School of Physics, Peking University

Lie Groups and Lie Algebras

Yiming Teng

Peking University

E-mail: tengyiming@stu.pku.edu.cn

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Contents

1 Groups, Rings and Fields

Definition 1.1 (Group). *A group G is defined as a set G with a binary operation $"\cdot" : G \times G \to G$ satisfying:*

1. *Associativity:* $\forall a, b, c \in G,\ a \cdot (b \cdot c) = (a \cdot b) \cdot c$
2. *There exists an unique identity element $e \in G$ such that $\forall a \in G,\ a \cdot e = e \cdot a = a$*
3. *$\forall a \in G$, there exists an unique inverse element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$*

Generally, the binary operation on G is not communtative. If the group operation is communtative for all elements in G, then G is an **abelian group**.

For additive groups, we may use "+" instead of "$\cdot$" to denote group operation. Meanwhile, sometimes we just omit the symbol of group operation:

$$g \cdot h \Rightarrow gh$$

Definition 1.2 (Congujacy Class). *$\forall a \in G$, its congujacy class is defined to be the set $g(a)$:*

$$g(a) := \{h^{-1}ah | h \in G\}$$

Definition 1.3 (Subgroup). *A subgroup of G is a subset $H \subset G$ and group operation on G is also the group operation on H.*

Definition 1.4 (Normal Subgroup). *The normal subgroup N of group G is defined to be the set of $h \in G$ such that $\forall g \in G,\ ghg^{-1} \in N$*

Definition 1.5 (Quotient Group). *For any normal subgroup N of G, the quotient group G/N is defined to be the set of cosets*

$$aN = \{a \cdot h | h \in N\}, \quad a \in G$$

The group operation on G/N is defined as follows:

$$(a_1N) \cdot (a_2N) = (a_1 \cdot a_2)N$$

Definition 1.6 (Center). *The center of group G is defined to be the set of elements in G which commute with all elements of G.*

Definition 1.7 (Product). *The product group $G \times H$ is defined to be the set of ordered pairs (g, h), where $g \in G,\ h \in H$. The group operation on $G \times H$ is defined as*

$$(g, h)(g', h') = (gg', hh')$$

Definition 1.8 (Homomorphism/Isomorphism). *A linear map between two groups $\phi : G \to H$ is a homomorphism if*

$$\forall g, h \in G,\ \phi(gh) = \phi(g)\phi(h)$$

If the inverse of a homomorphism is also a homomorphism, then the map is called an isomorphism.

For a homomorphism, it's easy to verify that

$$\phi(e_G) = e_H \quad \phi(g^{-1}) = \phi(g)^{-1}$$

We can treat isomorphic groups as the same, and classification of groups is usually up to isomorphism.

Definition 1.9 (Group Action). *The group action of G on a set X is a map $\psi : G \times X \to X$ such that $\forall a, b \in G, x \in X$, we have:*

1. $\psi(e, x) = x$

2. $\psi(a, \psi(b, x)) = \psi(a \cdot b, x)$

If G is a $n \times n$ matrix group, then elements of G are automatically actions on n dimensional vector space.

Here are some example of groups:

Example 1.1.

1. *Cyclic group $\mathbb{Z}_N$. Its identity element is denoted by $e = 1$ and oother elements are generated by an element a with $a^N = 1$. That is, the group is:*

$$\mathbb{Z}_N = \{1, a, a^2, ..., a^{N-1}\}$$

   *As a finite group, the number N is called the **order** of group $\mathbb{Z}_N$.*

2. *Addition Group of integers $\mathbb{Z}$. Group operation is denoted by "+", the identity element is 0 and the whole group is generated by the element 1.*

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$$

   Apparently it's an infinite discrete group.

3. *Additive group on rational numbers $\mathbb{Q}$: We can define suitbale topological structures on $\mathbb{Q}$ making it a continuous group, but $\mathbb{Q}$ is not smooth and thus not a Lie group.*

4. *Additive group on $\mathbb{R}$: A Lie group.*

Definition 1.10 (Ring)**.** *A ring R is defined to be a set with two binary operations $''\cdot''$: $R \times R \to R$ and $''+''$: $R \times R \to R$ satisfying:*

1. *R is an abelian group under $''+''$.*

2. *R is monoid under $''\cdot''$*

3. *Distribution law:*

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

For example, set of integers $\mathbb{Z}$ and set of all $n \times n$ real matrices $M_n(\mathbb{R})$ can form a ring.

Definition 1.11 (Field)**.** *A field F is a ring whose operation " $\cdot$ " form an abelian group structure on $F - \{0\}$.*

2 Matrix Groups

Definition 2.1 (Matrix Group). *A matrix group is a set of invertible groups and the group operation is matrix multiplication, inverse elements are inverse matrices and identity element is identity matrix I.*

Definition 2.2 (Dimension). *The dimension of a matrix group $\dim G$ can be defiend as the number of independent real parameters.*

Here are some example:

Example 2.1.

1. *General linear group on field F: $\mathsf{GL}(n;F)$ is defined as the set of all $n \times n$ invertible matrices whose elements are in F.*

   Since restrication on $\det M \neq 0$ doesn's provide an equation, it won't eliminate free parameters and we have:

   (a) $\dim \mathsf{GL}(n;\mathbb{R}) = n^2$

   (b) $\dim \mathsf{GL}(n;\mathbb{C}) = 2n^2$

   Obviously, all matrix groups are subgroups of $\mathsf{GL}(n;F)$.

2. *Special linear group on field F: $\mathsf{SL}(n;F)$ is defined as all invertible $n \times n$ matrices on F with determinants equal to 1. Since $\det M = 1$ reduces one degree of freedom for real matrices and two degree of freedom for complex matrices (equations on real part and imaginary part), thus:*

   (a) $\dim \mathsf{SL}(n;\mathbb{R}) = n^2 - 1$

   (b) $\dim \mathsf{SL}(n;\mathbb{C}) = 2n^2 - 2$

3. *Unitary group: $\mathsf{U}(n)$ is defined to be the set of all $n \times n$ unitary matrices, i.e.*

   $$M^\dagger M = I$$

   For the dimension of $\mathsf{U}(n)$, notice that if we decompose M as

   $$M = A + \mathrm{i}B,\ A, B \in M_n(\mathbb{R})$$

   Then

   $$M^\dagger M = (A^T - \mathrm{i}B^T)(A + \mathrm{i}B) = A^T A + \mathrm{i}A^T B - \mathrm{i}B^T A + B^T B$$

   Then $M^\dagger M = I$ indicating $A^T A + B^T B = I$ and $A^T B = B^T A$. $A^T B = B^T A$ indicating the real part and imaginary part are related, thus although $M^\dagger M = I$ gives $2n^2$ real equations, only n^2 equations are independent. Therefore, $\mathsf{U}(n)$ has n^2 independent parameters and

   $$\dim \mathsf{U}(n) = n^2$$

Notice that the determinant of $M \in \mathsf{U}(n)$ is indefinite: $M^\dagger M = I$ indicating

$$\overline{\det M} \det M = 1 \Rightarrow |\det M|^2 = 1$$

Thus we only know $\det M$ is a complex number with unit magnitude.

4. *Special unitary group: $\mathsf{SU}(n)$ is set of elements in $\mathsf{U}(n)$ whose determinant is 1. Obviously we have*

$$\dim \mathsf{SU}(n) = n^2 - 1$$

Elements of $\mathsf{U}(n)$ or $\mathsf{SU}(n)$ can be viewed as actions on $\mathbb{C}^n$ preserving $\|x\|^2$.

For instance, $\mathsf{SU}(2)$ is a 3 dimensional group whose elements can be written as

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}, \quad a^2+b^2+c^2+d^2 = 1$$

Thus we can topologically indentify $\mathsf{SU}(2)$ and S^3. We can allso use 3 parameters to represent elements of $\mathsf{SU}(2)$, according the following substitutions:

$$a = \cos\alpha \quad \beta = \sin\alpha\cos\beta \quad c = \sin\alpha\sin\beta\cos\gamma \quad d = \sin\alpha\sin\beta\sin\gamma$$

5. *Indefinite unitary group and indefinite special unitary group: $\mathsf{U}(p,q)$ is defined to be set of $(p+q)\times(p+q)$ matrices, preserving following metric:*

$$g = \mathrm{diag}(\underbrace{1,...,1}_{p},\underbrace{-1,...,-1}_{q}) \qquad M^\dagger g M = g$$

$\mathsf{SU}(p,q)$ *is set of elements in $\mathsf{U}(p,q)$ with determinants equal to 1.*

6. *Orthogonal group: $\mathsf{O}(n)$ is defined to be set of $n\times n$ real orthogonal matrices. That is, $\forall M \in \mathsf{O}(n)$ satisfies*

$$M^T M = I$$

Since $M^T M$ is symmetric, thus $M^T M = I$ only gives us $n(n+1)/2$ independent equations and we have

$$\dim \mathsf{O}(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

Unlike $\mathsf{U}(n)$, $M^T M = I$ gives much stronger restrcation on the determinants of matrices in $\mathsf{O}(n)$: $\forall M \in \mathsf{O}(n)$ is real matrix, thus $\det M \in \mathbb{R}$ and

$$M^T M = I \Rightarrow (\det M)^2 = 1 \Rightarrow \det M = \pm 1$$

Therefore, $\mathsf{U}(n)$ is composed by two disconnected parts, and only the part with $\det M = +1$ contains identity element so that it forms a subgroup $\mathsf{SO}(n)$. The we denote the subset of $\mathsf{O}(n)$ with $\det M = -1$ as $\mathsf{O}(n)^-$, and we claim $\forall M \in \mathsf{O}(n)^-$ can be represented as $M = RN$, where $N \in \mathsf{SO}(n)$ and R is spatial reflection.

According to the definition of $\mathsf{O}(n)$*, it's clear that action of* $\mathsf{O}(n)$ *on* n *dimensional Euclidean space* $\mathbb{R}^n$ *preserves Euclidean norm. Action of* $\mathsf{SO}(n)$ *corresponds to rotation in* $\mathbb{R}^n$*, while action of* $\mathsf{O}(n)^-$ *corresponds to rotation combined with reflection.*

Since restricting $\det M = +1$ *on* $\mathsf{O}(n)$ *doesn't eliminate a continuous degree of freedom, thus*

$$\dim \mathsf{SO}(n) = \dim \mathsf{O}(n) = \frac{n(n-1)}{2}$$

As a trivial example, elements of $\mathsf{SO}(2)$ *can be represented as*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi)$$

Then it's clear that $\mathsf{SO}(2) \cong \mathsf{U}(1) \cong S^1$*.*

Relation between $\mathsf{SO}(3)$ *and* $\mathsf{SU}(2)$ *is more interesting. Recall that elements of* $\mathsf{SU}(2)$ *can be represented as*

$$\begin{pmatrix} a+b\mathrm{i} & c+d\mathrm{i} \\ -c+d\mathrm{i} & a-b\mathrm{i} \end{pmatrix}, \quad a^2+b^2+c^2+d^2=1$$

While elements of $\mathsf{SO}(3)$ *can be written as*

$$\begin{pmatrix} 1-2c^2-2d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & 1-2b^2-2d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & 1-2b^2-2c^2 \end{pmatrix}, \quad a^2+b^2+c^2+d^2=1$$

We can verify $\mathsf{SU}(2)$ *and* $\mathsf{SO}(3)$ *are homomorphic, but they aren't isomorphic: Clearly, parameter tuples* (a,b,c,d) *and* $(-a,-b,-c,-d)$ *correspond to the same element in* $\mathsf{SO}(3)$ *but different elements in* $\mathsf{SU}(2)$*. Thus the homomorphism* $\phi : \mathsf{SU}(2) \to \mathsf{SO}(3)$ *is 2 to 1 map. Actually, it can be proved that*

$$\mathsf{PSU}(2) := \mathsf{SU}(2)/\mathbb{Z}_2 \cong \mathsf{SO}(3)$$

More generally, let's consider $\mathsf{PSU}(N) := \mathsf{SU}(N)/\mathbb{Z}_N$*.* $\mathbb{Z}_N$ *is defined to be*

$$\mathbb{Z}_N = \{1, e^{2\pi\mathrm{i}/N}I, ..., e^{2\pi\mathrm{i}(N-1)/N}I\}$$

Then $\mathbb{Z}_N$ *is clearly the center of* $\mathsf{SU}(N)$ *and elements of* $\mathsf{SU}(N)/\mathbb{Z}_N$ *are cosets labelled by* $a \in \mathsf{SU}(N)$*, i.e.* $a \in \mathsf{SU}(N)$ *mod out equivalence relation below:*

$$a \sim e^{2\pi\mathrm{i}/N}a$$

7. *Indefinite orthogonal group (Pseudo orthogonal group):* $\mathsf{O}(p,q)$ *is defined to be set of* $(p+q)\times(p+q)$ *matrices satisfying*

$$MgM^T = g, \quad g = \mathrm{diag}(\underbrace{1,...,1}_{p}, \underbrace{-1,...,-1}_{q})$$

$\mathsf{SO}(p,q)$ *is set of elements in* $\mathsf{O}(p,q)$ *with determinant is 1. Lorentz group is exactly* $\mathsf{O}(1,3)$*, preserving Minkowski metric* $\mathrm{diag}(1,-1,-1,-1)$*.*

8. *Euclidean group:* $\mathsf{E}(n)$*, also denoted by* $\mathsf{ISO}(n)$*, is defined to be the group of saptial rotation and translation in* $\mathbb{R}^n$*. Elements of* $\mathsf{E}(n)$ *can be expressed as an* $(n+1)\times(n+1)$ *matrix:*

$$\begin{pmatrix} \mathbf{R}_{n\times n} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

Here $\mathbf{R}_{n\times n} \in \mathsf{O}(n)$ *and* $\mathbf{a}$ *is n dimensional column vector,* $\mathbf{0}$ *is n dimensional zero row vector.* $\mathsf{E}(n)$ *acts on elements of* $(n+1)$ *dimensional vector space with the following form:*

$$\mathbf{v} = (x_1, ..., x_n, 1)^T$$

Direct computation shows

$$\begin{pmatrix} \mathbf{R}_{n\times n} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_{n\times n}\mathbf{x} + \mathbf{a} \\ 1 \end{pmatrix}$$

Clearly, $\mathsf{E}(n)$ *is a subgroup of* $\mathsf{GL}(n+1;\mathbb{R})$*. Besides,* $\mathsf{O}(n)$ *is a subgroup of* $\mathsf{E}(n)$*. If we introduce translation group* $\mathsf{T}(n)$ *whose elements can be expressed as*

$$\begin{pmatrix} \mathbf{I}_{n\times n} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

Then the intutiation that $\mathsf{O}(n)$ *is* $\mathsf{E}(n)$ *excluding translation leads to the relation below:*

$$\mathsf{O}(n) \cong \mathsf{E}(n)/\mathsf{T}(n)$$

Actually, $\mathsf{T}(n)$ *is a normal subgroup of* $\mathsf{E}(n)$*, we can verify it by direct calculation: For any element* g *of* $\mathsf{E}(n)$*, we have*

$$g = \begin{pmatrix} \mathbf{R}_{n\times n} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

It's inverse is

$$g^{-1} = \begin{pmatrix} \mathbf{R}^{-1}_{n\times n} & -\mathbf{R}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

For any element of $\mathsf{T}(n)$*, we express it as*

$$h = \begin{pmatrix} \mathbf{I}_{n\times n} & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix}$$

Then

$$\begin{aligned} ghg^{-1} &= \begin{pmatrix} \mathbf{R} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{R} & \mathbf{a}+\mathbf{R}\mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{R}\mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \in \mathsf{T}(n) \end{aligned}$$

9. *Poincaré group:* $\mathsf{ISO}(1, n-1)$ *is defined to be the indefinite version of* $\mathsf{E}(n)$. *For* $n = 4$, *we have*

$$\dim \mathsf{ISO}(1,3) = \dim \mathsf{O}(1,3) + \dim \mathsf{T}(4) = 6 + 4 = 10$$

10. *Symplectic group:* $\mathsf{Sp}(2n; F)$ *is defined to be the set of* $2n \times 2n$ *matrices* M *whose elements belong to field* F, *satisfying*

$$M\Omega M^T = \Omega, \quad \Omega = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{n\times n} \\ -\mathbf{I}_{n\times n} & \mathbf{0} \end{pmatrix}$$

We can express elements of $\mathsf{Sp}(2n)$ *as blocked matrices:*

$$M = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

Then group action preserving symplectic form indicates

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{D}^T \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$$

Thus we have following independent matrix equations:

$$\begin{aligned} \mathbf{A}\mathbf{B}^T - \mathbf{B}\mathbf{A}^T &= 0 \\ \mathbf{A}\mathbf{D}^T - \mathbf{B}\mathbf{C}^T &= \mathbf{I} \\ \mathbf{C}\mathbf{D}^T - \mathbf{D}\mathbf{C}^T &= 0 \end{aligned}$$

The first and the third equation give $n(n-1)/2$ *scalar equations respectively, and the second eqaution gives us* n^2 *independent scalar equations. Therefore, the dimension of a symplectic group is*

$$\dim \mathsf{Sp}(2n) = 4n^2 - n(n-1) - n^2 = n(2n+1)$$

$\mathsf{Sp}(2n; \mathbb{R})$ *naturally acts on* $\mathbb{R}^{2n}$, *whose coordinates is usually denoted as* $(q^1, ..., q^n, p_1, ..., p_n)$. *We can view* $\mathsf{Sp}(2n; \mathbb{R})$ *as a group composed by operations on* $\mathbb{R}^{2n}$ *preserving Poisson brackets:*

$$\{f, g\} = \sum_{\ell=1}^{n} \left(\frac{\partial f}{\partial q^\ell} \frac{\partial g}{\partial p_\ell} - \frac{\partial f}{\partial p_\ell} \frac{\partial g}{\partial q^\ell} \right)$$

Through the definition of $\mathsf{Sp}(2n, F)$, *we immediately know the determinant of a symplectic group element must be* $+1$ *or* -1. *However, it can be proved that* $\forall M \in \mathsf{Sp}(2n, F)$, $\det M = +1$.

11. *Unitary symplectic group, also called compact symplectic group:* $\mathsf{USp}(2n)$ *is defiend to be* $\mathsf{Sp}(2n; \mathbb{C}) \cap \mathsf{SU}(2n)$, *therefore any element* M *of* $\mathsf{USp}(2n)$ *simultaneously satisfies*

$$M\Omega M^T = \Omega, \qquad M^\dagger M = I$$

Its dimension is

$$\dim \mathsf{USp}(2n) = 2n(2n+1) - 2n^2 - n = n(2n+1)$$

For the proof, please refer to Homework 1.

3 Subgroups and Some Interesting Relations in Matrix Groups

At the beginning, let's review the definition of direct product: For two groups G and H, their direct product $G \times H$ is defined to be the set $\{(g,h)|g \in G,\ h \in H\}$ and group multiplication is defined to be $(g,h)\cdot(g',h') = (gg',hh')$. Based on group product, we have the following definition[1]:

Definition 3.1. *Consider a group K with subgroups G and H. Then $G \times H \cong K$ if:*

1. *Map $\phi : G \times H \to K,\ (g,h) \mapsto gh$ is isomorphism.*
2. *$\forall g \in G, h \in H,\ gh = hg$ in K.*
3. *$G \cap H$ is a trivial subgroup of K, i.e. $G \cap H = \{e_K\}$.*

If ϕ in the first criteria is not an isomorphism but a homomorphism, then we can only deduce $G \times H$ is isomorphic to a subgroup of K, i.e. $G \times H \subset K$.

Propsition 3.1. *If $G \times H \cong K$, then G and H are normal subgroups of K.*

Proof. When viewed as subgroups of K, we can denote elements of G, H as (g, e_H) and (e_G, h). For any element of K, it can be denoted as (a,b), where $a \in G$ and $b \in H$. Thus we have

$$(a,b)(g,e_H)(a^{-1},b^{-1}) = (aga^{-1},e_H) \in G$$
$$(a,b)(e_G,h)(a^{-1},b^{-1}) = (e_G,bhb^{-1}) \in H$$

□

Here we have some propsitions (as well as some example):

Propsition 3.2.

$$\mathsf{GL}(m;F) \times \mathsf{GL}(n-m;F) \subset \mathsf{SL}(n;F)$$
$$\mathsf{SL}(m;F) \times \mathsf{SL}(n-m;F) \subset \mathsf{SL}(n;F)$$
$$\mathsf{U}(m) \times \mathsf{U}(n-m) \subset \mathsf{U}(n)$$
$$\mathsf{SU}(m) \times \mathsf{SU}(n-m) \subset \mathsf{SU}(n)$$

Proof. Juts notice the elements of the product groups on the LHS can be represented as the blocked diagonal matrices below:

$$\begin{pmatrix} \mathbf{M}_{n\times n} & 0 \\ 0 & \mathbf{N}_{(n-m)\times(n-m)} \end{pmatrix}$$

Appraently this set of matrices can be viewed as elements in the corresponding group with dimension n. □

Propsition 3.3. $\mathsf{SL}(n;\mathbb{R}) \subset \mathsf{GL}(n;\mathbb{R})$

[1] We can make an analogy between these criterion and direct sum between vector spaces.

Proof. Obvious. □

Propsition 3.4. *If n is odd, then* $\mathsf{SL}(n;\mathbb{R}) \times \mathsf{GL}(1;\mathbb{R}) \cong \mathsf{GL}(n;\mathbb{R})$.

Proof. First of all, if n is odd, then we can construct a bijection $\phi : \mathsf{SL}(n;\mathbb{R}) \times \mathsf{GL}(1;\mathbb{R}) \to \mathsf{GL}(n;\mathbb{R})$ as follows:

$$\phi : (M, aI) \mapsto aM$$

Secondly, when embeded in $\mathsf{GL}(n;\mathbb{R})$, elements of $\mathsf{GL}(1;\mathbb{R})$ is aI, where $a \in \mathbb{R} - \{0\}$. Clearly, every matrices in $\mathsf{SL}(n;\mathbb{R})$ commute with aI.

Finally, we have to check if $\mathsf{SL}(n;\mathbb{R}) \cap \mathsf{GL}(1;\mathbb{R})$ trivial in $\mathsf{GL}(n;\mathbb{R})$. For odd n, the only element of $\mathsf{SL}(n;\mathbb{R}) \cap \mathsf{GL}(1;\mathbb{R})$ is I, and it's apparently trivial.

If n is even, then things will be different: On one hand, $\phi : (M; aI) \mapsto aM$ will no longer cover $\mathsf{GL}(n;\mathbb{R})$, as $\det(aM) = a^n$ is always postive. On the other hand, $\mathsf{SL}(n;\mathbb{R}) \cap \mathsf{GL}(1;\mathbb{R})$ won't be trivial, since $\det(-I) = \det(I) = 1$ when n is even and thus $\mathsf{SL}(n;\mathbb{R}) \cap \mathsf{GL}(1;\mathbb{R}) = \{+I, -I\}$. □

Propsition 3.5. $\mathsf{SU}(n) \subset \mathsf{U}(n)$

Proof. Obvious. □

Propsition 3.6. $\mathsf{SU}(n) \times \mathsf{U}(1) \not\cong \mathsf{U}(n)$, *while* $(\mathsf{SU}(n) \times \mathsf{U}(1))/\mathbb{Z}_n \cong \mathsf{U}(n)$.

Proof. First of all, elements of $\mathsf{SU}(n)$ and $\mathsf{U}(1)$ are clearly commutative in $\mathsf{U}(n)$, thus the second criteria is satisfied.

However, if we consider map $\phi : (M, e^{\mathrm{i}\theta}) \mapsto e^{\mathrm{i}\theta}M$ between $\mathsf{SU}(n) \times \mathsf{U}(1)$ and $\mathsf{U}(n)$, then we will find that it's not bijective: For any matrix $S \in \mathsf{U}(n)$, we denote its determinant as $e^{\mathrm{i}\Psi}$. Thus there exists a matrix $Q \in \mathsf{SU}(n)$ such that

$$S = e^{\mathrm{i}\psi}Q, \quad e^{\mathrm{i}n\psi} = e^{\mathrm{i}\Psi}$$

Thus there are actually n elements correspond to a single element in $\mathsf{U}(n)$:

$$(Q, e^{\mathrm{i}\Psi/n}),\ (Q, e^{\mathrm{i}\Psi/n+2\pi\mathrm{i}/n}), ...,\ (e^{\mathrm{i}\Psi/n+2\pi\mathrm{i}(n-1)/n})$$

Therefore the first criteria is violated, not to mention the third one: $\mathsf{SU}(n) \cap \mathsf{U}(1)$ is exactly isomorphic to $\mathbb{Z}_n$:

$$\mathsf{SU}(n) \cap \mathsf{U}(1) = \{I, e^{2\pi\mathrm{i}/n}, e^{4\pi\mathrm{i}/n}, ..., e^{2(n-1)\pi\mathrm{i}/n}\} \cong \mathbb{Z}_n$$

In the quotient group $(\mathsf{SL}(n;\mathbb{R}) \times \mathsf{U}(1))/\mathbb{Z}_n$, we have cosets as group elements and there's a one-to-one correspondence between $S \in \mathsf{U}(n)$ and

$$(Q, e^{\mathrm{i}\Psi/n})\mathbb{Z}_n = \{(Q, e^{\mathrm{i}\Psi/n}),\ (Q, e^{\mathrm{i}\Psi/n+2\pi\mathrm{i}/n}), ...,\ (e^{\mathrm{i}\Psi/n+2\pi\mathrm{i}(n-1)/n})\} \in (\mathsf{SL}(n;\mathbb{R}) \times \mathsf{U}(1))/\mathbb{Z}_n$$

There are two other ways taking quotient group: First, we can take $\mathsf{U}(1)/\mathbb{Z}_n$, in this way we identify $e^{\mathrm{i}\theta}$ and $e^{\mathrm{i}\theta+e\pi\mathrm{i}\ell/n}$ and clearly $\mathsf{SU}(n) \times (\mathsf{U}(1)/\mathbb{Z}_n) \cong \mathsf{U}(n)$. Or we can take $\mathsf{PSU}(n) = \mathsf{SU}(n)/\mathbb{Z}_n$, in this way $\mathsf{PSU}(n) \cap \mathsf{U}(1)$ is obviously trivial in $\mathsf{U}(n)$, and $(Q, e^{\mathrm{i}\Psi/n})$, $(Q, e^{\mathrm{i}\Psi/n+2\pi\mathrm{i}/n}), ...,$ $(e^{\mathrm{i}\Psi/n+2\pi\mathrm{i}(n-1)/n})$ will again be identified, making there exist bijection between $\mathsf{PSU}(n) \times \mathsf{U}(1)$ and $\mathsf{U}(n)$, thus $\mathsf{PSU}(n) \times \mathsf{U}(1) \cong \mathsf{U}(n)$. □

Propsition 3.7. $(\mathsf{SU}(m)\times\mathsf{SU}(n-m)\times\mathsf{U}(1))/\mathbb{Z}_{\mathrm{lcm}(m,n-m)}\subset\mathsf{SU}(n)$

Proof. Using blocked diagonal matrices we can immediately know $\mathsf{SU}(m)\times\mathsf{SU}(n-m)$ is subgroup of $\mathsf{SU}(n)$. However, elements of $(\mathsf{SU}(m)\times\mathsf{SU}(n-m))\cap\mathsf{U}(1)$ have the following form:

$$\mathrm{diag}(\underbrace{e^{2\pi\mathrm{i}k/m},...,e^{2\pi\mathrm{i}k/m}}_{m},\underbrace{e^{-2\pi\mathrm{i}k/(n-m)},...,e^{-2\pi\mathrm{i}k/(n-m)}}_{n-m}),\quad k\in[0,\mathrm{lcm}(m,n-m))$$

Thus it's clear $\mathsf{SU}(m)\times\mathsf{SU}(n-m)\times\mathsf{U}(1)$ is not a well-defined subgroup of $\mathsf{SU}(n)$.

But $(\mathsf{SU}(m)\times\mathsf{SU}(n-m)\times\mathsf{U}(1))/\mathbb{Z}_{\mathrm{lcm}(m,n-m)}$. We can view $\mathbb{Z}_{\mathrm{lcm}(m,n-m)}$ as the cyclic group generated by

$$\mathrm{diag}(\underbrace{e^{2\pi\mathrm{i}/m},...,e^{2\pi\mathrm{i}/m}}_{m},\underbrace{e^{-2\pi\mathrm{i}/(n-m)},...,e^{-2\pi\mathrm{i}/(n-m)}}_{n-m})$$

Taking $\mathsf{U}(1)/\mathbb{Z}_{\mathrm{lcm}(m,n-m)}$ will identify $k\sim k+1$. $\square$

Proposition 3.6 is of significance in physics: The gauge group of standard model is $\mathsf{SU}(3)\times\mathsf{SU}(2)\times\mathsf{U}(1)$, and now we know from mathematics that it's not a subgroup of the proposed GUT gauge group $\mathsf{SU}(5)$, thus $\mathsf{SU}(5)$ GUT theory is falsified mathematically.

4 Basic Topology and Topological Properties of Lie Groups

Definition 4.1 (Topological Space)**.** *A topological space is a set X with a collection of open set $\mathscr{T}=\{U_i\}$ satisfying:*

1. *$\varnothing$ and X itself belong to $\mathscr{T}$.*
2. *For any subcollection (finite or infinite) $S=\{U_j|j\in J\}\subset\mathscr{T}$, $\bigcup_J U_j\in\mathscr{T}$.*
3. *For any finite sbucollection $S=\{U_j|j\in J\}\subset\mathscr{T}$, $\bigcap_J U_j\in\mathscr{T}$.*

We can define various topologies for a set to make it a topological space. For example, in $\mathbb{R}^n$ we can use open cubes to define usual topology by choosing

$$\mathscr{T}=\{(a_1,b_1)\times(a_2,b_2)\times\cdots\times(a_n,b_n)\}$$

We can even define $\mathscr{T}$ is the collection of all susbets of X and get trivial topology. The following definition make it possible for a subset of a topological space to inherit topology from the mother space.

Definition 4.2 (Relative Topology)**.** *Suppose a topological space $(X,\mathscr{T})$ and a subset $Y\subset X$, the relative topology $\mathscr{S}$ on Y is defined to be*

$$\mathscr{S}=\{Y\cap U|U\in\mathscr{T}\}$$

Many sets we interested in are metric spaces, i.e. sets with metrics:

Definition 4.3 (Metric Space)**.** *A metric space is a set X with a metric map $d : X \times X \to \mathbb{R}$, satisfying:*

1. $\forall x, y \in X,\ d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$, *the equality holds if and only if* $x = y$.
3. $\forall x, y, z \in X,\ d(x, y) + d(y, z) \geq d(x, z)$

Only in metric spaces can we define discreteness:

Definition 4.4 (Discreteness)**.** *A metric space X is discrete if $\forall x \in X$ and $\forall Y \in X - \{x\}$, $\exists s \in \mathbb{R}^+$ such that $d(x, y) > s$ always holds.*

We can define a rather natural topology on metric spaces:

Definition 4.5 (Metric Topology)**.** *For a metric space (X, d), we define its matric topology by defining open subsets are open balls $U_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\}$ and their unions (finite or infinite).*

Definition 4.6 (Product Topology)**.** *Condiser two topological spaces $(X, \mathscr{T})$ and $(Y, \mathscr{O})$, the product topology on $X \times Y$ is defined by*

$$\mathscr{T} \times \mathscr{O} = \{U^x \times U^y | U^x \in \mathscr{T}, U^y \in \mathscr{O}\}$$

Recall the definition of closed sets in analysis: A subset $Y \subset X$ is closed if and only if any sequence $A_m \in Y$ converges to a single point in Y. Now we can give an equivalent definition via toppogy:

Definition 4.7. *Closed Set A subset $Y \subset X$ is closed is closed if and only if its complement $Y_c = X - Y$ is open.*

For example. whatever the set X is, X itself and $\varnothing$ are both open and closed. For $\mathbb{R}$, if we define its topology by defining all open intervals and their unions to be open, then closed sets in $\mathbb{R}$ are all closed intervals and their finite unions. Note: Intervals like $[a, b)$ are neither open nor closed. On a metric space with metric topology, $\overline{U_\varepsilon(x)} = \{y \in X | d(x, y) \leq \varepsilon\}$ is closed.

Continuity of maps can be redefined through topology too:

Definition 4.8 (Continuity)**.** *A map between two topological spaces $f : X \to Y$ is continuous if and only if for any open set $U_Y \in Y$, $f^{-1}(U_Y)$ is an open set in X.*

In topology, an important equivalence relation is homeomorphism:

Definition 4.9 (Homeomorphism)**.** *Two topological spaces X and Y are homeomorphic if and only if there exists a continuous mao $f : X \to Y$ such that $f^{-1} : Y \to X$ exists and is continuous.*

Usually, people prove two topological spaces are homeomorphic by constructive prooves. However, it's much harder to prove two topological spaces are not homeomorphic and one way to do this is using topological invariants, which we will introduce later.

Definition 4.10 (Topological Groups)**.** *A topological group is a topological space with a group structure, satisfying:*

1. *Group operation* $\cdot : G \times G \to G$ *and inverse* ${}^{-1} : G \to G$ *are continuous.*
2. $G \times G$ *has a product topology.*
3. *Isomorphism of topological groups is also group isomorphism and topollogical homeomorphism.*

A topological group is not necessarily a continuous group, even for point groups we can attach topologies to them, like trivial topology or meric topology. For the latter one, we can consider $\mathbb{Z}_N$ and express it as points on a circle so that we can measure distance between discrete group elements and have a metric topology.

Now we can define matrix Lie groups in topological sense:

Definition 4.11 (Matrix Lie Group, GTM 222, Page 4)**.** *A matrix Lie group* G *is a subgroup of* $\mathsf{GL}(n;\mathbb{C})$ *satisfying for any sequence of matrices* $\{A_m\}$ *in* G *which converges to* A*,* A *is either in* G *or non-invertible. In other word, a matrix Lie group is a closed subgroup of* $\mathsf{GL}(n;\mathbb{C})$*.*

Definition 4.12 (Compactness)**.** *If* $X \subset \mathbb{R}^n$*, then* X *is compact if and only if* X *is closed and bounded. Bounded means there exists* $s \in \mathbb{R}$ *such that* $\forall x, y \in X$*,* $d(x,y) < s$*.*

For example, $\mathbb{R}$ is non-compact, while $S^1 \cong \mathsf{U}(1) \cong \mathsf{SO}(2)$ and $\mathsf{SU}(2) \cong S^3$ are compact. $\mathsf{U}(n), \mathsf{SU}(n), \mathsf{O}(n), \mathsf{SO}(n), \mathsf{USp}(2n)$ are compact, too.

On the contrary, $\mathsf{GL}(n;\mathbb{R}), \mathsf{GL}(n;\mathbb{C}), \mathsf{SL}(n;\mathbb{R}), \mathsf{SL}(n;\mathbb{C}), \mathsf{Sp}(2n;\mathbb{R}), \mathsf{Sp}(2n;\mathbb{C}), \mathsf{O}(1,3)$ are non-compact.

Compactness is of significance in physics: In quantum gauge field theory, gauge group G is unitary only when G is compact.

Definition 4.13 (Connectedness)**.** *A topological space* X *is connected if it cannot be written as* $X = X_1 \cup X_2$*,* X_1, X_2 *are open subsets in* X *and* $X_1 \cap X_2 = \varnothing$*, otherwise* X *is disconnected.*

In practice, we usually check a generally stronger condition: path-connectedness. However, it can be proved for matrix groups that connectedness is equivalent to path-connectedness.

For example, $\mathsf{GL}(n;\mathbb{C}), \mathsf{SL}(n;\mathbb{C}), \mathsf{U}(n), \mathsf{SU}(n), \mathsf{SO}(n)$ are connected, while $\mathsf{O}(n)$ is disconnected: $\mathsf{O}(n) = \mathsf{SO}(n) \cup \mathsf{O}(n)^-$.

Definition 4.14 (Connected Components)**.** *Consider an topological space* X *and fix a point* $x \in X$*, the connected component containing* x *is the union of all connected open sets that contain* x*.*

Propsition 4.1. *If a topological space* X *has finitely many connected components, then each connected component is open and closed.*

For example, $\mathsf{O}(p,q)$ has 4 connected components, $\mathsf{SO}(p,q)$ has 2 connected components.

Definition 4.15 (Simply Connectedness)**.** *A topological space X is simply connected if and only if every loop in X can be continuously shrunk to a point. The definiiton of a loop is a map $f:[0,1]\to X$ satisfying $f(0)=f(1)$.*

For example, $\mathsf{U}(1)$ is not simply connected, and we can further infer for any Lie group G, $G\times\mathsf{U}(1)$ can't be simply connected, due to the $\mathsf{U}(1)$ part. $\mathsf{SU}(2)\cong S^3$ is simply connected, howeverr, since

$$\mathsf{SO}(3)\cong\mathsf{SU}(2)/\mathbb{Z}_2\cong\mathbb{R}P^3$$

Thus $\mathsf{SO}(3)$ is not connected: In $\mathbb{R}P^3$, a point on unit sphere and its antipodal point are indetified, for example, $(1,0,0)$ and $(-1,0,0)$, and a curve connecting the two points is a loop in $\mathbb{R}P^3$. However, it's obvious that the loop can't be shrunk to a point. Moreover, it can be proved that all $\mathsf{SO}(N)$ are not simply connected for $N>1$.

5 Differential Manifold

Definition 5.1 (Differential Manifold)**.** *A n-dimensional differential manifold M is a topological space staisfying:*

1. *M has a colloection of pairs $\{(U_i,\phi_i)\}$, $\{U_i\}$ is a collection of open sets covering M, and $\phi_i:U_i\to\mathbb{R}^n$ is a homeomorphism from U_i to an open subset $U_i'\in\mathbb{R}^n$.*
2. *Given U_i,U_j, $U_i\cap U_j\neq\varnothing$, then the map $\psi_{ij}:=\phi_i\circ\phi_j^{-1}:\phi_j(U_i\cap U_j)\to\phi_i(U_i\cap U_j)$ is infinitely differentiable.*

A pair (U_i,ϕ_i) is called a coordinate chart, ϕ is named as coordinate (map), ψ_{ij} is transition map. An atlas is defined to be the collection of all charts on a manifold.

Suppose we have a map $f:M\to N$, $\dim M=m$, $\dim N=n$, then $\forall p\in M$ is mapped to $f(p)\in N$. We can take a chart (U,ϕ) on M and (V,ψ), then f has a coordinate representation:

$$\psi\circ f\circ\phi^{-1}:\mathbb{R}^m\to\mathbb{R}^n$$

Denote x^μ is coordinate induced by (U,ϕ) on M while y^α is coordinate induced by (V,ψ) on N, then f can be represented as $y^\alpha(x^\mu)$. If $y^\alpha(x^\mu)$ is infinitely differentiable at p, then f is smooth at this point. Moreover, if f is smooth at every point of M, then f is smooth.

Definition 5.2 (Diffeomorphism)**.** *If a map $f:M\to N$ and its inverse are both smooth, then f is a diffeomorphism and M,N are diffeomorphic.*

Definition 5.3 (Product Manifold)**.** *Suppose a m dimensional manifold M with atlas $\{(U_i,\phi_i)\}$ and a n dimensional manifold N with atlas $\{(V_i,\psi_i)\}$, then $M\times N$ is a $(m+n)$ dimensional manifold, a point in $M\times N$ is an ordered pair (p,q), $p\in M$, $q\in N$, its atlas is $\{((U_i,V_j),(\phi_i,\psi_j))\}$. The coordinate map is $(p,q)\to(\phi(p),\psi(q))\in\mathbb{R}^{m+n}$.*

Definition 5.4 (Embeded Submanifold). *M is a n dimensional manifold, a k dimensional embeded submanifold of M is a set $S \subset M$, and $\forall p \in S$, there exists a chart (U, ψ) contains p such that $\phi(S \cap U)$ is the intersecton of a k dimensional plane with $\phi(U)$.*

Now we can talk about Lie groups in manifold terminology.

Definition 5.5 (Lie Group). *A Lie group is a differential manifold G with a group action $\cdot : G \times G \to G$ and inverse $^{-1} : G \to G$, satisfying group axioms and are smooth and differentiable.*

Product group corresponds to product manifold, so $\mathsf{U}(1)^n \cong T^n \cong (S^1)^n$. While for subgroups, we have the following theorem:

Theorem 5.1 (Cartan's Colsed Subgroup Theorem). *Any colsed subgroup $H \subset G$ is a embeded submanifold of G.*

The definition of Lie group homomorphism is the same as homomorphism for topological groups. It can be proved for maps on matix groups, smoothness is equivalent to continuity.

Definition 5.6 (Group Action). *Given a Lie group G, its group action on an manifold M is a smooth map $\cdot : G \times M \to M$ satisfying*

1. *For identity element $e \in G$, $\forall p \in M$, $e \cdot p = p$.*
2. *$\forall g_1, g_2 \in G,\ p \in M, g_1(g_2(p)) = (g_1 g_2)(p)$.*

6 Lie Algebras

6.1 Lie Algebra ABC

Definition 6.1 (Lie Algebra). *A finite dimensional Lie algebra $\mathfrak{g}$ is a finite dimensional vector space together with a map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying*

1. *Bilinearity.*
2. *Antisymmetry: $[X, Y] = -[Y, X],\ \forall X, Y \in \mathfrak{g}$.*
3. *Jacobi Identity:$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.*

Since the bilinear map $[\ ,\ ]$ is not associative, thus $\mathfrak{g}$ isn't a ring. Generators of Lie algebra can be viewed as basis of the vector space $\mathfrak{g}$. If $[X, Y] = 0$, we say X, Y are **commute**. If $\forall X, Y \in \mathfrak{g}$, we have $[X, Y] = 0$, then the Lie algebra is **abelian**.

Definition 6.2 (Real/Complex Lie algebra). *If $\mathfrak{g}$ is a vector space over $\mathbb{R}$, then $\mathfrak{g}$ is a real Lie algebra. If $\mathfrak{g}$ is a vector space over $\mathbb{C}$, then $\mathfrak{g}$ is a complex Lie algebra.*

Definition 6.3 (Complexification). *Take a real Lie algebra $\mathfrak{g}$, we can introduce complex coefficients to basis of $\mathfrak{g}$ and get its complexification $\mathfrak{g}_{\mathbb{C}}$. Elements of $\mathfrak{g}_{\mathbb{C}}$ has the form $v_1 + \mathrm{i} v_2$, where $v_1, v_2 \in \mathfrak{g}$.*

Definition 6.4 (Dimension of Lie Algebra)**.** *The dimension of a Lie algebra is defined to be its dimension as a vector space.*

Definition 6.5 (Lie Subalgebra)**.** *A Lie subalgebra of* $\mathfrak{g}$ *is a subspace of* $\mathfrak{g}$ *in the sense of linear algebra which is also closed under* $[\ ,\ ]$ *operation.*

Definition 6.6 (Center)**.** *Center is defined to be the set of all elements commuting with all elements in* $\mathfrak{g}$*. It's a Lie subalgebra.*

Definition 6.7 (Lie Algebra Homomorphism)**.** *A Lie algebra homomorphism is a linear map* $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *satisfying*

$$\phi([g,h]) = [\phi(g), \phi(h)], \quad \forall g, h \in \mathfrak{g}_1$$

If ϕ is also bijective, then it's an isomorphism.

Definition 6.8 (Direct Sum)**.** *For Lie algebra* $\mathfrak{g}_1, \mathfrak{g}_2$*, their direct sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ *is the direct sum of vector spaces* $\mathfrak{g}_1, \mathfrak{g}_2$*. Lie algebra on* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ *is defined to be*

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2])$$

Definition 6.9 (Structure Constant)**.** *Suppose a* n *dimensional Lie aglebra* $\mathfrak{g}$ *with basis* $X_1, ..., X_N$*, its structure constant is defined through*

$$[X_i, X_j] = \sum_{k=1}^{N} c_{ijk} X_k$$

Since $[\ ,\ ]$ *is anti-symmetric and satisfies Jacobi identity, we have*

1. $c_{ijk} = -c_{jik}$
2. $\displaystyle\sum_{\ell=1}^{N} (c_{jk\ell} c_{i\ell m} + c_{ij\ell} c_{k\ell m} + c_{ki\ell} c_{j\ell m}) = 0$

Here are some examples:

1. $\mathfrak{u}(1)$: It has a single generator, and $\mathfrak{u}(1) \cong \mathbb{R}$. Clearly it's abelian. What's more, acccording to the definition of direct sum, $\displaystyle\bigoplus_{\ell=1}^{n} \mathfrak{u}(1)$ is always abelian.
2. 3-dimensional vectors in $\mathbb{R}^3$ can also be a Lie algebra, Lie bracket is cross product and $c_{ijk} = \varepsilon_{ijk}$.
3. All $n \times n$ real/compelx matrices form a Lie algebra $\mathfrak{gl}(n; \mathbb{R})/\mathfrak{gl}(n; \mathbb{C})$. Lie bracket is defined to be $[X, Y] = XY - YX$.
4. Define $\mathfrak{sl}(n; \mathbb{R})/\mathfrak{sl}(n; \mathbb{C})$ to be all traceless $n \times n$ real/compelx matrices, it's a Lie subalgebra of $\mathfrak{gl}(n; \mathbb{R})/\mathfrak{gl}(n; \mathbb{C})$.

6.2 Lie Groups and Lie Algebras

Intuitively, Lie algebras are infinitesimal generators of Lie groups. And according to the following theorem, there is an intimate connection between Lie algebras and Lie groups:

Theorem 6.1 (Lie's Third Theorem)**.** *For any finite dimensional Lie algebra $\mathfrak{g}$, there exists a connected Lie subgroup G of $\mathsf{GL}(n;\mathbb{C})$ whose Lie algebra is isomorphic to $\mathfrak{g}$.*

However,G is not unique. For example, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, but obviously $\mathsf{SU}(2)$ is not isomorphic to $\mathsf{SO}(3)$.

To derive matrix Lie algebra from matrix Lie group, we'd better introduce matrix exponential first:

Definition 6.10 (Matrix Exponential)**.** *A matrix ponential is defined to be the series:*

$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n \qquad X^0 := I$$

It can be proved e^{tX} is well-defined and has the following properties:

1. $e^0 = I$
2. $e^{aX} e^{bX} = e^{(a+b)X} \Rightarrow e^X e^{-X} = T$
3. $\left(e^X\right)^\dagger = e^{X^\dagger}$
4. If $[X, Y] = 0$, then $e^X e^Y = e^{X+Y}$
5. For any $C \in \mathsf{GL}(n;\mathbb{C})$, $e^{CXC^{-1}} = Ce^X C^{-1}$.
6. $\dfrac{\mathrm{d}}{\mathrm{d}t} e^{tX} = X e^{tX}$

Here we have a new definition for matrix Lie algebra:

Definition 6.11. *The associated Lie algebra of matrix group G is the set of all matrices X such that $e^{tX} \in G, \forall t \in \mathbb{R}$. The Lie bracket is $[X, Y] = XY - YX$.*

Theorem 6.2 (Lie Product Formula)**.** *Consider a Lie group G and its Lie algebra $\mathfrak{g}$, then $\forall X, Y \in \mathfrak{g}$,*

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{X/m} e^{Y/m}\right)^m$$

According to Definition 6.11, the associated algebra of a matrix Lie group satisfies the following properties:

Propsition 6.1. *Associated Lie algebra is well-defined. i.e. Consider a matrix group G with associated Lie algebra $\mathfrak{g}$, then*[1]

1. $\forall s \in \mathbb{R},\ X \in \mathfrak{g},\ sX \in \mathfrak{g}$.

[1] The second property is not only a lemma needed to prove the fourth property, but it's also related to gauge transformation and is of physical interest in gauge theory.

2. $\forall C \in G,\ X \in \mathfrak{g},\ CXC^{-1} \in \mathfrak{g}$.

3. $\forall X, Y \in \mathfrak{g},\ X + Y \in \mathfrak{g}$.

4. $\forall X, Y \in \mathfrak{g},\ [X, Y] = XY - YX \in \mathfrak{g}$.

Proof. The first proposition is trivial, we may just focus on the rest ones. For the second proposition, we can just use the fifth property in Definition 6.10:

$$e^{CXC^{-1}} = Ce^X C^{-1}$$

As $C, C^{-1}, e^X \in G$, according to the axiom of groups, we have

$$e^{CXC^{-1}} = Ce^X C^{-1} \in G \Rightarrow CXC^{-1} \in \mathfrak{g}$$

For the third proposition, use Lie product formula and we have

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{X/m} e^{Y/m}\right)^m$$

As for any finite m, $e^{X/m}e^{Y/m} \in G$, besides the definition of matrix Lie groups assures the colseness of G under limition operations, thus

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{X/m} e^{Y/m}\right)^m \in G \Rightarrow X + Y \in \mathfrak{g}$$

The fourth one relies on the second one, we have to choose any two elements $X, Y \in \mathfrak{g}$ and consider $e^{tX}Ye^{-tX} \in \mathfrak{g}$. Since $\mathfrak{g}$, as a vector space, is colsed under limitation operations, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX}Ye^{-tX}\right)\bigg|_{t=0} = XY - YX = [X, Y] \in \mathfrak{g}$$

□

Here are some examples on associated Lie algebras:

Example 6.1.

1. $\mathfrak{gl}(n;\mathbb{R})$ *and* $\mathfrak{gl}(n;\mathbb{C})$*: Lie algebras associated to* $\mathsf{GL}(n;\mathbb{R})$ *and* $\mathsf{GL}(n;\mathbb{C})$*. According to matrix exponential, we clearly have*

$$\mathfrak{gl}(n;\mathbb{C}) = M_n(\mathbb{C}) \qquad \mathfrak{gl}(n;\mathbb{R}) = M_n(\mathbb{R})$$

Note that $\mathfrak{gl}(n;\mathbb{C})$ *should be considered as a complex algebra and we intuitively have*

$$\mathfrak{gl}(n;\mathbb{C}) = \mathfrak{gl}(n;\mathbb{R})_{\mathbb{C}}$$

Deonte $E_{k,\ell}$ *is a matrix whose the only nozero element is* 1 *at position* (k, ℓ)*, then we can construct a basis (find a set of generators) for* $\mathfrak{gl}(n)$ *(no matter it's on* $\mathbb{R}$ *or* $\mathbb{C}$ *):*

$$E_{k,\ell}, \quad k, \ell = 1, 2, ..., n$$

Therefore, $\mathfrak{gl}(n)$ is a n^2 dimensional vectoor space. The following job is to find commutation relations:

$$\begin{aligned}
[E_{k,\ell}, E_{m,n}]_{i,j} &= (E_{k,\ell})_{i,p}(E_{m,n})_{p,j} - (E_{m,n})_{i,p}(E_{k,\ell})_{p,j} \\
&= \delta_{ki}\delta_{\ell p}\delta_{mp}\delta_{nj} - \delta_{mi}\delta_{np}\delta_{kp}\delta_{j\ell} \\
&= \delta_{ik}\delta_{m\ell}\delta_{nj} - \delta_{im}\delta_{nk}\delta_{\ell j}
\end{aligned}$$

Thus

$$[E_{k,\ell}, E_{m,n}] = \delta_{m,\ell}E_{k,n} - \delta_{n,k}E_{m,j}$$

2. *$\mathfrak{sl}(n;\mathbb{R})$ and $\mathfrak{sl}(n;\mathbb{C})$: Lie algebras associated to $\mathsf{SL}(n;\mathbb{R})$ and $\mathsf{SL}(n;\mathbb{C})$. Compared to $\mathsf{GL}(n)$, there's only one more constraint: $\det M = +1$. According to*

$$\det e^X = e^{\operatorname{tr} X}$$

Thus $\mathfrak{sl}(n)$ is composed by $n\times n$ matrices with zero trace. We can easily find generators for it:

$$E_{k,\ell},\ k,\ell = 1,2,...,n, k \neq \ell, \qquad E_{k,k} - E_{k+1,k+1},\ k = 1,2,...,n-1$$

Counting the number of generators, we have

$$\dim \mathfrak{sl}(n) = n^2 - n + n - 1 = n^2 - 1$$

3. *$\mathfrak{u}(n)$: Lie algebra associated to $\mathsf{U}(n)$. As $\forall X \in \mathfrak{u}(n)$, $\forall t \in \mathbb{R}$, $e^{tX} \in \mathsf{U}(n)$, thus*

$$\left(e^{tX}\right)^\dagger e^{tX} = e^{tX^\dagger}e^{tX} = I$$

Differentiating by t then set $t = 0$, we have the constraint giving us $\mathfrak{u}(n)$:

$$X^\dagger + X = 0,\ \forall X \in \mathfrak{u}(n)$$

That is, $\mathfrak{u}(n)$ is composed by anti-hermitian matrices of order n. To explicitly write down a set if its generators, let's consider two cases:

- *If elements of X are real numbers, then $X^T + X = 0$, thus the corresponding generators are*
$$E_{k,\ell} - E_{\ell,k},\ k,\ell = 1,2,...,n,\ k < \ell$$
- *If elements of X are all complex numbers, then $X^\dagger + X = 0$ corresponds to $-X^T + X = 0$, thus the generators are*

$$\mathrm{i}(E_{k,\ell} + E_{\ell,k}),\ k,\ell = 1,2,...,n,\ k < \ell \qquad \mathrm{i}E_{k,k},\ k = 1,2,...,n$$

$\mathfrak{u}(n)$ itself is a real vector space, however, if we introduce complex coefficients to complexify it, we can easily have

$$\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}(n;\mathbb{C})$$

4. $\mathfrak{su}(n)$: *Lie algebra associated to* $\mathsf{SU}(n)$. *Clearly it's composed by elements of* $\mathfrak{u}(n)$ *with trace zero. Thus we can immediately write down its generators:*

$$E_{k,\ell} - E_{\ell,k},\ k,\ell = 1,2,...,n,\ k<\ell$$
$$\mathrm{i}(E_{k,\ell} + E_{\ell,k}),\ k,\ell = 1,2,...,n,\ k<\ell$$
$$\mathrm{i}(E_{k,k} - E_{k+1,k+1}),\ k = 1,2,...,n-1$$

It can be showed that

$$\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathfrak{u}(1) \qquad \mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n;\mathbb{C})$$

Note that after complexification, we can build relations between seemingly unrelated Lie algebras. That's the reason why complexification is very useful: Classifying complex Lie algebras is much easier than classifying real Lie algebras.

5. $\mathfrak{so}(n)$: *Lie algebra associated to* $\mathsf{SO}(n)$. *Using matrix exponentials, it's obvious that*

$$X^T + X = 0,\ \forall X \in \mathfrak{so}(n)$$

Thus one choice for its generators is

$$E_{k,\ell} - E_{\ell,k},\ k,\ell = 1,2,...,n,\ k<\ell$$

It can be proved that

$$\mathfrak{so}(3) \cong \mathfrak{su}(2)$$

6. $\mathfrak{o}(p,q)$: *Lie algebra associated to* $\mathsf{O}(p,q)$. *Denote*

$$g = \mathrm{diag}(\underbrace{1,...,1}_{p},\underbrace{-1,...,-1}_{q})$$

Then $\forall X \in \mathfrak{o}(p,q)$, *we should have*

$$e^{tX^T} g e^{tX} = g$$

Thus

$$X^T g + gX = 0$$

Denote

$$X = \begin{pmatrix} \mathbf{A}_{p\times p} & \mathbf{B}_{p\times q} \\ \mathbf{C}_{q\times p} & \mathbf{D}_{q\times q} \end{pmatrix}$$

Then

$$\begin{pmatrix} \mathbf{A}^T & -\mathbf{C}^T \\ \mathbf{B}^T & -\mathbf{D}^T \end{pmatrix} = -\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & -\mathbf{D} \end{pmatrix}$$

Thus

$$\mathbf{A} + \mathbf{A}^T = 0 \quad \mathbf{B} = \mathbf{C}^T \quad \mathbf{D} + \mathbf{D}^T = 0$$

And we have

$$\begin{aligned}
\mathbf{A} &= E_{k,\ell} - E_{\ell,k},\ k,\ell = 1,2,...,p,\ k<\ell \\
\mathbf{D} &= E_{k,\ell} - E_{\ell,k},\ k,\ell = p+1,...,p+q,\ k<\ell \\
\mathbf{B}^T = \mathbf{C} &= E_{k,\ell},\ k = 1,2,...,p,\ \ell = 1,2,...,q
\end{aligned}$$

Just counting the number of generators, we have

$$\dim \mathfrak{so}(p,q) = \dim \mathfrak{so}(p+q)$$

$\forall X \in \mathfrak{o}(p,q)$*, clearly* $\operatorname{tr} X = 0$*, thus* $\mathfrak{o}(p,q)$ *has the same Lie algebra generators as* $\mathfrak{so}(p,q)$*.*

Now consider $\mathfrak{so}(2,1)$*, we can explicitly write its generators down as*

$$L_1' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_2' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad L_3' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Direct calculation shows

$$\begin{aligned}
[L_1', L_2'] &= -L_3' \\
[L_2', L_3'] &= L_1' \\
[L_3', L_1'] &= -L_2'
\end{aligned}$$

Meanwhile, generators of $\mathfrak{so}(3)$ *can be written as*

$$L_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Then the commutation relation is

$$\begin{aligned}
[L_1, L_2] &= L_3 \\
[L_2, L_3] &= L_1 \\
[L_3, L_1] &= L_2
\end{aligned}$$

At a first glance, $\mathfrak{so}(2,1)$ *and* $\mathfrak{so}(3)$ *have different commutation relations, implying they are not likely to be isomorphic in the real sense, and it can be proved that we cannot construct a real isomorphic map between* $\mathfrak{so}(2,1)$ *and* $\mathfrak{so}(3)$*. However, if we consider the same problem on* $\mathbb{C}$*, it can be proved that* $\mathfrak{so}(2,1)_{\mathbb{C}} \cong \mathfrak{so}(3)_{\mathbb{C}}$*. Again, we have witnessed the power of complexification in bridging diffferent Lie algebras together.*

7. $\mathfrak{sp}(2n;\mathbb{R})$*: The associated Lie algebra of* $\mathsf{Sp}(2n;\mathbb{R})$*. For* $\forall X \in \mathfrak{sp}(2n;\mathbb{R})$*, we have*

$$e^{tX}\Omega e^{tX^T} = \Omega \Rightarrow X\Omega + \Omega X^T = 0 \Rightarrow X = \Omega X^T \Omega$$

Denote

$$X = \begin{pmatrix} \mathbf{A}_{n\times n} & \mathbf{B}_{n\times n} \\ \mathbf{C}_{n\times n} & \mathbf{D}_{n\times n} \end{pmatrix}$$

Then we have

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} -\mathbf{D}^T & \mathbf{B}^T \\ \mathbf{C}^T & -\mathbf{A}^T \end{pmatrix} \Rightarrow \begin{cases} \mathbf{A} = -\mathbf{D}^T \\ \mathbf{B} = \mathbf{B}^T \\ \mathbf{C} = \mathbf{C}^T \end{cases}$$

Therefore a set of generators of $\mathfrak{sp}(2n;\mathbb{R})$ *can be expressed as*

$$\begin{aligned} & E_{k,\ell} - E_{\ell+n,k+n}, \ k,\ell = 1,...,n \\ & E_{k,\ell+n} + E_{\ell,k+n}, \ k,\ell = 1,...,n, \ k<\ell \\ & E_{k,k+n}, \ k = 1,...,n \\ & E_{k+n,\ell} + E_{\ell+n,k}, \ k,\ell = 1,...,n, \ k<\ell \\ & E_{k+n,k}, \ k = 1,...,n \end{aligned}$$

6.3 Lie Algebras: A Differential Manifold Viewpoint

As we know, a Lie group G can be viewed as a differential manifold with smooth group structure. In this section, we will show how Lie algebra and Lie bracket naturally emerge from manifold structure.

Definition 6.12 (Tangent Vector)**.** *A tangent vector at a point $p \in M$ is defined to be an equivalence class of curves $c(t)$ in M. We define $c_1(t) \sim c_2(t)$ if*

1. $c_1(0) = c_2(0) = p$
2. *Assum p is in coordinate patch (U, x^μ), then*

$$\frac{\mathrm{d}}{\mathrm{d}t} x^\mu(c_1(t))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(c_2(t))\bigg|_{t=0}$$

The set of all tangent vectors at a point p has the structure of a vector space, and is named as tangent space, denoted as T_pM. Its basis is denoted as $e_\mu = \partial_\mu$, and

$$\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} T_pM$$

The subscript $\mathbb{R}$ indicates dimension in the real sense.

Definition 6.13 (Lie Agebra)**.** *The Lie algebra of a Lie group G is defined to be the tangent space T_eM at identity element e.*

Take $\mathsf{SU}(2) \cong S^3$ as example, as we know its elements can be expressed as

$$\begin{pmatrix} a+b\mathrm{i} & c+d\mathrm{i} \\ -c+d\mathrm{i} & a-b\mathrm{i} \end{pmatrix}, \quad a^2+b^2+c^2+d^2 = 1$$

The identity element corresponds to $(a,b,c,d) = (1,0,0,0)$, a curve in $\mathsf{SU}(2)$ is

$$(a(t), b(t), c(t), d(t)), \quad a^2(t) + b^2(t) + c^2(t) + d^2(t) = 1$$

Thus the tangent vector induced by the curve is

$$(a'(t), b'(t), c'(t), d'(t)), \quad a(t)a'(t) + b(t)b'(t) + c(t)c'(t) + d(t)d'(t) = 0$$

As $e = (1,0,0)$, it can be deduced that at e, the elements of T_eM has the form $(0, b'(t), c'(t), d'(t))$, where $b'(t), c'(t), d'(t)$ are unconstrained. Thus we can select a basis as $(0,1,0,0), (0,0,1,0), (0,0,0,1)$, corresponding to the generators of $\mathfrak{su}(2)$ as below:

$$\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

The emerging of Lie bracket is more tricky: It's related to Lie derivative of a vector field relative to another one.

Definition 6.14 (Vector Field)**.** *A vector field v is a section of tangent bundle TM which is smooth and for any differentiable function $f : M \to \mathbb{R}$, $v(f)$ is also a differentiable function.*

Definition 6.15 (Flow)**.** *A flow $\sigma(t,x) : \mathbb{R} \to M \to M$ is an action of real additive group on an manifold M, satisfying :*

1. *For fixed $t \in \mathbb{R}$, $\sigma(t,\cdot) : M \to M$ is a diffeomorphism and is called one-parameter group.*
2. *Under infinitisimal action $\sigma(\varepsilon,\cdot)$, x^μ is changed to $\sigma^\mu(\varepsilon,x) = x^\mu + \varepsilon X^\mu(x)$, where $X^\mu(x)$ is a vector field and is called infinitisimal generator of the flow.*

According to the definition of flow $\sigma(t,x)$, it can be deduced that $\sigma^\mu(0,x) = x^\mu$ and thus we have the equation that determines a flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^\mu(t,x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\sigma^\mu(t+\varepsilon,x) - \sigma^\mu(t,x)] = \lim_{\varepsilon\to 0}[\sigma^\mu(\varepsilon,\sigma(t,x)) - \sigma^\mu(t,x)] = X^\mu(\sigma(t,x))$$

Now, we consider two vector fields $X^\mu(x)$ and $Y^\mu(x)$ and we want to investigate the change of vector field $Y(x)$ along the flow generated by field $X(x)$. It's actually a tricky job, since on a general manifold different tangent spaces at different points are not identified and it makes no sense to make definition like

$$\lim_{\epsilon\to 0}\frac{1}{\varepsilon}[Y(\sigma(t+\epsilon,x)) - Y(\sigma(t,x))]$$

As the manifold does not necessarily have metric structure, we cannot define a way to translate a vector at point $\sigma(t,x)$ to the point $\sigma(t+\varepsilon,x)$.

Now, the only thing we can do is to recall the definition of tangent vectors. Let's generally consider a map between two manifolds $f : M \to N$. At a point $p \in M$, a tangent vector here is defined to be an equivalence class of curves. Meanwhile, each curve in this class is mapped to a curve in N passing through $f(p)$ by f. If we are so lucky that these curves in N also form an equivalence class, then we will have a correspondence between two vectors in dofferent tangent spaces. According to Definition 6.12, we are indeeed very lucky! The first criteria is obviously satisfied, we only have to check the second one: Suppose two curves $c_1(t), c_2(t)$ in M satisfying the second criteria, then in N, assuming the coordinate map is $y^\mu = f^\mu(x)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} y^\mu(f(c_1(t)))\bigg|_{t=0} = \frac{\partial y^\mu}{\partial x^\nu}\frac{\mathrm{d}}{\mathrm{d}t} x^\nu(c_1(t))\bigg|_{t=0} = \frac{\partial y^\mu}{\partial x^\nu}\frac{\mathrm{d}}{\mathrm{d}t} x^\nu(c_2(t))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} y^\mu(f(c_2(t)))\bigg|_{t=0}$$

This motivates us to formally define push forward map:

Definition 6.16 (Push Forward)**.** *Consider a map $f : M \to N$, the push forward map $f_* : T_pM \to T_{f(p)}N$ is defined by*

$$[f_*v](g) = v(g \circ f)$$

Where g is any differential function on N

This definition is compatible with our previous definition, since if we denote $v = \mathrm{d}c(t)/\mathrm{d}t$, then

$$v(g \circ f) = \frac{\mathrm{d}}{\mathrm{d}t} g(f(c(t)))\bigg|_{t=0}$$

Thus f_*v is exactly identical to the equivalence class $[f(c(t))]$. To write f_*v explicitly, consider $[f_*v](y^\mu)$:

$$(f_*v)^\mu = [f_*v](y^\mu) = v(y^\mu \circ f) = v(y^\mu(x)) = \frac{\partial y^\mu}{\partial x^\nu} v^\nu = \frac{\partial f^\mu}{\partial x^\nu} v^\nu$$

Now we have prepared to find a way to represent the change of vector field $Y(x)$ along the flow generated by field $X(x)$. $X(x)$ generates a flow $\sigma(t, x)$ on M and it can be viewed as a map $\sigma_t : M \to M$, therefore it induces a push forward map $(\sigma_t)_* : T_xM \to T_{\sigma_t(x)}M$. Therefore we have the following definition:

Definition 6.17 (Lie Derivative)**.** *The Lie derivative of vector field $Y(x)$ relative to the field $X(x)$ is defined to be*

$$\mathcal{L}_X Y = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[(\sigma_{-\varepsilon})_* Y(\sigma_\varepsilon(x)) - Y(x)] = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu := [X, Y]$$

Now let's spend some time on proving the explicit formula: Deonting the two vector fields as

$$Y(x) = Y^\mu \partial_\mu \quad X(x) = X^\mu \partial_\mu$$

Then

$$Y(\sigma_\varepsilon(x)) = Y(x^\mu + \varepsilon X^\mu(x)) = Y^\mu \partial_\mu + \varepsilon X^\nu \partial_\nu Y^\mu \partial_\mu + \mathcal{O}(\varepsilon^2)$$

As

$$\sigma^\mu_{-\varepsilon}(x) = x^\mu - \varepsilon X^\mu(x)$$

We have

$$(\sigma_{-\varepsilon})_* Y(\sigma_\varphi(x)) = (\delta^\nu_\mu - \varepsilon\partial_\mu X^\nu)(Y^\mu + \varepsilon X^\rho \partial_\rho Y^\mu)\partial_\nu = Y(x) + \varepsilon(X^\mu\partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu + \mathcal{O}(\varepsilon^2)$$

Thus

$$\mathcal{L}_X Y = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu$$

Propsition 6.2.

$$f_*([X,Y]) = [f_*X, f_*Y]$$

Proof.

$$\begin{aligned}
[f_*X, f_*Y] &= \frac{\partial f^\mu}{\partial x^\nu} X^\nu \overline{\partial}_\mu \left(\frac{\partial f^\rho}{\partial x^\sigma} Y^\sigma \right) \overline{\partial}_\rho - \frac{\partial f^\mu}{\partial x^\nu} Y^\nu \overline{\partial}_\mu \left(\frac{\partial f^\rho}{\partial x^\sigma} X^\sigma \right) \overline{\partial}_\rho \\
&= (X^\nu \partial_\nu Y^\sigma - Y^\nu \partial_\nu X^\sigma) \frac{\partial f^\rho}{\partial x^\sigma} \overline{\partial}_\rho \\
&= f_*([X,Y])
\end{aligned}$$

□

We can also verify that Lie bracket satisfies Jacobi identity.

Now consider a Lie group manifold G, we have the following definition based on the group structure:

Definition 6.18 (Left-translation). *For fixed $a \in G$, left-translation map $L_a : G \to G$ is defined as*

$$L_a g = ag, \ \forall g \in G$$

According to Lie group axiom, L_a is a smooth map and thus induces a push forward map $L_{a*} : T_g G \to T_{ag} G$. We then have the concept of left-invariance:

Definition 6.19 (Left-invariant Vector Field). *For a vector field X on G, it's left-invariant if*

$$L_{a*} X(g) = X(ag), \ \forall g \in G$$

For a Lie group manifold G, an element $v \in T_e G$ uniquely determines a left-invariant vector field as follows:

$$X_v|_g = L_{g*} v, \ g \in G$$

As

$$L_{a*} X_v|_g = L_{a*} L_{g*} v = L_{(ag)*} v = X_v|_{ag}$$

It's indeed a well-defined left-invariant vector field. Conversely, any left-invariant vector field also defines a unique vector in $T_e G$: Just consider $L_{g^{-1}*} X_g$. Thus $T_e G$ is ispmorphic to the set of left-invariant vector fields on G and we can alternatively define Lie algebra as the set of all left-invariant vector fields. This definition is convenient for defining Lie brackets:

According to Proposition 6.2, for left-invariant vector fields X, Y, we have

$$L_{a*}([X,Y]_g) = [L_{a*} X_g, L_{a*} Y_g] = [X_{ag}, Y_{ag}] = [X,Y]_{ag}$$

Thus $[X, Y]$ is also left-invariant and the set of left-invariant vector fields is colsed under Lie derivatives. Then we can finally give a formal definition of Lie algebra:

Definition 6.20 (Lie Algebra)**.** *The Lie algebra of a Lie group G is defined to be the set of left-invariant vector fields $\mathfrak{g}$ with the Lie bracket:*

$$[X, Y] = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu$$

7 Basics on Representation Theory

7.1 Linear Represnetations

Definition 7.1 (Lie Group Representation)**.** *A finite-dimensional complex representation of a Lie group G is a Lie group homomorphism $\Pi : G \to \mathsf{GL}(V)$, where V is a finite-dimensional complex vector space isomorphic to $\mathbb{C}^n$. The dimension of the representation Π is defined to be*

$$\dim \Pi = \dim V$$

$\forall g \in G$, it's action on V through representation Π is denoted as $\Pi(g)$.

Definition 7.2 (Lie Algebra Representation)**.** *A finite-dimensional complex representation of Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is again a finite-dimensional complex vector space. We also define*

$$\dim \pi = \dim V$$

Definition 7.3 (Faithful Representation)**.** *For Lie group or Lie algebra representationds, Π or π is faithful if it's injective.*

Above dfinitions are made for complex representations, based on complex vector space. If we demand V to be a real vector space isomorphic to $\mathbb{R}^n$, then we can make similar statements for real representations, whose dimension is also defined to be $\dim V$.

An important issue in group theory is doing calssifications. For Lie group/Lie algebra representations, we focus on classifying irreducible representations:

Definition 7.4 (Irreducible Representation)**.** *Consider a representation of Lie group $\Pi : G \to \mathsf{GL}(V)$. Define a subspace $W \subset V$ to be an invariant subspace if $\forall w \in W$, $\forall g \in G$, $\Pi(g)w \in W$. If the only invariant subspaces for the representation are V and $\{0\}$, then Π is irreducible.*

We can make similar definition for Lie algebra representations. Irriduciable representations are like bricks to build other representations, and is of great importance in classifying Lie groups. To do classifications, we have to define equivalence relations: In mathematics, equivalence is tied to isomorphism:

Definition 7.5 (Intertwining Map)**.** *Consider two represnetations $\Pi : G \to \mathsf{GL}(V)$, $\Sigma : G \to \mathsf{GL}(W)$, then Π, Σ are defined to be homomorphic if there exists a linear map $\phi : V \to W$ such that*

$$\phi(\Pi(g)v) = \Sigma(g)\phi(v), \quad \forall g \in G,\ v \in V$$

ϕ is called the inertwining map.

Definition 7.6 (Isomorphism)**.** *If the intertwining map ϕ is invertiable, the it's a representation isomorphism and they are isomorphic. We can make a similar definition for Lie algebra representations.*

Classifications of Lie group representations are up to isomorphism in this sense.

Theorem 7.1 (Schur's Lemma)**.**

1. *Let $\Pi : G \to \mathsf{GL}(V)$ and $\Sigma : G \to \mathsf{GL}(W)$ be irreducible real/complex representations of Lie group G or Lie algebra $\mathfrak{g}$. ϕ is intertwining map $\phi : V \to W$, then ϕ is either 0 or an isomorphism.*

2. *Let $\Pi : G \to \mathsf{GL}(V)$ be an irreducible complex representation for a Lie group or Lie algebra, let $\phi_1 : V \to V$ be an intertwining map to V itself, then $\phi = \lambda I,\ \lambda \in \mathbb{C}$.*

3. *Let $\Pi : G \to \mathsf{GL}(V)$, $\Sigma : G \to \mathsf{GL}(W)$ be irreducible complex representations of a Lie group or Lie algebra and let $\phi_1, \phi_2 : V \to W$ be nonzero intertwining maps. Then $\phi_1 = \lambda\phi_2, \lambda \in \mathbb{C}$.*

Proof. For the first clause: $\phi = 0$ is obvious. As ϕ satisfies

$$\phi(\Pi(g)v) = \Sigma(g)\phi(v), \quad \forall g \in G,\ v \in V$$

If ϕ is not an representation isomorphism, then $\ker\phi \neq \{0\}$.$\forall v_1 \neq 0 \in \ker\phi$,we have $\phi(v_1) = 0$. Then for any $g \in G$, we have

$$\phi(\Pi(g)v) = \Sigma(g)\phi(v) = \Sigma(g)0 = 0$$

Thus $\Pi(g)v \in \ker\phi$ and $\ker\phi$ is invariant subspace of representation Π, contradicating with the satatement that "Π is irreducible". Therefore, ϕ is either 0 or an isomorphism for Lie group representations. Following the same process, obviously it's the same for Lie algebra representations.

For the second clause, if $\phi : V \to V$ is the intertwining map to itself, then $\forall g \in G$, we have

$$\Pi(g)\phi = \phi\Pi(g)$$

Since we are considering complex representations, ϕ has at least one eigenvalue κ and corresponds to an eigenspace K. However, for any vector $k \in K$, we have

$$\phi(\Pi(g)k) = \Pi(g)\phi(k) = \kappa\Pi(g)k \Rightarrow \Pi(g)k \in K,\ \forall g \in G$$

Thus K is an invariant subspace for the representation. Since Π is irreducible, thus K is either $\{0\}$, which is impossible, or V, which is the only possible condition. If ϕ has the whole V as an eigenspace, then it must be proportional to identity matrix. Thus $\phi = \lambda I, \lambda \in \mathbb{C}$.

For the third clause: If $\phi_2 \neq 0$, then it's an isomorphism and $\phi_1 \circ \phi_2^{-1}$ is a nonzero intertwining map on W itself. Thus $\phi_1 \circ \phi_2^{-1} = \lambda I$ and $\phi_1 = \lambda\phi_2$. □

It it the first clause of Schur's lemma that assures us the legitimacy of classifying irreducible representations: Two irreducible representations are either isomorphic, or has nothing to do woth each other at all.

Corollary 7.1. *Two irreducible representations of the same Lie group/algebra can't be isomorphic if* $\dim V \neq \dim W$

Proof. According to the fundamental theorem of linear maps, there doesn't exist a bijection between vector spaces with different dimension. Thus if $\Pi : G \to \mathsf{GL}(V)$ and $\Sigma : G \to \mathsf{GL}(W)$ are two irreducible representations and $\dim V \neq \dim W$, then potential intertwining map $\phi : V \to W$ can only be trivial zero map. $\square$

Corollary 7.2. *If* Π *is an irreducible representation of* G *and* A *is the center of* G*, then* $\Pi(A) = \lambda I,\ \lambda \in \mathbb{C}$*. It's the same for Lie algebra representations.*

Proof. $\forall X \in A,\ g \in G$, we have

$$\Pi(X)\Pi(g) = \Pi(Xg) = \Pi(gX) = \Pi(g)\Pi(X)$$

Thus $\Pi(X)$ is a intertwining map on the vector space itself, thus $\Pi(A) = \lambda I$. $\square$

If we have a Lie group representation Π, then $\forall g \in G$, $\Pi(g) \in \mathsf{GL}(V)$ and thus can be represented in matrix exponential, indicating that it induces a Lie algebra representation via

$$\pi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Pi(e^{tX})\bigg|_{t=0}$$

This is actually one way to calculate Lie algebra representations based on Lie group representations.

Definition 7.7 (Unitary Representation)**.** *Lie group representation* $\Pi : G \to \mathsf{GL}(V)$ *is unitary, if* $\forall g \in G$*,*

$$\Pi(g)^\dagger \Pi(g) = I$$

Then unitary Lie algebra representation should satisfy

$$\pi(X)^\dagger + \pi(X) = 0$$

Which could be proved by differentiating $e^{t\pi(X)} = \Pi(e^{tX})$ *by* t*.*

Theorem 7.2. *A Lie group* G *has a faithful finite-dimensional unitary representation if and only if* G *is compact.*

For example, Lorentz group is non-compact and doesn't have a finite-dimensional faithful unitary representation. $\mathsf{GL}(n;\mathbb{R}), \mathsf{GL}(n;\mathbb{C}), \mathsf{SL}(n;\mathbb{R}), \mathsf{SL}(n;\mathbb{C})$ are non-compact and also has no faithful unitary representations. While $\mathsf{U}(1)$ has faithful unitary representations:

$$\forall e^{\mathrm{i}\theta} \in \mathsf{U}(1),\ \Pi_a(e^{\mathrm{i}\theta}) := e^{\mathrm{i}a\theta},\ a \in \mathbb{Z}$$

Note that $a \in \mathbb{Z}$ is necessary, as we should have

$$\Pi_a(e^{\mathrm{i}\theta}e^{\mathrm{i}(2\pi-\theta)}) = \Pi_a(1) = 1 = e^{2\pi a\mathrm{i}}$$

The integer a is called $\mathsf{U}(1)$ charge and is related to the quantization of electric charge.

As $\mathsf{U}(1)$ is abelian, according to Schur's lemma, only one dimensional representation of it is irreducible, as $\mathsf{U}(1)$ is center of itself and thus $\Pi(e^{\mathrm{i}\theta}) = \lambda I$, then any subspace of V is automatically invariant. Similarly, $\mathfrak{u}(1)$ is irreducible only when the dimension is 1. And unlike $\Pi_a(e^{\mathrm{i}\theta})$, we have $\pi_a(\mathrm{i}\theta) = \mathrm{i}a\theta,\ a \in \mathbb{R}$ and there's no 1-1 correspondence between $\pi(\mathfrak{u}(1))$ and $\Pi(\mathsf{U}(1))$, which is rlelated to the non-simply-connectness of $\mathsf{U}(1)$.

Propsition 7.1. *For a connected Lie group G, consider its Lie group rpresentation $\Pi : G \to \mathsf{GL}(V)$ and Lie algebra representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, then:*

1. *π is irreducible if and only if Π is irreducible.*
2. *π_1, π_2 are isomorphic if and only if Π_1, Π_2 are isomorphic.*
3. *If G is simply connected, then every Lie algebra representation π can be derived from Π by $\left.\dfrac{\mathrm{d}}{\mathrm{d}t}\Pi(e^{tX})\right|_{t=0}$.*

Proof. Just consider matrix exponentials. □

Now let's take a glance at examples on Lie group/algebra representations:

Example 7.1.

1. *Standard Representation (Fundamental Representation): For a matrix Lie group G, it's standard representation is just the inclusion map $\Pi : G \to \mathsf{GL}(n;\mathbb{C})$. $\mathsf{U}(n), \mathsf{SU}(n)$ have unitary representations as standard representations. $\mathsf{SO}(n)$, as subgroup of $\mathsf{SU}(n)$, is also unitary. While $\mathsf{Sp}(2n;\mathbb{R}/\mathbb{C})$ is non-compact, having no finite-dimensional faithful unitary representations. $\mathsf{USp}(2n)$ is compact, has finite-dimensional faithful unitary representations.*
2. *Trivial Representation: For a Lie group G, its tirvial representation is a homomorphism $\Pi : G \to \mathsf{GL}(1;\mathbb{C})$ satisfying*

$$\Pi(g) = 1, \quad \forall g \in G$$

The corresponding Lie algebra representation is

$$\pi(X) = 0, \quad \forall X \in \mathfrak{g}$$

Trivial representation is obviously non-faithful.

Definition 7.8 (Adjoint Representation)**.** *Consider a Lie group G, its adjoint representation is defined to be a homomorphism $\mathrm{Ad} : G \to \mathsf{GL}(\mathfrak{g})$ satisfying*

$$\mathrm{Ad}_A X = AXA^{-1}, \quad \forall A \in G,\ X \in \mathfrak{g}$$

Ad *naturally induces a Lie algebra representation* ad : $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$*:*

$$\mathrm{ad}_X Y = [X, Y], \quad \forall X, Y \in \mathfrak{g}$$

Obviously, we have

$$\dim(\mathrm{Ad}) = \dim(\mathfrak{g}) = \dim(G) = \dim(\mathrm{ad})$$

And

$$\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X}$$

Propsition 7.2. $\mathrm{Ad} : G \to \mathsf{GL}(\mathfrak{g})$ *and* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *are well-defined.*

Proof. Consider $\forall A, B \in G$ and $\forall X \in \mathfrak{g}$, we have

$$\mathrm{Ad}_A \mathrm{Ad}_B X = \mathrm{Ad}_A(BXB^{-1}) = ABXB^{-1}A^{-1} = \mathrm{Ad}_{AB} X$$

Thus Ad is indeed a group homomorphism. Meanwhile, $\forall X, Y, Z \in \mathfrak{g}$, we have

$$\mathrm{ad}_{[X,Y]} Z = [[X,Y],Z] = -[[Z,X],Y] - [[Y,Z],X] = [X,[Y,Z]] - [Y,[X,Z]] = [\mathrm{ad}_X, \mathrm{ad}_Y] Z$$

i.e. $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$ and ad is a Lie algebra homomorphism. □

To make the concept of adjoint representation concrete, let's calculate the adjoint representation of $\mathfrak{su}(2)$: As we know, $\dim \mathfrak{su}(2) = 3$ and we can choose a basis $\{X_1, X_2, X_3\}$ satisfying

$$[X_i, X_j] = \varepsilon_{ijk} X_k$$

Thus

$$\mathrm{ad}_{X_i} X_j = \varepsilon_{ijk} X_k$$

and thus

$$(\mathrm{ad}_{X_i})_{k,\ell} = \varepsilon_{i\ell k} = -\varepsilon_{ik\ell}$$

That is

$$\mathrm{ad}_{X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathrm{ad}_{X_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \mathrm{ad}_{X_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can immediately notice the similarity between above matrices and generators of $\mathfrak{so}(3)$. Actually, the adjoint representation of $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$. In fact, this phenomenon comes from the relation between $\mathfrak{su}(2)$ and its adjoint representation: They are isomorphic, as the center of $\mathfrak{su}(2)$ is trivial.

Besides, notice that $\varepsilon_{ik\ell}$ is exactly the structure constant of $\mathfrak{su}(2)$. Motivated by this phenomenon, we propose the following proposition:

Propsition 7.3. *For a general Lie algebra whose structure constant* $c_{ij,k}$ *is defined by*

$$[X_i, X_j] = c_{ij,k} X_k$$

The matrix elements of its adjoint representation is

$$(\mathrm{ad}_{X_i})_{j,k} = c_{ik,j}$$

In QCD, there are 8 kinds of gluons. It's related to the 8 dimensional adjoint representation of $\mathfrak{su}(3)$.

7.2 Constructing New Representations from the Old

Definition 7.9 (Direct Sum Representation)**.** *$\Pi_1, ..., \Pi_k$ are representations of G acting on $V_1, ..., V_k$, the direct sum representation $\Pi_1 \oplus ... \oplus \Pi_k : G \to \mathsf{GL}(V_1 \oplus ... \oplus V_k)$ is defined by*

$$(\Pi_1 \oplus ... \oplus \Pi_k)(A)(v_1, ..., v_k) = (\Pi_1(A)v_1, ..., \Pi_k(A)v_k), \quad \forall A \in G$$

Similarily, for Lie algebra $\mathfrak{g}$ and its representations $\pi_1, ..., \pi_k$ on $V_1, ..., V_k$, the direct sum representation is defined by

$$(\pi_1 \oplus ... \oplus \pi_k)(X)(v_1, ..., v_k) = (\pi_1(X)v_1, ..., \pi_k(X)v_k), \quad \forall X \in \mathfrak{g}$$

According to the definition of the dimension of a representation, clearly

$$\dim(\Pi_1 \oplus ... \oplus \Pi_k) = \dim(\pi_1 \oplus ... \oplus \pi_k) = \dim(V_1 \oplus ... \oplus V_k) = \dim V_1 + ... + \dim V_k$$

If we have a bunch of irreducible representations, we can construct new representations by direct sums. Conversely, for a Lie group/algebra representation, whether it can be decomposed into direct sums of irreducible representations is related to a thing called complete reducibility:

Definition 7.10 (Complete Reducibility)**.** *A finite-dimensional representation of G or $\mathfrak{g}$ is completely reducible if it's isomorphic to a direct sum of irreducible representations. A Lie group/algebra has complete reducibility if any finite-dimensional representation of it is completely reducible.*

Propsition 7.4. *If G is a matrix Lie group and Π is a finite-dimensional unitary representation, then Π is completely reduciable. It's the same for the representation of a real Lie algebra $\mathfrak{g}$.*

Theorem 7.3. *If G is a compact matrix Lie group, then every finite-dimensional representation of it is completely reducible.*

Theorem 7.4. *If a complex Lie algebra $\mathfrak{g}$ is semi-simple, then it's completely reducible.*

Definition 7.11 (Tensor Product Representation)**.** *There are two kinds of tensor product representations:*

1. *Let Π_1 be a representation of G_1 acting on V_1 and Π_2 is a representation of G_2 on V_2. The tensor product representation $\Pi_1 \otimes \Pi_2 : G_1 \times G_2 \to \mathsf{GL}(V_1 \otimes V_2)$ is defined by*

   $$(\Pi_1 \otimes \Pi_2)(A, B) = \Pi_1(A) \otimes \Pi_2(B), \ \forall A \in G_1, B \in G_2$$

   For Lie algebra representations $\pi_1 : \mathfrak{g}_1 \to \mathfrak{gl}(V_1)$ and $\pi_2 : \mathfrak{g}_2 \to \mathfrak{gl}(V_2)$, the tensor product representation is $\pi_1 \otimes \pi_2 : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{gl}(V_1 \otimes V_2)$:

   $$(\pi_1 \otimes \pi_2)(X_1, X_2) = \pi_1(X_1) \otimes I_2 + I_1 \otimes \pi_2(X_2), \quad \forall X_1 \in \mathfrak{g}_1, \ X_2 \in \mathfrak{g}_2$$

   Which could be derived by

   $$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_1(e^{tX_1}) \otimes \Pi_2(e^{tX_2})\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}e^{t\pi_1(X_1)} \otimes e^{t\pi_2(X_2)}\bigg|_{t=0}$$

2. If Π_1 and Π_2 are representations of the same group G acting on different vector spaces V_1, V_2, then the tensor product representation is defined to be $\Pi_1 \otimes \Pi_2 : G \to \mathsf{GL}(V_1 \otimes V_2)$:

$$(\Pi_1 \otimes \Pi_2)(A) = \Pi_1(A) \otimes \Pi_2(A), \quad \forall A \in G$$

The corresponding Lie algebra representation is $\pi_1 \otimes \pi_2 : \mathfrak{g} \to \mathfrak{gl}(V_1 \otimes V_2)$:

$$\pi_1 \otimes \pi_2(X) = \pi_1(X) \otimes I_2 + I_1 \otimes \pi_2(X), \quad \forall X \in \mathfrak{g}$$

Generally, $\pi_1 \otimes \pi_2$ is a reducible representation. Decomposing it into direct sums of irreducible representations is tackled by Clebsch-Gordon theory to be introduced later on.

Definition 7.12 (Dual Representation)**.** *Consider a representation $\Pi : G \to \mathsf{GL}(V)$, its dual representation Π^* is a representation of G on the dual space of V, denoted as V^*. In linear algebra, the dual of an operator $T : V \to V$ is defined to be $T^* : V^* \to V^*$ such that $\forall v \in V, w^* \in V^*$, we have*

$$(T^* w^*)(v) = w^*(Tv)$$

Then we have $T^* = T^T$. However, we can't define $\Pi^*(g) = [\Pi(g)]^T$, as we will have

$$\Pi^*(g_1)\Pi^*(g_2) = [\Pi(g_2)\Pi(g_1)]^T = [\Pi(g_2 g_1)]^T \neq \Pi^*(g_1 g_2)$$

Instead, to preserve group operation we have to define

$$\Pi^*(g) = [\Pi(g^{-1})]^T$$

The corresponding Lie algebra representation can be derived form

$$\pi^*(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Pi^*(e^{tX})\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}e^{-t\pi(X)^T}\bigg|_{t=0} = -\pi(X)^T$$

It's easy to check that $\pi^* : G \to \mathfrak{gl}(V^*)$ is a well-defined Lie algebra homomorphism:

$$\pi^*([X,Y]) = -[X,Y]^T = -[Y^T, X^T] = [X^T, Y^T] = [\pi^*(X), \pi^*(Y)]$$

For an unitary representation, we have

$$[\Pi(g^{-1})]^T = [\Pi(g)^\dagger]^T = \overline{\Pi(g)}$$

Thus sometimes this kind of adjoint representation is called conjugate representation. For fundamental representation of $\mathsf{SU}(n)$, denoted as $\mathtt{n}$, the conjugate representation $\bar{\mathtt{n}}$ is somesometimes called anti-fundamental representation.

If a representation's dual representation is isomorphic to the representation, then it is called a self-dual representation. A real representation is the one whose $\Pi(g)$ and $\pi(g)$ are all real, then we have the following proposition:

Propsition 7.5. *Every real represnetation is sefl-dual.*

Proof. To be completed... □

If a self-dual representation is not real, then it's called pseudo-real representation. Consider the proposition below:

Propsition 7.6. *The standard 2 dimensional representation of* $\mathsf{SU}(2)$ *is self-dual (pseudo-real).*

Proof. $\forall U \in \mathsf{SU}(2)$, its standard representation is $\Pi_s : \mathsf{SU}(2) \to \mathsf{GL}(\mathbb{C}^2)$:

$$\Pi_s(U) = \begin{pmatrix} a+b\mathrm{i} & c+d\mathrm{i} \\ -c+d\mathrm{i} & a-b\mathrm{i} \end{pmatrix}, \quad a^2+b^2+c^2+d^2=1$$

Its dual representation is $\Pi_s^* : \mathsf{SU}(2) \to \mathsf{GL}(\mathbb{C}^{2*})$:

$$\Pi_s^*(U) = [\Pi_s(U^{-1})]^T = \begin{pmatrix} a-b\mathrm{i} & c-d\mathrm{i} \\ -c+d\mathrm{i} & a+b\mathrm{i} \end{pmatrix}$$

Let's consider the map $\phi : \mathbb{C}^2 \to \mathbb{C}^{2*}$:

$$\phi : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Then we have

$$\Pi_s^*(U)\phi = \begin{pmatrix} c-d\mathrm{i} & -a+b\mathrm{i} \\ a+b\mathrm{i} & c-d\mathrm{i} \end{pmatrix} = \phi\Pi_s(U)$$

Therefore ϕ is an intertwining map. Besides, it's clear that ϕ is a bijection, thus ϕ is an isomorphism and the standard 2 dimensional representation of $\mathsf{SU}(2)$ is self-dual. ☐

Since $\mathsf{SU}(2)$ is sefl-dual, it's intuative that $\mathfrak{su}(2)$ is also self-dual. (As Lie algebra can be viewed as linearized Lie group near the identity element.) However, it can be proved that for a general $SU(n)$ with $n > 2$, it's nor self-dual.

We have introduced complexification in definition 6.3, the proposition below establishs the importance of this method in studying representations of a real Lie algebra:

Propsition 7.7. *If* $\mathfrak{g}$ *is a real Lie algebra and* $\mathfrak{g}_{\mathbb{C}}$ *is its complexification, then for every finite dimensional complex representation* π *of* $\mathfrak{g}$*, there is a unique extension of* π *into a complex representation* $\pi_{\mathbb{C}}$ *of* $\mathfrak{g}_{\mathbb{C}}$*, defiend by*

$$\pi_{\mathbb{C}}(X+\mathrm{i}Y) = \pi(X) + \mathrm{i}\pi(Y), \quad \forall X, Y \in \mathfrak{g}$$

$\pi_{\mathbb{C}}$ *is irreducible if and only if* π *is irreducible.*

This proposition assures the power of complexification in Lie algebra representation theory. Suppose we have a real Lie algebra $\mathfrak{g}$ with generators $X_1, ..., X_n$ whose representation is hard to construct, we can consider it's complexification $\mathfrak{g}_{\mathbb{C}}$ with generators $X_1', ..., X_n'$ and it's representation is usually easy to construct. As long as we know how to linearly combine $X_1', ..., X_n'$ to get $X_1, ..., X_n$, we will easily get representation of $\mathfrak{g}$ out of representation of $\mathfrak{g}_{\mathbb{C}}$.

7.3 Representations of Common Non-abelian Lie Groups

7.3.1 $\mathsf{SU}(2)$

Let V_m be the sapce of homogenous ploynomials of degree m in two complex variables z_1, z_2, elements of V_m can be expressed as

$$f(z) = f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + a_2 z_1^{m-2} z_2^2 + \cdots a_{m-1} z_1 z_2^{m-1} + a_m z_2^m, \quad a_i \in \mathbb{C}$$

Clearly

$$\dim V_m = m + 1$$

It can be verified $\mathsf{SU}(2)$ has a representation Π_m on V_m defined by

$$[\Pi_m(U)]f(z) = f(U^{-1}z), \quad \forall U \in \mathsf{SU}(2)$$

There are two ways of interpreting the seemingly unnatural inverse matrix U^{-1} in the above definition:

1. It's the restriction imposed by the definition of representation. $\Pi_m(U)$ should be viewed as an action on f in $f(z)$ rather than action on z. Therefore if we define $\Pi_m(U)f(z) = f(Uz)$, we will have

   $$\Pi_m(U_1)\Pi_m(U_2)f(z) = \Pi_m(U_2)f(U_1 z) = f(U_2 U_1 z) = \Pi_m(U_2 U_1 z)$$

   It contradicts with the definition of Lie group representation. On the other hand, according to the correct definition, we have

   $$\Pi_m(U_1)\Pi_m(U_2)f(z) = \Pi_m(U_2)f(U_1^{-1}z) = f(U_2^{-1}U_1^{-1}z = f((U_1U_2)^{-1}z)) = \Pi_m(U_1U_2)f(z)$$

   Which is consistent with the definition of group representation.

2. U^{-1} represents a sense of duality. We can think z as sort of contra-vectors while f is something like a co-vector dual to z. $\Pi_m(U)$ acting on f can thus be treated as something like operators acting on dual vectors.

   To make our statements more rigorous, let's consider representations of $\mathsf{SU}(2)$ on V_1. First note that any element in V_1 can be written as

   $$f(z) = a_0 z_1 + a_1 z_2 = \begin{pmatrix} a_0 & a_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

   $\forall U \in \mathsf{SU}(2)$, its action on a contra-vector $z = (z_1, z_2)^T$ can be written in matrix form :

   $$Uz = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

   Then representation of $\mathsf{SU}(2)$ on V_1 is

   $$\begin{aligned} [\Pi_1(U)]f(z) &= f(U^{-1}z) = a_0(U_{11}^{-1}z_1 + U_{12}^{-1}z_2) + a_1(U_{21}^{-1}z_1 + U_{22}^{-1}z_2) \\ &= (U_{11}^{-1}a_0 + U_{21}^{-1}a_1)z_1 + (U_{12}^{-1}a_0 + U_{22}^{-1}a_1)z_2 \\ &= \begin{pmatrix} a_0 & a_1 \end{pmatrix} \begin{pmatrix} U_{11}^{-1} & U_{12}^{-1} \\ U_{21}^{-1} & U_{22}^{-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned}$$

If we denote

$$[\Pi_1(U)]f(z) = \begin{pmatrix} a_0' & a_1' \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Then we should have

$$\begin{pmatrix} a_0' & a_1' \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \end{pmatrix} \begin{pmatrix} U_{11}^{-1} & U_{12}^{-1} \\ U_{21}^{-1} & U_{22}^{-1} \end{pmatrix}$$

However, row vectors represent dual vectors, it's contra-vector form corresponds to the Hermitian conjugate. Since the matrix is in $\mathsf{SU}(2)$ and satisfies $U = \left(U^{-1}\right)^\dagger$, we have

$$\begin{pmatrix} a_0'^* \\ a_1'^* \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} a_0^* \\ a_1^* \end{pmatrix}$$

Let's back to consider the general representation $\Pi_m : \mathsf{SU}(2) \to V_m$. As the induced Lie algebra representation can be derived from

$$\pi_m(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Pi_m\left(e^{tX}\right)\Big|_{t=0}$$

Thus $\pi_m : \mathfrak{su}(2) \to \mathfrak{gl}(V_m)$ can be derived by considering

$$[\pi_m(X)f](z) = \frac{\mathrm{d}}{\mathrm{d}t}[\Pi_m\left(e^{tX}\right)]f(z)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}f(e^{-tX}z)\Big|_{t=0}$$

Given that

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = e^{-tX}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = -X\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

We thus have

$$\begin{aligned} [\pi_m(X)f](z) &= \frac{\partial f}{\partial z_1}\frac{\mathrm{d}z_1}{\mathrm{d}t}\Big|_{t=0} + \frac{\partial f}{\partial z_2}\frac{\mathrm{d}z_2}{\mathrm{d}t}\Big|_{t=0} \\ &= -(X_{11}z_1 + X_{12}z_2)\frac{\partial f}{\partial z_1} - (X_{21}z_1 + X_{22}z_2)\frac{\partial f}{\partial z_2} \end{aligned}$$

Therefore, the Lie algebra representation of $\mathsf{su}(2)$ induced by Π_m on V_m can be expressed with differential operators[1] To write $\pi_m(X)$ explicitly, let's consider a basis of $\mathfrak{su}(2)$:

$$J_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$

Then we have a basis of $\pi_m(\mathfrak{su}(2))$:

$$\begin{aligned} \pi_m(J_1) &= -\mathrm{i}z_2\frac{\partial}{\partial z_1} - \mathrm{i}z_1\frac{\partial}{\partial z_2} \\ \pi_m(J_2) &= -z_2\frac{\partial}{\partial z_1} + z_1\frac{\partial}{\partial z_2} \\ \pi_m(J_3) &= -\mathrm{i}z_1\frac{\partial}{\partial z_1} + \mathrm{i}z_2\frac{\partial}{\partial z_2} \end{aligned}$$

[1] It's not hard to aware that a differential operator can be viewed as a linear map on ploynomial space.

Before we moving forward, let's take a break and investigate the relaton between $\pi_1(J_i)$ and the standard 2 dimensional representation of $\mathfrak{su}(2)$: Denote $f(z_1, z_2) = a_0 z_1 + a_1 z_2$, then direct calculation shows

$$\pi_1(J_1)f(z_1, z_2) = -\mathrm{i}a_0 z_1 - \mathrm{i}a_1 z_1 = \begin{pmatrix} a_0 & a_1 \end{pmatrix} (-J_1^T) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\pi_1(J_2)f(z_1, z_2) = -a_0 z_2 + a_1 z_1 = \begin{pmatrix} a_0 & a_1 \end{pmatrix} (-J_2^T) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\pi_1(J_3)f(z_1, z_2) = -\mathrm{i}z_1 a_0 + \mathrm{i}z_2 a_2 = \begin{pmatrix} a_0 & a_1 \end{pmatrix} (-J_3^T) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Therefore, π_1 is isomorphic to the dual of the standard 2 dimensional representation of $\mathfrak{su}(2)$.

For convenience, sometimes we don't study the original Lie algebra $\mathfrak{su}(2)$ but study its complexification $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2;\mathbb{C})$. According to proposition 7.7, determining the representations of complexified Lie algebra allows us to determine representations of the original algebra. For $\mathfrak{sl}(2;\mathbb{C})$, we can choose generators as follows:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then we have

$$\pi_m(X) = -z_2 \frac{\partial}{\partial z_1} \quad \pi_m(Y) = -z_1 \frac{\partial}{\partial z_2} \quad \pi_m(H) = -z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$$

(Note that by the time now we are studying representation of $\mathfrak{sl}(2;\mathbb{C})$ on V_m, but can easily recover representation of $\mathfrak{su}(2)$ by linearly reorginazing generaotrs.) Applying $\pi_m(X), \pi_m(Y), \pi_m(H)$ to $z_1^{m-k} z_2^k \in V_m$ shows

$$\begin{aligned} \pi_m(X)(z_1^{m-k} z_2^k) &= -(m-k) z_1^{m-k-1} z_2^{k+1} \\ \pi_m(Y)(z_1^{m-k} z_2^k) &= -k z_1^{m-k+1} z_2^{k-1} \\ \pi_m(H)(z_1^{m-k} z_2^k) &= (2k-m) z_1^{m-k} z_2^k \end{aligned}$$

Thus $z_1^{m-k} z_2^k$ is eigenstate of $\pi_m(H)$, corresponding to eigenvalue $(2k-m)$. In representation theory, $(2k-m)$ has a fancy name: weight. Clearly, in V_m there are $(m+1)$ eigenstates of $\pi_m(H)$ and the corresponding weight ranges from $-m$ to $+m$. Meanwhile, it's obvious that $\pi_m(X)(z_1^{m-k} z_2^k)$ has weight $(2k+2-m)$, while $\pi_m(Y)(z_1^{m-k} z_2^k)$ has weight $(2k-2-m)$. Therefore $\pi_m(X)$ and $\pi_m(Y)$ shifts the weight by ± 2 respectively, and they correspond to creation and annhailation operators in physics.

z_2^m has the highest weight $+m$ and is named as the highest weight state, while z_1^{-m} has the lowest weight $-m$ and is named as the lowest weight state. Later we will learn a systematic method to construct a group representation and the key to the method is to find the highest/lowest state at the beginning, then we can construct the whole representation by acting ladder operators. By the way, $\pi_m(H)$ can be interpreted as an angular momentum operator on a direction, but the corresponding magnetic quantum number s should be taken as half of the number m here. i.e. $s = m/2$.

Actually, the Lie algebra representation π_m is irreducible for all $m \in \mathbb{N}_+$, proof is as follows:

Proof. In the previous text we have used π_m to denote representations of $\mathfrak{su}(2)$ as well as $\mathfrak{sl}(2;\mathbb{C})$, but it doesn't matter, as $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2;\mathbb{C})$ so that $\mathfrak{su}(2)$ and $\mathfrak{sl}(2;\mathbb{C})$ have the same irreducibility. Thus, we can just focus on proving π_m is irreducible representation of $\mathfrak{sl}(2;\mathbb{C})$, as it's much more convenient to handle.

We have to show every non-zero invariant subspace of $\pi_m : \mathfrak{sl}(2;\mathbb{C}) \to \mathfrak{gl}(V_m)$ is equal to V_m. Assume W is a non-zero invariant subspace, then any element $w \in W$ has the form

$$w = a_0 z_1^m + a_1 z_1^{m-1} z_2 + \cdots + a_{m-1} z_1 z_2^{m-1} + a_m z_2^m$$

Of course not all a_i are nonzero, but there exist at least one integer k such that $a_k \neq 0$, let's denote k_0 as the minium k such that $a_k \neq 0$. Therefore, k_0 corresponds to the term $a_{k_0} z_1^{m-k_0} z_2^{k_0}$. Sine W is a invariant subspace of π_m, then we should have

$$\pi_m^{m-k_0}(X)^{m-k_0}(a_{k_0} z_1^{m-k_0} z_2^{k_0}) \propto z_2^m \in W$$

Then

$$\pi_m^j(Y)(z_2^m) \in W, \quad j = 0, 1, 2, ..., m$$

As j ranges from 0 to m, $\pi_m^j(Y)(z_2^m)$ generates $z_2^m, z_2^{m-1} z_1, ..., z_2 z_1^{m-1}, z_1^m$ and all of them belong to W, so we have $W = V_m$. Therefore, π_m irreducible for all $m \in \mathbb{N}_+$. ☐

In fact, it can be proved that every irreducible complex represnetation of $\mathfrak{sl}(2;\mathbb{C})$ is isomorphic to one of the π_m. Hence if an irreducible complex representation of $\mathfrak{sl}(2;\mathbb{C})$ is of dimension $(m+1)$, then it must isomorphic to π_m here.

Besides, as $\mathsf{SU}(2)$ is simply-connected, we can confirm there is a one-to-one correspondence between representation of $\mathfrak{su}(2)$ and Lie group representation of $\mathsf{SU}(2)$. Since π_m is irreducible, we conclude that Π_m is irreducible, too.

7.3.2 SO(3)

The only irreducible representations of $\mathsf{SO}(3)$ are realized on the space of homogeneous sphere harmonics[2] on $\mathbb{R}^3$ of degree ℓ, denoted as V_ℓ. $\Pi_\ell : \mathsf{SO}(3) \to \mathsf{GL}(V_\ell)$ is defined to be

$$[\Pi_\ell(R)]f(\vec{x}) = f(R^{-1}\vec{x}), \quad R \in \mathsf{SO}(3),\ \vec{x} = (x, y, z)^T$$

In quantum mechanics, the representation Π_ℓ of $\mathsf{SO}(3)$ describes a system with angular momentum quantum number $j = \ell$, while the magent quantum number is related to the Lie algebra representation $\pi_{2\ell}(H)/2$.

Propsition 7.8. $\mathsf{SO}(3)$ *has no even-dimensional irreducible representations.*

[2]WHY?

Proof. Consider to prove the Lie algebra representations π_m of $\mathfrak{su}(2)$ cannot be uplifted to a representation Σ_m of $\mathsf{SO}(3)$ when m is odd. (Remember that $\dim \pi_m = m+1$, thus it corresponds to even dimensional representations.)

Let's take the following generators X_i for $\mathfrak{su}(2)$ and L_i for $\mathfrak{so}(3)$:

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad X_3 = \frac{1}{2}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It's easy to verify they have the same commutation relations:

$$[X_i, X_j] = \varepsilon_{ijk} X_k \qquad [L_i, L_j] = \varepsilon_{ijk} L_k$$

As $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, a representatiom σ_m of $\mathfrak{so}(3)$ should isomorphic to a representaton π_m of $\mathfrak{su}(2)$. Without loss of generality, we have

$$\sigma_m(L_3) = \pi_m(X_3)$$

As

$$X_3 = -\frac{\mathrm{i}}{2}H, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

As there is a basis $u_0, u_1, ..., u_m$ for V_m such that u_j is an eigenvector for $\pi_m(H)$ with eigenvalue $(m-2j)$, so u_j should be an eigenvector for $\sigma_m(L_3)$ with eigenvalue $-\mathrm{i}(m-2j)/2$. Thus in the basis of u_j, $\sigma_m(L_3)$ is a diagonal matrix:

$$\sigma_m(L_3) = \mathrm{diag}\left(-\frac{\mathrm{i}}{2}m, -\frac{\mathrm{i}}{2}(m-2), ..., \frac{\mathrm{i}}{2}(m-2), \frac{\mathrm{i}}{2}m\right)$$

For the corresponding Lie group representation, we should have $\Sigma_m(e^{tL_3}) = \exp(t\sigma_m(L_3))$. Since

$$e^{2\pi L_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi) & -\sin(2\pi) \\ 0 & \sin(2\pi) & \cos(2\pi) \end{pmatrix} = I$$

We should expect to have $\Sigma_m(e^{2\pi L_3}) = \Sigma_m(I) = I$. However, calculation shows

$$e^{2\pi\sigma_m(L_3)} = \begin{cases} I, & m \text{ is even} \\ -I, & m \text{ is odd} \end{cases}$$

Thus for odd m we get a contradiction and $\mathsf{SO}(3)$ has no even-dimensional irreducible representations. $\square$

Therefore $\mathsf{SO}(3)$ only has odd dimensional irreducible representations with dimension $2\ell+1$, the corresponding Lie algebra representation is $\pi_{2\ell}$ on V_ℓ. $\mathsf{SO}(3)$ is an example that not every representation of its Lie algebra comes from the Lie group representation, due to $\mathsf{SO}(3)$ is not simply-connected.

In fact, all representations of $\mathsf{SO}(3)$ are real and thus all odd-dimensional irreducible representations of $\mathsf{SU}(2)$ and $\mathfrak{su}(2)$ are real (as they are isomorphic to representations of $\mathsf{SO}(3)$). On the other hand, even-dimensional irreducible representations of $\mathsf{SU}(2)$ and $\mathfrak{su}(2)$ are pseudo-real (they are not isomorphic to a real representation, but they are self-dual).

8 Semisimple Lie Algebras

8.1 Levi's Decomposition

Definition 8.1 (Lie Algebra Ideal). *A Lie algebra ideal $\mathfrak{i}$ of Lie algebra $\mathfrak{g}$ is a Lie subalgebra such that $\forall X \in \mathfrak{g}$ and $\forall Y \in \mathfrak{i}$, $[X, Y] \in \mathfrak{i}$.*

Definition 8.2 (Semisimple Lie Algebra). *A Lie algebra is semisimple if and only if it has no non-zero abelian ideals.*

Definition 8.3 (Simple Lie Algebra). *A Lie algebra $\mathfrak{g}$ is simple if it's not abelian and the only ideals are $\{0\}$ and $\mathfrak{g}$ itself.*

According to definition, a simple Lie algebra surely has no non-zero abelian ideals, thereby it's semi-simple. $\mathfrak{su}(2)$ and $\mathfrak{sl}(2;\mathbb{C})$ are simple, and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is semi-simple.

Definition 8.4 (Solvable Lie Algebra). *Consider the derived series*

$$\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}] \supset [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supset \cdots$$

If the sequence therminates at $\{0\}$, then $\mathfrak{g}$ is a solvable Lie algebra.

For example, $\mathfrak{su}(2)$ is not solvable, as $[\mathfrak{su}(2), \mathfrak{su}(2)] \cong \mathfrak{su}(2)$; but $\mathfrak{u}(1)$ is solvable, since $\mathfrak{u}(1)$ is abelian and $[\mathfrak{u}(1), \mathfrak{u}(1)] = \{0\}$.

Propsition 8.1. *A Lie algebra cannot be semisimple and solvable at the same time.*

Proof. Denote

$$\mathfrak{g}_0 = \mathfrak{g} \quad \mathfrak{g}_1 = [\mathfrak{g}_0, \mathfrak{g}_0] \quad \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] \ \cdots$$

Then $\mathfrak{g}_i$ are obviously Lie subalgebras of $\mathfrak{g}$. If $\mathfrak{g}$ is solvable, then there exists $\mathfrak{g}_\ell$ such that $[\mathfrak{g}_\ell, \mathfrak{g}_\ell] = 0$, i.e. $\mathfrak{g}_\ell$ is an abelian subalgebra of $\mathfrak{g}$. Besides, any element of $\mathfrak{g}_i$ is a commutator of two elements in $\mathfrak{g}_{i-1}$ and also belongs to $\mathfrak{g}_{i-1}$, therefore $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}_{i-1}$. Hence $\mathfrak{g}_\ell$ is a non-zero abelian ideal of $\mathfrak{g}_{\ell-1}$ and $\mathfrak{g}_{\ell-1}$ is not a semisimple Lie algebra. I'm not sure whether we can deduce $\mathfrak{g}$ is not semisimple based on one of its subalgebras $\mathfrak{g}_{\ell-1}$ is not semisimple. □

Of course there are Lie algebras which is neither semisimple nor solvable, for example, $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

Theorem 8.1 (Levi's Decomposition Theorem)**.** *Any finite dimensional Lie algebra $\mathfrak{g}$ can be decomposed as a semi-direct sum of a solvable Lie algebra $\mathfrak{s}$ and a semisimple Lie algebra $\mathfrak{l}$, $\mathfrak{g} = \mathfrak{s} \oplus_s \mathfrak{l}$. The semi-direct sum $\mathfrak{s} \oplus_s \mathfrak{l}$ of $\mathfrak{s}$ and $\mathfrak{l}$ is a Lie algebra $\mathfrak{s} \cup \mathfrak{l}$ where $\mathfrak{s} \cap \mathfrak{l} = \{0\}$, $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}$, $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}$, $[\mathfrak{s}, \mathfrak{l}] \subset \mathfrak{s}$.*

Above definitions and theorems are appliable for both real and complex Lie algebras. Later on we will introduce the classification of complex simple Lie algebras.

8.2 Cartan Subalgebras

Now we focus on semisimple complex Lie algebra $\mathfrak{g}$ which is a complexified real Lie algebra of a compact Lie group $\mathfrak{r}$, i.e. $\mathfrak{g} = \mathfrak{r}_{\mathbb{C}} = \mathfrak{r} + \mathrm{i}\mathfrak{r}$. Clearly, $\mathfrak{r}$ is a real subalgebra of $\mathfrak{g}$.

Till now we have treated complex Lie algebras as complex vector spaces attached with a binary Lie bracket operation, however, we have never considered what if there is inner product structure on it. In linear algebra, a inner product on complex vector space V is a bilinear map $V \times V \to \mathbb{C}$ satisfying:

- $\overline{\langle v_1, v_2 \rangle} = \langle v_2, v_1 \rangle$, $\forall v_1, v_2 \in V$
- $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$, $\forall v_1, v_2, w \in V$
- $\langle k v_1, v_2 \rangle = \overline{k} \langle v_1, v_2 \rangle$, $\forall v_1, v_2 \in V$, $\forall k \in \mathbb{C}$
- $\langle X, X \rangle \geq 0$, the equality holds if and only if $X = 0$

Surely the inner product we define on the semisimple Lie algebra $\mathfrak{g}$ should have the above properties, but merely these properties help us little. We demand more:

Propsition 8.2. *If a Lie algebra $\mathfrak{g}$ can be treated as a complexified real Lie algebra $\mathfrak{r}_{\mathbb{C}}$ (We don't even demand it to be semisimple!), then there exists a complex inner product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ such that :*

1. $\forall R_1, R_2 \in \mathfrak{r}, \langle R_1, R_2 \rangle \in \mathbb{R}$
2. *The adjoint action of $\mathfrak{r}$ on $\mathfrak{g}$ is anti-hermitian, that is*

$$\langle \mathrm{ad}_R(X), Y \rangle = -\langle X, \mathrm{ad}_R(Y) \rangle , \quad \forall R \in \mathfrak{r},\ \forall X, Y \in \mathfrak{g}$$

If we define an operation $X \mapsto X^*$ on $\mathfrak{g}$ by

$$(R_1 + \mathrm{i}R_2)^* = -R_1 + \mathrm{i}R_2, \quad \forall R_1, R_2 \in \mathfrak{r}$$

The the inner product we defined on $\mathfrak{g}$ satisfies

$$\langle \mathrm{ad}_X(Y), Z \rangle = \langle Y, \mathrm{ad}_{X^*}(Z) \rangle , \quad \forall X, Y, Z \in \mathfrak{g}$$

Proof. Firstly let's prove the existence of this inner product. It's a fact that by group integration there is a real-valued inner product on $\mathfrak{r}$ corresponding to a compact Lie group that is invariant under the adjoint action of R, which is the compact Lie group corresponds to the Lie algebra $\mathfrak{r}$. This real inner product extends to a complex inner product on $\mathfrak{g} = \mathfrak{r}_{\mathbb{C}}$ such that the adjoint action of R is unitary, i.e.

$$\langle \mathrm{Ad}_r(X), Y\rangle = \left\langle X, \mathrm{Ad}_r^{\dagger}(Y)\right\rangle = \langle X, \mathrm{Ad}_{r^{-1}} Y\rangle$$

Given that $r = e^{tR}, r^{-1} = e^{-tR}$ and $\mathrm{Ad}_r = e^{t\mathrm{ad}_r}$, thus differentiate the above formula by t then set $t = 0$ gives us

$$\langle \mathrm{ad}_R(X), Y\rangle = -\langle X, \mathrm{ad}_R(Y)\rangle, \quad R \in \mathfrak{r}$$

Secondly we shall prove $\langle \mathrm{ad}_X(Y), Z\rangle = \langle Y, \mathrm{ad}_{X^*}(Z)\rangle$. As $X \in \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{r}_{\mathbb{C}}$, there exists $R_1, R_2 \in \mathfrak{r}$ such that $X = R_1 + \mathrm{i}R_2$ and thus $X^* = -(R_1 - \mathrm{i}R_2)$. Hence

$$\begin{aligned}\langle \mathrm{ad}_X(Y), Z\rangle &= \langle \mathrm{ad}_{R_1}(Y) + \mathrm{ad}_{R_2}(Y)\mathrm{i}, Z\rangle = \langle \mathrm{ad}_{R_1}(Y), Z\rangle - \mathrm{i}\,\langle \mathrm{ad}_{R_2}(Y), Z\rangle \\ &= -\langle Y, \mathrm{ad}_{R_1}(Z)\rangle + \mathrm{i}\,\langle Y, \mathrm{ad}_{R_2}(Z)\rangle \\ &= \langle Y, -\mathrm{ad}_{R_1}(Z)\rangle + \langle Y, \mathrm{ad}_{R_2}(Z)\mathrm{i}\rangle \\ &= \langle Y, \mathrm{ad}_{-R_1+\mathrm{i}R_2}(Z)\rangle = \langle Y, \mathrm{ad}_{X^*}(Z)\rangle\end{aligned}$$

□

Actually the motivation of the definition of X^* is that if $\mathfrak{g} = \mathfrak{gl}(n;\mathbb{C})$ and $\mathfrak{r} = \mathfrak{u}(n)$, then $\forall X \in \mathfrak{g}$, $\exists U_1, U_2 \in \mathfrak{u}(n)$ such that $X = U_1 + \mathrm{i}U_2$ and

$$X^{\dagger} = U_1^{\dagger} - \mathrm{i}U_2^{\dagger} = -U_1 + \mathrm{i}U_2 = X^*$$

Besides, we can define the inner product on $\mathfrak{gl}(n;\mathbb{C})$ by

$$\langle X, Y\rangle = \mathrm{tr}\left(X^{\dagger}Y\right)$$

We can check that it has the weird property:

$$\begin{aligned}\langle \mathrm{ad}_U(X), Y\rangle &= \mathrm{tr}\left([U,X]^{\dagger}Y\right) = \mathrm{tr}\left([X^{\dagger},U^{\dagger}]Y\right) = \mathrm{tr}\left(UX^{\dagger}Y - X^{\dagger}UY\right) \\ &= \mathrm{tr}\left(X^{\dagger}YU - X^{\dagger}UY\right) = -\,\mathrm{tr}\left(X^{\dagger}[U,Y]\right) = -\langle X, \mathrm{ad}_U(Y)\rangle\end{aligned}$$

The strategy for classifying representations of semisimple Lie algebras and the structure we will need to carry out this strategy is as follows: We will look for commuting elements $H_1, ..., H_r$ in our Lie algebra and we will try to simultaneously diagonalize them in each represnetation. We want to find as much such elements as possible, and if they are going to be simultaneously diagonalizable in every representation, they must certainly be diagonalizable in the adjoint representqation. This leads to the definition of a Cartan subalgebra. The nonzero sets of simultaneous eigenvalues for $\mathrm{ad}_{H_1}, ..., \mathrm{ad}_{H_r}$ are called roots and the corresponding simultaneous eigenvectors are called root vectors. The root vectors will serve to raise and lower the eigenvalues of $\pi(H_1), ..., \pi(H_r)$ in each representation π. We will also have the Weyl group, which is an important symmetry of the roots and also of the weights in each representation.

We study semisimple Lie algebras because it has certain special subalgebras isomorphic to $\mathfrak{sl}(2;\mathbb{C})$ and we can use our knowledge of the representations of $\mathfrak{sl}(2;\mathbb{C})$.

Definition 8.5 (Killing Form)**.** *For any complex Lie algebra, the Killing form is a symmetric bilinear form*

$$K(X,Y) := \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y), \quad \forall X, Y \in \mathfrak{g}$$

If X_i is a basis of $\mathfrak{g}$ and the structure constant of $\mathfrak{g}$ is c_{ijk}, then

$$K_{ij} = K(X_i, X_j) = c_{ik\ell}c_{j\ell k}$$

Based on Killing form, we have the key to the whole theory of semi-simple Lie algebras: Cartan's criterion for semisimplicity:

Theorem 8.2. *A Lie algebra is semisimple if and only if its Killing form is non-degenerate. That is, $\mathfrak{g}$ is semi-simple if and only if $\det K \neq 0$.*

Corollary 8.1. *If $\mathfrak{g}$ is semi-simple, then its adjoint representation is faithful.*

Proof. Suppose ad is not faithful, the there exists two elements $X, Y \in \mathfrak{g}$ such that $X \neq Y$ but $\mathrm{ad}_X = \mathrm{ad}_Y$. Then $\mathrm{ad}_{X-Y} = 0$. Hence for any $Z \in \mathfrak{g}$, we have $\mathrm{ad}_{X-Y}\mathrm{ad}_Z = 0$ and thus $K(X-Y, Z) = 0$, then the Killing form is degenerate and hence $\mathfrak{g}$ cannot be semisimple. □

We need the previously defined inner product to separete the vector sapce of Lie algebra into different parts, and this process is based on Cartan subalgebras, which help us to identify certain special sorts of commutative subalgebras:

Definition 8.6 (Cartan Subalgebra)**.** *If $\mathfrak{g}$ is a complex semisimple Lie algebra, then the Cartan subalgebra of $\mathfrak{g}$ is a complex subspace $\mathfrak{h}$ of $\mathfrak{g}$ with the following properties:*

1. *$\forall H_1, H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$, indicating $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$.*
2. *If for some $X \in \mathfrak{g}$ we have $[H, X] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$, implying $\mathfrak{h}$ should be the largest abelian subalgebra of $\mathfrak{g}$.*
3. *For all H in $\mathfrak{h}$, ad_H should be diagonalizable.*

Propsition 8.3. ad_H*'s are simultaneously diagonalizable.*

Proof. Since $\forall H_1 \in \mathfrak{h}$, ad_{H_1} is diagonalizable, we can choose a eigenvector v such that

$$\mathrm{ad}_{H_1} v = \lambda v$$

Now we choose an arbitrary element $H_2 \in \mathfrak{h}$, since $[H_1, H_2] = 0$, we should have $\mathrm{ad}_{H_1}\mathrm{ad}_{H_2} = \mathrm{ad}_{H_2}\mathrm{ad}_{H_1}$ and thus

$$\mathrm{ad}_{H_1}\mathrm{ad}_{H_2} v = \mathrm{ad}_{H_2}\mathrm{ad}_{H_1} v = \lambda \mathrm{ad}_{H_2} v$$

Hence $\mathrm{ad}_{H_2} v$ is also an eigenvector of ad_{H_1} corresponding to the same eigenvalue as what v corresponds to. If the eigenvalue λ is nondegenerate, then we must have $\mathrm{ad}_{H_2} v \propto v$ so that v is also eigenvector of ad_{H_2}, thus we can simultaneously diagonalize ad_{H_1} and ad_{H_2}. If λ is a degenerate eigenvalue, then the eigenspace of eigenvalue λ is an invariant subspace of ad_{H_2} where we can diagonalize ad_{H_2}. Therefore, ad_{H_1} and ad_{H_2} are always simultaneously diagonalizable. □

Actually the definition of Cartan subalgebra makes sense in any Lie algebra, semisimple or not. However, if $\mathfrak{g}$ is not semisimple, then $\mathfrak{g}$ may not have any Cartan subalgebra. Fortunatelly, Cartan subalgebras exist in every semisimple Lie algebras:

Propsition 8.4. *Let $\mathfrak{g} = \mathfrak{r}_{\mathbb{C}}$ be a complex semisimple Lie algebra and let $\mathfrak{t}$ be any maximal commutative subalgebra of $\mathfrak{r}$. Define $\mathfrak{h} \subset \mathfrak{g}$ by*

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t}$$

Then $\mathfrak{h}$ is Cartan subalgebra of $\mathfrak{g}$.

We will consider only Cartan subalgebras of the form $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ where $\mathfrak{t}$ is a maximal commutative subalgebra of $\mathfrak{r}$. The Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is unique up to automorphism, hence its dimension makes sense:

Definition 8.7 (Rank)**.** *If $\mathfrak{g}$ is a complex semi-simple Lie algebra, its rank, denoted as* $\mathrm{rk}(\mathfrak{g})$*, is the dimension of any Cartan subalgebra.*

Obviously, for a semi-simple Lie algebra $\mathfrak{g}$, we have

$$\mathrm{rk}(\mathfrak{g}) = \dim \mathfrak{h} \leq \dim \mathfrak{g}$$

Here are some examples of rank:

1. $\mathrm{rk}(\mathfrak{u}^{\oplus N}(1)) = N$, as the Cartan subalgebra of $\mathfrak{u}^{\oplus N}(1)$ is itself.

2. $\mathrm{rk}(\mathfrak{su}(2)_{\mathbb{C}}) = 1$. Actually $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2;\mathbb{C})$ is the only rank 1 semisimple Lie algebra.

3. $\mathrm{rk}(\mathfrak{su}(n)_{\mathbb{C}}) = n-1$. The Cartan subalgebra of $\mathfrak{su}(n)$ is composed by $E_{k,k} - E_{k+1,k+1}$.

4. $\mathrm{rk}(\mathfrak{so}(2n)_{\mathbb{C}}) = n$. Its Cartan subalgebra is composed by blocked diagonal matrices whose the only nonzero diagonal element is the 2×2 submatrix:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Propsition 8.5.

$$\mathrm{rk}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \mathrm{rk}(\mathfrak{g}_1) + \mathrm{rk}(\mathrm{g}_2)$$

8.3 Roots

As the operators ad_H, $H \in \mathfrak{h}$ commute and ad_H's are simultaneously diagonalizable. If $X \in \mathfrak{g}$ is a simultaneous eigenvector for each ad_H, $H \in \mathfrak{h}$, then the corresponding eigenvalues depend linearly on $H \in \mathfrak{h}$. If this linear functional is nonzero, we call it a root. Besides, the linear functional is based on a finite dimensional vector space, thus according to Riesz represnetation theorem it's convenient to express this linear functional by inner product $H \mapsto \langle \alpha, H \rangle$ for some $\alpha \in \mathfrak{h}$, motivating us for the preceeding definition:

Definition 8.8 (Root)**.** *A nonzero $\alpha \in \mathfrak{h}$ is called a root if there exists a nonzero $X \in \mathfrak{g}$ such that*

$$[H, X] = \langle \alpha, X \rangle X, \quad \forall H \in \mathfrak{h}$$

Propsition 8.6. *Each root α belongs to $i\mathfrak{t} \subset \mathfrak{h}$.*

Proof. As each $\mathrm{ad}_H, H \in \mathfrak{t}$ is a skew self-adjoint on $\mathfrak{h}$, meaning ad_H has pure imaginary eigenvalues. Hence if α is a root, $\langle \alpha, H \rangle$ must be pure imaginary for $H \in \mathfrak{t}$. Since our inner product is real on $\mathfrak{t}$, it can only happen if α is in $i\mathfrak{t}$. □

Definition 8.9 (Root Space)**.** *If X satisfies $[H, X] = \langle \alpha, H \rangle X$, then aX ($a \in \mathbb{C}$) also satisfies $[H, aX] = \langle \alpha, H \rangle aX$, hence for a given root α, then the set of X forms a vector space $\mathfrak{g}_\alpha$ called the root space of the root α, while a nonzero element of $\mathfrak{g}_\alpha$ is called a root vector for α.*

More generally, if α is any element of $\mathfrak{h}$, we define $\mathfrak{g}_\alpha$ to be the space of all X in $\mathfrak{g}$ such that $[H, X] = \langle \alpha, H \rangle X, \forall H \in \mathfrak{h}$. If $\mathfrak{g}_\alpha$ is nontrivial, then it's naturally a root space, otherwise we don't call it a root space. We define $\mathfrak{g}_0 = \mathfrak{h}$.

As operators $\mathrm{ad}_H, H \in \mathfrak{h}$ are simultaneously diagonalizable, hence $\mathfrak{g}$ can be decomposed as the vector space direct sum of $\mathfrak{h}$ and the root spaces $\mathfrak{g}_\alpha$, denoted as

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_\alpha \mathfrak{g}_\alpha \right)$$

It provides us a way to calculate $\dim \mathfrak{g}$.

Propsition 8.7. *If $\alpha, \beta \in \mathfrak{h}$ and they are roots, we have $\forall X_\alpha \in \mathfrak{g}_\alpha, Y_\beta \in \mathfrak{g}_\beta$, $[X_\alpha, Y_\beta] \in \mathfrak{g}_{\alpha+\beta}$, if $\alpha + \beta$ is also a root.*

Proof. Consider Jacobi identity:

$$[H, [X_\alpha, Y_\beta]] + [X_\alpha, [Y_\beta, H]] + [Y_\beta, [H, X_\alpha]] = 0$$

Hence

$$[H, [X_\alpha, Y_\beta]] = [[H, X_\alpha], Y_\beta] + [X_\alpha, [H, Y_\beta]]$$

Since

$$[H, X_\alpha] = \langle \alpha, H \rangle X_\alpha \qquad [H, Y_\beta] = \langle \beta, H \rangle Y_\beta$$

Thus

$$[H, [X_\alpha, Y_\beta]] = (\langle \alpha, H \rangle + \langle \beta, H \rangle)[X_\alpha, Y_\beta] = \langle \alpha + \beta, H \rangle [X_\alpha, Y_\beta]$$

$$[X_\alpha, Y_\beta] \in \mathfrak{g}_{\alpha+\beta}$$

□

Corollary 8.2. *$X \in \mathfrak{g}_\alpha$ while $Y \in \mathfrak{g}_\alpha$. If $\alpha + \beta = 0$ then $[X, Y] \in \mathfrak{h}$. If $\alpha + \beta$ is neither 0 nor a root, then $[X, Y] = 0$.*

Proof.

$$[X,Y] \in g_{\alpha+\beta} = \mathfrak{g}_0 = \mathfrak{h}$$

While if $\alpha+\beta$ is not a root, then $\mathfrak{g}_{\alpha+\beta}$ is a trivial space $\{0\}$. □

Propsition 8.8. *1. If α is a root, then $-\alpha$ is also a root. Besides, if $X \in \mathfrak{g}_\alpha$, then X^* is in $g_{-\alpha}$.*

2. The roots span $\mathfrak{h}$.

Proof. For point 1, consider decomposing $X \in \mathfrak{g}_\alpha$ into $X = X_1 + \mathrm{i}X_2,\ X_1, X_2 \in \mathfrak{r}$. Let

$$\bar{X} = X_1 - \mathrm{i}X_2$$

Since $\mathfrak{r}$ is colsed under Lie bracket, then if $H \in \mathfrak{t} \subset \mathfrak{r}$ and $X \in \mathfrak{g}$, we have

$$\overline{[H,X]} = [H,\bar{X}] = [H,X_1] - \mathrm{i}[H,X_2] = [H,\bar{X}]$$

If X is a root vector with root $\alpha \in \mathrm{i}\mathfrak{t}$, then for all $H \in \mathfrak{h}$, we have

$$[H,\bar{X}] = \overline{\langle \alpha, H\rangle X} = -\langle \alpha, H\rangle \bar{X}$$

As $\langle \alpha, H\rangle$ is pure imaginary for $H \in \mathfrak{t}$. For all $H \in \mathfrak{h}$, it can be decomposed into $H = H_1 + \mathrm{i}H_2$,$H_1, H_2 \in \mathfrak{t}$. Hence for all $H \in \mathfrak{h}$, we have

$$[H,\bar{X}] = [H_1,\bar{X}] + \mathrm{i}[H_2,\bar{X}] = -\langle \alpha, H_1\rangle \bar{X} - \mathrm{i}\langle \alpha, H_2\rangle \bar{X} = -\langle \alpha, H\rangle \bar{X} = \langle -\alpha, H\rangle \bar{X}$$

Thus $\bar{X}$ is a root vector corresponding to root $-\alpha$. As $X^* = -\bar{X}$, X^* thus belongs to $\mathfrak{g}_{-\alpha}$.

For point 2, suppose the root didn't span $\mathfrak{h}$, then there would exist a nonzero $H \in \mathfrak{h}$ such that $\langle \alpha, H\rangle = 0$ for all α. (H is in the orthgonal complement of the space spanned by roots.) Then we will have $[H,H'] = 0$ for all $H' \in \mathfrak{h}$ and

$$[H,X] = \langle \alpha, H\rangle X = 0, \quad \forall X \in \mathfrak{g}_\alpha$$

Then $X \in \mathfrak{h}$ and $\mathfrak{h}$ would be a nontrivial abelian ideal, contradicting to the definition of semisimple Lie algebra. □

Theorem 8.3. *For each root α, we can find linearly independent elements $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and $H_\alpha \in \mathfrak{h}$ such that H_α is a multiple of α and*

$$\begin{aligned}
[H_\alpha, X_\alpha] &= 2X_\alpha \\
[H_\alpha, Y_\alpha] &= -2Y_\alpha \\
[X_\alpha, Y_\alpha] &= H_\alpha
\end{aligned}$$

Furthermore, Y_α can be chosen to equal to X_α^*. Besides, as $[H_\alpha, X_\alpha] = \langle \alpha, H_\alpha\rangle X_\alpha = 2X_\alpha$, we can choose

$$H_\alpha = \frac{2}{\langle \alpha, \alpha\rangle}\alpha$$

It's called the coroot associated to the root α.

Theorem 8.4. *1. For each root α, the only multiples of α that are roots are α and $-\alpha$.*

2. For each root α, the root space $\mathfrak{g}_\alpha$ is one dimensional.