P^\mu, P^\nu] &= 0, \\
[P^\mu, J^{\nu\rho}] &= g^{\mu\nu} P^\rho - g^{\mu\rho} P^\nu.
\end{aligned}$$

P^ρ and $J^{\mu\nu}$ just correspond to 4D translation and angular momentum operators (up to factors of i):

$$P^\mu = \partial^\mu, \quad J^{\mu\nu} = x^\mu \partial^\nu - x^\nu \partial^\mu.$$

In particular, we can identify $H \equiv \mathrm{i}P^0$ as the Hamiltonian operator, $\vec{P} = \mathrm{i}(P^1, P^2, P^3)$ is the momentum operator, and $\vec{J} = \mathrm{i}(J^{23}, J^{31}, J^{12})$ is the angular momentum operator.

To label different reps of $\mathsf{ISO}(1,3)$, we can consider the eigenvalues of $\mathrm{i}P^\mu$ operator the Casimir operators:

$$\begin{aligned}
P^2 &= -P^\mu P_\mu, \\
W^2 &= W^\mu W_\mu,
\end{aligned}$$

where

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma.$$

As for the eigenvalues of P^2, we have two types of reps that are physically relevant:

1. $P^2 = E^2 - |\vec{p}|^2 = m^2$, which corresponds to a massive particle. We can always perform a Lorentz transformation and bring $\mathrm{i}P^\mu$ to the form in a particular reference frame $(m, \vec{0})$. Spatial rotation wouldn't change $(m, \vec{0})$, so it's invariant under $\mathsf{SO}(3)$ subgroup of $\mathsf{ISO}(1,3)$, which is called the little group for the massive case.

   The rep of Poincaré group is labelled by m and j, j is the spin of the particle which also labels the rep under $\mathsf{SO}(3)$ little group and can be either an integer or a half-integer. In fact, the Casimir operator W^2 has eigenvalue $-m^2 j(j+1)$.

   Note that when the little group is $\mathsf{SO}(3)$, the spinor rep is a projective rep. To get a normal representation, we can replace $\mathsf{SO}(3)$ by its double cover $\mathsf{SU}(2)$.

2. $P^2 = 0$, it corresponds to a massless particle. In this case we can always perform a Lorentz transformation bringing $\mathrm{i}P^\mu$ to the form $(E, 0, 0, E)$. Now the little group is an $\mathsf{SO}(2)$ subgroup of Poincaré group, and the reps of Poincaré group is labelled by $\mathrm{i}P^\mu = (E, 0, 0, E)$ and the helicity h labelling the reps of little group $\mathsf{SO}(2)$, which can be a half-integer. For a massless scalar, it can only have helicity 0, while a massless Weyl spinor can have $h = \pm\frac{1}{2}$. A photon can have $h = \pm 1$ respectively correspond to two polarization degrees of freedom, and a graviton has $h = \pm 2$, which also corresponds to 2 polarizations.

11.3 Spinor Representations in Any Dimension

The notion of spinors is not confined in our 4-dimensional spacetime, but we can extend our imagination to any dimensions. Let's consider an n-D Minskowski metric

$$\eta^{\mu\nu} =\equiv \mathrm{diag}(1,-1,-1,\cdots,-1),$$

then we can define Dirac Γ-matrices by such Clifford algebra

$$\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}.$$

For even spacetime dimension $d = 2k+2$, following linear combinations of Dirac matrices can be introduced:

$$\Gamma^{0\pm} \equiv \frac{1}{2}(\Gamma^0 \pm \Gamma^1), \quad \Gamma^{m\pm} \equiv \frac{1}{2}(\mathrm{i}\Gamma^{2m} \pm \Gamma^{2m+1}), \quad m = 1,\cdots,k.$$

They satisfy

$$\{\Gamma^{a+},\Gamma^{b-}\} = \delta^{ab}, \quad \{\Gamma^{a+},\Gamma^{b+}\} = \{\Gamma^{a-},\Gamma^{b-}\} = 0.$$

Particularly,

$$(\Gamma^{a+})^2 = (\Gamma^{a-})^2 = 0.$$

Suppose Γ^μ act on a vector space V whose elements are called spinors, then we can intreprete $\Gamma^{a\pm}$ as creation/annihilation operators in a system of $k+1$ fermions. A spinor ζ is defined to be the satate annihilated by all Γ^{a-}:

$$\Gamma^{a-}\zeta = 0, \quad \forall a.$$

Starting from ζ, we can construct 2^{k+1}-dimensional rep by acting on Γ^{a+}:

$$\zeta^{(\boldsymbol{s})} \equiv (\Gamma^{0+})^{s_0+\frac{1}{2}}\cdots(\Gamma^{k+})^{s_k+\frac{1}{2}}\zeta, \quad s_i = \pm\frac{1}{2}, \ \boldsymbol{s} \equiv (s_0,\cdots,s_k).$$

Now we can iteratively write down Γ-matrices. Starting from $d=2$, where

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then we can define the Γ-matrices in $d = 2k+2$ using

$$\Gamma^\mu \equiv \gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ (\mu = 0,\cdots,d-3), \quad \Gamma^{d-2} \equiv \mathbb{1} \otimes \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \Gamma^{d-1} \equiv \mathbb{1} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here γ^μ are $2^k \times 2^k$ Dirac matrices in $2k$-dimensions, and $\mathbb{1}$ is a $2^k \times 2^k$ identity matrix.

For the generators of Lorentz algebra, we can define

$$\Sigma^{\mu\nu} \equiv \frac{1}{4}[\Gamma^\mu, \Game^\nu],$$

which satisfy $\mathfrak{so}(1,d-1)$ algebra

$$[\Sigma^{\mu\nu},\Sigma^{\rho\sigma}] = \eta^{\nu\rho}\Sigma^{\mu\sigma} + \eta^{\mu\sigma}\Sigma^{\nu\rho} - \eta^{\nu\sigma}\Sigma^{\mu\rho} - \eta^{\mu\rho}\Sigma^{\nu\sigma}.$$

In particular, generators $\Sigma^{2a,2a+1}(a=0,\cdots,k)$ commutes. Note

$$\frac{1}{4}[\Gamma^0,\Gamma^1]=\Gamma^{0+}\Gamma^{0-}-\frac{1}{2},$$
$$-\frac{\mathrm{i}}{4}[\Gamma^{2m},\Gamma^{2m+1}]=\Gamma^{m+}\Gamma^{m-}-\frac{1}{2},\quad m=1,\cdots,k.$$

We can hereby define

$$S_a\equiv\Gamma^{a+}\Gamma^{a-}-\frac{1}{2},\quad a=0,\cdots k,$$

as the generators of the Cartan subalgebra of $\mathfrak{so}(1,d-1)$. Spinors $\zeta^{\boldsymbol{s}}$ are simultaneous eigenstates of S_a with eigenvalue s_a, so $\zeta^{\boldsymbol{s}}$ is also a 2^{k+1}-D rep of $\mathfrak{so}(1,d-1)$.

In even dimensional spacetime, the Dirac spinor is a reducible rep. To see this, consider defining

$$\Gamma=\mathrm{i}^{-k}\Gamma^0\Gamma^1\cdots\Gamma^{d-1}=2^{k+1}S_0S_1\cdots S_k,$$

it satisfies

$$(\Gamma)^2=1,\quad\{\Gamma,\Gamma^\mu\}=0,\quad[\Gamma,\Sigma^{\mu\nu}]=0.$$

The state $\zeta^{\boldsymbol{s}}$ is an eigenstate of Γ with eigenvalue $2^k s_0 s_1\cdots s_k$. Thus the eigenvalue under Γ is $+1$ for an even number of $s_i=-1/2$, for the opposite case the eigenvalue is -1. Actually, the states with $+1$ eigenvalue form chiral/left-handed Weyl spinors with dinmension 2^k, while -1 eigenvalue gives rise to anti-chiral/right-handed Weyl spinors. One can also define projection operators

$$P_\pm\equiv\frac{1\pm\Gamma}{2},$$

then two kinds of Weyl spinors are produced by

$$\zeta_\pm=P_\pm\zeta.$$

In terms of jargons in the representation theory of $\mathfrak{so}(1,d-1)=D_{k+1}$, the decomposition of Dirac spinor into chiral and anti-chiral Weyl spinors in 4D spacetime is

$$\text{Dirac}=(\mathbf{2},\mathbf{1})_{\text{chiral}}\oplus(\mathbf{1},\mathbf{2})_{\text{anti-chiral}}.$$

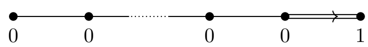
In 6D, it's

$$\text{Dirac}=\mathbf{4}_{\text{chiral}}\oplus\bar{\mathbf{4}}_{\text{anti-chiral}}.$$

In 10D, it's

$$\text{Dirac}=\mathbf{16}_{\text{chiral}}\oplus\bar{\mathbf{16}}_{\text{anti-chiral}}.$$

Now let's consider spinors in odd dimensional spacetime, denote the dimension as $d=2k+3$.In this case, we just need to add $\Gamma^{d-1}=\pm\mathrm{i}\Gamma$ as an additional Dirac matrix to those matrices for $d=2k+2$ case. Here Dirac spinor is a irrep with dimension 2^{k+1}. This is because for $\mathfrak{so}(1,2k+2)=B_{k+1}$, there's only one spinor rep with the heighest weight



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and its dimension is 2^{k+1}.

In some spacetime dimensions, we can introduce a notion of "real spinors" called Majorana spinors. Consider the case for $d = 2k+2$ first, here $\Gamma^2, \Gamma^4, \cdots, \Gamma^{d-2}$ are imaginary while the rest are real. Define

$$B_1 \equiv \Gamma^2\Gamma^4 \cdots \Gamma^{d-2}, \quad B_2 = \Gamma B_1,$$

then they satisfy

$$B_1\Gamma^\mu B_1^{-1} = (-1)^k \Gamma^{\mu *}, \quad B_2\Gamma^\mu B_2^{-1} = (-1)^{k+1}\Gamma^{\mu *}$$

and

$$B_i \Sigma^{\mu\nu} B_i^{-1} = \Sigma^{\mu\nu *}.$$

From the last equation, we find ζ and $B_i^{-1}\zeta$ transform in the same way under the Lorentz group, sine

$$B_i \Sigma^{\mu\nu} B_i^{-1} \zeta^* = \Sigma^{\mu\nu *}\zeta^*, \quad i = 1, 2.$$

Hence a Dirac spinor is the conjugate of itself.

For Weyl spinors, consider

$$B_1 \Gamma B_1^{-1} = B_2 \Gamma B_2^{-1} = (-1)^k \Gamma^*$$

for even k (here $d = 4n+2$), so the conjugation of B_i doesn't change the eigenvalues of Γ and Weyl spinors are also self-conjugate. For odd k in $d = 4n$, eigenvalues of Γ goes to its negative and two Weyl spinors are conjugate to each other.

Now we can choose a B_i and define the Majorana spinors as spinors satisfying

$$\zeta^* = B_i \zeta$$

or equivalently

$$\zeta = B_i^{-1}\zeta^*.$$

Taking a conjugate, we have

$$\zeta = B_i^* \zeta^* = B_i^* B_i \zeta.$$

Hence consistency requires $B_i^* B_i = 1$. Anyway, we can compute to verify

$$B_1^* B_1 = (-1)^{k(k-1)/2}, \quad B_2^* B_2 = (-1)^{k(k+1)/2},$$

so the Majorana condition using B_1 is only possible when

$$k = 0, 1 \mod 4.$$

If we use B_2, then Majorana condition would be

$$k = 0, 3 \mod 4.$$

Actually, the Majorana condition can only be imposed on Weyl spinors for even $k(d = 8n + 2)$. A spinor simultaneously satisfies Majorana and Weyl conditions is called a Majorana-Weyl spinor. For odd dimensions $d = 2k+3$, the last Dirac matrix is $\Gamma^{d-1} = \pm \mathrm{i}\Gamma$, so Majorana conditions only hold for B_2, again corresponds to $k = 0, , 3 \mod 4$. In short, Majorana spinors only exist for $d = 0, 1, 2, 3, 4 \mod 8$, and Majorana-Weyl spinors only exists for $d = 2 \mod 8$.

Finally, we summarize the existence of different kinds of spinors in any spacetime dimensions, which has a periodicity of 8. The minimal number of real components N of a spinor in that dimension is also listed.

d	Weyl	Majorana	Majorana-Weyl	N
2	√	√	√	1
3	—	√	—	2
4	√	√	—	4
5	—	—	—	8
6	√	—	—	8
7	—	—	—	16
8	√	√	—	16
9	—	√	—	16
10	√	√	√	16
11	—	√	—	32