A Literature Review on the Impact of Behavioral Finance on Asset Pricing Models

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Abstract

This paper provides a comprehensive review of classical asset pricing theories based on the achievements of behavioral finance research. The fundamental approach begins by explicating the basic framework of asset pricing—general equilibrium in exchange economies—then discusses the modifications made by subsequent researchers on this framework, based on the foundations of behavioral finance research. In other words, they explore the intuitive modifications that make the model's simulated results more congruent with reality. We mainly discuss two primary directions of modification: the amendment of non-random factors within the basic framework and the adjustment of random factors. The modification of non-random factors includes revisions to the utility function and discount factors, while the adjustment of random factors primarily involves introducing heterogeneity in probability measures. To accurately discern the implications of the models, we include key equations and derivations from the literature, striving to maintain consistency in the notation. Lastly, we compare the performance of various models in both types of modifications, briefly presenting an outlook on future research directions.

Keywords: Behavioral finance, asset pricing, general equilibrium

A Literature Review on the Impact of Behavioral Finance on Asset Pricing Models

Asset pricing is a major important area in modern financial theory, alongside corporate finance. The earliest theoretical research in this field can be traced back to Markowitz's (1952) static portfolio selection theory, which laid the foundation for the famous CAPM asset pricing model by Sharpe (1964) and Merton's (1969) dynamic pricing model in continuous time within the general equilibrium perspective of security market clearance. Thus far, asset pricing theory has been built upon the foundation of portfolio selection theory.

Nearly a decade later, Lucas (1978) proposed a general asset pricing theory that incorporated consumption optimization within an exchange economy, based on the consumption-portfolio selection problem. He introduced the concept of rational expectations equilibrium, which sparked the rational expectations revolution and preliminarily established the framework of stochastic discount factor (SDF) pricing. The rational expectations model, being a nearly perfect and ideal paradigm, became the basic framework for many subsequent theoretical research papers on asset pricing.

However, the good times did not last, as the rational expectations model's pricing predictions were quickly challenged by empirical evidence. A direct attack on the theory came from Mehra and Prescott (1985), who proposed the famous equity premium puzzle based on a simple Lucas parameterized model. Subsequently, a plethora of market anomalies emerged that contradicted the efficient market hypothesis and rational expectations, such as the volatility puzzle, the risk-free rate puzzle, the closed-end fund discount puzzle, intermediate-term momentum, and long-term reversals. As a result, the field of behavioral finance began to gain increasing attention.

One of the earliest research studies that emerged as a theoretical foundation for behavioral finance was prospect theory, proposed by Kahneman et al. in 1979. The introduction of this theory had a significant impact on rational expectations theory. Technically, the utility function was replaced by a period-inconsistent, relative gain-loss based value function, and probability measures no longer adhered to Bayes' rule. This period inconsistency and non-concavity, or ambiguity in probability measures, technically shook the compressive imaging operator in the rational expectations learning process, thereby affecting the stability of equilibrium.

Consequently, economists began to attempt modifications to Lucas's basic framework to accommodate classical market anomalies after incorporating the intuitions of behavioral finance. Some renowned models, albeit starting from completely different perspectives, converged in a sense and formed some interesting and unique intuitions. A summary of this work has been provided by Chen Yanbin and Zhou Ye'an (2004), but their work primarily focuses on the modification of non-random factors. This article attempts to include some models that address the modification of random factors, discussing their starting points, similarities, and differences, in order to establish a basic blueprint for modifying asset pricing theory based on behavioral finance.

Benchmark Model

Rational Expectations Framework

Model Setup

In 1978, Lucas presented a standard discrete-time model within a purely exchange economy with n sectors, where production is entirely exogenous, expressed as:

$$y_t \coloneqq (y_{1t}, \dots, y_{nt})$$

The dynamics of output are set as a Markov process, with output levels and transition probabilities encapsulating all information about the economy. The transition probabilities are given in a general form:

$$F(y', y) \coloneqq \mathbb{P}\{y_{t+1} \leq y' | y_t = y\}$$

There exists a capital market within the economy, comprising two types of assets: one risk-free asset with a total supply of 0 (implying no net demand in symmetric equilibrium) and a normalized risk-free interest rate of 0; and a second class of equity assets, with each sector having a measure of 1 for shares, and post-dividend prices as follows:

$$p_t \coloneqq (p_{1t}, \dots, p_{nt})$$

Consumer measure is also 1, with consumers holding shares each period as:

$$z_t \coloneqq (z_{1t}, \dots, z_{nt})$$

The consumers aim to maximize utility, considering the following problem:

$$\max_{c_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t) \right\}$$

$$s.t. 0 \leq c_t \leq \sum_{i=1}^{n} y_{it}$$

The policy function for consumers is denoted by:

$$\begin{cases} c_t = c(z_t, y_t, p_t) \\ z_{t+1} = g(z_t, y_t, p_t) \end{cases}$$

The pricing function within the economy is denoted by:

$$p_t = p(y_t)$$

Rational expectations refer to the notion that the actual pricing function $p(\cdot)$ and the pricing function $p^*(\cdot)$ that clears the market, given consumer policy functions $c(\cdot), g(\cdot)$, are consistent.

Equilibrium

In equilibrium, both quantity equilibrium and price equilibrium exist concurrently. Quantity equilibrium denotes the simultaneous clearing of both the goods market and the capital market:

$$\begin{cases} c_t = \sum_i y_{it} \\ z_t = (1, \ldots, 1) \end{cases}$$

Although not explicitly explained in the literature, in a fully private economy, the output of the production sector is equivalent to the dividends it distributes. Consequently, y_t also represents the dividend stream. In equilibrium, each consumer's consumption is entirely derived from the dividends received, a concept applied throughout all subsequent literature based on this foundation. The price equilibrium, also known as the rational expectations equilibrium, is defined as a combination of the pricing function $p(y)$ and the value function $v(z, y)$ that simultaneously satisfies the aforementioned quantity equilibrium and the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} & v(z, y) = \max_{c,x} \left\{ U(c) + \beta \int v(x, y') dF(y', y) \right\} \\ & s.t. \, c + p(y) \cdot x \leq y \cdot z + p(y) \cdot z \end{aligned}$$

In equilibrium, the stochastic Euler equation (SEE) for this model is:

$$U'\left(\sum_i y_i\right) p_i(y) = \beta \int U'\left(\sum_j y_j'\right) \big(y_i' + p_i(y')\big) dF(y', y) \tag{1}$$

Lucas demonstrates, through a series of contraction mapping theorems, that when $F(\cdot)$ has a stationary distribution, and $U(\cdot)$ is continuous, differentiable, monotonic, bounded, and strictly concave, there exists a unique solution for the pricing function, derived from the aforementioned SEE, and this equilibrium is stable. This implies that even if the economy initially starts from a non-equilibrium state (v, p) and instead lies in a combination of an alternate value function and pricing function (u, p), due to the contraction mapping operator M:

$$M[u(z,y)] \coloneqq \inf_{q \in E^{n+}} \left\{ \sup_{c,x} \left[U(c) + \beta \int u(x, y') dF(y', y) \right] \right\}$$
$$s.t.\, c + q \cdot x \leq y \cdot z + q \cdot z$$

$M^n v \to u$, The economy can still be guaranteed to converge from the initial state (u, q) to the equilibrium state (v, p) after repeated trial and error and myopic learning.

The intuition behind Lucas's paper lies in the consistent preferences of consumers and the desirable properties of their utility functions, which ensure the contraction mapping theorem holds at all times. Consequently, the actual pricing function aligns with the rational pricing function, and the equilibrium under rational expectations serves as a reasonable approximation for actual behavior.

Two-Period Model and Intuition

Complete Market Framework

Model Setup

Although the concept of a complete market was not first introduced by Shefrin and Statman (1994) —in fact, the theory of complete markets based on Arrow-Debreu state prices was established much earlier—this paper will later be referenced as an expository article. Therefore, its framework is presented here as the complete market framework.

Unlike Lucas, who incorporated information into the exogenous economic output, Shefrin and others treat information as a separate state variable in the economy. They assume that new information $s^t \in S = \{s_i\}$ is published at the beginning of each period. Thus, the public information (which can also be considered as history or a multi-branched tree) at the beginning of each period is denoted by:

$$x_t \coloneqq (s^0, \dots, s^t)$$

The information itself is dynamic and is described as a Markov process with a transition probability matrix (effectively converting Lucas's continuous-state model into a discrete-state multi-branched tree model, but without affecting the underlying intuition):

$$\Pi \coloneqq [\Pi_{ij}] \coloneqq [\mathbb{P}\{s^{t+1} = s_j | s^t = s_i\}]$$

The total population in the economy is denoted by H, with individual investors indicated by the subscript h. If an investor holds a portfolio ω_h during period t, they receive the actual dividend $\omega_h(x_t)$ after x_t is realized.

A financial asset is represented by a vector $Z = [Z(x_t)]$, where $Z(x_t)$ represents the dividends paid to the owner of one unit of security under state x_t (consistent with the notation of $\omega_h(x_t)$). In a complete market, since there are enough Arrow-Debreu securities to define state prices, the term $r(x_t)$ can be used to represent the price of one unit of contingent claim in state x_t at time 0 (with the price of the time 0 contingent claim normalized to $r(x_0) = 1$).

With state prices in place, security prices can be easily expressed. In the time 0 market, the value of security $Z = [Z(x_t)]$ can be defined as $q_Z(x_0) = r \cdot Z$; in the time t market, if we define $r'(x_t)$ as a vector, the number j value is equal to $r(x_j)$ for all values after t and equal to 0 for all values before t, then the uniform representation of security prices in the x_t market is:

$$q_Z(x_t) = \frac{r'(x_t) \cdot Z}{r(x_t)}$$

Each period, consumers use a portion of their wealth (dividends received) for consumption and the remainder to form their investment portfolio in state x_t. Assuming the trading volume for consumer in state $x_t$is $z_h(x_t)$ and the consumption sequence is $c_h \coloneqq [c_h(x_t)]$, while:

$$c_h = \omega_h + z_h$$

Then represent consumer's belief as a subjective probability distribution $P_h(x_t)$ for $\{x_t\}$. The optimization problem considered by consumer h is set up as follows:

$$\max \mathbb{E}U_h(x_t) \coloneqq \mathbb{E}\sum_{t=1}^{T} \beta_h^{t-1} \ln[c(x_t)]$$
$$s.t. r \cdot z_h \leq 0$$

Equilibrium

In equilibrium, the supply and demand for securities in the market are balanced, with r denoting the equilibrium state prices. Prices clear the market, and for each h, the wealth share is denoted by:

$$w_h \coloneqq \frac{W_h}{\sum_j W_j} \coloneqq \frac{r \cdot \omega_h}{\sum_j r \cdot \omega_j}$$

We can then define a representative investor for the market, characterized by:

Subjective probability distribution $\{\Gamma(x_t)\}$, defined as:

$$\Gamma(x_t) \coloneqq \sum_h \delta_h P_h(x_t) \coloneqq \sum_h \frac{w_h \beta_h^t}{\sum_j w_j \beta_j^t} P_h(x_t)$$

Integrated discount factors $\{\beta_R(t)\}$, defined as:

$$\beta_R(t) \coloneqq \sum_h w_h \beta_h^t$$

Using this representative investor, the equilibrium state prices for the entire market can be derived from the Euler equation:

$$r(x_t) = \frac{\beta_R(t)\Gamma(x_t)\omega(x_0)}{\omega(x_t)} \tag{2}$$

The intuition here is that the state price = market subjective probability/market total dividend growth discounted. This equation can serve as a necessary and sufficient condition for an efficient market. It is important to note that for rational expectations equilibrium, $P_h = \Gamma = \Pi$, meaning that the subjective probabilities of all economic agents align with the actual probabilities of the market state transitions. Here, we have three belief concepts: individual beliefs P_h, market beliefs Γ, and "true" beliefs Π.

In a complete market, assuming homogeneous discount rates (i.e., $\beta_h \equiv \beta$) and using $\omega = \sum \omega_h$ to represent the market portfolio, the total return (original value + net return) for purchasing the market portfolio Z_ω in the x_t market and holding it for j periods can be expressed as:

$$R_{\omega.j}(x_{t+j}) = \beta^{-j} \frac{\omega(x_{t+j})}{\omega(x_t)}$$

The equilibrium term structure of interest rates for bonds traded in the x_t market and maturing at $t + j$ can be expressed as:

$$\left(1 + i_j(x_t)\right)^j = \left\{\mathbb{E}_\Gamma\left[R_{\omega.j}(x_{t+j})^{-1} \middle| x_t\right]\right\}^{-1}$$

Basic Asset Pricing Equation

The SDF (stochastic discount factor) model can be seen as a unified form that encompasses existing asset pricing theories. As Campbell (2000) elaborates in his millennial asset pricing review paper, existing asset pricing equations can be summarized as:

$$1 = \mathbb{E}_t\left[M_{t+1}R_{i,t+1}\right] \tag{3}$$

In this equation, M_{t+1} is the SDF, $R_{i,t+1}$ is the total return of asset i in period $t \to t+1$ and the risk-free rate $R_{f,t}$ is defined as the total return of the risk-free asset $t \to t+1$. A rough definition is:

$$R_{i,t+1} \coloneqq \frac{P_{t+1} + D_{t+1}}{P_t}$$

The reason this equation is fundamental is that it is equivalent to the other two basic methods of asset pricing (Lucas paradigm's consumption-investment portfolio model and CAPM paradigm's risk premium pricing model). For instance, equation (1) can be rewritten as:

$$1 = \int \frac{\beta U'\left(\sum_j y_j'\right)}{U'(\sum_i y_i)}\left(\frac{y_i' + p_i(y')}{p_i(y)}\right) dF(y', y) \tag{1'}$$

This is formally consistent with (3), specifically:

$$M_{t+1} = \frac{\beta U'\left(\sum_j y_j'\right)}{U'(\sum_i y_i)}$$

Thus, the essence of the SDF equation is the Euler equation, with M_{t+1} interpreted as the intertemporal marginal rate of substitution (MRS). Equation (2), being a direct extension of the Lucas framework, can also be transformed into a similar form. For CAPM-like models, the SDF equation can be reformulated into an equivalent CAPM form:

$$\mathbb{E}_t(R_{i,t+1} - R_{f,t}) = \frac{-Cov(M_{t+1}R_{i,t+1} - R_{f,t})}{\mathbb{E}_t M_{t+1}}$$

Using the SDF model, it is easy to express some market anomalies. For instance, the equity premium puzzle and the risk-free rate puzzle can be considered under the CRRA utility with a relative risk aversion (RRA) parameter set to γ. The puzzles essentially state that there is no γ that satisfies the following two equations simultaneously:

$$\begin{cases} 0 = \mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}(R_{i,t+1} - R_{f,t})\right] \\ 1 = \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{f,t}\right] \end{cases}$$

The equity premium puzzle specifically refers to the γ obtained from the first equation not matching the actual economic situation, while the risk-free rate puzzle specifically refers to the risk-free rate obtained from the second equation being too high compared to reality when γ is high.

In the following discussions, we will discuss the equilibrium solutions of the models up to the point where explicit pricing functions, SDF equations, or risk premium equations appear. The work prior to this is theoretical

Non-random Factor Adjustments

Adjustments to $U(\cdot)$

General Form

In Lucas's standard framework, the consumer's objective function is:

$$\max_{c_t} \mathbb{E}_0 \left\{\sum_{t=0}^{\infty} \beta^t U(c_t)\right\}$$

This represents the simplest case of time-separable utility. Adjustments to this aspect have a unified form:

$$\max_{c_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t, v_t) \right\}$$

Here, v_t is a behavioral factor, generally a function of economic state variables, and therefore does not participate as a control variable in the maximization. The model acquires different meanings based on the settings of v_t.

Wealth Sentiment

Wealth sentiment implies that individuals derive pleasure not only from consumption but also from financial assets (or wealth levels). Notable studies include Bakshi, Chen (1996) and Barberis, Huang, Santos (1999).

Bakshi and colleagues propose a basic framework: Consider n risky asset prices following a geometric Brownian motion with undetermined parameters:

$$\frac{dp_{i,t}}{p_{i,t}} = \mu_{i,t} dt + \sigma_{i,t} d\omega_{i,t}$$

The consumer faces the problem of consumption C_t- and portfolio $\vec{\alpha}_t$ selection:

$$\max_{\{c_t, \vec{\alpha}_t\}} \sum_{t=0}^{\infty} e^{-\rho t} \mathbb{E}_0 U(c_t, S_t) \Delta t, \qquad t = 0, \Delta t, 2\Delta t, \ldots$$

$$s.t. \Delta W_t = \left\{ r_0 W_t - c_t + W_t \sum_{i=1}^{N} \alpha_{i,t} (\mu_{i,t} - r_0) \right\} \Delta t + W_t \sum_{i=1}^{N} \alpha_{i,t} \sigma_{i,t} \Delta \omega_{i,t}$$

Here, $S_t = f(W_t, V_t)$ represents social status, a function of individual total wealth level W_t and average social wealth level V_t. The optimal solution $(c^*, \vec{\alpha}^*)$ should satisfy the SDF equation:

$$1 = e^{-\rho \Delta t} \mathbb{E}_t \left\{ \left[\frac{U_{c,t+\Delta t} + U_{W,t+\Delta t} \Delta t}{U_{c,t}} \right] R_{i,t+\Delta t} \right\} \tag{4}$$

And the risk premium pricing equation:

$$\mu_{i,t} - r_0 = -\frac{C_t^* U_{cc}}{U_c}\sigma_{i,c} - \frac{W_t^* U_{cW}}{U_c}\sigma_{i,w} - \frac{V_t U_{cV}}{U_c}\sigma_{i,v} \tag{5}$$

Where:

$$\sigma_{i,x} \coloneqq Cov_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dX_t^*}{X_t^*}\right)$$

Based on this, parametrization can generate:

Absolute Wealth Model: Individuals derive pleasure based on absolute wealth levels:

$$U(c_t, W_t, V_t) = \frac{c_t^{1-\gamma}}{1-\gamma} W_t^{-\lambda}$$

Ratio Wealth Model: Individuals derive pleasure according to their relative position of wealth in society:

$$U(c_t, W_t, V_t) = \frac{c_t^{1-\gamma}}{1-\gamma}\left(\frac{W_t}{V_t}\right)^{-\lambda}$$

Self-evaluation Model: Individuals derive pleasure by comparing wealth with a subjective level:

$$U(c_t, W_t, V_t) = \frac{c_t^{1-\gamma}}{1-\gamma}(W_t - \kappa V_t)^{-\lambda}$$

By introducing wealth preference and the non-random nature of single-period risk-free interest rates in the general framework according to equation (4), we have:

$$R_{f,t} = \frac{1}{\mathbb{E}_t M_{t+1}} = \frac{U_{c,t}}{\mathbb{E}_t\left[U_{c,t+\Delta t} + U_{W,t+\Delta t}\right]}$$

The larger U_W is, the smaller the risk-free interest rate. As long as a sufficiently high wealth preference is set, a low risk-free interest rate can be generated even with relatively high risk aversion. However, in this paper, the explanatory power for the equity premium puzzle is very limited, as only some extreme parameter values can result in an acceptable range for the equity premium.

Building on this, Barberis adopted prospect theory from behavioral finance, proposing a new objective function form, generally referred to as the BHS model. First, assume total consumption exhibits iid exogenous growth (a common assumption in many papers):

$$\ln\left(\frac{\bar{c}_{t+1}}{\bar{c}_t}\right) = g + \sigma\epsilon_{t+1}, \qquad \epsilon \sim iidN(0,1)$$

Where: $\bar{c}$ represents per capita consumption in the economy, which should equal the dividends paid by risky assets each period in equilibrium. The lifetime optimization problem is set as:

$$\max \mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(\beta^t \frac{c_t^{1-\gamma}}{1-\gamma} + b_t\beta^{t+1}S_t\hat{v}(R_{t+1}, z_t)\right)\right]$$
$$s.t. W_{t+1} = (W_t - c_t)R_{f,t} + S_t\left(R_{t+1} - R_{f,t}\right)$$

Here, $b_t = b_0\bar{c}_t^{-\gamma}$ is a scaling parameter for wealth utility relative to consumption utility, which is set exogenously; S_t represents the value of risky assets held in period t (its placement outside $\hat{v}$ indicates that we are using a kinked linear value function from prospect theory), and the specific form of the value function $\hat{v}$ is as follows:

$$\hat{v}(R_{t+1}, z_t) = \begin{cases} R_{t+1} - R_{f,t}, & R_{t+1} \geq z_t R_{f,t} \\ \left(z_t R_{ft} - R_{f,t}\right) + \lambda\left(R_{t+1} - z_t R_{f,t}\right), & R_{t+1} < z_t R_{f,t} \end{cases}, \quad z_t \leq 1$$

$$\hat{v}(R_{t+1}, z_t) = \begin{cases} R_{t+1} - R_{f,t}, & R_{t+1} \geq R_{f,t} \\ \lambda(z_t)\left(R_{t+1} - R_{f,t}\right), & R_{t+1} < R_{f,t} \end{cases}, \quad z_t > 1$$

As can be seen, the risk-free interest rate serves as a basic reference point, and z_t is a quantity reflecting the prior reference point movement, defined as:

$$z_t \coloneqq \frac{Z_t}{S_t}$$

Z_t is an a priori benchmark value level. In the presence of prior gains, loss aversion is an increasing function of z_t:

$$\lambda(z_t) = \lambda + k(z_t - 1), \qquad z_t \geq 1$$

Furthermore, the dynamics of z_t are modeled as a process dependent on parameter η (cross-period reference point adjustment speed) and fixed constant $\bar{R}$:

$$z_{t+1} = \eta\left(z_t \frac{\bar{R}}{R_{t+1}}\right) + 1 - \eta$$

The SDF pricing equation for this model is:

$$\begin{cases} 1 = \beta R_{f,t}\mathbb{E}_t\left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{-\gamma}\right] \\ 1 = \beta\mathbb{E}_t\left[R_{t+1}\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{-\gamma}\right] + b_0\beta\mathbb{E}_t[\hat{v}(R_{t+1}, z_t)] \end{cases} \tag{6}$$

In a more realistic scenario with $\gamma = 0.9, \lambda = 2.25$, numerical simulations allowing g and $\sigma$to exhibit real-world characteristics reveal that this model can satisfactorily explain the risk-free interest rate puzzle, equity premium puzzle, and volatility puzzle simultaneously. The reason lies in the additional wealth effect in the R_{t+1} pricing equation, which can be qualitatively understood as loss aversion compensating for some risk aversion (especially downside risk aversion). Loss aversion is dynamic, generating fluctuations in the intertemporal substitution rate and consequently amplifying SDF fluctuations. This results in more severe equity return volatility (simulated volatility as high as 14%) and a higher risk premium. Although the equity premium does not reach the realistic level of around 6%, it is still substantial at 4%. Since a high γ is not required to support this risk premium, the risk-free interest rate puzzle is readily resolved.

Compared to Bakshi and colleagues' results, the BHS model, by incorporating findings from behavioral finance, performs better but is more complex and less intuitive, with many intuitions reliant on numerical simulations. Moreover, the BHS model does not utilize all the results of prospect theory. In prospect theory, the subjective probability measure $\pi(p)$ is a crucial component for explaining all empirical evidence but has not been considered within the model.

Despite its impressive numerical performance, the model's logical shortcoming is that we can easily replace the wealth level with another variable, as long as it can be reasonably translated into utility. In this way, the model loses its original intuition and can be used to study any issue (such as fertility or population), thereby failing to explain any problem specifically.

Consumption Habits

The basic idea of consumption habits is that an individual's utility in each period is influenced not only by current factors but also by past consumption. This makes the utility function time-inseparable. However, compared to introducing another variable into utility, the focus on consumption itself as a source of pleasure is the logical advantage of consumption habits over wealth sentiment.

The main theoretical work in this area has been done by Constanitinides (1990) and Campbell, Cochrane (1999). Constanitinides' model can be considered a continuous-time linear approximation of Campbell and colleagues' model. Both models share the same objective function settings but differ in their habit modeling, so we will focus on explaining Campbell and colleagues' model.

We still assume that total consumption exhibits iid exogenous growth, as follows:

$$\ln\left(\frac{c_{t+1}}{c_t}\right) = g + v_{t+1}, \qquad v \sim iidN(0, \sigma^2)$$

Suppose x_t is some habitual consumption level; then, the consumer maximization problem is:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{(c_t - x_t)^{1-\gamma} - 1}{1 - \gamma}$$

Define a surplus consumption ratio (as the state variable for the entire economy):

$$s_t := \frac{c_t - x_t}{c_t}$$

The general dynamic modeling of consumption habits can be described as an AR(1) process:

$$\ln s_{t+1} = (1 - \phi) \ln \bar{s} + \phi \ln s_t + \lambda(\ln s_t)(\ln c_{t+1} - \ln c_t - g)$$

where $\lambda(\cdot)$ is referred to as the sensitivity function. A linear approximation of this model can be expressed as follows,

$$\ln x_{t+1} \approx \left[h + \frac{g}{1 - \phi}\right] + (1 - \phi) \sum_{j=0}^{\infty} \phi^j c_{t-j}$$

with $h := \ln(1 - \bar{s})$ representing the steady-state value of $\ln x_t - \ln c_t$. This linear approximation is equivalent to Constantinides (1990), which takes the form of a continuous time process:

$$x(t) = e^{-at}x_0 + b\int_0^t e^{\alpha(s-t)}c(s)ds$$

Essentially, both equations represent habits as an exponentially weighted average of past consumption, with weights decreasing as the time interval increases. In Campbell's model, this representation allows for the derivation of the SDF pricing equation.

$$1 = \mathbb{E}_t[M_{t+1}R_{t+1}], \qquad M_{t+1} = \rho\left(\frac{S_{t+1}}{S_t}\frac{c_{t+1}}{c_t}\right)^{-\gamma} = \rho e^{-\gamma(\ln s_{t+1} - \ln s_t + v_{t+1} + g)} \tag{7}$$

Consumption habits can explain the equity premium puzzle and the risk-free rate puzzle. Boldrin, Christiano, and Fisher (2001) demonstrate excellent model performance in a simple parametric case. Their study focuses on a production economy with the following objective function:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t[\ln(c_t - bc_{t-1}) - h_t]$$

Here, h_t represents labor, and this is essentially the simplest case for habits: $x_t = bc_{t-1}$. Maximum likelihood estimation yields $b = 0.73$, which remarkably fits the risk-free rate, equity premium, and volatility of risky asset returns in a two-sector production friction model.

However, the consumption habit model has a common issue: it generates a new puzzle, the risk-free rate volatility puzzle. To illustrate this problem, consider the steady-state situation in Campbell's model:

$$\ln R_{f,t} = -\ln\rho + \gamma g - \left(\frac{\gamma}{\bar{s}}\right)^2 \frac{\sigma^2}{2}$$

This shows that the risk-free rate is still highly correlated with past consumption habits ($\bar{s}$) and is very sensitive to consumption. In fact, even in the well-fitted models of

Boldrin and others, the volatility of the risk-free rate is four times that of reality.

Herding Behavior

Herding behavior shares a similar starting point with consumption habits. Instead of introducing another variable in utility, it focuses on the consumption itself. If consumption habit research examines the effects induced by individual consumption time series correlation, herding behavior studies the effects of individual consumption cross-sectional externalities. That is, when an individual consumes, the overall consumption level of those around them also influences their utility. This can be simply understood as modeling a kind of herding or envious mentality.

The earliest work in this area is by Abel (1990), whose theory encompasses both consumption habits and herding behavior. Gali (1994) later removed the consumption habit component, focusing solely on herding behavior. Abel's objective function is as follows:

$$\max \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j \frac{[c_t/v_t]^{1-\alpha}}{1-\alpha} \right\}$$

Here, $v_t \coloneqq [c_{t-1}^D \bar{c}_{t-1}^{1-D}]^\gamma$, where $\bar{c}$ represents average consumption in the economy. The pricing equation is:

$$1 = \mathbb{E}_t \left\{ \beta R_{t+1} \frac{H_{t+2}}{\mathbb{E}_t H_{t+1}} x_t^{\gamma(\alpha-1)} x_{t+1}^{-\alpha} \right\} \tag{8}$$

With $H_{t+1} \coloneqq 1 - \beta\gamma D x_t^{\gamma(\alpha-1)}$, $x_t \coloneqq y_{t+1}/y_t = c_{t+1}/c_t = \bar{c}_{t+1}/\bar{c}_t$. This model can explain the equity premium puzzle and the risk-free rate puzzle, but like all consumption habit models, it also generates the risk-free rate volatility puzzle.

Gali's model is simpler in form:

$$\max \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} \bar{c}_t^{\sigma\alpha}}{1-\alpha} \right\}$$

Since $c_t = \bar{c}_t$ in symmetric equilibrium, the asset pricing equation takes a simple form:

$$1 = \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha(1-\sigma)} R_{t+1} \right] \tag{9}$$

If $\sigma > 0$, consumption externality will indeed affect asset pricing. However, without considering speculative bubbles, if the standard model's RRA parameter $\gamma = \alpha(1-\sigma)$, this model becomes indistinguishable from the standard model. As a result, $\alpha(1-\sigma)$ as a whole essentially plays the same role as the risk aversion coefficient γ. Thus, herding behavior alone cannot explain the equity premium puzzle or the risk-free rate puzzle.

Revisions to $\boldsymbol{\beta}$

Subjective Discount Factor

In general asset pricing literature, β is assumed to be a constant, serving as an exogenous parameter within the economy. In response to this assumption, Becker and Mulligan (1997) proposed an endogenous mechanism to determine the intertemporal discount factor. This model is quite generalized, with its practical implications outweighing its real-world applicability. The core idea is that the more time and effort people spend imagining future scenarios, the more significant the future becomes. Assuming the resource amount spent on imagination is S, the discount factor is set as a function of $\beta(S)$. Consider a simple optimization problem in a complete market:

$$\max V = \sum_{i=0}^{T} [\beta(S)]^i U_i(c_i)$$

$$s.t.\ \sum_{i=0}^{T} R_i c_i + \pi S = A_0$$

Here, U_i represents the different installment utility functions for each period, R_i is the market interest rate discount factor, π is the price of resources spent on imagination, and A_0 is the present value of all asset income, considered as an endowment. The first-order condition for the above problem is:

$$\beta'(S)\left[\sum_{i=1}^{T} i[\beta(S)]^{i-1}U_i(c_i)\right] = \lambda_0 = U_0'(c_0) \tag{10}$$

Within a two-period problem, this can be simply expressed as:

$$\beta'(S) = \frac{U_0'(c_0)}{U_1'(c_1)}$$

That is, the future utility MRS determines the optimal use of imaginative resources, which in turn determines the endogenous optimal discount rate.

Subjective Emotional Fluctuations

Since the discount factor can be endogenously determined, emotional fluctuations should introduce random disturbances to the discount factor. Thus, β can be treated as a random variable. Mehra and Sah (2002) conducted a numerical analysis based on this simple idea. They consider the most basic standard model of Prescott (1985):

$$\max \mathbb{E}_0 \left\{\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}\right\}$$

If we set the random growth rate of dividends to follow a geometric Brownian motion:

$$dx = \mu x dt + \sigma x dz$$

The pricing function can then be expressed explicitly:

$$p_t = y_t \frac{\beta e^{(1-\gamma)\left(\mu-\frac{1}{2}\gamma\sigma^2\right)}}{1-\beta e^{(1-\gamma)\left(\mu-\frac{1}{2}\gamma\sigma^2\right)}} \tag{11}$$

Comparing the static analysis of β within this expression yields:

$$\epsilon_\beta \coloneqq \frac{\partial \ln p}{\partial \ln \beta} = \frac{1}{1-\beta e^{(1-\alpha)\left(\mu-\frac{1}{2}\alpha\sigma^2\right)}} > 0$$

Thus, within a small local fluctuation range, we can approximate:

$$std\left[\frac{\tilde{p}-p}{p}\right] = \left|\epsilon_\beta\right| std\left[\frac{\tilde{\beta}-\beta}{\beta}\right]$$

This means that the standard deviation of β itself will be amplified by $\left|\epsilon_\beta\right|$ times, resulting in the standard deviation of the price. Numerical simulations show that $\left|\epsilon_\beta\right|$ is a large number under reasonable parameter settings, reaching several tens of times when the risk aversion coefficient is relatively small. This helps to explain the volatility puzzle to some extent.

Random Factor Corrections

Random Heterogeneous Expectations: DSSW Model

Previous studies on equilibrium have focused on symmetric general equilibrium, without considering the heterogeneity of investors' beliefs, and thus not generating heterogeneity in consumption-investment portfolios. However, Delong, Shleifer, Summers, and Waldmann (1990) introduced the concept of belief heterogeneity with their noise trader model. Although directly specifying different expectations is a simple and arbitrary approach, the results are quite satisfactory and have laid the groundwork for subsequent literature.

The DSSW model is set within a simplified overlapping generations (OLG) framework, where individuals live for two periods. In the first period, there is no consideration of labor allocation issues or consumption, only asset allocation. In the second period, assets are sold, dividends are received, and all wealth is consumed. There are two types of investors: rational investors (i) with a measure of $1-\mu$ and noise traders (n) with a measure of μ. Their utilities are the same, both being CARA utilities:

$$U = -e^{-(2\gamma)w}$$

Here, γ is the ARA, and $w \sim N(\overline{w}, \sigma_w^2)$. However, their beliefs are not consistent. Rational investors expect the correct asset distribution, but young noise traders in generation t have an expected mean future price of:

$$\rho_t \sim N(\rho^*, \sigma_p^2)$$

Here, ρ^* is interpreted as the average measure of bullishness. The normal distribution assumption of asset distribution and CARA utility simplifies the utility maximization process. Assuming the risky asset price is p_t, both risk-free and risky assets generate real dividends r, with no fundamental risk. Using the first-order conditions, the demand for risky assets by both types of investors can be easily derived:

$$\lambda_t^i = \frac{r + \mathbb{E}_t p_{t+1} - p_t(1+r)}{2\gamma\sigma_{t,p_{t+1}}^2}$$

$$\lambda_t^n = \frac{r + \mathbb{E}_t p_{t+1} - p_t(1+r)}{2\gamma\sigma_{t,p_{t+1}}^2} + \frac{\rho_t}{2\gamma\sigma_{t,p_{t+1}}^2}$$

$$\sigma_{t,p_{t+1}}^2 \coloneqq \mathbb{E}_t\{[p_{t+1} - \mathbb{E}_t(p_{t+1})]^2\}$$

This indicates that demand is proportional to excess returns and inversely proportional to risk. Since rational investors serve as a stabilizing factor (reducing volatility through arbitrage), all risk is attributed to noise traders.

Under market equilibrium, the total demand for risky assets should be 1. The final steady-state pricing function can be solved as:

$$p_t = 1 + \frac{\mu(\rho_t - \rho^*)}{1+r} + \frac{\mu\rho^*}{r} - \frac{2\gamma\mu^2\sigma_p^2}{r(1+r)^2} \tag{12}$$

The first term reflects the fundamental value; the second term indicates that the relative over- or underestimation of noise traders compared to the generational average will push asset prices up or down; the third term shows that the generational average estimation error itself will cause asset prices to deviate from fundamentals; the fourth term is the core of the model, demonstrating how noise traders create their own survival space.

On the one hand, the economy has no fundamental risk, and the only source of risk is the uncertainty of young noise traders' price estimation mean ρ_t in each period. This uncertainty leads both rational and noise trader camps to believe that prices are mispriced (deviating from fundamental values), but they do not dare to bet on this direction. Since the next period will have a new generation of noise traders, the realization of ρ_t is also uncertain. In other words, this risk limits investors' positions, preventing rational traders from making unlimited arbitrage transactions to eliminate price deviations.

On the other hand, noise traders' erroneous estimations generate σ_p^2, which, according to the pricing equation, depresses current prices, thereby raising the intertemporal return on holding risky assets. As a result, people are more willing to hold these risky assets. The combination of both factors allows for demand and clearing of risky assets, forming an equilibrium.

By equalizing the expected returns of noise traders and rational investors, an "equilibrium" noise trader measure can even be derived:

$$\mu^* = \frac{(1+r)^2\rho^{*2} + (1+r)^2\sigma_p^2}{2\gamma\rho^*\sigma_p^2}$$

The DSSW model can explain the equity premium puzzle, the closed-end fund discount puzzle, and the volatility puzzle through noise trader risk, without altering the risk aversion coefficient. Noise trader risk leads to stock prices being undervalued, returns becoming higher, or closed-end fund share prices being undervalued, resulting in a discount. The volatility of beliefs can directly translate into stock price volatility, all without the need to increase the risk aversion coefficient. As noise traders indeed create their own survival space, this mechanism is undeniable, even as a qualitative result.

The core intuition of DSSW is that people are overlapping generations, limiting the holding period and not allowing enough time to learn the steady-state distribution (or suggesting that there may not be a so-called steady-state distribution at all). This breaks an essential assumption in the Lucas framework, ultimately leading to a deviation of the pricing function from fundamentals and the inability to achieve rational expectations.

Non-Bayesian Learning: BAPM Model

We previously outlined the first half of the model proposed by Shefrin and Statman (1994) as our benchmark model, while the latter half incorporating behavioral finance factors contrasts with CAPM and is called BAPM, or the Behavioral Capital Asset Pricing Model.

The model is based on the complete market framework, and its core theorem is the essence of BAPM. In the x_t market, the cross-sectional risk over one period is priced by a mean-variance efficient factor R_{MV}. Let Z be a security in the market; then, by definition, its total single-period return can be written as:

$$R(Z, x_t) = \frac{Z(x_{t+1}) + q_Z(x_{t+1})}{q_Z(x_t)}$$

The equation, written as single-period net return, becomes the pricing equation:

$$\mathbb{E}_\Pi R(Z) - 1 = i_1 + \beta(Z)[\mathbb{E}_\Pi R_{MV}(x_{t+1}) - 1 - i_1] \tag{13}$$

Here, i_1 is the one-period risk-free interest rate $\beta(Z) \coloneqq Cov(R(Z), R_{MV})/Var(R_{MV})$, and it is a function of x_t. This form is similar to the familiar CAPM, but the meaning of the so-called R_{MV} factor is much more complex. In simple terms, it can be explained as follows. In a complete market equilibrium, if we define the weight:

$$\alpha_h(x_t) = \frac{\beta_h^{t-1}}{1 + \beta_h + \cdots + \beta_h^{T-1}} P_h(x_t)$$

By solving the dynamic programming problem using backward induction, the solution to the consumer optimization problem can be written as:

$$c_h(x_t) = \frac{\alpha(x_t) W_h}{r(x_t)}$$

Combining it with pricing equation (2) and using consumption equal to dividends in equilibrium, and considering the transition probability likelihood ratio $\Lambda \coloneqq \Pi/\Gamma$, then, since rational investors satisfy $P_\hbar = \Pi$, the return on the investment portfolio I held from $s^{t-1} = s_i$ to $s^t = s_j$ is a multiple of the market return:

$$R_I(x_t) = \Lambda_{ij} R_\omega(x_t)$$

The mean-variance efficient factor R_{MV} is a nonlinear expression of the return of rational investors:

$$R_{MV} = \frac{1 - \zeta/R_I(x_t)}{\nu}$$

$$\zeta \coloneqq \left[\left(1+i_1(x_{t-1})\right)^{-1}-\nu\right]\{\mathbb{E}_\Gamma[R_\omega(x_t)^{-1}R_I(x_t)^{-1}|x_{t-1}]\}^{-1}$$

Here, ν is a positive variable parameter, the variation of which generates the mean-variance efficient frontier.

We will first discuss the apparent differences between BAPM and CAPM. In the above discussion, although we assume that rational investors naturally satisfy $P_\hbar = \Pi$, we do not assume that the market is consistent with reality, i.e., $\Lambda = 1$ does not necessarily hold, and there may be noise traders in the market. In fact, the $\Lambda = 1$ condition only holds when the market is efficient and pricing is effective.

If we let R^* represent the mean-variance factor R_MV when prices are efficient ($\Gamma = \Pi$), which we call the market factor, then at this time $R_I = R_\omega$, so R^* is a function of R_ω. Let $\beta^*(Z) \coloneqq Cov(R(Z), R^*)/Var(R^*)$, called the market beta of Z, then according to the ordinary CAPM, expected returns should satisfy:

$$E^*R(Z) = i_1 + \beta^*(Z)(\mathbb{E}_\Pi R^* - 1 - i_1) + A(Z)$$

Here, $A(Z)$ is the expected excess return. If pricing is efficient, then obviously $A(Z) = 0$. However, if pricing is inefficient, we still use R^* for pricing (which is equivalent to using R_ω for pricing), resulting in non-zero excess returns. In fact, risk pricing is based on R_{MV}, which in turn is based on R_I, and $R_I(x_t) = \Lambda_{ij}R_\omega(x_t)$. Noting that R_MV is a nonlinear relationship with respect to R_I, even if R_I and R_ω have a multiple relationship, R_{MV} and R^* cannot be equivalent. When pricing is inefficient, $\Lambda \neq 1$, so ordinary CAPM pricing will exhibit bias, which can be written as:

$$A(Z) = \left[\frac{\beta(Z)}{\beta(R^*)} - \beta(Z)\right](\mathbb{E}_\Pi(R^*) - 1 - i_1)$$

It is evident that this is a positive number. At this time, the non-zero $A(Z)$ will generate a mean-variance efficient line that is higher than the Capital Market Line (CML) given by the original CAPM.

Next, let's discuss how this model describes noise traders. Its description of noise traders is more detailed than that of DSSW. As the theme of this chapter, probabilistic beliefs are the core way in which noise traders influence asset pricing (they cause $\Lambda \neq 1$, leading to inefficient pricing). We assume that rational investors follow the Bayesian rule, so even if their subjective probabilities P_h initially deviate from Π, they will eventually converge to Π. However, this is not the case for noise traders.

The psychological evidence cited in this article is representativeness bias. It has two meanings: first, base rate neglect (underweighting the base rate information), which means that noise traders will pay more attention to recent information and underestimate the importance of more distant information (base rate), so recent events will bring "positive feedback"; second, sample size neglect (probability mismapping), as if flipping a coin four times in a row and getting heads, people think the probability of tails appearing next is higher, a series of recent similar events will bring "negative feedback". These two non-Bayesian learning rules make noise traders' private beliefs inconsistent with market beliefs.

However, since the biases of noise traders are not of the same type, there is a possibility of cancellation, and they can even exist in efficient markets, or rather, efficient markets may protect their existence. As an intuitive illustration, we can directly consider the steady state, where rational investors hold the market portfolio under an efficient market, and the wealth of positive and negative feedback noise traders only transfers between them, so noise traders merely increase the trading volume of the securities market. This extends the conclusions of DSSW and provides noise traders with a broader survival space.

Thus, the core of this paper is that, compared to DSSW's learning and holding period restrictions, BAPM shows that even if economic agents are given enough time to learn, representativeness bias in psychology may prevent noise traders from learning the correct pricing function through the Bayesian rule. However, the possibility of efficient market existence is created. The form of psychological evidence is not important; what is important is the direct impact of its intuition on our technical perspective in solving problems. For this paper, the inefficiency of market pricing does not actually require noise traders to avoid using Bayesian learning. As long as their measures deviate from the actual measures in any form, it will lead to inefficient pricing, which is the most crucial intuition of this article.

Bayesian Learning I: BSV Model

This section and the next discuss relaxations of the BAPM model, showing that even if noise traders are allowed to use Bayesian learning, they may still be unable to learn the correct model in the economy due to certain psychological biases.

This section discusses the BSV model, an explanatory model proposed by Barberis, Shleifer, and Vishny (1998) based on representativeness bias and conservatism bias. Although the model setup is interesting, it is non-standard and not easily extendable to more general forms of research.

Suppose there is only one asset in the economy with future income following a random walk:

$$N_t = N_{t-1} + y_t$$

where y_t is a Bernoulli variable with equal probability of taking the values $y, -y$. There is only one investor in the economy whose view of the income shock y_t follows a regime-switching model, which consists of two sub-models, depending on their judgment of the economic mechanism. These two sub-models are described by the state-transition matrices:

Mean-reversion model

$$y_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} y \\ -y \end{bmatrix} \rightarrow y_{t+1} = \begin{bmatrix} y_{1,t+1} \\ y_{2,t+1} \end{bmatrix} = \begin{bmatrix} y \\ -y \end{bmatrix} : [\pi_H^{ij}] = \begin{bmatrix} \pi_H & 1-\pi_H \\ 1-\pi_H & \pi_H \end{bmatrix}$$

Trend-generating model

$$y_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} y \\ -y \end{bmatrix} \rightarrow y_{t+1} = \begin{bmatrix} y_{1,t+1} \\ y_{2,t+1} \end{bmatrix} = \begin{bmatrix} y \\ -y \end{bmatrix} : [\pi_L^{ij}] = \begin{bmatrix} \pi_L & 1-\pi_L \\ 1-\pi_L & \pi_L \end{bmatrix}$$

where: $\pi_H < 0.5 < \pi_L$. The subjective switching mechanism between mechanisms 1 and 2 is:

$$s_t = \begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow s_{t+1} = \begin{bmatrix} s_{1,t+1} \\ s_{2,t+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} : [\lambda_{ij}] = \begin{bmatrix} 1-\lambda_1 & \lambda_1 \\ \lambda_2 & 1-\lambda_2 \end{bmatrix}$$

To predict future income streams for asset valuation, the investor must judge which mechanism they are in by observing realized income. Suppose they observe the shock y_t at time t, they calculate the probability π_t that y_t is generated by Model 1, specifically:

$$\pi_t \coloneqq \mathbb{P}\{s_t = 1 | y_t, y_{t-1}, \pi_{t-1}\}$$

Assume that the investor updates using the Bayesian rule (meaning that if they observe $y_{t+1} = y_t$, then $\pi_{t+1} < \pi_t$; if they observe $y_{t+1} = -y_t$, then $\pi_{t+1} > \pi_t$). Although theoretically, after infinite periods of learning, people will eventually realize that this is a random walk model, numerical simulation results show that it may take decades to converge to the true model if people observe stock price data quarterly or semiannually. This demonstrates that under a regime-switching belief, even with a rational learning process, it is difficult to have rational expectations.

Under this model, the pricing equation is:

$$P_t = \mathbb{E}_t\left\{\sum_{i=1}^{\infty}\frac{N_{t+i}}{(1+\delta)^t}\right\} = \frac{N_t}{\delta} + y_t(p_1 - p_2\pi_t) \tag{14}$$

where: p_1, p_2 are constants that depend only on $\pi_H, \pi_L, \lambda_1, \lambda_2$. This equation is clearly deviated from the fundamental value (the first term is the fundamental value). If we formally define representativeness bias (or overreaction) and conservatism bias (or underreaction) as:

$$\mathbb{E}_t\big(P_{t+1} - P_t | y_t = \cdots = y_{t-j} = y\big) - \mathbb{E}_t\big(P_{t+1} - P_t | y_t = \cdots = y_{t-j} = -y\big) < 0$$
$$\mathbb{E}_t(P_{t+1} - P_t | y_t = +y) - \mathbb{E}_t(P_{t+1} - P_t | y_t = -y) > 0$$

As long as the parameters $\pi_H, \pi_L, \lambda_1, \lambda_2$ satisfy:

$$\underline{k}p_2 < p_1 < \bar{k}p_2$$

where: $\underline{k}, \bar{k}$ are normal numbers that depend only on $\pi_H, \pi_L, \lambda_1, \lambda_2$. Then this pricing equation can reflect both overreaction and underreaction, providing an interpretation of these two psychological biases from a rational perspective and laying a foundation for market price dynamics (although it does not specify when overreaction and underreaction occur). Conversely, the existence of these two psychological biases implies that many "smart" noise traders in the real market who learn using the Bayesian rule are likely to make security valuations based on this "fence-sitting" view formation mechanism.

The BSV model advances the intuition of the BAPM model, suggesting that even if noise traders are allowed to use the Bayesian rule, it is nearly impossible for them to learn the correct model if both representativeness bias and conservatism bias are present, thereby disrupting the subjective measures of the entire market.

Bayesian Learning II: DHS Model

In this section, we discuss the DHS model, another psychology-based model proposed by Daniel, Hirshleifer, and Subrahmanyam (1998). Compared to the BSV model, the DHS model provides further insight into the timing of overreaction and underreaction, offering model support for intermediate momentum and long-term reversals in asset prices.

We directly elaborate on the complete DHS dynamic model. This non-standard model assumes that there are two types of investors in the market: informed investors (I), who are risk-neutral, and uninformed investors (U), who are risk-averse. All individuals are overconfident, meaning they overestimate the accuracy of the signals they receive. Everyone also exhibits self-attribution bias.

The intrinsic unobservable value of a company's stock is given by:

$$\tilde{\theta} \sim N\left(0, \sigma_\theta^2\right)$$

Where σ_θ^2 is public information. Each period, a signal is generated in the market regarding the asset price, resulting in a random walk for the overall price. In the first period, private information is generated, which only informed investors (I) receive:

$$\tilde{s}_1 = \tilde{\theta} + \tilde{\epsilon}$$

Where $\tilde{\epsilon} \sim N(0, \sigma_\epsilon^2)$. $2 \sim T$ period, public signals are generated:

$$\tilde{\phi}_t = \tilde{\theta} + \tilde{\eta}_t$$

Where $\tilde{\eta}_t \sim iidN\left(0, \sigma_\eta^2\right)$, and σ_η^2 is also public knowledge. Based on these public signals, we can define a historical average signal:

$$\Phi_t := \frac{1}{t-1}\sum_{\tau=2}^{t}\tilde{\phi}_t = \theta + \frac{1}{t-1}\sum_{\tau=2}^{t}\tilde{\eta}_t$$

It is the sufficient statistic for all past $t-1$ public signals. Based on this historical average signal, we can define self-attribution bias. For informed investors (I) with private information, in the first period, they believe their signal's accuracy, $v_{C,1} := 1/\sigma_C^2$, is greater than the true accuracy, $v_\epsilon := 1/\sigma_\epsilon^2$. However, as time progresses, they adjust this belief using the Bayesian rule. Specifically, when confirmatory signals appear:

$$sgn(s_1 - \Phi_{t-1}) = sgn(\phi_t - \Phi_{t-1})$$
$$|s_1 - \Phi_{t-1}| < 2\sigma_{\Phi,t}$$

Their adjustment rule significantly increases their confidence:

$$v_{C,t} = (1+\bar{k})v_{C,t-1}$$

If a disconfirming signal appears, the aforementioned condition is not met, and their adjustment rule slightly reduces their confidence:

$$v_{C,t} = (1-\underline{k})v_{C,t-1}$$

Where $\sigma_{\Phi,t}$ denotes the standard deviation of Φ at time t, $\bar{k} > \underline{k} > 0$, and the ratio $(1+\bar{k})/(1-\underline{k})$ reflects the severity of self-attribution bias. In the market, both I and U trade each period, with the risk-neutral I determining the transaction price through the pricing equation:

$$\widetilde{P}_t = \frac{(t-1)v_\eta\Phi_t + v_{C,t}s_1}{v_\theta + v_\eta + v_{C,t}} \tag{15}$$

Where $v_\theta := 1/\sigma_\theta^2$, $v_\eta := 1/\sigma_\eta^2$. For this price, given a positive shock $s_1 = 1$, the impulse response function obtained through numerical simulation effectively captures the medium-term momentum and long-term reversal.

Like the BSV model, the DHS model does not assume that noise traders (I in this model) cannot use the Bayesian rule to update their beliefs. Unlike the BSV model, which only discusses public information, the intuition provided by the DHS model is that even if noise traders use the Bayesian rule, incorporating overconfidence and self-attribution bias in an economy with both private signals and subsequent public signals will prevent them from learning the correct model, as they will always be overly confident in their private information, thus underestimating the variance of that information. Furthermore, these two psychological effects allow our intuitions to better fit the dynamics of real-world prices.

Limited Information Set: HS Model

The previously mentioned DHS and BSV models, based on different psychological evidence, provide corresponding non-standard local equilibrium models for modification, thereby explaining some characteristics of price dynamics. Hong and Stein (1999) proposed the HS model, shifting the perspective. They argue that there is an abundance of psychological evidence, too much to be fully captured; perhaps another angle could yield a more general model.

In the HS model, they initially disregard the presence of rational investors. They hope to generalize the intuition provided by psychological evidence through the concept of "bounded rationality," meaning that people's information processing and abilities are limited. Noise traders are divided into two categories based on the type of information processed: news watchers and momentum traders.

Each news watcher can observe some private information but lacks the ability to extract information from prices or predict future price trends based on complex models. This limited rationality manifests as follows: each period t, they form their asset demand based on the principle of static optimization, holding the asset until maturity T; their conditional information set contains acquired private information but not current or past prices.

This setup is similar to the "fundamental analysts" in practice, who form valuations based on fundamental information rather than price itself. Their financial modeling costs are high, so after conducting a valuation, they hold the asset long-term based on the formed benchmark price, performing static optimization. Thus, the equilibrium discussed is a Walrasian equilibrium based on private valuations, not a fully dynamic rational expectations equilibrium.

Momentum traders, on the other hand, enter the market each period t, hold a position for j periods (until $t+j$), where j is exogenous. They submit market orders without knowing the execution price, which is determined by the competition among news watchers (who also act as market makers). Thus, momentum traders predict $P_{t+j} - P_t$ to determine the size of their orders. Their order volume (demand) is entirely accepted by news watchers, who treat it as an unexpected supply shock. They can only base their strategies on price data, forming simple models related to price (assuming they cannot handle overly complex models).

This setup resembles the "technical analysts" in practice, who predict price fluctuations by calculating technical indicators from observing candlestick charts. However, these indicators are simple functions of past prices, and they mainly focus on recent price movements. Technical analysts also typically have shorter holding periods.

Assuming there is only one risky asset in the market, with a fixed supply of Q, the asset pays a dividend at time T. The final value of this dividend is updated with each arrival of new information, following a random walk:

$$D_{t+1} = D_t + \epsilon_{t+1}, \qquad \epsilon \sim iidN(0, \sigma^2)$$

Now, consider how private information diffuses among news watchers. News watchers are divided into z completely symmetric groups. Each dividend update ϵ_j can be divided into z symmetric, independent sub-updates, each with a variance of σ^2/z, satisfying: $\epsilon_j = \epsilon_j^1 + \cdots + \epsilon_j^z$. At time t, the information ϵ_{t+z-1} is released. At this point, the first group observes ϵ_{t+z-1}^1, ..., and the z th group observes ϵ_{t+z-1}^z. At time $t+1$, the information circulates once, and the first group observes ϵ_{t+z-1}^z, ..., while the z th group observes ϵ_{t+z-1}^1. Until time $t+z-1$, the complete information ϵ_{t+z-1} is observed by all groups, becoming fully public. The parameter z represents the (linear) rate of information flow; the larger it is, the slower the information transmission.

Considering all individuals possess CARA utility, when only news observers exist, the price is equal to the terminal dividend value in period t, which depends on the sum of the information observed by all groups (note the perfect symmetry assumption). Thus, the pricing equation is:

$$P_t = D_t + \left[\frac{z-1}{z}\epsilon_{t+1} + \cdots + \frac{1}{z}\epsilon_{t+z-1}\right] - \theta Q \tag{16}$$

Here, γ is a function of ARA and σ^2. By adjusting ARA, we normalize θ to 1. Under rational expectations equilibrium and random walk assumptions, the rational pricing function should be:

$$P_t^* = D_{t+z-1} - Q$$

Hence, Equation (16) exhibits underreaction: the price will not exceed its long-term value, or given a positive unit shock, the price impulse response at time t is less than 1, thereby forming momentum to a certain extent.

Introducing momentum traders and assuming they predict the only variable for $P_{t+j} - P_t$ is the past one-period accumulated price change $P_{t-1} - P_{t-2} = \Delta P_{t-1}$, a univariate prediction model, we can write the order volume for t-period momentum traders as:

$$F_t = A + \phi \Delta P_{t-1}$$

Here, the constant A and the elasticity index ϕ are determined by the optimization on the momentum traders' side. As there are j-period momentum traders in the market at any time, the current pricing function is:

$$P_t = D_t + \left[\frac{z-1}{z}\epsilon_{t+1} + \cdots + \frac{1}{z}\epsilon_{t+z-1}\right] - Q + jA + \sum_{i=1}^{j} \phi \Delta P_{t-i} \qquad (16')$$

Ignoring the constants A and Q, the only value to optimize is ϕ, and the optimization condition for the momentum traders' side can be derived as:

$$\phi \Delta P_{t-1} = \frac{\gamma \mathbb{E}_M\left(P_{t+j} - P_t\right)}{Var_M\left(P_{t+j} - P_t\right)}$$

Here, γ is the ARA, and the moments in the conditions all rely on ΔP_{t-1}. We can rewrite it as:

$$\phi = \frac{\gamma Cov\left(P_{t+j} - P_t, \Delta P_{t-1}\right)}{Var(\Delta P_{t-1}) Var_M(P_{t+j} - P_t)}$$

Thus, we define the equilibrium discussed here as the covariance stationary equilibrium, which is the solution to the following system of equations:

$$\begin{cases} \text{Price Dynamics:} & P_t = D_t + \left[\frac{z-1}{z}\epsilon_{t+1} + \cdots + \frac{1}{z}\epsilon_{t+z-1}\right] - Q + jA + \sum_{i=1}^{j} \phi \Delta P_{t-i} \\ \text{FOC of M:} & \phi = \frac{\gamma Cov\left(P_{t+j} - P_t, \Delta P_{t-1}\right)}{Var(\Delta P_{t-1}) Var_M(P_{t+j} - P_t)} \\ \text{Covariance Stability:} & |\phi| < 1 \end{cases}$$

When γ is sufficiently small, this equilibrium has a unique solution and provides a good representation of many real-world price dynamics. Fundamentally, in a covariance stationary equilibrium, there must be $\phi > 0$, which implies that momentum traders act as trend-followers or positive feedback traders, thereby forming stronger medium-term momentum. This equilibrium, in addition to the underreaction reflected in Equation (16), also includes:

Property 1: Overreaction - In a covariance stationary equilibrium, given a positive unit shock ϵ_{t+z-1} at time t, the following holds:

1. The cumulative impulse response function of price peaks strictly greater than 1;
2. If $j \geq z - 1$, the peak occurs at $t + j$ and converges to 1 afterward;
3. If $j < z - 1$, the peak occurs no earlier than $t + j$ and converges to 1 afterward.

Property 1 indicates that in response to good news, prices continue to rise for at least j periods, exhibiting medium-term momentum.

Property 2: Unconditional zero profit - If there is short-term positive autocorrelation in prices, i.e., $cov(\Delta P_{t+1}, \Delta P_{t-1}) > 0$, under risk-neutral conditions for momentum traders ($\gamma = \infty$), negative autocorrelation in prices will occur within $j + 1$ periods, i.e., there exists i such that $cov(\Delta P_{t+i}, \Delta P_{t-1}) < 0,\ i \leq j$.

Property 1 demonstrates a conditional strategy - if momentum traders know there will be good news in period t, they can profit by buying in advance (however, this is not feasible as the news shock is not in the momentum traders' information set). Property 2 demonstrates an unconditional strategy: if momentum traders follow short-term price increases, interpreting them as good news, their purchases can lead to negative price externalities and eventual long-term reversals, resulting in an average profit of zero. Subsequent extensions by Hong and Stein indicate that even the presence of risk-averse, rational arbitrageurs cannot eliminate underreaction, overreaction, and momentum trading.

Mathematically, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consists of several elements, with Ω representing the basic state set under study, an immutable element. DSSW, BAPM, BSV, and DHS essentially manipulate the probability measure P. DSSW and DHS directly modify moments (since moment-generating functions are equivalent to probability measures or distributions), while BAPM and BSV directly alter $\mathbb{P}(\omega)$ values (albeit in the form of transition probabilities). The HS model, on the other hand, focuses on F and determines the conditional expectation of prices. While calculating the conditional expectation $\mathbb{E}_t\{p|\mathcal{F}_t\}$, the measure remains standard, but $\mathcal{F}_t$ is restricted within a narrow range, causing deviations from rational expectations in pricing models.

Conclusion

As budget constraints are typically standard and intuitive, recent behavioral finance adjustments have focused primarily on consumer objective functions. These functions can be divided into stochastic and non-stochastic components. Non-stochastic components include single-period utility functions $U(\cdot)$ and intertemporal discount factors β. Refinements in this area generally involve making these components more intricate, dynamic, and random. Models based on these adjustments operate within the Lucas standard framework, yielding homogenous general equilibrium and explaining static anomalies in asset pricing (e.g., equity premium puzzle, risk-free rate puzzle, and volatility puzzle). Although the introduction of wealth spirits is an early development, the latest work, the BHS model, employs prospect theory and outperforms its counterparts. Comparatively, the incorporation of consumption habits explains some static anomalies but generates new ones, while the introduction of herding phenomena fails to explain any anomalies, and subjective discount factors only clarify the volatility puzzle. Careful consideration is required when determining what to include in utility to avoid rendering the framework trivial and to ensure a solid theoretical foundation.

The stochastic component mainly refers to the subjective probability measure Γ used to generate expectations for consumers. General adjustments in this area introduce heterogeneity. DSSW, BAPM, BSV, and DHS incorporate heterogeneity from probability measure $\mathbb{P}$, while the HS model introduces it from information set $\mathcal{F}$. Except for BAPM, these models do not fit within the Lucas standard framework and generally result in partial equilibria or even static Walrasian equilibria. However, due to their strong psychological evidence, these models have robust explanatory power, particularly for dynamic anomalies in asset pricing (e.g., overreaction, underreaction, intermediate momentum, and long-term reversals). Among these models, DSSW serves as the foundation for many papers, and although its structure is simple, it can explain various static anomalies, an ability lacking in other models in this direction.

As observed, adjustments within the standard model become increasingly difficult to simulate and implement due to the refinement and complexity of assumptions, risking pandering to specific cases. In contrast, non-standard model adjustments entail simple, clear assumptions and models but face the issue of disparate opinions and a lack of unification. Future research may focus on developing general equilibrium models under non-standard assumptions, an idea of mine.

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